Descriptive Geometry

for Students in

Engineering Science

and

Architecture

A CAREFULLY GRADED COURSE OF INSTRUCTION

BY

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PREFACE

THE writer offers the contents of this Text Book as the result of over twenty years' constant teaching of Descriptive Geometry, and because he has found that a logical presentation of the subject, concerning itself largely with the best sequence possible, and with the introduction, at an early stage, of practical applications and well-graded exercises by which the first rules and principles can be practised, is far more preferable and satisfactory in its results than the mode of procedure and division of the subject matter commonly adopted in other text books on the subject.

Difficult and complex phases of Descriptive Geometry and problems beyond what the average student has time to assimilate during a limited college course, and beyond what is worth while in such a course, are avoided, because of the tendency for students to become thereby discouraged and to acquire a distaste for what should otherwise prove to be an interesting subject of study.

No pretense, therefore, has been made to place before students an exhaustive treatise on the subject, but rather to deal with the essential facts and methods readily arrived at and their useful application. To this end, the student is frequently required to work, independently, exercises on the problems of which illustrations are given and discussed, and himself to propose questions or data for new exercises to be worked.

It will be noticed that the study of the subject is conducted in such a way that Part I of the Text Book may serve as an Introductory Course, more or less complete in itself, and may be sufficient and suitable for a first term in a Science Course Curriculum, or for the upper forms in High Schools; or, Parts I and II may be taken in the First Year and Part III in the Second Year of a University or College Course in Science, if in the First Year there is not provided sufficient time for the whole to be undertaken.

By the study of this subject, the imaginative faculty of the mind, so little exercised in the ordinary school curriculum, is afforded a good training, while the practice in grasping a collection of detailed conditions before commencing the working out of a solution, proves invaluable to the would-be engineer or architect in his future career.

January, 1915.

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DESCRIPTIVE GEOMETRY

PART ONE

CHAPTER I

THE PROJECTION PLANES

SECTION 1. In Descriptive Geometry the object is chiefly to prepare drawings as follows:—

(*a*) Those which will display or describe by *different views* any object or arrangement of lines or figures discussed;
(*b*) Those which will, by various analytical and constructive methods and operations, discover or disclose facts as to shapes, inclinations, appearances, sizes, etc.; and
(*c*) Those which will represent planes and how they may be disposed to one another.

The views mentioned above in (*a*) are *projections*, and are made on what are called *planes of projection.* The same projection planes, two in number, are also made use of in the discussion of planes referred to in (*c*), lines being drawn over the planes of projection and made to represent other planes in various attitudes with respect to the projection planes.

The planes of projection are the *Horizontal Plane* and the *Vertical Plane*. These are considered as being *fixed*, and the lines, planes, figures or objects are considered as having a relation to them—near or otherwise as to *distance*, inclined or otherwise as to *attitude*.

The drawings made either represent points, lines, figures or objects by views thrown perpendicularly on to these planes of projection (the H.P. and the V.P. as they are commonly called), or they indicate the intersection of the planes of projection by lines and planes.

In order that the drawings may be descriptive of what is under consideration it is usually necessary to have one drawing of the same thing on each of the planes of projection, so that, whatever it is, it may be studied from different points of view, or better realized by the study of these drawings on the two planes of projection.

In the case of projections it is advisable and convenient to place one drawing representing the view projected perpendicularly on to the H.P. and another drawing representing the view projected perpendicularly on to the V.P. These drawings, projected perpendicularly from whatever is under discussion, will arrange themselves perpendicularly opposite each other.

For convenience we may, while dealing with simple and elementary problems, use one part of the drawing paper, or blackboard, upon which to make the drawings or projections thrown on to the H.P. These drawings are called the horizontal projections, or, more commonly, the *plans*. The other part of the drawing paper, or blackboard, may be used for making the drawings or projections thrown on to the V.P., called therefore the vertical projections, or, more commonly, the *elevations*.

In order to make use of the drawing paper effectively in this way, a line is drawn across it, and this is to represent the intersection line in which the V.P. and the H.P. meet. We name this line XY, and usually the lower part of the paper then represents the near part of the H.P., while the upper part represents the upper part of the V.P. The line XY, however, must not be thought to limit the extent of either of the planes, for it may sometimes be necessary to display something which would not project on to the near part of the H.P., and then the upper part of the paper would *also* represent the H.P., that is, the part of it beyond the V.P.'s position. Similarly, the lower part of the paper may sometimes be required for projections on the V.P. from things placed below the level of the H.P.

It must continually be remembered that the part of the drawing paper upon which vertical projections or elevations are made is to be considered actually vertical, and that the part of the paper used for the horizontal projections or plans must always be considered horizontal. Sometimes it may be well to fold the drawing paper along the XY line and then hold the V.P. part vertical, while the H.P. part remains horizontal. Try this.

The student must learn as soon as possible to picture to himself, or arrange, simple things in different positions and attitudes in respect to the two projection planes arranged as above decided upon, one vertical and the other horizontal, and imagine what should be the views projected to these planes and how the views or projections will change with a change of attitude.

When it is realized that this process of projection is perpendicular to the projection planes and not like that of the camera, then it will be seen that the size of the projection, plan or elevation, as the case may be, will not depend on distance.

The hand, resting on its edge on the table top, and arranged with the palm vertical, might be used to represent roughly the V.P. of projection, and the table top to represent the H.P. of projection. Now hold things with the other hand, and consider what views or appearances (elevations and plans) you might get, projected perpendicularly on to the palm of the hand and on to the table top respectively. The line across the table top where the vertical plane of the palm of the hand meets it, will be the *XY* line named above, and it will be seen that any two drawings of an object which is held still, the one drawing on the table top and the other on the palm of the hand, are perpendicularly opposite each other across the *XY* line, when the hand is thrown back about the *XY* line as a hinge.

We have seen that there may be plans or horizontal projections and elevations or vertical projections of points, lines, figures and objects, but it must be pointed out that there is no such thing as the plan or the elevation of a plane. A plane, however, other than the planes of projection, may be represented by a line or lines drawn on the projection planes to mark its intersection with them.

EXERCISE I

1. Distinguish between attitude and position.
2. What are plans and elevations, and how are they obtained?
3. What is the *XY*? What is meant by H.P. and by V.P.? What is the proper relation of the planes of projection to one another?
4. How can the same sheet of paper serve for both planes of projection?
5. How are planes, other than the projection planes, represented?
6. How are a plan and an elevation of an object arranged so as to show that they are views of the same object?

By further investigation on the same lines as suggested above, it will be seen that the distances of an object held away from the planes of projection are the same as the distances of the drawings or projections of it from *XY*. In other words, the plan at 3″ from *XY* will mean that the thing represented is 3″ from the V.P., and the elevation at 2″ from *XY* will mean that the thing is 2″ from the H.P.

When an object is on the near side of the V.P. it is said to be in front of the V.P., and its plan, on the paper, will be placed below the *XY* line.

EXERCISE II

1. Let the projection planes, for the purpose of this exercise, be represented by, say, a large book, standing on the table, for the V.P., and by the table top for the H.P., then arrange a book or other rectangular object, in such attitudes, with respect to the projection planes, as to satisfy the following descriptions:—

(1) The book, or flat rectangular box, placed horizontally with all its edges equally inclined to the V.P.
(2) The same, so that its true form (rectangle) will appear in elevation.
(3) The same, so that its appearance will be a rhomboid on both planes.
(4) The same, so that one edge is in the V.P. at, say, 45° to *XY*, and the surface of it is at, say, 60° to the V.P.

(5) The same, so that both projections are lines only, supposing the book to be very thin.

(6) The same, so that its plan and elevation are both rectangles.

2. Make sketch or freehand drawings, on a paper or blackboard, not carefully measured or scaled, as plans and elevations, to satisfy the descriptions above, and insert distances from the planes of projection, wherever possible.

3. How are distances from the planes of projection shown, of any points in an object whose plan and elevation are required or given?

LINES AND THEIR INCLINATIONS

SECTION 2. It will now be seen that any object, such as, for instance, a book, may have many different sets of drawings (a set = plan and elevation), differing according to its attitude or arrangement in relation to the projection planes, to represent it, and that, given any such set of drawings for the book, the book may

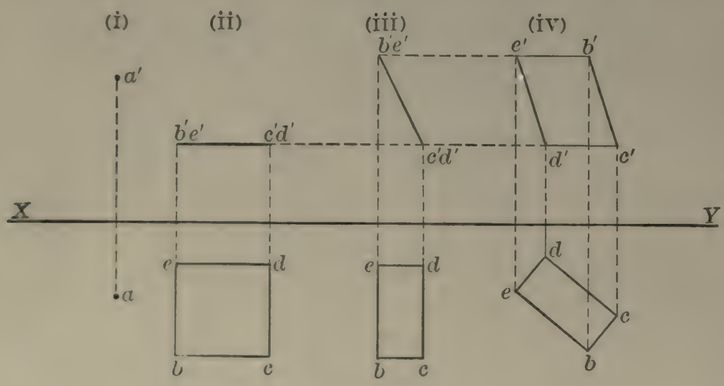


FIG. 1.

be held in the proper way to give, or provide for, that set of drawings or projections.

In order to distinguish plans from elevations, it is customary, for purposes of discussion, to use small letters a, b, c, etc., on the plans, and the same small letters, with a short stroke or dash as a', b', c', etc., for the same points named on the elevations. Capital letters A, B, C, etc., are used for the *names* of points regardless of their projections.

Illustrations of projections or plans and elevations are here given, which must be thoughtfully studied, while many facts are noted.

In Fig. 1 at (i) the point A is represented at a and a' and has a distance above the H.P. equal to twice that of its distance in front of the V.P. At (ii) the square $BCDE$ is arranged horizontally with edges BC and DE parallel to the V.P. The distance of the elevation above XY shows the distance of the square above the H.P. At (iii) the square still has its edges BC and DE parallel to the V.P. and its other edges still horizontal, i.e., parallel to the H.P., but the square is inclined to the H.P., and the angle the elevation line makes with *XY* is the angle the figure makes with the H.P. The figure is, of course, still perpendicular to the V.P. At (iv) the plan, the same shape and size as the plan at (iii), has been so turned that *BC* and *DE* are no longer parallel to the V.P., but as these edges are still inclined the same amount as before, to the H.P., the difference of level of their ends has not been altered, and so the elevation can be derived, partly, from the elevation at (iii). Realize that the edges *CD* and *BE* are still horizontal, but inclined to the V.P., the angle of inclination to the V.P. being that which *cd* makes with *XY*.

N.B.—A horizontal line may have any direction, for its plan, across the paper.

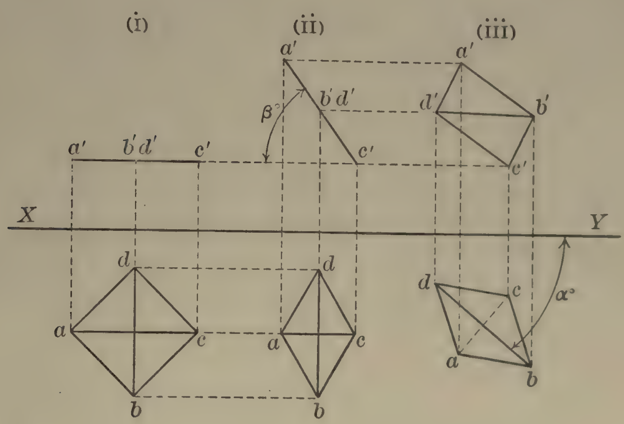


FIG. 2.

Consider next the group of drawings in Fig. 2. At (i) a square is represented horizontal, at a distance above the H.P. and a little in front of the V.P.; one diagonal is perpendicular to the V.P. and is represented in elevation at *b'd'*. The full length of the diagonal *AC* is shown in both the plan and the elevation. At (ii) the elevation of (i) has been inclined to give an inclination of the figure to the H.P., the point at *b'd'* still representing the horizontal diagonal whose plan *bd* therefore shows true length; the plan of the other diagonal, however, is now short of its true length, due to its inclination. This diagonal is said to be *foreshortened*, that is, arranged so that the view of it does not show its true size.

Now, if without altering the size and shape of the plan of this square as seen in Fig. 2 at (ii), it be moved so that the horizontal diagonal *BD* be at an inclination to the V.P. represented by the angle which *bd* makes with *XY*, it will be seen that the relative levels of the various points *ABCD* will remain as before, and hence a new elevation for set (iii) may be obtained from the plan as now arranged at (iii) and by making use of the levels from the elevation of (ii). The square may now be described as having a diagonal horizontal and at an angle ($\alpha°$) to the V.P., and the figure as having an inclination ($\beta°$) to the H.P. It will be seen that in order to obtain set (iii) it is necessary to first make sets (i) and (ii).

From further inspection of Fig. 1 and Fig. 2 it will be seen that a line, in order to show its true length in a projection, must be arranged parallel to the plane upon which the projection is thrown; and, similarly with regard to plane figures, i.e., for a figure to show its full size and shape in a projection, it must be arranged parallel to the plane upon which the projection is made.

This matter is still further illustrated and discussed in Fig. 3, where it will be seen that the plans in (i) and (ii) and (iii) remain full length because the line is horizontal in each case, whereas the elevations vary, because the relation of the line to the V.P. varies.

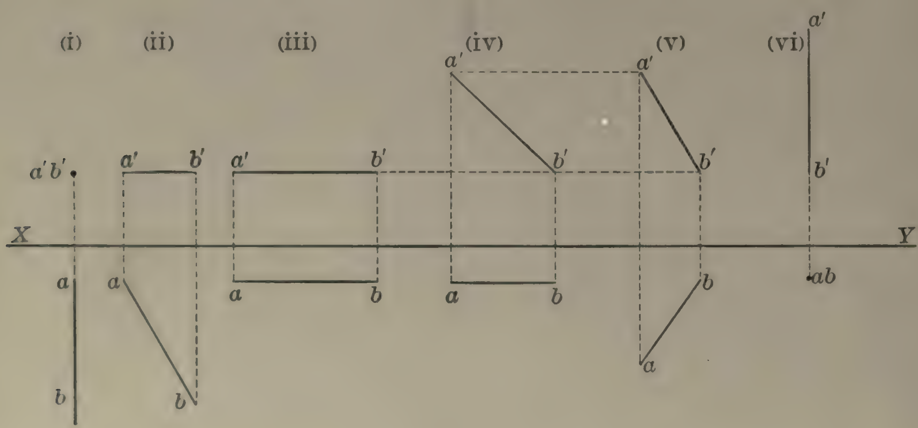


FIG. 3.

In (iii) the plan and elevation are both full size. In (iv) the elevation is full size because the line is still parallel to the V.P. as in (iii), and as shown by the arrangement of its plan, now foreshortened, however, because the line is inclined to the H.P. In (v) the plan is kept the same length as in (iv) and consequently the levels for the ends of the elevation may be taken from the elevation in (iv). At (vi) the line is vertical.

It will readily be seen that in Fig. 3 at (ii) the angle the line makes with the V.P. is the angle that ab makes with XY, and that in (iv) the angle the line makes with the H.P. is the angle $a'b'$ makes with XY.

Realize that although some of its projections are inclined to XY, the line AB itself is not inclined to XY.

If now it be required to discover the angles the line AB, in Fig. 3 as shown at (v), makes with the projection planes, it is evident that the angle it makes with the H.P. is not what its elevation at (v) makes with XY, but what its elevation at (iv) makes with XY, where the plan is arranged parallel to XY; and in order to find what inclination the line AB has with the V.P. its elevation $a'b'$ (v) must be arranged horizontally as at (ii), and then its plan, the full length of the line, will show with XY the angle required.

Let it be required to find the true lengths of any lines AB, Fig. 4, and CD, Fig. 5, and also their inclinations to the planes of projection. Their plans and elevations are shown at ab, $a'b'$, Fig. 4 and at cd, $c'd'$, Fig. 5. The line in each case must be arranged parallel to the V.P. by swinging its plan to become parallel to XY as at ab_2, Fig. 4, and at cd_2, Fig. 5, with the result, when the elevation

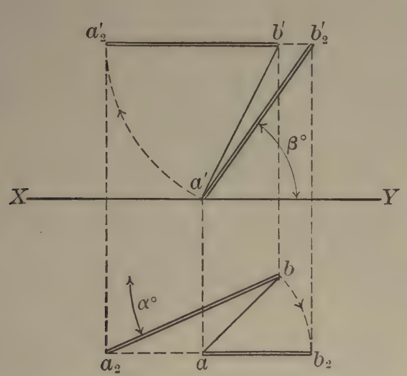


FIG. 4.

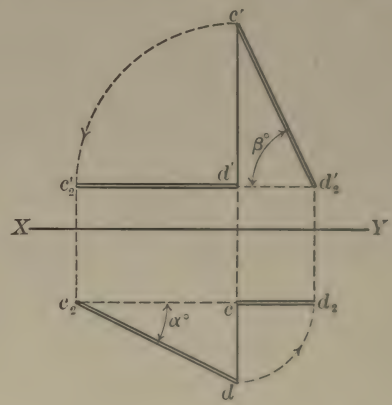


FIG. 5.

opposite this new arrangement of the plan in each case is set up, that the true length $a'b'_2$ and $c'd'_2$ respectively, is shown on the V.P., and the inclination $\beta°$, of the line, to the H.P. may now be seen; also, the line in each case must be arranged parallel to the H.P. by swinging its elevation parallel to XY, and so providing for a plan, a_2b and c_2d in the cases being considered, that will show true length and contain an angle, $\alpha°$, with XY, equal to the angle the line makes with the V.P.

N.B.—The secondary positions for the lines in Figs. 4 and 5 are shown by double lines to emphasize them.

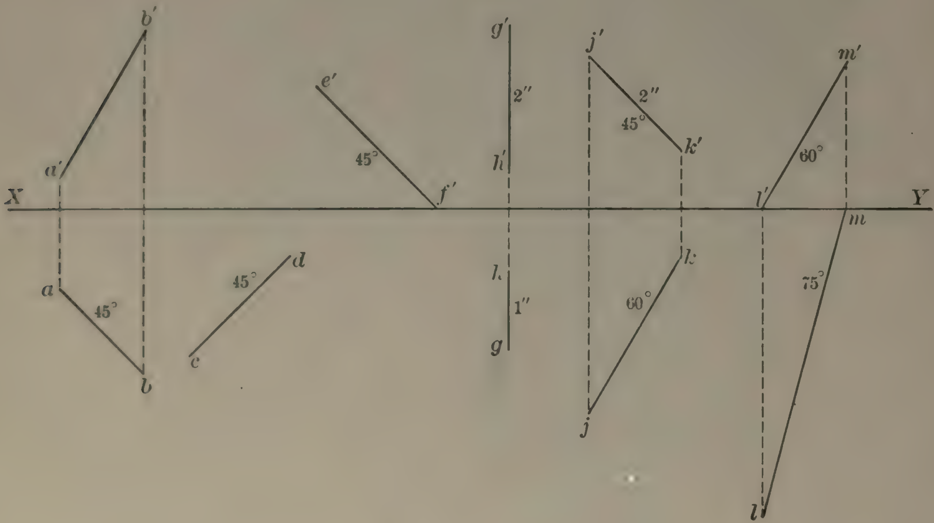
EXERCISE III

1. Place the plan and elevation of the line AB, making the plan ab 2″ long and the elevation 3″ long. Find the true length of this line, and also its inclinations to the planes of projection. Mark the angles as in Fig. 4 with arrow-headed arcs and use α to indicate the angle with the V.P., and β for the angle with the H.P.

2. The plan cd is 2″ long. The line CD is 3″ long. Draw an elevation for it, and show what angles the line CD makes with the planes of projection.

3. The elevation of EF is $2''$ long, and is at 45° to XY. The line is at 30° to the H.P. Find a plan for it, and also determine its true length.

4, 5 and 6. Find the true length of each of the given lines GH, JK and LM, and their inclinations to the planes of projection.



CHAPTER II

PLAN AND ELEVATION FORMS FOR PLANE RECTILINEAL FIGURES

SECTION 3. Suppose it be required to find the projections of any plane figure inclined at a given angle to one of the projection planes and having one of its edges in that projection plane at a given inclination to the XY. Illustrations of such a problem are shown in Fig. 6 and Fig. 7.

In Fig. 6 a square is chosen and is first placed, as at (i), in the V.P. Its plan is therefore part of the XY line. In order that it may swing out at the required

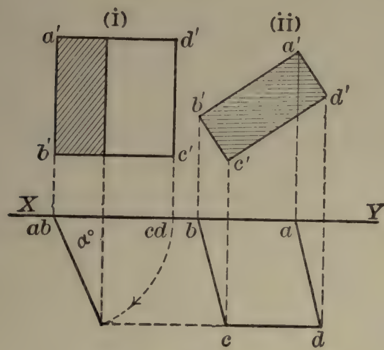


FIG. 6.

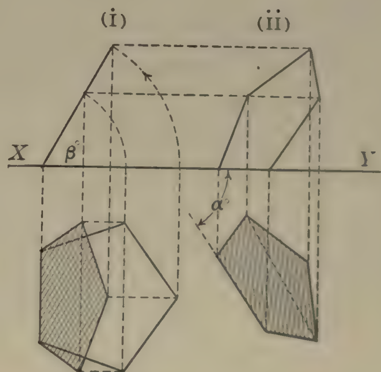


FIG. 7.

angle, and yet leave one edge in the V.P., it must be arranged so that when the line representing it in plan is moved to enclose the given angle $\alpha°$ with XY, one edge of it, say AB, represented in elevation at $a'b'$, is in plan a point, as at ab, and will remain as a point in XY. The shaded or "hatched" rectangle is now the elevation shape required. This projection shape is the first thing to obtain toward the solution of such a problem as the above.

If now this elevation shape be rearranged so that the edge AB is at the given angle to the XY, as at (ii), then it will be seen that a rhomboidal figure will be obtained for plan. The distance of the plan line cd in (ii) from XY will be the same as the distance it arrived at in (i), since the elevation shape is the same in both cases and therefore the relation of the figure to the V.P. is unchanged.

In Fig. 7 the case of the regular pentagon should be studied. The pentagon, as finally arranged at (ii), may be described as being inclined at β° to the H.P. and having one edge in the H.P. at α° to XY. The mode of procedure is similar to that for the square in Fig. 6.

N.B.—It is advisable that the student, in order to comprehend clearly this matter of projections, should frequently fold his drawing paper on the XY line and stand that part of the paper with elevations on it, vertically, so that the planes of projection are rightly related. He will then realize how the projections are accounted for.

Before working Exercise IV on paper, it may be well for the student to cut a square, and also a rectangle, out of a piece of cardboard, and practise placing them according to the questions. Realize that the projections or views required are thrown always perpendicularly on to the planes of projection.

EXERCISE IV

1. A rectangle $2''$ by $\frac{3}{4}''$ is the plan for a square. Show an elevation for it, (*a*) when an edge of the square is in the H.P. and perpendicular to XY, (*b*) when an edge in the H.P. is at 45° to XY.

2. A square of $2''$ edge is the plan for a rectangle having its long edges $3''$. Arrange the square plan so that edges representing the short edges of the rectangle are at 30° to the V.P., and then obtain an elevation.

3. Find the plan and elevation of a square, of $2''$ edge, inclined at 60° to the H.P and having one edge horizontal at a distance of $1''$ above the H.P. Place the square so that the horizontal edges are parallel to the V.P. and in front of it.

4. Find the plan and elevation of a $2''$ square when one diagonal of it is horizontal and at 45° to the V.P., while the figure is inclined at 60° to the H.P. (This is the same as saying that one diagonal being horizontal and at 45° to the V.P., the other is at 60° to the H.P.)

5. The plan of a square is a rectangle $2\frac{1}{2}''$ by $\frac{3}{4}''$, with the long edges parallel to XY. Find an elevation for this square.

6. A rectangle $2''$ by $3''$ has a long edge in the V.P. The figure is inclined to the V.P. at 60°. Show the elevation arranged so as to represent the rectangle with its long edges at 30° to the H.P., and obtain a plan for it.

PLAN AND ELEVATION FORMS FOR THE CIRCLE

SECTION 4. A similar process to that for obtaining projections of the pentagon in Fig. 7 is used for projections of the circle.

For the circle to appear circular in projection it must be placed parallel to the projection plane. If it is horizontally arranged, its plan is a circle, and if it is placed parallel to the V.P. its elevation is circular.

Hold some circular disc such as a coin or a piece of cardboard cut circular, and it will be observed that, like any other plane figure, it can be represented by a straight line on either one or the other, or on both at the same time, of the planes of projection; also, that when not perpendicular to a projection plane, nor parallel to it, the circle will appear in projection as an ellipse with the major axis equal to the length of the diameter of the circle. This major axis represents the diameter parallel to the projection plane upon which the ellipse appears.

Let it be required to find the projections of a circle inclined at, say, α° to the H.P. and having its horizontal diameter at, say, β° to the V.P.

In Fig. 8 at (i) a circle is shown, horizontal, with a line for its elevation. If the elevation line be rearranged, at an inclination of α° to XY as at (ii) it will represent the circle tilted or inclined at α° to the H.P. The horizontal diameter AB which is perpendicular to the V.P. will remain full length in plan at a_2b_2, while the diameter at right angles to it, viz., CD, will now be foreshortened to c_2d_2.

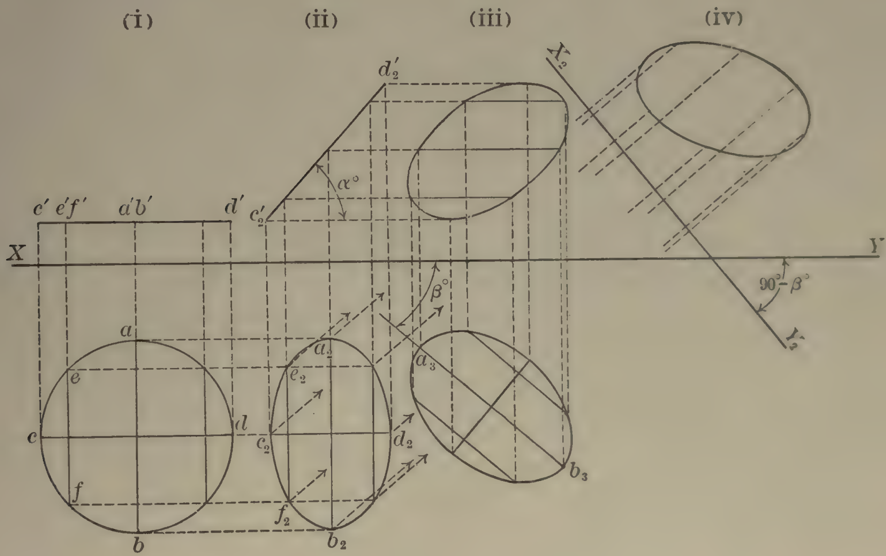


FIG. 8.

A chord EF made parallel to AB will appear in elevation as a point, and in plan at e_2f_2. Other horizontal chords may be made use of, and eventually sufficient points on the circumference will be thus secured in the plan at (ii) to insure a good shape, when drawn by passing a freehand curved line through them, to serve as the edge of the circle when the circle is tilted or inclined to the H.P.

Now if this plan, at (ii), be turned so that the horizontal diameter, a_2b_2, and horizontal chords, are no longer perpendicular to the V.P., but at β° to the V.P. as at (iii), then an elevation, in the shape of an ellipse instead of a line, will be obtained, and this is done by deriving the levels for the elevations of the various points in the curve, from the elevation line at (ii). Such an elevation, however, might also be obtained by making use of what is called a secondary elevation plane, represented as having its XY at any convenient distance, as at X_2Y_2. This being settled, without moving the plan around, as was necessary at (iii), the elevation (iv) is projected directly from the original plan at (ii), and heights for the various points are transferred by measurement from the elevation at (ii).

N.B.—This secondary XY must make with the plan of the horizontal diameter a_2b_2 at (ii) the same angle, $\beta°$, as that contained between the XY and the plan a_3b_3 at (iii). This will mean that X_2Y_2 makes with XY an angle $90° - \beta°$.

It will be seen that the solution of the problem requires a set of drawings to be made, either such as (i), (ii) and (iii), or a set such as (i), (ii) and (iv).

EXERCISE V

1. Arrange a regular pentagon, of $1\frac{1}{2}''$ edge, so that one edge is in the H.P. parallel to XY and the figure is inclined to the H.P. at 45°. (Note.—The angle at the corner of a pentagon is 108°.)

2. Incline a regular hexagon, of $1\frac{1}{4}''$ edge, at 50° to the H.P., leaving one of its edges in the H.P. Then turn the plan so that the edge in the H.P. is at 30° to XY, and show an elevation for it when so placed.

3. Find a plan and an elevation for an equilateral triangle inclined at 60° to the H.P., and having one edge, $2\frac{1}{2}''$, in the H.P. at 60° to the XY line.

4. Find the plan and elevation of a circle, $2\frac{1}{2}''$ diameter, when it is inclined at 50° to the H.P. Then arrange the plan so that the horizontal diameter (the major axis of the ellipse found for plan) is at 45° to the V.P., and obtain its elevation.

5. Find the plan of a circle $2\frac{1}{2}''$ diameter, inclined at 60° to the H.P., and find an elevation on a secondary vertical plane, (with secondary XY), arranging it so that the horizontal diameter of the circle makes an angle of 60° with the secondary plane.

6. Find the plan and elevation of a circle, $2\frac{1}{2}''$ diameter, inclined at 70° to the V.P., and having the diameter which is parallel to the V.P. at 45° to the H.P.

CHAPTER III

OBLIQUE PLANES, THEIR TRACES AND INCLINATIONS

SECTION 5. It has been previously seen that in order that a line might be set up at a given angle to the H.P., or in order to find the inclination to the H.P. of a given line, it is necessary to arrange it parallel to the V.P., or in it, and that then the *projection* in the V.P. makes with XY the same angle as the *line* makes with H.P. Also, that in order that a plane figure might be tilted or inclined to a definite angle with the H.P., it is necessary to arrange it in a perpendicular attitude to the V.P., that is, so that its appearance in elevation is a line only, and we have seen that the angle this line makes with the XY is the inclination the plane figure makes with the H.P.

Similarly, with respect to the inclination of a line or a plane figure to the V.P., the line must be arranged parallel to, or in, the H.P., and the figure must be arranged vertically, so that its plan is a straight line only, and the angle this straight line then makes with XY is the inclination of the given line or plane figure to the V.P.

If, now, the plane of any figure, that is, the imaginary plane of which the figure is a part, be considered, it is clear that when such a plane is vertical it will cut or intersect the V.P. in a vertical line, and at the same time cut or intersect the H.P. in a line that makes with XY the angle that the said plane, arranged in vertical attitude, makes with the V.P. (The room door opened will serve to illustrate this.)

Also, it will be seen that the plane of a figure arranged perpendicularly to the V.P., but inclined to the H.P., will cut or intersect the H.P. in a line perpendicular to the XY, and will cut or intersect the V.P. in a line which makes with XY the angle the plane makes with the H.P.

Two such planes are represented in Fig. 9 at RST and LMN. The intersection line RS is made by the vertical plane RST meeting the V.P., and the intersection line ST is made by the same plane meeting the H.P. So also the intersection lines LM and MN are made by the plane LMN, perpendicular to the V.P., but inclined to the H.P., meeting the planes of projection.

These intersection lines made by a plane meeting the planes of projection are called the *traces* of the plane, and it is by traces that the attitude or arrangement of any plane is expressed.

The traces found or marked in the V.P. are called Vertical Traces, though they are not necessarily vertical lines, as, for instance, LM; and the traces on the

H.P. are called Horizontal Traces, and these, of course, are always horizontal lines. In mentioning these traces the initial letters only are used as V.T. and H.T., instead of the whole words.

Planes having other attitudes than the two so far discussed, sometimes called oblique planes, are shown at OPQ and VWZ.

The student must realize that we are not now dealing with plans and elevations, but with intersections of the projection planes, made by planes of which there can be no plans and elevations. These imagined planes are indicated by their traces.

Reproduce on a paper with XY marked, these planes as in Fig. 9. Fold the paper along the line XY so that the V.P. is vertical while the H.P. remains horizontal, just as the projection planes are always supposed to be arranged in relation to one another, and it will be realized that RS is really at right angles to ST, and that LM is really at right angles to MN. In the other cases, also, the angle

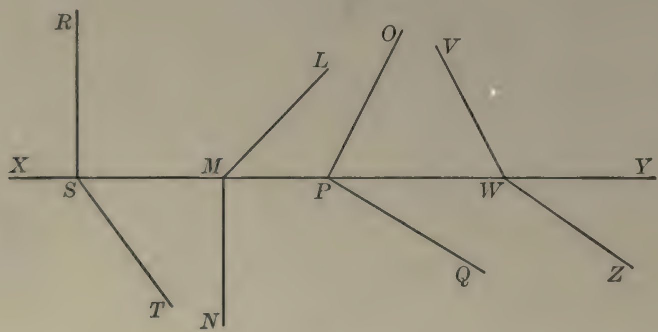


FIG. 9.

which OP makes with PQ is much less than what it appears to be when the paper is not so folded, and so also with VW and WZ.

In Fig. 9, if the Vertical Traces RS, LM, etc., be produced downward below XY, the productions will show where the planes cut the V.P. below the level of the H.P., and likewise, if the Horizontal Traces ST, MN, etc., be produced upward beyond the XY line, it will be seen where the planes RST, LMN, etc., cut the H.P. in that part of it beyond the XY where the V.P. passes through it. For general purposes, however, it is sufficient to represent the planes by two lines meeting in XY as in Fig. 9.

The student should now consider Fig. 10 and realize that whereas the *plane* RST is perpendicular to the V.P., as seen by the fact that its H.T. is perpendicular to XY, and shows its angle to the H.P. as $\alpha°$, the *line* AB is parallel to the V.P., its plan being parallel to XY, and therefore its elevation shows its inclination to the H.P. as $\beta°$.

The elevation $a'b'$, being arranged perpendicular to the trace RS, the angle β is complement to the angle α, that is to say, if the plane is inclined to the H.P.

at 40°, then the line perpendicular to it is inclined at 50° to the H.P.; and as *ST* shows the *plane* to be at 90° to the V.P., so, also, *ab* shows the line to be 0° to the V.P., and it is conclusively evident that the line *AB* is perpendicular to the plane *RST*.

In the case of the plane *LMN* and the line *CD*, similar conclusions will be arrived at, namely, that the line *CD* is perpendicular to the plane *LMN*, and that the inclination of the *line* to the V.P. is complement to the inclination of the *plane* to the V.P.

If the student will now take a pencil to serve for a line and arrange it perpendicularly to any oblique plane—the hand may be arranged to show the attitude of the oblique plane—he will notice that the projection on to the H.P., i.e., the *plan* of the line, is at right angles to the *H.T.* of the plane; and that the projection of the line on to the V.P., i.e., the *elevation* of the line, is at the same time per-

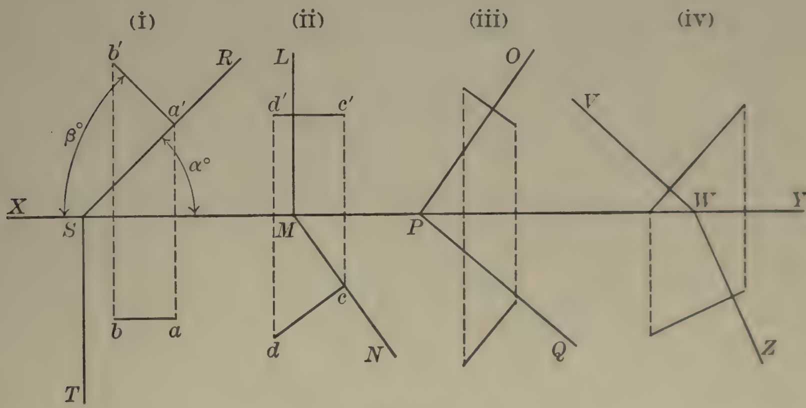


FIG. 10.

pendicular to the *V.T.* of the plane. Illustrations of perpendicular lines to oblique planes are shown at (iii) and (iv) in Fig. 10.

N.B.—The projections of the line need not cross the traces of the plane.

The method of finding the angles of inclination of any oblique plane whose traces are given is now obvious, and will depend upon the facts noted above, namely:—(1) A line perpendicular to an oblique plane can be shown by arranging its elevation at right angles to the V.T. of the plane, and its plan at right angles to the H.T. of the plane. (2) The inclination of the oblique plane to the V.P. is complement to the inclination of the line perpendicular to it, to the V.P., and the inclination of the oblique plane to the H.P. is complement to the inclination of the line perpendicular to it, to the H.P.

That is to say, to find the inclinations to the planes of projection, made by any plane whose traces are given, represent a line perpendicular to the plane by arranging its elevation perpendicular to the V.T. and its plan perpendicular to the H.T., and find the inclinations of this line by the method shown in Fig. 4, then subtract these inclinations from 90° in each case, and the inclinations of the plane are obtained.

An illustration of the method is shown in Fig. 11, where, to find the inclinations of the oblique plane *RST* a line perpendicular to it is represented at $a'b'$, ab.

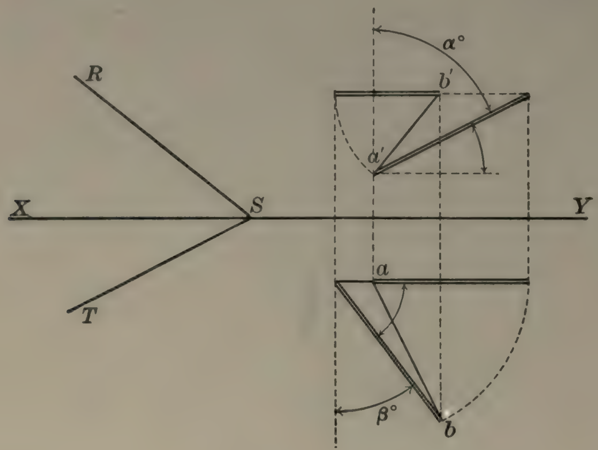
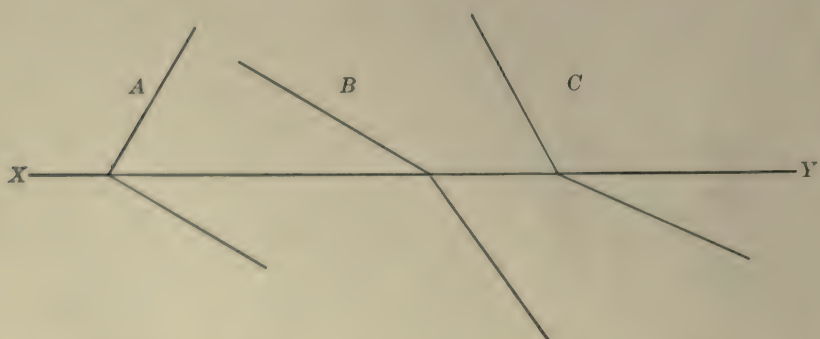


FIG. 11.

The angle $\alpha°$ is the complement of the angle the line *AB* makes with the H.P. Therefore $\alpha°$ is the inclination of the given plane *RST* to the H.P.; and, for similar reasons, the angle $\beta°$ is the inclination of the plane *RST* to the V.P

EXERCISE VI.

Find the inclinations to the planes of projection, of the planes represented at *A*, *B* and *C*, mark with arrow-headed arcs the angles required, and say which angles they are.



N.B.—Take each of the planes *A*, *B* and *C* as a separate problem and work to a large scale.

TRACES OF LINES

SECTION 6. A line AB is represented by its projections at (i) in Fig. 12. It will be observed that the A end of it is a point in the V.P. at a', and that the B end of it is a point in the H.P. at b. Any other point in the line may be taken, as at C, and it will readily be seen that C is at some distance in front of the V.P. and at some distance, also, above the H.P. If the point C could be moved in the line until it got down to the H.P. it would be at a definite place in the H.P., namely, at b, and b' would be the elevation of that place. Similarly, if the point were moved up the line until it met the V.P. it would arrive at a', a definite place in the V.P. of which a is the plan.

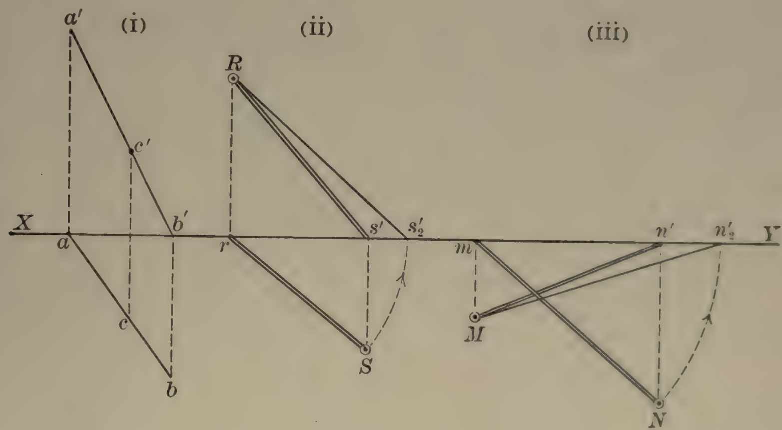


FIG. 12.

Now consider the points R and S at (ii). Let R be a place or point on the V.P. and S be another place or point on the H.P., and let it be required to find the actual distance from R to S.

R, being a point in the V.P., will have its plan directly opposite to it in XY at r, and S, being a point in the H.P., will have its elevation perpendicularly opposite to it on the V.P., in the XY line at s'. The double lines in the figure show the plan and elevation for a line joining R to S, and its length may now be found at Rs'_2, which, of course, represents the actual distance from R to S.

Another illustration is shown at (iii), where point M is on the V.P. and N is on the H.P. Following the same rule, the plan of point M is in XY at m, and the elevation of N is found in XY at n'. The line joining n' to M will give the elevation of the line extending from point M on the V.P. to the point N on the H.P., and the line joining m to N will give the plan of the same line. To obtain the true length of the line whose projections are so found, swing the plan line mN into XY, and the line Mn'_2, the length obtained, is the distance from M to N.

EXERCISE VII

1. Mark a point R on the V.P. 2″ above XY, and another point S on the H.P. perpendicularly opposite to it, at a distance of $1\frac{1}{2}''$ from XY. Find the real distance from R to S.
2. Mark a point M on the V.P. at a distance of 2″ below XY, and another point, N, on the H.P. 1″ in front of the V.P. and considerably to the right of M. Find the true distance between the two points.
3. A point V on the V.P. is 2″ above XY. Find a point H on the H.P., $1\frac{1}{2}''$ in front of the V.P. and 3″, real distance, from the point V.

In Fig. 13 the two projections of a line are given in each of five different arrangements for it. If, in each case, the line be imagined to continue in its direction,

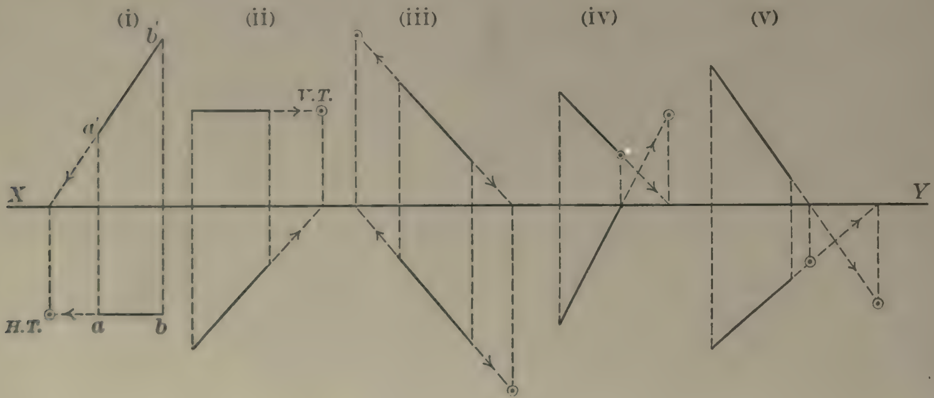


FIG. 13.

beyond the end of the given line, in other words, if the given line be produced, then, if its direction brings it toward a plane of projection, the point it strikes in the plane of projection, or the point of intersection it makes with that plane, is called its trace on that plane.

The trace or intersection point is called the Horizontal Trace (H.T.) or the Vertical Trace (V.T.) according to which plane it is found on. Thus at (i) the line AB has an H.T., but not a V.T., since the line is parallel to the V.P. and therefore cannot meet it in any point. At (ii) there is a V.T. and no H.T. At (iii) the line has both a V.T. and an H.T. So also with (iv) and (v).

The method of arriving at the traces of any line is no doubt apparent, namely:— Through points in XY where the projections of the line, or the projections produced, meet it, draw perpendiculars; then locate the H.T. in one of these where the plan line meets it, and the V.T. in the other perpendicular where the elevation line meets it.

If the given line has already met a plane of projection, as in (iv), before being produced, then the place in that projection plane must be located and marked as the trace of the line.

A special case presents itself when the projections of the line are both perpendicular to XY, as at ab, $a'b'$ in Fig. 14. To find the traces in this case, first swing the line down to a horizontal attitude so that a'_2b' represents it in elevation and a_2b in plan. Produce the plan ba_2 to c with its elevation c'. It will now be seen that if the line be lengthened or produced sufficiently to make it meet the V.P., its elevation will have to be added to, so that it will be from b' to c'_2, and this latter point will be the V.T. of the given line.

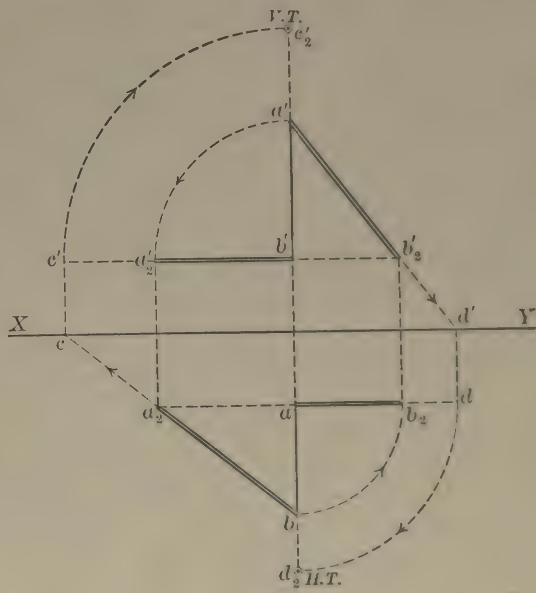
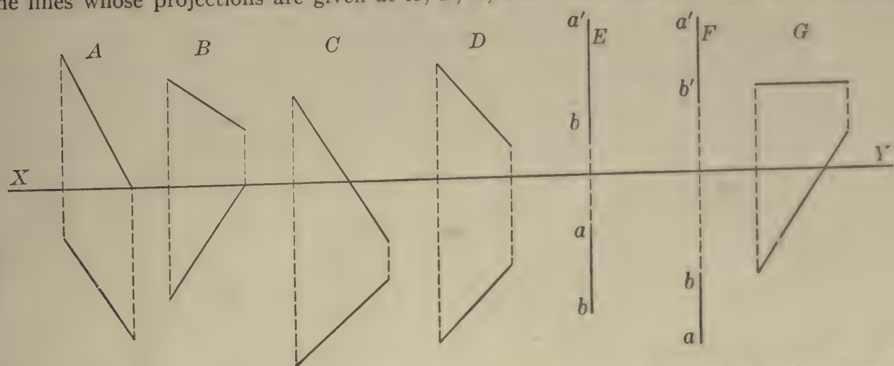


FIG. 14.

Similarly, turn now the plan ab to the position ab_2 and obtain the elevation $a'b'_2$ for it when the line is parallel to the V.P. Produce the elevation to d', and find that the lengthened line will need a proportionately lengthened plan. Carry d by an arc to d_2, which is the H.T. of the given line.

EXERCISE VIII

Find and mark clearly the true lengths, the angles to the planes of projection, and the traces of the lines whose projections are given at A, B, C, etc. The lower lines are, in all the cases,



plans. Separate the different cases so that no overlapping of the solutions will take place, and work to a large scale.

CHAPTER IV

SHADOWS OF LINES

SECTION 7. A ray of light may be represented by a line, and for our present purposes, such a line, with an arrow-head, will serve to indicate the direction of parallel rays of light and will be represented by projections, plan and elevation, such as may be seen at (i) in Fig. 15, the arrow-line below XY always being the plan.

Suppose a stick represented by a line AB be held or placed in various positions and attitudes in relation to the planes of projection as indicated by the projections for it at (ii), (iii), etc., Fig. 15. A ray of light parallel to the oblique line marked with arrow-heads at (i), striking the end B will be stopped there, and instead of the ray reaching one of the planes of projection, the *shadow* of B will be recorded on that plane. In other words, the shadow will be the H.T. or the V.T. as the case may be, of the ray. See H and V in (ii) and (iii) respectively.

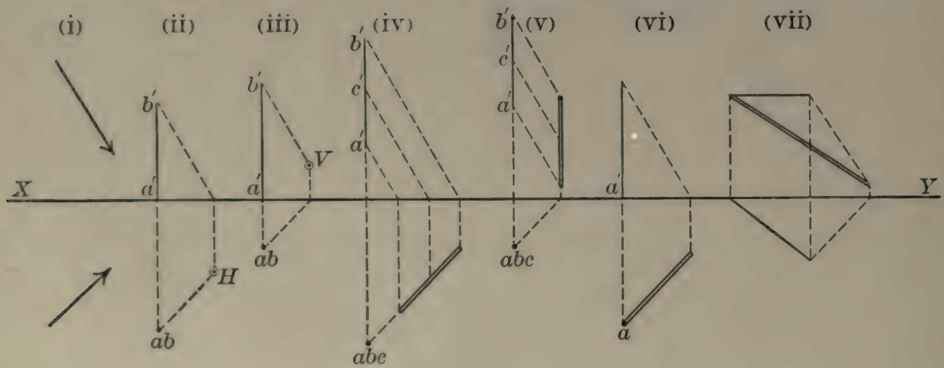


FIG. 15.

Do similarly, with regard to any other point, say C, in the line representing the stick at (iv) and at (v). Also, other points in the line may be dealt with to prove that a straight line will have a straight line for its shadow on any plane it is cast upon.

Notice that when an end of the line or stick touches a plane, as in (vi) and (vii), the shadow will start from that point, for, of course, no ray can reach the plane at the point where the line touches the plane, and hence that point is part of the shadow, and the shadow of any other point in the line will be found in a line continuing across the plane from the touching point.

An examination of cases (v), Fig. 15, and (i), Fig. 16, will reveal the fact that when a line is parallel to a plane the shadow on that plane is equal in length to the line itself, and parallel to it. Hence in cases shown at (ii) and (iii) in Fig. 16

this fact can be taken advantage of, and therefore the shadow of the point B having been found, from it a parallel to the given line is drawn for so much of the shadow as falls upon the plane to which the given line is parallel, and which catches the shadow of B. From the point in XY where this shadow line meets XY, join up to the shadow of A caught on the other projection plane.

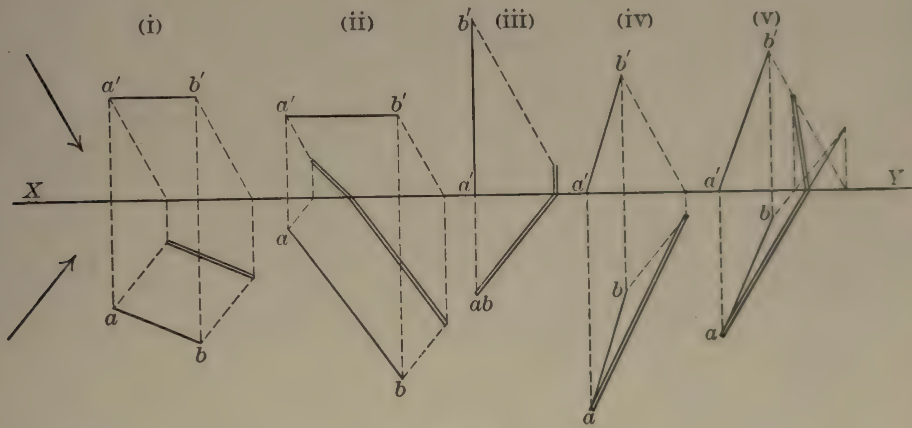


FIG. 16.

In Fig. 16, (v), it will be seen to be necessary, where the line is an oblique line, and part of its shadow is caught by the V.P., to first cast the shadow as if it fell completely on the H.P., part of it beyond the V.P., and then to join up from the point where it crosses XY to the shadow of its upper end on the V.P.

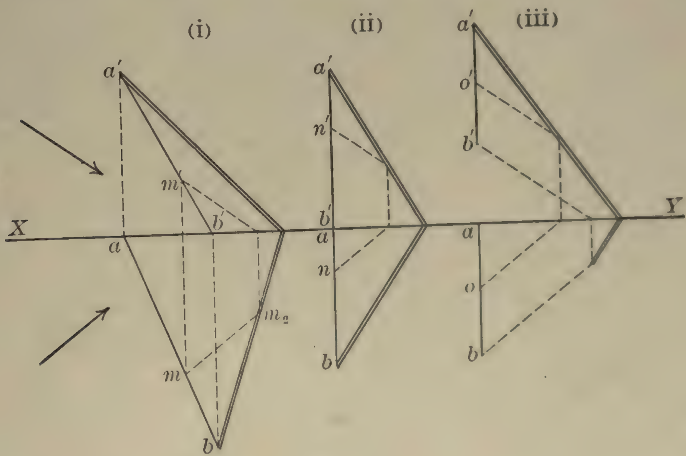


FIG. 17.

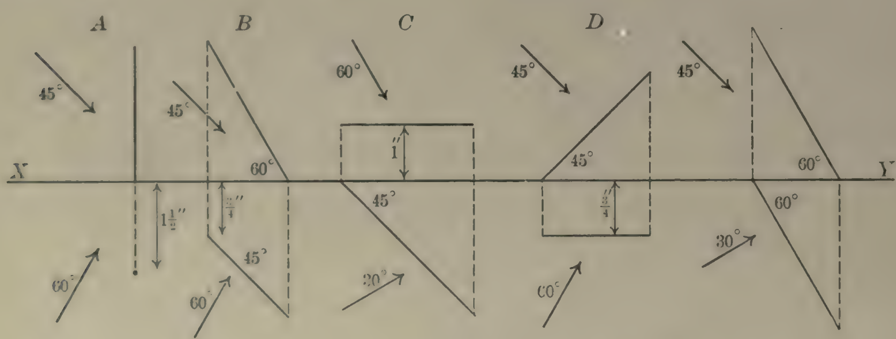
Three other cases are shown in Fig. 17, where the line is so arranged that it is neither parallel to one plane nor to the other. In all three cases the end A touches the V.P. and therefore that part of the shadow which falls on the V.P.

must start from a', while in (i) and (ii), for the same reason, that part of the shadow falling on the H.P. must start from b. The only way to obtain the direction of the shadow as it crosses one plane or the other, is to find the shadow of some intermediate point, chosen at convenience, such as M in (i), N, say, at one-third distance from A in (ii), and O, say, at half way between A and B in (iii).

The shadow of the portion BM in (i) is obviously bm_2, and because a straight line gives a straight line shadow on a plane, this shadow line must be continued until it meets XY, from which point in XY the balance of the shadow can be drawn to a'.

EXERCISE IX

1. Find the shadows on the planes of projection for the lines at A, B, C, D and E. The directions for the rays are indicated by the arrow lines, the number of degrees near the arrow line showing at what inclination the projection of the arrow line in each case must be made to XY. In working the exercise, make the elevations of the given lines about $2\frac{1}{2}''$ long, and arrange the cases so as to avoid overlapping.



2. Make another set of lines arranged in various attitudes somewhat similar to those here given, and find the shadows cast when the parallel rays of light are directed towards the planes of projection, their elevations making $45°$ with XY and their plans $30°$.

SHADOWS OF PLANE FIGURES

SECTION 8. The application of the method of finding shadows of lines to the problem of finding shadows of plane figures is a simple step. Illustrations are given in Fig. 18.

The rule for shadows of lines that are parallel to a projection plane upon which they are cast will save a lot of work, as it will only be necessary, if that rule is taken advantage of, to find the trace of a ray through one point (A) in cases (i), (ii), (iv), (v) and (vi), and the H.T. of the ray through A and the V.T. of the ray through B in case (iii).

Study of cases (iv) and (v) will show the obvious fact that a plane figure, parallel to a plane, will cast its real shape, and size also, for its shadow, on that plane to which it is parallel; therefore in case (vi) it is necessary only to find the trace of the ray through the centre A, and make a circle with this trace as centre and radius the same as that of the circle casting the shadow.

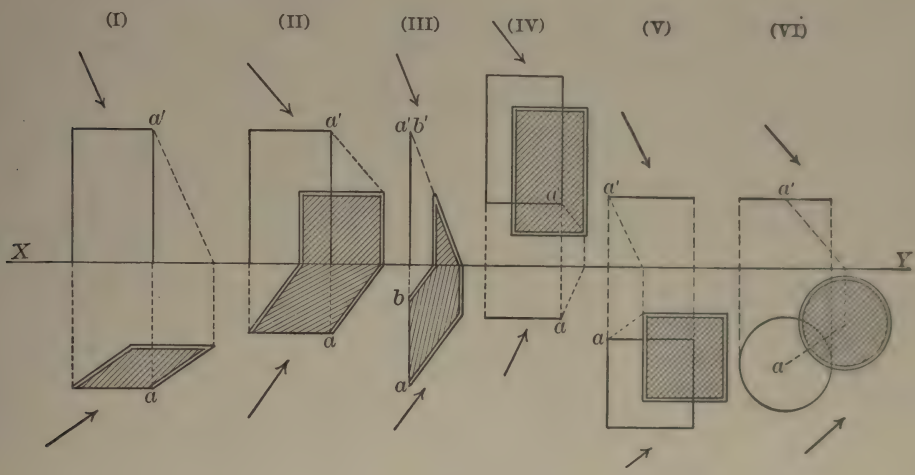


FIG. 18.

Further illustrations are shown in Figs. 19, 20 and 21, dealing with plane figures parallel to the H.P., but casting only part of the shadow, in each case, on the H.P.

In Fig. 19 a square figure, parallel to the H.P., touches the V.P. at one corner. First find the H.T. of the ray through A at a_2, and with lines parallel to the edges of the figure, make the shadow of the square on the H.P. so far as it is cast upon that plane. Next find the V.T. of the ray through B at b'_2 and finish the outline of the shadow, which must, of course, run up to the point where the figure touches the V.P. In Fig. 20 a circle, parallel to the H.P., casts most of its shadow on to the H.P., found by obtaining H, the H.T. of the ray through the centre A, and using it as a centre for a circle, or rather, the segment of a circle, on the H.P. The rest of the shadow, namely, the shadow of the segment BCD, will be caught by the V.P. To obtain it, find V.T.'s for a number of rays passing through points on the curve BDC. Some such points are marked at 1, 2, 3 and 4 in the figure. A free-hand curve is drawn on the V.P. through the V.T.'s of the rays passing into these points.

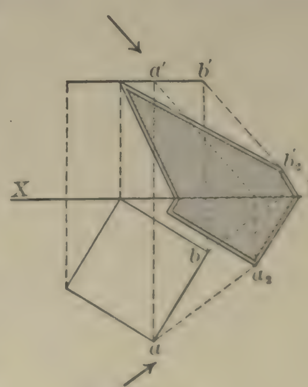


FIG. 19.

In Fig. 21 the H.T. of the ray through the centre A, falls beyond the XY, providing for only the smaller segment of a circle to be cast as shadow on the

H.P. Use this point H as centre, as before, and describe the circular part of the shadow appearing on the H.P. in front of the V.P., and then proceed to find the shadow cast on the V.P. as in Fig. 20.

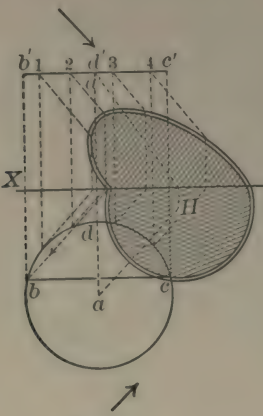


FIG. 20.

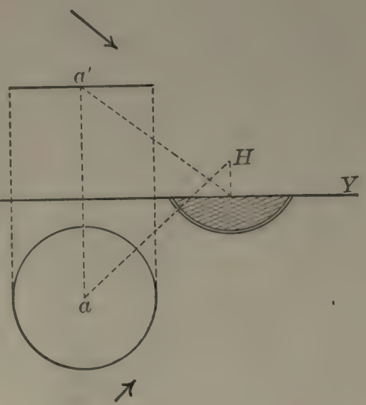
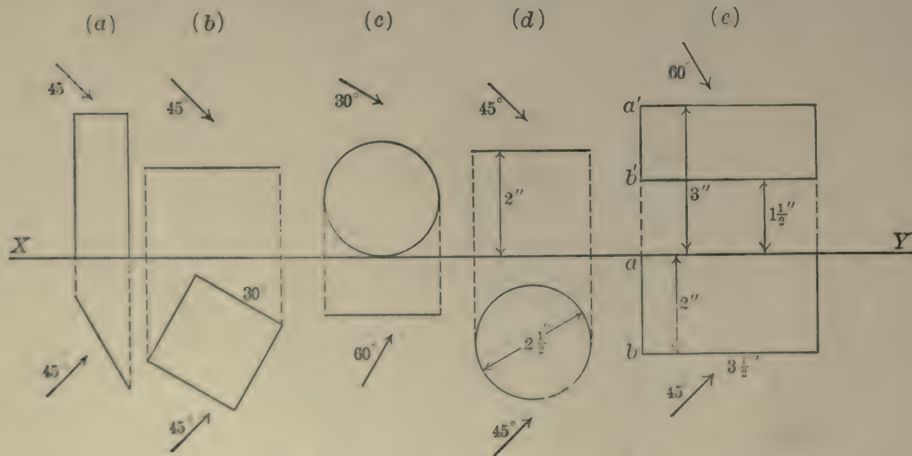


FIG. 21.

EXERCISE X

1. Find the shadows for the rectangle, the square and the circles shown at (*a*), (*b*), (*c*) and (*d*). Keep them well apart from each other.

2. Find the shadow of the awning represented in plan and elevation at (*e*).



3. Arrange other cases for the square and the circle, varying the direction of the rays of light. For example, let a horizontal circle, 2″ diameter, touch the V.P. $3\frac{1}{2}''$ above the XY, and let the rays have elevations at 45° and plans at 30° to XY. The result will be a shadow on the V.P. elliptical in shape.

N.B.—The student is urged to verify these solutions by actual sunlight, and models held in various relations to the planes of projection properly held at right angles to one another.

CHAPTER V

THE "COMPOUND ANGLE" FOR LINES

SECTION 9. Previously we have seen how to find the true length of a given line, and also its inclinations to the projection planes, when the projections, plan and elevation, are given; and it has been realized that the length made for the projection of a given line will influence its inclination to the projection plane. See Fig. 4. So, conversely, the projection length, for a given line, depends on its real length and its inclination to the projection plane. Illustrate this fact by the use of a stick or pencil to serve for a line.

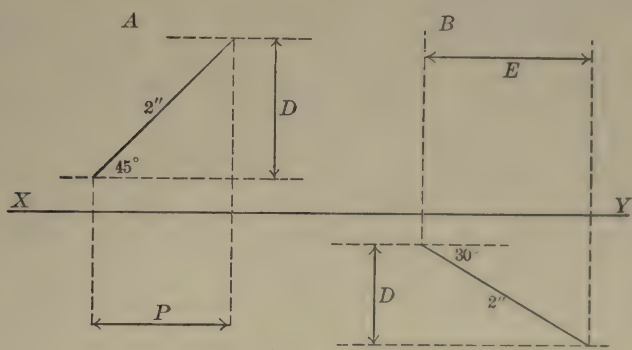


FIG. 22.

It must also now be realized that the conditions of its length and of its inclination to a projection plane, establish the difference of distance of the ends of the line from that projection plane.

Thus in Fig. 22 at *A*, a line arranged parallel to the V.P. will show its true length, say, 2″, and its true inclination to the H.P., say 45°, while its plan length, depending on these, is shown by the arrow-ended line *P* below *XY*, and also the difference of distance of its ends from the H.P. may be measured between the levels of the dotted lines passing horizontally through the ends of the elevation, and marked by another arrow-headed line, *D*.

A line of the same length arranged at, say, 30° to the V.P., will show that angle between its plan and *XY*, and also will show its true length in the plan if the line be placed as a parallel to the H.P., as in Fig. 22 at *B*, with similar results as to projection length, *E*, on the other plane, and difference of distance of its ends, *D*, from that other plane.

Note that, because, while the length of the line remains the same at B as at A, the angle is changed, therefore the length of projection is changed, and also the difference of distance of its ends from the projection plane is changed.

Hence it will be seen that there are four facts, which must be borne in mind, about any given line, namely:—(*a*) *Length;* (*b*) *Inclination*—attitude in relation to a projection plane; (*c*) *Projection length*, and (*d*) *Difference of distance of its ends* from the projection plane. Given any two of these facts the others may be found.

In further illustration of this, let a line, say 2″ long, have its lower end fixed in XY, and be inclined at, say, 45° to the H.P. By placing it in the V.P., as in a previous lesson, the line will show its true length for elevation and its true inclination to the H.P. in its projection on the V.P. It will be foreshortened in plan as seen in Fig. 23 at A to an extent depending on its true length, 2″, and its inclination, 45°, to the H.P. In other words, the *plan length* proper to this line is found, viz., ab, and also the *level* for the free end of the line is found to be at a height H above the level of the lower fixed end. Again, if a line, say 2″ long, be arranged to lie in the H.P. at, say, 30° to the V.P., with one end fixed in the XY as at B in the figure, then the elevation length, $a'b'$, and the proper distance, D, from the V.P., of the free end, are determined.

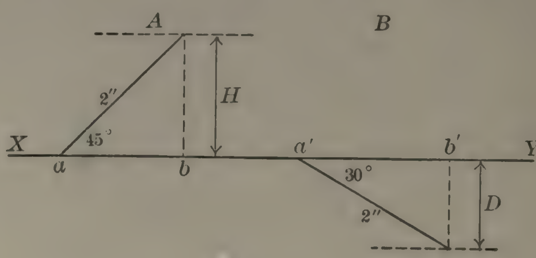
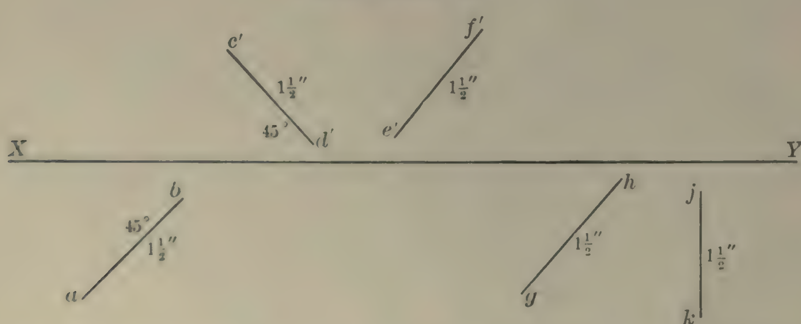


FIG. 23.

EXERCISE XI



Find the other projection for each of the given five lines, one projection of which is given, together with a description as follows:—

AB is 2½″ long and one end of it is in the H.P.
CD is inclined to the H.P. at 30°.
EF is at 60° to the V.P.
GH is at 45° to the H.P.
JK is at 60° to the V.P.

By combining or compounding the results we have obtained at *A* and at *B* in Fig. 23 we shall arrive at a method for the solution to the problem of finding the plan and elevation, properly placed in relation to each other and to *XY*, of any given line at given angles to the planes of projection.

Thus, as an illustration, let the line be 2″ long, with one end fixed in *XY*, and let the angle it makes with the H.P. be 45°, and that with the V.P. 30°.

In Fig. 24 at (i) the height or level for the upper end of the 2″ line inclined at 45° to the H.P. is marked by the dotted line *hh*, and the elevation length for a 2″ line at 30° to the V.P. is obtained and marked *e′e′*. This line is then moved by an arc into its proper position, namely, between the levels proper for a 2″ line at 45° to the H.P., i.e., between *XY* and the parallel *hh*. Now as the projections of any line are necessarily perpendicularly opposite each other across *XY*, it follows that the plan for this line at *A* must be opposite its elevation, and as it must also

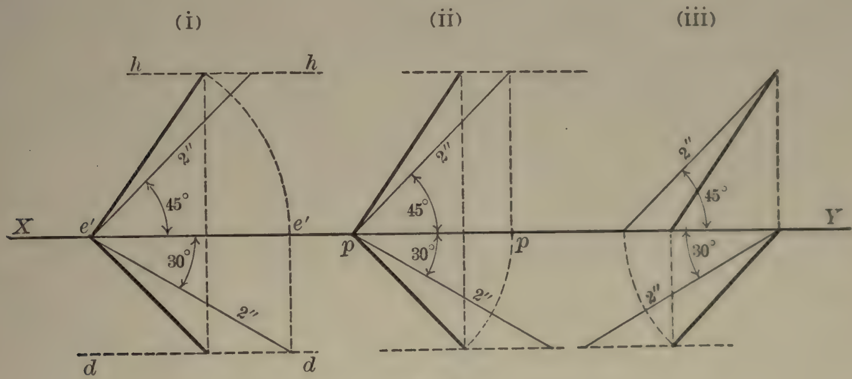


FIG. 24.

be limited to the space between *XY* and the parallel *dd* due to the 30° angle, a perpendicular from the free end, that is, the upper end of the elevation, drawn to this parallel *dd*, will decide its plan.

At (ii) the same result is arrived at by obtaining the plan length, marked *pp*, and moving it by an arc into its position due to the 30° angle, and from this plan the elevation, which must be opposite to it, is found.

If, instead of the arrangement of the line by which its projections meet in a point in *XY* as in Fig. 24 at (i) and (ii), an arrangement of it is desired, so that its upper end is, say, in the V.P. and its lower end is, say, in the H.P., then the solution will be obtained most readily by working it out as at (iii), in the following order:—(*a*) plan length due to 45°; (*b*) distance of lower end from *XY* due to 30°; (*c*) plan put into position, and (*d*) elevation found.

This problem, discussed in Fig. 24, is sometimes spoken of as the problem of the Compound Angle.

In Fig. 25 are shown two common cases of error made by beginners in their attempts to solve this problem without properly understanding what they are doing. The student is urged to make frequent use of some commonplace model of the planes of projection, arranged at right angles to one another, and to mark upon them projections of lines and to study these. Then open out the 90° angle between the planes, and so represent the flat paper upon which the projections are to be made. Eventually the student will be readily able to "read" sets of

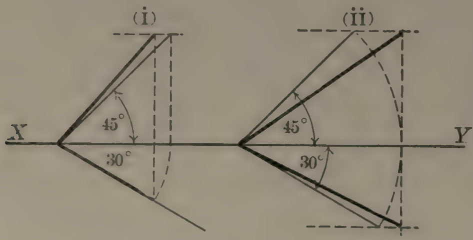


FIG. 25.

projections and also to imagine or picture them, for cases described, without having to handle a model.

EXERCISE XII

Find projections for the following lines:—

(1) *AB*, $2\frac{1}{2}''$ long, inclined at 45° to the H.P. and at 30° to the V.P.
(2) *CD*, $2\frac{1}{2}''$ long, inclined at 50° to the H.P. and at 20° to the V.P.
In both the above cases show two different ways of dealing with the construction.
(3) *EF*, $2\frac{1}{2}''$ long, at 40° to the H.P. and at 50° to the V.P.
(4) Find projections for a $2\frac{1}{2}''$ line, at 40° to the H.P. and at 30° to the V.P., by a method providing for one end of the line to be in the H.P. and the other in the V.P.
(5) Propose other cases, varying the position of, say, the lower end of the line.

N.B.—When proposing new cases for oblique lines, the student should realize that the sum of the two angles for the same line must not exceed 90°, and that in the case of the sum of the two angles being the limit of 90°, the plan and elevation will both appear as perpendicular to *XY*

It will now be recognized that upon this problem depends a method of finding or arranging traces for any oblique plane whose angles with the projection planes are given. For instance, if it be required to find the traces for a plane which is inclined at 50° to the H.P. and at 70° to the V.P., since the traces required are perpendicular to the projections of a line that is at right angles to the plane, it will be seen that a line at 40° to the H.P. and at 20° to the V.P. will be a perpendicular to this plane; therefore, find the projections of this 40°, 20° line, and then make traces for the required 50°, 70° plane, perpendicular to these projections.

EXERCISE XIII

Find the traces for the following oblique planes:—

(*a*) 60° to the H.P. and 50° to the V.P.
(*b*) 45° to the H.P. and 65° to the V.P.
(*c*) 30° to the H.P. and 60° to the V.P.

N.B.—Here it should be noticed, in proposing additional exercises, that the sum of the two angles for an oblique plane cannot be less than 90°, and in the case of the sum being equal to 90° the traces for the plane will run parallel to the *XY* line.

PROJECTIONS OF PLANE FIGURES, INVOLVING THE COMPOUND ANGLE

SECTION 10. Illustration may now be given of the application of the compound angle problem to cases of plane figures in which it is involved. And first let it be realized, experimentally, that when two lines meet at right angles, if one of them is in a plane of projection or parallel to it, the projection on that plane,

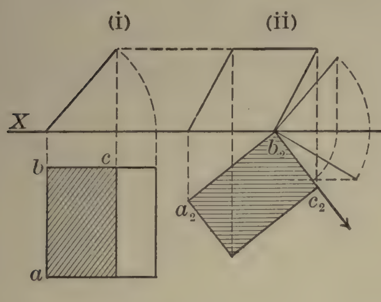


FIG. 26.

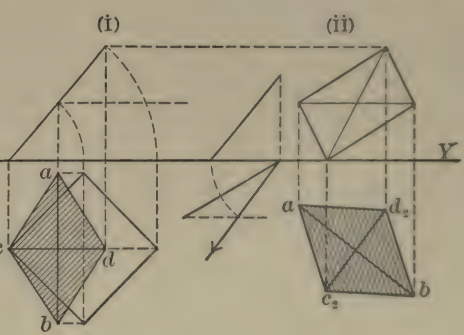


FIG. 27.

of the second line, will be perpendicular to the projection of the first line. Conversely, if one of them has to be inclined to the planes of projection, the arrangement of the other must follow it at right angles to the projection of the first, on the plane to which the second one is parallel. Consider, for example, that if one diagonal of a square be horizontal and the other be at 45° to the H.P. and 30° to the V.P., then it is impossible to say that the horizontal one makes any particular angle with the V.P., for its plan simply follows at right angles the plan of the inclined diagonal. It is only possible therefore to say of it that it is horizontal.

Study the cases shown in Figs. 26 and 27. The plan and elevation in Fig. 26 have been found for a square having one edge, *AB*, in the H.P., and an adjacent edge, *BC*, inclined at 50° to the H.P. and at 30° to the V.P.; and in Fig. 27 the plan and elevation have been found for a square when one diagonal, *AB*, is horizontal and the other diagonal is at a compound angle—50° to the H.P. and 30° to the V.P.

The first step is to find the plan shape, as at (i), quite irrespective of the compounding of angles for the inclined edge or diagonal. The next step is to find the compound angle for the oblique line—the edge *BC* in Fig. 26 and the diagonal *CD* in Fig. 27. For this, any length of line may be taken to serve the purpose, because all that is required is to obtain the *direction* of the plan, marked by the arrow line in each case. Now, either fit the plan shape to this arrow line, as at b_2c_2, Fig. 26, or make c_2d_2 parallel to it as in Fig. 27. The elevation, in each case, is the last thing to find, and it may be derived from the newly arranged plan, and the first elevation which provides levels.

Care must be taken, that, in working out the compound angle, the same *length* of line must be used (any length will do) to be placed at 30° to *XY*, as that which has been placed at 45° to *XY*.

EXERCISE XIV

1. Find the plan and elevation of a rectangle, 2″ by 3″, when a short edge is in the H.P. and the long edges are inclined at 40° to the H.P. and at 25° to the V.P.

2. Find the plan and elevation of a square 2″ edge, when one diagonal is horizontal and the other is at 50° to the H.P. and at 25° to the V.P.

3. Find the plan and elevation for a circle 2″ diameter, when the diameter perpendicular to the horizontal diameter (there is always a horizontal one) is at 55° to the H.P. and at 20° to the V.P.

4. Propose, and work out, other problems of plane figures to involve the compound angle, and realize that questions like Nos. 5 and 6, which follow, do not involve the problem of finding the compound angle.

5. Find the plan and elevation of a rectangle having its long edges horizontal and inclined at 30° to the V.P., and its short edges at 50° to the H.P.

6. Find the plan and elevation of a square when one diagonal is horizontal and at 50° to the V.P. and the other diagonal is at 25° to the V.P.

CHAPTER VI

PROJECTION OF SIMPLE SOLIDS

SECTION **11**. The student will now be able to deal with easy problems involving the use of simple solids, and as the solids used will be familiar types, the attention can be chiefly given to the descriptions with regard to position or place, and with regard to attitude while in any particular position. The student is advised to make use of small cardboard boxes or models of wood, and continu-

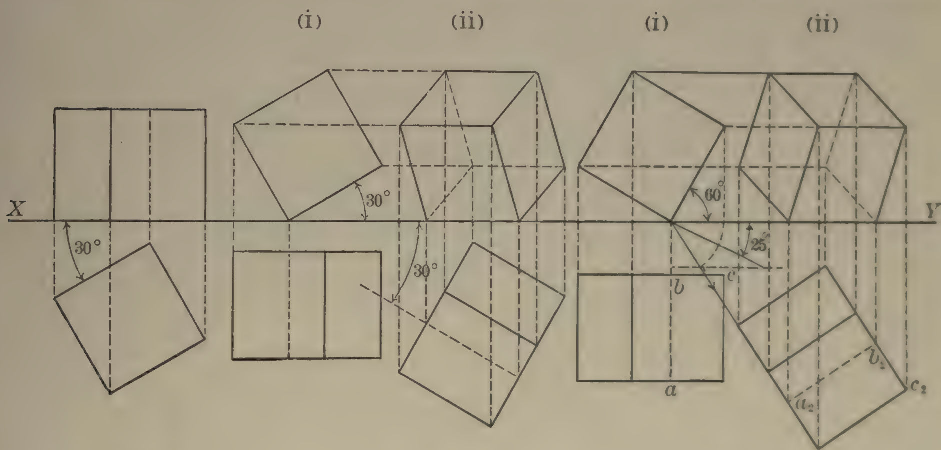


FIG. 28. FIG. 29. FIG. 30.

ally to remember that we are to represent these solids by projecting their appearances perpendicularly on to the two planes of projection. Therefore he should study the possible attitudes or arrangements of the solids in relation to a vertical plane and a horizontal plane held or placed conveniently for experimenting purposes.

In the illustrations Figs. 28, 29 and 30, the order of procedure may be studied for finding plans and elevations of such solids referred to above.

In Fig. 28 is represented a cube having one of its faces in the H.P. and another face at 30° to the V.P. The dotted line in the elevation represents an edge not seen, since the student is supposed to look toward the projection plane with the solid between him and the projection plane. This applies to both planes, hence the dotted lines appearing both in plans and in elevations in Figs. 29 and 30. Note that the same edges are not represented by dotted lines in both projections.

In Fig. 29 the cube has one edge in the H.P. at 30° to XY, and a face adjacent, that is, to which that edge belongs, is at 30° to the H.P. There must evidently be a preliminary set of projections at (i) before the plan and elevation required can be found at (ii). This also applies to Fig. 30, where the cube has one edge of a face in the H.P. and another edge of the same face is at 60° to the H.P. and at 25° to the V.P. The preliminary plan shape and elevation for levels are shown at (i), and the compound angle problem for the edge BC is worked out as in Fig. 26, the arrow line showing the direction for b_2c_2 in the required plan.

In the illustration Fig. 31 a regular hexagonal prism is arranged so that a short edge is in the H.P. and the long edges are at a compound angle, viz., 45° to the H.P. and 20° to the V.P. The upper end of the solid is turned away from

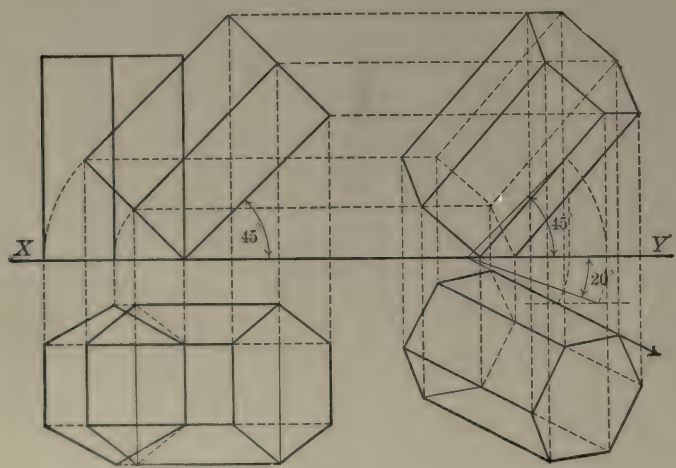


FIG. 31.

the V.P., and is therefore fully visible as a foreshortened hexagon, while dotted lines have to be used for part of the outline of the lower end. In working the compound angle problem it is only necessary to find the plan direction, marked with the arrow line in the figure, as the elevation will naturally assume its proper direction.

In Fig. 32 a right cylinder is arranged so that its axis is at 50° to the H.P. and at 15° to the V.P. This means that the circular end will be at 40° to the H.P. Its plan should be found by the method previously explained. The compound angle for any length of line RR is found in such a way (see Fig. 24, (iii)) that when the arrow line, which indicates the direction for the plan of the axis, is obtained, and the plan of the axis of the cylinder is made parallel to it, the lower end of the cylinder is turned away from the V.P. and appears in the vertical projection in full view, therefore is clear-lined all around. The upper end will be partly outlined by a dotted line. The elevation of the axis should be drawn, and, of course, will *not* be at 50° to XY. When the elevation ellipses have been correctly drawn, the width of the elevation of the cylinder, that is, at right angles to the axis, will be the same as that of the plan.

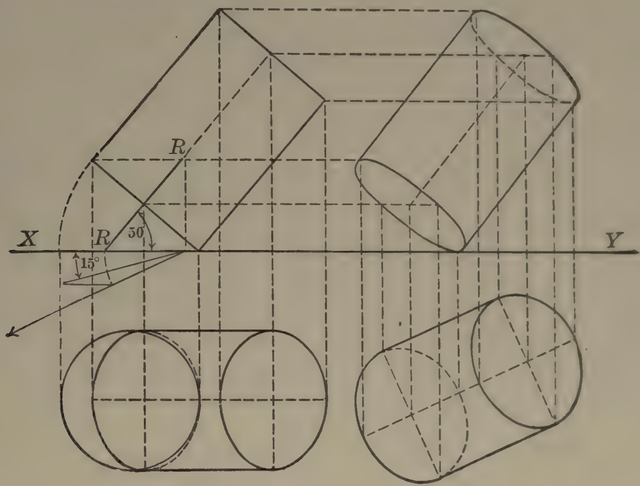


FIG. 32.

In Fig. 33 a regular pentagonal pyramid is made to rest with one of its triangular faces in the H.P. and the axis of the solid—the line from its apex to the centre of its base—is at 30° to the V.P. This means that the compound angle

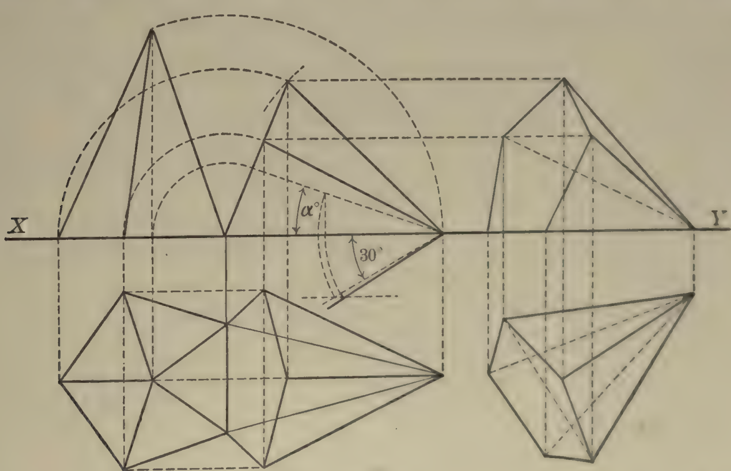


FIG. 33.

must be found for the axis, since there are two angles to be taken into consideration, namely, the angle $\alpha°$ which it makes with the H.P., and the 30° angle it makes with the V.P. In the final position, therefore, the plan of the axis is made parallel to the arrow line obtained by compounding these angles.

In Fig. 34 there is no compound angle problem required to be worked out. A right circular cone is represented with its axis at 45° to the H.P., and the horizontal diameter of its base is at 50° to the V.P. Notice that in drawing the lines for the sides of the inclined cone, where an ellipse is part of the drawing, these side lines are tangent to the curve of the ellipse, and do not run into the ends of the major axis of the ellipse. It will be advisable to draw the line *ee* the full length

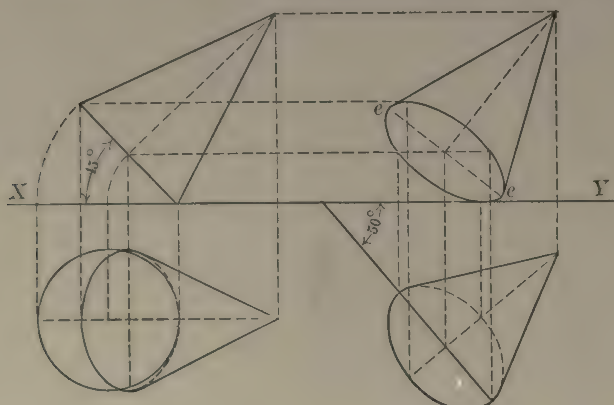


FIG. 34.

of the diameter of the base, at right angles to the elevation obtained of the axis of the cone. In finding the plans and elevations of circles inclined, not less than eight points in the curve should be found.

EXERCISE XV

1. Find the plan and elevation for a cube, 2″ edge, which has one edge in the H.P. at 60° to XY, and a face adjacent to that edge at 30° to the H.P.

2. A pentagonal prism has one short edge, $1\frac{1}{4}$″ long, in the H.P. The long edges, 3″ long, are at 45° to the H.P. and at 25° to the V.P. Find the plan and elevation.

3. A hexagonal pyramid, base edge 1″, axis 3″, has one triangular face in the H.P., and the horizontal diagonal of the base is at 45° to the V.P. Find the plan and elevation.

4. A right circular cone, diameter of base 2″, axis 3″, lies on its side on the H.P., with its apex therefore in the H.P. Find the projections for it when its axis is at 30° to the V.P.

5. Propose and work out other problems, for example, on the square prism, the cylinder, the pentagonal pyramid and the cube, after the manner of those already given, some to involve the problem of the compound angle and some not.

N.B.—This suggestion that the student himself shall be required to prepare in writing the exact description and data for additional exercises, and then work them out, is very important, for it is a good test of the student's grasp of the subject so far as the ground has been covered, and his powers of concentration.

SHADOWS OF SOLIDS AND OF GROUPS OF SOLIDS

SECTION **12**. The more difficult cases of shadow problems, involving the use of solids, may now be undertaken. Let the rays of light be parallel as previously, and directed toward the planes of projection, the direction to be shown by the plan and elevation of an arrow line.

Before commencing to work out the solutions for such shadow problems as now to be dealt with, the student must endeavor to picture or imagine the solid or group represented by the plan and elevation, or described verbally, with the light directed toward it, and then try to decide what edges will cause or are likely to cause the outline or contour of the shadow which will be cast.

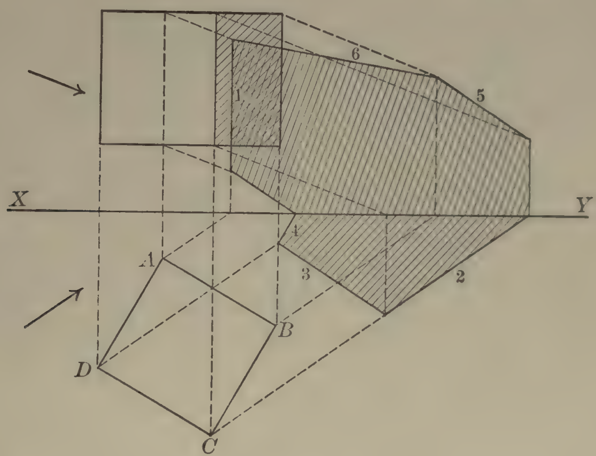


FIG. 35.

The student will not be able to make very satisfactory progress until he can readily conceive mental pictures of what he has to deal with, realizing the relation of what he is picturing to the vertical and horizontal planes of projection, and judging what the projections and other results might be like. Therefore it is important that the student should take advantage of every opportunity which presents itself to train himself in these matters.

In Fig. 35 it will be seen that a cube is represented, and by studying it as suggested above, it will be concluded that the light will fall on three of its faces, and that two vertical edges, those at A and at C, and four horizontal edges, the upper ones AB and BC, and the lower ones AD and DC, are the edges which will cast shadows to give the shadow shape. Deal with them in the order suggested by the numbers 1 to 6 in the figure. Notice that 3 and part of 4 will be parallel to DC and DA respectively.

In Fig. 36 it will be seen necessary to cast the shadow of the apex of the pyramid on to the H.P. beyond the XY line first, in order to draw from that point, on the H.P. the lines forming the contour for that part of the shadow falling upon the H.P. in front of the V.P. (see Fig. 16, (v)). These contour lines are then continued up the V.P. from where they break at XY, to the actual shadow, on the V.P., of the apex. Notice that as two triangular faces of the pyramid, visible

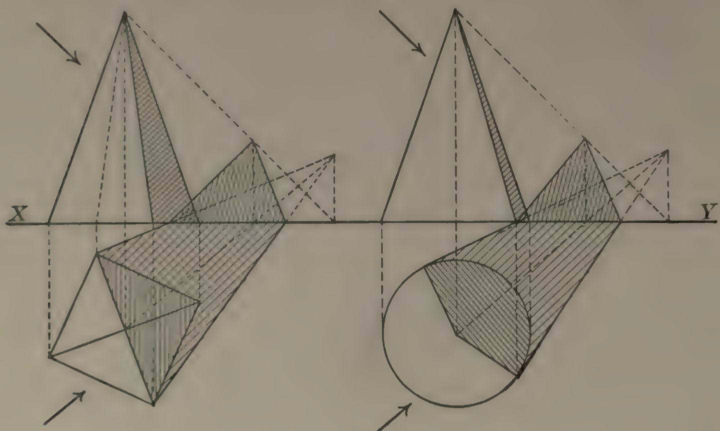


Fig. 36. Fig. 37.

in plan, have no light falling on them, they are shaded. They are said to be ***in*** *shade*, as distinguished from the cast shadow just found. In the case of the cone at Fig. 37, the part in shade is determined by the radii that are perpendicular to the tangent lines serving as contour or boundary lines for the shadow on the H.P.

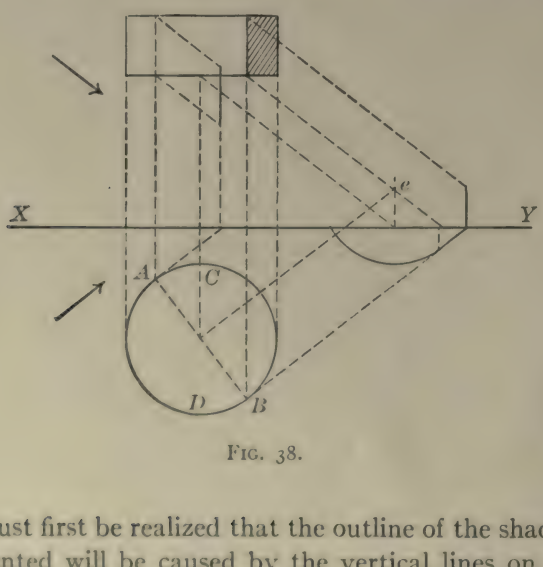


Fig. 38.

In Fig. 38 it must first be realized that the outline of the shadow of the circular block there represented will be caused by the vertical lines on its sides found in plan at A and at B, the upper rim ACB, and the lower rim ADB. The shadows of the vertical lines at A and B should first be found, then, by taking points on the rim lines mentioned, the curves can be obtained. The part of the lower rim casting its shadow on the H.P. will give a circular shadow with its centre at *e*. Only this, and the straight-line parts of the shadow have been found in Fig. 38; the remainder has been purposely left unfinished in order to emphasize the importance of first obtaining correctly the parts that are here found. In each of the cases, Figs. 35 to 38, the part of the surface in shade and visible in the elevation is shaded with hatching to indicate that fact.

Fig. 39 represents a flat block or slab resting on a vertical cylinder. First find the complete shadow of the group as it is cast on the planes of projection, and, in doing this, after drawing the shadow outlines on the H.P. and starting

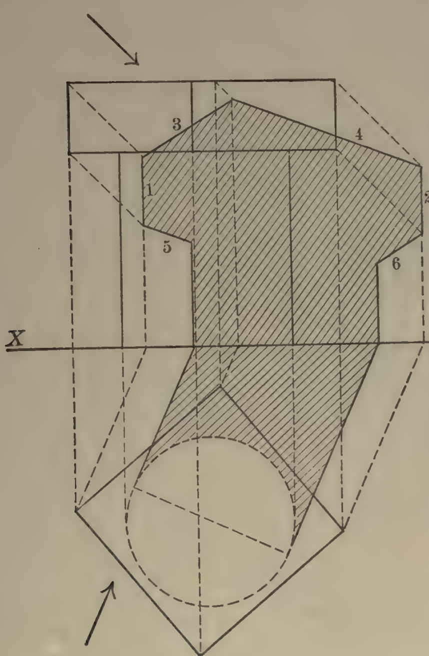


FIG. 39.

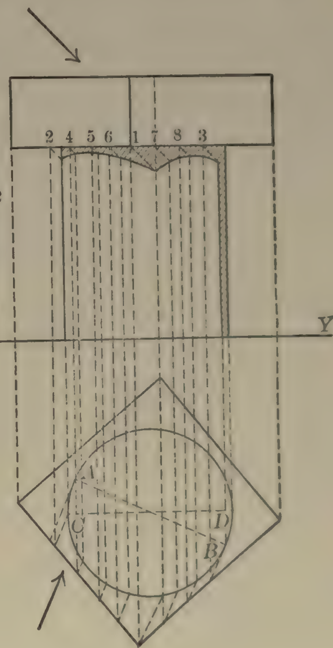


FIG. 40.

them vertically up the V.P., find the outline of that part of the shadow which falls on the V.P. The best order in which the parts of the outline are obtained is indicated by the numbers. Notice that 5 will be made parallel to 4, and 6 made parallel to 3, for reasons that are obvious.

After this shadow on the planes of projection is found, then find the shadow cast on the near or front half of the cylinder, by a portion of the upper solid. The group is reproduced in Fig. 40 in order to avoid confusion of lines, and to show only the shade and shadow seen on the near part of the cylindrical solid. The line AB should first be drawn, perpendicular to the plan direction of the rays. Then from B a vertical line on the near or front surface of the cylinder in elevation divides the light from the shade. Points in the order 1 to 8 should then be decided on, and shadows of these points cast on the surface of the cylinder. Freehand curves will then mark the outline of the shadow cast. Notice that point 2 is the one that casts a shadow on the outside generator whose plan is C.

In Fig. 41 a group consisting of a pyramid standing on a square block is represented. The shadow caught by the top surface of the block will be obtained by supposing the surface of it extended sufficiently to hold the whole shadow. This will give lines 1 and 2. The points a and b must now follow the shadows of the

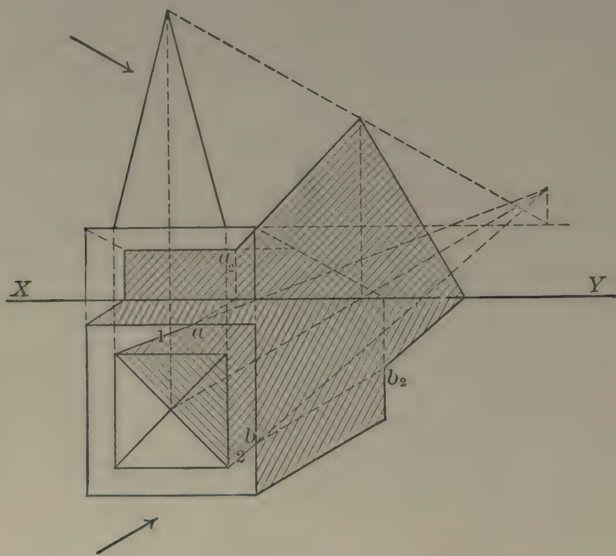


Fig. 41.

edges they are on, to a_2 and b_2. From b_2 the shadow of the slant edge of the pyramid continues on the H.P., parallel to 2 on the upper surface of the block, until it reaches the XY, when it runs up the V.P. to the shadow of the apex, on the V.P., and from a_2 a line on the V.P. to the shadow of the apex, completes the contour or outline of the shadow cast.

EXERCISE XVI

1. Work out carefully all the cases dealt with in Figs. 35 to 41, making much larger drawings.

2. Find shadows on the H.P., the V.P. and on the under solid, for a group consisting of a circular slab 2″ diameter, $\frac{3}{4}$″ thick, resting centrally on a cube, $1\frac{3}{8}$″ edge. The cube rests on the H.P. with its vertical faces equally inclined to the V.P., and one vertical edge $\frac{3}{4}$″ from the V.P. The rays of light have plans at 60° to XY, and elevations at 45° to XY.

3. A square block 2″ edge, and 1″ thick, rests with a square face centrally on a circular cylinder, $1\frac{3}{4}$″ diameter, $2\frac{1}{2}$″ high, standing on the H.P. The horizontal edges of the square

block are equally inclined to the V.P. A short edge of the upper solid is in the V.P. The real inclinations of the rays are 50° to the H.P. and 30° to the V.P. Find the shade and shadows.

4. A right circular cone, 2″ high, 2″ diameter of base, stands vertically on the centre of a square block, 1″ thick, sides 3″. The block has one rectangular face parallel to the V.P. and 1″ from it. The rays of light have plans at 45° to XY and elevations at 30° to XY. Find the shade and shadows.

5. Prepare other exercises, using your best judgment, and work them out for further practice.

PART TWO

CHAPTER VII

RABATTEMENT OF PLANES, AND PROJECTIONS OF FIGURES INVOLVING ITS USE

SECTION **13**. The process known as rabattement is that by which a plane is made to swing on one of its traces, as on a hinge, until whatever is represented as being in the plane, whether it be point, line or figure, is turned into one of the planes of projection, and its true form is made evident, also its relation to the trace of

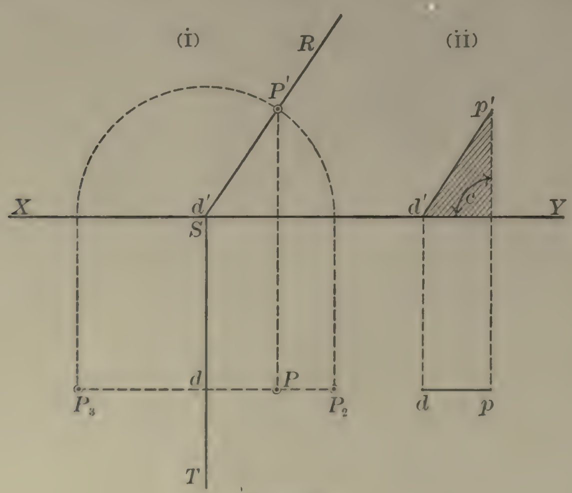


FIG. 42.

the plane. In Fig. 42 the plane *RST*, arranged perpendicularly to the V.P., has in it a point *P*. If the plane is made to swing about its trace *ST* as a hinge line until it is turned over into the H.P., either on the one side or on the other, it carries the point with it, and the point's rabattement is said to be at P_2 or at P_3. If *D* be a point perpendicularly opposite *P*, and in the H.T. or hinge line, then *DP* may be considered as the radius line for an arc in which the point *P* moves. This radius length, as shown in the figure at (ii), is the hypotenuse of a right-angled triangle having for its base the perpendicular distance from the plan *p* to the trace at d, and for its height the distance from XY to p'. The right angle at c in the figure should be specially noted.

In Fig. 43 an oblique plane is represented at RST and a point P in it with plan p and elevation p'. The construction for rabattement is identical with that of Fig. 42, and the point when rabatted appears in the H.P. at p_2 or at p_3. It should be noticed that whereas in Fig. 42 the true inclination between the two trace lines RS and ST is a right angle, in Fig. 43 the true inclination between the traces of the plane is the angle between ST and a line from S through p_3 or through p_2. This angle may be greater or less than a right angle according to the arrangement of the traces of the oblique plane given.

In Fig. 44 is shown the rabattement of a plane carrying with it a line which has previously been placed in the plane and arranged so as to be at 30° to the H.P. To arrange the line in the plane it must first be placed in the V.P. with

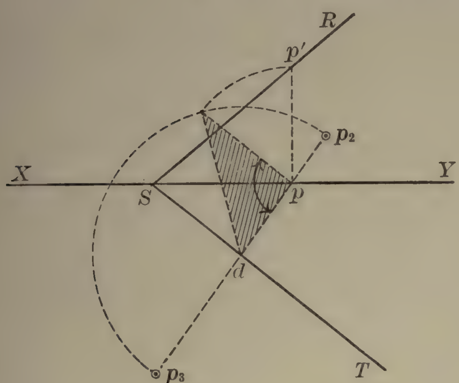


FIG. 43.

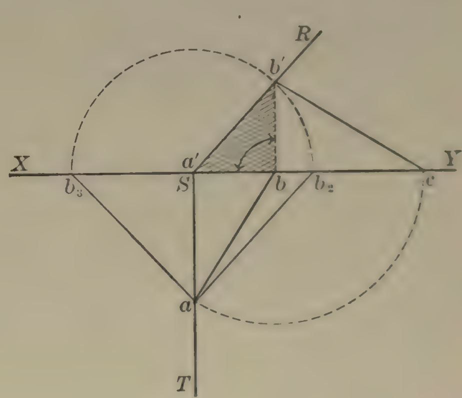


FIG. 44.

one end of it at any convenient point bb', the other end of it in XY at c. The plan bc is then moved, by an arc with b as centre, to bring the lower end of the line into the H.T. of the plane at a. The projections of the line are ab, $a'b'$, and its rabattement is at ab_2 or at ab_3.

An illustration of the use of this method is given in Fig. 45, where it is required to find the plan of a square when it is inclined to the H.P. at 45° and one edge of it is inclined to the same projection plane at 20°. On the rabattement ab_2 of the line AB the figure, true size and shape, is constructed at $c_2d_2e_2$. Any point in this rabattement figure will move across the H.T. at right angles until it reaches its plan position. Thus c_2 and d_2 will be carried across to their plans c and d respectively.

In order to get the plan of E from e_2, the line e_2d_2 is produced to the H.T. or hinge line at f, an immovable point, from which a line through d is drawn until a point opposite e_2 is found at e. Parallels to cd and dc will complete the plan required of the square.

When the figure is required to be found in an oblique plane, then the elevation obtainable will be a figure as at Fig. 46. In Fig. 45 since the plan only was called for, the plane of the figure, *RST*, was arranged perpendicular to the V.P., and this would give a line only for elevation, if it were needed.

In Fig. 46 the plan and elevation are obtained of a regular pentagon placed in a given oblique plane *RST* and having one of its edges at a given angle, say 20°, to the H.P. The line *ab*, *a′b′* is first found in place, as in the case of Fig. 45, and then rabatted over to a_2b (see Fig. 43). On this latter is constructed the pentagon. In constructing this pentagon, the angles of it at *c* and *d* are drawn, each 108°, and *de* and *cg* made equal to *cd*. The fifth corner of the pentagon is then obtained by intersecting diagonals from *c* and *d* made parallel to the sides *de* and *cg*

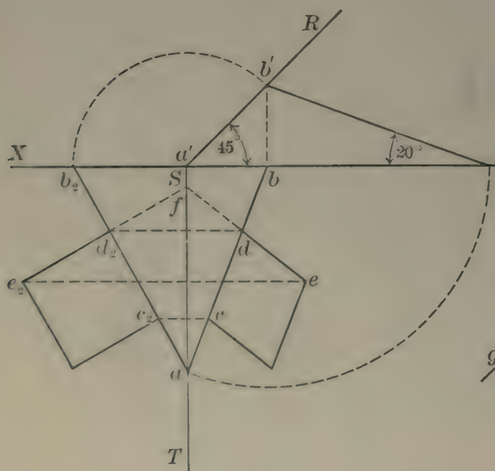


Fig. 45.

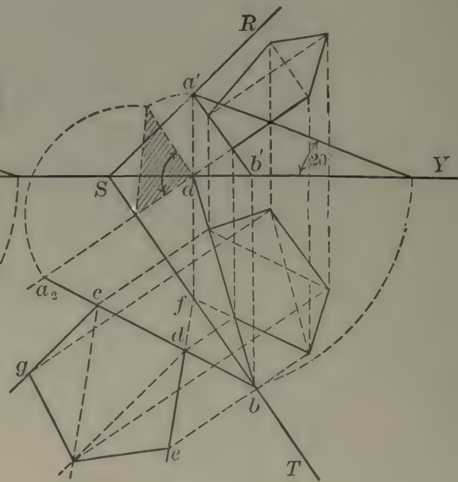


Fig. 46.

respectively. To obtain the plan of the edge whose rabattement is *de*, produce *de* to the H.T. at *f* and from that point draw a line through the plan of *D* till a point opposite *e* is found. For the remaining two corners of the plan draw diagonal lines parallel to the plans of the edges already found. The elevation of the point *E* may be obtained by drawing a line through the elevation of *D* from the elevation in *XY* of the point *F*.

The use of rabattement is again illustrated in Fig. 47, where two lines *AB* and *BC* enclose an angle the size of which it is required to find. As any two lines which meet each other have a common plane whose traces will contain the traces of the lines, it is only necessary to find the H.T. of each of the lines at *R* and *S* respectively, and the line passing through these is the H.T. of the plane of the lines. Use this H.T. as a hinge line about which to rabat the plane. Set up the usual right-angled triangle whose hypotenuse can be used to swing down *B* to its rabattement b_2. This point joined to the traces of the lines will give the rabattement of the angle, showing its true size. This is marked in the figure by the arrowed arc.

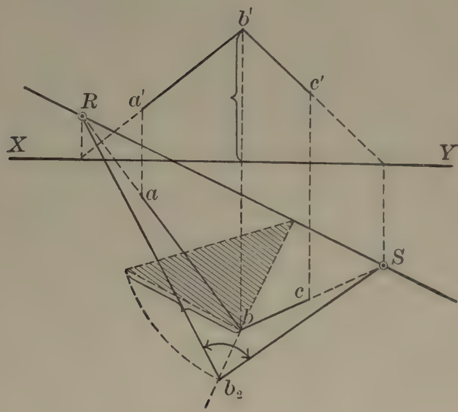
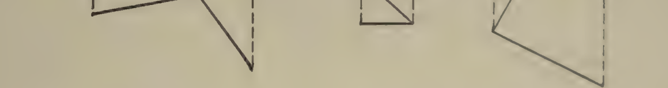


FIG. 47.

EXERCISE XVII

1. Find the plan of a regular pentagon, $1\frac{1}{4}''$ edge, whose plane is at 60° to the H.P. and which has an edge at 30° to the H.P.

2. Find the plan and elevation of a regular hexagon, 1″ edge, when it lies in an oblique plane with V.T. at 60° to *XY* and H.T. at 45° to *XY*, and when one edge of the figure is at 20° to the H.P.



3. Find the true inclinations between the lines meeting as at *A*, *B* and *C*.
4. Find the true inclination between the H.T. and the V.T. of any given oblique plane.

By experimenting with two sticks of pencil, realize that a line, inclined to a plane, makes an angle with it which is the complement of the angle the same line makes with a perpendicular to the plane from any point in the inclined line.

Thus, in Fig. 48 the line *AB* is evidently inclined to the plane *RST*. If a perpendicular to the plane *RST* be drawn from *A* by making its elevation perpendicular to *RS* and its plan perpendicular to *ST*, there will be contained by these two lines, *AB* and *AC*, an angle which is the complement of the angle which *AB* makes with *RST*. Find this angle by the method employed in Fig. 47. The angle indicated by the arrow-headed arc in Fig. 48 is the required angle the line *AB* makes with the oblique plane *RST*.

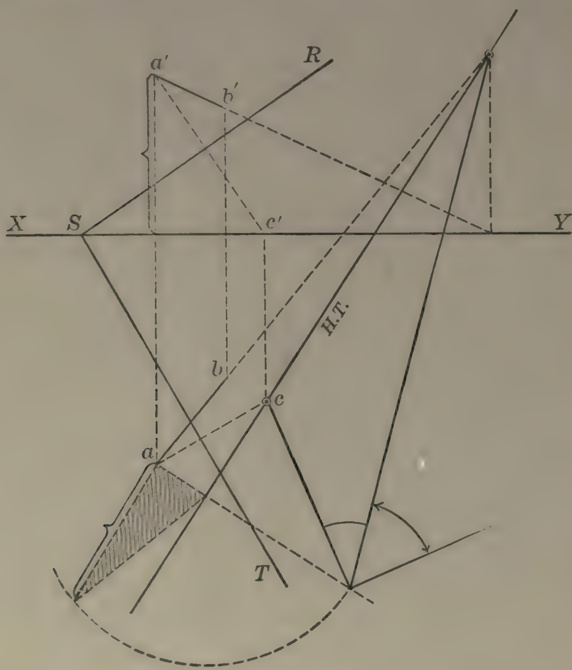
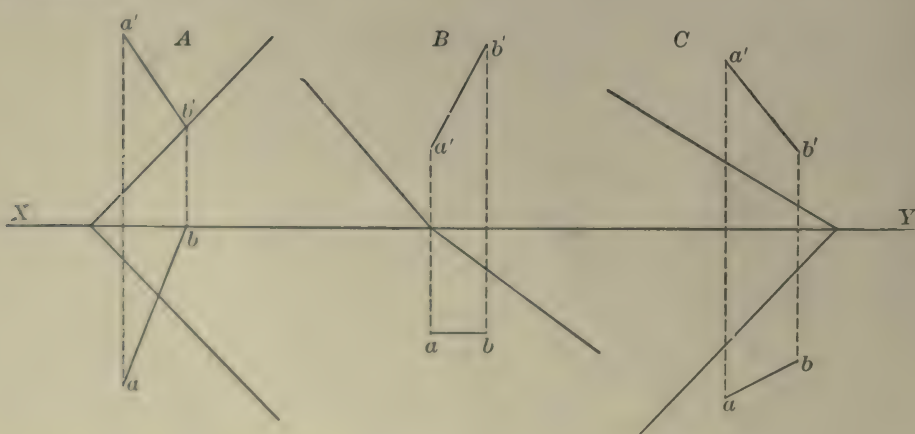


FIG. 48.

EXERCISE XVIII

Find the inclination to the given oblique plane, of the line *AB* in each of the cases shown at *A*, *B* and *C*.



POINTS AND LINES IN RELATION TO OBLIQUE PLANES

SECTION **14**. Realize by experiment the following facts:—

(1) Parallel lines must be projected as parallel lines, i.e., their plans will be parallel, and their elevations will be parallel, to each other.

(2) A horizontal line on an inclined plane must be parallel to any other horizontal line on the same plane and therefore parallel to the H.T. of that plane.

(3) A line on an oblique plane must have its V.T., if it has one, in the V.T. of that oblique plane.

(4) A line on an oblique plane and parallel to the V.P. is parallel to the V.T. of that plane.

(5) A line inclined to both planes of projection will have its traces in the traces of any plane containing it.

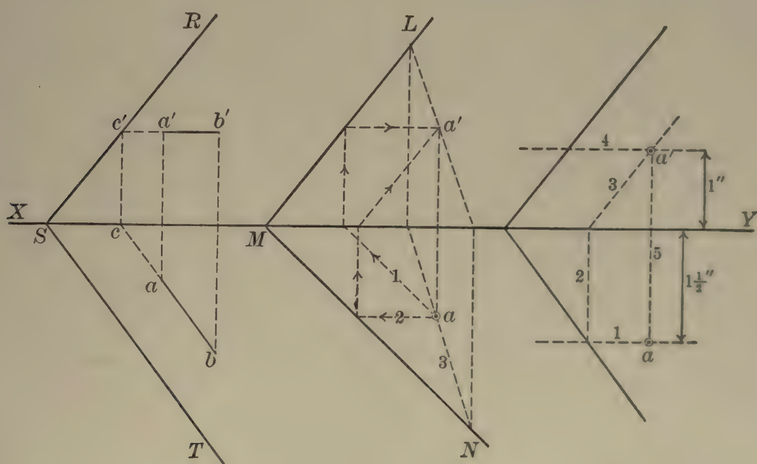


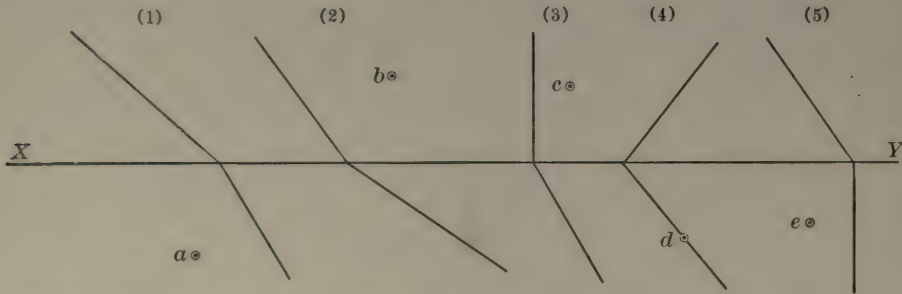
FIG. 49. FIG. 50. FIG. 51.

Thus the plan *ab* in Fig. 49 is made parallel to *ST* to represent a horizontal line in the plane of which *ST* is the H.T., and when *AB* is produced it will meet the V.P. in a point *cc′* in the V.T. of the plane *RST* in which it lies, and the elevation *a′b′* of the horizontal line can then be determined. This consideration helps us to find projections of definite points on oblique planes. For example, in Fig. 50 a point on the plane *LMN* has its plan given at *a*, and there are shown three different ways of obtaining its elevation:—1, by a horizontal line; 2, by a line parallel to the V.P., and 3, by an oblique line, the plan of the line in each case passing through the point, and the elevation of the line being obtained.

In Fig. 51 is shown how to locate, in plan and elevation, any point on a given plane and having given distances from the planes of projection, viz., say 1½″ from the V.P. and 1″ above the H.P. The solution lines may be drawn in the order indicated, resulting in first the elevation and then the plan.

EXERCISE XIX

1. Find the other projection, in each case, for the point in the given plane, one projection of the point being given.



2. Draw traces for any plane, and find the plan and elevation of a point on it ¾″ in front of the V.P. and 1¼″ above the H.P.

In Fig. 52 is shown how to set up, in plan and elevation, a perpendicular to a given plane from any point in it. It will be remembered that a perpendicular to a plane has its projections perpendicular to the traces of the plane. Let *aa′* be the point. First make a perpendicular of any convenient length, limited in the

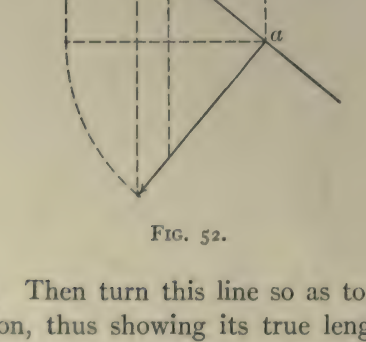


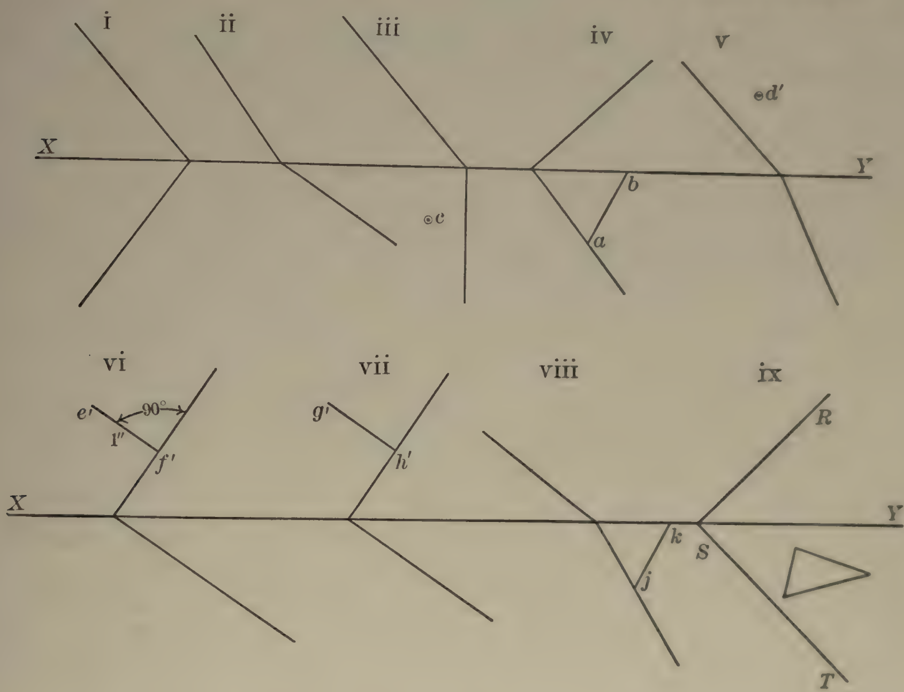
FIG. 52.

figure at the arrow-head. Then turn this line so as to arrange it parallel to one of the planes of projection, thus showing its true length. Cut off the required part on this true-length line, and by a parallel to *XY* the length of one of the projections is determined. A perpendicular across *XY* will define the other projection.

EXERCISE XX

1. Find the projections of a 2″ line perpendicular to a given plane and starting from it in point 1″ from both planes of projection. Let the plane be inclined at 40° to the H.P. and 65° to the V.P.

2. The traces of a plane are V.T. at 30° to *XY* and H.T. at 45° to *XY*. A line perpendicular to this plane has a plan 2″ long. Find the projections of a 2″ portion of the line.



3. Work out on a larger scale the following cases:—
   - i. Find plan and elevation of a point on the H.T. of this plane, 2″ from the *XY*, and set up a perpendicular line from it, 2″ long.
   - ii. Find a point on this plane $1\frac{1}{2}$″ above the H.P. and 1″ in front of the V.P.
   - iii. Find a 2″ line perpendicular to this plane from the point *C* in it.
   - iv. Find the elevation of *AB* and its true length. It lies in the given plane.
   - v. Find the plan of *D* which lies in the given oblique plane. Given *d′*.
   - vi. Find the plan of *EF* which is perpendicular to the given plane, and obtain its true length.
   - vii. Find the plan of the line *GH* which lies in the given oblique plane.
   - viii. Find the elevation of the line *JK* which is in the given plane, and show its inclinations to the H.P.
   - ix. The plan is given of a triangle lying in the oblique plane *RST*. Find its elevation then by rabattement of each corner in turn, on to the H.P., show the true shape and size of the triangle.

4. Find the true inclination between the traces of each of the planes at iii, iv and v.

INTERSECTION OF OBLIQUE PLANES WITH EACH OTHER, AND OF LINES WITH OBLIQUE PLANES

SECTION **15**. If any free line be taken, represented, for instance, by a pencil for purposes of experiment, it will be realized that any number of planes may contain it, and the condition of this will be, that the trace lines of such planes pass through or contain the trace points of the line.

In the illustration in Fig. 53 there are six such planes shown. Two of them are specially important and useful, owing to the fact that they can be drawn or

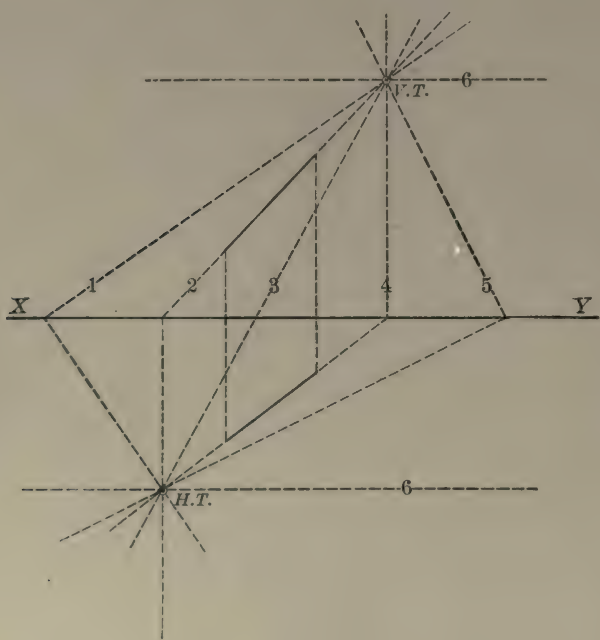


FIG. 53.

represented without the necessity of first discovering the trace points of the given line. They are Nos. 2 and 4—the one, a plane perpendicular to the V.P., and the other a vertical plane or plane perpendicular to the H.P.

The next thing to realize is that since planes intersect each other in straight lines, it will easily be possible to show the plan and elevation of an intersection line by joining the point on the V.P. common to the Vertical Traces of the two planes (i.e., where the two V.T.'s meet), to the point on the H.P. where the planes' H.T.'s meet. In Fig. 54 an illustration is given where plane *RST* intersects plane *LMN* in line *AB*. The plan of the intersection line is at *ab* and its elevation is *a′b′*.

Further realize that a free line directed to any plane, or passing through it, will have an intersection point on that plane, and this will be a point in the inter-

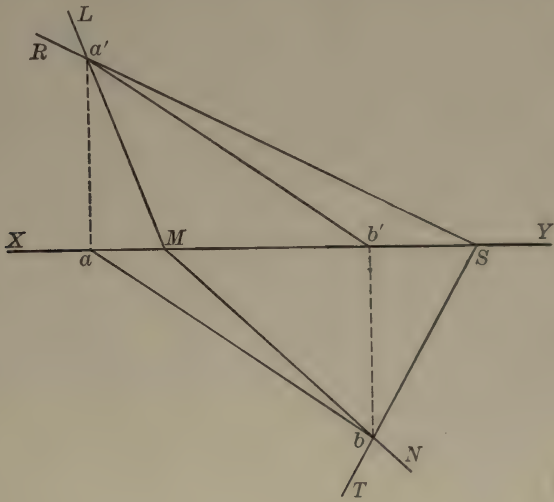


FIG. 54.

section line which a plane containing the line makes with the given plane. For illustration of this, see Fig. 55, where a line AB is directed toward a plane RST.

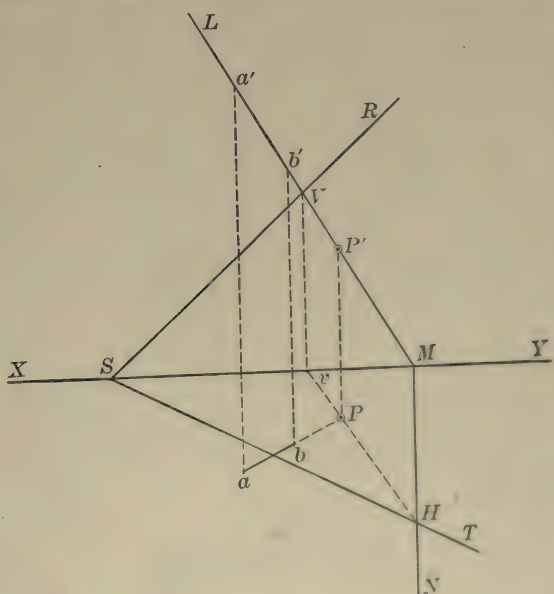


FIG. 55.

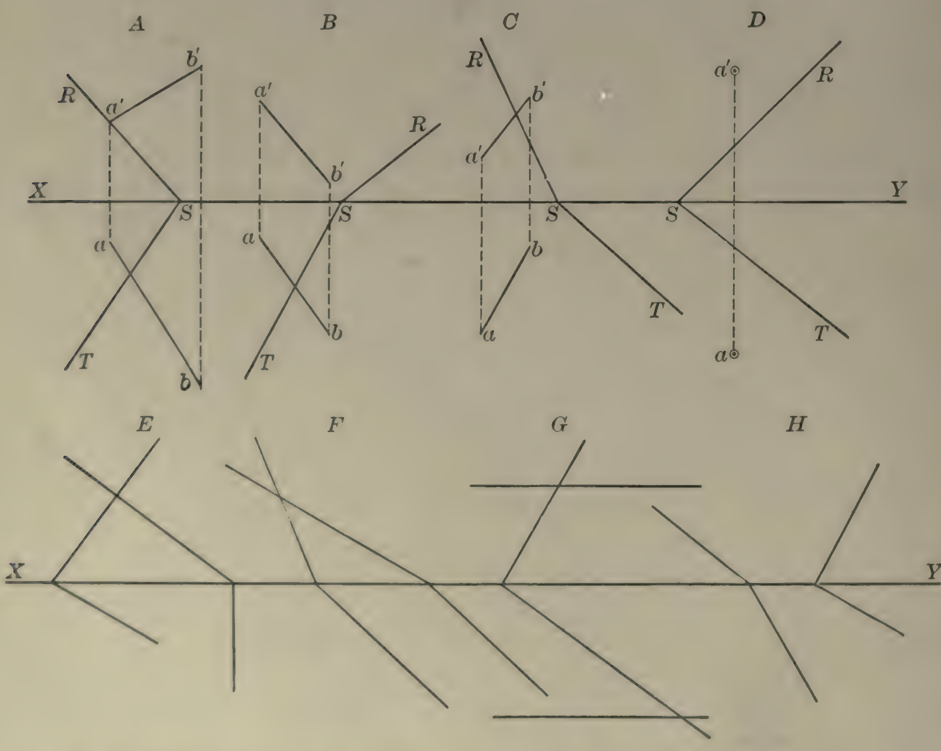
A plane LMN containing AB intersects RST in the line passing through V and H and shown in plan and elevation at vH and VM.

When the plan of the line AB, namely, ab, is produced to meet the plan of this intersection line, namely, vH, the plan of the intersection point is found at P, and P', the elevation of it, can be at once obtained.

Be careful to note that if a vertical plane is used in which to contain the given line, then the elevation of the intersection will meet, or cross, the elevation of the line, or the line produced, at the elevation of the intersection point required, and that if a plane perpendicular to the V.P. is used to contain the line, it will be the plan of the intersection line that will meet the plan of the given line in the plan of the intersection point required.

EXERCISE XXI

1. Find the intersection, showing it by plan and elevation, of the line AB with the plane RST in each of the cases A, B and C.



2. Find the distance of the point A from the plane RST at D.

3. Find the projections of the intersection line in each of the cases at E, F, G and H where planes are arranged so as to intersect each other. Work to a large scale.

CHAPTER VIII

PARALLEL PLANES

SECTION **16**. It needs no demonstration to realize that planes parallel to one another will meet the planes of projection in parallel trace lines.

In Fig. 56 at (i) two vertical planes are shown, and it is evident that the distance between these planes is not the distance between their Vertical Traces, which may be, for these planes, a varying distance apart according to the angle

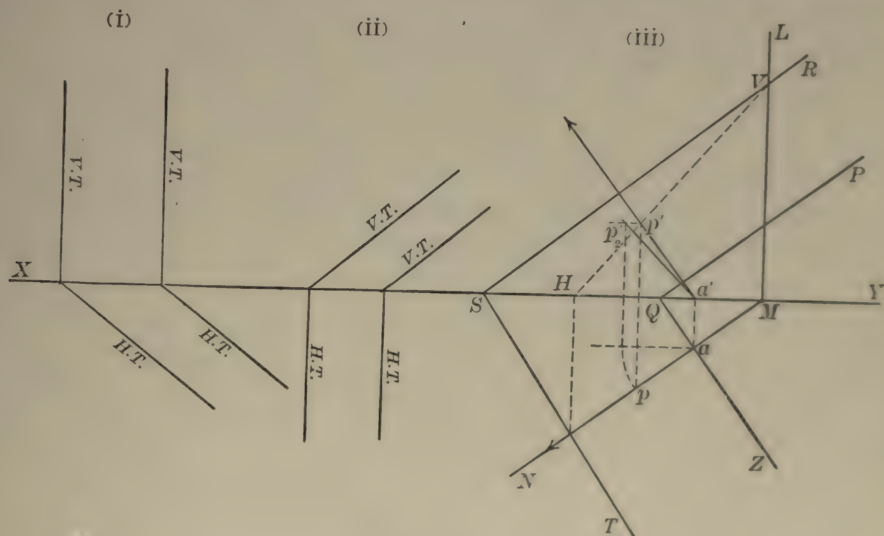


FIG. 56.

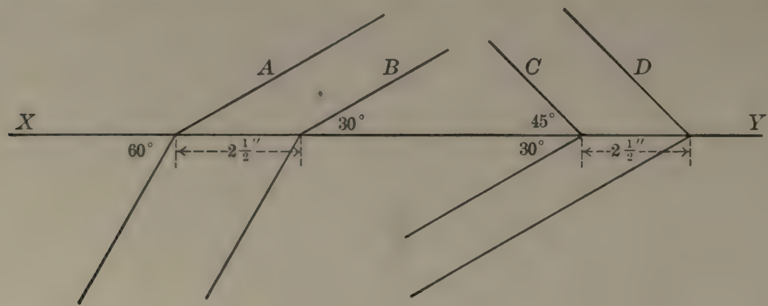
at which they meet the V.P., but the distance between them is the distance between their Horizontal Traces, because they meet the H.P. perpendicularly.

In the case at (ii) the distance between the planes is the same as that between their V.T.'s, because in this case the planes meet the V.P. perpendicularly.

At (iii) the parallel planes are oblique, and the method of finding the distance between them is to start a perpendicular line from some point in one of them, say, *PQZ*, and find where it intersects the other plane *RST*, then find the length of the line from one to the other. To do this, take any point in the H.T. of the plane *PQZ*, as at *aa'*, and from it set up a perpendicular, marked with arrow-heads in the figure. Now consider this line to be in a vertical plane *LMN*, and find the intersection of LMN with RST, giving a line whose elevation is VH and which contains point P shown at $p'p$. Find the true length of AP at $a'p'_2$. This is the distance between the two given planes.

EXERCISE XXII

Find the true distance between the parallel planes A and B, also between C and D.



It was previously seen that a horizontal line in a plane was parallel to the H.T. of that plane, and now it will be seen that *any* horizontal line parallel to a plane is parallel to the H.T. of that plane. And, as parallel lines have plans parallel to one another, it will be realized that when two planes are parallel to one another,

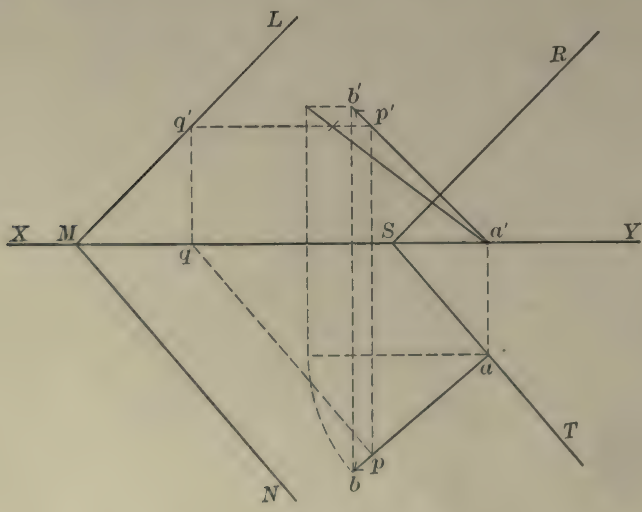


FIG. 57.

the H.T. of one of the planes is parallel to any horizontal line on the other plane. Hence the solution given in Fig. 57 for finding a plane parallel to a given one, and at a given distance from it.

From any point A in the given plane RST set up a perpendicular AB and cut off a part, AP, say, 1″ long. Through P draw a horizontal line parallel to the plane RST by making its plan pq parallel to ST and its elevation $p'q'$ parallel to XY. The V.T. of this line is at q', which is also a point in the V.T. of the required parallel plane. Draw LM parallel to RS, and MN parallel to ST. It should be noticed that the plane LMN has its traces perpendicular to the projections of the line AB, and that this plane has been passed through a certain definite point P, marked in the line AB.

So, likewise, if it be required to pass a plane through *any* given point P, Fig. 58, and to arrange it perpendicularly to any given line, AB, it is only necessary to make a second line, CP, through P, with its plan perpendicular to the plan of the given line AB, and to represent a horizontal line, in this way, on the required plane. This line CP will have its V.T. at V, which is a point in the V.T. of the required plane RST. RS will, of course, be made perpendicular to the elevation $a'b'$ of the given line.

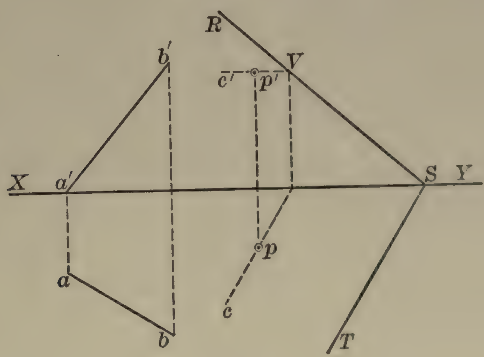


FIG. 58.

A problem illustrating the application of two or three recently discussed methods is worked out in Fig. 59, where it is required to find the centre of the sphere, which has upon its surface four points whose projections are given. The four points are shown in plan and elevation at aa', bb', cc' and dd'. Any point on the surface of a sphere, joined to any other point on the surface by a straight line, will give a chord of the sphere, and a plane bisecting this chord will cut the sphere into two equal parts, that is, will pass through the centre of the sphere. Notice also another fact, namely, that if three points only be chosen as points on the surface of a sphere, the size of the sphere, upon the surface of which these points may lie, may be any size provided that it is not less in diameter than the circle upon the circumference of which these points may have place. Consequently, in order to limit the size of the sphere to be dealt with, a fourth point is necessary, and all four of the points must be made use of in the solution.

By joining the given points, several chords may be obtained, as shown in plan and elevation, Fig. 59, and, according to the argument above, if planes are made to bisect them at right angles, each plane so found will pass through the centre of the sphere. Thus, by joining *ab* and *a′b′* the chord *AB* is represented, and from the centre point of it, *pp′*, a horizontal line *PQ* is drawn as a horizontal line on the plane perpendicular to *AB*, giving the point *q′* as its V.T. through which the V.T. of the *AB* plane may be drawn perpendicular to *a′b′*. Its H.T. is then drawn perpendicular to *ab*. So again in the two other cases, giving the *AC* plane and the *CD* plane. The plan and elevation of the intersection of plane *CD* with

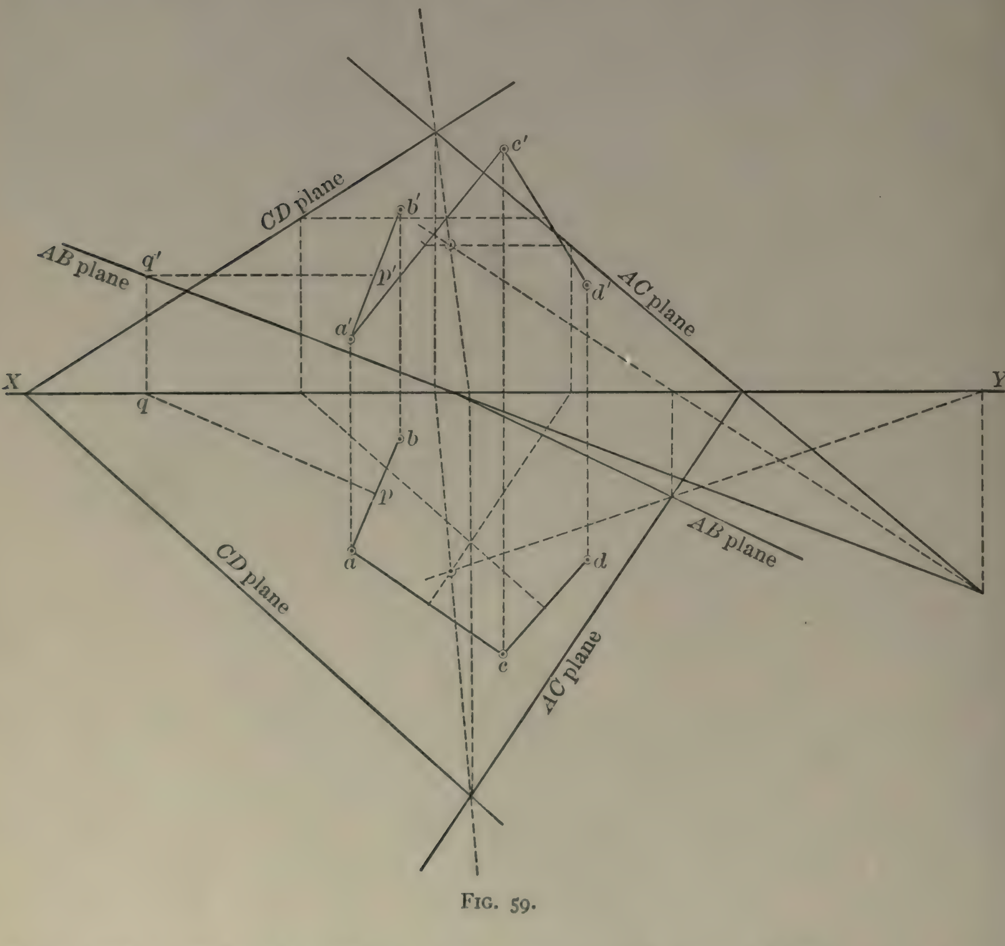


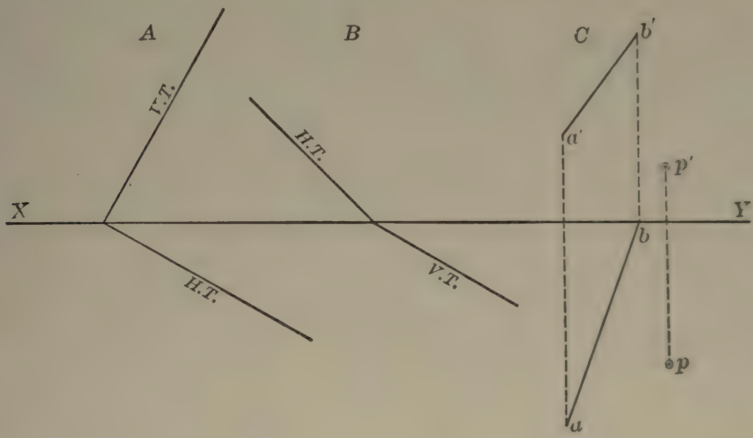
Fig. 59.

plane *AC* is next found, and then the intersection of plane *AB* with one of these (conveniently *AC* in this solution), and the *plans* of the intersections obtained cross each other in the *plan* of the centre of the sphere. Likewise the elevations of the intersections cross each other in the *elevation* of the centre of the sphere.

The radius of the sphere may now be found, if desired, by obtaining the true length of the line joining this centre point to any of the given points *A*, *B*, *C* or *D*. The sphere may then be represented by circles, one for plan and one for elevation, with the points found, as centres.

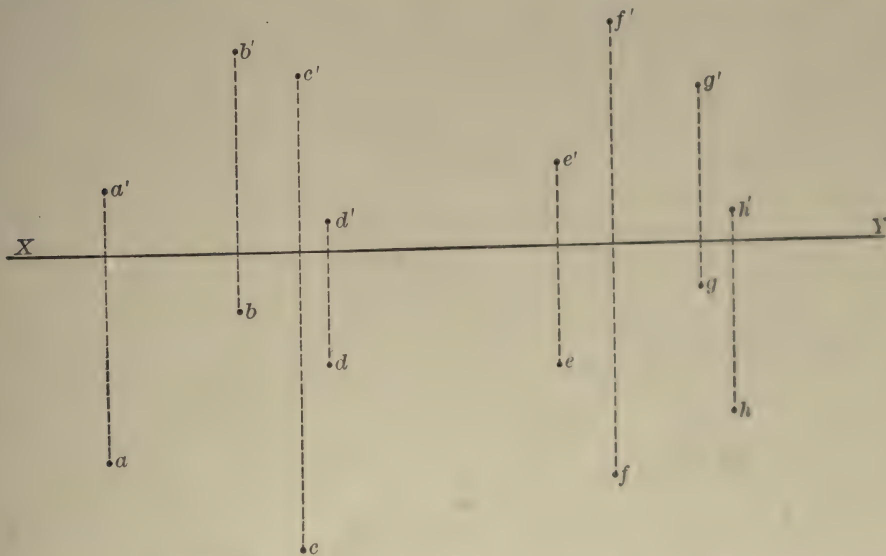
EXERCISE XXIII

1. Find the traces of a plane parallel to, and $1\frac{1}{2}''$ distant from, a given plane shown at *A*, whose V.T. is at 60° to *XY* and H.T. at 30° to *XY*.



2. Find the traces of a plane parallel to the given one shown at *B* and 1″ perpendicular distance from it.

3. Find a plane perpendicular to the given line shown at *C*, and passing through the given point *P*.



4. Find the centre of the sphere which has on its surface the four given points shown at *ABCD*.

5. Find the plan and elevation of the sphere which has on its surface the four given points *EFGH*

DIHEDRAL ANGLES CONTAINED BY OBLIQUE INTERSECTING PLANES, AND PLANES DIVIDING THESE ANGLES

SECTION **17**. In considering dihedral angles, or angles contained by planes as they incline to one another, the illustration at (i), Fig. 60, shows two planes *RST* and *LMN* meeting each other in a vertical line at *H*, and meeting the H.P. at right angles with the result that the angle and its supplement, named the dihedral angles contained by these two planes, are equalled by the angles *MHS* and *SHN* on the H.P.

Similarly, at (ii), the two planes *RST* and *LMN* meet the V.P. at right angles, and the dihedral angles contained by them are evident at *MVS* and *SVL*, because

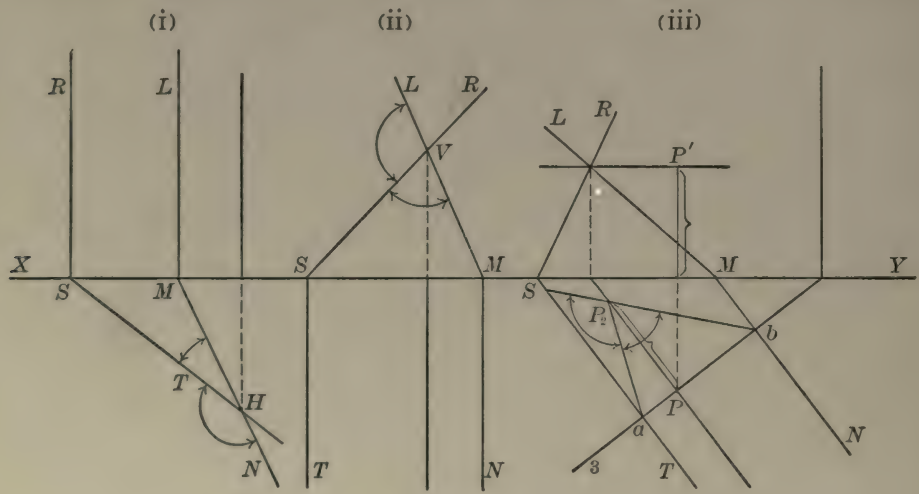


FIG. 60.

they are equalled by the rectilineal angles contained by the traces or intersections *RS* and *LM* on the V.P., that is, on the plane perpendicular to them.

At (iii) the planes are so arranged that they are not perpendicular to one of the planes of projection, and consequently a third plane, other than one of the planes of projection which served the purpose in cases (i) and (ii), must be made use of, in order that the intersections by the two given planes made on this third plane may be found, and the angles by the intersections on it measured.

Because at (iii) the planes are so arranged as to intersect each other in a horizontal line not perpendicular to the vertical plane of projection, the third plane, with its H.T. marked 3, is arranged so that it cuts the given planes at right angles. It cuts the intersection line of the given planes at *P*, and when this point is rabatted onto the H.P. at P_2 and joined to *a* and *b* where the third plane's H.T. crosses the H.T.'s of the given planes, the angle aP_2b and its supplement are obtained, to which the dihedral angles required are equal.

In Fig. 61 the two planes are so arranged that their common intersection line is inclined from the point v' where the vertical traces cross, to the point H where the horizontal traces cross. In this case a third plane, perpendicular to the two given ones, will therefore be inclined, and, being at right angles to the intersection line VH, its H.T. marked 3, must, of course, be made at right angles to the plan, vH, of the intersection line.

By turning vH, with centre v, into the XY, and carrying with it the point c, the real inclinations of the intersection line and of the third plane may be represented on the V.P., where the line $v'P$ is a view of the intersection line showing

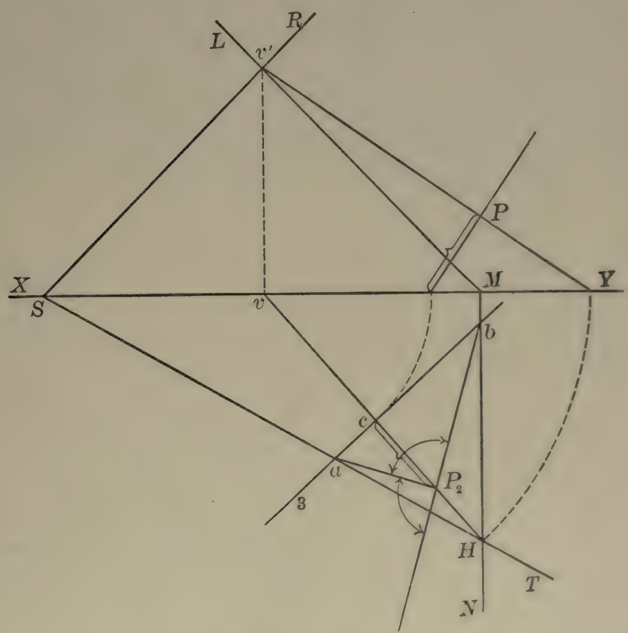


FIG. 61.

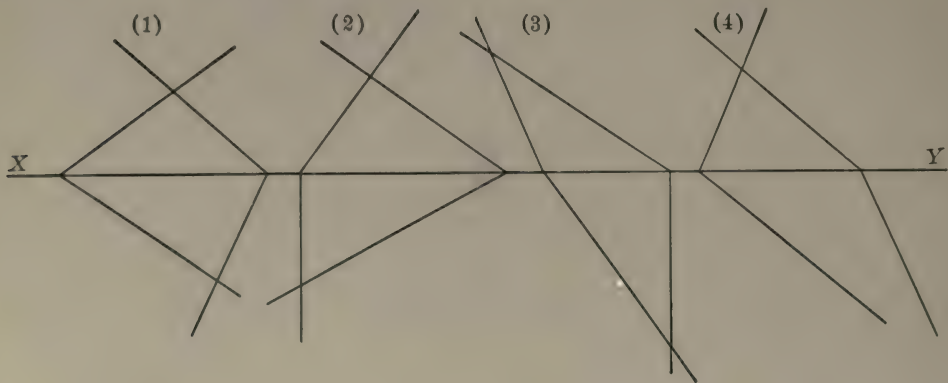
its inclination to the H.P. and the line at right angles to it through P shows the inclination of the plane 3. Point P marks the level at which they intersect. From the point P, where the inclined intersection line passes through the inclined plane 3, the distance, indicated by a bracket, down the plane 3 to the H.P. is then taken and transferred to cP_2. Join P_2 to a and to b, and these lines, which are the rabattements of intersections made by RST and LMN, respectively, with plane 3, as in case (iii), Fig. 60, will give angles to which the dihedral angles required are equal, namely, aP_2b and its supplement.

In working exercises, it is advisable, until the student is familiar with each step, to make hair lines and attach names to all the various lines and points while the solution is in progress.

(Notice that the third plane's H.T., marked 3, and arranged at any convenient place perpendicular to the plan vH of the intersection line, may need to be produced beyond XY in order to meet the H.T. of LMN at b, as in Fig. 62.)

EXERCISE XXIV

Find and mark the angles contained by the planes represented in pairs at (1), (2), (3) and (4). Make the drawings very much larger than what is shown here.



By experiment with an open book, it will be realized that any dihedral angle, contained by the covers of the book, may be divided by arranging a leaf or leaves of the book, so that from one cover to a leaf is a dihedral angle, and from that leaf to another, separated from it, or to the other cover, will be another dihedral angle. The leaf, representing a plane, divides the dihedral angle between the covers which represent other planes, and will be seen to have the same intersection line as the covers have, namely, the hinge edge of the book.

Now, let it be required to find the traces of planes which will bisect the dihedral angles contained by any two planes whose traces are given. It will be clear that such traces will pass through the traces of the intersection line made by the given planes, since they must contain the same intersection line.

In Fig. 62 the planes RST and LMN are the given planes with their intersection line passing through V and H. After proceeding as in the case explained in Fig. 61, the rabatted intersections of RST and LMN with plane 3 will give the angles aP_2b and aP_2d marked by arrow-headed arcs, showing the sizes of the dihedral angles contained by the given planes.

Now, bisectors of these angles, marked in the figure by straight lines with arrowheads, will serve as rabatted intersections made with plane 3 by planes bisecting the dihedral angles, and, of course, these intersection lines will have their H.T.'s in the H.T. of plane 3 at e and f respectively, and also it will be realized that these

H.T.'s will be in the H.T.'s of the required bisecting planes. Since the bisecting planes contain the intersection line passing through V and H, these points, V and H, are in the traces of the bisecting planes required. The result is, that the line through H and e is the H.T., and from where this meets XY a line through v' is the V.T., of one of the required planes, while fH is the H.T., and a line through v' to meet fH in XY, is the V.T. of the other required plane.

If, as is the case in this figure, there is not room enough to find the meeting of the H.T. and V.T. on XY for the last plane, then a point xx' should be chosen

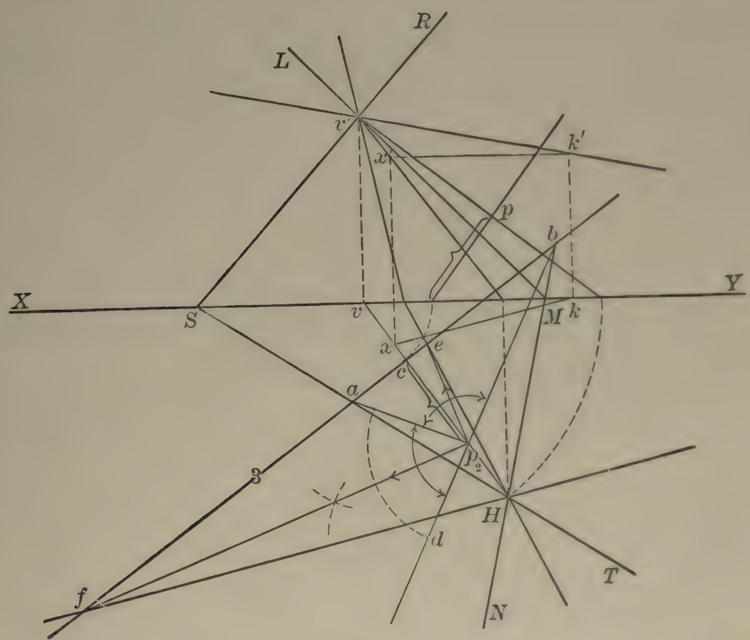


FIG. 62.

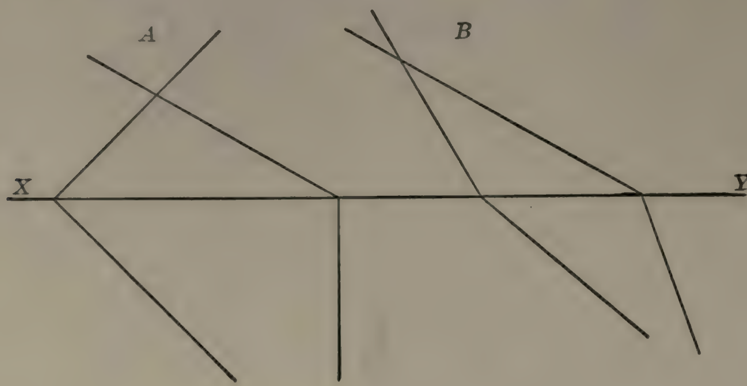
on the common intersection line, and a horizontal line on the required plane should be drawn to meet the V.P. at kk'. This latter point on the V.P. is in the V.T. required, which is obtained by joining v' to k'.

If the H.T. of this last plane, marked fH, is parallel to XY, of course the V.T. of it, passing through v' will also be parallel to XY.

Also, if the bisector P_2f runs nearly parallel to H.T. 3, so that there is not room enough to find f, then the direction of the line fH may be obtained by making use of an inclined line in the plane required.

EXERCISE XXV

Find the traces of planes which bisect the dihedral angles contained by the pairs of planes given at A and at B.



It will now be seen, that, by supposing the rabattement of the angle between two planes, and working conversely to the method in Fig. 61, it will be possible to find the traces of a second plane at a given angle to a first one, whose traces may be given, and intersecting it in a line of given inclination to the H.P.

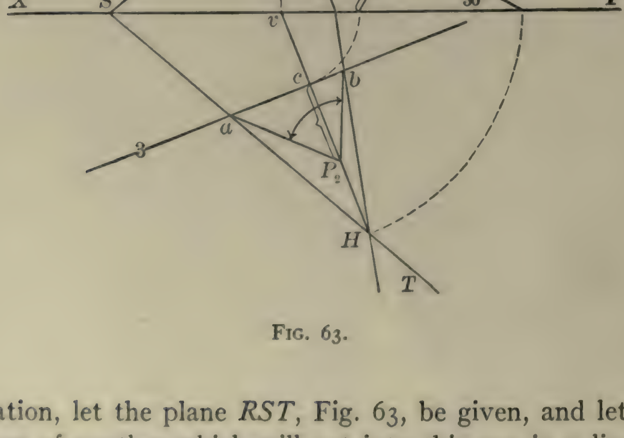


FIG. 63.

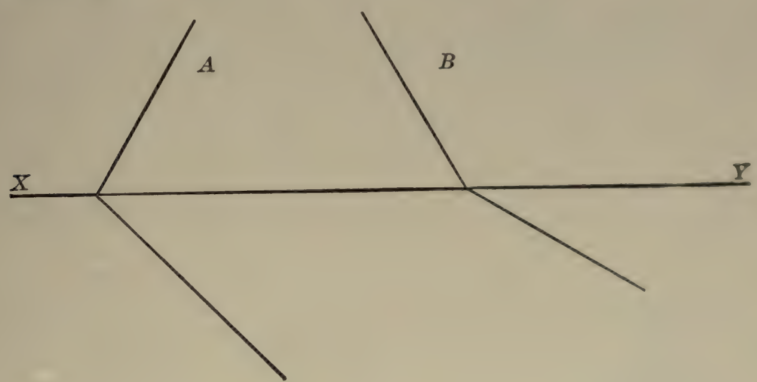
For illustration, let the plane RST, Fig. 63, be given, and let it be required to find the traces of another which will cut into this one in a line of 30° to the H.P., and make a dihedral angle of, say, 70° with this given plane.

From any point V in RS draw a line on the V.P. at 30° to XY. With v, the plan of V, as centre, bring the plan of this line around by an arc until its lower end is at H in ST. Then, at any convenient place, cross this plan of the intersection line by the H.T. of plane 3, previously made use of in Figs. 60 and 61, and carry the point c into XY so as to show from the point there obtained the inclination of plane 3 and its intersection of the given intersection line at P. The bracket line, now measured off at cP_2, will give a point which when joined to a will show the rabattement of intersection of the given plane RST with the plane 3. Apply to this rabattement the angle 70° and so obtain the rabatted intersection P_2b of the required plane with plane 3, and b, a point in its H.T.

By joining H and b and producing the line to XY the H.T. of the required plane is found, and its V.T. may then be found by drawing a line, from this point in XY, through v'.

EXERCISE XXVI

Find the traces of a plane which makes with the given plane an angle of 65° and intersects it in a line inclined at 30° to the H.P. Two cases, A and B.



CHAPTER IX

PROJECTION OF RECTILINEAL ANGLES, AND OF FIGURES AND SOLIDS INVOLVING THE SAME

SECTION **18**. Two inclined lines meeting each other at a point above the H.P. may be made to form any angle as they meet, and if, say, two sticks of pencil, to serve for lines, be used experimentally, it may be seen that the projections of the two lines and of the angle they contain will vary according to how the pair of lines is disposed in relation to the planes of projection. Notice that the plane

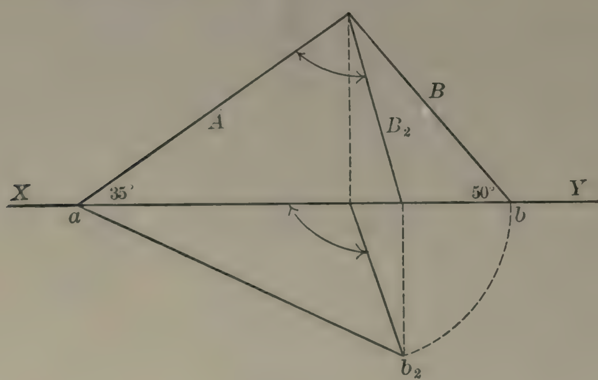


FIG. 64.

of the two lines, that is, the plane in which both have place, varies in its attitude, with the variation of the angle formed by the two lines, while the lines may still remain each at its own particular angle to the H.P.

As illustration, let there be two lines A and B. Let A be inclined at, say, 35° to the H.P., and B have an inclination of, say, 50° to the H.P. It must first be realized that the greatest possible angle that can be contained by these lines is 95°, and in order that this may be so, the plane of the lines would have to be vertical. Such an arrangement of them is shown in Fig. 64, where the plans of the lines are parts of the XY line. Realize that the lengths of their plans will not vary so long as the lines are not increased in length and their angles with the H.P. are not changed. The plans, however, may be moved the one toward the other so that instead of the distance between their lower ends or H.T.'s being as great as from a to b in the figure, it may be reduced to, say, the distance a to b_2, the line B being moved so as to have its H.T. at b_2. The angle contained by the two lines A and B is now reduced, since the distance a to b_2, subtending the angle, is reduced, but not the lengths of the lines. The new elevation of B is now at B_2, and the angle contained by A and B has been reduced to an angle of which the plan and elevation are indicated by the arrow-headed arcs. The true size of this angle can readily be obtained by rabattement about ab_2 as a hinge line, ab_2 being the H.T. of the plane of the two lines as now arranged.

As a specific case, let it be required to find plan and elevation for an angle contained by a pair of lines of any length when one of them is at 30° to the H.P. and the other is at 45° to the H.P., and the angle contained by them is 80°.

By placing both lines in the V.P. as in Fig. 65, (i), at A and B, starting them both from the same point and producing them downward to their H.T.'s at a and b respectively, their plans, proper for the particular angles they make with the H.P., are found. Allowing the 30° line A to remain in the V.P., the 45° line B must be moved toward A in order to reduce the angle from what it is at present (105°)

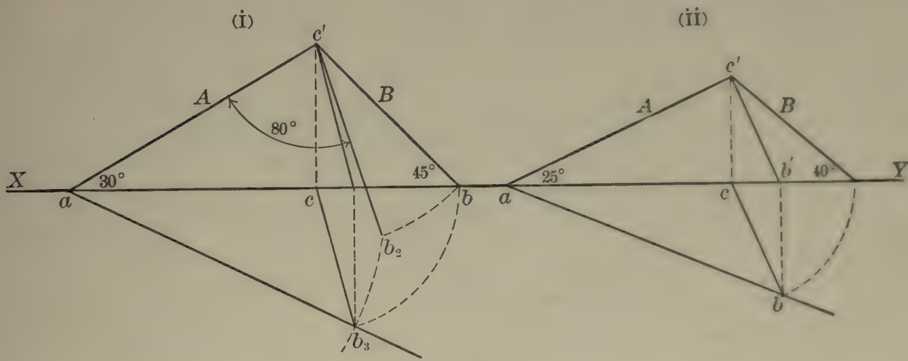


FIG. 65.

to 80°. This 80° will be subtended by a line, across the H.P. between the H.T.'s of the lines, the length of which must be found, therefore with c' as centre move B toward A until 80° is enclosed as shown by the arrow-headed arc. This gives the distance ab_2 required to subtend 80° when the lines are A and B. Now, with c as centre bring the plan cb round by an arc until the H.T. of B is at its proper distance from the H.T. of A. This will require the use of an intersecting arc with a as centre and ab_2 as radius. The new place for the H.T. of B is b_3. Join it to c for the new position of plan, and then obtain the elevation of the line as shown. By joining a to b_3 the H.T. of the plane of the two lines is obtained, and the V.T. of the plane of the two lines will be ac'.

In Fig. 65 at (ii) is shown a particular case which is very useful. The case is that of the right angle (90°) contained by two lines, the sum of whose angles to the H.P. is less than 90°. The two lines are first placed as at A and B in the V.P., with their plans therefore in XY. If, now, A be allowed to remain in the V.P. with ca as its plan, and B is to be placed so as to contain with A an angle of 90°, then the elevation of B will be a perpendicular to the elevation A, because, as we have seen previously, when a right angle is contained by two lines, and one of them is in a projection plane or parallel to it, then the other has its projection on the same plane perpendicular to the projection of the first. Therefore $c'b'$ is at once obtained as the elevation of the B line, and its plan can then be found by making a perpendicular from b' and intersecting it by an arc with c as centre, which carries down the plan length cb into its new place. Notice that by joining a to b the H.T. of the plane of the two lines is obtained, and, of course, the V.T. of the same plane is ac', coinciding with the elevation ac' of the line A which is in the V.P.

N.B.—When the sum of the angles made with the H.P. by the given lines is over 90°, the angle contained by them must be less than 90°. Consequently, for large contained angles the inclinations of the lines to the H.P. must be small accordingly.

EXERCISE XXVII

1. Find the plan and elevation of an angle of 45° contained by two lines when one of them is in the V.P. and inclined to the H.P. at 35°, and the other is at 50° to the H.P.

2. Find the plan and elevation of two lines, one at 25° to the H.P., and in the V.P., and the other at 45° to the H.P., when they contain an angle of 100°.

3. Find the plan and elevation of two lines, one at 45° to the H.P. and in the V.P., and the other at 30° to the H.P., when they contain an angle of 45°.

4. Find the plan and elevation of an angle of 90° when contained by two lines, one at 35° to the H.P., and the other at 20° to the H.P.

N.B.—In each case mark which is the plan of the contained angle, and which is the elevation of it.

An application of the problem above discussed may now be made to the finding of the projections of any rectilineal figure, triangle, square or polygon, when two adjacent sides, or a diagonal and an adjacent side, or two diagonals, are given at inclinations to the H.P. Instead of inclinations for edges, etc., levels may be given for corners, and the inclinations of edges or diagonals ascertained accordingly.

In Fig. 66 is represented in plan and elevation a square having one edge at 25° to the H.P. and in the V.P., and an adjacent edge at 40° to the H.P. As the angle contained by the two lines is 90°, the 40° line can be found in its proper place in relation to the 25° one, by the method of Fig. 65 (ii). The edge-length of the square must be marked off as $c'a'$ on the true length of the 25° line and as $c'b'_2$ on the true length of the 40° line. $a'c'$ gives ac as its plan, and from $c'b'_2$ may be obtained $c'b'$ and cb, the elevation and the plan respectively, of the 40° edge of the square. Opposite edges of the figure being parallels, their projections may be obtained by making them parallel to those already obtained. The square thus found may now be considered as the face of a cube, and it will be seen that by drawing a line through HH_1, which points are the H.T.'s of the two edges of the square, the H.T. of the plane of the square is obtained, and $Ha'c'$ will be the V.T. of the plane of the square. A perpendicular to this plane from, say, the point A may be made by the method explained in Fig. 52. If this perpendicular be made equal in length to the edge of the square, then it will be seen that the cube can be represented by drawing other perpendiculars with projections equal in length to the projections of this one and parallel to it; and by joining the upper ends of them the drawing will be completed. To correctly represent this cube as a solid the three edges in plan from the corner d should be made dotted lines, and the edge $a'c'$ in the elevation should also be made a dotted line.

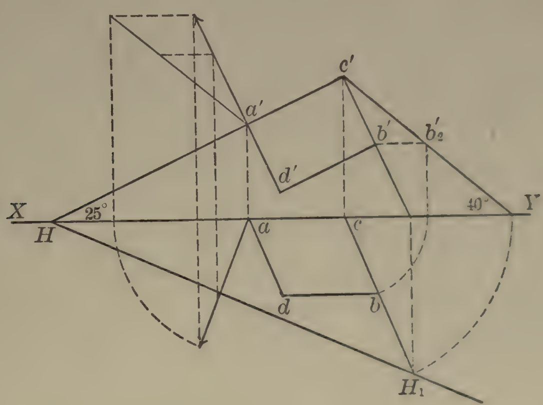


FIG. 66.

In Fig. 67 the plan and elevation of a regular pentagon (whose angle is 108°) is found, one edge being at 25° to the H.P. and an adjacent edge at 35° to the H.P. Following the construction shown in Fig. 65, (i), the inclined lines are found in position at ac, cb for plans, and ac', $c'b'$ for elevations. Point C is then rabatted to c_2, where the angle 108° appears at its true size, and the figure is formed with this angle for one of its corners. From the figure so formed in rabattement the corner points are carried over the H.T. line ab perpendicularly to the plan, and then from the plan the elevation is obtained. Advantage is taken of the fact that there is a diagonal of the pentagon parallel to each side of it.

The plan and elevation of the pentagon having been completed, the centre of it may be obtained by directing lines from the centres of any two sides to opposite corners and letting them intersect as at dd'.

The pentagon might be considered to be the base of a pyramid, and then a perpendicular from dd' may be found, of a given length, to serve as the axis of the solid, by the method of Fig. 52. The upper end of this axis, i.e., the apex of the pyramid, may then be joined by straight lines to the corners of the pentagon to complete the representation of the solid. Care must be taken to make certain lines dotted lines to represent them as being out of view, in the plan and in the elevation. In Fig. 67 the apex pp' is found, but the solid is not shown.

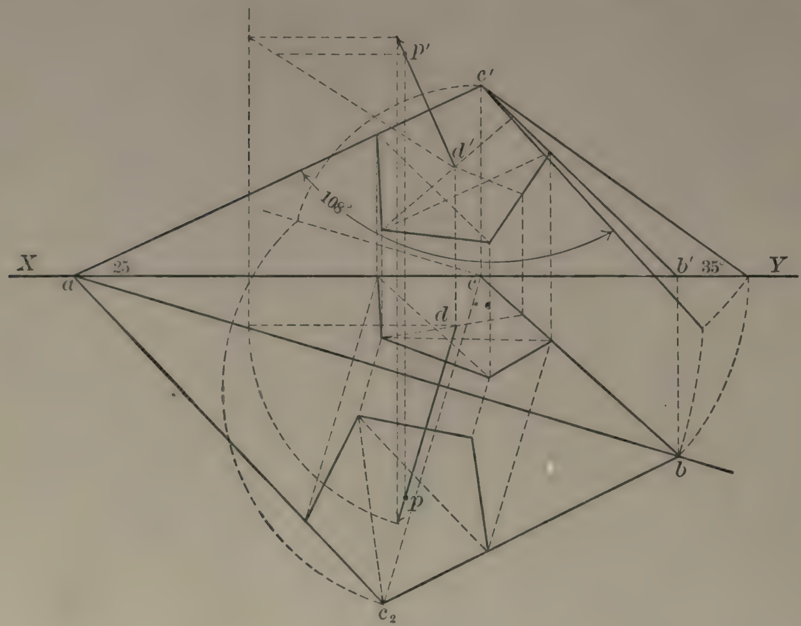


FIG. 67.

EXERCISE XXVIII

1. Find the plan and elevation of a regular pentagon $1\frac{1}{4}''$ edge, when one edge is at 20° to the H.P. and an adjacent edge is at 35° to the H.P.

2. Find the plan and elevation of a right hexagonal pyramid when one edge of the base is at 15° to the H.P. and an adjacent edge of the base is at 40° to the H.P. The length of the base edge is $1''$, and of the axis $3''$.

3. Find the plan and elevation of a cube $2''$ edge, when one edge is at 25° to the H.P. and an adjacent edge is at 50° to the H.P.

4. Find the plan and elevation of a pentagon, $1''$ edge, when three consecutive corners of it are at levels $\frac{1}{2}''$, $\frac{7}{8}''$ and $1\frac{3}{8}''$, above the H.P.

THE TETRAHEDRON AND THE OCTAHEDRON

SECTION **19**. Of the "regular" solids, we have already dealt with the *cube*, and we shall now proceed to consider two others, the *tetrahedron*, with four equal faces, each an equilateral triangle, and the *octahedron*, which has eight equal faces, each being an equilateral triangle.

The appearance and characteristics of these solids may be judged from the representations of them in Fig. 68 at (i) and (ii).

The tetrahedron has a corner perpendicularly opposite the centre of each face; while the perpendicular distance of a corner from the opposite face is the height of the solid. The simplest way by which to find the height of any tetrahedron is to place it on the H.P., as shown at (iii), and then, since all of its edges are equal, and the edge AB is parallel to the V.P., $a'b'$ shows true length, its plan being at ab. The height of the solid is therefore found to be $b'c'$.

The octahedron might be looked upon as two square pyramids, base to base, the height of each being equal to half the diagonal of its base. In other words,

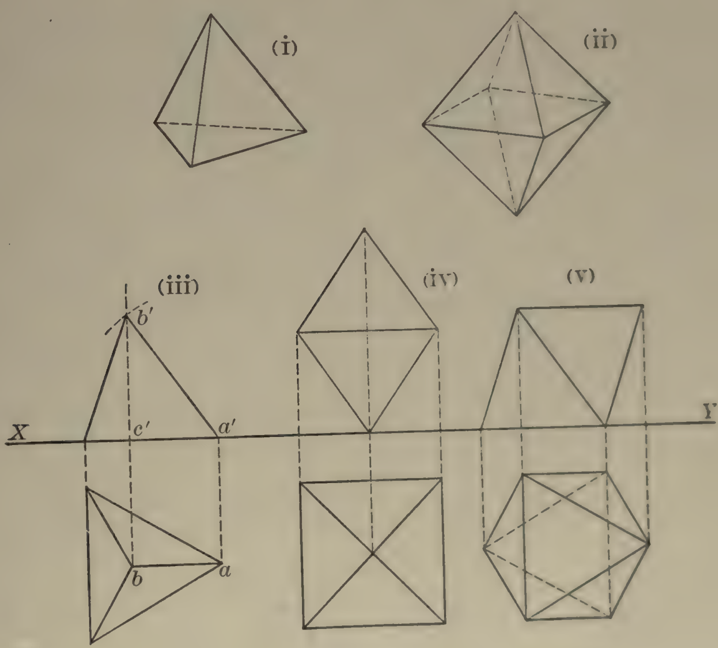


FIG. 68.

the solid has three diagonals at right angles to one another, and each one therefore at right angles to the plane of the two others.

The simplest way to arrange this solid is shown at (iv) in Fig. 68, where the full height in the elevation is made equal to a diagonal of the square representing the solid in plan.

At (v) the solid is shown cast over on to one of its faces, and in plan a dotted triangle for the under face, equilateral, and another, clear-lined for the upper face will be necessary. All the sloping edges are seen in plan and form a regular hexagon.

In Fig. 69 an octahedron, of given edge ab, is arranged so that one diagonal is in the V.P. at 25° to the H.P., and another diagonal is at 45° to the H.P.

From c' the half diagonal, obtained from the square on ab, is marked off on each of the inclined lines. When the plans and elevations of these have been determined, they are extended to make the complete diagonals. The third diagonal is then found, as a perpendicular to the plane of the two already found. Particular care must be taken in clear-lining the projections, so that the correct use of dotted lines is made for the edges not in view. It should be recognized that dd' is the near and upper end of the third diagonal.

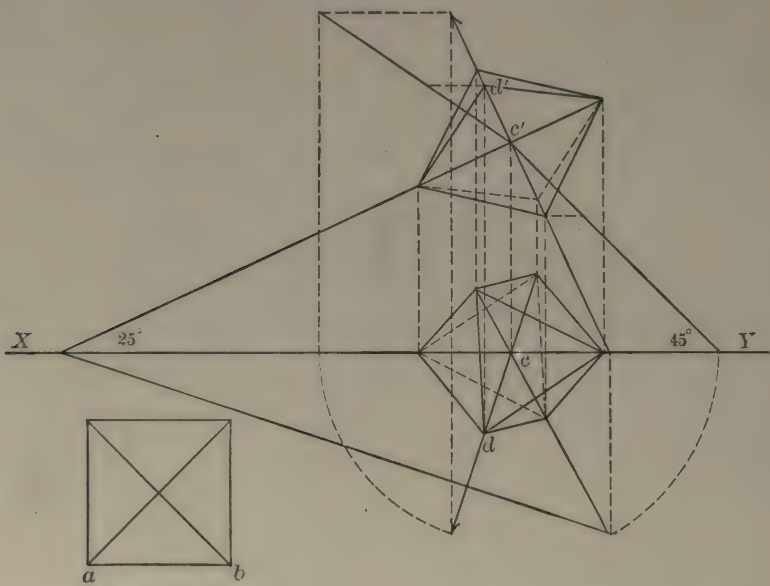


FIG. 69.

EXERCISE XXIX

1. Find the projections of a tetrahedron, 2″ edge, when one edge is at 25° to the H.P. and another edge is at 45° to the H.P.

2. Find the projections of an octahedron, 1½″ edge, when one diagonal is in the V.P. at 20° to the H.P., and another diagonal is at 35° to the H.P.

3. Find the projections of an octahedron 1½″ edge, when one edge is in the V.P. at 20° to the H.P. and a diagonal, adjacent to it, is at 40° to the H.P.

4. Find the plan of an octahedron, 1½″ edge, with one face in the H.P., and show an elevation of it, derived from this plan, on a plane to which no edge of the solid is parallel.

CHAPTER X

AXOMETRIC AND ISOMETRIC PROJECTION

SECTION **20**. Realize experimentally or otherwise the following facts:—

1. When two inclined lines meet, there is a plane of these lines which has its H.T. passing through the H.T.'s of the two lines;

2. When a line is perpendicular to a plane, the plan of the line and the H.T. of the plane are at right angles to one another;

3. Three lines may meet in a point, and each one be perpendicular to both of the others, that is, to the plane of the others; and

4. Three such lines, each perpendicular to the others, may either be equally inclined to the H.P. or all differently inclined.

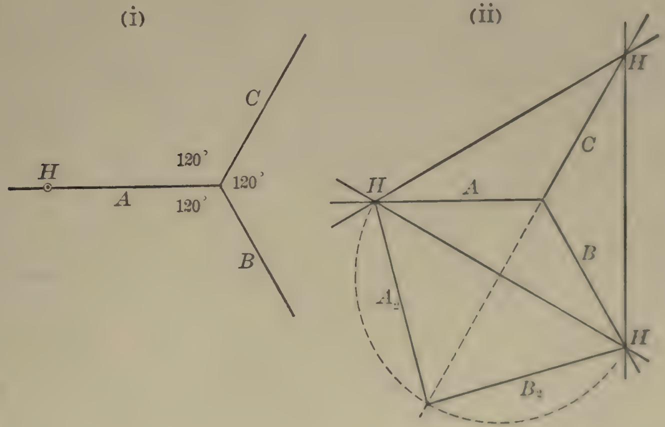


FIG. 70.

Let there be three lines, A, B and C, meeting in one point, and containing right angles with each other. If they be equally inclined to the H.P., the three right angles will appear in plan equal to one another, and the plans of the right angles will be 120° in each case, as at (i) in Fig. 70.

Now, if the point where A, B and C meet each other be at some distance above the H.P., then some point in one of them, say, H in A, will be the H.T. of line A, and the H.T. of B and the H.T. of C can then be placed similarly in the lines B and C at the same distance from the point where they all meet, as seen at (ii).

By drawing lines through these H.T.'s we have the H.T.'s of the planes of pairs of lines, and it will be seen that as the lines A and B contain a right angle, they will, when rabatted, expose that right angle, which is, of course, the angle of a semicircle; therefore, taking advantage of this fact, on HH make a semicircle, and by a perpendicular across the trace HH from the plan of the point where A and B meet, to the circumference of the semicircle, we obtain the rabattement of A and B as seen at A_2 and B_2, and so, true lengths are now shown for these lines A and B.

Next consider the case when the lines A, B and C are at different angles to the H.P., and let A be at 25° and B be at 45°, then by the method explained and illustrated in Fig. 65, (ii), the plans of A and B are found and their H.T.'s are at HH in Fig. 71. Through HH draw the H.T. of the plane of A and B, and the

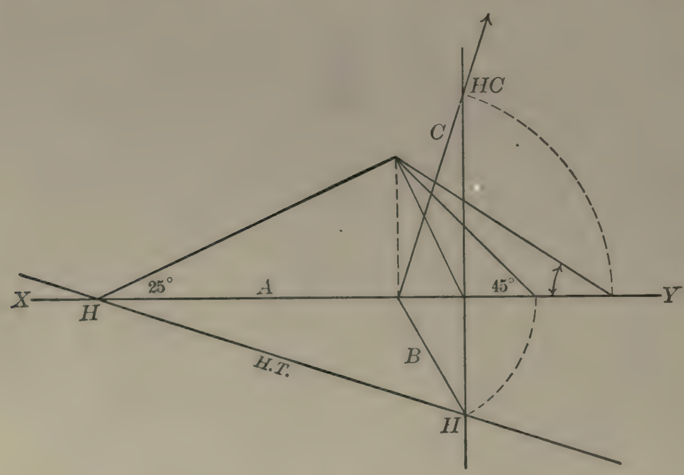


FIG. 71.

plan of C will be perpendicular to this H.T. line. This is shown by the arrow line in the figure. The H.T. of this arrow line can now be found by drawing the H.T. of the plane of B and C perpendicular to the plan A, or by drawing the H.T. of the plane of A and C perpendicular to plan B. This H.T. of C is indicated at HC.

From point HC to the point where the three lines meet is the plan of C, and this *plan* may be swung into XY and then the elevation of C, on the V.P., may be drawn, to show the angle, marked in the figure with an arrow-headed arc, just in the way that the lines A and B showed their inclinations, 25° and 45° respectively, when originally placed in the V.P.

As in the previous case, Fig. 70, (ii), since each pair of lines contains a right angle, true lengths for A, B and C may readily be arrived at by rabatting them into semicircles, as shown at A_2, B_2 and C_2, Fig. 72. Upon these rabattements any true length measurements may be made as at d_2 and then carried perpendicularly across to the plan line A at d.

This arrangement of three lines, A, B and C, provides a very useful means of making single projections or plans which show three dimensions to definite scales, and because of the three lines which serve as axes of direction for measurable distances, the method is known as *Axometric Projection.*

It will be recognized that the "scale" of measurement along each of the three lines, A, B and C, or *axes of projection*, as they are called, in Fig. 70 at (ii), is the same, because the axes are equally inclined to the projection plane. This is therefore referred to as *Isometric*—a special case of axometric projection. The scales of measurement along the axes A, B and C in Figs. 71 and 72, however, are all different from one another, so that, for instance, 2″ on the axis A is represented by a longer line than 2″ on the axis B.

As an illustration of the use of this method, let it be required to find the axometric projection of a skeleton cube, each face of which will appear as a foreshort-

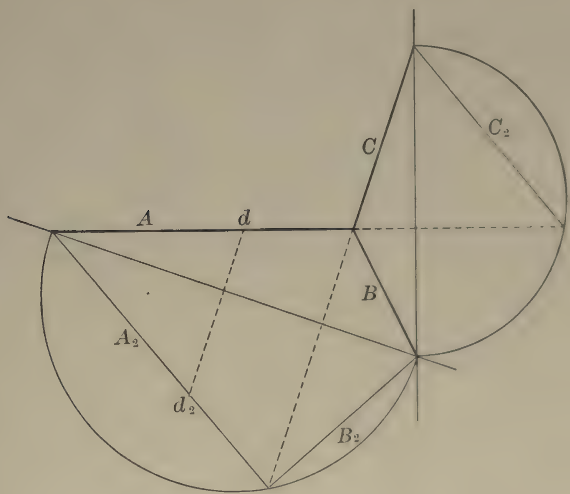


FIG. 72.

ened view of Fig. 73, (i). There will be twelve bars of square section, each one represented by three lines, as suggested at (ii). There are three directions for the edges of a cube, four edges to each direction, therefore the three axes of projection may each contain an edge. Let two of the axes be at 25° and 45° respectively, to the projection plane.

First find the axes A, B and C and rabattements of them at A_2, B_2 and C_2. On these rabattements mark off measurements taken from the edges of the square at (i), and transfer them to the axes by perpendiculars to the trace lines. Then by parallels the three upper faces of the cube, all foreshortened in appearance, may be found. Three of the bars are by this time represented, each by three lines. By the careful use of parallels each of the remaining nine bars may be found. All the twelve bars have square ends which may be shown as at (ii) in the figure.

The drawing here shown is not completed, but should be carried out on a large scale and completed by the student.

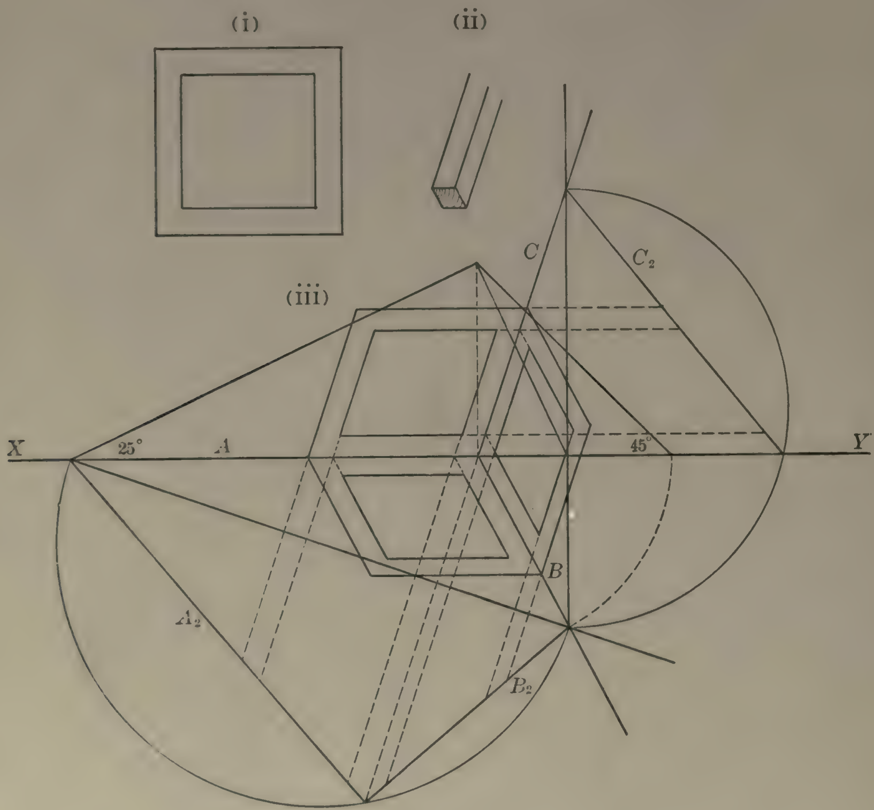
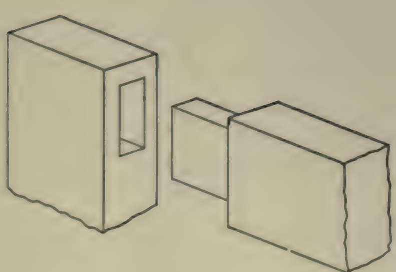


FIG. 73.

EXERCISE XXX



1. Find the projection of a 2″ cube when two of the axes of projection are inclined at 25° and 40° respectively to the projection plane.

2. Find the inclination to the projection plane of the third axis in 1.

3. Find by axometric projection a skeleton cube $3\frac{1}{2}''$ outside, $2\frac{1}{2}''$ inside, measurement. One axis is at 30° and another at 40° to the projection plane.

4. Find in isometric projection the two parts, separated as in the illustration herewith, of a mortise and tenon joint. Use dotted lines to represent those lines not in view.

AXOMETRIC PROJECTION, Continued

SECTION **21**. Since " scale," with respect to drawings, is the ratio of projection length to real length, it will be seen that the axes in axometric projection may be arranged in place when scales for any two of them are given, by setting up two lines at inclinations to the H.P. that will give those scales, and, having arranged their plans correctly, placing the third axis in proper relation to these. The scale of the third may then be obtained also.

Thus, suppose the scales for A and B are to be $\frac{3}{4}$ and $\frac{5}{6}$ respectively, the inclinations of A and B to the H.P. may be obtained as in Fig. 74 at (i), where ab, equal to 4 divisions, is represented in plan by ac equal to 3 of the same divisions;

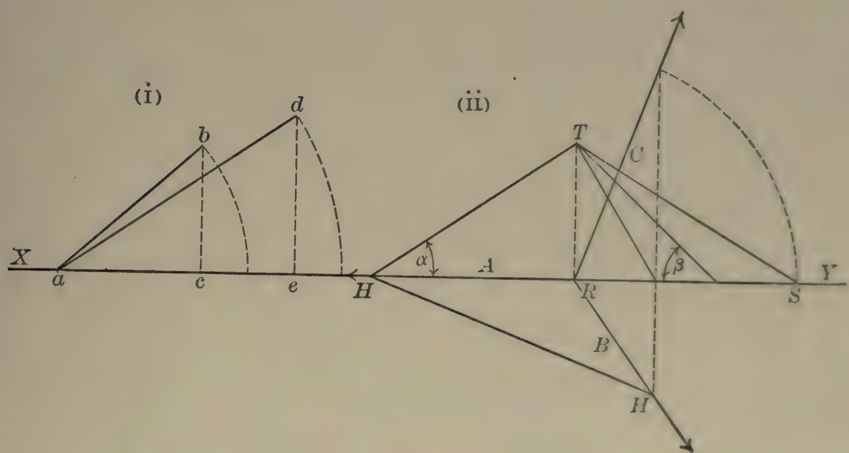


FIG. 74.

that is, ac is $\frac{3}{4}$ full length, and the inclination for this scale of $\frac{3}{4}$ is the angle bac. Again, ad, equal to 6 divisions, is represented in plan by ae, equal to 5 divisions, i.e., ae is $\frac{5}{6}$ full length and the angle for the axis which will have a scale of $\frac{5}{6}$ will be the angle dae.

Employing the method as shown in Fig. 71, set up at Fig. 74, (ii), the angle dae at α and the angle bac at β, and proceed to find axes A and B with their H.T.'s at HH. The third axes, C, will be found and its H.T. obtained as in Fig. 71. By swinging the plan of that portion of it above the H.P. into XY, the plan length RS and the real length TS are obtained, which will give the scale for the third axis C as $\frac{RS}{TS}$.

From what was seen previously (see Note in Section 18), the contained angle between the axes of projection being 90°, the inclinations of them to the projection plane must be small, so that the sum of them is less than 90°, and this limits the use of axometric projection to large scales only, if the drawings are to serve the purpose they are intended to serve.

Suppose the arrangement of the axes of projection to be given in plan as at (*i*), Fig. 75, and let it be required to find the projection of some solid having its edges or other lines in the directions of the axes given. First choose any point in one of the lines, say, H, in axis A at (i), and proceed to find points in B and C that are at the same level, by lines from H perpendicular to C and B respectively. Then by rabatting the right angle contained by axes A and B, also by A and C, into semicircles, true lengths are obtained at A_2, B_2 and C_2, and the measurements of the cube or other solid may be marked on these and transferred to A, B and C respectively, as in Fig. 73 at (iii).

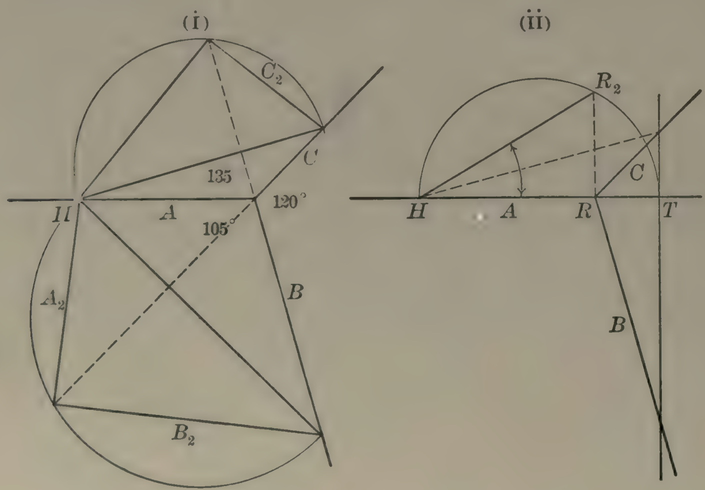


FIG. 75.

If the inclinations of these given axes, Fig. 75, are required to be found, it will be seen that a vertical plane containing one of them, say, axis A, will cut the plane of the others in the line RT, see (ii), and that HR is at right angles to RT, therefore the vertical semicircle on the line HT for diameter, rabatted as at HR_2T, will show the angle the axis A makes with the projection plane, namely, the angle R_2HR. This establishes the height of R above the projection plane. Hence, if the H.T.'s of B and C be brought around into the line HRT, their angles will likewise be seen.

A further illustration is given in Axometric Projection in Fig. 76. Let it be required to find the projection of a pentagonal pyramid when one edge of the base is at 25° to the projection plane and the axis of the solid is at 40° to the same plane.

First find the three axes of projection. As the axis of the solid has to be in the 40° direction, the base must be placed in the plane of the two others. The base, not being right angular, must be placed in a rectangle as at (ii). As AB at (ii) contains an edge of the base, the line AB must be fitted on to the 25° axis of projection. Therefore, place it in rabattement at A_2B_2 and draw the complete figure contained in the rectangle $B_2A_2C_2$, and proceed to carry it over to its position in plan. Having completed the base in plan, find its centre C, and from that centre draw a line in the direction of the 40° axis. Find, by the proper method, the projection length of the axis at c_2a_2 or c_2b_2, according to its length, and then mark this off on the axis line of the pyramid starting from c, thus giving the apex at a or b. Join the apex to the corners of the base and finish as usual with dotted lines for some edges as the case may require.

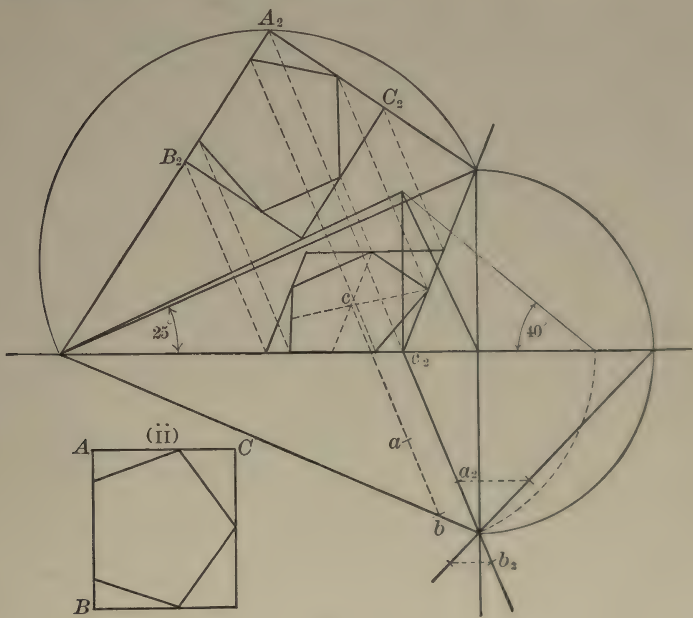


FIG. 76.

If a circular hole has to be represented, in a block, for instance, in axometric projection, points should be obtained in the circumference by using diagonals and parallels to the sides of a square made to contain the circle, the sides of the square being made to follow axis directions in the projection.

EXERCISE XXXI

1. The plans of the three axes of projection enclose angles of 110°, 120° and 130°. Find the inclinations of the three axes to the projection plane.

2. The scales of two of the axes for a projection are $\frac{3}{4}$ and $\frac{4}{5}$. Find the projection of the axes and represent the scale of the third axis.

3. Find by axometric projection the plan of a regular hexagonal pyramid, when the axis of the pyramid, 3″ long, is at 30° to the projection plane, and one of the edges of the base, 1″ long, is at 45° to the projection plane.

CHAPTER XI

SECTIONS OF SIMPLE SOLIDS

SECTION **22**. Sections of solids are made by passing planes through them. By rabatting these planes, carrying with them the outline points of the section shape, the true form of the particular section may be obtained. For convenience, the section plane is usually arranged either perpendicularly to the H.P. or perpendicularly to the V.P.

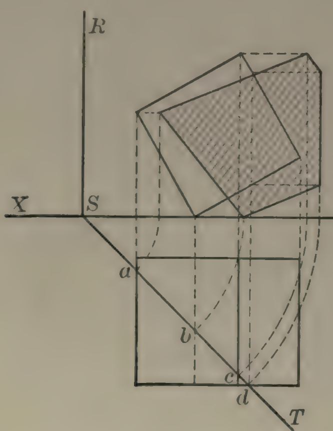


FIG. 77.

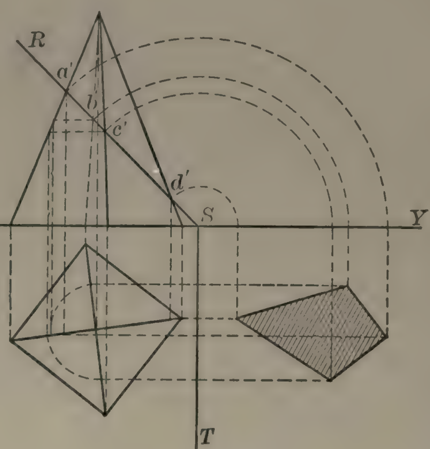


FIG. 78.

Illustrations of the process are shown in several figures now to be considered.

In Fig. 77 a cube has a vertical section plane *RST* passing through it. Horizontal edges are cut by it at *a*, *b* and *c*, while at *d* two of the inclined edges are cut. Realize that three inclined faces are cut and one vertical one. By using *S* as a centre and making rabattement of all the points in which the plane cuts the edges, the true form is obtained on the V.P. It is covered with hatching or shading in the figure.

In Fig. 78 a square pyramid has a section plane *RST* perpendicular to the V.P., passing through it. The slant edges of the pyramid are all cut in points whose elevations are marked $a'b'c'$ and d'. By using *S* as centre and rabatting all these points on to the H.P. the true form is obtained. Notice that it is necessary to have the plans of the four points. These are readily found in the case of *a* and of *d*, but as the projections of the edges in which *B* and *C* occur are so nearly perpendicular to XY these points are carried by level lines to the slant edge in which a occurs and from their plans in the plan of this edge they are carried by arcs to their proper places in the plans of the edges they belong to, and thence by perpendiculars across the H.T. line to their rabattements. The true form may now be drawn and is marked in the figure by hatching.

In Fig. 79 a right circular cylinder has a section plane perpendicular to the V.P., passing through it in such a way as to cut the top horizontal surface and cut also a considerable amount of its curved vertical surface. Since the section shape of a right circular cylinder by a plane inclined to its axis is an ellipse, the result in this case will be part of an ellipse, as the shape of the section. Points in the curved outline of the true form are obtained by choosing straight generator lines, as they are called, on the curved surface of the cylinder, marked in this example at a, b

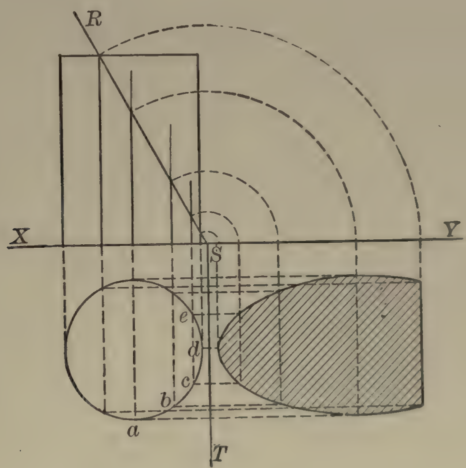


Fig. 79.

c, d, e, etc. Portions of the elevations of these generators are shown in order to obtain the elevations of the intersection points to be rabatted. In the figure the shape of section is shown covered with hatching.

In Fig. 80 a right circular cone has a section plane RST passing through it, and since this plane cuts all the generators of the curved surface it produces an ellipse for the shape of the section. This shape is rabatted on to the H.P. by using S as centre. Generators, taken in such places as will give a good distribution of points in the curve, are drawn at a, b, c, d, e, etc. These in elevation are seen to be cut by the section plane, and the points in which they are cut are carried over by arcs to the H.P. It is necessary to locate them in plan, which is straightforward work except in the case of generator bb'. The point in this generator must be carried horizontally to the generator e, and its plan carried back to the generator b by an arc as seen in the figure.

Fig. 81 shows a right circular cone with a section plane *RST* passing through it so as to give the hyperbola as a section. The plane for the hyperbola is parallel to the plane of two generator lines of the cone, whereas the plane for a parabola has only one generator line of the cone parallel to it. The two generators parallel to the plane of the hyperbola in this case are shown at *a* and *b*, and their Horizontal Traces are at *hh*. The curve of the hyperbola is shown in plan at *cc* with its true form by rabattement at c_2c_2. The H.T.'s of this curve are at *gg*.

Now, if the line *cc* which lies on the curved surface of the cone be continually produced in the same plane it becomes straight, at infinity, on opposite sides of the cone, and since the plane of the parallel generator lines *A* and *B* is at such

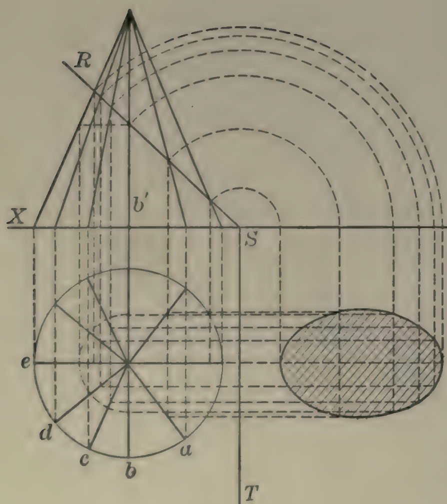


FIG. 80.

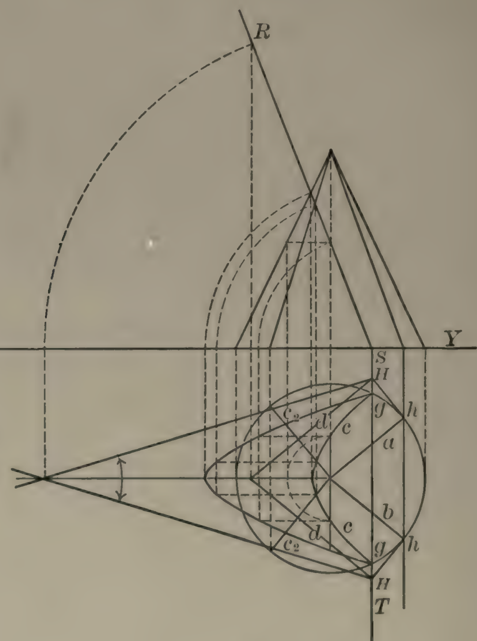


FIG. 81.

a small distance from the plane of the hyperbola, the tangents to the hyperbola lines, at infinity, and the generator lines near them are pairs of parallels. These tangents and generators on the surface of the cone at infinity will meet the circumference of the circular section of the cone at infinity, at right angles, and the H.T.'s of the planes of the pairs of parallels at infinity will be perpendiculars to the generators whose plans are *a* and *b* respectively.

Consequently, on the Horizontal Plane of projection, from points *h* perpendiculars to the plans *a* and *b* will pass through the H.T.'s of the tangents at infinity to the hyperbola, which tangent lines also lie in the plane of the hyperbola. Therefore *HH* are the H.T.'s of the tangents at infinity to the hyperbola, and the plans of them are the lines *dd*, made parallel in plan to the generators *a* and *b*.

The tangents at infinity to the hyperbola are called the *Asymptotes*, and meet each other at an angle. In Fig. 81 they are rabatted on to the H.P. and show the angle contained, which is marked by an arrow-headed arc.

Strictly speaking, the asymptotes, lying outside the two branches of a hyperbola, which are sections of two equal cones united at their apexes and having the same axis line, are lines which never really meet the curves of the hyperbola.

EXERCISE XXXII

1. A square prism, $1\frac{1}{2}''$ edge of end, $2\frac{1}{2}''$ long, has a long edge in the H.P. at 30° to XY, and the face adjacent to it is at 30° to the H.P. Find the true form of section by a vertical plane at 45° to the V.P., passing through the solid and cutting its axis $1''$ from one end. Two solutions.

2. Find the true form of section of a right hexagonal pyramid, base in H.P. with one diagonal of base parallel to XY, by a plane perpendicular to V.P., at 45° to the H.P., and passing through a point in the axis $\frac{5}{8}''$ from the base. Base edge $1''$. Axis $2\frac{1}{4}''$.

3. Find the true form of a parabola by a plane passing through a cone, when its base is $2''$ diameter, and height of apex of the cone above the base $2\frac{1}{2}''$. The plane cuts the axis in the centre.

4. Find the asymptotes and show the angle contained by them in the case of a hyperbola made by a plane at 15° to the axis of a cone, cutting the axis in a point $1''$ from the apex. The angle of the cone at the apex is 40°.

TRACES OF CURVED SURFACES

SECTION **23**. When a right circular cylinder is inclined to a projection plane, its curved surface, produced to meet the plane, will meet it in the curve of an ellipse, for the projection plane might be considered as a section plane passing through the circular cylinder at an angle to its axis. This applies also to the curved surface of a cone. The line in which the curved surface of the cone meets the projection plane will be an ellipse, if the cone is a right circular one, i.e., one that has a circular section in any plane at right angles to its axis.

The line of intersection in which the curved surface of a cylinder or of a cone, whether its axis be inclined or not, will meet the projection plane, if the curved surface is produced to do so, is called the trace of the curved surface, and may be the Vertical trace or the Horizontal trace according to whether it is found on the V.P. or on the H.P. Illustrations are shown in Figs. 82 and 83.

It will be seen that in the case of the right circular cylinder, the H.T. is an ellipse with its minor axis equal to the diameter of the cylinder, whereas in the case of the right circular cone the ellipse for H.T. has a minor axis which varies in length with the inclination of the cone or with the distance of its apex from the projection plane. The method of obtaining the trace is to choose any number of generator lines and produce them until their traces are found, then to draw, by freehand, the curve through these trace points.

A further illustration is shown in Fig. 84, where it is important to realize that the generators whose plans are at *aa* have their elevations at $a'a'$, and that

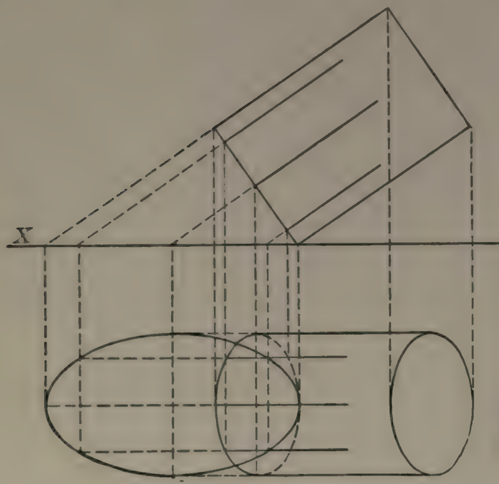


FIG. 82.

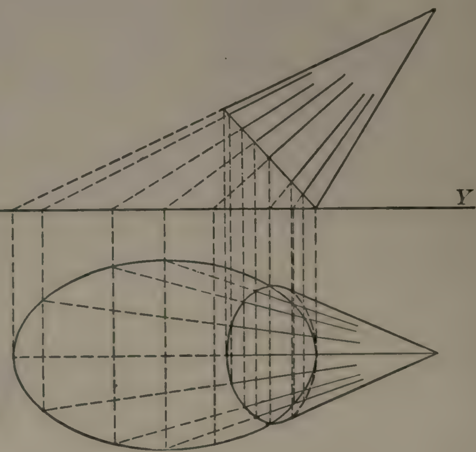


FIG. 83.

these are best located by recognizing the fact that they pass through the ends of the horizontal diameter of the end of the cylinder. The topmost generator

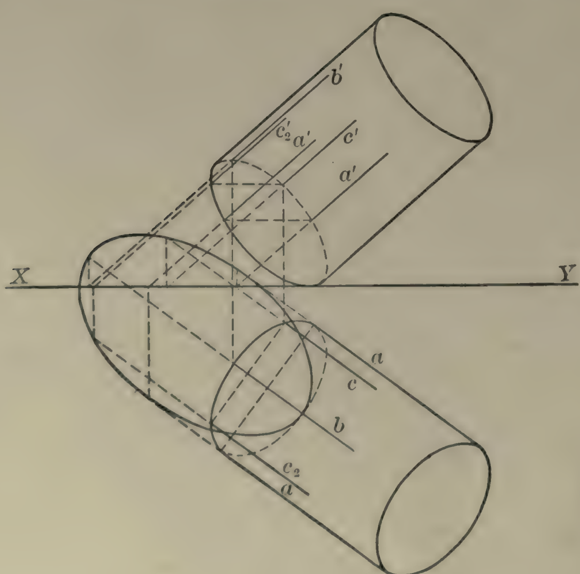


FIG. 84.

line, having its plan at *b*, has its elevation at b', which is not the topmost line of the elevation. Other generators, chosen at convenience for the purpose of securing points in the trace, such as those marked cc_2 in the figure, should meet the end of the cylinder in pairs, at the ends of horizontal chords, for then the elevation marked c'_2 can the more certainly be located by making use of the horizontal chord through the end of the one marked cc'.

N.B.—The projections for the circular ends of the solids should be carefully found by the method previously discussed. (See Section 4, Fig. 8.)

EXERCISE XXXIII

1. Find the H.T. of the curved surface of a right circular cylinder, $1\frac{1}{2}''$ diameter, when its axis is at $35°$ to the H.P. and the plan of it is at $25°$ to XY.

2. Find the H.T. of the curved surface of a right circular cone, whose base is $1\frac{1}{2}''$ diameter and axis $2\frac{1}{2}''$ long. Let the axis be parallel to the V.P., and inclined to the H.P. at $45°$.

3. Show the plan and elevation of a cone whose H.T. is a circle $2\frac{1}{2}''$ diameter, and whose axis is $3''$ long and inclined at $40°$ to the H.P. and $40°$ to the V.P. Show the V.T. of its curved surface when the centre of the circular H.T. is $1\frac{1}{2}''$ from XY.

4. Find the shadow of a sphere of $2''$ diameter, having its centre $2''$ above the H.P. and $2''$ in front of the V.P., when the rays of parallel light are inclined at $45°$ to the H.P. and $30°$ to the V.P.

N.B.—The shadow problem here given involves the method discussed in this section.

PART THREE

CHAPTER XII

TANGENT PLANES TO CONES AND CYLINDERS

SECTION **24**. Since the generator of a cylinder, and also of a cone, is a straight line, it is evident that a plane, tangent to or touching the curved surface

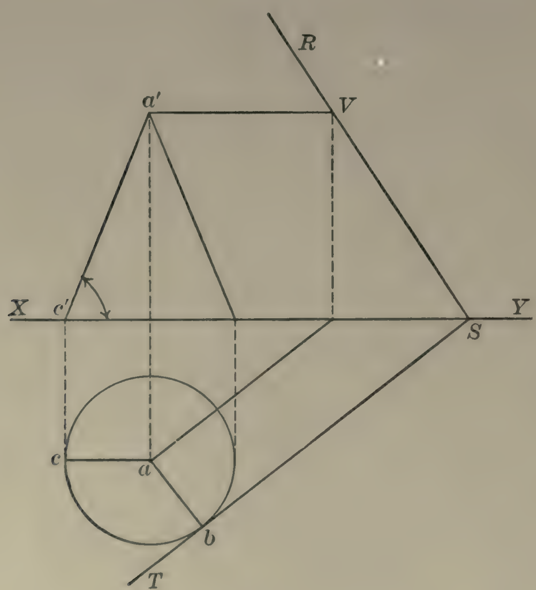


FIG. 85.

of one or of the other, will touch it along a straight line, and, in the case of the cone, will include its apex. Such a tangent plane may be shown, as usually planes are shown, by its traces, and its traces will be tangential to the curved traces of the curved surface of the solid.

In the illustration, Fig. 85, a right circular cone, that is, one whose section perpendicular to its axis is circular, stands vertically on the H.P., consequently, the circle, which is the circumference of its base, is the H.T. of its curved surface. A plane, tangent to this cone, will therefore have its H.T. tangent to this circular H.T., and this is shown at *ST*. As the apex *aa′* is in the tangent plane, a horizontal line may be drawn through it, lying in the tangent plane. This will have its plan parallel to *ST*, and its elevation will give its V.T. at *V*. *RS*, the V.T. of the plane, may then be drawn.

It will be realized next, that, as all the generators of the vertical cone meet the H.P. at the same inclination, therefore the generator whose plan is *ab*, is at the same inclination as the generator whose elevation is *a′c′*, and whose inclination is indicated by an arrow-headed arc in the figure; and, since the plane is at the same inclination as a line in it perpendicular to its trace, and *AB* is perpendicular to the trace *ST*, the tangent plane *RST* is at the angle indicated at *c′*.

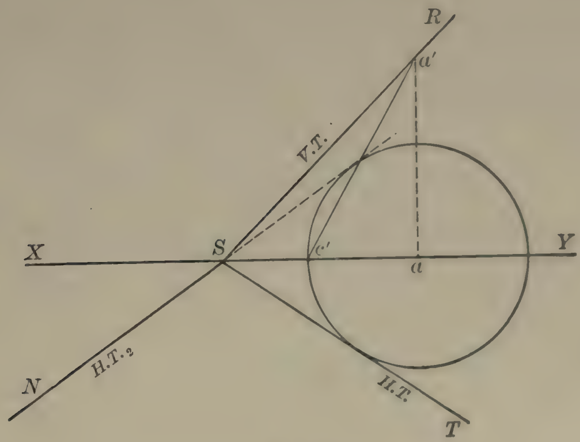


FIG. 86.

The converse of this problem is specially useful. For instance, if it be required to find the traces of a plane of given inclination to the H.P. having its H.T. in a given direction, it is only necessary to set up a right circular cone whose generators are at the given inclination for the required plane, and then make the H.T. of the plane tangent to the circular trace, the base of the cone is the H.P., and proceed as before to find the V.T. of the plane.

In Fig. 86 the V.T. of a plane is given at *RS*. Suppose the plane, whose V.T. is *RS*, to be inclined at 60° to the H.P., and let it be required to find its H.T. Then, because any point in this V.T. is in the plane and may be taken as the apex of a right circular vertical cone to which the plane is tangent, from any such point *A* draw the line *a′c′* in the V.P. at 60° to the H.P. The circumference of the circle drawn, with *a* as centre and *ac′* as radius, is the H.T. of the 60° cone, and the point *A*, its apex, is contained in every tangent plane to the cone. Therefore from the point, namely *S*, in *XY*, where the V.T. line meets it, draw the H.T.

line, ST, of the required 60° plane, making it tangent to the circular trace of the cone. The line SN shows the H.T. of another plane, having the same V.T., and tangent to the same 60° cone, therefore also at 60° to the H.P. There are, therefore, two planes, RST and RSN, satisfying the conditions.

In Fig. 87 a line AB is given, and it is shown how a plane of any particular inclination to the H.P., greater than the inclination of the given line to the H.P., may be found, when it has to contain the given line. For the solution, any point cc' in the given line may be taken, to serve as the apex of a right circular vertical cone.

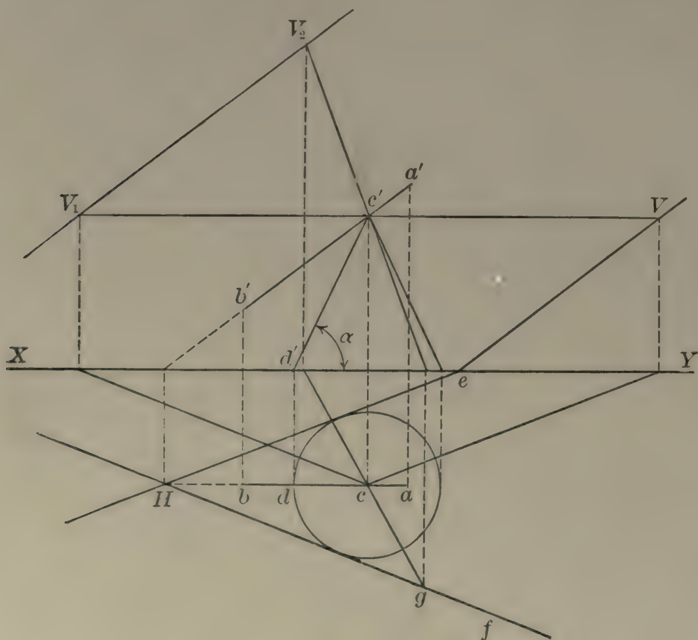


FIG. 87.

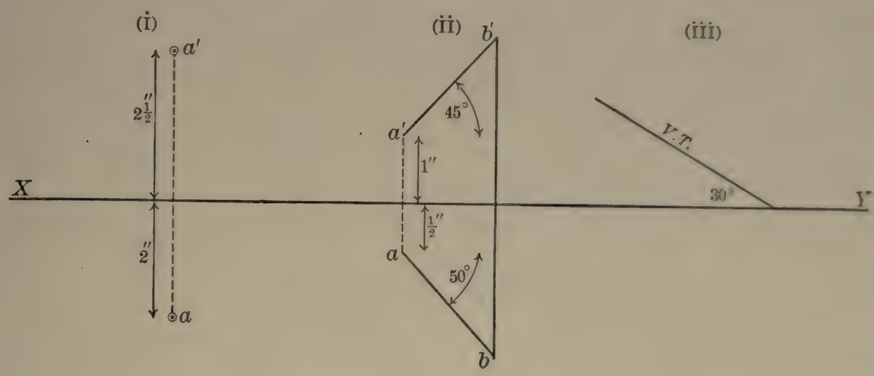
The cone may be represented by drawing the line $c'd'$ to make with XY an angle, α, equal to the given inclination of the required plane, and with cd, its plan, as radius, drawing the circular H.T. of its curved surface. Next, find the H.T. of the given line AB at H, and through H draw tangent lines to the circle. These are the horizontal traces of two tangent planes to the cone, each containing the given line. Horizontal lines on these planes, drawn through the apex C, will give points V and V_1 in the Vertical Traces required to more completely determine the planes. HeV is one of the planes. The other plane, whose H.T. is Hf, gives an opportunity of showing how to obtain the V.T. without first making Hf meet XY. Thus, from any point g, in Hf, draw the plan of a line through some point in the line HA—through the apex C will do—and produce this plan and its elevation till its V.T. is found at V_2. Join V_1V_2 and this line is part of the Veritcal Trace of the second plane, part of whose H.T. is the line Hf.

EXERCISE XXXIV

1. Find a plane at 50° to the H.P. containing the given point A at (i). Let the H.T. of the plane be at 45° to the XY.

2. Find a plane at 65° to the H.P. and containing the given line AB at (ii).

3. The V.T. of a plane is given at (iii). Show the H.T. of it when the plane is at 45° to the H.P. Two solutions.



By experiment with a cylindrical object—a roll of paper will do—it should be realized that if the cylinder is inclined, there may be any number of planes tangent to it, of angles to the H.P. not less than the angle of inclination of the axis of the cylinder to the H.P.

Thus, in Fig. 88 at (i) the plane RST is tangent to the inclined cylinder and is at the same inclination to the H.P. as that of the cylinder. Any plane of greater inclination may be made tangent to the same cylinder, until one is reached that is vertical, such as that whose H.T. is at LL. If the cylinder were inclined to the V.P. also, then this plane LL would have a V.T. perpendicular to the XY.

As previously noted, the condition for a plane to be tangent to a cylinder is, that it shall contain a straight-line generator of the curved surface, and as all such generators have their H.T.'s in the curve of the ellipse serving for the H.T. of the cylinder, the H.T. of any tangent plane to the cylinder must be a tangent line to the curved H.T. of the surface of the cylinder.

Now let ab, $a'b'$, at (ii), be the projections of a line parallel to the generators of the given cylinder, and from some point in it, serving as the apex of a cone, arrange a right circular vertical cone whose generators are at, say $\alpha°$ to the H.P. Two $\alpha°$ tangent planes may now be represented by their H.T.'s at ac and ad, to contain the line AB; and because parallel planes have their H.T.'s parallel, and AB is a parallel to all the tangent planes to the given cylinder, being parallel to all its generators, there will be four α° planes, two parallel to plane *ac* and two parallel to *ad*, and they will be shown by their H.T.'s, tangent to the curved H.T.

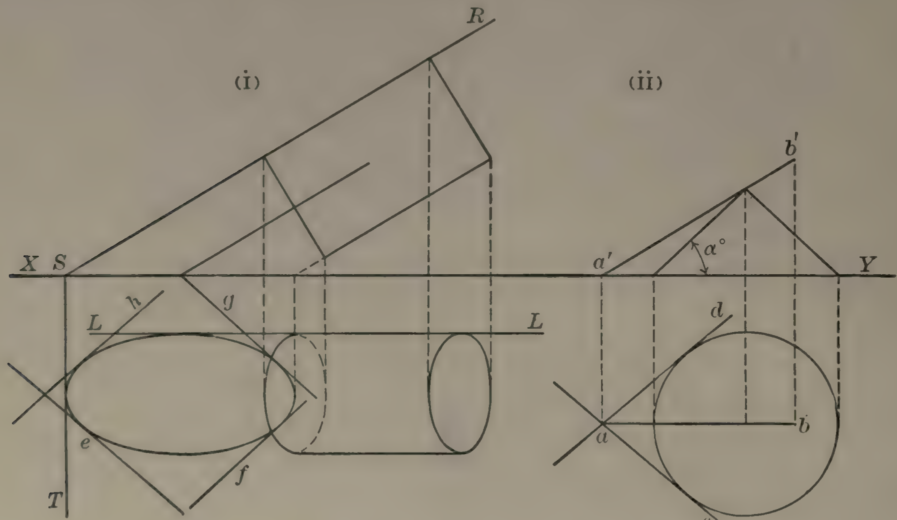


Fig. 88.

of the cylinder, as at *e*, *f*, *g* and *h*. Each of these four planes contains one generator of the cylinder, and therefore, since the generators of the cylinder are

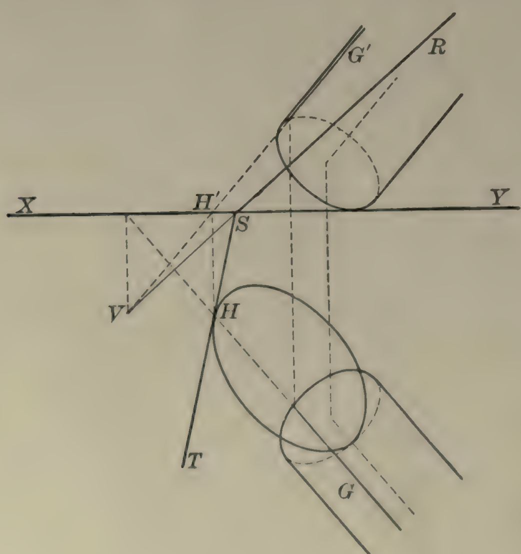


Fig. 89.

parallel to the V.P., the V.T.'s of these planes will be parallel to the elevations of the generators of the cylinder. Therefore, by producing the H.T. marked *g* to XY, the V.T. for it may be drawn as shown, parallel to the generators of the cylinder. So, also, with the V.T.'s of the other planes. N.B. The ellipse for H.T. should be obtained by the method shown in Section 23.

In Fig. 89 is shown a right circular cylinder whose axis is inclined to both planes of projection. Its H.T., an ellipse, is also shown. Suppose a tangent plane

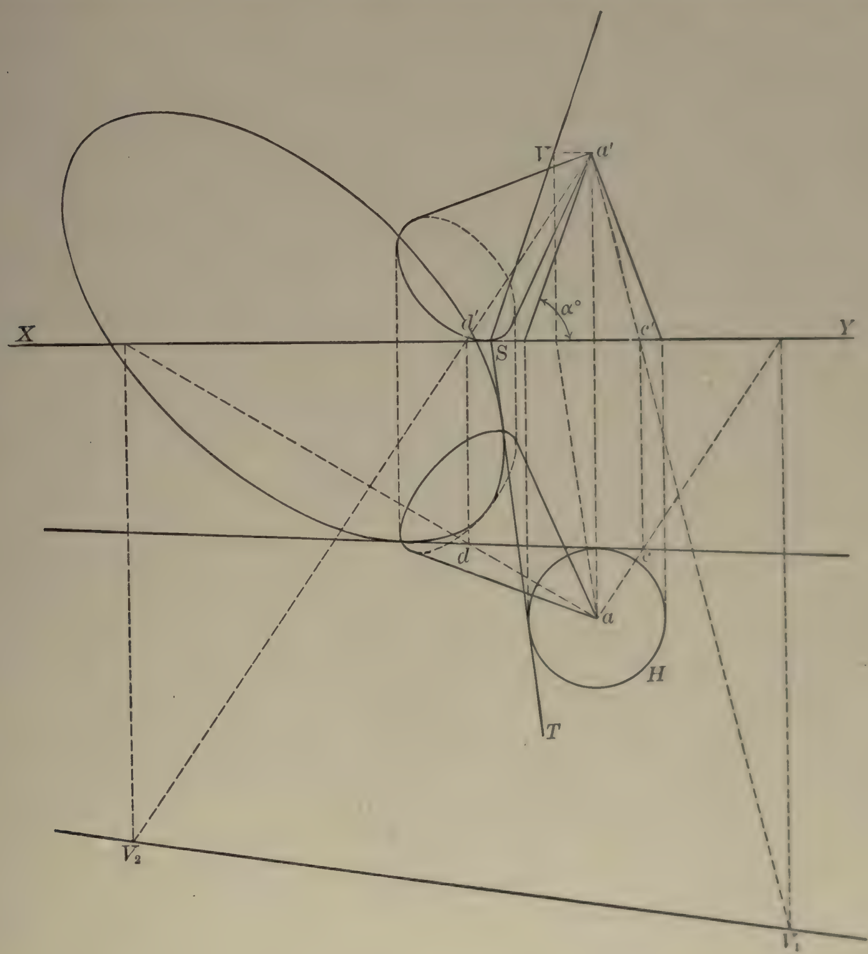


Fig. 90.

to this cylinder has ST for its H.T., obtained as in the previous case; then because the point H is the H.T. of a generator of the cylinder, and H' is the elevation of H, the generator having H for its H.T. can be located, as shown in the figure at GG', and produced until its V.T. is found at V. Now draw VSR, which is the V.T. of the tangent plane. So proceed with others.

In Fig. 90, a right circular cone is represented in plan and elevation, with axis inclined to both planes of projection. Its H.T., the large ellipse, is also shown.

Let it be required to find the traces of planes tangent to this inclined cone and having a given inclination to the H.P. Since all planes, tangent to the given cone, must pass through the apex of it, and a plane of a particular inclination, say $\alpha°$, must be tangent to a vertical right circular cone whose generators are at that particular inclination, it is necessary to make use of the apex of the given inclined cone to serve as the apex of such a vertical one. The H.T. of the vertical cone referred to is the circle H in the figure.

It should now be realized that a plane tangent to both cones at the same time, will have its H.T. as a common tangent to the H.T.'s of the two cones, and will contain the common apex A. The H.T. of such a plane is shown at ST, and to obtain the point V in its V.T., a horizontal line, on the tangent plane, may be drawn through the apex aa'. The line SV is the V.T. required.

In the case illustrated in Fig. 90, three other tangent planes of the same inclination may be found. If the H.T. of one of them is parallel, or nearly parallel, to the XY, or at such a small angle with it that there is not room for the point to be located in XY, where the H.T. and the V.T. meet, then two inclined lines on the plane required may be drawn through the apex and through any convenient points c and d in the H.T., and their V.T.'s found at V_1 and V_2. The line drawn through these V.T. points is the V.T. of the tangent plane whose H.T. is the line upon which the points c and d were chosen.

EXERCISE XXXV

1. Find the four tangent planes, each at 65° to the H.P., to a right circular cylinder whose axis is at 40° to the H.P. and at 25° to the V.P. Diameter of cylinder $1\frac{1}{2}''$. Let the cylinder rest on a point in the H.P. $2''$ from XY.

2. Find the traces of planes inclined to the H.P. at 65° and tangent to a right circular cone of $1\frac{1}{2}''$ base, and $2\frac{1}{2}''$ axis, when the axis is at 40° to the H.P. and the plan of the axis is at 45° to XY. Let the cone rest on the H.P. in a point $1\frac{1}{2}''$ from the XY.

PROJECTION OF SOLIDS DEPENDENT ON TANGENT PLANES TO RIGHT CIRCULAR CONES

SECTION **25**. In finding the projections of prisms, cubes and pyramids when different faces of the same solid are at different inclinations to the same projection plane, it is necessary to make use of the plane tangent to a right circular cone. Illustrations are given and explained in Figs. 91 and 92.

In Fig. 91 the method is shown for finding the plan of a cube, or of a right prism, when one face is at, say, 40° to the H.P. and another face is at, say, 70° to the H.P.

RST is the 40° plane in which one of the faces will be found, and is arranged perpendicularly to the V.P. for greater convenience. The line AB, shown in plan at ab and in elevation at $a'b'$, is an edge of the cube perpendicular to the 40° face and therefore may serve as an edge of the 70° face.

The 70° plane, to include this edge of the 70° face, is found, according to the method recently explained, by the use of a cone of 70° with its apex in the line AB or that line produced. This 70° plane, thus containing AB, has its H.T. at Hh.

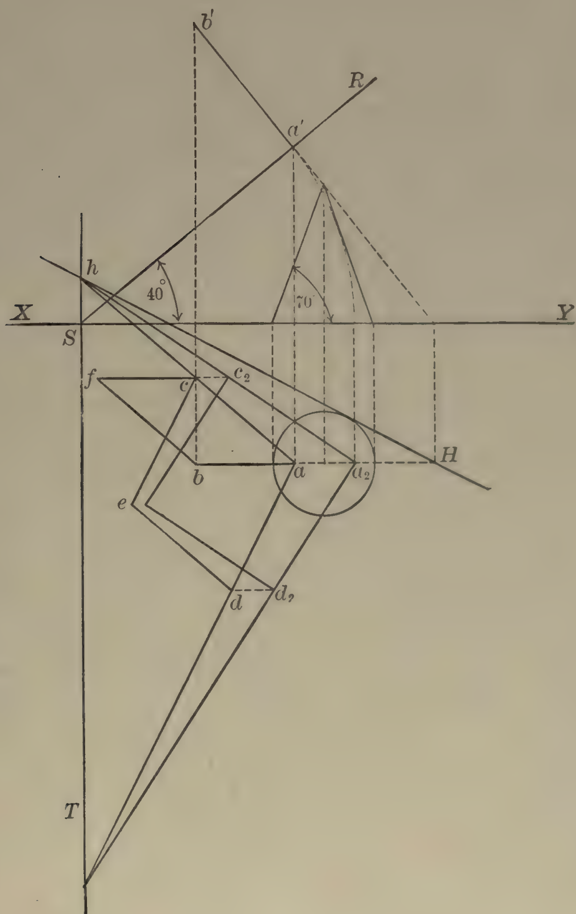


FIG. 91.

The point A is common to both planes, the 40° and the 70°; so, also, is the point h where the two planes' H.T.'s meet. Join h to a therefore, and so obtain the plan of the intersection of the 40° plane with the 70° plane. This line will contain the edge of the cube common to the two faces whose inclinations are given. Rabatte ha to ha_2, and using a_2 as the corner for a square, mark off a_2c_2 on the rabatted line, and also make a_2d_2 perpendicular to it. These measurements, in the case of a cube, will be the same as that of $a'b'$ previously chosen as the length of an edge of the cube, or they should be made the same as the edges of the 40° face of the prism, if it be a prism that has to be projected.

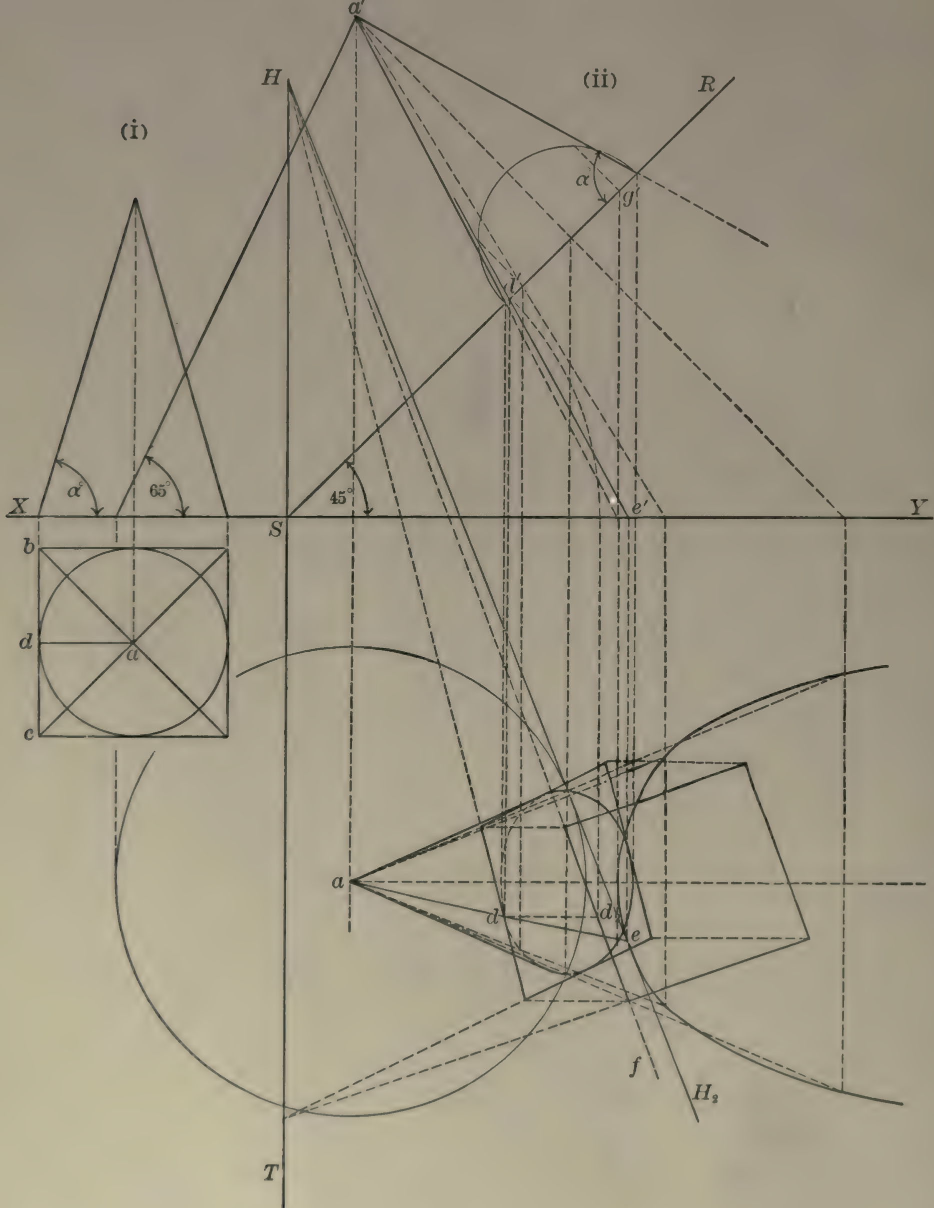


FIG. 92.

It will readily be seen that from this rabattement of the two edges the plans *ac* and *ad* may be obtained. Three edges are now found in plan, namely *AB*, *AC* and *AD*, and parallels to these will be necessary in order to complete the plan of the cube.

Of the two faces found in the figure, the face *dace* is a 40° face, and the face *bacf* is a 70° face.

In Fig. 92 is shown the case of a pyramid in which the method requires the employment of *two* right circular cones, since the triangular faces of the solid are inclined to the plane of the base.

Let it be required to find the plan of a square pyramid when the base is inclined at, say, 45° to the H.P. and one triangular face is at, say, 65° to the H.P. The base plane is marked *RST* in the figure, and is purposely arranged perpendicularly to the V.P. in order to simplify the work. At (i) in the figure the square pyramid is set up in plan and elevation in such a way as to find the angle, $\alpha°$, of the vertical cone to which each of the triangular faces of the solid is tangent. Each face is tangent on a generator which is at the same time the middle line of a triangular face and meets the middle point of a base edge of the pyramid, as at *d* in *bc*. At any convenient place on the plane *RST* at (ii) set up the right cone with base angle equal to that in (i), namely $\alpha°$. Then find its horizontal trace, the large ellipse, on the H.P. A tangent plane to this inclined cone, with its inclination according to that given in the problem for a triangular face of the pyramid, viz., 65°, is now obtained, with its H.T. at HH_2. This trace is made tangent to the large ellipse and at the same time tangent to the circular trace of a vertical cone of 65° from the same apex as that of the inclined cone, viz., the point aa'. The V.T. of this plane is not needed, so is therefore neglected.

The generator ae, $a'e'$, of the inclined cone, which at the same time lies in the 65° plane is now drawn, *e* being its H.T. Its elevation $a'e'$ crosses *RS*, giving the elevation d', of the point in which it passes through the base plane. The line *ad* in (ii) corresponds to the *ad* in (i), and by drawing a line through *d* and the point *H* where the H.T.'s of the 45° and 65° planes intersect, the plan of the intersection of the two planes, base plane and face plane, is obtained. The rabattement of this intersection line is *Hf*. On this line *Hf* mark off, on either side of the rabattement of point *d*, marked d_2, the measurements of *db*, *dc* from the base edge in (i), and construct the square in rabattement. Bring this up to its proper place in plan, where it will appear foreshortened, and the edges of it will be tangents to the small ellipse, the plan of the base of the inclined cone. Join the corners of the square to the apex *a*, and the plan will be completed when such edges as are not in view are shown by dotted lines. In the figure the plan of the solid is not completed, in order to save confusion of lines.

N.B.—In the figure the construction work for finding the ellipses is reduced or omitted as far as possible to save confusion of lines. In finding the plan of the base of the inclined cone, however, there is shown a method of finding plans of horizontal chords of it. Point g' is the elevation of such a chord. Its plan can be found by making a semicircle as shown, and a perpendicular from g' in the diameter of it, to the circumference, thus giving half the chord length.

EXERCISE XXXVI

1. Work out completely the problem in Fig. 91, and obtain an elevation of the solid on a vertical projection plane whose XY is at 45° to the XY given. Let the length of the edge of the cube be 2″.

2. Work out completely the problem in Fig. 92, making the base edge of the pyramid 2″ and its height $3\frac{1}{2}$″, with the triangular face at 70° to the H.P. instead of at 65°.

3. Find the plan and elevation of a square prism, when the square end of it is at 45° to the H.P. and a rectangular face is at 70° to the H.P. Short edges $1\frac{1}{4}$″, long edges $2\frac{1}{2}$″.

4. Find the plan of a regular pentagonal pyramid, 1″ edge of base, height 3″, when the base is inclined to the H.P. at 40°, and one triangular face is at 65° to the H.P.

CHAPTER XIII

TANGENT PLANES TO A SPHERE, AND THE FINDING OF AN OBLIQUE PLANE WITH GIVEN INCLINATIONS

SECTION **26.** In discussing tangent planes to spheres it must chiefly be noticed that the radius of the sphere, from the tangent point on its surface, is a perpendicular to the tangent plane, and its projections will therefore be perpendicular to the traces of the tangent plane.

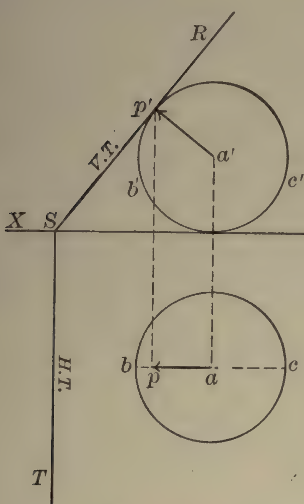


FIG. 93.

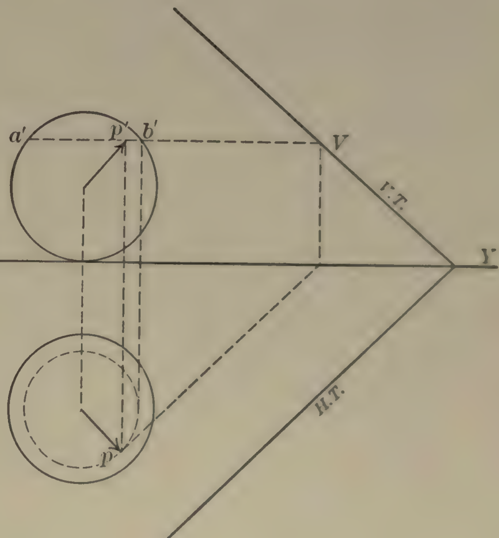


FIG. 94.

In the illustration, Fig. 93, a plane is shown tangent to a given sphere the centre of which is aa'. The radius, which is perpendicular to the plane, is also shown in elevation and plan, $a'p'$ and ap. A great circle of the sphere, that is, one whose plane passes through the centre of the sphere, is shown in plan by the dotted line bc and in elevation by the circle $b'c'$. It is perpendicular to the tangent plane, and has for its radius the line AP.

In Fig. 94, any point on the surface of the sphere is shown in plan at p. It is required to find the tangent plane to the sphere at this point. Any number of points on the surface of the sphere, and at the same level, would be in the circumference of a horizontal circle whose elevation would appear as a straight line

$a'b'$. Consequently, such a horizontal circle is represented and p' found in its elevation.

Similarly, if the point be thought of as being on the circumference of a vertical circle whose plan would be a straight line through p parallel to XY, a circle in elevation would need to be drawn in order to locate the elevation of the point. A tangent plane to the sphere at P may now be obtained by drawing a horizontal line through P in the direction of the H.T. required, i.e., perpendicular in plan, to the plan of the radius from P. Thus will be obtained a point V in the V.T.

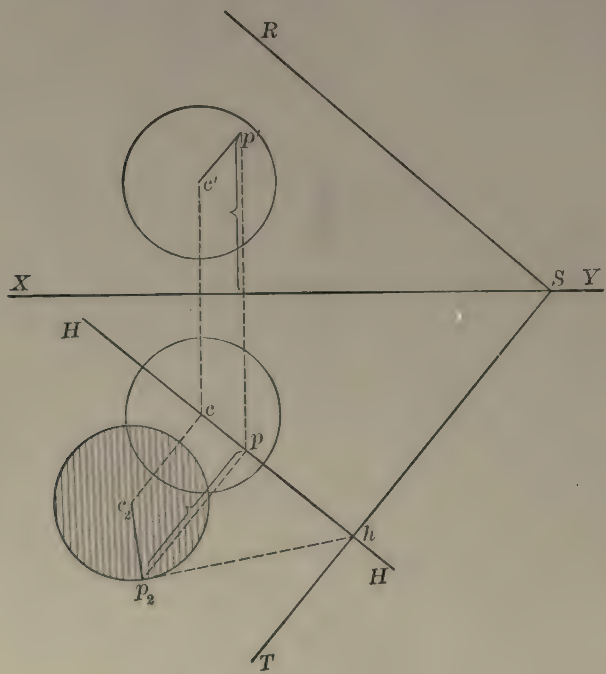


FIG. 95.

required, and the traces of the plane may now be drawn perpendicular to the radius from P.

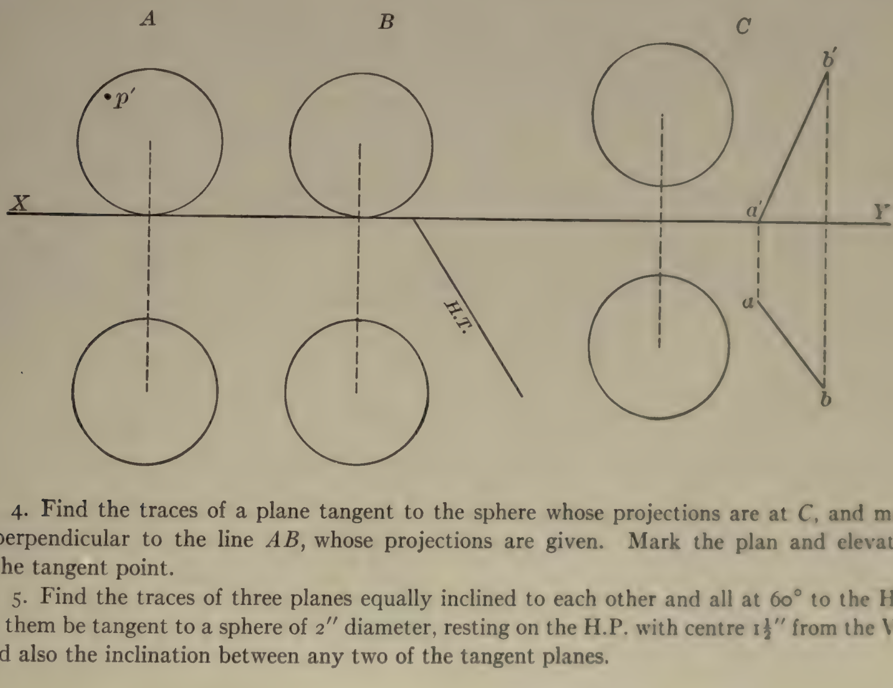
In Fig. 95 the same problem as in Fig. 94 is solved by the employment of a vertical plane containing the radius from the given point and therefore at right angles to the plane required. This vertical plane has HH for its H.T. By rabatting it on to the H.P., the centre of the sphere is carried over to c_2, and a rabatted great circle of the sphere is made on the H.P., shown in the figure covered with hatching. The point P will now appear rabatted to p_2, and the rabattement of the radius is seen at c_2p_2. At right angles to this draw p_2h as the rabattement of the intersection of the required plane with the vertical plane employed whose H.T. is the line HH. This gives h, a point in the H.T. of the required plane. Through h draw the H.T. marked ST, and through S draw RS at right angles to the elevation of the radius. *RST* is the plane required. The height of p' above XY is taken from the rabattement, as indicated by the bracket line. This construction, Fig. 95, will be necessary, used conversely, when the trace of a plane tangent to a sphere is given, and it is required to find the other trace of the plane and the projections of the tangent point.

EXERCISE XXXVII

1. Find the planes tangent to the sphere whose projections are at *A*, each containing a point on its surface, for which p' is the elevation.

2. A sphere, 2″ diameter, touches both planes of projection. Find the two planes, each touching the sphere in a point $1\frac{1}{2}''$ above the H.P. and $1\frac{3}{8}''$ from the V.P.

3. A sphere is given at *B*, and the H.T. of a plane tangent to it. Find the V.T. and also show the plan and elevation of the tangent point.



4. Find the traces of a plane tangent to the sphere whose projections are at *C*, and make it perpendicular to the line *AB*, whose projections are given. Mark the plan and elevation of the tangent point.

5. Find the traces of three planes equally inclined to each other and all at 60° to the H.P. Let them be tangent to a sphere of 2″ diameter, resting on the H.P. with centre $1\frac{1}{2}''$ from the V.P. Find also the inclination between any two of the tangent planes.

By experiment and examination it should be realized that a sphere may be enveloped by a right circular cone, having for its apex any point outside the sphere, and that if two cones envelope the same sphere, a tangent plane to the sphere may contain both apexes; also, that any two unequal spheres may be enveloped by a cone, the apex of which will lie in the line passing through their centres. The tangent lines on the surfaces of such spheres as are enveloped by cones are circumferences of circles, smaller than great circles of the spheres, and perpendicular to the axis of the cone in each case.

Fig. 96 is an illustration of the use of cones enveloping a single sphere, and is a second method for obtaining the traces of an oblique plane whose angles with the planes of projection are given. The previously discussed and more commonly used method was explained in Part I, Section 9, and was shown to depend on the projections of a line set up at angles complementary to those of the required plane, the traces of the required plane being then drawn perpendicular to the projections of the line.

In the method now discussed, a sphere with its centre in the XY, is represented by the circle with centre at c. This circle therefore serves for both plan and elevation of the sphere.

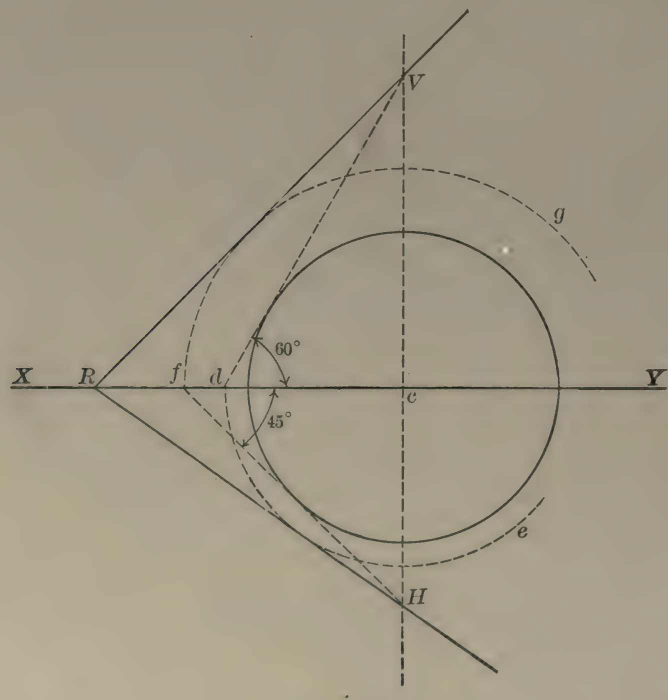


FIG. 96.

Suppose the required plane is to have angles of 60° to the H.P. and 45° to the V.P., then a vertical cone to envelope the sphere, with its generators at 60° to the H.P. and its apex therefore at V in the V.P., will have part of its circular H.T. represented by the arc de, and any plane, whose H.T. is a tangent to this arc, and whose V.T. contains the apex V, is therefore a plane at 60° to the H.P. Next, represent another enveloping cone, about the same sphere, with its axis perpendicular to the V.P., and its apex therefore at the point H in the H.P., and make its generators at 45° to the V.P. Part of its circular V.T. will be the arc fg. Any plane whose V.T. is a tangent to this arc and whose H.T. passes through

H is a plane at 45° to the V.P. because it is tangent to a right cone perpendicular to the V.P.

We now have two cones enveloping the same sphere, and a plane containing both apexes, and tangent to the sphere, will contain a generator of each cone. Such a plane may now be represented by drawing its H.T. through H, tangent to the circular H.T. of the 60° cone and its V.T. through V, tangent to the circular V.T. of the 45° cone. These traces, being traces of the same plane, meet in the XY at R.

EXERCISE XXXVIII

Find, by the method employing enveloping or tangent cones to a sphere, the traces of the following oblique plane:

(*a*) At 70° to the H.P. and at 55° to the V.P.
(*b*) At 35° to the H.P. and at 75° to the V.P.
(*c*) At 65° to the H.P. and at 50° to the V.P.

TANGENT PLANES COMMON TO THREE GIVEN SPHERES

SECTION **27**. In Fig. 97 is illustrated the method of finding tangent planes common to three unequal spheres, and the projections of the tangent points on their surfaces.

The three spheres A, B and C, are resting on the H.P., so that enveloping cones of any two of them have their apexes in the H.P. The enveloping cone of A and B has its apex at H_1, and one enveloping A and C has its apex at H_2. The line through H_1 and H_2 is therefore the H.T. of a plane tangent to the three spheres, and by passing a vertical plane, perpendicular to this tangent plane, through the centre of one of the spheres, say, C, and rabatting as in Fig. 95, the tangent point cc' is obtained. The V.T .of the tangent plane will be perpendicular to the elevation of the radius which ends in c'.

Just as the radius to point C is perpendicular to the tangent plane found, so, in like manner, the radius lines to the tangent points on the surfaces of A and B will be perpendicular to the plane, therefore plans of these radius lines made perpendicular to the H.T., and elevations of them made perpendicular to the V.T. may be drawn. These must be limited at aa' and bb' by generator lines on the surfaces of the cones, one from H_2 through cc' to limit the radius line from the centre of sphere A, and another from this point aa' found on sphere A, drawn to H_1, to limit the radius of the sphere B at bb'. The elevations of the points may be obtained by carrying perpendiculars across XY from the plans, instead of using elevations of generator lines. Both ways are shown in the figure. One checks the accuracy of the other.

EXERCISE XXXIX

1. Find two tangent planes to three spheres, the planes not to pass between any of the spheres, and mark the projections of the tangent points on their surfaces. The spheres have their centres all at the same level, and these centres are the corners of an equilateral triangle $2''$ side, with no side parallel to the V.P. Diameters of the spheres $1\frac{3}{4}''$, $1\frac{1}{8}''$ and $\frac{3}{4}''$ respectively

2. Three spheres rest on the H.P. and touch each other. Their centres are at different distances from the V.P. Their diameters are $1\frac{3}{4}''$, $\frac{7}{8}''$ and $\frac{5}{8}''$ respectively. Find the inclined plane tangent to all three spheres and mark the projections of the tangent points.

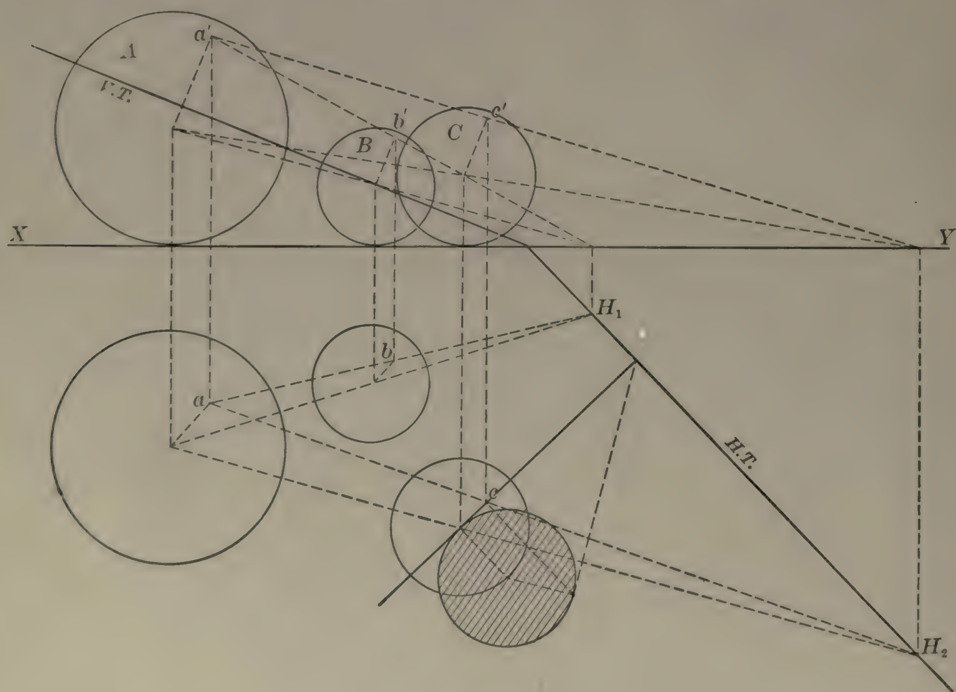


FIG. 97.

When the centres of three unequal spheres are at unequal distances from the planes of projection and the spheres are not all resting on the H.P., then the line joining the apex points of the enveloping cones will be oblique, and the problem becomes more involved. If, however, the line joining the apex points be considered in relation to one sphere only, then it becomes a matter of finding the planes containing this line and tangent to the one sphere. It will be seen that the tangent planes found will be tangent to the other spheres, the apex of whose common enveloping cone is a point in the line. The tangent points, also, having been obtained on the one sphere, the tangent points on the others may be found as in the case considered in Fig. 97.

Let ab, $a'b'$ in Fig. 98 be an inclined line, such as that referred to above, and cc' the centre of a sphere whose projections are given. Let it be required to find the tangent planes to this sphere, which contain the line AB. The order of procedure might be as follows:—(1) Find the traces, H and V, of the given line. These will be in the traces of the planes required. (2) Find a plane, RST, passing through the centre of the sphere and perpendicular to the line AB. (3) By the

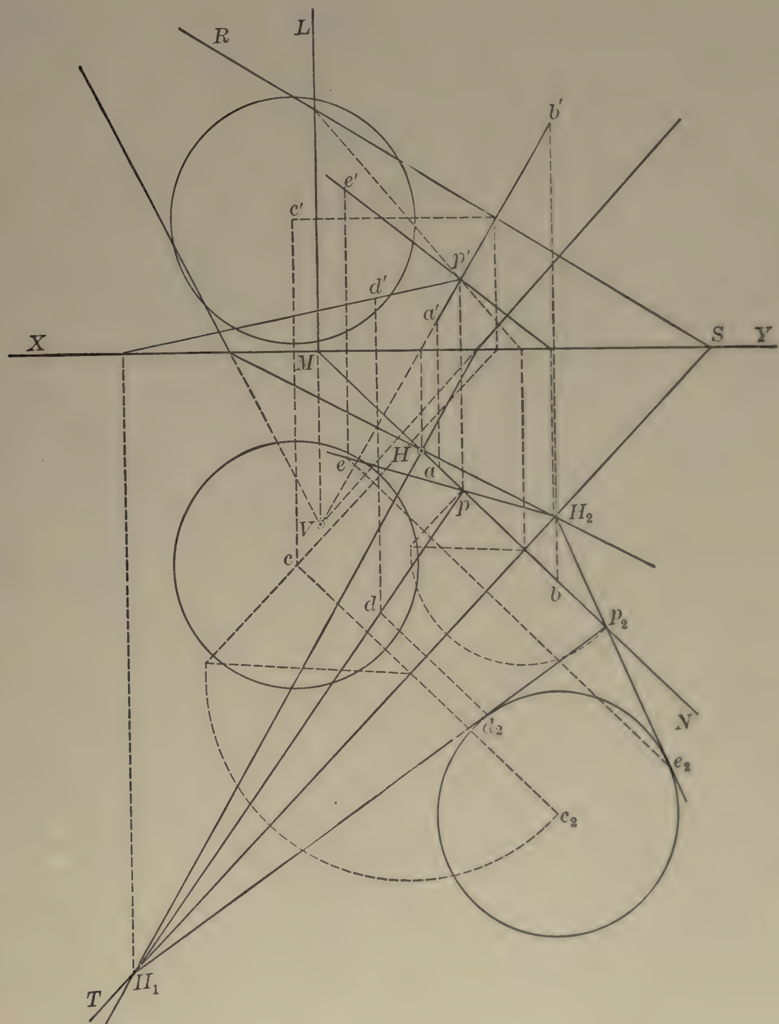


FIG. 98.

help of a plane LMN, containing the line AB, and intersecting plane RST, find the point of intersection which AB makes with RST at pp'. (4) Rabatte the point P about ST to p_2, and also rabatte the centre C to c_2, and draw the rabatted great circle of the sphere made by the intersection of the sphere by plane RST passing through its centre. (5) From p_2 draw the rabatted tangent lines to this great circle and find their H.T.'s at H_1 and H_2. (6) Join H_1 and H_2 to p and so obtain the plans of two lines which are tangent to the sphere and have a common point P in the given line AB. (7) The planes required, tangent to the sphere, will each contain one of these tangent lines and the given line, therefore draw the traces of the required planes by making their H.T's at H_1H and H_2H respectively. Their V.T.'s will pass through the V.T. of the given line AB, viz., V. (8) The projections of the tangent points may be obtained by passing them over from their rabattements at d_2 and e_2 to their plans at d and e respectively, and then obtaining their elevations in the usual way.

EXERCISE XL

1. Find the traces of the tangent planes to the sphere, of 2″ diameter, which has its centre $1\frac{1}{2}''$ from both planes of projection, and contains a line whose H.T. is 2″ from XY and 2″ from the plan of the centre of the sphere, while its V.T. is 3″ above XY and $1\frac{3}{4}''$ from the elevation of the centre of the sphere. Both these given traces are to the right of the centre of the sphere. Also, mark the projections of the tangent points on the surface of the sphere.

2. Arrange three spheres of different sizes and not far away from each other, placing them at different levels. Find two planes tangent to all the spheres and mark the tangent points on the spheres.

CHAPTER XIV

SIMPLE CASES OF INTERPENETRATIONS OF SOLIDS, AND THE DEVELOPMENTS OF SURFACES

SECTION **28**. The matter of finding intelligently the intersection of the surface or surfaces of one solid with the surface or surfaces of another, depends

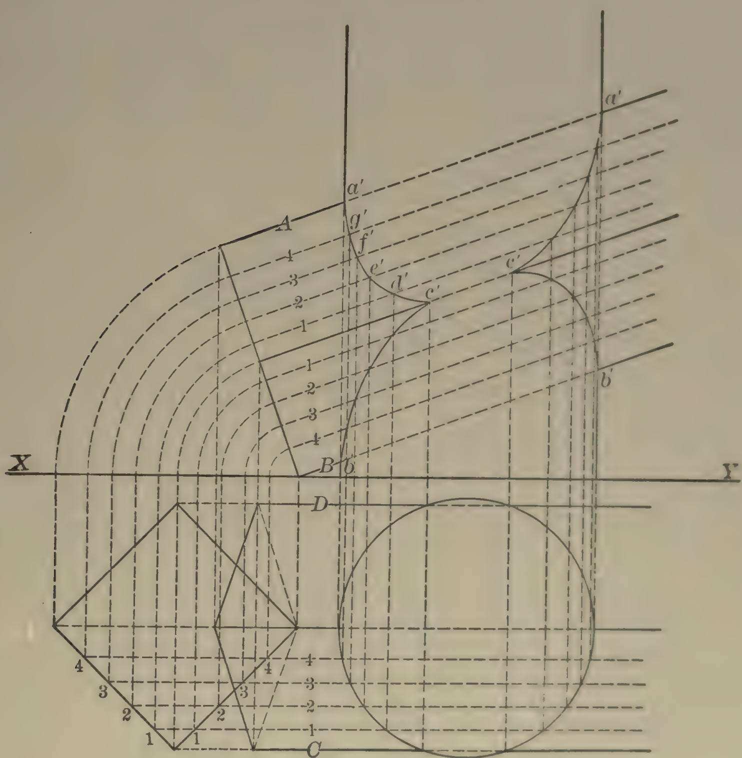


FIG. 99.

chiefly upon recognizing the use of sectional planes or of surfaces which will contain elements or generators of both of the given surfaces, meeting each other in points common to both given surfaces. The solids should be so arranged as to provide for the use of planes which will readily give plans and elevations of the generators or elements, and their intersection points common to both surfaces.

In Fig. 99 is shown a square prism penetrating a cylinder. In making the projections of the solids, or, rather, such parts of them as may be necessary, start by making the square which will serve as the rabattement of the end of the square prism.

Inspection of the figure will disclose the following facts:—(1), that a vertical section plane containing the axis of the cylinder contains two vertical generators of the cylinder, and may also contain the edges A and B of the prism which has been placed parallel to the V.P.; (2) that the elevations of the points of intersection of these two edges with the generators of the surface of the cylinder are $a'a'$ and $b'b'$ respectively; (3) that similarly $c'c'$ are the elevations of the intersections of the edge C with the surface of the cylinder; (4) that for other points, common to both surfaces, straight lines, of any convenient number, and placed conveniently

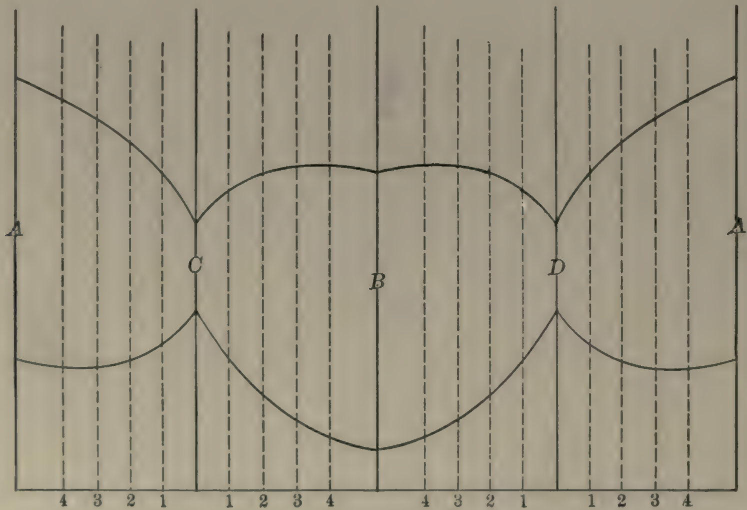


FIG. 100.

(i.e., not necessarily at equal distances from each other) to run parallel to the V.P., are drawn on the surfaces of the prism at 1, 2, 3, 4, first in plan and then in elevation, by taking advantage of the rabatted square end; (5) that vertical generator lines on the surface of the cylinder, and in the same vertical section planes as the lines on the prism, pass through the points where these lines 1, 2, 3 and 4 cut the curved surface of the cylinder. These are then projected, resulting in the elevations of points, common to both surfaces, and shown at d', e', f', g', etc.; (6) that these elevation points must be joined by freehand curved lines to indicate the elevation of the common section lines.

In Fig. 100 is shown what is called the development of the surfaces of the prism of Fig. 99, in order to show what the section lines on those surfaces appear like. The long lines marked $ACBDA$ are set up at distances from each other equal to the width of the faces of the prism. Then the straight lines which were drawn upon those faces in Fig. 99 are drawn at their proper distances from each other, taken from the rabatted end and numbered. The distances of the points of intersection on all these lines and upon the edges *ACBD* are now taken from the elevation in the figure, and joined by freehand curves, thus marking the exact way in which each face has been penetrated by the surface of the cylinder.

In a similar way the curved surface of the cylinder may be "developed" and the intersection lines upon its surface shown. The circumference of the end

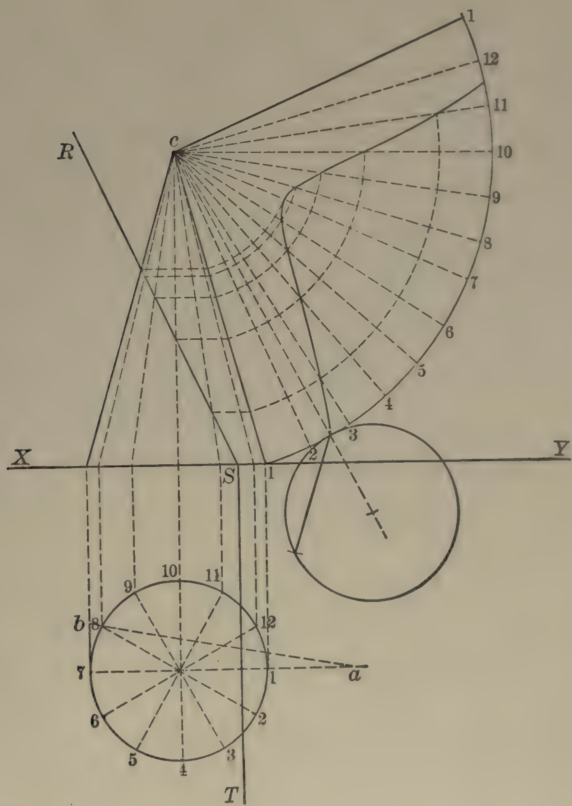


FIG. 101.

is laid out as a straight line, and perpendiculars are made from it to represent generators. These are cut at heights taken from the elevation in the figure and then the points so found are joined by curved lines, showing the exact way in which the surface has been cut.

A further illustration of development is given in Fig. 101, where it will be seen that the circular base of the solid is laid out as well as the curved surface, and the intersection lines made on them are displayed. For convenience the curved surface is divided into twelve equal parts. One-twelfth of the circumference of the base, that between 7 and 8 in the plan, is, by an approximate method, found to be equal to the tangent marked $7b$. The method is, to make the line from point 7 through the centre to a, equal to three times the radius of the circle, and from a to draw ab through the other end of the $30°$ arc to the tangent line from 7. Since all the division lines of the curved surface are equal, from c in the elevation, as centre, draw the arc 1, 2, 3, etc., to 1, marking on it the divisions, equal to the line $7b$ in the plan drawing. The intersections of the lines on the surface of the cone are now carried across, by parallels to XY, to the line $c1$, and from that line carried by arcs with c as centre to their development. Between the points 2 and 3 the circumference of the base is cut by the plane RST, and at this point, therefore, in the development, draw a tangent circle, so that the intersection of the base from this point may be marked on the circle representing it in the development.

EXERCISE XLI

1. Find the elevation of the intersection lines, made by the surfaces meeting each other of a square prism, $2''$ short edge, standing vertically with a vertical face at $30°$ to the V.P., and another square prism, $1\frac{3}{4}''$ short edge, with long edges horizontal and parallel to the V.P. A face of this prism is at $20°$ to the H.P., so that both of the diagonals of the end of it are inclined, the one at $65°$ and the other at $25°$, to the H.P. Arrange it so that three of the vertical edges of the first prism pass through the second prism, and one of the vertical edges of the first passes through the axis of the second. (N.B.—In working out the result, it must be seen that the vertical edges of the first prism penetrate faces of the second prism in points through which should be drawn, on the faces of the second prism, lines parallel to the long edges. The elevations of these should then be obtained.)

2. A right cylinder, $2''$ diameter, has its axis vertical, and a square prism, with one diagonal of end horizontal and at $70°$ to the V.P., has its axis inclined at $20°$ to the H.P. The axis of the prism passes through the axis of the cylinder. The edge of end of the prism is $1\frac{1}{4}''$ long. (N.B.—It is suggested that in setting up these solids the prism should be arranged correctly first, in relation to the planes of projection, and the cylinder placed in position after this has been done.)

3. Two right circular cylinders interpenetrate each other. Let one be placed with its axis vertical, its diameter being $2''$, and let the other, with diameter $1\frac{3}{4}''$, have its axis inclined at $20°$ to the H.P. and parallel to the V.P., $\frac{3}{16}''$ farther away from the V.P. than the axis of the other cylinder. Show the elevation of the intersection curve made by their surfaces. Make also a development of the curved surface of the $2''$ cylinder to show the intersection line upon it.

MORE DIFFICULT CASES OF INTERPENETRATION OF SOLIDS, AND THE PROJECTION OF THE INTERSECTIONS OF THEIR SURFACES

Section **29**. When the intersecting surfaces are those of the cone and the sphere, then all that is necessary, in order to obtain the projection of points common to both surfaces, is to make use of planes which give circular sections of the cone, and pass through the sphere. If the circular sections of the sphere, by these planes, intersect the circular sections of the cone by the same planes, then the points of intersection are points on both surfaces.

Another method of determining the intersection of the surfaces of these solids, is by the use of intersecting spherical surfaces instead of intersecting planes, and depends on the following facts:—

(1) That a sphere whose centre is in the axis of a right circular cone will intersect the surface of that cone, if it intersects at all, in a circle perpendicular to the axis of the cone;

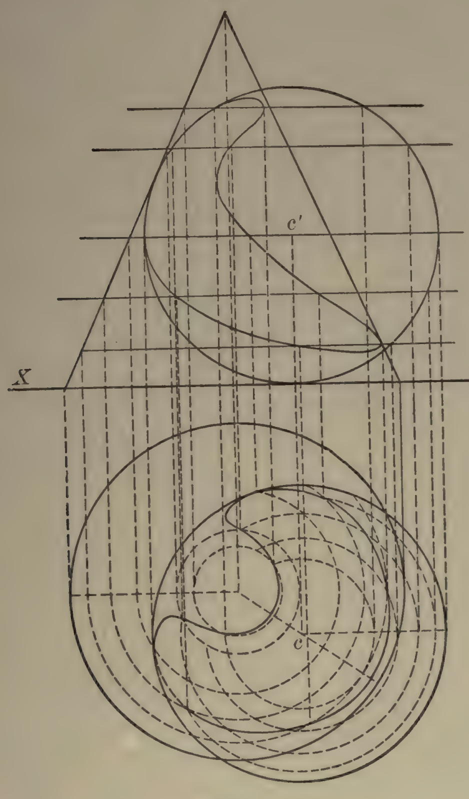


FIG. 102.

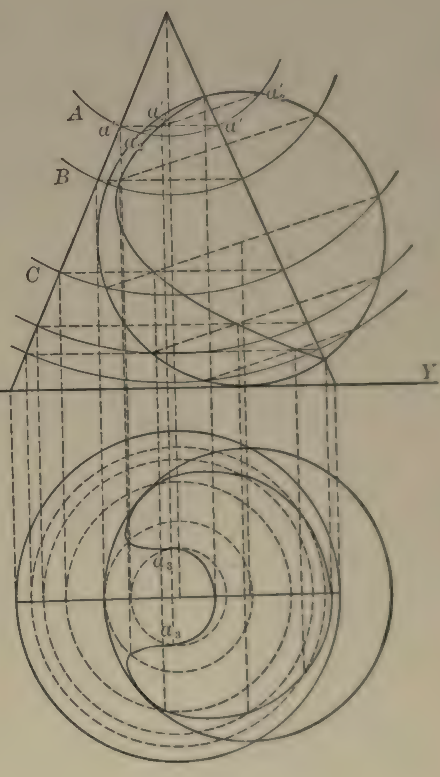


FIG. 103.

(2) That if a sphere intersects another sphere the intersection of their surfaces is the circumference of a circle, the plane of which is perpendicular to the line joining their centres; and

(3) That if a sphere, with its centre in the axis of a right circular cone interpenetrates at the same time another sphere, then the intersection of the circular sections it makes with both solids will be points common to the surfaces of both solids.

Illustrations of the use of both methods referred to are given in Figs. 102 and 103. In Fig. 102, a right circular cone with its axis vertical, and a sphere whose centre is cc', are arranged so that their surfaces intersect. The centre of the sphere is not necessarily at the same distance from the V.P. as the axis of the cone. By the use of horizontal section planes a number of points will be found in plan, where the circular sections of the solids intersect. Their elevations are readily discovered, and by joining the points by freehand curves, the projections of the section line or lines, are obtained.

In Fig. 103, concentric sectional spheres, having their common centre at the apex of the cone, are represented at A, B, C, etc. Since the centre of the given

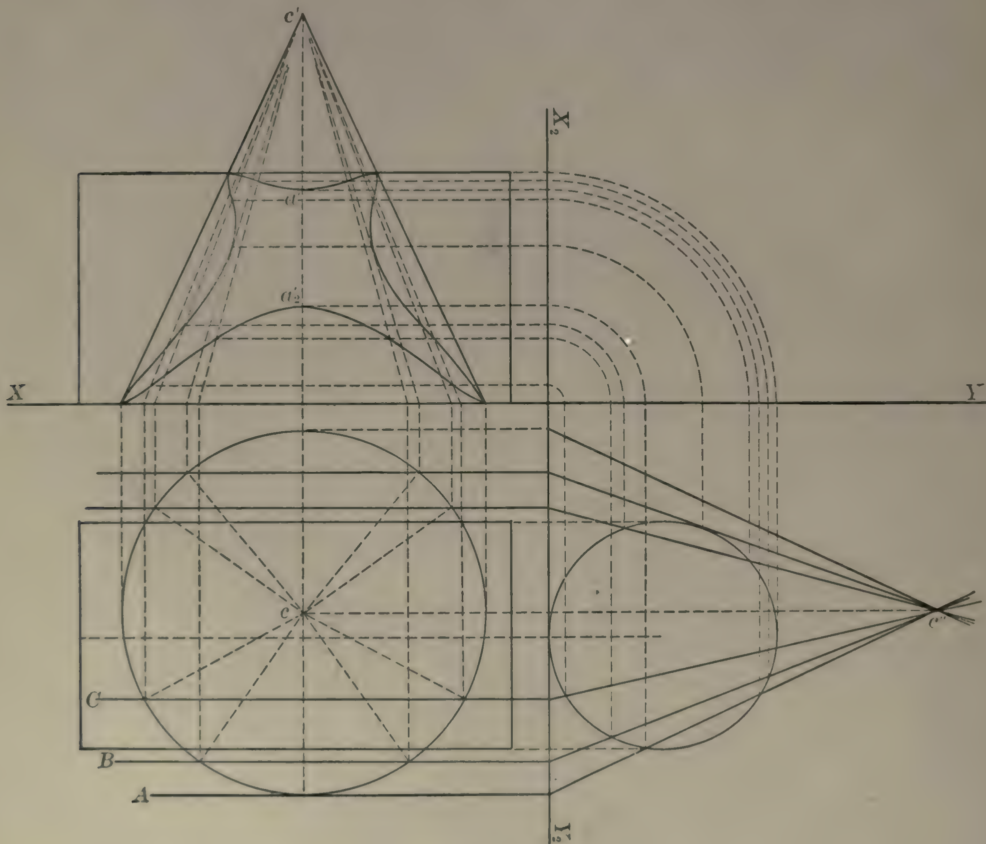


FIG. 104.

sphere is at the same distance from the V.P. as the axis of the given cone, the sections the given cone and sphere make with the concentric spheres will appear in elevation as straight lines, intersecting each other in the elevations of points common to their surfaces. These points may readily be found in plan in the plans of the circular sections of the cone. For example, the cone intersects the sphere A in the circle whose elevation is $a'a'$, and the given sphere intersects the sphere A in the circle whose elevation is $a'_2a'_2$. Where these cross each other is the point a'_3, which is the elevation for two points a_3 and a_3 in plan.

In Fig. 104 is shown how best to deal with the problem of finding the plan and elevation of the intersection of the surface of a cone with the surface of a cylinder, when the two solids interpenetrate.

The cylinder is placed with its axis parallel to the V.P., and a secondary elevation plane is represented, perpendicular to the ordinary V.P., with its XY marked X_2Y_2 in the figure. Upon this secondary vertical plane is drawn an end view of the cylinder, and another view of the cone.

Inclined planes perpendicular to this secondary vertical plane, and containing generators of the cylinder and of the cone, are represented with their H.T.'s at A, B, C, etc., and their V.T.'s, passing through the secondary elevation of the apex of the cone at c''. One generator of the cone and two generators of the cylinder are contained in the plane A. The intersections made by the generators result in two points whose elevations are a' and a'_2. Plane B contains two generators of each solid and consequently four points common to the two surfaces are found; and so on. After sufficient points are found the freehand curves may be drawn through them. In the figure, the elevation has been completed, in order to emphasize the fact that it is better to work it out first. The plan may now be obtained, either by projecting from the elevations of the points on to the plans of the generators of the cone, or by making plans of the generators of the cylinder to cross the plans of the generators of the cone in the points required. This latter method will be necessary in the case of the plans to be marked a and a_2.

EXERCISE XLII

1. Find plan and elevation of the intersection of the surfaces of a right circular cone and a sphere, by using the method involving spherical section surfaces. The cone has a base $2\frac{1}{2}''$ diameter, in the H.P. and a height $2\frac{1}{2}''$. The sphere, diameter $2''$, rests on the H.P., and has its centre in the curved surface of the cone.

2. Find, by the method of horizontal section planes, the plant and elevation of the intersection line, when a cone with $2\frac{1}{2}''$ circle for its H.T., and with an axis $2\frac{1}{2}''$ long from the centre of its H.T. to its apex, inclined at 70° to the H.P., interpenetrates with a sphere, $2''$ diameter, resting on the H.P. and having its centre in the shortest line from the apex to the H.P., on the surface of the cone.

3. A right circular cylinder, $2''$ diameter, lies with one of its generators in the H.P., parallel to the V.P. A cone, whose H.T. is a circle $3''$ diameter, has the centre of this circle in the plan of the axis of the cylinder. The axis of the cone is $4''$ long, and has a plan $\frac{3}{4}''$ long at 45° to XY. Find the plan and elevation of the intersection of the surfaces of the two solids.

To obtain the projections of the surface intersections when inclined cones and cylinders, not necessarily right circular ones, interpenetrate, it is necessary to make use of section planes which will have in them straight generator lines on both surfaces. Hence, when the surfaces of cones are cut by those of other cones, or of cylinders, the apex points of the cones must be in the section planes made use of; and in the case of cylinders, their axes must be parallel to the section planes. Thus, in Fig. 105, where two cylinders are represented whose given H.T.'s are circles and whose axes are directed as indicated by the plans and elevations of generators of their

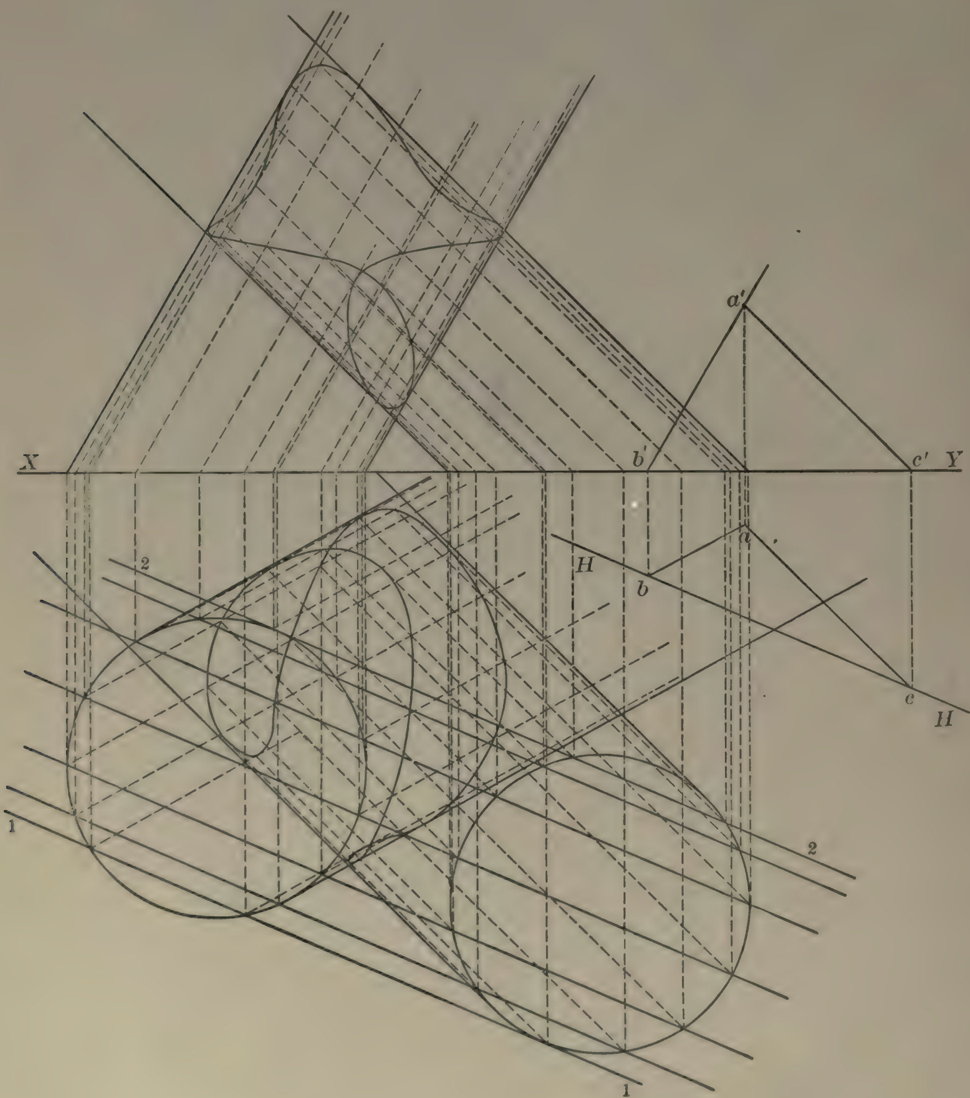


Fig. 105.

surfaces, it is necessary to take any generator of one cylinder, or a parallel to it as ab, $a'b'$, and from some point in that line to draw another line, say ac, $a'c'$, parallel to the generators of the second cylinder. The H.T.'s of these two lines are b and c respectively, and the plane of the two lines has the line HH for its H.T. All the generators of both surfaces are parallel to this plane, and hence any plane made parallel to this one, and having its H.T. crossing the circular H.T.'s of the given cylinders will therefore contain generators of the surfaces of both cylinders. The points in which these generators cut each other are points in the intersection line of the surfaces, and the plan and elevation of it can in this way be found, as shown. It will be found an advantage to arrange tangent planes to commence with. These in Fig. 105, are marked 1 and 2 respectively, and it will be there seen that a strip on each of the cylinders is not pierced by the surface of the other, and the curves will be turned back from these strips accordingly.

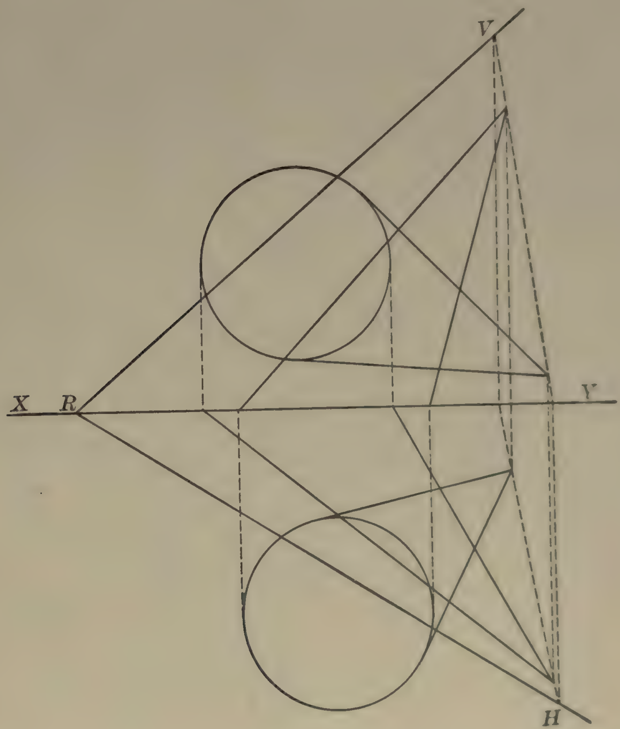


FIG. 106.

In Fig. 106 two cones are represented. A line through the apex points will be contained in the planes which have generators of both surfaces. Hence the V.T. and the H.T. of this line must be found, as at *V* and *H*, and the traces of planes passed through them. Those planes, such as *VRH*, which contain generators of both cones, will give points common to both surfaces. A number of such points must be found in order to obtain the projections of the curve or curves.

In Fig. 107, a cylinder and a cone are represented, and here it will be seen that since the apex of the cone must be in all the section planes that are of service in finding the intersection, and these section planes must contain generators of the cylinder also, therefore they must contain a line, through the apex point, parallel to the generators of the cylinder. Having drawn this line and obtained its traces, as at *V* and *H*, it is only necessary then to draw traces of section planes through *V* and *H* which will cross the H.T. of the cylinder and the V.T. of the

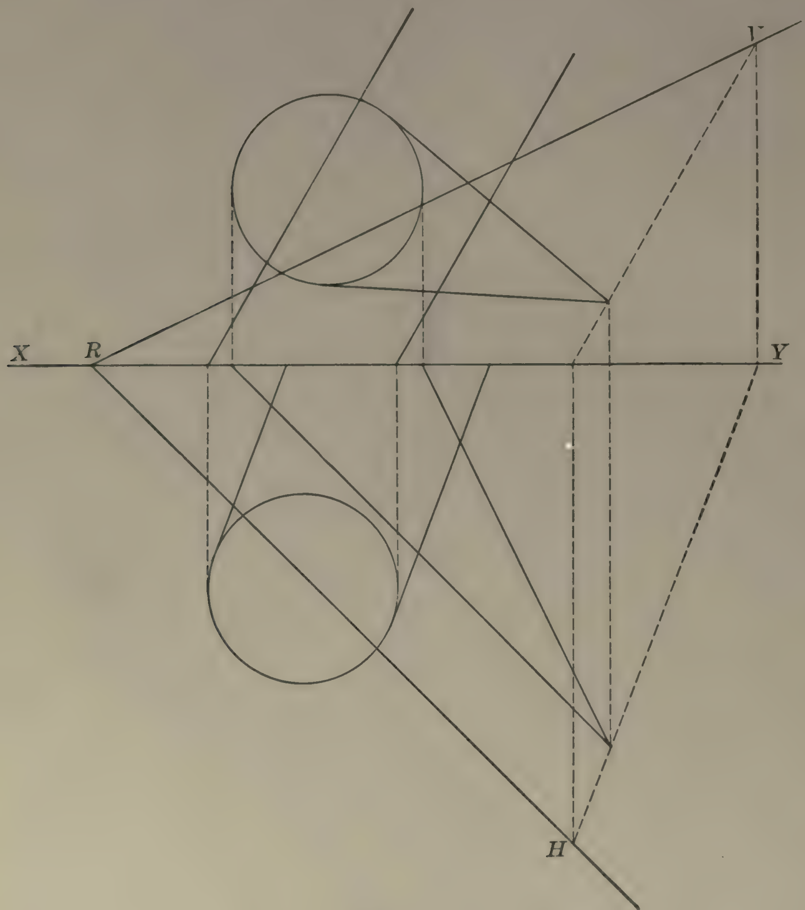


FIG. 107.

cone, as the plane *VRH* does, and obtain plans and elevations of generators of both surfaces, as they cross each other, thus finding projections of section points.

Other simple cases of interpenetration, not requiring for their solution any new principle, or serious change of method, will doubtless suggest themselves.

EXERCISE XLIII

Work to completion, in plan and elevation, the cases of interpenetration shown in Figs. 106 and 107.

SURFACES OF REVOLUTION, AND THE SCREW THREAD

SECTION **30**. Thus far we have dealt with three different surfaces of revolution, the cylinder, the cone and the sphere. The first is formed by revolving a straight line round an axis of rotation while maintaining a parallel attitude in relation to it. The second is formed by revolving a straight line round an axis of rotation, the revolving line having one point of it in the axis of rotation. The third is formed by revolving a semicircle, making use of its diameter as the axis of rotation. Others will now be considered, namely, the hyperboloid and the helicoid.

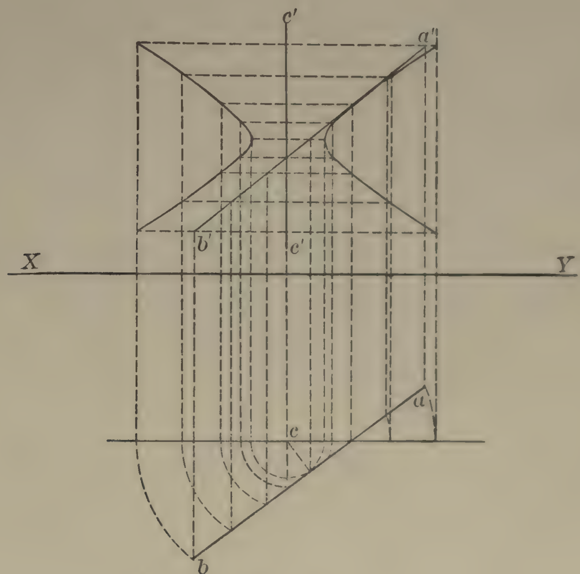


FIG. 108.

The hyperboloid of revolution is obtained as the result of a straight line revolving about an axis of rotation, but not in the same plane with it. Each point in the revolving line travels in the circumference of a circle for its locus, the plane of which is at right angles to the axis. This will be noted in Fig. 108, where c, $c'c'$ is the axis of rotation, and ab, $a'b'$ is the line revolving. Any points taken in this line revolve in paths which are the circumferences of circles whose planes are horizontal, the axis being vertical, and on joining the elevations of the extreme limits to which these points move from one side to the other, the freehand curve gives the projection of the contour of the surface of revolution generated, namely, the hyperboloid of revolution.

The helicoid, or surface of revolution generated by a straight line moving so that any point in it travels in a helicoidal line or helix, as a spiral, while the line is not in the same plane with the axis, but is constantly maintaining the same attitude in relation to it, is used in the making of screw-threads on bolts, etc.

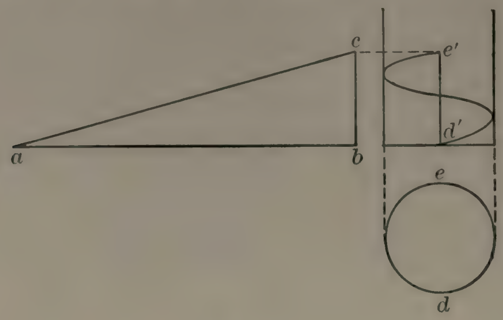


FIG. 109.

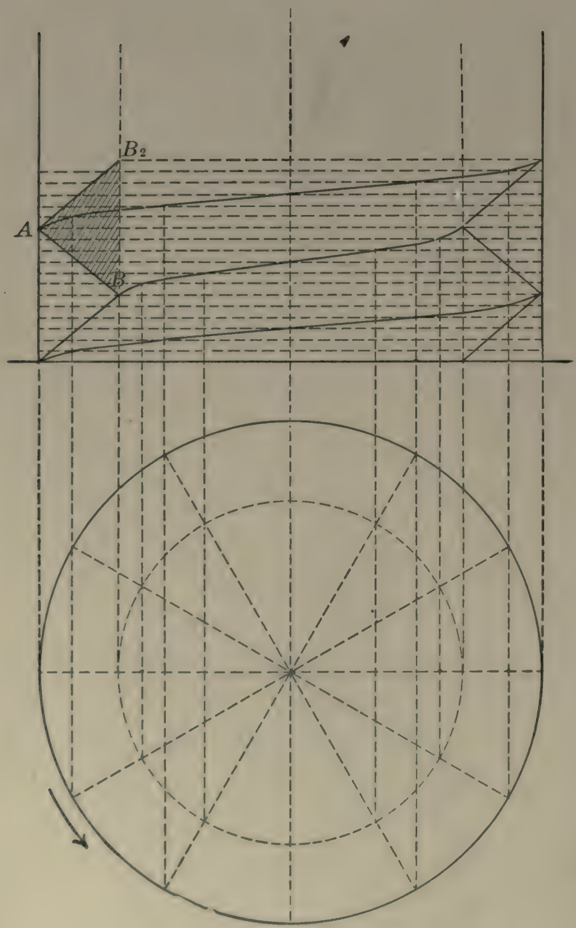


FIG. 110.

In projecting screw-threads, the curved edge lines of the threads are the paths of points in the generating lines of helicoids, and may be thought of as lines drawn on the surfaces of cylinders or drums. The development of such a cylindrical surface of a bolt, showing the thread line, would give that line as a straight one, as *ac* in the Fig. 109. The line *ab* is the length of the circumference of the circular section, and the distance *bc* represents the *pitch* or distance travelled in the direction of the axis during one revolution. *ac* is the straight line representing the helicoidal line developed as from the surface of a cylinder. These curved helicoidal lines may be projected on to the V.P., and in this way a screw-thread is represented.

As in illustration, in Fig. 110, let ABB_2 be the triangular section, made by a plane which has in it the axis of the bolt, of a screw-thread to be projected, when the large circle in the plan, represents the section of the bolt. From B to B_2 is the pitch of the thread. Two helicoidal lines must be found, one, on the outside cylindrical surface, and passing through the point A, and the other on an inner cylinder or drum upon which is situated the point B. Notice that the pitch is the same for both lines.

For convenience, 30° divisions of the circular plan are made, therefore twelve in all, and for each 30° division over which the line moves it rises $\frac{1}{12}$ of the pitch. For a right-hand screw the thread runs upwards in the direction of the arrow-headed line.

For a thread of square section there will be four helicoidal lines to draw. In such a case if the pitch be $\frac{3}{4}''$, the edge of the square section will be $\frac{3}{8}''$.

EXERCISE XLIV

1. Find the elevation of a screw-thread on a vertical bolt $2\frac{1}{2}''$ diameter, when the section of the thread by a plane including the axis, is an equilateral triangle. The thread to be single, and pitch $\frac{5}{8}''$.

2. Find the projection of a bolt screw-thread with square section. Diameter of bolt 3″, pitch $\frac{3}{4}''$.

3. Find the surfaces of revolution generated by the edges of a cube made to rotate on one of its solid diagonals, i.e., on a diagonal passing through its centre. Edge 2″.

CHAPTER XV

RADIAL PROJECTION. PERSPECTIVE PROJECTION

SECTION **31**. Things are made visible to us by means of light, and in order to obtain, in the eye, a picture of anything, rays of light, which travel in straight lines, must proceed from the thing looked at, to the eye. The retina, at the back of the eye, is the picture surface, a curved one, receiving the projection or view of the object. This projection or view is obtained by a process called Radial Projection. It is the same as that by which a plane surface, the plate or flat film, in a camera, receives a picture.

In Orthographical Projection, by which we have obtained plans and elevations of things, the projectors are at right angles to the projection plane. In Radial Projection the projectors are inclined to the projection plane. Radial Projection is used in different ways for the projection of maps, etc., but is most commonly used, by Architects and others, in what is called Perspective Projection.

In Perspective Projection a single projection plane is used, and is situated between the object and the eye. It can be seen through, by the spectator, as he looks at the object. The view or projection of the object is caught by this projection plane, or Picture Plane as it is called, which is, in attitude, at right angles to the direction of vision.

Since, in a natural way, and ordinarily, a person's sight is directed, as he stands, in a horizontal direction, the Picture Plane, in Perspective Projection, is arranged as a vertical plane, with a point, perpendicularly opposite the eye, marked upon it, as the Centre of Vision.

The eye-level exactly agrees with what one sees in the distance as the horizon, and is represented on the Picture Plane by a horizontal line, in which the Centre of Vision point is situated. This line, representing the eye-level and at the same time the horizon, is called the Horizon Line, and parallel to it, on the Picture Plane, another horizontal line is drawn, at a distance below the Horizon Line equal to the distance of the spectator's eye from the ground and is consequently called the Ground Line.

The distance from the Horizon Line to the Ground Line is decided upon for each case, and should be such as will best serve the purpose in viewing the object to be represented. Drawings, of course, are necessarily made strictly to some chosen scale—$\frac{1}{8}''$ to the foot, $\frac{1}{2}''$ to the foot, half size, etc., as may be suitable.

The distance between the object and the spectator is variable, and may be any distance chosen. The appearance of the object depends very much upon the distance chosen. This distance being settled, it is then necessary to fix the position, between the object and the spectator, of the Picture Plane, and upon this will depend the size of the drawing or projection of the object.

Let the student imagine the vertical transparent Picture Plane, between his eye and the object, and he will realize that, as the plane is placed nearer to the object the view of it which he could trace on the plane will be larger, and as he brings the Picture Plane towards the eye, the view of the object, traced on the plane, will be smaller.

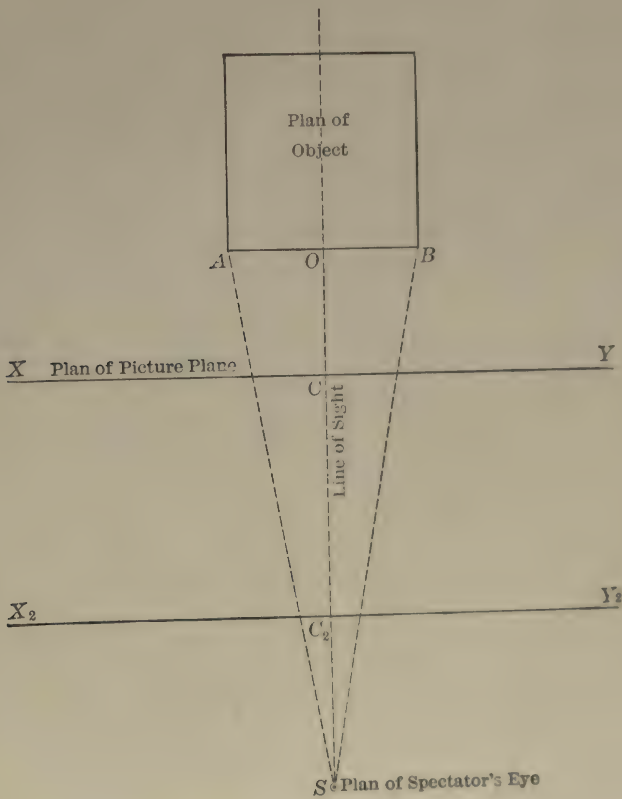


FIG. 111.

This and other matters will be better understood if reference is made to Fig. 111, where the proper arrangement for perspective projection is shown in plan. It will be seen that the spectator is at a distance OS from the object, and that XY, representing the vertical picture plane, at right angles to the direction of vision represented by the direction of the line OS, is placed not far away from the object in relation to the distance of the spectator from it, and consequently, the size of the picture of the object obtained by the interception of the rays, from the object to the eye, by the plane at XY, will be fairly large. On the other hand, if the picture plane be placed so as to have its plan at X_2Y_2, that is, rather close to the eye in relation to the distance of the object from it, then the picture obtained by the interception of the rays from the object as they proceed to the eye, will be correspondingly small. The point C is the plan of what was spoken of as the Centre of Vision, and is commonly marked C.V.

If a vertical plane be imagined as containing the horizontal line of sight or direction of vision, the line SC, then anything to the left of this vertical plane, as, for instance, the corner A of the object, is said to be to the left of the spectator, and anything to the right of this same vertical plane, as B, is said to be to the right of the spectator. The whole object might be to the left, or to the right, of the spectator, and of course that would mean that the plan of it would have to be placed to the left, or right, of the line SC, accordingly.

In order to make a perspective projection or drawing, the essential things are:—

(1) The placing of the spectator S, and the line of sight SC.
(2) The placing of the object, represented by plan, at its proper distance from the spectator, measured in the direction of the line of sight, and at its proper distance to the right or left of the spectator.
(3) The placing of the picture plane, represented by XY, and
(4) The decision as to the height of the eye above the horizontal plane of the ground, on which, or in relation to which, the object is placed.

To represent the two lines Ground Line and Horizon Line, two lines, so named, are drawn at their proper distances apart, across the paper parallel to XY, and at any convenient place between the point S and the line XY, as in Fig. 112.

Before proceeding to discuss specific cases of perspective projection, certain facts which have a bearing on the intelligent understanding of the method adopted, must be realized. These are as follows:—

(1) All parallel lines receding from the spectator appear to converge, regardless of attitude or direction.
(2) All horizontal lines regardless of level, when receding from the spectator approach the horizon, and when produced to infinity lose themselves on the horizon in points commonly spoken of as vanishing points, and
(3) Any number of parallel horizontal lines, regardless of levels or direction, have the same vanishing point in the horizon.

Consequently, therefore, lines on the ground plane, or other horizontal lines below the level of the eye, if receding from the spectator, will be represented by lines rising in the projection from their near ends towards the eye-level line which represents the horizon, and, similarly, horizontal receding lines above the level of the eye will be represented by lines drawn downwards from their near ends towards the eye-level line where their vanishing points are.

Consequently, also, horizontal lines that are perpendicular to the projection plane, that is, in the same direction as the line of sight, wherever they are, will have their vanishing point at *C*, or C.V., in the horizon line, that is, at the Centre of Vision.

An illustration is shown in Fig. 112, where a vertical shaft standing centrally on a square block is represented. The plan of this group is placed a little to the

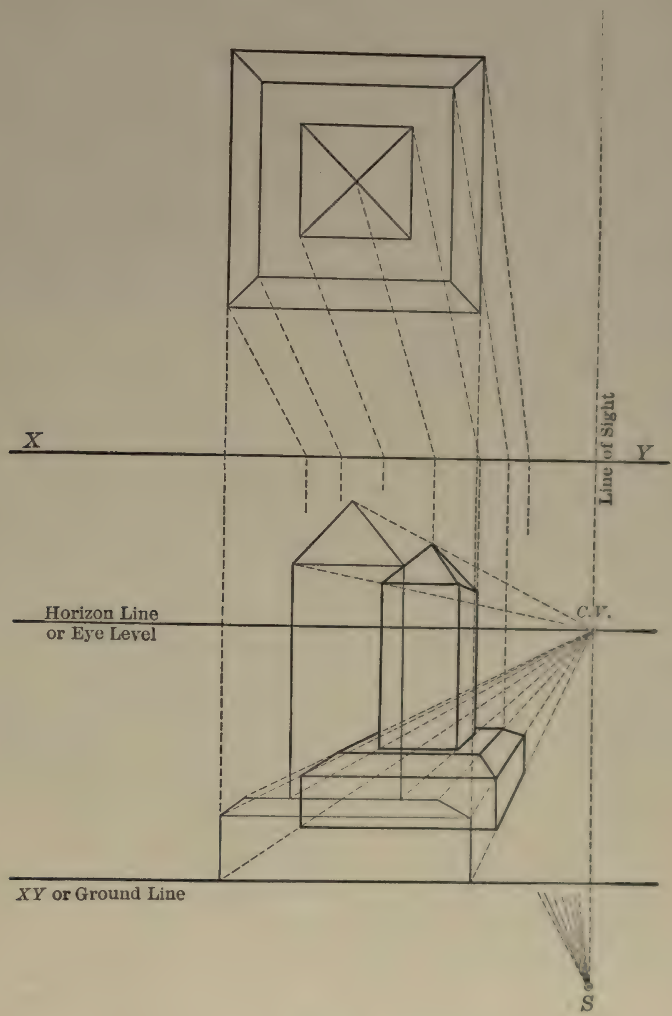


FIG. 112.

left of the spectator, measured to the left of the line of sight, and is at a distance, measured along the line of sight, from the spectator whose position is marked *S*. The picture plane, that is, the projection plane, is placed vertically at *XY*, which is its plan, and is then represented again, lower down on the paper, with *XY* or Ground Line parallel to the original position and the eye-level line or Horizon

Line represented at the height decided upon for the eye. In the figure the elevation of the group is shown, exactly opposite the plan of it, on this plane, and in this way the true heights on the picture plane, where horizontal lines through the points of the group, and perpendicular to the plane, meet it, are obtained.

To obtain the Perspective Projection it is now only necessary to represent all the horizontal lines that are really perpendicular to the Picture Plane, as proceeding to the Centre of Vision, C.V., and drop perpendiculars to them from the points in the plan XY of the picture plane, where rays from the points, to the eye, are intercepted by the picture plane. In the figure most of these perpendiculars are only started, as dotted lines, not carried all the way, in order to avoid confusion.

Fig. 112 illustrates what is often spoken of as Parallel Perspective, since the lines necessary for it are either parallel to the picture plane or parallel to the Line of Sight.

N.B.—Since the view or projection is not influenced, except in size, by the distance of the picture plane from the spectator, and the things represented are often large things drawn to a small scale, the picture plane is arranged as if near the object in order to get a reasonably large drawing for the projection.

EXERCISE XLV

1. Find the perspective projection of a box with a lid. Its measurements are 2 ft. $\times$ 3 ft. and it is $1\frac{1}{2}$ ft. high. Thicknesses may be omitted. Let the picture plane be 10 ft. from the spectator and the box 2 ft. beyond the plane, or 2 ft. "into the picture" as it is said, and the nearest corner 4 ft. to the left. Let one end be parallel to the picture plane and the front of the box be on the near side. Let the lid be opened to an angle of 45°.

2. Make a perspective projection of a double cross, that is having four arms at right angles to each other, with a pedestal and shaft similar to those in Fig. 112. Let the spectator's eye be 5 ft. above the ground and the total height of the group 10 ft., the pedestal being $1\frac{1}{2}$ ft. thick and 4 ft. square, one edge of the square being parallel to the picture plane and 3 ft. into the picture (i.e., beyond the picture plane). Make the shaft and crossbars 1 ft. square in section, each arm being $1\frac{1}{2}$ ft. long, and the distance from pedestal to crossbars 5 ft. Let the nearest corner of the pedestal be 3 ft. to the right of the spectator and the distance of the picture plane from the spectator 12 ft.

3. Make a perspective projection of any familiar object such as a book, a hut, a boat-house, a table, or any simple thing arranged in parallel perspective.

PERSPECTIVE PROJECTION, Continued

Section **32**. We have seen that the vanishing point, in the horizon, for horizontal lines perpendicular to the picture plane, is the same as that for the line of sight or line from the eye perpendicular to the picture plane, and marked C.V.—the central vanishing point, or Centre of Vision.

We shall now consider the determination of vanishing points for lines that are not perpendicular to the picture plane. For a horizontal line, the vanishing point must be in the horizon, and must, for our purposes of projection, be represented by a point on the horizon line drawn on the picture plane. This point, the given line's vanishing point, will be the same as for a line proceeding from the eye of the spectator parallel to the given line. Hence the method shown in Fig. 113, namely, from S draw a line parallel to the plan of the given horizontal line, ab, to meet XY, the projection plane, in a point whose plan is V, and whose place in the Horizon Line on the picture plane is V.P.

Since the line ab and the parallel to it SV are in the same plane, an oblique plane, it will be realized that the trace, on the picture plane, of the line ab, viz.,

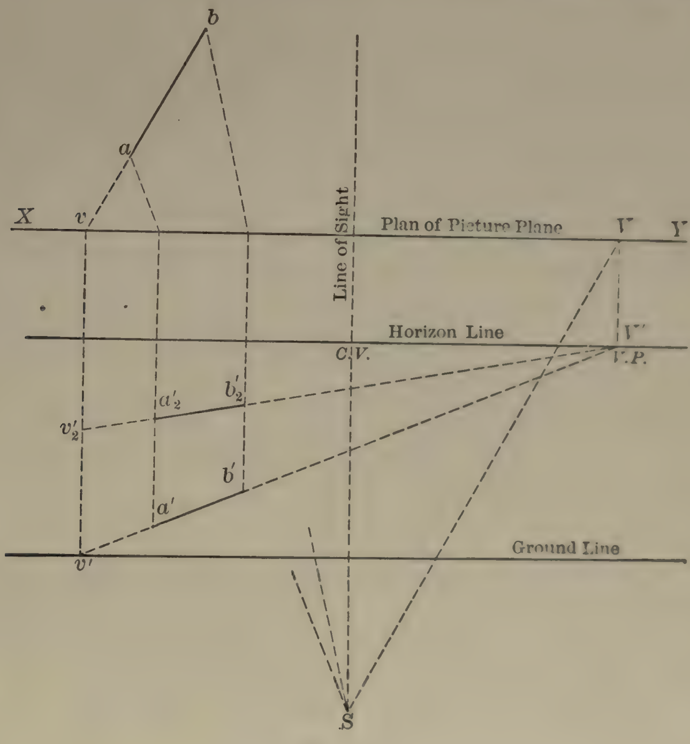


FIG. 113.

the point whose plan is v, joined to the trace of the line SV at the point whose plan is V, will give the intersection of the oblique plane with the picture plane. This is shown as the line, across the picture plane, marked $v'V'$, and so it will be realized that the perspective projection of the line ab, a line in the oblique plane, as seen from the point S, a point in the same plane, by radial projection in that oblique plane, must be a foreshortened line lying in the intersection line $v'V'$ in the projection plane. The plans of the rays are directed from a and b to S in the drawing, and are seen to meet the plan of the projection plane, XY, at points from which vertical projectors are drawn to determine the perspective projection $a'b'$. The point in the Horizon Line marked V' is the vanishing point for the line AB and for all other lines parallel to it. It is therefore also marked V.P. A second line, horizontal, and immediately above the line AB, is shown in projection at $a'_2b'_2$. Its vertical trace in the picture plane is v'_2. The intersection, of the plane of this second line and the eye point, with the picture plane, is the trace line v'_2V', and since, as before, the rays of light from this upper line

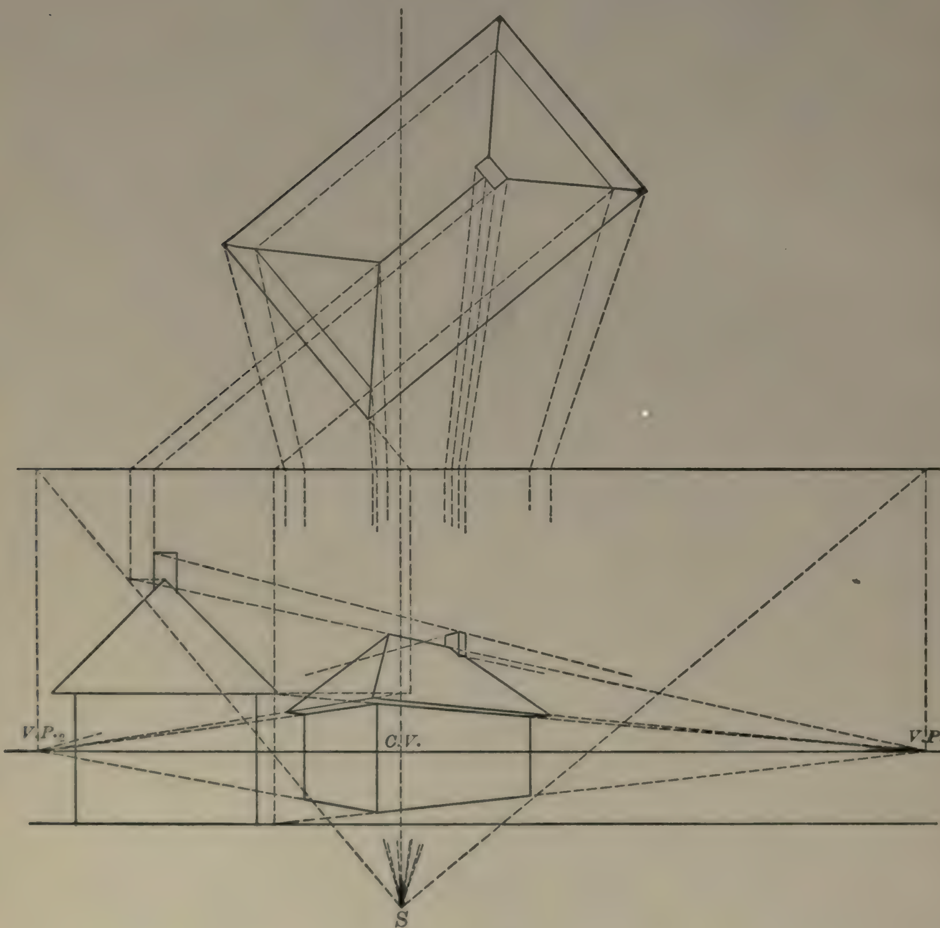


FIG. 114.

travel to the eye in the oblique plane whose intersection with the picture plane is v'_2V', the representation of the line in perspective is inclined, and is found to be $a'_2b'_2$.

In Fig. 114, an illustration is given involving the use of vanishing points other than the central vanishing point. These are marked $V.P_1$ and $V.P_2$.

It will be noticed that as there are no lines in the object perpendicular to the Picture Plane, the C.V. does not come into use as a vanishing point in this case.

The nearest corner of the cottage is at a certain distance into the picture, i.e., beyond the Picture Plane, and also at a distance to the left of the spectator, that is, to the left of the central line of vision, or Line of Sight.

The plan of the cottage is arranged so that the length way of it is at an angle of 40° to the Picture Plane. At a convenient place to the left, on the Ground Line as base, is erected an end elevation, from which levels for the vertical traces of lines, such as that for the roof ridge, may be found, from which to draw vanishing lines to the vanishing points. These vanishing lines on the Picture Plane, as was seen above, are intersections of the Picture Plane by oblique planes, each containing the eye and some line, such as the ridge line of the roof, and therefore containing the projection of it, by radial projection from the said line to the eye. In Fig. 114, the drawing has been made as economically as possible, in order to save confusion of lines. It will be readily seen that by continuing in the same manner, windows and door ways, etc., might be added.

The student will doubtless realize that, if it were necessary to find vanishing points for oblique lines, these would be found in the vertical traces, on the Picture Plane, of vertical planes through the eye, parallel to such oblique lines. For the purposes of our present study, however, sufficient has been undertaken.

EXERCISE XLVI

1. Work out to a scale, such as $\frac{1}{8}''$ to the ft., a plan and elevation for a structure such as that given in the Fig. 114, where the sizes are 15 ft. by 26 ft., with roof plan 19 ft. by 30 ft. The nearest corner of the roof is 4 ft. into the picture and 3 ft. to the left of the spectator. The distance of the spectator is 40 ft., that is, 36 ft. from the picture plane. Suitable heights and details should be chosen. Height of the eye 6 ft.

2. Numerous problems will readily suggest themselves, such as books lying unevenly upon one another, pieces of furniture, buildings, etc.

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