Faculty Working Papers

A Deterministic Theory of Individual Saving

and Portfolio Composition

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Introduction

The goal of this research is to develop a model that will show how an individual chooses his lifetime patterns of consumption and saving and how he simultaneously chooses the composition of his portfolio of assets at each point during his lifetime. The individual is assumed to know with certainty his future pattern of income, the rates of interest that he can earn on the various assets in his portfolio, and the rates of interest that he must pay if he goes into debt. The theory of portfolio composition presented here is not based on the usual trade-off between the expected yield and the riskiness of an asset. Instead it is based on the trade-off between the yield or the rate of interest that can be earned on each asset and the transactions costs involved in buying and selling that asset.

An asset with a low rate of interest and low transactions costs (for example, savings accounts) will only be held if the surplus is to last for a relatively short period of time. The individual is compensated for the low interest rate by the small costs involved in buying and selling that asset. If the surplus is to last for a relatively long period of time, an individual is better off holding the surplus in the form of an asset with a higher interest rate and undoubtedly higher transactions costs (for example, corporate bonds, real estate, etc.) Similarly at those points in an individual's lifetime when he is a net debtor, he will choose the type of loan to finance the deficit by comparing the rate of interest that he must pay against the cost of taking out and

paying back the loan. If his indebtedness is to last for a relatively short period of time, he is better off taking out a loan which has low transactions costs even though he will probably have to pay a higher rate of interest. For example, these loans might be in the form of charge accounts or credit cards. If the deficit is to last for a long period of time, he would be better off arranging for a loan with lower interest charges and undoubtedly higher transactions costs (for example, mortgages or other secured loans).

What this model will show is that if the individual is in a surplus position then he will probably be holding a number of different assets each with a different interest rate and transactions cost. In other words, at each point in time the individual's portfolio will usually contain a variety of different assets. Similarly, if the individual is in a net deficit position then he will have financed this deficit by taking out a variety of different types of loans each with a different interest rate and transactions costs. At each point during his lifetime, the individual will either be in a surplus position or a deficit position depending on whether his cumulative expenditures for consumption is greater than or less than his cumulative income up to that point. For simplicity, it is assumed that the individual does not borrow in order to buy an asset.

The Individual's Optimization Problem

An individual is assumed to maximize the discounted sum of instantaneous utility over his lifetime, i.e.,

$$(1) \qquad \max \int_0^T e^{-\theta t} u(c)dt$$

[illegible]

transactions [illegible] even though [illegible]

[illegible] for example [illegible] in the [illegible]

account [illegible] credit [illegible] the [illegible]

[illegible] for a loan [illegible]

[illegible]

[illegible]

[illegible] individual [illegible]

position [illegible] will [illegible]

[illegible] different [illegible]

[illegible] point in the [illegible] (final) period [illegible]

[illegible] different assets. Similarly, if [illegible]

[illegible] the [illegible]

[illegible] different [illegible]

[illegible] transactions costs. [illegible]

will either be [illegible] position [illegible]

[illegible]

[illegible] cumulative [illegible] to that point. [illegible]

[illegible] of the [illegible]

The individual's [illegible] problem

An individual [illegible] maximize [illegible] discounted [illegible]

[illegible] utility [illegible]

$$\max \int_0^{T} e^{-\delta t} u(c) \, dt$$

where Θ is his constant discount rate, $c(t)$ is his lifetime pattern of consumption, and $u(c)$ is a differentiable and concave utility function. The price level is constant and equal to one. His lifetime pattern of endowment income is denoted by $x(t)$, $0 \leq t \leq T$.

For simplicity, assume that there are only two types of assets or bonds that he may hold, denoted by $B_1(t)$ and $B_2(t)$. Each type of bond earns a rate of interest, denoted by r_1 and r_2 respectively; and these rates are constant over time. Similarly the individual may take out two different types of loans where the total indebtedness of each type is denoted by $D_1(t)$ and $D_2(t)$. On each type of debt, he must pay a rate of interest, denoted by i_1 and i_2 respectively, which are also constant over time. The stocks of bonds and the stocks of debt must be non-negative.

At any point in time, an individual's total receipts from all sources must equal his total expenditures. An individual's total receipts is the sum of his endowment income, interest income earned on positive stocks of bonds, receipts from the sale of bonds of either type, and the receipts from taking out additional loans of either type. His total expenditures is the sum of consumption expenditures, interest payments on his debts, expenditures for the purpose of adding to his stocks of bonds of either type, and expenditures for repaying loans of either type. The receipts from selling bonds are denoted by S_1 and S_2, and the receipts from taking out additional loans are denoted by L_1 and L_2. Expenditures for increasing the stocks of bonds are denoted by A_1 and A_2, and expenditures for the repayment of loans are denoted by R_1 and R_2. His budget constraint at each point in time is thus

(2) $$x + r_1B_1 + r_2B_2 + S_1 + S_2 + L_1 + L_2$$
$$- c - i_1D_1 - i_2D_2 - A_1 - A_2 - R_1 - R_2 = 0$$

[illegible] in $q_i(t)$ [illegible]

[illegible]

[illegible] ed, denote [illegible]

[illegible] only the [illegible]

diffe[illegible] to [illegible]

[illegible] ned by $q_1(t)$ and $q_2(t)$ [illegible] pay a rat[illegible]

[illegible] denoted [illegible] respectively [illegible] constant [illegible]

[illegible] of both [illegible]

[illegible] in the [illegible]

[illegible] test is [illegible]

[illegible] where the [illegible]

[illegible] types, and the [illegible]

[illegible] is the [illegible] expe[illegible]

[illegible] expenditure [illegible]

[illegible] types, and [illegible]

[illegible] and the [illegible]

[illegible] denoted [illegible] Appendix [illegible]

[illegible] the [illegible] constraint

[illegible]

On any changes in his stocks of bonds or debt, the individual must pay transactions costs proportional to the size of the change. The cost of changing the size of the stock of each type of bond by one dollar is given by v_1 and v_2 respectively. The cost of changing the size of the stock of each type of debt by one dollar is given by w_1 and w_2 respectively. Thus, the net change in the stock of each type of bond and debt is given by

(3) $\dot{B}_1 = (1 - v_1)A_1 - (1 + v_1)S_1$

(4) $\dot{B}_2 = (1 - v_2)A_2 - (1 + v_2)S_2$

(5) $\dot{D}_1 = (1 + w_1)L_1 - (1 - w_1)R_1$

(6) $\dot{D}_2 = (1 + w_2)L_2 - (1 - w_2)R_2$

where

(7) $B_1, B_2, D_1, D_2, A_1, S_1, A_2, S_2, L_1, R_1, L_2, R_2, \geq 0$

The individual is assumed to start his lifetime with initial stocks of bonds and initial stocks of debt which may or may not be positive, i.e.

(8) $B_1(0) = B_1^o, \qquad B_2(0) = B_2^o$

(9) $D_1(0) = D_1^o, \qquad D_2(0) = D_2^o.$

At the end of his lifetime, the individual is assumed to have no desire to leave an inheritance. It is also assumed that his creditors will not allow him to leave any outstanding debts. Therefore,

(10) $B_1(T) = B_2(T) = D_1(T) = D_2(T) = 0$

There are two features of this optimization problem that make it different from conventional problems in optimal control. The first is the inequality constraints on the state variables and the control variables given by condition (7). The second and much more important feature is

that the time path of the state variables may be discontinuous. In other words, it is possible in this model that the optimal stocks of bonds or debt may make instantaneous jumps in size.

For example, when might it be optimal for the individual to suddenly sell a part of his stock of one type of bond and use the proceeds to buy another type of bond or to pay off a debt? It is not difficult to convince oneself that this could never be optimal except at the initial time point, $t = 0$. If such jumps were to occur later in his lifetime, the individual must have made a mistake sometime in the past or his expectations of future income or interest rates have changed. If it is now optimal for an individual to suddenly sell one type of bond in order to buy another type then the question is why was this not done earlier when the surplus was accumulated in order to save transactions costs. However, the initial stocks of bonds and debt with which the individual begins his lifetime are assumed to be given to him exogenously, and it may be optimal to make an initial rearrangement of the composition of his portfolio of assets and debts.

To allow for this initial rearrangement of the individual's portfolio, let the initial changes in the stocks of bonds or debt be denoted by $\hat{A}_1$, $\hat{S}_1$, $\hat{A}_2$, $\hat{S}_2$, $\hat{L}_1$, $\hat{R}_1$, $\hat{L}_2$, and $\hat{R}_2$. The new stocks after the initial rearrangement are denoted by $B_1(0^+)$, $B_2(0^+)$, $D_1(0^+)$, and $D_2(0^+)$. The new values of the stocks are related to the old by

(11) $B_1(0^+) = B_1(0) + (1 - v_1)\hat{A}_1 - (1 + v_1)\,\hat{S}_1$

(12) $B_2(0^+) = B_2(0) + (1 - v_2)\hat{A}_2 - (1 + v_2)\,\hat{S}_2$

... the path ... [illegible] ... may ... [illegible]

... [illegible] ...

[illegible]

... [illegible] ... a part of ... [illegible] ...

... [illegible] ...

... [illegible] ... in the limit ... [illegible]

... [illegible] ...

... [illegible] ... by one this ... [illegible] ... the equation ...

... [illegible] ...

... [illegible] ... of his portfolio ... [illegible] ... and

... [illegible] ...

... [illegible] ...

(a) Let the ... [illegible] ...

... [illegible] ...

... are denoted by ... [illegible] ...

... [illegible] ...

(11) $a_1(0^+) = $ [illegible]

(12) $a_2(0^+) = a_2(0$ [illegible]

$$(13)\quad D_1(0^+) = D_1(0) + (1 + w_1)\hat{L}_1 - (1 - w_1)\hat{R}_1$$

$$(14)\quad D_2(0^+) = D_2(0) + (1 + w_2)\hat{L}_2 - (1 - w_2)\hat{R}_2$$

where as always transactions costs must be paid on any changes in a stock. Any change in the value of one stock must be matched by an appropriate change in another stock, i.e.

$$(15)\quad \hat{A}_1 - \hat{S}_1 + \hat{A}_2 - \hat{S}_2 - \hat{L}_1 + \hat{R}_1 - \hat{L}_2 + \hat{R}_2 = 0$$

where

$$(16)\quad \hat{A}_1,\ \hat{S}_1,\ \hat{A}_2,\ \hat{S}_2,\ \hat{L}_1,\ \hat{R}_1,\ \hat{L}_2,\ \text{and}\ \hat{R}_2 \geq 0$$

In conclusion, the individual's goal is to find the pattern of consumption, the patterns of bond holdings, and the patterns of indebtedness over his lifetime that maximize (1) subject to the constraints (2) through (16). Because of the non-negativety constraints on both the state variables and the control variables and the possibility of discontinuities in the state variables, it does not seem possible to use the conventional theory of optimal control to derive necessary conditions for an optimum for this problem. However, it is possible to state sufficient conditions for an optimum.

Sufficient Conditions for an Optimum

It is possible to give conditions that if satisfied will guarantee that the optimal solution to the above problem has been found. These conditions involve five shadow prices, $\eta(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\mu_1(t)$, and $\mu_2(t)$. The first shadow price, η, is equal to the discounted value of the marginal utility of consumption. The variables λ_1 and λ_2 are the implicit prices that the individual places on the two stocks of bonds. The variables μ_1 and μ_2 are the implicit prices that the individual places on the two stocks of debt.

The statement of these conditions is as follows: If there exists functions $c(t)$, $A_1(t)$, $S_1(t)$, $A_2(t)$, $S_2(t)$, $L_1(t)$, $R_1(t)$, $L_2(t)$, and $R_2(t)$ continuous for $0 \le t \le T$, functions $B_1(t)$, $B_2(t)$, $D_1(t)$ and $D_2(t)$ continuous for $0 < t \le T$, and values for $\hat{A}_1$, $\hat{S}_1$, $\hat{A}_2$, $\hat{S}_2$, $\hat{L}_1$, $\hat{R}_1$, $\hat{L}_2$, and $\hat{R}_2$ all satisfying relations (2) through (16) and functions $\eta(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\mu_1(t)$, and $\mu_2(t)$ continuous for $0 \le t \le T$ where $\dot{B}_1$, $\dot{B}_2$, $\dot{D}_1$, $\dot{D}_2$, $\dot{\eta}$, $\dot{\lambda}_1$, $\dot{\lambda}_2$, $\dot{\mu}_1$, and $\dot{\mu}_2$ are integrable such that at each point in time

(17) $\eta = e^{-\theta t}u'(c)$

(18) $[\dot{\lambda}_j + r_j\eta]B_j = 0 \quad j = 1,2$

(19) $\dot{\lambda}_j + r_j\eta \le 0 \quad j = 1,2$

(20) $[(1 - v_j)\lambda_j - \eta]A_j = 0 \quad j = 1,2$

(21) $(1 - v_j)\lambda_j - \eta \le 0 \quad j = 1,2$

(22) $[\eta - (1 + v_j)\lambda_j]S_j = 0 \quad j = 1,2$

(23) $\eta - (1 + v_j)\lambda_j \le 0 \quad j = 1,2$

(24) $[\dot{\mu}_j - i_j\eta]D_j = 0 \quad j = 1,2$

(25) $\dot{\mu}_j - i_j\eta \le 0 \quad j = 1,2$

(26) $[(1 + w_j)\mu_j + \eta]L_j = 0 \quad j = 1,2$

(27) $(1 + w_j)\mu_j + \eta \le 0 \quad j = 1,2$

(28) $[-\eta - (1 - w_j)\mu_j]R_j = 0 \quad j = 1,2$

(29) $-\eta - (1 - w_j)\mu_j \le 0 \quad j = 1,2$

and at time $t = 0$

(30) $[(1 - v_j)\lambda_j - \eta]\hat{A}_j = 0 \quad j = 1,2$

(31) $(1 - v_j)\lambda_j - \eta \le 0 \quad j = 1,2$

(32) $[\eta - (1 + v_j)\lambda_j]\hat{S}_j = 0 \quad j = 1,2$

(33) $\eta - (1 + v_j)\lambda_j \le 0 \quad j = 1,2$

(34) $[(1 + w_j)\mu_j + \eta]\hat{L}_j = 0 \quad j = 1,2$

(35) $(1 + w_j)\mu_j + \eta \leq 0 \quad j = 1,2$

(36) $[-\eta - (1 - w_j)\mu_j]\hat{R}_j = 0 \quad j = 1,2$

(37) $-\eta - (1 - w_j)\mu_j \leq 0 \quad j = 1,2$

then $c(t)$, $B_1(t)$, $B_2(t)$, $D_1(t)$, $D_2(t)$, $A_1(t)$, $S_1(t)$, $A_2(t)$, $S_2(t)$, $L_1(t)$, $R_1(t)$, $L_2(t)$, $R_2(t)$, $\hat{A}_1$, $\hat{S}_1$, $\hat{A}_2$, $\hat{S}_2$, $\hat{L}_1$, $\hat{R}_1$, $\hat{L}_2$, and $\hat{R}_2$ will maximize the functional (1) subject to the conditions (2) through (16).

Proof

If we denote by an asterisk those functions and variables which satisfy the previously stated conditions then for any other feasible set of functions and variables it must be shown that

(38) $\int_0^T e^{-\Theta t}u(c^*)dt \geq \int_0^T e^{-\Theta t}u(c)dt$

or that

(39) $\int_0^T e^{-\Theta t}[u(c^*) - u(c)]dt \geq 0$

The following string of equalities and inequalities will prove this result. The explanation of each step is given in brackets.

(40) $\int_0^T e^{-\Theta t}[u(c^*) - u(c)]dt$

[since u(c) is assumed to be concave]

(41) $\geq \int_0^T e^{-\Theta t} u'(c^*)(c^* - c)\, dt$

[from equation (17)]

(42) $= \int_0^T \eta(c^* - c)\, dt$

[from equation (2)]

$$(43)\quad = \int_0^T \eta\left[S_1^* + S_2^* + L_1^* + L_2^* + r_1B_1^* + r_2B_2^* - i_1D_1^* - i_2D_2^*\right.$$

$$- A_1^* - A_2^* - R_1^* - R_2^* - S_1 - S_2 - L_1 - L_2 - r_1B_1$$

$$\left.-r_2B_2 + i_1D_1 + i_2D_2 + A_1 + A_2 + R_1 + R_2\right] dt$$

[from conditions (20) through (29)]

$$(44)\quad \geq \int_0^T \left[(1 + v_1)\lambda_1S_1^* + (1 + v_2)\lambda_2S_2^* - (1 + w_1)\mu_1L_1^* - (1 + w_2)\mu_2L_2^* + r_1\eta B_1^*\right.$$

$$+r_2\eta B_2^* - i_1\eta D_1^* - i_2\eta D_2^* - (1 - v_1)\lambda_1A_1^* - (1 - v_2)\lambda_2A_2^*$$

$$+ (1 - w_1)\mu_1R_1^* + (1 - w_2)\mu_2R_2^* - (1 + v_1)\lambda_1S_1 - (1 + v_2)\lambda_2S_2$$

$$+ (1 + w_1)\mu_1L_1 + (1 + w_2)\mu_2L_2 - r_1\eta B_1 - r_2\eta B_2 - r_2\eta B_2 + i_1\eta D_1$$

$$+ i_2\eta D_2 + (1 - v_1)\lambda_1A_1 + (1 - v_2)\lambda_2A_2 - (1 - w_1)\mu_1R_1$$

$$\left.- (1 - w_2)\mu_2R_2\right] dt$$

[from conditions (3) through (6)]

$$(45)\quad = \int_0^T\left[-\lambda_1\dot{B}_1^* + \lambda_1\dot{B}_1 - \lambda_2\dot{B}_2^* + \lambda_2\dot{B}_2 + r_1\eta B_1^* + r_2\eta B_2^* - r_1\eta B_1\right.$$

$$- r_2\eta B_2 - \mu_1\dot{D}_1^* + \mu_1\dot{D}_1 - \mu_2\dot{D}_2^* + \mu_2\dot{D}_2 - i_1\eta D_1^* - i_2\eta D_2^*$$

$$\left.+ i_1\eta D_1 + i_2\eta D_2\right]dt$$

[integrating by parts]

$$(46)\quad = -\lambda_1B_1^*\Big|_0^T + \int_0^T(\dot{\lambda}_1 + r_1\eta)B_1^*dt + \lambda_1B_1\Big|_0^T + \int_0^T (-\dot{\lambda}_1 - r_1\eta)B_1dt$$

$$-\lambda_2 B_2^*\Big|_0^T + \int_0^T(\dot{\lambda}_2 + r_2\eta)B_2^*dt + \lambda_2 B_2\Big|_0^T + \int_0^T(-\dot{\lambda}_2 - r_2\eta)B_2dt$$

$$-\mu_1 D_1^*\Big|_0^T + \int_0^T(\dot{\mu}_1 - i_1\eta)D_1^*dt + \mu_1 D_1\Big|_0^T + \int_0^T(-\dot{\mu}_1 + i_1\eta)D_1dt$$

$$-\mu_2 D_2^*\Big|_0^T + \int_0^T(\dot{\mu}_2 - i_2\eta)D_2^*dt + \mu_2 D_2\Big|_0^T + \int_0^T(-\dot{\mu}_2 + i_2\eta)D_2dt$$

[from conditions (18), (19), (24), (25) and (10)]

(47) $$\geq + \lambda_1(0)B_1^*(0^+) - \lambda_1(0)B_1(0^+) + \lambda_2(0)B_2^*(0^+) - \lambda_2(0)B_2(0^+)$$

$$+ \mu_1(0)D_1^*(0^+) - \mu_1(0)D_1(0^+) + \mu_2(0)D_2^*(0^+) - \mu_2(0)D_2(0^+)$$

[from conditions (8), (9), (11), (12), (13), and (14)]

(48) $$= (1 - v_1)\lambda_1(0)\hat{A}_1^* - (1 + v_1)\lambda_1(0)\hat{S}_1^* - (1 - v_1)\lambda_1(0)\hat{A}_1 + (1 + v_1)\lambda_1(0)S_1$$

$$+ (1 - v_2)\lambda_2(0)\hat{A}_2^* - (1 + v_2)\lambda_2(0)\hat{S}_2^* - (1 - v_2)\lambda_2(0)\hat{A}_2 + (1 + v_2)\lambda_2(0)\hat{S}_2$$

$$+ (1 + w_1)\mu_1(0)\hat{L}_1^* - (1 - w_1)\mu_1(0)\hat{R}_1^* - (1 + w_1)\mu_1(0)\hat{L}_1$$

$$+ (1 - w_1)\mu_1(0)\hat{R}_1 + (1 + w_2)\mu_2(0)\hat{L}_2^* - (1 - w_2)\mu_2(0)\hat{R}_2^*$$

$$- (1 + w_2)\mu_2(0)\hat{L}_2 + (1 + w_2)\mu_2(0)\hat{R}_2$$

[from conditions (30) through (37)]

(49) $$\geq \eta[\hat{A}_1^* - \hat{S}_1^* + \hat{A}_2^* - \hat{S}_2^* - \hat{L}_1^* + \hat{R}_1^* - \hat{L}_2^* + \hat{R}_2^*]$$

$$-\eta[\hat{A}_1 - \hat{S}_1 + \hat{A}_2 - \hat{S}_2 - \hat{L}_1 + \hat{R}_1 - \hat{L}_2 + \hat{R}_2]$$

[from condition (16)]

(50) $$= 0$$

<u>Example</u>

In order to illustrate the basic features of an optimal solution, the upper half of Figure 1 gives a hypothetical lifetime pattern of income. This pattern of income shows a low level of income during the individual's youth, a higher level of income during middle age, and again a low level of income during old age. In this example, let us also assume that the individual's utility function is of the specific form

(51) $u(c) = \frac{1}{1-\sigma}c^{1 - \sigma}$

Let the interest rates on debt (i_1 and i_2), the interest rates on bonds (r_1 and r_2), and the individual's rate of discount (σ) have the following relationship,

(52) $i_2 > i_1 > \Theta > r_1 > r_2$

Also let the transactions costs coefficients for bonds (w_1 and w_2) and debt (v_1 and v_2) have the following relationship,

(53) $w_2 < w_1$ and $v_1 > v_2$

For simplicity, assume that $B_1^o = B_2^o = D_1^o = D_2^o = 0$. Given these assumptions, the resulting optimal patterns of consumption, bond holdings, and indebtedness might look as depicted in Figure 1.

In this example, the individual's lifetime can be divided into ten subperiods. Some of the important characteristics of each subperiod are as follows:

$[0, t_1]$ In the first subperiod, the individual's level of consumption is greater than his income; and the deficit is financed by taking out loans of the first type ($D_1 \geq 0$ and $L_1 > 0$). Since these loans

will not be paid off for a relatively long period of time (after point t_4), the high transactions costs are compensated for by the low interest charges. From conditions (24) and (26) and equation (51), the optimal pattern of consumption is defined by the differential equation

(54) $$\dot{c}/c = [(1 + w_1)i_1 - \Theta]/\sigma.$$

The level of consumption will be increasing during this subperiod since $i_1 > \Theta$.

$[t_1, t_2]$ In the second subperiod, the individual now takes out loans of the second type because these loans will be repaid in a relatively short period of time. The higher interest charges are compensated for by the lower transactions costs. Again it is not difficult to show that consumption will be increasing during this subperiod.

$[t_2, t_3]$ In the third subperiod, it is not optimal to use use either type of debt to alter the consumption path; and

(55) $$c = x - i_1D_1 - i_2D_2.$$

The gain from rearranging the pattern of consumption is not worth the combined interest charges and transactions costs.

$[t_3, t_4]$ In the fourth subperiod, the level of consumption is less than income; and the surplus is used to pay back loans of the second type which have the higher interest charges ($R_2 > 0$). From conditions (24) and (28) and equation (51), the optimal path of consumption is defined by

(56) $$\dot{c}/c = [(1 - w_2)i_2 - \Theta]/\sigma.$$

Again consumption will be increasing if w_2 is not very large.

$[t_4, t_5]$ In the fifth subperiod, the surplus is used to pay back loans of the first type; and at $t = t_5$, all loans have been repaid.

$[t_5, t_6]$ In the sixth subperiod, consumption is still less than income; and the individual begins to save for his old age. The surplus is invested in bonds of the first type. Because these bonds will be held for a relatively long period of time, the high transactions costs are compensated for by the higher interest income. From conditions (19) and (20) and equation (51), the optimal path of consumption is defined by

(57) $$\dot{c}/c = [(1 - v_1)r_1 - \theta]/\sigma.$$

Consumption will be decreasing during this subperiod.

$[t_6, t_7]$ In the seventh subperiod, the surplus is now invested in bonds of the second type because of the lower transactions costs.

$[t_7, t_8]$ In the eighth subperiod, it is not optimal to save at all because of the high transactions costs relative to the small amount of interest that could be earned on either type of bond during this short period and

(58) $$c = x + r_1B_1 + r_2B_2.$$

$[t_8, t_9]$ In the ninth subperiod, the individual sells his bonds of the second type in order to pay for consumption during his old age. From conditions (19) and (22) and equation (51), consumption is defined by

(59) $$\dot{c}/c = [(1 + v_2)r_2 - \theta]/\sigma.$$

$[t_9, T]$ In the last subperiod, the individual sells his stock of bonds of the first type to pay for consumption. At point $t = T$, the stock of bonds is exhausted.

FIGURE 1

Optimal Patterns of Consumption, Bonds Holdings, and Indebtedness for a Typical Life-Time Pattern of Income