DIFFERENTIAL AND INTEGRAL CALCULUS

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DIFFERENTIAL AND INTEGRAL

CALCULUS

BY

CLYDE E. LOVE, PH.D.

ASSISTANT PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF MICHIGAN

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PREFACE

This book presents a first course in the calculus substantially as the author has taught it at the University of Michigan for a number of years. The following points may be mentioned as more or less prominent features of the book.

In the treatment of each topic, the text is intended to contain a precise statement of the fundamental principle involved, and to insure the student's clear understanding of this principle, without distracting his attention by the discussion of a multitude of details. The accompanying exercises are intended to present the problem in hand in a great variety of forms and guises, and to train the student in adapting the general methods of the text to fit these various forms. The constant aim is to prevent the work from degenerating into mere mechanical routine, as it so often tends to do. Wherever possible, except in the purely formal parts of the course, the summarizing of the theory into rules or formulas which can be applied blindly has been avoided. For instance, in the chapter on geometric applications of the definite integral, stress is laid on the fact that the basic formulas are those of elementary geometry, and special formulas involving a coördinate system are omitted.

Where the passage from theory to practice would be too difficult for the average student, worked examples are inserted.

It seems clear that so-called applications in which the student is made to use a formula without explanation of its meaning and derivation, are of little value. In the present text the non-geometric applications are taken systematically from one subject, mechanics, and the theory is developed as fully as in the calculus proper.

A feature of the book is its insistence on the importance of checking the results of exercises, either directly or by solving in more than one way. The latter method is largely used in the integral calculus, on account of the variety of elementary transformations possible with definite integrals.

The answers to many of the exercises are given, but seldom where a knowledge of the answer would help in the solution, or where a simple means of checking the answer exists.

Topics of minor importance are presented in such a way that they may be omitted if it is desired to give a short course.

The chapter on curve tracing is introduced as early as possible, so that the results are available for use throughout the course.

Some instructors will wish to begin the use of integral tables immediately after the chapters on formal integration. This of course can easily be done.

In spite of obvious difficulties, a chapter embodying a first treatment of centroids and moments of inertia is introduced before multiple integrals have been defined. By this arrangement the student is brought to realize the fact that in most cases of practical importance mass-moments of the first and second orders can be found by simple integration, whereas from the usual treatment he gets exactly the opposite idea.

In the chapters on differential equations, emphasis is laid on those types most likely to be met by the student of engineering or the mathematical sciences. In the last chapter the average student will doubtless require considerable help from the instructor, but it is hoped that, if properly presented, the chapter may give the student some facility in writing and solving the simpler differential equations of mechanics and in interpreting the results.

To Professor Alexander Ziwet, who has read the entire manuscript, the author makes grateful acknowledgment, not only for valuable advice and criticism, but for his unfailing encouragement and support. Thanks are also due to Professor T. H. Hildebrandt, who has kindly assisted in reading the proofs, and has made a number of useful suggestions.

CLYDE E. LOVE.

Ann Arbor,
August, 1916.

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DIFFERENTIAL AND INTEGRAL CALCULUS

CALCULUS

CHAPTER I

FUNCTIONS. LIMITS. CONTINUITY

1. Functions. If a variable y depends upon a variable x so that to every value of x there corresponds a value of y, then y is said to be a *function of x*.

For example, (*a*) the area of a circle is determined by the radius and is therefore a function of the radius; (*b*) the attraction (or repulsion) between two magnetic poles is a function of the distance between them; (*c*) the volume of a given mass of gas at a constant temperature is a function of the pressure upon the gas.

A complete study of the properties of a function is possible in general only when the function is given by a definite mathematical expression. For this reason we shall be concerned almost entirely with functions defined in this way. Thus, in the examples above, we have

(*a*) $A = \pi r^2$,

(*b*) $F = \dfrac{k}{r^2}$,

(*c*) for a "perfect gas,"

$$v = \frac{k}{p}.$$

But the existence of a functional relation between two quantities does not imply the possibility of giving this relation a mathematical formulation. If by any means whatever a value of y is determined corresponding to every value of x under consideration, then y is a function of x. For example, the temperature of the air at any point of the earth's surface is a function of the time at which the thermometer is read, although no mathematical law connecting the two variables is known.

We often wish to express merely the fact that y is a function of x, without assigning the particular form of the function. This is done by writing

$$y = f(x)$$

(read y equals f of x). Other letters may of course be used in the functional symbol, as $F(x)$, $\phi(x)$, $\psi(x)$, etc.

The value of $f(x)$ when $x = a$ is denoted by the symbol $f(a)$. Thus, if

$$f(x) = x^2 - 3x - 1,$$

then

$$f(a) = a^2 - 3a - 1,$$
$$f(2) = -3,$$
$$f(x+h) = (x+h)^2 - 3(x+h) - 1.$$

Except where the contrary is explicitly stated, the variables and functions with which we shall have to deal are restricted to real values. This restriction is introduced for the sake of simplicity, and also because in the elementary applications only real quantities are of importance.

2. Geometric representation. The student is already familiar with the geometric representation of a function as the ordinate of a plane curve. Thus in (a) of § 1 the graph is a parabola; in (b) it is a certain cubic curve; in (c) it is an equilateral hyperbola.

Even though no mathematical expression for the function is known, it may still be represented graphically. For instance, to represent the temperature at a point of the earth's surface as a function of the time, let a large number of readings be taken, the corresponding points be plotted on coördinate paper with time as abscissa and temperature as ordinate, and a smooth curve be drawn through the points. This curve will represent approximately the variation of temperature throughout the time-interval in question.

3. Independent variable. We usually think of x as varying arbitrarily — *i.e.* we assign values to x at pleasure, and compute the corresponding values of y. The variable x is then called the *independent variable*, or *argument*. But it is clear that if y is a function of x, x is likewise a function of y, and in general either one may be chosen as the independent variable.

The values assigned to x must of course be compatible with the conditions of the problem in hand. In most cases x is restricted to a definite *range* or *interval;* for instance, if the function we are dealing with is $y = \sqrt{x}$, we restrict x to positive values.

4. Kinds of functions. We shall have to deal with both *algebraic* and *transcendental* functions. The algebraic functions are *rational integral functions*, or *polynomials; rational fractions*, or quotients of polynomials; and *irrational functions*, of which the simplest are those formed from rational functions by the extraction of roots. The elementary transcendental functions are *trigonometric* and *inverse trigonometric functions; exponential functions*, in which the variable occurs as an exponent; and *logarithms*.

Function

- *algebraic*
  - *rational*
    - *integral*
    - *fractional*
  - *irrational*
- *transcendental*
  - *elementary*
    - *trigonometric*
    - *inverse trig'c*
    - *exponential*
    - *logarithmic*
  - *higher*

5. One-valued and many-valued functions. A function $y = f(x)$ is said to be *one-valued*, if to every value of x corresponds a single value of y; *two-valued*, if to every value of x correspond two values of y, etc.

In the case of a many-valued function it is usual to group the values in such a way as to form a number of one-valued functions, called the *branches* of the original function. Thus the equation

$$y^2 = x$$

defines a two-valued function whose branches are

$$y = \sqrt{x},$$
$$y = -\sqrt{x}.$$

In dealing with many-valued functions, *we shall in general confine our attention to a particular branch.*

EXERCISES

1. Express the surface and volume of a sphere as functions of the radius; the radius as a function of the surface and of the volume.

2. Express the surface and volume of a cube as functions of the length of its edge.

3. Represent geometrically each of the functions of Ex. 2.

4. Find $f(t), f(3), f(-1), f(0), f(x+h)$, if

(*a*) $f(x) = 2x + 5$; (*b*) $f(x) = x^3 - 3x + 3$;
(*c*) $f(x) = \sin \pi x$; (*d*) $f(x) = 2^x$.

5. Exhibit graphically each of the functions of Ex. 4.

6. Plot the graph of each of the functions (*a*), (*b*), (*c*) of § 1.

7. Restate the examples (*a*), (*b*), (*c*) of § 1 both in words and by an equation, with the independent and the dependent variable interchanged.

8. Plot the graph of each of the following functions:

(*a*) $y = (1 - x^2)^2$; (*b*) $y = \frac{1}{1 + x^2}$;

(*c*) $y = \frac{x^2}{1 + x^2}$; (*d*) $y = \frac{x}{1 - x}$.

9. Show that (*a*) the graph of a one-valued function is met by any parallel to the y-axis in not more than one point; (*b*) the graph of a many-valued function consists of a number of branches (not necessarily disconnected), each of which has this same property. Give examples.

10. Show that the equation $y^2 = x^2 - a^2$ defines y as a two-valued function of x, and draw the graph.

11. The freezing point of water is 32° Fahrenheit, 0° Centigrade; the boiling point, 212° F., 100° C. Express temperature in degrees F. as a function of temperature in degrees C., both analytically and graphically.

12. A sum of money is placed at simple interest. Express the amount at any time as a function of the time, and draw the graph.

6. Rate of change; slope. A fundamental problem in studying the nature of a function is the determination of its *rate of change*.

Let $P : (x, y)$ be a point on the graph of the function

$$y = f(x).$$

Assign to x an arbitrary change, or *increment*, Δx (read delta x, *not* delta times x), usually taken positive, and denote by Δy the corresponding change in y, so that the point $P' : (x + \Delta x, y + \Delta y)$ is a second point on the curve. The ratio $\frac{\Delta y}{\Delta x}$ is the *average rate of change* of y with respect to x in the interval Δx; geometrically this ratio is the slope of the chord PP'. If now we let Δx approach 0, the ratio $\frac{\Delta y}{\Delta x}$ in general approaches a definite limiting value, which is defined as the *rate of change* of y with respect to x *at the point* P.

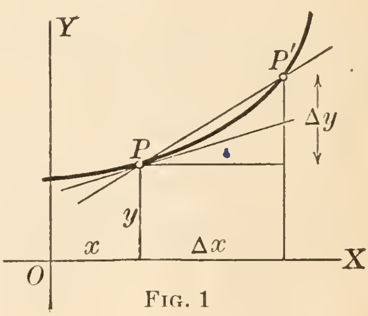


FIG. 1

The geometric interpretation is obvious: when Δx is taken smaller and smaller, P' approaches P along the curve, the chord PP' approaches the tangent at P as its limiting position, and $\frac{\Delta y}{\Delta x}$ approaches as its limit the slope of the tangent. Hence *the rate of change of a function is the slope of its graph.*

7. Limits. From what has just been said, it appears that the determination of the rate of change of a function, or the slope of a curve, requires the evaluation of a certain *limit*. It will therefore be well to introduce at this point a brief discussion of the subject of limits.

When the successive values of a variable x approach nearer and nearer a fixed number a, in such a way that the difference $a - x$ *becomes* and *remains* numerically less than any preassigned positive number however small, the constant a is called the *limit* of x, and x is said to *approach the limit* a — in symbols,

$$\lim x = a.$$

Examples are easily found in elementary work:

(*a*) If a regular polygon be inscribed in a circle, the difference between the area A_P of the polygon and the area A_C of the circle becomes arbitrarily small (less than any preassigned number) as the number of sides increases indefinitely. Hence

$$\lim A_P = A_C.$$

(*b*) We know from elementary algebra that the sum S_n of the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$$

is

$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

The difference between 2 and S_n is

$$2 - S_n = \frac{1}{2^{n-1}}.$$

This difference becomes arbitrarily small as the number of terms increases indefinitely; hence

$$\lim S_n = 2.$$

(*c*) If a steel spring of length l suspended vertically be stretched to a length $l + a$ and then released, the end of the spring will oscillate about its original position. The length x of the spring will be alternately greater and less than the original length l, but as the oscillations become smaller the difference between x and l will become and remain arbitrarily small. Thus

$$\lim x = l.$$

In this example, the variable actually reaches its limit, since the spring soon ceases to oscillate at all. In many cases, however, the variable never reaches its limit. This is true in (a) above, since no matter how many sides the polygon may have, its area is always less than that of the circle.

8. Theorems on limits. We shall have occasion to use the following theorems on limits, which we assume without formal proof.

THEOREM I*: *The limit of the sum of two variables is equal to the sum of their limits.*

THEOREM II: *The limit of the product of two variables is equal to the product of their limits.*

THEOREM III: *The limit of the quotient of two variables is equal to the quotient of their limits, provided the limit of the denominator is not* 0.

THEOREM IV: *If a variable steadily* $\begin{Bmatrix}\textit{increases}\\ \textit{decreases}\end{Bmatrix}$ *but never becomes* $\begin{Bmatrix}\textit{greater}\\ \textit{less}\end{Bmatrix}$ *than some fixed number A, the variable approaches a limit which is not* $\begin{Bmatrix}\textit{greater}\\ \textit{less}\end{Bmatrix}$ *than A.*

Theorems I and II may evidently be extended to the case of any number of variables.

* In theorems I, II, III it is of course implied that the limits of the two variables exist. We shall see later (§§ 139, 140) that the sum of two variables, for instance, may approach a limit when neither of the two variables taken by itself approaches a limit.

9. Limit of a function. We have frequently to observe the behavior of a function $f(x)$ as the argument x approaches a limit. If, as x approaches a, the difference between $f(x)$ and some fixed number l ultimately becomes and remains numerically less than any preassigned constant however small, the function $f(x)$ is said to *approach the limit* l, and we write

$$\lim_{x \to a} f(x) = l.$$

Unless otherwise specified it is supposed that the same limit is approached whether x comes up to a from the positive or the negative direction. If we wish to consider what happens when x approaches a from the positive side only, we write $\lim\limits_{x \to a^+} f(x)$; from the negative side only, $\lim\limits_{x \to a^-} f(x)$.

10. Infinitesimals. An *infinitesimal* is a *variable whose limit is* 0. Thus a constant, however small, is not an infinitesimal. An infinitesimal is not necessarily small at all stages of its variation; the only thing necessary is that *ultimately* it must become and remain numerically less than any assignable constant however small.

If one infinitesimal is a function of another, the independent variable is called the *principal infinitesimal.*

In the problem of § 6, both Δx and Δy are infinitesimals, with Δx as the principal infinitesimal.

11. Limit of the ratio of two infinitesimals. We return to the exceptional case of theorem III, § 8, in which the denominator is infinitesimal. Given any fraction $\frac{v}{u}$ in which u approaches 0, two cases are to be distinguished:

(*a*) v also approaches 0;

(*b*) v does not approach 0.

It is clear that in case (*b*) the fraction $\frac{v}{u}$ may be made to assume values greater than any assignable constant by taking u sufficiently small; hence the fraction can approach no limit. But consider case (a), in which both u and v are infinitesimal. Theorem III does not apply; the ratio of the limits is $\frac{0}{0}$, which is quite meaningless; nevertheless the *limit of the ratio* may exist, as we shall find in many cases in the next few chapters.

The determination of the limit of the ratio of two infinitesimals is a problem of the greatest importance; in fact, it is clear from the discussion of § 6 that this problem always arises in finding the rate of change of a function, or the slope of a curve.

EXERCISES

1. Determine (a) $\lim\limits_{x \to -1} (x^3 - 3x^2 - 5x - 5)$;

(b) $\lim\limits_{x \to 2} \dfrac{x^2 - x - 1}{x + 3}$.

Which of the theorems of § 8 are needed?

2. Determine (a) $\lim\limits_{x \to 0} \dfrac{x^2 - 3x + 2}{x + 1}$;

(b) $\lim\limits_{x \to 0} (\sin x + \cos x)$;

(c) $\lim\limits_{x \to 0} \dfrac{x + \frac{1}{x}}{x^2 - \frac{1}{2x}}$. *Ans.* ($c$) -2

Which of the theorems of § 8 are needed?

3. Determine $\lim\limits_{x \to 1} \dfrac{x^2 - 3x + 2}{x - 1}$. Which of the theorems of § 8 are used? *Ans.* -1.

4. Evaluate $\lim\limits_{x \to 1} \dfrac{1 - x}{\sqrt{1 - x^2}}$.

5. Evaluate $\lim\limits_{x \to 1} \dfrac{\sqrt{1 - x^3}}{\sqrt{1 - x^2}}$. *Ans.* $\frac{1}{2}\sqrt{6}$.

6. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{\tan x}$. *Ans.* 1.

7. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\tan x}$. *Ans.* 2.

8. Evaluate $\lim\limits_{x\to 0}\dfrac{\tan 2x}{\sin x}$.

9. Show that, if n is a positive integer,

$$\lim x^n = (\lim x)^n.$$

10. Show that, if $P(x)$ is a polynomial in x,

$$\lim_{x\to a} P(x) = P(a).$$

11. Show that, if $P_1(x)$ and $P_2(x)$ are polynomials,

$$\lim_{x\to a}\frac{P_1(x)}{P_2(x)} = \frac{P_1(a)}{P_2(a)},$$

provided $P_2(a) \neq 0$.

12. Under what circumstances may the limit in Ex. 11 exist when $P_2(a) = 0$? Give an example.

13. Does the limit in Ex. 11 always exist when $P_1(a) = P_2(a) = 0$? Give examples.

12. Continuity. An important idea in the study of functions is that of *continuity*.

A function $f(x)$ is said to be *continuous* at the point $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

This means, first, that the function is defined when $x = a$, and second, that the difference between $f(x)$ and $f(a)$ becomes and remains arbitrarily small (numerically less than any assignable constant) as x approaches a. The curve $y = f(x)$ passes through the point $x = a$ without a gap or break.

A function is said to be *continuous in an interval* of values of the argument if it is continuous at all points of the interval.

In the discussion of § 6, it is tacitly assumed that the function is continuous in an interval including the point P; this assumption is an essential part of the argument.

All the functions treated in this book are continuous, except perhaps for certain particular values of the variable, and such values are either excluded or subjected to special investigation.

13. Infinity. The most important type of discontinuity is that in which the function increases numerically without limit, or, as we say, *becomes infinite*, as x approaches a. In this case we write

$$\lim_{x \to a} f(x) = \infty.$$

But it must be noted that this equation is merely symbolic, for the reason that *the symbol* ∞ *does not represent a number.* The symbolic equation tells us, not that $f(x)$ approaches some vague, indefinite, very large limiting value, but that it increases beyond any limit whatever.

Graphically the occurrence of such a discontinuity means that the curve $y = f(x)$ approaches nearer and nearer the line $x = a$, usually without ever reaching it, at the same time receding indefinitely from the x-axis.

Examples: (a) As x approaches 0, the function

$$y = \frac{1}{x^2}$$

becomes infinite (Fig. 2):

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

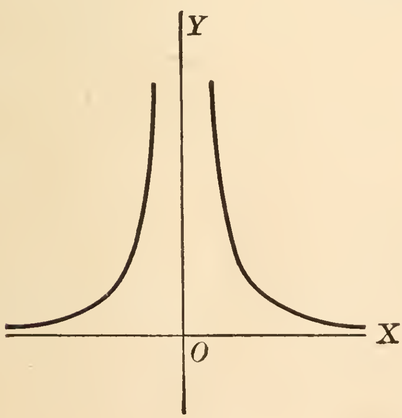


FIG. 2

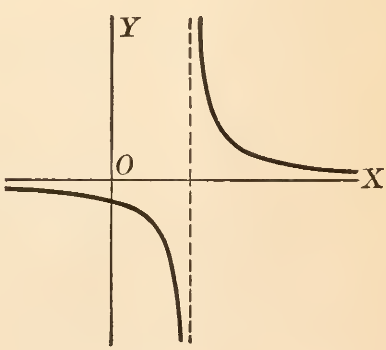


FIG. 3

(b) The function

$$y = \frac{1}{x - 2}$$

becomes positively or negatively infinite according as x approaches 2 from the right or the left (Fig. 3):

$$\lim_{x \to 2^+} \frac{1}{x-2} = +\infty, \qquad \lim_{x \to 2^-} \frac{1}{x-2} = -\infty.$$

14. Function with infinite argument. We have frequently to investigate the behavior of a function as the argument becomes infinite.

If when x increases indefinitely the difference between $f(x)$ and some fixed number l ultimately becomes and remains numerically less than any preassigned constant however small, we write

$$\lim_{x \to \infty} f(x) = l.$$

Graphically this means that the curve $y = f(x)$ approaches nearer and nearer the line $y = l$, usually without ever reaching it, at the same time receding indefinitely from the y-axis.

Examples: (*a*) As x increases indefinitely in either direction, the function $y = \frac{1}{x^2}$ approaches 0 (Fig. 2):

$$\lim_{x \to \infty} \frac{1}{x^2} = 0.$$

(*b*) $$\lim_{x \to \infty} \frac{x+1}{x-1} = \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = 1.$$

EXERCISES

1. Show that a polynomial is continuous for all values of x (see Ex. 10, p. 10).

2. For what values of x is a rational fraction discontinuous?

3. For what values of x is the function $\frac{x}{x^2 - 4}$ discontinuous?

4. Evaluate $\lim_{x \to 3} \frac{1}{x^2 - 9}$. Trace the curve $y = \frac{1}{x^2 - 9}$.

5. Evaluate (*a*) $\lim_{x \to 0^+} \frac{x+1}{x}$; (*b*) $\lim_{x \to 0^-} \frac{x+1}{x}$.

6. Evaluate $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{(x-2)^2}$.

7. Evaluate

(*a*) $\lim\limits_{x \to \infty} \dfrac{3x^2 + 5x}{x^2 - 3x - 1}$; (*b*) $\lim\limits_{x \to \infty} \dfrac{x - \dfrac{4}{x}}{3x^2 - 4}$;

(*c*) $\lim\limits_{x \to -\infty} 10^x$; (*d*) $\lim\limits_{x \to +\infty} 10^x$;

(*e*) $\lim\limits_{x \to \infty} \dfrac{x^2 + 3x + 1}{x - 5}$; (*f*) $\lim\limits_{x \to \infty} \tan x$.

Ans. (*a*) 3; (*c*) 0; (*f*) non-existent.

8. Does $\sin x$ approach any limit as x becomes infinite? Does $\dfrac{\sin x}{x}$? Does $\dfrac{\tan x}{x}$?

9. Show that as x approaches 0, the function $\sin \dfrac{\pi}{x}$ oscillates between -1 and 1, without approaching any limit.

10. Discuss the behavior of $\tan \dfrac{\pi}{x}$ near the origin.

11. Discuss the behavior of $10^{\frac{1}{x}}$ near the origin.

12. Evaluate $\lim\limits_{x \to 0} x \sin \dfrac{\pi}{x}$.

13. Is the function

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 4}$$

continuous at $x = 2$? Can $f(2)$ be so *defined* as to make $f(x)$ continuous?

14. If $f(x)$ is continuous, is its square continuous? Is its reciprocal?

15. Given two continuous functions, what can be said of the continuity of their sum? Their product? Their quotient?

16. Are the trigonometric functions continuous for all values of the argument? Discuss fully.

CHAPTER II

THE DERIVATIVE

15. The derivative. We return now to the problem (§ 6) of finding the rate of change of a function, or the slope of a curve.

Given a function

$$y = f(x),$$

continuous at the point $P : (x, y)$, let us assign to x an arbitrary increment Δx, and compute the corresponding increment Δy of y. We have

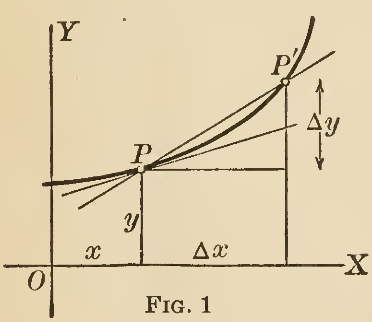


FIG. 1

$$y + \Delta y = f(x + \Delta x),$$

so that

$$\Delta y = f(x + \Delta x) - f(x).$$

Now form the ratio

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The limit of the ratio $\frac{\Delta y}{\Delta x}$ *as* Δx *approaches* 0 *is called the* ***derivative*** *of* y *with respect to* x.

The derivative is designated by the symbol $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Other commonly used symbols for the derivative are y', $D_x y$, $f'(x)$.

The operation of finding the derivative is called *differentiation*.

It follows from § 6 that *the derivative of a function is identical with its rate of change. Geometrically the derivative of a function is the slope of its graph.*

Only differentiable functions (*i.e.* those having a derivative) *are considered in this book.* In some cases the derivative may fail to exist for particular values of the argument, but such values are either excluded or subjected to special investigation.

To find the derivative of a given function, we have merely to *build up the "difference-quotient"* $\frac{\Delta y}{\Delta x}$ *and then pass to the limit as* Δx *approaches* 0. It will be remembered that this is essentially the method used in analytic geometry to find the slope of a curve. Since Δx and Δy approach 0 together, our problem is to find the limit of the ratio of two infinitesimals (cf. § 11). In general, this limit cannot be evaluated until some suitable transformation, algebraic or otherwise, has been applied to the quotient $\frac{\Delta y}{\Delta x}$.

The process of finding the derivative is illustrated by the following

Examples: (*a*) Find the slope of the parabola

$$y = 2x^2 - 6x + 4$$

at the point (x, y); at the point $(1, 0)$.

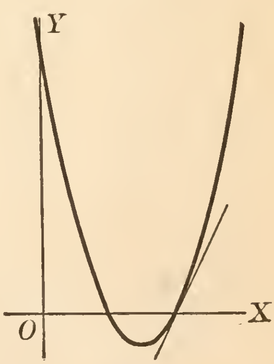


FIG. 4

If

$$y = f(x) = 2x^2 - 6x + 4,$$

then

$$y + \Delta y = f(x + \Delta x)$$
$$= 2(x + \Delta x)^2 - 6(x + \Delta x) + 4,$$
$$\Delta y = 4x\Delta x + 2\overline{\Delta x}^2 - 6\Delta x,$$
$$\frac{\Delta y}{\Delta x} = 4x + 2\Delta x - 6,$$
$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 4x - 6.$$

Hence the slope at any point (x, y) is $4x - 6$; in particular, the slope at the point $(1, 0)$ is -2.

(*b*) Given $s = \frac{1}{t}$, find $\frac{ds}{dt}$.

We have

$$s = \frac{1}{t},$$

$$s + \Delta s = \frac{1}{t + \Delta t},$$

$$\Delta s = \frac{1}{t + \Delta t} - \frac{1}{t}.$$

Reducing the fractions in the right member to a common denominator, we find

$$\Delta s = \frac{t - (t + \Delta t)}{(t + \Delta t)\,t} = \frac{-\Delta t}{(t + \Delta t)t}.$$

Whence

$$\frac{\Delta s}{\Delta t} = \frac{-1}{(t + \Delta t)t},$$

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = -\frac{1}{t^2}.$$

Geometrically this means that the slope of the hyperbola $s = \frac{1}{t}$ at the point (t, s) is $-\frac{1}{t^2}$.

(*c*) Find the rate of change of the function $y = \sqrt{x}$ at the point (x, y); at the point $(4, 2)$.

If

$$y = \sqrt{x},$$

then

$$y + \Delta y = \sqrt{x + \Delta x},$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

$$= (\sqrt{x + \Delta x} - \sqrt{x}) \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{(x + \Delta x) - x}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}},$$

$$\frac{\Delta y}{\Delta x}=\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}},$$

$$\frac{dy}{dx}=\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x}=\frac{1}{2\sqrt{x}}.$$

At the point (4, 2), the rate of change is

$$\left.\frac{dy}{dx}\right]_{x=4}=\frac{1}{4}.$$

EXERCISES

Find the slopes of the following curves at the points indicated.

1. $y = x - x^2$ at (x, y); at $x = 2$. Trace the curve.

2. $y = x^3 + 1$ at (x, y). Trace the curve.

3. $y = x^3 - x^2$ at the points where the curve crosses the x-axis. Trace the curve.

4. $y = \dfrac{1}{x+1}$ at $x = 2$. *Ans.* $-\frac{1}{9}$.

5. $y = \dfrac{1}{x^2}$ at $x = 2$. *Ans.* $-\frac{1}{4}$.

6. $y = x^4 - 3x^2 + 2$ at (x, y).

7. $y = x + \dfrac{1}{x}$ at $x = 2$. *Ans.* $\frac{3}{4}$.

8. If $y = \dfrac{1}{x^3}$, find $\dfrac{dy}{dx}$.

9. If $y = \sqrt{3 - x}$, find y'. *Ans.* $\dfrac{-1}{2\sqrt{3-x}}$.

10. If $f(x) = \dfrac{1}{(1-x)^2}$, find $f'(x)$. *Ans.* $\dfrac{2}{(1-x)^3}$.

11. If s is measured in feet and t in seconds, find the rate at which s is changing at the end of 2 seconds when

(a) $s = \dfrac{1}{1-t^2}$; (b) $s = \sqrt{t+1}$. *Ans.* (a) $\frac{4}{9}$ ft. per second.

12. At what points does the curve $y = \dfrac{x}{x+1}$ have the slope $\dfrac{1}{4}$?

Ans. $(1, \frac{1}{2})$, $(-3, \frac{3}{2})$.

13. Differentiate $y = \dfrac{1}{\sqrt{x}}$. *Ans.* $\dfrac{-1}{2x^{\frac{3}{2}}}$.

14. Find $\dfrac{dr}{d\theta}$ if $r = \theta^{\frac{3}{2}}$. *Ans.* $\frac{3}{2}\theta^{\frac{1}{2}}$.

15. Differentiate $y = \frac{x}{(x-1)^2}$.

16. If $f(x) = \sqrt{a^2 - x^2}$, find $f'(x)$. *Ans.* $\frac{-x}{\sqrt{a^2 - x^2}}$.

17. Find the angle between the curve $y = \frac{2x^2 - x}{x+1}$ and the line $y = x$ at each point of intersection.

16. Higher derivatives. The derivative of y with respect to x is itself a function of x. The derivative of the first derivative is called the *second derivative*, and is written $\frac{d^2y}{dx^2}$ (read d second y over dx square); the derivative of the second derivative is called the third derivative, written $\frac{d^3y}{dx^3}$; etc.

Other symbols for the higher derivatives are y'', y''', $\cdots$; D_x^2y, D_x^3y, $\cdots$; $f''(x)$, $f'''(x)$, $\cdots$.

Example: In example (a), § 15, we found

$$y' = 4x - 6.$$

Hence

$$y' + \Delta y' = 4(x + \Delta x) - 6,$$

$$\Delta y' = 4\,\Delta x,$$

$$\frac{\Delta y'}{\Delta x} = 4,$$

$$y'' = \lim_{\Delta x \to 0} \frac{\Delta y'}{\Delta x} = 4.$$

In this case all the higher derivatives are 0.

EXERCISES

1. Find y'' and y''' in Exs. 2, 3, 5, p. 17.

2. In example (b), § 15, find $\frac{d^2s}{dt^2}$.

3. In Ex. 10, p. 17, find $f''(x)$.

4. In Ex. 11, p. 17, find how fast $\frac{ds}{dt}$ is changing when $t = 2$ seconds.

CHAPTER III

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

17. Introduction. In this and a later chapter (V) we develop certain *standard formulas* by means of which any elementary function may be differentiated. The use of these formulas effects a great saving of time, and obviates the necessity of evaluating a special limit in every problem.

The formulas of §§ 19–20 are direct consequences of the definition of the derivative, and are valid for all functions (*i.e.* all functions that are *continuous*, *one-valued*, and *differentiable;* see §§ 12, 5, 15).

18. Derivative of a constant. We note first that the derivative of a constant is 0:

(1) $$\frac{dc}{dx} = 0.$$

For, if $y = c$, then no matter what the values of x and Δx may be, y will remain unchanged, and hence $\Delta y = 0$:

$$\frac{\Delta y}{\Delta x} = 0, \quad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.$$

The line $y = c$ is parallel to OX; its slope is everywhere 0.

19. Derivative of a sum; a product; a quotient. If u and v are functions of x, the following formulas are true by the definition of the derivative:

(2) $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx},$$

(3) $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$

(4) $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

These formulas may be stated in words as follows:

(2) *The derivative of the sum of two functions is equal to the sum of their derivatives.*

(3) *The derivative of the product of two functions is equal to the first function times the derivative of the second plus the second times the derivative of the first.*

(4) *The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, divided by the square of the denominator.*

Proof of (2): Let x assume an increment Δx, and denote by Δu and Δv the corresponding increments of u and v. Then

$$y = u + v,$$
$$y + \Delta y = u + \Delta u + v + \Delta v,$$
$$\Delta y = \Delta u + \Delta v,$$
$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x},$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{du}{dx} + \frac{dv}{dx}.$$

Proof of (3):

$$y = uv,$$
$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$
$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v,$$
$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x},$$
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof of (4):

$$y = \frac{u}{v},$$
$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v},$$
$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{uv + v\Delta u - uv - u\Delta v}{(v + \Delta v)v},$$

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{(v+\Delta v)v},$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

Formulas (2) and (3) can be extended to the case where n functions are involved. For three functions, (3) becomes

$$\frac{d}{dx}uvw = vw\frac{du}{dx} + wu\frac{dv}{dx} + uv\frac{dw}{dx}.$$

In the special case when $u = c$, a constant, (3) and (4) become

(3′) $$\frac{d}{dx}cv = c\frac{dv}{dx},$$

(4′) $$\frac{d}{dx}\frac{c}{v} = -\frac{c\dfrac{dv}{dx}}{v^2}.$$

20. Derivative of a function of a function. A function is sometimes expressed in terms of an auxiliary variable which in turn is a function of the independent variable; for example,

$$y = 5u^2 + 2u, \text{ where } u = x^3 + 3x + 7.$$

The variable u in the first equation may of course be replaced by its value in terms of x, and $\dfrac{dy}{dx}$ can then be determined directly; but it is desirable to have a formula by which $\dfrac{dy}{dx}$ can be found without eliminating u.

Let $$y = f(u), \text{ where } u = \phi(x).$$

Assign to x an increment Δx, and denote by Δu and Δy the corresponding changes in u and y. Then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$

and, passing to the limit, we find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x},$$

or

(5) $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

This very important formula is easily remembered from the fact that *in form* it is a mere identity.

21. Derivative of x^n, n a positive integer. If

$$y = x^n,$$

where n is a positive integer, then

(1) $$\frac{dy}{dx} = nx^{n-1}.$$

For,

$$y + \Delta y = (x + \Delta x)^n$$

$$= x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n,$$

$$\Delta y = nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\overline{\Delta x}^2 + \cdots + \overline{\Delta x}^n,$$

$$\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + \overline{\Delta x}^{n-1},$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}.$$

In particular, if $n = 1$, *i.e.* if $y = x$,

$$\frac{dx}{dx} = 1,$$

which is obvious geometrically.

By means of (4′), it can be shown that (1) is true when n is a negative integer.

Examples: (*a*) Find the derivative of

$$y = 3x^3 + 7x^2 + 2.$$

$$y' = \frac{d}{dx}3x^3 + \frac{d}{dx}7x^2 + \frac{d}{dx}2$$

$$= 9x^2 + 14x.$$

(*b*) Differentiate $$y = \frac{x^2}{x + 3}.$$

$$y' = \frac{(x+3)\dfrac{d}{dx}x^2 - x^2\dfrac{d}{dx}(x+3)}{(x+3)^2}$$

$$= \frac{(x+3)2\,x - x^2}{(x+3)^2} = \frac{x^2+6\,x}{(x+3)^2}.$$

EXERCISES

Differentiate the following functions.

1. (*a*) $y = 5\,x^3 - 2\,x$; (*b*) $s = t - 3\,t^2 + t^3$.

2. (*a*) $y = x^4 - 3\,x^3 - 2\,x^2 - 1$; (*b*) $y = x^2(5\,x^3 + 3)$.

3. (*a*) $y = 1 - 2\,x - 3\,x^5$; (*b*) $y = (x^2 - 1)(x^2 + 3\,x + 2)$.

4. $y = \dfrac{x}{x^2 - 1}$. *Ans.* $-\dfrac{1+x^2}{(x^2-1)^2}$.

5. $y = \dfrac{1 + 2\,x - x^2}{x^2 + 3}$.

6. $y = (x^3 - 1)(x^3 + 1)$.

7. $y = \dfrac{x^3}{1 - x}$.

8. $y = \dfrac{(1 + x)(1 - 2\,x)}{x}$.

9. If $y = \dfrac{2\,x^2 - 1}{x}$, find $\dfrac{d^2y}{dx^2}$. *Ans.* $-\dfrac{2}{x^3}$.

10. Find $\dfrac{d}{dx}(5\,x^3 + 7\,x^2 + 8\,x)(x^2 + 3\,x + 4)$.

11. Differentiate $y = (x + 1)(x + 2)(x + 3)$.

12. If $x = \dfrac{1}{t^2}$, find $\dfrac{d^3x}{dt^3}$.

13. If $F(t) = \left(\dfrac{t}{1-t}\right)^m$, find $F'(t)$. *Ans.* $\dfrac{mt^{m-1}}{(1-t)^{m+1}}$.

14. Find the rate of change of $s = t - \dfrac{2}{t} + \dfrac{3}{t^4}$.

15. In the proof of (1), § 21, why is n assumed to be a positive integer?

16. Show directly from the definition of the derivative that formula (1) of § 21 holds when $n = \pm \frac{1}{2}$.

17. Find $\dfrac{dy}{dx}$ if $y = 2\,u^2 - 4$, $u = 3\,x^2 + 1$.

18. Find the slope of the curve $y = x(x + 1)(x + 2)$ at the points where it crosses the x-axis. Trace the curve.

19. At what points is the tangent to the curve $y = (x - 3)^2(x - 2)$ parallel to OX? Trace the curve.

20. Prove formula (1) of § 21 when n is a negative integer.

21. Given a polynomial of the n-th degree, prove that all the derivatives after the n-th are identically 0.

22. Derivative of x^n, n fractional. By means of formula (**5**), the power formula (1) of § 21 can be extended at once to the case when n is a rational fraction.

If

$$y = x^{\frac{p}{q}},$$

where p and q are positive integers, then

$$y^q = x^p.$$

Differentiating each member of this equation with respect to x, we find, by (**5**),

$$qy^{q-1}\frac{dy}{dx} = px^{p-1},$$

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{y^q}\cdot y$$

$$= \frac{p}{q}\frac{x^{p-1}}{x^p}\cdot x^{\frac{p}{q}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

This shows that formula (1) of § 21 holds even when n is a positive rational fraction.

By using (**4′**), the formula can be shown to hold when n is a negative rational fraction.

In more advanced texts it is proved that the power formula holds when n is irrational, and hence is valid for all values of n.

23. The general power formula. Suppose

$$y = u^n, \text{ where } u = \phi(x).$$

Then

$$\frac{dy}{du} = nu^{n-1},$$

and we have by (**5**)

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \qquad (6)$$

An important special case of this formula is the case $n = \frac{1}{2}$:

$$\text{(6')} \qquad \frac{d}{dx}\sqrt{u} = \frac{\frac{du}{dx}}{2\sqrt{u}}.$$

Example: Find the derivative of

$$y = (3x^2 + 1)^4.$$

This function is of the form u^n, with $u = 3x^2 + 1$, $n = 4$. Hence (6) gives

$$\frac{dy}{dx} = 4(3x^2 + 1)^3 \cdot 6x = 24x(3x^2 + 1)^3.$$

EXERCISES

Differentiate the following functions.

1. $y = x^{\frac{1}{3}} + 3x^{\frac{5}{2}}$.

2. $y = \dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{2}} + 1}$.

3. $y = (2x + 1)^5$. *Ans.* $y' = 10(2x + 1)^4$.

4. $y = (x^3 + 5x^2 + 7)^2$. *Ans.* $y' = (6x^2 + 20x)(x^3 + 5x^2 + 7)$.

5. $y = \sqrt{1 - 3x^2}$.

6. $y = \dfrac{1}{\sqrt{x^2 - x}}$.

7. $y = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + 3x$.

8. $y = (x^2 - 5x)^3(8x - 7)^2$.

9. $y = \dfrac{(3x^3 - 7x + 1)^3}{(1 - x)^2}$.

10. $y = (x^2 + x^{\frac{1}{2}})^{\frac{1}{3}}$.

11. $y = (3x + 2)^3 + (5x + 7)^2$.

12. $y = \sqrt{\dfrac{1 + 2x}{1 - 2x}}$.

13. $y = \sqrt[3]{a^3 - x^3}$.

14. $y = (1 - x^2)(1 + x^2)^2$.

15. $y = \dfrac{x}{\sqrt{(a^2 - x^2)^3}}$. *Ans.* $y' = \dfrac{a^2 + 2x^2}{(a^2 - x^2)^{\frac{5}{2}}}$.

16. Find the slope of the hyperbola $x^2 - y^2 = 12$ at $(4, -2)$. *Ans.* -2.

17. If $\phi(v) = \dfrac{1}{\sqrt{1 - v}}$, find $\phi'(v)$, $\phi''(v)$, $\phi'''(v)$. *Ans.* $\phi'''(v) = \dfrac{15}{8(1 - v)^{\frac{7}{2}}}$.

18. Find $\dfrac{d}{ds}\sqrt{1 - s} \cdot \sqrt{1 + s^2}$.

19. If $y = 2\sqrt{ax}$, find y''.

20. Find $\dfrac{d}{dx}(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$.

21. Differentiate $y = \left(\frac{x}{1+\sqrt{1-x^2}}\right)^n$. *Ans.* $y' = \frac{ny}{x\sqrt{1-x^2}}$.

22. Find the slope of the curve $y = (x^2 - 1)^2$ at each of the points where $y'' = 0$. Trace the curve.

23. Draw the graph of the function $y = x^n$ for $n = \frac{1}{2}, 1, \frac{3}{2}, 2, 3$.

24. Implicit functions. Up to this point we have been concerned with functions defined explicitly by an equation of the form

$$y = f(x).$$

It may happen, however, that x and y are connected by an equation not solved for y; for example,

$$x^2 + y^2 = a^2.$$

In such a case y is called an *implicit function* of x, and the relation is expressed by writing

$$F(x, y) = 0.$$

The definition becomes explicit if we solve for y; in the above example,

$$y = \pm\sqrt{a^2 - x^2}.$$

25. Differentiation of implicit functions. To find the derivative of an implicit function, we proceed as follows:

Differentiate each term of the equation

$$F(x, y) = 0,$$

bearing in mind that, owing to the equation, y is a function of x.

Example: Find the slope of the ellipse

$$x^2 - xy + y^2 + x = 1$$

at the point (x, y).

We have $$\frac{d}{dx}(x^2 - xy + y^2 + x) = 0,$$

or by (6) and (5)

$$2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} + 1 = 0,$$

$$\frac{dy}{dx} = \frac{2x - y + 1}{x - 2y}.$$

26. Inverse functions. The equation

(1) $$y = f(x)$$

defines x implicitly as a function of y; when solved for x, it takes the form

(2) $$x = \phi(y).$$

The function $\phi(y)$ is called the *inverse* of the *direct* function $f(x)$.

For example,

(*a*) if $y = x^2$, then $x = \pm\sqrt{y}$;

(*b*) if $y = \pm\sqrt{a^2 - x^2}$, then $x = \pm\sqrt{a^2 - y^2}$;

(*c*) if $y = a^x$, then $x = \log_a y$.

In each case the second function is the inverse of the first.

To construct the graph of the inverse function from that of the direct function, we have only to *interchange* x *and* y, which amounts to a *reflection in the line* $y = x$. This is shown in the figure for example (*a*) above.

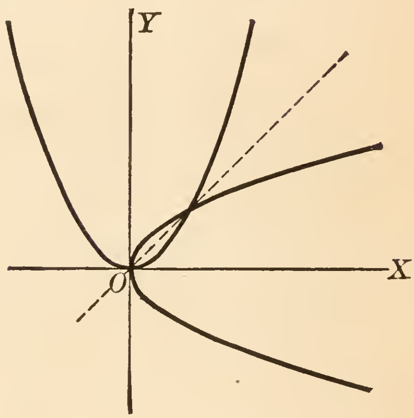


FIG. 5

If $f(x)$ and $\phi(y)$ are inverse functions, then

(3) $$f'(x) = \frac{1}{\phi'(y)} \qquad (\phi'(y) \neq 0).$$

For, since

$$\frac{\Delta y}{\Delta x} = \frac{1}{\dfrac{\Delta x}{\Delta y}},$$

we have, passing to the limit,

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}},$$

which by (1) and (2) is the desired formula.

EXERCISES

1. Express y explicitly as a function of x, when

(*a*) $x^3 - y^3 = 1$; (*b*) $2xy + y^2 = 4$;

(*c*) $\sin(x+y) = 1$; (*d*) $xa^y = 1$.

Find the slopes of the following curves at the points indicated.

2. $x^2 + y^2 = 25$ at (x, y); at $(3, 4)$. *Ans.* $-\frac{x}{y}$; $-\frac{3}{4}$.

3. $x^2 + xy + y^2 = 3$ at $(1, 1)$. *Ans.* -1.

4. $2x^2 + 2y^3 - 9xy = 0$ at $(1, 2)$.

5. $xy^3 = 3$ at $(3, 1)$. Do this in two ways.

Find $\frac{dy}{dx}$ in the following cases.

6. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Ans.* $-\frac{b^2x}{a^2y}$.

7. $(x - y)(x + y)^2 = a^3$. **8.** $x^3 + y^3 - 3axy = 0$.

9. $x + \sqrt{(2x - 3y)^3} = 0$. **10.** $x^2y^2 + 2x^2y - xy^2 + 2 = 0$.

11. $x = (1 - 3y)^2$. Solve in two ways.

12. If $y^2 = 4ax$, find y''. Cf. Ex. 19, p. 25. *Ans.* $-\frac{4a^2}{y^3}$.

13. Prove that a tangent to a circle is perpendicular to the radius drawn to the point of contact.

14. If $x^2 + y^2 = a^2$, find y''.

15. Show that

$$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} \qquad \left(\frac{dy}{dx} \neq 0\right).$$

16. Find the inverse of each of the following functions:

(*a*) $y = 3x - 4$; (*b*) $y = \frac{x-1}{x-2}$;

(*c*) $y = (x^2 - 1)^2$; (*d*) $y = \log_{10} x$.

17. In Ex. 16, are the direct functions one-valued? Are the inverse functions?

18. In Ex. 16 (*a*), (*c*), verify formula (3), § 26.

19. In Ex. 16 (*a*), (*c*), verify that the graph of the inverse function may be found from that of the direct function by reflection in the line $y = x$.

20. In Ex. 16 (*b*), (*c*), discuss the continuity of the direct and inverse functions.

CHAPTER IV

GEOMETRIC APPLICATIONS

27. Tangents and normals to curves. It is known from analytic geometry that the equation of a line through the point (x_0, y_0) with slope m is

(1) $$y - y_0 = m(x - x_0).$$

Let $P : (x_0, y_0)$ be a point on the curve

$$F(x, y) = 0,$$

and denote by y_0' *the value of the derivative at the point* (x_0, y_0). The equation of the tangent to the curve at P can then be written, by (1), in the form

$$y - y_0 = y_0'(x - x_0).$$

The equation of the normal — *i.e.* the line perpendicular to the tangent at the point of contact — can be found at once from that of the tangent, by recalling that if two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other.

Examples: (*a*) Find the tangent and normal to the ellipse

$$4x^2 + 9y^2 = 40$$

at the point $(1, -2)$.

We have

$$8x + 18yy' = 0,$$

hence

$$m = y_0' = -\frac{4x}{9y}\Big]_{(1,-2)} = \frac{2}{9}.$$

Therefore the equation of the tangent is

$$y + 2 = \tfrac{2}{9}(x - 1),$$

that of the normal is

$$y + 2 = -\tfrac{9}{2}(x - 1).$$

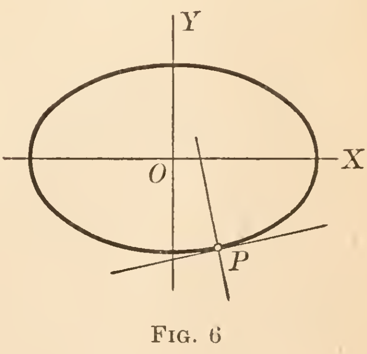


FIG. 6

(*b*) Find the equation of a tangent to the curve $y = x^3$ parallel to the line $y = 3x + 1$.

The slope of the required tangent is 3. But the slope at any point (x, y) of the curve $y = x^3$ is

$$y' = 3x^2.$$

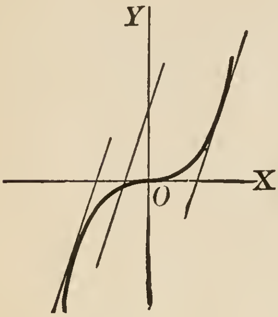


Fig. 7

Hence the coördinates of the point of contact are found by solving the simultaneous equations

$$3x^2 = 3, \quad y = x^3.$$

This gives the points (1, 1), (−1, −1), and the required tangents are

$$y - 1 = 3(x - 1), \quad y + 1 = 3(x + 1).$$

28. Length of tangent, subtangent, normal, and subnormal. Let $P : (x, y)$ be a point on the curve

$$F(x, y) = 0.$$

The segment TP of the tangent intercepted between the point of tangency and the x-axis is called the *length of the tangent;* its projection TQ on OX is called the *length of the subtangent.* The segment NP of the normal intercepted between P and the x-axis is called the *length of the normal;* its projection QN on the x-axis is called the *length of the subnormal.*

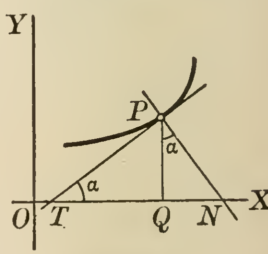


Fig. 8

It is customary to consider all these lengths as essentially positive. They are evidently determined by the coördinates (x, y) of the point P and the slope at P.

EXERCISES

Find the tangent and normal to each of the following curves at the points indicated.

1. (*a*) $y = 1 - x - x^2$ at $(1, -1)$; (*b*) $xy = 2a^2$ at $(a, 2a)$;
   (*c*) $x^2 + y^2 = 25$ at $(-3, 4)$; (*d*) $y = x + \frac{1}{x}$ at (1, 2).

   Ans. (*a*) $y + 1 = -3(x - 1)$, $y + 1 = \frac{1}{3}(x - 1)$.

2. $x^2 - 2xy + 2y^2 - x = 0$ at the points where $x = 1$.

Ans. At $(1, 0)$: $2y = x - 1$, $y + 2x = 2$; at $(1, 1)$: $2y = x + 1$, $y + 2x = 3$.

3. $y = \dfrac{8a^3}{x^2 + 4a^2}$ at $x = 2a$. *Ans.* $x + 2y = 4a$, $y = 2x - 3a$.

4. Find the equation of the tangent to

(a) $y^2 = 4ax$ at (x_0, y_0); (b) $\dfrac{x^2}{a^2} \pm \dfrac{y^2}{b^2} = 1$ at (x_0, y_0).

Ans. (a) $y_0 y = 2a(x + x_0)$; (b) $\dfrac{x_0 x}{a^2} \pm \dfrac{y_0 y}{b^2} = 1$.

5. Find the subtangent, subnormal, tangent, and normal lengths in each of the cases of Ex. 1. Draw a figure in each case.

6. Find the angle between the parabolas $y^2 = x$, $y = x^2$ at each of their points of intersection.

7. Find the tangent and normal to the curve $y^2 = 2x^2 - x^3$ at the points $x = 1$. *Ans.* At $(1, 1)$: $2y = x + 1$, $y + 2x = 3$; at $(1, -1)$: $x + 2y + 1 = 0$, $y = 2x - 3$.

8. Show that the subtangent to the parabola $y^2 = 4ax$ is bisected at the vertex, and that the subnormal is constant. Hence give a geometric construction for drawing the tangent and normal; also show how to find the focus of a parabola if the axis is given.

9. Find a tangent to the parabola $y^2 = 4ax$ making an angle of 45° with the x-axis. *Ans.* $y = x + a$.

10. Find the tangents to the hyperbola $4x^2 - 9y^2 + 36 = 0$ perpendicular to the line $2y + 5x = 10$. *Ans.* $2x - 5y = \pm 8$.

11. Find a tangent to the curve $y = 1 - x^4$ parallel to the line $x - 2y = 5$.

12. Find a normal to the parabola $y = x^2$ perpendicular to the line $3x - 2y = 1$.

13. Show that the portion of the tangent to the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intercepted between the axes is constant.

14. Find the tangents to the circle $x^2 + y^2 = 5$ which are parallel to the line $x + 3y = 0$. Draw the figure.

Ans. $y \pm \frac{3}{2}\sqrt{2} = -\frac{1}{3}(x \pm \frac{1}{2}\sqrt{2})$.

15. Find the tangents to the curve

$$y = \tfrac{1}{4}x^4 - x^3 + 5x$$

which make an angle of 45° with the x-axis. Plot the curve.

16. Find the angle between the line $y = -2x$ and the curve $y = x^2(1 - x)$ at each point of intersection.

17. Find the equation of a tangent to the curve $y^3 = 1 - x$ parallel to the y-axis. Trace the curve.

18. Show that the area of the triangle formed by the coördinate axes and the tangent to the hyperbola $2\,xy = a^2$ is constant.

19. Show that the length of the normal is constant (equal to a) in the circle

$$(x-c)^2 + y^2 = a^2,$$

where c is arbitrary, and explain geometrically.

20. Show that the sum of the intercepts on the axes of the tangent to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is constant.

21. Show that, in the curve $y^n = ax$, the subtangent is n times the abscissa of the point of contact. Hence show how to draw the tangent at any point of the curve $y = ax^n$.

22. Find the length of the tangent, subtangent, normal, and subnormal to the curve $y = f(x)$ at the point (x, y). *Ans.* Tangent, $\frac{y}{y'}\sqrt{1+y'^2}$; subtangent, $\frac{y}{y'}$; normal, $y\sqrt{1+y'^2}$; subnormal, yy'.

29. Increasing and decreasing functions. In studying the properties of a function

$$y = f(x),$$

it is usually of great assistance to represent the function graphically. In tracing a curve, it is well to begin by locating several points, *e.g.* the intersections with the axes, and finding the slope at those points; it is also useful to note the behavior of y for large positive and negative values of x.

In addition to giving the slope at any point, the differential calculus is of assistance in a variety of other ways, as will be shown in the next few articles.

We shall assume as usual that the function in question is one-valued, continuous, and differentiable.

We note first that, as x increases, *the curve rises if the slope is positive*, as on the arc AB (Fig. 9); *it falls if the slope is negative*, as along BD:

If $y' > 0$, y increases ;

If $y' < 0$, y decreases.

Of course this also appears at once from the fact that y' is the rate of change of y.

30. Maxima and minima. At a point such as B (Fig. 9), where the function is algebraically greater than at any neighboring point, the function is said to have a *maximum value*, and the point is called a *maximum point*. Similarly, at a point such as D the function has a *minimum value*. It is evident that *at such a point the tangent is parallel to OX; i.e.*

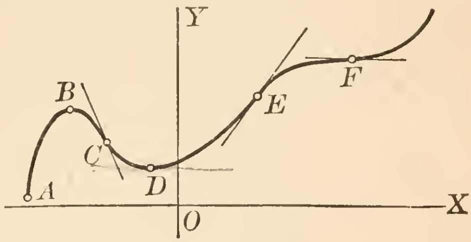


FIG. 9

$$y' = 0.$$

But the vanishing of the derivative does not mean that the function is necessarily a maximum or a minimum; the tangent is parallel to OX at F, yet the function is neither a maximum nor a minimum there. It appears from the figure that the test is as follows:

At a point where $y' = 0$, if y' changes from positive to negative (as x increases), y is a maximum; if y' changes from negative to positive, y is a minimum; if y' does not change sign, y is neither a maximum nor a minimum.

Since the function is continuous, the maxima and minima must alternate: between two maxima there must be a minimum, and vice versa.

The points at which $y' = 0$ are called *critical points*, and the corresponding values of x are the *critical values* of x.

31. Concavity. The second derivative is the rate of change of the first derivative. It follows that when y'' is positive, y' is increasing: as x increases the tangent turns in counterclockwise sense and the curve is *concave upward*.

When y'' is negative, y' decreases: the curve is *concave downward.*

At a maximum point the curve is concave downward, and hence y'', if it is not 0, must be negative. At a minimum y'', if not 0, must be positive. If the second derivative is easily obtained and if it does not happen to be 0 at the critical point in question, it is usually more convenient to determine whether we have a maximum or a minimum by finding the sign of y''; but the test of § 30 has the advantage of being perfectly general. However, in practice other considerations usually enable us to distinguish between maxima and minima *without the application of either of these tests.*

Example: Find the maximum and minimum values of the function

$$y = x^3 - 3x,$$

and trace the curve.

This curve crosses the x-axis at $x = 0, \pm\sqrt{3}$. Since

$$y' = 3x^2 - 3,$$

the slope at (0, 0) is -3, at $(\pm\sqrt{3}, 0)$ it is 6.

Setting

$$y' = 0,$$

we find the critical points $(-1, 2)$, $(1, -2)$. When x is large and negative, y is large and negative; when x is large and positive, y is large and positive. It is therefore clear that the curve must rise to a maximum at $(-1, 2)$, fall to a minimum at $(1, -2)$, and then rise indefinitely. These conclusions may be verified as follows. The second derivative,

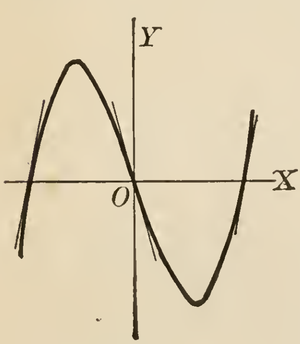


FIG. 10

$$y'' = 6x,$$

is negative at $x = -1$, so that the point $(-1, 2)$ is a maximum; y'' is positive at $x=1$, hence $(1, -2)$ is a minimum. The curve is shown in the figure.

EXERCISES

Examine the following functions for maxima and minima, and trace the curves.

1. $y = x(x+5)$.
2. $y = x^3 - 2x^2 + x$.
3. $y = x^2(x^2 - 8)$.
4. $y = (x^2 - 4)^2$.
5. $y = x^4 + 1$.
6. $y = x^3$.
7. $y = \dfrac{a^3}{a^2 + x^2}$.
8. $y = \dfrac{a^2 x}{a^2 + x^2}$.
9. $s = \dfrac{t^2}{1 + t^2}$.
10. $x = (t+1)^3$.

32. Points of inflection. A point at which the curve changes from concave upward to concave downward, or vice versa, is called a *point of inflection.* At a point of inflection the tangent reverses the sense in which it turns, so that y'' changes sign. Hence at such a point y'', *if it is continuous, must vanish.* In Fig. 9 the points C, E, F are points of inflection.

Since y'' — *i.e.* the rate of change of the slope — vanishes at a point of inflection, the tangent is sometimes said to be *stationary for an instant* at such a point, and in the neighborhood of the point it turns very slowly. Hence the inflectional tangent agrees more closely with the curve near its point of contact than does an ordinary tangent; it is therefore especially useful in tracing the curve to *draw the tangent at each point of inflection.*

33. Summary of tests for maxima and minima, etc. The results of §§ 29–32 may be summarized as follows:

Let the function $y = f(x)$ and its first and second derivatives be one-valued and continuous.

(*a*) *In an interval where* $y' > 0$, *the curve rises; where* $y' < 0$, *the curve falls.*

(*b*) *A point where* $y' = 0$ *is a maximum or a minimum point, unless at the same point* $y'' = 0$, *in which case see* (*e*) *below. If $y' > 0$ at the left* of this point and $y' < 0$ at the right, y is a maximum; if $y' < 0$ at the left and $y' > 0$ at the right, y is a minimum. Or, if $y'' < 0$ at the point, y is a maximum; if $y'' > 0$, y is a minimum.*

(c) In an interval where $y'' > 0$, the curve is concave upward; where $y'' < 0$, the curve is concave downward.

(d) A point at which $y'' = 0$ is a point of inflection, provided y'' changes sign as the curve passes through the point.

(e) A point at which both $y' = 0$ and $y'' = 0$ is a maximum or a minimum if y' changes sign as the curve passes through the point; it is a point of inflection with a horizontal tangent if y' does not change sign.

Example: Trace the curve

$$y = x(x-1)^3.$$

This curve crosses the x-axis at (0, 0), (1, 0). For large positive or negative values of x, y is large and positive. The derivatives are

$$y' = (x-1)^3 + 3x(x-1)^2 = (x-1)^2(4x-1),$$
$$y'' = 2(x-1)(4x-1) + 4(x-1)^2 = 6(x-1)(2x-1).$$

The slope at (0, 0) is -1; at (1, 0) it is 0.

The critical values are $x = 1, \frac{1}{4}$. When $x = \frac{1}{4}$, y'' is positive; hence $(\frac{1}{4}, -\frac{27}{256})$ is a minimum point. When $x = 1$, $y'' = 0$ and the test by the second derivative fails. Since y' does not change sign as x passes through 1, the function has neither a maximum nor a minimum at that point.

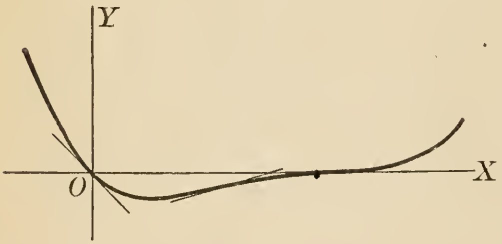


FIG. 11

The second derivative vanishes at $(\frac{1}{2}, -\frac{1}{16})$, (1, 0); these are points of inflection. The slope at $(\frac{1}{2}, -\frac{1}{16})$ is $\frac{1}{4}$. The curve is shown in the figure.

* That is, *immediately* at the left.

EXERCISES

Trace the following curves. Where possible, find the points of intersection with the axes, determine the behavior of y for large values of x, find the maxima and minima and points of inflection, and draw the tangent at each point of inflection.

1. $y = x^3 - 6x^2 + 9x + 3.$

2. $y = 4 + 3x - x^2.$

3. $y = x^3 - 3x^2 + 6x + 10.$

4. $y = (x - 3)^2(x - 2).$

5. $y = (1 - x^2)^3.$

6. $y = (4 - x^2)^2.$

7. $y = (x - 1)^3(x + 2)^2.$

8. $y = x^3 - 3x^2 - 9x + 5.$

9. $y = x^4.$

10. $y = x^5.$

11. $y = x(x - 1)(x - 2).$

12. $y = \dfrac{8a^3}{x^2 + 4a^2}.$

13. $y = \dfrac{1}{1 + x^4}.$

14. $y = \dfrac{1}{(1 + x^2)^2}.$

15. Show that the curve $y = \dfrac{a^2 x}{x^2 + a^2}$ has three points of inflection lying on a straight line. Trace the curve.

16. Show that, for the curve $y = x^n$, where $n = 2, 3, 4, \cdots$, the origin is a minimum or a point of inflection according as n is even or odd.

34. Applications of maxima and minima. The theory of maxima and minima finds application in a great variety of problems. In the applications it is rarely necessary to use either of the tests of § 33 to distinguish between maxima and minima; the critical value that gives the desired result can usually be selected by inspection of the conditions of the problem.

It frequently happens that the function to be tested for maxima or minima can be most simply expressed in terms of two variables. When this is done, a relation between the two variables must be found from the conditions of the problem. By means of this relation one of the variables can be eliminated, after which the maxima and minima can be found in the usual way. However, it is often more convenient not to eliminate, but to proceed as in example (b) below.

Examples: (*a*) A box is to be made of a piece of cardboard 4 in. square by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be made in this way.

Let x be the length of the side of each of the squares cut out. Then the volume of the box is

(1) $$V = x(4 - 2x)^2.$$

The derivative is

$$\frac{dV}{dx} = (4 - 2x)^2 - 4x(4 - 2x)$$
$$= (4 - 2x)(4 - 6x).$$

Setting

$$\frac{dV}{dx} = 0,$$

we find

$$x = 2 \text{ or } \tfrac{2}{3}.$$

Since V vanishes when $x = 0$ and again when $x = 2$, it must reach a maximum at some intermediate point; it therefore follows without the application of further tests that the critical value $x = \frac{2}{3}$ gives the required maximum volume:

$$V_{max.} = \tfrac{2}{3}(4 - \tfrac{4}{3})^2 = \tfrac{128}{27} \text{ cu. in.}$$

The graph of equation (1) is shown in the figure, the V-scale being one fourth as large as the x-scale. Since by the conditions of the problem x is restricted to values between 0 and 2, the dotted portions of the curve have no meaning in the present case.

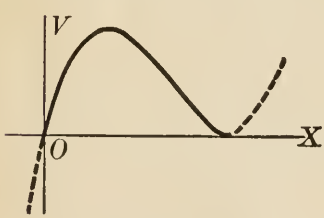


FIG. 12

(*b*) Find the dimensions of the largest rectangle that can be inscribed in a given circle.

Take the coördinate axes parallel to the sides of the rectangle. The area of the rectangle is

(2) $$A = 4xy.$$

This can be expressed in terms of x (or y) by means of the relation

(3) $$x^2 + y^2 = a^2,$$

which gives

$$y = \sqrt{a^2 - x^2},$$
$$A = 4\,x\sqrt{a^2 - x^2}.$$

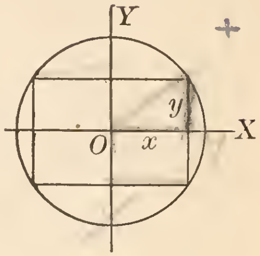


FIG. 13

From this point the method is the same as in (a).

The problem can be solved without eliminating x or y, as follows: Differentiating equation (2) with respect to x and setting the derivative equal to 0, we have, since y is a function of x,

$$\frac{dA}{dx} = 4\left(x\frac{dy}{dx} + y\right) = 0,$$

or

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Differentiating equation (3), we get

$$2\,x + 2\,y\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Equating values of $\frac{dy}{dx}$, we find

$$-\frac{y}{x} = -\frac{x}{y},$$

whence

$$y = x:$$

the maximum rectangle is a square.

EXERCISES

1. What is the largest rectangular area that can be inclosed by 800 yd. of fencing?

2. For a rectangle of given area, what shape has the minimum perimeter?

+3. Find the most economical proportions for a cylindrical tin cup of given volume. *Ans.* Radius = height.

4. A rectangular field is to be fenced off along the bank of a straight river. If no fence is needed along the river, what is the shape of the rectangle requiring the least amount of fencing?

5. The equal sides of an isosceles triangle are 10 in. long. Find the length of the base if the area is a maximum.

6. Find the rectangle of maximum perimeter inscribed in a given circle.

7. Find the most economical proportions for a box with an open top and a square base.

+8. Find the most economical proportions for a covered box whose base is a rectangle with one side twice the other.

Ans. Altitude = $\frac{4}{3}\times$ shorter side.

+9. Find the dimensions of the largest right circular cylinder that can be inscribed in a given sphere. *Ans.* Diameter = $\sqrt{2}\times$ height.

10. In Ex. 9, find the form of the cylinder if its convex surface is a maximum.

+11. Find the dimensions of the largest rectangle that can be inscribed in a given right triangle. *Ans.* $x = \frac{1}{2}a$.

+12. Find the most economical proportions for a conical tent of given capacity. *Ans.* $h = \sqrt{2}\,r$.

13. A man in a rowboat 6 miles from shore desires to reach a point on the shore at a distance of 10 miles from his present position. If he can walk 4 miles per hour and row 3 miles per hour, where should he land in order to reach his destination in the shortest possible time?

Ans. 1.2 miles from his destination.

14. A rectangular field of given area is to be inclosed, and divided into two lots by a parallel to one of the sides. What must be the shape of the field if the amount of fencing is to be a minimum?

15. A Norman window consists of a rectangle surmounted by a semicircle. What shape gives the most light for a given perimeter?

Ans. Breadth = height.

+16. Find the most economical proportions for a quart can.

Ans. Diameter = length.

17. The strength of a rectangular beam is proportional to the breadth and the square of the depth. Find the shape of the strongest beam that can be cut from a log of given diameter.

Ans. Depth $= \sqrt{2} \times$ breadth.

18. Find the volume of the largest box that can be made by cutting equal squares out of the corners of a piece of cardboard 6×16 in. and turning up the sides. *Ans.* $\frac{1600}{27}$ cu. in.

19. A gutter is to be made of a strip of tin 12 in. wide, the cross section having the form shown in the figure. What depth gives a maximum carrying capacity?

Fig. 14

20. Find the most economical proportions for a cylindrical cup of given capacity, if the bottom is to be three times as thick as the sides.

21. Find the most economical proportions for an A-tent of given volume, whose sides slope at 45° to the horizontal.

22. Find the dimensions of the largest right circular cylinder that can be inscribed in a given right circular cone. *Ans.* Altitude $= \frac{1}{3} h$.

23. Solve Ex. 22 if the convex surface of the cylinder is to be a maximum.

24. Find the right circular cone of maximum volume inscribed in a given sphere.

25. Find the cone of minimum volume circumscribed about a given sphere.

26. The base of a box is a rectangle with one side twice the other. The top and front are to be made of oak, the remainder of pine. If oak is twice as valuable as pine, find the most economical proportions.

27. A wall tent 12×16 ft. is to contain a given volume. Find the most economical proportions. *Ans.* Height above eaves $= 2\sqrt{3}$ ft.

28. An oil can is made in the shape of a cylinder surmounted by a cone. If the radius of the cone is three fourths of its height, find the most economical proportions.

Ans. Altitude of cylinder = altitude of cone.

29. A cupboard 5 ft. high and having shelves 1 ft. apart is to be made from a given amount of material. If a front, but no back, is required, what shape gives the greatest amount of shelf room?

Ans. Width = twice depth.

30. A silo is made in the form of a cylinder, with a hemispherical roof; there is a floor of the same thickness as the wall and roof. Find the most economical shape. *Ans.* Diameter = total height.

31. Solve Ex. 30 if the floor is twice as thick as the wall and roof.

Ans. Height of cylinder = diameter.

32. The cost of erecting an office building is \$100,000 for the first floor, \$105,000 for the second, \$110,000 for the third, etc.; other expenses (lot, plans, excavation, etc.) are \$700,000. The net annual income is \$10,000 for each story. How high should the building be, to return the maximum rate of interest on the investment?

Ans. 17 stories.

35. Derived curves. The curves $y = f'(x)$, $y = f''(x)$, $y = f'''(x)$, $\cdots$ are called the *first*, *second*, *third*, $\cdots$ *derived curves*, corresponding to the curve $y = f(x)$.

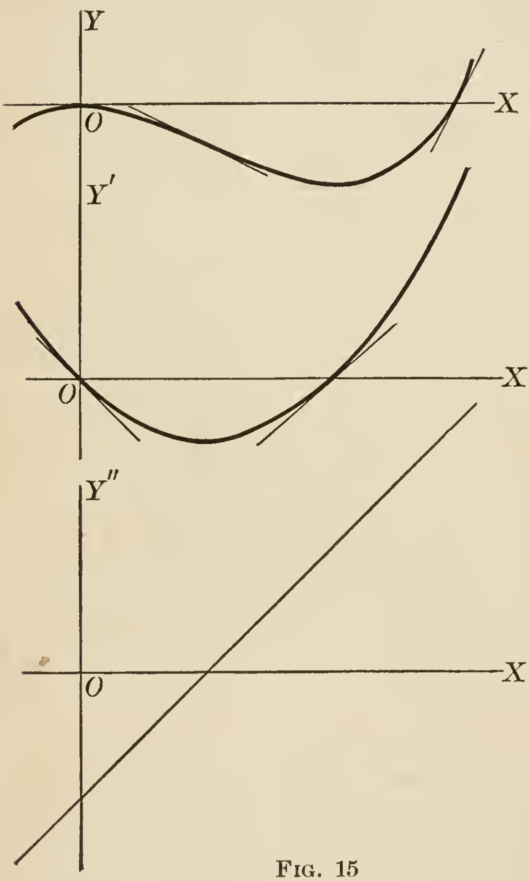


FIG. 15

The relations between the given function and its first and second derivatives, which have been formulated analytically in § 33, are well brought out graphically by drawing the original curve and its first and second derived curves. This is shown in Fig. 15 for the curve

$$y = \frac{x^3}{3} - x^2.$$

The work should be arranged with the several axes of ordinates lying in the same vertical line.

It often happens in practical work that a function is defined in such a way — for instance by experimental data

— that no mathematical expression for it is known. If then we wish to examine the behavior of one of the derivatives, as is frequently the case, we plot the graph of the original function from the given data, and construct the required derived curve graphically. The process is obvious. We can measure the slope at any point of the original curve ; the number thus obtained is the ordinate of the corresponding point on the first derived curve. In this way as many points as desired may be plotted on the first derived curve and a smooth curve drawn through them, after which the second derived curve may be constructed in a similar way; etc.

EXERCISES

1. What can be said of the first derived curve

(*a*) in an interval where the original curve is $\left\{\begin{matrix}\text{rising}\\ \text{falling}\end{matrix}\right\}$?

(*b*) in an interval where the original curve is concave $\left\{\begin{matrix}\text{upward}\\ \text{downward}\end{matrix}\right\}$?

(*c*) at a point where the original curve has a $\left\{\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right\}$?

(*d*) at a point where the original curve has a point of inflection?

2. What can be said of the second derived curve

(*a*) in an interval where the original curve is concave $\left\{\begin{matrix}\text{upward}\\ \text{downward}\end{matrix}\right\}$?

(*b*) at a point where the original curve has a point of inflection?

(*c*) in an interval where the first derived curve is $\left\{\begin{matrix}\text{rising}\\ \text{falling}\end{matrix}\right\}$?

(*d*) at a point where the first derived curve has a $\left\{\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right\}$?

3. What can be said of the original curve at a point where the second derived curve touches the x-axis without crossing it?

4. Plot the curve $y = \frac{x^4}{4} - x^3 + x^2$ and its first, second, and third derived curves.

5. Plot the curve $y = \sin x$, and construct the first derived curve. What well-known curve does the latter resemble?

6. Draw a smooth curve, on a large scale, through the points

x	-4	-2	0	2	4	6	8	10	12	14	16	18	20	22	24
y	0	-1	$-\frac{3}{2}$	0	6	10	3	0	-1	-1	$-\frac{1}{2}$	1	9	20	35

and construct the first and second derived curves.

7. The national debt of the United States at the indicated dates is given in the accompanying table, the unit being $100,000,000. Construct the curve showing the rate at which the debt has increased or decreased.

Date . . .	1850	'55	'60	'61	'62	'63	'64	'65	'68	'70
Debt . . .	0.6	0.4	0.6	0.9	5.0	11.1	17.1	26.8	24.8	23.3

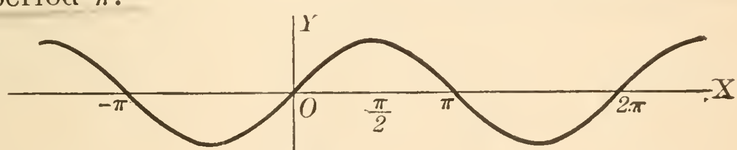
Date . . .	'75	'80	'85	'90	'95	'99	1900	'05	'10
Debt . . .	20.9	19.2	13.8	8.9	9.0	11.6	11.1	9.9	10.5

CHAPTER V

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS

I. Trigonometric and Inverse Trigonometric Functions

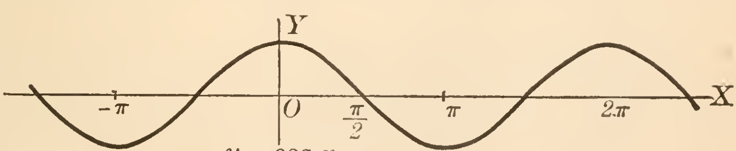
36. Trigonometric functions. The student is already familiar with the elementary properties of the trigonometric functions. They are one-valued and continuous for all values of the argument x, except that the tangent and secant become infinite when $x = \pm(2n+1)\frac{\pi}{2}$, the cotangent and cosecant become infinite when $x = \pm n\pi$, where n is a positive integer. The sine and cosine, and their reciprocals, the cosecant and secant, are periodic with the period 2π; the tangent and cotangent are periodic with the period π.



$y = \sin x$

Fig. 16

The properties just mentioned are well exhibited by the graphs of the various functions. The graphs of the sine,



$y = \cos x$

Fig. 17

cosine, and tangent are shown in Figs. 16–18; the student should draw the graphs of the other functions.

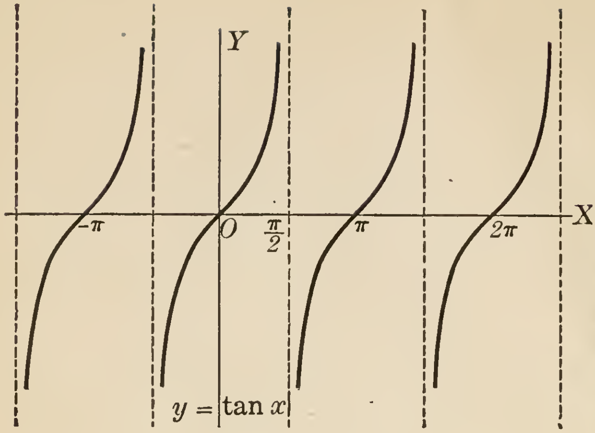


FIG. 18

The derivative is an important aid in the further study of these functions. Since all the functions can be expressed in terms of the sine, it will be sufficient to find the derivative of this one function by the general method; from this result all the others can be obtained.

37. Differentiation of sin *x*. The derivative of $\sin x$ may be obtained directly from the definition of the derivative (§ 15). We have

$$y = \sin x,$$

$$y + \Delta y = \sin(x + \Delta x),$$

$$\Delta y = \sin(x + \Delta x) - \sin x,$$

$$\frac{\Delta y}{\Delta x} = \frac{\sin(x + \Delta x) - \sin x}{\Delta x}.$$

Expanding $\sin(x + \Delta x)$ by the addition formula of trigonometry, we get

$$\frac{\Delta y}{\Delta x} = \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}.$$

By trigonometry,

$$\cos \Delta x = 1 - 2\sin^2 \tfrac{1}{2}\Delta x,$$

so that

$$\frac{\Delta y}{\Delta x} = \frac{\cos x \sin \Delta x - 2 \sin x \sin^2 \frac{1}{2}\Delta x}{\Delta x}$$

(1) $$= \cos x \cdot \frac{\sin \Delta x}{\Delta x} - \sin x \cdot \frac{\sin \frac{1}{2}\Delta x}{\frac{1}{2}\Delta x} \cdot \sin \tfrac{1}{2}\Delta x.$$

It will be shown in the next article that

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

Assuming this result for the moment, we see that

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} = 1, \quad \lim_{\Delta x \to 0} \frac{\sin \frac{1}{2} \Delta x}{\frac{1}{2} \Delta x} = 1.$$

Hence, passing to the limit in equation (1), we find

$$\frac{dy}{dx} = \frac{d}{dx} \sin x = \cos x.$$

In some applications, it is convenient to write this formula in the form

$$\frac{d}{dx} \sin x = \sin \left(x + \frac{\pi}{2}\right).$$

If u is any function of x, it follows from formula (5) of Chapter III that

$$\frac{d}{dx} \sin u = \frac{d}{du} \sin u \cdot \frac{du}{dx},$$

or

(7) $$\frac{d}{dx} \sin u = \cos u \frac{du}{dx} = \sin \left(u + \frac{\pi}{2}\right) \frac{du}{dx}.$$

Example: Differentiate $\sin 5x^2$.

By (7), with $u = 5x^2$,

$$\frac{d}{dx} \sin 5x^2 = 10x \cos 5x^2.$$

38. Limit of $\sin \alpha / \alpha$ as α approaches 0. In the differentiation of $\sin x$ we had to make use of the fact that

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

This result may be obtained as follows.

Let P, Q be two points on a circle such that the chord PQ subtends an angle $2\alpha < \pi$. As α approaches 0, the ratio of the chord to the arc approaches unity:

$$\lim_{\alpha \to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

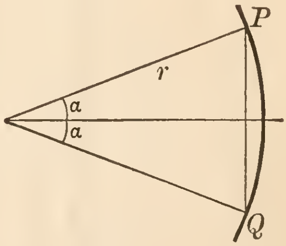


FIG. 19

But

$$\text{chord } PQ = 2\,r \sin \alpha,$$

and, if α is measured in radians,

$$\text{arc } PQ = 2\,r\alpha.$$

Hence

$$\lim_{\alpha \to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = \lim_{\alpha \to 0} \frac{2\,r \sin \alpha}{2\,r\alpha} = \lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

When α is in degrees, the length of the arc is

$$\text{arc } PQ = 2\,r \cdot \frac{\pi}{180}\,\alpha,$$

and the formula for $\frac{d}{dx} \sin x$ is much less simple than when radians are used (see Ex. 26, p. 50). For this reason *angles in the calculus are always measured in radians* unless the contrary is stated.

39. Differentiation of cos *x*, tan *x*, etc. The derivatives of the other trigonometric functions can also be obtained directly from the definition of the derivative, but they are more easily found from (7).

To differentiate $\cos x$, we write

$$\cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right)$$

$$= - \sin x.$$

If u is any function of x, we find by formula (5) of Chapter III,

(8) $$\frac{d}{dx} \cos u = - \sin u \frac{du}{dx} = \cos\left(u + \frac{\pi}{2}\right) \frac{du}{dx}.$$

The remaining trigonometric functions may be differentiated by expressing them in terms of the sine and cosine. The results are as follows:

(1) $$\frac{d}{dx} \tan x = \sec^2 x,$$

(2) $$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x,$$

(3) $$\frac{d}{dx}\sec x = \sec x \tan x,$$

(4) $$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x.$$

If u is any function of x, we find by formula (**5**) of Chapter III,

(**9**) $$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx},$$

$$\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u\frac{du}{dx},$$

$$\frac{d}{dx}\sec u = \sec u \tan u\frac{du}{dx},$$

$$\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u \cot u\frac{du}{dx}.$$

EXERCISES

1. Trace the curve $y = \sin x$, finding maxima and minima and points of inflection, and drawing the inflectional tangents.

2. Proceed as in Ex. 1 with the curves

(*a*) $y = \cos x$; (*b*) $y = \tan x$; (*c*) $y = \sec x$.

Differentiate the following functions.

3. (*a*) $\sin 2x$; (*b*) $\cos\frac{1}{x}$; (*c*) $\tan(\pi + x)$;

(*d*) $x\sec x$; (*e*) $x^2\cot x$; (*f*) $(3\theta + 1)\cos 3\theta$;

(*g*) $\frac{\sin\theta}{\theta}$; (*h*) $\sin^2 x$; (*i*) $\cos^3 2\theta$.

Ans. (*b*) $\frac{1}{x^2}\sin\frac{1}{x}$; (*h*) $2\sin x\cos x$; (*i*) $-6\cos^2 2\theta\sin 2\theta$.

4. $y = x\tan 2x + \sqrt{1 + x^2}$.

5. $y = \sqrt{1 + \sin x}$.

6. $y = \frac{4x}{\sin x}$.

7. $y = \cot^3 4x$. *Ans.* $-12\cot^2 4x\operatorname{cosec}^2 4x$.

8. $y = \sin^3 3\theta$.

9. $r = \sec(2\theta + 1)$.

Find $\frac{dy}{dx}$ in the following cases.

10. $\cos 2y = x^2 + 4$.

11. $y\sin x = \cos 2x$.

12. $y = \dfrac{\sin 2x}{1 + \cos 2x}$.

13. $y = \sqrt{1 + \tan^3 x}$.

14. $y^2 = \sin 2x$.

15. $y^3 - y = \tan \dfrac{x}{3}$.

16. If $y = \sin x$, find y'', y''', $\cdots$, $y^{(n)}$.

17. If $x = \cos \omega t$, find $\dfrac{dx}{dt}$, $\dfrac{d^2x}{dt^2}$.

18. If $x = A \sin kt + B \cos kt$, show that $\dfrac{d^2x}{dt^2} = -k^2x$.

19. Obtain each of the formulas (1)–(4), § 39.

20. From the trigonometric formula for $\sin(x + a)$, deduce by differentiation the formula for $\cos(x + a)$.

21. Find the tangent and normal to the curve $y = \sin x$ at $x = \dfrac{\pi}{6}$.

22. Find tangents to the curve $y = \tan x$ parallel to the line $y = 2x + 5$.

23. If $f(x) = \cos 2x$, find $f''(x)$, $f'''(x)$, $\cdots$, $f^{(n)}(x)$.

24. If $y = x \sin x$, find $\dfrac{d^2y}{dx^2}$.

25. If $x = t \cos kt$, find $\dfrac{d^2x}{dt^2}$.

26. Show that if x is measured in degrees, the formula for the derivative of $\sin x$ becomes

$$\frac{d}{dx} \sin x = \frac{\pi}{180} \cos x.$$

27. Differentiate $\cos x$ directly from the definition of the derivative.

28. Writing $\tan x$ in the form $\tan x = \dfrac{\sin x}{\cos x}$, obtain the derivative of $\tan x$ directly from the definition of the derivative.

29. Find the maximum rectangle inscribed in a circle, using trigonometric functions.

30. Find the rectangle of maximum perimeter inscribed in a circle.

31. Find the right circular cylinder of maximum volume inscribed in a sphere.

32. Find the largest right circular cone that can be inscribed in a given sphere. *Ans.* $V = \frac{32}{81}\pi a^3$.

33. A steel girder 30 ft. long is carried along a passage 10 ft. wide and into a corridor at right angles to the passage. The thickness of the girder being neglected, how wide must the corridor be in order that the girder may go round the corner?

34. A wall 8 ft. high is 27 ft. from a house. Find the length of the shortest ladder that will reach the house when one end rests on the ground outside the wall.

40. Inverse trigonometric functions. The symbol arcsin x, or $\sin^{-1} x$, denotes the *angle whose sine is* x:

$$y = \arcsin x \text{ if } x = \sin y.$$

That is, the function $\arcsin x$ is the *inverse* (§ 26) of the function $\sin x$. The graph of

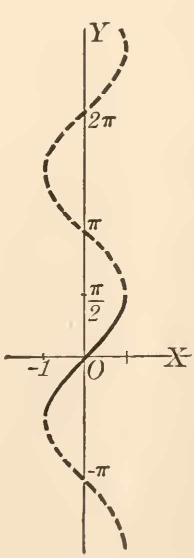
$$y = \arcsin x$$

is as shown in Fig. 20. It is of course the same as that of $\sin x$, with the coördinate axes interchanged; *i.e.* it is the reflection of the sine curve in the line $y = x$.

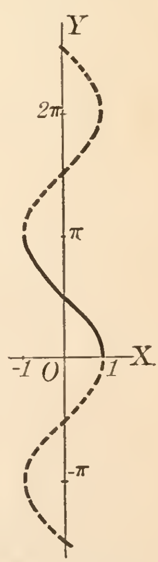
The functions $y = \arccos x$, $y = \arctan x$, etc., are defined in a similar way.

In §§ 41–42 we consider only the three principal functions $\arcsin x$, $\arccos x$, $\arctan x$. The other three functions may be treated similarly.

41. Restriction to a single branch. The trigonometric functions are one-valued: to a given value of the argument there corresponds one and but one value of the function. The inverse trigonometric functions, on the other hand, are *infinitely many-valued;* corresponding to a given value of the variable there are infinitely many values of the function. Geometrically this means that a line $x = x_0$, if it meets the curve at all, meets it in infinitely many points; the truth of this statement is evident from a glance at Figs. 20–22.

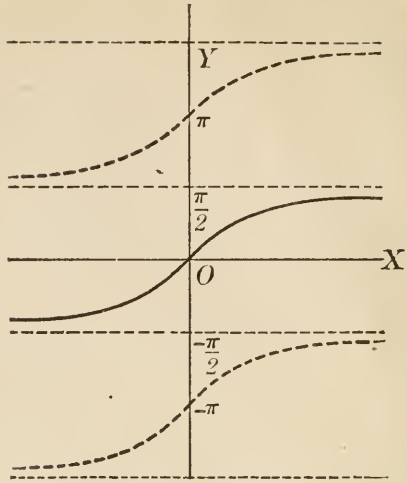


$y = \arcsin x$
FIG. 20



$y = \arccos x$
FIG. 21

Following the rule of § 5, *we shall confine our attention*



$y = \arctan x$

FIG. 22

to a single branch of each of these functions; the branch chosen is the one drawn full in each figure. Thus in our future work the function arcsin x, for example, is *restricted to the interval*

$$-\frac{\pi}{2} \leqq \arcsin x \leqq \frac{\pi}{2}.$$

This means that

$$\arcsin(-1) = -\frac{\pi}{2},$$

not $\frac{3\pi}{2}$; etc. Similarly,

$$-\frac{\pi}{2} \leqq \arctan x \leqq \frac{\pi}{2};$$

therefore

$$\arctan(-1) = -\frac{\pi}{4}, \arctan(-\infty) = -\frac{\pi}{2};$$

etc.

EXERCISES

In the following, the restrictions laid down in § 41 are assumed to hold.

1. Find (*a*) arcsin $\frac{1}{2}$, (*b*) arcsin $(-\frac{1}{2})$, (*c*) arctan $(-\sqrt{3})$, (*d*) arctan ∞, (*e*) arccos $(-\frac{1}{2})$, (*f*) arccos (-1).

2. Show that $\arcsin x + \arcsin(-x) = 0$.

3. Show that $\arccos x + \arccos(-x) = \pi$.

4. Show that

(*a*) $\arccos x = \frac{\pi}{2} - \arcsin x$;

(*b*) $\operatorname{arccot} x = \frac{\pi}{2} - \arctan x = \arctan \frac{1}{x}$;

(*c*) $\operatorname{arcsec} x = \arccos \frac{1}{x} = \frac{\pi}{2} - \arcsin \frac{1}{x}$;

(*d*) $\operatorname{arccosec} x = \arcsin \frac{1}{x}$.

42. Differentiation of the inverse trigonometric functions. To differentiate the function

$$y = \arcsin x,$$

let us pass to the direct form

$$\sin y = x.$$

Differentiating by the rule for finding the derivative of an implicit function (§ 25), we find

$$\cos y \frac{dy}{dx} = 1,$$

hence

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}},$$

or

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

It should be noticed that $\cos y$ is put equal to $\sqrt{1 - \sin^2 y}$ rather than $-\sqrt{1 - \sin^2 y}$. This is correct because, as Fig. 20 shows, the slope of the curve $y = \arcsin x$ is positive at all points of the branch that we are considering.

In a similar way we find

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}},$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

By formula (5) of Chapter III we find that if u is any function of x,

$$\frac{d}{dx}\arcsin u = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}, \tag{10}$$

$$\frac{d}{dx}\arccos u = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}},$$

$$\frac{d}{dx}\arctan u = \frac{\frac{du}{dx}}{1+u^2}. \tag{11}$$

While in the above discussion we confine our attention to a single branch of the function, it appears from Figs. 20–22 that if we know the slope at every point of one branch, we can at once find the slope at every point of any other branch.

EXERCISES

Find the derivatives of the following functions.

1. $y = \arcsin 2x.$
2. $y = \arccos \frac{1}{x}.$
3. $y = \arctan (1 + 2x).$
4. $y = \arcsin \sqrt{x}.$
5. $y = \text{arccot } (2x + 5)^2.$
6. $y = \text{arccosec } \frac{1}{2x}.$
7. $s = t \arcsin 3t.$
8. $p = \sqrt{1 - \arcsin v}.$
9. $y = (\arcsin x)^2.$ *Ans.* $\frac{2 \arcsin x}{\sqrt{1-x^2}}.$
10. $y = \frac{1}{\arctan x}.$
11. $y = \arctan \frac{1}{x}.$

12. $s = \sqrt{1-2t} \arccos \sqrt{2t}.$

13. $y = x \arctan 4x.$

14. $y = \arcsin \dfrac{x}{\sqrt{1+x^2}}\cdot$ *Ans.* $\dfrac{1}{1+x^2}.$

15. $y = t^2 \arcsin \dfrac{t}{2}\cdot$

16. $y = \dfrac{1}{\sqrt{\arcsin 2x}}\cdot$

17. $y = \arcsin x + \arccos x.$ Explain the meaning of the result.

18. If $y^3 \sin x + y = \arctan x$, find y'.

19. Find tangents to the curve $y = \arctan x$ perpendicular to the line $4x + y = 0$.

20. Obtain $\dfrac{d}{dx} \arccos x$ from the relation

$$\arccos x = \frac{\pi}{2} - \arcsin x.$$

21. Show that

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}},$$

and obtain $\dfrac{d}{dx} \arctan x$ from this fact.

22. If $y = \arcsin x$, find $\dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}\cdot$

23. If $y = \arctan x$, find $\dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}\cdot$

24. Show that

$$\frac{d}{dx} \operatorname{arccot} x = \frac{-1}{1+x^2},$$

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}},$$

$$\frac{d}{dx} \operatorname{arccosec} x = \frac{-1}{x\sqrt{x^2-1}}\cdot$$

25. Trace the curve $y = \operatorname{arccot} x$.

26. Trace the curve $y = \operatorname{arcsec} x$.

27. Trace the curve $y = \operatorname{arccosec} x$.

II. Exponential and Logarithmic Functions

43. Exponentials and logarithms. The number $a^n (a > 0)$ is defined in algebra for all *rational* values of n. In the calculus it becomes necessary to attach a meaning to the function

$$y = a^x \qquad (a > 0)$$

as x varies *continuously*.

Let x_0 be any given irrational number. It can be shown that when x approaches x_0 passing through rational values, the function a^x approaches a definite limit. This limit is denoted by a^{x_0}:

$$\lim_{x \to x_0} a^x = a^{x_0}.$$

The function a^x thus becomes defined for all values of x. This function is one-valued and continuous, and obeys the ordinary laws of exponents, viz.:

(1) $$a^x \cdot a^t = a^{x+t},$$
$$(a^x)^t = a^{xt}.$$

The inverse of the exponential function is the *logarithm*, defined by the statement that

$$y = \log_a x \text{ if } x = a^y \qquad (a > 1\,*).$$

This function is one-valued and continuous for all *positive* values of x. The number a is called the *base* of the system of logarithms.

The graph of the function

$$y = e^x,$$

where $e = 2.718 \cdots$ (see § 46), is shown in Fig. 23; the graph of its inverse

$$y = \log_e x$$

is shown in Fig. 24.

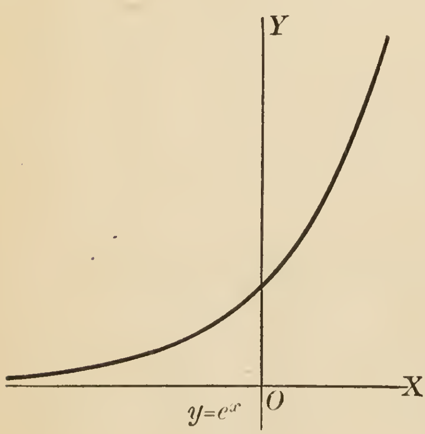


Fig. 23

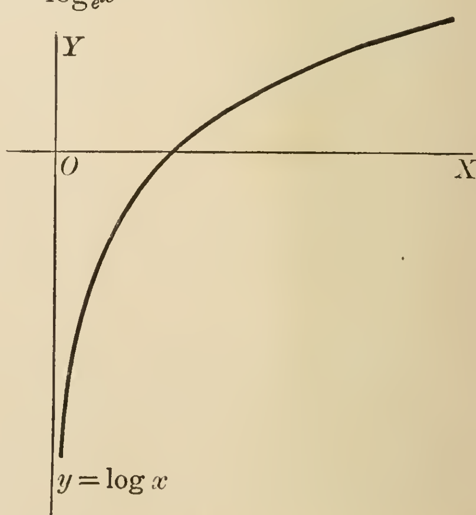


Fig. 24

* The assumption $a > 1$ is introduced for simplicity; this condition is satisfied in all cases of practical importance.

44. Properties of logarithms. For convenient reference we recall the fundamental properties of logarithms :

(1) $$\log_a xy = \log_a x + \log_a y,$$

(2) $$\log_a \frac{x}{y} = \log_a x - \log_a y,$$

(3) $$\log_a x^n = n \log_a x,$$

(4) $$\log_a a^x = x,$$

(5) $$a^{\log_a x} = x,$$

(6) $$\log_a x = \log_b x \cdot \log_a b,$$

(7) $$\log_b a = \frac{1}{\log_a b}.$$

To prove (1), let

$$\log_a xy = p, \ \log_a x = m, \ \log_a y = n:$$

then we must show that

$$p = m + n.$$

Passing to the direct form, we have

$$xy = a^p, \ x = a^m, \ y = a^n,$$

so that

$$a^p = a^m \cdot a^n.$$

Hence, by (1) of § 43,

$$p = m + n.$$

Formulas (2) and (3) may be proved in a similar way.

Formula (4) is merely a restatement of the definition of the logarithm; formula (5) is the converse. To prove (5), set

$$a^{\log_a x} = t,$$

and take logarithms to the base a on each side:

$$\log_a x = \log_a t,$$

whence

$$t = x.$$

To prove (6), let $m = \log_a x$ and $n = \log_b x$; then

$$x = a^m = b^n.$$

If we take logarithms to the base a on each side of the equation

$$a^m = b^n,$$

it appears that

$$m = \log_a x = n \log_a b = \log_b x \cdot \log_a b.$$

As a special case take $x = a$: the formula gives (7),

$$\log_b a = \frac{1}{\log_a b}.$$

EXERCISES

1. Find x, if (*a*) $\log_{10} x = 2$, (*b*) $\log_{10} x = -\frac{1}{2}$, (*c*) $\log_2 x = 4$, (*d*) $\log_{10} x^3 = 4$, (*e*) $\log_a x = 0$, (*f*) $\log_a x = 1$.

2. Simplify (*a*) $a^{\log 3}$, (*b*) $a^{-\log 3}$, (*c*) $a^{2 \log 5}$, (*d*) $a^{2 \log x}$, (*e*) $a^{x + \log x}$, (*f*) $a^{x - \frac{1}{2} \log x}$, the logarithms being taken to the base a in each case.
Ans. (*b*) $\frac{1}{3}$.

3. Prove formulas (2) and (3) of § 44.

4. Show that negative numbers have no (real) logarithms.

5. Show that numbers between 0 and 1 have negative logarithms; numbers greater than 1, positive logarithms.

6. Show that

$$\lim_{x \to 0^+} \log_a x = -\infty.$$

7. Find the inverse of the function

$$y = \frac{a^x - a^{-x}}{a^x + a^{-x}}.$$

8. For what two values of x is $a^{x^x} = (a^x)^x$?

45. The derivative of the logarithm. To obtain the derivative of the logarithm we proceed by the general method of § 15:

$$y = \log_a x,$$

$$y + \Delta y = \log_a (x + \Delta x),$$

$$\Delta y = \log_a (x + \Delta x) - \log_a x = \log_a \frac{x + \Delta x}{x}$$

$$= \log_a \left(1 + \frac{\Delta x}{x}\right),$$

by property (2) of § 44. Hence,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right).$$

Let us multiply and divide by x and then make use of (3), § 44:

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \frac{x}{\Delta x} \log_a\left(1 + \frac{\Delta x}{x}\right)$$

$$= \frac{1}{x} \log_a\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}.$$

Hence,

(1) $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{x} \lim_{\Delta x \to 0} \log_a\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

(2) $$= \frac{1}{x} \log_a\left[\lim_{\Delta x \to 0}\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right].$$

It will appear in the next article that the limit $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$ exists and is a number lying between 2 and 3. This number is denoted by the letter e; we shall find later (Ex. 4, p. 230) that

$$e = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = 2.71828 \cdots.$$

Now, in the limit occurring in (1), let us put $\frac{x}{\Delta x} = n$. Since x is supposed to be different from 0, it follows that when Δx approaches 0, n becomes infinite, and

$$\lim_{\Delta x \to 0}\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e.$$

Hence, assuming for the moment the existence of e, we have from (2)

(3) $$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

In case the base a of the system of logarithms is the number e, the numerical factor $\log_a e$ in formula (3) reduces to unity, and the formula takes a particularly simple form. For that reason *logarithms to the base e are used almost exclusively in the calculus.*

Logarithms to the base e are called *natural logarithms*, or *Napierian logarithms*. In our future work the symbol $\log x$, in which no base is indicated, will be understood to mean the *natural logarithm* of x. Thus we have from (3)

$$\frac{d}{dx}\log x = \frac{1}{x}.$$

By formula (5) of Chapter III, if u is any function of x,

$$\frac{d}{dx}\log_a u = \frac{\frac{du}{dx}}{u}\cdot\log_a e,$$

and

$$\frac{d}{dx}\log u = \frac{\frac{du}{dx}}{u}. \tag{12}$$

Example: Find the slope of the curve

$$y = \log\sqrt{1+3x}$$

at the point (x, y).

Let us write y in the form

$$y = \tfrac{1}{2}\log(1+3x).$$

Then

$$y' = \frac{1}{2}\cdot\frac{3}{1+3x} = \frac{3}{2+6x}.$$

46. The limit *e*. It will now be shown that the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

exists and that e is a number between 2 and 3. For the sake of simplicity we shall prove this result only for the case when n becomes infinite passing through positive integral values, referring for the general proof to more advanced texts.

When n is a positive integer, we can expand the quantity $\left(1+\frac{1}{n}\right)^n$ by the binomial theorem:

$$(1)\quad \left(1+\frac{1}{n}\right)^n$$

$$= 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2\,!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3\,!}\left(\frac{1}{n}\right)^3 + \cdots$$

$$+ \frac{n(n-1)\cdots(n-\overline{n-1})}{n\,!}\left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \frac{1-\frac{1}{n}}{2\,!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3\,!} + \cdots$$

$$+ \frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}{n\,!}.$$

As n increases the number of terms in the expansion increases, and every term (except the first two) becomes larger. Hence the quantity $\left(1+\frac{1}{n}\right)^n$ steadily increases with n.

On the other hand, this quantity is always less than 3. For, the $n+1$ terms in the expansion (1) are each less than (or, for the first two terms, equal to) the corresponding terms of the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}.$$

Remembering that, by elementary algebra, the sum of the geometric progression (cf. § 7)

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

is $2 - \frac{1}{2^{n-1}}$, we find

$$\left(1+\frac{1}{n}\right)^n < 3 - \frac{1}{2^{n-1}} < 3.$$

We have now shown that $\left(1+\frac{1}{n}\right)^n$ steadily increases with n, but never becomes greater than 3. It follows by theorem IV, § 8, that as n increases the quantity $\left(1+\frac{1}{n}\right)^n$ *approaches a limit* e which is not greater than 3.

Since, in (1), the sum of the first two terms is 2 and the succeeding terms are all positive, it follows that $e > 2$. Hence e lies between 2 and 3.

As already stated, we shall see later that

$$e = 2.71828 \cdots.$$

47. Differentiation of the exponential function. The derivative of the exponential function a^x may be found as follows.

If

$$y = a^x,$$

then

(1) $$\log_a y = x.$$

Differentiating (1) by the rule for implicit functions (§ 25), we find

$$\frac{1}{y}\frac{dy}{dx}\log_a e = 1,$$

$$\frac{dy}{dx} = \frac{y}{\log_a e} = y \log_e a,$$

by (7), § 44; hence

$$\frac{d}{dx} a^x = a^x \log_e a.$$

For the case $a = e$, this formula becomes simply

$$\frac{d}{dx} e^x = e^x.$$

If u is a function of x, we have

$$\frac{d}{dx} a^u = a^u \log_e a \cdot \frac{du}{dx}.$$

This formula, too, becomes simpler when $a = e$:

(13) $$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

EXERCISES

1. Show that common logarithms are transformed into natural logarithms by the formula

$$\log_{10} x = \log_{10} e \cdot \log_e x$$
$$= 0.4343 \log_e x.$$

2. Show that

$$\log_e x = 2.3026 \log_{10} x.$$

3. By means of a table of common logarithms, show that

$$\log 2 = 0.693, \quad \log 3 = 1.099, \quad \log 5 = 1.609.$$

4. Using the results of Ex. 3, find $\log \frac{1}{2}$, $\log \sqrt{3}$, $\log 6$, $\log 0.1$, $\log \sqrt[3]{9}$.

Find the derivatives of the following functions.

5. $y = \log 2x.$

6. $y = \log(1 + x^2).$

7. $y = \log\sqrt{5 - x}.$

8. $y = \log\dfrac{(1-x)^2}{1+2x}.$

9. $y = \log\sqrt{\dfrac{1+x}{1-x}}.$

10. $y = \log_{10} 2x.$

11. $y = \log_{10}(x^2 - 1).$

12. $y = \log_a x^2.$

13. $y = \log\sin x.$

14. $y = x\log x.$

15. $y = \dfrac{\log x}{x}.$

16. $y = (1 - x^2)\log x.$

17. $y = \log^3 x.$

18. $y = \log\log x.$

19. $y = \log\log(1 - x).$

20. $y = e^{2x}.$

21. $y = e^{x^2}.$

22. $y = xe^{-x}.$

23. $y = e^x \log x.$

24. $y = 10^x.$

25. $y = 2^{x^2}.$

26. $y = e^{e^x}.$

27. $r = e^{a\theta}.$

28. $r = e^\theta \cos 2\theta.$

29. $y = \log\cos 2x.$

30. $y = \arcsin\log x.$

31. $y = e^{\cos^2 t}.$

32. $y = \log\dfrac{e^x - e^{-x}}{e^x + e^{-x}}.$

33. $y = \sin^3 e^x.$

34. $y = e^{\tan\frac{x}{2}}.$

35. $y = \sqrt{1 + \log x}.$

36. $y = \arctan e^x.$

37. $y = (1 - e^{2x})^3.$

38. $y = \log e^{2x}.$

39. If $y = e^{-\frac{1}{2}x^2}$, find y''.

40. If $f(\theta) = \log\log\sin 2\theta$, find $f'(\theta)$.

41. Given $\log(x+y) = x^2 + y^2$, find $\frac{dy}{dx}$.

42. If $y = \log x$, find y'', y''', $\cdots$, $y^{(n)}$.

43. If $y = e^{ax}$, find y'', y''', $\cdots$, $y^{(n)}$. *Ans.* $y^{(n)} = a^n e^{ax}$.

44. Find the inverse of the function $y = e^{\sin x}$.

45. Find the inverse of the function $y = \log\cos 2x$.

46. Find the tangent and normal to the curve $y = \log x$ at (*a*) $y = 0$; (*b*) $y = -\frac{1}{2}$; (*c*) $x = e^2$.

47. Show that the curve $y = e^x$ has a constant subtangent. Hence devise a simple geometric construction for drawing the tangent to $y = e^x$ at any point.

48. Show how to draw the tangent to the curve $y = \log x$.

49. Find the maximum and minimum points on the curve $y = x\log x$. Trace the curve.

50. Trace the curve $y = e^{-\frac{1}{2}x^2}$.

51. If $y = xe^x$, find y'', y''', $\cdots$, $y^{(n)}$.

52. Trace the curve $y = xe^x$.

53. Find the equation of a tangent to the curve $y = x\log x$ parallel to the line $3x - 2y = 5$.

54. In passing from (1) to (2), § 45, we make use of the principle that

$$\lim(\log x) = \log(\lim x).$$

From which one of our assumptions concerning the logarithm does this principle follow?

48. Hyperbolic functions. A class of exponential functions of frequent occurrence in some applications are known as *hyperbolic functions.* They are denoted by the symbols $\sinh x$ (read hyperbolic sine of x), $\cosh x$, and $\tanh x$, and are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}.$$

The reciprocals of these are cosech x, sech x, and coth x respectively. Tables of hyperbolic functions have been computed; see, for example, Peirce's Short Table of Integrals (Ginn and Co.).

The inverses of the hyperbolic functions are called *anti-hyperbolic functions:*

$$y = \sinh^{-1} x \text{ if } x = \sinh y, \text{ etc.}$$

The fundamental properties of the hyperbolic functions are easily obtained from the definitions; their derivation is left to the student in the exercises below.

EXERCISES

1. Show that

$$\cosh^2 x - \sinh^2 x = 1,$$
$$1 - \tanh^2 x = \operatorname{sech}^2 x,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x.$$

2. Show that

$$\frac{d}{dx} \sinh x = \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x$$

3. Show that

$$\sinh^{-1} x = \log (x + \sqrt{1 + x^2}).$$

Fundamental Differentiation Formulas

(1) $$\frac{dc}{dx} = 0,$$

(2) $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx},$$

(3) $$\frac{d}{dx} uv = u\frac{dv}{dx} + v\frac{du}{dx},$$

(3′) $$\frac{d}{dx} cv = c\frac{dv}{dx},$$

(4) $$\frac{d}{dx} \frac{u}{v} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

(4′) $\frac{d}{dx}\frac{c}{v} = -\frac{c}{v^2}\frac{dv}{dx},$

(5) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$

(6) $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx},$

(6′) $\frac{d}{dx}\sqrt{u} = \frac{\frac{du}{dx}}{2\sqrt{u}},$

(7) $\frac{d}{dx}\sin u = \cos u\frac{du}{dx} = \sin\left(u + \frac{\pi}{2}\right)\frac{du}{dx},$

(8) $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx} = \cos\left(u + \frac{\pi}{2}\right)\frac{du}{dx},$

(9) $\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx},$

(10) $\frac{d}{dx}\arcsin u = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}},$

(11) $\frac{d}{dx}\arctan u = \frac{\frac{du}{dx}}{1 + u^2},$

(12) $\frac{d}{dx}\log u = \frac{\frac{du}{dx}}{u},$

(13) $\frac{d}{dx}e^u = e^u\frac{du}{dx}.$

MISCELLANEOUS EXERCISES

Find the derivatives of the following functions.

1. $\sin^3\frac{\theta}{3}$.
2. $\log\tan 3x$.
3. $\arctan x^3$.
4. $(1 - x)^2(2x + 3)^3$.
5. $e^{\cos x}$.
6. $\sqrt{1 - \cot x}$.
7. $x \arcsin\frac{x}{2}$.
8. $\frac{x}{\sqrt{1 - x^2}}$.
9. $\log^2 \sin\theta$.
10. $\arctan(1 - t^2)$.

11. $\cos^4 2x$.

12. $\dfrac{(3x^2-4)^2}{x^2+1}$.

13. $\arcsin \dfrac{x}{4}$.

14. $\tan^2(1-x)$.

15. $\dfrac{\sqrt{3-4x}}{2}$.

16. $\dfrac{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}{2}$.

17. $\log\log\cos x$.

18. $\cos^2\left(\dfrac{\pi}{4}-x\right)$.

19. 2^{2x}.

20. $\sqrt{\sin x^2}$.

21. $x\log\sqrt{1-x}$.

22. $(e^{2x}-1)^4$.

23. $\sin x\cos 2x$.

24. $\arccos\log x$.

25. $\dfrac{x^2-1}{x\sqrt{x^2+1}}$.

26. $(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}$.

27. $\dfrac{(x-1)^3}{(x^2+3x+3)^2}$.

28. $\dfrac{\sin^2 2\theta}{(1-\cos 2\theta)^2}$.

29. $\log(e^{2x}+1)$.

30. $\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$.

Find y' in the following cases.

31. $\sin(x+y)=\cos(x-y)$.

32. $e^{xy}=x+y$.

33. $\dfrac{y^2}{x^2}+\dfrac{2y}{x}=3$.

34. $x-y=\tan(x-y)$.

35. Find y'', if $ay^2=x^3$.

36. Find y''', if $x^2-y^2=a^2$.

Find the slope of each of the following curves at the point indicated.

37. $(x-y)^2=3x+4y-14$ at $(2, 2)$.

38. $y=\log x$ at the point where $y=-2$.

39. $y=e^x$ (*a*) at the point where $y=2$; (*b*) at the point where $x=\log 3$.

40. $\arcsin x+xy=0$, at the point $x=-1$.

CHAPTER VI

THE DIFFERENTIAL

49. Order of infinitesimals. We have found that in the problem of differentiation the increments Δx and Δy are infinitesimals, with Δx as the principal infinitesimal (§ 10).

An idea of fundamental importance in the study of infinitesimals is that of *order*. Let an infinitesimal v be defined as a function of a principal infinitesimal u. If

$$\lim_{u \to 0} \frac{v}{u} = k,$$

where $k \neq 0$, then u and v are said to be *infinitesimal of the same order*; if

$$\lim_{u \to 0} \frac{v}{u} = 0,$$

v is said to be of *higher order* than u. More precisely, if a number n can be found such that

$$\lim_{u \to 0} \frac{v}{u^n} = k,$$

where $k \neq 0$, v is said to be *infinitesimal of the n-th order* with respect to u.

If u and v are of the same order, we may write

$$v = ku + \epsilon u,$$

where ϵ is an infinitesimal. It is clear that when u approaches 0, the term ϵu approaches 0 more rapidly than does ku, so that for small values of u the term ku is numerically the larger. For this reason the term ku is called the *principal part* of v.

Example: When the side l of a square increases by an amount Δl, the area increases by an amount

$$\Delta A = (l + \Delta l)^2 - l^2 = 2\,l\Delta l + \overline{\Delta l}^2.$$

If Δl approaches 0, ΔA does also. The two infinitesimals are of the same order, since

$$\lim_{\Delta l \to 0} \frac{\Delta A}{\Delta l} = \lim_{\Delta l \to 0} (2\,l + \Delta l) = 2\,l.$$

The principal part of ΔA is $2\,l\Delta l$. The figure illustrates the fact that ΔA consists of a term of the first and a term of the second order.

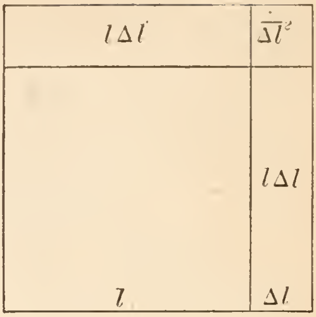


FIG. 25

EXERCISES

1. What is the increase ΔV in the volume of a cube of edge l when the side increases by an amount Δl? Show that if Δl is infinitesimal, ΔV is infinitesimal of the same order, and find the principal part of ΔV. Illustrate by a figure. *Ans.* $\Delta V = 3\,l^2\Delta l + 3\,l\overline{\Delta l}^2 + \overline{\Delta l}^3$.

2. Of the functions $\sin\theta$, $\sec\theta$, $\tan\theta$, $1 - \cos\theta$, which are infinitesimal with respect to θ as the principal infinitesimal?

3. As the radius of a right circular cylinder of given altitude approaches 0, the volume and the total surface do likewise. Show that the volume is infinitesimal of higher order than the total surface.

4. Given a right circular cylinder and a right circular cone of the same base and altitude, show that

(*a*) if the altitude is infinitesimal, the lateral surface of the cylinder is infinitesimal of a higher order than that of the cone;

(*b*) if the radius is infinitesimal, the lateral surfaces of the cylinder and cone are of the same order and (for small values of the radius) the former is approximately twice the latter.

5. Is the sum of two infinitesimals itself infinitesimal? Is the product? Is the quotient?

50. The differential. It follows from the above definition that Δy and Δx are in general infinitesimals of the same order. For, the limit of their ratio is $k = y'$, and this in general exists and is different from 0. Further, the principal part of Δy is evidently $y'\Delta x$. It is easily seen from Fig. 26 that the principal part of $\Delta y = QP'$ is QR, the segment of Δy cut off by the tangent at P. For, the slope at P is

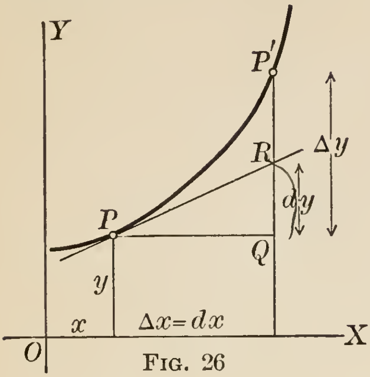


FIG. 26

$$y' = \frac{QR}{PQ} = \frac{QR}{\Delta x},$$

so that

$$QR = y'\Delta x.$$

The principal part of Δy (the length QR in Fig. 26) *is called the **differential*** * *of* y and is written dy*:*

$$dy = y'\Delta x.$$

Hence the increment Δy consists in general of the differential dy plus an infinitesimal of higher order. This is illustrated by the example of § 49.

In particular, let $y = x$; then $y' = 1$, and

$$dy = dx = \Delta x;$$

i.e. *the differential of the independent variable is the increment of the variable.* We may therefore write

$$dy = y'dx.$$

Thus *the differential of any function is equal to its derivative multiplied by the differential of the independent variable.*

The derivative of y with respect to x may now be thought of as a quotient — the differential of y divided by the differential of x. This is the reason for using the symbol $\frac{dy}{dx}$ to denote the derivative. The symbol $\frac{dy}{dx}$ may thus be considered as representing an actual division — the ratio $dy \div dx$. It must be kept clearly in mind, however, that the derivative is a certain limit, viz.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

If $y = f(x)$, instead of writing

$$\frac{dy}{dx} = f'(x),$$

* In case $y' \neq 0$. If $y' = 0$, then $dy = 0$.

we may, and often do, write

$$dy = f'(x)dx.$$

Thus the fundamental formulas of differentiation are often written in this so-called differential notation; *e.g.*

$$d(x^n) = nx^{n-1}dx,\ d(\log u) = \frac{du}{u},\ \text{etc.}$$

Examples: (*a*) If $y = \sin 2\theta$, then

$$dy = 2\cos 2\theta d\theta.$$

(*b*) Find an approximate formula for the area of a narrow circular ring.

The area of a circle of radius r is

$$A = \pi r^2.$$

If the radius be increased by an amount Δr, the area is increased by an amount ΔA whose principal part is

$$dA = 2\pi r\, dr.$$

Hence the area A_r of a narrow circular ring is approximately the product of the circumference * by the width w:

$$A_r = 2\pi r w.$$

EXERCISES

Find the differential of each of the following functions.

1. (*a*) x^2; (*b*) $\cos\theta$; (*c*) $t^3 - 1$; (*d*) $\log x$;
(*e*) $\arcsin y$; (*f*) $\tan 2\alpha$; (*g*) $\frac{y-1}{y+1}$; (*h*) $\sin^2 v$.

Ans. (*a*) $2x\,dx$; (*b*) $-\sin\theta\, d\theta$.

2. (*a*) $(1 - 3x^2)^2$; (*b*) $\log(1 - \cos 2\theta)$; (*c*) ue^u;
(*d*) $\arctan e^t$; (*e*) $x\sqrt{a + bx}$; (*f*) $\frac{x}{\sqrt{x - 1}}$.

3. $y = x(1 - x^2)^3$.

4. $y = \frac{\sqrt{1 - x}}{4x}$.

5. $v = u\sin^2 u$.

6. $x = y\log y$.

7. $y = \frac{1}{x^2}$.

8. $s = \arcsin(1 - t)$.

* Either the inner or the outer circumference.

9. $r = \frac{\cos \theta}{\theta}$.

10. $y = e^{-x} \sin kx$.

11. $V = \frac{4}{3} \pi r^3$.

12. $x = t \sin at$.

13. $y = (1 + \alpha^2) \arctan \alpha$.

14. $y = \cos^3 2x$.

15. Find the difference between dy and Δy, if $y = x^3$. Draw the figure.

16. Proceed as in Ex. 15 for the function $y = x^4 - x^2$.

17. If $y = 4x$, find Δy and dy and show geometrically why they are equal.

18. If $s = 16t^2 + 25t$, find the difference between Δs and ds when $t = 12$ and $\Delta t = .02$.

19. Draw figures to show that dy may be equal to, greater than, or less than Δy.

20. Show that the error committed in using the approximate formula of example (*b*), § 50, is πw^2. When $r = 10$ ft., what is the greatest allowable value of w if accuracy to within 5% is required?

Ans. About 1 ft.

21. If A is the area of a rectangle one of whose sides is twice the other, draw a figure showing the difference between dA and ΔA when the length of the side changes (cf. Fig. 25).

22. If V is the volume of a cube, draw a figure showing the difference between dV and ΔV when the length of the edge of the cube changes.

23. Find an approximate formula for the volume of a thin cylindrical shell of thickness t. *Ans.* $2\pi rht$.

24. Find an approximate formula for the volume of a thin spherical shell. What is the greatest allowable thickness for a radius of 5 ft. if accuracy to 1% is required? *Ans.* About 0.6 in.

25. Find approximately the volume of wood required to make a covered cubical box of edge 3 ft., using half-inch boards.

Ans. $2\frac{1}{4}$ cu. ft.

26. Work Ex. 25 if the dimensions of the box are 6, 4, and 2 ft.

51. Parametric equations; implicit functions. A curve is frequently not determined by an equation between x and y, but by two equations giving x and y in terms of a third variable, or *parameter*. These equations are called *parametric equations* of the curve.

For instance, the coördinates of a point moving in a plane are functions of the time:

$$x = \phi(t),\ y = \psi(t).$$

These two equations may be considered as parametric equations of the path. Again, the equations of an ellipse in terms of the eccentric angle ϕ are

$$x = a \cos \phi,\ y = b \sin \phi.$$

While it may be possible to eliminate the parameter, thus obtaining the ordinary cartesian equation of the curve, it is often more convenient not to do so.

When dealing with parametric equations, it is convenient to use differentials in finding derivatives, particularly the derivatives of higher order. The method is illustrated by example (a) below.

Differentials can also be used conveniently in finding derivatives when the relation between the variables is an implicit one.

Examples: (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when

$$x = 3\,t,\ y = t^2 - 4.$$

We have

$$dx = 3\,dt,\ dy = 2\,t\,dt,\ \frac{dy}{dx} = \frac{2\,t}{3}.$$

To find $\frac{d^2y}{dx^2}$, put (for convenience) $\frac{dy}{dx} = y'$. Then

$$dy' = \tfrac{2}{3}\,dt,$$

hence

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{2}{3}\,dt}{3\,dt} = \tfrac{2}{9}.$$

(b) Find y' and y'' when $x^2 + y^2 = a^2$.

Differentiating both sides of the equation, we get

$$2\,x\,dx + 2\,y\,dy = 0,\quad y' = -\frac{x}{y};$$

$$dy' = -\frac{y\,dx - x\,dy}{y^2},$$

$$y'' = \frac{dy'}{dx} = \frac{-y + x\dfrac{dy}{dx}}{y^2} = \frac{-y - \dfrac{x^2}{y}}{y^2}$$

$$= \frac{-y^2 - x^2}{y^3} = -\frac{a^2}{y^3}.$$

EXERCISES

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the following cases.

1. (*a*) $x = t^2,\ y = t - 3$; (*b*) $x = t^2 + 1,\ y = t^3$;
(*c*) $x = \cos 2\theta,\ y = \sin 2\theta$; (*d*) $x = a\cos^3\theta,\ y = a\sin^3\theta$;
(*e*) $x = e^{2t},\ y = e^t + 1$;
(*f*) $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$.

Ans. (*a*) $\frac{d^2y}{dx^2} = -\frac{1}{4t^3}$.

2. (*a*) $y^2 = 4ax$; (*b*) $x^2 - y^2 = 1$; (*c*) $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$;
(*d*) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$; (*e*) $x - y^2 = y^3$; (*f*) $x^3 + y^3 = 3axy$.

Ans. (*a*) $\frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}$.

Find $\frac{dy}{dx}$, using differentials.

3. $3x^3y^2 - xy + x^2 - y - 5 = 0$.

4. $y = \cos(x - y)$.

5. $e^{x^2+y^2} = xy$.

6. $\frac{x - y}{x + y} + y^2 = 5$.

7. $\log\sqrt{x^2 + y^2} = x$. *Ans.* $\frac{dy}{dx} = \frac{x^2 + y^2 - x}{y}$.

8. $xy - x^2y^2 + 5y = 5$.

9. $x^4 - 3x^3 + xy^2 - y^2 = 0$.

CHAPTER VII

CURVATURE

52. Differential of arc. Let s denote the length of the arc of the plane curve

$$y = f(x)$$

counted from some initial point P_0 up to the point $P:(x, y)$, and suppose for definiteness that s increases as x increases. The arc s can be regarded as a function of x. Its derivative $\frac{ds}{dx}$ may be found as follows:

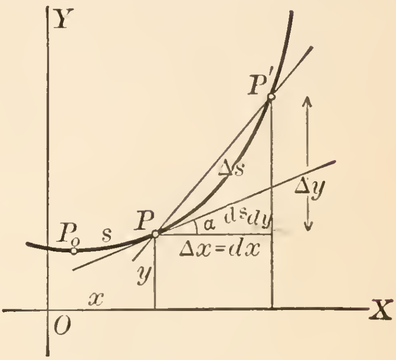


FIG. 27

$$\frac{\Delta s}{\Delta x} = \frac{\Delta s}{\overline{PP'}} \cdot \frac{\overline{PP'}}{\Delta x} = \frac{\Delta s}{\overline{PP'}} \cdot \frac{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}}{\Delta x}$$

$$= \frac{\Delta s}{\overline{PP'}} \cdot \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

where Δs is the length of the arc, $\overline{PP'}$ the length of the chord, from $P:(x, y)$ to $P':(x + \Delta x, y + \Delta y)$. Since, as pointed out in § 38,

$$\lim_{\Delta x \to 0} \frac{\Delta s}{\overline{PP'}} = 1,$$

we have

(1) $$\frac{ds}{dx} = \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

If s increases as x decreases, then

$$\frac{\Delta s}{\Delta x} = -\frac{\Delta s}{\overline{PP'}} \cdot \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2},$$

and

(2) $$\frac{ds}{dx} = -\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

After squaring and clearing of fractions, equation (1) (or (2)), becomes

$$\overline{ds}^2 = \overline{dx}^2 + \overline{dy}^2;$$

i.e. ds is the hypotenuse of the right triangle whose sides are dx and dy.

If the tangent to the curve at P makes an angle α with OX, then

$$\cos\alpha = \frac{dx}{ds}, \ \sin\alpha = \frac{dy}{ds}.$$

53. Curvature. We say in ordinary language that a curve whose direction changes rapidly has great *curvature*, or is sharply curved. Thus a circular arc is said to have greater curvature when the radius is small than when it is large. This somewhat vague idea may be made precise as follows.

Consider, first, two points P, P' on a circle, and denote the arc PP' by Δs, the angle between the tangents at P, P' by $\Delta\alpha$. The quotient $\frac{\Delta\alpha}{\Delta s}$ is evidently the change in the direction of the curve, per unit of arc*; it is called the *curvature* of the circle.

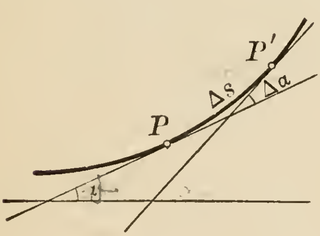


FIG. 28

If now the curve in question is not a circle, the direction of the curve no longer changes uniformly, and the quotient $\frac{\Delta\alpha}{\Delta s}$ represents merely the *average curvature* of the arc Δs. But if P' be made to approach P along the curve, so that Δs and $\Delta\alpha$ approach 0, the quantity $\frac{\Delta\alpha}{\Delta s}$ in general approaches a limit $\frac{d\alpha}{ds}$, which

* It is easily seen that, in the case of the circle, this quotient is constant.

is called the *curvature at the point P*:

$$\kappa = \lim_{\Delta s \to 0} \frac{\Delta \alpha}{\Delta s} = \frac{d\alpha}{ds}.$$

The definition is of course independent of the particular coördinate system used; the angle α is the angle made by the tangent at P with *any* fixed line in the plane of the curve. When the equation of the curve is given in cartesian coördinates, it is convenient to take α as the slope-angle of the tangent — *i.e.* the angle between the tangent and the x-axis. The curvature κ is then easily expressed in terms of the coördinates. For,

$$\tan \alpha = \frac{dy}{dx} = y',$$

$$\alpha = \arctan y',$$

$$d\alpha = \frac{dy'}{1 + y'^2} = \frac{y'' dx}{1 + y'^2}.$$

Also, by § 52,

$$ds = \sqrt{1 + y'^2}\, dx.$$

Hence

(1) $$\kappa = \frac{d\alpha}{ds} = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}}.$$

It is customary to consider κ as essentially positive, so that, strictly speaking, we should write

$$\kappa = \left|\frac{d\alpha}{ds}\right| = \frac{|y''|}{(1 + y'^2)^{\frac{3}{2}}},$$

where the symbol $|a|$ means the absolute or numerical value of a.

It should be noted that when $y' = 0$, formula (1) reduces to

$$\kappa = y''.$$

Thus the value of the second derivative at any point is equal to the curvature at that point when the coördinate axes are so chosen that the first derivative is 0.

54. Radius of curvature. The reciprocal of the curvature is called the *radius of curvature*, and is denoted by ρ:

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\alpha} = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}.$$

This quantity is also to be considered as essentially positive.

If a length equal to the radius of curvature ρ at the point P be laid off on the normal from P toward the concave side of the curve, the extremity Q of this segment is called the *center of curvature.* It can be shown that the circle with radius ρ and center Q represents the curve near P more closely than any other circle. This circle is called the *osculating circle*, or *circle of curvature.*

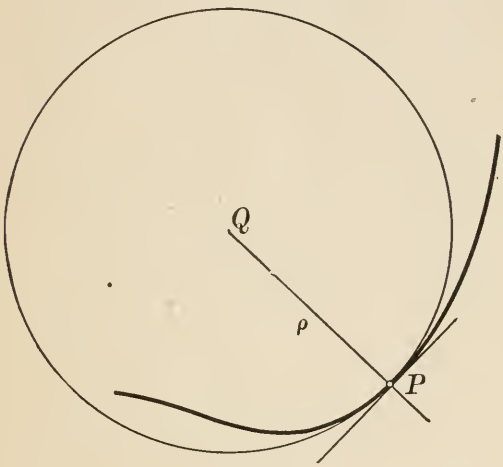


FIG. 29

In general, the circle of curvature crosses the curve at P, as is the case in Fig. 29.

EXERCISES

1. Show that the curvature of a straight line is everywhere 0.

2. Show that the radius of curvature of a circle is the radius of the circle.

Find the radius of curvature of the following curves.

3. $y = x^2$ (*a*) at any point; (*b*) at the vertex.

4. $y^2 = 4ax$. *Ans.* $\dfrac{(4a^2+y^2)^{\frac{3}{2}}}{4a^2}$.

5. The equilateral hyperbola $2xy = a^2$ at $(a, \frac{1}{2}a)$. *Ans.* $\frac{5}{8}\sqrt{5}\,a$.

6. $y = x^3 + 5x^2 + 6x$ at $(0, 0)$. *Ans.* 22.51.

7. The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* $\dfrac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$.

8. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

9. The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $3(axy)^{\frac{1}{3}}$.

10. The ellipse $x = a \cos \phi$, $y = b \sin \phi$.

11. The curve $x = t^2$, $y = 1 - t^4$.

12. The curve $x = 3t^2$, $y = 3t - t^3$. *Ans.* $\frac{3}{2}(1 + t^2)^2$.

13. The catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ at the point $(0, a)$. *Ans.* a.

14. Show that the curvature at a point of inflection is 0.

15. Find the point of maximum curvature on the curve $y = e^x$.

Ans. $(-0.347, 0.707)$.

16. At what points of the curve $y = x^3$ is the curvature greatest?

17. Plot the parabola $x^2 = 4y$ accurately, on a large scale, in the interval from $x = -\frac{1}{2}$ to $x = \frac{3}{2}$, and draw the osculating circles at the points $x = 0$, $x = \frac{1}{2}$, $x = 1$.

CHAPTER VIII

APPLICATIONS OF THE DERIVATIVE IN MECHANICS

55. Velocity and acceleration in rectilinear motion. If a point P moving in a straight line describes equal spaces in equal times, its motion is said to be *uniform*. Its distance x from the starting point O is evidently proportional to the time:

$$x = v_0 t.$$

The constant factor v_0 is called the *velocity* of the moving point; it is equal to the space passed over per unit time.

If the motion is not uniform, we introduce the idea of *velocity at a point* or *instant*. Suppose that a distance Δx including the point P is described in time Δt: then the quotient $\frac{\Delta x}{\Delta t}$ is the *average velocity* during that interval of time. If now Δt approaches 0 in such a way that P always remains in Δx, the quotient $\frac{\Delta x}{\Delta t}$ approaches a limit which is called the *velocity at the point P*. This limit is of course the derivative of x with respect to t:

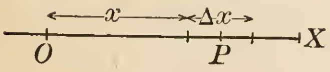


FIG. 30

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}.$$

Thus the velocity is the time-rate of change of space described (cf. § 6).

The rate of change of the velocity is called the *acceleration*:

$$j = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

If the acceleration j is constant, the motion is said to be *uniformly accelerated.* An important case of uniformly accelerated motion is that of a body falling toward the earth from a point near the earth's surface, all resistances being neglected. The attraction of the earth gives the body an acceleration g, called the *acceleration of gravity*, equal to 32 ft. per second per second approximately.

In any problem, it is instructive to draw the graphs of x, v, and j as functions of t. In doing this, it should be remembered that the graph of v is the first derived curve (§ 35), the graph of j is the second derived curve, corresponding to the graph of x.

EXERCISES *

1. A stone is thrown upward with a velocity of 64 ft. per second. The distance from the starting point at the time t (in seconds) is

$$y = 16\,t^2 - 64\,t,$$

the positive sense being downward. Find the velocity and the acceleration. How high will the stone rise and for how long a time? Where is the stone and what is its velocity after 5 seconds of motion? What distance is covered in the sixth second?

2. In Ex. 1, draw the graphs of y, v, and j.

3. A particle slides down an inclined plane. The distance from the starting point at any time t is

$$x = 4\,t^2 - 20\,t.$$

Discuss the motion.

4. A point moves according to the law $x = 5 \cos 2\,t$. Discuss the motion. Draw the graphs of x, v, and j.

5. A point moves according to the law $x = 32\,(1 - e^{-t})$. Discuss the motion.

6. A point moves according to the law $x = \log\,(1 + 2\,t)$. Discuss the motion. Draw the graphs of x, v, and j.

7. A point moves according to the law $x = e^{-t} \sin 2\,t$. Discuss the motion.

* The types of motion considered here will be discussed more in detail in Chapter XXVII.

8. The positions of a point at the ends of successive seconds are observed as follows:

t	0	1	2	3	4	5	6	7	8
x	0	-1	$-\frac{4}{3}$	-1	0	$\frac{5}{3}$	4	7	$\frac{32}{3}$

Draw the graphs of v and j, and find an approximate expression for v and j in terms of t.

56. Vectors. A right line segment of definite *length*, *direction*, and *sense* is called a *vector*. Vectors are of great importance in physics because they can be used to represent velocities, accelerations, forces, and other fundamental quantities.

The *resultant* of two vectors AB, AC is the diagonal AD of the parallelogram having AB, AC as adjacent sides.

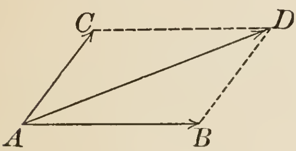


FIG. 31

Two forces acting on the same particle are equivalent to a single force, their resultant; similarly for other vectors. This is the *parallelogram law.* The operation of finding the resultant by the parallelogram law is called *geometric addition.*

The original vectors AB, AC are called *components* of AD. It is evident that any vector may be resolved into components in an infinite number of ways.

57. Velocity in curvilinear motion. If a moving point describes a plane curve, its coördinates are functions of the time:

$$x = \phi(t),\ y = \psi(t).$$

The distance s passed over along the curve is also a function of the time.

The velocity at any point P is defined as the vector, laid off on the tangent to the path from P, of magnitude

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}.$$

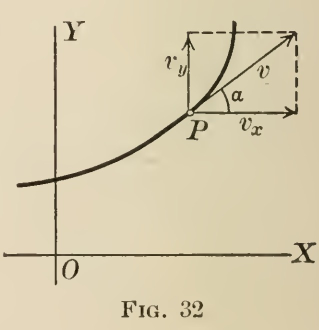


FIG. 32

The components of the velocity parallel to the coördinate axes are

$$v_x = v \cos \alpha, \ v_y = v \sin \alpha,$$

where α is the angle between OX and the tangent at P. By § 52,

$$v \cos \alpha = \frac{ds}{dt} \cdot \frac{dx}{ds} = \frac{dx}{dt}, \ v \sin \alpha = \frac{ds}{dt} \cdot \frac{dy}{ds} = \frac{dy}{dt},$$

so that

$$v_x = \frac{dx}{dt}, \ v_y = \frac{dy}{dt}.$$

Thus the rectangular components of the velocity of P are the velocities of the projections P_x and P_y of P on the coördinate axes.

By § 56, the total velocity v is

$$v = \sqrt{v_x^2 + v_y^2},$$

inclined to the x-axis at an angle

$$\alpha = \arctan \frac{v_y}{v_x}.$$

The equations

$$x = \phi(t), \ y = \psi(t)$$

may be regarded as parametric equations (§ 51) of the path in terms of the parameter t. The cartesian equation may be obtained by eliminating the parameter.

58. Rotation. If a point moves in a circle at a uniform rate, so that equal angles are swept out in equal times, the angle θ swept out by the radius vector in the time t is

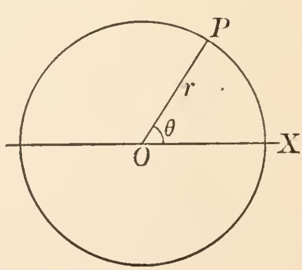


FIG. 33

$$\theta = \omega_0 t.$$

The constant ω_0 is called the *angular velocity.*

If the motion is not uniform, we are led as in § 55 to define the *angular velocity at a particular instant* as

$$\omega = \frac{d\theta}{dt}.$$

Since $ds = r\,d\theta$, where r is the radius of the circle, it follows that the linear velocity $v = \frac{ds}{dt}$ and the angular velocity ω are connected by the relation

$$v = \omega r.$$

The rate of change of the angular velocity is called the *angular acceleration*, and is denoted by α:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

EXERCISES

1. A man can row a boat 5 mi. per hour. If he pulls at right angles to the course of a river 2 mi. wide having a current of 3 mi. per hour, where and when will he reach the opposite shore?

2. In Ex. 1, if the man wishes to land directly opposite his starting point, in what direction must he row and how long will it take him to cross? *Ans.* 30 min.

3. A steamship is moving at the rate of 12 mi. per hour. A man walks across the deck at right angles to the ship's course, at the rate of 5 mi. per hour. If the deck is 40 ft. wide, how far is he finally from his starting point?

4. If a point moves so that

$$x = a\cos t,\ y = a\sin t,$$

find the total velocity in magnitude and direction at the time t. What is the path described?

5. Find the path and discuss the motion of a point whose coordinates are

$$x = 3t,\ y = t - 7.$$

6. The equations of the path of a moving body in terms of the time are

$$x = 20t,\ \ y = 16t^2.$$

Find the position of the body, its distance from the starting point, and the magnitude and direction of the velocity when $t = 2$.

7. A flywheel 2 ft. in diameter makes 100 revolutions per minute. Find its angular velocity in radians per second, and the linear velocity of a point on the rim. What constant angular retardation (negative acceleration) would bring it to rest in 10 seconds?

8. A point moves in a circle in such a way that

$$\theta = 4\,t^2 - 3\,t.$$

Find ω and α, and draw the graphs of θ, ω, and α as functions of t.

9. Find the angular velocity in Ex. 4.

10. In Ex. 8, find v_x and v_y when $t = 1$ if the radius of the circle is 10 ft. *Ans.* $v_x = -42.1$, $v_y = 27.0$ ft. per sec.

59. Acceleration in curvilinear motion. Suppose the velocity of the moving point P at the time t is v, at the time $t + \Delta t$ is $v' = v + \Delta v$, where Δv is the *vector* which, geometrically added to v, produces v'. If v and v' be laid off from a common origin O, the third side VV' of the triangle (Fig. 35) is evidently Δv. Now as Δt approaches 0, Δv does likewise; but in general the ratio $\dfrac{\Delta v}{\Delta t}$ approaches a definite limit, and the direction of Δv approaches a definite limiting direction.

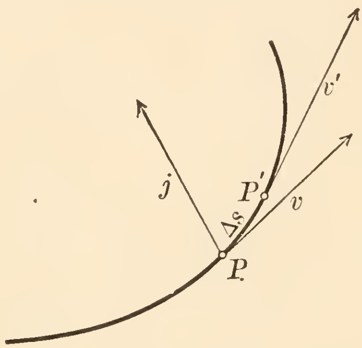


FIG. 34

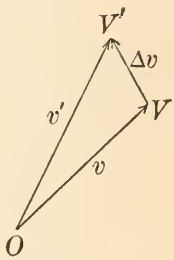


FIG. 35

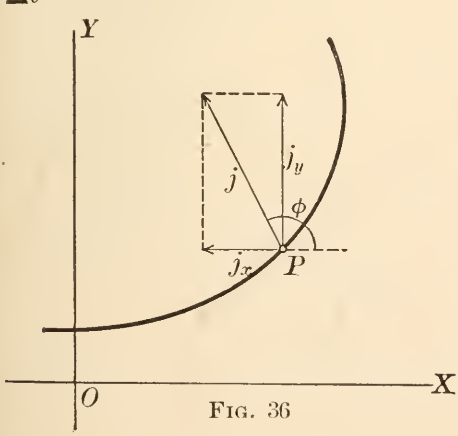


FIG. 36

The vector of length

$$j = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t},$$

laid off in the limiting direction of Δv, is called the *acceleration* of P at the time t. It is the so-called *geometric derivative*, or *vector derivative*, of v with respect to t.

To find an expression for j in terms of the coördinates of P, we may resolve j into components j_x and j_y parallel to the coördinate axes. Denoting by ϕ' the angle between Δv and the x-axis, let us project the triangle OVV' on OX:

$$\Delta v \cos \phi' = v_x' - v_x = \Delta v_x.$$

Dividing by Δt, we get

$$\frac{\Delta v}{\Delta t} \cos \phi' = \frac{\Delta v_x}{\Delta t},$$

whence, in the limit when Δt approaches 0,

$$j \cos \phi = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},$$

where ϕ, the limiting value of ϕ', is the angle between j and the x-axis. Similarly

$$j \sin \phi = \frac{dv_y}{dt} = \frac{d^2x}{dt^2}.$$

Thus

$$j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},$$

$$j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}.$$

The total acceleration j is

$$j = \sqrt{j_x^2 + j_y^2},$$

inclined to the x-axis at an angle

$$\phi = \arctan \frac{j_y}{j_x}.$$

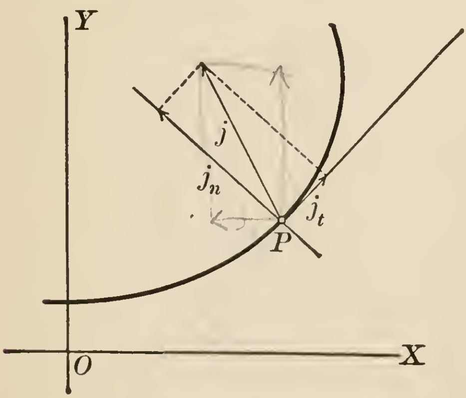


Fig. 37

It is often more convenient to resolve j into components j_t and j_n along the tangent and the normal to the curve at P. These components can be found directly from the definition of j; for variety, however, we will find them by projecting the components j_x and j_y on the tangent and normal.

If the tangent at P makes an angle α with the x-axis, then

$$\begin{aligned} \dot{j}_t &= \dot{j}_x \cos\alpha + \dot{j}_y \sin\alpha \\ &= \frac{dv_x}{dt}\cdot\frac{dx}{ds} + \frac{dv_y}{dt}\cdot\frac{dy}{ds} \\ &= \frac{\frac{dv_x}{dt}\cdot\frac{dx}{dt} + \frac{dv_y}{dt}\cdot\frac{dy}{dt}}{\frac{ds}{dt}} \\ &= \frac{v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt}}{v}. \end{aligned}$$

But, differentiating the equation

$$v = \sqrt{v_x^2 + v_y^2}$$

with respect to t, we find

$$\frac{dv}{dt} = \frac{2\,v_x\frac{dv_x}{dt} + 2\,v_y\frac{dv_y}{dt}}{2\sqrt{v_x^2 + v_y^2}} = \frac{v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt}}{v}.$$

Hence,

$$\dot{j}_t = \frac{dv}{dt}.$$

Again,

$$\begin{aligned} \dot{j}_n &= \dot{j}_y \cos\alpha - \dot{j}_x \sin\alpha \\ &= \frac{dv_y}{dt}\cdot\frac{dx}{ds} - \frac{dv_x}{dt}\cdot\frac{dy}{ds}. \end{aligned}$$

By § 54, the radius of curvature of the path is

$$\rho = \frac{ds}{d\alpha},$$

so that

$$\begin{aligned} \dot{j}_n &= \frac{1}{\rho}\left(\frac{dv_y}{dt}\cdot\frac{dx}{d\alpha} - \frac{dv_x}{dt}\cdot\frac{dy}{d\alpha}\right) \\ &= \frac{1}{\rho}\left(v_x\frac{dv_y}{d\alpha} - v_y\frac{dv_x}{d\alpha}\right). \end{aligned}$$

Now, differentiating both sides of the equation (§ 57)

$$\alpha = \arctan\frac{v_y}{v_x}$$

with respect to α, we find

$$1 = \frac{1}{1 + \frac{v_y^2}{v_x^2}} \cdot \frac{v_x \frac{dv_y}{d\alpha} - v_y \frac{dv_x}{d\alpha}}{v_x^2}$$

$$= \frac{v_x \frac{dv_y}{d\alpha} - v_y \frac{dv_x}{d\alpha}}{v^2},$$

so that

$$v_x \frac{dv_y}{d\alpha} - v_y \frac{dv_x}{d\alpha} = v^2.$$

Hence,

$$j_n = \frac{v^2}{\rho}.$$

Thus the acceleration j is equal to $\frac{dv}{dt}$ *only in the case of rectilinear motion;* in curvilinear motion $\frac{dv}{dt}$ represents the *tangential component* of the acceleration.

EXERCISES

1. In Ex. 4, p. 84, find j_x, j_y, j_t, j_n. Find j, (*a*) as the resultant of j_x and j_y, (*b*) as the resultant of j_t and j_n.

2. In Ex. 6, p. 84, find the total acceleration in magnitude and direction when $t = 2$.

3. Show that in uniform circular motion the acceleration is directed toward the center and is proportional to the radius of the circle.

4. In Ex. 8, p. 85, find j_t and j_n, if the radius of the circle is 10 ft.

5. A point describes the parabola $y^2 = 4x$ with a constant velocity of 6 ft. per second. Find v_x, v_y, j_x, and j_y at the point (1, 2).

60. Time-rates. The question of determining time-rates arises in a variety of problems beside those that have been considered.

If in any problem the quantity whose rate of change is to be found can be expressed directly as a function of the time, the result can of course be obtained at once by differentiating with respect to the time. Frequently, however, the problem is solved by expressing the quantity in question in terms of another quantity whose rate of change is known, and then differentiating the equation connecting them. The method is best explained by an

Example: Water is flowing into a conical reservoir 20 ft. deep and 10 ft. across the top, at the rate of 15 cu. ft. per minute. Find how fast the surface is rising when the water is 8 ft. deep.

The volume of water is

$$V = \tfrac{1}{3}\pi r^2 h.$$

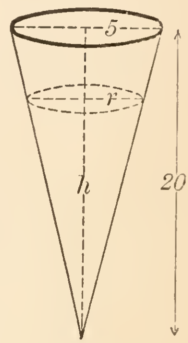


FIG. 38

By similar triangles,

$$\frac{r}{h} = \frac{5}{20}, \quad r = \frac{1}{4}h.$$

Hence

$$V = \frac{\pi h^3}{48},$$

$$dV = \frac{\pi h^2}{16}\,dh,$$

$$\frac{dV}{dt} = \frac{\pi h^2}{16}\frac{dh}{dt}.$$

But we have given that

$$\frac{dV}{dt} = 15,$$

so that

$$\frac{\pi h^2}{16}\frac{dh}{dt} = 15, \quad \frac{dh}{dt} = \frac{240}{\pi h^2}.$$

When $h = 8$,

$$\frac{dh}{dt} = \frac{15}{4\pi} = 1.19 \text{ ft. per minute.}$$

EXERCISE

1. Water is flowing into a cylindrical tank of radius 5 ft. at the rate of 20 gallons per second. Find how fast the surface is rising.

2. In the example of § 60, find how fast the water is flowing in if, when the water is 5 ft. deep, the surface is rising 2 ft. per minute.

3. Water is flowing into an inverted conical tank 32 ft. deep and 12 ft. across at the bottom, at the rate of 4 cu. ft. per second. How fast is the surface rising?

4. Two trains start from the same point at the same time, one going due east at the rate of 40 mi. per hour, the other north 60 mi. per hour. At what rate do they separate? *Ans.* 72.1 mi. per hour.

5. Two railroad tracks intersect at right angles. At noon there is a train on each track approaching the crossing at 40 mi. per hour, one being 100 mi., the other 200 mi. distant. Find (*a*) how fast they are approaching each other, (*b*) when they will be the nearest together, and (*c*) what will be their minimum distance apart.

Ans. (*b*) 3 : 45 P.M.; (*c*) 70.7 mi.

6. A ladder 20 ft. long leans against a vertical wall. If the lower end is being moved away from the wall at the rate of 2 ft. per second, how fast is the top descending when the lower end is 12 ft. from the wall?

7. A man 6 ft. tall walks away from a lamp-post 10 ft. high at the rate of 4 mi. per hour. (*a*) How fast is the further end of his shadow moving? (*b*) How fast is the shadow lengthening?

8. A man on a wharf 20 ft. above the water pulls in a rope, to which a boat is attached, at the rate of 4 ft. per second. At what rate is the boat approaching the shore when there is 25 ft. of rope out?

9. A kite is 120 ft. high, with 130 ft. of cord out. If the kite moves horizontally 4 mi. per hour directly away from the boy flying it, how fast is the cord being paid out?

10. A stone dropped into a pond sends out a series of concentric ripples. If the radius of the outer ripple increases steadily at the rate of 6 ft. per second, how fast is the disturbed area increasing at the end of 2 seconds? *Ans.* 452 sq. ft. per sec.

11. The path traced by a moving point is the parabola $y = x^2 + 2x + 3$. If $v_x = 3$ ft. per second, find v_y and the total velocity v.

Ans. $v_y = 6x + 6$.

12. A point moves on the hyperbola $x^2 - y^2 = 144$ with $v_x = 15$ ft. per second. Find v at the point (13, 5).

+ **13.** As a man walks across a bridge at the rate of 5 ft. per second, a boat passes directly beneath him at 10 ft. per second. If the bridge is 30 ft. above the water, how fast are the man and the boat separating 3 seconds later? *Ans.* $8\frac{1}{3}$ ft. per sec.

14. A light is placed on the ground 30 ft. from a building. A man 6 ft. tall walks from the light toward the building, at the rate of 5 ft. per second. Find the rate at which his shadow on the wall is shortening when he is 15 ft. from the building. *Ans.* 4 ft. per sec.

15. Solve Ex. 14 if the light is 8 ft. above the ground.

16. An elevated train on a track 30 ft. above the ground crosses a street at the rate of 20 ft. per second, at the instant that an automobile, approaching at the rate of 30 ft. per second, is 40 ft. up the street. Find how fast the train and the automobile are separating 2 seconds later.

17. In Ex. 16, find when the train and the automobile are nearest together. *Ans.* $\frac{12}{13}$ sec.

18. A light stands 60 ft. from a building. A man walks along a path 20 ft. from the building, at the rate of 5 ft. per second. How fast does his shadow move on the building?

19. An arc light hangs at a height of 30 ft. above the center of a street 60 ft. wide. A man 6 ft. tall walks along the sidewalk at the rate of 4 ft. per second. How fast is his shadow lengthening when he is 40 ft. up the street? *Ans.* 0.8 ft. per sec.

20. In Ex. 19, how fast is the tip of the shadow moving?

21. A light stands 30 ft. from a house, and 20 ft. from the path leading from the house to the street. A man walks along the path at 5 ft. per second. How fast does his shadow move on the wall when he is 20 ft. from the house?

CHAPTER IX

CURVE TRACING IN CARTESIAN COÖRDINATES

I. Algebraic Curves

61. Introduction. In Chapter IV we learned how to trace simple curves whose equations are given in the explicit form

$$y = f(x),$$

and for which y, y', and y'' are one-valued and continuous. In the present chapter we shall attempt a more general treatment of the subject of curve tracing.

In §§ 62–67 we confine our attention to *algebraic* curves — *i.e.* curves for which the ordinate y is an algebraic function of x.

62. Singular points. If y is defined implicitly as a function of x by the equation

$$F(x, y) = 0,$$

the derivative in general takes the form of a fraction whose numerator and denominator are functions of both x and y: say

$$y' = \frac{A(x, y)}{B(x, y)}.$$

If $A(x, y)$ and $B(x, y)$ both vanish at the point $P : (x, y)$ on the curve, the slope at that point assumes the indeterminate form $\frac{0}{0}$. A point at which the derivative takes the form $\frac{0}{0}$ is called a *singular point.*

To find the singular points of a curve we must therefore find the values of x and y that satisfy the three equations

$$F(x, y) = 0,$$
$$A(x, y) = 0,$$
$$B(x, y) = 0.$$

As we have but two unknowns x and y to satisfy three equations, it appears that a curve will have singular points only under certain conditions.

It will be sufficient to consider an algebraic curve having a singular point at the origin. If a singularity occurs at any other point (h, k), the origin may be transferred to that point by the substitutions

$$x = x_1 + h,$$
$$y = y_1 + k.$$

63. Determination of tangents by inspection. Let the equation of the curve be written in the form

$$F(x, y) = a_0 + b_0 x + b_1 y + c_0 x^2 + c_1 xy + c_2 y^2 + \cdots + g_n y^n = 0.$$

Differentiating, we find

$$(b_0 + 2c_0 x + c_1 y + \cdots)dx + (b_1 + c_1 x + 2c_2 y + \cdots)dy = 0,$$

$$\frac{dy}{dx} = -\frac{b_0 + 2c_0 x + c_1 y + \cdots}{b_1 + c_1 x + 2c_2 y + \cdots}.$$

The origin is on the curve only if $a_0 = 0$. In that case the equation of the tangent at (0, 0) is found by the usual method (§ 27) to be

$$b_0 x + b_1 y = 0,$$

provided b_0 and b_1 are not both 0; *i.e.* the equation of the tangent at the origin may be found by simply equating to 0 the group of terms of the first degree.

In case a_0, b_0 and b_1 are all 0, the origin is on the curve and the derivative is indeterminate at that point; hence the origin is a singular point. In this case the equation of the curve evidently contains no terms of lower degree than the second.

For convenience let us put*

$$c_0x^2 + c_1xy + c_2y^2 = c_2\left(y^2 + \frac{c_1}{c_2}xy + \frac{c_0}{c_2}x^2\right)$$
$$= c_2(y - m_1x)(y - m_2x).$$

Then

$$F(x, y) = c_2(y - m_1x)(y - m_2x) + d_0x^3 + \cdots = 0.$$

The abscissas of the points of intersection of the line

$$y = mx$$

with this curve are given by the equation

$$c_2x^2(m - m_1)(m - m_2) + x^3(d_0 + \cdots) + \cdots = 0.$$

Two roots of this equation are 0: *every* line $y = mx$ intersects the curve in two coincident points at the origin. But the above equation in x also shows that if we let m approach either m_1 or m_2, the coefficient of x^2 approaches 0; *i.e.* a *third* point of intersection of the curve with the line $y = mx$ approaches the origin, and the lines

$$y = m_1x, \quad y = m_2x$$

are both tangent to the curve at the singular point. These lines may of course be real and distinct, real and coincident, or imaginary.

Since

$$c_2(y - m_1x)(y - m_2x) = c_0x^2 + c_1xy + c_2y^2,$$

we see that the equations of the two tangents are obtained by *equating the second degree terms to* 0, and factoring the left member of the resulting equation.

The argument we have used can be extended to show that if $F(x, y)$ has no terms of degree lower than the kth, any line through the origin meets the curve there in k points, and the k tangents to the curve at the origin are obtained by equating the group of terms of lowest degree to 0.

* The argument is readily modified to take care of the case $c_2 = 0$.

64. Kinds of singular points. A point at which there are two tangents (whether distinct, coincident, or imaginary) is called a *double point;* one at which there are three tangents is a *triple point;* etc. It follows from § 63 that the origin is a double point if the equation $F(x, y) = 0$ has terms of the second, but none of lower, degree; a triple point if the equation has terms of the third but none of lower degree; etc.

If the tangents at a double point are real and different, the point is called a *node:* two branches of the curve cross each other, as in Fig. 39. If the tangents are imaginary, the point is called an *isolated* or *conjugate* point: there are no other points of the curve in its vicinity. Such a point is P in Fig. 40.

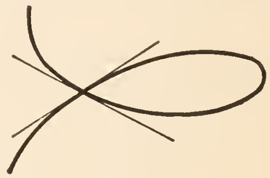


Fig. 39

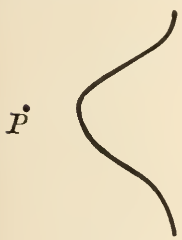


Fig. 40

If the tangents are real and coincident, there are several possibilities. The simplest singularity in this case is the *cusp* of the *first kind:* two branches of the curve touch each other, coming up on opposite sides of the cuspidal tangent, as in Fig. 41. At a cusp of the *second kind* the two branches lie on the same side of the tangent, as in Fig. 42. Frequently the point is a *double cusp*, or *point of osculation*,

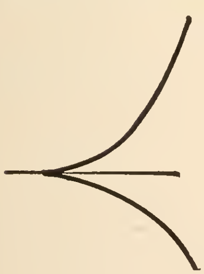


Fig. 41

Fig. 42

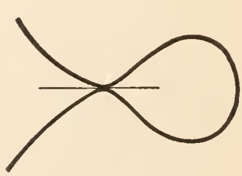


Fig. 43

the commonest form of which is shown in Fig. 43. And in some cases the point may be an isolated point.

Example: Examine the curve $y^2 = x^3 - x^2$ for singular points.

Since there are no terms of lower than the second degree, the curve has a singular point at the origin. The tangents at that point are given by the equation

$$y^2 = -x^2;$$

i.e. they are the lines

$$y = ix,$$
$$y = -ix.$$

These lines are imaginary, and the origin is an isolated point.

There are no other singular points. For,

$$y' = \frac{3x^2 - 2x}{2y},$$

and the coördinates of the singular point must satisfy the three equations

$$3x^2 - 2x = 0,$$
$$2y = 0,$$
$$y^2 = x^3 - x^2.$$

The first two equations are satisfied by the coördinates $(0, 0)$, $(\frac{2}{3}, 0)$, but the second pair do not satisfy the last equation. (See also Ex. 20 below.)

EXERCISES

Show that the origin is a singular point for each of the following curves, write the equations of the tangents there, and determine the nature of the singularity.

1. The folium $x^3 + y^3 = 3axy$. *Ans.* Node.
2. $x^2y^2 = a^2(x^2 + y^2)$.
3. $y = \dfrac{x^3}{x + y}$.
4. The cissoid $y^2 = \dfrac{x^3}{2a - x}$. *Ans.* Cusp of the first kind.
5. $y^3 = 2ax^2 - x^3$.
6. $y^2(x^2 + y^2) = a^2x^2$. *Ans.* Double cusp.
7. $y^2(x^2 - a^2) = x^4$. *Ans.* Isolated point.

8. $(y - x^2)^2 = x^5$. *Ans.* Cusp of the second kind.

9. $y^2 = x^4 + x^5$.

10. $x^3 - xy^2 = y^4$. *Ans.* Triple point.

11. The lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

12. Show that the conchoid $x^2y^2 = (a + y)^2(b^2 - y^2)$ has a node at $(0, -a)$ if $b > a$.

13. Show that the curve $a(y - x)^2 = (x - a)^3$ has a cusp at (a, a).

Find the singular points of the following curves.

14. $y^2 = x(x - 1)^2$.

15. $y^2 = x(2x + 1)^2$.

16. $y^2 = x(x^2 - 1)$. *Ans.* None.

17. $ay^3 = x^2(x - a)^2$. *Ans.* Cusps at $(0, 0)$, $(a, 0)$.

18. $y^3 + y^2 = (x^2 - 1)^2$.

19. $xy^2 + x^2y = a^3$.

20. Prove that a cubic curve cannot have more than one singular point.

21. Prove that the graph of a one-valued algebraic function

$$y = f(x)$$

cannot have any singular points.

65. Asymptotes. As the point of contact of a tangent to a curve recedes indefinitely from the origin, the tangent may or may not approach a limiting position. If it does, the line approached is called an *asymptote*.

For example, the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has the lines

$$\frac{x}{a} \pm \frac{y}{b} = 0$$

as asymptotes. On the other hand, the parabola has no asymptotes, since as the point of tangency recedes the tangent does not approach any limiting position.

Although general methods for finding asymptotes exist, they are frequently difficult to apply. The following tests are sufficient in ordinary cases.

(*a*) To find the conditions that must be satisfied in order that the line

$$y = mx + k$$

shall be an asymptote to the algebraic curve

$$(1) \qquad F(x, y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n + b_0x^{n-1} + b_1x^{n-2}y + \cdots = 0,$$

let us substitute $mx + k$ for y in the equation of the curve. This gives an equation of the nth degree in x whose roots are the abscissas of the n (real or imaginary) points of intersection of the line with the curve.

It is shown in algebra that one root of the equation

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_n = 0$$

becomes infinite if A_0 approaches 0; two roots become infinite if both A_0 and A_1 approach 0; etc. Hence if we equate to 0 the coefficients of x^n and x^{n-1}, we shall in general determine values of m and k such that the line

$$y = mx + k$$

will intersect the curve in two infinitely distant points. Such a line is in general an asymptote. Of course if the coefficients of x^n and x^{n-1} cannot both vanish, there are no asymptotes (except such as may be given by (*b*) below).

Example: (*a*) Find the asymptotes of the hyperbola

$$x^2 - y^2 - 2x - 2y + 1 = 0.$$

Substituting $y = mx + k$, we get

$$x^2 - (mx + k)^2 - 2x - 2(mx + k) + 1 = 0,$$

or

$$(1 - m^2)x^2 + (-2mk - 2m - 2)x + \cdots = 0.$$

Equating to 0 the coefficients of the two highest powers of x, we find

$$1 - m^2 = 0,$$
$$-2mk - 2m - 2 = 0.$$

Whence

$$m = 1, \ k = -2,$$
$$m = -1, \ k = 0,$$

and the asymptotes are

$$y = x - 2,$$
$$y = -x.$$

The curve is shown in Fig. 44.

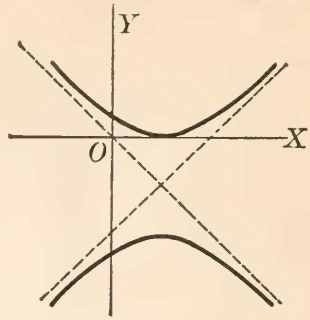


FIG. 44

(*b*) Asymptotes parallel to the y-axis are not given by test (a), since their equations cannot be written in the slope form.

Let us arrange the equation of the curve in descending powers of y:

$$F(x, y) = \alpha_0 y^n + (\alpha_1 x + \beta_1) y^{n-1} + (\alpha_2 x^2 + \beta_2 x + \gamma_2) y^{n-2} + \cdots = 0.$$

If the term in y^n is present, every value of x gives n finite (real or imaginary) values of y, and no line $x = k$ can intersect the curve in infinitely distant points. But if $\alpha_0 = 0$, then *every* line $x = k$ intersects the curve in one infinitely distant point. If now the coefficient of y^{n-1} involves x (*i.e.* if $\alpha_1 \neq 0$), that coefficient equated to 0 gives us the equation of a line parallel to the y-axis which intersects the curve in *two* infinitely distant points, and is an asymptote.

This result can be extended to the case when the coefficient of the highest power of y is a polynomial of higher degree in x. By equating this polynomial to 0 we find an asymptote parallel to OY corresponding to each real linear factor of the polynomial.

Similarly, asymptotes parallel to OX may be found by equating to 0 the coefficient of the highest power of x. Such asymptotes are detected in general by test (a), but it may be easier to find them by the present method.

Example: (*b*) Test the curve $(x^2 - 1) y^2 = x^3$ for asymptotes.

Equating to 0 the coefficient of the highest power of y, we find the lines

$$x^2 - 1 = 0, \text{ i.e. } x = \pm 1$$

as asymptotes parallel to OY. The coefficient of the highest power of x cannot be equated to 0, so there are no asymptotes parallel to OX.

To test for asymptotes oblique to the axes, put $y = mx + k$:

$$(x^2 - 1)(m^2x^2 + 2\,mkx + k^2) = x^3.$$

Equating to 0 the coefficients of the two highest powers of x, we find

$$m^2 = 0,$$
$$2\,mk = 1.$$

These equations are incompatible; hence there are no oblique asymptotes.

EXERCISES

Test the following curves for asymptotes.

1. $xy + x = 5$.

2. $x^3 + y^3 = 1$. *Ans.* $x + y = 0$.

3. $y^3 = ax^2 + x^3$.

4. The cissoid $y^2 = \dfrac{x^3}{2\,a - x}$.

5. $x^2 + 7\,xy + 12\,y^2 + x + 4\,y - 16 = 0$. *Ans.* $x + 3\,y + 1 = 0$, $x + 4\,y = 0$.

6. $x^2 - xy + y^2 + 5\,x = 0$. *Ans.* None.

7. $x^2y^2 = a^2(x^2 + y^2)$.

8. The folium $x^3 + y^3 = 3\,axy$.

9. $x^2y^2 + 4\,x^2 + y^2 + x = 0$.

10. $x^2y + xy^2 = a^3$.

11. $x^3 - 4\,xy^2 - 3\,x^2 + 12\,xy - 12\,y^2 + 8\,x + 2\,y + 4 = 0$. *Ans.* $x + 3 = 0$, $x - 2\,y = 0$, $x + 2\,y = 6$.

12. $xy^2 - ay^2 = x^3 + ax^2 + a^3$. *Ans.* $x = a$, $x \pm y + a = 0$.

13. $ay^2 = x^3 + xy^2$.

14. Prove that a parabola has no asymptotes, but that every line parallel to the axis meets the curve in one infinitely distant point.

15. Prove that every line parallel to an asymptote meets the curve in one infinitely distant point.

16. In example (*b*), § 65, prove that every line parallel to the x-axis meets the curve in one infinitely distant point.

17. Show that a curve of the n-th degree cannot have more than n asymptotes.

18. Show that the curve

$$y = P(x),$$

where $P(x)$ is any polynomial in x, has no asymptotes.

66. Exceptional cases. Exceptions may arise to the theory of § 65. For instance, it may happen that the coefficient of x^{n-1} vanishes *identically* for some value of m for which the term in x^n disappears, so that *all* lines having this value of m as their slope meet the curve in two points at infinity. In this case there are in general two or more parallel asymptotes having the given slope, and the values of k are determined by equating to 0 the coefficient of the highest power of x that does not disappear identically.

The exceptional cases are rare and unimportant in elementary work, and a fuller discussion of them is unnecessary.

EXERCISES

1. Show that the curve

$$(x+y)^2(x^2+xy+y^2) = a^2y^2 + a^3(x-y)$$

has the pair of parallel asymptotes $x + y = \pm a$.

2. Show that every line parallel to the x-axis meets the curve

$$y^4 = x^2 + x$$

in two infinitely distant points, but that the curve has no asymptotes.

67. General directions for tracing algebraic curves. The following questions should be answered as fully as possible before trying to trace an algebraic curve.

(1) *Is the curve symmetric with respect to the coördinate axes?* (It is symmetric with respect to OY if the equation is unchanged when x is changed to $-x$; etc.)

(2) *Where does it intersect the axes?*

(3) *Has it any asymptotes?* If so, locate each of the points where the curve intersects its asymptotes.

(4) *Is it possible to determine certain regions of the plane within which the curve must lie?*

(5) *Has the curve any singular points?* If so, determine the tangents at each point, and the nature of the singularity; draw the tangents if they are real.

(6) *Has it any maximum and minimum points?*

The above is only a general outline of the process to be followed; other steps will often suggest themselves. In many cases the points of inflection should be found and the inflectional tangents drawn, but this is not worth while if the second derivative is complicated. Translation or rotation of axes is occasionally useful. The elementary method of tracing the curve by plotting points is too laborious to be used extensively, but it is frequently advisable to plot a few points, merely as a check on the analysis.

Examples: (*a*) Trace the curve $y^3 = 3\,ax^2 - x^3$.

(1) The curve is not symmetric with respect to either axis.

(2) When $x = 0$, $y = 0$; when $y = 0$, $x = 0$ or $3\,a$.

(3) By § 65, the line

$$y = a - x$$

is an asymptote. Substituting $a - x$ for y in the equation of the curve, we find

$$a^3 - 3\,a^2x + 3\,ax^2 - x^3 = 3\,ax^2 - x^3.$$

The highest powers of x drop out, as they should, and we find that the curve crosses its asymptote at $x = \frac{a}{3}$.

(4) Writing the equation in the form

$$y^3 = x^2(3\,a - x),$$

we see that $y > 0$ if $x < 3\,a$, $y < 0$ if $x > 3\,a$. Hence the curve is above the x-axis at the left of $x = 3\,a$, below at the right of that point.

(5) Since there are no terms of lower than the second degree, the origin is a singular point. The tangents are given by the equation

$$3\,ax^2 = 0:$$

they coincide in the y-axis. Since, by (4), the curve near the origin cannot go below OX, the point is a cusp. For small values of x on either side of 0, y is real, hence there is a branch on each side of OY, and the origin is a cusp of the first kind.

Since the curve is a cubic, there can be no other singular points (Ex. 20, p. 97).

(6) The derivative is

$$\frac{dy}{dx} = \frac{6\,ax - 3\,x^2}{3\,y^2}.$$

The numerator vanishes when $x = 0$ or $2\,a$. Rejecting the value $x = 0$, which gives the singular point, we have $x = 2\,a$ as the only critical value. It will appear presently that the point $(2\,a, \sqrt[3]{4}\,a)$ is a maximum point.

To trace the curve, let us begin at the extreme left. In that region, the curve must be just below its asymptote, since it has to pass through the origin and can cross the asymptote only at $x = \frac{a}{3}$. It comes down to the origin tangent to the y-axis, turns back on the other side, and crosses the asymptote at $\left(\frac{a}{3}, \frac{2\,a}{3}\right)$. It is now clear that the critical value $x = 2\,a$ corresponds to a maximum point. Returning from the maximum, the curve crosses OX at $(3\,a, 0)$ and again approaches the asymptote.

The curve is shown in the figure.

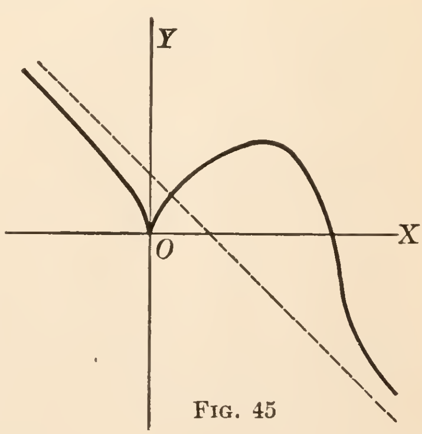


FIG. 45

(*b*) Trace the curve $a^2y^2 = x^2(a^2 - x^2)$.

(1) The curve is symmetric with respect to both axes.

(2) When $x = 0$, $y = 0$; when $y = 0$, $x = 0, \pm a$.

(3) There are no asymptotes.

(4) When x is numerically greater than a, y^2 is negative and y is imaginary. Hence, the curve lies entirely between the lines $x = \pm a$.

(5) The origin is a singular point. The tangents are real and different,

$$a^2y^2 = a^2x^2, \ y = \pm x,$$

hence the point is a node. There are no other singular points.

(6) The first derivative is

$$\frac{dy}{dx} = \frac{2\,a^2x - 4\,x^3}{2\,y} = \frac{x(a^2 - 2\,x^2)}{y}.$$

This vanishes when

$$a^2 - 2\,x^2 = 0, \ x = \pm\frac{a}{\sqrt{2}}.$$

Corresponding to each of these values the curve has, on account of its symmetry, a maximum $y = \frac{a}{2}$ and a minimum $y = -\frac{a}{2}$.

The student may draw the curve.

EXERCISES

Trace the following curves.

1. $y = \frac{1-x}{1+x^2}$.
2. $y = x(x-1)^2$.
3. $y = \frac{a^2x}{(x-a)^2}$.
4. $y = \frac{x^2+a^2}{x}$.
5. $y^2 = \frac{4-x}{x}$.
6. $y^2 = \frac{x^2(1+x)}{1-x}$.
7. $y = 2\,x^3 - 9\,x^2 + 12\,x - 3$.
8. $y = x^3 - 3\,x^2 + 6\,x$.
9. $y^2(x^2 + a^2) = a^2x^2$.
10. $y(x^2 - y^2) = x + 1$.
11. $x^3 + y^3 = 3\,axy$.
12. $y^2 = x(x^2 - 1)$.
13. $y = \frac{a^2x}{x^2 - a^2}$.
14. $y^2 = \frac{x}{4 - x^2}$.

15. $y^2(x^2 + y^2) = a^2x^2$.

16. $y^2 = \frac{x^2(x^2 - a^2)}{x^2 + a^2}$.

17. $(y - x)^2 = x^3$.

18. $(x^2 - a^2)y^2 = ax^3$.

19. $y = \frac{(2 - x)^3}{2 - 2x}$.

20. $xy^2 + x^2y = a^3$.

21. $x^2y^2 = a^2(x^2 + y^2)$.

22. $x^2y^2 = (x + a)^2(4a^2 - x^2)$.

23. $(x^2 - y^2)(x - 3y) = x$.

24. $(y - x^2)^2 = x^5$.

25. $y^2 = \frac{1 - x}{1 + x^2}$.

26. $ay^3 = (x^2 - a^2)^2$.

27. $x^3y = a^2(x + a)^2$.

28. $y = \frac{4x}{(x + 1)^2}$.

II. Transcendental Curves

68. Tracing of transcendental curves. In tracing transcendental curves, we follow much the same procedure as in § 67, except in the matter of asymptotes and singular points. While the definitions of §§ 62–65 hold for all curves, the tests there given apply only to algebraic curves.

We shall in this article confine our attention to transcendental curves having no singular points. To find asymptotes, the following rule may be used:

In general, if y becomes infinite as x approaches a definite limit a, the line $x = a$ is an asymptote; if x becomes infinite as y approaches b, the line $y = b$ is an asymptote.

In rare instances, the derivative may behave in such a way that although the conditions of the rule are satisfied, the tangent does not approach any limiting position, and hence there is no asymptote; but the rule holds in all cases that are apt to arise in practice.

Example: Trace the curve $y = xe^x$.

(1) There is no symmetry.

(2) The curve crosses the axes at (0, 0).

(3) As x becomes large and negative, y approaches 0 *;

* This statement is easily made plausible; a strict proof will be given later (§ 140).

hence the negative x-axis is an asymptote. When x is large and positive, y is large and positive.

(4) Since e^x is always positive, y has always the same sign as x: the curve lies in the first and third quadrants.

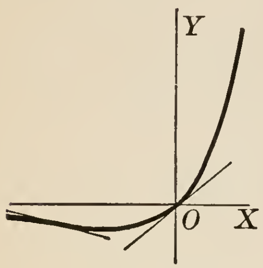


FIG. 46

(5) Since $y' = xe^x + e^x$, the only critical point is $\left(-1, -\frac{1}{e}\right)$. The slope at (0, 0) is 1.

(6) $y'' = xe^x + 2e^x$. There is a point of inflection at $\left(-2, -\frac{2}{e^2}\right)$; the slope of the inflectional tangent is

$$-e^{-2} = -0.14.$$

The curve is shown in Fig. 46.

EXERCISES

Trace the following curves.

1. $y = e^{-x^2}$.
2. $y = xe^{-x^2}$.
3. $y = \tan x$.
4. $y = \sec x$.
5. $y = \frac{\log x}{x}$.
6. $y = e^{x^2}$.
7. $y = \sin^2 x$.
8. $y^2 = \sin x$.
9. $y = \frac{x}{\log x}$.
10. $y^2 = \frac{\log x}{x}$.
11. $y^2 = \log x$.
12. $y = \frac{e^x}{x}$.
13. $y = \frac{1}{\log x}$.
14. $y = e^{-x} \sin x$.
15. $y = x \log x$.
16. Show that

$$\lim_{x \to \infty} \frac{2 + \sin x^2}{x} = 0;$$

then show that the curve

$$y = \frac{2 + \sin x^2}{x}$$

furnishes an exception to the rule of § 68.

69. Curve tracing by composition of ordinates. The curve

$$y = f(x)$$

can be traced very readily if $f(x)$ has the form

$$f(x) = \phi_1(x) + \phi_2(x),$$

where

$$y = \phi_1(x), \ y = \phi_2(x)$$

are curves whose form is easily obtained. We have only to add the ordinates of the two latter curves to obtain the required curve.

Less frequently curves may be conveniently traced by multiplying or dividing ordinates in a similar way.

EXERCISES

Trace the following curves.

1. $y = x + \log x.$

2. $y = x + \frac{1}{x}.$

3. $y = e^x - x.$

4. $y = \sin x + \cos x.$

5. $y = x + \sin x.$

6. $y = \sinh x = \frac{e^x - e^{-x}}{2}.$

7. $y = \frac{\sin x}{x}.$

8. $y = \frac{\cos x}{x}.$

9. The *catenary* is the curve in which a homogeneous cord or chain hangs when suspended from two of its points under its own weight. The equation is

$$y = a \cosh \frac{x}{a} = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

Trace the curve.

70. Graphic solution of equations. The roots of the equation

$$f(x) = 0$$

are the abscissas of the points where the curve $y = f(x)$ crosses the x-axis. Hence if we trace the curve $y = f(x)$ and measure its intercepts on OX, we have a graphic solution of the equation $f(x) = 0$. It is usually best to get the general form of the curve by the methods of the preceding articles, and then plot it carefully, on a large scale, in the neighborhood of each of its x-intersections.

The roots of the equation

(1) $$f(x) = \phi(x)$$

are the abscissas of the points of intersection of the curves

$$y = f(x),$$
$$y = \phi(x).$$

In case these two curves are easily traced, we thus obtain with little labor a graphic solution of (1). This method is frequently preferable to the first one mentioned above.

Such methods may be useful in various ways. If no high degree of approximation is required, the graphical result may be sufficient in itself; it may be used as a rough check on a more accurate result obtained in some other way; it gives a first approximation that may be needed as a starting point for more elaborate methods, or it may suggest some value of the variable which by substitution is found to satisfy the equation exactly.

EXERCISES

Solve the following equations graphically.

1. $x^4 - 3x^3 + 3 = 0.$

2. $3x^4 - 2x^3 - 21x^2 - 4x + 11 = 0.$

3. $x + 10^x = 0.$

4. $x + 2\cos x = 0.$

5. $x + \log_{10} x = 0.$

6. $x + \cos x = 1.$

7. Trace the curve $y = x \sin x$, locating maxima and minima graphically.

8. Solve the equation $x \log x = 1.$

9. A gutter whose cross-section is an arc of a circle is to be made by bending into shape a strip of tin of width 8 inches. Find the radius of the cross-section when the carrying capacity of the gutter is a maximum. *Ans.* 2.55 in.

71. The cycloid. In the remainder of this chapter we consider several special transcendental curves.

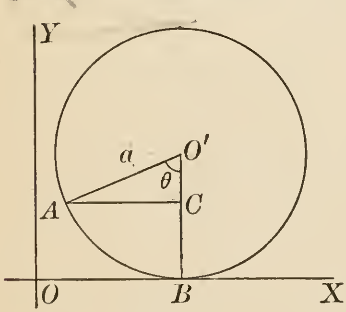


FIG. 47

The path traced by any point A on the rim of a wheel that rolls without slipping along a straight track is called a *cycloid*.

Let the circle of radius a roll along the x-axis, and take the initial position of A as origin. Then, if (x, y) are the coördinates of A,

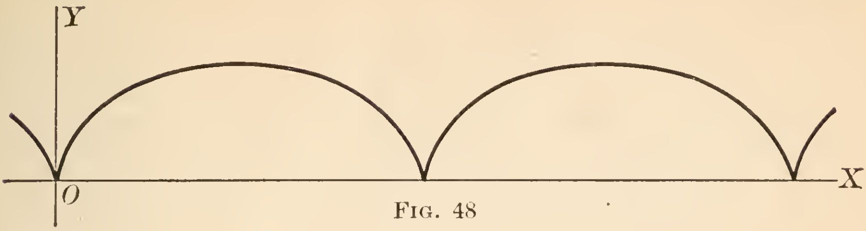


FIG. 48

$$OB = \text{arc } AB = a\theta,$$
$$x = OB - AC = a\theta - a\sin\theta = a(\theta - \sin\theta),$$
$$y = BO' - CO' = a - a\cos\theta = a(1 - \cos\theta).$$

These are the parametric equations of the cycloid in terms of the angle θ through which the circle has rolled. The coördinates of the center of the rolling circle are $(a\theta, a)$. The rate at which the center is advancing is

$$\frac{d}{dt}(a\theta) = a\frac{d\theta}{dt} = a\omega,$$

where ω is the angular velocity.

The curve is shown in Fig. 48.

72. The epicycloid. If a circle rolls without slipping on the outside of a fixed circle, a point on the circumference of the rolling circle generates an *epicycloid*.

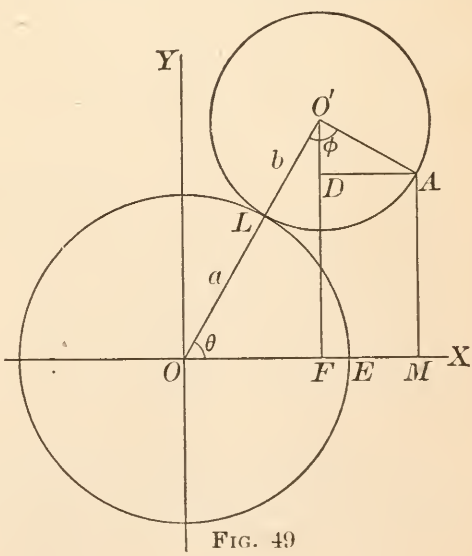


FIG. 49

Let a and b be the radius of the fixed circle and that of the rolling circle respectively, and suppose the point A was originally at E. Then

$$\text{arc } AL = \text{arc } EL,$$

or

$$b\phi = a\theta.$$

The equations of the path of A in terms of the parameter θ may be obtained as follows:

$$x = OM = OF + FM = OF + DA$$
$$= (a+b)\cos\theta + b\sin\left[\phi - \left(\frac{\pi}{2} - \theta\right)\right]$$
$$= (a+b)\cos\theta - b\cos(\theta + \phi)$$
$$= (a+b)\cos\theta - b\cos\left(\theta + \frac{a}{b}\theta\right)$$
$$= (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta;$$
$$y = MA = FD = FO' - DO'$$
$$= (a+b)\sin\theta - b\cos\left[\phi - \left(\frac{\pi}{2} - \theta\right)\right]$$
$$= (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

73. The hypocycloid. A point on the circumference of a circle that rolls on the inside of a fixed circle generates a *hypocycloid*.

Its equations are obtained in the same manner as those of the epicycloid. They are

$$x = (a-b)\cos\theta + b\cos\frac{a-b}{b}\theta,$$
$$y = (a-b)\sin\theta - b\sin\frac{a-b}{b}\theta.$$

EXERCISES

1. Show that the tangent to the cycloid passes through the highest point of the rolling circle.

2. A wheel of radius 2 ft. rolls on a straight track with a velocity of 6 ft. per second. Find j_x, j_y, j, j_t, and j_n at the points $\theta = 0$, $\theta = \frac{\pi}{4}$, $\theta = \pi$.

3. The highest point on an arch of the cycloid is called its *vertex*. Show that by taking the origin at the vertex and replacing θ by $\theta' = \theta - \pi$, the equations of the cycloid become

$$x' = a(\theta' + \sin\theta'),$$
$$y' = -a(1 - \cos\theta'),$$

or, if we change the sense of the y-axis and drop subscripts,

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta).$$

4. Sketch the epicycloid for which the rolling circle and the fixed circle have the same radius. If $\frac{d\theta}{dt} = \frac{\pi}{2}$ radians per second, find v and j, and also find between what limits these quantities will vary.

5. Show that the hypocycloid for which $b = \frac{a}{2}$ is a diameter of the fixed circle.

6. Show that the equations of the hypocycloid of four cusps, for which $b = \frac{a}{4}$, may be written

$$x = a\cos^3\theta,$$
$$y = a\sin^3\theta.$$

Hence find its cartesian equation. Trace the curve.

7. Give the cartesian equation of the cycloid.

8. Find the radius of curvature of the cycloid. *Ans.* $4a\sin\frac{\theta}{2}$.

9. Trace the epicycloid for which $b = \frac{a}{2}$.

CHAPTER X

CURVE TRACING IN POLAR COÖRDINATES

74. Slope of a curve in polar coördinates. We have seen that in sketching a curve it is helpful to know the direction of the curve at any point. In cartesian coördinates the direction at the point $P_0 : (x_0, y_0)$ is most easily determined by giving the inclination of the curve to the straight line $y = y_0$ — *i.e.* the "slope" of the curve — since this is found by a mere differentiation. Similarly, given the equation of a curve in polar coördinates, the direction at the point $P_0 : (r_0, \theta_0)$ is best found by means of the inclination to the curve $r = r_0$, which is of course a circle through P_0 with center at O. For this reason the quantity $\tan \phi$, where ϕ is the angle between the curve and the circle just mentioned, will be called the *polar slope*.

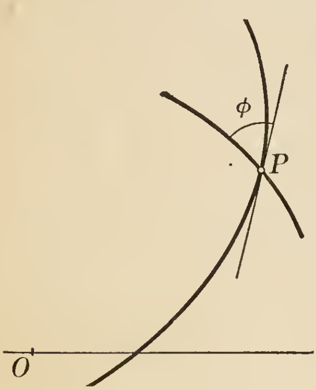


FIG. 50

To find the polar slope we proceed as follows. Consider a fixed point $P : (r, \theta)$ and a neighboring point $P' : (r + \Delta r, \theta + \Delta\theta)$ on the curve (Fig. 51), and drop a perpendicular PN from P upon OP'. Let $\phi' = \angle NPP'$. Then

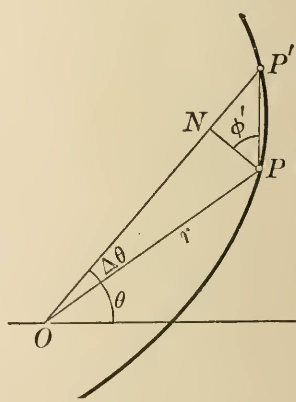


FIG. 51

$$\tan \phi' = \frac{NP'}{PN} = \frac{OP' - ON}{PN}.$$

But

$$OP' = r + \Delta r,\ ON = r\cos\Delta\theta,\ PN = r\sin\Delta\theta.$$

Hence

$$\tan\phi' = \frac{r + \Delta r - r\cos\Delta\theta}{r\sin\Delta\theta}$$

$$= \frac{\Delta r + r(1-\cos\Delta\theta)}{r\sin\Delta\theta}$$

$$= \frac{\Delta r + 2\,r\sin^2\frac{1}{2}\,\Delta\theta}{r\sin\Delta\theta}$$

$$= \frac{\dfrac{\Delta r}{\Delta\theta} + \dfrac{r\sin\frac{1}{2}\,\Delta\theta}{\frac{1}{2}\,\Delta\theta}\sin\frac{1}{2}\,\Delta\theta}{r\,\dfrac{\sin\Delta\theta}{\Delta\theta}}.$$

When P' approaches P along the curve, ϕ' approaches ϕ. By § 38,

$$\lim_{\alpha \to 0}\frac{\sin\alpha}{\alpha} = 1.$$

Hence

$$\tan\phi = \lim_{\Delta\theta \to 0}\tan\phi' = \frac{\dfrac{dr}{d\theta}}{r},$$

or

$$\boldsymbol{\tan\phi = \frac{dr}{rd\theta}}.$$

The formula for $\tan\phi$ may be remembered as follows. Strike through P a circular arc PM with center at O (Fig. 52). Then

$$\text{arc } PM = rd\theta,$$

$$MP' = dr \text{ (approximately)},$$

and

$$\tan\phi = \frac{MP'}{\text{arc } PM} \text{ (approximately)}.$$

This at once suggests the formula.

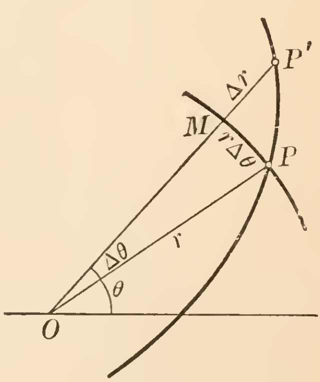


FIG. 52

75. Maxima and minima. When $\frac{dr}{d\theta} = 0$, $\phi = 0$, and in general r is a maximum or a minimum, as at A and B, Fig. 53. Just as in cartesian coördinates, there is a possible exception: the curve may have the form shown at C. However, the exceptional case is rare in the simpler curves.

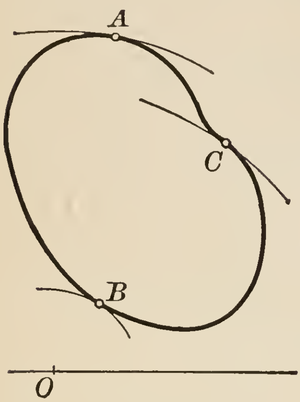


FIG. 53

76. Curve tracing. Before sketching a curve whose equation is given in polar coördinates, the following questions should be considered* :

(1) *Is the curve symmetric with respect to the initial line?* (It is, if the equation is unchanged when θ is replaced by $-\theta$; other tests may frequently be used.)

(2) *Is it possible to determine any particular regions of the plane within which the curve must lie?*

(3) *At what points is the polar slope 0? Is the radius vector a maximum or a minimum at each of these points?*

The above discussion is frequently insufficient to determine the general form of the curve, in which case additional points must be plotted.

Example: Trace the lemniscate $r^2 = a^2 \cos 2\theta$.

(1) The curve is symmetric about the initial line.

(2) $\cos 2\theta$ is negative, and r is imaginary, when

$$\frac{\pi}{2} < 2\theta < \frac{3\pi}{2}, \text{ i.e. } \frac{\pi}{4} < \theta < \frac{3\pi}{4};$$

also when

$$\frac{5\pi}{2} < 2\theta < \frac{7\pi}{2}, \text{ or } \frac{5\pi}{4} < \theta < \frac{7\pi}{4}.$$

Hence the curve lies entirely within the sectors AOB, COD (Fig. 54).

* For brevity, the discussion of singular points and asymptotes is omitted.

(3) Since

$$r\,dr = -a^2 \sin 2\theta\, d\theta,$$

$$\tan\phi = \frac{-a^2 \sin 2\theta}{a^2 \cos 2\theta} = -\tan 2\theta.$$

From this the direction of the curve at any point may be found.

When

$$\tan\phi = -\tan 2\theta = 0,$$
$$2\theta = 0,\ \pi,\ 2\pi,\ 3\pi,$$

and

$$\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}.$$

Only the values 0 and π give real values of r; at each of these points r is a maximum, viz. $r = a$.

The curve passes through the origin whenever $r = 0$; *i.e.* when $\cos 2\theta = 0$, or $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$, etc. The curve is shown in Fig. 54.

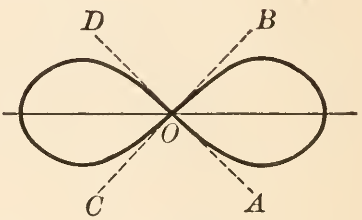


FIG. 54

EXERCISES

Trace the following curves.

1. $r = 2a\cos\theta$.
2. The spiral of Archimedes $r = a\theta$.
3. $r = a\sec\theta$.
4. $r = a\cos 2\theta$.
5. $r^2 = a^2\sin\theta$.
6. $r = a\cos 3\theta$.
7. $r^2 \sin 2\theta = a^2$.
8. The limaçon $r = b - a\cos\theta$, (*a*) when $b = 2a$; (*b*) when $b = a$; (*c*) when $b = \frac{1}{2}a$.
9. The conic $r = \frac{l}{1 - e\cos\theta}$, where e is the eccentricity, (*a*) if $e < 1$; (*b*) if $e = 1$; (*c*) if $e > 1$.
10. The logarithmic spiral $r = e^{k\theta}$. Show that $\tan\phi$ is constant.
11. What is the form of the curve $r = a\cos n\theta$, (*a*) when n is even; (*b*) when n is odd?

CHAPTER XI

THE INDEFINITE INTEGRAL

77. Integration. We have been occupied up to this point with the problem: Given a function, to find its derivative. Many of the most important applications of the calculus lead to the inverse problem: Given the derivative of a function, to find the function. The required function is called an *integral* of the given derivative, or *integrand*, and the process of finding it is called *integration*.

If $f(x)$ is a given function and $F(x)$ is a function whose derivative is $f(x)$, the relation between them is expressed by writing

$$F(x) = \int f(x)\,dx,$$

where the "integral sign" $\int$ indicates that we are to perform the operation of integration upon $f(x)\,dx$. For reasons that will appear later, it is customary to write after the integral sign the differential $f(x)\,dx$, rather than the derivative $f(x)$.

Examples: (*a*) Find the equation of a curve whose slope at every point is equal to twice the abscissa of the point.

We have to find a function y such that

$$\frac{dy}{dx} = 2\,x, \text{ or } dy = 2\,x\,dx;$$

hence

$$y = \int 2\,x\,dx.$$

It appears at once that $2x$ is the derivative of x^2. Thus a curve having the desired property is the parabola

$$y = x^2.$$

But it is clear that if

$$(1) \qquad y = x^2 + C,$$

where C is any constant whatever, we still have

$$dy = 2x\,dx,$$

and our data are satisfied by any one of the *family* of parabolas represented by (1). In order to obtain a unique answer to our problem, we must have some additional information about the curve. Thus, if it is to pass through the point $P : (1, \frac{7}{4})$, we substitute these coördinates in (1):

$$\tfrac{7}{4} = 1 + C, \quad C = \tfrac{3}{4},$$

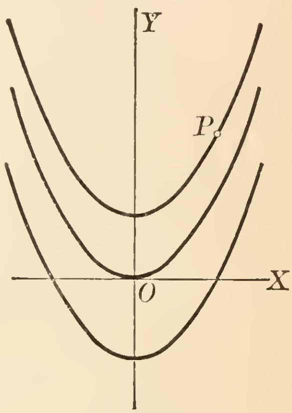


FIG. 55

and the answer is

$$y = x^2 + \tfrac{3}{4}.$$

(*b*) Find the velocity of a body falling freely under gravity at the end of 5 seconds, if the initial velocity is 20 ft. per second upward.

Taking motion downward as positive, we have to find v from the relation (see § 55)

$$j = \frac{dv}{dt} = g.$$

Hence

$$v = \int g\,dt = gt + C.$$

Making use of the fact that $v = -20$ when $t = 0$, we get

$$-20 = 0 + C,$$

and the velocity at the time t is

$$v = gt - 20.$$

At the end of 5 seconds we have, taking $g = 32$,

$$v = 140 \text{ ft. per second.}$$

(*c*) Find the space covered in the fifth second of the motion in (*b*).

Here

$$v = \frac{dx}{dt} = 32\,t - 20,$$

$$x = \int (32\,t - 20)dt = 16\,t^2 - 20\,t + C.$$

No data are given for determining C. But if we denote by x_t the space covered in t seconds, the space described in the fifth second is

$$\begin{aligned} x_5 - x_4 &= (16 \cdot 25 - 20 \cdot 5 + C) - (16 \cdot 16 - 20 \cdot 4 + C) \\ &= 124 \text{ ft.}, \end{aligned}$$

the unknown constant having disappeared. In fact, C is merely x_0, the distance of the starting point from some arbitrarily chosen origin, so that the distance passed over between any two instants must necessarily be independent of C.

78. Integration an indirect process. Differentiation is a direct process; by means of the fundamental formulas the derivative of any elementary function may be found. On the other hand, to find an integral of a given function, we must be able to discover a function whose derivative is the given integrand, and this is always in the last analysis a matter of trial. The problem can by no means always be solved; in fact, there are many comparatively simple functions whose integrals cannot be expressed in terms of elementary functions*.

79. Constant of integration. In each of the examples of § 77, an arbitrary constant presented itself. It is clear that this will be the case in general; *i.e.* a function whose

* It will be shown in § 81 that for every continuous function *an integral exists*, although it may not be an elementary function.

derivative is given is not completely determined, since it contains an arbitrary additive constant, the *constant of integration.*

On account of the presence of this undetermined constant, the function $\int f(x)dx$ is called the *indefinite integral* of $f(x)$.

80. Functions having the same derivative. In § 79 it was tacitly assumed that if the derivative of a function is given, the function is determined aside from an additive constant. That this is true follows from the

THEOREM: *Two functions having the same derivative differ only by a constant.*

The theorem is almost self-evident. Let $\phi(x)$ and $\psi(x)$ be the two functions, and place

$$y = \phi(x) - \psi(x).$$

By hypothesis,

$$\frac{dy}{dx} = \phi'(x) - \psi'(x) = 0.$$

The rate of change of y with respect to x is everywhere 0, hence y is constant.

EXERCISES

Evaluate the following integrals, checking the answer in each case by differentiation.

1. (*a*) $\int x^3 dx$; (*b*) $\int (2x - x^2)dx$; (*c*) $\int (1 - 4t^4)dt$;

(*d*) $\int (1+y)^2 dy$; (*e*) $\int \frac{dx}{x^2}$; (*f*) $\int \left(\sqrt{x} - \frac{2}{\sqrt{x}}\right)dx$.

2. $\int \frac{dx}{x}$. **3.** $\int \sin\theta\, d\theta$. **4.** $\int \sin 2\theta\, d\theta$.

5. $\int \sqrt{x+1}\, dx$. **6.** $\int \sqrt{1-x}\, dx$. **7.** $\int e^{2x} dx$.

8. $\int (1 + 2x)^3 dx$. **9.** $\int x\sqrt{a^2 + x^2}\, dx$. *Ans.* $\frac{1}{3}(a^2 + x^2)^{\frac{3}{2}} + C$.

10. Find the equation of the family of curves whose slope at every point is equal to the square of the abscissa of the point. Exhibit graphically.

11. Find the equation of that one of the curves of Ex. 10 that passes through the point (3, – 5). *Ans.* $y = \frac{x^3}{3} - 14.$

12. A body falls from rest under gravity. Find the velocity at the end of 3 seconds, and the distance traveled in that time.

13. Find the equation of the curve for which $y'' = 4$ at every point, if the curve touches the line $y = 3\,x$ at (2, 6).

Ans. $y = 2\,x^2 - 5\,x + 8.$

14. A body moves under an acceleration numerically equal to the time. If the initial velocity is 10 ft. per second in the direction of the acceleration, find v and x at the end of 4 seconds, x being measured from the starting point.

15. In Ex. 14, find the initial velocity if the body moves 10 ft. in the first second.

16. Find the equation of the curve for which $y'' = -\frac{1}{x^2}$, if the curve makes an angle of 45° with OX at the point (1, 0).

17. Find the equation of the curve through (1, 2) and (2, 3) (*a*) if $y'' = 0$; (*b*) if $y'' = 6\,x$; (*c*) if $y'' = \frac{1}{x^3}$. Trace the curve in each case. *Ans.* (*b*) $y = x^3 - 6\,x + 7.$

81. Geometric interpretation of an integral. Consider the area A bounded by the curve $y = f(x)$, the x-axis, the fixed ordinate $x = a$, and a variable ordinate $x = x$. This area is evidently a function of x. We proceed to find the derivative of A with respect to x.

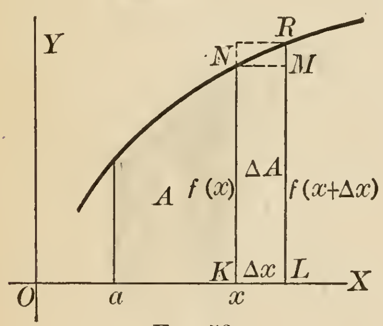


Fig. 56

When x is increased by an amount Δx, A assumes an increment ΔA, the area $KLRN$ in Fig. 56. It appears from the figure that ΔA is greater than the area $f(x)\Delta x$ of the inscribed rectangle $KLMN$, and less than the area $f(x + \Delta x)\Delta x$ of the circumscribed rectangle*:

$$f(x)\Delta x < \Delta A < f(x + \Delta x)\Delta x.$$

* The argument is readily modified to fit the case when $f(x)$ is a decreasing function.

Hence

$$f(x) < \frac{\Delta A}{\Delta x} < f(x + \Delta x).$$

If now Δx approaches 0, $f(x + \Delta x)$ approaches $f(x)$, and since $\frac{\Delta A}{\Delta x}$ always lies between $f(x)$ and $f(x + \Delta x)$, it must also approach $f(x)$. Thus

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = f(x).$$

Since the derivative of A is $f(x)$, it follows by the definition of the integral that

$$A = \int f(x) dx.$$

In case the position of the fixed ordinate $x = a$ is given, the constant of integration may be determined by the fact that $A = 0$ when $x = a$.

We have thus proved the following result:

The indefinite integral $\int f(x) dx$ *represents the area bounded by the curve* $y = f(x)$, *the x-axis, a fixed ordinate, and a variable ordinate.*

It is evident that if $f(x)$ is continuous, this area always exists; hence *every continuous function has an integral* (cf. § 78).

Since

$$y = f(x),$$

the formula for the area under the curve is frequently written

$$A = \int y \, dx.$$

Example: Find the area bounded by the parabola $y = x^2$, the x-axis, and the lines $x = 1$, $x = 4$.

The area from $x = 1$ to any variable ordinate is

$$A = \int y \, dx = \int x^2 \, dx = \frac{x^3}{3} + C.$$

Since $A = 0$ when $x = 1$, we have

$$0 = \tfrac{1}{3} + C, \quad C = -\tfrac{1}{3},$$

or

$$A = \frac{x^3}{3} - \frac{1}{3}.$$

In particular, the area from $x = 1$ to $x = 4$ is

$$A\Big]_{x=4} = \tfrac{64}{3} - \tfrac{1}{3} = 21.$$

EXERCISES

In the following, find the area bounded by the x-axis, the given curve, and the indicated ordinates. Check roughly by drawing the figure on coördinate paper and estimating the area.

1. $y = x^3, \quad x = 0, \quad x = 4$. *Ans.* 64.

2. The parabola $y^2 = 4x$ and its latus rectum.

3. The hyperbola $y = \dfrac{1}{x}, \quad x = 1, \quad x = 3$. *Ans.* 1.099.

4. Find the area of one arch of the sine curve. *Ans.* 2.

5. Find the area bounded by the parabola $y = 1 - x^2$ and the x-axis.

82. Variable of integration. In the last article we had occasion to use the symbol $\int y\,dx$. In order that such a symbol shall have any meaning, y must be directly or indirectly a function of x. The variable whose differential occurs is called the *variable of integration*, any other variables appearing under the integral sign must be functions of the variable of integration, and their values in terms of that variable must be introduced before the integral can be evaluated.

The fact that the differential occurring tells us which variable is the variable of integration is one of the reasons for using the notation $\int f(x)dx$ rather than the notation $\int f(x)$.

83. Change of the variable of integration. If x is so related to y that

$$\text{(1)} \qquad \phi(x)dx = \psi(y)dy,$$

we may replace $\int \phi(x)dx$ by $\int \psi(y)dy$. For, let

$$\Phi(x) = \int \phi(x)dx,$$

$$\Psi(y) = \int \psi(y)dy.$$

Now

$$\frac{d}{dx}\Phi(x) = \phi(x),$$

$$\frac{d}{dx}\Psi(y) = \frac{d}{dy}\Psi(y) \cdot \frac{dy}{dx} = \psi(y)\frac{dy}{dx}.$$

The two functions therefore have the same derivative, by (1), and hence differ only by a constant. Since each function contains a constant of integration, these constants may be so chosen that

$$\Phi(x) = \Psi(y).$$

The device of replacing a given integral by an equivalent integral in a different variable is very useful in many problems.

84. Integration by substitution. A change of variable is usually brought about by means of an explicit substitution

$$x = \phi(u),$$

$$dx = \phi'(u)du.$$

The process is called *integration by substitution*, and is highly important. It is to be remembered that *not merely x, but dx as well*, must be replaced by the proper value in terms of the new variable.

Example: Evaluate $\int x\sqrt{1-x}\,dx$.

Let us put

$$1 - x = u,$$

so that

$$x = 1 - u, \ dx = -du.$$

Then

$$\int x\sqrt{1-x}\,dx = -\int(1-u)u^{\frac{1}{2}}\,du = -\int(u^{\frac{1}{2}} - u^{\frac{3}{2}})du$$
$$= -\tfrac{2}{3}u^{\frac{3}{2}} + \tfrac{2}{5}u^{\frac{5}{2}} + C$$
$$= -\tfrac{2}{3}(1-x)^{\frac{3}{2}} + \tfrac{2}{5}(1-x)^{\frac{5}{2}} + C.$$

EXERCISES

1. Work the above example by placing $1 - x = u^2$.

2. Evaluate $\int x(3 - 2x^2)dx$ by substituting $3 - 2x^2 = u$. Check by expanding the given integrand and integrating directly.

Evaluate the following integrals by means of the indicated substitution, and check the results by differentiation.

3. $\int\sqrt{a + bx}\,dx$, $\quad a + bx = u$.

4. $\int\sin^2\theta\cos\theta\,d\theta$, $\quad \sin\theta = u$.

5. $\int\frac{dx}{1 + 4x^2}$, $\quad 2x = u$.

6. $\int\frac{x\,dx}{1 - x^2}$, $\quad 1 - x^2 = u$.

7. $\int\frac{\sec^2\theta\,d\theta}{1 + \tan\theta}$, $\quad 1 + \tan\theta = u$.

8. $\int\frac{x\,dx}{\sqrt{1 - x^4}}$, $\quad x^2 = u$.

9. If the velocity of a point moving in a straight line is given as a function of the time, show that the distance covered may be found by the formula

$$x = \int v\,dt.$$

Given $v = 10t + 20$, find x in terms of t by substituting the value of v in the above integral; also find x in terms of v by substituting for dt. Show that the two values of x are equivalent.

10. Given $x = t^2$, $y = 3t$, find

$$A = \int y\,dx$$

as a function of t by substituting for y and dx their values in terms of t. *Ans.* $A = 2t^3 + C$.

11. In Ex. 10, find y in terms of x by eliminating t. Then express A as a function of x by substituting for y and integrating with respect to x. *Ans.* $A = 2x^{\frac{3}{2}} + C$.

12. In Ex. 11, find A as a function of y by substituting for dx. *Ans.* $A = \frac{2}{27}y^3 + C$.

13. Show that the three values of A found in Exs. 10–12 are equivalent.

14. By the formula

$$A = \int y\,dx,$$

find the area under the curve $y^2 = x$ from $x = 0$ to $x = 4$, (*a*) by substituting for y in terms of x; (*b*) by substituting for dx in terms of y and dy.

15. Proceed as in Ex. 14 for the area under the curve $y = e^x$ from $x = 0$ to $x = 1$.

16. Proceed as in Ex. 14 for the area of half an arch of the curve $y = \frac{1}{3}\sin 4x$. *Ans.* $\frac{1}{12}$.

CHAPTER XII

STANDARD FORMULAS OF INTEGRATION

85. Standard formulas. If we were to try at present to solve any but the simplest applied problems involving integration, we must fail through inability to evaluate the indefinite integrals involved. We shall therefore devote this and the following chapter to the technique of integration — the formal evaluation of indefinite integrals — after which the question of applications will be treated at some length.

As the first step toward facility in integration, the student must become thoroughly familiar with the following

Fundamental Integration Formulas

(1) $\int du = u + C,$

(2) $\int (du + dv) = \int du + \int dv,$

(3) $\int c\, du = c \int du,$

(4) $\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad (n \neq -1)$

(5) $\int \frac{du}{u} = \log u + C,$

(6) $\int e^u du = e^u + C,$

(7) $\int \cos u\, du = \sin u + C = \cos\left(u - \frac{\pi}{2}\right) + C,$

(8) $\int \sin u\, du = -\cos u + C = \sin\left(u - \frac{\pi}{2}\right) + C,$

(9) $\int \sec^2 u\, du = \tan u + C,$

(10) $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C,$

(11) $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C,$

(12) $\int u\, dv = uv - \int v\, du.$

It is strongly recommended that each of the formulas be written out by the student in words, and memorized in that form.

The test of the correctness of an integral is that its derivative must be the given integrand. The above formulas may be verified at once by differentiation.

86. Formulas (1)–(3). Formula (1) merely embodies the definition of an integral.

Formula (2) is readily extended to the case of any number of terms. The formula shows that if the integrand consists of a sum of terms each term may be integrated separately.

Formula (3) says that if the integrand contains a constant factor, that factor may be written before the integral sign. As a corollary, we may *introduce a constant factor into the integrand*, provided we place its reciprocal before the integral sign. But the student must beware of introducing *variable* factors by this rule.

87. Formula (4): Powers. This formula evidently fails when $n = -1$. The exceptional case $\int \frac{du}{u}$ is taken care of by formula (5).

Examples:

(*a*) $$\int\left(3x^3+1+\frac{1}{2x^2}\right)dx = 3\int x^3\,dx + \int dx + \frac{1}{2}\int x^{-2}\,dx$$
$$= \frac{3x^4}{4} + x - \frac{1}{2}x^{-1} + C$$
$$= \frac{3x^4}{4} + x - \frac{1}{2x} + C.$$

(*b*) $$\int x(1+x^2)^2\,dx = \int (x+2x^3+x^5)dx$$
$$= \frac{x^2}{2} + \frac{x^4}{2} + \frac{x^6}{6} + C.$$

A better method: Introducing a factor 2 after the integral sign and its reciprocal in front, we have

$$\int x(1+x^2)^2\,dx = \frac{1}{2}\int (1+x^2)^2 \cdot 2x\,dx.$$

Since $2x\,dx$ is the differential of $1+x^2$, formula (**4**) applies with $u = 1+x^2$, $n = 2$. Hence

$$\frac{1}{2}\int (1+x^2)^2 \cdot 2x\,dx = \frac{1}{2}\frac{(1+x^2)^3}{3} + C = \frac{1}{6}(1+x^2)^3 + C.$$

In substance we have introduced a new variable $u = 1+x^2$, as in § 84. But with a little practice one is able to think of the quantity $1+x^2$ directly as the variable of integration, without writing out a formal substitution, thus effecting a great saving of time.

The student should compare the two answers that have been obtained in this example.

(*c*) $$\int \sqrt{a+bx}\,dx = \frac{1}{b}\int \sqrt{a+bx}\cdot b\,dx$$
$$= \frac{1}{b}\frac{(a+bx)^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$= \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C.$$

Here the quantity $a+bx$ is taken as the variable of integration, the factor b is introduced to give the proper differential, and (**4**) then applies with $n = \frac{1}{2}$.

EXERCISES

Evaluate the following integrals; check the results by differentiation.

1. $\int\left(1+\frac{1}{x^3}\right)dx.$

2. $\int\left(\sqrt{x}+2\sqrt{x^3}+\frac{1}{\sqrt[3]{x}}\right)dx.$

3. $\int\frac{x^4+x^3+1}{x^3}\,dx.$

4. $\int 4(x+1)^4\,dx.$

5. $\int(1-2x)^5\,dx.$

6. $\int(x+1)(x+2)\,dx.$

7. $\int\frac{dx}{(1-x)^2}.$

8. $\int x^2(1-x^2)\,dx.$

9. $\int x\sqrt{a^2+x^2}\,dx.$

10. $\int\frac{y^2\,dy}{\sqrt{1-y^3}}.$

11. $\int(a^2-x^2)^2\,dx.$

12. $\int\sqrt{(1+kt)^3}\,dt.$

13. $\int e^y\sqrt{1+e^y}\,dy.$

14. $\int(x+1)(x^2+2x+6)\,dx.$

15. $\int(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}\frac{dx}{x^{\frac{1}{3}}}.$ *Ans.* $-\frac{3}{5}(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{5}{2}}+C.$

16. $\int\sin^2\theta\cos\theta\,d\theta.$

17. $\int\frac{4x\,dx}{(a^2-x^2)^2}.$

18. $\int(a^{\frac{1}{2}}-x^{\frac{1}{2}})^2\,dx.$

19. $\int\frac{dx}{\sqrt{3-4x}}.$

20. $\int x\sqrt{ax}\,dx.$

21. $\int\frac{dt}{(1-t)^3}.$

88. Formulas (5)–(6): Logarithms and exponentials.

Examples: (*a*) Evaluate $\int\frac{x\,dx}{a^2+x^2}.$

If the factor 2 be inserted under the integral sign, the numerator becomes the differential of the denominator, and (5) applies:

$$\int\frac{x\,dx}{a^2+x^2}=\frac{1}{2}\int\frac{2x\,dx}{a^2+x^2}=\frac{1}{2}\log(a^2+x^2)+C.$$

(*b*) Evaluate $\int\frac{x^2-x}{x+1}\,dx.$

By division we find

$$\frac{x^2 - x}{x+1} = x - 2 + \frac{2}{x+1}.$$

Whence

$$\int \frac{x^2 - x}{x+1}\, dx = \int \left(x - 2 + \frac{2}{x+1}\right) dx$$

$$= \frac{x^2}{2} - 2x + 2 \log (x+1) + C.$$

(*c*) Evaluate $\int e^{3x}\, dx.$

If the factor 3 be inserted, this is integrable by (**6**):

$$\int e^{3x}\, dx = \tfrac{1}{3} \int e^{3x} \cdot 3\, dx = \tfrac{1}{3} e^{3x} + C.$$

In (*b*) we have used a device that finds frequent application:

As the first step toward integrating a rational fraction, carry out the indicated division until the numerator is of lower degree than the denominator.

EXERCISES

Evaluate the following integrals; check the results.

1. $\int \frac{dx}{x - 7}.$

2. $\int \frac{dx}{\sqrt{1 - 4x}}.$

3. $\int \frac{dx}{3 - x}.$

4. $\int \frac{t\, dt}{a^2 - t^2}.$

5. $\int \frac{3\, dx}{1 - 4x}.$

6. $\int e^{\frac{x}{2}} dx.$

7. $\int \frac{x^2 dx}{\sqrt{1 + x^3}}.$ *Ans.* $\frac{2}{3}\sqrt{1 + x^3} + C.$

8. $\int \tan \theta\, d\theta.$ *Ans.* $-\log \cos \theta + C.$

9. $\int u e^{u^2 - 3} du.$

10. $\int \frac{2x - 1}{2x + 3} dx.$

11. $\int \frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2}{4} dx.$

12. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx.$

13. $\int \frac{x^3 - x^2 + 2x - 1}{1 + x^2} dx.$

14. $\int \frac{\sin x \, dx}{\cos^2 x}.$

15. $\int \frac{\sec^2\phi \, d\phi}{\sqrt{1 + \tan \phi}}.$

16. $\int \frac{x^3 - 5x + 7}{3x - 4} dx.$

17. $\int \frac{dx}{e^x}.$

18. $\int \frac{e^{2x} dx}{(1 - e^{2x})^2}.$

19. Find the equation of a curve through (1, 1), if the slope at every point is inversely proportional to the abscissa of the point $\left(i.e. \text{ if } y' = \frac{k}{x}\right)$.

89. Formulas (7)-(9): Trigonometric functions.

EXERCISES

Evaluate the following integrals.

1. $\int \sin 2x \, dx.$

2. $\int \sec^2 \frac{\theta}{2} d\theta.$

3. $\int x \sin x^2 dx.$

4. $\int \cos kt \, dt.$

5. $\int \cos (\log x) \frac{dx}{x}.$

6. $\int \frac{\cos x \, dx}{\sin^3 x}.$

7. $\int \tan^2 \theta \, d\theta.$ *Ans.* $\tan\theta - \theta + C.$

8. $\int \frac{\sin 2x \, dx}{a + b \cos 2x}.$

9. $\int e^x \cos e^x \, dx.$

10. $\int \sin^3 u \, du.$ *Ans.* $-\cos u + \frac{1}{3}\cos^3 u + C.$

11. $\int \frac{d\theta}{1 + \sin\theta}.$

12. $\int \sin\theta \cos\theta \, d\theta.$

13. $\int e^{\cos x} \sin x \, dx.$

14. $\int \tan 3t \, dt.$

15. $\int \sec^4 z \, dz.$

16. $\int \frac{\sin\theta \, d\theta}{\cos^5\theta}.$

17. If the acceleration of a moving particle is

$$\frac{d^2x}{dt^2} = -k^2 \cos kt,$$

the particle has *simple harmonic motion* (§ 229). Find v and x in terms of t if $v = 0$ and $x = 1$ when $t = 0$. Show that x vanishes periodically, hence the particle oscillates about the origin. Find the maximum velocity.

90. Formulas (10)–(11): Inverse trigonometric functions.

Example:

$$\int \frac{dx}{x^2 + 2x + 5} = \int \frac{dx}{(x+1)^2 + 4} = \frac{1}{2} \arctan \frac{x+1}{2} + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{\sqrt{9 - x^2}}$.

2. $\int \frac{x\,dx}{\sqrt{1 - x^4}}$.

3. $\int \frac{dx}{4x^2 + 9}$. *Ans.* $\frac{1}{6} \arctan \frac{2x}{3} + C$.

4. $\int \frac{dx}{1 - 2x + 2x^2}$. *Ans.* $\text{Arctan}\,(2x - 1) + C$.

5. $\int \frac{x + 1}{x^2 + 4}\,dx$.

6. $\int \frac{x\,dx}{\sqrt{1 + x^2}}$.

7. $\int \frac{x^2 + x}{x^2 + 1}\,dx$.

8. $\int \frac{\sin 3x\,dx}{1 + \cos^2 3x}$.

9. $\int \frac{e^v dv}{\sqrt{1 - e^{2v}}}$.

10. $\int \frac{e^{2v}\,dv}{\sqrt{1 - e^{2v}}}$.

11. $\int \frac{dx}{\sqrt{x}\sqrt{1 - x}}$.

12. $\int \frac{dx}{x(1 + \log^2 x)}$.

13. $\int \frac{x\,dx}{\sqrt{1 - x^2}}$.

14. $\int \frac{x\,dx}{(1 + x^2)^2}$.

91. Formula (12): Integration by parts. From the formula for the differential of a product,

$$d(uv) = u\,dv + v\,du,$$

we find, integrating both sides,

$$uv = \int u\,dv + \int v\,du.$$

Transposing, we obtain formula (**12**):

$$\int u\,dv = uv - \int v\,du.$$

Integration by this formula is called *integration by parts.* The use of the formula will be explained by the following

Examples: (*a*) Evaluate $\int xe^x\,dx$.

Let

$$u = x,\ dv = e^x\,dx,$$

whence

$$du = dx,\ v = \int e^x\,dx = e^x.$$

Then $$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

(*b*) Evaluate $\int \log x\,dx$.

Let

$$u = \log x,\ dv = dx,$$

whence

$$du = \frac{dx}{x},\ v = x.$$

Then

$$\int \log x\,dx = x\log x - \int x \cdot \frac{dx}{x} = x\log x - x + C.$$

Integration by parts is highly important, as it succeeds in many cases when the methods of the preceding articles fail. The success of the method depends as a rule on our ability to choose u and dv so that the integral $\int v\,du$ is easier to evaluate than the given one. No general directions can be given for choosing u and dv; if the new integral is no simpler than the original, we should begin over again with a different choice of u and dv.

EXERCISES

Evaluate the following integrals.

1. $\int xe^{2x}dx.$

2. $\int t\sin t\,dt.$

3. $\int x^2e^{2x}dx.$ *Ans.* $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$

4. $\int \arcsin x\,dx.$ *Ans.* $x\arcsin x + \sqrt{1-x^2} + C.$

5. $\int x^3\log x\,dx$ *Ans.* $\frac{x^4}{4}\left(\log x - \frac{1}{4}\right) + C.$

6. $\int u\sqrt{1-u}\,du.$

7. $\int x^2\arcsin x\,dx.$

8. $\int x \tan^2 x\, dx.$ *Ans.* $x \tan x - \frac{x^2}{2} + \log \cos x + C.$

9. $\int x^3 \sqrt{a^2 - x^2}\, dx.$ *Ans.* $-\frac{1}{3}x^2(a^2 - x^2)^{\frac{3}{2}} - \frac{2}{15}(a^2 - x^2)^{\frac{5}{2}} + C.$

10. $\int x^3 \sqrt{a^4 - x^4}\, dx.$

11. $\int \cos\theta \log \sin\theta\, d\theta.$

12. $\int \sec^4 \theta\, d\theta.$

13. $\int \cos^3 \theta\, d\theta.$

14. $\int \frac{x^2 dx}{(a^2 - x^2)^{\frac{3}{2}}}.$

15. $\int \frac{x^2\, dx}{(1 + x^2)^2}.$

16. $\int \frac{y^3 dy}{\sqrt{a^2 - y^2}}.$ *Ans.* $-y^2(a^2 - y^2)^{\frac{1}{2}} - \frac{2}{3}(a^2 - y^2)^{\frac{3}{2}} + C.$

92. Integration by substitution.* An integral that cannot be reduced directly to one of the standard forms may often be evaluated by the formal substitution of a new variable. If the integrand is algebraic, and rational except for the presence of a single radical, it may frequently be integrated by placing the radical equal to a new variable. In general, if any simple function is especially conspicuous in the integrand, substitution of a new variable for that function is worth trying. However, no general rules can be laid down; skill in the use of substitutions comes only with practice.

Examples: (*a*) Evaluate $\int \frac{\sqrt{x}\, dx}{1 + x}.$

Let

$$\sqrt{x} = u, \quad x = u^2, \quad dx = 2\, u\, du.$$

Then

$$\int \frac{\sqrt{x}\, dx}{1 + x} = 2\int \frac{u^2 du}{1 + u^2} = 2\int \left(1 - \frac{1}{1 + u^2}\right) du$$
$$= 2\, u - 2 \arctan u + C$$
$$= 2\sqrt{x} - 2 \arctan \sqrt{x} + C.$$

(*b*) Evaluate $\int \frac{\sin x \cos x\, dx}{1 + \sin x}.$

* At this point the student should re-read § 84.

Let

$$\sin x = u, \quad \cos x\, dx = du,$$

whence

$$\int \frac{\sin x \cos x\, dx}{1 + \sin x} = \int \frac{u\, du}{1 + u} = \int \left(1 - \frac{1}{1 + u}\right) du$$
$$= u - \log(1 + u) + C$$
$$= \sin x - \log(1 + \sin x) + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{1 - \sqrt{x}}$.
2. $\int \frac{x + 3}{\sqrt{1 + 2x}} dx$.
3. $\int \frac{x\, dx}{(x + 1)^2}$.
4. $\int e^{2x} \sin e^x\, dx$.
5. $\int x^3 \sqrt{a^2 - x^2}\, dx$.
6. $\int x^3 e^{x^2}\, dx$.
7. $\int \frac{x^3\, dx}{(a^2 + x^2)^2}$.
8. $\int \sin \sqrt{x}\, dx$.
9. $\int x^2 \sqrt{1 - x}\, dx$.
10. $\int \frac{\sec^2 \theta \tan^2 \theta\, d\theta}{\sqrt{1 + \tan \theta}}$.

MISCELLANEOUS EXERCISES

Evaluate the following integrals.

1. $\int \cot \theta\, d\theta$.
2. $\int \frac{x\, dx}{\sqrt{a^2 + x^2}}$.
3. $\int \frac{x^3 - x}{2 + 3x} dx$.
4. $\int x \cos 3x\, dx$.
5. $\int \frac{e^x dx}{e^{2x} + 1}$.
6. $\int \frac{\log x}{x} dx$.
7. $\int \left(1 + \cos \frac{\theta}{2}\right)^3 \sin \frac{\theta}{2}\, d\theta$.
8. $\int x(1 - x^4) dx$.
9. $\int x e^{-x^2} dx$.
10. $\int \sin^2 2x \cos 2x\, dx$.
11. $\int \frac{\sec^2 2\theta\, d\theta}{1 - \tan 2\theta}$.
12. $\int \frac{1 - e^{-x}}{1 + e^{-x}} dx$.
13. $\int \left(1 - \frac{hy}{a}\right)^2 dy$.
14. $\int x(4 - x^2)^4 dx$.
15. $\int \frac{dx}{(1 - x)^4}$.
16. $\int \frac{x^2 dx}{(1 - x)^4}$.

17. $\int \frac{dx}{x^2 + 8x + 20}$.

18. $\int \cot x \log \sin x \, dx$.

19. $\int \frac{dx}{x \log^2 x}$.

20. $\int \frac{x^2 dx}{\sqrt{1 - x^6}}$.

21. $\int e^{2x} \sqrt{1 - e^{2x}} \, dx$.

22. $\int \cos \frac{x}{2} \, dx$.

23. $\int e^{\tan t} \sec^2 t \, dt$.

24. $\int \frac{x^3 + x^2 + x}{x^2 + 9} \, dx$.

25. $\int (a^{\frac{2}{3}} - y^{\frac{2}{3}})^3 \, dy$.

26. $\int \frac{(x + x^3) \, dx}{\sqrt{4 - x^4}}$.

27. $\int \frac{x^3 \, dx}{(1 - x^2)^2}$.

28. $\int x \sec^2 x \, dx$.

29. $\int \arctan x \, dx$.

30. $\int \frac{\arctan x}{1 + x^2} dx$.

31. $\int e^{\sqrt{x}} dx$.

32. $\int y \left(b - \frac{by}{a}\right)^3 dy$.

33. $\int \frac{x - x^2}{1 + x^2} \, dx$.

34. $\int e^{-x}(1 + e^{-x})^3 \, dx$.

35. $\int \frac{x(1 + x^2)}{1 + x} dx$.

36. $\int \sec^2 \theta \tan^3 \theta \, d\theta$.

37. $\int (1 - x^3)^2 \, dx$.

38. $\int (a - x)^{\frac{5}{2}} \, dx$.

39. $\int x(a - x)^{\frac{5}{2}} \, dx$.

40. $\int \frac{dx}{x^2 - 6x + 10}$.

41. $\int \frac{dx}{x \log x}$.

42. $\int e^x \log(1 + e^x) dx$.

43. $\int \frac{x^3 \, dx}{1 + x^8}$.

44. $\int \frac{e^x \, dx}{1 - 3e^x}$.

45. $\int \frac{dx}{(3x - 4)^2}$.

46. $\int \frac{\sqrt{1 + \log x}}{x} dx$.

47. $\int \log^2 x \, dx$.

48. $\int \frac{dx}{\sqrt{x}(1 + x)}$.

49. $\int x^5 e^{-x^3} \, dx$.

50. $\int \frac{x^2 \, dx}{1 - 3x^3}$.

51. Using the formula $\cos 2x = 1 - 2 \sin^2 x$, show that

$$\int \sin^2 x \, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C,$$

$$\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

52. By putting $x = a \sin \theta$, show that

$$\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \arcsin \frac{x}{a} + C.$$

CHAPTER XIII

INTEGRATION OF RATIONAL FRACTIONS

93. Preliminary step. In dealing with the integral of a rational fraction, or quotient of two polynomials, the first step (cf. § 88) is to *carry out the indicated division until the numerator is of lower degree than the denominator.*

94. Partial fractions. If the denominator is of the first degree, or of the second degree with complex roots,* the integration can be performed by the fundamental formulas. Many examples have already arisen: *e.g.* Exs. 17, 33, 35, 40, p. 136. In other cases the integral may sometimes be evaluated either directly or after a suitable substitution, as in Exs. 15, 16, 27, 50, p. 135.

In general, however, we must resort to the method of "partial fractions." The first step (after the numerator has been made of lower degree than the denominator) is to resolve the denominator into real linear or quadratic factors.† If this can be done, the given fraction can then be expressed as a sum of *partial fractions* whose denominators are the factors of the original denominator. We proceed to show how to do this.

95. Distinct linear factors. The simplest case is that in which the denominator can be broken up into real

* That is, the denominator of the form $ax^2 + bx + c$ with $b^2 - 4ac < 0$.

† The term "quadratic factor" means here a factor of the second degree whose linear factors are imaginary; *i.e.* a factor of the form $ax^2 + bx + c$ with $b^2 - 4ac < 0$.

linear factors, none of which are repeated. The process will be explained by an

Example: Evaluate $\int \frac{x^3+2}{x^3-x}dx.$

By division we find

$$\frac{x^3+2}{x^3-x} = 1 + \frac{x+2}{x^3-x}.$$

The factors of the denominator are x, $x+1$, $x-1$. Assume

$$\frac{x+2}{x^3-x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1},$$

where A, B, C are constants to be determined. Clearing of fractions, we find

$$x + 2 = A(x^2-1) + Bx(x-1) + Cx(x+1).$$

Since this relation is an *identity*, it must hold for all values of x. Hence,

putting $x = 0$, we find $A = -2$,

$x = -1$, $B = \frac{1}{2}$,

$x = 1$, $C = \frac{3}{2}$.

Thus

$$\int \frac{x^3+2}{x(x^2-1)}dx = \int \left(1 - \frac{2}{x} + \frac{1}{2}\cdot\frac{1}{x+1} + \frac{3}{2}\cdot\frac{1}{x-1}\right)dx$$

$$= x - 2\log x + \tfrac{1}{2}\log(x+1) + \tfrac{3}{2}\log(x-1) + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{1-x^2}.$

2. $\int \frac{dx}{x^2-x}.$

3. $\int \frac{x^2\,dx}{x^2-4}.$ *Ans.* $x - \log\frac{x+2}{x-2} + C.$

4. $\int \frac{(2x^3+x-1)\,dx}{x^3+x^2-4x-4}.$

Ans. $2x + \frac{17}{12}\log(x-2) - \frac{19}{4}\log(x+2) + \frac{4}{3}\log(x+1) + C.$

5. $\int \frac{x\,dx}{1-x^4}.$

6. $\int \frac{x^3\,dx}{x^2+3x+2}.$

7. $\int \frac{x^4\,dx}{x^3 + 2x^2 - x - 2}$.

Ans. $\frac{x^2}{2} - 2x + \frac{1}{6}\log(x-1) - \frac{1}{2}\log(x+1) + \frac{16}{3}\log(x+2) + C.$

8. $\int \operatorname{cosec}\theta\, d\theta$. $\left(\text{Note: } \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\sin\theta}{\sin^2\theta} = \frac{\sin\theta}{1-\cos^2\theta}.\right)$

Ans. $\frac{1}{2}\log\frac{1-\cos\theta}{1+\cos\theta} + C.$

9. $\int \frac{e^{3x}dx}{1-e^{2x}}$.

10. $\int \sec ax\, dx$.

96. Repeated linear factors. If the denominator contains a factor $(x-\alpha)^r$, the above method fails, since there would be r partial fractions with denominator $x-\alpha$, and these could be combined into a single one. We can obtain the desired result in this case by assuming, corresponding to the factor $(x-\alpha)^r$, r partial fractions

$$\frac{A}{(x-\alpha)^r} + \frac{B}{(x-\alpha)^{r-1}} + \cdots + \frac{D}{x-\alpha}.$$

Example: Evaluate $\int \frac{x^3-1}{x(x+1)^3}dx$

Assume

$$\frac{x^3-1}{x(x+1)^3} = \frac{A}{x} + \frac{B}{(x+1)^3} + \frac{C}{(x+1)^2} + \frac{D}{x+1},$$

$$x^3 - 1 = A(x+1)^3 + Bx + Cx(x+1) + Dx(x+1)^2.$$

Put $\qquad x = 0 : A = -1,$

$$x = -1 : B = 2.$$

To find C and D, we may assign any two other values to x, say $x=1$ and $x=2$, thus obtaining two simultaneous equations to solve for C and D; or we may equate the coefficients of like powers of x in the two members of the identity.

Equate coefficients of x^3: $A + D = 1,\ D = 2.$

Equate coefficients of x^2: $3A + C + 2D = 0,\ C = -1.$

Whence

$$\int \frac{x^3-1}{x(x+1)^3}dx = \int\left(-\frac{1}{x} + \frac{2}{(x+1)^3} - \frac{1}{(x+1)^2} + \frac{2}{x+1}\right)dx$$

$$= -\log x - \frac{1}{(x+1)^2} + \frac{1}{x+1} + 2\log(x+1) + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{x^3 - x^2}$.

2. $\int \frac{dx}{(1 - x)^5}$.

3. $\int \frac{x\,dx}{(1 - x)^5}$.

4. $\int \frac{(x^3 - 1)\,dx}{x(x - 2)^2}$.

5. $\int \frac{dx}{(x^2 + x)(x - 1)^2}$.

Ans. $\log x - \frac{1}{4}\log(x + 1) - \frac{3}{4}\log(x - 1) - \frac{1}{2(x - 1)} + C.$

6. $\int \frac{x^2\,dx}{x^4 + 12\,x^3 + 52\,x^2 + 96\,x + 64}$.

Ans. $\frac{-5\,x - 12}{x^2 + 6\,x + 8} + 2\log\frac{x + 4}{x + 2} + C.$

7. $\int \frac{dx}{e^{2x} - 2\,e^x}$. *Ans.* $\frac{1}{2}e^{-x} - \frac{x}{4} + \frac{1}{4}\log(e^x - 2) + C.$

97. Quadratic factors. Corresponding to a factor in the denominator of the form* $ax^2 + bx + c$ where $b^2 - 4\,ac < 0$, we assume the terms $\frac{A(2\,ax + b)}{ax^2 + bx + c} + \frac{B}{ax^2 + bx + c}$.

The case in which the denominator has repeated quadratic factors is of less importance, and will be omitted.

Example: Evaluate $\int \frac{x^2 + 4\,x + 10}{x^3 + 2\,x^2 + 5\,x}dx.$

Assume

$$\frac{x^2 + 4\,x + 10}{x^3 + 2\,x^2 + 5\,x} = \frac{A}{x} + \frac{B(2\,x + 2)}{x^2 + 2\,x + 5} + \frac{C}{x^2 + 2\,x + 5},$$

$$x^2 + 4\,x + 10 = A(x^2 + 2\,x + 5) + Bx(2\,x + 2) + Cx.$$

Put $x = 0$: $5\,A = 10,\ A = 2.$

Equate coefficients of x^2 : $A + 2\,B = 1,\ B = -\frac{1}{2}.$

Equate coefficients of x : $2\,A + 2\,B + C = 4,\ C = 1.$

* Of course such a factor might be broken up into complex linear factors, after which the process of § 95 would apply. The present method has the advantage of avoiding imaginaries.

Whence

$$\int \frac{x^2+4x+10}{x^3+2x^2+5x}dx$$

$$= \int\left(\frac{2}{x} - \frac{1}{2}\cdot\frac{2x+2}{x^2+2x+5} + \frac{1}{x^2+2x+5}\right)dx$$

$$= 2\log x - \frac{1}{2}\log(x^2+2x+5) + \frac{1}{2}\arctan\frac{x+1}{2} + C.$$

EXERCISES

Evaluate the following integrals.

1. $\int \frac{x^2\,dx}{x^2-4x+5}$. *Ans.* $x + 2\log(x^2-4x+5) + 3\arctan(x-2) + C$.

2. $\int \frac{x\,dx}{1+x^4}$.

3. $\int \frac{dx}{x^3+4x^2+8x}$.

4. $\int \frac{x^2\,dx}{a^4-x^4}$. *Ans.* $\frac{1}{4a}\log\frac{a+x}{a-x} - \frac{1}{2a}\arctan\frac{x}{a} + C$.

5. $\int \frac{x\,dx}{1-x^8}$.

6. $\int \frac{x\,dx}{x^2+x+1}$.

7. $\int \frac{x\,dx}{x^3+x^2+4x+4}$.

   Ans. $\frac{1}{10}\log(x^2+4) - \frac{1}{5}\log(x+1) + \frac{2}{5}\arctan\frac{x}{2} + C$.

8. $\int \frac{x^3\,dx}{(1+x^2)^2}$.

9. $\int \frac{x\,dx}{x^2-2x+2}$.

10. $\int \frac{\arctan x}{x^3}dx$.

11. $\int \frac{\arctan x\,dx}{x^2(1+x^2)}$.

12. $\int \frac{d\theta}{1+\tan\theta}$. *Ans.* $\frac{1}{2}\theta + \frac{1}{2}\log(\sin\theta + \cos\theta) + C$.

MISCELLANEOUS EXERCISES

Evaluate the following integrals.

1. $\int \frac{dx}{(a+bx)^2}$.

2. $\int \frac{x\,dx}{a+bx}$.

3. $\int \frac{x\,dx}{(a+x)^2}$.

4. $\int \frac{x^2\,dx}{(a+x)^2}$.

5. $\int \frac{x\,dx}{(a+x)^2(b+x)}$.

6. $\int \frac{x\,dx}{(a^2+x^2)^2}$.

7. $\int \frac{x^3 + x^2}{a^2 + x^2} dx.$

8. $\int \frac{dx}{x^2(a^2 + x^2)}.$

9. $\int \frac{x^3\,dx}{a^2 - x^2}.$

10. $\int \frac{x\,dx}{a^4 + x^4}.$

11. $\int \frac{x^3\,dx}{a^4 - x^4}.$

12. $\int \frac{x + 2}{2\,x^2 + 8\,x + 1} dx.$

13. $\int \frac{\sqrt{x}\,dx}{1 - x^2}.$

14. $\int \frac{x\,dx}{(2 - x)^6}.$

15. $\int \frac{dx}{1 - e^x}.$

16. $\int \frac{d\theta}{\tan\theta - \cot\theta}.$

17. $\int \frac{(x^2 - 2)dx}{1 + 6\,x - x^3}.$

18. $\int \frac{x^4 + 1}{(1 + x^2)^2} dx.$

CHAPTER XIV

THE DEFINITE INTEGRAL

98. The definite integral. Let $f(x)$ be a given function, $F(x)$ an integral of $f(x)$, and $x = a$ and $x = b$ two given values of x. The *change in the value of the integral* $F(x)$ as x changes from a to b is called the *definite integral of* $f(x)$ *between the "limits"* a *and* b, or simply the definite integral from a to b, and is denoted by the symbol $\int_a^b f(x)dx$. Its value is evidently $F(b) - F(a)$.

This change in the value of the integral between two values of the variable is required in many important problems. It is called the *definite* integral because its value is independent of the constant of integration.

To evaluate a definite integral, we have merely to find the indefinite integral, and then subtract its value at the "lower limit" a from its value at the "upper limit" b. It is customary to use the symbol $F(x)\Big]_a^b$ as meaning $F(b) - F(a)$. Thus

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a).$$

Since the constant of integration disappears, there is no object in writing it at all.

Examples: (*a*) In (*c*), § 77, we have for the space in the fifth second

$$\int_4^5 (32\,t - 20)dt = 16\,t^2 - 20\,t\Big]_4^5$$
$$= 300 - 176 = 124.$$

(*b*) $$\int_0^1 (x+1)^5\,dx = \frac{(x+1)^6}{6}\Big]_0^1 = \frac{64}{6} - \frac{1}{6} = \frac{21}{2}.$$

EXERCISES

Evaluate the following definite integrals.

1. $\int_{-2}^{2} x^2(x+2)\,dx.$ *Ans.* $\frac{32}{3}$.

2. $\int_{-1}^{0} \frac{dx}{1-x}.$ *Ans.* $\log 2$.

3. $\int_{0}^{\pi} \cos\frac{\theta}{2}\,d\theta.$ *Ans.* 2.

4. $\int_{0}^{2} \frac{x\,dx}{4+x^2}.$ *Ans.* $\frac{1}{2}\log 2$.

5. $\int_{-1}^{0} \frac{dx}{1+x^2}.$ *Ans.* $\frac{\pi}{4}$.

6. $\int_{0}^{\pi} \sin\phi\,d\phi.$ *Ans.* 2.

7. $\int_{-a}^{0} \frac{dx}{\sqrt{a^2-x^2}}.$ *Ans.* $\frac{\pi}{2}$.

8. $\int_{1}^{2} xe^x\,dx.$ *Ans.* e^2.

9. $\int_{1}^{e^2} \frac{dx}{x}.$

10. $\int_{0}^{2} \frac{x^3\,dx}{x+1}.$ *Ans.* 1.568.

11. $\int_{0}^{\frac{\pi}{2}} \frac{\cos\theta\,d\theta}{1+\sin^2\theta}.$ *Ans.* $\frac{\pi}{4}$.

12. $\int_{0}^{1} \arcsin x\,dx.$ *Ans.* $\frac{\pi}{2}-1$.

13. $\int_{-\pi}^{\pi} x\sin 2x\,dx.$

14. $\int_{0}^{\log 3} e^{2x}\,dx.$ *Ans.* 4.

15. A body falls from rest under gravity. Find the velocity at the end of 3 seconds, and the space described in the third second.

16. A flywheel, starting from rest, rotates under an angular acceleration of π radians per second per second. Find the number of revolutions made in the fourth second of motion.

17. A point describes a plane curve, the components of its velocity at the time t being

$$v_x = 5, \quad v_y = 24 - 32\,t.$$

Find the distance of the point from its original position at the end of 2 seconds. *Ans.* 18.9 ft.

99. Geometric interpretation of a definite integral. It was shown in § 81 that the indefinite integral $\int f(x)dx$ represents the area under the curve $y=f(x)$ between a certain fixed ordinate and a variable ordinate $x=x$. In particular, the *change* in this area as x changes from a to b is the *definite* integral $\int_a^b f(x)\,dx$. Hence:

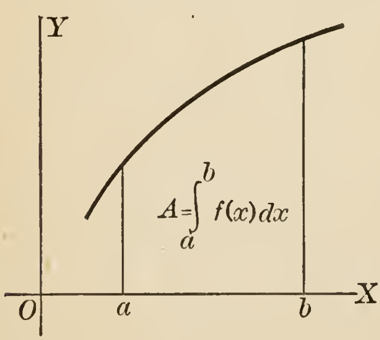


FIG. 57

The definite integral $\int_a^b f(x)dx$ *may be interpreted as the area bounded by the curve* $y=f(x)$, *the x-axis, and the lines* $x=a$, $x=b$.

EXERCISES

1. Find the area bounded by the parabola $y^2 = 4ax$, the x-axis, and the lines $x = 4a$, $x = 9a$. *Ans.* $\frac{76}{3}a^2$.

2. Find the area between the curve $y = 4 - x^2$ and the x-axis.

3. Find the area bounded by the curve $y = \log x$, the x-axis, and the line $x = 2$.

4. Find the area of one arch of the sine curve.

100. Interchanging limits. The effect of interchanging the limits in a definite integral is to change the sign of the integral. For,

$$\int_a^b f(x)dx = F(b) - F(a),$$

$$\int_b^a f(x)dx = F(a) - F(b);$$

i.e.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

101. Change of limits corresponding to a change of variable. In the definite integral $\int_a^b f(x)dx$ it is of course implied that a and b are the limiting values of the variable of integration x. If we change the variable by a substitution

$$x = \phi(z),$$

we must either *return to the original variable* before substituting the limits, or *change the limits to correspond with the change of variable.* The latter method is usually preferable.

The new limits are found, of course, from the equation of substitution

$$x = \phi(z),$$

as in the following

Example: Evaluate $\int_{-1}^{1} x\sqrt{1-x}\,dx.$

Let

$$1 - x = z,\ x = 1 - z,\ dx = -dz.$$

When

$$x = -1, \quad z = 2;$$

when

$$x = 1, \quad z = 0.$$

Hence

$$\int_{-1}^{1} x\sqrt{1-x}\,dx = -\int_{2}^{0}(1-z)z^{\frac{1}{2}}\,dz$$

$$= \int_{2}^{0}(z^{\frac{3}{2}} - z^{\frac{1}{2}})dz = \tfrac{2}{5}z^{\frac{5}{2}} - \tfrac{2}{3}z^{\frac{3}{2}}\Big]_{2}^{0} = -\tfrac{4}{15}\sqrt{2}.$$

EXERCISES

1. Work the above example, putting $1 - x = z^2$.

Evaluate the following integrals.

2. $\int_{1}^{6}\frac{x\,dx}{\sqrt{3+x}}.$ *Ans.* $\frac{20}{3}$.

3. $\int_{1}^{2}x^3\sqrt{x^2-1}\,dx.$ *Ans.* $\frac{14}{5}\sqrt{3}$.

4. $\int_{0}^{4}\frac{dx}{1+\sqrt{x}}.$ *Ans.* 1.802.

5. $\int_{0}^{\log 2}\sqrt{e^x-1}\,dx.$ *Ans.* $2 - \frac{\pi}{2}$.

6. $\int_{0}^{1}\frac{x\,dx}{(x+1)^4}.$ *Ans.* $\frac{1}{12}$.

7. $\int_{0}^{a}\frac{x^3\,dx}{(a^2+x^2)^{\frac{3}{2}}}.$ *Ans.* $(\frac{3}{2}\sqrt{2}-2)a$.

8. $\int_{2}^{3}\frac{x\,dx}{1-x^4}.$

9. $\int_{0}^{8}\frac{dx}{x^{\frac{1}{3}}+1}.$

10. The area bounded by the parabola $y^2 = 4\,ax$, the x-axis, and the latus rectum is, by § 99,

$$A = \int_{0}^{a} y\,dx.$$

Evaluate this integral (*a*) by substituting for y; (*b*) by substituting for dx and changing limits.

11. Find the area under the curve $y = e^x$ from $x = 0$ to $x = 1$ by the two methods of Ex. 10.

12. Find the area of half an arch of the cosine curve by the two methods of Ex. 10.

13. Given $y = \sin x$, evaluate $\int_0^1 x\,dy$ in two ways.

14. The velocity of a point moving in a straight line is

$$v = 4 \cos \frac{t}{2}.$$

Find in two ways the distance from the starting point at the end of $\frac{\pi}{2}$ seconds.

CHAPTER XV

THE DEFINITE INTEGRAL AS THE LIMIT OF A SUM

102. Area under a curve. We have seen in § 99 that the area $ABCD$ bounded by the plane curve $y = f(x)$, the x-axis, and the lines $x = a$, $x = b$ is given by the definite integral $\int_a^b f(x)dx$. We will now obtain a new expression for the same area.

In what follows, the function $f(x)$ is assumed to be one-valued and continuous, and to have only a finite number of maxima and minima in the interval from $x = a$ to $x = b$; in fact, we may suppose for definiteness that the curve rises throughout the interval. The argument is readily modified to fit the case when the curve steadily falls, or rises and falls alternately.

We may evidently get an *approximate* expression for the area A by dividing the base AB into n equal intervals Δx, erecting the ordinates at the points of division, and taking the sum of the inscribed rectangles $AEFD$, etc. The areas of these rectangles are respectively

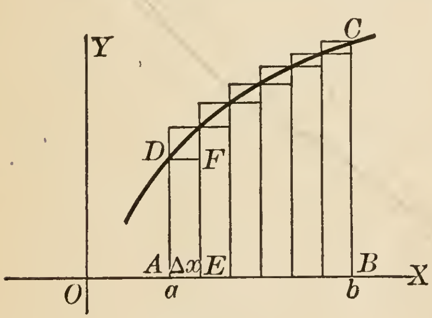


FIG. 58

$$f(x_1)\Delta x,\ f(x_2)\Delta x, \cdots,\ f(x_n)\Delta x,$$

where

$$\begin{aligned} x_1 &= a, \\ x_2 &= a + \Delta x, \\ x_3 &= a + 2\,\Delta x, \\ &\vdots \\ x_n &= a + (n-1)\Delta x = b - \Delta x. \end{aligned}$$

Hence, *approximately*,

$$A = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

Now it is geometrically evident that this sum of rectangles may be made to represent the area A with an error less than any preassigned constant however small, by taking n sufficiently large. Hence, by the definition of § 14, we have *exactly*

(1) $$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

A formal proof of this statement may be given as follows. Let A_1 denote the sum of the inscribed rectangles, A_2 the sum of the circumscribed rectangles (Fig. 59). Then for all values of n

$$A_1 < A < A_2.$$

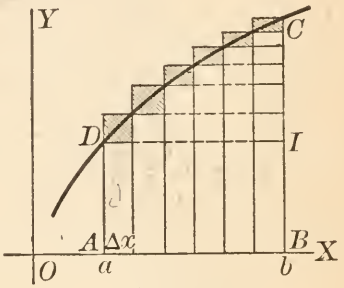


FIG. 59

Now the difference between A_2 and A_1 is the sum of the shaded rectangles. Sliding all these across into the last column, we see that

$$A_2 - A_1 = IC \cdot \Delta x = [f(b) - f(a)]\Delta x.$$

As n increases indefinitely Δx approaches 0, so that $A_2 - A_1$ also approaches 0. Since A always lies between A_1 and A_2, it follows that A_1 and A_2 approach A as their common limit. Hence we have formula (1).*

The rectangles $f(x_i)\Delta x$ are called *elements* of area. As n increases indefinitely, each of the elements approaches 0: *i.e.* they are infinitesimals.

* The n parts into which AB is divided need not be taken all equal; the same limit is obtained provided the width of each rectangle approaches 0 as the number of divisions is indefinitely increased. Further, it is clear that we may take the limit of the sum of either the inscribed or the circumscribed rectangles, or of any set intermediate between these two.

Questions like that of the present article, in which we have to deal with the limit of a sum of infinitesimal elements, will arise many times in this and later chapters. As in every case the existence of the limit will be evident by geometric intuition, we shall in future omit formal proofs.

103. Evaluation of the limit. Equating the values of A found in §§ 99 and 102, we find

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx.$$

Thus the limit occurring in § 102 can always be evaluated by a definite integration.*

The fact that the quantity $f(x)dx$ appearing under the integral sign represents the area of a rectangle of altitude $f(x)$ and base $dx = \Delta x$, and thus suggests the sum from which the integral was derived, is the chief reason for using the notation $\int f(x)dx$ (see § 77). In fact, the integral sign $\int$ is historically a somewhat conventionalized S, meaning sum.

104. The fundamental theorem. In § 102 we have expressed the area under a plane curve as the limit of a sum of rectangles; in § 99 we have found the same area as a definite integral. It is clear that the arguments used will hold *no matter what may be the geometric or physical meaning of the given function*, for any function whatever may be interpreted as the ordinate of a point on a plane curve. We therefore have at once the following

FUNDAMENTAL THEOREM FOR DEFINITE INTEGRALS: *Given a function $f(x)$, continuous in the interval from $x = a$ to $x = b$, divide this interval into n equal parts Δx,*

* More precisely, the limit can always be *expressed* as a definite integral; the actual evaluation of the integral is often impossible (see § 78).

and form the sum $\sum_{i=1}^{n} f(x_i)\Delta x$, *where* $x_1 = a$, $x_2 = a +$ $\cdots$, $x_n = a + (n-1)\Delta x$. *Then*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx.$$

Hence, if in any problem an arbitrarily close approximation to the required quantity can be found by adding up terms of the type* $f(x)\Delta x$ *from* $x = a$ *to* $x = b$, *that quantity is given exactly by the definite integral* $\int_a^b f(x)dx$.

We have now presented the definite integral in two distinct aspects: first, as the change in the value of the indefinite integral between two values of the variable; second, as the *limit of a sum of infinitesimal elements.* The great advantage of this latter point of view will become apparent as we proceed.

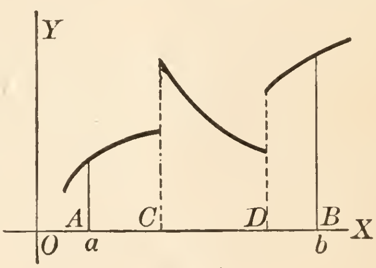


FIG. 60

It may be remarked that if the function $f(x)$ has a finite number of finite discontinuities in the interval from a to b, as in Fig. 60, it can still be integrated. For it will be continuous in a number of subintervals such as AC, CD, DB, to each of which the fundamental theorem can be applied and the results added.

105. Plane areas in cartesian coördinates. Not only the area considered in § 102, but *any* plane area bounded by curves whose equations are given in cartesian coördinates, can be found as follows. Imagine inscribed in the area a set of n elementary rectangles of altitude h_i and width Δl, in such a way that the sum $\sum_{i=1}^{n} h_i\Delta l$ may be made to repre-

* That is, an approximation in which the error may be made less than any preassigned constant.

Que[illegible]e area to any desired degree of approximation by hav[illegible]asing n. Then at once, by the fundamental theorem, r[illegible]

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} h_i \Delta l = \int h\, dl,$$

the limits being chosen in such a way as to extend the integration over the whole area. Of course in any particular problem h and dl must be expressed in terms of the coördinates.

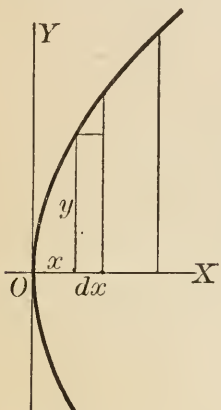


Fig. 61

In every problem the student should make a sketch of the area to be found, draw an element in a general position, and obtain the area of the element directly from the figure, as in the following

Examples: (*a*) Find the area in the first quadrant bounded by the parabola $y^2 = 4ax$, the x-axis, and the line $x = a$.

An arbitrarily close approximation to this area can be found by forming a sum of rectangles of altitude y, base dx, and area $y\,dx$, as shown in Fig. 61. Hence, we have *exactly*

$$A = \int_0^a y\, dx = 2\int_0^a \sqrt{ax}\, dx = \frac{2}{a} \cdot \frac{2}{3}(ax)^{\frac{3}{2}}\Big]_0^a = \frac{4}{3}a^2.$$

(*b*) Find the area in the first quadrant between the parabolas

(1) $y^2 = 4\,ax,$

(2) $y^2 = 8\,ax - 4\,a^2.$

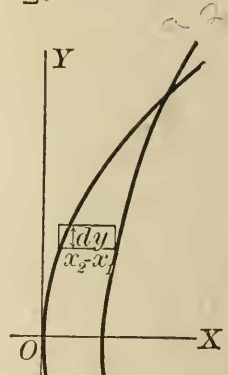


Fig. 62

Let us take as the element a rectangle parallel* to OX. The area of the rectangle is evidently $(x_2 - x_1)dy$, where x_1 and x_2 are the abscissas of the points on the curves (1) and (2) respectively. The

* That is, with its finite side parallel to OX.

curves are found to intersect at $(a, 2a)$. Hence

$$A = \int_0^{2a} (x_2 - x_1)\, dy = \int_0^{2a} \left(\frac{y^2}{8a} + \frac{a}{2} - \frac{y^2}{4a}\right) dy$$

$$= \int_0^{2a} \left(\frac{a}{2} - \frac{y^2}{8a}\right) dy = \left[\frac{ay}{2} - \frac{y^3}{24a}\right]_0^{2a} = \frac{2}{3} a^2.$$

EXERCISES

1. Find the area bounded by the curve $y = x^3$ and (a) the lines $y = 0$, $x = 2$, (b) the lines $x = 0$, $y = 1$.

2. Solve example (a), § 105, taking the element parallel to OX. Evaluate the integral in two ways.

3. Find the area bounded by the curve $ay^2 = x^3$ and the line $x = 4a$. *Ans.* $\frac{128}{5} a^2$.

4. Find the area of a circle (see Ex. 52, p. 136).

5. Find the area of an ellipse, using the cartesian equation; check by using the equations $x = a \cos \phi$, $y = b \sin \phi$ (see Ex. 51, p. 136). *Ans.* πab.

6. Find the area of half an arch of the curve $y = \frac{1}{2} \cos 2x$.

7. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. *Ans.* $\frac{16}{3} a^2$.

8. Show that the area bounded by a parabola and any chord perpendicular to the axis is two thirds of the circumscribing rectangle.

9. Find in two ways the area bounded by the parabola $y = x^2$, the y-axis, and the lines $y = 1$, $y = 4$.

10. Find the area bounded by the curve $y = \log x$, the axes, and the line $y + 1 = 0$.

11. Find the area bounded by the curve $y = \log x$, the x-axis, and the line $x = 2$. *Ans.* 0.386.

12. Find the area under the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, from $x = -a$ to $x = a$. *Ans.* $a^2\left(e - \frac{1}{e}\right)$.

13. Find the area between the curve $x^2 = 4a^2 - ay$ and the x-axis, taking the element (a) parallel to OY; (b) parallel to OX.

14. Find the area bounded by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes. *Ans.* $\frac{1}{6} a^2$.

15. Find the area of a circular segment of height h. Check by putting $h = 2r$.

16. Find the area of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. *Ans.* $3\pi a^2$.

17. Find the area bounded by the curves $y = x$, $y = 2x$, $y = x^2$. *Ans.* $\frac{7}{6}$.

18. Find in two ways the area in the first quadrant bounded by the curves $y = x^3$, $x^2 = 2 - y$, $y = 0$.

19. Find in two ways the area bounded by the curve $y = (1 - x^2)^2$ and the x-axis.

20. Find the area bounded by the curve $y = (x - 3)^2(x - 2)$ and the x-axis.

21. Find the area bounded by the curve $y = \frac{2x}{1 + x^2}$, its asymptote, and the maximum ordinate. *Ans.* 0.693.

22. Trace the curve $y^2(x^2 + a^2) = a^2x^2$, and find the area bounded by the curve and the line $x = a$. *Ans.* $0.83\,a^2$.

23. Trace the curve $ay^2 = ax^2 + x^3$, and find the area of the loop.

24. Find the area bounded by the curve $y^3 = x^3(1 + x)$ and the x-axis. Why is the answer negative?

25. Trace the curve $y = \frac{3 - 2x}{4 + x^2}$, and find the area under the curve from $x = -2$ to $x = 0$. *Ans.* 1.87.

26. Find in two ways the area bounded by the coördinate axes and the curve $y^3 = 1 - 2x - xy$.

27. Find the area of the circle $x = a\sin 2\theta$, $y = a\cos 2\theta$.

28. Find the area bounded by the curve $y = \frac{\log x}{x}$, the x-axis, and the maximum ordinate. *Ans.* $\frac{1}{2}$.

29. Find the area in the first quadrant under the curve $y^2 = \frac{x^2}{x - 1}$, between the minimum ordinate and the ordinate at $x = 3$. *Ans.* 2.05.

30. Trace the curve $y^2 = x^4(x + 4)$, and find the area inclosed by it. *Ans.* $\frac{2^{12}}{105}$.

106. Plane areas in polar coördinates. Given the equation

$$r = f(\theta)$$

of a plane curve in polar coördinates, let us try to find the area bounded by the curve and the radii vectores corresponding to $\theta = \alpha$, $\theta = \beta$. We can obtain an arbitrarily close approximation to the area by inscribing in it n circular sectors of radius r_i and angle $\Delta\theta$, hence of area* $\frac{1}{2} r_i^2 \Delta\theta$, and forming the sum $\sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta$. Hence, by the fundamental theorem,

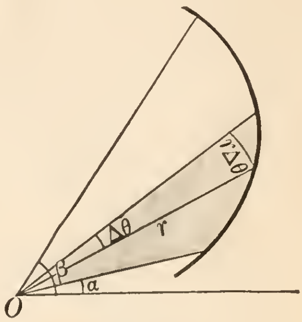


FIG. 63

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta.$$

By means of a theorem to be proved in § 109, it can be shown that this result may also be obtained by choosing as the element a *triangle* of altitude r_i, base $r_i \Delta\theta$, and area $\frac{1}{2} r_i^2 \Delta\theta$.

EXERCISES

1. Find the area swept out by the radius vector of the spiral of Archimedes $r = a\theta$, in the interval from $\theta = 0$ to $\theta = 2\pi$.

2. Solve Ex. 1 for the logarithmic spiral $\log r = a\theta$.

3. Find by integration the area of the triangle bounded by the lines $r = a \sec \theta$, $\theta = 0$, $\theta = \frac{\pi}{4}$.

4. Find the area inside the lemniscate $r^2 = a^2 \cos 2\theta$. *Ans.* a^2.

5. Find the area of the curve $r^2 = a^2 \cos \theta$.

6. Find the entire area of the cardioid $r = a(1 + \cos \theta)$. *Ans.* $\frac{3}{2} \pi a^2$.

7. Find the area between the parabola $r = a \sec^2 \frac{\theta}{2}$ and its latus rectum. *Ans.* $\frac{8}{3} a^2$.

8. Find the area of the curve $r = a \sin 2\theta$. *Ans.* $\frac{1}{2} \pi a^2$.

9. Show that the area of one loop of the curve $r = a \cos n\theta$ is $\frac{\pi a^2}{4n}$, hence the total area inside the curve is one fourth or one half the area of the circumscribed circle, according as n is odd or even.

* By elementary geometry, the area of a circular sector of radius r and angle α is

$$A = \frac{1}{2} r^2 \alpha.$$

107. Volumes of revolution. The volume of a solid of revolution may be found very readily by the fundamental theorem.

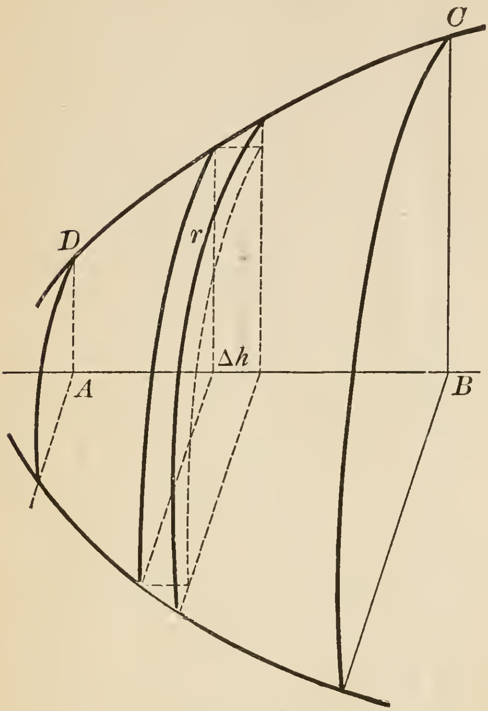


FIG. 64

Suppose the volume is generated by revolving the area $ABCD$ about the line AB. If we inscribe in the revolving area a set of n rectangles of altitude r_i and base Δh, each rectangle will generate in its rotation a *circular disk*, or cylinder, of radius r_i, altitude Δh, and volume $\pi r_i^2 \Delta h$. Further, as n increases the sum of these cylindrical volumes approaches as its limit the required volume. Hence, by the fundamental theorem,

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi r_i^2 \Delta h = \pi \int r^2 dh,$$

the limits being chosen so as to include the whole volume. Of course in any problem both r and dh must be expressed in terms of the coördinates.

When the axis of revolution does not form part of the boundary of the revolving area, we may choose as elements a set of *circular rings*, as in example (b) below.

Examples: (a) The area in example (a), § 105, revolves about OX. Find the volume generated.

Dividing the area into elements as in Fig. 61, we see that each rectangle generates a cylindrical volume-element of radius y, altitude dx, and volume $\pi y^2 dx$. Hence

$$V = \pi \int_0^a y^2\, dx = 4\pi a \int_0^a x\, dx = 2\pi a^3.$$

(*b*) The above area rotates about OY. Find the volume generated.

If we divide the area into elements as in the figure, each element generates a *circular ring* of outer radius a, inner radius x, thickness dy, and volume $\pi(a^2 - x^2)dy$. Further, the limit of the sum of these volumes is the required volume. Hence

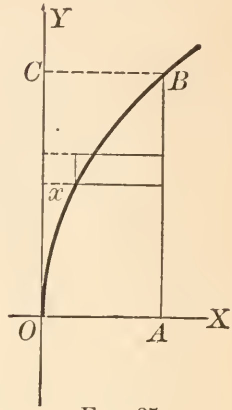


FIG. 65

$$V = \pi \int_0^{2a} (a^2 - x^2) dy = \pi \int_0^{2a} \left(a^2 - \frac{y^4}{16\,a^2}\right) dy$$
$$= \pi \left[a^2 y - \frac{y^5}{80\,a^2} \right]_0^{2a} = \frac{8}{5}\pi a^3.$$

This result could have been obtained equally well by finding the volume generated by rotating the area OBC about OY, and subtracting this from the volume of the cylinder formed by revolving the rectangle $OABC$. But in case it is possible to simplify the integral before integrating, as often happens, the first method is to be preferred.

108. Volumes of revolution: second method. The following method for finding volumes of solids of revolution is often preferable to that of § 107.

Let us take as an element of the area ABC (Fig. 66) a rectangle of length h_i parallel to the axis of revolution AB, and of width Δr. This rectangle generates by its rotation a *cylindrical shell* of inner radius r_i, altitude h_i, and thickness Δr. The volume of the shell is evidently

$$\pi(r_i + \Delta r)^2 h_i - \pi r_i^2 h_i = 2\,\pi r_i h_i \Delta r + \pi h_i \overline{\Delta r}^2,$$

and the limit of the sum of these elementary shells is evidently the required volume. Now it will be shown in the next article that, in passing to the limit, we may *neglect the infinitesimal of higher order** $\pi h_i \overline{\Delta r}^2$. Hence, by the fundamental theorem,

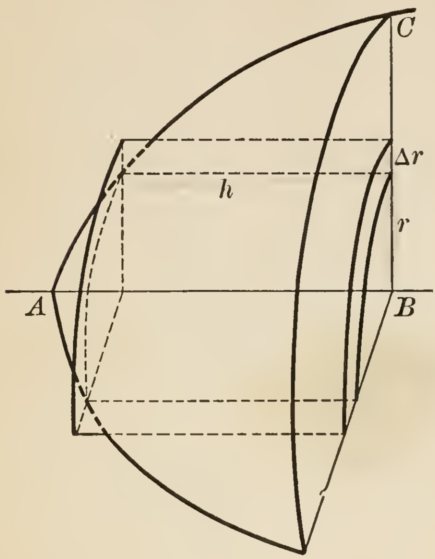


FIG. 66

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi r_i h_i \Delta r$$

$$= 2\pi \int rh\, dr,$$

the integration being extended through the whole region.

This result is easily remembered from the fact that the integrand is the differential of volume of a right circular cylinder, the altitude being constant:

$$V = \pi r^2 h, \quad dV = 2\pi rh dr.$$

Example: Solve example (*b*), § 107, by the present method.

Divide the rotating area into rectangles parallel to OY, as in Fig. 61. Each rectangle generates a cylindrical shell of radius x, altitude y, and thickness dx. Hence

$$V = 2\pi \int_0^a xy\, dx = 4\pi a^{\frac{1}{2}} \int_0^a x^{\frac{3}{2}}\, dx = \tfrac{8}{5}\pi a^3.$$

109. A theorem on infinitesimals. It often happens, as in the preceding article, that in applying the fundamental

* This is easily shown directly. The quantity neglected is $\sum_{i=1}^{n} \pi h_i \Delta r^2$. This may be written $\pi \Delta r \sum_{i=1}^{n} h_i \Delta r$. When Δr approaches 0, the sum $\sum_{i=1}^{n} h_i \Delta r$ approaches a finite limit, viz. the generating area, so that the whole quantity approaches 0.

theorem we have to replace the element as originally chosen by another element differing from the first one by an infinitesimal of higher order. That this is allowable appears from the following

THEOREM: *The limit of a sum of positive infinitesimals is unchanged when each infinitesimal is replaced by another that differs from it by an infinitesimal of higher order.*

It follows that, in taking the limit of such a sum, *all infinitesimals of higher order may be neglected*, as was done in § 108.

In this connection it should perhaps be mentioned explicitly that two infinitesimals differ from each other by an infinitesimal of higher order whenever the limit of their ratio is 1, and conversely.

To prove the theorem, let $u_1, u_2, \cdots, u_n$ be a set of positive infinitesimals such that $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} u_i$ exists; and let $v_1, v_2, \cdots, v_n$ be another set such that v_i differs from u_i by an infinitesimal of higher order:

$$v_i = u_i + w_i u_i,$$

where w_i is infinitesimal. Then

$$\lim_{n\to\infty}\sum_{i=1}^{n} v_i = \lim_{n\to\infty}\sum_{i=1}^{n} u_i + \lim_{n\to\infty}\sum_{i=1}^{n} w_i u_i.$$

Denoting by $\overline{w}$ the absolute value of the largest of the w's, we have

$$-\overline{w}\sum_{i=1}^{n} u_i \leqq \sum_{i=1}^{n} w_i u_i \leqq \overline{w}\sum_{i=1}^{n} u_i.$$

Since the first and third of these quantities both approach 0, the second must do likewise. Hence

$$\lim_{n\to\infty}\sum_{i=1}^{n} v_i = \lim_{n\to\infty}\sum_{i=1}^{n} u_i,$$

and the theorem is proved.

EXERCISES

1. Find the volume of a sphere.

2. Find the volume of a right circular cone.

3. The hyperbola $x^2 - y^2 = a^2$ revolves about its transverse axis. Find the volume of a segment of height a of the hyperboloid generated. *Ans.* $\frac{4}{3}\pi a^3$.

4. Find the volume generated by revolving the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about OX. *Ans.* $\frac{32}{105}\pi a^3$.

5. Find the volume generated by revolving the area under the curve $y = e^x$ from $x = 0$ to $x = 1$ (*a*) about OX; (*b*) about OY; (*c*) about the line $x = 1$. *Ans.* (*c*) $2\pi(e - 2)$.

6. The area under one arch of the sine-curve revolves (*a*) about OX; (*b*) about OY. Find the volume generated.

Ans. (*a*) $\frac{\pi^2}{2}$; (*b*) $2\pi^2$.

7. Find the volume obtained by revolving about OX the area under the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, from $x = -a$ to $x = a$.

Ans. $\frac{\pi a^3}{4}\left(e^2 + 4 - \frac{1}{e^2}\right)$.

8. The area OBC of Fig. 65 revolves (*a*) about the line CB; (*b*) about AB. Find the volume generated; check by solving in two ways.

9. Find the volume of an oblate spheroid, using (*a*) the ordinary equation of the ellipse; (*b*) the parametric equations $x = a\cos\phi$, $y = b\sin\phi$. Solve each part in two ways. *Ans.* $\frac{4}{3}\pi a^2 b$.

10. The area in example (*b*), § 105, revolves about OX. Find the volume generated.

11. Find the volume of a spherical segment of height h. Check by putting $h = 2r$.

12. Trace the curve $a^2y^2 = x^3(2a - x)$, and find the volume generated by revolving the curve about the x-axis.

13. Find the volume generated by revolving (*a*) about OX, (*b*) about OY, the area between the curves $2y = x^3$, $y = x^2$. Check the results by solving in two ways.

14. Trace the curve $(x - 4a)y^2 = ax(x - 3a)$, and find the closed volume generated by revolving it about the x-axis.

Ans. $6.12\,a^3$.

15. The curve $y^2 = x(x-1)(x-2)$ rotates about the x-axis. Find the closed volume generated. *Ans.* $\frac{\pi}{4}$.

16. Find the volume generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about the x-axis. *Ans.* $5\pi^2a^3$.

17. Trace the curve $y(x^2+y^2) = a(x^2-y^2)$, and find the volume generated by revolving the loop about OY. *Ans.* $0.053\,\pi a^3$.

18. Find the volume of a torus. Solve in two ways. *Ans.* $2\pi^2a^2b$.

19. The area bounded by the curve $y = (1-x^2)^2$ and the x-axis revolves about the y-axis. Find the volume generated, (a) by the method of § 107; (b) by the method of § 108. In (a) evaluate the integral in two ways, first by substituting for x^2, next by substituting for dy.

20. Trace the curve $a^4y^2 = a^2x^4 - x^6$, and find the volume generated by revolving one loop about OY. *Ans.* $\frac{8}{15}\pi a^3$.

21. A round hole of radius a is bored through the center of a sphere of radius $2a$. Find the volume cut out.

22. Find the closed volume generated by revolving the curve $y^2 = \frac{x^2(x^2-a^2)}{x^2-4a^2}$ about the x-axis. Trace the curve. *Ans.* $0.072\,\pi a^3$.

23. Find the volume inside the cylinder $x^2+y^2 = 2a^2$ and outside the hyperboloid $x^2+y^2-z^2 = a^2$. *Ans.* $\frac{4}{3}\pi a^3$.

24. Find the closed volume generated by revolving the curve $y^2 = x^4(x+4)$, (a) about the x-axis, (b) about the y-axis.

25. Find in two ways the volume generated by revolving about the y-axis the area bounded by the curve $y = \frac{\sin x}{x}$ and the coördinate axes.

110. Other volumes. The volume of any solid can be expressed as a definite integral, provided we know the area of every plane section parallel to some fixed plane. Let us divide the volume into slices of thickness Δh by

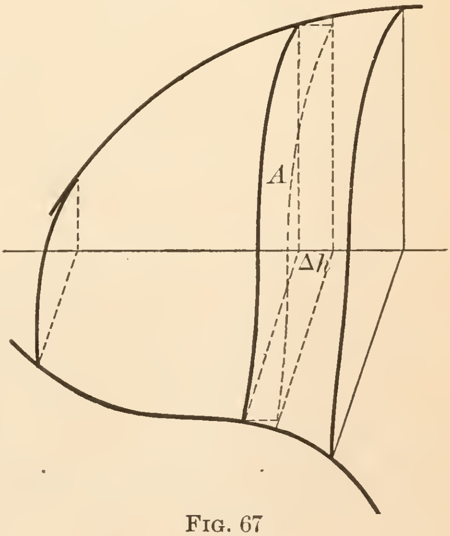


FIG. 67

means of n planes parallel to this fixed plane. If on each of these plane sections we erect a cylinder of altitude Δh and base A, where A is the area of the section, the sum of the n cylindrical volumes thus formed will be approximately the required volume, and the limit of this sum will be exactly the volume. Hence

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A_i \, \Delta h = \int A \, dh,$$

with properly chosen limits.

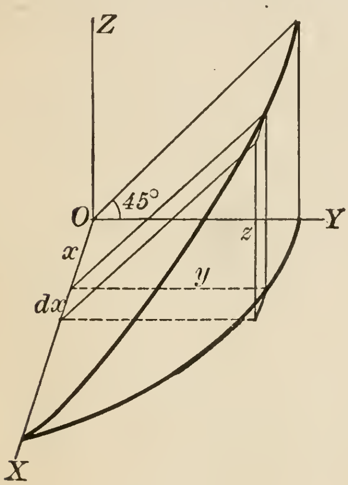


FIG. 68

Example: A woodsman chops halfway through a tree 4 ft. in diameter, one face of the cut being horizontal, the other inclined at 45°. Find the volume of wood cut out.

The figure shows one half of the required volume. If we slice up the volume by planes parallel to the yz-plane, the element of volume is a triangular plate of width y, altitude z, and thickness dx. Hence

$$V = 2\int_0^2 \tfrac{1}{2}\, yz \, dx.$$

But

$$z = y, \text{ and } y = \sqrt{4 - x^2},$$

so that

$$V = \int_0^2 (4 - x^2)\, dx = 5\tfrac{1}{3} \text{ cu. ft.}$$

EXERCISES

1. Solve the above example in a different way.

2. Find the volume of a tetrahedron with three mutually perpendicular faces. *Ans.* $\frac{1}{6}abc$.

3. Find the volume sliced off from a right circular cylinder by a plane through a diameter of one base and tangent to the other base. *Ans.* $\frac{2}{3}a^2h$.

4. Find the volume of a right pyramid with a square base.

5. Find the volume of an ellipsoid, using the answer to Ex. 5, p. 153. *Ans.* $\frac{4}{3}\pi abc$.

6. Find the volume of an elliptic cone. *Ans.* $\frac{1}{3}\pi abh$.

7. Find the volume of a spherical wedge. *Ans.* $\frac{2}{3}\alpha a^3$.

8. Find the volume of a wedge cut from a right circular cone by two planes through the axis. *Ans.* $\frac{1}{6}\alpha a^2 h$.

9. Obtain a formula for the volume of a wedge cut from any solid of revolution by two planes through the axis.

10. A carpenter chisels a square hole of side 2 in. through a round post of radius 2 in., the axis of the hole intersecting that of the post at right angles. Find the volume of wood cut out. *Ans.* 15.3 cu. in.

11. Find the volume cut from the cylinder $x^2 + y^2 = a^2$ by the planes $z = mx$, $z = nx$. Solve in two ways.

12. A right circular conoid is generated by a straight line which moves always parallel to the xy-plane and passes through the line $y = h$ in the yz-plane and the circle $x^2 + z^2 = a^2$ in the xz-plane. Find the volume of the conoid. *Ans.* $\frac{1}{2}\pi a^2 h$.

13. Find the volume in the first octant bounded by the hyperbolic paraboloid generated by a straight line moving always parallel to the xy-plane and passing through the lines $y + z = a$ in the yz-plane and $x = b$ in the xz-plane. *Ans.* $\frac{1}{4}a^2 b$.

14. A banister cap is bounded by two equal cylinders of revolution 8 in. in diameter, whose axes intersect at right angles in the plane of the base of the cap. Find the volume of the cap in two ways.

15. Find the volume of a right pyramid whose base is a regular hexagon.

16. Find the volume in the first octant under the plane $z = x$ and inside a cylinder standing on the parabola $y = 4 - x^2$ as a base. Solve in two ways.

17. Solve Ex. 12 if the line $y = h$ is replaced by the line $y + z = h$ $(h > a)$. *Ans.* $\frac{1}{2}\pi a^2 h$.

18. Solve Ex. 13 if the line $x = b$ is replaced by the line $x = z$.

19. Find the volume in the first octant bounded by the planes $x + z = a$, $x + 2y + 2z = 2a$.

111. Line integrals. The ordinary definite integral depends on all the values of a given function $f(x)$ along a straight line segment — the segment of the x-axis from

$x = a$ to $x = b$. It happens frequently that we have to compute a quantity that depends in a similar way on the values of a function F along a *curvilinear arc* C. The function F is in general dependent on both coördinates of the point on the curve:

$$F = F(x, y).$$

But since y is given as a function of x by the equation of the curve, the function $F(x, y)$ reduces at once to a function of one variable.

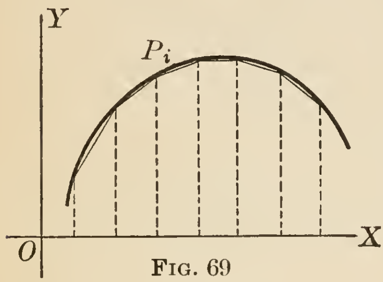


FIG. 69

Given a function $F(x, y)$ defined at all points of a plane curve C, let us inscribe in C a broken line of n segments $\Delta s_i'$ having equal projections Δx on the x-axis, multiply each segment by the value of $F(x, y)$ at the corresponding point of division* P_i (Fig. 69), and form the sum of these products, $\sum_{i=1}^{n} F(x_i, y_i)\,\Delta s_i'$. The limit of this sum, as the number of divisions becomes infinite, is called the *line integral* of $F(x, y)$ along the arc C, and is denoted by the symbol $\int_C F(x, y)\,ds$:

$$\lim_{n \to \infty} \sum_{i=1}^{n} F(x_i, y_i)\,\Delta s_i' = \int_C F(x, y)\,ds.$$

112. Geometric interpretation of the line integral. The existence of the limit last written may be made evident geometrically. Let us interpret the function $F(x, y)$ as the z-coördinate of a point on a surface in space:

$$z = F(x, y). \tag{1}$$

* It is of course merely for convenience that the broken-line segments are drawn with equal projections on OX. The division may be made in any manner provided in the limit every segment approaches 0. Further, $\Delta s_i'$ may be multiplied by the value of $F(x, y)$ at either end-point of $\Delta s_i'$ or at any point on the subtended arc.

On the curve C as directrix erect a cylindrical surface with generators perpendicular to the xy-plane. Each of the quantities $F(x_i, y_i)\ \Delta s_i'$ represents the area of a rectangle inscribed in this cylinder, having a base $\Delta s_i'$ and an altitude $F(x_i, y_i)$, and the sum of these rectangles evidently approaches as its limit that part of the cylindrical surface lying between the xy-plane and the surface (1).

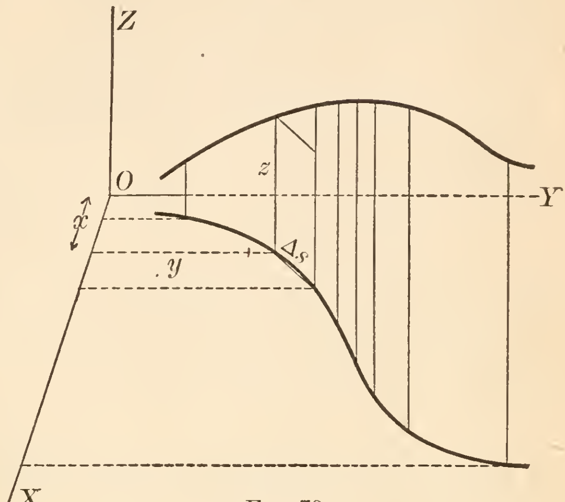


FIG. 70

113. Fundamental theorem for line integrals. The theorem of § 104 takes the following form for line integrals:

Given a function $F(x, y)$ defined at all points of an arc C, inscribe in the arc a broken line of n segments $\Delta s_i'$ having equal projections on OX, and form the sum $\sum_{i=1}^{n} F(x_i, y_i)\ \Delta s_i'$, where (x_i, y_i) is the i-th point of division on C. Then

$$\lim_{n\to\infty} \sum_{i=1}^{n} F(x_i, y_i)\Delta s_i' = \int_C F(x, y)\, ds.$$

Hence, if in any problem an arbitrarily close approximation to the required quantity can be found by adding up terms of the type $F(x, y)\ \Delta s'$, the quantity is given exactly by $\int_C F(x, y)\, ds$.

114. Evaluation of line integrals. To evaluate a line integral, we have in general to express both $F(x, y)$ and ds in terms of x, y, or some other suitable variable, and then integrate between limits in such a way as to extend the integration over the given arc.

Thus, if x is chosen as the variable of integration, we replace ds by its value (§ 52)

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

and obtain

$$\int_C F(x, y) ds = \int_a^b F(x, y) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

where a and b are the abscissas of the end points of C,* and where y must be replaced by its value in terms of x from the equation of the curve.†

* It is assumed that no parallel to the y-axis can meet C in more than one point. If this condition is not satisfied, C must consist of several portions for each of which the condition holds, and each portion may be considered separately.

† This transformation of the line integral into an ordinary integral may be justified by the theorem of § 109. We have evidently

$$\Delta s_i' = \sqrt{\overline{\Delta x}^2 + \overline{\Delta y_i}^2} = \sqrt{1 + \frac{\overline{\Delta y_i}^2}{\overline{\Delta x}^2}}\, \Delta x.$$

Now the limit of the sum $\sum_{i=1}^{n} F(x_i, y_i) \sqrt{1 + \frac{\overline{\Delta y_i}^2}{\overline{\Delta x}^2}}\, \Delta x$ cannot be expressed directly as a definite integral by the theorem of § 104, since the summand depends not only on x_i but on Δx as well. But the infinitesimals $F(x_i, y_i)\sqrt{1 + \frac{\overline{\Delta y_i}^2}{\overline{\Delta x}^2}}\, \Delta x$ and $F(x_i, y_i)\sqrt{1 + y'^2}\, \Delta x$ differ from each other by an infinitesimal of higher order, since the limit of their ratio is evidently 1, and hence the latter may be substituted for the former, by § 109. Therefore

$$\begin{aligned}\int_C F(x, y)\, ds &= \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i, y_i) \sqrt{1 + \frac{\overline{\Delta y_i}^2}{\overline{\Delta x}^2}}\, \Delta x \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i, y_i) \sqrt{1 + y_i'^2}\, \Delta x \\ &= \int_a^b F(x, y) \sqrt{1 + y'^2}\, dx,\end{aligned}$$

by the fundamental theorem of § 104.

To integrate with respect to y, we put

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy,$$

and express the entire integrand in terms of y.

If x and y are given in terms of a parameter t, we use

$$ds = \sqrt{\overline{dx}^2 + \overline{dy}^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In the discussion of line integrals, we have spoken in terms of cartesian coördinates, but the argument is evidently independent of the particular coördinate system used.

Some simple types of line integrals are considered in the next three sections; other examples will be met with later.

115. Length of a curvilinear arc. To find the length of an arc of a plane curve C, we proceed as follows: Inscribe in C a broken line of n segments $\Delta s_i'$ as in § 111, and form the sum $\sum_{i=1}^{n} \Delta s_i'$. This sum is of course the length of the broken line, and its limit is the length s of the arc. It is evidently the line integral $\int_C ds$, the given function in this case being $F(x, y) = 1$:

$$s = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i' = \int_C ds.$$

The process of finding the length of a curve is sometimes called *rectification* of the curve.

Example: Find the circumference of the circle

$$x^2 + y^2 = a^2.$$

Here

$$\frac{dy}{dx} = -\frac{x}{y},$$

so that

$$s = \int_C ds = 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 4\int_0^a \sqrt{1 + \frac{x^2}{y^2}}\, dx$$

$$= 4\int_0^a \sqrt{\frac{x^2 + y^2}{y^2}}\, dx = 4\int_0^a \frac{a}{y}\, dx$$

$$= 4\,a\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = 4\,a \arcsin \frac{x}{a}\Big]_0^a = 2\,\pi a.$$

EXERCISES

1. Find the circumference of the circle $x = a\cos\theta$, $y = a\sin\theta$.

2. Rectify the semicubical parabola $ay^2 = x^3$ from $x = 0$ to $x = 5a$.
Ans. $\frac{335}{27}a$.

3. Trace the curve $9\,y^2 = 4(1 + x^2)^3$, and find its length from $x = 0$ to $x = 2$.
Ans. $\frac{22}{3}$.

4. Rectify the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ from $x = 0$ to $x = x$.
Ans. $\frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$.

5. Find the length of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.
Ans. $6\,a$.

6. Rectify the curve $x = t^2$, $y = t^3$ from $t = 0$ to $t = \sqrt{5}$.

7. Find the length of the curve $y = \frac{4}{5}x^{\frac{5}{4}}$ between the origin and the point $x = 4$.
Ans. $\frac{8}{15}(1 + 6\sqrt{3})$.

8. Find the length of one arch of the cycloid.
Ans. $8\,a$.

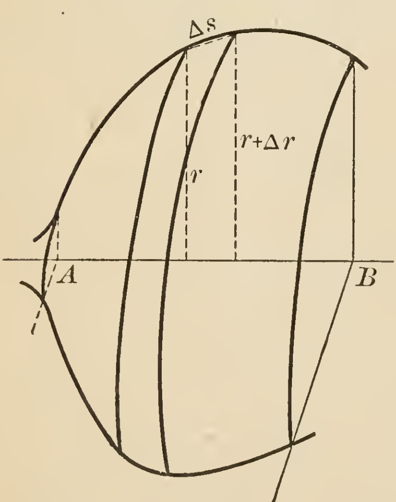


FIG. 71

9. Trace the curve

$$9\,ay^2 = x(x - 3\,a)^2,$$

and find the circumference of the loop.
Ans. $4\,a\sqrt{3}$.

10. Find the length of the curve $y = e^x$ from $x = 0$ to $x = \frac{3}{2}\log 2$.

116. Surfaces of revolution. The surface generated by the rotation of a plane curve about a line AB in its plane is easily expressed as a line integral.

Let us inscribe in the curve

C a broken line of n segments $\Delta s_i'$. Each of the segments $\Delta s_i'$ generates in the rotation the frustum of a right circular cone, the radii of whose bases may be called r_i and $r_i + \Delta r_i$. The surface of this conical frustum is, by elementary geometry, the circumference of the middle section multiplied by the slant height, or $2\pi(r_i + \frac{1}{2}\Delta r_i)\Delta s_i'$. The sum of these surfaces, $\sum_{i=1}^{n} 2\pi(r_i + \frac{1}{2}\Delta r_i)\Delta s_i'$, approaches as its limit the required surface of revolution.

Hence, by the theorems of § 113 and § 109, we have

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi r_i \Delta s_i' = 2\pi \int_C r\, ds.$$

Example: Find the surface of a paraboloid of revolution bounded by a right section through the focus.

Given the equation of the generating parabola $y^2 = 4ax$, we have

$$S = 2\pi \int_C y\, ds = 2\pi \int_0^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= 2\pi \int_0^a y \sqrt{1 + \frac{4a^2}{y^2}} dx = 2\pi \int_0^a \sqrt{y^2 + 4a^2}\, dx$$

$$= 2\pi \int_0^a \sqrt{4ax + 4a^2}\, dx = \frac{4\pi}{a} \cdot \frac{2}{3}(ax + a^2)^{\frac{3}{2}}\Big]_0^a$$

$$= \tfrac{8}{3}\pi a^2 (2\sqrt{2} - 1).$$

EXERCISES

1. Work the above example, using y as the variable of integration.

2. Find the surface of a sphere, using polar coördinates.

3. Find the surface of a sphere, using cartesian coördinates. Evaluate the integral in various ways (cf. Ex. 10, p. 146).

4. Find the surface generated by revolving the cubical parabola $a^2y = x^3$ about OX, from $x = 0$ to $x = a$. *Ans.* $\frac{\pi a^2}{27}(10\sqrt{10} - 1)$.

5. Find the surface generated by revolving (a) about OX, (b) about OY, the arc of the curve $y = \frac{x^4 + 3}{6x}$ between the minimum point and the point $x = 2$. *Ans.* (a) $\frac{47\pi}{16}$; (b) $\frac{\pi}{4}(15 + 4\log 2)$.

6. Trace the curve $y = \frac{1}{2}\log x - \frac{x^2}{4}$ (cf. § 69), and obtain the surface generated by revolving the curve about OY from $x = 1$ to $x = 2$.
Ans. 10.47.

7. Find the surface generated by revolving the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$, (*a*) about OX, (*b*) about OY, from $x = 0$ to $x = a$.
Ans. (*b*) $2\pi a^2\left(1 - \frac{1}{e}\right)$.

8. Find the surface generated by the revolution of an arch of the cycloid about its base. *Ans.* $\frac{64}{3}\pi a^2$.

9. Find the surface of a torus. *Ans.* $4\pi^2 ab$.

10. Find the entire surface generated by revolving the curve $8a^2y^2 = x^2(a^2 - x^2)$ about OX. *Ans.* $\frac{1}{2}\pi a^2$.

11. Find the surface formed by revolving the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about OX. *Ans.* $\frac{12}{5}\pi a^2$.

12. Find the surface cut from a sphere by a circular cone of half-angle α with its vertex at the center of the sphere.

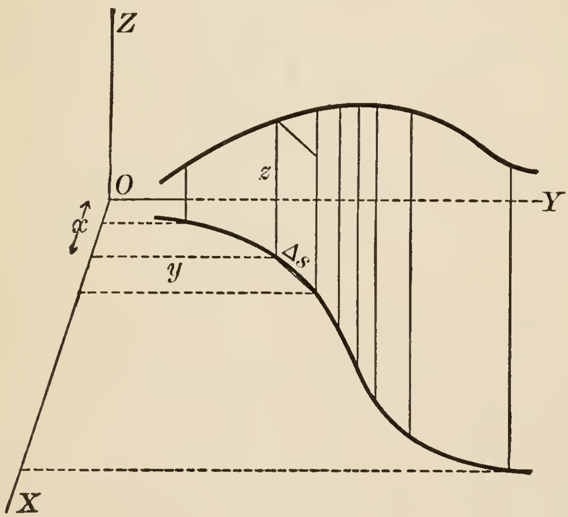


FIG. 70

117. Cylindrical surfaces. Given a cylinder whose directrix is a plane curve C, the area of any portion of the cylinder may be found as follows: Inscribe in C a broken line of segments $\Delta s_1'$, ⋯, $\Delta s_n'$, and inscribe in the required area a set of rectangles of altitude h_i and base $\Delta s_i'$. The limit of the sum of these rectangular areas is the area on the cylinder:

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} h_i \Delta s_i' = \int_C h\, ds.$$

Example: Find the area on the cylinder $x^2 + z^2 = a^2$ included between the planes $y = 0$, $y = mx$.

Denoting by C the circular arc APB, we have

$$S = 4\int_C y\,ds = 4\int_0^a mx\sqrt{1 + \left(\frac{dz}{dx}\right)^2}\,dx$$

$$= 4m\int_0^a x\sqrt{1 + \frac{x^2}{z^2}}\,dx$$

$$= 4\,ma\int_0^a \frac{x\,dx}{z}$$

$$= -\,4\,ma\int_a^0 dz = 4\,ma^2.$$

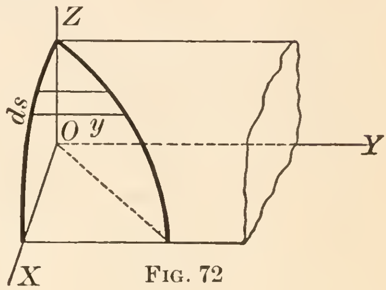


FIG. 72

EXERCISES

1. In the above example, find the area of the section cut by the plane $y = mx$.

2. Find the surface of the cap in Ex. 14, p. 163. *Ans.* 128 sq. in.

3. Find the surface of the cylinder $x^2 + y^2 = a^2$ included between the planes $z = x$, $z = 3\,x$.

4. Find the surface cut off on the cylinder $x^2 + y^2 = a^2$ by the paraboloid of revolution $x^2 + y^2 = bz$. *Ans.* $2\,\pi\frac{a^3}{b}$.

5. Find the area, in the first octant, of the section of the cone $x^2 - y^2 + z^2 = 0$ by the plane $x + y = a$.

6. The center of a sphere of radius $2\,a$ is on the surface of a cylinder of radius a. Find the surface of the cylinder intercepted by the sphere. *Ans.* $16\,a^2$.

7. Find the surface on the cylinder $z^2 = 4\,ax$ inside the cylinder $y^2 = 4\,ax$, from $x = 0$ to $x = 3\,a$. *Ans.* $\frac{112}{3}\,a^2$.

8. Work the example of § 117, using polar coördinates.

CHAPTER XVI

INTEGRAL TABLES

118. Use of tables. In the solution of problems involving integration, the work may frequently be much shortened by the use of a table of integrals. Many such tables have been prepared; the references below are to B. O. Peirce's *Short Table of Integrals* (Ginn & Co.).

The chief object in using a table is to save time. The student is therefore not making the best use of the table unless he is so familiar with its contents and arrangement that he can tell at a glance whether the desired formula is likely to be given and where it is to be found. Further, it should be remembered that in many cases the result may be found by the methods of Chapter XII in less time than would be required if the table were used.

Examples: (*a*) Evaluate $\int \frac{dx}{x(1-x^2)}$.

Let us use formula 55 of the table, with $a = 1$, $b = -1$:

$$\int \frac{dx}{x(1-x^2)} = \frac{1}{2} \log \frac{x^2}{1-x^2}.$$

(*b*) Evaluate $\int_0^2 x^3 e^{x^2} dx.$

This integral is not given explicitly in the table, but it resembles formula 402. Making the substitution

$$x^2 = z,\ 2\,x\,dx = dz,$$

we find

$$\int_0^2 x^3 e^{x^2} dx = \tfrac{1}{2}\int_0^4 z e^z dz = \tfrac{1}{2} e^z(z-1)\Big]_0^4 = \tfrac{3}{2}\,e^4 + \tfrac{1}{2}.$$

EXERCISES

Evaluate the following integrals, using a table whenever a saving of time may be effected by so doing.

1. $\int \frac{dx}{1 + x + x^2}$. *Ans.* $\frac{2}{\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + C$.

2. $\int \frac{dx}{\sqrt{1 + x + x^2}}$. *Ans.* $\log(\sqrt{1 + x + x^2} + x + \frac{1}{2}) + C$.

3. $\int \frac{(x + 2a)dx}{(x^2 + 4ax + a^2)^{\frac{3}{2}}}$. *Ans.* $-\frac{1}{\sqrt{x^2 + 4ax + a^2}} + C$.

4. $\int \frac{dx}{(1 + x^2)^2}$.

5. $\int \frac{x\,dx}{(1 - 2x)^2}$.

6. $\int x\sqrt{1 + x^2}dx$.

7. $\int \frac{dx}{(1 + 3x)^4}$.

8. $\int x\sqrt{1 - x}\,dx$.

9. $\int \sin^4 x\,dx$.

10. $\int \frac{dx}{1 + \cos x}$.

11. $\int x \cos x^2 dx$.

12. $\int x^2 e^{x^3}\,dx$.

13. $\int x^2 e^{-x}\,dx$.

14. $\int \arcsin x\,dx$.

15. $\int \log^2 x\,dx$.

16. $\int_0^a \frac{x^2 dx}{\sqrt{a^2 + x^2}}$.

17. $\int_0^a \frac{dy}{\sqrt{a^2 + y^2}}$.

18. $\int_0^1 \sqrt{2t + t^2}\,dt$. *Ans.* $\sqrt{3} - \frac{1}{2}\log(2 + \sqrt{3})$.

19. $\int_0^{\log 2} \sqrt{e^x - 1}\,dx$. *Ans.* $2 - \frac{\pi}{2}$.

20. Find the area bounded by the hyperbola $x^2 - y^2 = a^2$ and the line $x = 2a$.

21. Find the length of the arc of a parabola from the vertex to the end of the latus rectum. *Ans.* $2.29\,a$.

22. Find the surface generated by revolving the curve $y = e^x$ about OX from $x = 0$ to $x = 1$.

23. Find the area inside the four-cusped hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$. *Ans.* $\frac{3}{8}\pi a^2$.

24. Find the area of the ellipse $r = \frac{l}{1 + e\cos\theta}$, where e is the eccentricity.

25. Find the volume generated by revolving one arch of the cycloid about its base. *Ans.* $5\pi^2a^3$.

26. Find the surface generated by revolving the curve $3a^2x + y^3 = 0$ about OY from $y = 0$ to $y = a$.

27. Find the surface generated by revolving one arch of the sine curve about the x-axis. *Ans.* $2\pi[\sqrt{2} + \log(1 + \sqrt{2})]$.

CHAPTER XVII

IMPROPER INTEGRALS

119. Definitions. Definite integrals in which either or both of the limits of integration are infinite, and also those in which the integrand becomes infinite within the interval of integration, are called *improper integrals.* Such integrals have no meaning under the definitions so far laid down (see §§ 98, 102); we proceed to show how they may arise, and to find under what conditions a meaning can be assigned to them.

Examples: (*a*) The area under the curve $y = \frac{1}{x^2}$ from $x = 1$ to $x = b$ is evidently

$$A = \int_1^b \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^b = 1 - \frac{1}{b}.$$

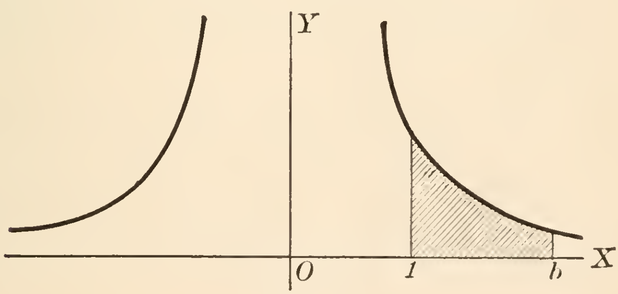


FIG. 73

When b becomes infinite, the area approaches the limit 1. This limit we *define* as the area "bounded" by the curve, the x-axis, and the line $x = 1$, although it is not properly a bounded area in the usual sense of the term. Symbolically we write

$$\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^\infty = 1.$$

The first thought might be that the area in the figure would increase indefinitely as the right-hand boundary recedes. Our result shows that this is not the case — the area is always less than 1.

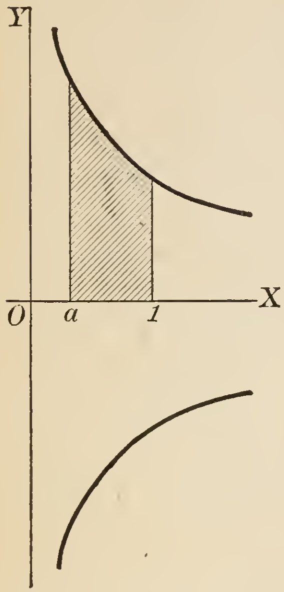


Fig. 74

(*b*) The area under the curve $xy^2 = 1$ from $x = a$ $(a > 0)$ to $x = 1$ is

$$A = \int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}.$$

When a approaches 0, the initial ordinate becomes infinite; the area approaches the limit 2. This limit we define as the area in the first quadrant "bounded" by the curve, the axes, and the line $x = 1$. For brevity, we write merely

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_0^1 = 2,$$

but it must be borne in mind that the geometric interpretation is quite different from that of the ordinary integral.

(*c*) Let us try to find the area under the curve $y = \frac{1}{x^2}$ from $x = -1$ to $x = 1$. If we were to work carelessly, without noticing that the integrand becomes infinite within the interval, we might write

$$A = \int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\Big]_{-1}^1 = -2,$$

which is obviously absurd. But if we write

$$A = \lim_{a' \to 0} \int_{-1}^{-a'} \frac{dx}{x^2} + \lim_{a'' \to 0} \int_{a''}^1 \frac{dx}{x^2}$$

$$= \lim_{a' \to 0} \left[-\frac{1}{x}\right]_{-1}^{-a'} + \lim_{a'' \to 0} \left[-\frac{1}{x}\right]_{a''}^1,$$

it is clear that the limits do not exist, and the area-integral has no meaning.

These examples suggest the following *definitions:*

1. $$\int_{a}^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_{a}^{b} f(x)\,dx;$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx,$$

provided the limits exist.

2. If $f(x)$ becomes infinite as $x \to b^-$,

$$\int_{a}^{b} f(x)\,dx = \lim_{b' \to b} \int_{a}^{b'} f(x)\,dx;$$

if $f(x)$ becomes infinite as $x \to a^+$,

$$\int_{a}^{b} f(x)\,dx = \lim_{a' \to a} \int_{a'}^{b} f(x)\,dx,$$

provided the limits exist.

3. If $f(x)$ becomes infinite as $x \to c$, where $a < c < b$,

$$\int_{a}^{b} f(x)\,dx = \lim_{c' \to c^-} \int_{a}^{c'} f(x)\,dx + \lim_{c'' \to c^+} \int_{c''}^{b} f(x)\,dx,$$

provided the limits exist.

120. Geometric interpretation. It is obvious that an integral with an infinite limit may be interpreted in general as the area under a curve which is asymptotic to the x-axis; an integral whose integrand becomes infinite may usually be thought of as the area between a curve and a vertical asymptote. Of course, as in example (c) of § 119, these integrals may not have any meaning in a given case.

EXERCISES

Evaluate the following integrals.

1. (a) $\int_{1}^{\infty} \frac{dx}{x^3}$; ($b$) $\int_{1}^{\infty} \frac{dx}{x}$; ($c$) $\int_{0}^{\infty} \cos x\,dx$; ($d$) $\int_{-\infty}^{0} e^{2x}\,dx$.

Ans. (a) $\frac{1}{2}$; (b) meaningless; (c) meaningless; (d) $\frac{1}{2}$.

2. $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$. *Ans.* π. **3.** $\int_{0}^{2} \frac{dx}{x}$. *Ans.* Meaningless.

4. $\int_{0}^{1} \frac{dx}{\sqrt{x}}$. *Ans.* 2. **5.** $\int_{0}^{2a} \frac{dt}{\sqrt{2a-t}}$. *Ans.* $2\sqrt{2a}$.

6. Trace the curve $y = \dfrac{8a^3}{x^2 + 4a^2}$, and find the area between the curve and its asymptote. *Ans.* $4\pi a^2$.

7. Find the volume generated by revolving the area of Ex. 6 about the asymptote. *Ans.* $4\pi^2 a^3$.

8. Find the area in the fourth quadrant bounded by the curve $xy^2 = (x-1)^2$ and the coördinate axes.

9. Find the area in the second quadrant under the curve $y = e^x$.

10. Find the volume generated by revolving (*a*) about OX, (*b*) about OY, the area in the second quadrant under the curve $y = e^x$.

Ans. (*a*) $\frac{\pi}{2}$; (*b*) 2π.

11. The area in example (*b*), § 119, revolves about the y-axis. Find the volume generated.

12. Find the surface generated by revolving about OX that portion of the curve $y = e^x$ which lies to the left of the y-axis. *Ans.* 2.29π.

13. Trace the curve $x(x-y)^2 = a^3$, and find the area bounded by the curve, the y-axis, and the line $x = 4a$. *Ans.* $2a^2$.

14. Find the volume generated by revolving about OY the area under the curve $y = e^{-\frac{1}{2}x^2}$. Check by solving in two ways.

CHAPTER XVIII

CENTROIDS. MOMENTS OF INERTIA

I. Centroids

121. Mass; density. The student is assumed to be familiar with the idea of *mass* as introduced in physics.

A mass is said to be *homogeneous* if the masses contained in any two equal volumes are equal. In all other cases the mass is *heterogeneous.* In the present chapter we confine our attention to homogeneous masses.

The *density* δ of a homogeneous mass is the ratio of the mass M to the volume V that it occupies :

$$\delta = \frac{M}{V}.$$

That is, the density is the mass per unit volume.

Although every physical mass occupies a certain *volume* or three-dimensional portion of space, nevertheless it is frequently desirable to introduce the idea of the *material particle*, or geometric point endowed with mass.* The mass-point may be imagined as the limiting form approached by a body whose dimensions approach 0, while the density increases in such a way that the mass remains finite.

Similarly we may think of masses of one dimension and of two dimensions — *i.e.* of material curves and surfaces. Such masses are represented approximately, for example, by slender wires and thin sheets of metal. In these cases we define the density as "linear density," or mass per unit length,

$$\delta = \frac{M}{s},$$

* This notion is fundamental in studying the motion of a rigid body.

and "surface density," or mass per unit area,

$$\delta = \frac{M}{S},$$

respectively.

122. Moment of mass. The product of a mass m, concentrated at a point P, by the distance l of P from a given point, line, or plane, is called the *moment** of m with respect to the point, line, or plane. Denoting this moment by G, we have

$$G = ml.$$

If a system of points $P_1, P_2, \cdots, P_n$, having masses $m_1, m_2, \cdots, m_n$ respectively, be referred to cartesian coördinate axes, the moments of the system with respect to the three coördinate planes are respectively

$$G_{yz} = \sum_{i=1}^{n} m_i x_i,$$

$$G_{zx} = \sum_{i=1}^{n} m_i y_i,$$

$$G_{xy} = \sum_{i=1}^{n} m_i z_i.$$

In case the particles all lie in one of the coördinate planes, the moments with respect to coördinate planes reduce to moments with respect to coördinate axes.

The idea of mass-moment may be extended to the case of a continuous mass by thinking of the mass as composed of an indefinitely large number of particles. A precise definition will be laid down in § 187. The actual computation of such a moment is usually effected by means of definite integrals; we return to this question presently.

123. Centroid. Given any mass M, let G_{yz}, G_{zx}, G_{xy} denote the moments of the mass with respect to the coördinate planes. The point C whose coördinates $\bar{x}, \bar{y}, \bar{z}$, are given by the formulas

$$M\bar{x} = G_{yz}, \quad M\bar{y} = G_{zx}, \quad M\bar{z} = G_{xy}$$

* More precisely, the *simple moment*, or *moment of first order.*

clearly has the property that the moment of the mass with respect to each of the coördinate planes is the same as if the whole mass were concentrated at that point.

It is easily shown that this property holds for moments with respect to any other plane. The proof for the general case requires the use of multiple integrals (Chapter XXIII); for a system of mass-particles the proof is as follows. Let

(1) $$ax + by + cz = p$$

be the equation of any plane in the normal form; let $\bar{p}, p_1, p_2, \cdots, p_n$ be the distances of the points $C, P_1, P_2, \cdots, P_n$ from this plane. Now

$$p_1 = ax_1 + by_1 + cz_1 - p,$$
$$\vdots$$
$$p_n = ax_n + by_n + cz_n - p,$$

so that

$$\begin{aligned}\sum_{i=1}^{n} m_i p_i &= a\sum_{i=1}^{n} m_i x_i + b\sum_{i=1}^{n} m_i y_i + c\sum_{i=1}^{n} m_i z_i - p\sum_{i=1}^{n} m_i \\ &= aM\bar{x} + bM\bar{y} + cM\bar{z} - Mp \\ &= M(a\bar{x} + b\bar{y} + c\bar{z} - p) \\ &= M\bar{p}.\end{aligned}$$

That is, the moment of the system with respect to the plane (1) is the same as if the whole mass were concentrated at C.

The point C is called the *center of mass*, or *centroid**:

The centroid of a mass is a point such that the moment of the mass with respect to any plane is the same as if the whole mass were concentrated at that point.

By § 122, the coördinates of the centroid of a *system of particles* are given by the formulas

$$M\bar{x} = \sum_{i=1}^{n} m_i x_i, \quad M\bar{y} = \sum_{i=1}^{n} m_i y_i, \quad M\bar{z} = \sum_{i=1}^{n} m_i z_i.$$

* The centroid coincides with the *center of gravity*, and is frequently so-called; but the term centroid is in some respects preferable.

In the actual determination of centroids, the following considerations are often useful (the first two apply only to homogeneous masses):

(*a*) If the body has a geometrical center, that point is the centroid.

(*b*) Any plane or line of symmetry must contain the centroid.

(*c*) If the body consists of several portions for each of which the centroid can be found, each portion may be imagined concentrated at its centroid: the problem thus reduces to the consideration of a set of particles.

124. Centroids of geometrical figures. It is clear that, for a homogeneous body of given size and shape, both the mass and its moment with respect to any plane are proportional to the density δ. Hence, in the formulas for $\bar{x}$, $\bar{y}$, $\bar{z}$, δ cancels out from both members, leaving the coordinates of the centroid independent of the density. We may therefore without loss of generality take $\delta = 1$, and are thus led to speak of centroids of geometrical figures — volumes, areas, and lines — without reference to the idea of mass.

EXERCISES

1. Find the centroid of the following plane systems of particles:

(*a*) Equal particles at (0, 0), (4, 2), (3, – 5), (– 2, – 3).

(*b*) A mass of 2 units at (0, 1), one of 3 units at (3, – 3), one of 6 units at (4, 1).

2. Four particles of mass 2, 4, 6, 8 units are placed at the points (0, 0, 0), (0, 2, 2), (4, 1, 5), (– 3, 2, – 1) respectively. Find the centroid.

3. Show that the centroid of two particles divides the line joining them into segments inversely proportional to the masses.

4. Show that the centroid of three equal particles lies at the intersection of the medians of the triangle having the three points as vertices.

5. Equal particles are placed at five of the six vertices of a regular hexagon. Find the centroid.

6. Particles of mass 1, 2, ⋯, 8 units are placed at the successive vertices of a regular octagon. Find the centroid.

7. Find the centroid of the cross section of an angle-iron, the sides being 5 in. and 8 in., and the thickness of each flange 1 in.

Ans. $\left(\frac{17}{6}, \frac{4}{3}\right)$.

8. Find the centroid of the T-iron section (*a*) of Fig. 75, (*b*) of Fig. 76.

9. Find the centroid of a wire frame in the shape of the perimeter (*a*) of Fig. 75, (*b*) of Fig. 76.

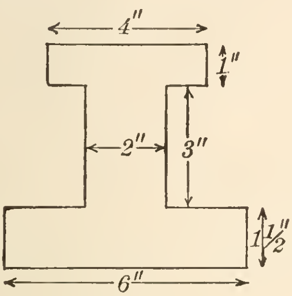


FIG. 75

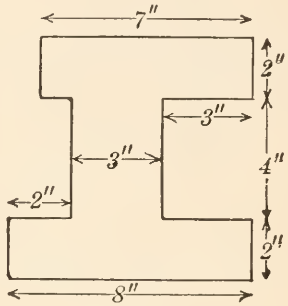


FIG. 76

10. From a circular disk a round hole is punched out, the two circles being tangent internally. Find the centroid of the remaining figure.

11. From a circular plate of radius 4 in. a hole 2 in. square is cut out, one corner of the square being at the center of the plate. Find the centroid of the remainder.

12. Find the centroid of a cylindrical basin of radius 4 in. and depth 3 in., if the bottom is twice as thick as the sides.

13. A monument is composed of a block of stone of base 4 ft. by 3 ft. and height 2 ft. 6 in., surmounted by a cube of side 2 ft., this in turn supporting a sphere of radius 1 ft. Find the centroid of the whole figure.

125. Determination of centroids by integration. To find the centroid of a continuous mass, we must in general resort to integration. In the most general case multiple integrals (see Chapter XXIII) must be used, but in most cases of practical importance the result may be obtained by a single integration.

In the following discussion we restrict ourselves to one-, two-, or three-dimensional bodies of the forms considered in Chapter XV.* Let us choose, as in that chapter, a suitable geometrical element (of volume, area, or length), and denote the mass contained in this element by Δm_i. Let x_i, y_i, z_i be the coördinates of the *centroid*† of Δm_i. Then the sum $\sum_{i=1}^{n} x_i \Delta m_i$ represents approximately the moment of the mass with respect to the yz-plane (or the y-axis, in the case of a plane mass in the xy-plane), and the limit of this sum as n becomes infinite is exactly the moment in question. In this way we obtain the following formulas for the coördinates of the centroid:

$$M\bar{x} = \lim_{n \to \infty} \sum_{i=1}^{n} x \Delta m_i,$$

$$M\bar{y} = \lim_{n \to \infty} \sum_{i=1}^{n} y \Delta m_i,$$

$$M\bar{z} = \lim_{n \to \infty} \sum_{i=1}^{n} z \Delta m_i.$$

Now upon recalling the meaning of Δm_i, we see that in any given case the above limits may be expressed as definite integrals by the fundamental theorem of § 104. The result may be written in the following form:

$$(1) \qquad M\bar{x} = \int x\, dm, \quad M\bar{y} = \int y\, dm, \quad M\bar{z} = \int z\, dm,$$

where x, y, z are the coördinates ‡ *of the centroid of the mass-element.* In any problem each integrand must be expressed in terms of a single variable, and limits are to be assigned in such a way as to extend the integration over the whole mass.

In the next few articles we explain more in detail the application of the above formulas.

* Nevertheless the formulas obtained are applicable, with proper interpretation, to *all* masses, with no restriction whatever.

† It follows from the theorem of § 109 that in the expressions for these coördinates, *all infinitesimals may be neglected.*

‡ Apart from infinitesimals.

126. Centroids of plane areas.* To find the centroid of a plane area (thin sheet or plate) we choose an element of area as in § 105 (or § 106, if polar coördinates are used), and find the centroid from the formulas

$$A\bar{x} = \int x\,dA,\quad A\bar{y} = \int y\,dA,$$

where x and y are the coördinates of the *centroid* of the element.

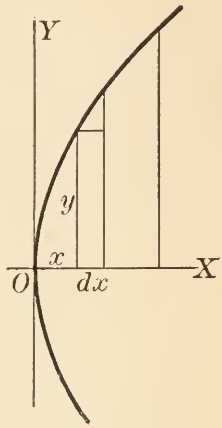


FIG. 61

Example: Find the centroid of the area in the first quadrant bounded by the parabola $y^2 = 4\,ax$ and its latus rectum.

With the element of area chosen as in Fig. 61, we have

$$A = \int_0^a y\,dx = \tfrac{4}{3}\,a^2;$$

$$A\bar{x} = \int_0^a xy\,dx = 2\sqrt{a}\int_0^a x^{\frac{3}{2}}\,dx = \tfrac{4}{5}\,a^3;$$

$$A\bar{y} = \int_0^a \frac{y}{2}\cdot y\,dx = 2\,a\int_0^a x\,dx = a^3.$$

Hence

$$\bar{x} = \tfrac{3}{5}\,a,\quad \bar{y} = \tfrac{3}{4}\,a.$$

We may also find $\bar{y}$ very easily by taking the element parallel to OX. Thus

$$A\bar{y} = \int_0^{2a} y(a-x)dy = 2\,a\int_0^a (a-x)dx = a^3, \text{ etc.}$$

EXERCISES

Find the centroid of the following areas. In each case, draw a figure and estimate the coördinates of the centroid, thus obtaining a rough check on the result.

1. An isosceles triangle.

2. A semicircular area. Evaluate the integral in two ways.

* The problem of this article is of particular importance in the theory of the flexure or bending of beams.

3. One quadrant of an ellipse, using (*a*) the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (*b*) the equations $x = a\cos\phi$, $y = b\sin\phi$. *Ans.* $\left(\frac{4a}{3\pi}, \frac{4b}{3\pi}\right)$.

4. Any triangle. *Ans.* At the intersection of the medians.

5. Half an arch of the sine curve. *Ans.* $\left(1, \frac{\pi}{8}\right)$.

6. The area between the curves $2y = x^2$, $y = x^3$. Get each coördinate in two ways.

7. A circular sector. *Ans.* $\bar{x} = \frac{2}{3}a\frac{\sin\alpha}{\alpha}$.

8. One arch of the cycloid. *Ans.* $(\pi a, \frac{5}{6}a)$.

9. A semicircular area, using polar coördinates.

10. A circular segment. Check by putting $h = a$.

11. A trapezoid.

12. One loop of the curve $r = a\cos 2\theta$.

13. The area under the curve $y = e^x$ from $x = 0$ to $x = 1$.

14. The area bounded by the curve $y = \frac{\log x}{x}$, the x-axis, and the maximum ordinate.

15. The area bounded by the parabola $y = x^2$, the x-axis, and the line $x = 3$.

16. The area bounded by the curves $y = x$, $y = 2x$, $y = x^2$.

17. The area bounded by the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, the axes, and the line $x = a$.

18. The area swept out by the radius vector of the spiral of Archimedes $r = a\theta$ in the first revolution.

19. Half the area of the cardioid $r = a(1 - \cos\theta)$.

20. The upper half of the loop of the curve $ay^2 = ax^2 - x^3$.

21. The area of Ex. 7, p. 153.

22. From one corner of a square of side a, a triangle of sides $\frac{1}{4}a$, $\frac{1}{2}a$ is cut off. Find the centroid of the remaining area.

23. From a circle an inscribed isosceles right triangle is cut out. Find the centroid of the remainder.

24. Prove the *second proposition of Pappus:*
The volume of any solid of revolution is equal to the product of the generating area into the circumference of the circle described by the centroid of the area.

Solve the following by using the second proposition of Pappus.

25. Find the volume of a torus. *Ans.* $2\pi^2a^2b$.

26. Find the centroid of a right triangle.

27. Find the centroid of a semicircular area.

127. Centroids of volumes. The centroid of a volume of revolution evidently lies on the axis of revolution, so that a single coördinate determines its position. Taking the axis of revolution as axis of x, we have

$$V\bar{x} = \int x\, dV,$$

where the element of volume is chosen as in § 107 or § 108, and where x is the x-coördinate of the centroid of the element.

In certain special cases the centroids of other solids may be found by a simple integration, but in general we must resort to multiple integrals.

EXERCISES

Find the centroid of the following volumes. Draw a figure in each case and estimate the coördinates of the centroid.

1. A hemisphere. Solve in two ways. *Ans.* $\bar{x} = \frac{3}{8}a$.

2. A right circular cone. *Ans.* $\bar{x} = \frac{1}{4}h$.

3. A paraboloid of revolution bounded by a right section through the focus.

4. Half an ellipsoid of revolution. Solve in various ways.

5. A spherical segment of height h. Check by putting $h = r$.

6. The volume generated by revolving (a) about OX, (b) about OY, the area under the curve $y = e^x$ from $x = 0$ to $x = 1$.

7. The volume generated by revolving half an arch of the cycloid about its base.

8. The volume formed by rotating the area under the parabola $y^2 = 4ax$ from $x = 0$ to $x = a$, (a) about the y-axis; (b) about the latus rectum; (c) about the line $y = 2a$.

9. The volume in Ex. 13, p. 163. *Ans.* $(\frac{1}{3}b, \frac{2}{9}a, \frac{1}{3}a)$.

10. The volume in Fig. 68. Check all three coördinates by solving again with the element chosen in a different way.

11. An elliptic cone. *Ans.* $\bar{x} = \frac{1}{4}h$.

12. One quarter of a right circular conoid (see Ex. 12, p. 163).

13. One quadrant of the banister cap in Ex. 14, p. 163.

14. A tetrahedron three of whose faces are mutually perpendicular.
Ans. $(\frac{1}{4}a, \frac{1}{4}b, \frac{1}{4}c)$.

15. The volume of Ex. 16, p. 163. Solve in two ways.

16. One quarter of a right pyramid with a square base.

17. The volume of Ex. 18, p. 163.

18. The volume of Ex. 19, p. 161.

19. The volumes of Ex. 6, p. 160.

20. The volume formed by revolving about the y-axis the area under the curve $y = e^{-\frac{1}{2}x^2}$. Take as the element a cylindrical shell, and evaluate the integrals in two ways.

21. Obtain formulas for the coördinates of the centroid of a wedge cut from any solid of revolution by two planes through the axis.

128. Centroids of lines. In the case of an arc of a plane curve, the fundamental limits of § 125 may be expressed at once as line integrals, by the theorem of § 113:

$$s\bar{x} = \int_C x\,ds, \quad s\bar{y} = \int_C y\,ds,$$

where s is the length of the arc C.

Example: Find the centroid of a semicircular wire.

Taking the bounding diameter as axis of y, we have

$$s\bar{x} = \int_C x\,ds = 2\int_0^a x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_0^a x\sqrt{1 + \frac{x^2}{y^2}}\,dx = 2\,a\int_0^a \frac{x\,dx}{\sqrt{a^2 - x^2}}$$

$$= -2\,a\sqrt{a^2 - x^2}\Big]_0^a = 2\,a^2.$$

Hence

$$\bar{x} = \frac{2\,a^2}{s} = \frac{2\,a^2}{\pi a} = \frac{2a}{\pi}.$$

By symmetry, $\bar{y} = 0$.

129. Centroids of curved surfaces. The coördinates of the centroid of a surface of revolution (§ 116), or of a cylindrical surface (§ 117), may be expressed in terms of line integrals.

The required integrals are easily built up in each problem.

EXERCISES

1. In the example of § 128, evaluate the integral by expressing the integrand in terms of y.

Find the centroid of each of the following figures.

2. A semicircular wire, using polar coördinates.

3. The arc of the curve $ay^2 = x^3$, from $x = 0$ to $x = 5a$.

4. The arc of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ between two symmetric points.

5. The arc of the semicycloid (from cusp to vertex).

Ans. $(\frac{4}{3}a, \frac{4}{3}a)$.

6. A hemispherical surface, using (*a*) cartesian, (*b*) polar coordinates. *Ans.* $\bar{x} = \frac{1}{2}a$.

7. The lateral surface of a right circular cone. *Ans.* $\bar{x} = \frac{1}{3}h$.

8. The total surface of a right circular cone.

9. The cylindrical surface in Fig. 72.

10. The area in Ex. 5, p. 171.

11. The surface in Ex. 7, p. 171.

12. The surface of a paraboloid of revolution bounded by a right section through the focus.

13. Prove the *first proposition of Pappus:*

The surface of a solid of revolution is equal to the length of the generating arc multiplied by the circumference of the circle described by the centroid of the arc.

14. Find the surface of a torus. *Ans.* $4\pi^2 ab$.

15. Find the centroid of a semicircular wire by the first proposition of Pappus.

II. Moments of Inertia

130. Moment of inertia. The product of a mass m, concentrated at a point P, by the square of the distance r of P from a fixed line, or *axis*, is called the moment of the second order, or *moment of inertia*, of m with respect to the given axis:

$$I = mr^2.$$

The moment of inertia of a system of such masses is of course the sum

$$I = \sum_{i=1}^{n} m_i r_i^2.$$

If we think of a "continuous" mass as composed ultimately of particles, the meaning of moment of inertia of such a mass becomes clear. An analytic definition will be given in § 187.

The moment of inertia of a homogeneous body is proportional to the density. Taking $\delta = 1$, we may speak of "moment of inertia" of areas, volumes, etc., no idea of mass being involved.

131. Radius of gyration. The moment of inertia of any mass M may always be written in the form

$$I = MR^2,$$

where the quantity R is called the *radius of gyration*, or *radius of inertia*, of M with respect to the given axis of moments. The meaning of the radius of gyration is obvious: it is the distance from the axis at which a *particle* of mass M must be placed in order to have the same moment of inertia as the original mass.

132. Determination of moment of inertia by integration. The actual computation of the moment of the second order of a continuous mass is effected by integration in much the same way that the moment of the first order (§ 125) is determined. In the general case, we must have recourse to double or triple integrals (Chapter XXIII); but for the simple cases of practical importance the result can usually be found by a single integration.

For the present we consider only such bodies as were studied in Chapter XV. Let us choose a geometrical element (of volume, area, or length) in some suitable way, and denote the mass of this element by Δm_i. *The element must be so chosen that its radius of gyration is known* *; let r_i denote this radius. Then the sum

$$\sum_{i=1}^{n} r_i^2 \Delta m_i$$

is approximately the moment of inertia of the mass with respect to the given axis, and the limit of this sum is exactly the required moment:

$$I = \lim_{n \to \infty} \sum_{i=1}^{n} r_i^2 \Delta m_i.$$

Now if we apply the fundamental theorem of § 104, the above limit appears as the definite integral

(1) $$I = \int r^2\, dm,$$

where *r is the radius of gyration of the mass element* with respect to the axis of moments. Of course the integrand must be expressed in terms of a single variable and the integration must be extended over the whole mass.

Just as, in finding centroids, we must take an element the position of whose centroid is known, so here the essential point is to choose an element whose radius of gyration is known. Thus the moment of inertia of a plane area (or of a thin sheet of mass) with respect to a line in its plane may be found by taking as the element a rectangle with its finite side parallel to the axis of

* In the expression for the radius of gyration, infinitesimals may as usual be neglected.

moments, since then the radius of gyration of the element is simply its distance from the axis.

To find the moment of inertia of a volume of revolution about the axis of revolution it is usually best to choose the element as in § 108.

Examples: (*a*) Find the moment of inertia, with respect to the y-axis, of the area bounded by the parabola $y^2 = 4\,ax$, the x-axis, and the latus rectum.

Taking the element parallel to OY, we find

$$I_y = \int_0^a x^2 y\,dx = 2\sqrt{a}\int_0^a x^{\frac{5}{2}}\,dx = \tfrac{4}{7}\,a^4.$$

Since the mass, or area, is

$$M = \int_0^a y\,dx = \tfrac{4}{3}\,a^2,$$

we may write

$$I_y = \tfrac{3}{7}\,Ma^2,$$

which shows that the square of the radius of gyration is

$$R^2 = \tfrac{3}{7}\,a^2.$$

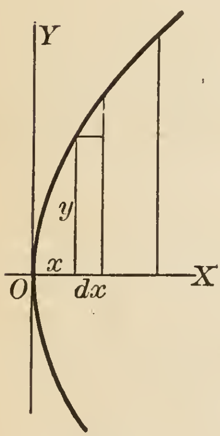


FIG. 61

(*b*) The area in Fig. 61 revolves about the y-axis. Find the moment of inertia of the volume generated, with respect to the axis of revolution.

Take as element of volume the cylindrical shell generated by the rectangle shown in Fig. 61, so that

$$dV = 2\,\pi xy\,dx.$$

The radius of gyration of this shell about the y-axis is evidently x. Hence

$$I_y = 2\,\pi\int_0^a x^2 \cdot xy\,dx = 4\,\pi\sqrt{a}\int_0^a x^{\frac{7}{2}}\,dx$$
$$= \tfrac{8}{9}\,\pi a^5.$$

The mass, or volume, is

$$M = 2\,\pi\int_0^a xy\,dx = \tfrac{8}{5}\,\pi a^3,$$

whence

$$I_y = \tfrac{5}{9}\,Ma^2.$$

EXERCISES

Find the following moments of inertia.

1. A particle of mass 3 units at (0, 0), one of 4 units at (2, 2), and one of 5 units at (– 1, – 3), with respect to each of the coördinate axes.

2. Equal particles at (0, 0, 0), (0, 5, 0), (3, 4, 3), with respect to each coördinate axis.

3. Equal particles at each corner of a cube, (*a*) with respect to an edge of the cube, (*b*) with respect to a diagonal of one face.

Ans. (*a*) Ma^2; (*b*) $\frac{3}{4}Ma^2$.

4. A straight rod or wire with respect to a perpendicular through one end. *Ans.* $\frac{1}{3}Ml^2$.

5. A rectangle about one side. *Ans.* $\frac{1}{3}Ma^2$.

6. A circular disk with respect to a diameter. *Ans.* $\frac{1}{4}Ma^2$.

7. The area in the example of § 132 with respect to the x-axis, (*a*) taking the element parallel to OX, (*b*) taking the element parallel to OY and using the result of Ex. 5.

8. (*a*) An isosceles triangle, (*b*) any triangle, with respect to the base.

9. A circular disk of radius 4 in. with a square of side 2 in. cut out of the center, with respect to a diameter parallel to a side of the square.

10. An ellipse with respect to each of its axes, using (*a*) the cartesian equation, (*b*) the equations $x = a \cos \phi$, $y = b \sin \phi$.

11. The area bounded by the parabola $y^2 = 4\,ax$, the y-axis, and the line $y = 2\,a$, with respect to each coördinate axis.

12. The area in Fig. 75, p. 183, (*a*) with respect to the base, (*b*) with respect to the line of symmetry.

13. The area in Fig. 76, with respect to the base.

14. A sphere with respect to a diameter. *Ans.* $\frac{2}{5}Ma^2$.

15. A cylinder of revolution, with respect to its axis. *Ans.* $\frac{1}{2}Ma^2$.

16. A right circular cone with respect to its axis. *Ans.* $\frac{3}{10}Ma^2$.

17. A paraboloid of revolution bounded by the right section through the focus, with respect to the axis.

18. An ellipsoid generated by revolving an ellipse about its major axis, with respect to the axis of revolution. Use (*a*) the cartesian equation of the ellipse, (*b*) the equations $x = a \cos \phi$, $y = b \sin \phi$.

19. The volume formed by revolving the area of Fig. 61 about the latus rectum, with respect to the axis of revolution.

20. A circular disk about its axis — *i.e.* the line through the center of the disk perpendicular to its plane. *Ans.* $\frac{1}{2}Ma^2$.

21. A wire bent in the form of a square, with respect to (*a*) a side, (*b*) a diagonal.

22. A circular wire with respect to a diameter, using (*a*) polar, (*b*) cartesian coördinates.

23. The arc of the curve $ay^2 = x^3$, from $x = 0$ to $x = 5a$, with respect to the y-axis.

24. A spherical surface about a diameter, using (*a*) polar, (*b*) cartesian coördinates.

25. The lateral surface of a cone of revolution, about its axis. *Ans.* $\frac{1}{2}Ma^2$.

26. A torus, with respect to its axis.

27. The surface of a torus, about its axis.

28. The surface of a paraboloid of revolution bounded by a right section through the focus, with respect to the axis.

29. The surface in Ex. 5, p. 171, about the z-axis.

30. The surface in Ex. 7, p. 171, about the y-axis.

31. The arc of the curve $ay^2 = x^3$ from $x = 0$ to $x = a$ revolves about OY. Find the moment of inertia of the surface generated, with respect to the y-axis.

32. Find the moment of inertia of the volumes in Exs. 14, 16, 17, 18, by using the result of Ex. 20.

33. The area under the curve $y = e^{-\frac{1}{2}x^2}$ revolves about the y-axis. Find the moment of inertia of the volume generated, with respect to the y-axis. *Ans.* $2M$.

133. Moment of inertia with respect to a plane. In most applications we are concerned with moment of inertia with respect to a line. Nevertheless it is frequently useful, as we shall see presently, to introduce the idea of moment of inertia with respect to a *plane*. The definitions and discussion of §§ 130–132 hold at once for the moment of the second order with respect to a plane if we replace the word "line" (or "axis") throughout by the word "plane."

Example: Find the moment of inertia of a sphere with respect to a diametral plane.

Taking the plane of moments as yz-plane and choosing as the element of volume a circular disk parallel to the yz-plane, we have

$$I_{yz} = \pi \int_{-a}^{a} x^2 y^2 \, dx = \pi \int_{-a}^{a} x^2 (a^2 - x^2) dx = \tfrac{4}{15} \pi a^5 = \tfrac{1}{5} Ma^2.$$

EXERCISES

Find the following moments of inertia.

1. The following system of particles, with respect to each of the coördinate planes: 3 units at (0, 0, 2), 2 units at (4, 3, 2), 4 units at (– 2, 2, 1), 1 unit at (3, – 3, 0).

2. A right circular cylinder with respect to the plane of the base.

3. A paraboloid of revolution bounded by a plane through the focus at right angles to the axis, (*a*) for that plane; (*b*) for the plane tangent at the vertex.

4. A spherical surface for a diametral plane, using (*a*) cartesian, (*b*) polar coördinates.

5. A right circular cone with respect to the plane of the base.

Ans. $\frac{1}{10} Mh^2$.

6. An ellipsoid of revolution with respect to the plane through the center perpendicular to the axis.

7. The lateral surface of a right circular cone, for the plane through the vertex at right angles to the axis.

8. The volume in Fig. 68, with respect to each coördinate plane.

9. The banister cap of Ex. 14, p. 163, with respect to the plane of the base.

10. A right pyramid with a square base, with respect to the plane of the base.

11. An ellipsoid with respect to the three principal planes.

134. General theorems on moments of inertia. We proceed to state certain theorems by means of which the work of finding moments of inertia may in many cases be greatly simplified.

THEOREM I: *The moment of inertia of any mass with respect to a line is equal to the sum of the moments with respect to two perpendicular planes through the line.*

For example,

$$I_x = I_{xy} + I_{zx}.$$

COROLLARY: *The moment of inertia of a plane mass with respect to a line perpendicular to its plane is equal to the sum of the moments with respect to two lines in the plane intersecting at right angles in the foot of the perpendicular.*

For example, for a mass in the xy-plane,

$$I_z = I_x + I_y.$$

THEOREM II: *The moment of inertia of any mass with respect to a line (or plane) is equal to the moment with respect to the parallel centroidal line* * *(or plane) plus the product of the mass by the square of the distance between the lines (or planes).*

That is, if l is any line, $\bar{l}$ the parallel centroidal line, d the distance between them, then

$$I_l = I_{\bar{l}} + Md^2.$$

We shall prove these theorems at present only for a system of particles, returning to the general case later (§ 187).

To prove theorem I, let us take the two perpendicular planes as the xy-plane and the zx-plane. Then, for a system of n particles,

$$I_x = \sum_{i=1}^{n} m_i(y_i^2 + z_i^2),$$

$$I_{xy} = \sum_{i=1}^{n} m_i z_i^2,$$

$$I_{zx} = \sum_{i=1}^{n} m_i y_i^2.$$

Hence

$$I_x = I_{xy} + I_{zx}.$$

* That is, the parallel line through the centroid.

The proof of the corollary is left to the student.

To prove theorem II for any line l and the parallel centroidal line $\bar{l}$, we note that in Fig. 77, by the cosine law,

$$d_i^2 = \bar{d}_i^2 + d^2 - 2\, d\bar{d}_i \cos \theta_i$$
$$= \bar{d}_i^2 + d^2 - 2\, dp_i,$$

where p_i is the distance of m_i from the plane through $\bar{l}$ perpendicular to the plane of l and $\bar{l}$. Hence

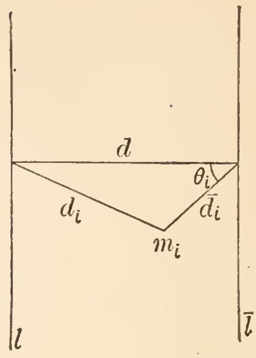


FIG. 77

$$\sum_{i=1}^{n} m_i d_i^2 = \sum_{i=1}^{n} m_i \bar{d}_i^2 + d^2 \sum_{i=1}^{n} m_i - 2\, d \sum_{i=1}^{n} m_i p_i.$$

But the quantity $\sum_{i=1}^{n} m_i p_i$ is the mass-moment of first order of the system with respect to the plane through $\bar{l}$ perpendicular to the plane determined by l and $\bar{l}$, and since this perpendicular plane contains the centroid, the moment in question is 0. Hence

$$I_l = I_{\bar{l}} + Md^2.$$

The proof of theorem II for two parallel planes is still simpler. It is left to the student.

EXERCISES

Find the following moments of inertia.

1. A right circular cylinder with respect to (*a*) a plane through the axis; (*b*) a generator; (*c*) a diameter of the middle section; (*d*) a line tangent to the base.

2. A cube with respect to (*a*) one face; (*b*) an edge.

3. A circular disk (*a*) for a tangent, (*b*) for a perpendicular through a point in the circumference. Solve (*b*) in two ways.

4. An isosceles triangle about a line (*a*) parallel to the base bisecting the altitude, (*b*) through the vertex perpendicular to the plane.

5. A sphere with respect to a tangent. Solve in two ways.

6. A square plate for a line perpendicular to its plane (*a*) through a corner, (*b*) through the center.

7. A right pyramid with a square base, with respect to the axis.

8. The area in Fig. 75 with respect to a line through the centroid (*a*) parallel to the base, (*b*) perpendicular to the plane.

9. A wire frame in the shape of an isosceles triangle, with respect to a line (*a*) through the centroid parallel to the base, (*b*) through the vertex perpendicular to the plane.

10. An ellipsoid of revolution about a diameter of the middle cross section.

11. The area in Fig. 76 with respect to a line through the centroid parallel to the base.

12. A right circular cone with respect to (*a*) a diameter of the base, (*b*) a line through the vertex perpendicular to the axis, (*c*) a diameter of the middle cross section.

Ans. (*b*) $(\frac{3}{5}h^2 + \frac{3}{20}a^2)M$; (*c*) $(\frac{1}{10}h^2 + \frac{3}{20}a^2)M$.

13. The volume in Fig. 68, with respect to each coördinate axis.

14. An ellipse for a line through the center perpendicular to the plane.

15. An ellipsoid for each of its axes.

135. Kinetic energy of a rotating body. The *kinetic energy* of a particle of mass m moving with a velocity v is defined as

$$E = \frac{1}{2} mv^2.$$

If a particle at a distance r from a fixed line rotates with an angular velocity ω about that line as an axis, its linear velocity is (§ 58)

$$v = \omega r;$$

hence its kinetic energy is

$$E = \frac{1}{2} m\omega^2 r^2.$$

The kinetic energy of a system of n particles rotating with angular velocity ω is

$$E = \frac{1}{2} \omega^2 \sum_{i=1}^{n} m_i r_i^2 = \frac{1}{2} I\omega^2,$$

where I is the moment of inertia of the system with respect to the axis of rotation. This formula holds in general.

A discussion of the various systems of units in actual use is outside the scope of this book. In the exercises below the so-called engineering system is used. In this system the mass m is replaced by the value

$$m = \frac{w}{g},$$

where w is the "weight" in pounds, and g the acceleration of gravity (32 ft. per second per second approximately). If then v is expressed in feet per second, the energy is measured in "foot-pounds."

EXERCISES

1. A straight rod 10 ft. long, weighing 20 lbs., rotates about a perpendicular through one end at the rate of 2 R. P. S. Find its kinetic energy. *Ans.* 1650 ft.-lbs.

2. A flywheel 12 ft. in diameter whose rim weighs 10 tons makes 50 R. P. M. Neglecting the mass of the spokes, find the kinetic energy of the wheel. *Ans.* 156 ft.-tons.

3. A flywheel 1 ft. in diameter, weighing 50 lbs., makes 100 R. P. M. If the wheel can be considered as a uniform circular disk, find its kinetic energy.

4. A wheel 4 ft. in diameter has 8 spokes weighing 20 lbs. each. The rim weighs 600 lbs. Find the kinetic energy of the wheel when it is making 20 R. P. M.

5. The kinetic energy of a solid sphere 1 ft. in diameter making 60 R.P.M. about an axis through its center is 5 ft.-lbs. Find the weight of the sphere. *Ans.* 80 lbs.

6. A hollow cast-iron sphere (sp. gr. 7.5) 4 ft. in diameter rotates about an axis through its center at the rate of 1 radian per second. Its kinetic energy is 375 ft.-lbs. Find the inner radius. *Ans.* 1 ft.

7. Find the inner radius in Ex. 6 if the kinetic energy is 200 ft.-lbs.

CHAPTER XIX

LAW OF THE MEAN. EVALUATION OF LIMITS

136. Rolle's theorem. Let there be given a continuous, one-valued, and differentiable function $\phi(x)$, which vanishes at $x=a$ and $x=b$. In order that the function, starting with the value 0 at $x=a$, shall assume again the value 0 at $x=b$, it must first increase up to some point P, and then begin to decrease, or vice versa.

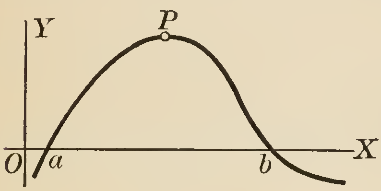


FIG. 78

At P there is either a maximum or a minimum, and the derivative is 0 at that point.

We have thus

ROLLE'S THEOREM: *If $\phi(x)$ vanishes when $x=a$ and $x=b$, then $\phi'(x)$ vanishes for at least one value of x between a and b.*

If the fundamental assumptions of continuity and differentiability are not satisfied, the theorem may not hold. In Fig. 79 it fails because $\phi(x)$ is discontinuous at one point; in Fig. 80 it fails because the derivative is undefined at one point.

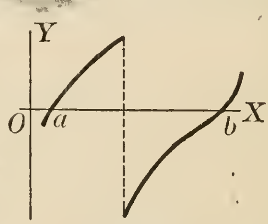


FIG. 79

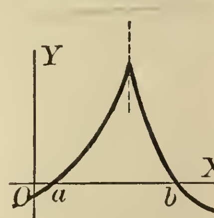


FIG. 80

137. The law of the mean. Let $f(x)$ be a continuous, one-valued, and differentiable function whose graph in the interval $x=a$ to $x=b$ is shown in Fig. 81. It is geometrically obvious that at some point P the tangent must be parallel to the secant SQ. Now the slope of the secant is

$$\frac{RQ}{SR} = \frac{f(b) - f(a)}{b - a};$$

the slope of the tangent at P is $f'(x_1)$, where x_1 is the abscissa of P. Hence

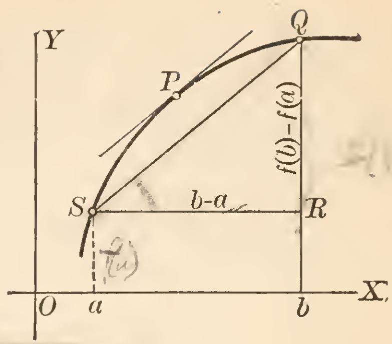


FIG. 81

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

or

(1) $$f(b) - f(a) = (b - a)f'(x_1), \qquad a < x_1 < b.$$

This relation is called the *law of the mean.*

138. Other forms of the law of the mean. It is often necessary to apply formula (1) above with $b = x$, thus making the length of the interval variable:

(1) $$f(x) = f(a) + (x - a)f'(x_1), \qquad a < x_1 < x.$$

It is to be noted that x_1 is here a function of x.

Again, placing

$$a = x, \quad b = a + \Delta x,$$

and denoting by θ a positive number less than unity, we obtain

$$f(x + \Delta x) = f(x) + \Delta x f'(x + \theta\Delta x), \qquad 0 < \theta < 1.$$

EXERCISES

For each of the following functions, show why Rolle's theorem does not hold in the indicated interval.

1. $y = 1 - \frac{1}{x^2}$, $\qquad -1 < x < 1.$

2. $(y + 4)^3 = x^2$, $\qquad -8 < x < 8.$

3. $y = \tan x$, $\qquad 0 < x < \pi.$

4. Draw curves showing that the law of the mean may fail when $f(x)$ is discontinuous or non-differentiable in the interval.

5. At what point on the parabola $y = x^2$ is the tangent parallel to the secant through the points $x = 0$, $x = 2$?

6. At what point on the curve $y = x^3 - x$ is the tangent parallel to the secant through (1, 0) and (2, 6)? Draw the figure.

7. Find the point on the curve $y = \log x$ where the tangent is parallel to the secant through the points $x = 1$, $x = 2$.

139. The indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$. If two functions $f(x)$, $F(x)$ both vanish at $x = a$:

$$f(a) = 0, \; F(a) = 0,$$

their quotient $\frac{f(x)}{F(x)}$ assumes the "indeterminate form" $\frac{0}{0}$ at $x = a$, and is undefined at that point. Nevertheless the *limit* $\lim\limits_{x \to a} \frac{f(x)}{F(x)}$ may exist. This fact is illustrated in the derivation of the fundamental differentiation formulas, where in each case both numerator and denominator of the difference quotient $\frac{\Delta y}{\Delta x}$ approach 0, yet the derivative, which is the limit of that quotient, exists.

In case the fraction $\frac{f(x)}{F(x)}$ does approach a limit when x approaches a, we lay down the following *definition:*

$$\left.\frac{f(x)}{F(x)}\right]_{x=a} = \lim_{x \to a} \frac{f(x)}{F(x)}.$$

The function $\frac{f(x)}{F(x)}$ thus becomes continuous at the point $x = a$, by the definition of continuity (§ 12).

It may be possible to evaluate the limit by means of more or less obvious transformations of $\frac{f(x)}{F(x)}$, as was done in deriving the differentiation formulas. In many cases the limit may be obtained by a method that will now be developed.

By the law of the mean, using the form (1) of § 138 we may write

$$f(x) = f(a) + (x - a)f'(x_1),$$
$$F(x) = F(a) + (x - a)F'(x_2),$$

where x_1 and x_2 lie between a and x. But by hypothesis

$$f(a) = F(a) = 0.$$

Hence

$$\frac{f(x)}{F(x)} = \frac{(x-a)f'(x_1)}{(x-a)F'(x_2)} = \frac{f'(x_1)}{F'(x_2)}.$$

As x approaches a, x_1 and x_2 must do likewise, and we have, by theorem III of § 8,

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x_1)}{F'(x_2)} = \lim_{x \to a} \frac{f'(x)}{F'(x)} = \frac{f'(a)}{F'(a)},$$

provided $f'(a)$ and $F'(a)$ exist and $F'(a) \neq 0$.

If $f(x)$ and $F(x)$ both increase indefinitely as x approaches a:

$$\lim_{x \to a} f(x) = \infty, \quad \lim_{x \to a} F(x) = \infty,$$

the fraction $\dfrac{f(x)}{F(x)}$ is said to assume the indeterminate form $\frac{\infty}{\infty}$ at $x = a$. Here again it may happen that $\lim\limits_{x \to a} \dfrac{f(x)}{F(x)}$ exists, and it can be shown that the same method may be applied in this case as in the case just treated.

It may happen that the fraction $\dfrac{f'(x)}{F'(x)}$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. In this case we may differentiate numerator and denominator again, and repeat as many times as necessary.

Finally, we may have $a = \infty$: *i.e.* $\dfrac{f(x)}{F(x)}$ approaches the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ when x increases indefinitely. In this case the rule holds also.

The results of this article may be summarized in the following

THEOREM: *If the fraction* $\frac{f(x)}{F(x)}$ *assumes the indeterminate form* $\frac{0}{0}$ or $\frac{\infty}{\infty}$ *when* $x = a$, *then*

$$\lim_{x \to a} \frac{f(x)}{F(x)} = \lim_{x \to a} \frac{f'(x)}{F'(x)},$$

provided the latter limit exists.

Thus we may differentiate the numerator and the denominator *separately*, and take the limit of the new fraction thus formed. It must be borne clearly in mind, however, that the theorem applies only to *fractions* in which the numerator and the denominator *both approach* 0 *or both increase indefinitely.*

Example: Evaluate $\lim_{x \to 0} \frac{\tan x}{x}$.

This fraction takes the form $\frac{0}{0}$ when $x = 0$, so that the theorem applies:

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = 1.$$

140. The indeterminate forms $0 \cdot \infty$, $\infty - \infty$. Given the product of two functions $f(x) \cdot F(x)$, suppose that as x approaches a one function approaches 0 while the other increases indefinitely. The product is then said to take the indeterminate form $0 \cdot \infty$.

If we write

$$f(x) \cdot F(x) = \frac{f(x)}{\frac{1}{F(x)}},$$

it appears that the quotient last written assumes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and the theorem of § 139 may be applied.

If, as x approaches a, each of two functions $f(x)$, $F(x)$ increases indefinitely, their difference $f(x) - F(x)$ is said to assume the indeterminate form $\infty - \infty$. Here also we express $f(x) - F(x)$ as a fraction which takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then apply the theorem.

Example: Evaluate $\lim\limits_{x\to 0^+} x \log x$.

This takes the form $0 \cdot \infty$. If we write it in the form $\dfrac{\log x}{\frac{1}{x}}$, the theorem of § 139 becomes applicable:

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0.$$

EXERCISES

Evaluate the following limits, when they exist.

1. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cos x}{x - \frac{\pi}{2}}$.

2. $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$.

3. $\lim\limits_{x\to 4} \dfrac{x^2 - 16}{x^2 + x - 20}$.

4. $\lim\limits_{x\to -1} \dfrac{\log(x+2)}{x+1}$.

5. $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x}$.

6. $\lim\limits_{x\to\infty} xe^{-x}$.

7. $\lim\limits_{x\to 0^+} x \log \sin x$.

8. $\lim\limits_{x\to 0} \dfrac{\log \cos x}{x}$.

9. $\lim\limits_{x\to\infty} \dfrac{3x^2 - 4x}{2x^2 - 3x + 1}$.

10. $\lim\limits_{x\to\infty} \dfrac{x - 1}{x^2 + 1}$.

11. $\lim\limits_{x\to\infty} \dfrac{x^4 - 4x^3}{x^2 + 1}$.

12. $\lim\limits_{x\to\frac{\pi}{2}} (\sec x - \tan x)$.

13. $\lim\limits_{\theta\to 0} \dfrac{\theta - \arcsin\theta}{\sin^3\theta}$. *Ans.* $-\frac{1}{6}$.

Trace the following curves.

14. $y = x \log x$.

15. $y = xe^{-x}$.

16. $y = \dfrac{\log x}{x^2}$.

17. $y = x^2 \log x$.

18. Find the area in the fourth quadrant bounded by the curve $y = \log x$ and the coördinate axes.

19. Find the centroid of the area in the second quadrant under the curve $y = e^x$. Obtain each coördinate in two ways.

20. Find the moment of inertia of the area in Ex. 19, about each coördinate axis.

21. Find the area bounded by the curve $y = x \log x$ and the x-axis.

141. General remarks on evaluation of limits. While the methods of §§ 139–140 are frequently very useful in investigating the limit of a function at a point where the function ceases to be defined, they are by no means always applicable. In the first place, the function may lose its meaning in some other way than by taking the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (or a form reducible to one of these), so that the theorem of § 139 cannot be brought into play, yet it may be possible to show the existence of a limit by other methods. Even when the function $\frac{f(x)}{F(x)}$ does take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ the theorem may fail to apply because $\frac{f'(x)}{F'(x)}$ approaches no limit, yet the limit of the original quotient may exist. Again, the function $\frac{f(x)}{F(x)}$ may take the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and at the same time $\frac{f'(x)}{F'(x)}$ may approach a limit, yet it may be impossible to obtain any result by the use of the theorem. Finally, there is always the possibility that a function undefined at $x = a$ may fail to approach any limit as x approaches a.

Each of these cases is illustrated by the following

Examples: (*a*) Evaluate $\lim\limits_{x \to \infty} \frac{\sin x}{x}$.

Here the denominator increases indefinitely, while the numerator oscillates between -1 and 1, without approaching any fixed value. Nevertheless, since $\frac{\sin x}{x}$ is never numerically greater than $\frac{1}{x}$, it clearly approaches 0.

(*b*) Evaluate $\lim\limits_{x\to 0}\dfrac{x^2\cos\frac{1}{x}}{\sin x}$.

Since $\cos\frac{1}{x}$ lies always between -1 and 1, the numerator approaches 0 and the fraction takes the form $\frac{0}{0}$. Differentiating numerator and denominator separately, we obtain a new fraction $\dfrac{2x\cos\frac{1}{x}+\sin\frac{1}{x}}{\cos x}$. This fraction approaches no limit, since $\sin\frac{1}{x}$ oscillates between -1 and 1, and the theorem is therefore inapplicable. But the original fraction has a limit, which may be found directly:

$$\lim_{x\to 0}\frac{x^2\cos\frac{1}{x}}{\sin x}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\lim_{x\to 0}x\cos\frac{1}{x}=0,$$

since $\lim\limits_{x\to 0}\dfrac{x}{\sin x}=1$ and $\lim\limits_{x\to 0}x\cos\dfrac{1}{x}=0$.

(*c*) Evaluate $\lim\limits_{x\to\infty}\dfrac{2^x}{3^x}$.

This fraction takes the form $\frac{\infty}{\infty}$. By § 139,

$$\lim_{x\to\infty}\frac{2^x}{3^x}=\lim_{x\to\infty}\frac{2^x\log 2}{3^x\log 3}=\lim_{x\to\infty}\frac{2^x\log^2 2}{3^x\log^2 3}=\cdots.$$

No matter how many times we differentiate, we cannot get rid of the quotient $\frac{2^x}{3^x}$. Yet if the function be written in the form $(\frac{2}{3})^x$, it is seen to approach the limit 0.

(*d*) Evaluate $\lim\limits_{x\to\infty}\dfrac{\tan x}{x}$.

In this case no limit is approached. For no matter how large x be taken, as x varies from $n\pi$ to $(n+1)\pi$

the function $\tan x$, and hence $\frac{\tan x}{x}$, ranges through all possible values from $-\infty$ to $+\infty$.

EXERCISES

Evaluate the following limits, when they exist.

1. $\lim\limits_{n\to\infty} \frac{n}{2^n}$. *Ans.* 0.

2. $\lim\limits_{\phi\to\pi} \sin\phi \cot\phi$. *Ans.* -1.

3. $\lim\limits_{x\to\infty} \frac{\cos x}{x}$. *Ans.* 0.

4. $\lim\limits_{x\to\infty} (x^2 - x)$.

5. $\lim\limits_{x\to\infty} \frac{\log(1+e^x)}{x}$.

6. $\lim\limits_{n\to\infty} \frac{3^{n+1}}{\pi^{n-1}}$.

7. $\lim\limits_{x\to 0} \frac{\tan x - x}{x - \sin x}$.

8. $\lim\limits_{x\to 0} x \sin\frac{1}{x}$.

9. $\lim\limits_{x\to\infty} \frac{\sin x}{\cos 2x}$.

10. $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{\tan\theta}{\tan 3\theta}$.

11. $\lim\limits_{x\to a} \frac{\sin x - \sin a}{x - a}$.

12. $\lim\limits_{x\to 1} \frac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}$.

13. $\lim\limits_{x\to 0} \frac{x\cos\frac{1}{x}}{\sin x}$.

14. $\lim\limits_{x\to 0} (\cot x - \operatorname{cosec} x)$.

15. $\lim\limits_{x\to\infty} \sqrt{\frac{2x+1}{x+1}}$.

16. $\lim\limits_{n\to\infty} \frac{2^n}{n!}$. *Ans.* 0.

Trace the following curves.

17. $y = \frac{\sin x}{x}$.

18. $y = \frac{\tan x}{x}$.

19. $y = x^2 e^{-x}$.

20. $y = \frac{e^x}{x}$.

21. Find the area bounded by the curve $y = \frac{\log x}{x}$, the x-axis, and the maximum ordinate. Trace the curve. *Ans.* $\frac{1}{2}$.

22. Trace the curve $y = \frac{1}{x \log x}$, and find the area under the curve from $x = 2$ to $x = e$. *Ans.* 0.367.

23. Trace the curve $y = xe^{-\frac{1}{2}x^2}$, and find the area under the curve in the first quadrant.

24. Find the moment of inertia of the area in Ex. 23, with respect to each coördinate axis.

CHAPTER XX

INFINITE SERIES. TAYLOR'S THEOREM

I. Series of Constant Terms

142. Series of n terms. A *series* of n terms is an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n,$$

where each term is formed from the preceding one by some definite law. Examples are the *arithmetic series*

$$a + [a + d] + [a + 2d] + \cdots + [a + (n-1)d],$$

in which each term is formed from the preceding by the addition of a constant d, and the *geometric series*

$$a + ar + ar^2 + \cdots + ar^{n-1},$$

in which each term is equal to r times the one before it.

143. Infinite series. When the number of terms increases indefinitely, a series of n terms becomes an *infinite series*, denoted by the symbol

$$a_1 + a_2 + a_3 + \cdots.$$

The series is defined by the *law of formation* of successive terms, or, what amounts to the same thing, by the *n-th or general term.*

The general term may frequently be written down by inspection of the first few terms, as in the following

Examples: (*a*) In the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots,$$

the general term is $\dfrac{1}{n}\cdot$

(*b*) In the geometric series

$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots,$$

the general term is $(-\tfrac{1}{2})^{n-1}$.

(c) In the series

$$\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2\cdot 4}\cdot\frac{1}{5}+\frac{1}{2\cdot 4\cdot 6}\cdot\frac{1}{7}+\frac{1}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{1}{9}+\cdots,$$

the general term is $\dfrac{1}{2^n n!}\cdot\dfrac{1}{2n+1}$.

144. Sum of an infinite series. It is shown in elementary algebra that the sum of a geometric series of n terms is

$$(1) \qquad S_n=\frac{a-ar^n}{1-r};$$

of an arithmetic series of n terms,

$$S_n=\frac{n}{2}(a+l),$$

where l is the last term — *i.e.*

$$S_n=\frac{n}{2}[2a+(n-1)d].$$

Similarly the sum of any series of a finite number of terms can be found.

On the other hand, an infinite series has no sum in the ordinary sense of the term, since no matter how many terms we might add up, there would always be an infinite number left over. We may, however, give a meaning to the term "sum" even in this case by laying down the following *definition:*

The sum of an infinite series is defined as the limit, as n increases indefinitely, of the sum of the first n terms:

$$S=\lim_{n\to\infty}S_n,$$

provided the limit exists.

Thus the "sum" of an infinite series is the *limit* of an ordinary sum.

Example: By (1), the sum of the first n terms of the infinite geometric series

$$a+ar+ar^2+\cdots+ar^{n-1}+\cdots$$

is

$$S_n=\frac{a-ar^n}{1-r}.$$

Hence the sum of the series, if the sum exists, is

$$S = \lim_{n \to \infty} \frac{a - ar^n}{1 - r}.$$

When r is numerically less than 1, the quantity ar^n approaches 0 as n increases, and

$$S = \frac{a}{1 - r}.$$

When r is numerically greater than 1, the quantity ar^n increases indefinitely, and the above limit does not exist; the series has no sum.

145. Convergence and divergence. If the series has a sum S, *i.e.* if S_n approaches a limit when n increases, the series is said to be *convergent*, or to *converge to the value* S; if the limit does not exist, the series is *divergent*.

It follows from the above example that a geometric series converges to the value $\frac{a}{1-r}$ if $|r| < 1$; it diverges if $|r| > 1$.

A series may diverge, as in the case of a geometric series for which $r > 1$, because S_n increases indefinitely as n increases; or it may diverge because S_n increases and decreases alternately, or *oscillates*, without approaching any limit. In the latter case the series is called *oscillatory*.

EXERCISES

1. Show that every infinite arithmetic series is divergent.

2. Find the sum of a geometric series of n terms for which $r = 1$, (*a*) directly, (*b*) by applying the theorem of § 139 to formula (1), § 144.

3. Show that the infinite geometric series for which $r = 1$ is divergent.

4. Show that the infinite geometric series for which $r = -1$, viz.

$$a - a + a - a + \cdots,$$

is oscillatory.

146. Tests for convergence. In the elementary applications divergent series are of no importance. Before being able to use a given series we must determine whether it converges or diverges. If S_n can be expressed explicitly as a function of n, as in the case of the arithmetic and geometric series, we can in general determine the convergence or divergence of the series directly, and find the sum if it exists; but S_n cannot be so expressed in most cases.

A *necessary* condition for convergence is that the general term approach 0 as its limit:

$$\lim_{n\to\infty} a_n = 0.$$

For, when this condition is not satisfied, each term that is added changes S_n by an amount that does not approach 0, so that the difference between S_n and a fixed number S obviously cannot become and remain arbitrarily small.

This condition, though necessary, is not sufficient; *i.e.* if the condition is not satisfied, the series diverges, but if it is satisfied, the series still may diverge. This is illustrated by the *harmonic series*

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots,$$

which will be shown in the next article to be divergent, although

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Many special tests for convergence have been devised, applicable to more or less broad classes of series. Several of the simplest are considered in the next few articles.

147. Cauchy's integral test. We will begin with an *Example:* Prove that the *harmonic series*

(1) $$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

is divergent.

Here the general term is

$$a_n = f(n) = \frac{1}{n}.$$

Let us draw the curve

$$y = f(x) = \frac{1}{x},$$

erect the ordinates at $x = 1$, 2, 3, $\cdots$, n, and complete the circumscribed rectangles as shown in the figure. Then the areas of the rectangles are, respectively,

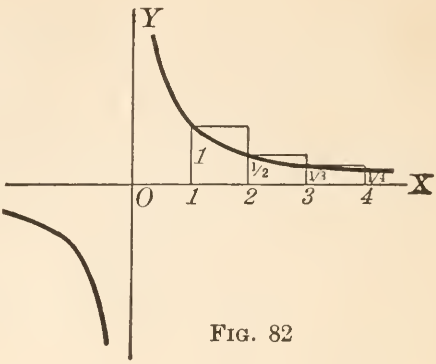


FIG. 82

$$1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n},$$

so that the sum of these areas is the sum of the first n terms of the series (1):

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

But the sum of the rectangles is clearly greater than the area under the curve from $x = 1$ to $x = n$:

$$S_n > \int_1^n y\,dx = \int_1^n \frac{dx}{x} = \log n.$$

When n increases, the area $\log n$ under the curve becomes infinite, hence S_n does likewise and the series diverges.

This example illustrates *Cauchy's integral test:*

Given a series of *positive terms*

(2) $$a_1 + a_2 + a_3 + \cdots,$$

put $$a_n = f(n).$$

If the function $f(x)$ is defined not only for positive integral values, but for all positive values of x, and if $f(x)$ never increases with x, then the series (2) converges or diverges according as the integral $\int_1^\infty f(x)\,dx$ does or does not exist.*

The proof of this test is easily written out by drawing the curve $y = f(x)$ and following the process suggested by the above example. The details are left to the student.

* For the definition of this improper integral, see § 119.

EXERCISES

1. Write out the proof of Cauchy's integral test.

2. In the statement of the integral test, why is it assumed that $f(x)$ never increases with x? Show that it would be sufficient to assume that $f(x)$ never increases with x after some fixed point $x = x_0$.

3. Prove that the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$$

is convergent.

4. Prove that the series

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$

converges if $p > 1$, diverges if $p \leqq 1$.

Test the following series for convergence.

5. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots.$

6. $1+\frac{1}{1+2^2}+\frac{1}{1+3^2}+\frac{1}{1+4^2}+\cdots.$

7. $1-2+3-4+\cdots.$

8. $1-\frac{3}{4}+\frac{4}{6}-\frac{5}{8}+\cdots.$

9. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots.$

10. $1+\frac{2}{1+2^2}+\frac{3}{1+3^2}+\frac{4}{1+4^2}+\cdots.$

11. Test the geometric series for convergence by the integral test.

148. Comparison test. Let

$$u_1+u_2+u_3+\cdots$$

be a series of *positive terms* to be tested.

(*a*) *If a series*

$$a_1+a_2+a_3+\cdots$$

of positive terms, known to be convergent, can be found such that

$$u_n \leqq a_n,$$

then the series to be tested is convergent.

(*b*) *If a series*

$$b_1+b_2+b_3+\cdots$$

of positive terms, known to be divergent, can be found such that

$$u_n \geqq b_n,$$

then the series to be tested is divergent.

To prove (*a*), let $S_n(u)$ be the sum of the first n terms of the u-series, $S_n(a)$ the sum of the first n terms of the a-series, and $S(a)$ the sum of the a-series. Since all the terms u_n are positive, $S_n(u)$ always increases with n. On the other hand, we have

$$S_n(u) < S_n(a) < S(a).$$

Since $S_n(u)$ always increases, but never exceeds the fixed number $S(a)$, it approaches a limit, by theorem IV of § 8, which is not greater than $S(a)$.

The proof of (*b*) is left to the student.

The success of the test depends on our ability to find a *convergent* series whose terms are *greater* than the corresponding terms of the series to be tested, or a *divergent* series whose terms are *less* than those of the series to be tested. To show that the terms of the u-series are greater than those of some convergent series, or less than those of some divergent series, proves nothing.

It is clear that the convergence of a series is not affected by discarding any finite number of terms from the series. Hence the conditions of the test do not need to be satisfied from the very beginning of the series, but only *after a certain point*, all the terms up to that point being neglected.

Example: Test the series

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots.$$

The series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is known to converge (Ex. 4, p. 214). Discarding the first term of the series to be tested, we have

$$u_n = \frac{1}{n(n+1)}.$$

The general term of the known series is

$$a_n = \frac{1}{n^2}.$$

Since

$$u_n < a_n,$$

the series in question converges.

EXERCISES

Test the following series as to convergence or divergence.

1. $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$

2. $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots.$

3. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots.$

4. $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots.$

5. $\frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots.$

6. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$

149. Ratio test. There are many "ratio tests"; the simplest is the following:

Given the series

$$u_1 + u_2 + u_3 + \cdots$$

to be tested for convergence, form the ratio $\frac{u_{n+1}}{u_n}$ of a general term * to the one preceding it.

(*a*) *If* $\lim\limits_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$, *the series converges.*

(*b*) *If* $\lim\limits_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1$, *or if* $\left| \frac{u_{n+1}}{u_n} \right|$ *increases indefinitely, the series diverges.*

(*c*) *If* $\lim\limits_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1$, *the test fails.*

* We may divide the $(n+1)$-th term by the n-th, the $(n+10)$-th by the $(n+9)$-th — *any* general term by the one before it, since the question of convergence is not affected by dropping any finite number of terms.

This test holds for any series whatever, not merely for series of positive terms.

Suppose we have case (a): $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L < 1$. At present we shall consider only the case in which all the terms are positive, and show later how the proof may be completed. Let us choose some number r between L and 1. By the definition

FIG. 83

of limit, the difference between the ratio $\frac{u_{n+1}}{u_n}$ and its limit L ultimately becomes and remains as small as we please; therefore a number m can be found such that for all values of $n \geqq m$ we have

$$\frac{u_{n+1}}{u_n} < r.$$

Hence

$$\begin{aligned} u_{m+1} &< u_m r, \\ u_{m+2} &< u_{m+1} r < u_m r^2, \\ u_{m+3} &< u_{m+2} r < u_m r^3, \\ &\vdots \quad\quad \vdots \quad\quad \vdots \end{aligned}$$

Discarding the first m terms of our series, we see that the remaining terms are less than the corresponding terms of the series

$$u_m r + u_m r^2 + u_m r^3 + \cdots.$$

But this latter series, being a geometric series with ratio $r < 1$, is convergent; hence the given series is convergent, by the comparison test.

Case (b) may be proved by showing that the general term u_n does not approach 0 when n increases indefinitely. The details are left to the student.

The test may be shown to fail in case (c) by the following example: For the series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

the test ratio is

$$\frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \left[\frac{n}{n+1}\right]^p = \left[\frac{1}{1+\frac{1}{n}}\right]^p,$$

and

$$\lim_{n \to \infty} \left[\frac{1}{1+\frac{1}{n}}\right]^p = 1.$$

But, by Ex. 4, p. 214, when $p \leqq 1$, this series diverges; when $p > 1$, the series converges. Hence there are both convergent and divergent series for which the limit of the test ratio is 1.

150. Alternating series. A series whose terms are alternately positive and negative is called an *alternating* series. Such series are of frequent occurrence. Their most important properties are contained in the following theorems.

THEOREM I: *If after a certain point the terms of an alternating series never increase numerically, and if the limit of the n-th term is 0, the series is convergent.*

THEOREM II: *In a convergent alternating series, the difference between the sum of the series and the sum of the first n terms is not greater numerically than the $(n+1)$-th term:*

$$|S - S_n| \leqq |u_{n+1}|.$$

To prove theorem I, let us write the series in the form

$$u_1 - u_2 + u_3 - u_4 + \cdots,$$

where all the u's are positive. When n is even, say $n = 2m$, we may write S_n in the two forms

(1) $\quad S_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}),$

(2) $\quad S_{2m} = u_1 - (u_2 - u_3) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}.$

Equation (1) shows that S_{2m} always increases, since each of the parentheses is positive. Equation (2) shows that S_{2m} is always less than u_1. Since S_{2m} always increases, but never exceeds the fixed number u_1, it approaches a limit S not greater than u_1, by theorem IV of § 8. Further, the sum of an odd number of terms S_{2m+1} approaches the same limit, since

$$\lim_{n \to \infty} (S_{2m+1} - S_{2m}) = \lim_{n \to \infty} u_{2m+1} = 0.$$

Theorem II follows at once. For, if n is even, the difference $S - S_n$ is the alternating series

$$u_{n+1} - u_{n+2} + u_{n+3} - \cdots,$$

and we have just shown that the sum of an alternating series is not greater than the first term. Similarly if n is odd.

151. Absolute convergence. A series is said to be *absolutely convergent* if the series formed from it by replacing all its terms by their absolute values is convergent. It can be shown that a series always converges if the series of absolute values converges. From this fact the proof in § 149 is easily completed for the case when the terms are not all positive.

EXERCISES

Determine whether the following series are convergent or divergent.

1. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots.$

2. $1 - \frac{1}{2!} + \frac{1}{3!} - \cdots.$

3. $\frac{e^2}{2} + \frac{e^3}{3} + \frac{e^4}{4} + \cdots.$

4. $\frac{\pi}{4} + 2\left(\frac{\pi}{4}\right)^2 + 3\left(\frac{\pi}{4}\right)^3 + \cdots.$

5. $\frac{1!}{10} + \frac{2!}{10^2} + \frac{3!}{10^3} + \cdots.$

6. $\frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \cdots.$

7. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$

8. $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots.$

9. $1 - \frac{2}{2^2} + \frac{3}{2^3} - \frac{4}{2^4} + \cdots.$

10. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$

11. Are the series in Exs. 7–10 absolutely convergent?

12. Carry out the proof of case (b), § 149.

II. Power Series

152. Power series. Up to this point we have considered only series whose terms are constants. The case of greatest practical importance, however, is that in which the terms are functions of a variable. In what follows, we shall be chiefly concerned with the class known as power series.

A series of the form

$$a_0 + a_1 x + a_2 x^2 + \cdots,$$

where x is a variable and a_0, a_1, a_2, $\cdots$ are constants, is called a *power series*. Such series are of especial importance in practice.

A power series may converge for all values of the variable x, or for no values except 0; but usually it will converge for all values in some finite interval, and diverge for all values outside that interval. The interval of convergence always extends equal distances on each side of the point $x = 0$.

The interval of convergence can usually be determined by the ratio test. We illustrate the process by an

Example: Find the interval of convergence of the series

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots.$$

Here

$$u_n = \frac{x^n}{n},$$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{\dfrac{x^{n+1}}{n+1}}{\dfrac{x^n}{n}} \right| = \lim_{n \to \infty} \frac{n}{n+1} \cdot |x| = |x|,$$

by the theorem* of § 139.

* Objection may be raised to the use of this theorem in the present instance, on the ground that n is not a continuous variable. The objection, however, is easily disposed of. For, if we can prove by the theorem that a given function of n approaches a certain limit when n varies continuously, it is certain that the same limit will be approached when n varies through positive integral values.

(*a*) The series converges when $|x| < 1$, *i.e.* $-1 < x < 1$.

(*b*) The series diverges when $|x| > 1$.

(*c*) The test fails when $x = \pm 1$. But when $x = 1$ the series is the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots,$$

and therefore diverges; when $x = -1$, the series is

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots,$$

which converges by § 150.

Hence the interval of convergence is $-1 \leqq x < 1$.

EXERCISES

Find the interval of convergence of the following series.

1. $1 + x + x^2 + x^3 + \cdots$.

2. $1 - 2x + 3x^2 - 4x^3 + \cdots$.

3. $1 - \frac{x}{3} + \frac{x^2}{9} - \frac{x^3}{27} + \cdots$.

4. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$. *Ans.* All values of x.

5. $1 + 10x + 2 \cdot 100x^2 + 3 \cdot 1000x^3 + \cdots$.

6. $1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$.

7. $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$.

8. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$. *Ans.* All values of x.

9. If $a_1 + a_2 + a_3 + \cdots$ is an absolutely convergent series and $b_1, b_2, b_3, \cdots$ a set of numbers that remain finite as n increases: $|b_n| < M$, where M is a constant, show that the series

$$a_1b_1 + a_2b_2 + a_3b_3 + \cdots$$

converges absolutely.

10. Prove that the series

$$\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots$$

converges absolutely for all values of x.

11. If $a_1 + a_2 + a_3 + \cdots$ is an absolutely convergent series and if $u_1 + u_2 + u_3 + \cdots$ is a series such that $\frac{u_n}{a_n}$ approaches a limit when n increases, show that the u-series converges absolutely.

12. State and prove a theorem for divergent series analogous to that of Ex. 11.

153. Maclaurin's series. It is shown in algebra that the quantity $(1+x)^m$, where m is not a positive integer, may be developed into an infinite series in powers of x by the binomial theorem :

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2\,!}x^2 + \frac{m(m-1)(m-2)}{3\,!}x^3 + \cdots,$$

the expansion being valid for all values of x numerically less than 1.

Consider now the problem of developing any given function $f(x)$ in powers of x. We will assume for the present that such a development is possible, and write

$$(1) \qquad f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots,$$

where the coefficients c_0, c_1, c_2, $\cdots$ are constants to be determined. Letting $x=0$, we get $f(0)=c_0$; *i.e.* c_0 is the value of the given function at $x=0$. Differentiating (1),

$$f'(x) = c_1 + 2\,c_2x + 3\,c_3x^2 + \cdots,$$

and setting $x=0$, we find

$$f'(0) = c_1.$$

Proceeding in this way, we find

$$\begin{aligned} f''(0) &= 2 \cdot 1\, c_2, \\ f'''(0) &= 3 \cdot 2 \cdot 1\, c_3, \\ &\vdots \\ f^{(n)}(0) &= n\,!\, c_n, \\ &\vdots \end{aligned}$$

Hence (1) takes the following form, called *Maclaurin's series :*

$$(2) \quad f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2\,!}x^2 + \frac{f'''(0)}{3\,!}x^3 + \cdots.$$

It must be remembered that as yet we have not proved the validity of this result; we have merely shown that, if a development in powers of x is possible, it must have the form (2). Evidently a *necessary* condition for the existence of Maclaurin's series is that the function and its successive derivatives be defined at $x = 0$.

Example: Expand e^x in Maclaurin's series.

Here

$$\begin{aligned} f(x) &= e^x, & \text{hence}\quad f(0) &= 1,\\ f'(x) &= e^x, & f'(0) &= 1,\\ f''(x) &= e^x, & f''(0) &= 1,\\ f'''(x) &= e^x, & f'''(0) &= 1,\\ &\vdots & &\vdots \end{aligned}$$

Therefore the development is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

154. Taylor's series. More generally, let it be required to develop a function $f(x)$ in powers of $x - a$, where a is a given number. Assuming

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots,$$

and setting $x = a$, we find

$$f(a) = c_0.$$

Proceeding as in § 153, we obtain finally *Taylor's series:*

$$(1)\quad f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots.$$

Thus Maclaurin's series is a special case of Taylor's series, viz.: the case $a = 0$.

When a function is represented by the series (1), we say that it has been *developed* or *expanded* in Taylor's series *about the point* $x = a$.

If, in (1), we replace x by $a + h$, we obtain another important form of Taylor's series:

$$f(a + h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \cdots.$$

It is clear that the Taylor series for a function $f(x)$ can always be *formally* written down if the function and its derivatives of all orders are defined at $x = a$. But it by no means follows from this that the series represents the function for any particular value of x. The series may diverge, or, if convergent, its sum may not be $f(x)$. The development in Taylor's series is valid only for those values of x for which the series *converges to the value* $f(x)$.

In the next article we show under precisely what circumstances a function may be developed in Taylor's series. However, for all functions that we shall consider, the series, if it converges at all, converges to the value $f(x)$; hence for those functions *the interval within which Taylor's series is valid coincides with the interval of convergence of the series.*

Example: Expand the function $\log x$ in Taylor's series about the point $x = 1$, and find the interval of convergence of the series.

In this case $a = 1$:

$$
\begin{aligned}
f(x) &= \log x, & f(1) &= 0,\\
f'(x) &= \frac{1}{x}, & f'(1) &= 1,\\
f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1,\\
f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2,\\
&\vdots & &\vdots
\end{aligned}
$$

Hence, by (1),

$$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \cdots.$$

The general term is

$$u_n = (-1)^{n-1}\frac{(x-1)^n}{n}.$$

Applying the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(x-1)^{n+1}}{n+1}}{\dfrac{(x-1)^n}{n}}\right| = |x-1|.$$

Thus the series converges when $|x-1|<1$, *i.e.* when $0<x<2$. When $x=0$, the series is

$$-1-\tfrac{1}{2}-\tfrac{1}{3}-\cdots,$$

which is divergent (§ 147). When $x=2$, the series is

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\cdots,$$

which is convergent (§ 150). Hence, finally, the interval of convergence is $0<x\leqq 2$.

EXERCISES

1. In the Maclaurin series for e^x (§ 153), show that the series converges for all values of x.

In each of the following, determine the interval of convergence of the series.

2. Expand $\sin x$ in powers of x.

Ans. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, all values of x.

3. Expand $\cos x$ about the origin.

Ans. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$, all values of x.

4. Expand e^x in Taylor's series about the point $x=2$.

5. By replacing x by $1+x$ in the example of § 154, obtain the development of $\log(1+x)$ in powers of x.

Ans. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1<x\leqq 1.$

6. Expand $\log(1-x)$ about the origin.

Ans. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \ -1\leqq x<1.$

7. Obtain the binomial theorem

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots.$$

8. Expand $\sin x$ about the point $x = \frac{\pi}{4}$.

9. Show that $\log x$ cannot be expanded in powers of x.

10. Expand $\arctan x$ about the origin.

Ans. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots.$

11. Show that, if $P(x)$ is a polynomial of the n-th degree in x,

$$P(x) = P(a) + P'(a)(x-a) + \frac{P''(a)}{2!}(x-a)^2 + \cdots + \frac{P^{(n)}(a)}{n!}(x-a)^n,$$

whatever may be the values of a and x.

155. Taylor's theorem. Let the function $f(x)$ and its first $n+1$ derivatives be continuous in an interval including the point $x=a$, and let $x=a+h$ be a second point of the interval. Let R_n denote the difference between $f(a+h)$ and the sum of the first $n+1$ terms of the corresponding Taylor's series (2), § 154; *i.e.* set

$$(1)\qquad f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\cdots+\frac{f^{(n)}(a)}{n!}h^n+R_n.$$

For convenience, write R_n in the form

$$R_n=\frac{h^{n+1}}{(n+1)!}P_n,$$

so that

$$(2)\quad f(a+h)=f(a)+f'(a)h+\cdots+\frac{f^{(n)}(a)}{n!}h^n+\frac{h^{n+1}}{(n+1)!}P_n.$$

Consider now the auxiliary function

$$\phi(x)=f(a+h)-f(x)-(a+h-x)f'(x)-\frac{(a+h-x)^2}{2!}f''(x)-\cdots-\frac{(a+h-x)^n}{n!}f^{(n)}(x)-\frac{(a+h-x)^{n+1}}{(n+1)!}P_n.$$

This function evidently vanishes when $x=a+h$, and, by (2), it also vanishes when $x=a$. Further, it results from our hypotheses that $\phi(x)$ has a derivative $\phi'(x)$ in the interval from $x=a$ to $x=a+h$. Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem (§ 136) in that interval, and its derivative must vanish at some point x_1 of that interval. Differentiating $\phi(x)$, we find after simplifying that

$$\phi'(x)=\frac{(a+h-x)^n}{n!}[P_n-f^{(n+1)}(x)].$$

By Rolle's theorem,

$$\phi'(x_1)=0,$$

hence

$$P_n=f^{(n+1)}(x_1).$$

Substituting this value of P_n in (2), we get

$$(3)\quad f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\cdots+\frac{f^{(n)}(a)}{n!}h^n + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x_1),$$

or, writing $x-a$ for h,

$$(4)\quad f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(x_1),$$

where x_1 lies between a and x.

Formula (4), or its equivalent (3), is called *Taylor's theorem with a remainder.* The last term is called the *remainder after* $n+1$ *terms:*

$$R_n=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(x_1).$$

For $n=0$, Taylor's theorem reduces to the law of the mean (§ 137):

$$f(x)=f(a)+(x-a)f'(x_1).$$

If n increases indefinitely, the right member of (4) becomes an infinite series, the Taylor's series for $f(x)$. The *necessary and sufficient condition* that the series shall converge to the value $f(x)$, and hence that the function shall be developable in Taylor's series, is that

$$\lim_{n\to\infty} R_n=0.$$

Example: Prove that the function e^x can be developed in powers of x for all values of x.

Here $a=0$ and

$$f(x)=e^x,\ f'(x)=e^x,\ \cdots,\ f^{(n+1)}(x)=e^x,$$

so that the remainder in Taylor's theorem has the form

$$R_n=\frac{x^{n+1}}{(n+1)!}e^{x_1},$$

where x_1 is between 0 and x.

By Ex. 8, p. 221, the quantity $\frac{x^{n+1}}{(n+1)!}$ is the general term of a series which converges for all values of x, so that, by § 146, this quantity converges to 0 when n becomes infinite. Hence, for all values of x,

$$\lim_{n \to \infty} R_n = 0,$$

and the proof is complete.

EXERCISES

1. In the Maclaurin series for $\sin x$, prove that the remainder converges to 0 for all values of x.

2. Prove that $\cos x$ can be expanded in powers of x for all values of x.

156. Approximate computation by series. We have found in the preceding article that any function whatever, provided certain conditions regarding continuity are satisfied, can be represented by Taylor's theorem as a *polynomial* of arbitrary degree, with a certain *remainder* R_n. It is clear that R_n is the *error* committed if we replace the function by the polynomial.

This suggests a method for computing approximately the numerical value, for any given value of the argument, of functions such as the sine, cosine, logarithm, etc., whose value cannot be found directly. We have only to build up the Taylor polynomial for the function in question, and show that the error R_n is less than the allowable limit of error for the problem in hand.

If now our function can be developed in Taylor's series, we know at once that its value *to any desired degree of accuracy* can be found by merely adding up a sufficient number of terms at the beginning of the series. For, by § 155, the remainder, or error, R_n converges to 0 as n increases indefinitely, and hence, by the definition of § 14, can be made as small as we please by taking n sufficiently large.

An upper limit for the error committed by stopping at any point may frequently be found from the general properties of series. Thus in the case of an *alternating series* the error is less than the first term neglected, by theorem II of § 150.

Example: Compute sin 3° correct to five decimal places.

Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

it follows that

$$\sin 3° = \sin\frac{\pi}{60} = \frac{\pi}{60} - \frac{1}{6}\left(\frac{\pi}{60}\right)^3 + \frac{1}{120}\left(\frac{\pi}{60}\right)^5 - \cdots$$
$$= 0.052365 - 0.000024 + \cdots.$$

Since this is an alternating series, the error committed by stopping with any term is less than the next term. Without computing the third term, we see that it is much too small to affect the fifth decimal place, hence we need keep only two terms:

$$\sin 3° = 0.05234.$$

To be of practical use in computation, a series should converge rapidly, as in the above example, so that a few terms are enough to give the desired degree of accuracy. In this connection the following point should be noted. Our choice of a in Taylor's theorem is governed only by the necessity of knowing at that point the value of $f(x)$ and its derivatives. Since the remainder R_n contains the factor $(x-a)^{n+1}$, it is clear that, in general, the smaller the difference $x - a$, the faster the remainder will approach 0. Hence in general, of all possible values for a, we should choose that one lying nearest to the value of x in question. Thus to compute sin 3°, we took $a = 0$; if we had to compute sin 47°, we would take* $a = \frac{\pi}{4}$, *i.e.* we would expand $\sin x$ in powers of $x - \frac{\pi}{4}$; etc.

* Assuming of course that we know the value of the sine and cosine only for the "principal angles" 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, etc.

EXERCISES

1. Draw on the same axes, on a large scale, the curve $y = \sin x$ and the first and second "approximation curves" $y = x$, $y = x - \frac{x^3}{6}$, in the interval $0 \leqq x \leqq \pi$. Estimate the interval within which each of the approximating polynomials is correct to one decimal place.

2. Proceed as in Ex. 1 with the curve $y = e^x$ and the successive approximation curves $y = 1 + x$, $y = 1 + x + \frac{x^2}{2}$, $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, in the interval $-2 \leqq x \leqq 2$.

3. Compute (*a*) sin 1° to five places; (*b*) sin 9° to three places; (*c*) cos 3° to four places.

4. Find the value of e to five decimal places. *Ans.* $e = 2.71828 \cdots$.

5. Compute (*a*) sin 47°, (*b*) cos 31°, each to four places.

6. Find the tenth root of e to five decimal places.

7. Find the value of $e^{0.97}$ to five decimal places.

8. Show that an arc of a great circle of the earth $2\frac{1}{2}$ miles long recedes 1 ft. from its chord.

9. Taking the circumference of the earth as 40,000,000 meters, show that the difference between the circumference and the perimeter of a regular inscribed polygon of 1,000,000 sides is less than one fifteenth of a millimeter.

10. Within what interval can $\sin\theta$ be replaced by θ, if accuracy to three decimal places is required?

157. Operations with power series. Operations that can always be performed upon series of a finite number of terms, such as rearrangement of terms, multiplication of one series by another, term-by-term differentiation or integration, etc., cannot be assumed offhand to be allowable with infinite series, and in fact it is easily shown that they are not allowable in all cases.

In dealing with developments in Taylor's series, it is frequently desirable to know just when such elementary operations are permissible. We therefore state, without proof, the following theorems regarding power series.

THEOREM I: ADDITION. *Two power series may be added together for all values of x for which both series are convergent.*

That is, if the series

$$\phi(x) = a_0 + a_1x + a_2x^2 + \cdots,$$
$$\psi(x) = b_0 + b_1x + b_2x^2 + \cdots$$

are both convergent, the series obtained by adding these together will converge to the value $\phi(x) + \psi(x)$:

$$\phi(x) + \psi(x) = a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots.$$

THEOREM II : MULTIPLICATION. *Two power series may be multiplied together for all values of x for which both series are absolutely convergent.*

That is, if the series

$$\phi(x) = a_0 + a_1x + a_2x^2 + \cdots,$$
$$\psi(x) = b_0 + b_1x + b_2x^2 + \cdots$$

are both absolutely convergent, then

$$\phi(x) \cdot \psi(x) = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots.$$

THEOREM III: DIVISION. *One convergent power series may be divided by another, provided the constant term in the denominator is not* 0. The result holds within a certain interval, the determination of which is beyond the scope of this discussion.

THEOREM IV : SUBSTITUTION. *If the series*

$$z = a_0 + a_1y + a_2y^2 + \cdots$$

converges for all values of y, and the series

$$y = b_0 + b_1x + b_2x^2 + \cdots$$

converges for all values of x, the series for y may be substituted in the series for z and the result arranged in powers of x. This result holds for all values of x.

THEOREM V : DIFFERENTIATION. *A power series may be differentiated term by term for all values of x within its interval of convergence.**

THEOREM VI : INTEGRATION. *A power series may be integrated term by term between any limits lying within the interval of convergence.**

* The endpoints of the interval are excluded.

These theorems enable us to obtain many Taylor expansions in which the evaluation of the successive derivatives would be very tedious, and in which the law of formation of the coefficients is so complicated that the interval of convergence could not be determined directly.

Example: Expand $e^{\sin x}$ in powers of x, to x^4 inclusive.

By Exs. 1, 2, p. 225, we have

$$e^{\sin x} = 1 + \sin x + \frac{\sin^2 x}{2\,!} + \frac{\sin^3 x}{3\,!} + \frac{\sin^4 x}{4\,!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3\,!} + \cdots.$$

Since both these series converge for all values of x, the series for $\sin x$ may, by theorem IV, be substituted in the series for $e^{\sin x}$:

$$e^{\sin x} = 1 + \left(x - \frac{x^3}{3\,!} + \cdots\right) + \left(\frac{1}{2\,!}x - \frac{x^3}{3\,!} + \cdots\right)^2 + \frac{1}{3\,!}\left(x - \frac{x^3}{3\,!} + \cdots\right)^3 + \frac{1}{4\,!}\left(x - \frac{x^3}{3\,!} + \cdots\right)^4 + \cdots.$$

Expanding the parentheses by theorem II and collecting terms, we find

$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots.$$

By theorem IV, this series converges for all values of x.

EXERCISES

Expand the following functions in powers of x, and determine the interval of convergence in each case.

1. $\sin^2 x$. *Ans.* $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots$, all values.

2. $\cos^2 x$. Compare this result with that of Ex. 1.

3. $e^x \sin x$. *Ans.* $x + x^2 + \dfrac{x^3}{3} - \dfrac{x^5}{30} + \cdots$, all values.

4. $\log \dfrac{1+x}{1-x}$.

5. $e^{x \cos x}$.

6. By integrating the series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

between the limits 0 and x, obtain the Maclaurin series for $\log(1+x)$.

7. Expand $\arcsin x$ in powers of x by integrating the binomial expansion of $\frac{1}{\sqrt{1-x^2}}$.

8. Expand $\frac{1}{1+x^2}$ by division, and integrate the resulting series term by term. Cf. Ex. 10, p. 225.

9. By differentiating the Maclaurin series for $\log(1-x)$ (Ex. 6, p. 225), prove the formula of elementary algebra for the sum of an infinite geometric series.

10. By means of series, prove the formula $\sin 2x = 2 \sin x \cos x$.

Expand each of the following in powers of x; the interval of convergence is as indicated in each case.

11. $\frac{e^x}{1-x}$. $\qquad -1 < x < 1$.

12. $\sec x$. *Ans.* $1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.

13. $\tan x$. *Ans.* $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$.

14. Show that, for values of x so small that the fourth and higher powers of $\frac{x}{a}$ may be neglected, the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ may be replaced by a parabola.

15. Find the area under the curve $y = \frac{\sin x}{x}$ from $x = 0$ to $x = 1$. Draw the figure. *Ans.* 0.9461.

16. Find the centroid of the area in Ex. 15.

17. Find the area under the curve $y = e^{-\frac{1}{2}x^2}$, from $x = 0$ to $x = 1$. *Ans.* 0.86.

18. The area in Ex. 17 revolves about the x-axis. Find the centroid of the volume generated.

19. Raise 1.03 to the fifth power. *Ans.* 1.16.

20. Show that, in leveling, the correction for the curvature of the earth is 8 in. for one mile.

21. A mountain peak 1 mile high, situated on an island, is just visible from the mainland. If there is no refraction, how far out at sea is the island? Solve in two ways. *Ans.* 90 miles.

22. Show that the length of the arc of the hyperbola $xy = 1$ from $x = 100$ to $x = 200$ differs from the length of the chord by one part in 5,000,000,000.

23. Find the difference between the circumference of the earth and the perimeter of a regular circumscribed polygon of 1,000,000 sides. Cf. Ex. 9, p. 230.

24. Find the surface generated by revolving the curve $y = \frac{x^4}{4}$ about the x-axis, from $x = 0$ to $x = \frac{1}{2}$.

158. Computation of logarithms. We have found

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots,$$

each series holding for values of x numerically less than 1. From these we may deduce a series that is better adapted to numerical computation than either of the above series.

Subtracting the second equation from the first (by theorem I, § 157), we find

$$\log \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

for values of x numerically less than unity. Let us put

$$\frac{1+x}{1-x} = \frac{m+1}{m},$$

or

$$x = \frac{1}{2m+1},$$

where m may have any positive value. Then

$$(1)\quad \log(1+m) = \log m + 2\left[\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right].$$

This series converges rapidly, and is therefore well adapted to computation. It is easily shown that for values of $m \geqq 1$ the error committed by stopping at any point is only slightly greater than the first term neglected.

Example: Taking $m = 1$, we have

$$\log 2 = 2\left[\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right]$$
$$= 2[0.3333 + 0.0123 + 0.0008 + \cdots]$$
$$= 0.693.$$

From this the logarithms of 4, 8, $\cdots$ may be found directly.

With $m = 2$, we find log 3; from this and the previous result we may obtain the logarithms of all numbers whose only prime factors are 2 and 3. In fact, it is clear that only the logarithms of prime numbers need be computed by the series.

EXERCISES

1. Compute to three decimal places the natural logarithms of all integers from 3 to 10 inclusive.

2. After finding log 10, obtain $\log_{10} 2$, $\log_{10} 3$.

3. By comparison with the geometric series

$$\frac{1}{2n+1} \cdot \frac{1}{(2m+1)^{2n+1}}\left[1 + \frac{1}{(2m+1)^2} + \frac{1}{(2m+1)^4} + \cdots\right],$$

prove that the error committed by stopping with the n-th term of the series (1) is less than $\dfrac{1}{1 - \dfrac{1}{(2m+1)^2}}$ times the first term neglected.

CHAPTER XXI

FUNCTIONS OF SEVERAL VARIABLES

I. Partial Differentiation

159. Functions of several variables. Up to this point we have been concerned with functions of a single argument. A function may, however, depend upon several independent variables. For example, the volume of a circular cylinder is a function of its radius and altitude; the acceleration of a moving particle is a function of all the forces acting on it; the strength of a rectangular beam is a function of its breadth and depth.

If z is a function of two variables x and y, we write

$$z = f(x, y),$$

with a similar notation for functions of more than two variables.

Geometrically a function of two variables may be represented as the ordinate of a surface in space. Thus the equation

$$z = ax + by + c$$

represents a plane; the equation

$$z = x^2 - y^2$$

represents a hyperbolic paraboloid, etc.

A thorough study of functions of several variables is beyond the scope of a first course in the calculus. In the present chapter we set forth a few of the most important definitions and theorems, confining our attention chiefly to functions of two arguments.

160. Limits; continuity. Suppose we have given a function of two variables

(1) $$z = f(x, y)$$

representing a surface in space. When x and y approach the respective values x_0, y_0, the function z is said to *approach a limit* z_0 if the point (x, y, z) of the surface (1) approaches a definite limiting point (x_0, y_0, z_0). In other words, if when x is sufficiently near x_0 and y is sufficiently near y_0 the difference between z and z_0 becomes and remains numerically less than any preassigned quantity however small, then z is said to approach the limit z_0: in symbols,

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = z_0.$$

A function $f(x, y)$ is said to be *continuous* at the point (x_0, y_0) if

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0).$$

Similar definitions are laid down for functions of more than two variables.

In what follows, it is supposed that all functions occurring are continuous at all points under consideration.

161. Partial derivatives. If y be kept *fixed*, the function

$$z = f(x, y)$$

becomes a function of x alone, and its derivative may be found by the ordinary rules. This derivative is called the *partial derivative of z with respect to x*, and is denoted by any one of the symbols

$$\frac{\partial z}{\partial x}, \quad \frac{\partial f}{\partial x}, \quad f_x(x, y).$$

The partial derivative with respect to y has a similar meaning.

The idea of partial differentiation may be extended at once to functions of any number of variables. We have only to remember that in differentiating with respect to any one variable, *all the other variables are treated as constants.*

162. Geometric interpretation of partial derivatives. To keep y constant, say $y = y_0$, in the equation

$$z = f(x, y)$$

means geometrically that we cut the surface by the plane $y = y_0$. The partial derivative $\frac{\partial z}{\partial x}$ is therefore the slope of the curve of intersection of the surface and the plane, *i.e.* of the curve whose equations are

$$z = f(x, y), \quad y = y_0.$$

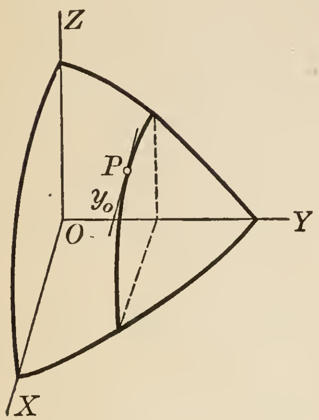


FIG. 84

The partial derivative $\frac{\partial z}{\partial y}$ may be interpreted similarly.

163. Higher derivatives. The derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are themselves functions of x and y, and their partial derivatives can in turn be found. They are denoted by the following symbols:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{x^2}(x, y),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{y^2}(x, y).$$

The process can of course be repeated to find still higher derivatives.

It can be shown that the two "cross-derivatives" $\frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$ are identical:

$$\frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial^2 z}{\partial x \partial y}.$$

That is, *the order of differentiation is immaterial.* This is true for derivatives of all orders, and for functions of any number of variables.

EXERCISES

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ for the following functions.

1. $z = x^2 + xy - 3x + 5$. *Ans.* $\frac{\partial z}{\partial x} = 2x + y - 3$; $\frac{\partial z}{\partial y} = x$.

2. $z = x^3 + 3x^2y - xy + 2y - 3$.

3. $z = (x^2 - 2xy)^3$.

4. $z = \frac{(x-y)^2}{x^2+y^2}$.

5. $z = e^{\cos^2(x-y)}$.

6. $z = \log\sqrt{x^2 + y^2}$.

7. $z = \arctan\frac{y}{x}$. *Ans.* $\frac{\partial z}{\partial x} = \frac{-y}{x^2+y^2}$.

8. Given $f(x, y, z) = xyz + 3x^2y + z^3$, find f_x, f_y, f_z.
Ans. $f_x = y(z + 6x)$.

9. If $u = x^2 + y^2 - z^2 + x + 2y$, find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

10. Find the slope of the curve cut from the hyperbolic paraboloid $z = x^2 - 2y^2$ by the plane $y = 3$, at the point $(4, 3, -2)$. *Ans.* 8.

11. Find the equations of the tangent to the parabola

$$z = 3x^2 + 4y^2,\ x = 2$$

at the point (2, 1, 16). *Ans.* $x = 2$, $z = 8y + 8$.

12. If $u = x^3y^2 - 2xy^4 + 3x^2y^3$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 5u$.

13. If $u = (y-z)(z-x)(x-y)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

14. Given $z = x + x^3y^2 + 2x^2y^4$, find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$.

Ans. $\frac{\partial^2 z}{\partial x^2} = 6xy^2 + 4y^4$; $\frac{\partial^2 z}{\partial y \partial x} = 6x^2y + 16xy^3$.

15. Given $z = x^2y^2 + 3x^3y^3 - x^2y$, verify that $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$.

16. If $z = \cos(x - y)$, show that $\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial^3 z}{\partial x \partial y \partial x} = \frac{\partial^3 z}{\partial x^2 \partial y}$.

17. If $u = e^{x-y-2z}$, verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x};\quad \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}.$$

18. If $z=\frac{1}{2}\log(x^2+y^2)$, show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

19. If $z=x^2y$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=3z$.

20. If $u=x^2+y^2+yz$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2u$.

21. If $f(x, y, z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$, show that $f_{x^2}+f_{y^2}+f_{z^2}=0$.

22. Prove that if two functions u and v are so related that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

then

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

164. Total differentials. When x and y change by amounts Δx and Δy, the function

$$z=f(x, y)$$

changes by an amount Δz. It can be shown that Δz may be expressed in the form

$$\Delta z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y+\epsilon\Delta x+\eta\Delta y,$$

where ϵ and η are infinitesimals.

The quantity $\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y$ is called the *principal part* (cf. § 49) of the infinitesimal Δz. The *total differential of* z is defined as the principal part of Δz:

$$dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y.$$

In particular, if $z=x$, $\frac{\partial z}{\partial x}=1$ and $\frac{\partial z}{\partial y}=0$, so that

$$dx=\Delta x.$$

Similarly

$$dy=\Delta y.$$

Hence we may write

(1) $$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

For functions of more than two arguments a similar formula holds. Thus, if

$$u = f(x, y, z),$$

(2) $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz.$$

If x, y, z are functions of a fourth variable t, then u becomes a function of t alone, and its differential has been defined in § 50. It can be shown that the value of du as given by (2) agrees with the earlier definition, so that (2) still holds even when x, y, z are functions of a single variable.

Example: Find approximately the increase in the area of a rectangle if each of its dimensions increases by a small amount.

We have

$$A = ab,$$

hence

$$dA = bda + adb.$$

The actual increase in the area is

$$\begin{aligned} \Delta A &= (a + da)(b + db) - ab \\ &= bda + adb + dadb. \end{aligned}$$

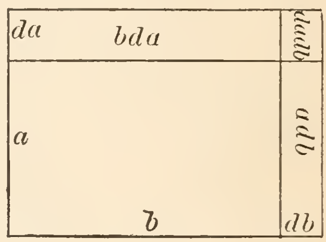


FIG. 85

If da and db are so small that their product can be neglected in comparison with the other terms occurring, the total differential dA represents the actual change ΔA with sufficient accuracy.

165. Differentiation of implicit functions. Let y be defined as a function of x by the equation

$$f(x, y) = 0.$$

Let us for an instant put

$$z = f(x, y);$$

then by (1), § 164,

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

But in the present instance

$$z = 0,$$

hence

$$dz = \frac{\partial f}{\partial x} dx + \frac{f}{\partial y} dy = 0,$$

or

(1) $$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \left(\frac{\partial f}{\partial y} \neq 0\right).$$

The value of $\frac{dy}{dx}$ as given by this formula is of course identical with that given by the method of § 25.

Again, let z be defined implicitly as a function of the two independent variables x and y by the equation

$$F(x, y, z) = 0.$$

Put

$$u = F(x, y, z);$$

then

$$du = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz.$$

But since $u = 0$, $du = 0$ likewise, and

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0.$$

Further, since z is a function of x and y, we may write

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

Eliminating dz between these two equations, we find

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right) dy = 0.$$

To find $\frac{\partial z}{\partial x}$, keep y fixed, so that $dy = 0$. Then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0,$$

or

$$(2) \qquad \frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \left(\frac{\partial F}{\partial z} \neq 0\right).$$

Similarly

$$(3) \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}} \qquad \left(\frac{\partial F}{\partial z} \neq 0\right).$$

EXERCISES

Find the total differential of each of the following functions.

1. $z = x^2 - 3\,xy + y^2 + 2\,y$.

2. $z = \cos^2(x - y)$.

3. $u = x + y + z$.

4. $u = \log \tan \frac{yz}{x}$.

5. Let V be the volume, S the total surface, of a right circular cylinder. If r and h change by an amount Δr and Δh respectively, find dV, ΔV, dS, ΔS. Draw a figure.

6. In Ex. 5, if $r = 5$ ft., $h = 10$ ft., $\Delta r = \Delta h = 2$ in., compute the percentage of error made by using dV in place of ΔV and dS in place of ΔS.

7. The dimensions x, y, z of a rectangular parallelepiped change by amounts Δx, Δy, Δz. Find dV, ΔV. Also obtain dV and ΔV directly by inspection of a figure.

Find $\frac{dy}{dx}$ in the following cases, using formula (1) of § 165.

8. $3\,x - 4\,y + 2\,xy = 1$.

9. $(2\,x^2 - 3\,y^2)^2 + 1 - xy = 0$.

10. $\text{Arctan}\,\frac{y}{x} = x$.

11. Given $x^2 + y^2 + z^2 = 1$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ by formulas (2) and (3) of § 165.

12. Find the equations of the tangent to the circle $x^2 + y^2 + z^2 = 36$, $y = 4$ at the point (2, 4, 4).

13. Find the equations of the tangent to the ellipse $x^2 + 3\,y^2 = z^2$, $z = 4$ at the point (2, 2, 4).

II. Applications to Solid Analytic Geometry

166. Tangent plane to a surface. It can be shown that all the lines tangent to a surface

$$z = f(x, y)$$

at a point $P : (x_0, y_0, z_0)$ lie in a plane,* the *tangent plane* to the surface at that point. This plane is of course determined by any two of the tangent lines. We have already learned (§ 162) how to find the equations of the tangent lines lying in the planes $x = x_0$, $y = y_0$. Let us assume the equation of the tangent plane in the form

$$z - z_0 = m_1(x - x_0) + m_2(y - y_0),$$

where m_1 and m_2 are to be determined. Now the line of intersection of this plane with the plane $y = y_0$ has the slope m_1. But this line is the tangent lying in the plane $y = y_0$, and, by § 162, its slope is the value of $\frac{\partial z}{\partial x}$ at P, which value we shall denote by the symbol $\left.\frac{\partial z}{\partial x}\right]_P$. Hence

$$m_1 = \left.\frac{\partial z}{\partial x}\right]_P.$$

Similarly we find

$$m_2 = \left.\frac{\partial z}{\partial y}\right]_P.$$

Thus the equation of the plane tangent to the surface

$$z = f(x, y)$$

at (x_0, y_0, z_0) is

$$(1) \qquad z - z_0 = \left.\frac{\partial z}{\partial x}\right]_P (x - x_0) + \left.\frac{\partial z}{\partial y}\right]_P (y - y_0).$$

More generally, let the equation of the surface be given in the implicit form

$$(2) \qquad F(x, y, z) = 0,$$

* Provided z, $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$ are continuous at P.

where the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ do not all vanish at $P : (x_0, y_0, z_0)$. Suppose for definiteness that $\left.\frac{\partial F}{\partial z}\right]_P \neq 0$. We may imagine equation (2) solved for z, and may then write the equation of the tangent plane by formula (1). But, by (2) and (3) of § 165,

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Substituting these values in (1), we find

$$z - z_0 = -\frac{\left.\frac{\partial F}{\partial x}\right]_P}{\left.\frac{\partial F}{\partial z}\right]_P}(x - x_0) - \frac{\left.\frac{\partial F}{\partial y}\right]_P}{\left.\frac{\partial F}{\partial z}\right]_P}(y - y_0),$$

or

$$(3) \quad \left.\frac{\partial F}{\partial x}\right]_P (x - x_0) + \left.\frac{\partial F}{\partial y}\right]_P (y - y_0) + \left.\frac{\partial F}{\partial z}\right]_P (z - z_0) = 0.$$

167. Normal line to a surface. The *normal* to a surface at a point P is the line through P perpendicular to the tangent plane.

It will be recalled from solid analytic geometry that the direction cosines of any line perpendicular to the plane

$$Ax + By + Cz + D = 0$$

are *proportional to the coefficients* A, B, C. Hence, since the normal is perpendicular to the tangent plane (3) of § 166, we have at once the following

THEOREM: *The direction cosines of the normal to the surface*

$$F(x, y, z) = 0$$

at any point are proportional to the values of $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ *at that point.*

This theorem is fundamental in the geometry of surfaces.

By analytic geometry, the equations of a line through x_0, y_0, z_0 with direction cosines proportional to a, b, c are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

From this the equations of the normal at any point may be written down at once.

168. Angle between two surfaces; between a line and a surface. The angle between two surfaces at a point of intersection is defined as the *angle between the tangent planes* at that point, and this in turn is equal to the angle between the normals. This angle may be found by the theorem of analytic geometry that, if two lines have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively, the angle between them is given by the formula

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

The angle at which a line pierces a surface is defined as the *angle between the line and the tangent plane* at the piercing-point. This is evidently the complement of the angle between the line and the normal.

Example: Find the angle between the cylinder $y^2 = 4x$ and the ellipsoid $2x^2 + y^2 + z^2 = 7$ at the point (1, 2, 1).

For the ellipsoid, the partial derivatives are

$$\frac{\partial F}{\partial x} = 4x, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 2z,$$

hence the direction cosines of the normal at (1, 2, 1) are proportional to 4, 4, 2, and their actual values are $\frac{2}{3}$, $\frac{2}{3}$, $\frac{1}{3}$. For the cylinder,

$$\frac{\partial F}{\partial x} = -4, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = 0,$$

hence the direction cosines of the normal are $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$. Therefore

$$\cos\theta = -\frac{1}{\sqrt{2}} \cdot \frac{2}{3} + \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = 0:$$

the surfaces intersect at right angles.

EXERCISES

Find the equations of the tangent plane and normal line to each of the following surfaces at the point indicated.

1. The cone $x^2 + 3y^2 = z^2$ at $(2, 2, 4)$; draw the figure.

Ans. $x + 3y - 2z = 0;\ \frac{x-2}{1} = \frac{y-2}{3} = \frac{z-4}{-2}.$

2. The paraboloid $z = x^2 - y^2$ at $(1, 1, 0)$.

3. The cylinder $y^2 = 4ax$ at $(a, 2a, a)$; draw the figure.

4. The paraboloid $x = yz$ at the origin.

5. The sphere $x^2 + y^2 + z^2 = a^2$ at (x_0, y_0, z_0).

Ans. $x_0x + y_0y + z_0z = a^2.$

6. The surface $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$ at (x_0, y_0, z_0).

7. Find the equations of the tangent to the circle $x^2 + y^2 + z^2 = 9$, $x + y + z = 5$ at the point $(1, 2, 2)$; draw the figure.

8. Find the angle between the sphere $x^2 + y^2 + z^2 = 14$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 20$ at the point $(-1, -2, 3)$. *Ans.* $23° 33'$.

9. Show that at any point on the z-axis there are two tangent planes to the surface $a^2y^2 = x^2(b^2 - z^2)$.

10. Show that the sum of the squares of the intercepts on the axes made by a tangent plane to the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$ is constant. Sketch this surface.

11. Prove that the tetrahedron formed by the coördinate planes and a tangent plane to the surface $xyz = a^3$ is of constant volume.

12. Find the angle at which the normal to the hyperboloid $y^2 - x^2 + 4z^2 = 16$ at the point $(2, 2, 2)$ intersects the xy-plane. Draw the figure.

13. Find the equations of the projections on the coördinate planes of the normal to the cylinder $x = y + z^2$ at $(2, 1, 1)$.

14. Find the equations of the normal to the surface $x^2y + y^2 + z^2 = 3$ at the point $(1, 1, 1)$.

15. Show that the sphere $x^2 + y^2 + z^2 = 2a^2$ and the hyperbolic cylinder $xy = a^2$ are tangent to each other at the point $(a, a, 0)$.

16. Determine a and b so that the ellipsoid $x^2 + 2y^2 + z^2 = 7$ and the paraboloid $z = ax^2 + by^2$ may intersect at right angles at $(1, 1, 2)$.

Ans. $a = 3,\ b = -1.$

17. Find the angle between the normal to the oblate spheroid $x^2 + y^2 + 2z^2 = 10$ at $(2, 2, 1)$ and the line joining the origin to that point. *Ans.* Arccos $\frac{5}{9}\sqrt{3}$.

18. In Ex. 17, find the shortest distance from the origin to the normal in question. *Ans.* $\frac{1}{3}\sqrt{6}$.

19. Find the angle at which the line $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ pierces the ellipsoid $2x^2+4y^2+z^2=25$. *Ans.* 67° 48′.

20. Prove that every line through the center of a sphere intersects the sphere at right angles.

169. Space curves. Two surfaces

$$(1) \qquad \Phi(x, y, z)=0, \quad \Psi(x, y, z)=0$$

intersect in general in a curve in space. The curve is determined by the equations of the two surfaces considered as simultaneous.

Since there are an infinite number of surfaces through a given curve, and since the equations of any two of these surfaces in general determine the curve, it follows that the equations of the curve may be given in an infinite number of ways. A particularly simple way is to give the equations of two of the "projecting cylinders" — *i.e.* the cylinders through the curve with generators perpendicular to the coördinate planes. Eliminating y and z in turn between the equations (1), we find two equations of the form

$$(2) \qquad \phi(x, z)=0, \quad \psi(x, y)=0.$$

These equations represent cylinders through the curve (1) with generators perpendicular to the xz-plane and the xy-plane respectively.

We have seen that the coördinates of a point on a plane curve are frequently given in terms of a parameter t. The same device is often employed with curves in space: the curve is given by the three parametric equations

$$(3) \qquad x=f(t), \quad y=g(t), \quad z=h(t).$$

By eliminating the parameter between two different pairs of these equations, we obtain equations of the form (2).

170. Tangent line and normal plane to a space curve. The tangent to the curve (1) of § 169 at the point $P:(x_0, y_0, z_0)$ is the intersection of the tangent planes to the two surfaces

$$\Phi(x, y, z) = 0, \quad \Psi(x, y, z) = 0.$$

Hence its equations can be written down at once.

The *normal plane* is the plane through P perpendicular to the tangent. To find its equation, we have only to transform the equations of the tangent line to the "symmetric form"

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

after which the equation of the normal plane can be written directly.

Example: Find the equations of the tangent line and the normal plane to the curve

$$x^2 + y^2 + z^2 = 3,$$
$$4z = 3x^2 + y^2,$$

at the point (1, 1, 1).

The equations of the tangent are found by formula (3), § 166, to be

$$x + y + z = 3,$$
$$3x + y - 2z = 2.$$

To put these equations in the symmetric form, let us eliminate y and z in turn, thus representing the line by two of its projecting planes:

$$2x - 3z = -1,$$
$$5x + 3y = 8.$$

Equating the values of x from these two equations, we find

$$x = \frac{-3y + 8}{5} = \frac{3z - 1}{2},$$

or

$$\frac{x}{3} = \frac{y - \frac{8}{3}}{-5} = \frac{z - \frac{1}{3}}{2}.$$

The equation of the normal plane is therefore

$$3(x - 1) - 5(y - 1) + 2(z - 1) = 0.$$

171. Direction cosines of the tangent. Let us draw a secant (Fig. 86) through $P:(x_0, y_0, z_0)$ and a second point $P':(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)$, and denote by s the length of the arc from a fixed point to P, by Δs the length of the arc PP'. The direction cosines of the secant are

$$\cos\alpha' = \frac{\Delta x}{\overline{PP'}},\quad \cos\beta' = \frac{\Delta y}{\overline{PP'}},\quad \cos\gamma' = \frac{\Delta z}{\overline{PP'}},$$

those of the tangent are

$$(1)\quad \begin{cases} \cos\alpha = \lim\limits_{\overline{PP'}\to 0}\dfrac{\Delta x}{\overline{PP'}} = \lim\limits_{\overline{PP'}\to 0}\dfrac{\Delta x}{\Delta s}\cdot\dfrac{\Delta s}{\overline{PP'}} = \dfrac{dx}{ds}, \\ \cos\beta = \dfrac{dy}{ds}, \\ \cos\gamma = \dfrac{dz}{ds}. \end{cases}$$

Hence the direction cosines of the tangent are proportional to dx, dy, dz.

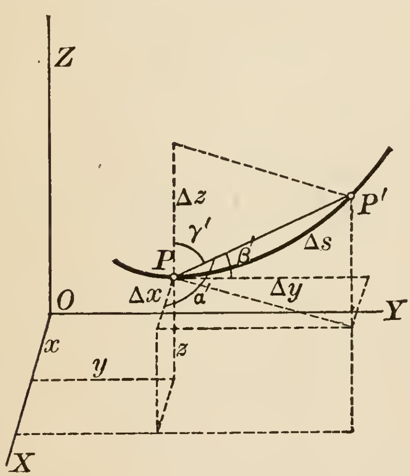


FIG. 86

From this fact we obtain at once the equations of the tangent to the curve (3) of § 169. They are

$$\frac{x-x_0}{f'(t_0)} = \frac{y-y_0}{g'(t_0)} = \frac{z-z_0}{h'(t_0)}.$$

The equation of the normal plane may be written down at once.

172. Length of a space curve. Since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

it follows from the formulas (1) of the preceding article that

$$\overline{ds}^2 = \overline{dx}^2 + \overline{dy}^2 + \overline{dz}^2.$$

Hence the *length of the arc* of the curve (3) of § 169 between any two points $P_0: (x_0, y_0, z_0)$ and $P_1: (x_1, y_1, z_1)$ is

$$s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

For the curve (2) of § 169 this becomes

$$s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx.$$

EXERCISES

1. Find the equations of the projecting cylinders of the curve $x^2 + y^2 = 2a^2$, $x^2 - y^2 + z^2 = a^2$; also find the equations of the tangent line and the normal plane at the point (a, a, a).

Ans. Normal plane: $x - y - 2z + 2a = 0$.

2. Find the equations of the tangent line and the normal plane to the circle $x^2 + y^2 + z^2 = 9$, $y + z = 3$ at the point (2, 2, 1).

3. Find the equations of the tangent line and the normal plane to the helix $x = a \cos\theta$, $y = a \sin\theta$, $z = b\theta$ at the point $\theta = \theta_0$.

4. Find the length of one turn of the helix in Ex. 3.

5. Find the angle between the curves

$$x^2 + y^2 + z^2 = 3,\quad z = xy,$$

and

$$x^2 - y^2 + z^2 = 1,\quad x + y + z = 3,$$

at the point (1, 1, 1).

6. Find the circumference of the circle $4x^2 + 3y^2 + 2z^2 = 1$, $z = x$.

7. Find the condition that the surfaces $\Phi(x, y, z) = 0$, $\Psi(x, y, z) = 0$ intersect at right angles in a point (x_0, y_0, z_0).

8. Find the centroid of the arc of the curve $x^2 = 2ay$, $x^3 = 6a^2z$ from (0, 0, 0) to $\left(a, \frac{a}{2}, \frac{a}{6}\right)$.

CHAPTER XXII

ENVELOPES. EVOLUTES

173. Envelope of a family of plane curves. The equation

(1) $$f(x, y, a) = 0,$$

where a is arbitrary, represents a *family* of plane curves: a is constant for any one curve, but varies when we pass from one curve of the family to another. Thus the equation

$$(x - a)^2 + y^2 = 1$$

represents all the unit circles having their centers on the x-axis; the equation

$$y = x + k$$

represents the family of straight lines making an angle of 45° with OX.

It may happen that there exists a curve to which each member of the family (1) is tangent. Such a curve is called the *envelope* of the family. The family of circles mentioned above have the lines $y = \pm 1$ as their envelope, since each of the circles is tangent to these lines. On the other hand, the family of straight lines $y = x + k$ have no envelope.

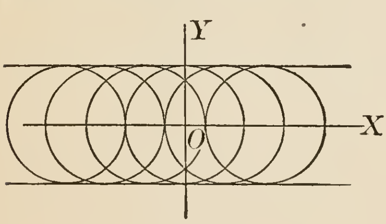


FIG. 87

174. Determination of the envelope. Suppose that the curves

(1) $$f(x, y, a) = 0$$

have an envelope. Let (x, y) be the point of tangency of the envelope with a curve C of the family; then the coördinates x and y are functions of a alone, and they satisfy equation (1). Differentiating (1), we find (§ 165)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial a}da = 0. \tag{2}$$

We have not yet made use of the fact that the envelope and the curve C have a common tangent at (x, y). The slope of the tangent to C at (x, y) is determined by the equation (§ 165)

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0, \tag{3}$$

and this gives the slope of the envelope also. Combining (2) and (3), we find

$$\frac{\partial f}{\partial a}da = 0.$$

But since x and y are functions of a, a is the independent variable and we may take $da \neq 0$. We thus find

$$\frac{\partial f}{\partial a} = 0$$

as a second equation, in addition to (1), that is satisfied by the coördinates x, y. Hence the equations

$$\begin{cases} f(x, y, a) = 0, \\ \dfrac{\partial f}{\partial a} = 0, \end{cases} \tag{4}$$

taken together constitute *parametric equations of the envelope.* The equation in cartesian coördinates can be found by eliminating the parameter a.

In the above discussion the existence of the envelope was assumed. It can be shown, conversely, that the curve (4) is an envelope, provided $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not both vanish along the curve.

Example: Find the envelope of the family of straight lines $y = mx + \frac{a}{m}$, where m is the variable parameter.

Differentiating partially with respect to m, we find

$$0 = x - \frac{a}{m^2},$$

or

$$m = \pm \sqrt{\frac{a}{x}}.$$

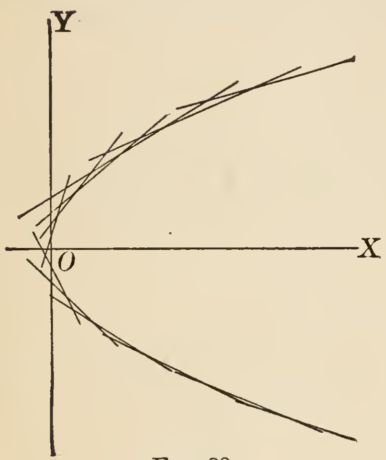


FIG. 88

Substituting this value of m in the original equation, we get

$$y = \pm 2\sqrt{ax},$$

or

$$y^2 = 4\,ax.$$

This agrees with the result of analytic geometry that the straight line $y = mx + \frac{a}{m}$ is tangent to the parabola $y^2 = 4\,ax$ for all values of m.

175. Envelope of tangents. Every curve may be considered as the envelope of its tangents, as appears at once from the definition of the envelope. This is illustrated by the example of the previous article, where the parabola was found as the envelope of its tangents.

EXERCISES

Find the envelope of each of the following families of curves. In each case draw several curves of the family, and the envelope.

1. The circles of radius a with their centers on the y-axis.

2. The family of straight lines $y = 2\,mx + m^4$.

Ans. $16\,y^3 + 27\,x^4 = 0$.

3. The family of parabolas $y^2 = a(x - a)$. *Ans.* $x \pm 2\,y = 0$.

4. The family of circles whose diameters are double ordinates of a parabola.

5. The family of circles tangent to the x-axis and having their centers on the parabola $y = x^2$. *Ans.* $y = 0,\ 2x^2 + 2y^2 - y = 0$.

6. The circles with centers on a parabola and passing through the vertex of the parabola. *Ans.* $y^2(x + 2a) + x^3 = 0$.

7. The circles through the origin with their centers on the hyperbola $x^2 - y^2 = a^2$. *Ans.* $(x^2 + y^2)^2 = 4a^2(x^2 - y^2)$.

8. The family of ellipses whose axes coincide and whose area is constant. *Ans.* Two conjugate rectangular hyperbolas.

9. A straight line segment of constant length moves with its ends in two perpendicular straight lines. Find its envelope.

Ans. A hypocycloid of four cusps.

10. A straight line moves so that the sum of its intercepts on the axes is constant. Find its envelope. *Ans.* The parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

11. When a projectile is fired from a gun with an initial velocity v_0 inclined at an angle α to the horizontal, the equation of its path, all resistances being neglected, is (§ 235)

$$y = x \tan \alpha - \frac{gx^2}{2 v_0^2 \cos^2 \alpha}.$$

Find the envelope of all possible trajectories when the angle of elevation α varies. *Ans.* The parabola $y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$.

12. The sides of a variable right triangle lie along two fixed lines. If the area of the triangle is constant, find the envelope of the hypotenuse.

13. Find the equation of the curve tangent to the lines

$$y = mx - am^2,$$

where m is the parameter.

14. Find the equation of the curve which is tangent to the line

$$y_0 y = 2ax + \tfrac{1}{2} y_0^2$$

for all values of y_0.

15. Find the equation of the curve tangent to the family of straight lines

$$x \cos \alpha + y \sin \alpha = p,$$

where α is the variable parameter.

16. Find the equation of the curve tangent to the straight line

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

for all values of m. *Ans.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

176. The evolute. When a point P moves along a curve, the center of curvature Q (§ 54) describes a second curve, called the *evolute* of the original curve.

It can be shown that the normal PQ to the original curve is tangent to the evolute; *i.e. the evolute is the envelope of the normals.* Its equation may therefore be found by writing the equation of the normal to the given curve in terms of a parameter, and then applying the method of § 174.

Example: Find the evolute of the parabola $y^2 = 4\,ax$.

It is shown in analytic geometry that the line

(1) $$y = mx - 2\,am - am^3$$

is normal to this parabola for all values of m. We have therefore to find the envelope of the family (1), regarding m as the variable parameter.

Differentiating partially with respect to m, we get

$$0 = x - 2\,a - 3\,am^2,$$

or

$$m^2 = \frac{x - 2\,a}{3\,a}.$$

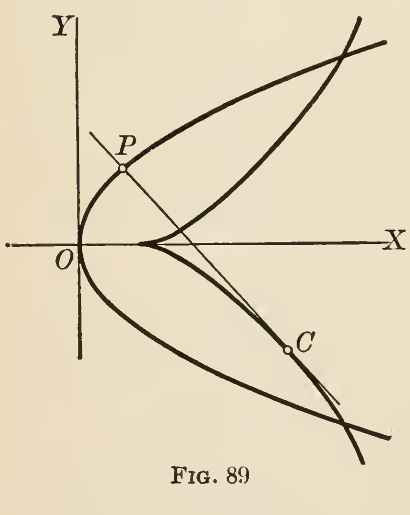


Fig. 89

Equation (1) may be written in the form

$$y = m(x - 2\,a - am^2),$$

or

$$y^2 = m^2(x - 2\,a - am^2)^2.$$

Substituting for m^2, we find the equation of the evolute

$$ay^2 = \frac{4}{27}(x - 2\,a)^3,$$

a "semi-cubical parabola" with a cusp at $(2\,a, 0)$.

EXERCISES

1. Find the equation of the evolute of the parabola $y^2 = 4\,ax$ by writing the equation of the normal in terms of the ordinate of the point at which the normal meets the curve.

2. In Ex. 1, show that the distance from any point P on the parabola to the corresponding point on the evolute is equal to the radius of curvature, thus verifying that the locus of the center of curvature and the envelope of the normals are the same curve.

3. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, given that the equation of the normal in terms of the eccentric angle ϕ is

$$by = ax \tan\phi - (a^2 - b^2)\sin\phi.$$

Ans. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$

4. Find the evolute of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the equation of whose normal is

$$y\cos\alpha - x\sin\alpha = a\cos 2\,\alpha.$$

Ans. $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2\,a^{\frac{2}{3}}.$

CHAPTER XXIII

MULTIPLE INTEGRALS

177. Volume under a surface. Let us try to find the volume V bounded by a portion T of the surface

$$z = f(x, y),$$

the area S into which T projects in the xy-plane, and the cylindrical surface through the boundaries of S and T.

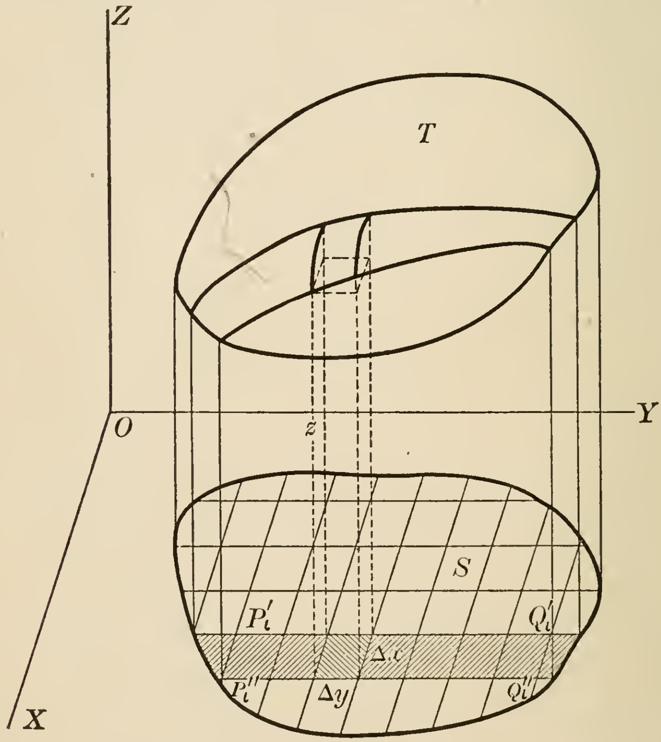


FIG. 90

We can get an approximate expression for the required volume as follows. Draw in S a set of n lines parallel to the y-axis and a set of m lines parallel to the x-axis, as in Fig. 90, thus dividing S into rectangles of area $\Delta y\ \Delta x$, together with a number of irregular portions around the boundary. By passing through each line of the two sets a plane perpendicular to the xy-plane we divide V into vertical rectangular columns, together with smaller irregular columns. The upper boundary of each rectangular column is a portion of the surface T. Through that point of the upper boundary of each column which is nearest the xy-plane, pass a horizontal plane, thus forming a set of rectangular prisms lying wholly within V. The sum of the volumes of these prisms is evidently an approximation to the required volume, the error committed being the sum of the irregular columns around the outside, together with the portions lying above the upper bases of the rectangular prisms. That is, *approximately*,

$$V = \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta y\, \Delta x,$$

where $f(x_i, y_j)$ is the altitude of the prism.

It is obvious that the error in this approximation may be made arbitrarily small by taking both Δx and Δy sufficiently small. Hence the required volume is *exactly*

$$(1) \qquad V = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta y\, \Delta x.$$

The "double limit" (1) may be evaluated by two successive applications of the fundamental theorem of § 104, as follows.

Let us fix our attention on the rectangle $P_i'P_i''Q_i''Q_i'$ in S (Fig. 90). The volume $\Delta V_i'$ whose base is this rectangle may be found approximately by adding the volumes of all the included elementary prisms. Hence, by the fundamental theorem of § 104, $\Delta V_i'$ is given exactly by the formula

$$\Delta V_i' = \lim_{m \to \infty} \sum_{j=1}^{m} f(x_i, y_j) \Delta y \Delta x$$
$$= \left[\int_{y_i'}^{y_i''} f(x_i, y) dy \right] \Delta x,$$

x_i and Δx remaining *constant* as we pass to the limit.

Now if we add all the volumes of this type, we have approximately the required volume. It is to be noticed that in the expression for $\Delta V_i'$ the coefficient of Δx is a function of x_i alone, since the limits y_i' and y_i'' are functions of x_i alone. Thus we may apply again the theorem of § 104, and find that the required volume under the surface $z = f(x, y)$ is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \left[\int_{y_i'}^{y_i''} f(x_i, y) dy \right] \Delta x = \int_a^b \left[\int_{y'}^{y''} f(x, y) dy \right] dx,$$

where a and b are the extreme values of x on the boundary of S.

The quantity just found is usually written without the brackets, thus:

(2) $$V = \int_a^b \int_{y'}^{y''} f(x, y) dy\, dx.$$

It is called a *double integral*, or more properly an *iterated integral*, being merely an integral of an integral. It is to be noted that the inner integral sign belongs with the inner differential, and that during the integration with respect to y, x remains constant. Further, the first or inner limits of integration are in general variables, but the outer limits are always constants.

Of course we might integrate first with respect to x, then with respect to y. The same argument as before would lead to the formula

(3) $$V = \int_c^d \int_{x'}^{x''} f(x, y)\, dx\, dy,$$

y remaining constant during the first integration.

In the foregoing argument, we have assumed our volume to be divided into rectangular columns perpendicular to the xy-plane. Frequently, however, it is more convenient to erect columns perpendicular to one of the other coördinate planes (cf. example (b) below). Such variations offer no difficulty provided the geometric meaning of the successive integrations be kept clearly in mind. In every problem, a sketch of the required volume should be made, and the required double integral built up by inspection of the figure.

Examples: (a) Find the volume in the first octant bounded by the plane $z = x + y$ and the cylinder $y = 1 - x^2$.

Integrating in the order y, x, we have

$$V = \int_0^1 \int_0^{1-x^2} z \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} (x + y) dy \, dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big]_0^{1-x^2} dx$$

$$= \int_0^1 \left(x - x^3 + \frac{(1 - x^2)^2}{2} \right) dx$$

$$= \tfrac{31}{60}.$$

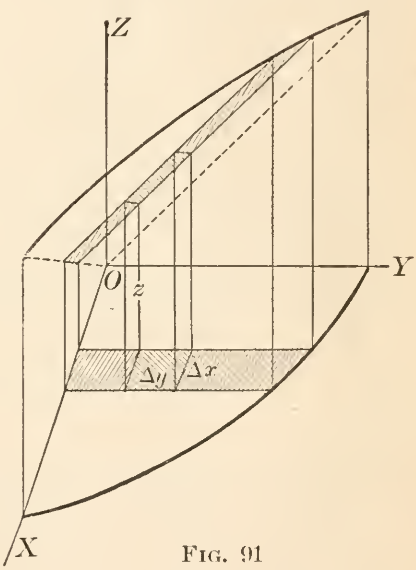


FIG. 91

(b) Find the volume common to the circular cylinder $y^2 + z^2 = ay$ and the sphere $x^2 + y^2 + z^2 = a^2$.

Let us divide the volume into columns perpendicular to the yz-plane:

$$V = 4 \int_0^a \int_0^{\sqrt{ay - y^2}} x \, dz \, dy$$

$$= 4 \int_0^a \int_0^{\sqrt{ay - y^2}} \sqrt{a^2 - y^2 - z^2} \, dz \, dy, \text{ etc.}$$

178. Volume under a surface: second method. The result of § 177 may be obtained by a somewhat different method. The area of the section by a particular one of the planes parallel to the yz-plane is evidently

$$A(x) = \int_{y'}^{y''} f(x, y)dy.$$

Hence, by § 110, the volume is

$$V = \int_a^b A(x)dx = \int_a^b \int_{y'}^{y''} f(x, y)dy\,dx.$$

The actual work of obtaining the volume in any particular case is therefore the same by the two methods — the only difference is in the geometric interpretation of the successive steps. The great advantage of the method of § 177 is that it lends itself readily to the discussion of a great variety of other problems besides the computing of volumes, as we shall see in the next few articles.

Of course when $A(x)$ is known to start with, the volume may be found by a single integration as in § 110. This is the case in several of the exercises below.

EXERCISES

In each of the following exercises, the limits of integration should be obtained directly from a figure.

1. Find the volume in the first octant bounded by the planes $x = 1$, $z = x + y$ and the cylinder $y^2 = x$. *Ans.* $\frac{13}{20}$.

2. Find the volume in the first octant bounded by the cylinder $x^2 + y^2 = a^2$ and the plane $z = x + y$. *Ans.* $\frac{2}{3}a^3$.

3. Find the volume of a cylindrical column standing on the area common to the two parabolas $x = y^2$, $y = x^2$ as base and cut off by the surface $z = 1 + y - x^2$. Check the result by integrating in two ways: first in the order y, x; next in the order x, y.

4. Find the volume in the first octant bounded by the plane $y + z = 1$ and the surface $x = 4 - z - y^2$. Check as in Ex. 3.

5. Find the volume cut off from the paraboloid $y = 1 - \frac{x^2}{4} - \frac{z^2}{9}$ by the xz-plane.

6. Find the volume in the first octant under the surface $z = xy$ bounded by the cylinder $y = x^2$ and the plane $y = 1$. Solve in two ways.

7. Find the volume bounded by the surface $z = xy$, the cylinder $y^2 = ax$, and the planes $x + y = 2a$, $y = 0$, $z = 0$.

8. Find the volume sliced off from the paraboloid $az = a^2 - x^2 - y^2$ by the plane $y + z = a$.

9. Find the volume cut out of the first octant by the cylinders $z = 1 - x^2$, $x = 1 - y^2$. *Ans.* $\frac{18}{35}$.

10. Find the volume of a segment of an elliptic paraboloid bounded by a right section.

11. Find the volume bounded by the surfaces $4y^2 + 4z^2 = x^2$, $x = 4y$, $x = 2a$, $z = 0$. *Ans.* $\frac{1}{36}(4\pi - 3\sqrt{3})a^3$.

12. Find in two ways the volume in the first octant bounded by the paraboloid $y = xz$ and the planes $z = x$, $z = 2 - x$.

13. Find the volume of a segment of a hyperboloid of two nappes bounded by a right section.

14. Find the volume bounded by the surface $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$. Sketch this surface. *Ans.* $\frac{4}{35}\pi a^3$.

15. Find the volume in the first octant inside the cylinder $x^2 + y^2 = 2ax$ and outside the paraboloid $x^2 + y^2 = az$. *Ans.* $\frac{3}{4}\pi a^3$.

16. Find the volume in the first octant bounded by the surface $\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} + \left(\frac{z}{c}\right)^{\frac{1}{2}} = 1$. *Ans.* $\frac{abc}{90}$.

17. Show that the volume of any cone or pyramid is one third the area of the base times the altitude.

18. Write out six different double integrals for the volume in the first octant bounded by the cylinders $y = x^2$, $x^2 + z^2 = 1$.

19. Find the volume in Ex. 18 by simple integration.

179. Interpretation of the given function. Any function $f(x, y)$ of two independent variables may be interpreted as the z-coördinate of a point on a surface in space. If, then, in any problem, we can express the required quantity as a double limit of the form (1), § 177, *no matter what may be the geometric or physical meaning of the given function* $f(x, y)$, the limit may be evaluated by an iterated integration as in § 177. Thus the result of that article is by no means confined to the determination of volumes — we shall, as was mentioned in § 178, apply it to the study of a variety of problems.

180. The double integral. In the argument of § 177 it is not necessary that the function f be expressed in terms of cartesian coördinates x and y; further, the area S need not be divided into elements in the particular way there adopted. The essential points are, first, that we have a function f of two independent variables defined at all points of the region S; second, that we divide S into n elements ΔS which are infinitesimal of the *second order*.*

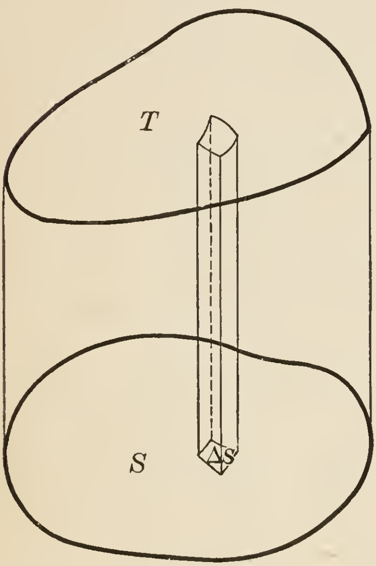


FIG. 92

When S is divided in this way, the double limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f_i\,\Delta S$$

is called the *double integral* of the function f over the region S, and is denoted by the symbol $\iint_S f\,dS$:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f_i\,\Delta S = \iint_S f\,dS.$$

As noted in § 177, the integral $\int_a^b\int_{y'}^{y''} f(x,\ y)\,dy\,dx$ is often called a double integral, and it is evidently equal to $\iint_S f\,dS$; but it is clear that the latter integral is the

* That is, such that the maximum distance between two points on the boundary of ΔS approaches 0.

more general, since it does not tie us down to a particular coördinate system, or to a particular mode of division of S. The integrals (2) and (3) of § 177 are merely two special forms of the double integral.

181. The double integral in polar coördinates. Given a function $f(r, \theta)$ of the polar coördinates r, θ, the double integral $\iint_S f\,dS$ may be evaluated as follows. Divide S into elements by a set of circles with center at the origin and a set of lines radiating from the origin, as in Fig. 93. Then ΔS is the difference between two circular sectors of angle $\Delta\theta$ and radius r and $r + \Delta r$ respectively; *i.e.*

$$\Delta S = \tfrac{1}{2}(r + \Delta r)^2\Delta\theta - \tfrac{1}{2}r^2\Delta\theta$$
$$= (r\Delta r + \tfrac{1}{2}\overline{\Delta r}^2)\Delta\theta.$$

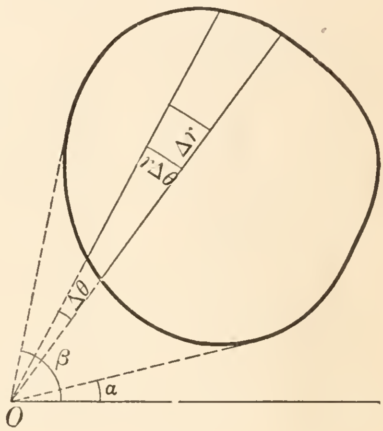


FIG. 93

We may now repeat the argument of § 177, integrating first with respect to r, and noting that, by § 109, the infinitesimal of higher order $\tfrac{1}{2}\overline{\Delta r}^2\Delta\theta$ may be neglected. This leads to the result

$$\iint_S f\,dS = \lim \sum\sum f(r, \theta)\,r\,\Delta r\,\Delta\theta = \int_\alpha^\beta\int_{r'}^{r''} f(r, \theta)r\,dr\,d\theta.$$

EXERCISES

1. A round hole is bored through the center of a sphere. Find the volume cut out, using polar coördinates.

2. A cylinder is erected on the circle $r = a\cos\theta$ as a base. Find the volume of the cylinder inside a sphere of radius a with center at the origin.

3. Find the volume above the xy-plane common to the paraboloid $z = 4 - x^2 - y^2$ and the cylinder $x^2 + y^2 = 1$, using polar coördinates.

4. A square hole of side 2 whose axis is the z-axis is cut through the paraboloid of Ex. 3. Find the volume cut out.

5. Find the volume of a spherical wedge by double integration.

6. Prove that when a curve $r = f(\theta)$ revolves about the initial line, the volume of revolution generated is given by the formula

$$V = 2\pi \int_{\alpha}^{\beta}\int_{r'}^{r''} r \sin\theta \cdot r\,dr\,d\theta.$$

Solve the following by the method of Ex. 6.

7. Find the volume of a sphere.

8. Find the volume generated by revolving the cardioid $r = a(1 - \sin\theta)$ about its line of symmetry. *Ans.* $\frac{8}{3}\pi a^3$.

9. The curve $r^2 = a^2 \sin\theta$ revolves about the y-axis. Find the volume generated.

10. Find the volume generated by revolving one loop of the curve $r = a\cos 2\theta$ about its line of symmetry.

11. Find the volume generated by revolving a circle about one of its tangents.

12. Find the volume cut from a sphere by a cone of half-angle $\frac{\pi}{3}$ with its vertex at the center of the sphere. Check by using cartesian coördinates.

13. Find the volume of the prolate spheroid generated by revolving about its major axis the ellipse

$$r = \frac{l}{1 - e\cos\theta},$$

where e is the eccentricity. *Ans.* $\frac{4\pi l^3}{3(1-e^2)^2}$.

14. Find the volume of a paraboloid of revolution bounded by a right section through the focus, taking the equation of the generating parabola in the form $r = \frac{2a}{1 - \cos\theta}$. *Ans.* $2\pi a^3$.

182. Transformation of double integrals. We have seen that the integrals (2) and (3) of § 177 are merely different forms of the double integral $\int\int_S f dS$. It may happen that an integral given in the form (2) is difficult or impossible to evaluate, but that when transformed to the form (3), it becomes simple. Or sometimes after evaluating the form (2) we change to the form (3) and evaluate again, merely as a check on the result. The process of changing the form of an integral from (2) to (3), or *vice versa*, is called *inverting the order of integration.*

Another transformation of importance is the change from one coördinate system to another — for instance, from cartesian to polar coördinates.

Example: Evaluate $\int_0^a\int_x^a \frac{e^y}{y}\, dy\, dx.$

This integral cannot be evaluated directly, since the function $\frac{e^y}{y}$ is not integrable in terms of elementary functions. But a study of the limits shows that the field of integration is the triangle bounded by the lines $x = 0$, $y = x$, $y = a$. Hence

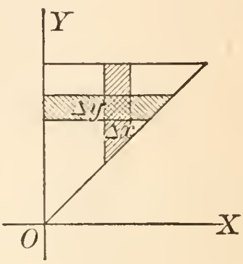


FIG. 94

$$\int_0^a\int_x^a \frac{e^y}{y}\, dy\, dx = \int_0^a\int_0^y \frac{e^y}{y}\, dx\, dy = \int_0^a \left[\frac{e^y}{y}\cdot x\right]_0^y dy$$

$$= \int_0^a e^y dy = e^a - 1.$$

EXERCISES

1. Check the result in example (a), § 177, by inverting the order of integration.

2. Invert the order of integration in Exs. 1 and 2, p. 262.

3. Find the volume bounded by the cylinder $x^2 = 4\,ay$ and the planes $x + y + z = a$, $z = 0$, $x = 0$, integrating in two different ways.

4. Express the volume of Ex. 3 as a double integral in two other ways.

5. Interpret the integral $\int_0^a\int_0^{2\sqrt{ax}} \sqrt{4\,a^2 - y^2}\, dy\, dx$ as a volume, and write out five other double integrals (all in cartesian coördinates) for this same volume.

6. Evaluate $\int_0^a\int_a^{2a-x} (x + y) dy\ dx$, and check by inverting the order of integration. Interpret geometrically.

7. Evaluate $\int_0^1\int_x^1 e^{y^2}\, dy\, dx.$ *Ans.* 0.859.

8. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$ by transforming to polar coördinates. *Ans.* 1.35.

9. Evaluate $\int_0^1 \int_{2\sqrt{x}}^2 \frac{\sin \frac{\pi y}{2}}{y^2}\,dy\,dx.$ *Ans.* $\frac{4}{\pi}$.

10. Express $\iint_S f\,dS$ (*a*) in cartesian coördinates, (*b*) in polar coördinates, where S is the triangle bounded by the lines $x = a$, $y = 0$, $y = x$.

11. Transform $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} e^{r\sin\theta}\, r\,dr\,d\theta$ to cartesian coördinates.

12. Compute the value of $\iint_S \cos(x^2 + y^2)dS$ extended over the interior of the circle $x^2 + y^2 = 1$. *Ans.* 2.644.

13. Find the area in the first quadrant under the curve $y = e^{-\frac{1}{2}x^2}$ by noting that

$$\left(\int_0^\infty e^{-\frac{1}{2}x^2}dx\right)^2 = \left(\int_0^\infty e^{-\frac{1}{2}x^2}dx\right)\cdot\left(\int_0^\infty e^{-\frac{1}{2}y^2}dy\right) = \int_0^\infty \int_0^\infty e^{-\frac{1}{2}(x^2+y^2)}dy\,dx.$$

Ans. $\sqrt{\frac{\pi}{2}}$.

14. Find the centroid of the area in Ex. 13.

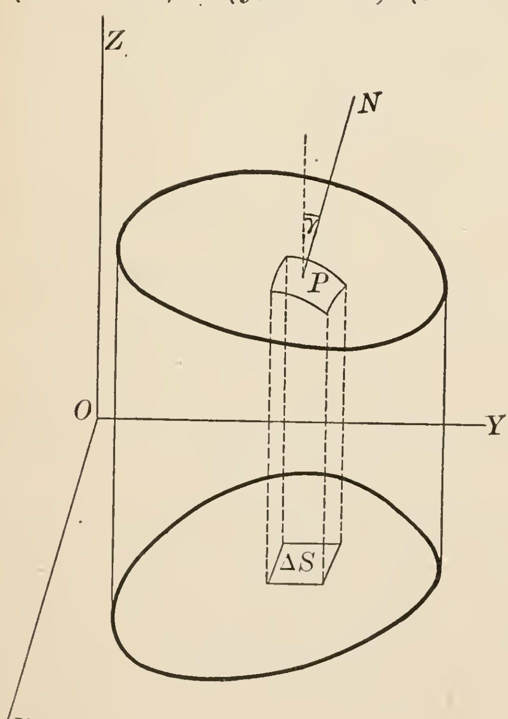


FIG. 95

183. Area of a surface. Let us try to find the area σ of a portion of the surface

$$z = f(x, y).$$

Suppose that the xy-projection of σ is the region S. Let us divide S into elements ΔS in any suitable way, and fix our attention on a particular one of these elements. This element is the horizontal projection of the portion $\Delta\sigma$ of σ. If we draw the tangent plane at some point P of $\Delta\sigma$, then ΔS will be the horizontal projection of a certain area $\Delta\sigma'$ on the tangent plane. Hence

$$\Delta S = \Delta\sigma' \cos\gamma,$$

where γ is the angle between the z-axis and the normal PN to σ at P, and

$$\Delta\sigma' = \frac{\Delta S}{\cos\gamma}$$

If now we form the sum of the quantities $\Delta\sigma'$ and pass to the limit, we have

$$\sigma = \lim_{\Delta S \to 0} \sum\sum \frac{\Delta S}{\cos\gamma} = \int\int_S \frac{dS}{\cos\gamma}.$$

In case it is more convenient to project the area on the xz- or the yz-plane, the corresponding formula is readily developed.

Example: Find the area of that part of the surface $z = y + x^2$ whose projection on the xy-plane is the triangle bounded by the lines $y = 0$, $y = x$, $x = 1$.

Writing the equation of the surface in the form

$$z - y - x^2 = 0,$$

we have for the partial derivatives the values

$$\frac{\partial F}{\partial x} = -2x, \quad \frac{\partial F}{\partial y} = -1, \quad \frac{\partial F}{\partial z} = 1.$$

Hence by the theorem of § 167, the direction cosines of the normal are proportional to $-2x$, -1, 1, and

$$\cos\gamma = \frac{1}{\sqrt{4x^2 + 1 + 1}}.$$

Therefore the required area is

$$\sigma = \int_0^1 \int_0^x \sqrt{4x^2 + 2}\, dy\, dx$$

$$= \int_0^1 x\sqrt{4x^2 + 2}\, dx$$

$$= \tfrac{1}{8} \cdot \tfrac{2}{3}(4x^2 + 2)^{\frac{3}{2}}\Big]_0^1 = \tfrac{1}{12}(6^{\frac{3}{2}} - 2^{\frac{3}{2}}).$$

EXERCISES

1. Find the area cut out of the plane $x + y + 2z = 2a$ by the cylinder $x^2 + y^2 = a^2$. *Ans.* $\frac{1}{2}\sqrt{6}\,\pi a^2$.

2. Find the area of that part of the surface $z = y + \frac{2}{3}x^{\frac{3}{2}}$ whose projection in the xy-plane is the triangle bounded by the lines $y = 0$, $y = x$, $x = 2$. *Ans.* $\frac{16}{15}(2 + \sqrt{2})$.

3. The center of a sphere of radius a is on the surface of a cylinder of diameter a. Find the surface of the cylinder intercepted by the sphere. *Ans.* $4\,a^2$.

4. In Ex. 3, find the surface of the sphere intercepted by the cylinder. *Ans.* $2(\pi - 2)a^2$.

5. How much of the conical surface $z^2 = x^2 + y^2$ lies above a square of side $2\,a$ in the xy-plane whose center is the origin?

6. How much of the surface $az = xy$ lies within the cylinder $x^2 + y^2 = a^2$? (Use polar coördinates.)

7. A square hole is cut through a sphere, the axis of the hole coinciding with a diameter of the sphere. Find the area cut from the surface of the sphere.

Ans. $16\,ab\arcsin\dfrac{b}{\sqrt{a^2 - b^2}} - 8\,a^2\arcsin\dfrac{b^2}{a^2 - b^2}$.

184. Triple integrals. We have seen that the integral of a function of one variable, extended over a given interval, may be interpreted as the area under a plane curve. Again, the integral of a function of two variables extended over a plane region may be interpreted as the volume under a surface. If now we have a function of three variables

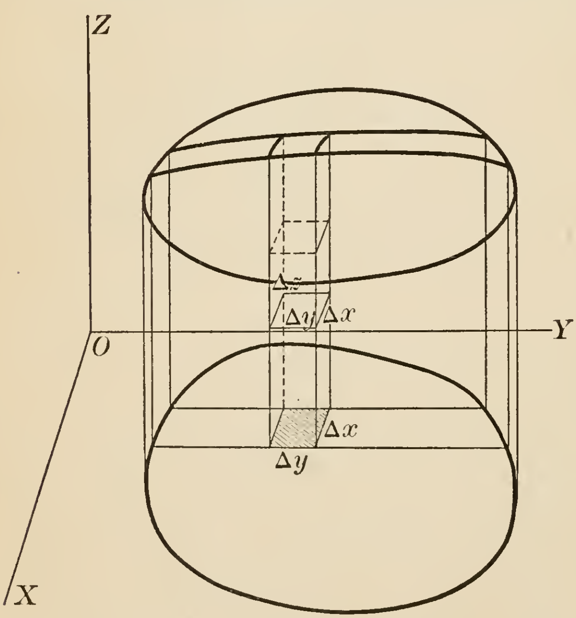


FIG. 96

defined at all points of a portion of space, no similar geometric interpretation for the integral of the function over the given region is possible, since geometric intuition fails in space of four dimensions. Nevertheless the meaning of such an integral may be made plain by analogy with the earlier cases.

Suppose we have given a function $f(x, y, z)$ defined at all points of a three-dimensional region V. Let us pass through V three sets of planes parallel to the coördinate planes, thus dividing V into elementary rectangular parallelepipeds of volume $\Delta x\,\Delta y\,\Delta z$, together with smaller irregular portions around the boundary. Now multiply the volume of each element by the value of the function at some point within the element, say at its center, and form the sum of these products. The triple limit

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum\sum\sum f(x, y, z)\Delta x\,\Delta y\,\Delta z$$

is defined as the value of the *triple integral* of $f(x, y, z)$ over the region V.

This limit may be evaluated by three successive integrations (cf. §177):

$$\begin{aligned} T &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum\sum\sum f(x, y, z)\Delta x\,\Delta y\,\Delta z \\ &= \int_a^b \int_{y'}^{y''} \int_{z'}^{z''} f(x, y, z)dz\,dy\,dx. \end{aligned}$$

The first integration extends over a vertical column of base $\Delta y\,\Delta x$; the limits z', z'' are the extreme values of z in this column, and are in general functions of both x and y. The integration with respect to y is extended over a slice parallel to the yz-plane; the limits y' and y'' are the extreme values of y in this slice, and are functions of x alone. In the final integration the limits are of course the extreme values of x in the whole region.

More generally, the function f may be given in terms of any system of coördinates, and the region V may be divided into elements in any suitable way.* We write in general, for the value of the triple integral of f over the region V,

$$T = \iiint_V f\,dV.$$

It is hardly necessary to say that such transformations as inversion of order and change from one coördinate system to another are allowable and useful with triple integrals, just as with double integrals.

It may be well to observe at this point that applications of triple integration are comparatively rare in elementary work. In the problems treated in the next two articles, triple integrals are sometimes required.

The volume V itself may be expressed as a triple integral, the given function f being taken equal to unity:

$$V = \iiint_V dV.$$

It is true that the volume may be found more directly by methods previously studied; nevertheless it may be worth while to solve a few exercises by the present method for the sake of practice in determining the limits in triple integration.

Example: Find the volume cut off from the paraboloid

$$z = 1 - x^2 - \frac{y^2}{4}$$

by the xy-plane.

In this case

$$V = 4\int_0^1 \int_0^{2\sqrt{1-x^2}} \int_0^{1-x^2-\frac{y^2}{4}} dz\,dy\,dx$$

$$= 4\int_0^1 \int_0^{2\sqrt{1-x^2}} \left(1 - x^2 - \frac{y^2}{4}\right) dy\,dx$$

* The element must of course be infinitesimal of the *third order*.

$$= 4\int_0^1 \left[(1-x^2)y - \frac{y^3}{12}\right]_0^{2\sqrt{1-x^2}} dx$$

$$= \frac{16}{3}\int_0^1 (1-x^2)^{\frac{3}{2}} dx$$

$$= \frac{16}{3}\cdot\frac{3}{8}\cdot\frac{\pi}{2} = \pi.$$

EXERCISES

Find the following volumes by triple integration, drawing a figure in each case.

1. The tetrahedron bounded by the coördinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

2. The volume bounded by the paraboloid $x^2+y^2=az$, the cylinder $x^2+y^2=2ax$, and the plane $z=0$. *Ans.* $\frac{3}{2}\pi a^3$.

3. Interpret the triple integral $\int_0^a \int_0^{3a-x} \int_0^{\sqrt{a^2-x^2}} dz\,dy\,dx$ geometrically, and express the same volume as a triple integral in several other ways, drawing a figure for each case.

185. Heterogeneous masses. The density of a homogeneous mass has been defined in § 121 as the ratio of the mass to the volume it occupies:

$$\delta = \frac{M}{V}.$$

For a heterogeneous mass, *i.e.* one whose density varies from point to point, we must introduce the idea of *density at a point.*

Consider an element of volume ΔV including a point P, and let ΔM denote the mass contained in ΔV. Then the ratio $\frac{\Delta M}{\Delta V}$ is the *average* density in ΔV. If ΔV approaches 0 in such a way that P is always included, the ratio $\frac{\Delta M}{\Delta V}$ in general approaches a limit* δ, called the *density at the*

* In general this is true only if ΔV is infinitesimal of the third order, as in § 184.

T

point P:

$$\delta = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV}.$$

The mass of a heterogeneous body whose density at any point is given as a function of the coördinates of the point can be found by integration. We have only to choose a suitable mass-element and integrate over the whole body. The great point to be noted is that in general *the element itself must be homogeneous*,* since otherwise the mass of the element cannot be computed and hence the integral cannot be built up.

In many cases it is possible to choose an element as in Chapter XV and obtain the result by a simple integration; in more complicated problems double or triple integration may be necessary.

We give the argument in full only for the general case where triple integrals are employed. Given a mass M occupying a volume V, divide V into elements as in § 184, and multiply each element ΔV by the density δ at one of its points. Then the sum $\sum\sum\sum \delta \Delta V$ is an arbitrarily close approximation to the mass M if ΔV be taken sufficiently small, and the mass is therefore given exactly by the formula

$$M = \lim_{\Delta V \to 0} \sum\sum\sum \delta \Delta V = \int\int\int_V \delta dV.$$

For a mass distributed over a surface S, the idea of "surface density" must be introduced:

$$\delta = \lim_{\Delta S \to 0} \frac{\Delta M}{\Delta S} = \frac{dM}{dS}.$$

whence, by argument now familiar,

$$M = \int\int_S \delta dS.$$

* By this is meant that the density at different points of the element varies only by *infinitesimal* amounts; cf. example (a) below. By the theorem of § 109, the infinitesimal variations may be neglected.

Similarly, for a mass distributed along a curve C, the "linear density" is

$$\delta = \lim_{\Delta s \to 0} \frac{\Delta M}{\Delta s} = \frac{dM}{ds},$$

and

$$M = \int_C \delta ds.$$

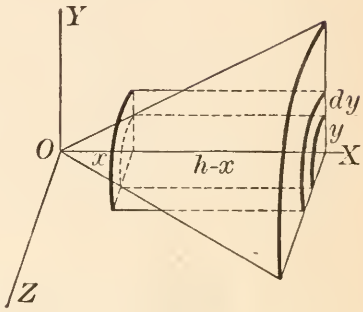


FIG. 97

Examples: (*a*) Find the mass of a circular cone whose density varies as* the distance from the axis.

Let us take the vertex of the cone at the origin and its axis along OX. If we divide the mass into cylindrical shells about the axis, each element will be "homogeneous" of density

$$\delta = kr = ky.$$

We have

$$dV = 2\pi y(h - x)dy,$$

$$M = \int dM = \int \delta dV = 2\pi k \int_0^a y^2(h - x)dy$$

$$= 2\pi k \int_0^a y^2\left(h - \frac{h}{a}y\right)dy = \frac{1}{6}\pi k a^3 h.$$

(*b*) The density at any point of a cube is proportional to the sum of the distances from three adjacent faces. Find the mass of the cube.

Taking the three faces mentioned as coördinate planes, and choosing the element as in § 184, we have

$$M = k\int_0^a \int_0^a \int_0^a (x + y + z)dz\, dy\, dx.$$

* To say that *a varies as b*, or *a is proportional to b*, means that $a = kb$, where k is constant.

EXERCISES

Determine the following masses.

1. A straight rod whose density is proportional to the n-th power of the distance from one end.

2. A semicircular wire whose density varies as the distance from the diameter joining the ends. *Ans.* $2\,ka^2$.

3. A circular plate whose density varies (a) as the distance from the center; (b) as the distance from a fixed diameter. *Ans.* (b) $\frac{4}{3}\,ka^3$.

4. A spherical surface whose density varies as the distance (a) from a fixed diameter; (b) from a diametral plane. *Ans.* (a) $k\pi^2a^3$.

5. A sphere whose density is proportional to the distance from the center. *Ans.* $k\pi a^4$.

6. A rectangle whose density is proportional to the sum of the distances from two adjacent sides.

7. A circular plate whose density varies as the distance from a point on the circumference.

8. A square whose density is proportional to the distance from one corner. *Ans.* $.765\,ka^3$.

9. The tetrahedron bounded by the coördinate planes and the plane $x + y + z = a$, if the density is proportional to the sum of the distances from the coördinate planes.

186. Centroids and moments of inertia: the general case. We are now in position to lay down precise definitions of the moment of first order, and the moment of inertia, of any mass. Divide the mass into elements ΔV as in § 184, and multiply each element by the density δ at a point $P:(x, y, z)$ of the element. Then the *moment of the first order* with respect to the yz-plane is defined as

$$\lim_{\Delta V \to 0} \sum\sum\sum x\delta\Delta V = \iiint\limits_V x\delta dV,$$

with a similar formula for the moment with respect to any other plane. The centroid is defined as the point $(\bar{x}, \bar{y}, \bar{z})$ whose coördinates are given by the formulas

$$M\bar{x} = \iiint\limits_V x\,\delta\,dV,\quad M\bar{y} = \iiint\limits_V y\,\delta\,dV,\quad M\bar{z} = \iiint\limits_V z\,\delta\,dV.$$

Similarly, the moment of inertia with respect to any axis is defined as

$$I = \iiint\limits_V r^2 \,\delta \, dV,$$

where r is the distance of a point of the element from the axis.

While the above formulas are important from the theoretical standpoint on account of their generality, it must not be forgotten that in the actual computation of moments of the first order and moments of inertia multiple integrals are very rarely needed, at least for homogeneous masses.

It will be remembered that the theorems of § 134 have been proved only for a set of particles. The reader will now have no difficulty in extending the proof to the general case.

EXERCISES

1. Find the centroid of the volume in the first octant bounded by the paraboloid $az = x^2 + y^2$ and the planes $y = x$, $x = a$.

Ans. $(\frac{4}{5}a, \frac{9}{20}a, \frac{7}{15}a)$.

2. Find the moment of inertia of the volume in Ex. 1, with respect to the z-axis.

3. Find the centroid of the volume in Ex. 3, p. 262.

4. Find in two ways the centroid of the volume in Ex. 9, p. 263.

5. Find the moment of inertia, with respect to the x-axis, of the volume in Ex. 4, p. 262. Check by inverting the order of integration.

In Exs. 6–10, use polar coördinates.

6. Find the moment of inertia, with respect to the z-axis, of the volume in Ex. 2, p. 265.

7. Find the centroid of a hemisphere (cf. Ex. 6, p. 266).

8. Find the moment of inertia of a sphere about a diameter.

9. Find the centroid of a spherical wedge of half-angle α. Check by putting $\alpha = \frac{\pi}{2}$.

10. For the wire of Ex. 2, p. 276, find (a) the centroid; also the moment of inertia with respect to (*b*) the diameter joining the ends, (*c*) the radius perpendicular to this diameter.

Ans. (*a*) $\bar{x} = \frac{1}{4}\pi a$; (*b*) $\frac{2}{3}Ma^2$; (*c*) $\frac{1}{3}Ma^2$.

11. Find the moment of inertia of a circular disk whose density varies as the distance from the center, (*a*) about the axis of the disk, (*b*) about a diameter. *Ans.* (*a*) $\frac{3}{5}Ma^2$.

12. Find the centroid of a rectangle whose density is proportional to the sum of the distances from two adjacent sides.

13. Find the moment of inertia with respect to (*a*) the xy-plane, (*b*) the x-axis, of the volume bounded by the planes $z = x + y$, $x + y = a$, and the coördinate planes. *Ans.* (*a*) $\frac{1}{5}Ma^2$.

14. Find the moment of inertia, with respect to the yz-plane, of the volume bounded by the planes $x = 0$, $y = 0$, $z = a$, $z = x + y$, integrating in the order z, y, x; check by integrating in the order x, y, z.

15. Find the centroid of a straight rod whose density is proportional to the n-th power of the distance from one end.

Ans. $\bar{x} = \dfrac{n+1}{n+2}l$.

16. By dividing a triangle into strips parallel to the base and concentrating the mass of each strip at its center, show that in finding the centroid of the triangle we may replace the triangle by a straight line lying along the median and having a density proportional to the distance from the vertex. Hence find the centroid of any triangle by the result of Ex. 15.

17. By a method analogous to that of Ex. 16, find the centroid of any cone or pyramid.

18. Prove theorems I and II of § 134 for the general case of any continuous mass.

CHAPTER XXIV

FLUID PRESSURE

187. Force. If a particle of mass m moves with an acceleration j, the product of the mass by the acceleration is called *force:*

$$F = mj,$$

and the motion is said to be due to the action of the force.

Since force is a mere numerical multiple of acceleration, it follows that force is a *vector* (§ 56). If several forces act on the same particle, their combined effects are equivalent to that of a single force, their *resultant.* Usually the resultant is most easily found analytically by resolving each force into components parallel to the coördinate axes and summing in each direction to get the rectangular components of the resultant, after which the resultant is found by compounding these rectangular components (see the example below).

If there is no force acting on the particle, or, what is the same thing, if the resultant of all the forces is 0, the particle is said to be *in equilibrium.* It follows from § 59 that a particle in equilibrium is either at rest or moving uniformly in a straight line.

If several forces act at various points of a body, it is not always possible to compound them into a single resultant. In what follows, we shall consider only cases in which this is possible.

Example: Find the resultant of a plane system of forces

$$F_1 = 10 \text{ lbs.}, \quad F_2 = 20 \text{ lbs.}, \quad F_3 = 8 \text{ lbs.}, \quad F_4 = 15 \text{ lbs.}$$

acting as in the figure, where $\alpha = \arctan \frac{3}{4}$.

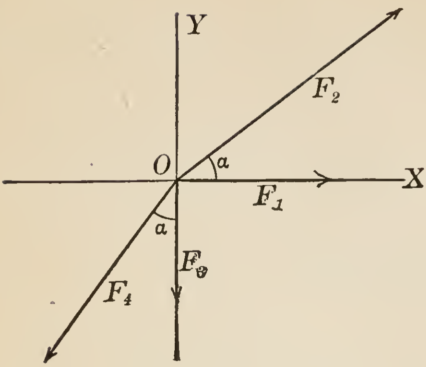


FIG. 98

The components parallel to OX are, of F_1, 10; of F_2, $20 \cos \alpha = 16$; of F_3, 0; of F_4, $-15 \sin \alpha = -9$. Hence the x-component R_x of the resultant is

$$R_x = 10 + 16 + 0 - 9 = 17.$$

Similarly,

$$R_y = 0 + 12 - 8 - 12 = -8.$$

Hence

$$R = \sqrt{17^2 + 8^2} = \sqrt{353} = 18.8 \text{ lbs.},$$

inclined to the x-axis at an angle

$$\arctan\left(-\tfrac{8}{17}\right) = -25° \, 12'.$$

188. Force distributed over an area. We have frequently to consider a force not acting at a single point, but distributed over an area. Examples are the pressure of a body of water upon a dam, that of a carload of sand against the sides of the car, the attraction of an electric point-charge upon an electrified plate, etc. If the mass upon which the force acts be thought of as composed ultimately of particles, such a distributed force may be regarded as comprising the totality of forces acting on the separate particles. We shall consider only the case in which all these separate forces taken together are equivalent to a single resultant; the resultant is the *total force* acting on the body.

Consider a force acting in the same direction at all points of a plane surface S, and suppose for concreteness that the force is normal to the surface. If we denote by ΔF the total force acting on an element of area ΔS chosen as in § 180, then the ratio $\dfrac{\Delta F}{\Delta S}$ is called the *average pressure* on ΔS. Now, if ΔS approaches 0 in such a way that a certain point Q is always included, the ratio $\frac{\Delta F}{\Delta S}$ in general approaches a limit, called the *pressure at the point Q:*

$$p = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} = \frac{dF}{dS}.$$

When the pressure at every point is given as a function of the coördinates, the total force F can be found by integration. In the most general case the force appears as a double integral, by § 180:

$$F = \lim_{\Delta S \to 0} \sum \sum p\Delta S = \int\int_S p dS,$$

but in most cases of practical importance the element of area can be so chosen that a single integration is sufficient.

189. Fluid pressure. An example of force acting normally to a surface is furnished by the pressure of a fluid against a retaining wall.

The pressure, at any point of an incompressible fluid, due to the weight of the fluid is equal to the weight per unit volume times the depth h of the point below the surface of the fluid:

$$p = wh.$$

We will assume the retaining area to be plane and vertical. Let us divide this area into horizontal rectangular elements of area $l_i \Delta h$ as in the figure. If we denote by p_i the pressure at the depth h_i, the force acting on the rectangle $l_i \Delta h$ is approximately *

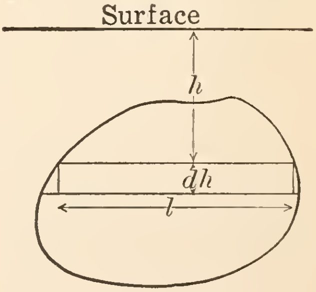


FIG. 99

$$p_i l_i \Delta h = wh_i l_i \Delta h.$$

Then the sum $\sum_{i=1}^{n} wh_i l_i \Delta h$ is approximately the total force,

* The actual force on the rectangle evidently differs from the quantity $p_i l_i \Delta h$ by an infinitesimal of higher order, which may be neglected.

or *total pressure* * P, on the whole area, and the limit of this sum is exactly P. Hence, by the fundamental theorem of § 104,

$$P = \lim_{n \to \infty} \sum_{i=1}^{n} wh_i l_i \Delta h = w \int hl\, dh,$$

where limits of integration are to be assigned in such a way as to extend the integration over the whole area.

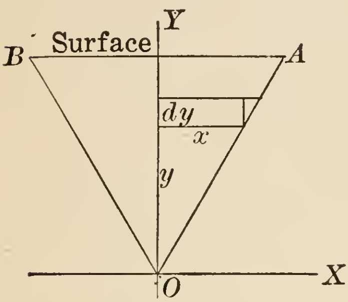


FIG. 100

Example: A trough, whose cross-section is an equilateral triangle of side 2 ft., is full of water. Find the total pressure on one end.

Let us take the origin at the lower vertex of the triangle. Then the equation of the line OA is

$$y = \sqrt{3}\, x.$$

The total pressure on the triangle is

$$P = 2w \int_0^{\sqrt{3}} (\sqrt{3} - y) x\, dy = 2w \int_0^1 (\sqrt{3} - \sqrt{3}\, x) x \sqrt{3}\, dx$$

$$= 6w \int_0^1 (1 - x) x\, dx = 6w \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= w = 62 \text{ lbs., nearly.}$$

EXERCISES

1. A particle is acted on by two forces F_1, F_2 lying in the same vertical plane and inclined to the horizon at angles α_1, α_2. Find their resultant in magnitude and direction, if $F_1 = 527$ lbs., $F_2 = 272$ lbs., $\alpha_1 = 127° 52'$, $\alpha_2 = 32° 13'$.

Ans. 569 lbs., inclined to the horizon at 99° 26′.

2. Six forces, of 1, 2, 3, 4, 5, 6 lbs. respectively, act at the same point, making angles of 60° with each other. Find their resultant.

Ans. 6 lbs., acting along the line of the 5-lb. force.

* Care must be taken not to confuse the *total pressure*, P, with the *pressure at a point*, p. The former is a force, the latter, a force per unit area.

3. Work the example of § 189 with the origin at B.

4. Find the total pressure on one side of a plank 2×8 ft. submerged vertically with its upper end (a) in the surface, (b) 4 ft. below the surface.

5. A horizontal cylindrical boiler 4 ft. in diameter is half full of water. Find the total pressure on one end. *Ans.* 330 lbs.

6. Work Ex. 5 if the boiler is full of water.

7. What force must be withstood by a vertical dam 100 ft. long and 20 ft. deep?

8. Work Ex. 7 if the dam is a trapezoid 100 ft. long at the top and 80 ft. long at the bottom, taking the origin at an upper corner. Check by solving again with the origin in a different position.

9. Find the total pressure on one side of a right triangle of sides $AB = 3$ ft., $AC = 4$ ft., submerged with AC vertical and (a) A in the surface, (b) A 2 ft. deep, (c) C 2 ft. deep. In each case check as in Ex. 8.

10. Find the total pressure on one face of a square 2 ft. on a side, submerged with one diagonal vertical and one corner in the surface.

11. Find the force on one end of a parabolic trough full of water, if the depth is 2 ft. and the width across the top 2 ft. *Ans.* $\frac{32}{15}w$.

12. A trough 4 ft. deep and 6 ft. wide has semi-elliptical ends. If the trough is full of water, find the pressure on one end.

13. Find the force that must be withstood by a bulkhead closing a watermain 4 ft. in diameter, if the surface of the water in the reservoir is 40 ft. above the center of the bulkhead. *Ans.* 16 tons.

14. Show that the problem of § 189 is analytically equivalent to the following: To find the mass of a thin plate, if the density is proportional to the distance from a line in the plane of the plate.

190. Resultant of parallel forces. Suppose we have given a set of parallel forces $f_1, f_2, \cdots, f_n$, whose resultant (algebraic sum) F is not 0. The problem of finding the line of action of the resultant is analogous to that of finding the centroid of a set of mass particles.

Let us take the xy-plane perpendicular to the given forces, and let (x_i, y_i) be the point where the line of action of f_i pierces this plane. The moment of the resultant about each coördinate axis must equal the sum of the moments of the forces about the same axis. Hence the line of action of F pierces the xy-plane at the point whose coördinates $\bar{x}$, $\bar{y}$ are given by the formulas

$$F\bar{x} = \sum_{i=1}^{n} f_i x_i, \quad F\bar{y} = \sum_{i=1}^{n} f_i y_i.$$

191. Center of pressure. More generally, consider again the case of a force acting normally at all points of a plane area. Take the given plane as xy-plane, and divide the surface into elements ΔS as in § 180; then the force on ΔS is approximately $p\Delta S$, where p is the pressure at a point (x, y) of ΔS, and the moment of this force about the y-axis is $xp\Delta S$. The sum of these moments is approximately the moment of the whole force, and the limit of the sum is exactly that moment. Similarly, we can find the moment about the x-axis. Hence the resultant acts at the point whose coördinates $\bar{x}$, $\bar{y}$ are given by the formulas

$$F\bar{x} = \iint_S xp\, dS, \quad F\bar{y} = \iint_S yp\, dS,$$

where F is the total force. The point $(\bar{x}, \bar{y})$ is called the *center of pressure*.

As usual, it happens in many problems that the double integrals reduce to simple integrals, if the element be properly chosen. In particular, in the problem of fluid pressure it is easily seen that *the depth of the center of pressure* below the surface is given by the formula

$$P\bar{h} = w\int h^2 l\, dh,$$

where P is the total pressure.

EXERCISES

1. A straight beam AB 50 ft. long bears loads as follows: 100 lbs. at A, 100 lbs. at C, 200 lbs. at D, 50 lbs. at B; $AC = 10$ ft., $AD = 20$ ft. Find the point of application of the resultant.

2. Work Ex. 1 if the segment AD bears a uniformly distributed load of 5 lbs. per foot.

3. Work Ex. 2 if the segment DB bears a distributed load which increases uniformly from 5 lbs. per foot at D to 15 lbs. per foot at B.

4. A platform $ABCD$ 20 ft. square bears a single concentrated load. The reactions are, at A, 50 lbs.; at B, 80 lbs.; at C, 100 lbs.; at D, 70 lbs. Where is the load?

5. Find the most advantageous length for a lever to lift a weight of 100 lbs., if the distance from the weight to the fulcrum is 4 ft. and the lever weighs 4 lbs. per foot.

Find the depth of the center of pressure in the following cases.

6. A rectangle submerged vertically (a) with one edge in the surface, (b) with its upper edge at a depth c. *Ans.* (a) $\frac{2}{3}a$.

7. An isosceles triangle submerged with the line of symmetry vertical and (a) the vertex, (b) the base, in the surface.

Ans. (a) $\frac{3}{4}h$; (b) $\frac{1}{2}h$.

8. Any triangle submerged with one side in the surface.

9. One end of the parabolic trough of Ex. 11, p. 283.

10. A semicircle submerged with its bounding diameter in the surface.

11. In each case of Ex. 9, p. 283, if the pressure is removed from one side of the triangle, at what point must a brace be applied in order to hold the triangle in position?

Ans. (a) With AB, AC as axes, $(\frac{3}{4}, 2)$.

12. Show that the problem of § 191 is analytically equivalent to that of finding the centroid of a plane mass of variable density p.

CHAPTER XXV

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

I. General Introduction

192. Differential equations. A *differential equation* is an equation that involves derivatives or differentials. Various examples have arisen in our previous work, of which the following may be mentioned:

(1) $\frac{ds}{dt} = -\frac{1}{t^2}$. (§ 15)

(2) $y'' = 4$. (§ 16)

(3) $dy = 2\cos 2\theta\, d\theta$. (§ 50)

(4) $y^{(n)} = a^n e^{ax}$. (Ex. 43, p. 64)

(5) $x\,dx + y\,dy = 0$. (§ 51)

(6) $\frac{d^2x}{dt^2} = -k^2x$. (Ex. 18, p. 50)

(7) $\frac{d^2y}{dt^2} = g$. (Ex. 1, p. 81)

(8) $\frac{(1+y'^2)^{\frac{3}{2}}}{y''} = a$. (Ex. 2, p. 78)

(9) $\frac{\partial z}{\partial x} = 2x + y - 3$. (Ex. 1, p. 239)

(10) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. (Ex. 22, p. 240)

Equations containing partial derivatives, such as examples (9) and (10), are called *partial differential equations.* Such equations are of great importance, but a study of them is beyond the limits of this book.

193. Order of a differential equation. The *order* of a differential equation is the order of the highest derivative that occurs in it. Thus, in § 192, examples (1), (3), (5) are of the first order, (2), (6), (7), (8) are of the second order, (4) is of the n-th order.

In the applications, equations of the first and second orders are of predominant importance, and we shall be chiefly concerned with these two types.

194. Solutions of a differential equation. A *solution* of a differential equation is any relation between x and y by virtue of which the differential equation is satisfied. Thus equation (1) of § 192 is true if

$$s = \frac{1}{t} + c,$$

where c is arbitrary; hence this relation is a solution of the equation. A solution of (2) is easily seen to be

$$y = 2x^2 + c_1 x + c_2.$$

It appears from these examples that a solution of a differential equation may involve one or more arbitrary constants; we shall find this to be true in general. It follows that each equation has an infinity of solutions, obtained by assigning different values to the arbitrary constants.

By analogy with the integral calculus, a solution of a differential equation is often called an *integral* of the equation, and the arbitrary constants are called *constants of integration.*

II. Equations of the First Order

195. The general solution. Suppose there is given a relation (free of derivatives) between x, y and an arbitrary constant:

(1) $$F(x, y, c) = 0.$$

Geometrically this equation represents a *family of curves*, whose individual members are obtained by assigning particular values to c.

If we differentiate (1) with respect to x, the arbitrary constant c may be eliminated from the equation thus formed and the original equation. The result of this elimination is evidently an equation involving x, y, and y'; *i.e.* it is a differential equation of the first order:

$$(2) \qquad \Phi(x, y, y') = 0.$$

As this equation does not contain c, it represents a property common to *all the curves* of the above-mentioned family.

Since equation (2) is true by virtue of equation (1), it follows that (1) is a solution of (2).

If a solution of a differential equation of the first order contains an arbitrary constant, it is called the *general solution:* hence (1) is the general solution of (2). It can be shown that, in general, corresponding to every differential equation of the form (2) there exists a general solution (1); methods of finding this solution in various cases will be considered presently.

It may be worth while to point out that, if the differential equation has the simple form

$$\frac{dy}{dx} = f(x),$$

the integral calculus gives us the general solution at once:

$$y = \int f(x)\, dx + c.$$

It should also be noted that while in the integral calculus the constant of integration always appears as an additive constant, this is not true in general in the solution of a differential equation; the constant often enters in other ways.

Examples: (*a*) Find the differential equation whose general solution is

$$y = ce^{2x}.$$

Differentiating, we find

$$dy = 2\,ce^{2x}\,dx;$$

eliminating c by division, we get

$$\frac{dy}{y} = 2\,dx.$$

This example illustrates the fact that the arbitrary constant is not always additive.

(*b*) Find by inspection the general solution of the equation

$$x\,dy + y\,dx = 0.$$

The answer is seen at once to be

$$xy = c.$$

196. Particular solutions. A solution obtained from the general solution by assigning a particular value to the arbitrary constant is called a *particular solution* of the differential equation. Thus in example (*b*), § 195, the equations $xy = 0$, $xy = 5$, etc., are particular solutions.

In applied problems involving differential equations we are often concerned with a particular solution. Nevertheless the determination of the general solution is usually a necessary preliminary step, after which the required particular solution is found by determining the arbitrary constant from given *initial conditions*. The process is illustrated by the examples of § 77, which should be reviewed at this point.

Differential equations involving y' to a degree higher than the first may in some cases have a so-called *singular solution*, which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. As such solutions are of little importance in most of the elementary applications, we shall omit a discussion of them.

EXERCISES

In the following cases, find the differential equation whose general solution is the given equation.

1. $y = x^3 + c$.

2. $y = cx$.

3. $y = ce^x$.

4. $y = cx + c^2$.

5. $\log r = k\theta$.

6. $xy + cy = 1$.

7. $s = \sin t + c \cos t$.

8. $c^2 + 2\,cy = x^2$.

Find by inspection the general solution of each of the following differential equations.

9. $dy - \sin x\, dx = 0$.

10. $x\, dx + y\, dy = 0$.

11. $\frac{dy}{y} = x\, dx$.

12. $\frac{dy}{y} = \frac{dx}{x}$.

13. $x\, dy + y\, dx + 2\, dy = 0$.

14. Find the equation of a curve whose slope at any point is equal * to the abscissa of the point. How many such curves are there? Draw several of them.

15. In Ex. 14, find the curve that passes through $(4, -3)$.

16. Solve Ex. 14, reading "ordinate" instead of "abscissa."

17. A point, starting with a velocity of 10 ft. per second, moves under a constant acceleration of 8 ft. per second per second. Find (*a*) the velocity, (*b*) the distance from the starting point, after t seconds of motion.

18. A point moves under an acceleration

$$\frac{dv}{dt} = -4 \cos 2\,t.$$

If $v = 0$ and $x = 1$ when $t = 0$, find v and x in terms of t.

197. Geometrical interpretation. In analytic geometry we find that the locus of a point whose coördinates x, y are connected by an equation

$$y = f(x)$$

is a certain curve, the *graph* of the equation. In general, any value whatever may be assigned to x, and the corresponding value of y determined.

* That is, the number representing the slope is the same as that representing the abscissa. It is only in this sense that a ratio, such as the slope of a curve, can be equal to a length, such as the abscissa of a point.

If now we have given a differential equation of the first order, and of the first degree in y', *i.e.* a relation between x, y, and y' of the form

(1) $$y' = F(x, y),$$

it is clear that, in general, any values whatever may be assigned to x and y provided we associate with them the value of y' given by the equation. Thus, equation (1) is satisfied by the coördinates of *any* point (x, y) provided the point is *moving in the proper direction.* Starting with any assumed initial position, and moving always in the direction required by the given equation, the point describes a curve; the values of x, y, y' at any point of the curve satisfy the differential equation. Further, since the initial position is entirely arbitrary, it is clear that the point may be made to describe any one of a family of curves, the so-called *integral curves.* The equation of this family is, of course, the general solution of the differential equation; it contains, as it should, an arbitrary parameter, viz., the constant of integration. The graph of any particular solution is merely one of the family of integral curves.

Example: Interpret geometrically the differential equation

$$x\,dx + y\,dy = 0.$$

Writing the equation in the form

$$\frac{dy}{dx} = -\frac{x}{y},$$

we see that the point (x, y) must always be moving in a direction perpendicular to the line joining it to the origin. Its path is therefore any one of the family of circles with center at the origin. This may be verified by observing that the general solution of the differential equation is

$$x^2 + y^2 = c.$$

EXERCISES

In each of the following cases find the equation of the family of integral curves and draw several curves of the family.

1. $y' = 0$.

2. $y' = 5$.

3. $y' = 4x$.

4. $\frac{dy}{dx} = \frac{y}{x}$.

5. $\frac{dy}{dx} = -\frac{y}{x}$.

6. $y' = y$.

7. Find the differential equation of the family of circles through the origin with centers on the x-axis. *Ans.* $2xyy' = y^2 - x^2$.

8. Find the differential equation of the family of parabolas with foci at the origin and axes coinciding with the x-axis.

9. Interpret geometrically the equations in Exs. 9, 12, and 14, p. 290.

198. Separation of variables. In the remainder of this chapter we show how to find the general solution of a differential equation of the first order in some of the simpler cases.

Every differential equation of the first order, and of the first degree in y', can evidently be written in the form

$$M\,dx + N\,dy = 0,$$

where in general M and N are functions of both x and y. It is often possible to transform the equation so that M is a function of x alone and N is a function of y alone; this transformation is called *separation of variables*. When the variables have been separated, the differential equation may be solved by a simple integration, as in the following

Example: Solve the equation

$$xy\,dx + (x^2 + 1)\,dy = 0.$$

After division by $y(x^2 + 1)$ the equation takes the form

$$\frac{x\,dx}{x^2 + 1} + \frac{dy}{y} = 0.$$

Integrating, we get

$$\tfrac{1}{2}\log(x^2+1)+\log y = c,$$

or

$$\log y\sqrt{x^2+1} = c,$$
$$y\sqrt{x^2+1} = e^c,$$
$$y^2(x^2+1) = c',$$

where

$$c' = e^{2c}.$$

EXERCISES

Solve the following differential equations.

1. $(1+x)y\,dx+(1-y)x\,dy=0.$ *Ans.* $\log(xy)+x-y=c.$

2. $y'=axy^2.$ *Ans.* $ax^2y+cy+2=0.$

3. $\sin x\cos y\,dx=\cos x\sin y\,dy.$ *Ans.* $\cos y = c\cos x.$

4. $\dfrac{dy}{dx}+y^2=a^2.$ *Ans.* $\dfrac{y+a}{y-a}=ce^{2ax}.$

5. $\dfrac{dy}{1+y}=\dfrac{dx}{1-x}.$

6. $(1+x)y^2\,dx-x^3\,dy=0.$

7. $\sqrt{1-y^2}\,dx+\sqrt{1-x^2}\,dy=0.$ *Ans.* $x\sqrt{1-y^2}+y\sqrt{1-x^2}=c.$

8. $\dfrac{dv}{dt}=-kv^2.$

9. $\dfrac{ds}{dt}=-\cos 2t.$

10. Show that the function

$$y=ce^x$$

is the only function that is unchanged by differentiation.

11. Find a function whose first derivative is equal to the square of the original function.* Interpret geometrically.

12. Determine the family of curves whose slope at any point is equal to the product of the coördinates of the point. Find the curve of this family that passes through the point (0, 1), and trace it.

13. A particle falls under gravity, the resistance of the air being neglected. If the initial velocity is v_0, find v and x in terms of t.

* Cf. footnote, p. 290.

14. Determine the family of curves represented by the equation

$$\frac{dy}{dx} = e^y.$$

15. In Ex. 14, find the curve (*a*) that passes through (0, 0); (*b*) that crosses the line $x = 1$ at an angle of 45°. Trace these curves.

199. Coefficients homogeneous of the same degree. A polynomial in x and y is said to be *homogeneous* if all the terms are of the same degree in x and y. More generally, any function of x and y is said to be *homogeneous of the n-th degree* if, when x and y are replaced by kx and ky respectively, the result is the original function multiplied by k^n. Thus the function

$$x + \sqrt{x^2 - y^2} + y \log \frac{y}{x}$$

is homogeneous of the first degree.

If, in the equation

$$M\,dx + N\,dy = 0,$$

the coefficients M and N are homogeneous functions *of the same degree*, it is easily seen that the equation when solved for y' takes the form

$$y' = f\left(\frac{y}{x}\right):$$

i.e. y' is a function of $\frac{y}{x}$ alone. This suggests the substitution of a new variable v for the ratio $\frac{y}{x}$; *i.e.* the substitution

$$y = vx,\ dy = v\,dx + x\,dv.$$

This substitution always produces a differential equation in v and x in which *the variables are separable.*

Example: Solve the equation

$$(x + y)dx - x\,dy = 0.$$

Substituting

$$y = vx,\ \ dy = v\,dx + x\,dv,$$

we find

$$(x + vx)dx - x(v\,dx + x\,dv) = 0,$$

or

$$dx - x\,dv = 0.$$

The variables can now be separated :

$$\frac{dx}{x} - dv = 0,$$

$$\log x - v + c = 0,$$

or, since

$$v = \frac{y}{x},$$

$$y = x \log x + cx.$$

EXERCISES

Solve the following differential equations.

1. $(x + y)y' + x - y = 0.$ *Ans.* $\arctan \frac{y}{x} + \frac{1}{2} \log (x^2 + y^2) = c.$

2. $(x^2 + y^2)dx - 2\, xy\, dy = 0.$ *Ans.* $x^2 - y^2 = cx.$

3. $(xy - x^2)\frac{dy}{dx} = y^2.$ *Ans.* $y = ce^{\frac{y}{x}}.$

4. $x^3\, dy + y^3\, dx = 0.$

5. $x^3\, dx + y^2\, dy = 0.$

6. $u\, dv - v\, du - \sqrt{u^2 + v^2}\, du = 0.$ *Ans.* $u^2 = c^2 + 2\, cv.$

7. $x\, dx + \sqrt{x^2 + 1}\, dy = 0.$

8. $2\, uv\, du + (v^2 - 3\, u^2)dv = 0.$ *Ans.* $v^3 = c(u^2 - v^2).$

9. $v\frac{dv}{dx} = - x - v.$

10. Show that, if M and N are homogeneous of the same degree, the equation

$$M\, dx + N\, dy = 0$$

can always be put in the form

$$y' = f\left(\frac{y}{x}\right).$$

11. Give a general proof of the fact that, in the problem of § 199, the substitution $y = vx$ always leads to an equation in which the variables are separable.

200. Exact differentials. The differential of a function u of two variables x and y is given by formula (1) of § 164 :

$$(1) \qquad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

The quantity

(2) $$M\,dx + N\,dy$$

is called an *exact differential* if it is precisely the differential of some function u. Thus, the quantity $x\,dy + y\,dx$ is an exact differential, viz. $d(xy)$; on the other hand, the quantity $x\,dy - y\,dx$ is not an exact differential.

If the quantity (2) is an exact differential, it appears by comparison with (1) that there must exist a function u such that

(3) $$\frac{\partial u}{\partial x} = M,$$

(4) $$\frac{\partial u}{\partial y} = N.$$

Differentiating (3) with respect to y and (4) with respect to x, we find

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial N}{\partial x}.$$

Equating values of $\frac{\partial^2 u}{\partial y\,\partial x}$ and $\frac{\partial^2 u}{\partial x\,\partial y}$, by § 163, we get the relation

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

as a *necessary* condition that (2) be an exact differential. It can be shown that this condition is not only necessary but sufficient: *i.e. the quantity* $M\,dx + N\,dy$ *is an exact differential if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

201. Exact differential equations. The equation

(1) $$M\,dx + N\,dy = 0$$

is called an *exact differential equation* if its left member is an exact differential.

Since equation (1), when exact, has the form

$$du = 0,$$

its general solution is evidently

$$u = c.$$

While a general method can be given for finding the function u, we shall consider only cases in which this function is readily found by inspection.

202. Integrating factors. If the equation

(1) $$M\,dx + N\,dy = 0$$

is not exact, its solution can still be put in the form

(2) $$u = c$$

by merely solving for the arbitrary constant. By differentiating (2) we obtain an equation of the first order that is satisfied whenever (1) is satisfied: this equation must therefore have the form

$$v(M\,dx + N\,dy) = 0,$$

where v is in general a function of both x and y. Thus for every differential equation* (1) there exists a function v, called an *integrating factor*, whose introduction renders the equation exact.

It can be shown that every differential equation has not merely one, but infinitely many, integrating factors; nevertheless it is frequently impossible to find one of them. In various cases, some of which will be considered presently, an integrating factor can be found by direct processes; in other cases it is best found by inspection.

It should be noticed that in separating variables, as in § 198, we are really introducing an integrating factor. Thus, in the example of that article, the integrating factor is

$$\frac{1}{y(x^2+1)}.$$

* Assuming the existence of the general solution. Cf. § 195.

Example: Solve the differential equation

$$x\,dy - y\,dx = 0.$$

If we note that the differential of $\frac{y}{x}$ is $\frac{x\,dy - y\,dx}{x^2}$, it appears that $\frac{1}{x^2}$ is an integrating factor in the present instance:

$$\frac{x\,dy - y\,dx}{x^2} = 0,$$

$$\frac{y}{x} = c,$$

$$y = cx.$$

Other integrating factors are $\frac{1}{xy}$ (which merely separates the variables), $\frac{1}{y^2}$, $\frac{1}{x^2 \pm y^2}$.

EXERCISES

1. Solve the above example by using each of the integrating factors there mentioned, and compare the results.

2. Solve Ex. 1, p. 295, by means of an integrating factor.

Solve the following equations.

3. $x\,dy - (x + y)\,dx = 0.$

4. $(2x + 2y)\,dx + (2x + y^2)\,dy = 0.$

5. $(x - y^2)\,dx + 2xy\,dy = 0.$

6. $x\,dy - y\,dx = (x^2 + y^2)\,dx.$

7. $(x + y + 1)\,dx + (x - y)\,dy = 0.$

8. $x\,dx + y\,dy + x\,dy - y\,dx = 0.$

9. $xy' = y + \sqrt{x^2 - y^2}.$

10. $u(u + 2v)\,du + (u^2 - v^2)\,dv = 0.$

11. $v\frac{dv}{ds} = -\frac{1}{s^2}.$

12. $(\sin y + 2x)\,dx + x\cos y\,dy = 0.$

203. The linear equation. A differential equation of the first order is said to be *linear* if it is of the first degree in y and y'. Every such equation may evidently be written in the form

(1) $$y' + Py = Q,$$

where P and Q are functions of x alone. We shall find that the linear equation is of especial importance in the applications.

Before undertaking to solve equation (1), let us consider the special case

(2) $$y' + Py = 0.$$

Here the variables are separable, and the solution may be obtained at once:

$$\frac{dy}{y} + P\,dx = 0,$$

whence

$$\log y + \int P\,dx = c,$$

(3) $$ye^{\int P\,dx} = c'.$$

Now, differentiating (3), we get

$$e^{\int P\,dx}(dy + Py\,dx) = 0,$$

which shows that $e^{\int P\,dx}$ is an integrating factor for equation (2). But since Q is a function of x alone, it follows that $e^{\int P\,dx}$ *is likewise an integrating factor for equation* (1).

Examples: (*a*) Solve the equation

$$dy + 2\,y\,dx = x\,dx.$$

Here

$$P = 2, \quad \int P\,dx = 2\,x, \quad e^{\int P\,dx} = e^{2x}.$$

Introducing the integrating factor e^{2x}, and integrating, we find

$$ye^{2x} = \int xe^{2x}\,dx = \tfrac{1}{2}\,xe^{2x} - \tfrac{1}{4}\,e^{2x} + c,$$

whence

$$y = \tfrac{1}{2}\,x - \tfrac{1}{4} + ce^{-2x}.$$

(*b*) Solve the equation

$$xy' - x^3 - y = 0.$$

Writing this in the form

(4) $$dy - \frac{y}{x}\,dx = x^2 dx,$$

we have

$$P = -\frac{1}{x}, \int P\,dx = -\log x,$$

whence

$$e^{\int P\,dx} = e^{-\log x} = \frac{1}{x},$$

by formula (5) of § 44. Hence, dividing equation (4) by x and integrating, we get

$$\frac{y}{x} = \int x\,dx = \frac{x^2}{2} + c,$$

$$2\,y = x^3 + c'x.$$

204. Equations linear in $f(y)$. The equation

(1) $$f'(y) + Pf(y) = Q,$$

where P and Q are functions of x alone, is evidently *linear in* $f(y)$, and may be solved by the method of the preceding article.

An equation not given directly in the form (1) may sometimes be reduced to that form by a simple transformation. In particular, this is always possible with the equation

$$\frac{dy}{dx} + Py = Qy^n.$$

The process is as shown in the following

Example: Solve the equation

$$y' + \frac{y}{x} = \frac{1}{y^2}.$$

Let us write the equation in the form

$$y^2\,dy + \frac{y^3}{x}\,dx = dx.$$

If we multiply through by 3, so that the first term becomes $d(y^3)$, this equation is seen to be linear in y^3:

$$3\,y^2\,dy + \frac{3\,y^3}{x}\,dx = 3\,dx.$$

Here

$$P = \frac{3}{x}, \quad e^{\int P\,dx} = e^{3\log x} = x^3,$$

whence the solution of the equation is

$$x^3y^3 = 3\int x^3\,dx = \tfrac{3}{4}\,x^4 + c,$$

or

$$y^3 = \tfrac{3}{4}\,x + cx^{-3}.$$

EXERCISES

Solve the following equations.

1. $\dfrac{dy}{dx} + y = x.$ *Ans.* $y = x - 1 + ce^{-x}.$

2. $(x+1)\,dy - 2\,y\,dx = (x+1)^4 dx.$ *Ans.* $2\,y = (x+1)^4 + c(x+1)^2.$

3. $y' - xy = x.$

4. $x\dfrac{dy}{dx} + (1+x)y = e^x.$ *Ans.* $2\,xye^x = e^{2x} + c.$

5. $(x - 2\,y + 5)\,dx + (2\,x + 4)dy = 0.$

6. $y' \sin y + \sin x \cos y = \sin x.$

7. $dy + y(1 - xy^2)\,dx = 0.$ *Ans.* $\dfrac{1}{y^2} = x + \dfrac{1}{2} + ce^{2x}.$

8. $3\,y^2y' - 2\,y^3 = x + 1.$ *Ans.* $y^3 = ce^{2x} - \frac{1}{2}\,x - \frac{3}{4}.$

9. $\dfrac{dv}{dt} = g - kv.$ Solve in two ways.

10. $\dfrac{dy}{dx}\cos x + y \sin x = 1.$

11. $\dfrac{dy}{dx} + y \cos x = \sin 2\,x.$

12. $(1 + x^2)\dfrac{dy}{dx} + y = \arctan x.$

13. $\dfrac{dv}{dt} = -v + \cos t.$

14. $(xy^2 + y)dx - x\,dy = 0.$

15. $3\,y\,dx + (x + xy^2)\,dy = 0.$

16. $y\,dy + (xy^2 - x)\,dx = 0.$ Solve in two ways.

17. $x\,dy + (xe^y - 1)dx = 0.$

205. Geometric applications. Many of the properties of a curve depend not only on the coördinates x, y, but on the slope y' as well. When a curve is defined by such properties, the analytic expression of the given data leads to a relation between x, y, and y' — in other words, to a differential equation of the first order. The general solution of this equation represents the family of "integral curves," as seen in § 197; in many cases additional data are given that enable us to determine the constant of integration.

Example: Find the equation of the curves whose normal always passes through a fixed point.

Let us take the fixed point as origin of coördinates. The slope of the normal at (x, y) is $-\frac{1}{y'}$; but since the normal passes through the origin, its slope is $\frac{y}{x}$. Hence the differential equation of the required curves is

$$-\frac{1}{y'}=\frac{y}{x},$$

or

$$x\,dx+y\,dy=0.$$

Solving, we get

$$x^2+y^2=c.$$

The only curves having the given property are circles with center at the given fixed point.

EXERCISES

1. Find the equation of the curves whose subnormal is constant. Draw the figure. (See Ex. 22, p. 32; cf. also Ex. 8, p. 31.)

2. Find the equation of the curves whose subtangent is constant and equal to a. Draw the figure. *Ans.* $y=ce^{\frac{x}{a}}$.

3. Determine the curves in which the normal at any point is perpendicular to the radius vector (*i.e.* the line joining the point to the origin).

4. Determine the curves in which the perpendicular from the origin upon the tangent is equal to the abscissa of the point of contact.

5. Determine the curves in which the area inclosed between the tangent and the coördinate axes is equal to a^2.

6. Determine the curves such that the area included between the curve, the coördinate axes, and any ordinate is proportional to the ordinate.

Ans. $y = ce^{\frac{x}{a}}$.

7. Find the curve of Ex. 6 that crosses the y-axis (a) at $(0, 2a)$; (b) at an angle of 45°.

MISCELLANEOUS EXERCISES

Solve the following equations.

1. $x^3\,dy - (1 + x^2y)dx = 0$.

2. $v\dfrac{dv}{dx} = 1 - v^2$. Solve in two ways.

3. $dy - \sin x\,dx = 2y\,dx$.

4. $y\,dx + dy = y^2\,dx$. Solve in two ways.

5. $(x - y)dx + (1 - x - 2y)dy = 0$.

6. $dy + x^2y\,dx = 0$. Solve in two ways.

7. $\dfrac{dv}{dt} = a - \cos kt$.

8. $\dfrac{dy}{y} + (\log y - 1)dx = 0$. Solve in two ways.

9. $\dfrac{dv}{dt} = 1 - v + \sin t$.

10. $(x^2 - 4xy)dx + y^2\,dy = 0$.

11. $x\dfrac{dy}{dx} - y = x\sqrt{x^2 + y^2}$.

12. $(1 + x^2)dy - (1 + xy)dx = 0$.

13. $\dfrac{dv}{dt} = a^2 - k^2v^2$.

14. $v\dfrac{dv}{dy} = y - v$.

CHAPTER XXVI

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

I. INTRODUCTION

206. General and particular solutions. Being given a relation between x, y, and n arbitrary constants, say

$$F(x, y, c_1, \cdots, c_n) = 0, \tag{1}$$

let us differentiate this relation n times in succession. The equations thus obtained form with the original equation a set of $n+1$ equations from which the n constants may be eliminated. The result is a differential equation of the n-th order,

$$\Phi(x, y, y', \cdots, y^{(n)}) = 0. \tag{2}$$

Conversely, corresponding to a differential equation of the form (2), there exists in general a relation of the form (1) which satisfies the differential equation. Equation (1) is called the *general solution* of equation (2). Thus *the general solution of a differential equation of the n-th order involves n arbitrary constants.*

It is understood that the general solution contains n *essential* constants: *i.e.* that it cannot be replaced by an equally general form containing a smaller number of constants. Thus the equation

$$y = c_1 e^{x+c_2}$$

appears at first sight to contain two constants, but there is really only one. For, writing the equation in the form

$$y = c_1 e^x \cdot e^{c_2}$$

and setting

$$c_1 e^{c_2} = C,$$

we see that the equation

$$y = Ce^x$$

is equally general.

A *particular solution* is one that is obtained from the general solution by assigning particular values to one or more of the arbitrary constants. Thus a particular solution may contain any number of constants less than the maximum number, n.

For example, it follows from Ex. 18, p. 50, that the equation

$$\frac{d^2x}{dt^2} = -k^2x$$

is satisfied by the equation

$$x = A \cos kt + B \sin kt,$$

where A and B are arbitrary. Since the differential equation is of the second order, the solution here given, containing two constants, is the general solution. Particular solutions are

$$x = A \cos kt,$$
$$x = A(\cos kt + \sin kt),$$
$$x = 2 \sin kt,$$
$$x = 0,$$

etc.

207. Geometric interpretation. Given a differential equation of the second order, and of the first degree in y'',

$$y'' = f(x, y, y'),$$

we may in general assign values at pleasure to x, y, and y', and compute the corresponding value of y''. The equation is satisfied by the coördinates of any point (x, y) moving in any direction, provided its direction is *changing at the proper rate*. Or, since the value of y'', together with the assumed value of y', determines the curvature of the path, we may also say that the differential equation is satisfied by the coördinates of any point moving in any direction, provided its path has always the proper curvature.

The paths of the point (x, y) moving in the manner just described are called, as in § 197, the *integral curves* of the given differential equation. The ordinary equation of the family of integral curves is of course the general solution of the differential equation; since this solution contains two arbitrary constants, or parameters, it follows that the integral curves form a *doubly-infinite system.* The point (x, y) may start from any assumed initial position in any direction; hence through any point in the plane there pass infinitely many integral curves.

The above discussion is readily extended to differential equations of the third and higher orders.

EXERCISES

Find the differential equation whose general solution is as follows.

1. $y = c_1 + c_2 e^{2x}$. *Ans.* $y'' - 2y' = 0$.

2. $y = c_1 e^x + c_2 e^{-x}$.

3. $y = c_1 e^x + c_2 x e^x$.

4. $y = c_1 \sin x + c_2 \cos x$. *Ans.* $y'' + y = 0$.

5. $y = c_1 + c_2 x + x^2$.

6. $y = c_1(1 + x)^2 + c_2$.

Solve the following differential equations, and discuss the nature of the integral curves.

7. $y'' = 0$.

8. $y'' = 1$.

9. $y'' = 6x$.

10. $y'' = y'$.

11. $\dfrac{[1 + y'^2]^{\frac{3}{2}}}{y''} = a$. (Cf. Ex. 2, p. 78.)

12. Solve Ex. 8, (*a*) if the curve touches the line $y = 2x$ at (1, 2); (*b*) if the curve passes through the points (1, 2), (3, 3); (*c*) if the curve passes through (1, 1); (*d*) if the curve intersects the y-axis at right angles. Draw the curve (or several of the curves) in each case.

13. Solve Ex. 9 for each of the cases of Ex. 12.

II. THE LINEAR EQUATION WITH CONSTANT COEFFICIENTS

208. The linear equation. We have already (§ 203) defined the linear equation of the first order as an equation that is of the first degree in y and y'. More generally, a differential equation of the n-th order is said to be linear if it is of the first degree in $y, y', \cdots, y^{(n)}$. Thus every linear differential equation of the n-th order can be written in the form

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = X,$$

where the coefficients $p_1, \cdots, p_n$ and the right member X are functions of x.

In what follows, we shall be concerned entirely with the important special case in which the functions $p_1, \cdots, p_n$ are *constants*:

(1) $$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = X.$$

209. The homogeneous linear equation. A linear differential equation whose right-hand member is 0 is said to be *homogeneous*.* Thus the general form of the homogeneous linear equation with constant coefficients is

(1) $$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.$$

This equation is important not only in itself but because its solution must be determined before that of the non-homogeneous equation (1) of § 208 can be found.

If $y = y_1$ is a particular solution of equation (1), then $y = c_1 y_1$, where c_1 is arbitrary, is also a solution, as appears at once by substitution in (1). Further, if $y = y_2$ is a second particular solution,† then not only $y = c_2 y_2$ but also

$$y = c_1 y_1 + c_2 y_2$$

* That is, it is homogeneous in y and its derivatives. See § 199.

† That is, a solution not of the form $y = c_1 y_1$.

is a solution. Finally, if

$$y = c_1 y_1,$$
$$y = c_2 y_2,$$
$$\vdots \qquad \vdots$$
$$y = c_n y_n$$

are n distinct particular solutions, then

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is a solution, and since it contains n arbitrary constants, it is the general solution.

We proceed to show that the general solution of equation (1) can always be written down, provided a certain algebraic equation of the n-th degree can be solved. The theory will be developed in detail only for the equation of the second order.

210. The characteristic equation. The homogeneous linear equation of the first order, viz.,

$$y' + a_1 y = 0,$$

is evidently satisfied by

$$y = e^{-a_1 x}.$$

This suggests the possibility of *determining* m so that

$$y = e^{mx}$$

will be a solution of the equation

(1) $$y'' + a_1 y' + a_2 y = 0.$$

Substituting in (1) the values

$$y = e^{mx},\ y' = me^{mx},\ y'' = m^2 e^{mx},$$

and bracketing out the factor e^{mx}, we find that the differential equation is satisfied, provided

(2) $$m^2 + a_1 m + a_2 = 0.$$

Equation (2) is called the *characteristic equation** corresponding to (1). Thus

$$y = e^{mx}$$

is a solution of equation (1) *if and only if m is a root of the characteristic equation.*

* Also called the *auxiliary equation.*

211. Distinct roots. If the roots m_1, m_2 of the characteristic equation are distinct, we obtain at once two distinct particular solutions of the differential equation, viz.,

$$y = e^{m_1 x}, \quad y = e^{m_2 x}.$$

Hence, by § 209, the general solution is

(1) $$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example: Solve the differential equation

$$y'' - y' - 2y = 0.$$

The characteristic equation is

$$m^2 - m - 2 = 0,$$

whence $$m = 2 \text{ or } -1.$$

Thus the general solution of the given equation is

$$y = c_1 e^{2x} + c_2 e^{-x}.$$

212. Repeated roots. When the characteristic equation has equal roots, the method of the previous article does not give the general solution. For, if $m_1 = m_2$, equation (1) above becomes

$$\begin{aligned} y &= c_1 e^{m_1 x} + c_2 e^{m_1 x} \\ &= (c_1 + c_2) e^{m_1 x} = c' e^{m_1 x}; \end{aligned}$$

hence the solution contains only a single constant, and is a particular solution.

To find a second particular solution, let us try

$$y = x e^{m_1 x},$$

whence $$y' = e^{m_1 x}(m_1 x + 1),$$

$$y'' = e^{m_1 x}(m_1{}^2 x + 2m_1).$$

Substituting in the differential equation, we find that

$$y = x e^{m_1 x}$$

will be a solution, provided

(1) $$(m_1{}^2 + a_1 m_1 + a_2)x + 2m_1 + a_1 = 0.$$

Now the coefficient of x vanishes because m_1 is a root of the characteristic equation. Further, since $m_1 = m_2$, it follows that

$$m_1 = -\frac{a_1}{2},$$

or $$2\,m_1 + a_1 = 0.$$

Thus (1) holds, and $y = xe^{m_1x}$ is a second particular solution.

Therefore the general solution of the differential equation is

$$y = c_1e^{m_1x} + c_2xe^{m_1x}.$$

213. Complex roots. If the characteristic equation has complex roots $\alpha \pm i\beta$, the general solution takes the form

$$y = c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x}$$

(1) $$= e^{\alpha x}(c_1e^{i\beta x} + c_2e^{-i\beta x}).$$

Up to this point the exponential function has not been defined for imaginary values of the exponent. If, however, we expand e^{ix} *formally* in Maclaurin's series, and compare with the series for $\sin x$ and $\cos x$, we obtain the relation

(2) $$e^{ix} = \cos x + i\sin x.$$

In the theory of functions of a complex variable, this formula is taken as the definition of the imaginary exponential function.

By means of (2), the right member of (1) may be simplified. For,

$$e^{i\beta x} = \cos\beta x + i\sin\beta x,$$
$$e^{-i\beta x} = \cos\beta x - i\sin\beta x.$$

Whence (1) becomes

$$y = e^{\alpha x}[(c_1 + c_2)\cos\beta x + i(c_1 - c_2)\sin\beta x],$$

or, if we place

$$c_1 + c_2 = c_1', \quad i(c_1 - c_2) = c_2'$$

and drop the accents,

(3) $$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

Changing again the meaning of c_1 and c_2, we may write (3) in the form

$$y = c_1e^{\alpha x}\cos(\beta x + c_2),$$

as is easily verified. This form is to be preferred in certain applications.

EXERCISES

Solve the following differential equations.

1. $y'' - 5y' + 6y = 0.$

2. $y'' = y.$

3. $y' - 5y = 0.$

4. $6\frac{d^2y}{dx^2} = \frac{dy}{dx} + y.$

5. $y'' + 3y' = 0.$

6. $\frac{d^2x}{dt^2} = k^2x.$

7. $\frac{d^2x}{dt^2} + k\frac{dx}{dt} = 0.$

8. $y'' + 11y' + 28y = 0.$

9. $y'' - 4y' + 4y = 0.$ *Ans.* $y = c_1e^{2x} + c_2xe^{2x}.$

10. $\frac{d^2\theta}{dt^2} = 0.$

11. $4\frac{d^2r}{dt^2} + 4\frac{dr}{dt} + r = 0.$

12. $9y'' + 12y' + 4y = 0.$

13. $y'' - 5y' = 5y.$

14. $y'' + 2y' + 5y = 0.$ *Ans.* $y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x).$

15. $y'' - 4y' + 6y = 0.$

16. $\frac{d^2x}{dt^2} = -k^2x.$

17. $y'' + 9y = 0.$

18. $y'' + 2y' + y = 0.$

19. $8y'' + 16y' + 9y = 0.$

20. Find the equation of a curve for which $y'' = y$, if it crosses the y-axis at right angles at (0, 1).

21. Find the equation of a curve for which $y'' = -y$, if it touches the line $y = x + 1$ at (0, 1).

22. Determine the curves for which the rate of change of the slope is equal to the slope.

23. In Ex. 22, find the curve that touches the line $y = 2x$ at the origin. *Ans.* $y = 2e^x - 2.$

24. In Ex. 22, find the curves that cross the y-axis at 45°.

25. In Ex. 22, find the curve that passes through (0, 1) and approaches the negative x-axis asymptotically.

26. Show that $e^{\frac{\pi i}{2}} = i$, $e^{\pi i} = -1$, $e^{2\pi i} = 1.$

27. Derive formula (2) of § 213 by comparison of the Maclaurin series for e^{ix}, $\sin x$, and $\cos x$.

28. Show that, if the characteristic equation has equal roots m_1, the equation

$$y'' + a_1y' + a_2y = 0$$

can be reduced to the form $z'' = 0$ by the substitution $y = ze^{m_1x}$, and derive the result of § 212 from this fact.

214. Extension to equations of higher order. The theory of §§ 210–213 is readily extended to equations of higher than the second order. We give the results without proof:

Let there be given a differential equation

(1) $$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.$$

(*a*) If the roots $m_1, m_2, \cdots, m_n$ of the characteristic equation

$$m^n + a_1 m^{n-1} + \cdots + a_n = 0$$

are all distinct, the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

(*b*) Corresponding to a *double root* m_1, the terms in the general solution are

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x};$$

corresponding to a triple root,

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x};$$

etc.

(*c*) A pair of complex roots $\alpha \pm i\beta$ give rise to the terms

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x);$$

a pair of *double roots* $\alpha \pm i\beta$ give rise to the terms

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x + c_3 x \cos \beta x + c_4 x \sin \beta x);$$

etc.

EXERCISES

Solve the following equations.

1. $y''' - 7y' + 6y = 0.$

2. $y''' = 4y'.$

3. $y''' = y'' + 6y'.$

4. $y^{(4)} - 12y'' + 27y = 0.$

5. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} - y = 0.$

Ans. $y = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x}.$

6. $y''' = 0.$

7. $y^{(4)} - 2y'' + y = 0.$

8. $\dfrac{d^3y}{dx^3} = \dfrac{d^2y}{dx^2}.$

9. $4y''' - 3y' + y = 0.$

10. $y''' - 6y'' + 13y' = 0.$ *Ans.* $y = c_1 + e^{3x}(c_2 \cos 2x + c_3 \sin 2x).$

11. $\dfrac{d^3x}{dt^3} = x.$

12. $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = 0.$

13. $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = 0.$

14. $y''' + 3y'' + 3y' + y = 0.$ **15.** $y''' - 2y'' - y' + 2y = 0.$

16. Prove the results of § 214 for the equation of the third order.

215. The non-homogeneous linear equation. Let us consider now the *non-homogeneous* linear equation

(1) $$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = X.$$

In solving this equation, the first step is to write down the general solution

$$Y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

of the *homogeneous* equation obtained from (1) by *making the right member* 0. The quantity Y is called the *complementary function.*

The next step is to obtain, by any means whatever, a *particular integral* of (1),

$$y = \bar{y}.$$

Then the equation

$$y = Y + \bar{y}$$

is a solution of (1), as appears at once by substitution, and since it contains n arbitrary constants, it is the general solution.

Various methods are known for finding the particular solution

$$y = \bar{y}.$$

The method given below, though not entirely general, is usually the best method when it applies, and it is available in nearly all cases that arise in the simpler applications.

We begin with an

Example: Solve the equation

(2) $$y'' - 5y' + 6y = x + e^{2x}.$$

The complementary function, *i.e.* the solution of the equation

$$y'' - 5y' + 6y = 0,$$

is

$$Y = c_1 e^{2x} + c_2 e^{3x}.$$

To obtain a particular integral of (2), proceed as follows: Differentiating twice, we obtain

(3) $$y^{(4)} - 5y''' + 6y'' = 4e^{2x}.$$

Differentiating again, we get

(4) $$y^{(5)} - 5y^{(4)} + 6y''' = 8e^{2x}.$$

Multiplying equation (3) by 2 and subtracting from (4), we get the *homogeneous* equation

(5) $$y^{(5)} - 7y^{(4)} + 16y''' - 12y'' = 0.$$

It is easily seen that the complementary function Y forms part of the solution of this equation; hence two of the roots of the characteristic equation

$$m^5 - 7m^4 + 16m^3 - 12m^2 = 0$$

are $m = 2, 3$. The other roots are 2, 0, 0. Thus the general solution of (5) is

(6) $$y = c_1e^{2x} + c_2e^{3x} + c_3 + c_4x + c_5xe^{2x}.$$

Let us substitute y in the original equation as a trial solution, noting, however, that the terms arising from the complementary function must disappear identically after the substitution, so that it is sufficient to substitute*

(7) $$y = c_3 + c_4x + c_5xe^{2x}.$$

We have

$$y' = c_4 + 2c_5xe^{2x} + c_5e^{2x},$$
$$y'' = 4c_5xe^{2x} + 4c_5e^{2x}.$$

Substituting in (2), we find that (7) will be a particular integral provided the equation

$$4c_5xe^{2x} + 4c_5e^{2x} - 5c_4 - 10c_5xe^{2x}$$
$$- 5c_5e^{2x} + 6c_3 + 6c_4x + 6c_5xe^{2x} = x + e^{2x}$$

holds identically — *i.e.* for all values of x. The terms in xe^{2x} destroy each other. Equating coefficients of the other functions, we find the following:

Coefficients of e^{2x}: $4c_5 - 5c_5 = 1.$

Coefficients of x: $6c_4 = 1.$

Constant terms: $-5c_4 + 6c_3 = 0.$

* That is, we place, temporarily, $c_1 = c_2 = 0$.

This gives

$$c_5 = -1, \quad c_4 = \tfrac{1}{6}, \quad c_3 = \tfrac{5}{36}.$$

Substituting in (6), we get as the general solution of (2)

$$y = c_1 e^{2x} + c_2 e^{3x} + \tfrac{5}{36} + \tfrac{1}{6}x - xe^{2x}.$$

Thus the method consists of the following steps:

(*a*) Write down the complementary function.

(*b*) Differentiate both members of the given equation successively until the right member becomes 0, either directly or by elimination. The original equation is thus replaced by a derived *homogeneous* equation of higher order (equation (5) in the example).

(*c*) Write down, by § 214, the general solution (equation (6) above) of this derived equation. The complementary function will always be a part of this solution, so that certain of the roots of the characteristic equation are known beforehand; these should be removed at once by synthetic division.

(*d*) Of the arbitrary constants occurring in this general solution, those belonging to the complementary function (c_1, c_2 above) will remain arbitrary in the final result; they may therefore be placed temporarily equal to 0, since we are trying to find merely a particular solution of the original equation. The other constants, the so-called *superfluous constants*, are determined by substituting the value of y in the original equation as a trial solution and equating coefficients.

It is clear that the success of the method depends on our ability to reduce the right-hand member to 0 by differentiation and elimination, as in the above example. Hence X and its successive derivatives must contain only a finite number of distinct functions of x. The method therefore applies whenever X contains only constants or terms of the form x^n, e^{ax}, $\sin^n ax$, $\cos^n ax$, or products of these, n being a positive integer.

EXERCISES

1. Check the result of the above example by differentiation.

Solve the following equations.

2. $y'' - 7y' + 12y = x$. *Ans.* $y = c_1e^{3x} + c_2e^{4x} + \frac{1}{12}x + \frac{7}{144}$.

3. $y'' - 5y' + 6y = e^{4x}$. *Ans.* $y = c_1e^{3x} + c_2e^{2x} + \frac{1}{2}e^{4x}$.

4. $y'' + y = \cos 2x$.

5. $y'' - 5y' + 4y = 2x - 3$.

6. $y'' + y' = (1 + x)^2$.

7. $\dfrac{d^2x}{dt^2} = \cos t - x$.

8. $y'' - 5y' + 6y = \cos x - e^{2x}$. *Ans.* $y = c_1e^{2x} + c_2e^{3x} + \frac{1}{10}\cos x - \frac{1}{10}\sin x + xe^{2x}$.

9. $\dfrac{d^2x}{dt^2} + 4x = \sin 3t + t^2$.

Ans. $x = c_1\cos 2t + c_2\sin 2t - \frac{1}{5}\sin 3t + \frac{1}{4}t^2 - \frac{1}{8}$.

10. $y'' - 2y' + y = xe^x$. *Ans.* $y = e^x(c_1 + c_2x + \frac{1}{6}x^3)$.

11. $y'' + y = 1 + 2\cos t$.

12. $\dfrac{d^2y}{dx^2} + y = x\sin x$.

Ans. $y = c_1\cos x + c_2\sin x - \frac{1}{4}x^2\cos x + \frac{1}{4}x\sin x$.

13. $\dfrac{d^2u}{dv^2} = u$.

14. $\dfrac{d^2u}{dv^2} - v = 0$. *Ans.* $u = \frac{1}{6}v^3 + c_1v + c_2$.

15. $y''' - 3y'' + 2y' = 3x - 4$.

16. $y''' - 2y'' + y' = e^x$. *Ans.* $y = c_1 + e^x(c_2 + c_3x + \frac{1}{2}x^2)$.

17. $y' + \dfrac{1}{x}y = e^{2x}$.

18. $y' - y = \dfrac{e^x}{x}$.

19. Prove the statement that the complementary function corresponding to the original equation is always a part of the solution of the derived homogeneous equation.

III. Miscellaneous Equations of the Second Order

216. The equation $y'' = f(x)$. In this section we consider various classes of equations of the second order which can be solved by special devices.

The simplest case is that in which the second derivative is a function of the *independent* variable :

$$y'' = f(x).$$

This equation can be solved directly by two successive integrations. In fact, it is obvious that the equation

$$y^{(n)} = f(x)$$

can be solved by n successive integrations.

217. The equation $y'' = f(y)$. An equation in which the second derivative is a function of the *dependent* variable,

$$y'' = f(y),$$

can always be rendered exact by introducing the integrating factor $2\,y'\,dx$ in the left member, and its equivalent $2\,dy$ in the right member:

$$2\,y'\,y''\,dx = 2\,f(y)\,dy.$$

Integrating, we find

$$y'^2 = 2\int f(y)\,dy + c_1.$$

After extracting the square root of both sides, we have a differential equation of the first order, and of the first degree in y', in which the variables can be separated.

Example: Solve the equation

$$y'' = \frac{1}{y^3}.$$

Multiplying through by $2\,y'\,dx$, we get

$$2\,y'\,y''\,dx = \frac{2\,dy}{y^3},$$

whence

$$y'^2 = -\frac{1}{y^2} + c_1$$

$$= \frac{c_1 y^2 - 1}{y^2},$$

$$y' = \pm\frac{\sqrt{c_1 y^2 - 1}}{y}.$$

Separating variables, we have

$$\frac{y\,dy}{\pm\sqrt{c_1y^2-1}} = dx,$$

whence

$$\pm\sqrt{c_1y^2-1} = c_1x + c_2,$$

$$c_1y^2 - 1 = (c_1x + c_2)^2.$$

218. Dependent variable absent. An equation of the second order in which the *dependent* variable y does not occur is an equation of the first order in y'; it may therefore be solved for y' by the methods of Chapter XXV. The result is of course an equation of the first order in y, which in turn may be solved for y.

The problem of § 216 is evidently a special case of the present one.

Example: Solve the equation

$$(1+x)y'' - y' = 0.$$

Setting

$$y' = v,\ y'' = \frac{dv}{dx},$$

we have

$$(1+x)dv - vdx = 0,$$

or

$$\frac{dv}{v} - \frac{dx}{1+x} = 0.$$

Hence

$$\log v - \log(1+x) = \log c_1,$$

or

$$v = c_1(1+x).$$

Replacing v by y', and integrating again, we find

$$y = \frac{c_1}{2}(1+x)^2 + c_2,$$

or, with c_1 in place of $\frac{c_1}{2}$,

$$y = c_1(1+x)^2 + c_2.$$

219. Independent variable absent. An equation of the second order from which the *independent* variable x is absent may be written as an equation of the first order in the variables y and v by putting

$$\frac{dy}{dx} = v, \quad \frac{d^2y}{dx^2} = v\frac{dv}{dy}.$$

The truth of this last formula is obvious :

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = v\frac{dv}{dy}.$$

It should be noted that the problem of § 217 is merely a special case of the present one.

Example : Solve the equation

$$y'' = yy'.$$

With

$$y' = v, \quad y'' = v\frac{dv}{dy},$$

this becomes

$$v\frac{dv}{dy} = yv,$$

or

$$dv = y\,dy,$$

$$v = \tfrac{1}{2}y^2 + c_1.$$

Whence

$$\frac{2\,dy}{y^2 + 2\,c_1} = dx,$$

and, if $c_1 > 0$,

$$\sqrt{\frac{2}{c_1}}\arctan\frac{y}{\sqrt{2\,c_1}} = x + c_2.$$

This may be simplified by writing $\frac{c_1^{\,2}}{2}$ in place of c_1 :

$$\frac{2}{c_1}\arctan\frac{y}{c_1} = x + c_2.$$

EXERCISES

Solve the following equations.

1. $\frac{d^2y}{dx^2} = x^3.$

2. $t\frac{d^2x}{dt^2} = 1.$

3. $y'' = a^2y.$ *Ans.* $ax = \log(y + \sqrt{y^2 + c_1}) + c_2.$

4. $y'' = \frac{1}{\sqrt{y}}.$ *Ans.* $3x = 2(y^{\frac{1}{2}} + 2c_1)(y^{\frac{1}{2}} - c_1)^{\frac{1}{2}} + c_2.$

5. $\frac{d^2s}{dt^2} = -\frac{1}{t^2}.$

6. $\frac{d^2s}{dt^2} = -\frac{1}{s^2}.$

7. $x^2y'' = y'.$

8. $y^2y'' = y'.$

9. $y'' = 1 - y'.$

10. $y'' = xe^x.$

11. $y'' + yy' = 0.$ *Ans.* $\frac{c_1 + y}{c_1 - y} = c_2e^{c_1x}.$

12. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 0.$ *Ans.* $y = c_1 \log x + c_2.$

13. $\frac{d^2y}{dt^2} = t + 4y.$

14. $t\frac{d^2x}{dt^2} + \frac{dx}{dt} = 1.$ *Ans.* $x = t + c_1 \log t + c_2.$

15. $\frac{y''}{(1 + y'^2)^{\frac{3}{2}}} = \frac{1}{a}.$ *Ans.* $(x - c_1)^2 + (y - c_2)^2 = a^2.$

16. $\frac{d^2x}{dt^2} = -k^2x.$

17. $\frac{d^2x}{dt^2} = -k^2t.$

18. Of the above exercises, which ones can be solved by the methods of section II?

19. Solve Exs. 16, 17 by the methods of section II.

20. Show how to solve the equation

$$y'' + Py' + Qy'^2 = 0,$$

where P and Q are functions of x alone.

CHAPTER XXVII

APPLICATIONS OF DIFFERENTIAL EQUATIONS IN MECHANICS

I. Rectilinear Motion

220. Rectilinear motion. Consider a point P moving in a straight line: for instance, the centroid of a falling body, of the piston of a steam engine, or of a train running on a straight track. The position of the point at any instant is determined by its abscissa $OP = x$, counted from an arbitrarily chosen origin O on the line, a definite sense along the line being selected as positive.

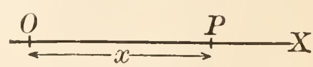


Fig. 101

As the point moves, its abscissa x is a function of the time:

$$x = \phi(t).$$

If this function is known, the motion of the point is completely determined. The velocity v is found as the first derivative $\frac{dx}{dt}$, and the acceleration j as the second derivative $\frac{d^2x}{dt^2}$, of the abscissa x with respect to the time (see § 55).

In most applications, however, it is the converse problem that presents itself. Thus, the velocity may be given as a function of t or x or both, say

$$\frac{dx}{dt} = \psi(t, x),$$

so that in order to determine the position of the point at any time it is necessary to solve this differential equation of the first order. Or, and this is the most common case, the acceleration may be given as a function of t, x, and v (or of any one or two of these), say

(1) $$\frac{d^2x}{dt^2}=f(t, x, v).$$

The abscissa x is found in terms of t by *solving this differential equation of the second order.*

It should be noted that when the acceleration (or the velocity) is given, the motion is not completely determined unless "initial conditions" are also given by means of which the constants of integration can be determined.

221. Motion of a particle under given forces. Suppose the "point" whose motion was discussed in the preceding article is a material particle moving under given forces. If the particle is free to move in any direction, the motion will be rectilinear only if the resultant F of all the applied forces lies in the same straight line with the initial velocity. The product of the mass by the acceleration is equal to the resultant force, by § 187. If we multiply both members of equation (1) above by m, and write $F(t, x, v)$ in place of $mf(t, x, v)$, that equation takes the form

$$m\frac{d^2x}{dt^2}=F(t, x, v).$$

This equation and equation (1) of § 220 are mathematically equivalent, since one is a mere constant multiple of the other. The difference lies in the physical meaning of the quantities involved.

It should be noted that the term "particle" as here used does not mean necessarily a mere mass-point. The "particle" may be a body of any size or shape, provided that all the forces acting may be regarded as applied at a single point, and that the motion of one point determines the motion of the whole mass, as in the case of a rigid body moving without rotation.

222. The equation of motion. The equation

(1) $$m\frac{d^2x}{dt^2} = F(t, x, v),$$

or its equivalent

(2) $$\frac{d^2x}{dt^2} = f(t, x, v),$$

is called the *equation of motion.* It follows from what has been said that the rectilinear motion of a particle is determined by the equation of motion together with the initial conditions.

In each problem there are in general three steps: first, to write the equation of motion; second, to solve this equation, determining the constants of integration in accordance with given initial conditions; third, to interpret the results.

When the forces acting are given, the equation of motion can be written at once: we have only to *equate* $m\frac{d^2x}{dt^2}$ *to the sum of the components of the forces in the direction of motion.*

In the most general case, the equation of motion may be expressed as a differential equation of the second order in x and t by substituting

$$v = \frac{dx}{dt}.$$

Special cases, however, are common. If the force is a function of t only, the method of § 216 evidently applies. If F is a function of t and v, we may use the method of § 218, writing

(3) $$\frac{d^2x}{dt^2} = \frac{dv}{dt}.$$

If F is a function of x and v, the method of § 219 applies: in this case, since

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt},$$

we substitute

$$(4) \qquad \frac{d^2x}{dt^2} = v\frac{dv}{dx}.$$

We shall find that in many cases a variety of methods may be used.

In any problem we may desire to know the position of the particle at any time, the velocity at any time, and the velocity at any position. We should therefore try to obtain three equations, giving * x in terms of t, v in terms of t, and v in terms of x, respectively. The (x, t)-equation is of course obtained by solving the equation of motion (1) (or (2)) as an equation in x and t, and determining the constants. The (v, t)-equation may be found by differentiation of the (x, t)-equation, after which the (v, x)-equation may be obtained (theoretically at least) by eliminating t between the other two. If it is possible to introduce (3) and apply the method of § 218, the (v, t)-equation results directly from the first integration; if formula (4) and § 219 can be used, the (v, x)-equation is obtained directly.

223. Uniformly accelerated motion. A motion is said to be *uniformly accelerated* if the applied force, and hence the acceleration, is constant (cf. § 55). If the constant acceleration be denoted by k, the equation of motion is simply

$$\frac{d^2x}{dt^2} = k.$$

EXERCISES

1. Write the differential equation of *uniform* rectilinear motion (§ 55), and find x in terms of t, v in terms of t, and v in terms of x, if $x = 2$ and $v = 4$ when $t = 0$. Solve the equation of motion in three ways, by the methods of §§ 212, 216, and 219, and obtain the (v, t)-equation and the (v, x)-equation in each of the ways suggested in § 222. Draw the graph of each equation.

* Explicitly if possible.

2. Solve Ex. 1 if $x = 10$ when $t = 5$ and $x = 22$ when $t = 9$. Find the values of x and v when $t = 0$.

3. The velocity of a particle at the time t is

$$v = 6\,t - 5.$$

Find (*a*) the acceleration; (*b*) the space covered in 4 seconds; (*c*) the velocity when $x = 6$ (x being measured from the starting point). Describe the motion in words.

4. The velocity of a particle at the distance x from the starting point is

$$v = \sqrt{x + 10}.$$

Find x in terms of t; also find the acceleration.

5. A particle falls under gravity, all resistances being neglected. Write the equation of motion, taking motion downward as positive, and solve it by three methods. Explain the meaning of the constants of integration.

6. Determine the constants of integration in Ex. 5 if the particle falls from rest, the starting point being taken as origin. Draw the graph of the (x, t)- and (v, t)-equations, noting that the latter is the first derived curve of the former (§ 35).

7. (*a*) Solve Ex. 5 if the initial velocity is 10 ft. per second upward. (*b*) How far and how long does the particle rise? (*c*) Find v and t when the particle is 20 ft. below the starting point.

Ans. (*c*) $v = 37.1$ ft. per second.

8. Solve Ex. 5 if $x = 10$ when $t = 1$ and $x = 100$ when $t = 3$. Does the particle at first move upward or downward? Find the velocity at the end of 1 second. *Ans.* 13 ft. per second.

9. If a stone dropped from a balloon while ascending at the rate of 20 ft. per second reaches the ground in 10 seconds, what was the height of the balloon when the stone was dropped? With what velocity does the stone strike the ground?

10. Solve Ex. 5 if the velocity 2 ft. below the starting point is 23 ft. per second. If the starting point is 500 ft. above the earth's surface, when and with what velocity does the particle reach the earth? *Ans.* $t = 5$ or $6\frac{1}{4}$ seconds.

11. Show that the velocity acquired by a body falling from rest through a height h is

$$v = \sqrt{2\,gh}.$$

Derive the formula in two ways.

12. A body falls 50 ft. in the third second of its motion. Find the initial velocity.

13. A body falls under gravity. Find the distance covered in 6 seconds if at the end of 2 seconds the distance below the starting point is 84 ft.

14. The motion of a railroad train is uniformly accelerated. If when the train is 250 ft. from a station the velocity is 30 ft. per second, when 600 ft. from the station it is 40 ft. per second, find the acceleration, and the velocity when passing the station.

Ans. $v_0 = 20$ ft. per second.

15. A stone is thrown vertically upward from the top of a tower. At the end of 2 seconds it is 400 ft. above the ground, and is still rising, with a velocity of 10 ft. per second. Find the height of the tower. *Ans.* 316 ft.

16. A stone thrown upward from the top of a tower with a velocity of 100 ft. per second reaches the ground with a velocity of 140 ft. per second. Discuss the motion. What is the height of the tower?

Ans. 150 ft.

224. Momentum; impulse. When a particle of mass m is moving with a velocity v, the product mv of the mass by the velocity is called the *momentum* of the particle.

When a particle moves under a *constant* force F from the time t_0 to the time t_1, the product $F(t_1 - t_0)$ of the force by the time during which it acts is called the *impulse* of the force for that time-interval. More generally, if F varies from instant to instant, let us divide the time from t_0 to t_1 into n equal intervals Δt, multiply each Δt by the value of F at the beginning (or any other instant) of the interval, and form the sum of the products thus obtained. The limit of this sum, as Δt approaches 0, is the *impulse* of the variable force F during the interval from t_0 to t_1:

$$I = \lim_{\Delta t \to 0} \sum F \Delta t = \int_{t_0}^{t_1} F\, dt.$$

225. The principle of impulse and momentum. Let us write the equation

$$m \frac{d^2x}{dt^2} = F$$

in the form

$$m\frac{dv}{dt} = F.$$

Multiplying by dt and integrating from the time t_0, when the velocity is v_0, to the time t, when the velocity is v, we find

(1) $$mv - mv_0 = \int_{t_0}^{t} F\,dt.$$

By § 224, the left member of (1) is the change of momentum in the time-interval from t_0 to t_1, the right member is the impulse of the force F. Hence we have the

THEOREM: *If a particle moves in a straight line, the change of momentum in any time-interval is equal to the impulse of the force during that interval.*

This theorem will be referred to as the *principle of impulse and momentum.*

It should be observed that what we have really done here is to find a first integral of the equation of motion by the method of § 218. Since the force F is always either directly or indirectly a function of t, the above theorem is true in general; but in order actually to compute the impulse directly in a given case, the force must of course be given explicitly as a function of t:

$$F = F(t).$$

If the force F is *constant*, equation (1) becomes simply

$$mv - mv_0 = Ft - Ft_0.$$

226. Work. When a particle moves in a straight line under the action of a *constant* force F, the *work* done is defined as the product of the force by the distance passed over:

$$W = Fx.$$

When the force is variable, we proceed as follows: Take the line of motion as x-axis, and suppose the body moves from $x = a$ to $x = b$. Divide the interval into segments Δx, and multiply each segment Δx by the value of F at some point of Δx. The limit of the sum of the products thus obtained is defined as the *work* of the variable force during the motion:

$$W = \lim_{\Delta x \to 0} \sum F\,\Delta x = \int_a^b F\,dx.$$

227. The principle of kinetic energy and work. Let us write the equation

$$m\frac{d^2x}{dt^2} = F$$

in the form

$$mv\frac{dv}{dx} = F.$$

Multiplying by dx and integrating between the x-limits x_0 and x and the corresponding v-limits v_0 and v, we find

$$(1) \qquad \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \int_{x_0}^{x} F\,dx.$$

By § 135, the quantity $\frac{1}{2}mv^2$ is the kinetic energy of the particle, hence the left member of (1) is the change in kinetic energy from x_0 to x. By § 226, the right member is the work done during the motion. Hence we have the

THEOREM: *If a particle moves in a straight line, the change of kinetic energy in any space-interval is equal to the work done by the force in that interval.*

This is the *principle of kinetic energy and work.*

Here we have merely applied to the equation of motion the method of § 219. In order to compute the work directly, the force must of course be given explicitly as a function of x:

$$F = F(x).$$

If the force is *constant*, equation (1) reduces to

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = Fx - Fx_0.$$

EXERCISES

1. Verify the principle of impulse and momentum in Exs. 7, 8, p. 325.

2. Verify the principle of kinetic energy and work in Exs. 7, 8, p. 325.

3. Solve Ex. 14, p. 326, by the principle of kinetic energy and work.

4. Solve Ex. 15, p. 326, by the principles of §§ 225, 227.

5. A ball of mass $5\frac{1}{4}$ oz. strikes a bat with a velocity of $12\frac{1}{2}$ ft. per second, and returns in the same line with a velocity of 32 ft. per second. If the blow lasts $\frac{1}{20}$ second, what force is exerted by the batter? *Ans.* 9 lbs.

6. A ball of mass $5\frac{1}{4}$ oz. moving at 50 ft. per second is caught and brought to rest in a distance of 6 in. What is the average pressure on the hand? *Ans.* 26 lbs.

228. Constrained motion. The motion of a body sometimes depends on other conditions than the given forces. Thus, the piston of a steam engine can move only along the cylinder, a body sliding down an inclined plane cannot fall through the plane, etc. The motion in such cases is said to be *constrained.*

In the case of constrained motion, let the applied force be resolved into components along, and at right angles to, the path. The *component in the direction of motion* is the "effective force"; the motion is due entirely to this component, and hence it is *only this component that appears in the equation of motion.* For example, when a particle slides down a smooth inclined plane, the effective force is the component of gravity parallel to the plane.*

Further, it is evident that, in the definitions and theorems of §§ 224–227, the force F must be taken as *merely the effective component.* The component normal to the path cannot do work, or contribute to a change of momentum.

* The motion is supposed to take place along a "line of greatest slope" — *i.e.* a line at right angles to a horizontal line in the plane.

EXERCISES

1. Write the equation of motion down an inclined plane, and solve it in a variety of ways. Explain the meaning of the constants.

2. Determine the constants in Ex. 1 if the angle of inclination to the horizon is 30°, and the initial velocity is (*a*) 0; (*b*) 10 ft. per second up the plane. In (*b*), how far and how long will the body move up the plane? *Ans.* (*b*) $3\frac{1}{8}$ ft.

3. A bead is strung on a smooth straight wire inclined at 45° to the horizontal. What initial velocity must the bead be given to raise it to a vertical height of 10 ft.?

4. A railroad train is running up a grade of 1 in 200 at the rate of 20 miles per hour when the coupling of the last car breaks. Friction being neglected, (*a*) how far will the car have gone after 2 minutes from the point where the break occurred? (*b*) When will it begin moving down the grade? (*c*) How far will it be behind the train at that moment? (*d*) If the grade extends 1500 ft. below the point where the break occurred, with what velocity will it arrive at the foot of the grade? *Ans.* (*a*) 2368 ft.; (*b*) 3 minutes 3 seconds; (*c*) 2689 ft.; (*d*) 25 miles per hour.

5. Show that it takes a body twice as long to slide down a plane of 30° inclination as it would take to fall through the "height" of the plane.

6. Show that in sliding down a smooth inclined plane a body acquires the same velocity as in falling vertically through the height of the plane.

7. A mass of 12 lbs. rests on a smooth horizontal table. A cord attached to this mass runs over a pulley on the edge of the table; from the cord a mass of 4 lbs. is suspended. Discuss the motion. If the 12 lb. mass is originally 5 ft. from the edge of the table, find when and with what velocity it reaches the edge. Check by the principles of §§ 225, 227.

8. A cord hangs over a vertical pulley and carries equal weights of 10 lbs. at each end. If a 1-lb. weight be added at one end, discuss the motion of the system. Find v when the system has moved 6 ft.

229. Simple harmonic motion. If a point P moves in a circle with constant angular velocity ω, the motion of the projection P_x of P on a diameter of the circle is called *simple harmonic motion.* As P moves in the circle uniformly, P_x *oscillates* from A through O to B and back again.

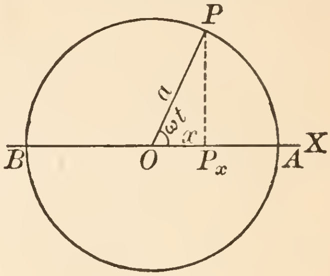


FIG. 102

Suppose P_x is at A at the time $t=0$. Then in time t the angle AOP swept out by the radius vector of P is equal to ωt, hence the distance x of P_x from O is

(1) $$x = a \cos \omega t,$$

where a is the radius of the circle.

If when $t=0$ the point P is not at A, but at some point P' such that the angle AOP' is equal to ϵ, the equation (1) is evidently replaced by

(2) $$x = a \cos(\omega t + \epsilon).$$

The abscissa x is called the displacement of P_x; the maximum displacement a is the *amplitude* of the motion.

The time of completing one whole oscillation from A to B and back is called the *period;* it is evidently equal to the time required for P to make one complete revolution, and is therefore

$$T = \frac{2\pi}{\omega}.$$

The number of oscillations per unit time is called the *frequency;* it is obviously the reciprocal of the period:

$$n = \frac{1}{T} = \frac{\omega}{2\pi}.$$

The angle $\omega t + \epsilon$ is called the *phase-angle*, or simply the *phase*, of the motion.

Differentiating (2), we get the velocity

$$v = \frac{dx}{dt} = -a\omega \sin(\omega t + \epsilon),$$

and the acceleration

(3) $$j = \frac{d^2x}{dt^2} = -a\omega^2 \cos(\omega t + \epsilon).$$

Combining (2) and (3), we may write the acceleration in the form

$$\frac{d^2x}{dt^2} = -\omega^2 x:$$

i.e. the acceleration is proportional to the displacement, and is always directed opposite to it.

230. Attraction proportional to the distance. If a particle moves in a straight line under the action of a force directed toward a fixed point O in the line of motion, and proportional to the distance x from that point, the equation of motion can evidently be written in the form

(1) $$m\frac{d^2x}{dt^2} = -mk^2x,$$

where k is a constant, the minus sign being chosen because the force is always directed opposite to the displacement x. The fixed point toward which the force is directed is called the *center of force.*

Integrating equation (1) by the method of § 213, we get

$$x = c_1 \cos kt + c_2 \sin kt,$$

whence

$$v = \frac{dx}{dt} = -kc_1 \sin kt + kc_2 \cos kt.$$

Take $v = 0$ and $x = a$ when $t = 0$. Then

$$a = c_1, \quad 0 = kc_2,$$

whence $$c_1 = a, \quad c_2 = 0,$$

and finally

(2) $$x = a \cos kt,$$
$$v = -ak \sin kt.$$

Since x has here the same form as in equation (1) of § 229, it follows that a particle moving under the conditions of this article performs simple harmonic oscillations about the center of force O. This is a fact of great importance, as forces directed toward a fixed point and proportional to the distance from that point are of frequent occurrence in nature.

231. Hooke's law. When a spiral steel spring of length $AO = l$ is stretched to a length $AP = l + x$, the tension in the spring, or the force tending to restore it to its natural length, is *proportional to the extension* x. This law, known as *Hooke's law*, is obeyed very closely (provided the extension is not too great) by all so-called *elastic* materials.

Suppose a steel spring of negligible mass is placed on a smooth horizontal table with one end fast at A. Let the natural length of the spring be $AO = l$. A particle of mass m attached to the free end is drawn out to the position P and then released. The only force acting is the tension in the spring, which by Hooke's law is directed toward the position of equilibrium O and is proportional to the distance from O. If the spring offers the same resistance to compression as to extension, it follows from § 230 that the particle performs simple harmonic oscillations about O. The equation of motion is

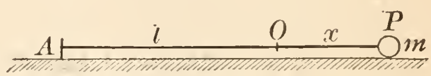


FIG. 103

$$m\frac{d^2x}{dt^2} = -mk^2x.$$

Of course if the resistance to compression is not the same as to extension, a different equation comes into play as soon as the particle passes through O.

EXERCISES

1. In the problem of simple harmonic motion, trace the curves showing x, v, and j as functions of t, remembering that the graph of v is the first derived curve, the graph of j the second derived curve, of the graph of x. Take $a = 1$, $\omega = 2$, $\epsilon = 0$.

2. Show that, if x performs periodic oscillations as in § 229, v and j do likewise. Prove the following from the equations of § 229, and verify by the curves of Ex. 1: the periods of all three are the same; the amplitude of v is ω times that of x, the amplitude of j is ω times that of v; in phase, v differs from x by $\frac{\pi}{2}$ and j differs from v by $\frac{\pi}{2}$.

3. In the problem of § 230, obtain the $(v, \dot{x})$-equation by two methods. *Ans.* $v = \pm k\sqrt{a^2 - x^2}$.

4. A particle has simple harmonic motion. Proceeding from equation (1) of § 230, find x in terms of t, v in terms of t, and v in terms of x, if $v = v_0$ and $x = 0$ when $t = 0$.

5. Show directly from equation (2) of § 230 that the particle performs periodic vibrations about the center, and find the amplitude and the period. Find when and where the velocity is a maximum, and find the magnitude of the maximum velocity.

6. A steel spring offering the same resistance to compression as to extension is placed on a smooth horizontal table with one end fixed. The spring is stretched to a length 6 in. greater than the natural length and then released. Discuss the subsequent motion of a mass attached to the free end. Take $k^2 = 4$. Find the period.

Ans. $T = \pi$ seconds.

7. In Ex. 6, find the work done by the force in a quarter-oscillation. Check by the theorem of § 227.

8. Work Ex. 6 if the steel spring is replaced by a rubber band of natural length 1 ft. *Ans.* $T = 7.14$ seconds.

9. In Exs. 6 and 8, discuss the effect of increasing the constant k^2.

10. Work Ex. 8 if $k^2 = 512$. *Ans.* $T = 0.6$ second.

Fig. 104

11. A rubber band of natural length $AB = l$ is suspended vertically with a weight attached. The effect of the weight is to stretch the band to a length $AO = l + h$. The weight is given a displacement $OP = a$ and then released. Write the equation of motion and solve it completely. Show that the particle performs simple harmonic oscillations about O, provided $a < h$.

12. Solve Ex. 11 if $a > h$.

13. In Ex. 11, find in two ways the work done by the forces as the particle moves from P to O.

14. In Ex. 11, the weight is let fall from a height b above B. Determine the greatest extension of the rubber band.

15. A bead is strung on a smooth straight wire, and is attached by a rubber band of very short natural length to a point in the perpendicular bisector of the wire. Taking the wire as axis of y, show that, if gravity can be neglected, the equation of motion of the bead is approximately

$$\frac{d^2y}{dt^2} = - k^2 y.$$

Discuss the motion completely.

16. A particle is acted upon by a force of *repulsion* from a point O proportional to the distance from O. Neglecting gravity, write the equation of motion and solve it completely, taking $x = 0$ and $v = v_0$ when $t = 0$. Discuss the solution.

17. In Ex. 16, find the work done in the first 10 ft. of the motion. Check by the principle of kinetic energy and work.

18. In Ex. 16, find the impulse of the force during the first second. Check by the principle of impulse and momentum.

19. It is shown in the theory of attraction that the attraction of a spherical mass on a particle within the mass is directed toward the center of the sphere and is proportional to the distance from the center. Discuss the motion of a particle moving in a straight tube through the center of the earth, if the velocity at the surface is 0. Determine the proportionality constant k^2 from the fact that the force at the surface is $-mg$.

20. In Ex. 19, how long does it take the particle to pass through the earth? *Ans.* $42\frac{1}{2}$ minutes.

21. A straight tube is bored through the earth connecting two points of its surface. Show that the equation of motion of a particle sliding in this tube is

$$m\frac{d^2x}{dt^2} = - \frac{mg}{R}x,$$

where R is the radius of the earth and x is the distance of the particle from the midpoint of the tube. Discuss the motion. Show that the time of passing through such a tube is independent of the position of the endpoints.

II. Plane Curvilinear Motion

232. Rotation. In discussing circular motion, it is usually convenient to take as dependent variable the angle θ swept out in the time t.

The problem of *uniformly accelerated* circular motion (§ 58) is closely analogous to that of uniformly accelerated rectilinear motion. The equation of motion is evidently

$$\frac{d^2\theta}{dt^2} = k,$$

where k is the constant angular acceleration.

233. The simple pendulum. A *simple pendulum* is a point swinging in a vertical circle under the acceleration of gravity.

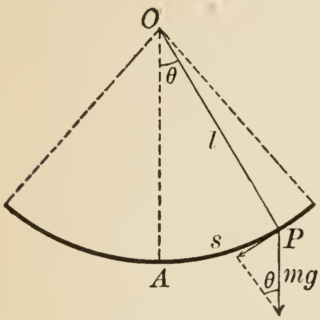


FIG. 105

Let P be a particle of mass m connected to the point O by a cord or rod of length l, and denote by θ the angle between OP and the vertical, by s the length of the arc AP. The effective force acting on P is the component of gravity tangent to the circle; since this is directed opposite to s, it must be given the minus sign. The equation of motion of P is therefore

(1) $$m\frac{d^2s}{dt^2} = -mg\sin\theta.$$

But

$$s = l\theta,$$

so that (1) may be written

(2) $$ml\frac{d^2\theta}{dt^2} = -mg\sin\theta.$$

A first integration of (2) can be performed by the method of § 219; the general solution, however, cannot be expressed in terms of elementary functions. We shall therefore consider only the case in which the oscillations are so small that $\sin\theta$ may be replaced by θ (see § 156), and (2) written in the form

(3) $$ml\frac{d^2\theta}{dt^2} = -mg\theta.$$

This equation shows that for small oscillations the motion is approximately simple harmonic. The remainder of the discussion is left to the student.

EXERCISES

1. Write the equation of *uniform* circular motion, and solve it in a variety of ways, explaining the meaning of the constants. Exhibit the results graphically.

2. Proceed as in Ex. 1 for uniformly accelerated circular motion.

3. A wheel is making 400 R. P. M. when a resistance begins to retard its motion at the rate of 10 radians per second. When will it come to rest? How many revolutions will it make before stopping?

4. Solve equation (3), § 233, taking $\theta = \theta_0$ and $v = 0$ when $t = 0$. For convenience, put $\frac{g}{l} = k^2$.

5. Show that, in the problem of the simple pendulum, the time of one *swing* or *beat* is

$$T = \pi\sqrt{\frac{l}{g}}.$$

6. Find the length of the "seconds pendulum" — *i.e.* a pendulum making one swing per second — at a place where $g = 32.17$.

Ans. 3.2595 ft.

7. Find the angular velocity ω in terms of θ if the oscillations are so large that (2), § 233, must be used.

8. Study the motion of a pendulum making small oscillations, if the resistance of the air is proportional to the velocity.

234. The equations of motion. In the general case of motion in a plane curve, it is convenient to resolve all the applied forces into components parallel to the coördinate axes. The product $m\frac{d^2x}{dt^2}$ of the mass by the x-component of the acceleration (see § 59) is equal to the sum F_x of the x-components of all the forces; similarly for the y-components. We thus have the two *equations of motion :*

$$m\frac{d^2x}{dt^2} = F_x,$$

$$m\frac{d^2y}{dt^2} = F_y.$$

In the most general case, both F_x and F_y are functions of x, y, t, and the velocity-components $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$. We shall, however, confine our attention to the case in which F_x is a function only of x, v_x, and t, and F_y is a function of y, v_y, and t. In this case the two equations of motion may be integrated separately. We thus obtain two equations giving respectively x and y in terms of t; these are parametric equations of the path of the moving point. By the same methods as those already used we find equations giving v_x and v_y in terms of t, and in terms of x and y respectively. The total velocity v may be found by § 57.

235. Projectiles. A simple example of curvilinear motion is furnished by a projectile moving under gravity alone — *i.e.* in a medium whose resistance can be neglected.

Let a particle be projected with an initial velocity v_0 inclined at an angle α to the horizontal. With the starting point as origin and the y-axis positive upward, the initial conditions are

$$x = 0,\ y = 0,\ v_x = v_0 \cos \alpha,\ v_y = v_0 \sin \alpha \text{ when } t = 0.$$

The force of gravity acts vertically downward; there is no horizontal force. Hence the equations of motion are

$$m\frac{d^2x}{dt^2} = 0, \qquad m\frac{d^2y}{dt^2} = -mg.$$

These may be integrated and the constants determined precisely as in our earlier work.

EXERCISES

1. Solve the problem of § 235 completely, finding x, y, v_x, and v_y in terms of t, v_x in terms of x, and v_y in terms of y.

2. In Ex. 1, by eliminating t from the (x, t)- and (y, t)-equations, show that the path is a parabola.

3. Show that, in the ideal case of § 235, where all resistances are negligible, a projectile whose initial velocity is horizontal will strike the ground in the same time as a body let fall from rest from the same height.

4. The *range* of a projectile is the distance from the starting point to the point where it strikes the ground. Show that the range on a horizontal plane is

$$R = \frac{v_0^2}{g} \sin 2\alpha.$$

5. What elevation gives the greatest range on a horizontal plane?

6. The *time of flight* is the time from the starting point until the projectile strikes the ground. Show that on a horizontal plane the time of flight is

$$T = \frac{2v_0}{g} \sin \alpha.$$

7. A stone is thrown horizontally from the top of a tower 400 ft. high, with a velocity of 20 ft. per second. (a) When, (b) where, and (c) with what velocity does it strike the ground?

Ans. (a) 5 seconds; (c) 161.2 ft. per second, at 7° 8′ to the vertical.

8. Find the work done by gravity in Ex. 7.

9. A stone slides down a roof sloping 30° to the horizon, through a distance of 12 ft. If the lower edge of the roof is 50 ft. high, (a) when, (b) where, (c) with what velocity does the stone strike the ground? *Ans.* (b) 25.1 ft. from the building; (c) 59.7 ft. per second, at 16° 30′ to the vertical.

10. A pitcher throws a ball with a speed of 100 ft. per second, the ball leaving his hand horizontally at a height of 5 ft. Show that under the assumptions of § 235 the ball would strike the ground before reaching the batter 60 ft. away.

11. A particle slides on a smooth roof inclined at 45° to the horizontal. If the initial velocity is 10 ft. per second parallel to the edge of the roof and the starting point is 20 ft. above the edge, find when, where, and with what velocity the particle leaves the roof.

12. A particle moves under the action of a force directed toward the origin O and proportional to the distance from O (cf. § 230). If the initial conditions are $x = 10$, $y = 0$, $v_x = 0$, $v_y = 20$ ft. per second, discuss the motion completely. Take $k = 1$.

13. Find the cartesian equation of the path in Ex. 12.

14. In Ex. 12, find the work done in one quarter of the period. Check by the principle of kinetic energy and work.

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