NCSL 144-72

WHOI
DOCUMENT
COLLECTION

INFORMAL
REPORT
NCSL 144-72

Formerly NSRDL/PC 3472

DECEMBER 1972

THE DIRECTIONAL ANALYSIS OF OCEAN WAVES: AN INTRODUCTORY DISCUSSION

(SECOND EDITION)

CARL M. BENNETT

Approved for Public Release;
Distribution Unlimited

NAVAL COASTAL SYSTEMS LABORATORY
PANAMA CITY, FLORIDA 32401

GC
213.7
.M3
B4
1972

COPY- 66

ABSTRACT

An introductory discussion of the mathematics behind the directional analysis of ocean waves is presented. There is sufficient detail for a reader interested in applying the methods; further, the report can serve as an entry into the theory. The presentation is basically tutorial but does require a reasonably advanced mathematical background. Results of a program for the measurement of directional ocean wave bottom pressure spectra are included as an appendix. This second edition makes corrections to the first and adds some details of an iterative directional analysis method.

ADMINISTRATIVE INFORMATION

This report was prepared to document the mathematical methods used in connection with work done in support of Task SWOC SR 004 03 01, Task 0582, and applied on Task ZR 000 01 01, Work Unit 0401-40.

This report was originally issued in September 1971 as NSRDL/PC Report 3472.

s of ocean waves:
ion. (2nd ed.)
oratory
NCSL 144-72)
RETURNED
9 Oct 73

TABLE OF CONTENTS

MBL/WHOI
0 0301 0043880 0

LIST OF ILLUSTRATIONS

1. INTRODUCTION

The report presents an introductory discussion of the mathematics pertaining to the directional analysis of ocean waves. The presentation is tutorial in form but does require a reasonably complete mathematical background; a background equivalent to that required in reading Kinsman's textbook Wind Waves (1965).

The level of the presentation is moderate at the beginning. The level picks up rapidly toward the middle but there should be sufficient detail and redundancy in the mathematics to allow the reader to follow the development without having to rediscover too many omitted steps. It is in this sense that the report is tutorial. In some places the mathematical development is intuitive rather than rigorous. This is deliberate in order to provide insight and understanding. In most such cases, references to rigorous reports are given.

The development is reasonably detailed so that the interested reader may apply the methods presented and use the report as an entry point into the rigorous theory of the directional analysis of ocean waves. In this respect, if the report serves as a bridge across the gap between a handbook and a rigorous and sparse theory on the subject then the objective of the report will have been fulfilled.

The report first presents an intuitive development of a sea surface model that assumes the sea surface to be a two-dimensional random process definable in terms of a directional power spectrum. A discussion of the space and time covariance function and its relationship to the directional power spectrum follows. Both one- and two-sided power spectra are discussed; however, the main development is in terms of the two-sided spectrum. Next, the relationship between the power and cross power spectrum for two fixed locations and the sea surface directional spectrum is developed. Explicit relationships for the special cases of an isotropic sea and a single wave of a given direction and frequency are then obtained. The related topic of the directional resolving power of an array of wave transducers is then presented.

Using the preliminary developments as a basis, several methods for the directional analysis of ocean waves based on the information obtainable from an array of wave transducers are presented. The methods are basically a direction finder technique, a least square single-wave train fit, and a Fourier-Bessel expansion fit. In conclusion, a generalized Fourier expansion method is suggested. Extensive results of the application of the least square single-wave train fit are presented in Appendix A. Appendix B is a FORTRAN II listing of a program for this analysis.

2. WAVE MODELS

In its simplest form an ocean wave can be thought of as a single frequency, sinusoidal, infinitely long crested wave of length λ, moving in time over the ocean surface from a given direction θ. Such a wave is illustrated in Figure 1.

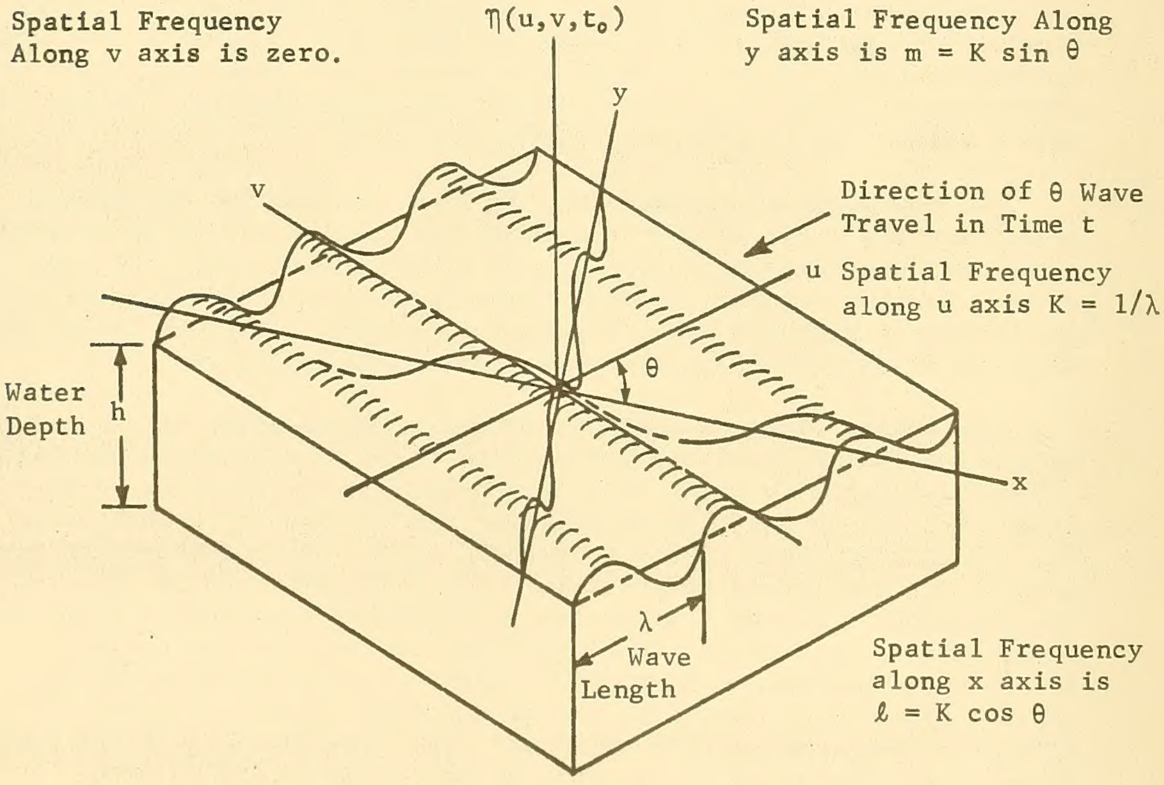


FIGURE 1. SIMPLE OCEAN WAVE

Assume that the wave is frozen in time over the surface (the x,y spacial plane). The coordinates (u,v) are a θ degree rotation of the (x,y) coordinates. The positive u axis lies along the direction from which the wave is traveling. The wave surface $\eta(u,v)$, shown frozen in time in Figure 1 can be described mathematically by

$$\eta(u,v) = \cos(2\pi Ku + 2\pi\phi) \tag{2.1}$$

where $K = 1/\lambda$ is the wave number of spacial frequency in cycles per unit length along the u axis, and $2\pi\phi$ is a spacial phase shift.

To make the wave move in time across the spacial plane with a time frequency $f = 1/p$, where p is the wave period, it is necessary to add a time part to the argument of the cosine function in the model above. The time part is a phase shift dependent only upon time. As time passes, the time part changes causing the cosine wave to move across the (u,v) plane, in this case the ocean surface. Adding the time part we get (where $2\pi\psi$ is a fixed time phase shift)

$$\eta(u,v,t) = \cos(2\pi Ku + 2\pi\phi + 2\pi\ ft + 2\pi\psi).$$

If we combine the effect of the ϕ and ψ phase shifts as $\alpha = \phi + \psi$, we get

$$\eta(u,v,t) = \cos(2\pi(Ku + ft + \alpha)) \tag{2.2}$$

as a simple model of a sinusoidal wave moving in time over the ocean surface.

Since the coordinates (u,v) are a rotation of the coordinates (x,y) through an angle of θ degrees, we know

$$u = x \cos\theta + y \sin\theta$$

$$v = -x \sin\theta + y \cos\theta.$$

Using the above relations, and letting $\ell = k \cos\theta$ and $m = K \sin\theta$ be the spacial frequencies along the x and y axes, respectively, we have

$$\eta(x,y,t) = A \cos(2\pi(\ell x + my + ft + \alpha)) \tag{2.3}$$

as a model for a wave of height 2A moving from a direction

$$\theta = \arctan\ (m/\ell)$$

with a phase shift of $2\pi\alpha$. A wave crest of such a wave system is infinite in length. A crest occurs at a set of points (x,y,t) which satisfy the relation

$$\ell x + my + ft = \text{a constant} = (n - \alpha)$$

where n = 0, -1, +1, -2, +2, ... Each value of the index n relates to a particular crest. The intersections of the crests with the x and y axes move along the respective axes with time velocities $V_x = -f/\ell$ and $V_y = -f/m$. This follows from the differential expressions

$$D_t(x) = D_t\left[\frac{n-\alpha}{\ell} - \frac{my}{\ell} - \frac{ft}{\ell}\right] = -\frac{f}{\ell} \tag{2.4}$$

$$D_t(Y) = D_t\left[\frac{n-\alpha}{m} - \frac{\ell x}{m} - \frac{ft}{m}\right] = -\frac{f}{m} \tag{2.5}$$

obtained from the wave crest relationship given above.

From Euler's equation we know that $\cos \gamma = (\mathrm{Exp}(i\gamma) + \mathrm{Exp}(-i\gamma))/2$. If we consider γ as $2\pi(\ell x + my + ft + \alpha)$ we can write

$$\eta(x,y,t) = 1/2A\ \mathrm{Exp}(i2\pi(\ell x + my + ft + \alpha) + 1/2A\ \mathrm{Exp}\ (i2\pi(-\ell x - my - ft - \alpha)) \tag{2.6}$$

where $-\infty < \ell \leq +\infty$, $-\infty < m \leq +\infty$ and $-\infty < f \leq +\infty$.

In the above we have introduced the notion of negative time frequencies. This makes it possible to express an elementary wave in the mathematically convenient form

$$\eta(x,y,t) = a\ \mathrm{Exp}\ (i2\pi(\ell x + my + ft + \alpha)) \tag{2.7}$$

where a = 1/2A. In the real world a complex wave of this type implies the existence of another wave $\eta^*(x,y,t)$ which is the complex conjugate of $\eta(x,y,t)$ above. This complex conjugate is given by

$$\eta^*(x,y,t) = a\ \mathrm{Exp}\ (-i2\pi(\ell x + my + ft + \alpha))$$

$$= a\ \mathrm{Exp}\ (i2\pi[(-\ell)x + (-m)y + (-f)t + (-\alpha)]). \tag{2.8}$$

The fact that negative frequencies are considered is explicit in the above relation.

A property of the above model, which will be used later in connection with the directional analysis of waves from measurements obtained from an array of detectors, is expressed by the equation for the phase difference of two measurements made at two different points in space and time. Assume we know the value of $\eta(x,y,t)$ at the three-dimensional coordinates (x_o,y_o,t_o) and $(x_o+X,\ y_o+Y,\ t_o+T)$, where X, Y, and T are constants. The phases at the two points are given by

$$\phi(x_o, y_o, t_o) = \ell x_o + my_o + ft_o + \alpha, \tag{2.9}$$

$$\phi(x_o + X, y_o + Y, t_o + T) = \ell(x_o + X) + m(y_o + Y) + f(t_o + T) + \alpha \tag{2.10}$$

This gives a phase difference of

$$\Delta\phi = (\ell X + mY + fT). \tag{2.11}$$

To obtain a more complicated wave system consisting of many waves of various frequencies and directions, we can linearly superimpose (add up) many waves of the form given above. If we do this, we can write

$$\eta(x,y,t) = \sum_{n=1}^{N} a_n \operatorname{Exp}(i2\pi(\ell_n x + m_n y + f_n t + \alpha_n)). \tag{2.12}$$

For this wave system to be real, the terms must occur in complex conjugate pairs as indicated above.

For completeness, consider a model for an infinite but countable number of distinct (discrete) waves and write

$$\eta(x,y,t) = \sum_{n=1}^{\infty} a_n \operatorname{Exp}(i2\pi(\ell_n x + m_n y + f_n t + \alpha_n)). \tag{2.13}$$

Again the terms must occur in complex conjugate pairs for the wave system to be real. This will be assumed to be the case in future discussions.

A model for a wave system in the case where energy exists for continuous intervals of frequency and direction should be considered. In particular, consider the general case of continuous direction from 0 to 2π radians and continuous frequency in the interval $(-f_n, +f_n)$, or even the interval $(-\infty, \infty)$. In theory the above model does not hold for the continuous case. The power spectrum for the infinite but countable case would be a set of Dirac delta functions of amplitude a_n^2 standing on the points (ℓ_n, m_n, f_n) of a three-dimensional frequency space. The continuous case produces a power spectrum, $S_o(\ell,m,f)$, which is everywhere nonnegative and in general continuous over the region of three-dimensional frequency space where power is assumed to exist. A reasonable model for $\eta(x,y,t)$ in the continuous case must be determined. Consider a single wave element

$$a_n \operatorname{Exp}(i2\pi(\ell_n x + m_n y + f_n t + \alpha_n)). \tag{2.14}$$

The energy or mean square in this element is a_n^2. Assume the element is a part of a continuum of elements for $-\infty < f < +\infty$ and $0 \leq \theta < 2\pi$. In this case a_n^2 must be an infinitesimal energy associated with the frequency differential, df, and space frequency differentials, $d\ell$, and dm, which are related to the direction θ of the wave element as before. Let the power spectrum, $S(\ell,m,f)$, be defined with units of amplitude squared and divided by unit spacial frequency, ℓ, unit spacial frequency, m, and unit time frequency, f. The power spectrum is then a spectral density value at (ℓ,m,f). In this case we must have the infinitesimal energy, a_n^2, defined by

$$a_n^2 = S(\ell_n, m_n, f_n)\, d\ell\, dm\, df \tag{2.15}$$

The real valued, nonnegative function $S(\ell,m,f)$ is a power (energy density) spectrum of the standard type in three-dimensional frequency space (ℓ,m,f). Intuitively, we can write an infinitesimal wave element as

$$\left[\mathrm{Exp}\left(i2\pi\left(\ell_n x + m_n y + f_n t + \alpha_n\right)\right)\right]\sqrt{S(\ell_n, m_n, f_n)\, d\ell\, dm\, df}\,, \tag{2.16}$$

where the positive square root is assumed. To arrive at a model of $\eta(x,y,t)$ for the continuous case, we need only form a triple "sum" of the infinitesimals or, to be precise, the triple integral $\eta(x,y,t) =$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{Exp}\left(i2\pi\left(\ell x + my + ft + \alpha(\ell,m,f)\right)\right)\sqrt{S(\ell,m,f)\, d\ell\, dm\, df} \tag{2.17}$$

For different sets of values of the phase relation $\alpha(\ell,m,f)$, the wave system, $\eta(x,y,t)$, has a different shape, even when $S(\ell,m,f)$ is fixed. In fact there is a wide range of possible shapes of $\eta(x,y,t)$ for a given $S(\ell,m,f)$. The above development is more intuitive than mathematically rigorous. It has been shown by Pierson (1955 - pp. 126-129) that if $2\pi\alpha(\ell,m,f)$ is a random function such that for fixed (ℓ,m,f) phase values of the form $2\pi\alpha$ MOD 2π between 0 and 2π, are equally probable and all phase values are independent, then Equation (2.17) represents an ensemble (collection of all probable) $\eta(x,y,t)$ for a given $S(\ell,m,f)$. The random process represented by the ensemble is then a stationary Gaussian process indexed by the three dimensions (x,y,t). Detail discussions of the above can be found in St. Dennis and Pierson (1953 - pp. 289-386) and Kinsman (1965 - pp. 368-386). The fact that a particular sea-way can be considered as a realization of a stationary three-dimensional Gaussian process has been verified. Refer to Pierson and Marks (1952).

The model in Equation (2.17) as a stationary random process will be assumed in following discussion.

3. A DIRECTIONAL WAVE SPECTRUM

The power spectrum $S(\ell,m,f)$ is the directional wave spectrum of $\eta(x,y,t)$. If $\eta(x,y,t)$ is to be real, every infinitesimal of the form given in the continuous case above presumes the existence of its complex conjugate. Let us consider the one-sided power spectral density, $S'(\ell_o,m_o,f_o)$, of a single real wave where $0 \leq f_o < \infty$. For such a real wave element of length λ_o, from a direction θ_o, $0 \leq \theta_o < 2\pi$, the value $S'(\ell_o,m_o,f_o)$, $= S(\ell_o,m_o,f_o) + S(-\ell_o,-m_o,-f_o)$, where $\ell_o = K_o \cos\theta_o$, $m_o = K_o \sin\theta_o$, and $K_o = 1/\lambda_o$. If the above real wave came from a direction $(2\pi - \theta)$, the power density would be $S'(\ell_o,-m_o,f_o) = S(\ell_o,-m_o,f_o) + S(-\ell_o,m_o,-f_o)$.

Figure 2 illustrates a real wave of length λ_o, from a direction θ_o, in two-dimensional spacial frequency (wave number) space.

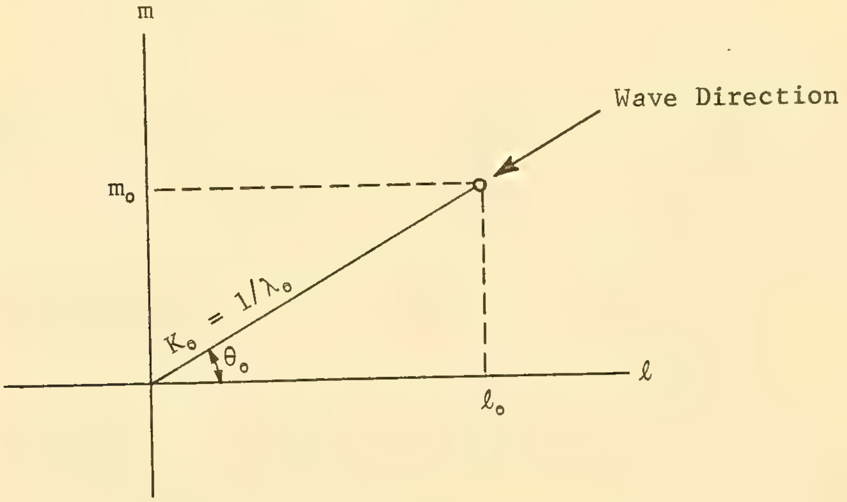


FIGURE 2. REAL WAVE IN WAVE NUMBER SPACE

If the wave number relation

$$K = 2\pi f^2/g \, \mathrm{Tanh}(2\pi K h) \tag{3.1}$$

holds, refer to Kinsman (1965 - p. 157) and Munk et al (1963 - p. 527), where h = water depth, g = acceleration of gravity, and K = wave number $= 1/\lambda$, λ being the wave length; then a relationship between f and (ℓ,m) is implied that requires a wave frequency f_o to have a unique wave number k_o. From this we have the general one-sided spectral form for waves where $f = f_o$ of

$$S'(\ell,m,f_o) = \begin{cases} \text{zero where } \ell^2 + m^2 \neq K_o^{\,2} \\ \text{a power density} \geq 0 \text{ for } \ell^2 + M^2 = K_o^{\,2}. \end{cases}$$

$S'(\ell,m,f_o)$ thus defines power density at $f = f_o$ for wave energy over $0 \leq \theta < 2\pi$. Figure 3 illustrates this case in wave number space.

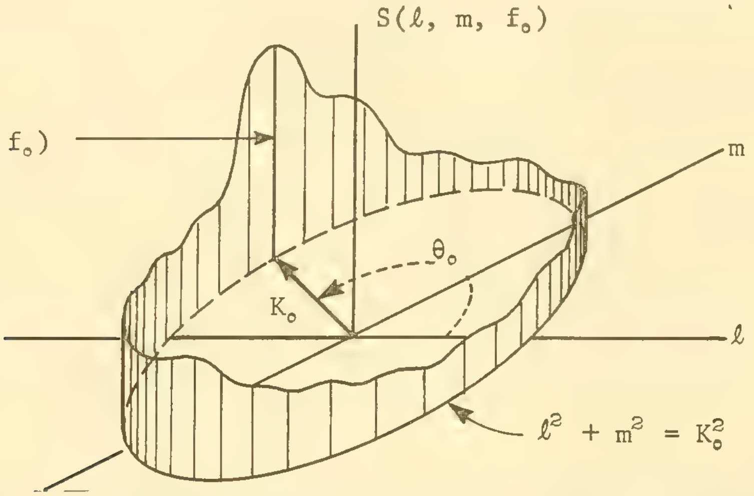


FIGURE 3. DIRECTIONAL WAVE SPECTRUM AT A FIXED FREQUENCY, f_o

We want to estimate the shape of $S'(\ell,m,f_o)$ above the circle $\ell^2 + m^2 = K_o^{\,2}$ in a directional wave train analysis. Remember, the $S'(\ell,m,f_o)$ above is restricted to $f_o \geq 0$ and is, in fact, equal to

$$[\,S(\ell,m,f_o) + S(-\ell,-m,-f_o)],$$

where

$$S(\ell,m,f_o) = S(-\ell,-m,-f_o) \tag{3.2}$$

if $\eta(x,y,t)$ is to be real.

Let us see how $S'(\ell,m,f)$ might be found: we have said that $\eta(x,y,t)$ can be assumed to be a stationary Gaussian process. One characteristic of such a process is that for fixed values (x_o, y_o, t_o) of the process indices, $\eta(x_o, y_o, t_o)$ is random variable with a Gaussian distribution; i.e.,

$$\text{Prob}\left(\eta(x_o, y_o, t_o) < \eta_o\right) = \int_{-\infty}^{\eta_o} \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{1}{2}\left(\frac{\eta-\mu}{\sigma}\right)^2} d\eta$$

where μ is the arithmetic mean of η and σ^2 is the variance. Intuitively η is as likely to be positive as negative, so let us assume that Prob $(\eta(x_o, y_o, t_o) < 0) = 1/2$. Since η is Gaussian distributed, and is thus symmetric about its mean, we have $\text{Prob}(\eta(x_o, y_o, t_o) < \mu) = 1/2$ or that $\mu = 0$. For σ^2, we have (using expected value notation)

$$\sigma^2 = E\left[(\eta - \mu)^2\right] = E\left[\eta^2(x_o, y_o, t_o)\right]$$

where we are thinking of η as a random variable.

A Gaussian process is completely defined statistically if we know the form of the mean

$E(\eta(x,y,t))$ and the covariances

$$E\left\{\left[\eta(x,y,t) - E(\eta(x,y,t))\right]\left[\eta(x+X, y+Y, t+T) - E(\eta(x+X, y+Y, t+T))\right]\right\},$$

where X, Y, and T are space and time separations, respectively. Refer to Parzen (1962 - pages 88-89). We have assumed $E(\eta(x,y,t)) = \mu = 0$ and that the process is stationary (only weakly stationary is necessary). Hence, by definition of weak stationarity, we have for each (x,y,t), and yet independent of the particular x,y,t values, the covariance form

$$R(X,Y,T) \equiv E\left[\eta(x,y,t)\,\eta(x+X, y+Y, t+T)\right] \quad . \qquad (3.3)$$

All of the properties of the stationary Gaussian process $\eta(x,y,t)$ are implicit in $R(X,Y,T)$, just as a knowledge of μ and σ^2 for a single Gaussian random variable completely defines such a random variable. Here it is important to understand that we are discussing expected values across all possible realizations at a point (x,y,t); i.e., across the ensemble of all possible sea wave shapes at (x,y,t) for a given $S(\ell,m,f)$.

There is a simple and unique relationship between $R(x,y,t)$ and $S(\ell,m,f)$. Consider a single real wave element (from Equation (2.16) and (3.2)) as a random process and write $\eta(x,y,t) = [\text{EXP}(i2\pi(\ell x+my+ft+\alpha))$

$$+ \text{Exp}(-i2\pi(\ell x + my + ft + \alpha))]\sqrt{S(\ell,m,f)\, d\ell\, dm\, df}\ .$$

Form the covariance function

$$R(XYT) = E\left[\eta(x,y,t)\,.\,\eta(x+X,y+Y,t+T)\right]$$
$$= E\Big[\mathrm{Exp}\left(i2\pi\left(l(2x+X)+m(2y+Y)+f(2t+T)+2\alpha\right)\right.$$
$$+\ \mathrm{Exp}\left(-i2\pi\left(l(2x+X)+m(2y+Y)+f(2t+T)+2\alpha\right)\right.$$
$$+\ \mathrm{Exp}\left(i2\pi\left(lX+mY+fT\right)\right)$$
$$+\ \mathrm{Exp}\left(-i2\pi\left(lX+mY+fT\right)\right)\Big]\,S(l,m,f)\,dl\,dm\,df$$

where E is the expected value over the ensemble for any fixed x,y,t. Note that

$$\sqrt{S(l,m,f)\,dl\,dm\,df}$$

is a constant with respect to the expected value.

Consider the following problem. Let u be a random variable with uniform probability density function

$$f(u) = \begin{cases} K & 0 \leq u < 2\pi \text{ radians} \\ 0 & \text{elsewhere} \end{cases}$$

Define a random variable $Z = e^{iu}$. The expected value of Z is defined as

$$E(Z) = \int_{-\infty}^{\infty} e^{iu} f(u)\,du$$
$$= K\int_{0}^{2\pi} e^{iu}\,du = 0$$

Considering the random variable nature of the phase $2\pi\alpha$ as described following Equation (2.17) at the end of Section 2, and applying the above notion to the cross product terms of R(X,Y,T) we obtain

$$R(XYT) = \Big[\mathrm{Exp}\left(i2\pi(lX+mY+fT)\right) + \mathrm{Exp}\left(-i2\pi(lX+mY+fT)\right)\Big]S(l,m,f)\,dl\,dm\,df.$$

For a real wave element we have, where $S(\ell,m,f) = S(-\ell,-m,-f)$, see Equation (3.2), $R(X,Y,T) = \text{Exp}\left(i2\pi(\ell X + mY + fT\right) S(\ell,m,f)\,d\ell\,dm\,df$

$$+ \text{Exp}\left(-i2\pi(\ell X + mY + fT)\right) S(-\ell,-m,-f)\,d\ell\,dm\,df$$

which is simply the sum of the covariance functions of two complex wave elements which are conjugate pairs. It also follows that R(X,Y,T) is real valued.

Reverting to the complex wave element form, and noting that the expected ensemble value of cross products between different wave elements is zero in a manner similar to the case of cross products shown above, we obtain the composite general relationship

$$R(X,Y,T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Exp}\left(i2\pi(\ell X + mY + fT\right) S(\ell,m,f)\,d\ell\,dm\,df \qquad (3.4)$$

We have demonstrated, but not rigorously proven, that the covariance function R(X,Y,T) is the three-dimensional Fourier transform of the directional power spectrum $S(\ell,m,f)$.

We cannot hope to be able to estimate R(X,Y,T) for continuous values of X, Y, and T. However, there is a way around this problem, we can write the above as $R(X,Y,T) =$

$$\int_{-\infty}^{\infty} \text{Exp}(i2\pi fT)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Exp}\left(i2\pi(\ell X + mY\right) S(\ell,m,f)\,d\ell\,dm\right] df \qquad (3.5)$$

which is in the form of a single dimension (variable f) Fourier transform of the term in brackets [ ]'s. Note this term is not a function of T. It depends only on the value of (X,Y). Further, by Fourier transform pairs we can write this expression as

$$[\ \ ]'s = \int_{-\infty}^{\infty} R(X,Y,T)\,\text{Exp}(-i2\pi fT)\,dT \qquad (3.6)$$

In general, assuming that the term in [ ]'s is complex, we can write

$$[C(X,Y,f) - i\,Q(X,Y,f)] = \int_{-\infty}^{\infty} R(X,Y,T)\,\text{Exp}(-i2\pi fT)\,dT\,. \qquad (3.7)$$

To find [C(X,Y,f) - iQ(X,Y,f)] we need only know R(X,Y,T) for continuous T for the given value of X,Y). Further, we have just stated that

$$[C(X,Y,f) - i\,Q(X,Y,f)] = [\ ]'_s = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\ell,m,f)\,\mathrm{Exp}\left(i2\pi(\ell X + mY)\right) d\ell\, dm \qquad (3.8)$$

This is in the form of a Fourier transform; thus, we can write from transform pairs

$$S(\ell,m,f) = \iint [C(X,Y,f) - i\,Q(X,Y,f)]\,\mathrm{Exp}\left(-i2\pi(\ell X + mY)\right) dX\, dY \qquad (3.9)$$

As has been stated, we cannot hope to have a continuous set of values of (X,Y). The solution is to find R(X,Y,T) for continuous T and selected values of (X,Y), and then employ the above to estimate $S(\ell,m,f)$. This is described in the next section.

4. CROSS SPECTRAL MATRIX OF AN ARRAY

Let us look at $\eta(x,y,t)$ at two fixed points in space, say (x_o, y_o) and (x_1, y_1). This would give two stationary Gaussian processes indexed on time alone because (x_o, y_o) and (x_1, y_1) are fixed. Thus, we may write

$$\eta_o(t) = \eta(x_o, y_o, t) \quad \text{and} \quad \eta_1(t) = \eta(x_1, y_1, t)$$

If $X = (x_1 - x_o)$ and $Y = (y_1 - y_o)$ Then we can say, since $\eta(x,y,t)$ is assumed weakly stationary (see Equation (3.3)), that

$$R(X,Y,T) = E[\eta_o(t) \cdot \eta_1(t+T)]$$

where the expected value is over the ensemble for some specific value of t, where $-\infty < t < \infty$. Let us extend this idea by a change of notation and let N(X,Y,t) be a two-dimensional (vector) process, double-indexed on time; i.e., let N be a vector function

$$N(X,Y,t) = (\eta_o(t), \eta_1(t)) = N(t) \quad \text{for } -\infty < t < \infty \qquad (4.1)$$

We can then write a generalized covariance function (assumed to be finite) as the matrix equation

$$R(T) = E\left[N^T(t)\ N(t+T)\right]$$
$$= \begin{bmatrix} E(\eta_0(t)\,\eta_0(t+T)) & E(\eta_0(t)\,\eta_1(t+T)) \\ E(\eta_1(t)\,\eta_0(t+T)) & E(\eta_1(t)\,\eta_1(t+T)) \end{bmatrix}$$
$$= \begin{bmatrix} R(0,0,T) & R(X,Y,T) \\ R(-X,-Y,T) & R(0,0,T) \end{bmatrix} \qquad (4.2)$$

Now R(-T) is

$$R(-T) = \begin{bmatrix} R(0,0,-T) & R(X,Y,-T) \\ R(-X,-Y,-T) & R(0,0,-T) \end{bmatrix}$$

and R(-T) transpose is

$$R^T(-T) \begin{bmatrix} R(0,0,-T) & R(-X,-Y,-T) \\ R(X,Y,-T) & R(0,0,-T) \end{bmatrix} \qquad (4.3)$$

We have by Equation (3.3) and stationarity that

$$R(-X,-Y,-T) = E(\eta(x_1,y_1,t+T)\,\eta(x_0,y_0,t))$$
$$= E(\eta(x_0,y_0,t)\,\eta(x_1,y_1,t+T)) = R(X,Y,T) \qquad (4.4)$$

It then follows that

$$R(T) = R^T(-T) \qquad (4.5)$$

From Equation (3.7),

let $$P^*(X,Y,f) = [C(X,Y,f) - iQ(X,Y,f)]$$

and we get

$$P^*(X,Y,f) = \int_{-\infty}^{\infty} R(X,Y,T)\,Exp(-i2\pi fT)\,dT \,.$$

Now R(-X,-Y,T) = R(X,Y,-T) by Equation (4.5). Thus, we have from Equation (3.4)

$$R(X,Y,-T) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Exp(i2\pi(\ell X + mY + f(-T)))S(\ell,m,f)\,d\ell\, dm\, df$$

or, by the same procedure, that Equation (3.6) was obtained from Equation (3.4), we have

$$P^*(X,Y,f) = \int_{-\infty}^{\infty} R(X,Y,-T)\,Exp(i2\pi fT)\,dT \qquad (4.6)$$

Now, since R(X,Y,-T) is real and Fourier transform pairs are unique, we must have from Equation (4.6) that

$$\int_{-\infty}^{\infty} R(X,Y,-T)\,Exp(-i2\pi fT)\,dT = P(X,Y,f)$$

a parallel form of Equation (3.6). Therefore, the Fourier transform of R(-X,-Y,T) = R(X,Y,-T) is the complex conjugate of the transform of R(X,Y,T); i.e., (see Equation (3.7))

$$P^*(-X,-Y,f) = [P^*(X,Y,f)]^* = P(X,Y,f)$$

In general, we have that the Fourier transform of $R(T) = R^T(-T)$ is

$$\begin{bmatrix} P(0,0,f) & P^*(X,Y,f) \\ P(X,Y,f) & P(0,0,f) \end{bmatrix} \qquad (4.7)$$

Note: P*(0,0,f) = P(0,0,f) is real.

By definition it follows that

$$C(X,Y,f) = \mathrm{Re}[P(X,Y,f)]$$

$$Q(X,Y,f) = \mathrm{Im}[P(X,Y,f)]$$

where $-\infty < f < \infty$.

The function $C(X,Y,f)$ is called the cospectrum and $Q(X,Y,f)$ is called the quadrature spectrum. Both are spectral density functions.

Explicitly, we can think of (x_o, y_o) and (x_1, y_1) as being the location of elements of a probe-array with space separation (X,Y).

The first step to find (or estimate) $S(\ell,m,f)$ (see Equation (3.9)) is to find $P(X,Y,f)$ related to a pair of array elements. This is a problem of estimating the cospectrum and quadrature spectrum of a two-dimensional (vector) stationary Gaussian process. Goodman (Mar 1967 - Chapter 3) has an excellent treatment of this subject, which we will discuss. Kinsman (1965 - Chapters 7-9) also discusses the subject. The essence of the problem is that if $P(X,Y,f)$ is continuous and negligible for $|f| > f_n$, then $R(X,Y,T)$ can be obtained by a time average over a particular realization $N(X,Y,t)$ instead of having to average (find the expected value) over the ensemble. This says that we can find $R(X,Y,T)$ by obtaining two time series (realizations) $r_o(t)$ and $r_1(t)$ measured over time at only two points; e.g., (x_o, y_o) and (x_1, y_1) where $(x_1 - x_o) = X$ and $(y_1 - y_o) = Y$. The relationship between $(r_o(t), r_1(t))$ and $R(X,Y,T)$ $-\infty < T < \infty$ is

$$R(X.Y.T) = \underset{t_o \to \infty}{\mathrm{Limit}} \frac{1}{t_o} \int_{-t_o/2}^{t_o/2} r_o(t) \cdot r_1(t+T)\, dt \qquad (4.8)$$

where $r_o(t)$ is a realization measured over time at (x_o, y_o) and $r_1(t)$ is measured at (x_1, y_1). We can simplify the notation by an expression for a time average given by

$$R(X,Y,T) = R_{01}(T) = \overline{r_o(t) r_1(t+T)}.$$

It follows from Equation (3.7) that $C(X,Y,f) = C_{01}(f) =$

$$\int_{-\infty}^{\infty} R_{01}\,(T)\,\cos(2\pi fT)\,dt,$$

and

$$Q(X,Y,f) = Q_{01}(f) = \int_{-\infty}^{\infty} R_{01}(T)\ \mathrm{Sin}(2\pi fT)dT. \tag{4.9}$$

We also have (see Equations (4.2) and (4.7)) $C_{00}(f) = C_{11}(f) = P_{00}(f) = P_{11}(f)$,

$$Q_{00}(f) = Q_{11}(f) = 0;$$

$$P^*_{01}(f) = C_{01}(f) - i\ Q_{01}(f). \tag{4.10}$$

The phase of $P_{01}(f)$ is given by

$$Q_{01}(f) = \mathrm{Arctan}\ \frac{Q_{01}(f)}{C_{01}(f)}\ . \tag{4.11}$$

This is the expected phase lead of the signal at (x_o, y_o) over the signal at (x_1, y_1) for f where $-\infty < f < \infty$.

For an array of N detectors located at (x_1, y_1), (x_2, y_2), (x_3, y_3) ...(x_N, y_N) we can find N^2 spectra $[P_{ij}]$ i = 1, N, j = 1, N. This gives a unique spectrum P_{ii} ($P_{11} = \ldots = P_{NN}$) and since $P_{ij} = P^*_{ji}$ (see Equation (4.6) and 4.7))

we have

$$\frac{N(N-1)}{2} \tag{4.12}$$

unique cross spectra or a total of $[N(N-1) + 2]/2$ unique spectra. Thus, for real $\eta(x,y,t)$ we have $P_{ij}(f) = P^*_{ji}(f)$ and $P_{ij}(f) = P^*_{ij}(-f)$ allowing us to define the information about $P(X,Y,f)$ obtainable from an array by a cross spectral matrix

$$\begin{pmatrix} P_{11}(f) & C_{12}(f) & \cdots & C_{1N}(f) \\ Q_{12}(f) & P_{22}(f) & \cdots & C_{2N}(f) \\ \vdots & & \ddots & \vdots \\ Q_{1N}(f) & & \cdots & P_{NN}(f) \end{pmatrix} \quad \text{WHERE } 0 \leq f < \infty \tag{4.13}$$

This information can be used together with Equation (3.9) to obtain an approximation to $S(\ell,m,f)$. Numerical details for finding the spectral matrix are given in Bennett, et al (June 1964).

It should be pointed out that negative frequencies are still considered in the relationships being discussed. We do not know $P^*(X,Y,f)$ for continuous values of (X,Y). We do know from the spectral matrix the values of

$$P^*(X_{ij}, Y_{ij}, f) \quad \text{for} \quad i = 1,\cdots,N \; ; \; j = i,\cdots,N$$

where $X_{ij} = (x_j - x_i)$

$$Y_{ij} = (y_j - y_i) \ .$$

We also have from Equation (4.7) that

$$P^*(-X_{ij}, -Y_{ij}, f) = P(X_{ij}, Y_{ij}, f) \ . \qquad (4.14)$$

From Equation (3.9) we get

$$S(\ell,m,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big\{P^*(X,Y,f)\cos[2\pi(\ell X+mY)] - iP^*(X,Y,f)\sin[2\pi(\ell X+mY)]\Big\}\,dX\,dY$$

or, since $S(\ell,m,f)$ is real

$$S(\ell,m,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big\{C(X,Y,f)\cos[2\pi(\ell X+mY)] - Q(X,Y,f)\sin[2\pi(\ell X+mY)]\Big\}\,dX\,dY \qquad (4.15)$$

Let us consider treating the points of (X,Y) where we know $C(X,Y,f)$ and $Q(X,Y,f)$ as weighted Dirac delta functions; e.g., at (X_{12}, Y_{12}) we get

$$C(X,Y,f) = b_{12}C(X_{12},Y_{12},f)\,\delta(X-X_{12})\,\delta(Y-Y_{12}) \ .$$

Reverting to the C_{ij}, Q_{ij} notation of Equation (4.13), we have, where $X_{ji} = -X_{ij}$ and $Y_{ji} = -Y_{ij}$, $C_{ji} = C_{ij}$ and $Q_{ji} = -Q_{ij}$. The numerical form of Equation (4.15) then becomes

$$S(\ell,m,f) = b_{ii}\, C_{ii}(f) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} b_{ij}\Big\{C_{ij}(f)\cos\left[2\pi\left(\ell X_{ij} + mY_{ij}\right)\right] - Q_{ij}(f)\sin\left[2\pi\left(\ell X_{ij} + mY_{ij}\right)\right]\Big\}. \tag{4.16}$$

Choice of b_{ij} values is arbitrary. A reasonable choice is $b_{ij} = [N(N-1) + 1]^{-1}$ for all (i,j) (refer to following section). We now have a basis for an approximation of $S(\ell,m,f)$. Before exploiting this result, we need a few side results.

5. SPECIAL CROSS SPECTRAL MATRICES

Assume that we have a real sea wave of frequency $f_o > 0$ moving from direction θ_o where $\ell_o = K_o \cos\theta_o$ and $m_o = K_o \sin\theta_o$, K_o being the wave number from Equation (3.1). We can write the wave as

$$\eta(x,y,t) = A\cos\left(2\pi\left(\ell_o x + m_o y + f_o t + \alpha\right)\right) \tag{5.1}$$

Since the root-mean-square (rms) value of a cosine wave is $A/\sqrt{2}$, we have for the two-sided $(-\infty < f < \infty)$ directional power spectrum of the wave in Equation (5.1)

$$S(\ell,m,f) = \frac{A^2}{4}\Big[\delta(\ell-\ell_o)\,\delta(m-m_o)\,\delta(f-f_o) + \delta(\ell+\ell_o)\,\delta(m+m_o)\,\delta(f+f_o)\Big] \tag{5.2}$$

or in polar form where $K = \sqrt{\ell^2 + m^2}$; $\theta = \arctan(\frac{m}{\ell})$

$$S(K,\theta,f) = \frac{A^2}{4}\Big[\delta(\theta-\theta_o)\,\delta(f-f_o) + \delta(\theta-(\theta_o-\pi))\,\delta(f+f_o)\Big]\delta(K-K_o)$$

From Equations (3.8) and (5.2) we have for a single wave of frequency $f_o > 0$ from a direction θ_o that the two-sided

$$P^*_{ij}(f) = \frac{A^2}{4}\Big[\text{EXP}\big(i2\pi(l_o X_{ij} + m_o Y_{ij})\big)\delta(f - f_o) + $$
$$+ \text{EXP}\big(i2\pi(-l_o X_{ij} - m Y_{ij})\big)\delta(f + f_o)\Big]$$

or where $P^*_{ij}(f) = C_{ij}(f) - iQ_{ij}(f)$ that

$$C_{ii}(f_o) = C_{ii}(-f_o) = \frac{A^2}{4},$$
$$C_{ij}(f_o) = C_{ij}(-f_o) = \frac{A^2}{4}\cos\big(2\pi(l_o X_{ij} + m_o Y_{ij})\big),$$
$$Q_{ij}(f_o) = -Q_{ij}(-f_o) = -\frac{A^2}{4}\text{SIN}\big(2\pi(l_o X_{ij} + m Y_{ij})\big) \qquad (5.3)$$

describe the elements for the spectral matrix of a single wave. Recall that, in general, $C_{ij}(f) = C_{ji}(f)$ and $Q_{ij}(f) = -Q_{ij}(-f)$. Substituting into Equation (4.16), we get where $b_{ij} = [N(N-1)+1]^{-1} = 1/M$ (5.4)

and $f_o > 0$ is assumed

$$S(l, m, f_o) \doteq \frac{A^2}{4M}\Big\{1 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\Big[\cos\big(2\pi(l_o X_{ij} + m Y_{ij})\big)\cos\big(2\pi(l X_{ij} + m Y_{ij})\big) +$$
$$+ \text{SIN}\big(2\pi(l_o X_{ij} + m_o Y_{ij})\big)\text{SIN}\big(2\pi(l X_{ij} + m Y_{ij})\big)\Big] \quad \text{OR}$$

$$S(l, m, f_o) \doteq \frac{A^2}{4M}\Big\{1 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\cos\big(2\pi[X_{ij}(l - l_o) + Y_{ij}(m - m_o)]\big)\Big\} \qquad (5.5)$$

The choice of $b_{ij} = 1/M = [N(N-1)+1]^{-1}$ and observing that $\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} 1 = \frac{N(N-1)}{2}$, gives the convenient result (Note: For $-f_o < 0$ we would use $(-\ell_o,-m_o)$ in place of (ℓ_o,m_o))

$$S(\ell_o,m_o,f_o) = S(-\ell_o,-m_o,-f_o)$$

$$\doteq \frac{A^2}{4m}\left\{1 + N(N-1)\right\} = \frac{A^2}{4}$$

which agrees with the values of Equation (5.2) at the points (ℓ_o,m_o,f_o) and $(-\ell_o,-m_o,-f_o)$. Note that there is not general agreement elsewhere. In fact, Equation (4.16) may (and does) give negative values for the approximation of $S(\ell,m,f)$. This is a problem of probe array design and is directly related to the directional resolving power of a probe array. This problem is discussed later in another section.

Consider the case of a single frequency, $f_o > 0$, real sea with equal wave energy from all directions; i.e., isotropic waves. In this case, assuming the wave equation (Equation (3.1)) holds, we have

$$S(\ell,m,f) = \left[\tilde{A}\,\delta\left((\ell^2+m^2)-(\ell_o^2+m_o^2)\right)\right]\left[\delta(f-f_o)+\delta(f+f_o)\right]$$

OR WHERE $K^2 = \ell^2 + m^2$, $K = +\sqrt{K^2}$, AND $\theta = \text{ARCTAN}\,(m/\ell)$

$$S(K,\theta,f) = \tilde{A}\,\delta(K^2-K_o^2)\left[\delta(f-f_o)+\delta(f+f_o)\right] \qquad (5.6)$$

as the two-sided directional power spectrum for such an isotropic wave.

Since $\ell = K\cos\theta$ and $m = K\sin\theta$, Equation (3.8) can be expressed in polar coordinate form as

$$\overset{*}{P}_{ij}(f) = \int_{-\pi}^{\pi}\int_{0}^{\infty} S(K,\theta,f)\,\text{EXP}\left(i\,2\pi K\left(X_{ij}\cos\theta + Y_{ij}\sin\theta\right)\right)K\,dK\,d\theta \qquad (5.7)$$

Letting $D_{ij} = (X_{ij}^2 + Y_{ij}^2)$

and $\phi_{ij} = \mathrm{ARCTAN}\left(\frac{Y_{ij}}{X_{ij}}\right)$

we can write

$$P^{*}_{ij}(f) = \int_{-\pi}^{\pi}\int_{0}^{\infty} S(K,\theta,f)\,\mathrm{EXP}\left(i\,2\pi K D_{ij}\cos(\theta-\theta_{ij})\right)K\,dK\,d\theta \qquad (5.8)$$

Using Equation (5.6) for $S(K,\theta,f)$ we get

$$\begin{aligned} P^{*}_{ij}(f_0) &= \tilde{A}K_0\int_{-\pi}^{\pi}\mathrm{EXP}\left(i\,2\pi K_0 D_{ij}\cos(\theta-\phi_{ij})\right)d\theta \\ &= \tilde{A}K_0\int_{-\pi}^{\pi}\cos\left(2\pi K_0 D_{ij}\cos(\theta-\phi_{ij})\right)d\theta\ + \\ &\quad + i\tilde{A}K_0\int_{-\pi}^{\pi}\mathrm{SIN}\left(2\pi K_0 D_{ij}\cos(\theta-\phi_{ij})\right)d\theta \end{aligned} \qquad (5.9)$$

Now, departing from the above development, consider the following integral where

$$Z = 2\pi K_0 D_{ij}\ ;\ \psi = \phi_{ij}$$

$$\int_{-\pi}^{\pi}\cos(n\theta)\cos\left(z\cos(\theta-\psi)\right)d\theta$$

Let $\phi = \theta - \psi$ and we get

$$\int_{-\pi-\psi}^{\pi-\psi}\cos(n\phi+n\psi)\cos(z\cos\phi)\,d\phi$$

Expanding $\cos(n\theta + n\psi)$ we get

$$\cos n\psi \int_{-\pi-\psi}^{\pi-\psi} \cos n\phi \cos(z\cos\phi)\,d\phi$$
$$-\sin n\psi \int_{-\pi-\psi}^{\pi-\psi} \sin n\phi \cos(z\cos\phi)\,d\phi\ .$$

We have $(\pi - \psi) - (-\pi - \psi) = 2\pi$ so that the integrands are over 2π allowing us to write the above as

$$\cos n\psi \int_{-\pi}^{\pi} \cos n\phi \cos(z\cos\phi)\,d\phi$$
$$-\sin n\psi \int_{-\pi}^{\pi} \sin n\phi \cos(z\cos\phi)\,d\phi\ .$$

From Ryzhik and Gradshteyn (1965 - page 402) we have (since sine is odd and cosine is even) the result

$$\int_{-\pi}^{\pi} \cos(n\theta)\cos(z\cos(\theta-\psi))\,d\theta$$
$$= \cos n\psi\left[2\pi\cos\left(\frac{n\pi}{2}\right)J_n(z)\right]\ . \tag{5.10}$$

In a similar way we get

$$\int_{-\pi}^{\pi} \cos(n\theta)\sin(z\cos(\theta-\psi))\,d\theta$$
$$= \cos n\psi\left[2\pi\sin\left(\frac{n\pi}{2}\right)J_n(z)\right], \tag{5.11}$$

where $J_n(Z)$ is a Bessel function of the first kind of integer order n. Returning to Equation (5.9), the above results give, for $n = 0$ where $f_o > 0$, the result

$$P_{ij}(\pm f_o) = \tilde{A}\, 2\pi K_o J_o(2\pi K_o D_{ij}) \quad .$$

Since the above is real

$$\begin{aligned} C_{ii}(\pm f_o) &= 2\pi K_o \tilde{A} \\ C_{ij}(\pm f_o) &= 2\pi K_o \tilde{A} J_o(2\pi K_o D_{ij}) \\ Q_{ij}(\pm f_o) &= 0 \end{aligned} \qquad (5.12)$$

describe the elements for the spectral matrix of a single frequency isotropic sea.

The above two special cases for sea waves are the extremes of directionality of real sea waves of frequency $f_o > 0$. These results will have important applications later.

6. A MEASURE OF ARRAY DIRECTIONAL RESOLVING POWER

From Equation (3.9) we have the Fourier transform

$$S(l.m.f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^*(X,Y,f) \, \mathrm{EXP}\left(-i\, 2\pi (l X + m Y)\right) dx\, dY \qquad (6.1)$$

In practice we use a probe array with elements at $(x_1, y_1) \ldots, (x_k, y_k)$ to obtain $P^*(X,Y,f)$ at the separation points (0.0), (X_{12}, Y_{12}), $(-X_{12}, -Y_{12}) \ldots, (X_{k-1,k}, Y_{k-1,k}), (-X_{k-1,k}, -Y_{k-1,k})$.

We do not then know $P^*(X,Y,f)$ but the product

$$[P^*(X,Y,f)\ g(X,Y)] \qquad (6.2)$$

where g(X,Y) is a set of Dirac delta functions standing on the separation points of the probe array, and zero elsewhere.

Thus, we have the estimate

$$\hat{S}(\ell,m,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[P^{*}(X,Y,f)\, g(X,Y)\right] \mathrm{EXP}\left(-i2\pi(\ell X + mY)\right) dX\, dY$$

Let

$$G(\ell,m) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(X,Y)\, \mathrm{EXP}\left(-i2\pi(\ell X + mY)\right) dX\, dY \qquad (6.3)$$

Using this and Equation (3.8) we have for a given (ℓ_0, m_0, f_0) that

$$\hat{S}(\ell_0,m_0,f_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\ell,m,f_0)\, \mathrm{EXP}\left(i2\pi(\ell X + mY)\right) d\ell\, dm\right] g(X,Y)$$

$$\mathrm{EXP}\left(-i2\pi(\ell_0 X + m_0 Y)\right) dX\, dY$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\ell,m,f_0)\, g(X,Y)\, \mathrm{EXP}\left(-i2\pi(X(\ell_0-\ell) + Y(m_0-m))\right) dX\, dY\right] d\ell\, dm$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\ell,m,f_0)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(X,Y)\, \mathrm{EXP}\left(-i2\pi(X(\ell_0-\ell) + Y(m_0-m))\right) dX\, dY\right] d\ell\, dm$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\ell,m,f_0)\, G(\ell_0-\ell, m_0-m)\, d\ell\, dm \qquad (6.4)$$

As expected, $\hat{S}(\ell,m,f)$ is a two-dimensional convolution of the true directional spectrum $S(\ell,m,f)$ with $G(\ell,m)$ the Fourier transform of $g(X,Y)$. We see then that $\hat{S}(\ell,m,f)$ is a weighted average of $S(\ell,m,f)$

and that $G(\ell,m)$ is a measure of the directional resolving power of the assumed probe array. By the nature of $g(X,Y)$ we have from Equation (6.3)

$$G(\ell,m) = 1 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\cos\left(2\pi\left(\ell X_{ij} + m Y_{ij}\right)\right) \tag{6.5}$$

If we assume that the wave equation (Equation (3.1)) holds, we find that $S(\ell,m,f_o)$ is zero when $\ell^2 + m^2 \neq k_o^2$, the wave number for f_o. From this we have for a given $(\ell_o m_o)$ that the directional resolving power, DRP, is

$$DRP(\ell,m,f_o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta\left((\ell^2+m^2)-K_o^2\right)G(\ell_o-\ell, m_o-m)\,d\ell\,dm$$

LET $\ell = K_o\cos\theta$, $m = K_o\sin\theta$, WHICH IMPLIES $\ell^2+m^2=K_o$

and we get, for energy coming from a direction θ_o at frequency f_o as a function of $0 \leq \theta \leq 2\pi$, that

$$DRP(\theta|\theta_o,f_o) = G\left(K_o(\cos\theta_o-\cos\theta), K_o(\sin\theta_o-\sin\theta)\right).$$

From Equation (6.5) we get

$$DRP(\theta|\theta_o,f_o) = 1 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\cos\left[2\pi\left(X_{ij}K_o(\cos\theta_o-\cos\theta) + Y_{ij}K_o(\sin\theta_o-\sin\theta)\right)\right]$$

$$= 1 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\cos\left[2\pi\left((\ell_o-K_o\cos\theta)X_{ij} + (m_o-K_o\sin\theta)Y_{ij}\right)\right]$$

NOTE THAT $DRP(\theta_o|\theta_o f_o) = N(N-1)+1$ (6.6)

Compare Equation (6.6) and (5.5). Except for the amplitude term, $A^2/4M$, in Equation (5.5) the equations give identical results. The choice of $b_{ij} = [N(N-1) + 1]^{-1} = 1/M$ is again found convenient.

7. DIRECTIONAL ANALYSIS FROM THE CROSS SPECTRAL MATRIX

First, we consider a fundamental approach. We have for a pair of detectors I and J located at (x_i, y_i) and (x_j, y_j) respectively, the cross spectral matrix, $P_{ij}(f) = C_{ij}(f) + iQ_{ij}(f)$ and more importantly $\phi_{ij}(f)$ the phase lead of I over J given by,

$$\phi_{ij}(f) = \text{ARCTAN}\left[\frac{Q_{ij}(f)}{C_{ij}(f)}\right] . \tag{7.1}$$

The actual phase lead may differ from this value since the true phase lead θ is some one of the values

$$\phi_{ij}(f) + h\,2\pi$$

where

$$h = 0, \pm 1, \pm 2, \cdots .$$

Consider a single wave of frequency f_o with corresponding wave length λ_o and wave number $K_o = 1/\lambda_o$, and find the direction the wave must travel; i.e., fit a single wave to the spectral matrix results for the detectors I and J.

Let D_{ij} be the distance between I and J. The distance between I and J in wave lengths is $K_o D_{ij}$. In radians this is $2\pi K_o D_{ij}$. From this relation we get

$$-2\pi K_o D_{ij} \leq \phi \leq 2\pi K_o D_{ij}$$

or that only values of h such that

$$-2\pi K_o D_{ij} \leq \phi_{ij}(f) + h\,2\pi \leq 2\pi K_o D_{ij} \tag{7.2}$$

give physically acceptable candidates for the value of ϕ. If $D_{ij} < \lambda_o/2$ only one h value is valid. If $\lambda_o/2 < D_{ij} < \lambda_o$ at most two values of h are valid, etc.

The problem is how to find the true direction, θ_o, of a single wave given the above possible value(s) of ϕ. Consider a given value of ϕ in terms of wave length units and we obtain

$$L_0 = \frac{\phi}{2\pi}\lambda_0 = \frac{\phi}{2\pi K_0} \quad . \qquad (7.3)$$

Figure 4 illustrates a case for $\phi \geq 0$ (and thus $L_o \geq 0$). From the figure we have, where the true direction is θ_o and true phase is ϕ, that

$$\theta_0 = \psi_{ij} + \frac{\pi}{2} + \alpha$$

WHERE $$\sin\alpha = \frac{\phi}{2\pi K_0 D_{ij}} \qquad (7.4)$$

OR $$\alpha = \arcsin\left[\frac{\phi}{2\pi K_0 D_{ij}}\right] = \arcsin\left[\frac{L_0}{D_{ij}}\right]$$

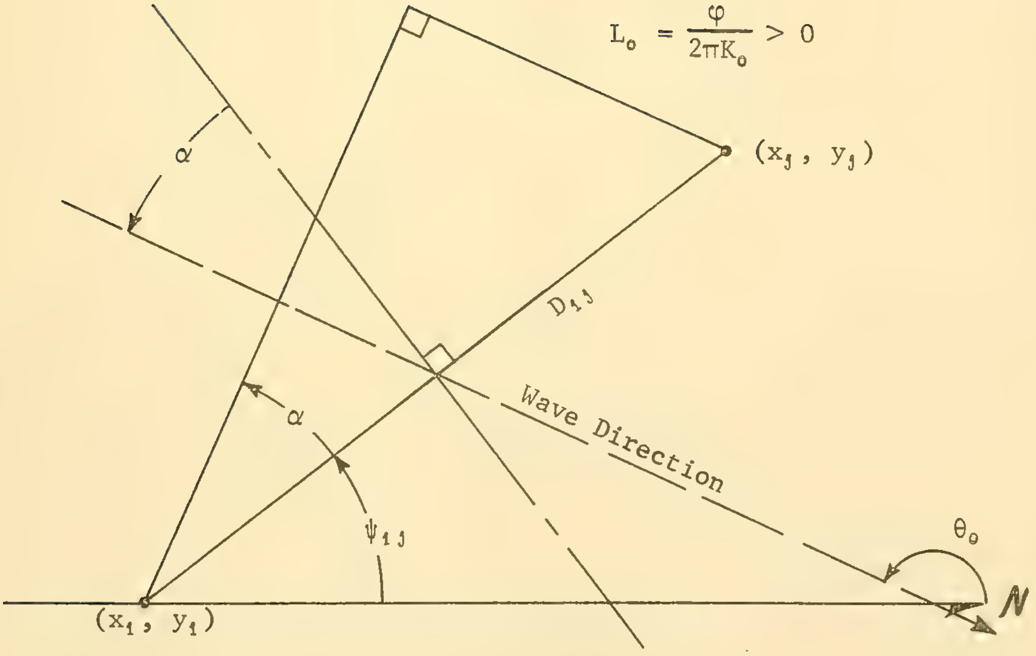


FIGURE 4. WAVE DIRECTION ANALYSIS

Recall that $\sin(\alpha) = \sin(\underline{+}\pi-\alpha)$. Thus for a given $\phi \geq 0$ we get, since α must be obtained from an arcsin relationship, two possible values of wave direction, the true value θ_o and its image

$$\theta' = \left[\psi_{ij} + \frac{\pi}{2} \right] + \left[-\pi - \alpha \right]$$

$$= \psi_{ij} - \frac{\pi}{2} - \alpha = \psi_{ij} - \left[\frac{\pi}{2} + \alpha \right] \tag{7.5}$$

This is illustrated in Figure 5. In actual practice we do not know the true direction, thus a given value of $\phi \geq 0$ gives the direction as

$$\theta_o = \psi_{ij} \pm \left[\frac{\pi}{2} + \alpha \right]$$

where $0 \leq \alpha \leq \pi/2$ is obtained from the principle value of the arcsin. If $\phi < 0$, then I actually lags J by $|\phi| > 0$ and $L_o' = L_o/D_{ij} < 0$, so that arcsin (L_o') gives $-\pi/2 \leq \alpha < 0$.

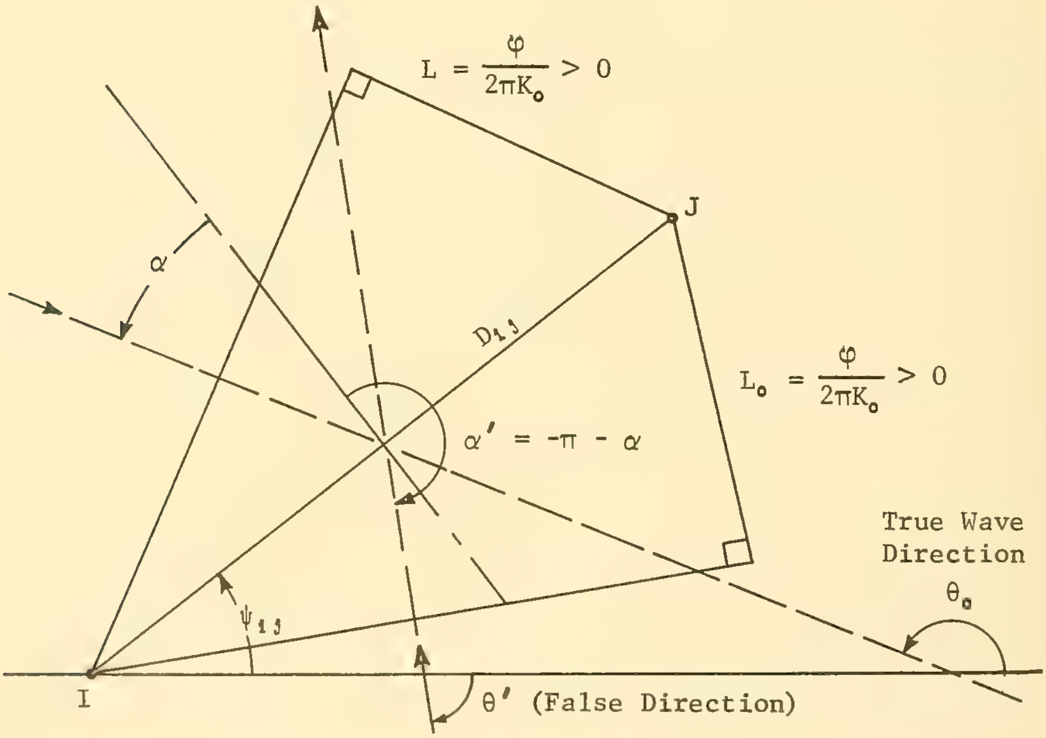


FIGURE 5. DIRECTION ANALYSIS FROM A PAIR OF ARRAY ELEMENTS

The implied directions, for $\phi < 0$, are directly opposite from those for $\phi \geq 0$ so that for a given value of $\phi \leq 0$ we get directions ($\alpha < 0$)

and $$\theta_o = \psi_{ij} - \frac{\pi}{2} - \alpha$$

or $$\theta_o = \psi_{ij} + \frac{\pi}{2} + \alpha$$

$$\theta = \psi_{ij} \pm \left[\frac{\pi}{2} + \alpha\right]$$

as before.

Thus, where the principle value of arcsin is assumed, we get a set of possible directions

$$\theta_{o_h} = \psi_{ij} \pm \left[\frac{\pi}{2} + \alpha_h\right]$$

where

$$\alpha_h = \text{ARCSIN}\left[\frac{\Phi_{ij}(f) + h2\pi}{2\pi K_o D_{ij}}\right] \qquad (7.6)$$

h being constrained by

$$\left|\frac{\Phi_{ij}(f) + h2\pi}{2\pi K_o D_{ij}}\right| \leq 1 \quad .$$

An example of this type analysis, for an array pair, is illustrated in Figure 6. Thus several estimates of θ_o are available (at least two).

The estimates of true direction, θ_o, often vary from one array element pair to another, making the selection of a true θ_o value difficult. The selection is also hindered because half of the estimates of θ_o are of the image type; i.e., false estimates.

While the above directional method leaves something to be desired, it does illustrate the basic directional information produced by an array of detectors.

A better method, suggested in Munk et al (April 1963), of using the spectral matrix directional information to fit a single wave at each

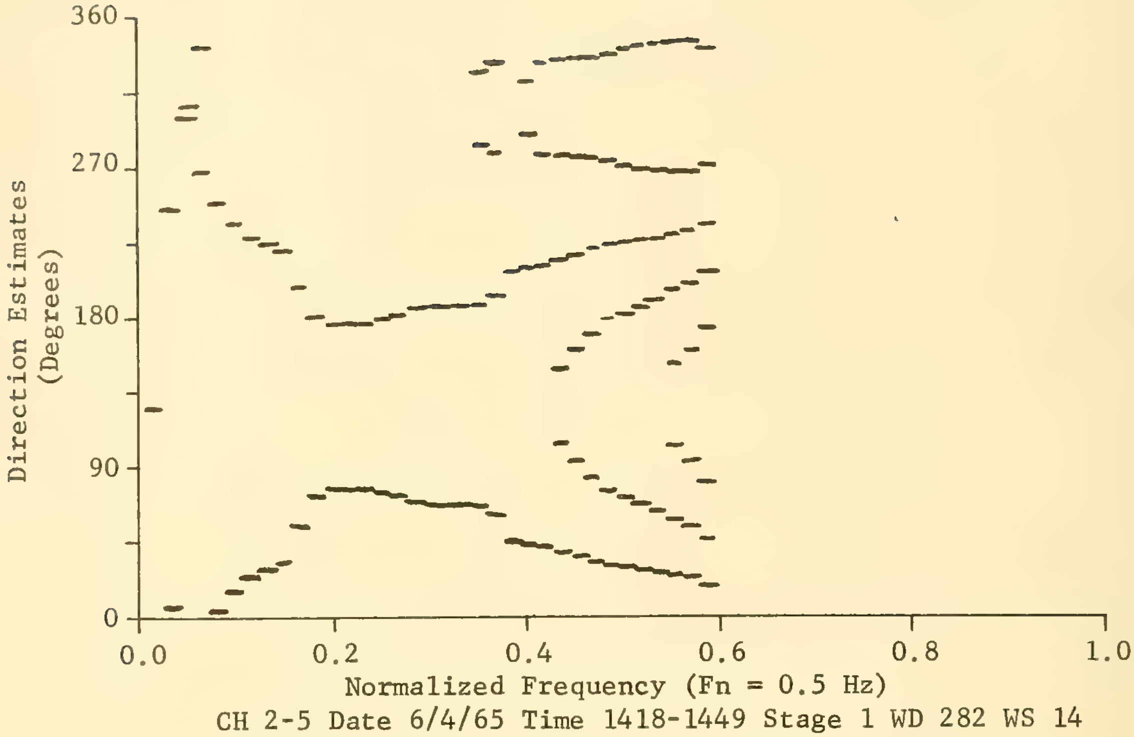


FIGURE 6. DIRECTIONAL ESTIMATES FOR A PAIR OF ARRAY ELEMENTS

frequency is given below. It is based on Equation (4.16) in the form

$$S(\ell,m,f) \doteq \frac{1}{M}\left[C_o(f) + 2\sum_{i=1}^{N}\sum_{j=i+1}^{N}\left[C_{ij}(f)\cos\left(2\pi(\ell X_{ij} + mY_{ij})\right) - Q_{ij}(f)\sin\left(2\pi(\ell X_{ij} + mY_{ij})\right)\right]\right] \tag{7.7}$$

where M = [N(N-1) + 1], $C_o(f) = \frac{1}{N}\sum_{i=1}^{N} C_{ii}(f)$

and N = number of array elements.

Recall that for a single real wave of frequency $f_o > 0$ and known direction θ_o, $C_{ij}(f_o)$, $Q_{ij}(f_o)$ are known (see Equation (5.3)). These values give

$$S(K_o, \theta_o, f_o) = S(K_o, [\theta_o - \pi], -f_o) = \frac{A^2}{4} \quad .$$

When a single, well-directed swell is expected, it is reasonable to assume a single wave for a given frequency exists, and to select θ_o and $A_o = A^2/4$ such that the least square error between the theoretical cross spectral matrix for a single wave (see Equations (5.1) and (5.3)) and an observed cross spectral matrix is a minimum. Accordingly, using the expressions of Equation (5.3) and an observed cross spectral matrix for a given frequency, f_o, we can form the squared error

$$H = (C_o - A_o)^2$$

$$+ 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[C_{ij} - A_o \cos\left(2\pi\left(l_o X_{ij} + m_o Y_{ij}\right)\right)\right]^2$$

$$+ 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left[Q_{ij} + A_o \sin\left(2\pi\left(l_o X_{ij} + m_o Y_{ij}\right)\right)\right]^2$$

$$\text{NOTE: } A_o = \frac{A^2}{4} \;:\; C_o = \frac{1}{N}\sum_{i=1}^{N} C_{ii} \qquad (7.8)$$

Expanding and collecting terms we get

$$H = C_o^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(C_{ij}^2 + Q_{ij}^2\right) + \left[N(N-1)+1\right]A_o^2$$

$$- 2A_o\left[C_o + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} C_{ij}\cos\left(2\pi\left(l_o X_{ij} + m_o Y_{ij}\right)\right)\right.$$

$$\left. - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} Q_{ij}\sin\left(2\pi\left(l_o X_{ij} + m_o Y_{ij}\right)\right)\right] \qquad (7.9)$$

or using results in Equation (7.7)

$$H = C_0^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(C_{ij}^2 + Q_{ij}^2) + [N(N-1)+1][A_0^2 - 2A_0 S(\ell,m,f_0)] \tag{7.10}$$

To find A_o that minimizes H, consider

$$\frac{\partial H}{\partial A_0} = 2[N(N-1)+1][A_0 - S(\ell,m,f_0)] = 0 \tag{7.11}$$

This requires that A_o be of the form $A_o = S(\ell,m,f_o)$ and a resulting value of H of the form

$$H = C_0^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(C_{ij}^2 + Q_{ij}^2) - [N(N-1)+1][S(\ell,m,f_0)]^2 \tag{7.12}$$

Since C_o^2, C_{ij}^2, and Q_{ij}^2 are all nonnegative, a minimum H results when $S(\ell,m,f_o)$ is a maximum. A choice of ℓ_o and m_o that maximizes $S(\ell,m,f_o)$ implies a $\theta_o = \arctan m_o/\ell_o$ which is optimum. Remember that we are assuming $K_o^2 = \ell_o^2 + M_o^2$ holds, along with the wave equation. The results then for each f_o is a two-sided energy spectrum estimate $A_o(f_o,\theta_o)$. Appendix B contains a listing of a FORTRAN II program for finding $A_o(f_o,\theta_o)$, the least square wave fit, from a set of spectral matrices obtained from the task SWOC data collection and analysis system described in Bennett et al (June 1964).

Examples of least square single-wave train fit analysis from Bennett (March 1968) are shown in Figure 7.

A more complete collection of the directional spectra calculated from data collected off Panama City, Florida, is given in Appendix A, and in Bennett (November 1967), and Bennett and Austin (September 1968), an unpublished Laboratory Technical Note TN160.

We would actually like a continuous estimate of $S'(\ell,m,f)$. Consider then a third method. From Equation (3.8) we have for a pair of detectors, as illustrated in Figure 8, that

$$P^*(X,Y,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\ell,m,f)\, \mathrm{EXP}[i2\pi(\ell X + mY)]\, d\ell\, dm \; .$$

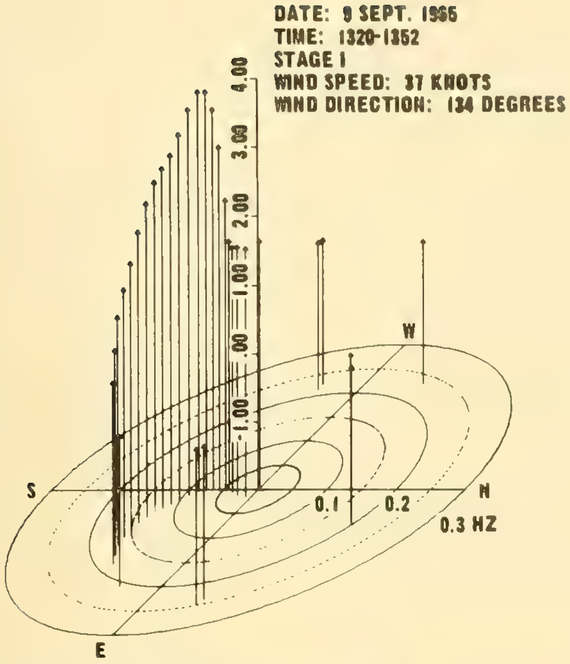


FIGURE 7A DIRECTIONAL SPECTRUM $A_o(f,\theta)$

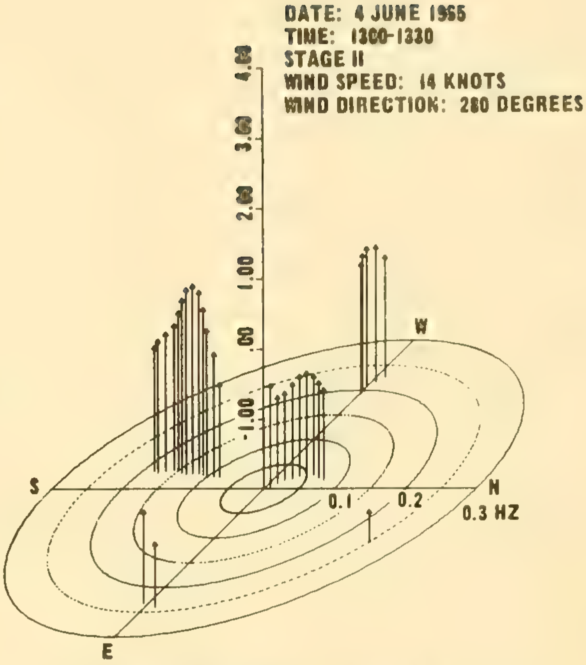


FIGURE 7B DIRECTIONAL SPECTRUM $A_o(f,\theta)$

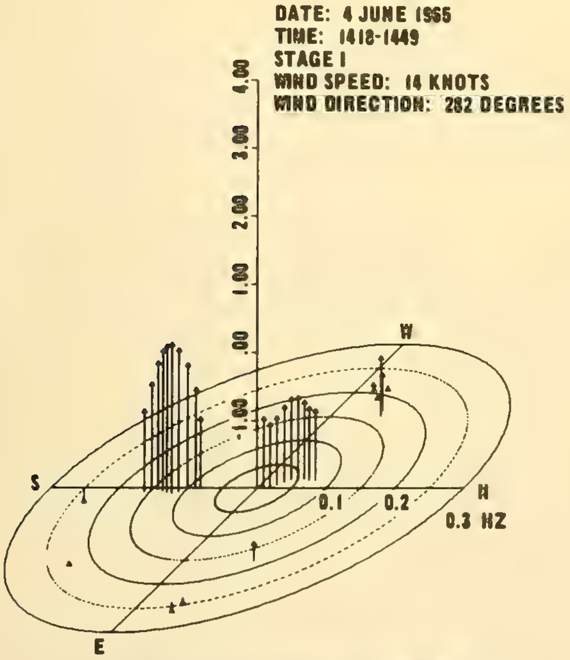


FIGURE 7C DIRECTIONAL SPECTRUM $A_o(f,\theta)$

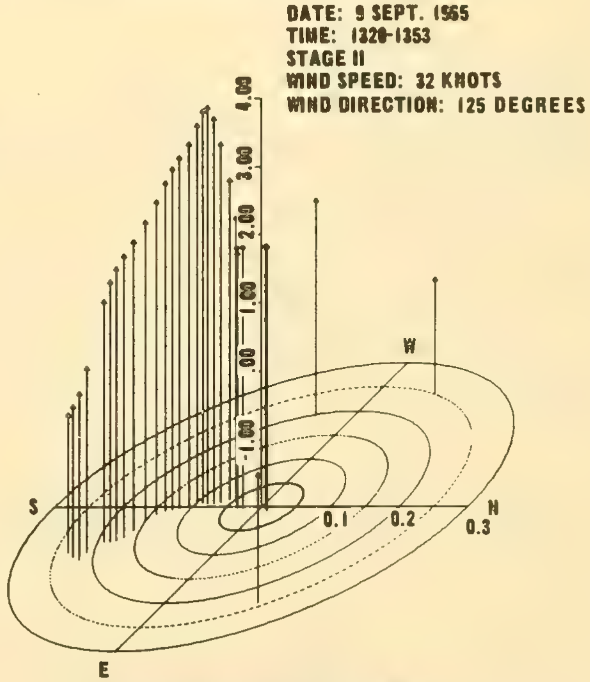


FIGURE 7D DIRECTIONAL SPECTRUM $A_o(f,\theta)$

FIGURE 7. LEAST SQUARE SINGLE WAVE FIT DIRECTIONAL SPECTRA $A_o(f,\theta_o)$

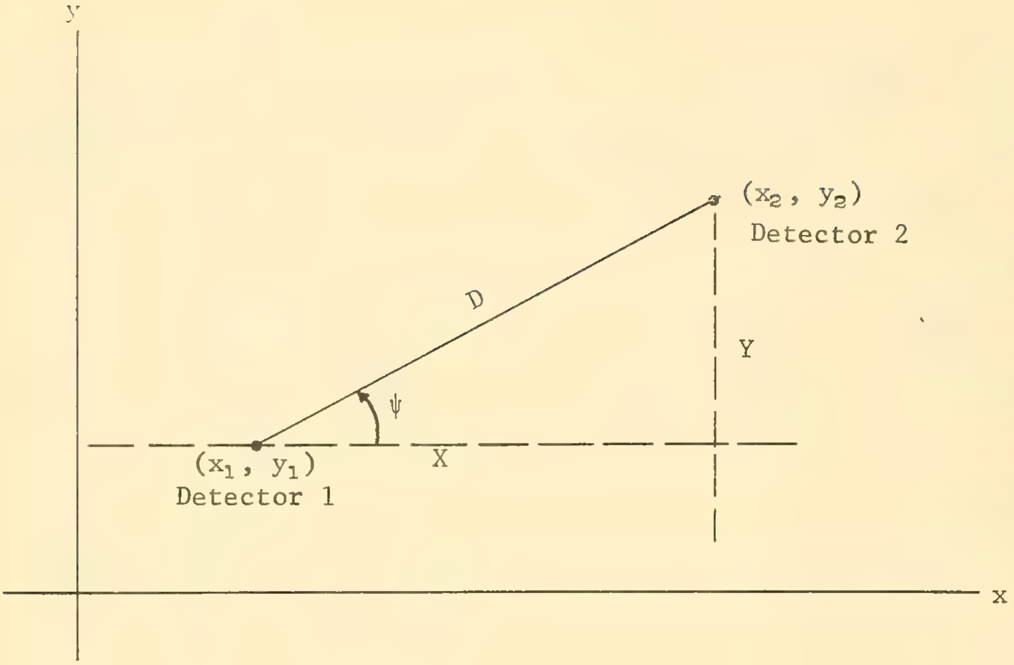


FIGURE 8. DETECTOR GEOMETRY

Let $\ell = K\cos\theta$ and $m = K\sin\theta$

and

$$D = \sqrt{X^2 + Y^2},\ \psi = \text{Arctan}\,\frac{Y}{X}$$

or

$$X = D\cos\psi \text{ and } Y = D\sin\psi.$$

Using the above changes of variable, we get

$$P^*(X,Y,f) = \int_{-\pi}^{\pi}\int_{0}^{\infty} S(K\,\theta\,f)\,\text{EXP}\left[i2\pi\left(KD\cos\theta\cos\psi + KD\sin\theta\sin\psi\right)\right] K\,dK\,d\theta$$

$$P^*(X,Y,f) = \int_{-\pi}^{\pi}\int_{0}^{\infty} S(K,\theta,f)\,\text{EXP}\left[i2\pi KD\cos(\theta-\psi)\right] K\,dK\,d\theta \quad (7.13)$$

We have from Equation (3.2) that

$$S(K,\theta,f_o) = S(K,\theta-\pi,-f_o)$$

If we think in terms of $f_o > 0$

$$S'(K,\theta,f_o) = 2S(K,\theta,f_o) \qquad (7.14)$$

We are assuming that the wave number relation of Equation (3.1) holds. Thus Figure 3 is applicable, and we can write the one-sided spectral density as

$$S'(K,\theta,f_o) = 2a(\theta,f_o)\,\delta(K-K_o)$$

$$\text{WHERE} \quad \delta(K-K_o) = \begin{cases} 1 & K = K_o \\ 0 & K \neq K_o \end{cases}$$

This allows us to write, $-\infty < f_o < +\infty$,

$$P^*(X,Y,f_o) = \int_{-\pi}^{\pi}\int_{0}^{\infty} a(\theta,f_o)\,\delta(K-K_o)\,\mathrm{EXP}\left[i2\pi KD\cos(\theta-\psi)\right]K\,dK\,d\theta$$

$$P^*(X,Y,f_o) = \int_{-\pi}^{\pi} a(\theta,f_o)\,\mathrm{EXP}\left[i2\pi K_oD\cos(\theta-\psi)\right]K_o\,d\theta \qquad (7.15)$$

We have reduced the problem to finding $[a(\theta,f_o \cdot K_o]$.

From Equation (7.15) we see that

$$P^*(0,0,f) = P(f) = \int_{-\pi}^{\pi} a(\theta,f_o)\,K_o\,d\theta$$

where $P(f_o)$ is the power spectral density of frequency f_o. For better comparison of cases where $P(f_1) = P(f_o)$, $|f_1| \neq |f_o|$, it is convenient to express $a(\theta,f_o)$ in a normalized form

$A(\theta,f_o) = a(\theta,f_o)K_o$

where we get

$$P(f) = \int_{-\pi}^{\pi} A(\theta, f)\, d\theta \qquad (7.16)$$

Thus if the energy distribution as a function of direction is the same for $P(f_1) = P(f_o)$ then we also get

$$A(\theta, f_1) = A(\theta, f_o).$$

Consider now (assuming $A(\theta,f)$ can be so expressed) a Fourier series expansion of $A(\theta,f)$ for fixed f. Clearly it is periodic in θ with period 2π. Thus for any given $f = f_o$ we can write $A(\theta,f)$ as

$$A(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos n\theta + b_n \sin n\theta\right] \qquad (7.17)$$

Substituting this expansion into Equation (7.15) we get

$$P^*(X, Y, f) = \frac{a_0}{2}\int_{-\pi}^{\pi} \exp i\, 2\pi KD \cos(\theta - \psi)\, d\theta + \sum_{n=1}^{\infty}\Big[a_n \int_{-\pi}^{\pi} \cos n\theta$$

$$\exp i\, 2\pi KD \cos(\theta-\psi)\, d\theta + b_n \int_{-\pi}^{\pi} \sin n\theta \exp i\, 2\pi KD \cos(\theta-\psi)\, d\theta\Big]$$

$$\text{or } P^*(X, Y, f) = \left[\frac{a_0}{2}\int_{-\pi}^{\pi} \cos\big(2\pi KD \cos(\theta-\psi)\big)\, d\theta + \sum_{n=1}^{\infty}\Big(a_n \int_{-\pi}^{\pi} \cos n\theta \cos\big(2\pi KD\right.$$

$$\left.\cos(\theta-\psi)\big)\, d\theta + b_n \int_{-\pi}^{\pi} \sin n\theta \cos\big(2\pi KD \cos(\theta-\psi)\big) d\theta\Big)\right]$$

$$+ j\left[\frac{a_0}{2}\int_{-\pi}^{\pi} \sin\big(2\pi KD \cos(\theta-\psi)\big) d\theta + \right.$$

$$+ \sum_{n=1}^{\infty}\Big(a_n \int_{-\pi}^{\pi} \cos n\theta \sin\big(2\pi KD \cos(\theta-\psi)\big) d\theta +$$

$$\left. b_n \int_{-\pi}^{\pi} \sin n\theta \sin\big(2\pi KD \cos(\theta-\psi)\big) d\theta\Big)\right] \qquad (7.18)$$

Thus we can express P*(X,Y,f) as complex infinite series with unknown coefficients a_0, a_1, a_2,... and b_1, b_2, ... and constants defined by the integrals (let $Z = 2\pi KD$; $n = 0, 1, 2, \ldots$) of the form

$$\int_{-\pi}^{\pi} \cos n\theta \cos(z \cos(\theta-\psi))\, d\theta \tag{7.19}$$

$$\int_{-\pi}^{\pi} \sin n\theta \cos(z \cos(\theta-\psi))\, d\theta \tag{7.20}$$

$$\int_{-\pi}^{\pi} \cos n\theta \sin(z \cos(\theta-\psi))\, d\theta \tag{7.21}$$

and

$$\int_{-\pi}^{\pi} \sin n\theta \sin(z \cos(\theta-\psi))\, d\theta \tag{7.22}$$

Consider Equation (7.19) where $\phi = (\theta-\psi)$ and $d\phi = d\theta$, ψ being a constant. We then have on changing variables

$$\begin{aligned}\int_{-\pi-\psi}^{\pi-\psi} & \cos(n\phi + n\psi)\cos(z\cos\phi)\, d\phi \\ &= \cos n\psi \int_{-\pi-\psi}^{\pi-\psi} \cos n\phi \cos(z\cos\phi)\, d\phi \\ &\quad - \sin n\psi \int_{-\pi-\psi}^{\pi-\psi} \sin n\phi \cos(z\cos\phi)\, d\phi\end{aligned} \tag{7.23}$$

Since the integrands in Equation (7.23) are both of period 2π and the interval $[-\pi-\psi, \pi-\psi]$ is of length 2π we can write the equivalent of Equation (7.23) as

$$\begin{aligned}&\cos n\psi \int_{-\pi}^{\pi} \cos n\phi \cos(z\cos\phi)\, d\phi \\ &- \sin n\psi \int_{-\pi}^{\pi} \sin n\phi \cos(z\cos\phi)\, d\phi\end{aligned} \tag{7.24}$$

From Ryzhik and Gradshteyn (1965 - page 402) and noting that the second integrand is odd we get Equation (7.23) equivalent to

$$\cos n\psi \left[2\pi \cos\left(\frac{n\pi}{2}\right) J_n(z)\right] \qquad (7.25)$$

where $J_n(Z)$ is the Bessel function of the first kind. Employing a similar procedure for Equations (7.20), (7.21), and (7.22) we get

$$P^*(X,Y,f) = \frac{a_0}{2}\left[2\pi\, J_0(2\pi KD)\right] + \sum_{n=1}^{\infty}\left[a_n \cos n\psi\, 2\pi \cos\left(\frac{n\pi}{2}\right) J_n(2\pi KD) + b_n \sin n\psi\, 2\pi \cos\left(\frac{n\pi}{2}\right) J_n(2\pi KD)\right]$$
$$+ j\sum_{n=1}^{\infty}\left[a_n \cos n\psi\, 2\pi \sin\left(\frac{n\pi}{2}\right) J_n(2\pi KD) + b_n \sin n\psi\, 2\pi \sin\left(\frac{n\pi}{2}\right) J_n(2\pi KD)\right] \qquad (7.26)$$

Now $$\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2} & n \text{ even} \end{cases}$$

and $$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

Thus we get $$P^*(X,Y,f) = \frac{a_0}{2}\left[2\pi\, J_0(2\pi KD)\right]$$
$$+ \sum_{n=2,4,6,\cdots}^{\infty}\left[2\pi\, J_n(2\pi KD)(-1)^{\frac{n}{2}}\left(a_n \cos n\psi + b_n \sin n\psi\right)\right]$$
$$+ j\sum_{n=1,3,5,\cdots}^{\infty}\left[2\pi\, J_n(2\pi KD)(-1)^{\frac{n-1}{2}}\left(a_n \cos n\psi + b_n \sin n\psi\right)\right] \qquad (7.27)$$

From a spectral matrix of the form in Equation (4.13), $M = N(N-1) +1$ different equations can be set up using Equation (7.27). This allows us to get a system of equations for any m of the unknown coefficients $a_0, a_1, a_2, \ldots ; b_1, b_2, b_3, \ldots$ while assuming the rest of the coefficients are negligible. We can then solve for the m desired coefficient values. This has not worked well in practice for two reasons. The inverse of the matrix of constants obtained is sparse and often ill-conditioned. Further, if the wave energy is from a narrow beam width (30 degrees or less), the first 100 harmonics in the Fourier series expansion can be significant. There is perhaps a more efficient orthogonal set of functions than the sines and cosines of the standard Fourier expansion. Search for such an orthogonal set should prove fruitful. One might start with Walsh or Haar functions. See Hammond and Johnson (February 1960).

8. SUMMARY

It is believed that the least square method of using the information in a spectral matrix is the best method presently available. Examples of such analysis can be found in several of the papers in the bibliography. A collection of ocean-wave induced, bottom pressure directional spectra from these papers is given in Appendix A.

An iterative extension of the least square method can be found in an excellent paper by Munk et al (April 1963). Some details of this method are given in Appendix C along with an example result and a FORTRAN program for the method.

There is merit to using the coherency,

$$R_{ij}(f) = \frac{|P_{ij}(f)|}{(P_{ii}\ P_{jj})^{\frac{1}{2}}} ,$$

to form the weights b_{ij} in Equation (4.16). One idea being explored is

$$b_{ij} = \frac{R_{ij}}{1+2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} R_{ij}} = \frac{R_{ij}}{R} .$$

REFERENCES

Bennett, C. M., "A Directional Analysis of Sea Waves from Bottom Pressure Measurements," *Transactions: Ocean Sciences and Engineering of the Atlantic Shelf,* Marine Technology Society, Washington, D. C., pp. 71-87 (March 1968), Unclassified.

U.S. Navy Mine Defense Laboratory Report 344, *Power Spectra of Bottom Pressure Fluctuations in the Nearshore Gulf of Mexico During 1962 and 1963,* by C. M. Bennett, November 1967, Unclassified.

Bennett, C. M., Pittman, E. P., and Austin, G. B., "A Data Processing System for Multiple Time Series Analysis of Ocean Wave Induced Bottom Pressure Fluctuations," *Proceedings of the First U.S. Navy Symposium on Military Oceanography,* U.S. Navy Oceanographic Office, Washington, D. C., pp. 379-415, June 1964.

David Taylor Model Basin Report No. 8, *On the Joint Estimation of the Spectra, Cospectrum and Quadrature Spectra of a Two-Dimensional Stationary Gaussian Process,* by N. R. Goodman, March 1957, Unclassified.

N. Y. University Technical Report No. 8, (EES Project No. A-366), *Two Orthogonal Classes of Functions and Their Possible Applications,* by J. L. Hammond and R. S. Johnson, February 1960, Unclassified.

Kinsman, Blair, *Wind Waves - Their Generation and Propagation on the Ocean Surface,* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965 (pp. 368-386).

Munk, W. H., Miller, G. R., Snodgrass, F. E., and Barber, N. F., "Directional Recording of Swell from Distant Storms," *Philosophical Transactions of the Royal Society of London,* Series A. Vol. 255, No. 1062, pp. 505-584, April 1963.

Parzen, Emanuel, *Stochastic Processes,* Holden-Day, Inc., San Francisco, California (1962).

Pierson, W. J. Jr., *Wind Generated Gravity Waves,* Academic Press, Inc., New York (1955), v. 2, pp. 126-129, 1955.

Pierson, W. J. Jr., and Marks, Wilbur, "The Power Spectrum Analysis of Ocean Wave Records," *Transactions, American Geophysical Union,* v. 36, No. 6 (1952).

Ryzhik, I. M. and Gradshteyn, I. S., *Table of Integrals, Series, and Products,* Academic Press, New York, 1965.

St. Denis, M., and Pierson, W. J. Jr., "On the Motions of Ships in Confused Seas," *Transactions of the Society of Naval Architects and Marine Engineers,* v. 61, pp. 280-357 (1953).

BIBLIOGRAPHY

Barber, N. F., "Design of 'Optimum' Arrays for Direction Finding," *Electronic and Radio Engineer,* New Series 6, v. 36, pp. 222-232, June 1959.

Bennett, C. M., "Digital Filtering of Ocean Wave Pressure Records to Records of Prescribed Power Spectral Content," *Proceedings of the Fifth Annual Southeastern Regional Meeting of Association for Computing Machinery,* June 1966.

Bennett, C. M., "An Annual Distribution of the Power Spectra of Ocean Wave Induced Bottom Pressure Fluctuations in the Near Shore Gulf of Mexico," (abstract), *Transactions, American Geophysical Union,* v. 48, No. 1, p. 140, April 1967.

Texas A&M College Reference 62-1T, *Instrumentation and Data Handling System for Environmental Studies off Panama City, Florida,* by Roy D. Gaul, February 1962, Unclassified.

Texas A&M University Reference 66-12T, *Automated Environmental Data Collected off Panama City, Florida, January 1965 - April 1966,* by A. Kirst, Jr., and C. W. McMath, June 1966, Unclassified.

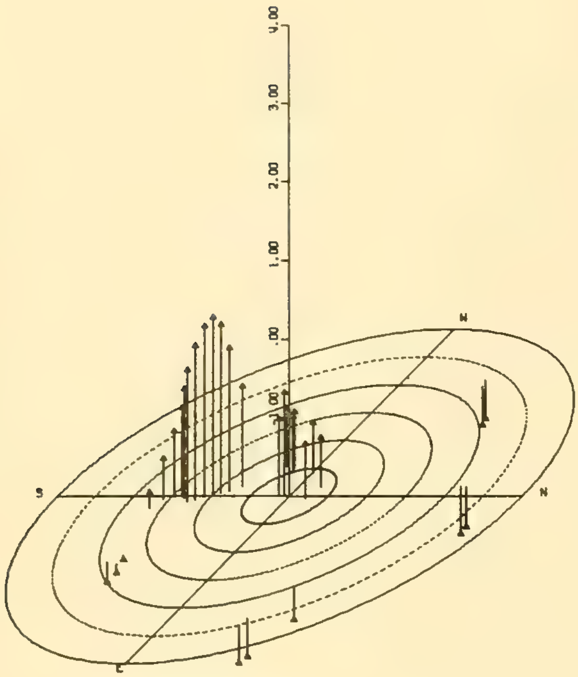
Scripps Institute of Oceanography Bulletin, *Spectra of Low-Frequency Ocean Waves,* by W. H. Munk, F. E. Snodgrass, and J. J. Tucker, v. 7, No. 4, pp. 283-362, 1959.

APPENDIX A

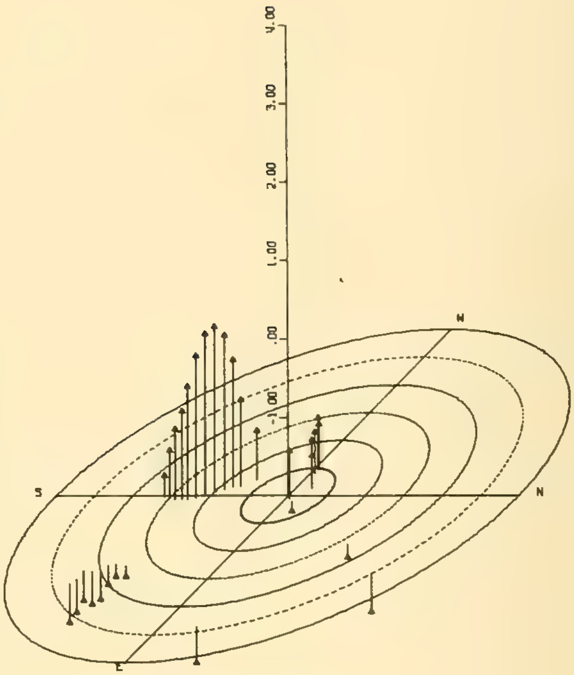
A COLLECTION OF DIRECTIONAL OCEAN WAVE BOTTOM PRESSURE POWER SPECTRA

This appendix is a collection of the results of a least square, directional, single-wave train analysis of the cross power spectral matrix resulting from the analysis of ocean bottom pressure data. The data were collected at Stages I and II offshore from Panama City, Florida, during 1965. The data collection system and the estimation of the cross power spectral matrix associated with a set of data are described in Bennett, et al (June 1964). Augmented pentagonal arrays, containing six pressure transducers each, were located seaward of each of the stages. Stage I is 11 miles offshore in approximately 103 feet of water, and Stage II is 2 miles offshore in approximately 63 feet of water.

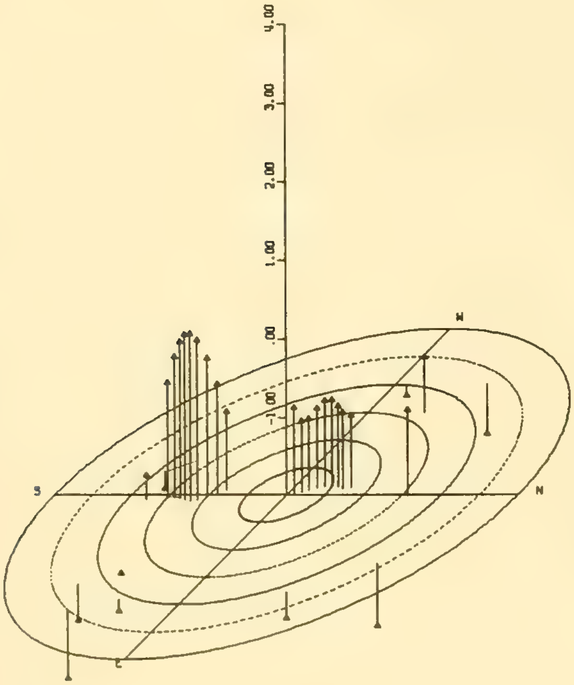
Certain parameter values are pertinent to the directional analyses presented: the number of data points in each pressure data set is N = 1800; the sampling rate is once per second, Δt = 1 second. On the cylindrical polar plots frequency is the radial variable and compass bearing the angular variable. The vertical axis is $\log_{10}$ of power spectral density in inches^2-seconds of water pressure. The frequency axis range is 0 to 0.3 Hz in 0.05 Hz increments. This is illustrated in Figure 7 of the report. In each plot title, the date, time, and location (stage) is indicated. The value WD is wind direction in compass degrees, and WS is wind speed in knots. Appendix B gives a listing of the FORTRAN II computer program used to produce the plots.



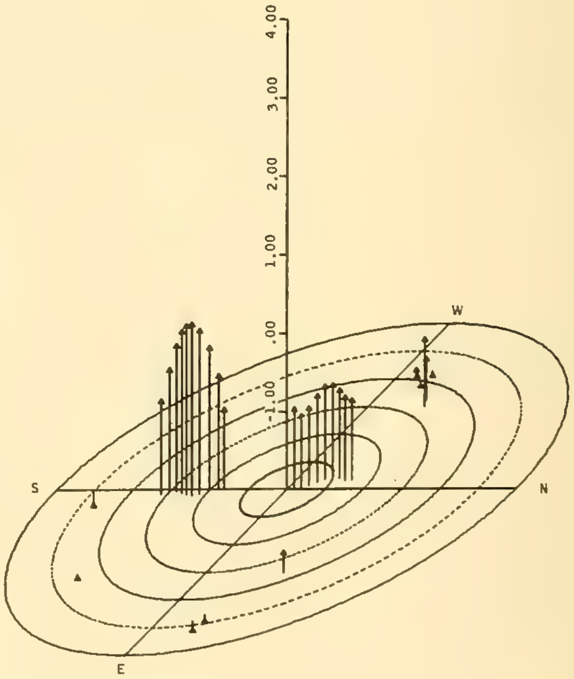
DATE06/01/65 TIME1154-1226 STAGE1 W0165



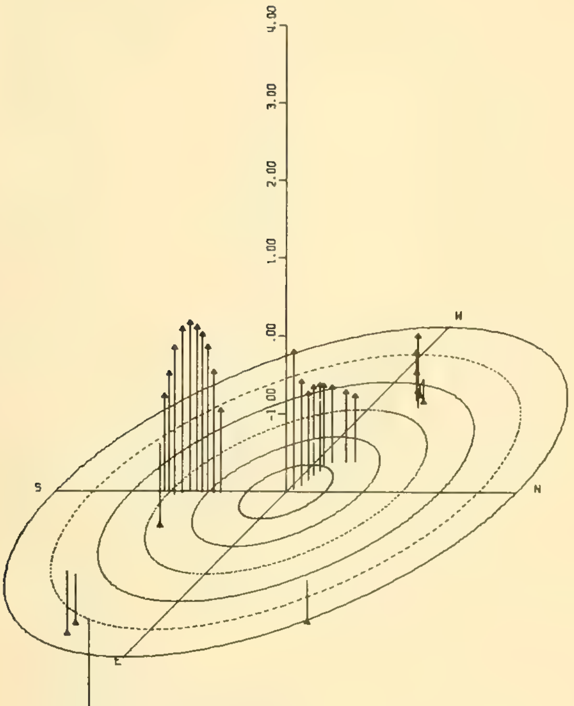
DATE06/01/65 TIME1227-1259 STAGE1 W0170



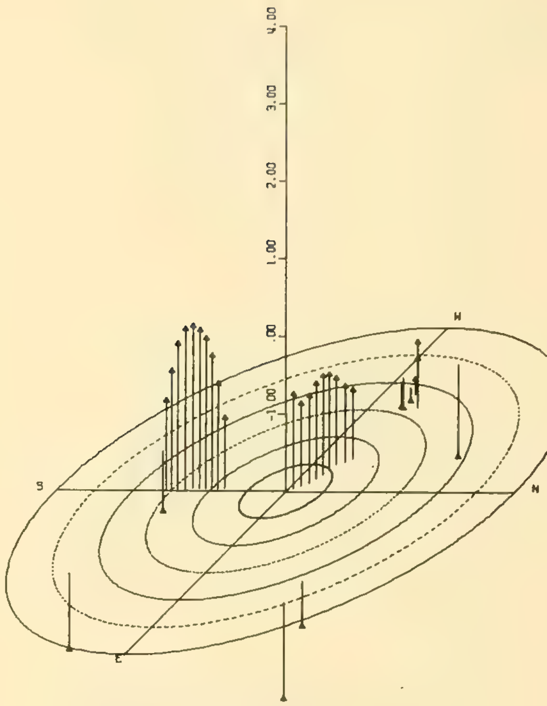
DATE06/04/65 TIME1415 1449 STAGE1 W0282



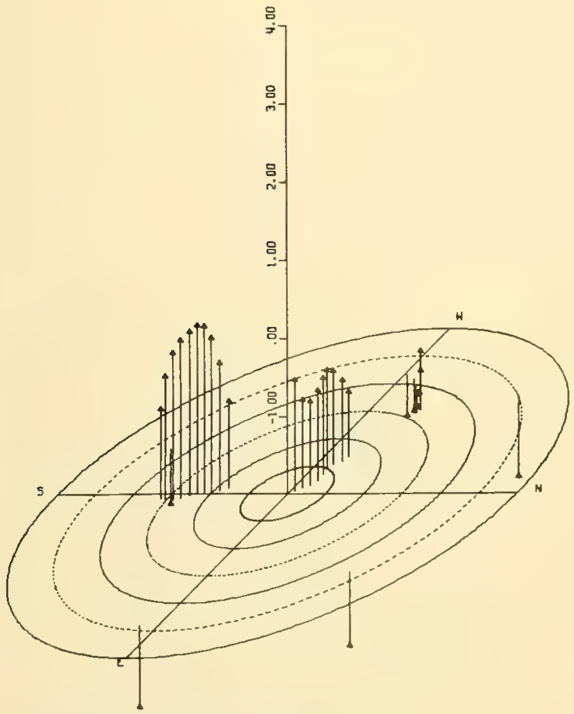
DATE06/04/65 TIME1418-1449 STAGE1 Wo282



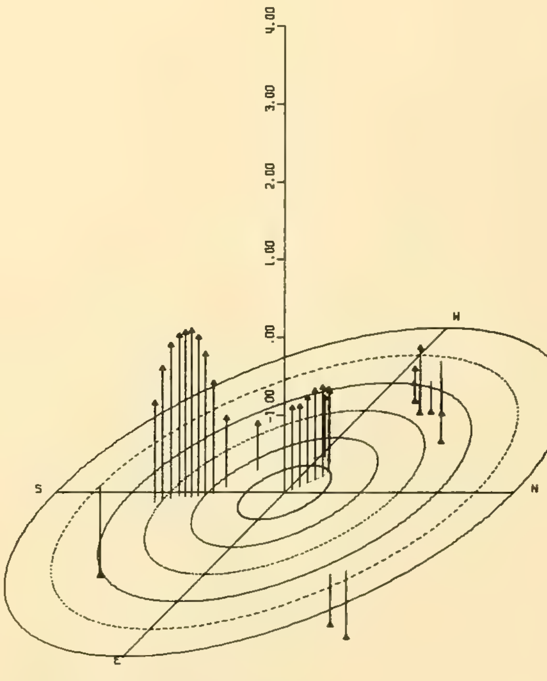
DATE06/04/65 TIME1608-1639 STAGE1 WD360



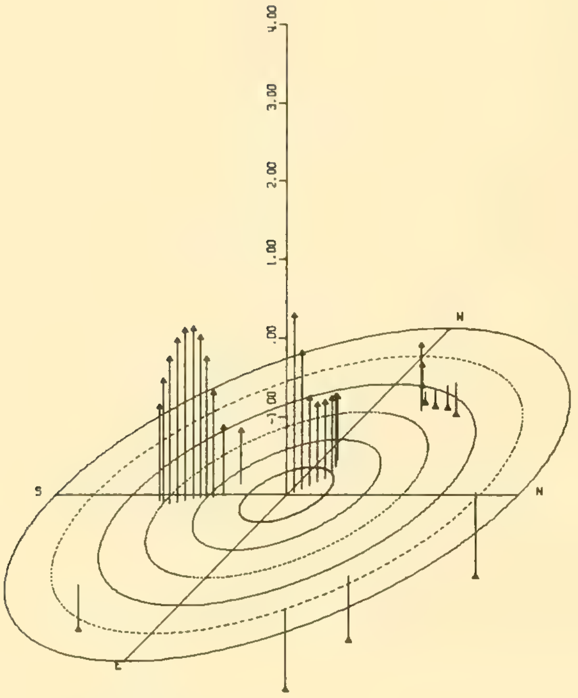
DATE06/04/65 TIME1640-1708 STAGE1 WD335



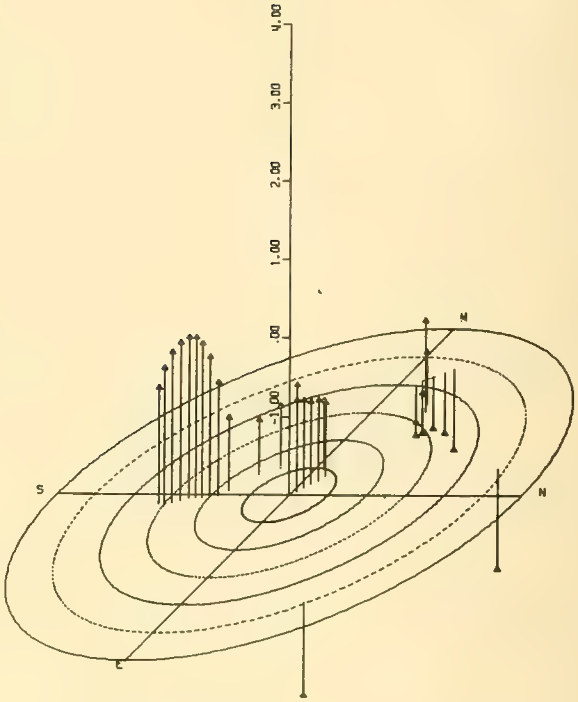
DATE06/04/65 TIME1709-1731 STAGE1 WD315



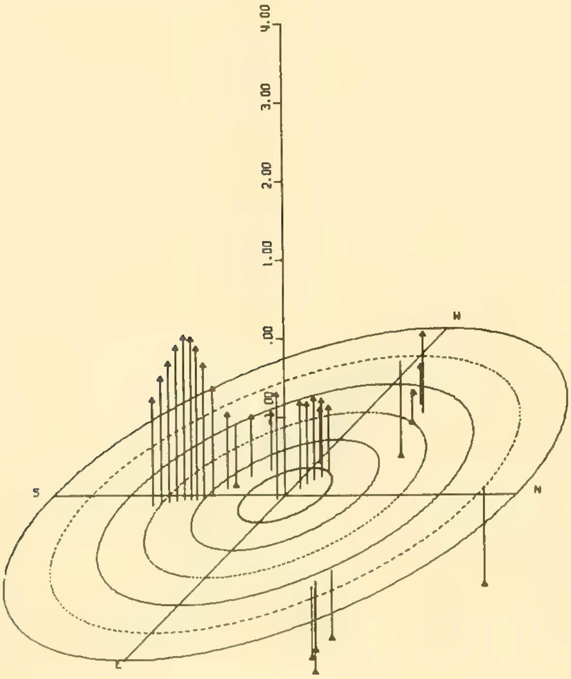
DATE06/04/65 TIME2007-2035 STAGE1 WD 45



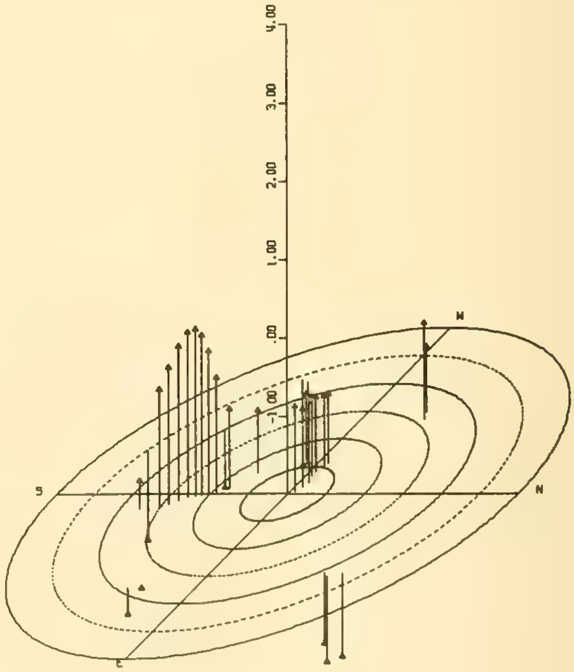
DATE06/04/65 TIME2105-2133 STAGE1 W0 80



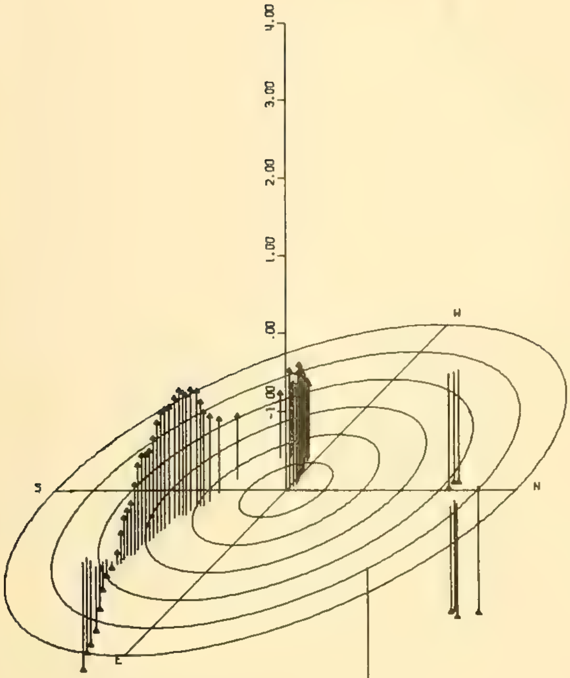
DATE06/05/65 TIME0003-0032 STAGE1 W0100



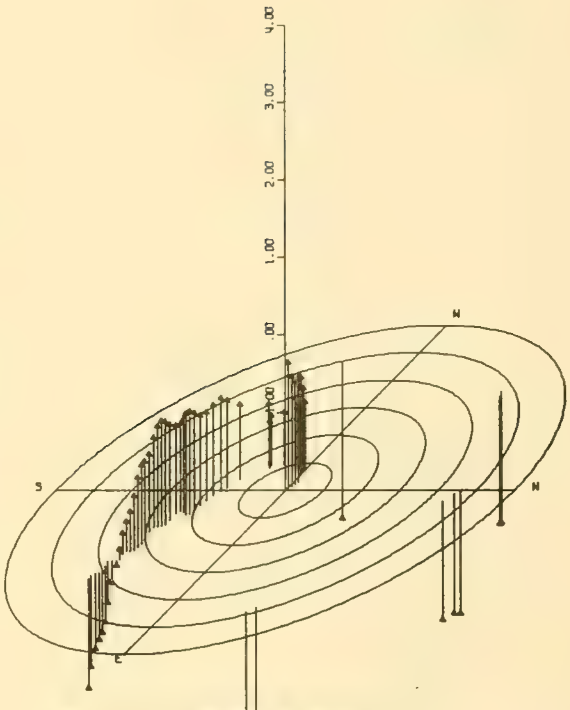
DATE 06/05/65 TIME0033-0101 STAGE1 W0100



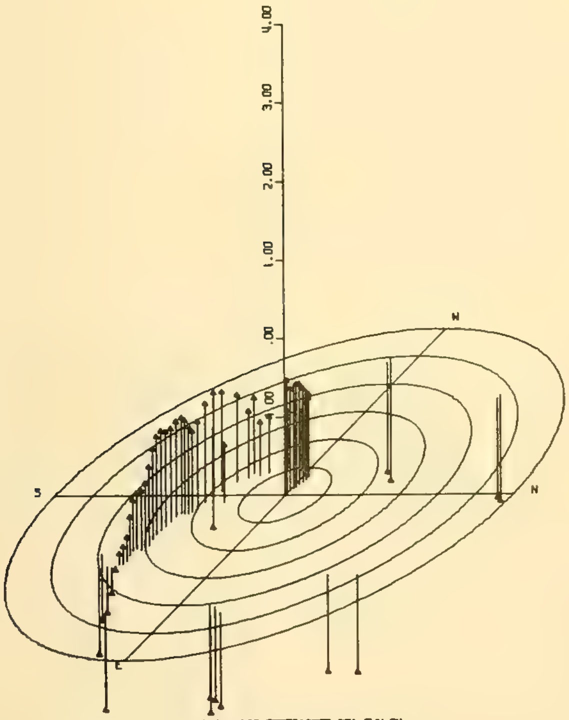
DATE06/05/65 TIME0102-0130 STAGE1 W0100



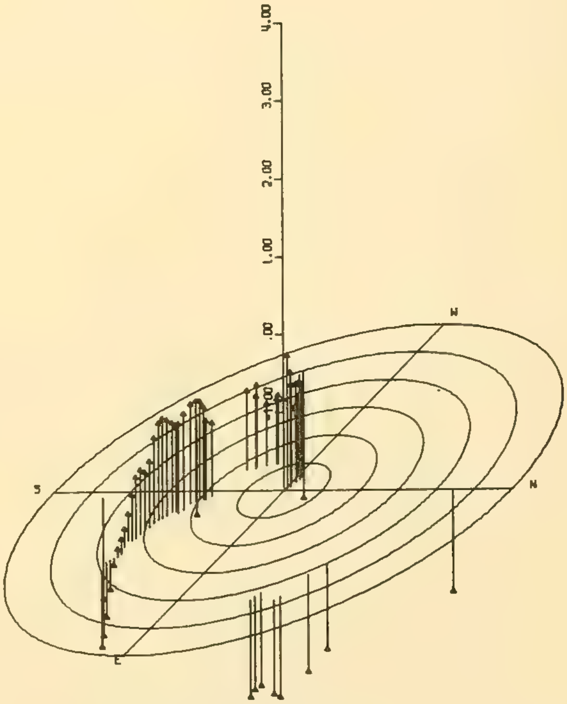
1701-1730 07SEP65 31 WD073 WS16 T324F1A145-174 RUN 22



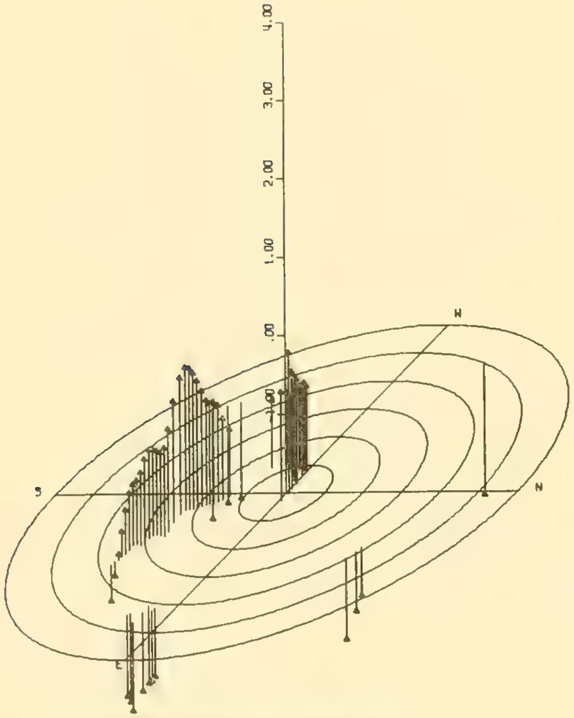
1901-1930 07SEP65 31 WD060 WS16 T324F1A265-194 RUN 23



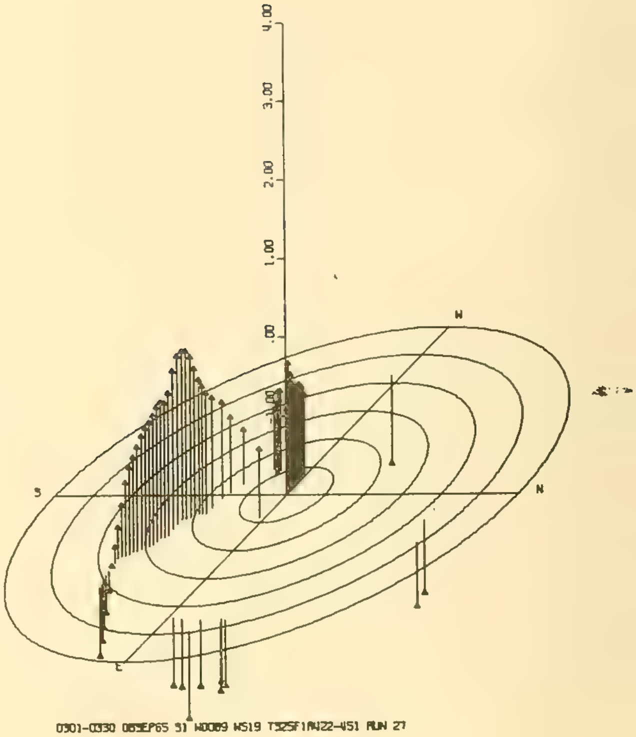
2101-2130 07SEP65 31 WD073 WS22 T325F1A062-091 RUN 24



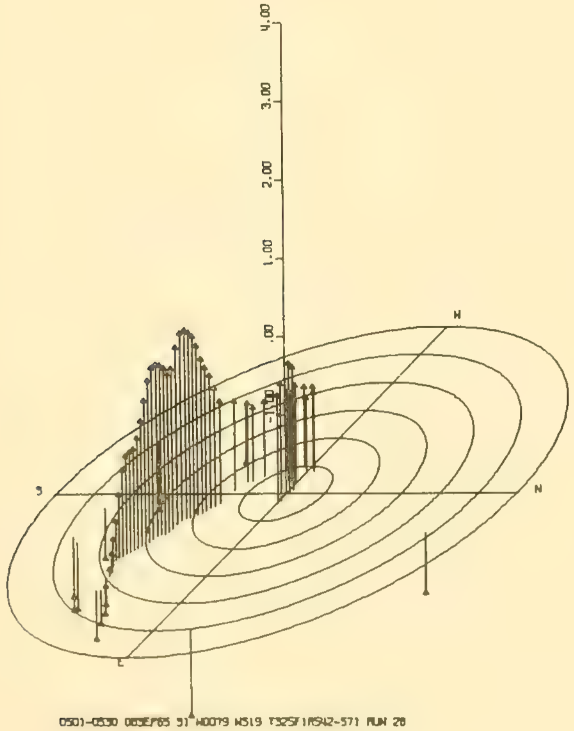
2301-2330 07SEP65 31 WD082 WS19 T325F1A182-211 RUN 25



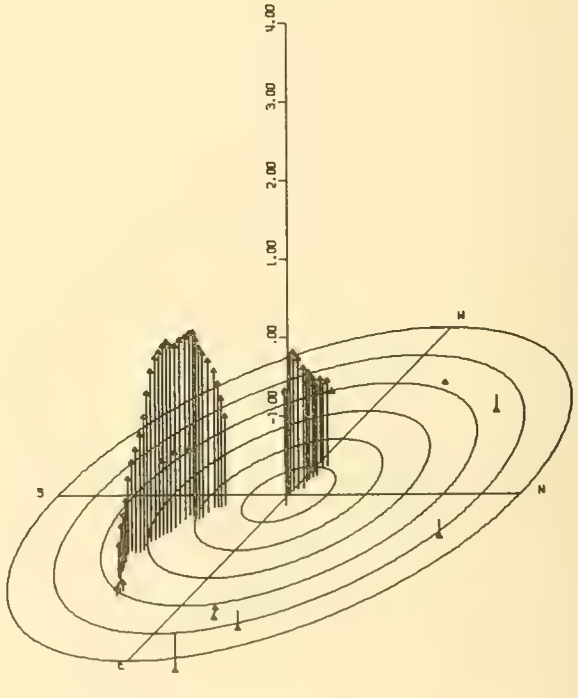
0101-0130 08SEP65 31 W0095 WS21 T32SF1R302-331 RUN 26



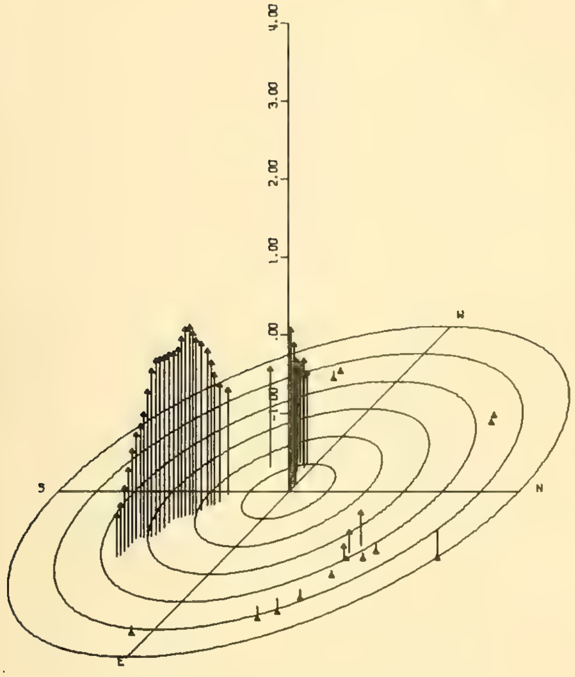
0301-0330 08SEP65 31 W0089 WS19 T32SF1R422-451 RUN 27



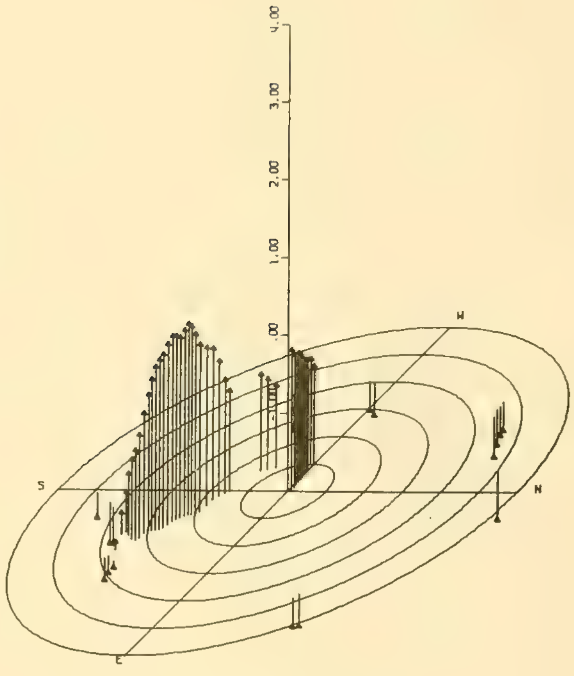
0501-0530 08SEP65 31 W0079 WS19 T32SF1R542-571 RUN 28



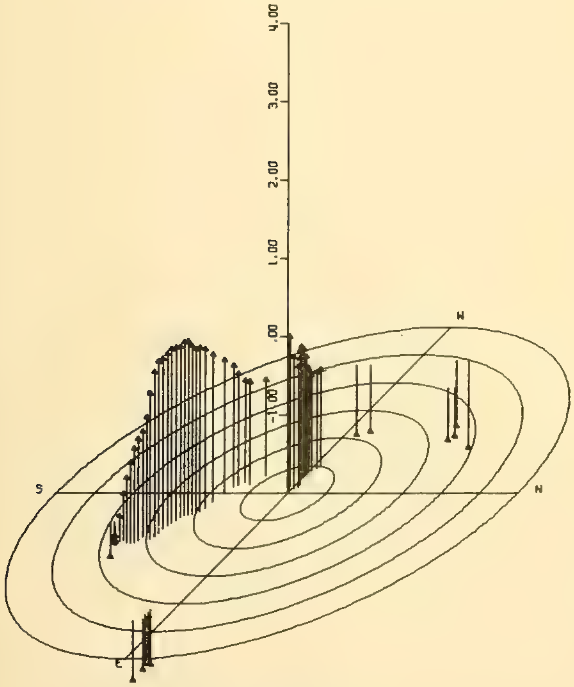
0701-0730 08SEP65 31 W0083 WS22 T32SF1R662-691 RUN 29



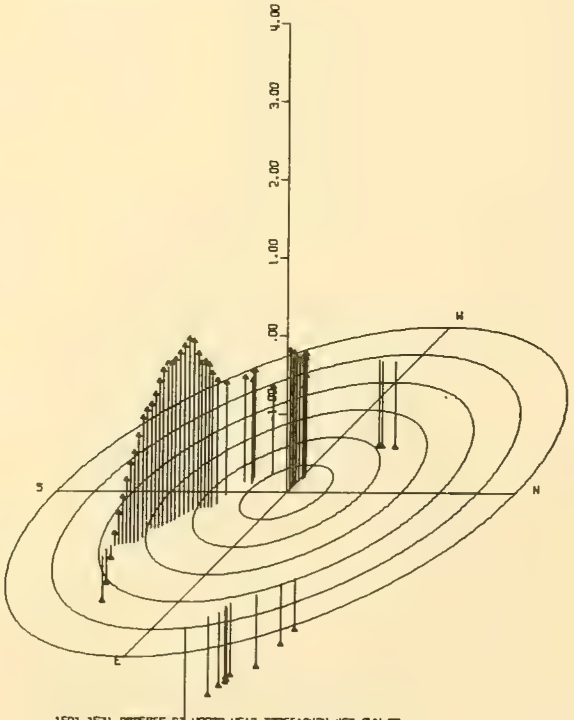
0901-0930 08SEP65 51 W0077 WS18 T326F1W06N-093 RUN 30



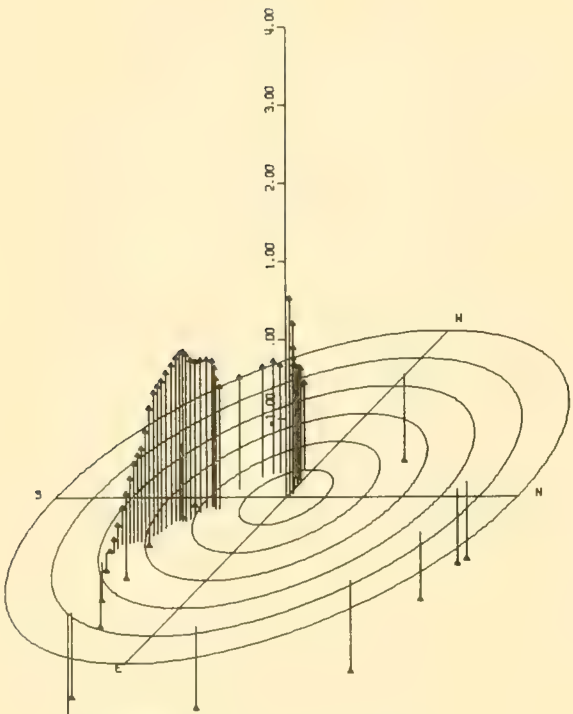
1101-1131 08SEP65 51 W0(V) WS16 T326F1A16N-213 RUN 31



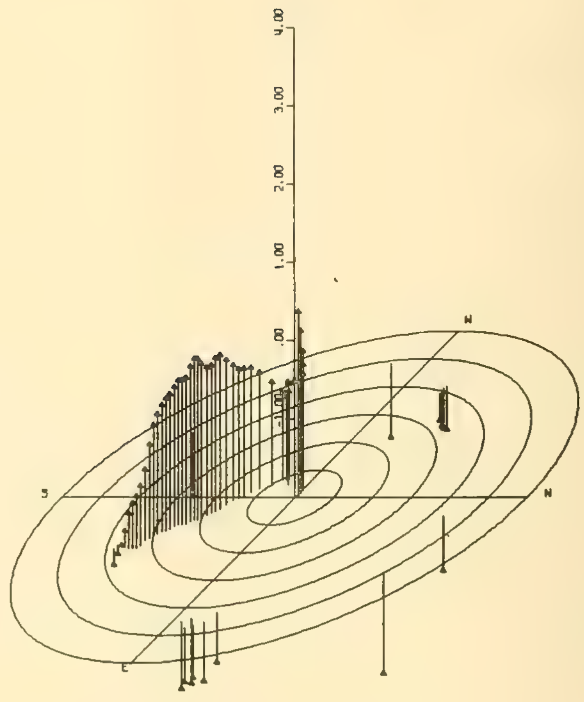
1301-1331 08SEP65 51 W0055 WS17 T326F1A30N-333 RUN 32



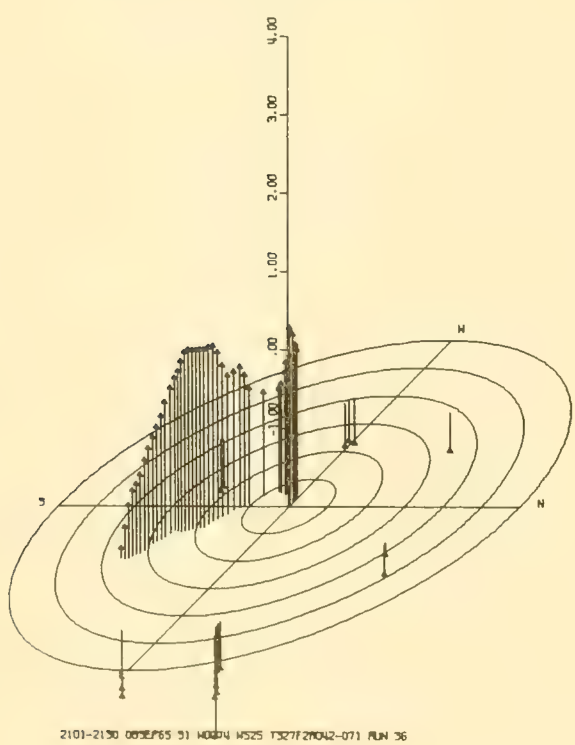
1501-1531 08SEP65 51 W0070 WS17 T326F1A42N-453 RUN 33



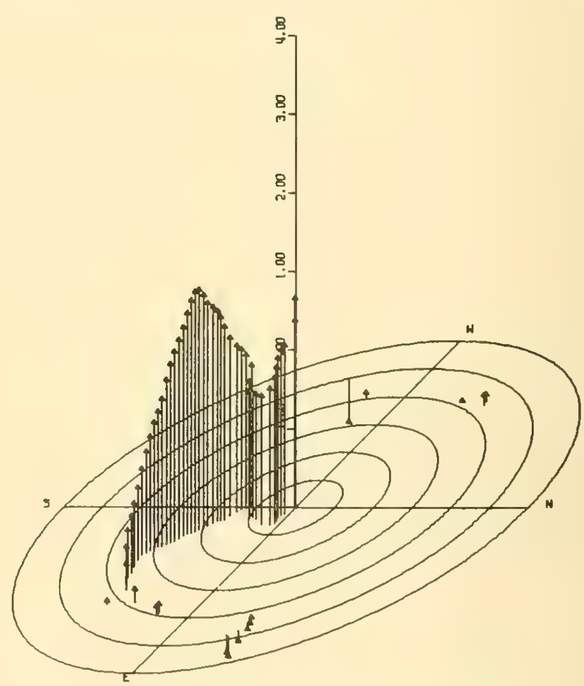
1701-1731 08SEP65 31 W0072 W521 T326F1A544-573 RUN 34



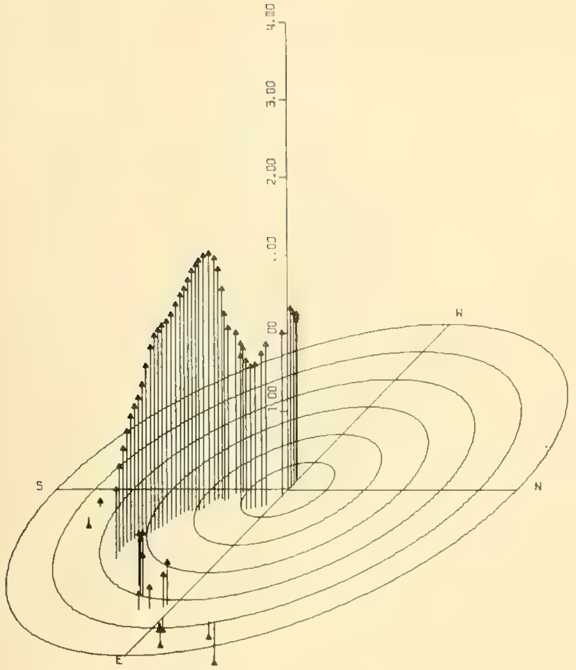
1901-1931 08SEP65 31 W0071 W522 T326F1A664-693 RUN 35



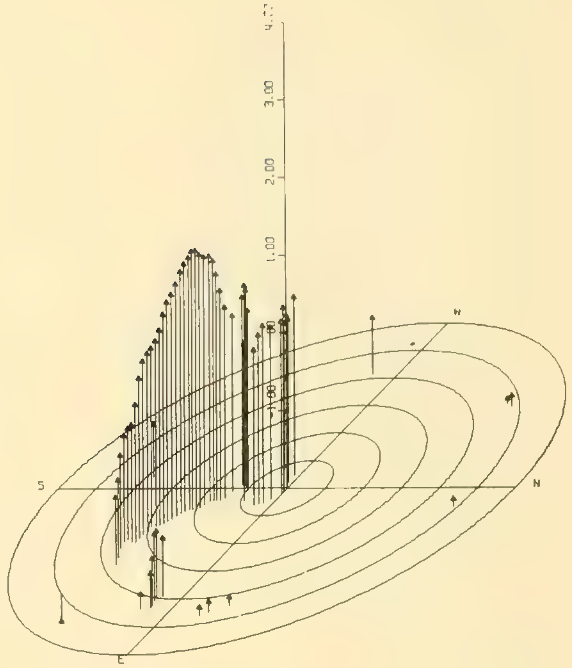
2101-2130 08SEP65 31 W0074 W525 T327F2A042-071 RUN 36



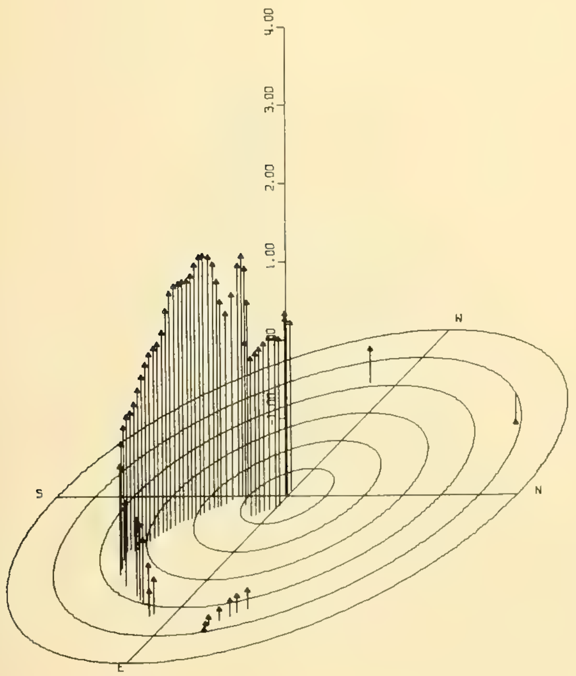
2301-2330 08SEP65 31 W5083 W526 T327F2A162-191 RUN 37



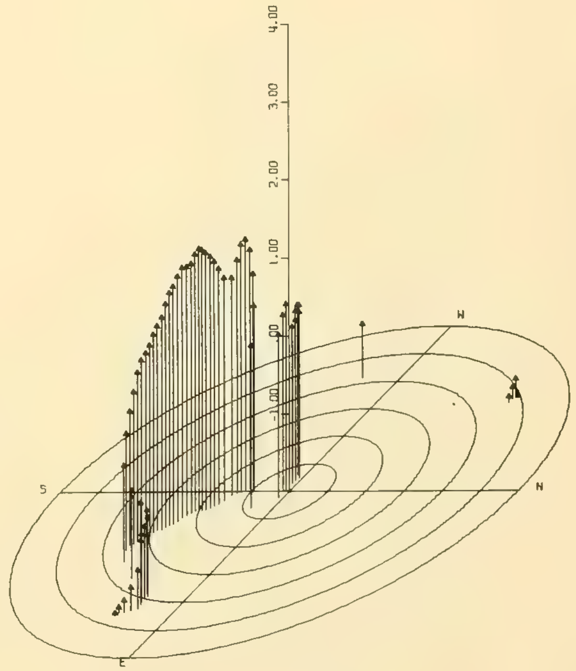
0101-0130 09SEP65 S1 W0081 WS28 T327F2A282-311 RUN 38



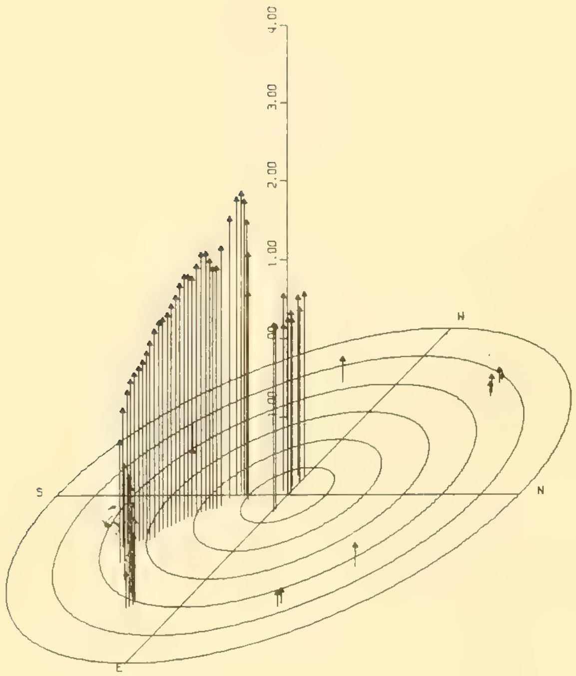
0201-0230 09SEP65 S1 W5085 WS27 T327F2A342-371 RUN 39



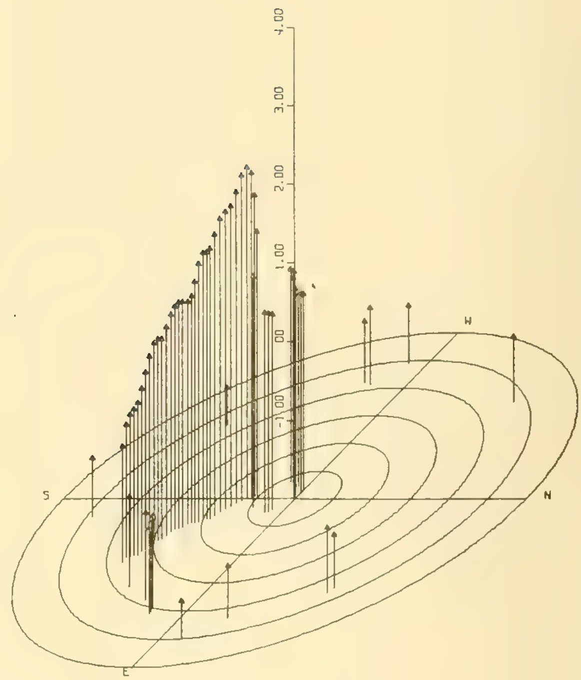
0301-0330 09SEP65 S1 W0080 WS27 T327F2A402-431 RUN 40



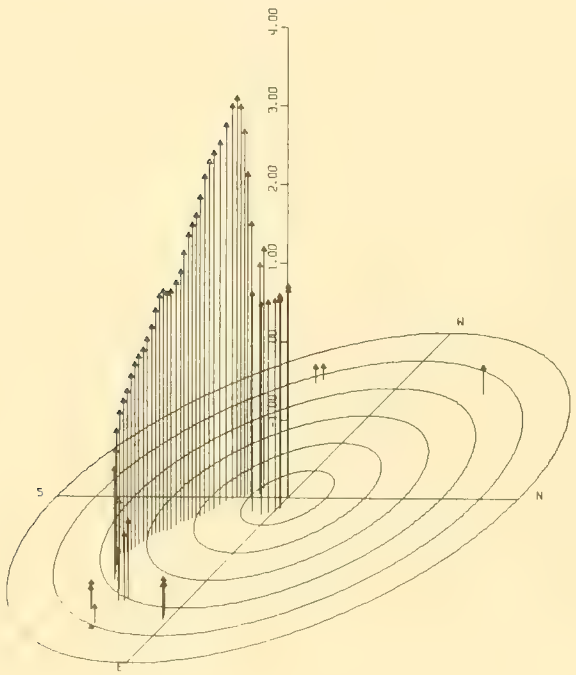
0401-0430 09SEP65 S1 W0082 WS26 T327F2A462-491 RUN 41



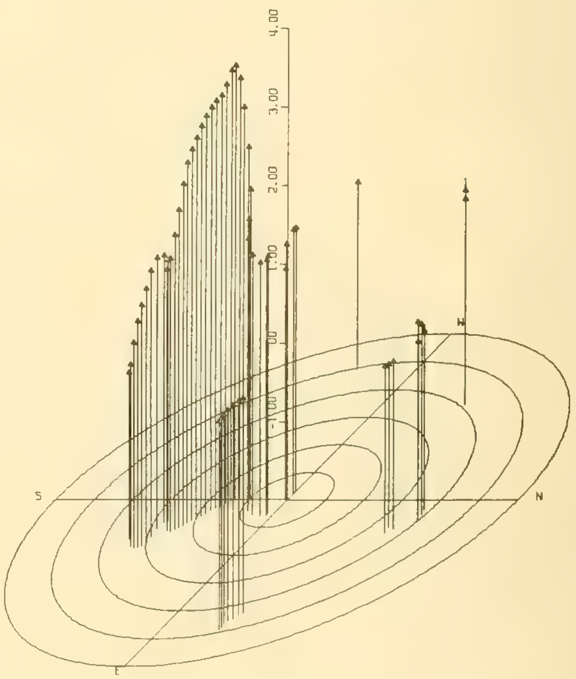
0501-0530 09SEP65 S1 W0085 WS28 T327F2RS22-551 RUN 42



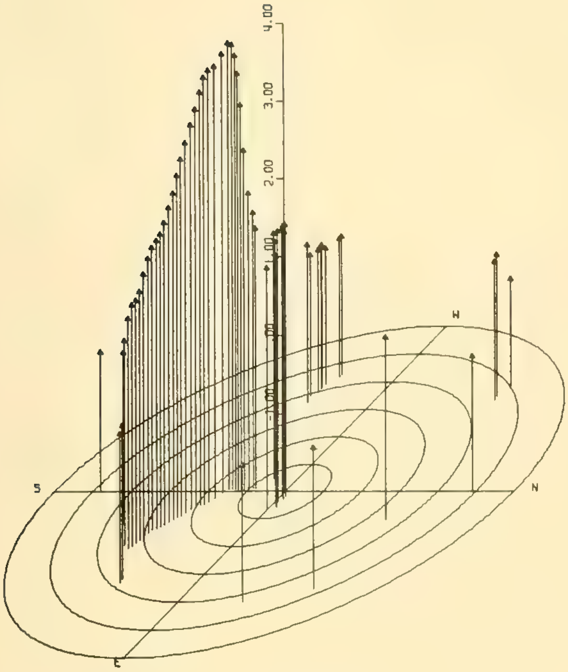
0601-0630 09SEP65 S1 W0088 WS28 T327F2RS82-611 RUN 43



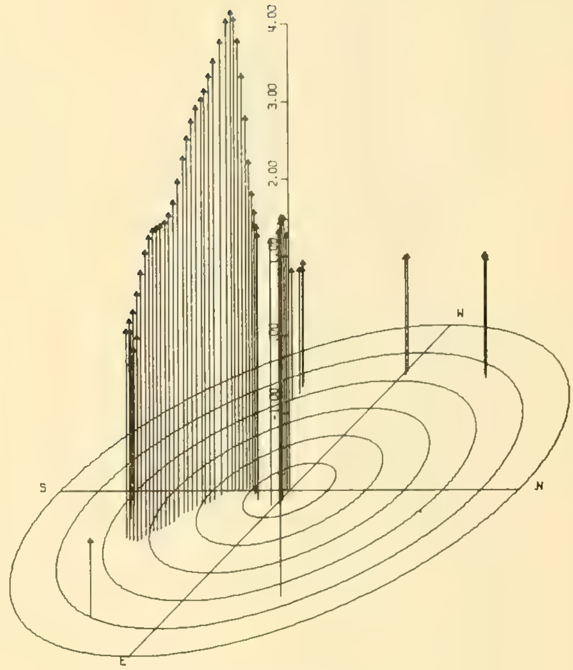
0701-0730 09SEP65 S1 W0092 WS28 T327F2R642-771 RUN 44



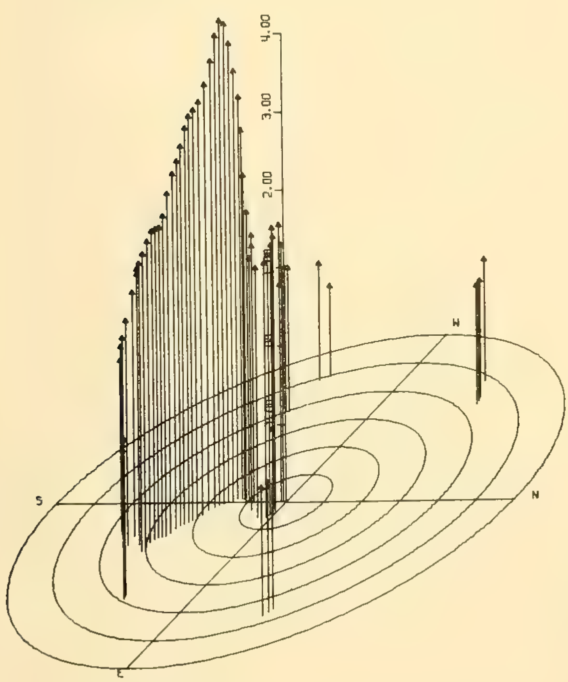
0801-0830 09SEP65 S1 W0103 WS27 T083F1R003-032 RUN 45



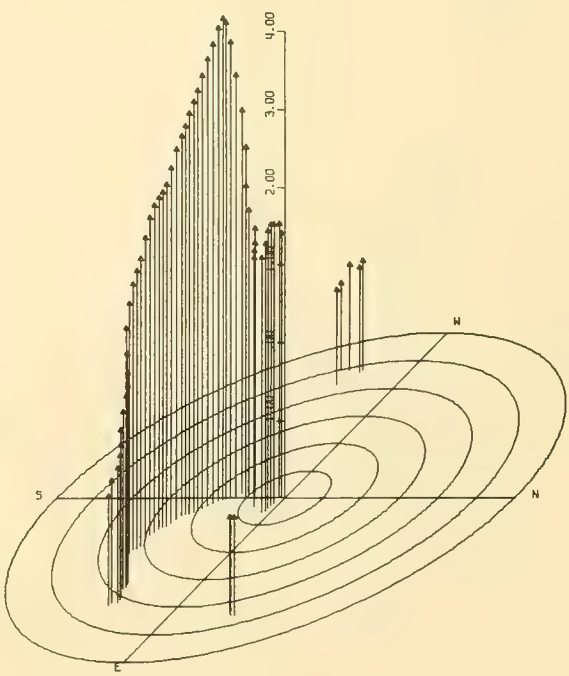
0901-0930 09SEP65 S1 W0117 WS28 TOB3F1R063-092 RUN 46



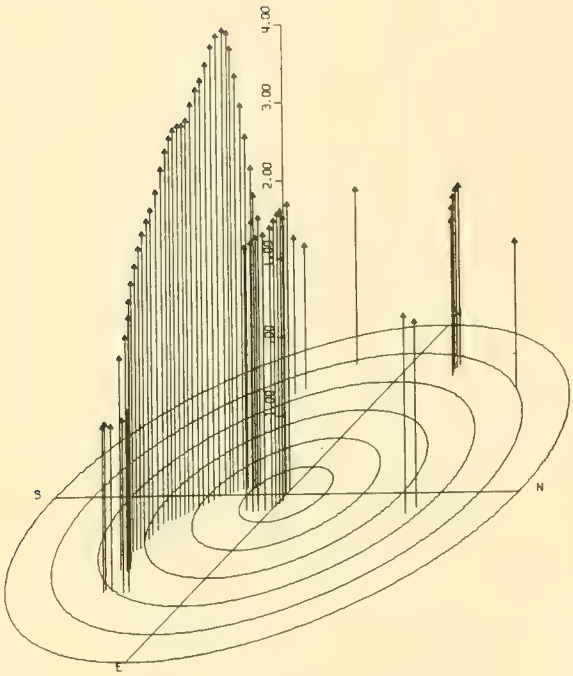
1001-1030 09SEP65 S1 W0123 WS33 TOB3F1R123-152 RUN 47



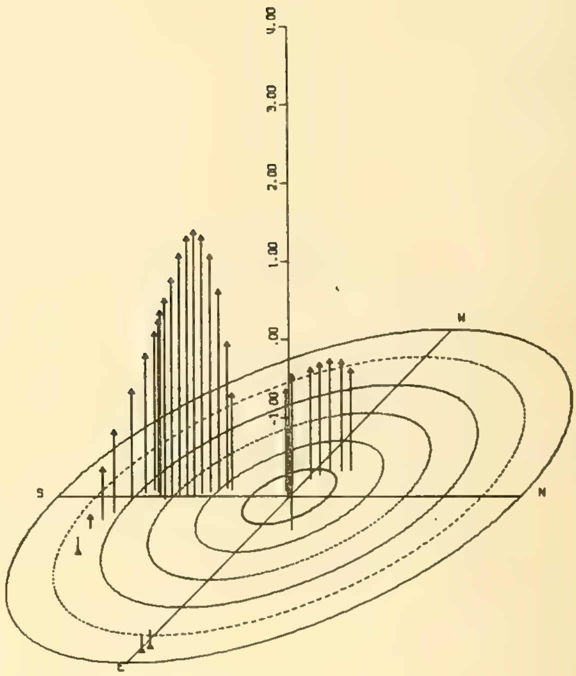
1101-1131 09SEP65 S1 W0125 WS34 TOB3F1R183-212 RUN 48



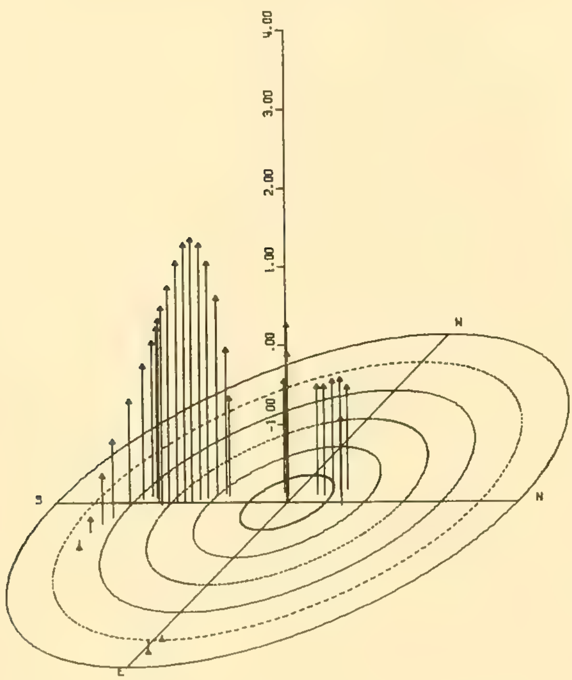
1201-1230 09SEP65 S1 W0130 WS34 TOB3F1R243-272 RUN 49



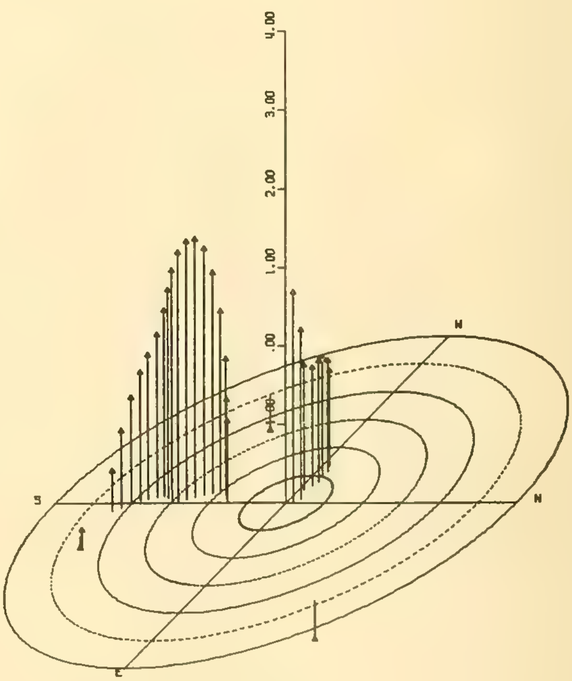
1310-1339 09SEP65 S1 W0133 W535 T0B3F2R001-030 RUN 50



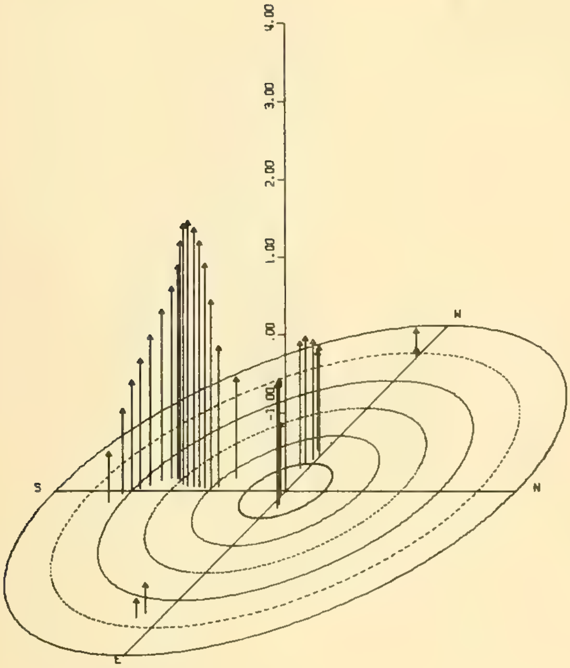
DATE05/04/65 TIME1200-1229 STAGE2 WD 82



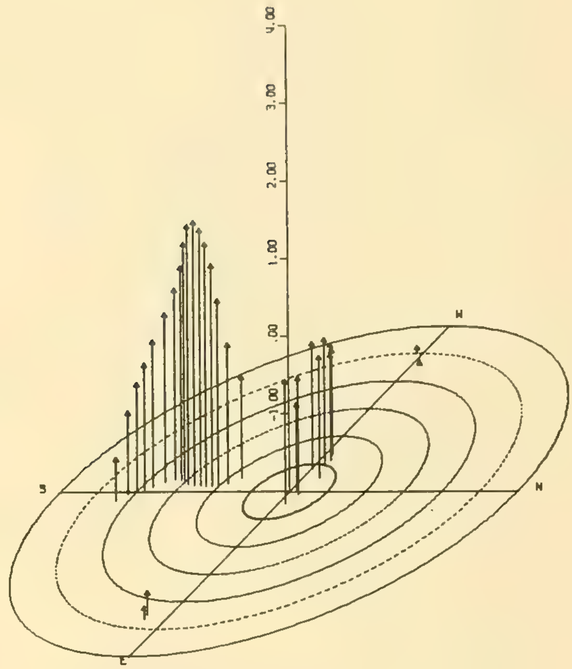
DATE05/04/65 TIME1200-1229 STAGE2 WD 82



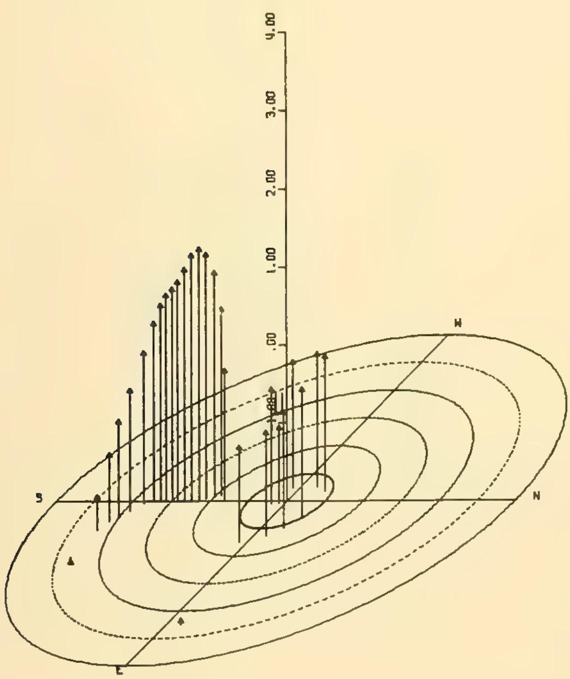
DATE05/04/65 TIME1301-1333 STAGE2 WD 66



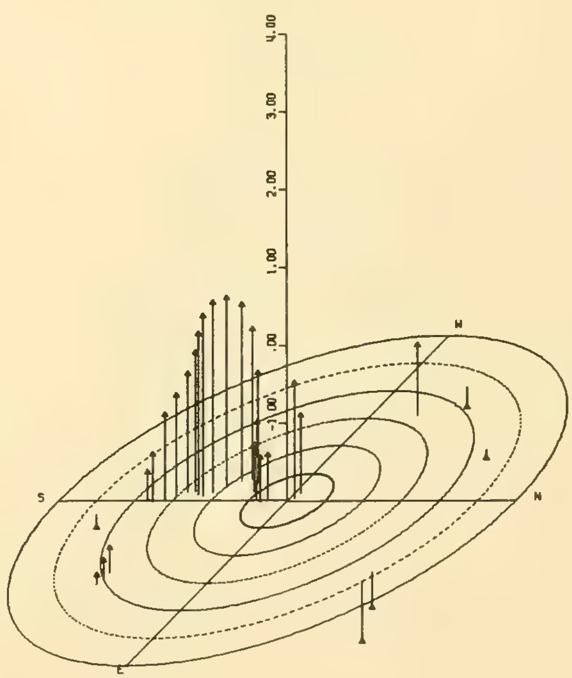
DATE05/04/65 TIME1401-1425 STAGE2 W0113



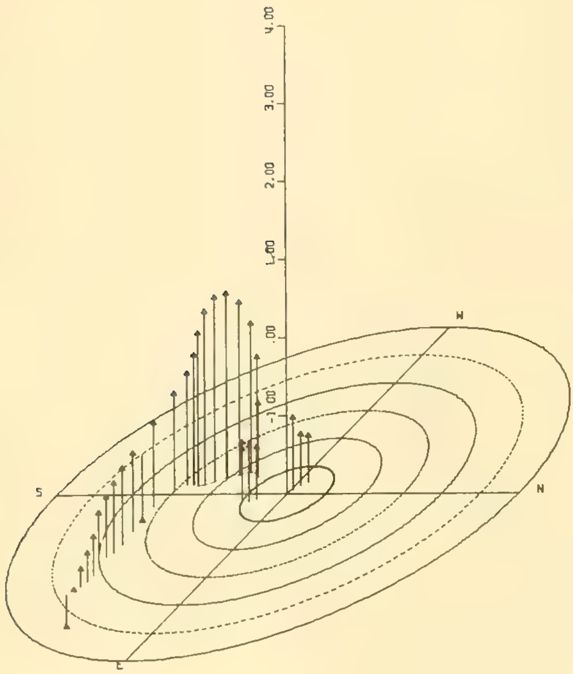
DATE05/04/65 TIME1401-1425 STAGE2 W0113



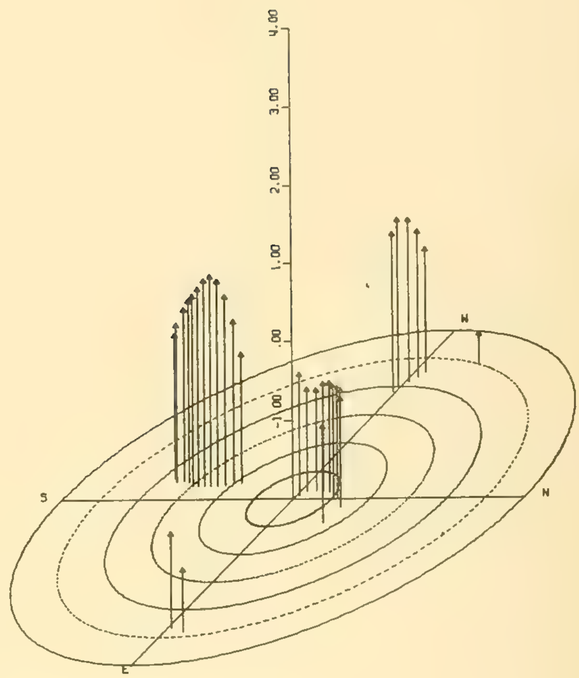
DATE05/07/65 TIME1130-1256 STAGE2 W0120



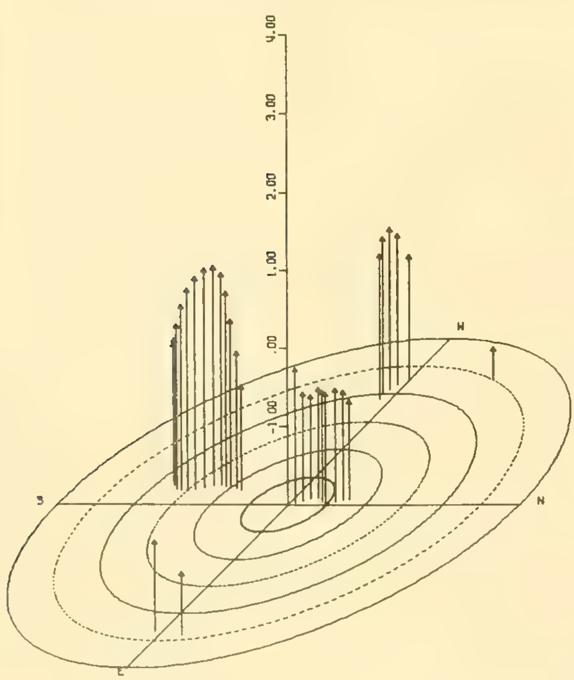
DATE06/01/65 TIME1154-1220 STAGE2 W0165



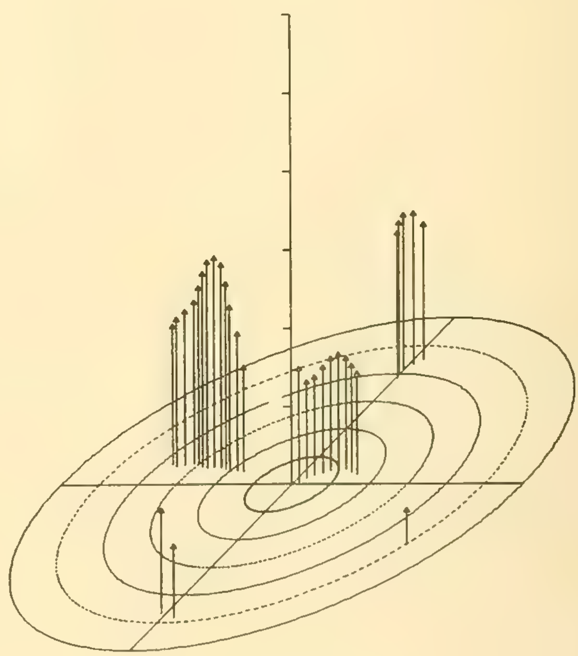
DATE06/01/65TIME1220-1245STAGE2 W0173



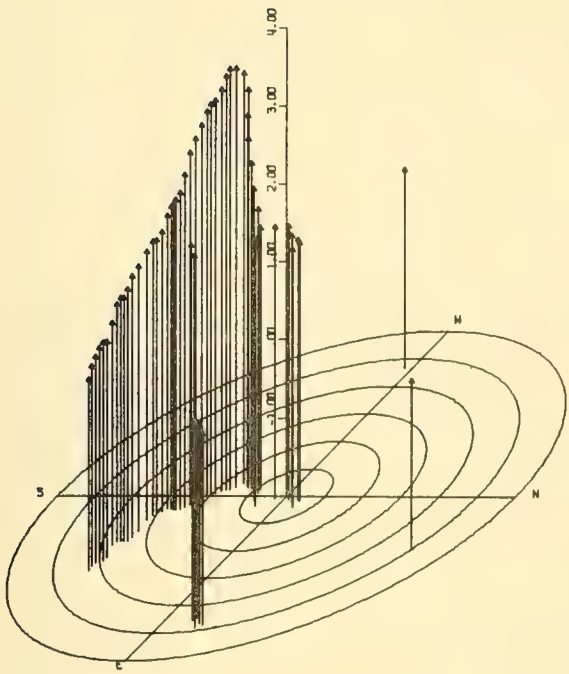
DATE06/04/65 TIME1049-1119 STAGE2 W0250



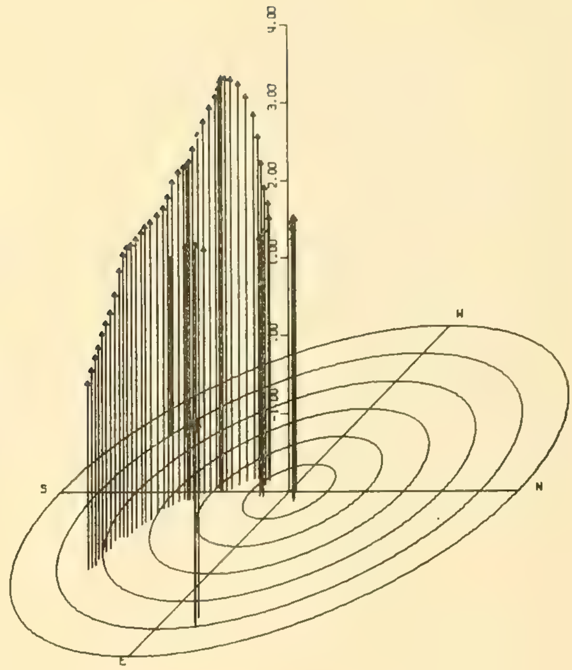
DATE06/04/65 TIME1200-1230 STAGE2 W0265



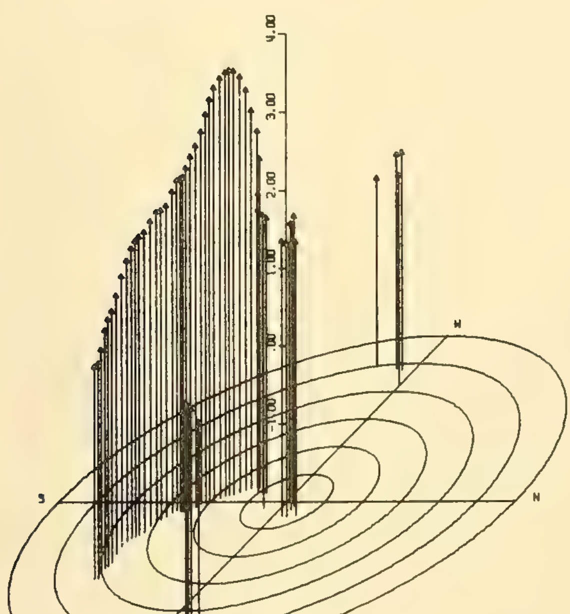
DATE06/04/65 TIME1300-1330 STAGE 2 W0280



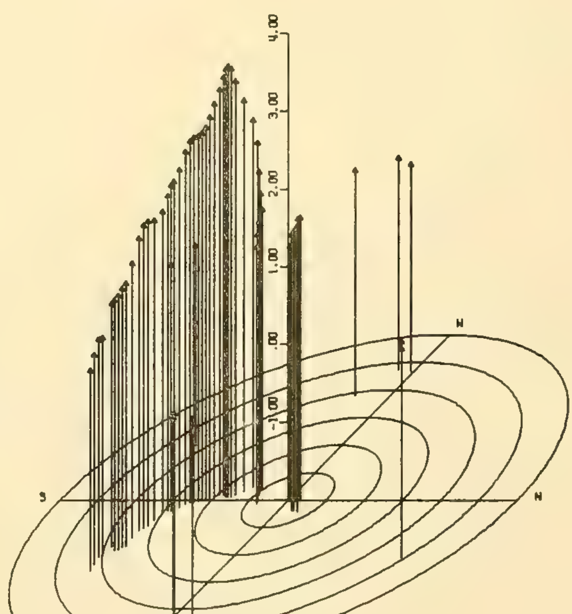
1001-1030 09SEP65 S2 W0110 WS29 T31SF1R122-151 RUN14



1101-1130 09SEP65 S2 W0118 WS34 T31SF1R182-211 RUN15



1201-1230 09SEP65 S2 W0123 WS30 T31SF1R242-271 RUN16



1301-1330 09SEP65 S2 W0120 WS35 T31SF1R302-331 RUN17

APPENDIX B

A FORTRAN II PROGRAM FOR SINGLE-WAVE TRAIN ANALYSIS

The FORTRAN II listing of an IBM 704 program for the least square single-wave train analysis of a spectral matrix obtained from an array of ocean wave bottom pressure transducers is included. The listing is from the FORTRAN to ALGOL translator of the Burroughs B-5500 and is syntax free at the FORTRAN II level. The mathematics of the single-wave train analysis is described in the body of this report. The data collection system and calculation of the required cross spectral matrix is described in Bennett et al (June 1964). The plotter subroutines GPHPVW and PFB3D are also included. Bennett (March 1968) describes the plotting technique in PFB3D which was used to produce the plots in Appendix A.

FORTRAN TO ALGOL TRANSLATOR
PHASE 1 FORTRAN STATEMENTS

```
C LISTING OF AN IBM 704 FORTRAN PROGRAM FOR DIRECTIONAL WAVE ANALYSIS.
C 1335D-2 C BENNETT JULY 1967 SWOC 78-301-8210
C     REQUEST-0268
C     LEAST SQUARE SINGLE WAVE TRAIN FIT AFTER MUNK
      DIMENSION ID(11),FQ(100),WN(100),PERIOD(100),WAVLGH(100),THMAX(100
     X),DEPATN(100),AVEP(100),P(100,6,6),EMAX(100),TEM(146)
      DIMENSION D(6,6),PSI(6,6),E(100,72),THETA(72),DE(73),THZRO(146),
     XH(100),H1(100),H2(100) ,GOOD(6)
      DIMENSION DATA (1000)
      E2TEN=0.43429448
      TWOPI=6.281853
      RTD=57.29578
      DTR=0.0174532925
      FIV=0.087266465
      REWIND 3
      REWIND 9
      CALL PLOTS(DATA(1000),1000)
      CALL PLOT(0.0,-30.0,-3)
      CALL PLOT(2.5,2.5,-3)
      DO 5 I=1,72
      ZI=(I-1)*5
    5 THETA(I)=ZI*DTR
```

```
      DO 10 I=1,6
      DO 10 J=1,6
      D(I,J)=0.0
   10 PSI(I,J)=0.0
      D(1,2)=100.0
      D(1,3)=100.0
      D(1,4)=100.0
      D(1,5)=100.0
      D(1,6)=100.0
      D(2,3)=117.558
      D(2,4)=190.212
      D(2,5)=190.212
      D(2,6)=117.558
      D(3,4)=117.558
      D(3,5)=190.212
      D(3,6)=190.212
      D(4,5)=117.558
      D(4,6)=190.212
      D(5,6)=117.558
C     PSI IS TRIG ANGLE
C     D IS DISTANCE
      PSI(1,2)=0.0
      PSI(1,3)=72.0*DTR
      PSI(1,4)=144.0*DTR
```

```
      PSI(1,5)=216.0*DTR
      PSI(1,6)=288.0*DTR
      PSI(2,3)=126.0*DTR
      PSI(2,4)=162.0*DTR
      PSI(2,5)=198.0*DTR
      PSI(2,6)=234.0*DTR
      PSI(3,4)=198.0*DTR
      PSI(3,5)=234.0*DTR
      PSI(3,6)=270.*DTR
      PSI(4,5)=270.0*DTR
      PSI(4,6)=306.0*DTR
      PSI(5,6)=342.0*DTR
C     GOOD(I)= 1 FOR USABLE CHANNEL DATA AND 0 FOR BAD CHANNEL
   15 READ 2, ISKIP   ,(GOOD(I),I=1,6)
    2 FORMAT(I2 ,6F1.0 )
      IF(ISKIP)100,30,20
   20 DO 25 I=1,ISKIP
      READ TAPE 3,ID,FQ(1),WN(1),M,K
      IF(K)21,22,21
   21 PAUSE 20202
C     TAPE OUT OF PHASE WITH DATA READ DESIRED
      GO TO 15
   22 MP=M+1
      DO 23 L=2,MP
```

```
23 READ TAPE 3
25 CONTINUE
   GO TO 15
C     FQ(1)=0.0 CAN NOT BE MEANINGLYFULLY PROCESSED
30 READ TAPE 3,ID,FQ(1),WN(1),M,K
   IF(K)21,31,21
31 MP=M+1
   DO 32 L=2,MP
32 READ TAPE 3,ID,FQ(L),WN(L),M,K,DELTAT,DEPTH,PERIOD(L),WAVLGH(L),
  XDEPATN(L),AVEP(L),((P(L,I,J),J=1,6),I=1,6)
C     P(L,I,J)=-0.0 IF CHANNEL I OR J IS NO GOOD
C     WN(L) IS WAVE NUMBER=2PI/WAVE LENGTH IN FEET
C     PI=3.1415927...
C     VALUE K NO LONGER NEEDED
   GOODCH=0.0
   DO 40 I=1,6
   GOOD(I)=GOOD(I)*P(2,I,I)
   IF(GOOD(I))41,40,41
41 GOODCH=GOODCH+1.0
   GOOD(I)=1.0
40 CONTINUE
   TERMS=1.0+(GOODCH*(GOODCH-1.0))
   DO 70 L=2,MP
   SUM=0.0
```

```
      DO 50 I=1,6
      IF(GOOD(I))51,50,51
   51 SUM=SUM+P(L,I,I)
   50 CONTINUE
      AVEP(L)=SUM/GOODCH
      DO 60 K=1,72
      SUM=0.0
      DO 90 I= 1,5
      IF(GOOD(I) )90,90,91
   91 IP = I+1
      DO 95 J= IP,6
      IF(GOOD(J) )95,95,92
   92 SUM = SUM+P(L,I,J)*COSF(WN(L)*D(I,J)*COSF(THETA(K)-PSI(I,J)))
     X          -P(L,J,I)*SINF(WN(L)*D(I,J)*COSF(THETA(K)-PSI(I,J)))
   95 CONTINUE
   90 CONTINUE
   60 E(L,K)=(AVEP(L)+2.0*SUM)/TERMS
      IF(SENSE SWITCH 1)62,63
   62 PRINT6,FQ(L),AVEP(L),(ID(I),I=1,11),(E(L,K),K=1,72)
    6 FORMAT(1X,1PE11.4,1PE12.4,11A6/(10(1X,1PE11.4)))
   63 G=0.1/DTR
C     DELTA THETA=5DEG OR H=1/2*DEL THETA
      DE(1)=G*(E(L,2)-E(L,72))
      DE(72)=G*(E(L,1)-E(L,71))
```

```
      DO 61 ITHETA=2,71
   61 DE(ITHETA)=G*(E(L,ITHETA+1)-E(L,ITHETA-1))
      DE(73)=DE(1)
      K=1
      DO 65 I=1,72
      IF(DE(I)*DE(I+1))66,67,65
   66 ZI=I-1
C     FIV IS 5 DEG IN RADIANS
      THZRO(K)=FIV*(ZI+(DE(I)/(DE(I)-DE(I+1))))
      K=K+1
      GO TO 65
   67 IF(DE(I))68,69,68
   69 THZRO(K)=FIV*FLOATF(I-1)
      K=K+1
   68 IF(DE(I+1))65,64,65
   64 THZRO(K)=FIV*FLOATF(I)
      K=K+1
   65 CONTINUE
      NZEROS=K-1
      DO 75 K=1,NZEROS
C     TERMS SAME AS ABOVE
      SUM=0.0
      DO 76 I=1,5
      IF(GOOD(I) )76,76,77
```

```
77 IP=I+1
   DO 79 J=IP,6
   IF(GOOD(J) )79,79,78
78 SUM = SUM+P(L,I,J)*COSF(WN(L)*D(I,J)*COSF(THZRO(K)-PSI(I,J)))
  X          -P(L,J,I)*SINF(WN(L)*D(I,J)*COSF(THZRO(K)-PSI(I,J)))
79 CONTINUE
76 CONTINUE
75 TEM(K)=(AVEP(L)+2.0*SUM)/TERMS
   EMAX(L)=TEM(1)
   THMAX(L)=RTD*(TWOPI-THZRO(1))
   DO 74 K=2,NZEROS
   IF(EMAX(L)-TEM(K))73,74,74
73 EMAX(L)=TEM(K)
   THMAX(L)=RTD*(TWOPI-THZRO(K))
C     THMAX IS BEARING FROM MAGNITIC NORTH
74 CONTINUE
   SUM=0.0
   DO 80 I=1,5
   IF(GOOD(I) )80,80,81
81 IP=I+1
   DO 83 J=IP,6
   IF(GOOD(J) )83,83,82
82 SUM=SUM+P(L,I,J)*P(L,I,J)+P(L,J,I)*P(L,J,I)
83 CONTINUE
```

```
80 CONTINUE
   SSQ=AVEP(L)*AVEP(L)+2.0*SUM
   H(L)=SSQ-TERMS*EMAX(L)*EMAX(L)
   ATILDA=AVEP(L)/TWOPI
   HTILDA=SSQ-ATILDA*ATILDA*TERMS
   H1(L)=1.0-(H(L)/HTILDA)
   H2(L)=(EMAX(L)-ATILDA)/(AVEP(L)-ATILDA)
   IF(SENSE SWITCH 1)99,70
99 PRINT7, TERMS,EMAX(L),ATILDA,HTILDA
 7 FORMAT(1X,F5.1,1P3E11.4 )
70 CONTINUE
   PRINT 3,(ID(I),I=1,11)  ,(GOOD(I),I=1,6),M
 3 FORMAT(1H1,39HLEAST SQUARE SINGLE WAVE TRAIN FIT OF  ,11A6/
  X1X,6F2.0,2X,2HM=I3//
  X3X,9HFREQUENCY,5X,6HPERIOD,3X,11HWAVE LENGTH,1X,11HATTENUATION,4X,
  X5HAVE P,8X,1HA,10X,5HBRNG ,8X,1HH,12X,2HH1,10X,2HH2)
   PRINT 4,(FQ(L),PERIOD(L),WAVLGH(L),DEPATN(L),AVEP(L),EMAX(L),
  XTHMAX(L),H(L),H1(L),H2(L),L=2,MP)
 4 FORMAT(1X,0PF11.7,3X,0PF9.5,1X,0PF11.4,1X,1PE11.4,
  X 1X,1PE11.4,1X,1PE11.4,3X,0PF7.2,3X,1PE11.4,4X,0PF6.3,6X,0PF6.3 )
   DO 110 L=2,MP
   AVEP(L)=E2TEN*LOGF(ABSF(AVEP(L)))
   EMAX(L)=E2TEN*LOGF(ABSF(EMAX(L)))
110 WAVLGH(L)=EMAX(L)-E2TEN*LOGF(ABSF(DEPATN(L)))
```

```
      AVEP(1)=AVEP(2)
      EMAX(1)=EMAX(2)
      WAVLGH(1)=WAVLGH(2)
      CALL SYMBOL(2.0,0.75,0.1,14HWAVE TRAIN.FN=     ,0.0,14)
      CALL NUMBER(3.5,0.75,0.1,FQ(MP),0.0,2)
      CALL GPHPVW(MP,ID,AVEP,EMAX,WAVLGH)
C     WAVLGH IS SURFACE WAVE STAFF SPECTRUM EST FROM BOTTOM PRESSURE.
C     AVE=P  EMAX=V   WAVLGH=W  ON THE PLOTS.
      ZM=M
      NP=DELTAT*ZM*0.5+1.0
      DELX=0.0
      DO 115 I=1,6
      CALL NUMBER(DELX,0.1,0.1,GOOD(I),0.0,-1)
  115 DELX=DELX+0.1
      CALL PFB3D(NP,ID,EMAX,FQ,THMAX)
      GO TO 15
  100 REWIND 3
      CALL EXIT
      PAUSE 70707
      GO TO 15
C     END(0,1,1,0,1)
      END
```

```
      SUBROUTINE GPHPVW(L,ID,ZLOGP,ZLOGV,ZLOGW)
C     SPECIAL FORM OF GPHPVW FOR 13350-2 AUGUST 1967
      DIMENSION ZLOGP(2),ZLOGV(2),ZLOGW(2),ID(2)
      CALL PLOT(0.0,0.0,3)
      CALL PLOT (0.0,10.5,2)
      CALL PLOT (8.0,10.5,2)
      CALL PLOT (8.0,0.0,2)
      CALL PLOT (0.0,0.0,2)
      CALL PLOT (2.0,1.5,-3)
      CALL SYMBOL (0.0,-1.0,-.1,ID(1),0.0,66)
      CALL AXIS (0.0,0.0,20HNORMALIZED FREQUENCY,-20,5.0,0.0,0.0,0.2)
      ZMAX=ZLOGP(1)
      DO 10 I=2,L
      COMPAR=ZLOGP(I)
   10 ZMAX=MAX1F(ZMAX,COMPAR)
      MAX=ZMAX+3.0
      BE=MAX-8
      CALL AXIS(0.0,0.0,17HLOG POWER DENSITY,17,8.0,90.0,BE,1.0)
      DX=5.0/FLOATF(L-1)
      Y=ZLOGP(1)-BE
      X=0.0
      CALL PLOT(X,Y,3)
```

```
      DO 20 I=2,L
      X=X+DX
      Y=ZLOGP(I)-BE
   20 CALL PLOT (X,Y,2)
      CALL SYMBOL(X,Y,0.1,1HP,0.0,1)
      X=5.0
      Y=ZLOGV(L)-BE
      CALL SYMBOL(X,Y,0.1,1HA,0.0,1)
      CALL PLOT(X,Y,3)
      M=L-1
      DO 21 I=1,M
      X=X-DX
      II=L-I
      Y=ZLOGV(II)-BE
   21 CALL PLOT(X,Y,2)
      WMAX=MAX
      X=0.0
      Y=ZLOGW(1)-BE
      CALL PLOT(X,Y,3)
      DO 22 I=2,L
      X=X+DX
      IF(ZLOGW(I)-WMAX)24,23,23
   23 Y=8.0
      GO TO 22
```

```
   24 Y=ZLOGW(I)-BE
   22 CALL PLOT(X,Y,2)
      CALL SYMBOL(X,Y,0.1,1HS,0.0,1)
      IF (SENSE LIGHT 1)40,30
   30 CALL PLOT (-2.0,9.0,-3)
      SENSE LIGHT 1
      GO TO 50
   40 CALL PLOT (6.0,-12.0,-3)
   50 RETURN
C     END(0,1,1,0,0)
      END
```

```
      SUBROUTINE PFB3D(N,ID,P,R,B)
C     P=FUNCTION OF R = POWER DENSITY
C     B= FUNCTION OF R = COMPASS BEARING
C     R= FREQUENCY IN HZ
      DIMENSION ID(2),P(2),R(2),B(2),C(360),S(360)
      T=10.0
      D=0.70710678
      DTH=0.017453293
C     DTH IS ONE DEGREEE IN RADIANS IE RADIANS PER 1 DEGREE
      A=0.0
      DO 10 I=1,360
      A=A+DTH
      C(I)=COSF(A)
   10 S(I)=SINF(A)
      CALL PLOT(0.0,0.0,3)
      CALL PLOT(0.0,10.5,2)
      CALL PLOT(8.0,10.5,2)
      CALL PLOT(8.0,0.0,2)
      CALL PLOT(0.0,0.0,2)
      CALL SYMBOL(1.0,0.5,-0.1,ID(1),0.0,66)
      CALL PLOT(4.0,3.5,-3)
      CALL PLOT(3.0,0.0,2)
```

```
      CALL SYMBOL(3.25,0.0,0.1,1HN ,0.0,1)
      CALL AXIS(0.0,1.0,0H ,0,5.0,90.0,-1.0,1.0)
      CALL PLOT(0.0,1.0,3)
      CALL PLOT(0.0,0.0,2)
      CALL PLOT(-3.0,0.0,2)
      CALL SYMBOL(-3.26,0.0,0.1,1HS ,0.0,1)
      X=-3.0*D
      Y=X
      CALL SYMBOL(X-0.1,Y-0.1,0.1,1HE ,0.0,1)
      CALL PLOT(X,Y,3)
      CALL PLOT(-X,-Y,2)
      CALL SYMBOL(0.1-X,0.1-Y,0.1,1HW ,0.0,1)
      CALL PLOT(0.0,0.0,-3)
      DO 20 I=1,6
      A=0.5*FLOATF(I)
      CALL PLOT(A,0.0,3)
      DO 30 J=1,360
      Y=A*S(J)*D
      X=A*C(J) + Y
   30 CALL PLOT(X,Y,2)
   20 CONTINUE
      CALL PLOT(0.0,0.0,-3)
      D=10.0*D
C     10*D NEEDED TO SCALE 0.0-0.3 TO 0-3 INCHES ON PLOTS
```

```
      DO 40 I=2,N
      RAD=DTH*B(I)
      Y=R(I)*D*SINF(-RAD)
      X=R(I)*T*COSF(RAD) + Y
      IUD=3
      CALL PLOT(X,Y,IUD)
      AY=P(I) + Y + 1.96
C     2 IS NOT ADDED SO TOP SYMBOL WILL BE THE REFERENCE IN .08 SYMBOL
      CALL PLOT(X,AY,2)
      CALL SYMBOL(X,AY,0.08,2,0.0,-2)
      CALL PLOT(X,AY,3)
   40 CALL PLOT(X,Y,2)
      IF(SENSE LIGHT 1)41,42
   42 CALL PLOT(-4.0,7.0,-3)
      SENSE LIGHT 1
      GO TO 50
   41 CALL PLOT(4.0,-14.0,-3)
   50  RETURN
C     END(0,1,1,0,1)
      END
```

APPENDIX C

A FORTRAN PROGRAM FOR ITERATIVE WAVE TRAIN ANALYSIS

The FORTRAN listing of a Burroughs B-5500 program for the iterative least square multiple wave train analysis of a spectral matrix is presented. The mathematics is an iterative utilization of the single wave train analysis described in the body of this report. Following the single wave train analysis of the measured spectral matrix the resulting values of single wave train power, A, and wave bearing, θ, are used to find the spectral matrix that would occur for such a wave. Details for this are given in Section 5 of this report. From the above, a residual measured spectral matrix is formed by subtracting a fractional portion of the single wave spectral matrix from the previously used spectral matrix. The residual spectral matrix is then single wave train analyzed. The above procedure is continued iteratively until a specified number of iterations have been completed or the residual spectral matrix total power gets smaller than a specified value. Table C1 shows numerical results for several frequencies. Figure C1 is a plot of the bearing, θ, and the ratio of A to the total power available in the original measure spectral matrix for the frequency band 0.00833 to 0.24187 Hz. The iteration parameters were set for a maximum of five iterations, a residual power ratio of 0.1, and a fractional portion value of 0.1. Both results are for data collected for task SWOC at Stage II between 1220 and 1249 hours on 4 June 1965. The wind speed was 12 knots from a bearing of 280 degrees. In Table C1 A, H, and bearing are as previously defined. AVE P is the average $C_{ii}(f)$ for the residual spectral matrix and P is the AVE P for the first iteration; i.e., the original measured average power. The H1 and H2 values are measures of the isotropicity of the energy represented by the residual spectral matrix. H1 compares the least square value of H with the value H' that would have been obtained if the wave energy were isotropic:

$$H1 = 1 - (H/H') .$$

H2 compares the power A of a single wave fit to the total power, AVE P, and the value A' that would have been obtained for isotropic energy:

$$H2 = (A-A')/(AVE\ P-A').$$

Both H1 and H2 are between 0 and 1, the lower limit is for the isotropic case and the upper for a plane wave from a single direction.

TABLE C1

SOME NUMERICAL RESULTS OF ITERATIVE DIRECTIONAL ANALYSIS

FREQUENCY	ITERATION	A	AVE P	BEARING	A/P	H	H1	H2
0.058333	1	2.7023E-01	4.5520E-01	4.59	0.594	7.0425E-01	0.749	0.517
	2	2.4821E-01	4.2817E-01	4.59	0.534	7.0425E-01	0.706	0.486
	3	2.1839E-01	4.0385E-01	4.59	0.481	7.0425E-01	0.658	0.455
	4	1.9700E-01	3.8196E-01	4.59	0.433	7.0425E-01	0.607	0.424
	5	1.7720E-01	3.6226E-01	54.19	0.394	6.8325E-01	0.566	0.379
FREQUENCY	ITERATION	A	AVE P	BEARING	A/P	H	H1	H2
0.066667	1	2.3621E-01	4.6934E-01	11.05	0.503	7.8311E-01	0.665	0.409
	2	2.1259E-01	4.4572E-01	11.05	0.453	7.8311E-01	0.614	0.378
	3	1.9133E-01	4.2446E-01	11.05	0.408	7.8311E-01	0.559	0.347
	4	1.7219E-01	4.0533E-01	11.04	0.367	7.8311E-01	0.502	0.316
	5	1.5715E-01	3.8811E-01	56.13	0.335	7.6205E-01	0.459	0.292
FREQUENCY	ITERATION	A	AVE P	BEARING	A/P	H	H1	H2
0.075000	1	1.5356E-01	4.6426E-01	5.29	0.342	7.7033E-01	0.442	0.217
	2	1.4270E-01	4.4840E-01	5.29	0.307	7.7033E-01	0.331	0.189
	3	1.2343E-01	4.3413E-01	5.29	0.277	7.7033E-01	0.320	0.163
	4	1.1359E-01	4.2129E-01	10.43	0.249	7.7032E-01	0.263	0.137
	5	1.0255E-01	4.0973E-01	194.55	0.236	7.3312E-01	0.247	0.129
FREQUENCY	ITERATION	A	AVE P	BEARING	A/P	H	H1	H2
0.083333	1	2.0211E-01	5.5432E-01	193.72	0.365	7.2767E-01	0.585	0.244
	2	1.8190E-01	5.3411E-01	193.72	0.328	7.2767E-01	0.524	0.216
	3	1.6371E-01	5.1592E-01	193.72	0.295	7.2767E-01	0.461	0.188
	4	1.4734E-01	4.9955E-01	193.72	0.266	7.2767E-01	0.396	0.161
	5	1.3260E-01	4.8481E-01	193.72	0.239	7.2767E-01	0.331	0.136
FREQUENCY	ITERATION	A	AVE P	BEARING	A/P	H	H1	H2
0.091667	1	4.7266E-01	9.1547E-01	192.42	0.524	9.7927E-01	0.869	0.434
	2	4.3170E-01	8.6750E-01	192.42	0.472	9.7927E-01	0.841	0.403
	3	3.8853E-01	8.2433E-01	192.42	0.424	9.7927E-01	0.809	0.371
	4	3.4967E-01	7.8548E-01	192.42	0.382	9.7927E-01	0.771	0.340
	5	3.1471E-01	7.5051E-01	192.42	0.344	9.7927E-01	0.729	0.309
FREQUENCY	ITERATION	A	AVE P	BEARING	A/P	H	H1	H2
0.100000	1	1.2352E 00	1.8884E 00	192.65	0.654	3.1192E 00	0.934	0.589
	2	1.1117E 00	1.7649E 00	192.65	0.589	3.1192E 00	0.920	0.560
	3	1.0005E 00	1.6537E 00	192.65	0.530	3.1192E 00	0.903	0.530
	4	9.0047E-01	1.5537E 00	192.65	0.477	3.1192E 00	0.882	0.500
	5	8.1042E-01	1.4636E 00	192.65	0.429	3.1192E 00	0.857	0.469

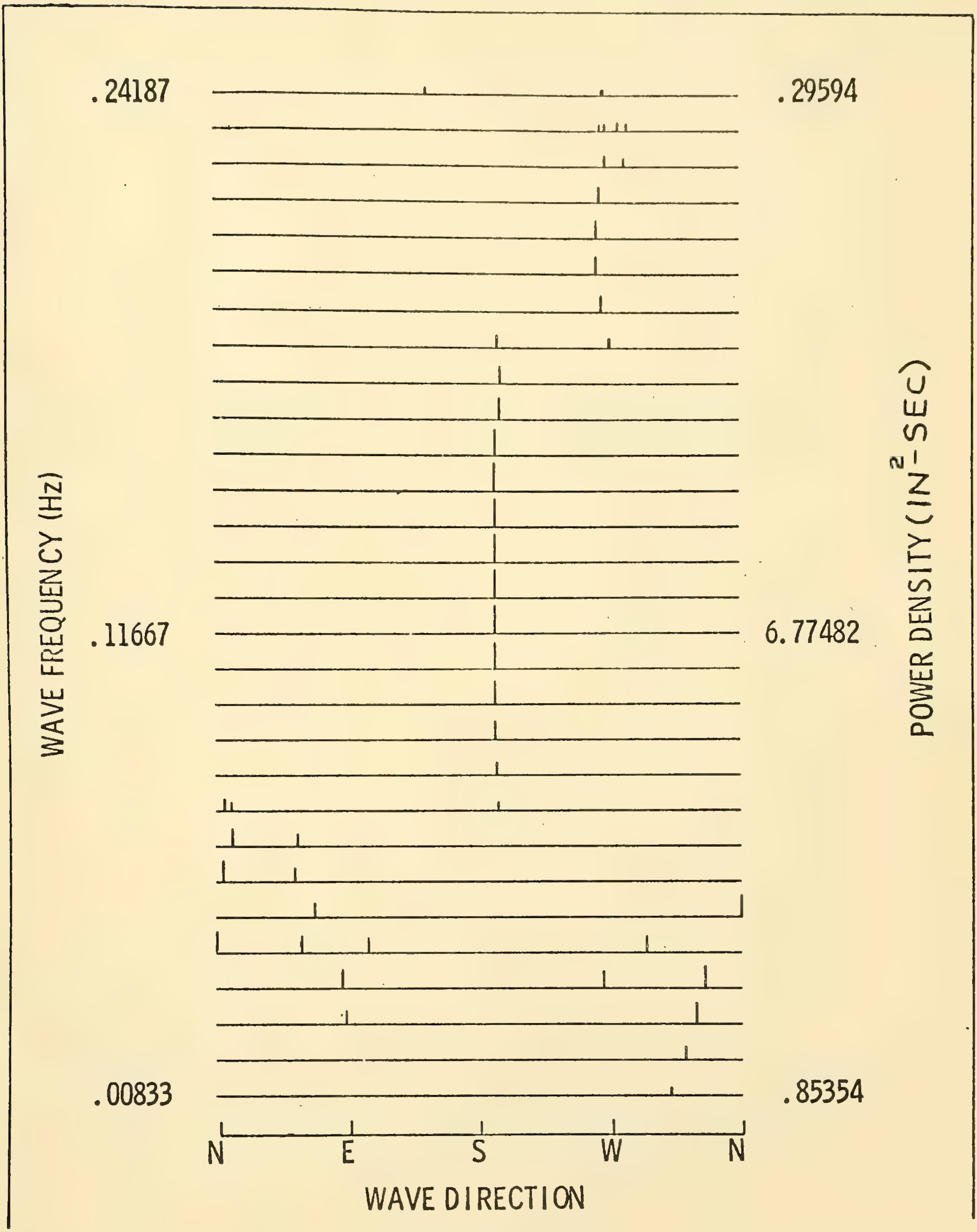


FIGURE C1. ITERATIVE LEAST SQUARE WAVE FIT DIRECTIONAL ANALYSIS

TABLE C2

B 5 7 0 0   F O R T R A N   C O M P I L A T I O N   XIII.411,

```
FILE  3=FORSWOC/EDITC,SAVE=30,LOCK,RECORD=64
FILE  6=F1335, INIT=READER,RECORD=10
FILE  9=FORP/INP,SAVE=30
C      SWOCFOR/F1335: ITERATION OF MULTIPLE WAVE TRAIN FIT FOR AN
C      INTERVAL OF FREQUENCY BINS, BY CARL M BENNETT MARCH 1972.
C      BASED ON ANALYSIS BY ITERATION OF LEAST SQUARE WAVE TRAIN FIT
C      AFTER MUNK,MILLER,SNODGRASS,AND BARBER  APRIL 1963
                                                              START OF SEGMENT
      DIMENSION  IO(11),FQ(100),WN(100), P(100,6,6),TEM(146),THZRO(146),
     1           D(6,6),PSI(6,6),F(72),THETA(72),DF(73),GOOD(6),
     2           CONTRL(8),HCD(12)
      DIMENSION  PERIOD(100),WAVLGH(100),DEPATN(100),AVP(100)
      DATA CONTRL / 48HCC EXECUTE SWOCPLT/PLOT ;END.574321202 01 CMH   /
      E2TEN=0.43429448
      TWOPI=6.281853
      RTD=57.29578
      DTR=0.01745329225
      FIV=0.087266465
      CALL PLOT(2.0,30.0,-3)
C     SET PLOTTER ORIGAN TO THE LEFT
      DO 5 I=1,72
      ZI=(I-1)*5
    5 THETA(I)=ZI*DTR
      DO 10 I=1,6
      DO 10 J=1,6
      D(I,J)=0.0
   10 PSI(I,J)=0.0
      D(1,2)=100.0
      D(1,3)=100.0
      D(1,4)=100.0
      D(1,5)=100.0
      D(1,6)=100.0
      D(2,3)=117.558
      D(2,4)=190.212
      D(2,5)=190.212
      D(2,6)=117.558
      D(3,4)=117.558
      D(3,5)=190.212
      D(3,6)=190.212
      D(4,5)=117.558
      D(4,6)=190.212
      D(5,6)=117.558
C     PSI IS TRIG ANGLE
C     D IS DISTANCE
      PSI(1,2)=0.0
```

TABLE C2 (CONTINUED)

```
      PSI(1,3)=72.0*DTR
      PSI(1,4)=144.0*DTR
      PSI(1,5)=216.0*DTR
      PSI(1,6)=288.0*DTR
      PSI(2,3)=126.0*DTR
      PSI(2,4)=162.0*DTR
      PSI(2,5)=198.0*DTR
      PSI(2,6)=234.0*DTR
      PSI(3,4)=198.0*DTR
      PSI(3,5)=234.0*DTR
      PSI(3,6)=270.*DTR
      PSI(4,5)=270.0*DTR
      PSI(4,6)=306.0*DTR
      PSI(5,6)=342.0*DTR
C     GOOD(I)= 1 FOR USABLE CHANNEL DATA AND 0 FOR BAD CHANNEL
      READ(6,2) (GOOD(I),I=1,6),LF,MF,MAXITR,RATIO,PERCENT,DXFQ
    2 FORMAT(6F1.0,3I6,3F6.0)
      CLOSE 6
      IF ( DXFQ .LT.  0.2 )  DXFQ = 0.2
      DXP=2.0*DXFQ
      READ (3)     ID,FQ(1),WN(1),M,K

      MP=M+1
      DO 32 L=2,MP
   32 READ (3)     ID,FQ(L),WN(L),M,K,DELTAT,DEPTH,PERIOD(L),WAVLGH(L),
     1 DEPATN(L),AVP(L),((P(L,I,J),J=1,6),I=1,6)
      CLOSE 3
C     P(L,I,J)=-0.0 IF CHANNEL I OR J IS NO GOOD
C     WN(L) IS WAVE NUMBER=2PI/WAVE LENGTH IN FEET
C     PI=3.1415927...
C     VALUE K NO LONGER NEEDED
      GOODCH=0.0
      DO 40  I=1,6
      GOOD(I)=GOOD(I)*P(2,I,I)
      IF(GOOD(I))41,40,41
   41 GOODCH=GOODCH+1.0
      GOOD(I)=1.0
   40 CONTINUE
      TERMS=1.0+(GOODCH*(GOODCH-1.0))
C     AXIS AND LABEL FOR BEARING VS FREQUENCY PLOT OF (EMAX,THMAX).
      DELY=-0.2
      DO 42  I=1,6
      CALL NUMBER(0.1,DELY,0.07,GOOD(I),-90.0,-1)
   42 DELY=DELY-0.1
      CALL SYMBOL(0.5,-1.25,0.14,ID(1),-90.0,60)
      CALL SYMBOL(0.5,-9.63,0.14,ID(11),-90.0,6)
      BCD(1)="WAVE D"
```

TABLE C2 (CONTINUED)

```
      BCD(2)="IRECTI"
      BCD(3)="ON     "
      CALL SYMBOL(1.0 ,-4.0,0.14,BCD(1),-90.0,14)
      CALL PLOT(1.5,-1.75,-3)
      CALL PLOT(-0.1,0.0,2)
      CALL PLOT( 0.0,0.0,2)
      CALL PLOT( 0.0,-1.8,2)
      CALL PLOT(-0.1,-1.8,2)
      CALL PLOT( 0.0,-1.8,2)
      CALL PLOT( 0.0,-3.6,2)
      CALL PLOT(-0.1,-3.6,2)
      CALL PLOT( 0.0,-3.6,2)
      CALL PLOT( 0.0,-5.4,2)
      CALL PLOT(-0.1,-5.4,2)
      CALL PLOT( 0.0,-5.4,2)
      CALL PLOT( 0.0,-7.2,2)
      CALL PLOT(-0.1,-7.2,2)
      BCD(1)="N      "
      CALL SYMBOL(-0.25,-7.1,0.14,BCD(1),-90.0,1)
      BCD(1)="W      "
      CALL SYMBOL(-0.25,-5.3,0.14,BCD(1),-90.0,1)
      BCD(1)="S      "
      CALL SYMBOL(-0.25,-3.5,0.14,BCD(1),-90.0,1)
      BCD(1)="E      "
      CALL SYMBOL(-0.25,-1.7,0.14,BCD(1),-90.0,1)
      BCD(1)="N      "
      CALL SYMBOL(-0.25, 0.1,0.14,BCD(1),-90.0,1)
      BCD(1)= "WAVE F"
      BCD(2)= "REQUEN"
      BCD(3)= "CY(HZ)"
      CALL SYMBOL(5.0,1.0,0.14,BCD(1),0.0,18)
      BCD(1)="POWER ";BCD(2)="DENSIT";BCD(3)="Y     "
      CALL SYMBOL(5.0,-8.75,0.14,BCD(1),0.0,13)
      PRINT 8,(ID(I),I=1,11),(GOOD(I),I=1,6),MAXITR,RATIO,FQ(LF),FQ(MF)
     1 , PERCENT
    8 FORMAT(" ITERATIVE LEAST SQUARE MULTIPLE WAVE TRAIN FIT OF ",11A6/
     1 1X,6F2.0," MAXIMUM ITERATION=",I4," MINIMUM RATIO=",F7.3,1X,
     2"FREQUENCY RANGE (", 2F9.6,")" , " PERCENT=",F7.3)
      DO 70 L=LF,MF
      ITR=0
      PRINT 9, FQ(L)
    9 FORMAT(" FREQUENCY  ITERATION      A",11X,"AVE P    BEARING   ",
     1 " A/P        H          H1      H2"/(1X,F9.6))
  700 SUM=0.0; ITR= ITR +1; IF(ITR .GT. MAXITR) GO TO 70
      DO 50 I=1,6
      IF(GOOD(I))51,50,51
   51 SUM=SUM+P(L,I,I)
```

TABLE C2 (CONTINUED)

```
   50 CONTINUE
      AVEP=SUM/GOODCH
      IF( AVEP .LE. 0) GO TO 70
      IF(ITR .NE. 1)  GO TO 600
          PMAX=AVEP
          CALL PLOT(DXFQ,0.0,-3)
          CALL NUMBER(0.0,0.8,0.07,FQ(L),-90.0,  5)
      CALL PLOT(0.0,0.0,3)   ; CALL PLOT(0.0,-7.2,2)
          CALL NUMBER(0.0,-7.5,0.07,PMAX,-90.0,  5)
  600 CONTINUE
      DO 60 K=1,72
      SUM=0.0
      DO 90 I= 1,5
      IF(GOOD(I) )90,90,91
   91 IP = I+1
      DO 95 J= IP,6
      IF(GOOD(J) )95,95,92
   92 SUM = SUM+P(L,I,J)*COS(WN(L)*D(I,J)*COS(THETA(K)-PSI(I,J)))
     X         -P(L,J,I)*SIN(WN(L)*D(I,J)*COS(THETA(K)-PSI(I,J)))
   95 CONTINUE
   90 CONTINUE
   60 E(K)=(AVEP + 2.0*SUM)/TERMS
   62 G=0.1/DIR
C     DELTA THETA=5DEG OR H=1/2*DEL THETA
      DE(1)=G*(E(  2)-E(  72))
      DE(72)=G*(E(  1)-E(  71))
      DO 61 ITHETA=2,71
   61 DE(ITHETA)=G*(E(  ITHETA+1)-E(  ITHETA-1))
      DE(73)=DE(1)
      K=1
      DO 65 I=1,72
      IF(DE(I)*DE(I+1))66,67,65
   66 ZI=I-1
C     FIV IS 5 DEG IN RADIANS
      THZRO(K)=FIV*(ZI+(DE(I)/(DE(I)-DE(I+1))))
      K=K+1
      GO TO 65
   67 IF(DE(I))68,69,68
   69 THZRO(K)=FIV*FLOAT(I-1)
      K=K+1
   68 IF(DE(I+1))65,64,65
   64 THZRO(K)=FIV*FLOAT(I)
      K=K+1
   65 CONTINUE
      NZEROS=K-1
      DO 75 K=1,NZEROS
C     TERMS SAME AS ABOVE
```

```
      SUM=0.0
      DO 76 I=1,5
      IF(GOOD(I) )76,76,77
   77 IP=I+1
      DO 79 J=IP,6
      IF(GOOD(J) )79,79,78
   78 SUM = SUM+P(L,I,J)*COS(WN(L)*D(I,J)*COS(THZRO(K)-PSI(I,J)))
     X         -P(L,J,I)*SIN(WN(L)*D(I,J)*COS(THZRO(K)-PSI(I,J)))
   79 CONTINUE
   76 CONTINUE
   75 TEM(K)=(AVEP + 2.0*SUM)/TERMS
      EMAX = TEM(1)
      ROMAX = THZRO(1)
C     TRIG TO COMPASS CONVERSION OF THETA
      THMAX = RTD*(TWOPI-THZRO(1))
      DO 74 K=2,NZEROS
      IF(EMAX-TEM(K))73,74,74
   73 EMAX = TEM(K)
      ROMAX = THZRO(K)
      THMAX = RTD *(TWOPI-THZRO(K))
C     THMAX IS BEARING FROM MAGNITIC NORTH
   74 CONTINUE
      TEST = EMAX/PMAX
      SUM=0.0
      DO 80 I=1,5
      IF(GOOD(I) )80,80,81
   81 IP=I+1
      DO 83 J=IP,6
      IF(GOOD(J) )83,83,82
   82 SUM=SUM+P(L,I,J)*P(L,I,J)+P(L,J,I)*P(L,J,I)
   83 CONTINUE
   80 CONTINUE
      SSQ = AVEP * AVEP + 2.0*SUM
      H = SSQ -TERMS* EMAX*EMAX
      ATILDA = AVEP/TWOPI
      HTILDA=SSQ-ATILDA*ATILDA*TERMS
      H1 = 1.0 -(H /HTILDA)
      H2 = (EMAX-ATILDA)/(AVEP-ATILDA)
      Y = -7.2 * (THMAX/ 360.0);  X =  TEST * DXFQ
      CALL PLOT(0.0,Y,3); CALL PLOT(X,Y,2)
C           ITERATION A    AVE P   BEARING   RATIO   H    H1   H2
      PRINT 7, ITR, EMAX, AVEP , THMAX  , TEST , H , H1 ,H2
    7 FORMAT( 14X, I4,5X,1PE11.4,1PE13.4,0PF9.2,F8.3,1PE12.4,0P2F7.3)
      IF(TEST .LT. RATIO) GO TO 70
      DO 710 I= 1,6
      IF(GOOD(I) .NE. 0)  P(L,I,I) = P(L,I,I) - (EMAX* PERCENT)
  710 CONTINUE
```

TABLE C2 (CONTINUED)

```
      DO 730 I= 1,5 ;IF(GOOD(I) .EQ. 0) GO TO 730
      IP = I+1
      DO 720 J= IP,6; IF(GOOD(J) .EQ. 0) GO TO 720
      P(L,I,J) = P(L,I,J) - (PERCENT *(EMAX* COS(WN(L)*D(I,J)*
     1                                         COS(RDMAX - PSI(I,J)))))
      P(L,J,I)= P(L,J,I) + (PERCENT *(EMAX* SIN(WN(L)*D(I,J)*
     1                                         COS(RDMAX - PSI(I,J)))))
                                                                   SEGMENT
                                                          START OF SEGMENT
  720 CONTINUE
  730 CONTINUE
      GO TO 700
   70 CONTINUE
100         CALL PLOT(0,0,999)

      CLOSE 9
      CALL ZIP ( CONTRL)
        STOP
      END
                                                                   SEGMENT
```

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Coastal Systems Laboratory Panama City, Florida 32401	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP ---

3. REPORT TITLE

THE DIRECTIONAL ANALYSIS OF OCEAN WAVES: AN INTRODUCTORY DISCUSSION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Informal

5. AUTHOR(S) (First name, middle initial, last name)

Carl M. Bennett

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1972	83	12

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. Task SWOC SR 004 03 01, Task 0582 d. ZR 000 01 01 (0401-40)	9a. ORIGINATOR'S REPORT NUMBER(S) NCSL 144-72 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) First Edition NSRDL/PC 3472

10. DISTRIBUTION STATEMENT

Approved for Public Release; Distribution Unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
Second edition. This report was originally issued in September 1971 as NSRDL/PC Report 3472.	Commander Naval Ship Systems Command (00V1K)

13. ABSTRACT

An introductory discussion of the mathematics behind the directional analysis of ocean waves is presented. There is sufficient detail for a reader interested in applying the methods; further, the report can serve as an entry into the theory. The presentation is basically tutorial but does require a reasonably advanced mathematical background. Results of a program for the measurement of directional ocean wave bottom pressure spectra are included as an appendix. This second edition makes corrections to the first and adds some details of an iterative directional analysis method.

DD FORM 1 NOV 65 1473

UNCLASSIFIED
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Ocean waves Spectrum analysis Fourier analysis Matrix methods Convariance Bibliographies IDENTIFIERS Cross spectral analysis Directional analysis SWOC (Shallow Water Oceanography) Ocean bottom pressure data						

INITIAL DISTRIBUTION
NCSL 144-72

000100	Chief of Naval Material	
	(MAT-03L4)	(Copy 1)
000400	Commander, Naval Ship Systems Command	
	(SHIPS 00V1K)	(Copies 2-4)
	(SHIPS 2052)	(Copies 5-6)
	(SHIPS 03542)	(Copy 7)
000500	Commander, Naval Ship Engineering Center	(Copy 8)
003100	Assistant Secretary of the Navy	(Copy 9)
005600	Chief of Naval Research	
	(ONR 410)	(Copy 10)
	(ONR 414)	(Copy 11)
	(ONR 420)	(Copy 12)
	(ONR 438)	(Copy 13)
	(ONR 460T)	(Copy 14)
	(ONR 461)	(Copy 15)
	(ONR 462)	(Copy 16)
	(ONR 463)	(Copy 17)
	(ONR 466)	(Copy 18)
	(ONR 468)	(Copy 19)
028900	Director, Office of Naval Research (Boston)	(Copy 20)
029000	Director, Office of Naval Research (Chicago)	(Copy 21)
029100	Director, Office of Naval Research (Pasadena)	(Copy 22)
028500	Oceanographer of the Navy	(Copies 23-24)
021000	Commander, Naval Oceanographic Office	(Copies 25-26)
023900	Commander, Naval Ship Research and Development Center	
	(Code 561)	(Copies 27-29)
	(Code 01)	(Copy 30)
	(Code 01B)	(Copy 31)
027200	Commander, Naval Weapons Center, China Lake	(Copy 32)
015900	Commander, Naval Air Development Center	(Copy 33)
019800	Commander, Naval Electronics Laboratory Center, San Diego	(Copies 34-35)
026900	Commander, Naval Undersea Center, San Diego	(Copy 36)
	(Dr. Robert H. Riffenburg)	(Copy 37)
	(Dr. W. J. MacIntire)	(Copy 38)
018800	Commanding Officer, Naval Civil Engineering Laboratory	(Copy 39)
021200	Commander, Naval Ordnance Laboratory	(Copy 40)
028100	Officer in Charge, New London Laboratory, NUSC	(Copy 41)
036100	Officer in Charge, Annapolis Laboratory,NSRDC	(Copy 42)

022600	Director, Naval Research Laboratory	
	(ONR 481)	(Copy 43)
	(ONR 483)	(Copy 44)
	(ONR 102 OS)	(Copy 45)
	Ocean Sciences Division (NRL)	(Copy 46)
	Library (NRL)	(Copy 47)
029600	Commander, Pacific Missile Range	(Copy 48)
027900	Officer in Charge, Environmental Prediction Research Facility	(Copy 49)
015600	Superintendent, Naval Academy	(Copy 50)
	(Professor Paul R. Van Mater, Jr.)	(Copy 51)
022500	Superintendent, Naval Postgraduate School	(Copy 52)
002700	Army Research, Office of the Chief of Research and Development, Dept. of the Army	(Copy 53)
002500	Director, Army Engineers Waterways Experiment Station	(Copy 54)
002300	Army Coastal Engineering Laboratory	(Copy 55)
014900	District Engineer, Mobile District Corps of Engineers	
	(W. W. Burdin)	(Copy 56)
007900	Director of Defense, Research and Engineering	
	(Ocean Control)	(Copy 57)
015400	Director, National Oceanographic Data Center	(Copies 58-59)
000600	Director, Advanced Research Projects Agency	(Copy 60)
	(Nuclear Test Detection Office)	(Copy 61)
	(Dr. R. W. Slocum)	(Copy 62)
028600	Chief, Oceanographic Branch, CERC	(Copy 63)
	(Dr. D. Lee Harris)	(Copy 64)
	(Dr. Cyril J. Galvin, Jr.)	(Copy 65)
034700	Director, Woods Hole Oceanographic Inst.	(Copy 66)
	(Dr. John C. Beckerle)	(Copy 67)
	(Dr. N. N. Panicker)	(Copy 68)
015300	National Oceanic & Atmospheric Administration, U. S. Department of Commerce	(Copy 69)
	(Dr. Moe Ringenbach)	(Copy 70)
009600	Environmental Science Services Administration U. S. Department of Commerce	(Copy 71)
013400	Chief, Marine Science Center, Coastal Geodetic Survey, U.S. Dept. of Commerce	(Copy 72)
004200	Director, Bureau of Commercial Fisheries, U. S. Fish and Wildlife Service	(Copy 73)
000700	Allan Hancock Foundation	(Copy 74)
010700	Gulf Coast Research Laboratory, Ocean Springs	(Copy 75)
012300	Director, Lamont-Doherty Geological Observatory, Columbia University, Palisades	(Copies 76-77)
	(Dr. Arnold L. Gordon)	(Copy 78)
	(Dr. Leonard E. Alsop)	(Copy 79)
	(Dr. John E. Nafe)	(Copy 80)

(Dr. Keith McCamy) (Copy 81)
(Dr. John T. Kuo) (Copy 82)
(Dr. Tom Herron) (Copy 83)
(Dr. Manik Talwani) (Copy 84)

030700 Director, Scripps Institute of Oceanography, University of California (Copy 85)
(Dr. Walter H. Munk) (Copy 86)
(Dr. D. L. Inman) (Copy 87)
(John D. Isaacs) (Copy 88)

008500 Department of Geotechnical Engineering, Cornell University (Copy 89)

008700 Department of Oceanography, Florida Institute of Technology (Copy 90)

008800 Department of Oceanography, Florida State University
(Dr. K. Warsh) (Copy 91)

008200 Chairman, Department of Coastal Engineering, University of Florida (Copy 92)

031800 University of West Florida
(Dr. A. Chaet) (Copy 93)

009200 Department of Physics, Georgia Southern College
(Dr. Arthur Woodrum) (Copy 94)

011800 Institute of Geophysics, University of Hawaii (Copy 95)

009300 Division of Engineering and Applied Physics, Harvard University, Cambridge (Copy 96)

004900 Director, Chesapeake Bay Institute, Johns Hopkins University (Copy 97)
(W. Stanley Wilson) (Copy 98)

031200 Officer in Charge, Applied Physics Laboratory, Johns Hopkins University (Copy 99)

013500 Director, Marine Science Center, Lehigh Univ. (Copy 100)

005700 Coastal Studies Institute, Louisiana State University (Copies 101-102)

012000 Institute of Marine Sciences, University of Miami
(Dr. W. Duing) (Copy 103)

010600 Great Lakes Research Division, University of Michigan (Copy 104)

008600 Department of Meteorology and Oceanography, New York University (Copy 105)

009500 Environmental Science Center, Nova University
(Dr. W. S. Richardson) (Copy 106)

008900 Head, Department of Oceanography, Oregon State University (Copy 107)

029900 Pell Marine Science Library, University of Rhode Island (Copy 108)

009100	Department of Oceanography and Meteorology, Texas A&M University	(Copies 109-110)
002000	Applied Physics Laboratory, University of Washington	(Copy 111)
009000	Head, Department of Oceanography, University of Washington	(Copy 112)
010500	Chairman, Department of Oceanography, University of South Florida	(Copy 113)
015200	National Institute of Oceanography, Wormley, Godalming, Surrey, England	
	(Director)	(Copy 114)
	(J. Ewing)	(Copy 115)
	(Dr. L. Draper)	(Copy 116)
011700	Institute fur Meereskunde Under Universitat, West Germany	
	(Dr. Wolfgang Krauss)	(Copy 117)
	(Dr. F. Schott)	(Copy 118)
007700	Director, Defense Documentation Center	(Copies 119-130)
	Dr. William P. Raney, Special Assistant for Research, Navy Department, Washington, D.C. 20350	(Copy 131)
	National Oceanic and Atmospheric Administration, Boulder, Colorado 80302	
	(Earth Sciences Lab)	(Copy 132)
	(Wave Propagation Lab)	(Copy 133)
	Director, National Weather Service, NOAA, 8060 13th Street, Silver Spring, MD 20910	(Copy 134)
	(Dr. William Kline, Sys Dev O)	(Copy 135)
	Director, National Ocean Survey, NOAA, Rockville, MD 20852	(Copy 136)
	Director, Pacific Marine Laboratory, NOAA Seattle, Washington 98102	(Copy 137)
	Earthquake Mechanism Laboratory, NOAA, 390 Main Street, San Francisco, CA 94105	(Copy 138)
	Ports and Waterways Staff, Office of Marine Environment and Systems, U. S. Coast Guard Headquarters, 400 7th St., SW, Washington, D.C. 20591	(Copy 139)
	Director, National Center for Earthquake Research, U. S. Geological Survey, Menlo Park, CA 94025	(Copy 140)
	Director, Seismological Laboratory, California Institute of Technology, Pasadena, CA 91109	(Copy 141)

Director, Seismic Data Laboratory, Geotech-Teledyne, 314 Montgomery St., Alexandria, VA 22314 (Copy 142)

Director, Thomas J. Watson Research Center, Yorktown Heights, NY 10598 (Copy 143)

Director, Institute for Storm Research, Houston, TX 77006 (Copy 144)

The Offshore Company, P.O. Box 2765, Houston, TX 77001
(Crane E. Zumwalt) (Copy 145)

Tetra Tech, Inc., 630 N. Rosemead Blvd., Pasadena, CA 91107
(Dr. J. I. Collins) (Copy 146)
(Dr. Bernard LeMehaute) (Copy 147)

Director, Geophysical Institute, University of Alaska, College Br., Fairbanks, Alaska 99701 (Copy 148)

Chairman, Department of Geological Sciences, Brown University, Providence, RI 02912 (Copy 149)

Chairman, Dept. of Geophysics, University of California, Berkeley, CA 94720 (Copy 150)

Director, Institute of Geophysics and Planetary Physics, University of California, Los Angeles, CA 90024 (Copy 151)

Director, Institute of Geophysics and Planetary Physics, University of California, Riverside, CA 92502 (Copy 152)

University of California, Hydraulic Engineering Division, Berkeley, CA 94720
(Dr. R. L. Wiegel) (Copy 153)

Chairman, Division of Fluid, Thermal, Aerospace Sciences, Case Western Reserve University, Cleveland, Ohio 44106 (Copy 154)

Central Michigan University, Brooks Science Hall, Box 12, Mt. Pleasant, MI 48858
(Dr. Kenneth Uglum) (Copy 155)

Chairman, Department of Geophysical Sciences The University of Chicago, Chicago IL 60637 (Copy 156)

Columbia University, Dept. of Physics, New York City, NY 10027
(Dr. Gerald Feinberg) (Copy 157)

Columbia University, 202 Haskell Hall, BASR, 605 W. 115th St., New York City, NY 10025
(Dr. Alan G. Hill) (Copy 158)

Chairman, Department of Geological Sciences, Cornell University, Ithaca, NY 14850 (Copy 159)

Chairman, College of Marine Studies, Univ. of Delaware, Newark, DE 19711 (Copy 160)

Chairman, Dept. of Oceanography, Duke Univ., Durham, NC 27706 (Copy 161)

Geophysical Fluid Dynamics Institute, Florida State University, Tallahassee, FL 32306
(Dr. Ivan Tolstoy) (Copy 162)
(Dr. Joe Lau) (Copy 163)

Chairman, Center for Earth and Planetary Physics, Harvard University, Cambridge, MA 02138 (Copy 164)

University of Hawaii, Department of Ocean Engineering, Honolulu, Hawaii 96822
(Dr. Charles L. Bretschneider) (Copy 165)

University of Idaho, Department of Physics, Moscow, Idaho 83843
(Dr. Michael E. Browne) (Copy 166)

University of Iowa, Institute of Hydraulic Research, Iowa City, Iowa 52240
(Dr. Hunter Rouse) (Copy 167)

Chairman, Department of Earth and Planetary Sciences, Johns Hopkins University, Baltimore, MD 21218 (Copy 168)

Chairman, Department of Earth and Planetary Sciences, Massachusetts Institute of Technology, Boston, MA 02139 (Copy 169)

Massachusetts Institute of Technology, Dept. of Naval Architecture and Marine Engineering, Boston, MA 02139
(Attn: Dr. J. N. Newman) (Copy 170)

Division of Physical Oceanography, School of Marine and Atmospheric Science, University of Miami, 10 Rickenbacker Causeway, Miami, FL 33149
(Dr. Christopher N. K. Mooers) (Copy 171)
(Dr. Claes Rooth) (Copy 172)

Chairman, Department of Natural Science,
Michigan State University, East Lansing,
MI 48823 (Copy 173)

New York University, Institute of Mathematical
Sciences, New York City, NY 10003
(Dr. J. J. Stoker) (Copy 174)

Chairman, Dept. of Geosciences, North Carolina
State University, Raleigh, NC 27607 (Copy 175)

Chairman, Institute of Oceanography, Old
Dominion University, Norfolk, VA 23508 (Copy 176)

Chairman, Department of Geosciences, Geophysics
Section, The Pennsylvania State University,
University Park, PA 16802 (Copy 177)

Chairman, Dept. of Earth and Planetary Sciences,
University of Pittsburg, Pittsburg, PA 15213 (Copy 178)

Chairman, Dept. of Geological and Geophysical
Sciences, Princeton, New Jersey 08540 (Copy 179)

University of Rhode Island, Graduate School
of Oceanography, Kingston, RI 02881
(Dr. Kern Kenyon) (Copy 180)

Chairman, Dept. of Geology, Rice University,
Houston, TX 77001 (Copy 181)

Chairman, Dept. of Earth and Atmospheric
Sciences, St. Louis University, St. Louis,
MO 63103 (Copy 182)

Director, Dallas Geophysical Laboratory,
Southern Methodist Univ., Dallas, TX 75222 (Copy 183)

Director, Center for Radar Astronomy,
Stanford University, Stanford, CA 94305 (Copy 184)

Chairman, Department of Geophysics, Stanford
University, Stanford, CA 94305 (Copy 185)

Chairman, Dept. of Geological Sciences,
University of Texas, Austin, TX 78712 (Copy 186)

Chairman, Dept. of Geological and Geophysical
Sciences, University of Utah, Salt Lake
City, Utah 84112 (Copy 187)

Director, Marine Research Laboratory, University
of Wisconsin, Madison, WI 53706 (Copy 188)

University of Wisconsin, Center for Great
Lakes Studies, Milwaukee, WI 53201
(Dr. David L. Cutchin) (Copy 189)

Chairman, Dept. of Geology and Geophysics,
Yale University, New Haven, CT 06520 (Copy 190)

Director, Navy Hydrographic Office, Buenos
Aires, Argentina (Copy 191)

Chairman, Department of Geophysics and
Geochemistry, Australian National University,
Canberra, 2600, Australia (Copy 192)

Monash University, Geophysical Fluid Dynamics
Lab., Clayton, Victoria, Australia 3168
(Dr. B. R. Morton) (Copy 193)

Flinders University of South Australia, Horace
Lamb Center for Oceanographic Research,
Bedford Park, Adeleide, South Australia
5042
(Prof. Bye) (Copy 194)

Chairman, Dept. of Oceanography, Dalhousie
University, Halifax, Nova Scotia, Canada (Copy 195)

Director, Geophysics Laboratory, University
of Toronto, Toronto, Canada (Copy 196)

Director, Institute of Earth and Planetary
Physics, University of Alberta, Edmonton 7,
Alberta, Canada (Copy 197)

Director, Institute of Oceanography, University
of British Columbia, Vancouver 8, British
Columbia, Canada (Copy 198)

Chairman, Department of Geophysics and
Planetary Physics, The University, Newcastle
Upon Tyne, NE 1 7 RU, England (Copy 199)

Cambridge University, Madlingley Rise,
Madingley, Cambridge CB3 OEZ, England
(Dr. M. S. Longuet-Higgins) (Copy 200)

Chairman, Department of Geodesy and
Geophysics, Cambridge University, Madlingley
Rise, Madingley Road, Cambridge CB3 OEZ
England (Copy 201)

Director, Institute for Coastal Oceanography
and Tides, Birkenhead, Cheshire, England (Copy 202)

Director, Oceanographic Research Institute of
the Defense Dept., Kiel, West Germany (Copy 203)

Abteilung fur Theoretische Geophysik,
Universitat Hamburg, Hamburg, West Germany (Copy 204)

Institute for Advanced Studies, 64 Merrion Square, Dublin, Ireland
(Dr. John Lighton Synge) (Copy 205)

Director, NATO Saclant ASW Research Centre, La Spezia, Italy (Copy 206)

Director, Geophysical Institute, University of Tokyo, Tokyo, Japan (Copy 207)

University of Auckland, Dept. of Physics, Auckland, Dept. of Physics, Auckland, New Zealand
(Professor A. C. Kibblewhite) (Copy 208)

Chairman, Dept. of Marine Science, University of Puerto Rico, Mayaguez, Puerto Rico 00708 (Copy 209)