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Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

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S. N. KARP and T. S. CHU

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ABSTRACT

We investigate the diffracted field that arises when an incident electromagnetic surface wave strikes the edge of a right-angle bend at an oblique angle. One face of the bend supports the surface wave and the other face is a perfect conductor. This diffraction problem is related to the scanning surface wave antennas. An analysis of oblique surface waves on an infinite impedance plane is first presented graphically. We treat two cases. In one case the edge of the bend is parallel, and, in the other case, perpendicular, to the perfectly conducting direction of the impedance plane. When the surface wave is incident normally, these special cases reduce to the normal TM and TE surface waves respectively. The mathematical formalism of the first case may be reduced to that of a two dimensional TM problem which has been solved previously. The second case involves two coupled mixed boundary conditions and is exactly analyzed in detail. The effect of the obliquity on the reflection coefficient and the radiation pattern under various conditions will be discussed.

1. Introduction

In this paper we discuss the diffracted field that arises when an incident electromagnetic surface wave strikes the edge of a right-angle bend at an oblique angle. One face of the bend which supports the surface wave is an impedance plane and the other face is a perfector conductor. The edge of the bend may be either parallel or perpendicular to the perfectly conducting direction of the impedance plane. The diffraction of normally incident surface waves by a right-angle bend and its implication for surface wave antennas have been analyzed in detail.[1,2] The case of oblique incidence bears a similar relationship to scanning surface wave antennas.[3] In Section 2 we make a general analysis of oblique surface waves on an infinite impedance plane. The relationships among the propagation constants, the decay constant, and other parameters are graphically presented. When the edge of the bend is parallel to the perfectly conducting direction of the impedance plane, the mathematical formalism may be reduced to that of normal incidence. This analogy will be demonstrated in Section 3. Section 4 contains an analysis of the case when the edge of the bend is perpendicular to the perfectly conducting direction of the impedance plane. This case involves two coupled mixed boundary conditions. The edge conditions, the jump conditions, and the boundary conditions on the conducting face are used to determine the various coefficients. Furthermore, the fields satisfy the divergence condition and are regular in the infinity. In Section 5 simplified expressions for the coefficients of the latter case are obtained for those special incident directions which are nearly normal to the edge and which have $k_S^2 \approx k^2$. In Section 6 we summarize the results which include discussions on the reflected surface waves and the radiation fields under various conditions.

2. Oblique Surface Waves

In this preliminary section, we examine the oblique surface waves on an infinite impedance plane. It is of interest to study the possible directions of propagation. In a sense we are therefore looking at the "geometrical optics" of the propagation of waves in the surfaces. The impedance boundary conditions on the plane $x = 0$ in Fig. 1a may be defined by

$$\text{(2.1)} \qquad \begin{cases} E_T = 0 \\ E_S = (-iX)H_T \end{cases}$$

where T is the perfectly conducting direction, S is the direction perpendicular to T, and $-i$ implies the time dependence $e^{-i\omega t}$. The surface waves in the region $x < 0$ may be characterized by a scalar wave function u

$$\text{(2.2)} \qquad u = e^{ik_S S + ik_T T + \alpha x}$$

where k_S is the propagation constant in the S direction, k_T is the propagation constant in the T direction, and α is a positive decay constant.

$$\text{(2.3)} \qquad \begin{cases} \bar{E} = \text{curl } U_T \\ \bar{H} = \dfrac{1}{i\omega\mu} \text{ curl curl } U_T \end{cases}$$

Then U satisfies the following condition

$$(2.4) \qquad \frac{\partial u}{\partial x} - \alpha U = 0 \qquad \text{where} \quad \alpha = \frac{\omega \epsilon X}{k^2} (k^2 - k_T^2) \; .$$

The wave equation for the surface waves becomes

$$(2.5) \qquad k_S^2 + k_T^2 - \alpha^2 - k^2 = 0 \; .$$

Substituting α into the above equation, we have an equation for k_S^2 and k_T^2:

$$(2.6) \qquad \frac{k^4}{(\omega \epsilon X)^2} \left[k_S^2 + \frac{k^4}{4(\omega \epsilon X)^2} \right] = \left[k_T^2 - k^2 - \frac{k^4}{2(\omega \epsilon X)^2} \right]^2$$

where k, ω, ϵ and X are constants for a definite frequency and a definite impedance plane. This parabolic curve has been drawn in Fig. 2. It should be noticed that only those portions of the curve where both k_S^2 and k_T^2 are positive may indicate the propagation of surface waves. Furthermore, since α must be positive, k_T^2 should be less than k^2 when X is positive, and k_T^2 should be greater than k^2 when X is negative. α has been plotted as linear function of k_T^2 for both positive and negative X. The identity $k^2/-\omega \epsilon X = \omega \mu / -X$ has been used in the indicated coordinates of the points C and D. It is interesting to point out that the points A and B correspond to the normal TM mode surface wave and the points C and D correspond to the normal TE mode surface wave.

3. The parallel case

Now we proceed to discuss the diffraction of oblique surface waves by terminations. If we make a cut along the plane S = 0, and impose a perfectly conducting condition on the half plane $X > 0$, $S = 0$, then the edge of the right-angle bend is parallel to the perfectly conducting direction of the impedance plane. The geometry of the bend is illustrated in Fig. 1b, where the free space is defined by the angular region $0 \leq \varphi \leq \frac{3}{2}\pi$. The boundary conditions on the surfaces may be defined by

$$\left.\begin{aligned} E_T &= 0 \\ E_S &= (iX)H_T \end{aligned}\right\} \qquad X = 0, \quad S < 0 \tag{3.1}$$

and

$$\left.\begin{aligned} E_X &= 0 \\ E_T &= 0 \end{aligned}\right\} \qquad S = 0, \quad X > 0 . \tag{3.2}$$

For this geometry the total electromagnetic field may be divided into one part which contains E_T and a second part without this field component. If there is no E_T in the incident field, the vanishing of E_T on both surfaces implies a general representation of E_T as follows:

$$E_T = \sum_{n=1}^{\infty} C_n \, H^{(1)}_{\frac{2n}{3}} (\kappa r) \sin \frac{2n}{3} \varphi \, e^{ik_T T} \tag{3.3}$$

where $\kappa = \sqrt{k^2 - k_T^2}$. However, all the coefficients C_n in (3.3) must vanish in order to satisfy the edge condition. Therefore all the field components can be determined by a scalar wave function U.

$$\bar{E} = \text{curl } U\hat{T}, \qquad \bar{H} = \frac{1}{i\omega\mu} \text{ curl curl } U\hat{T} \tag{3.4}$$

and U satisfies the wave equation:

$$(3.5) \qquad (\nabla^2 + k^2)U = 0 \qquad \text{or} \qquad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial S^2} + \kappa^2\right] U = 0$$

The incident wave may be written as

$$(3.6) \qquad U_{inc} = e^{ik_T T + ik_S S + \alpha x} \qquad x \leq 0$$

where α is a positive real number. Now the boundary conditions are transformed by (3.4) into the following form:

$$(3.7) \qquad \begin{cases} \dfrac{\partial U}{\partial X} - \alpha U = 0 & x = 0, \quad S \leq 0 \\ \dfrac{\partial U}{\partial S} = 0 & S = 0, \quad X \geq 0 \end{cases}$$

where $\alpha = \frac{\omega \epsilon X}{k^2}(k^2 - k_T^2)$. Since the T-dependence of the field is always $e^{ik_T T}$, the solution for U may be obtained by properly substituting κ for k in the solution of the two dimensional TM case.

The reflected oblique surface wave should be

$$(3.8) \quad U_{ref} = \left[\frac{3i\sqrt{\kappa^2 - \alpha^2}\, I_{\frac{1}{3}} + \sqrt{3}\left(I_{\frac{1}{3}}\alpha - I_{\frac{2}{3}}\kappa\, e^{i\frac{2}{3}\pi}\right)}{3i\sqrt{\kappa^2 + \alpha^2}\, I_{\frac{1}{3}} - \sqrt{3}\left(I_{\frac{1}{3}}\alpha - I_{\frac{2}{3}}\kappa\, e^{i\frac{2}{3}\pi}\right)}\right] e^{-ik_S S + ik_T T + \alpha x}$$

$$\text{where } I_\nu = \int_0^\infty e^{-\alpha\xi} H_\nu^{(1)}(\kappa\xi)d\xi$$

$$(3.9) \qquad = \frac{\left(\sqrt{\kappa^2+\alpha^2}-\alpha\right)^\nu}{\kappa^\nu\sqrt{\kappa^2+\alpha^2}}\left\{1+\frac{i}{\sin\nu\pi}\left[\cos\nu\pi-\frac{\left(\alpha+\sqrt{\kappa^2+\alpha^2}\right)^{2\nu}}{\kappa^{2\nu}}\right]\right\} .$$

The far-field expression for the radiation part of U is

$$(3.10) \qquad U = A_o\sqrt{\frac{2}{\pi\kappa r}}\; e^{i(k_T T+\kappa r-\frac{5}{12}\pi)}\; \frac{\cos\frac{1}{3}\varphi}{i\kappa\cos\varphi-\alpha}$$

where

$$A_o = \frac{-4i\sqrt{\kappa^2+\alpha^2}}{3i\sqrt{\kappa^2+\alpha^2}\, I_{\frac{1}{3}} - \sqrt{3}\left(I_{\frac{1}{3}}\alpha - I_{\frac{2}{3}}\kappa e^{i\frac{2}{3}\pi}\right)} .$$

It is convenient to find the Poynting vector in terms of cylindrical coordinates. If the terms of higher order than $\frac{1}{\sqrt{r}}$ are neglected,

$$(3.11) \qquad \begin{cases} E_\varphi = -i\kappa U , & E_r = 0, \quad E_T = 0 \\ H_\varphi = 0 , & H_r = \frac{i\kappa k_T}{\omega\mu} U, \quad H_T = \frac{\kappa^2}{i\omega\mu} U \end{cases}$$

$$(3.12) \qquad \begin{cases} P_T = \frac{1}{2}\mathrm{Re}(E_r H_\varphi^* - E_\varphi H_r^*) = \frac{1}{2}\,\frac{\kappa^2 k_T}{\omega\mu}\,|U|^2 \\ P_r = \frac{1}{2}\mathrm{Re}(E_\varphi H_T^* - E_T H_\varphi) = \frac{1}{2}\,\frac{\kappa^3}{\omega\mu}\,|U|^2 . \end{cases}$$

The far field power flows in the directions $\tan\theta = \frac{k_T}{\kappa}$, where θ is an angle from the T-axis, with the following density

$$(3.13) \qquad P = (P_T^2 + P_r^2)^{\frac{1}{2}} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\ \kappa^2|U|^2$$

Since the ratio $\frac{\alpha}{\kappa} = \frac{\omega\,\epsilon X}{k}\ \frac{\sqrt{k^2 - k_T^2}}{k}$ in this oblique case, where $\frac{\omega\,\epsilon X}{k}$ is the corresponding ratio in the case of TM normal incidence, this ratio becomes smaller when the obliquity of the incident surface wave increases. It follows from the results of the two dimensional TM problem that for a positive X the greater obliquity gives less reflected surface wave and sharper radiation pattern, and this case reduces to the normal incidence of a TM surface wave when the obliquity vanishes. For a negative X, k_T^2 must be greater k^2 and κ becomes imaginary. Substitution of an imaginary κ into Eq. (3.8) reveals a total reflection of the oblique surface wave in this case of negative X.

4. The perpendicular case

If we make a cut along the plane T = 0, and impose a perfectly conducting condition on the half plane $x > 0$, $T = 0$, then the edge of the right-angle bend is perpendicular to the perfectly conducting direction of the impedance plane. The geometry of the bend is illustrated in Fig. 1C, where the free space is defined by the angular region $0 \leq \varphi \leq \frac{3\pi}{2}$. It is convenient to use the coordinate system xTS' in this case where S' = -S. The boundary conditions on the surfaces may be defined by

$$(4.1) \qquad \left.\begin{aligned} E_T &= 0 \\ E_{S'} &= -(-iX)H_T \end{aligned}\right\} \qquad x = 0, \quad T < 0$$

and

$$(4.2) \qquad \left.\begin{aligned} E_S' &= 0 \\ E_X &= 0 \end{aligned}\right\} \qquad T = 0, \quad x > 0 .$$

The divergence condition $\nabla \cdot \bar{E} = 0$ requires the following additional boundary conditions

$$(4.3) \qquad \frac{\partial E_T}{\partial T} = 0 , \qquad T = 0, \quad x > 0$$

$$(4.4) \qquad \frac{\partial E_x}{\partial x} + ik_{S'}E_{S'} = 0, \quad x = 0, \quad T \leq 0$$

where the S'-dependence has been assumed as $e^{ik_{S'}S'}$. The incident surface wave is given by Eqs. (2.2) and (2.3) where $S = -S'$, $k_S = -k_S'$, and $\alpha = \alpha_1$. The second condition of Eq. (4.1) may be transformed into

$$E_{S'} = -(-iX) \frac{1}{i\omega\mu} \left(\frac{\partial E_X}{\partial S'} - \frac{\partial E_{S'}}{\partial x}\right)$$

or

$$(4.5) \qquad \frac{\partial E_{S'}}{\partial x} - \lambda E_{S'} - ik_{S'}E_x = 0 \qquad x = 0, \quad T \leq 0$$

where $\lambda = \frac{\omega\mu}{-X}$. $E_{S'}$ or E_x may be eliminated between Eqs. (4.4) and (4.5) to yield second order boundary conditions

$$(4.6) \qquad \left(\frac{\partial^2}{\partial X^2} - \lambda \frac{\partial}{\partial X} - k_{S'}^2\right)E_X = 0$$

and

$$(4.7) \qquad \left(\frac{\partial^2}{\partial X^2} - \lambda \frac{\partial}{\partial X} - k_{S'}^2\right)E_{S'} = 0 .$$

However, we may use the following linear combinations:

(4.8) $$\psi_{\frac{1}{2}} = E_{S'} + i a_{\frac{1}{2}} E_x$$

where

(4.9) $$a_{\frac{1}{2}} = \frac{-\lambda \pm \sqrt{\lambda^2 + 4k_{S'}^2}}{2k_{S'}}$$

and

(4.10) $$\begin{cases} E_x = \dfrac{\psi_1 - \psi_2}{i(a_1 - a_2)} \\[2ex] E_{S'} = \dfrac{a_1\psi_2 - a_2\psi_1}{a_1 - a_2} \end{cases} .$$

Then $\psi_{\frac{1}{2}}$ satisfies the following boundary conditions:

(4.11) $$\begin{aligned} \frac{\partial \psi_{\frac{1}{2}}}{\partial x} - \alpha_{\frac{1}{2}} \psi_{\frac{1}{2}} &= 0 \qquad x = 0, \quad T < 0 \\ \psi_{\frac{1}{2}} &= 0 \qquad T = 0, \quad x > 0 \end{aligned}$$

where

(4.12) $$\alpha_{\frac{1}{2}} = \frac{\lambda}{2} \pm \sqrt{\lambda^2 + 4k_{S'}^2} .$$

The incident wave may be written as

(4.13) $$V_{inc} = e^{ik_{S'}S' + ik_T T + \alpha_1 x} \qquad x \leq 0$$

where $k_{S'}^2 + k_T^2 - \alpha_1^2 - k^2 = 0$ and $\alpha_1^2 - \lambda\alpha_1 - k_{S'}^2 = 0$.

The field components are

$$\bar{E}_{inc} = \text{curl}\ V_{inc}\ \hat{T}, \qquad \bar{H}_{inc} = \frac{1}{i\omega\mu}\ \text{curl curl}\ V_{inc}\ \hat{T}\ . \tag{4.14}$$

Let

$$f = \frac{\partial\psi_1}{\partial x} - \alpha_1\psi \tag{4.15}$$

Then f satisfies the wave equation and vanishes on both faces of the wedge; therefore the general representation of f is

$$f = \sum_{n=1}^{\infty} A_n\, H^{(1)}_{\frac{2n}{3}}(\kappa r)\sin\frac{2n}{3}\varphi\, e^{ik_{S'}S'} \tag{4.16}$$

where $\kappa = \sqrt{k^2 - k_{S'}^2}$ and ψ_1 may be obtained from Eq. (4.15).

$$\psi_1 = -e^{\alpha_1 x}\int_x^{\infty} e^{-\alpha_1\xi} f\,d\xi + e^{\alpha_1 x} F(S',T). \tag{4.17}$$

Here the limits of integration are chosen for $\alpha_1 > 0$, and the arbitrary function $F(S',T)$ will be determined by the continuity conditions later. Let

$$g = \frac{\partial\psi_2}{\partial x} - \alpha_2\psi_2\ . \tag{4.18}$$

Then g satisfies the wave equation, and vanishes on both faces of the wedge; therefore the general representation of g is

$$g = \sum_{n=1}^{\infty} B_n\, H^{(1)}_{\frac{2n}{3}}(\kappa r)\sin\frac{2n}{3}\varphi\, e^{ik_{S'}S'} \tag{4.19}$$

and ψ_2 may be obtained from Eq. (4.18).

$$(4.20) \qquad \psi_2 = e^{\alpha_2 x} \int_{-\infty}^{x} e^{-\alpha_2 \xi} g \, d\xi \, .$$

Here the limits of integration are chosen for $\alpha_2 < 0$. The general expression for E_T satisfying Eqs. (4.1) and (4.3) is

$$(4.21) \qquad E_T = \sum_{m=0}^{\infty} C_m \; H^{(1)}_{\frac{2m+1}{3}} (\kappa r) \cos \frac{2m+1}{3} \varphi \; e^{ik_{S'} S'}$$

However, any singularity of the field components at the origin must be less than $\frac{1}{r}$; hence $A_n = B_n = 0$ when $n \geq 3$ and $C_m = 0$ when $m \geq 1$. Now the fields must satisfy the following edge condition: the current density flowing perpendicularly towards the edge must be finite, i.e., $H_{S'}$ is finite at the edge.

$$(4.22) \qquad H_{S'} = \frac{1}{i\omega\mu} \left[\frac{\partial E_T}{\partial x} - \frac{\partial E_X}{\partial T} \right]$$

$$(4.23) \qquad \frac{\partial E_T}{\partial x} = \frac{\partial}{\partial x} \left[C_o \; H^{(1)}_{\frac{1}{3}} (\kappa r) \cos \frac{1}{3} \varphi \right] e^{ik_{S'} S'}$$

$$= \left[\frac{1}{2} \kappa e^{i \frac{2}{3} \pi} C_o \; H^{(1)}_{\frac{2}{3}} (\kappa r) \cos \frac{2}{3} \varphi \; - \frac{1}{2} \kappa C_o H^{(1)}_{\frac{4}{3}} (\kappa r) \cos \frac{4}{3} \varphi \right] e^{ik_{S'} S'}$$

Using the identities in Eqs. (4.29) and (4.32), we obtain

$$(4.24)\qquad \frac{\partial E_X}{\partial T} = \frac{1}{i(a_1-a_2)}\frac{\partial}{\partial T}(\psi_1-\psi_2)$$

$$= e^{ik_{S'}S'}\Bigg\{-\frac{A_1}{i(a_1-a_2)}H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi + \frac{A_1\alpha_1}{i(a_1-a_2)}e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1\xi}H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi\, d\xi$$

$$-\frac{\kappa e^{i\frac{\pi}{3}}}{i(a_1-a_2)}A_1 e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1\xi}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{\varphi}{3}\,d\xi - \frac{A_2}{i(a_1-a_2)}H^{(1)}_{\frac{4}{3}}(\kappa r)\cos\frac{4}{3}\varphi$$

$$+\frac{A_2\alpha_1}{i(a_1-a_2)}\frac{2}{\kappa}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{1}{3}\varphi - \frac{A_2\alpha_1^2}{i(a_1-a_2)}\frac{2}{\kappa}e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1\xi}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{\varphi}{3}\,d\xi$$

$$+\frac{A_2\alpha_1 e^{i\frac{2\pi}{3}}}{i(a_1-a_2)}e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1\xi}H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi\,d\xi - \frac{\kappa A_2}{i(a_1-a_2)}e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1\xi}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{\varphi}{3}\,d$$

$$+\frac{B_1}{i(a_1-a_2)}H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi + \frac{B_1\alpha_2}{i(a_1-a_2)}e^{\alpha_2 x}\int_{-\infty}^x e^{-\alpha_2\xi}H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi\,d\xi$$

$$-\frac{\kappa B_1 e^{i\frac{\pi}{3}}}{i(a_1-a_2)}e^{\alpha_2 x}\int_{-\infty}^x e^{-\alpha_2\xi}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{1}{3}\varphi\,d\xi + \frac{B_2}{i(a_1-a_2)}H^{(1)}_{\frac{4}{3}}(\kappa r)\cos\frac{4}{3}\varphi$$

$$-\frac{B_2\alpha_2}{i(a_1-a_2)}\frac{2}{\kappa}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{1}{3}\varphi - \frac{B_2\alpha_2^2}{i(a_1-a_2)}\frac{2}{\kappa}e^{\alpha_2 x}\int_{-\infty}^x e^{-\alpha_2\xi}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{\varphi}{3}\,d\xi$$

$$+\frac{B_2\alpha_2 e^{i\frac{2}{3}\pi}}{i(a_1-a_2)}e^{\alpha_2 x}\int_{-\infty}^x e^{-\alpha_2\xi}H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi\,d\xi - \frac{\kappa B_2}{i(a_1-a_2)}e^{\alpha_2 x}\int_{-\infty}^x e^{-\alpha_2\xi}H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{\varphi}{3}\,d\xi\Bigg\}$$

Substituting Eqs. (4.23) and (4.24) into Eq. (4.22) and an examination of the singularities reveal the following conditions:

$$(4.25)\qquad \frac{1}{i(a_1-a_2)}(A_1-B_1) + \frac{1}{2}\kappa C_o e^{i\frac{2}{3}\pi} = 0$$

$$(4.26) \qquad \frac{1}{i(a_1 - a_2)}\,(A_2 - B_2) \;-\; \frac{1}{2}\,\kappa C_o \;=\; 0$$

$$(4.27) \qquad A_2\alpha_1 - B_2\alpha_2 \;=\; 0$$

A check of the divergence condition $\nabla\cdot\bar{E} = 0$ shows the necessity of the conditions (4.25) and (4.26). These two conditions also follow from the regularity requirement of the fields in the infinity.

Since the incident field of ψ_1 is $(\alpha_1 + k_{S'}a_1)e^{\alpha_1 x + ik_T T + ik_{S'}S'}$, and the reflected surface wave is of the form $R(\alpha_1 + k_{S'}a_1)e^{\alpha_1 x - ik_T T + ik_{S'}S'}$, we propose the following version of Eq. (4.17)

$$(4.28) \quad \psi_1 = -e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1 \xi}\left[A_1 H^{(1)}_{\frac{2}{3}}(\kappa r)\sin\frac{2}{3}\,\varphi + A_2 H^{(1)}_{\frac{4}{3}}(\kappa r)\sin\frac{4}{3}\,\varphi\right] e^{ik_{S'}S'}\,d\xi$$
$$+ \begin{cases} (\alpha_1 + k_{S'}a_1)e^{\alpha_1 x + ik_T T + ik_{S'}S'} + R(\alpha_1 + k_{S'}a_1)e^{\alpha_1 x - ik_T T + ik_{S'}S'} & T < \\ 0 & T > \end{cases}$$

Using

$$(4.29) \quad \frac{\partial}{\partial T}\left[H_\nu^{(1)}(\kappa r)\sin\nu\varphi\right] + \frac{\partial}{\partial x}\left[H_\nu^{(1)}(\kappa r)\cos\nu\varphi\right] = \kappa H_{\nu-1}^{(1)}(\kappa r)\cos(\nu - 1)\varphi\ ,$$

we have

$$(4.30)\quad \frac{\partial\psi_1}{\partial T} = -A_1 H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi\, e^{ik_{S'}S'} - A_2 H^{(1)}_{\frac{4}{3}}(\kappa r)\cos\frac{4}{3}\varphi\, e^{ik_{S'}S'}$$

$$+ A_1\alpha_1 e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1\xi} H^{(1)}_{\frac{2}{3}}(\kappa r)\cos\frac{2}{3}\varphi\; e^{ik_{S'}S'}\,d\xi$$

$$+ A_2\alpha_1 e^{\alpha_1 x}\int_x^\infty e^{\alpha_1\xi} H^{(1)}_{\frac{4}{3}}(\kappa r)\cos\frac{4}{3}\varphi\; e^{ik_{S'}S'}\,d\xi$$

$$- e^{\alpha_1 x}\int_x^\infty e^{-\alpha_1\xi}\left[A_1\kappa e^{i\frac{\pi}{3}} H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{\varphi}{3} + A_2\kappa H^{(1)}_{\frac{1}{3}}(\kappa r)\cos\frac{\varphi}{3}\right] e^{ik_{S'}S'}\,d\xi$$

$$+ \begin{cases} ik_T(\alpha_1 + k_{S'}a_1)\, e^{\alpha_1 x + ik_T T + ik_{S'}S'} - ik_T(\alpha_1 + k_{S'}a_1) e^{\alpha_1 x - ik_T T + ik_{S'}S'} & y < 0 \\ 0 & y > 0 \end{cases}$$

Imposing the continuity conditions

$$(4.31)\quad \begin{cases} \psi_1(x,+0) - \psi_1(x,-0) = 0 \\ \dfrac{\partial\psi_1}{\partial T}(x,+0) - \dfrac{\partial\psi_1}{\partial T}(x,-0) = 0 \end{cases} \qquad T = 0,\ x \leq 0$$

and using the identities

$$(4.32)\quad \begin{cases} H^{(1)}_{\nu+1}(\kappa r)\cos(\nu+1)\varphi - H^{(1)}_{\nu-1}(\kappa r)\cos(\nu-1)\varphi = -\frac{2}{\kappa}\frac{\partial}{\partial x}\left[H^{(1)}_{\nu}(\kappa r)\cos\nu\varphi\right] \\ H^{(1)}_{\nu+1}(\kappa r)\sin(\nu+1)\varphi - H^{(1)}_{\nu-1}(\kappa r)\sin(\nu-1)\varphi = -\frac{2}{\kappa}\frac{\partial}{\partial x}\left[H^{(1)}_{\nu}(\kappa r)\sin\nu\varphi\right] \end{cases}$$

yield the following two conditions

$$(4.33)\quad \begin{cases} -A_1(-\frac{\sqrt{3}}{2})I_{\frac{2}{3}} + A_2\frac{2\alpha_1}{\kappa}(\frac{\sqrt{3}}{2})I_{\frac{1}{3}} + A_2 e^{i\frac{2}{3}\pi}(-\frac{\sqrt{3}}{2})I_{\frac{2}{3}} + (\alpha_1 + k_{S'}a_1) + R(\alpha_1 + k_{S'}a_1) = 0 \\ \frac{3}{2}A_1\alpha_1 I_{\frac{2}{3}} - \frac{3}{2}A_2\frac{2\alpha_1^2}{\kappa} I_{\frac{1}{3}} + \frac{3}{2}A_2\alpha_1 e^{i\frac{2}{3}\pi} I_{\frac{2}{3}} - \frac{3}{2}A\,\kappa e^{i\frac{\pi}{3}} I_{\frac{1}{3}} - \frac{3}{2}A_2\kappa I_{\frac{1}{3}} - ik_T(\alpha_1 + k_{S'}a_1) \\ \qquad + ik_T R(\alpha_1 + k_{S'}a_1) = 0 \end{cases}$$

where I_ν is defined in Eq. (3.9) with $\alpha = \alpha_1$. Since $A_1 = DA_2$ [see Eq. (4.40)], we may solve Eqs. (4.33) for A_2 and R.

$$(4.34)\quad A_2 = \frac{-i4k_T(\alpha_1 + k_{S'}a_1)}{\sqrt{3}\left[DI_{\frac{2}{3}} + 2\frac{\alpha_1}{\kappa}I_{\frac{1}{3}} - e^{i\frac{2}{3}\pi}I_{\frac{2}{3}}\right]i\beta \quad 3\left[D\alpha_1 I_{\frac{2}{3}} - D\kappa e^{i\frac{\pi}{3}}I_{\frac{1}{3}} - \frac{2}{\kappa}\alpha_1^2 I_{\frac{1}{3}} + \alpha_1 e^{i\frac{2}{3}\pi}I_{\frac{2}{3}} - \kappa I_{\frac{1}{3}}\right]}$$

$$(4.35)\quad R = \frac{\sqrt{3}\left[DI_{\frac{2}{3}} + 2\frac{\alpha_1}{\kappa}I_{\frac{1}{3}} - e^{i\frac{2}{3}\pi}I_{\frac{2}{3}}\right]ik_T + 3\left[D\alpha_1 I_{\frac{2}{3}} - D\kappa e^{i\frac{\pi}{3}}I_{\frac{1}{3}} - \frac{2}{\kappa}\alpha_1^2 I_{\frac{1}{3}} + \alpha_1 e^{i\frac{2}{3}\pi}I_{\frac{2}{3}} - \kappa I_{\frac{1}{3}}\right]}{\sqrt{3}\left[DI_{\frac{2}{3}} + 2\frac{\alpha_1}{\kappa}I_{\frac{1}{3}} - e^{i\frac{2}{3}\pi}I_{\frac{2}{3}}\right]ik_T - 3\left[D\alpha_1 I_{\frac{2}{3}} - D\kappa e^{i\frac{\pi}{3}}I_{\frac{1}{3}} - \frac{2}{\kappa}\alpha_1^2 I_{\frac{1}{3}} + \alpha_1 e^{i\frac{2}{3}\pi}I_{\frac{2}{3}} - \kappa I_{\frac{1}{3}}\right]} .$$

Since the incident field of ψ_2 is identically zero, no reflected surface wave will arise from this part of the fields. An examination of Eq. (4.20) shows that ψ_2 and its normal derivative are continuous across the surface T = 0, x < 0; however, we must impose the vanishing of ψ_2 over the surface T = 0, x $\geq$ 0, i.e.,

$$(4.36) \qquad \psi_2 = e^{\alpha_2 x} \int_{-\infty}^{x} e^{-\alpha_2 \xi} \left[B_1 H^{(1)}_{\frac{2}{3}}(\kappa r) \sin \tfrac{2}{3}\varphi + B_2 H^{(1)}_{\frac{4}{3}}(\kappa r) \sin \tfrac{4}{3}\varphi \right] e^{ik_{S'}S'} d\xi = 0$$

$$T = 0, \; x > 0 \; .$$

Using Eq. (4.32), we obtain

$$(4.37) \qquad \psi_2 = e^{ik_{S'}S'} \Bigg\{ e^{\alpha_2 x} \int_{-\infty}^{x} e^{-\alpha_2 \xi} B_1 H^{(1)}_{\frac{2}{3}}(\kappa r) \sin \tfrac{2}{3}\varphi \, d\xi - \frac{2B_2}{\kappa} H^{(1)}_{\frac{1}{3}}(\kappa r) \sin \frac{\varphi}{3}$$

$$- \frac{2B_2\alpha_2}{\kappa} e^{\alpha_2 x} \int_{-\infty}^{x} e^{-\alpha_2 \xi} H^{(1)}_{\frac{1}{3}}(\kappa r) \sin \frac{\varphi}{3} \, d\xi - B_2 e^{i\frac{2}{3}\pi} e^{\alpha_2 x} \int_{-\infty}^{x} e^{-\alpha_2 \xi} H^{(1)}_{\frac{2}{3}}(\kappa r) \sin \tfrac{2}{3}\varphi \, d\xi \Bigg\} .$$

Imposing the condition (4.36) yields

$$(4.38) \quad B_1 e^{\alpha_2 x} \int_{-\infty}^{0} e^{-\alpha_2 \xi} H^{(1)}_{\frac{2}{3}}(\kappa|\xi|) \sin \tfrac{2}{3}\pi \, d\xi - \frac{2B_2\alpha_2}{\kappa} e^{\alpha_2 x} \int_{-\infty}^{0} e^{-\alpha_2 \xi} H^{(1)}_{\frac{1}{3}}(\kappa|\xi|) \sin \frac{\pi}{3} \, d\xi$$

$$- B_2 e^{i\frac{2}{3}\pi} e^{\alpha_2 x} \int_{-\infty}^{0} e^{-\alpha_2 \xi} H^{(1)}_{\frac{2}{3}}(\kappa|\xi|) \sin \tfrac{2}{3}\pi \, d\xi = 0 .$$

It follows that

$$(4.39) \quad \frac{B_1}{B_2} = \frac{2\frac{\alpha_2}{\kappa} \int_{0}^{\infty} e^{\alpha_2 \xi} H^{(1)}_{\frac{1}{3}}(\kappa\xi) d\xi + e^{i\frac{2}{3}\pi} \int_{0}^{\infty} e^{\alpha_2 \xi} H^{(1)}_{\frac{2}{3}}(\kappa\xi) d\xi}{\int_{0}^{\infty} e^{\alpha_2 \xi} H^{(1)}_{\frac{2}{3}}(\kappa\xi) d\xi} .$$

The ratio A_1/A_2 which is needed in Eqs. (4.34) and (4.35) may be found from Eqs. (4.25), (4.26), (4.27) and (4.39):

$$(4.40)\quad \frac{A_1}{A_2} = \frac{\left[2\frac{\alpha_1}{\kappa}\int_0^\infty e^{\alpha_2\xi} H_{\frac{1}{3}}^{(1)}(\kappa\xi)d\xi + \frac{\alpha_1}{\alpha_2}\, e^{i\frac{2}{3}\pi}\int_0^\infty e^{\alpha_2\xi} H_{\frac{2}{3}}^{(1)}(\kappa\xi)d\xi\right]}{\int_0^\infty e^{\alpha_2\xi} H_{\frac{2}{3}}^{(1)}(\kappa\xi)d\xi} - \left[1 - \frac{\alpha_1}{\alpha_2}\right] e^{i\frac{2}{3}\pi} .$$

Now the explicit expressions for the six coefficients A_2, R, A_1, B_2, B_1, and C_o have been obtained in the six equations, (4.34), (4.35), (4.40), (4.27), (4.38) and (4.25) respectively. This observation completes the formal solution of this case.

It is of interest to determine the far-zone forms of the radiated field. We first w ite down the asymptotic forms of the functions f and g.

$$(4.41)\quad f \cong e^{ik_{S'}S'}\left[A_1\sqrt{\frac{2}{\pi\kappa r}}\, e^{i(\kappa r - \frac{7\pi}{12})}\sin\frac{2}{3}\varphi + A_2\sqrt{\frac{2}{\pi\kappa r}}\, e^{i(\kappa r - \frac{11\pi}{12})}\sin\frac{4}{3}\varphi\right]$$

$$(4.42)\quad g \cong e^{ik_{S'}S'}\left[B_1\sqrt{\frac{2}{\pi\kappa r}}\, e^{i(\kappa r - \frac{7\pi}{12})}\sin\frac{2}{3}\varphi + B_2\sqrt{\frac{2}{\pi\kappa r}}\, e^{i(\kappa r - \frac{11\pi}{12})}\sin\frac{4}{3}\varphi\right] .$$

The far-zone forms of ψ_1 and ψ_2 are expected to be

$$(4.43)\quad \psi_{\substack{1\\2}} = \frac{M_{\substack{1\\2}}(\varphi)}{\sqrt{r}}\, e^{i\kappa r + ik_{S'}S'} .$$

Now

$$(4.44)\qquad \frac{\partial}{\partial x} = \cos\varphi\,\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\,\frac{\partial}{\partial r} \approx \cos\varphi\,\frac{\partial}{\partial r}\ .$$

Substituting Eqs. (4.41), (4.43) and (4.44) into Eq. (4.15) gives

$$(4.45)\qquad \psi_1 = \frac{1}{i\kappa\ \cos\ \varphi - \alpha_1}\sqrt{\frac{2}{\pi\kappa r}}\ e^{i\kappa r + ik_{S'}S'}\left[A_1\sin\frac{2}{3}\varphi\ e^{-i\frac{7\pi}{12}} + A_2\sin\frac{4}{3}\varphi\, e^{-i\frac{11\pi}{12}}\right].$$

Substituting Eqs. (4.42), (4.43) and (4.44) into Eq. (4.18) gives

$$(4.46)\qquad \psi_2 = \frac{1}{i\kappa\ \cos\ \varphi - \alpha_2}\sqrt{\frac{2}{\pi\kappa r}}\ e^{i\kappa r + ik_{S'}S'}\left[B_2\sin\frac{2}{3}\varphi\ e^{-i\frac{7\pi}{12}} + B_2\sin\frac{4}{3}\varphi\, e^{-i\frac{11\pi}{12}}\right].$$

Then substituting the above two expressions into Eq. (4.10) yields

$$(4.47)\quad E_x = \frac{1}{i(a_1-a_2)}\sqrt{\frac{2}{\pi\kappa r}}\ e^{i(\kappa r - \frac{7\pi}{12} + k_{S'}S')}\left\{\frac{1}{i\kappa\ \cos\varphi - \alpha_1}\left[A_1\sin\frac{2}{3}\varphi + A_2\sin\frac{4}{3}\varphi\, e^{-i\frac{\pi}{3}}\right]\right.$$

$$\left. + \frac{1}{i\kappa\ \cos\ \varphi - \alpha_2}\left[B_1\sin\frac{2}{3}\varphi + B_2\sin\frac{4}{3}\varphi\, e^{-i\frac{\pi}{3}}\right]\right\}$$

$$(4.48)\quad E_{S'} = \frac{1}{a_1-a_2}\sqrt{\frac{2}{\pi\kappa r}}\ e^{i(\kappa r - \frac{7\pi}{12} + k_{S'}S')}\left\{\frac{a_1}{i\kappa\ \cos\ \varphi - \alpha_2}\left[B_1\sin\frac{2}{3}\varphi + B_2\sin\frac{4}{3}\varphi\, e^{-i\frac{\pi}{3}}\right]\right.$$

$$\left. - \frac{a_2}{i\kappa\ \cos\varphi - \alpha_1}\left[A_1\sin\frac{2}{3}\varphi + A_2\sin\frac{4}{3}\varphi\, e^{-i\frac{\pi}{3}}\right]\right\}.$$

The far-zone form of E_T may be obtained directly from Eq. (4.21).

(4.49) $$E_T = C_o \sqrt{\frac{2}{\pi\kappa r}}\; e^{i(\kappa r - \frac{5\pi}{12} + k_{S'}S')} \cos\frac{\varphi}{3} \; .$$

Using the following relations for the far-zone cylindrical components:

(4.50) $$E_r = E_x \cos\varphi + E_T \sin\varphi$$

(4.51) $$E_\varphi = -E_x \sin\varphi + E_T \cos\varphi$$

(4.52) $$\bar{H} = \frac{1}{i\omega\mu} \nabla \times \bar{E} = \frac{1}{\omega\mu}\left[-k_{S'}E_\varphi\hat{r} + (k_{S'}E_r - \kappa E_{S'})\hat{\varphi} + \kappa E_\varphi \hat{S}'\right] \; .$$

We may obtain the power flow densities:

(4.53) $$P_r = \frac{1}{2}\,\mathrm{Re}\,\left[E_\varphi H^*_{S'} - E_{S'}H^*_\varphi\right]$$

$$= \frac{\kappa}{2\omega\mu}\left[|E_\varphi|^2 + \left(1 + \frac{k^2_{S'}}{\kappa^2}\right)|E_{S'}|^2\right]$$

(4.54) $$P_{S'} = \frac{1}{2}\,\mathrm{Re}\,\left[E_r H^*_\varphi - E_\varphi H^*_r\right]$$

$$= \frac{k_{S'}}{2\omega\mu}\left[|E_\varphi|^2 + \left(1 + \frac{k^2_{S'}}{\kappa^2}\right)|E_{S'}|^2\right]$$

The radiated far-field power flows in the directions $\tan\theta = \frac{k_{S'}}{\kappa}$, where θ is an angle from the S'-axis with the following density:

(4.55) $$P = \sqrt{P_r^2 + P_{S'}^2} = \frac{k}{2\omega\mu}\left[|E_\varphi|^2 + \left(1 + \frac{k^2_{S'}}{\kappa^2}\right)|E_{S'}|^2\right]$$

5. Particular incidences for the Perpendicular Case

It is of interest to find simplified expressions for the coefficients in those special incident directions which are nearly normal to the edge end which correspond to $k_{S'} \approx k$.

Let us make the substitutions

$$\sin \Theta_{S_2} = \frac{1}{\kappa}\sqrt{\alpha_2^2 + \kappa^2} \tag{5.1}$$

$$\cos \Theta_{S_2} = i\,\frac{-\alpha_2}{\kappa} \tag{5.2}$$

where α_2 is negative using Eqs. (3.9) and (4.39), we find

$$\int_0^\infty e^{\alpha_2 \xi} H_\nu^{(1)}(\kappa\xi)d\xi = \frac{2e^{-i(\frac{\nu\pi}{2})}}{\kappa \sin\nu\pi} \; \frac{\sin \nu\,\Theta_{S_2}}{\sin \Theta_{S_2}} \tag{5.3}$$

$$\frac{B_1}{B_2} = 2\cos\frac{2}{3}\Theta_{S_2}\; e^{i\frac{2}{3}\pi} \; . \tag{5.4}$$

We first consider the incident direction which is nearly normal to the edge,

$$\frac{\alpha_1}{k} = \frac{\lambda}{2k} + \frac{1}{2k}\sqrt{\lambda^2 + 4k_{S'}^2} \approx \frac{\lambda}{k}\left(1 + \frac{k_{S'}^2}{\lambda^2}\right) = \gamma(1 + \frac{\delta^2}{\gamma^2}) \tag{5.5}$$

$$\frac{\alpha_2}{k} = \frac{\lambda}{2k} - \frac{1}{2k}\sqrt{\lambda^2 + 4k_{S'}^2} \approx \frac{\lambda}{k}\left(- \frac{k_{S'}^2}{\lambda^2}\right) = \gamma(- \frac{\delta^2}{\gamma^2}) \; , \tag{5.6}$$

where $\delta = \frac{k_{S'}}{k}$, $\gamma = \frac{\lambda}{k}$, and $\frac{\delta}{\gamma}$ is small. Now $-\frac{\alpha_2}{\kappa}$ is small, $\theta_{S_2} \to \frac{\pi}{2} + i\frac{\alpha_2}{\kappa}$,

where $\kappa \approx k(1 - \frac{\delta^2}{2})$ Eq. (3.40) becomes

$$(5.7) \qquad D = \frac{A_1}{A_2} \to e^{i\frac{2}{3}\pi}\left[\frac{2\alpha_1}{\alpha_2} - 1 - i\frac{2\alpha_1}{\sqrt{3}\,\kappa}\right]$$

and

$$(5.7A) \qquad \frac{B_1}{B_2} \to e^{i\frac{2}{3}\pi}\left[1 - i\frac{2\alpha_2}{\sqrt{3}\,\kappa}\right]$$

$$(5.7B) \qquad \frac{A_1}{B_2} \to e^{i\frac{2}{3}\pi}\left[2 - \frac{\alpha_2}{\alpha_1} - i\frac{2\alpha_2}{\kappa}\right]$$

$$(5.7C) \qquad \frac{A_2}{B_2} = \frac{\alpha_2}{\alpha_1} \quad .$$

Substituting (5.7) into Eqs. (4.34) and (4.35), and imposing the condition that $\gamma(= \frac{\lambda}{k}) \to 0$, we obtain

$$(5.8) \qquad A_1 = \alpha_1 \frac{k}{\sqrt{3}} e^{-i\frac{2}{3}\pi}$$

$$(5.9) \qquad R = \frac{\sqrt{3}}{2}\frac{\lambda}{k}\left[1 + \frac{8}{3}\frac{\delta^2}{\gamma^2}\right] e^{i\frac{3}{2}\pi} .$$

It is interesting to compare Eqs. (5.8) and (5.9) with Eqs. (26) and (28) of Reference 2 respectively. (Please notice that the notation λ in this paper corresponds to α in Reference 2, the factor α_1 in Eq. (5.8) is introduced by the assumed form Eq. (4.14) of the incident surface wave.) Eq. (5.9) indicates that the obliquity of the incident field tends to increase the reflection.

Substituting Eq. (5.7) into Eqs. (4.34) and (4.35) and imposing the condition that $\gamma(=\frac{\lambda}{k}) \to \infty$, we obtain

(5.10) $$A_1 = -\alpha_1(\alpha_1)^{\frac{1}{3}} (\frac{\kappa}{2})^{\frac{2}{3}} e^{i\frac{\pi}{6}}$$

(5.11) $$R = e^{-i\frac{2}{3}\pi} .$$

We may compare the above two equations with Eqs. (32) and (34) of Reference 2 respectively. Furthermore, Eqs. (5.10) and (5.11) are valid in all oblique directions where κ is not too small, because either small δ or large γ may give rise to small $(-\frac{\alpha_2}{\kappa})$. The magnitude of the reflected oblique surface wave approaches that of the incident oblique surface wave as $\frac{\lambda}{k}$ approaches infinity. Substituting Eq. (5.10) into Eq. (4.55) may check the expected result that the radiation field vanishes as $\frac{\lambda}{k}$ becomes infinitely large.

We next consider another special incident direction where $k_{S'} \to k$ (i.e. $\kappa \approx 0$). Now $-\frac{\alpha_2}{\kappa}$ is large. $\theta_S \to \frac{\pi}{2} - i \log \frac{-2\alpha_2}{\kappa}$. Using Eq. (5.3), we may approximate Eq. (4.40) by

(5.12) $$D = \frac{A_1}{A_2} = -\frac{\alpha_1}{\alpha_2} (-\frac{2\alpha_2}{\kappa})^{\frac{2}{3}}$$

and

(5.12A) $$\frac{B_1}{B_2} = -(-\frac{2\alpha_2}{\kappa})^{\frac{2}{3}}$$

(5.12B)
$$\frac{A_1}{B_2} = -\left(-\frac{2\alpha_2}{\kappa}\right)^{\frac{2}{3}}$$

(5.12C)
$$\frac{A_2}{B_2} = \frac{\alpha_2}{\alpha_1} \quad .$$

Substituting (5.12) into Eqs.(4.34) and (4.35), we obtain

(5.13)
$$A_1 = \frac{i2k_T \frac{\alpha_1}{\alpha_2}(\alpha_1 - \alpha_2)\left(-\frac{\alpha_2}{\alpha_1}\right)^{\frac{2}{3}}}{\left(\frac{2\alpha_1}{\kappa}\right)^{\frac{2}{3}}\left\{\left[1 + \left(-\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{3}}\right] - i\sqrt{3}\left[1 - \left(-\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{3}}\right]\right\}}$$

(5.14)
$$R = e^{i2\psi}$$

where
$$\tan\psi = \frac{\sqrt{3}\left[1 - \left(-\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{3}}\right]}{\left[1 + \left(-\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{3}}\right]}$$

When the incident direction approaches the condition that $k_{S'} \approx k$, the magnitude of the reflected oblique surface wave again approaches that of the incident oblique surface wave. Substituting Eq. (5.13) into Eq. (4.53) shows that the radial component of the radiation power vanishes as $k_{S'} \to k$.

When the obliquity increases beyond the point that $k_{S'}^2 = k^2$, i.e. $k_{S'}^2 > k^2$, i.e. κ becomes imaginary, an inspection of Eq. (4.35) reveals a total reflection of the oblique surface wave. This phenomenon is analogous to the total reflection at the dielectric interface.

When the obliquity vanishes in the perpendicular case, the incident field becomes the two dimensional TE surface wave for a negative X, and the incident field becomes a plane wave for a positive X.

6. Conclusions

A graphical analysis of oblique surface waves on an infinite impedance plane has been presented. Then the diffractions of oblique surface waves by terminations in the form of right-angle bends are treated. When the edge of the bend is parallel to the perfectly conducting direction of the impedance plane, the mathematical formalism may be reduced to that of the TM normal incidence. For a positive surface reactance X in this case, the amplitude of the reflection coefficient becomes smaller and the radiation power pattern becomes more directive, when the obliquity of the incident direction increases. The incident field becomes a TM surface wave when the obliquity vanishes. For a negative surface reactance X in this geometry, we expect a total reflection of the oblique surface wave.

When the edge of the bend is perpendicular to the perfectly conducting direction of the impedance plane, the mathematical analysis involves two coupled mixed boundary conditions. In addition to the jump conditions and the boundary conditions, the edge condition must be used to determine the coefficients in the field expressions. Various limiting cases are considered. For a negative surface reactance X in this geometry, the amplitude of the reflection coefficient tends to increase when the obliquity increases. The incident field becomes a TE surface wave when the obliquity vanishes. A total reflection of the oblique surface wave will occur when the obliquity reaches the condition $k_{s'}^2 \geq k^2$. For a positive surface reactance in this case, the incident field becomes a plane wave when the obliquity vanishes; however, the amplitude of the reflection coefficient also tends to increase when the obliquity increases, and a total reflection will occur when the obliquity satisfies. The condition $k_{s'}^2 \geq k^2$.

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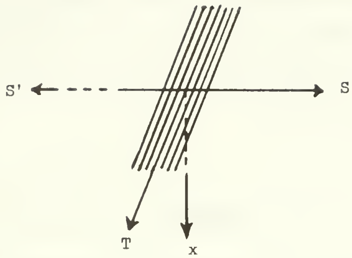


Fig. 1a

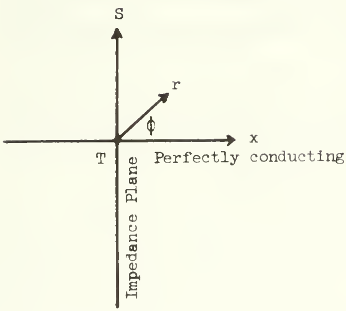


Fig. 1b

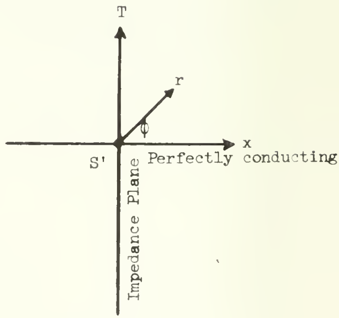
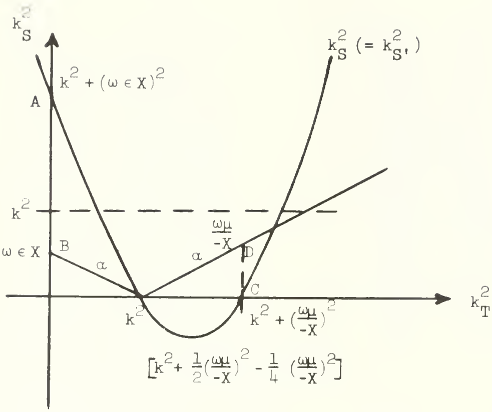


Fig. 1c



k_S^2
$k_S^2 \ (= k_{S'}^2)$
$k^2 + (\omega \epsilon X)^2$
A
k^2
$\frac{\omega\mu}{-X}$
$\omega \epsilon X$
B
α
α
D
C
k_T^2
k^2
$k^2 + (\frac{\omega\mu}{-X})^2$
$[k^2 + \frac{1}{2}(\frac{\omega\mu}{-X})^2 - \frac{1}{4}(\frac{\omega\mu}{-X})^2]$

Fig. 2

Courant Institute of Mathematical Sciences, New York University, New York, N.Y., THE DIFFRACTION OF OBLIQUE SURFACE WAVES BY A RIGHT-ANGLE BEND by S.N. Karp and T.S. Chu. Air Force Cambridge Research Laboratories, Bedford, Mass. Scientific Report-Number AFCRL-64-599, 28 pages, July, 1964.
Unclassified Report

The diffracted field arising when an incident electromagnetic surface wave strikes the edge of a right-angle bend at an oblique angle is investigated. One face of the bend supports the surface wave and the other face is a perfect conductor. This diffraction problem has implications for scanning surface wave antennas.

An analysis of oblique surface waves on an infinite impedance plane is first presented graphically. Two cases are treated. In one case the edge of the bend is parallel, and, in the other case, perpendicular, to the perfectly conducting direction of the impedance plane.

1. Diffraction of oblique surface waves on a corrugated surface
2. Anisotropic impedance
3. Termination of anisotropic impedance surfaces

I. Project No. 5635
Task No. 563502
II. Contract AF 19(628)3868
III. S.N. Karp and T.S. Chu
IV. In DDC collection

Courant Institute of Mathematical Sciences, New York University, New York, N.Y., THE DIFFRACTION OF OBLIQUE SURFACE WAVES BY A RIGHT-ANGLE BEND by S.N. Karp and T.S. Chu. Air Force Cambridge Research Laboratories, Bedford, Mass. Scientific Report-Number AFCRL-64-599, 28 pages, July, 1964.
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Unclassified Report

The diffracted field arising when an incident electromagnetic surface wave strikes the edge of a right-angle bend at an oblique angle is investigated. One face of the bend supports the surface wave and the other face is a perfect conductor. This diffraction problem has implications for scanning surface wave antennas.

An analysis of oblique surface waves on an infinite impedance plane is first presented graphically. Two cases are treated. In one case the edge of the bend is parallel, and, in the other case, perpendicular, to the perfectly conducting direction of the impedance plane.

1. Diffraction of oblique surface waves on a corrugated surface
2. Anisotropic impedance
3. Termination of anisotropic impedance surfaces

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