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THE DIRECTIONAL CALCULUS,

BASED UPON THE METHODS OF

HERMANN GRASSMANN.

BY

E. W. HYDE,

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO.

BOSTON, U.S.A.:
PUBLISHED BY GINN & COMPANY.
1890.

Typography by J. S. Cushing & Co., Boston, U.S.A.

Presswork by Ginn & Co., Boston, U.S.A.

PREFACE.

THE wonderful and comprehensive system of Multiple Algebra invented by Hermann Grassmann, and called by him the *Ausdehnungslehre* or *Theory of Extension,* though long neglected by the mathematicians even of Germany, is at the present time coming to be more and more appreciated and studied. In order that this system, with its intrinsic naturalness, and adaptability to all the purposes of Geometry and Mechanics, should be generally introduced to the knowledge of the coming generation of English-speaking mathematicians, it is very necessary that a text-book should be provided, suitable for use in colleges and universities, through which students may become acquainted with the principles of the subject and its applications.

The following pages present, in part, the results of eight or nine years of study and experience in lecturing to university classes upon this subject, and the Author hopes that they may aid in some measure to bring about that general study and use of directional methods, which, in his opinion, will ultimately cause the comparatively awkward and roundabout methods of Cartesian coordinates to be superseded by them, for many of the purposes of analysis.

As the great generality of Grassmann's processes — all results being obtained for n-dimensional space — has been one of the main hindrances to the general cultivation of his system, it has been thought best to restrict the discussion to space of two and three dimensions.

The Author, though formerly an enthusiastic admirer of Hamilton's Quaternions, has been brought, by study and experience in teaching both, to a firm belief in the great practical, as well as theoretical, superiority of Grassmann's system. This superiority consists, according to the judgment of the writer, first, and largely, in the fact that Grassmann's system is founded upon, and absolutely consistent with, the idea of geometric dimensions. Second, in the fact that *all* geometric quantities appear as independent units, viz.: the point; the point at ∞, or line direction; the definite line; the line at ∞, or plane direction; the definite plane; and, finally, the plane at ∞, equivalent to a volume, which is scalar. The same holds for space of any number of dimensions; in fact, it seems scarcely possible that any method can ever be devised, comparable with this, for investigating n-dimensional space.

Now Quaternions deals only with the vector, or line direction, and the scalar — for a quaternion is only the sum of these two; it knows nothing of a vector having a definite *position*, which is the complete representative of the space qualities of a force. Further, Hamilton's vector is not a vector pure and simple, but a *versor-vector* — a fact which gives its peculiar character to his system, as being really a calculus of directed imaginaries. This, which Tait regards as "one of the main elements of the singular simplicity of the Quaternion Calculus," appears to the writer in a very different light. It gives rise to such equations as $\frac{i}{j} = ji$, *i.e.* multiplying two quantities together is the same as dividing one by the other, which the author has found a great stumbling-block to the student. It can hardly be regarded as a *natural* geometrical conception, that the product of two vectors should be another vector perpendicular to each of them; still less, that it should be, as in the general case, a scalar plus such a vector. In fact, the Quaternion system practically throws overboard the idea of geometric dimensions.

It has been deemed advantageous, however, to make use of certain terms and symbols introduced by Hamilton, such as *scalar*, *tensor*, with its symbol T, etc.

Although this work is based upon the principles and methods of Grassmann, yet much matter will be found in it that is believed to be original with the author.

Thus the idea of the *complement in a point system*, with its geometric interpretation, as developed in Arts. 40–44 and 63, does not occur in Grassmann's works, either of 1844 or 1862. This idea is of great value in geometric applications, and corresponds to that of *duality* in Modern Geometry.

In Arts. 68 and 69 are treated some properties of what I have called *screws*, which, as such, are not discussed by Grassmann.

One of the most beautiful and valuable theories developed by Hamilton in his great works is that of linear and vector functions. This theory I have found to be equally capable of application to *point* functions in n-dimensional space. In Chapters IV. and VI. this linear, point function has been used in the discussion of the conic and quadric, giving rise to a treatment of these loci bearing the same analogy to trilinear and quadriplanar methods that the vector treatment of these curves and surfaces bears to ordinary Cartesian methods, and possessing likewise the same superiority in clearness, conciseness, and intelligibility which the vector methods have over the Cartesian.

It appears to the writer that Hamilton's method of dealing with linear functions much excels that of Grassmann, as developed in the second part of his work of 1862, for practical utility and convenience.

It is hoped and believed that the exposition of the fundamental ideas and principles of the system in the first two chapters will be found sufficiently full and explicit to give the student a working knowledge and grasp of the subject, such as will enable him to take up without difficulty the work of the succeeding chapters, or to read, with comparative ease, the original treatises of Grassmann. A large number of exercises have been inserted, in the belief that only by the repeated application of the principles which he has learned to the solution of actual problems can the student acquire any real command of any branch of Mathematics. Eight or nine blank pages will be found at the end of each chapter, for the reception of notes, solutions, etc., the Author's experience having convinced him of the advantages of such an arrangement.

E. W. HYDE.

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DIRECTIONAL CALCULUS.

CHAPTER I.

ADDITION AND SUBTRACTION.

1. The quantities to be treated in this book are such as possess one or more of the qualities *magnitude*, *direction*, *position*.

A quantity possessing magnitude only, though it may be either positive or negative, is a *scalar* quantity.

A quantity possessing magnitude and direction is a *directed*, or *vector*, quantity. Thus a line of given length and direction is a *line-vector*, or simply a *vector;* a plane area of given direction and magnitude is a *plane-vector*.

If a vector pass through some definite point, it has *position*, as well as magnitude and direction, and may be called a *point-vector*. Similarly, when a plane-vector passes through a definite point, it may be called a *point-plane-vector*. We shall, however, often substitute for point-vector and point-plane-vector the simpler terms *line* and *plane*, especially when, as is often the case, the question of magnitude does not concern us. These terms correspond respectively to the terms *Linientheil* and *Flächentheil*, as used by Grassmann.

2. *Equality.* Two quantities are said to be *equal* when their qualities are identical. Thus two vectors having the same length and direction are equal. Of course the term *direction* must include the idea of *sense*.

3. *The point.* A point, as used in this Calculus, may be defined as a position in space, of no magnitude, but endowed with a certain value which will be called its *weight.* This value, or weight, is a *scalar* quantity, and, as such, obeys, in connection with the point to which it belongs, the ordinary laws of multiplication; that is, if p be a point and m and n weights, $mp = pm$ and $mp + np = (m + n)p$. Letters, such as p, p_1, etc., will be used to designate the *positions* of points, and these will be multiplied by other letters, such as k, l, m, n, representing their weights. If the weight of a point be unity, the figure 1 representing this weight will be omitted, so that a letter without coefficient, representing a point, will be regarded as representing a point of unit weight, or a unit point.

4. *Equality of points.* If two weighted points m_1p_1 and m_2p_2 are to be equal, their qualities must be identical. These are their weights (*quasi*-magnitudes) and their positions. Hence, in order that m_1p_1 and m_2p_2 may be equal, we must have $m_1 = m_2$, and p_1 coincident with p_2.

5. *Difference of unit points.* Suppose $m_1 = m_2 = 1$, but p_1 and p_2 *not* coincident. Then the only difference between p_1 and p_2 is one of *position.* This difference is naturally expressed by saying that it is a certain distance in a certain direction, which corresponds precisely with the definition of a *vector* in Art. 1. Thus a difference of position, which is the *only* difference between two unit points, is naturally represented by a vector. Now we may extend the meaning of the algebraic sign for a difference, *i.e.* the minus sign, so that it shall indicate the difference of position of two unit points, provided that this use of the sign leads to no inconsistencies or contradictions. Hence, if p_1 and p_2 are unit points, and ϵ the vector from p_1 to p_2, we write

$$p_2 - p_1 = \epsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the development of the subject will show this to be in accordance with the above proviso.

If $p_2 - p_1 = \epsilon = 0$, or $p_2 = p_1$, there is no difference of position between the points; *i.e.* they coincide, as previously stated.

Adding p_1 to both sides of (1), we have

$$p_2 = p_1 + \epsilon, \qquad \text{(2)}$$

in which, of course, the meaning of the sign $+$ is extended in a manner similar to that of the minus sign.

Eq. (2) shows that the sum of a point and a vector is a *point*, distant from the first point by the length of the vector, and in the direction of the same.

6. Let e_1, e_2, e_3 be any three unit points, and ϵ_1, ϵ_2, ϵ_3 vectors ‖ and equal in length to the sides of the triangle $e_1e_2e_3$ taken in order, so that $\epsilon_1 = e_3 - e_2$, $\epsilon_2 = e_1 - e_3$, $\epsilon_3 = e_2 - e_1$: then

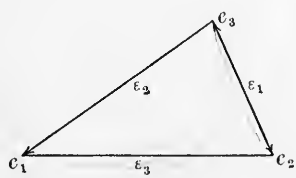
$$\epsilon_1 + \epsilon_2 + \epsilon_3 = e_3 - e_2 + e_1 - e_3 + e_2 - e_1 = 0; \qquad \text{(3)}$$

that is, the sum of the vector sides of a triangle, taken positively in the same sense around the triangle, is zero.

We have also

$$\epsilon_1 + \epsilon_2 = e_1 - e_2 = -\epsilon_3; \qquad \text{(4)}$$

so that the sum of two vectors is a vector which represents the difference of the initial and final positions of a point which moves along the two vectors successively.



Write $\epsilon_1' = e_2 - e_3 = -\epsilon_1$;

then $$\epsilon_1' - \epsilon_2 = e_2 - e_3 - e_1 + e_3 = e_2 - e_1 = \epsilon_3; \qquad \text{(4 a)}$$

that is, the difference of two vectors is the vector joining their extremities when they are drawn outwards from a common point.

Similarly, if we take n unit points $e_1, e_2, \cdots e_n$, we have

$$(e_2 - e_1) + (e_3 - e_2) + (e_4 - e_3) + \cdots + (e_n - e_{n-1}) = e_n - e_1;$$

that is, the sum of any number of vectors is the vector representing the difference between the initial and final positions of a point which moves along all the vectors in succession. The result is evidently independent of the order in which the vectors are taken.

7. *Addition of points.* In order to interpret the meaning of such an expression as $m_1p_1 + m_2p_2 + \text{etc.} \equiv \Sigma mp$, we will assume, —

1st. That the sum of any number of points will be itself a point, which we shall call the mean point of the system.

2d. That the weight of the sum will be equal to the algebraic sum of the weights of the points.

These assumptions are allowable if they lead to no contradictory or inconsistent results. In accordance with them we write for two points,

$$m_1p_1 + m_2p_2 = (m_1 + m_2)\bar{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

Transposing, we have

$$m_2(p_2 - \bar{p}) = m_1(\bar{p} - p_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which shows that $\bar{p}$ is on the straight line joining p_1 and p_2, at distances from these points inversely proportional to their weights.

Let e be any point whatever, and subtract from both sides of (5) $(m_1 + m_2)e$;

$$\therefore\ m_1(p_1 - e) + m_2(p_2 - e) = (m_1 + m_2)(\bar{p} - e),$$

or

$$\bar{p} - e = \frac{m_1}{m_1 + m_2}(p_1 - e) + \frac{m_2}{m_1 + m_2}(p_2 - e), \quad . \quad . \quad (7)$$

by which $\bar{p}$ is easily found.

For three points we have

$$m_1p_1 + m_2p_2 + m_3p_3 = (m_1 + m_2 + m_3)(\bar{p})', \quad . \quad . \quad (8)$$

or, by (5),

$$(m_1 + m_2)\bar{p} + m_3p_3 = (m_1 + m_2 + m_3)(\bar{p})';$$

so that $(\bar{p})'$ is on the line joining $\bar{p}$ and p_3 at distances from them inversely proportional to their weights.

Subtracting from both sides of (8) $(m_1 + m_2 + m_3)e$, we have, on dividing by Σm,

$$(\bar{p})' - e = \frac{m_1(p_1 - e) + m_2(p_2 - e) + m_3(p_3 - e)}{m_1 + m_2 + m_3}, \quad (9)$$

by which $(\bar{p})'$ is easily constructed.

Similarly, for any number of points we have

$$\Sigma mp = \bar{p}\Sigma m, \quad \ldots \quad (10)$$

and
$$\bar{p} - e = \frac{1}{\Sigma m}\Sigma[m(p - e)]. \quad \ldots \quad (11)$$

The reader will notice the analogy to "center of parallel forces."

8. *Case in which $\Sigma m = 0$.* In (5) and (6) let

$$m_1 + m_2 = 0,$$

and these equations become

$$m_2(p_2 - p_1) = 0 \cdot \bar{p} \quad \ldots \quad (12)$$

and
$$\bar{p} - p_2 = \bar{p} - p_1. \quad \ldots \quad (13)$$

If m_2 be not zero, and p_1 not coincident with p_2, these equations can only be satisfied when $\bar{p}$ is at an *infinite distance* on the line p_1p_2. Eq. (6) shows at once that, when m_2 is negative, $\bar{p}$ is not *between* p_1 and p_2, and that, as the numerical value of m_2 approaches that of m_1, $\bar{p}$ recedes farther and farther, until, when $m_1 + m_2 = 0$, $\bar{p}$ is at ∞. Thus the meaning of eq. (12) is, *that a point of zero weight at an infinite distance is equivalent to a vector of definite length directed towards this point.* It will often be convenient to regard vectors as points at ∞.

Consider next the general case of n points. Eqs. (10) and (11) become, when $\Sigma_1^n m = 0$,

$$\left.\begin{aligned} &\Sigma_1^n mp = 0 \cdot \bar{p} \\ \text{and} \quad &\Sigma_1^n [m(p - e)] = 0 \cdot (\bar{p} - e). \end{aligned}\right\} \quad \ldots \quad (14)$$

Let

$$\Sigma_2^n mp = (\bar{p})' \Sigma_2^n m = m'(\bar{p})', \text{ say:}$$

then, since

$$m_1 + m' = \Sigma_1^n m = 0,$$

$$\left.\begin{aligned} \Sigma_1^n mp &= m_1 p_1 + m'(\bar{p})' = m'((\bar{p})' - p_1) = 0 \cdot \bar{p}, \\ \text{and}\quad \Sigma_1^n [m(p-e)] &= m_1(p_1 - e) + m'((\bar{p})' - e) \\ &= m'((\bar{p})' - p_1) = 0 \cdot (\bar{p} - e). \end{aligned}\right\} \quad . \quad . \ (15)$$

Hence the direction of the point at ∞ is found by constructing the mean point of all the points except one, when the vector from the excepted point to this mean point has the required direction. As $\bar{p}$ is a unique point for any given system of weighted points, and p_1 may be taken as any point of the system, it follows that, in any system of points whose total weight is zero, a line drawn from any point of the system to the mean point of all the rest is parallel to any other such line.

Finally, suppose $\quad \Sigma mp = \bar{p} \cdot \Sigma m = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (16)$

which gives as a necessary consequence $\Sigma m = 0$. Then, by (15), if m' is not zero, we must have

$$(\bar{p})' = p_1;$$

that is, p_1 is the mean point of the system consisting of the remaining points. But, as p_1 may be *any* point of the system, it appears that any one of the points is, in this case, the mean of all the rest. For example, let $n = 3$; then (16) becomes $m_1p_1 + m_2p_2 + m_3p_3 = 0$, which requires also $m_1 + m_2 + m_3 = 0$; whence

$$m_1p_1 + m_2p_2 = -m_3p_3 = (m_1 + m_2)p_3,$$

or p_3 is the mean point of m_1p_1 and m_2p_2. Now we have seen in Art. 7 that the mean of two points is collinear with them; hence

$$m_1p_1 + m_2p_2 + m_3p_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (17)$$

is the condition that the three points shall be collinear.

If three points p_1, p_2, p_3 are *not* collinear, and yet are connected by a linear relation such as (17), this equation can only be satisfied when *each weight is separately zero*, that is

$$m_1 = m_2 = m_3 = 0; \quad \ldots \ldots \quad (17a)$$

for, as we have just seen, for all values of the weights different from zero the three points are collinear, but when each weight is zero, (17) is satisfied independently of the positions of the points.

Similarly, $m_1p_1 + m_2p_2 + m_3p_3 + m_4p_4 = 0$, . . . (18)

which requires also $m_1 + m_2 + m_3 + m_4 = 0$, is the condition that four points shall be *coplanar*, and if the points are *not* in one plane, then we must have

$$m_1 = m_2 = m_3 = m_4 = 0. \quad \ldots \ldots \quad (19)$$

A similar condition between *five* points would imply that the points were all in one tri-dimensional space, which leads to the consideration of space of higher dimensions than three.

9. Let ϵ_1 and ϵ_2 be two vectors, and n_1 and n_2 scalars; then

$$n_1\epsilon_1 + n_2\epsilon_2 = 0 \quad \ldots \ldots \quad (20)$$

is the condition that the vectors shall be *parallel*, as appears at once by Art. 2, if we write the equation $n_1\epsilon_1 = -n_2\epsilon_2$.

Similarly, $n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 = 0$ . . . . (21)

requires the three vectors to be *parallel to one plane;* for, by Art. 6, the vectors $n_1\epsilon_1$, $n_2\epsilon_2$, $n_3\epsilon_3$, may be represented by the three sides of a triangle. The same appears from (17) if we take p_1, p_2, p_3 as points at ∞.

If we have the additional condition

$$n_1 + n_2 + n_3 = 0, \quad \ldots \ldots \quad (22)$$

the extremities of the vectors, if drawn outwards from a point will be in one right line. For let e_0, e_1, e_2, e_3 be four points so taken that $e_1 - e_0 = \epsilon_1,\ e_2 - e_0 = \epsilon_2,\ e_3 - e_0 = \epsilon_3$; then (21) becomes

$$n_1(e_1 - e_0) + n_2(e_2 - e_0) + n_3(e_3 - e_0) = 0,$$

or, by (22), $$n_1 e_1 + n_2 e_2 + n_3 e_3 = 0;$$

which, by (17), requires e_1, e_2, e_3 to be collinear.

Similarly, it may be shown that the eqs.

$$\left.\begin{aligned} n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 + n_4\epsilon_4 = 0 \\ n_1 + n_2 + n_3 + n_4 = 0 \end{aligned}\right\} \quad \ldots \ldots \quad (23)$$

are the conditions that the extremities of the vectors ϵ_1, ϵ_2, etc., drawn outwards from one point shall be coplanar.

10. If the equation $\Sigma mp = 0$ can be satisfied by finite values of the m's different from zero, then, as we have seen, any one of the points can be expressed in terms of the others, and is, therefore, dependent on them; thus the points are mutually *dependent.* If, however, the equation can only be satisfied when all the m's are *zero,* no relation of this kind exists between the points; *i.e.* they are *independent.* This is evidently as true of vectors as of points.

Thus, by Art. 4, two different points are always independent.

Three non-collinear points are *independent,* while three collinear points are mutually *dependent.*

Four non-coplanar points are *independent,* while four coplanar points are mutually *dependent.*

Any five, or more, points are always mutually dependent.

Similarly, two non-parallel vectors are *independent,* but two parallel vectors are *dependent.*

Three vectors not parallel to one plane are *independent,* but when all are parallel to one plane they are mutually *dependent.*

Four, or more, vectors are always mutually dependent.

In a system of independent points or vectors no one can be expressed linearly in terms of the others.

11. The *tensor* of a directed quantity is its numerical magnitude, taken always as a positive quantity. It will be denoted, as in Quaternions, by T; as, $T\epsilon =$ tensor of ϵ.

That portion of a directed quantity whose magnitude is unity will be called its *unit*, and will be denoted by U; as, $U\epsilon =$ unit of ϵ.

Hence we have

$$T\epsilon \cdot U\epsilon = \epsilon. \qquad (24)$$

12. *Reference systems.* Let p be any point, and e_0, e_1, e_2 three fixed reference points; then writing

$$p = xe_0 + ye_1 + ze_2, \qquad (25)$$

we must have, if p is to be a unit point,

$$x + y + z = 1, \qquad (26)$$

and p is the mean point of xe_0, ye_1, and ze_2. Eliminating x between (25) and (26), we have

$$p = e_0 + y(e_1 - e_0) + z(e_2 - e_0), \qquad (27)$$

from which it appears that, by varying y and z, p may be made to occupy any position whatever in the plane of e_0, e_1, e_2. Hence any three unit points e_0, e_1, e_2 may be taken as a reference system for plane space, in terms of which all points of this plane space may be expressed.

Writing $p - e_0 = \rho$, $e_1 - e_0 = \epsilon_1$, $e_2 - e_0 = \epsilon_2$, eq. (27) becomes

$$\rho = y\epsilon_1 + z\epsilon_2, \qquad (28)$$

and any vector in plane space may be expressed in terms of two reference vectors ϵ_1, ϵ_2 in that plane. When ϵ_1 and ϵ_2 are of unit length and at right angles, the system will be called a unit normal reference system, and ι_1 and ι_2 will be substituted in this case for ϵ_1 and ϵ_2.

Similarly, in solid space any point p may be expressed in terms of four non-coplanar points e_0, e_1, e_2, e_3, by the equation

$$p = we_0 + xe_1 + ye_2 + ze_3, \qquad (29)$$

which requires, if p is a unit point,

$$w + x + y + z = 1, \qquad (30)$$

whence $$p = e_0 + x(e_1 - e_0) + y(e_2 - e_0) + z(e_3 - e_0). \qquad (31)$$

Thus any four unit points may be taken as a reference system for solid space.

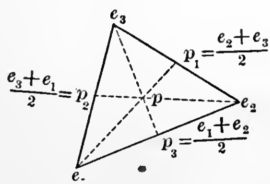
Putting, as before, $\rho = p - e_0$, $\epsilon_1 = e_1 - e_0$, etc., (31) becomes

$$\rho = x\epsilon_1 + y\epsilon_2 + z\epsilon_3, \qquad (32)$$

so that any vector may be expressed in terms of three given vectors $\epsilon_1, \epsilon_2, \epsilon_3$. When $T\epsilon_1 = T\epsilon_2 = T\epsilon_3 = 1$, and the three vectors are at right angles to each other, we have a unit normal system, and, in this case, substitute $\iota_1, \iota_2, \iota_3$ for $\epsilon_1, \epsilon_2, \epsilon_3$.

13. A number of exercises will now be given illustrative of the application of the preceding principles.

(1) The mean point of unit points at the vertices of a triangle coincides with the center of gravity of its area. The center of gravity of the area will be at the common point of lines through the vertices and the middle points of the opposite sides. Thus, if p be the point,



$$p = xe_1 + yp_1 = x'e_2 + y'p_2 = xe_1 + \frac{y}{2}(e_2 + e_3) = x'e_2 + \frac{y'}{2}(e_3 + e_1).$$

$$\therefore\ (x - \tfrac{1}{2}y')e_1 + (\tfrac{1}{2}y - x')e_2 + \tfrac{1}{2}(y - y')e_3 = 0;$$

by eqs. (17) and (17 a) this gives

$$x - \tfrac{1}{2}y' = \tfrac{1}{2}y - x' = y - y' = 0.$$

But, since p is a unit point, we have

$$x + y = 1 = x' + y'.$$

From these equations we find

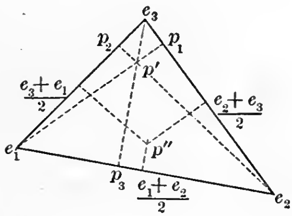
$$x = x' = \tfrac{1}{3}, \quad y = y' = \tfrac{2}{3};$$

whence $$p = \tfrac{1}{3}e_1 + \tfrac{2}{3}p_1 = \tfrac{1}{3}(e_1 + e_2 + e_3),$$

which shows that p is the mean point of e_1, e_2, e_3, and trisects the distance from e_1 to p_1.

(2) To find the common point of the perpendiculars from the vertices of a triangle on the opposite sides.

Let l, m, n be the ratios of the sides of the triangle to the cosines of the opposite angles; *i.e.*



$$l = \frac{T(e_3 - e_2)}{\cos(\angle \text{ at } e_1)}, \text{ etc.};$$

then $$p_1 = \frac{me_2 + ne_3}{m+n},$$

$$p_2 = \frac{ne_3 + le_1}{n+l}, \quad p_3 = \frac{le_1 + me_2}{l+m}.$$

Proceeding now precisely as in the first problem, with the above values of p_1, p_2, p_3 instead of those there used, we find

$$p' = \frac{le_1 + me_2 + ne_3}{l+m+n}.$$

(3) Find the common point of perpendiculars to the sides of a triangle through the middle points of the respective sides.

By the figure for the last problem we have

$$p'' = \tfrac{1}{2}(e_1 + e_2) + x(e_3 - p') = \tfrac{1}{2}(e_2 + e_3) + y(e_1 - p')$$

$$= \tfrac{1}{2}(e_1 + e_2) + x \cdot \frac{l(e_3 - e_1) + m(e_3 - e_2)}{l+m+n}$$

$$= \tfrac{1}{2}(e_2 + e_3) + y \cdot \frac{m(e_1 - e_2) + n(e_1 - e_3)}{l+m+n}.$$

Hence, equating to zero coefficients of e_1 and e_2,

$$\frac{1}{2} - \frac{xl}{l+m+n} - \frac{y(m+n)}{l+m+n} = 0 = \frac{1}{2} - \frac{xm}{l+m+n}$$

$$-\frac{1}{2} + \frac{ym}{l+m+n}.$$

$$\therefore\ x = y = \frac{1}{2}, \text{ and } p'' = \frac{(m+n)e_1 + (n+l)e_2 + (l+m)e_3}{2(l+m+n)}.$$

(4) Show that p, p', p'' of examples 1, 2, 3 are collinear.

It is easily seen that $3p - p' - 2p'' = 0$, which, by (17), proves the collinearity of the points. It is evident also that p, p', and p'' are collinear *whatever* values be assigned to l, m, n; for nothing in the demonstration depends on l, m, n having the values assigned in ex. 2.

(5) The center of gravity of the volume of a tetraedron coincides with

(*a*) The mean of unit points at the vertices;

(*b*) The center of gravity of the tetraedron whose vertices are the centers of gravity of its faces;

(*c*) The mean of the middle points of its edges.

(6) Find a point such that the sum of the vectors drawn from it to n given points shall be zero.

Ans. $p = \frac{1}{n}\Sigma_1^n e$ = mean of points.

(7) Construct by eq. (11) the mean point of points at the four corners of a square, whose weights are respectively 1, 2, 3, and 4.

(8) Let e_1, e_2, e_3, e_4, be the corners of a parallelogram, and $e_5 = \frac{1}{2}(e_1 + e_2)$: show that e_1e_3 and e_4e_5 will trisect each other.

(9) Let e_1, e_2, e_3, e_4 be the corners of a parallelogram, and let p_1, p_2 be points on a line ∥ to e_1e_2: if q_1 is the common point of p_1e_1 and p_2e_2, and q_2 the common point of p_1e_4 and p_2e_3, show that q_1q_2 is ∥ to e_1e_4.

(10) Let e_1, e_2, e_3, e_4 be four non-coplanar points, and let p_1, p_2, p_3, p_4 be points taken on e_1e_2, e_2e_3, e_3e_4, and e_4e_1 respectively: find the condition to be fulfilled in order that p_1p_2, e_1e_3, and p_3p_4 may have a common point.

Write $p_1 = m_1e_1 + n_1e_2$, $p_2 = m_2e_2 + n_2e_3$, etc.; then the p's must be coplanar, which leads to the condition

$$m_1m_2m_3m_4 = n_1n_2n_3n_4.$$

(11) If through any point within the triangle $e_1e_2e_3$ lines be drawn ∥ to the sides and terminated by them, and if l, m, n, be the respective ratios of these lines to the sides to which they are ∥; then $l+m+n=2$.

(12) The center of gravity of the sides of a triangle coincides with the center of the circle inscribed in the triangle formed by joining the middle points of the sides.

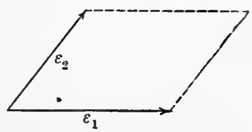
(13) Find the center of gravity of the faces of a tetraedron; also of the edges.

CHAPTER II.

MULTIPLICATION.

14. Grassmann's first conception of a *geometrical product* is that it is what is *produced* or generated by the first factor, as it moves over a distance determined by the second.

Thus $\epsilon_1\epsilon_2$, if ϵ_1 and ϵ_2 are two vectors, signifies the *directed plane area bounded by the parallelogram whose sides are* ‖ *and equal in length to* ϵ_1 *and* ϵ_2; that is, the plane area generated by ϵ_1 as it moves, ‖ to itself, along ϵ_2 from its initial to its final point.



This is a *plane-vector* as defined in Art. 1. The product will evidently have the same value whether the *initial point* of ϵ_1 moves in the direction ϵ_2 or in any other path, provided that ϵ_1 *itself* moves in the plane of ϵ_1 and ϵ_2 from the initial to the terminal point of ϵ_2.

Evidently $\epsilon_2\epsilon_1$, interpreted in the same way, gives a generation of the same parallelogram in the *reverse sense*, and should therefore be the negative of $\epsilon_1\epsilon_2$.

Similarly, if p_1 and p_2 are two unit points, p_1p_2 is that which is generated by p_1 in moving from its position to that of p_2 in a right line; thus p_1p_2 is a line of definite magnitude, direction, and position; *i.e.* a point-vector, according to Art. 1. Evidently $p_2p_1 = -p_1p_2$.

Such products as the above are called *combinatory*, because the factors *combine* to form a new geometric quantity different from either of the component factors.

15. The term *posited* will be applied to such geometric quantities as have definite positions; as, for instance, a right line or plane passing through a definite point not at ∞.

Any two geometric quantities which differ only in *magnitude* are said to be *congruent.*

16. Another conception of combinatory products of posited quantities will now be given, which will be found to be consistent with that given in Art. 14.

(*a*) *The product of two posited quantities which have no common figure is some multiple of the connecting figure.*

(*b*) *The product of two posited quantities which* MUST *have a common figure in the space under consideration, is the common figure, multiplied by a scalar quantity.*

Examples. — Under (*a*), the product of two points is the connecting straight line as in Art. 14.

Under (*b*), the product of two point-plane-vectors, which must necessarily have a common line, is that common line (point-vector) multiplied by a scalar to be determined hereafter.

(*c*) *A continued product of several posited quantities is to be interpreted by taking together first the two factors on the right, then the result of this multiplication with the next factor towards the left, then this result with the next, etc.; but if at any stage of the process the product of the factors treated up to this point becomes a scalar quantity, then this scalar will not form a* COMBINATORY *product with the next factor to the left, but is to be treated like any other merely numerical factor, obeying as a* WHOLE *the laws of ordinary algebraic multiplication.*

This statement cannot well be illustrated here, but its meaning will appear in the sequel.

Grassmann calls a product of the kind (*a*) *progressive,* because it is of a geometric order higher than that of either factor; while one of the kind (*b*) is *regressive,* because it is of lower order than either factor. If in a continued product of factors, as in (*c*), some of the successive products are progressive and some regressive, then the product as a whole is said to be *mixed.*

17. *Laws of combinatory multiplication of any number of points or vectors not exceeding the number of* INDEPENDENT *points or vectors possible in the space under consideration.*

These are:

The associative law;

$$p_1p_2p_3 = p_1 \cdot p_2p_3 = p_1p_2 \cdot p_3. \qquad (33)$$

The distributive law;

$$p_1(p_2 + p_3) = p_1p_2 + p_1p_3. \qquad (34)$$

The alternative law;

$$p_1p_2 = -p_2p_1. \qquad (35)$$

If $p_2 = p_1$ in (35), we have $p_1p_1 = 0$, which agrees with the meaning of the product of two points as given in Art. 14.

Also $\qquad p_1p_2p_1 = p_1 \cdot p_2p_1 = -p_1 \cdot p_1p_2 = -p_1p_1 \cdot p_2 = 0,$

by (33) and (35); hence, *if there occur two identical factors in a product of the kind stated at the head of this article, the product is zero.*

The meaning of certain products will be different according as they are interpreted in two- or three-dimensional space; hence we shall apply the term *planimetric* to such a product when it is to be taken in two-dimensional space, and the term *stereometric* when it is to be taken in three-dimensional space.

We proceed now to a detailed discussion of all the geometric products possible in two- and three-dimensional space.

18. *Product of two points.* Let p_1 and p_2 be two unit points. Their product, p_1p_2, has already, in Art. 14, been stated to be the portion of the right line fixed by p_1 and p_2 extending *from* p_1 *to* p_2, and it will be called a *point-vector*, or simply a *line* for brevity.

Let $\epsilon = p_2 - p_1$; then

$$p_1(p_2 - p_1) = p_1\epsilon = p_1p_2 - p_1p_1 = p_1p_2, \qquad (36)$$

by the last article; so that the product of two points is equivalent to that of the first point into the vector from the first to the second. Hence multiplying a vector by a point changes it into a point-vector; *i.e.* fixes its position. Let ϵ' be any vector whatever; then, by eq. (2), $p_1 + x\epsilon'$ may be any point in space by suitably choosing x and ϵ': multiply into ϵ; therefore

$$(p_1 + x\epsilon')\epsilon = p_1\epsilon + x\epsilon'\epsilon. \qquad (37)$$

Hence the point-vector obtained by multiplying the vector ϵ by the point $p_1 + x\epsilon'$ is not in general equal to $p_1\epsilon$, but differs from it by the quantity $x\epsilon'\epsilon$. If, however, we have $\epsilon' = \epsilon$, then

$$(p_1 + x\epsilon)\epsilon = p_1\epsilon + x\epsilon\epsilon = p_1\epsilon; \qquad (38)$$

so that, when ϵ is multiplied by any point on the line through p_1 and p_2, the resulting point-vector is equal to $p_1\epsilon$.

Hence, in order that two point-vectors may be *equal,* they must have *the same length, the same direction,* and must be situated *upon the same straight line,* while their position on this line is indifferent.

19. *Product of two vectors.* Let ϵ_1 and ϵ_2 be any two vectors: their product $\epsilon_1\epsilon_2$ has already, in Art. 14, been stated to be a plane area ∥ to ϵ_1 and ϵ_2 and equal to the area of the parallelogram whose sides are parallel and equal in length to ϵ_1 and ϵ_2. Write

$$\epsilon = x_1\epsilon_1 + x_2\epsilon_2 \text{ and } \epsilon' = y_1\epsilon_1 + y_2\epsilon_2;$$

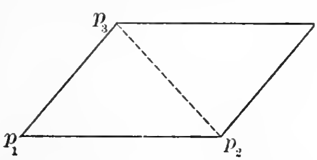
then ϵ and ϵ' may be *any* two vectors ∥ to the plane-vector $\epsilon_1\epsilon_2$, by giving suitable values to x_1, x_2, y_1, y_2. Multiplying, we have

$$\epsilon\epsilon' = (x_1\epsilon_1 + x_2\epsilon_2)(y_1\epsilon_1 + y_2\epsilon_2) = (x_1y_2 - x_2y_1)\epsilon_1\epsilon_2 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \epsilon_1\epsilon_2. \qquad (39)$$

Hence the product of *any* two vectors ∥ to $\epsilon_1\epsilon_2$ only differs from $\epsilon_1\epsilon_2$ by a scalar factor. If $x_1y_2 - x_2y_1 = 1$, then $\epsilon\epsilon' = \epsilon_1\epsilon_2$. Thus in order that two plane-vectors should be equal, it is only necessary that their plane directions and areas should be the same, without regard to the directions of the component vectors. Of course direction as applied to a plane-vector must include the *sense* in which the generation takes place. See Art. 14.

Regarding ϵ_1 and ϵ_2 as points at ∞, we see that, just as a zero point at ∞ is a vector, so the product of two such, that is a point-vector at ∞, is a plane-vector; or, in other words, just as a point at ∞ gives a line direction, so a line at ∞ gives a plane direction.

20. *Product of three points.* Let p_1, p_2, p_3 be three unit points. By Art. 16, (a) and (c), the product should be a multiple of the connecting triangle whose vertices are p_1, p_2, p_3. By Art. 17 the product obeys the associative law, so that



$$p_1p_2p_3 = p_1p_2 \cdot p_3.$$

Hence, by Art. 14, the product is what is generated by the point-vector p_1p_2 in moving, $\parallel$ to itself, in the plane of the three points, from its original position, till it passes through p_3; that is, a *parallelogram* whose area is *twice* that of the connecting triangle. Let $p_2 - p_1 = \epsilon$, and $p_3 - p_1 = \epsilon'$; then

$$p_1p_2p_3 = p_1p_2\epsilon' = p_1\epsilon\epsilon'. \qquad (39)$$

The product is thus a *posited and directed plane area of given magnitude;* that is, a *point-plane-vector*, or simply a *plane* for brevity, especially when the magnitude is a matter of indifference. Eq. (39) shows that multiplying a plane-vector by a point fixes its position by making it pass through the point, since by Art. 14, $\epsilon\epsilon'$ is a plane-vector. Let ϵ'' be a vector in any direction; then, as in Art. 18, $p_1 + x\epsilon''$ may be any point in space. Now

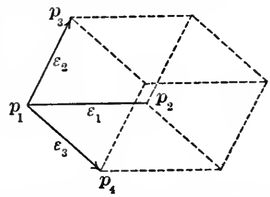
$$(p_1 + x\epsilon'')\epsilon\epsilon' = p_1\epsilon\epsilon' + x\epsilon''\epsilon\epsilon'; \qquad (40)$$

so that the point-plane-vector obtained by multiplying $\epsilon\epsilon'$ by the point $p_1 + x\epsilon''$ is, in general, different from $p_1\epsilon\epsilon'$. If, however, we have $\epsilon'' = y\epsilon + z\epsilon'$, so that ϵ'' is $\parallel$ to $\epsilon\epsilon'$, and $p_1 + x\epsilon''$ is a point of the plane $p_1p_2p_3$, then

$$(p_1 + x\epsilon'')\epsilon\epsilon' = (p_1 + x(y\epsilon + z\epsilon'))\epsilon\epsilon' = p_1\epsilon\epsilon'. \qquad (41)$$

Hence two point-plane-vectors are equal when they have the same *area* and *direction*, or sense, and lie in the *same plane*, without regard to position *in* that plane.

21. *Product of three vectors.* Let ϵ_1, ϵ_2, ϵ_3 be any three vectors not ∥ to the same plane; then $\epsilon_1\epsilon_2\epsilon_3 = \epsilon_1\epsilon_2 \cdot \epsilon_3$, and, by Art. 14, the product is the parallelopiped generated by the plane-vector $\epsilon_1\epsilon_2$ as it moves, ∥ to itself, from the initial to the terminal point of ϵ_3.



Let $\quad \epsilon = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 = \Sigma_1^3 x\epsilon,$

$\epsilon' = \Sigma_1^3 y\epsilon, \quad \epsilon'' = \Sigma_1^3 z\epsilon;$

then ϵ, ϵ', ϵ'' may be any three vectors whatever with suitable values of the scalar coefficients, and

$$\epsilon\epsilon'\epsilon'' = \Sigma x\epsilon\Sigma y\epsilon\Sigma z\epsilon = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \epsilon_1\epsilon_2\epsilon_3.* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

Hence two triple products of vectors can only differ in magnitude and sign, and two such products will be equal when their magnitudes and order of generation, or sense, are the same.

It follows therefore that the combinatory product of three vectors is always a *scalar quantity*, by the definition given in Art. 1. If the three vectors are parallel to one plane, the volume of the parallelopiped becomes *zero*, and the product therefore *vanishes*. Hence the *planimetric combinatory product of three vectors is always zero.* If we regard ϵ_1, ϵ_2, ϵ_3 as points at ∞, we see that a point-plane-vector at ∞ is equivalent to a solid, which carries out the analogy mentioned at the end of Art. 19.

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* For the benefit of any reader who may not be familiar with determinants, it may be stated that the coefficient of $\epsilon_1\epsilon_2\epsilon_3$ in (42) is an abbreviated way of writing

$$x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1),$$

which expression will be obtained by multiplying out the values of ϵ, ϵ', ϵ'', and remembering that terms containing repeated factors vanish.

22. *Product of four points.* Let p_1, p_2, p_3, p_4 be four unit points; then, by Arts. 17 and 14, we have

$$p_1p_2p_3p_4 = p_1p_2p_3 \cdot p_4,$$

and the product is the volume generated by the point-plane-vector $p_1p_2p_3$ when it is moved ∥ to itself from its initial position till it passes through p_4; that is, a *parallelopiped,* of which p_1p_2, p_1p_3, and p_1p_4 are three conterminous edges. This volume is six times the connecting tetrahedron of the four points, which accords with Art. 16. (See figure of last Art.) If

$$p_2 - p_1 = \epsilon,\ p_3 - p_1 = \epsilon', \text{ and } p_4 - p_1 = \epsilon'',$$

then $$p_1p_2p_3p_4 = p_1p_2p_3\epsilon'' = p_1p_2\epsilon'\epsilon'' = p_1\epsilon\epsilon'\epsilon''. \quad . \quad . \quad . \quad (43)$$

The point $p_1 + x\epsilon + y\epsilon' + z\epsilon''$ may be any point whatever, with suitable values of x, y, z, and we have

$$(p_1 + x\epsilon + y\epsilon' + z\epsilon'')\epsilon\epsilon'\epsilon'' = p_1\epsilon\epsilon'\epsilon'', \quad . \quad . \quad . \quad (44)$$

so that any point whatever may be substituted for p_1 in (43) without changing the value. By the above, and by the preceding article, it appears that a product of four points is equal to the product of any other four points having the same magnitude and intrinsic sign, or order of generation: thus such a product is a *scalar;* and, in fact, differs in no manner whatever from a product of three *vectors* which has the same magnitude and sign. We may therefore write

$$p_1\epsilon\epsilon'\epsilon'' = \epsilon\epsilon'\epsilon''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

If the four points are in one plane, the product is zero, because the volume of the connecting tetrahedron is zero. Thus the *planimetric* progressive product of four points is always zero.

23. The *stereometric,* progressive product of four or more vectors is always zero. But we may have $\epsilon_1 \cdot \epsilon_2\epsilon_3\epsilon_4$, meaning the vector ϵ_1 times the scalar $\epsilon_2\epsilon_3\epsilon_4$, etc.

The *stereometric,* progressive product of five or more points is always zero. But we may have such products as $p_1p_2 \cdot p_3p_4p_5p_6$, meaning the ordinary algebraic product of the point-vector p_1p_2 into the scalar $p_3p_4p_5p_6$.

24. From the preceding articles we have the following conditions:

$$p_1p_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

is the condition that two points shall *coincide.*

$$p_1p_2p_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

is the condition that three points shall be in *one right line.*

$$p_1p_2p_3p_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

is the condition that four points shall be in *one plane.*

$$\epsilon_1\epsilon_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

is the condition that two vectors shall be *parallel.*

$$\epsilon_1\epsilon_2\epsilon_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

is the condition that three vectors shall be parallel to one plane.

In the further development of the subject it will be convenient to treat separately two- and three-dimensional space, considering the former first.

Planimetric Products.

25. In two-dimensional, or plane, space two plane-vectors, or two point-plane-vectors, cannot differ from each other except in magnitude and sign, since both are restricted to one plane.

Hence they become *scalar* quantities. Furthermore, by Arts. 19 and 20, the product of two vectors is now identical in meaning with the product of three points. Thus, if $p_2 - p_1 = \epsilon$ and $p_3 - p_1 = \epsilon'$, we have, in *plane space,*

$$p_1\epsilon\epsilon' = \epsilon\epsilon'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Plane space is the locus of all points dependent on three fixed reference points. We shall call these e_0, e_1, e_2, *and shall always take the area* $e_0e_1e_2$ *as the unit of measure of area,* when dealing with a point system. That is, we write always in plane space,

$$e_0e_1e_2 = 1. \quad \ldots \ldots \quad (52)$$

This is a great practical convenience, and in no way affects the generality of results.

The *sides* of the reference triangle taken around in order are called the *complements* of the opposite *vertices;* thus

$$\left.\begin{aligned} e_1e_2 &= (\text{complement of } e_0) = |e_0 \\ e_2e_0 &= (\text{complement of } e_1) = |e_1 \\ e_0e_1 &= (\text{complement of } e_2) = |e_2 \end{aligned}\right\}, \quad \ldots \quad (53)$$

the vertical line before the point being called the *sign of the complement.*

In dealing with a vector system we shall usually refer to a *unit normal system* of vectors ι_1, ι_2, as stated in Art. 12. We shall then have in *plane space*

$$\iota_1\iota_2 = 1. \quad \ldots \ldots \quad (54)$$

26. Let

$$p_1 = l_0e_0 + l_1e_1 + l_2e_2 = \Sigma_0^2 le, \;\; p_2 = \Sigma_0^2 me, \;\; p_3 = \Sigma_0^2 ne;$$

then

$$\left.\begin{aligned} p_1p_2 &= \begin{vmatrix} l_0 & l_1 \\ m_0 & m_1 \end{vmatrix} e_0e_1 + \begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix} e_1e_2 + \begin{vmatrix} l_2 & l_0 \\ m_2 & m_0 \end{vmatrix} e_2e_0 \\ &\equiv \left\|\begin{matrix} l_0 & l_1 & l_2 \\ m_0 & m_1 & m_2 \end{matrix}\right\| [e_0, e_1, e_2] \equiv \begin{vmatrix} e_0 & |e_1 & |e_2 \\ l_0 & l_1 & l_2 \\ m_0 & m_1 & m_2 \end{vmatrix} \end{aligned}\right\} \quad . \quad (55)$$

The third and fourth members of (55) are simply different ways of expressing the second. The fourth member is especially noticeable for its symmetry. Eq. (55) shows that any point-vector is expressible in terms of the sides of the reference triangle.

Again, $$p_1p_2p_3 = \Sigma le \cdot \Sigma me \cdot \Sigma ne = \begin{vmatrix} l_0 & l_1 & l_2 \\ m_0 & m_1 & m_2 \\ n_0 & n_1 & n_2 \end{vmatrix}; \quad \ldots \quad (56)$$

as will be found on multiplying out and putting $e_0e_1e_2 = 1$, by (52).

27. EXERCISES. — 1. Find by eq. (47) the condition that the three points $p+\epsilon_1$, $p+\epsilon_2$, $p+\epsilon_3$, shall be collinear, and illustrate geometrically. *Ans.* $\epsilon_1\epsilon_2+\epsilon_2\epsilon_3+\epsilon_3\epsilon_1=0$.

2. Show that if a line be expressed in terms of the sides of the reference triangle, and the sum of the coefficients be zero, the line passes through the mean point of the reference points.

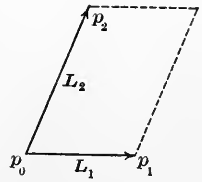
28. Since the product of three points obeys the associative law, it can be regarded as the product of a point into a point-vector, or of a point-vector into a point. Thus, if $L=p_2p_3$,

$$p_1p_2p_3=p_1\cdot p_2p_3=p_1L=p_2p_3p_1=Lp_1, \quad . \quad . \quad . \quad (57)$$

so that *this* product is commutative.

29. *Product of two lines, or point-vectors.* The products hitherto considered have all been *progressive;* we now come to one which is *regressive.* Since two lines in plane space *must* intersect, they come under Art. 16, (*b*). Let the lines be L_1 and L_2, let p_0 be their common point, and let p_1 and p_2 be so taken that

$$L_1=p_0p_1, \text{ and } L_2=p_0p_2.$$



We may also write,

$$L_1=p_0(p_1-p_0),\ L_2=p_0(p_2-p_0);$$

now the product of the *vectors* $(p_1-p_0)(p_2-p_0)$ is the area of the parallelogram on these vectors, and is scalar; the product of the *point*-vectors should certainly give *this* result, and, in addition, the *point* fixed by them, viz. their intersection. This is in accordance with Art. 16, (*b*). The product

$$(p_1-p_0)(p_2-p_0)$$

is equivalent to $p_0p_1p_2$; hence we may write

$$L_1L_2=p_0p_1\cdot p_0p_2=p_0p_1p_2\cdot p_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

It is to be carefully noted that the third member of eq. (58) *is not derived from the second by interchanges, according to the associative and alternative laws, but is an independent expression,* *which gives the meaning of the product of two lines.* It may be regarded as a *model form* for the treatment of regressive products. Thus, if AB and AC are *any* two quantities whose product is regressive, and if A is their common figure, we shall always have

$$AB \cdot AC = ABC \cdot A. \qquad (59)$$

We have accordingly

$$L_2L_1 = p_0p_2 \cdot p_0p_1 = p_0p_2p_1 \cdot p_0 = -p_0p_1p_2 \cdot p_0 = -L_1L_2, \qquad (60)$$

or the product of two lines is non-commutative.

30. *Product of a point and two lines.* Let L_1 and L_2 be as in the last article, and p be some point; then

$$pL_1L_2 = p \cdot p_0p_1 \cdot p_0p_2 = p \cdot p_0p_1p_2 \cdot p_0 = p_0p_1p_2 \cdot pp_0. \qquad (61)$$

$p_0p_1p_2$ can be placed first because it is a scalar. This is a *mixed* product, that of the two lines being regressive, and that of p into their common point, p_0, being progressive. Also,

$$pL_1L_2 = -pL_2L_1 = L_2L_1 \cdot p. \qquad (62)$$

The period is necessary in the last member of this equation to preserve the meaning; that is, the product of the points L_1L_2 and p. Without the period the expression would mean the line L_2 multiplied into the scalar L_1p.

31. *Product of three lines.* Let L_1, L_2, L_3 be three lines, and p_1, p_2, p_3 their common points, and take scalar factors n_1, n_2, n_3 so that $L_1 = n_1p_2p_3$, $L_2 = n_2p_3p_1$, $L_3 = n_3p_1p_2$; then

$$\begin{aligned} L_1L_2L_3 &= n_1n_2n_3p_2p_3 \cdot p_3p_1 \cdot p_1p_2 = -n_1n_2n_3p_2p_3 \cdot p_1p_3 \cdot p_1p_2 \\ &= -n_1n_2n_3p_2p_3 \cdot p_1p_3p_2 \cdot p_1 = n_1n_2n_3(p_1p_2p_3)^2. \qquad (63) \end{aligned}$$

It thus appears that in plane space lines obey the same laws of multiplication as points.

32. Let two points be given each as the common point of two lines, viz. $p_1 = L_0L_1$ and $p_2 = L_0L_2$, then

$$p_1p_2 = L_0L_1 \cdot L_0L_2 = L_0L_1L_2 \cdot L_0, \qquad (64)$$

which is a reciprocal equation to (58). Similarly, if

$$p_1 = m_1 L_2 L_3,\ p_2 = m_2 L_3 L_1,\ p_3 = m_3 L_1 L_2,$$

we have

$$p_1 p_2 p_3 = m_1 m_2 m_3 L_2 L_3 \cdot L_3 L_1 \cdot L_1 L_2 = m_1 m_2 m_3 (L_1 L_2 L_3)^2. \qquad (65)$$

33. The condition

$$L_1 L_2 = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (66)$$

requires that L_1 and L_2 shall be *congruent*, that is, be situated on the *same right line;* for, by (58), this gives $p_0 p_1 p_2 = 0$.

The condition

$$L_1 L_2 L_3 = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (67)$$

requires that the three lines shall *pass through a common point;* for $L_2 L_3$ is some point, say p, and $L_1 p = 0$ makes L_1 pass through p, by eq. (47).

34. *Product of parallel lines.* Let the lines be $L_1 = p_1 \epsilon$ and $L_2 = n p_2 \epsilon$; then

$$L_1 L_2 = n p_1 \epsilon \cdot p_2 \epsilon = n \epsilon p_1 \cdot \epsilon p_2 = n \epsilon p_1 p_2 \cdot \epsilon = n p_1 p_2 \epsilon \cdot \epsilon. \ . \ . \quad (68)$$

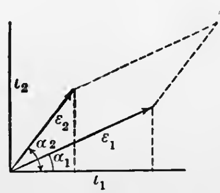
Thus the product is the common vector, or point at ∞, multiplied by the scalar $n p_1 p_2 \epsilon$.

35. *Product of two vectors.* Let

$$\epsilon_1 = m_1 \iota_1 + m_2 \iota_2,\ \ \epsilon_2 = n_1 \iota_1 + n_2 \iota_2;$$

then, since $\iota_1 \iota_2 = 1$,

$$\epsilon_1 \epsilon_2 = \begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}. \ . \ . \ . \ . \ . \ . \quad (69)$$



But, since $\epsilon_1 \epsilon_2$ is a parallelogram whose sides are ∥ to ϵ_1 and ϵ_2, we have

$$\epsilon_1 \epsilon_2 = T\epsilon_1 T\epsilon_2 \sin <_{\epsilon_1}^{\epsilon_2}.$$

Also, $m_1 = T\epsilon_1 \cos <_{\iota_1}^{\epsilon_1} = T\epsilon_1 \cos a_1$, say; $m_2 = T\epsilon_1 \sin a_1$;

$$n_1 = T\epsilon_2 \cos a_2; \quad n_2 = T\epsilon_2 \sin a_2.$$

Therefore

$$\epsilon_1\epsilon_2 = T\epsilon_1 T\epsilon_2 \begin{vmatrix}\cos a_1 & \sin a_1\\ \cos a_2 & \sin a_2\end{vmatrix} = T\epsilon_1 T\epsilon_2 \sin <^{\epsilon_2}_{\epsilon_1}; \quad . \quad . \quad (70)$$

which affords a proof of the trigonometrical formula for the sine of the difference of two angles.

36. *Sum of point-vectors*. Using L_1, L_2, p_0, p_1, p_2 as in Art. 29, we have

$$L_1 + L_2 = p_0p_1 + p_0p_2 = p_0(p_1 + p_2) = 2p_0\bar{p}, \quad . \quad . \quad (71)$$

if $\bar{p}$ is the mean point of p_1 and p_2. Thus, $L_1 + L_2$ passes through the common point of the two lines and is equal in length to that diagonal of the parallelogram on L_1 and L_2 which passes through p_0. Similarly,

$$L_1 - L_2 = p_0(p_1 - p_2); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

so that the difference of L_1 and L_2 passes through p_0, and is equal in length to the other diagonal of the parallelogram.

Suppose L_1 and L_2 to be parallel, and equal respectively to $np_1\epsilon$ and $p_2\epsilon$; then

$$L_1 + L_2 = (np_1 + p_2)\epsilon = (n+1)\bar{p}\epsilon; \quad . \quad . \quad . \quad . \quad . \quad (73)$$

that is, the sum is a ∥ point-vector passing through the mean point of np_1 and p_2. Finally, let $n = -1$, so that

$$L_1 + L_2 = (p_2 - p_1)\epsilon, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

and the sum is, in this case, the product of two vectors; that is, in plane space, a scalar.

The reader will notice the exact correspondence between the results of this article and the resultant of forces in plane space.

37. *Sum of sides of a polygon*. Let 1, 2, 3 be any three points; then

$$\begin{aligned}12 + 23 + 31 = 23 - 21 + 11 - 13 &= 2(3-1) - 1(3-1)\\ &= (2-1)(3-1).\end{aligned}$$

Hence the sum of the three point-vector sides of a triangle taken around in order is equal to twice the area of the triangle. Similarly, let 1, 2, 3, 4, ⋯ n, be n points, the vertices of a polygon of n sides; then

$$12 + 23 + 31 = (2-1)(3-1)$$
$$13 + 34 + 41 = (3-1)(4-1)$$
$$\cdots\cdots\cdots\cdots$$
$$1(n-1) + (n-1)\mathrm{n} + \mathrm{n}1 = [(n-1)-1](\mathrm{n}-1)$$

and, adding,

$$12 + 23 + 34 + \cdots + \mathrm{n}1 = \text{twice the area of the polygon.}$$

38. COMPLEMENT. — (*a*) *The complement of a reference unit is the product of the other reference units, so taken that the product of the unit into its complement shall be positive unity.*

This definition is perfectly general, and applies to either a point or vector system in space of any number of dimensions. We have already had examples in eq. (53,) as $|e_0 = e_1e_2$, whence $e_0|e_0 = e_0e_1e_2 = 1$.

(*b*) *The complement of a scalar quantity is the quantity itself.*

Thus, $|n = n$; $|\iota_1\iota_2 = \iota_1\iota_2 = 1$.

(*c*) *The complement of the product of several factors is equal to the product of the complements of the factors.*

Thus, $|(np) = |n|p = n|p$; $|\iota_1\iota_2 = |\iota_1|\iota_2$.

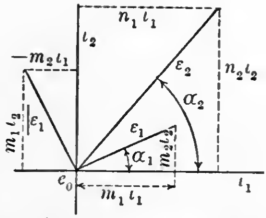
(*d*) *The complement of the sum of several quantities is the sum of the complements of the quantities.*

39. *Complement in a plane vector system.* Taking a unit normal system ι_1, ι_2, we have, according to the previous article,

Complement of ι_1, written $|\iota_1 = \iota_2$, for $\iota_1|\iota_1 = \iota_1\iota_2 = 1$;

Complement of ι_2, written $|\iota_2 = -\iota_1$, for $\iota_2|\iota_2 = -\iota_2\iota_1 = \iota_1\iota_2 = 1$;

$||\iota_1 = |\iota_2 = -\iota_1$; $||\iota_2 = -|\iota_1 = -\iota_2$.



Let $\qquad \epsilon_1 = m_1\iota_1 + m_2\iota_2$

and $\qquad \epsilon_2 = n_1\iota_1 + n_2\iota_2$; then

$$|\epsilon_1 = m_1|\iota_1 + m_2|\iota_2 = m_1\iota_2 - m_2\iota_1. \quad (75)$$

By the figure it is evident that $|\epsilon_1$ is a vector of the *same length* as ϵ_1, and *perpendicular* to it, or, in other words, *taking the complement of a vector in plane space rotates it positively through* 90°.

The product $\epsilon_1|\epsilon_2$ is the parallelogram whose sides are ϵ_1 and $|\epsilon_2$; if ϵ_1 is parallel to $|\epsilon_2$, the area of the parallelogram vanishes, or $\epsilon_1|\epsilon_2 = 0$; but, since $|\epsilon_2$ is $\perp$ to ϵ_2, ϵ_1 must, in this case, be $\perp$ to ϵ_2; hence the equation

$$\epsilon_1|\epsilon_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

is the condition that the two vectors ϵ_1 *and* ϵ_2 *shall be perpendicular.*

The product $\epsilon_1|\epsilon_1$ is the area of a *square* each side of which is of the length $T\epsilon_1$; hence

$$\epsilon_1|\epsilon_1 = T^2\epsilon_1 = \epsilon_1^2, \text{ say.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

The form ϵ_1^2 is merely another way of writing $\epsilon_1|\epsilon_1$, which is often convenient. If n is a scalar, we have

$$n|n = nn = n^2,$$

whence the analogy is apparent.

Grassmann calls $\epsilon_1|\epsilon_2$ the "inner product" of ϵ_1 and ϵ_2, regarding the complement sign as a species of *multiplication sign*, and accordingly calls ϵ^2 the "inner square" of ϵ. It seems preferable to the author not to introduce a new species of multiplication, but to regard $\epsilon_1|\epsilon_2$ as simply the combinatory product of ϵ_1 into $|\epsilon_2$, a way of looking at it which is practically far more simple, and renders interpretation easier. Somewhat after the analogy of the word *cosine* for *sine of the complement*, we may call ϵ^2 the *co-square* of ϵ, and, just as we read a^2, a square, we may read ϵ^2, ϵ co-square. Similarly, $\epsilon_1|\epsilon_2$ may be called the co-product of ϵ_1 and ϵ_2.

We have also,

$$\epsilon_1|\epsilon_2 = (m_1\iota_1 + m_2\iota_2)|(n_1\iota_1 + n_2\iota_2) = m_1n_1 + m_2n_2 = \epsilon_2|\epsilon_1. \quad . \qquad (78)$$

We obtain the third member because

$$\iota_1|\iota_1 = \iota_1\iota_2 = 1, \quad \iota_1|\iota_2 = -\iota_1\iota_1 = 0, \quad \iota_2|\iota_1 = \iota_2\iota_2 = 0,$$

and $\iota_2|\iota_2 = -\iota_2\iota_1 = \iota_1\iota_2 = 1$;

the fourth member is apparent from the symmetry of the third. It will be found that we have always, when A and B are *quantities of the same order* in the reference units,

$$A|B = B|A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (79)$$

Since $T\epsilon = T|\epsilon$, as shown above, we have, as in Art. 35,

$$\epsilon_1|\epsilon_2 = T\epsilon_1 T\epsilon_2 \times \sin(\text{ang. bet. } \epsilon_1 \text{ and } |\epsilon_2) = T\epsilon_1 T\epsilon_2 \cos \angle_{\epsilon_1}^{\epsilon_2}. \qquad (80)$$

Also, taking the values of m_1, m_2, n_1, n_2 as in Art. 35, we have

$$\epsilon_1|\epsilon_2 = m_1n_1 + m_2n_2 = T\epsilon_1 T\epsilon_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$$
$$= T\epsilon_1 T\epsilon_2 \cos \angle_{\epsilon_1}^{\epsilon_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (81)$$

which affords a proof of the trigonometrical formula for the cosine of the difference of two angles.

If $\epsilon_2 = \epsilon_1$, we have

$$\left.\begin{aligned} \epsilon_1^2 &= m_1^2 + m_2^2 = T^2\epsilon_1 \\ \therefore\ T\epsilon_1 &= +\sqrt{m_1^2 + m_2^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

Square and add eqs. (70) and (80); therefore

$$T^2\epsilon_1 T^2\epsilon_2 = \epsilon_1^2\epsilon_2^2 = (\epsilon_1\epsilon_2)^2 + (\epsilon_1|\epsilon_2)^2. \quad . \quad . \quad . \qquad (83)$$

If ι be any unit vector, and ρ any other vector, we have

$$\iota \cdot \rho|\iota = \iota \cdot T\rho \cdot \cos \angle_{\iota}^{\rho} = \text{projection of } \rho \text{ on direction of } \iota,$$

and $\iota\rho \cdot |\iota = |\iota \cdot T\rho \cdot \sin \angle_{\iota}^{\rho} =$ projection of ρ on direction $\perp$ to ι;

hence we may write

$$\rho = \iota \cdot \rho|\iota + \iota\rho \cdot |\iota. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (84)$$

This equation may be verified by multiplying successively by ι and $|\iota$.

40. *Complement in a plane point system.* Taking as reference points e_0, e_1, e_2, with the condition $e_0e_1e_2 = 1$, we have, in accordance with Art. 38,

$$\left.\begin{aligned} &|e_0 = e_1e_2, \qquad |e_1e_2 = ||e_0 = e_0 \\ &|e_1 = e_2e_0, \qquad |e_2e_0 = ||e_1 = e_1 \\ &|e_2 = e_0e_1, \qquad |e_0e_1 = ||e_2 = e_2 \\ &|(e_0e_1e_2) = |e_0 \cdot |e_1 \cdot |e_2, \text{ by Art. 38, (c),} \\ &\qquad\qquad = e_1e_2 \cdot e_2e_0 \cdot e_0e_1 = (e_0e_1e_2)^2 = 1 = e_0e_1e_2 \end{aligned}\right\} \quad . \quad . \quad (85)$$

Note that the complement of the complement of a reference point is here the *point itself*, while in the vector system the complement of the complement was *negative*. The general law is that when the number of reference units is *even*, the complement of the complement of one of them is *negative*, and when the number is *odd*, it is *positive*.

Let $p_1 = l_0e_0 + l_1e_1 + l_2e_2$, and $p_2 = m_0e_0 + m_1e_1 + m_2e_2$ be any two points; then

$$\begin{aligned} |p_1 &= l_0|e_0 + l_1|e_1 + l_2|e_2 = l_0e_1e_2 + l_1e_2e_0 + l_2e_0e_1 \\ &= \frac{1}{l_0}(l_0e_1 - l_1e_0)(l_0e_2 - l_2e_0)\,; \end{aligned} \quad . \quad (86)$$

so that the *complement of any point is a point-vector.* The fourth member of (86) expresses this point-vector as the product of the points in which it cuts two of the sides of the reference triangle, so that it may be easily constructed. We have also

$$\begin{aligned} p_1|p_2 &= (l_0e_0 + l_1e_1 + l_2e_2)(m_0e_1e_2 + m_1e_2e_0 + m_2e_0e_1) \\ &= l_0m_0 + l_1m_1 + l_2m_2 = p_2|p_1, \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

so that eq. (79) is here verified. Note that the product is *scalar*. We have, however,

$$\left.\begin{aligned} &e_0e_1|e_1 = e_0e_1 \cdot e_2e_0 = -\,e_0e_1e_2 \cdot e_0 = -\,e_0 \\ \text{and} \quad &e_1|e_0e_1 = e_1e_2 = |e_0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (88)$$

so that, when the quantities on each side of the complement sign are of *different order*, the product is *not scalar, nor commutative about the sign.*

If $L_1 = |p_1$ and $L_2 = |p_2$, we have $p_1 = |L_1$ and $p_2 = |L_2$, and

$$L_1|L_2 = |p_1 \cdot ||p_2 = |p_1 \cdot p_2 = p_2|p_1 = p_1|p_2 = L_2|L_1, \quad (89)$$

which also agrees with eq. (79). Again,

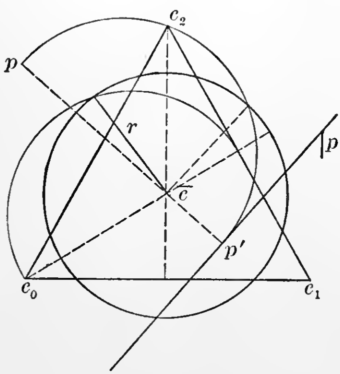
$$L_1 L_2 = |p_1 \cdot |p_2 = |p_1 p_2; \quad \dots \dots \quad (90)$$

so that the common point of the lines L_1 and L_2 is the complement of the line $p_1 p_2$. Also,

$$L_1|p_2 = ||(L_1|p_2) = |(|L_1 \cdot p_2) = |(-p_2|L_1) = -|(p_2|L_1), \quad . \quad (91)$$

of which eqs. (88) are a special case.

41. *Geometric interpretation.* For the sake of simplicity, suppose e_0, e_1, e_2 to be the corners of an *equilateral* triangle. With $\bar{e}$, the mean point of the triangle, as a center, draw a circle of radius $\frac{a}{\sqrt{6}}$, a being the side of the triangle; then, p being any point whatever, $|p$ is its *anti-polar* with reference to this circle; that is, a line parallel to its polar, and equidistant with it from the center of the circle, but on the opposite side.



It is evident that with reference to the above circle each vertex is the anti-pole of the opposite side of the reference triangle; for the respective distances of a vertex and its opposite side from the center are

$$\frac{2}{3} \cdot \frac{a\sqrt{3}}{2} = \frac{a}{\sqrt{3}} \text{ and } \frac{1}{3} \cdot \frac{a\sqrt{3}}{2} = \frac{a}{2\sqrt{3}},$$

and the radius of the circle is a mean proportional between these. The figure shows the construction for the radius of the circle, and also for the anti-polar of any point p.

If the reference triangle is not equilateral, it can be obtained by projection from an equilateral one, and the *circle* corresponding to that will be projected into an *ellipse* such that, with reference to it, $|p$ is always the anti-polar of p. Hence it will only be necessary to prove the property in the case of an equilateral reference triangle. Before proceeding to the proof, however, it will be necessary to find an expression for the distance between two points in terms of their coefficients.

42. *Distance between two points.* Let the points be p_1 and p_2 as in Art. 40. Since they are *unit* points, we have

$$\Sigma_1^3 l = 1 = \Sigma_1^3 m\,; \quad \therefore\ l_0 = 1 - l_1 - l_2, \quad m_0 = 1 - m_1 - m_2.$$

Hence

$$p_1 = \Sigma le = e_0 + l_1(e_1 - e_0) + l_2(e_2 - e_0) = e_0 + l_1\epsilon_1 + l_2\epsilon_2, \text{ say,}$$

and $p_2 = \Sigma me = e_0 + m_1(e_1 - e_0) + m_2(e_2 - e_0) = e_0 + m_1\epsilon_1 + m_2\epsilon_2.$

Thus, $$p_2 - p_1 = (m_1 - l_1)\epsilon_1 + (m_2 - l_2)\epsilon_2,$$

and the required distance

$$= T(p_2 - p_1) = \sqrt{[(m_1 - l_1)\epsilon_1 + (m_2 - l_2)\epsilon_2]^2}, \text{ by eq. (82),}$$

$$= \sqrt{(m_1-l_1)^2\epsilon_1^2 + (m_2-l_2)^2\epsilon_2^2 + 2(m_1-l_1)(m_2-l_2)\epsilon_1|\epsilon_2}. \quad (92)$$

In the case of an equilateral reference triangle, whose side is a, we have $\epsilon_1^2 = \epsilon_2^2 = a^2$, and $\epsilon_1|\epsilon_2 = a^2 \cos 60° = \frac{1}{2}a^2$; so that eq. (92) becomes

$$T(p_2 - p_1) = a\sqrt{(m_1-l_1)^2 + (m_2-l_2)^2 + (m_1-l_1)(m_2-l_2)}. \quad (93)$$

43. *Proof of the anti-polar property of the complement in a point system.* Referring to the figure of Art. 41, we propose to show that

$$T(p - \bar{e})\,T(\bar{e} - p') = \frac{a^2}{6} = r^2,$$

which will establish the proposition. We have, since p' is the common point of $|p$ and $p\bar{e}$, p' congruent with

$$p\bar{e} \cdot |p = xp + y\bar{e}.$$

Multiply both members by pp_1; then,

$$pp_1 \cdot p\bar{e} \cdot |p = pp_1\bar{e} \cdot p|p = ypp_1\bar{e};$$

whence $\quad y = p^2.$

Multiply the same equation by $p_1\bar{e}$; therefore

$$p_1\bar{e} \cdot p\bar{e} \cdot |p = \bar{e}p_1p \cdot \bar{e}|p = xp_1\bar{e}p = -x\bar{e}p_1p;$$

whence $\quad x = -\bar{e}|p.$

Substituting, we have

$$p' = \frac{p^2 \cdot \bar{e} - \bar{e}|p \cdot p}{p^2 - p|\bar{e}}. \qquad (94)$$

Hence we have

$$\bar{e} - p' = \bar{e} - \frac{p^2 \cdot \bar{e} - \bar{e}|p \cdot p}{p^2 - \bar{e}|p} = \frac{\bar{e}|p \cdot (p - \bar{e})}{p^2 - \bar{e}|p},$$

and
$$T(p - \bar{e})\,T(\bar{e} - p') = \frac{\bar{e}|p \cdot T^2(p - \bar{e})}{p^2 - \bar{e}|p}. \qquad (95)$$

Now, by eq. (93), taking l_0, l_1, l_2 as the coefficients for p, and $m_0 = m_1 = m_2 = \frac{1}{3}$ as those for $\bar{e}$, we have

$$T^2(p - \bar{e}) = [(l_1 - \tfrac{1}{3})^2 + (l_2 - \tfrac{1}{3})^2 + (l_1 - \tfrac{1}{3})(l_2 - \tfrac{1}{3})]a^2$$
$$= [l_1^2 + l_2^2 + l_1l_2 - l_1 - l_2 + \tfrac{1}{3}]a^2.$$

Also,

$$p^2 = p|p = l_0^2 + l_1^2 + l_2^2 = (1 - l_1 - l_2)^2 + l_1^2 + l_2^2$$
$$= 2(l_1^2 + l_2^2 - l_1 - l_2 + l_1l_2 + \tfrac{1}{2}),$$

and
$$\bar{e}|p = \tfrac{1}{3}(l_0 + l_1 + l_2) = \tfrac{1}{3},$$

so that

$$p^2 - \bar{e}|p = 2(l_1^2 + l_2^2 - l_1 - l_2 + l_1l_2 + \tfrac{1}{3}).$$

Hence (95) becomes

$$T(p - \bar{e})\,T(\bar{e} - p') = \frac{\frac{1}{3}a^2(l_1^2 + l_2^2 + l_1l_2 - l_1 - l_2 + \frac{1}{3})}{2(l_1^2 + l_2^2 + l_1l_2 - l_1 - l_2 + \frac{1}{3})} = \frac{a^2}{6}. \quad \text{Q.E.D.}$$

44. The conception of the complement in a point system, as developed Arts. 40–43, *is not found in Grassmann's works.* He deals exclusively with the complement in a *unit normal vector system.* See *Die Ausdehnungslehre,* 1862, Art. 330. In a remark at the end of Art. 337 he shows how the idea of the complement might be extended to a point system, but in a way entirely different from mine, and one which he himself evidently considered of no practical value, since he has made no application of it. On the contrary, the method above developed is of great utility, giving at once reciprocal properties, according to the principle of duality.

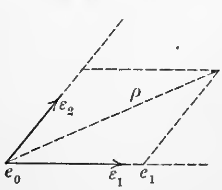
In a subsequent chapter the anti-polar property of the complement will be established in a different manner, directly, for a reference triangle of any shape.

45. A multiplication table for a point system in plane space is given on page 44. The product of any quantity at the left into any at the top is found at the intersection of the corresponding row and column. Thus, $e_2e_0 \cdot e_1e_2 = -e_2$, $e_2|e_0 = 0$, etc. Algebraically considered this system forms a *seven-fold* algebra, seven reference quantities being required to express all quantities of the system, including scalars.

46. *Projections.* If we write the equation

$$\rho = x_1\epsilon_1 + x_2\epsilon_2, \qquad (96)$$

$x_1\epsilon_1$ is evidently the projection of ρ on $\epsilon_1 \parallel$ to ϵ_2, and $x_2\epsilon_2$ is the projection of ρ on $\epsilon_2 \parallel$ to ϵ_1. Multiply (96) into ϵ_2; therefore $\rho\epsilon_2 = x_1\epsilon_1\epsilon_2$, since $\epsilon_2\epsilon_2 = 0$; thus, $x_1 = \dfrac{\rho\epsilon_2}{\epsilon_1\epsilon_2}$. Similarly, multiplying into ϵ_1, we have $x_2 = \dfrac{\rho\epsilon_1}{\epsilon_2\epsilon_1}$; whence



$$\rho = \frac{\epsilon_1 \cdot \rho\epsilon_2}{\epsilon_1\epsilon_2} + \frac{\epsilon_2 \cdot \rho\epsilon_1}{\epsilon_2\epsilon_1}. \qquad (97)$$

Hence the projections of ρ on $\epsilon_1 \parallel$ to ϵ_2 and on $\epsilon_2 \parallel$ to ϵ_1 are

A MULTIPLICATION TABLE FOR A POINT SYSTEM IN PLANE SPACE.

	e_0	e_1	e_2	$e_0e_1 = \|e_2$	$e_2e_0 = \|e_1$	$e_1e_2 = \|e_0$	$e_0e_1e_2 = 1$
e_0	0	$\|e_2$	$-\|e_1$	0	0	1	e_0
e_1	$-\|e_2$	0	$\|e_0$	0	1	0	e_1
e_2	$\|e_1$	$-\|e_0$	0	1	0	0	e_2
$e_0e_1 = \|e_2$	0	0	1	0	$-e_0$	e_1	$\|e_2$
$e_2e_0 = \|e_1$	0	1	0	e_0	0	$-e_2$	$\|e_1$
$e_1e_2 = \|e_0$	1	0	0	$-e_1$	e_2	0	$\|e_0$
$e_0e_1e_2 = 1$	e_0	e_1	e_2	$\|e_2$	$\|e_1$	$\|e_0$	1

respectively $\frac{\epsilon_1 \cdot \rho\epsilon_2}{\epsilon_1\epsilon_2}$ and $\frac{\epsilon_2 \cdot \rho\epsilon_1}{\epsilon_2\epsilon_1}$. These are particular cases of a general proposition which may be stated as follows.

Let B be a quantity of the nth order in the reference units, and C one of the mth order, and let the number of reference units be $m + n$, so that BC is scalar: then *the projection on B of any quantity A, directed by C, is*

$$\frac{B \cdot AC}{BC}.$$

Similarly, the projection of A on C, directed by B, is

$$\frac{C \cdot AB}{CB}.$$

We shall have also

$$A = \frac{B \cdot AC}{BC} + \frac{C \cdot AB}{CB}. \qquad (98)$$

Since we have restricted ourselves to space of two and three dimensions, we shall not give a general proof, but shall verify and explain the proposition in such cases as arise under this restriction. See *Die Ausdehnungslehre*, 1844, Chap. 5, and the same, 1862, §§ 127–129.

If in (97) ϵ_1, ϵ_2 are a unit normal system, replace them by ι_1 and ι_2; then, since $\iota_1\iota_2 = 1$,

$$\rho = \iota_1 \cdot \rho\iota_2 - \iota_2 \cdot \rho\iota_1 = \iota_1 \cdot \rho|\iota_1 + \iota_1\rho \cdot |\iota_1, \qquad (99)$$

which agrees with eq. (84).

47. Consider next the point equation

$$p = x_0 p_0 + x_1 p_1 + x_2 p_2; \qquad (100)$$

multiply successively by p_1p_2, p_2p_0, and p_0p_1, and we find

$$x_0 = \frac{pp_1p_2}{p_0p_1p_2}, \quad x_1 = \frac{pp_2p_0}{p_1p_2p_0}, \quad x_2 = \frac{pp_0p_1}{p_2p_0p_1},$$

whence $$p = \frac{1}{p_0p_1p_2}[p_0 \cdot pp_1p_2 + p_1 \cdot pp_2p_0 + p_2 \cdot pp_0p_1]. \qquad (101)$$

Each term in the brackets taken with the outside factor is of the typical form (98), and is the projection of p on one of the points on which it depends. Write again,

$$p = x_0|p_1p_2 + x_1|p_2p_0 + x_2|p_0p_1,$$

and multiply into $|p_0$, $|p_1$, $|p_2$ successively, and we find

$$p = \frac{1}{p_0p_1p_2}[|p_1p_2 \cdot p|p_0 + |p_2p_0 \cdot p|p_1 + |p_0p_1 \cdot p|p_2]. \quad (102)$$

We might also have obtained (102) from (101) by putting $|p_1p_2$ for p_0, etc. The terms of the right-hand member of (102) are again of the typical form, and are the projections of p on the anti-poles of p_1p_2, p_2p_0, and p_0p_1. Note that

$$|p_1p_2 \cdot |p_0 = |(p_1p_2p_0) = p_1p_2p_0 = p_0p_1p_2.$$

If in (101) we take the last two points together, their sum is some point on p_1p_2 and also on pp_0, since p is expressed in terms of p_0 and this point. Hence this point is congruent with $p_1p_2 \cdot pp_0$, and we may write

$$p = xp_0 + yp_1p_2 \cdot pp_0.$$

Multiply into p_1p_2, and we have

$$pp_1p_2 = xp_0p_1p_2.$$

Multiply into p_0p_1, and we have

$$pp_0p_1 = yp_1p_2 \cdot pp_0 \cdot p_0p_1 = yp_1p_2p_0 \cdot pp_0p_1,$$

or
$$y = \frac{1}{p_0p_1p_2}.$$

Substituting values of x and y, we obtain

$$p = \frac{p_0 \cdot pp_1p_2}{p_0p_1p_2} + \frac{p_1p_2 \cdot pp_0}{p_1p_2p_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

an equation of the same form as (98). The second term of the right-hand member of (103) is the projection of p on p_1p_2, directed by p_0.

The second members of (101) and (103) must be identically equal; hence we have

$$p_1p_2 \cdot pp_0 = p_1 \cdot pp_2p_0 + p_2 \cdot pp_0p_1,$$

or, writing p_3 instead of p, and p_4 instead of p_0, for symmetry,

$$\left.\begin{aligned} p_1p_2 \cdot p_3p_4 &= -p_1 \cdot p_2p_3p_4 + p_2 \cdot p_3p_4p_1 \\ &= \quad p_3 \cdot p_4p_1p_2 - p_4 \cdot p_1p_2p_3 \end{aligned}\right\} \quad . \quad . \quad . \qquad (104)$$

The last expression is obtained by interchanging the suffixes 1 and 2 with 3 and 4. If in (101) and (102) we put the reference points e_0, e_1, e_2 for the p's, the equations become identical, viz.,

$$p = e_0 \cdot p|e_0 + e_1 \cdot p|e_1 + e_2 \cdot p|e_2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (105)$$

Similarly, (103) becomes

$$p = e_0 \cdot p|e_0 + |e_0 \cdot pe_0 = e_0 \cdot p|e_0 + e_0p \cdot |e_0, \quad . \quad . \quad . \qquad (106)$$

a form analogous to eq. (99), expressing p in terms of its projections on e_0 and e_1e_2.

48. The operations of the last article would have been precisely the same if *lines* (point-vectors) had been used throughout instead of points. Hence, substituting L's for p's in eqs. (101) to (106), we have

$$L = \frac{1}{L_0L_1L_2}[L_0 \cdot LL_1L_2 + L_1 \cdot LL_2L_0 + L_2 \cdot LL_0L_1], \quad . \quad . \qquad (107)$$

$$L = \frac{1}{L_0L_1L_2}[|L_1L_2 \cdot L|L_0 + |L_2L_0 \cdot L|L_1 + |L_0L_1 \cdot L|L_2], \qquad (108)$$

$$L = \frac{L_0 \cdot LL_1L_2}{L_0L_1L_2} + \frac{L_1L_2 \cdot LL_0}{L_1L_2L_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (109)$$

$$\begin{aligned} L_1L_2 \cdot L_3L_4 &= -L_1 \cdot L_2L_3L_4 + L_2 \cdot L_3L_4L_1 \\ &= L_3 \cdot L_4L_1L_2 - L_4 \cdot L_1L_2L_3, \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (110)$$

$$L = e_1e_2 \cdot Le_0 + e_2e_0 \cdot Le_1 + e_0e_1 \cdot Le_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (111)$$

$$L = e_1e_2 \cdot Le_0 + e_0 \cdot Le_1e_2 = e_1e_2 \cdot L|e_1e_2 + (e_1e_2 \cdot L) \cdot |e_1e_2. \qquad (112)$$

In equations (107), (108), and (111) L appears as equal to the sum of its projections on three given lines; in (109) and (112) it appears as the sum of its projections on a line and point. The projection on the point in (109), viz. $\frac{L_1L_2 \cdot LL_0}{L_1L_2L_0}$, is a certain portion of the line joining the common point of L_1 and L_2 with the common point of L and L_0, the reciprocal idea to that of the projection of p on p_1p_2 in (103).

49. In eq. (104) put $p_3p_4 = |q_1$, and we have

$$p_1p_2 \cdot |q_1 = -p_1 \cdot p_2|q_1 + p_2 \cdot p_1|q_1. \quad \ldots \ldots \quad (113)$$

In (110) put $L_3L_4 = |M_1$, and we have

$$L_1L_2 \cdot |M_1 = -L_1 \cdot L_2|M_1 + L_2 \cdot L_1|M_1. \quad \ldots \quad (114)$$

Note that q_1 is a point, the anti-pole of p_3p_4, and M_1 is a line, the anti-polar of the point L_3L_4.

Again, in (110) put $L_1L_2 = p_2$, $L_3 = |q_1$, $L_4 = |q_2$; then, since $L_3L_4 = |q_1 \cdot |q_2 = |q_1q_2$, we have

$$p_2|q_1q_2 = |q_1 \cdot p_2|q_2 - |q_2 \cdot p_2|q_1, \quad \ldots \ldots \quad (115)$$

and similarly from (104),

$$L_2|M_1M_2 = |M_1 \cdot L_2|M_2 - |M_2 \cdot L_2|M_1. \quad \ldots \quad (116)$$

Multiply (115) and (116) respectively by p_1 and L_1, or (113) and (114) respectively into* $|q_2$ and $|M_2$, and we have

$$p_1p_2|q_1q_2 = \begin{vmatrix} p_1|q_1 & p_1|q_2 \\ p_2|q_1 & p_2|q_2 \end{vmatrix}. \quad \ldots \ldots \quad (117)$$

$$L_1L_2|M_1M_2 = \begin{vmatrix} L_1|M_1 & L_1|M_2 \\ L_2|M_1 & L_2|M_2 \end{vmatrix}. \quad \ldots \ldots \quad (118)$$

If $q_1 = p_1$ and $q_2 = p_2$, $M_1 = L_1$ and $M_2 = L_2$, (117) and (118) become

$$\left.\begin{aligned} p_1p_2|p_1p_2 &\equiv (p_1p_2)^2 = p_1^2p_2^2 - (p_1|p_2)^2 \\ L_1L_2|L_1L_2 &\equiv (L_1L_2)^2 = L_1^2L_2^2 - (L_1|L_2)^2 \end{aligned}\right\} \ldots \quad (119)$$

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* A multiplied *by* B means BA; A multiplied *into* B means AB.

Put q_2 for p in (102), and multiply by $p_0p_1p_2 \cdot q_0q_1$; then,

$$p_0p_1p_2 \cdot q_0q_1q_2 = q_0q_1|p_1p_2 \cdot q_2|p_0 + q_0q_1|p_2p_0 \cdot q_2|p_1 + q_0q_1|p_0p_1 \cdot q_2|p_2$$

$$= p_0|q_2 \cdot \begin{vmatrix} p_1|q_0 & p_1|q_1 \\ p_2|q_0 & p_2|q_1 \end{vmatrix} + p_1|q_2 \cdot \begin{vmatrix} p_2|q_0 & p_2|q_1 \\ p_0|q_0 & p_0|q_1 \end{vmatrix} + p_2|q_2 \cdot \begin{vmatrix} p_0|q_0 & p_0|q_1 \\ p_1|q_0 & p_1|q_1 \end{vmatrix}$$

$$= \begin{vmatrix} p_0|q_0 & p_0|q_1 & p_0|q_2 \\ p_1|q_0 & p_1|q_1 & p_1|q_2 \\ p_2|q_0 & p_2|q_1 & p_2|q_2 \end{vmatrix} \quad \ldots \ldots \quad (120)$$

Of course L's and M's may be written in (120) for p's and q's; *i.e.* lines may be substituted for points.

Finally, a point may be expressed in terms of two points in plane space as follows. Write

$$p = xp_1 + yp_2,$$

and multiply into $p_1|p_1p_2$ and $p_2|p_1p_2$ successively.

$$\therefore\ pp_1|p_1p_2 = yp_1p_2|p_1p_2 = -y(p_1p_2)^2,$$

$$pp_2|p_1p_2 = x(p_1p_2)^2;$$

whence
$$p = \frac{1}{(p_1p_2)^2}(p_1 \cdot pp_2|p_1p_2 - p_2 \cdot pp_1|p_1p_2). \quad . \quad (121)$$

Note that, since all these point equations are homogeneous in all the points involved, these points may have any weights we please.

50. It can be easily seen, from the geometric interpretation given to combinatory products, that the equation

$$AB = AC$$

does *not* imply that $B = C$; or, in other words, the quotient of $A(B - C)$ divided by A is *not*, in general, $B - C$. Thus, in plane space

$$pL_1 = pL_2$$

simply means that the two quantities are the same in magnitude and sign, and L_1 and L_2 may have an infinite number of relative positions and lengths. The algebraic reason for this is that a product can be zero without either factor vanishing, so that division is *indeterminate*. Thus,

$$\frac{pL_1}{p} = L_1 + (x_1p_1 + x_2p_2)p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

because, on multiplying both sides into p, the equation becomes an identity. As the subject of division has no great importance in the developments or applications proposed in this work, it will not be further discussed.*

51. EXERCISES. — (1) To show that

$$\epsilon_1 T\epsilon_2 \pm \epsilon_2 T\epsilon_1 \text{ and } p_0(p_1 Tp_0p_2 \pm p_2 Tp_0p_1)$$

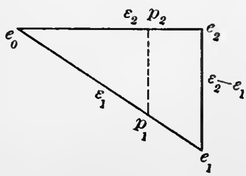
are the respective bisectors of the angles between the vectors ϵ_1 and ϵ_2 and the point-vectors p_0p_1 and p_0p_2, the upper signs corresponding to the internal bisectors and the lower to the external.

The sum and difference of two *equal* vectors, being ‖ to the two diagonals of a rhombus, evidently bisect the two angles between the vectors; hence

$$\text{Bisector} = U\epsilon_1 \pm U\epsilon_2 = \frac{\epsilon_1}{T\epsilon_1} \pm \frac{\epsilon_2}{T\epsilon_2} = \frac{1}{T\epsilon_1 T\epsilon_2}(\epsilon_1 T\epsilon_2 \pm \epsilon_2 T\epsilon_1),$$

which is the first expression above except as to length. The point expression is found in the same way.

(2) A parallel to a side of a triangle cuts the other sides proportionally.



Let ϵ_1 and ϵ_2 be ‖ and equal to two of the sides; then $\epsilon_2 - \epsilon_1$ is ‖ and equal to the other side. Let $p_1 - e_0 = x_1\epsilon_1$ and $p_2 - e_0 = x_2\epsilon_2$. Then, by given conditions, $x_2\epsilon_2 - x_1\epsilon_1 = n(\epsilon_2 - \epsilon_1)$. Multiply by ϵ_1 and ϵ_2 successively, and we have $x_2\epsilon_1\epsilon_2 = n\epsilon_1\epsilon_2$, and $x_1\epsilon_1\epsilon_2 = n\epsilon_1\epsilon_2$. $\therefore\ x_1 = x_2 = n$. Q.E.D.

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* A treatment of the matter will be found in the fourth chapter of the *Ausdehnungslehre* of 1844, and a more extended one in an article by the Author in No. 1, Vol. IV. of the "Annals of Mathematics," published at the University of Virginia.

(3) The bisectrix of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.

With the figure of the last proposition the bisectrix of the angle at e_0 is $e_0(e_1 T\epsilon_2 \pm e_2 T\epsilon_1)$, which gives the proof immediately, for the point $e_1 T\epsilon_2 \pm e_2 T\epsilon_1$ is on the line $e_1 e_2$ at distances from these two points inversely as the weights, *i.e.* directly as $T\epsilon_1$ and $T\epsilon_2$, and between them or outside according as we use the upper or lower sign.

(4) If a, b, c are the three sides of a triangle, to show that

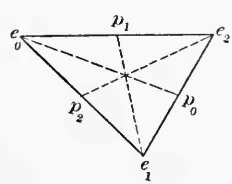
$$a^2 = b^2 + c^2 - 2\,bc \cos \angle \frac{c}{b}.$$

With the figure above let

$$T(\epsilon_2 - \epsilon_1) = a, \quad T\epsilon_1 = c, \quad T\epsilon_2 = b.$$

Then,
$$a^2 = T^2(\epsilon_2 - \epsilon_1) = (\epsilon_2 - \epsilon_1)^2 = \epsilon_2^2 + \epsilon_1^2 - 2\,\epsilon_1|\epsilon_2$$
$$= b^2 + c^2 - 2\,bc \cos \angle \frac{c}{b}.$$

(5) Find the condition that lines through the three vertices of a triangle shall have a common point.



By the figure the condition is

$$e_0 p_0 \cdot e_1 p_1 \cdot e_2 p_2 = 0.$$

Let
$$p_0 = m_0 e_1 + n_0 e_2,$$
$$p_1 = n_1 e_2 + l_1 e_0,$$
$$p_2 = l_2 e_0 + m_2 e_1;$$

then,

$$e_0 p_0 \cdot e_1 p_1 \cdot e_2 p_2 \equiv e_0(m_0 e_1 + n_0 e_2) \cdot e_1(n_1 e_2 + l_1 e_0) \cdot e_2(l_2 e_0 + m_2 e_1)$$
$$\equiv m_0 n_1 l_2 - n_0 l_1 m_2 = 0.$$

This is equivalent to

$$p_0 e_2 \cdot p_1 e_0 \cdot p_2 e_1 - e_1 p_0 \cdot e_2 p_1 \cdot e_0 p_2 = 0.$$

(6) Find the condition that three points on the respective sides of a triangle shall be collinear. This case is the recip-

rocal of the preceding. Let the points be as in Ex. (5). Then the condition is

$$p_0p_1p_2 = 0 = (m_0e_1 + n_0e_2)(n_1e_2 + l_1e_0)(l_2e_0 + m_2e_1)$$
$$= m_0n_1l_2 + n_0l_1m_2,$$

which is equivalent to

$$p_0e_2 \cdot p_1e_0 \cdot p_2e_1 + e_1p_0 \cdot e_2p_1 \cdot e_0p_2 = 0.$$

(7) Show, by Ex. (5), that the following sets of lines in a triangle have a common point.

1st. Lines through the vertices and the middle points of the opposite sides.

2d. A line through one vertex and the middle of the opposite side, and two lines through the other vertices ∥ to the sides opposite to them.

3d. The bisectors of the angles; all internal, or one internal and two external.

4th. The perpendiculars from the vertices on the opposite sides.

5th. The perpendiculars to the sides at their middle points.

(8) Show, by Ex. (6), that the points where the bisectors of the angles of a triangle cut the opposite sides are collinear, if two of them are internal and one external, or if all are external.

(9) From the values of x_0, x_1, x_2, given just before eq. (101), determine the effect upon the position of p of giving a negative value to one or more of these coefficients.

(10) If in the result of Ex. (6) lines be substituted for points, — say (L_0, L_1, L_2) for (e_0, e_1, e_2) and (L_0', L_1', L_2') for (p_0, p_1, p_2), — interpret the resulting equation.

(11) If a quadrilateral be divided by a right line into two quadrilaterals, show that the common points of the three pairs of diagonals are collinear.

(12) By substituting lines for points, in the equation of condition of the last exercise, derive the reciprocal proposition.

(13) If two triangles are so situated that the lines joining their vertices two by two meet in a point, then will their corresponding sides meet each other in three points lying in one right line.

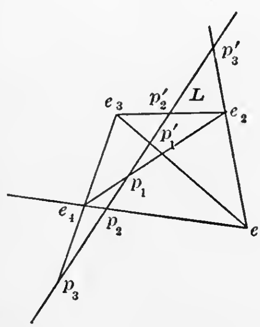
(14) Show that, if a line L cut the six lines that can be drawn through four points e_1, e_2, e_3, e_4 in the six points

$$p_1, p_2, p_3, p_1', p_2', p_3',$$

as in the figure; then the relation

$$Tp_1p_2 \cdot Tp_3p_1' \cdot Tp_2'p_3' = Tp_1'p_2' \cdot Tp_3'p_1 \cdot Tp_2p_3$$

holds. These points are said to be in *involution.*



(15) By substituting lines for points, and a point p for L, obtain the reciprocal theorem, and interpret it.

(16) Let p_1, p_2, p_3, p_4 be four fixed points, and let p_2' and p_3' vary subject to the conditions

$$p_1p_2'p_3' = p_2p_2'p_4 = p_3p_3'p_4 = 0;$$

find the locus of p, the common point of p_2p_3' and $p_2'p_3$.

If $p_1p_2p_3 = 0$, show, by eq. (104), that the locus becomes two straight lines, one of which passes through p_4.

We have at once $p_2' = pp_3 \cdot p_2p_4$ and $p_3' = pp_2 \cdot p_3p_4$; whence, by substitution in above condition, we have

$$p_1(pp_3 \cdot p_2p_4)(pp_2 \cdot p_3p_4) = 0,$$

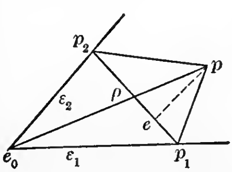
the equation of the locus, which, being of the second degree in p, represents a conic. On applying (104), this will separate into two factors of the first degree in p, if $p_1p_2p_3 = 0$.

(17) Interpret the reciprocal results obtained by putting L's for p's.

(18) If e_1, e_2, e_3, e_4 are four coplanar points, and e_5 and e_6 are the common points of e_1e_2 and e_3e_4, and of e_4e_1 and e_2e_3 respectively, show that the middle points of e_1e_3, e_2e_4, and e_5e_6 are collinear.

(19) Lines through the vertices of any triangle and the corresponding vertices of its complementary triangle meet in a point; and, reciprocally, the corresponding sides cut each other in three collinear points.

(20) A triangle whose sides are of constant length moves so that two of its vertices remain on two fixed straight lines: find the locus of the other vertex. Let $e_0\epsilon_1$ and $e_0\epsilon_2$ be the two fixed lines, and p_1p_2p the triangle. Also let $p_1 - e_0 = x\epsilon_1$ and $p_2 - e_0 = y\epsilon_2$; then $p_2 - p_1 = y\epsilon_2 - x\epsilon_1$, and we have the condition



$$T(y\epsilon_2 - x\epsilon_1) = c.$$

Let pe be $\perp$ to p_1p_2, $Tp_1e = mc$, $Tep = nc$; then,

$$p - e_0 = \rho = x\epsilon_1 + m(y\epsilon_2 - x\epsilon_1) + n|(y\epsilon_2 - x\epsilon_1).$$

This equation in ρ and the scalar variables x and y, with the condition above, which is really of the second degree in x and y, is that of a conic section, which must evidently be an ellipse. The student should eliminate x and y by multiplying successively by ϵ_1 and ϵ_2, thus obtaining a scalar equation in ρ of the second degree.

(21) Show that the expression $pp_1L_1p_2L_2p_3p'$, interpreted according to Art. 16, (c), is identically equal to

$$(pp_1 \cdot L_1)p_2(L_2 \cdot p_3p'),$$

and from this that it is also equal to $-p'p_3L_2p_2L_1p_1p$.

By eq. (104),

$$pp_1L_1p_2L_2p_3p' = pp_1L_1p_2(p_3 \cdot p'L_2 - p' \cdot p_3L_2)$$
$$= (pp_1 \cdot L_1 \cdot p_2p_3) \cdot p'L_2 - (pp_1 \cdot L_1 \cdot p_2p') \cdot p_3L_2$$
$$= (p_1p_2p_3 \cdot pL_1 - pp_2p_3 \cdot p_1L_1) \cdot p'L_2$$
$$- (p_1p_2p' \cdot pL_1 - pp_2p' \cdot p_1L_1) \cdot p_3L_2$$
$$= (p_1 \cdot pL_1 - p \cdot p_1L_1)p_2(p_3 \cdot p'L_2 - p' \cdot p_3L_2)$$
$$= (pp_1 \cdot L_1)p_2(L_2 \cdot p_3p').$$

The second part is left to the student.

(22) If, as in eq. (86),

$$|p = L = \frac{1}{l_0}(l_0e_1 - l_1e_0)(l_0e_2 - l_2e_0),$$

show that

$$T|p = TL = \sqrt{(l_0 - l_1)^2\epsilon_2^2 + (l_0 - l_2)^2\epsilon_1^2 - 2(l_0 - l_1)(l_0 - l_2)\epsilon_1|\epsilon_2},$$

in which $\epsilon_1 = e_1 - e_0$ and $\epsilon_2 = e_2 - e_0$.

(23) If $L_1 = p_1p_1'$, $L_2 = p_2p_2'$, $L_3 = p_3p_3'$, then show that

$$L_1L_2L_3 = \begin{vmatrix} p_1\,L_2 & p_1\,L_3 \\ p_1'L_2 & p_1'L_3 \end{vmatrix} = \begin{vmatrix} p_2\,L_3 & p_2\,L_1 \\ p_2'L_3 & p_2'L_1 \end{vmatrix} = \begin{vmatrix} p_3\,L_1 & p_3\,L_2 \\ p_3'L_1 & p_3'L_2 \end{vmatrix}.$$

(24) By eq. (119) prove that

$$(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) = (x_1x_2 + y_1y_2 + z_1z_2)^2$$
$$+ \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}^2 + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}^2 + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^2.$$

(25) Show by eq. (120) how the product of two determinants of the third order may be expressed as a determinant of the same order.

(26) If $L_1 = \Sigma_0^2(l|e)$ and $L_2 = \Sigma_0^2(m|e)$, show that when

$$\begin{vmatrix} l_0 & l_1 \\ m_0 & m_1 \end{vmatrix} + \begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix} + \begin{vmatrix} l_2 & l_0 \\ m_2 & m_0 \end{vmatrix} = 0,$$

then L_1 and L_2 are parallel.

Stereometric Products.

52. Three-dimensional space is the locus of all points dependent on four fixed points. Let these four reference points be e_0, e_1, e_2, e_3, so situated relatively to each other that $e_0e_1e_2e_3 = 1$, always; *i.e.* the unit of volume is six times the volume of the reference tetraedron. Let four points be taken, viz.:

$$p_1 = \Sigma_0^3 ke,\ p_2 = \Sigma_0^3 le,\ p_3 = \Sigma_0^3 me,\ p_4 = \Sigma_0^3 ne;$$

then,

$$\left.\begin{aligned} p_1p_2 = L_1 &= \begin{vmatrix} k_0 & k_1 \\ l_0 & l_1 \end{vmatrix} e_0e_1 + \begin{vmatrix} k_0 & k_2 \\ l_0 & l_2 \end{vmatrix} e_0e_2 + \begin{vmatrix} k_0 & k_3 \\ l_0 & l_3 \end{vmatrix} e_0e_3 \\ &\quad + \begin{vmatrix} k_2 & k_3 \\ l_2 & l_3 \end{vmatrix} e_2e_3 + \begin{vmatrix} k_3 & k_1 \\ l_3 & l_1 \end{vmatrix} e_3e_1 + \begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} e_1e_2 \\ &\equiv \left\|\begin{matrix} k_0 & k_1 & k_2 & k_3 \\ l_0 & l_1 & l_2 & l_3 \end{matrix}\right\| [e_0, e_1, e_2, e_3]. \end{aligned}\right\} \quad (123)$$

The first result will be obtained by actual multiplication of the values of p_1 and p_2, and the second result is simply an abbreviated way of writing the other as in eq. (55). It appears thus that any point-vector in space is expressible in terms of the six edges of the reference tetraedron.

Again,

$$\left.\begin{aligned} p_1p_2p_3 = P_4 &= \left\|\begin{matrix} k_0 & k_1 & k_2 & k_3 \\ l_0 & l_1 & l_2 & l_3 \\ m_0 & m_1 & m_2 & m_3 \end{matrix}\right\| [e_0, e_1, e_2, e_3] \\ &= \begin{vmatrix} |e_0 & |e_1 & |e_2 & |e_3 \\ k_0 & k_1 & k_2 & k_3 \\ l_0 & l_1 & l_2 & l_3 \\ m_0 & m_1 & m_2 & m_3 \end{vmatrix} \end{aligned}\right\}; \quad . \quad . \quad . \quad (124)$$

in which the third member means the sum of the four third-order determinants that can be formed of the columns taken three at a time, each multiplied into its corresponding triple product of the reference points, with the same order of suffixes. In the fourth member $|e_0 = e_1e_2e_3$, $|e_1 = -e_2e_3e_0$, $|e_2 = e_3e_0e_1$, and $|e_3 = -e_0e_1e_2$. Thus any point-plane-vector is expressible in terms of the four faces of the reference tetraedron. Of course the product of three points is not scalar in solid space.

Finally,

$$p_1p_2p_3p_4 = \begin{vmatrix} k_0 & k_1 & k_2 & k_3 \\ l_0 & l_1 & l_2 & l_3 \\ m_0 & m_1 & m_2 & m_3 \\ n_0 & n_1 & n_2 & n_3 \end{vmatrix}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

because $e_0e_1e_2e_3 = 1$; thus the product of four points is scalar, as was shown in Art. 22.

As an exercise let the student find the condition that the plane P_4, in (124), shall pass through the mean of the reference points.

53. Since, by Art. 17, the continued product of four points obeys the associative law, we have

$$p_1p_2p_3p_4 = p_1p_2p_3 \cdot p_4 = P_4p_4 = p_1p_2 \cdot p_3p_4 = L_1L_2$$
$$= -p_4P_4 = L_2L_1. \quad . \quad . \quad . \quad . \quad (126)$$

Thus the product of a point and plane is *non*-commutative, while that of two lines in solid space is commutative. The *stereometric* product of two lines is according to Art. 16, (a), while the *planimetric* product is according to Art. 16, (b).

54. *Product of a line and a plane.* Let L be the line, and P the plane, and let p_0 be the point where the line pierces the plane. Take p_1, p_2, p_3, so that

$$L = p_0p_1 \text{ and } P = p_0p_2p_3.$$

$$\therefore\ LP = p_0p_1 \cdot p_0p_2p_3 = p_0p_1p_2p_3 \cdot p_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

This is in accordance with Art. 16, (b), and the model form of eq. (59). Also,

$$PL = p_0p_2p_3 \cdot p_0p_1 = p_0p_2p_3p_1 \cdot p_0 = p_0p_1p_2p_3 \cdot p_0 = LP; \quad (128)$$

so that this product is *commutative*, like pL.

If L is parallel to P, p_0 is at ∞, and, replacing it by ϵ, we have for this case

$$PL = LP = \epsilon p_1 \cdot \epsilon p_2p_3 = \epsilon p_1p_2p_3 \cdot \epsilon. \quad . \quad . \quad . \quad . \quad . \quad (129)$$

55. *Product of two planes.* Let them be P_1 and P_2, and let L be their common line, while p_1 and p_2 are so taken that

$$P_1 = Lp_1 \text{ and } P_2 = Lp_2.$$

Then,

$$\left.\begin{aligned} P_1P_2 &= Lp_1 \cdot Lp_2 = Lp_1p_2 \cdot L \\ P_2P_1 &= Lp_2 \cdot Lp_1 = Lp_2p_1 \cdot L = -P_1P_2 \end{aligned}\right\} ; \qquad (130)$$

so that the product of two planes, like that of two points, is non-commutative.

If P_1 and P_2 are parallel, L is at ∞ and becomes a plane-vector; call it η, and substitute in (130); then we have for the product of two planes, having a common line η at ∞,

$$P_1P_2 = \eta p_1 \cdot \eta p_2 = \eta p_1 p_2 \cdot \eta = -P_2P_1. \quad . \quad . \quad (131)$$

56. *Product of three planes.* Let p_0 be the common point of P_1, P_2, and P_3, and take p_1, p_2, p_3 on the common lines of these planes, so that

$$P_1 = p_0p_2p_3, \quad P_2 = p_0p_3p_1, \quad P_3 = p_0p_1p_2;$$

then,

$$\left.\begin{aligned} P_1P_2P_3 &= p_0p_2p_3 \cdot p_0p_3p_1 \cdot p_0p_1p_2 = -023 \cdot 013 \cdot 012 \\ &= -023 \cdot 0132 \cdot 01 = 0123 \cdot 023 \cdot 01 \\ &= 0123 \cdot 0231 \cdot 0 = (p_0p_1p_2p_3)^2 \cdot p_0 \end{aligned}\right\}. \qquad (132)$$

In this equation we have used 0 for p_0, 1 for p_1, etc., for convenience. This we may frequently do when no ambiguity will result. In eq. (132) we have worked according to Art. 16, (c), by which $P_1P_2P_3 = P_1 \cdot P_2P_3$; but if we had combined P_1 and P_2 first, and the result with P_3, we should have obtained the same result. Hence planes obey the same laws* of multiplication as points, in solid space.

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* We have here assumed the distributive law to hold, as, in fact, it does, for all products, progressive, regressive, or mixed; but it is easy to prove the law for planes or lines, assuming it to be true for points. Thus, taking the planes as above,

$$\begin{aligned} P_1P_2 + P_1P_3 &= 023 \cdot 031 + 023 \cdot 012 = 0123 \cdot (03 - 02), \text{ because 0123 is scalar,} \\ &= 0123 \cdot 0(3-2) = 01(2-3)3 \cdot 0(3-2) \\ &= 0(3-2)13 \cdot 0(3-2) = 0(3-2)1 \cdot 0(3-2)3 \\ &= 023 \cdot 01(2-3) = 023 \cdot (031 + 012) = P_1(P_2 + P_3). \end{aligned}$$

If the three planes are parallel to one right line, the common point is at ∞, and ϵ may be substituted for p_0 in (132).

57. *Product of four planes.* Let p_1, p_2, p_3, p_4 be the four common points of four planes P_1, P_2, P_3, P_4 taken three by three, and take four coefficients $n_1, \cdots n_4$, so that $P_1 = n_1 p_2 p_3 p_4$, etc.; then

$$\left.\begin{aligned} P_1P_2P_3P_4 &= n_1n_2n_3n_4 \cdot 234 \cdot 341 \cdot 412 \cdot 123 \\ &= n_1n_2n_3n_4(p_1p_2p_3p_4)^3 \end{aligned}\right\} \quad . \quad . \quad (133)$$

Mixed products are to be interpreted according to Art. 16, (*c*). Thus,

$$L_1P_1L_2P_2L_3 = L_1\{P_1[L_2(P_2L_3)]\}$$

has this meaning. P_2L_3 is a point; this, multiplied by L_2, gives a plane; this, by P_1, a line; and this, by L_1, a scalar quantity.

58. *Products of plane-vectors.* Let η_1 and η_2 be two plane-vectors (lines at ∞), and let ϵ be parallel to each of them, while ϵ_1 and ϵ_2 are so taken that $\eta_1 = \epsilon\epsilon_1$ and $\eta_2 = \epsilon\epsilon_2$; then,

$$\eta_1\eta_2 = \epsilon\epsilon_1 \cdot \epsilon\epsilon_2 = \epsilon\epsilon_1\epsilon_2 \cdot \epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

This result may be obtained directly from eq. (58) by regarding the points and lines of that equation as all at ∞, and therefore necessarily in the plane at ∞.

The product of two plane-vectors appears as a vector parallel to each of them, multiplied by a scalar quantity. We have at once

$$\eta_2\eta_1 = -\eta_1\eta_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

Next take a third plane-vector η_3, and let ϵ_1 be $\parallel$ to η_2 and η_3, ϵ_2 $\parallel$ to η_3 and η_1, ϵ_3 $\parallel$ to η_1 and η_2, while the tensors of ϵ_1, etc., are such that $\eta_1 = \epsilon_2\epsilon_3$, $\eta_2 = \epsilon_3\epsilon_1$, $\eta_3 = \epsilon_1\epsilon_2$; then

$$\eta_1\eta_2\eta_3 = \epsilon_2\epsilon_3 \cdot \epsilon_3\epsilon_1 \cdot \epsilon_1\epsilon_2 = (\epsilon_1\epsilon_2\epsilon_3)^2. \quad . \quad . \quad . \quad . \quad (136)$$

As an exercise let the student discuss ηP, the product of a plane-vector and a point-plane-vector.

59. *Equations of condition.* By eq. (48),

$$pP = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)$$

makes p lie on P, or P pass through p;

$$L_1L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

makes the two lines intersect;

$$LP = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

makes L lie in P, or P pass through L;

$$P_1P_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

makes the two planes coincide;

$$P_1P_2P_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

makes the three planes pass through a common line; for P_2P_3 is a line, say L, and, by (139), $P_1L=0$ makes P_1 pass through L;

$$P_1P_2P_3P_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (142)$$

makes the four planes pass through one point, for $P_1P_2P_3$ is some point, say p, and pP_4, by (137), makes P_4 pass through p;

$$P_1LP_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

makes P_1 and P_2 cut L at the same point; for, writing $L = P_3P_4$, the result follows from (142);

$$\eta_1\eta_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

makes the two plane-vectors parallel;

$$\eta_1\eta_2\eta_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145)$$

makes the three plane-vectors all parallel to one straight line.

Equations (138), (140), (141), (142), (144), (145), should be compared with equations (66), (46), ··· (50), respectively.

60. *Addition of planes and plane-vectors.* Let P_1 and P_2 be two planes intersecting in L, and let p_1 and p_2 be so taken that $P_1 = Lp_1$ and $P_2 = Lp_2$; then

$$P_1 + P_2 = Lp_1 + Lp_2 = L(p_1 + p_2) = 2L\bar{p}, \quad (146)$$

in which $\bar{p}$ is the mean of p_1 and p_2. Thus the sum is that diagonal plane of the parallelopiped, of which two adjacent faces are P_1 and P_2, which passes through L; the parallelograms P_1 and P_2 being so placed as to have a common side L.

If the two planes are parallel, let η be a plane-vector parallel to each of them, *i.e.* their common line at ∞, and let p_1 and p_2 be points of the respective planes; then we may write

$$P_1 = n_1 p_1 \eta \text{ and } P_2 = n_2 p_2 \eta;$$

whence

$$\left.\begin{aligned} P_1 + P_2 &= n_1 \eta p_1 + n_2 \eta p_2 = \eta(n_1 p_1 + n_2 p_2) \\ &= (n_1 + n_2)\bar{p}\eta \end{aligned}\right\} \quad (147)$$

If $n_1 + n_2 = 0$, then

$$P_1 + P_2 = n_2(p_2 - p_1)\eta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

so that the sum, in this case, becomes a volume, and is scalar. Cf. eq. (74).

Take P_1, P_2, P_3, p_0, p_1, p_2, p_3 as in Art. 56; then

$$P_1 + P_2 + P_3 = p_0(p_2 p_3 + p_3 p_1 + p_1 p_2) = p_0(p_2 - p_1)(p_3 - p_1). \quad (149)$$

Thus the sum is a plane through the common point parallel to the plane $p_1 p_2 p_3$.

If p_0 is at ∞, call it ϵ; then each plane is $\parallel$ to ϵ, and the sum becomes the product of three vectors, and therefore scalar.

If $$P_1 + P_2 = 0, \text{ or } P_1 = -P_2, \quad . \quad . \quad . \quad . \quad . \quad (150)$$

the two planes are coincident.

If $$P_1 + P_2 + P_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

the three planes pass through one right line, as appears by comparison with eq. (146).

Similarly, $$P_1 + P_2 + P_3 + P_4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

causes the four planes to pass through a common point, as appears from eq. (149).

Take η_1, η_2, ϵ, ϵ_1, ϵ_2 as in Art. 58; then

$$\eta_1 + \eta_2 = \epsilon\epsilon_1 + \epsilon\epsilon_2 = \epsilon(\epsilon_1 + \epsilon_2), \quad . \quad . \quad . \quad . \quad (153)$$

so that the sum is a plane-vector parallel to ϵ.

61. *Addition of point-vectors, or lines.* Take n point-vectors $p_1\epsilon_1, p_2\epsilon_2, \cdots p_n\epsilon_n$, and call their sum S; then

$$\left.\begin{aligned} S = p_1\epsilon_1 + p_2\epsilon_2 + \cdots + p_n\epsilon_n \equiv \Sigma_1^n p\epsilon \equiv e_0\Sigma\epsilon - e_0\Sigma\epsilon + \Sigma p\epsilon \\ \equiv e_0\Sigma\epsilon + \Sigma(p - e_0)\epsilon \end{aligned}\right\}. \quad (154)$$

It appears that S is, in general, composed of two parts, of which one is a *point*-vector, and the other a *plane*-vector. If this plane-vector is *parallel* to the point-vector, *i.e.* capable of expression as the product of some vector α into $\Sigma\epsilon$, then their sum can be expressed as a point-vector only; for we have, in this case,

$$S = e_0\Sigma\epsilon + \alpha\Sigma\epsilon = (e_0 + \alpha)\Sigma\epsilon,$$

a point-vector of the same length as $e_0\Sigma\epsilon$, $\parallel$ to it, and distant from it by the amount $T\alpha \sin \angle{}^{\Sigma\epsilon}_{\alpha}$.

S being composed of two parts which cannot be equal to each other, if we have the equation $S = 0$, it can only be satisfied by making each part separately zero, so that $S = 0$ implies $\Sigma\epsilon = 0$ and $\Sigma(p - e_0)\epsilon = 0$. The quantity S may be called a *screw*,* and we shall hereafter consider some of its properties.

62. *The complement in three-dimensional space.* Following the definitions of Art. 38, we have for a unit normal vector system $\iota_1, \iota_2, \iota_3$,

$$\left.\begin{aligned} |\iota_1 = \iota_2\iota_3, \quad |\iota_2\iota_3 = ||\iota_1 = \iota_1 \\ |\iota_2 = \iota_3\iota_1, \quad |\iota_3\iota_1 = ||\iota_2 = \iota_2 \\ |\iota_3 = \iota_1\iota_2, \quad |\iota_1\iota_2 = ||\iota_3 = \iota_3 \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

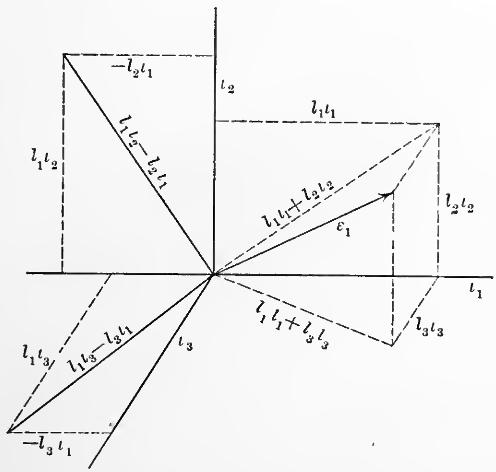
Let $\epsilon_1 = l_1\iota_1 + l_2\iota_2 + l_3\iota_3$

and $\epsilon_2 = m_1\iota_1 + m_2\iota_2 + m_3\iota_3$;

then $|\epsilon_1 = l_1\iota_2\iota_3 + l_2\iota_3\iota_1 + l_3\iota_1\iota_2 = \frac{1}{l_1}(l_1\iota_2 - l_2\iota_1)(l_1\iota_3 - l_3\iota_1)$, (156)

* See "The Theory of Screws," by R. S. Ball, Dublin, Hodges, Foster & Co.; and also a paper by the author on "The Directional Theory of Screws," Annals of Mathematics, Vol. IV., No. 5.

so that $|\epsilon_1$ is a *plane-vector*. The third member of (156) is the product of two vectors; the first, $l_1\iota_2 - l_2\iota_1$, is easily seen, by the figure, to be $\perp$ to $l_1\iota_1 + l_2\iota_2$, the projection of ϵ_1 on the plane



$\iota_1\iota_2$, and hence $\perp$ also to ϵ_1, because $\perp$ to the plane that projects ϵ_1 on $\iota_1\iota_2$; similarly, $l_1\iota_3 - l_3\iota_1$ is $\perp$ to $l_1\iota_1 + l_3\iota_3$, hence to the plane that projects ϵ_1 on $\iota_3\iota_1$, and therefore to ϵ_1 itself.

Hence $|\epsilon_1$ is a plane-vector *perpendicular* to ϵ_1. Since $||\epsilon_1 = \epsilon_1$, it follows that the converse is true; that is, the complement of a plane-vector is a line-vector perpendicular to it.

It is evident from the figure that ϵ_1 is a diagonal of the rectangular parallelopiped whose edges are l_1, l_2, l_3 in length; hence,

$$T\epsilon_1 = \sqrt{l_1^2 + l_2^2 + l_3^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Multiply (156) by ϵ_1; therefore

$$\epsilon_1|\epsilon_1 = \epsilon_1^2 = l_1^2 + l_2^2 + l_3^2 = T^2\epsilon_1, \quad . \quad . \quad . \quad . \quad (158)$$

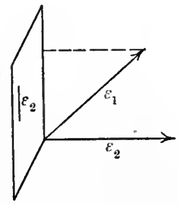
so that, as in plane space, the co-square of a vector is equal to the square of its tensor. The product $\epsilon_1|\epsilon_1$ is that of the vector ϵ_1 into a $\perp$ plane-vector, as has just been shown; it is therefore a volume which is equivalent to $T\epsilon_1$ times the area of $|\epsilon_1$; hence, by (158), the area of $|\epsilon_1$ is numerically equal to $T\epsilon_1$, or

$$T|\epsilon_1 = T\epsilon_1. \quad (159)$$

Thus the *complement of a vector in solid space is a perpendicular plane-vector having the same tensor.*

We have

$$\left.\begin{aligned}\epsilon_1|\epsilon_2 &= (l_1\iota_1 + l_2\iota_2 + l_3\iota_3)(m_1\iota_2\iota_3 + m_2\iota_3\iota_1 + m_3\iota_1\iota_2) \\ &= l_1m_1 + l_2m_2 + l_3m_3 = \epsilon_2|\epsilon_1\end{aligned}\right\}. \quad (159)$$



Now $\epsilon_1|\epsilon_2$, being the product of the vector ϵ_1 into the plane-vector $|\epsilon_2$, is equivalent to

$$T\epsilon_1 \cdot T|\epsilon_2 \cdot \sin \angle_{\epsilon_1}^{|\epsilon_2} = T\epsilon_1 T\epsilon_2 \cos \angle_{\epsilon_1}^{\epsilon_2};$$

that is,

$$\epsilon_1|\epsilon_2 = \epsilon_2|\epsilon_1 = l_1m_1 + l_2m_2 + l_3m_3 = T\epsilon_1 T\epsilon_2 \cos \angle_{\epsilon_1}^{\epsilon_2}. \quad (160)$$

If ϵ_1 and ϵ_2 were *unit* vectors, l_1, l_2, l_3, m_1, m_2, m_3 would be direction cosines, and thus (160) gives a proof of the formula for the cosine of the angle between two lines in terms of the direction cosines of the lines.

By (160) the condition that ϵ_1 and ϵ_2 shall be at right angles is

$$\epsilon_1|\epsilon_2 = 0. \quad (161)$$

Let $\eta_1 = |\epsilon_1$ and $\eta_2 = |\epsilon_2$; then

$$\eta_1|\eta_2 = |\epsilon_1 \cdot \epsilon_2 = \epsilon_2|\epsilon_1 = \epsilon_1|\epsilon_2 = T\epsilon_1 T\epsilon_2 \cos \angle_{\epsilon_1}^{\epsilon_2} = T\eta_1 T\eta_2 \cos \angle_{\eta_1}^{\eta_2}, \quad (162)$$

and

$$\eta_1|\eta_2 = 0 \quad (163)$$

is the condition of perpendicularity of two plane-vectors.

63. *Complement in a point system in three-dimensional space.* Let e_0, e_1, e_2, e_3 be four unit reference points, so taken that the product $e_0e_1e_2e_3 = 1$; then

$$\begin{array}{llll} |e_0 = e_1e_2e_3, & ||e_0 = |e_1e_2e_3 = -e_0, & |e_0e_1 = e_2e_3, & ||e_0e_1 = |e_2e_3 = e_0e_1, \\ |e_1 = -e_2e_3e_0, & ||e_1 = -|e_2e_3e_0 = -e_1, & |e_0e_2 = e_3e_1, & ||e_0e_2 = |e_3e_1 = e_0e_2, \\ |e_2 = e_3e_0e_1, & ||e_2 = |e_3e_0e_1 = -e_2, & |e_0e_3 = e_1e_2, & ||e_0e_3 = |e_1e_2 = e_0e_3. \\ |e_3 = -e_0e_1e_2, & ||e_3 = -|e_0e_1e_2 = -e_3, & & \end{array}$$

Note that the complement of the complement of a reference *point* is the point with *negative* sign, but that the complement of the complement of a reference *line*, or edge of the reference tetraedron, is the line with *positive* sign. We have

$$|e_0 \cdot |e_1 \cdot |e_2 = -e_1e_2e_3 \cdot e_2e_3e_0 \cdot e_3e_0e_1 = (e_0e_1e_2e_3)^2 \cdot e_3 = e_3 = |(e_0e_1e_2),$$

which agrees with Art. 38, (*c*).

Let $p_1 = \Sigma_0^3 ke$, and $p_2 = \Sigma_0^3 le$; then

$$\left.\begin{aligned} |p_1 &= \Sigma_0^3 k|e = k_0e_1e_2e_3 + \text{etc.}, \\ &= k_0k_1k_2k_3\left(\frac{e_1}{k_1} - \frac{e_0}{k_0}\right)\left(\frac{e_2}{k_2} - \frac{e_0}{k_0}\right)\left(\frac{e_3}{k_3} - \frac{e_0}{k_0}\right) \end{aligned}\right\}, \quad . \quad . \quad (164)$$

so that the complement of any point is a point-plane-vector, or plane, and any plane may be expressed in terms of the four faces of the reference tetraedron.

From eq. (123) it follows that the complement of any point-vector or line is another point-vector. Again,

$$p_1|p_2 = \Sigma ke|\Sigma le = l_0m_0 + l_1m_1 + l_2m_2 + l_3m_3 = p_2|p_1. \quad . \quad . \quad (165)$$

Let $P_1 = |p_1$, $P_2 = |p_2$; therefore,

$$P_1|P_2 = |p_1 \cdot ||p_2 = -|p_1 \cdot p_2 = p_2|p_1 = p_1|p_2 = P_2|P_1. \quad . \quad (166)$$

Let $\quad L_1 = k_1e_0e_1 + k_2e_0e_2 + k_3e_0e_3 + k_1'e_2e_3 + k_2'e_3e_1 + k_3'e_1e_2$

and $\quad L_2 = l_1e_0e_1 + l_2e_0e_2 + \text{etc.}$;

then $\quad L_1|L_2 = k_1l_1 + k_2l_2 + k_3l_3 + k_1'l_1' + k_2'l_2' + k_3'l_3' = L_2|L_1. \quad (167)$

Also $\quad L_1L_2 = k_1l_1' + k_2l_2' + k_3l_3' + k_1'l_1 + k_2'l_2 + k_3'l_3. \quad . \quad (168)$

If $L_2 = L_1$, we have

$$L_1^2 = 2(k_1k_1' + k_2k_2' + k_3k_3'). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

But if L_1 is a *point-vector*, its square must be zero, and as the second member of (169) is not necessarily zero, it follows that L_1 and L_2 are not, in general, point-vectors; in fact, they are *screws*, as shown in Art. 61. We have then for the condition that L_1 shall be a *point-vector*,

$$k_1k_1' + k_2k_2' + k_3k_3' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

From (165), (166), and (167) it appears that a co-product in which the factors on opposite sides of the sign are of the *same order* is commutative about that sign, and always scalar. If the factors are not of the same order, this is not the case; for example,

$$P|L = -\,||(P|L) = -\,|(|P \cdot L) = -\,|(L|P). \quad . \quad . \quad (171)$$

Proceeding in a similar manner to that of Art. 43, it may be easily shown that $|p$ is the *anti-polar plane* of p, with reference to an ellipsoid so situated that each vertex of the reference tetraedron is the anti-pole of the opposite face. If the reference tetraedron is *regular*, and a be one of its equal edges, the ellipsoid becomes a sphere whose radius is easily found to be $\frac{a}{2\sqrt{2}}$.

With this geometric interpretation

$$p_1|p_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

causes p_1 to be in the anti-polar plane of p_2 with reference to the reciprocating ellipsoid, and *vice versâ;*

$$P_1|P_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

causes P_1 to pass through the anti-pole of P_2, and *vice versâ;*

$$L_1|L_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

causes L_1 to intersect the anti-polar line of L_2, and *vice versâ;*

$$p|P = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

makes p the anti-pole of P.

64. All the quantities we have to deal with in three-dimensional space — viz. scalars, points, lines, screws, and planes — are expressible in terms of fifteen quantities, which are all either the reference points or products of them of different orders; they are the four reference points; their six products, two by two, *i.e.* the edges of the reference tetraedron; their four products, three by three, or the faces of the reference tetraedron; and the product of the four, which is numerical unity. A multiplication table can be easily constructed similar to that in Art. 45. Considered as an algebra it appears that this system is *fifteen-fold.*

65. *Projections.* We have the same fundamental formula for projection as in Art. 46, viz.:

$$(\text{Projection on } B \text{ of } A, \text{ directed by } C) = \frac{B \cdot AC}{BC}, \qquad (176)$$

in which BC is scalar, while B and C separately are not. If we substitute in the equations of Art. 47 vectors for points, and plane-vectors for point-vectors, we shall obtain a set of corresponding formulæ for a vector system in solid space, as follows:

$$\rho = \frac{1}{\epsilon_1\epsilon_2\epsilon_3}(\epsilon_1 \cdot \rho\epsilon_2\epsilon_3 + \epsilon_2 \cdot \rho\epsilon_3\epsilon_1 + \epsilon_3 \cdot \rho\epsilon_1\epsilon_2), \qquad (177)$$

$$= \frac{1}{\epsilon_1\epsilon_2\epsilon_3}(|\epsilon_2\epsilon_3 \cdot \rho|\epsilon_1 + |\epsilon_3\epsilon_1 \cdot \rho|\epsilon_2 + |\epsilon_1\epsilon_2 \cdot \rho|\epsilon_3), \qquad (178)$$

$$= \frac{\epsilon_1 \cdot \rho\epsilon_2\epsilon_3}{\epsilon_1\epsilon_2\epsilon_3} + \frac{\epsilon_2\epsilon_3 \cdot \rho\epsilon_1}{\epsilon_2\epsilon_3\epsilon_1}. \qquad (179)$$

These are derived from eqs. (101), (102), and (103), and the last one gives ρ in terms of its projections on ϵ_1 parallel to $\epsilon_2\epsilon_3$, and on $\epsilon_2\epsilon_3$ parallel to ϵ_1. Also, from (104), (105), and (106), we have

$$\epsilon_1\epsilon_2 \cdot \epsilon_3\epsilon_4 = -\epsilon_1 \cdot \epsilon_2\epsilon_3\epsilon_4 + \epsilon_2 \cdot \epsilon_3\epsilon_4\epsilon_1 = \epsilon_3 \cdot \epsilon_4\epsilon_1\epsilon_2 - \epsilon_4 \cdot \epsilon_1\epsilon_2\epsilon_3, \qquad (180)$$

$$\rho = \iota_1 \cdot \rho|\iota_1 + \iota_2 \cdot \rho|\iota_2 + \iota_3 \cdot \rho|\iota_3, \qquad (181)$$

$$= \iota_1 \cdot \rho|\iota_1 + |\iota_1 \cdot \rho\iota_1 = \iota_1 \cdot \rho|\iota_1 + \iota_1\rho \cdot |\iota_1. \qquad (182)$$

If η, η_1, etc., are plane-vectors, we have from the equations of Art. 48,

$$\eta = \frac{1}{\eta_1\eta_2\eta_3}(\eta_1 \cdot \eta\eta_2\eta_3 + \eta_2 \cdot \eta\eta_3\eta_1 + \eta_3 \cdot \eta\eta_1\eta_2), \qquad (183)$$

$$= \frac{1}{\eta_1\eta_2\eta_3}(|\eta_1\eta_2 \cdot \eta|\eta_3 + |\eta_2\eta_3 \cdot \eta|\eta_1 + |\eta_3\eta_1 \cdot \eta|\eta_2), \qquad (184)$$

$$= \frac{\eta_1 \cdot \eta\eta_2\eta_3}{\eta_1\eta_2\eta_3} + \frac{\eta_2\eta_3 \cdot \eta\eta_1}{\eta_2\eta_3\eta_1}, \qquad (185)$$

$$\eta_1\eta_2 \cdot \eta_3\eta_4 = -\eta_1 \cdot \eta_2\eta_3\eta_4 + \eta_2 \cdot \eta_3\eta_4\eta_1 = \eta_3 \cdot \eta_4\eta_1\eta_2 - \eta_4 \cdot \eta_1\eta_2\eta_3, \qquad (186)$$

$$\eta = \iota_2\iota_3 \cdot \eta\iota_1 + \iota_3\iota_1 \cdot \eta\iota_2 + \iota_1\iota_2 \cdot \eta\iota_3, \qquad (187)$$

$$= \iota_2\iota_3 \cdot \eta\iota_1 + \iota_1 \cdot \eta\iota_2\iota_3. \qquad (188)$$

Finally, from the equations of Art. 49, we have

$$\epsilon_1\epsilon_2 \cdot |\epsilon_1' = -\epsilon_1 \cdot \epsilon_2|\epsilon_1' + \epsilon_2 \cdot \epsilon_1|\epsilon_1', \quad \ldots \quad (189)$$

$$\eta_1\eta_2 \cdot |\eta_1' = -\eta_1 \cdot \eta_2|\eta_1' + \eta_2 \cdot \eta_1|\eta_1', \quad \ldots \quad (190)$$

$$\epsilon_2|\epsilon_1'\epsilon_2' = |\epsilon_1' \cdot \epsilon_2|\epsilon_2' - |\epsilon_2' \cdot \epsilon_2|\epsilon_1', \quad \ldots \quad (191)$$

$$\eta_2|\eta_1'\eta_2' = |\eta_1' \cdot \eta_2|\eta_2' - |\eta_2' \cdot \eta_2|\eta_1', \quad \ldots \quad (192)$$

$$\epsilon_1\epsilon_2|\epsilon_1'\epsilon_2' = \begin{vmatrix} \epsilon_1|\epsilon_1' & \epsilon_1|\epsilon_2' \\ \epsilon_2|\epsilon_1' & \epsilon_2|\epsilon_2' \end{vmatrix}, \quad \ldots \quad (193)$$

$$\epsilon_1\epsilon_2|\epsilon_1\epsilon_2 = (\epsilon_1\epsilon_2)^2 = \epsilon_1^2\epsilon_2^2 - (\epsilon_1|\epsilon_2)^2, \quad \ldots \quad (194)$$

$$\epsilon_1\epsilon_2\epsilon_3 \cdot \epsilon_1'\epsilon_2'\epsilon_3' = \begin{vmatrix} \epsilon_1|\epsilon_1' & \epsilon_1|\epsilon_2' & \epsilon_1|\epsilon_3' \\ \epsilon_2|\epsilon_1' & \epsilon_2|\epsilon_2' & \epsilon_2|\epsilon_3' \\ \epsilon_3|\epsilon_1' & \epsilon_3|\epsilon_2' & \epsilon_3|\epsilon_3' \end{vmatrix}, \quad \ldots \quad (195)$$

$$\rho \cdot (\epsilon_1\epsilon_2)^2 = \epsilon_1 \cdot \rho\epsilon_2|\epsilon_1\epsilon_2 - \epsilon_2 \cdot \rho\epsilon_1|\epsilon_1\epsilon_2. \quad \ldots \quad (196)$$

From eq. (192) on, the plane-vector equations have not been written; to obtain them we have only to substitute plane-vectors for vectors in eqs. (193)–(196).

66. *Projections in a point system.* Write

$$p = x_0p_0 + x_1p_1 + x_2p_2 + x_3p_3$$

and multiply successively by $p_1p_2p_3$, $p_2p_3p_0$, etc., and we find

$$x_0 = \frac{pp_1p_2p_3}{p_0p_1p_2p_3}, \quad x_1 = \frac{pp_2p_3p_0}{p_1p_2p_3p_0}, \quad \text{etc.,}$$

whence

$$p = \frac{1}{p_0p_1p_2p_3}(p_0 \cdot pp_1p_2p_3 - p_1 \cdot pp_2p_3p_0 + p_2 \cdot pp_3p_0p_1 - p_3 \cdot pp_0p_1p_2), \quad \ldots \quad (197)$$

which gives p in terms of its projections on any four points. Write next

$$p = x_0p_0 + x_1p_1 + x_2p_2p_3 \cdot pp_0p_1,$$

and multiply successively by $p_1p_2p_3$, $p_2p_3p_0$, and $p_3p_0p_1$; the values of x_0 and x_1 will be the same as before, but that of x_2 will be $x_2 = \dfrac{1}{p_0p_1p_2p_3}$; hence

$$p = \frac{1}{p_0p_1p_2p_3}(p_0 \cdot pp_1p_2p_3 - p_1 \cdot pp_2p_3p_0 + p_2p_3 \cdot pp_0p_1), \quad (198)$$

which gives p as the sum of its projections on p_0, p_1, and the line p_2p_3. Similarly, we have for the expression of a point in terms of its projections on any point p_1 and any plane P_1,

$$p=\frac{p_1 \cdot pP_1}{p_1P_1}+\frac{P_1 \cdot pp_1}{P_1p_1}, \qquad (199)$$

and for the expression in terms of its projections on any two lines L_1 and L_2,

$$p=\frac{L_1 \cdot pL_2}{L_1L_2}+\frac{L_2 \cdot pL_1}{L_1L_2}. \qquad (200)$$

In equations (197)–(200) *the projected point p may be replaced by a screw S, or a plane P. We may also write in* (197) *and* (198) *planes for points throughout.*

Let $\quad P_1=p_2p_3p_0,\ \ P_2=p_3p_0p_1,$ etc.;

the projection of p on $|P_1$ directed by $|p_1$ is $\dfrac{|P_1 \cdot p|p_1}{P_1p_1}$, and we may write

$$p=\frac{|P_0 \cdot p|p_0}{P_0p_0}+\frac{|P_1 \cdot p|p_1}{P_1p_1}+\frac{|P_2 \cdot p|p_2}{P_2p_2}+\frac{|P_3 \cdot p|p_3}{P_3p_3}; \quad (201)$$

or, taking the complement of both sides,

$$|p=P=\frac{1}{p_0p_1p_2p_3}(P_0 \cdot p|p_0-P_1 \cdot p|p_1+P_2 \cdot p|p_2-P_3 \cdot p|p_3). \quad (202)$$

Let there be two planes, $P=p_1p_2p_3$ and $Q=q_1q_2q_3$; then

$$PQ=p_1p_2p_3 \cdot q_1q_2q_3=xq_2q_3+yq_3q_1+zq_1q_2, \text{ say.}$$

Multiply both sides into q_1p_1; then

$$p_1p_2p_3 \cdot q_1q_2q_3 \cdot q_1p_1=p_1p_2p_3q_1 \cdot q_1q_2q_3p_1=xq_2q_3q_1p_1,$$

and $\quad x=Pq_1.$

Similarly, $y=Pq_2$ and $z=Pq_3$.

Hence,

$$\left.\begin{aligned} PQ&=Pq_1 \cdot q_2q_3+Pq_2 \cdot q_3q_1+Pq_3 \cdot q_1q_2 \\ &=p_2p_3 \cdot p_1Q+p_3p_1 \cdot p_2Q+p_1p_2 \cdot p_3Q \end{aligned}\right\}, \qquad (203)$$

the second value being found in the same way as the first. This equation expresses the common line of P and Q in terms of three points of P or Q.

The terms of the second and third members of (203) may also be obtained by the model form of eq. (176). For instance, the projection of PQ on q_2q_3 directed by q_1p_1 is

$$\frac{q_2q_3 \cdot PQq_1p_1}{q_2q_3q_1p_1} = \frac{q_2q_3 \cdot (P \cdot q_1q_2q_3 \cdot q_1p_1)}{q_2q_3q_1p_1} = q_2q_3 \cdot Pq_1.$$

Again let P be as in (203) and $L = q_1q_2$; then

$$\left.\begin{aligned} PL = LP &= -q_1 \cdot q_2P + q_2 \cdot q_1P, \\ &= p_1 \cdot p_2p_3L + p_2 \cdot p_3p_1L + p_3 \cdot p_1p_2L \end{aligned}\right\}, \quad . \quad (204)$$

the results being found as in previous cases.

Multiply the first of (204) into Q; then

$$LPQ = \begin{vmatrix} q_1P & q_1Q \\ q_2P & q_2Q \end{vmatrix}. \quad . \quad . \quad . \quad (205)$$

Let Q and R be two planes intersecting in L, and substitute QR for L in (204); then, by (205),

$$\left.\begin{aligned} PQR &= p_1 \cdot p_2p_3QR + p_2 \cdot p_3p_1QR + p_3 \cdot p_1p_2QR \\ &= \begin{vmatrix} p_1 & p_1Q & p_1R \\ p_2 & p_2Q & p_2R \\ p_3 & p_3Q & p_3R \end{vmatrix} \end{aligned}\right\}. \quad . \quad (206)$$

If S be a fourth plane, multiply (206) into it, and we have

$$PQRS = \begin{vmatrix} p_1Q & p_1R & p_1S \\ p_2Q & p_2R & p_2S \\ p_3Q & p_3R & p_3S \end{vmatrix}. \quad . \quad . \quad . \quad (207)$$

Of course three other equal expressions could be written in terms of points in the other planes.

In (205) let $P = |p_1$ and $Q = |p_2$, and we have

$$q_1q_2|p_1p_2 = p_1p_2|q_1q_2 = \begin{vmatrix} p_1|q_1 & p_1|q_2 \\ p_2|q_1 & p_2|q_2 \end{vmatrix}, \quad . \quad . \quad . \quad (208)$$

as in eq. (117). In (206) let $Q = |q_1$ and $R = |q_2$; then

$$\left.\begin{aligned} p_1p_2p_3|q_1q_2 &= p_1 \cdot p_2p_3|q_1q_2 + p_2 \cdot p_3p_1|q_1q_2 + p_3 \cdot p_1p_2|q_1q_2 \\ &= \begin{vmatrix} p_1 & p_1|q_1 & p_1|q_2 \\ p_2 & p_2|q_1 & p_2|q_2 \\ p_3 & p_3|q_1 & p_3|q_2 \end{vmatrix} \end{aligned}\right\}. \quad (209)$$

Finally in (207) give Q and R the above values, and let $S = |q_3$; then

$$p_1p_2p_3|q_1q_2q_3 = \begin{vmatrix} p_1|q_1 & p_1|q_2 & p_1|q_3 \\ p_2|q_1 . & p_2|q_2 & p_2|q_3 \\ p_3|q_1 & p_3|q_2 & p_3|q_3 \end{vmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (210)$$

By eqs. (201) and (210) it may be shown that we have

$$p_1p_2p_3p_4 \cdot q_1q_2q_3q_4 = [p_1|q_1, p_2|q_2, p_3|q_3, p_4|q_4], \quad . \quad . \quad . \quad . \qquad (211)$$

in which the second member is a determinant formed on the plan of (210), of which the quantities given make up the first diagonal.

In all these equations points may be put for planes and planes for points without affecting their validity. Also, because of the homogeneity of the equations in all the points involved, these points may have any weights we please.

67. *Normal form of the screw.* Returning now to the subject of Art. 61, we propose to show that by properly choosing the position of the line part of S, the screw can be reduced to a line and a *perpendicular* plane-vector. The *complement* as used in treating screws will refer to a *unit normal vector system*, so that $|\epsilon$ will be a plane-vector $\perp$ to ϵ and having the same tensor. We have, from Art. 61,

$$S \equiv e_0\Sigma\epsilon + \Sigma(p - e_0)\epsilon \equiv q\Sigma\epsilon - (q - e_0)\Sigma\epsilon + \Sigma(p - e_0)\epsilon. \quad (212)$$

Write, for convenience,

$$\Sigma\epsilon = \alpha, \; q - e_0 = \rho, \text{ and } \Sigma(p - e_0)\epsilon = |\beta;$$

$$\therefore S \equiv q\alpha - \rho\alpha + |\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (213)$$

The condition that the plane-vector $|\beta - \rho\alpha$ shall be perpendicular to α is

$$(|\beta - \rho\alpha)|\alpha = 0 = |\beta\alpha - \rho\alpha|\alpha = |\alpha \cdot \rho\alpha - |\alpha\beta,$$

whence $$\frac{|\alpha \cdot \rho\alpha}{\alpha^2} = \frac{|\alpha\beta}{\alpha^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (214)$$

Comparing the first member of (214) with eq. (176), it appears that it is the orthogonal projection of ρ, or $q - e_0$, on a plane $\perp$ to α. Hence the second member gives the length and direction of this projection in terms of known quantities; that is, it is the vector perpendicular between the lines $e_0\Sigma\epsilon$ and $q\Sigma\epsilon$. We have by (189) and (214)

$$\frac{\alpha(|\alpha\cdot\rho\alpha)}{a^2}=\frac{\alpha(\alpha\rho\cdot|\alpha)}{a^2}=\frac{\alpha(-\alpha\cdot\rho|\alpha+\rho\cdot a^2)}{a^2}$$

$$=\alpha\rho=\frac{\alpha|\alpha\beta}{a^2}=\frac{|\alpha\cdot\alpha|\beta}{a^2}-|\beta.$$

Substituting this value of $\alpha\rho$ in (213), it becomes

$$S\equiv q\alpha+\frac{\alpha|\beta}{a^2}\cdot|\alpha\equiv q\Sigma\epsilon+\frac{\Sigma\epsilon\Sigma(p-e_0)\epsilon}{(\Sigma\epsilon)^2}\cdot|\Sigma\epsilon, \quad . \quad . \quad . \quad (215)$$

and the required reduction is accomplished.

68. *Product of two screws.* Let the screws be

$$S_1=e_1\epsilon_1+a_1\eta_1=e_1\epsilon_1+a_1|\epsilon_1 \text{ and } S_2=e_2\epsilon_2+a_2\eta_2=e_2\epsilon_2+a_2|\epsilon_2,$$

in which a_1 and a_2 are scalars, called by Ball the *pitches* of the respective screws. Then

$$S_1S_2=(e_1\epsilon_1+a_1\eta_1)(e_2\epsilon_2+a_2\eta_2)=e_1\epsilon_1e_2\epsilon_2+a_2e_1\epsilon_1\eta_2+a_1e_2\epsilon_2\eta_1+a_1a_2\eta_1\eta_2.$$

Now this is a *progressive* product, each term being the product of two lines, and scalar; the two lines in the last term being in the plane at ∞, they intersect, and their product is therefore zero. (See also Art. 23.) Further, by eq. (45),

$$e_1\epsilon_1\eta_2=\epsilon_1\eta_2=\epsilon_1|\epsilon_2=\epsilon_2|\epsilon_1=\epsilon_2\eta_1=e_2\epsilon_2\eta_1;$$

hence the product becomes

$$S_1S_2=e_1\epsilon_1e_2\epsilon_2+(a_1+a_2)\epsilon_1|\epsilon_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

If $S_2=S_1$, we have

$$S_1^2=2a_1\epsilon_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217)$$

If S_1 reduce to a *line*, a_1 must be zero, as appears from the value at the beginning of the article; hence

$$S^2=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

is the condition that a screw shall reduce to a line.

69. The product $pS = pe\epsilon + ap\eta$ is evidently a plane through the common line of the planes $pe\epsilon$ and $p\eta$. We wish to show that we have

$$Sp_1 \cdot Sp_2 \equiv Sp_1p_2 \cdot S - ap_1p_2 \cdot \epsilon^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (219)$$

Take two lines pp' and qq' whose sum is S; then

$$\begin{aligned} Sp_1 \cdot Sp_2 &\equiv (pp' + qq')p_1 \cdot (pp' + qq')p_2 \\ &\equiv pp'p_1 \cdot pp'p_2 + pp'p_1 \cdot qq'p_2 + qq'p_1 \cdot pp'p_2 + qq'p_1 \cdot qq'p_2 \\ &\equiv pp'p_1p_2 \cdot pp' + qq'p_1p_2 \cdot qq' + pp' \cdot p_1qq'p_2 + p'p_1 \cdot pqq'p_2 \\ &\quad + p_1p \cdot p'qq'p_2 + qq' \cdot p_1pp'p_2 + q'p_1 \cdot qpp'p_2 + p_1q \cdot q'pp'p_2 \\ &\equiv pp'p_1p_2 \cdot S + qq'p_1p_2 \cdot S \\ &\quad + p_1(p \cdot p'qq'p_2 - p' \cdot pqq'p_2 + q \cdot q'pp'p_2 - q' \cdot qpp'p_2) \\ &\equiv Sp_1p_2 \cdot S - p_1p_2 \cdot pp'qq' \\ &\equiv Sp_1p_2 \cdot S - p_1p_2 \cdot ae\epsilon|\epsilon \\ &\equiv Sp_1p_2 \cdot S - ap_1p_2 \cdot \epsilon^2. \end{aligned}$$

In the above we have used eqs. (197) and (204), and also the fact that $pp'qq'$ is constant whatever the lines may be so long as $pp' + qq' = S$, for $S^2 = 2pp'qq' = 2a\epsilon^2$, by (217).

We easily find in the same way,

$$S \cdot pS \equiv a\epsilon^2 p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220)$$

$$Sp_1 \cdot Sp_2 \cdot Sp_3 \equiv a\epsilon^2(p_1 \cdot p_2p_3S + p_2 \cdot p_3p_1S + p_3 \cdot p_1p_2S). \quad (221)$$

In all these equations planes may be substituted for points.

70. We give here the Quaternion equivalents of some of our vector expressions; of course there are no such equivalents for point expressions.

$$\epsilon_1|\epsilon_2 = -S\epsilon_1\epsilon_2, \qquad |\epsilon_1\epsilon_2 = V\epsilon_1\epsilon_2,$$

$$\epsilon_1\epsilon_2\epsilon_3 = -S\epsilon_1\epsilon_2\epsilon_3, \qquad \epsilon_2\epsilon_3|\epsilon_1 = -V\epsilon_1 V\epsilon_2\epsilon_3,$$

$$\epsilon_1\epsilon_2|\epsilon_3\epsilon_4 = -S \cdot \epsilon_1\epsilon_2 V\epsilon_3\epsilon_4, \qquad \epsilon_1\epsilon_2 \cdot \epsilon_3\epsilon_4 = V \cdot V\epsilon_1\epsilon_2 V\epsilon_3\epsilon_4,$$

$$\epsilon_1\epsilon_2 \cdot \epsilon_3\epsilon_4 \cdot \epsilon_5\epsilon_6 = -S \cdot V\epsilon_1\epsilon_2 V\epsilon_3\epsilon_4 V\epsilon_5\epsilon_6.$$

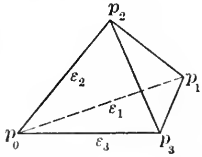
The superior simplicity of Grassmann's notation is evident at a glance, and the interpretation of the expressions is as much simplified as their form.

71. EXERCISES. — (1) If η_1 and η_2 are two plane-vectors, and P_1 and P_2 are two point-plane-vectors, show that the bisectors of the diedral angles between them are

$$\eta_1 T\eta_2 \pm \eta_2 T\eta_1 \text{ and } P_1 TP_2 \pm P_2 TP_1$$

respectively. If $\eta_1 = \epsilon_1\epsilon_2$, $\eta_2 = \epsilon_1\epsilon_3$, $P_1 = p_0p_1p_2$, and $P_2 = p_0p_1p_3$, these become

$$\epsilon_1(\epsilon_2 T\epsilon_1\epsilon_3 \pm \epsilon_3 T\epsilon_1\epsilon_2) \text{ and } p_0p_1(p_2 Tp_0p_1p_3 \pm p_3 Tp_0p_1p_2);$$



or, if we write A_0, A_1, A_2, A_3 for the double areas of the faces of the tetraedron opposite p_0, p_1, etc., the expressions become

$$\epsilon_1(A_2\epsilon_2 \pm A_3\epsilon_3) \text{ and } p_0p_1(A_2p_2 \pm A_3p_3).$$

(2) Show that $p_0(A_1p_1 \pm A_2p_2 \pm A_3p_3)$ and $A_1\epsilon_1 \pm A_2\epsilon_2 \pm A_3\epsilon_3$ are trisectors of the triedral angle at p_0; that is, that the first expression is the common line of the bisecting planes through p_0p_1, p_0p_2, and p_0p_3, while the second is ‖ to it.

(3) The trisector of a triedral angle of a tetraedron pierces the opposite face in a point such that, if it be joined by right lines to the vertices of the tetraedron that are in this face, the triangles thus formed are proportional to the adjacent faces.

(4) The bisecting plane through one edge of a tetraedron divides the opposite edge into segments which are proportional to the adjacent faces.

(5) The twelve bisecting planes of the diedral angles of a tetraedron pass six by six through eight points which are the centers of the inscribed and escribed spheres.

(6) The twelve points in which the edges of a a tetraedron are cut by the bisecting planes of the opposite diedral angles fix eight planes, each of which passes through six of them.

(7) Using A_0, A_1, etc., as in exercise (1), show that

$$A_0^2 = A_1^2 + A_2^2 + A_3^2 - 2A_2A_3\cos\angle\frac{A_3}{A_2} - 2A_3A_1\cos\angle\frac{A_1}{A_3} - 2A_1A_2\cos\angle\frac{A_2}{A_1}.$$

(8) Show that if e_0, e_1, e_2, e_3 are non-coplanar points, and e_0', e_1', e_2', e_3' divide the lines e_0e_1, e_1e_2, e_2e_3, and e_3e_0, so that

$$Te_0e_0' \cdot Te_1e_1' \cdot Te_2e_2' \cdot Te_3e_3' = Te_0'e_1 \cdot Te_1'e_2 \cdot Te_2'e_3 \cdot Te_3'e_0;$$

then will e_0', e_1', e_2', e_3' be co-planar.

(9) If a plane cut the faces of a tetraedron $e_0e_1e_2e_3$ in the lines L_0, L_1, L_2, L_3, L_0 lying in the face opposite to e_0, etc., then we shall have the relations

$$\begin{aligned} e_0L_1 \cdot e_1L_2 \cdot e_2L_3 \cdot e_3L_0 &= e_0L_2 \cdot e_1L_3 \cdot e_2L_0 \cdot e_3L_1 \\ &= e_0L_3 \cdot e_1L_0 \cdot e_2L_1 \cdot e_3L_2. \end{aligned}$$

(10) By interchanging planes and points derive the reciprocal propositions to (8) and (9).

(11) If two tetraedra $e_0e_1e_2e_3$ and $e_0'e_1'e_2'e_3'$ are so related that the right lines through corresponding vertices all meet in one point, then will the corresponding faces cut each other in four coplanar right lines.

(12) If n_1, n_2, etc., are scalars, and L_1, L_2, etc., lines, and we have the relation

$$n_1L_1 + n_2L_2 + n_3L_3 + n_4L_4 \equiv 0,$$

then any straight line that cuts three of these lines will also cut the fourth, and, consequently, L_1, L_2, L_3, L_4 are generators of the same system of a skew quadric.

(13) The perpendiculars from the vertices of a tetraedron on the opposite faces are generators of the same system of a skew quadric.

(14) The six planes through the middle points of the edges of a tetraedron $\perp$ to the respective edges meet in one point. If ϵ_1, ϵ_2, ϵ_3 are the vector edges of the tetraedron drawn outward from e_0, and ρ is the vector from e_0 to the common point of the planes, then

$$\rho = \frac{1}{2\,\epsilon_1\epsilon_2\epsilon_3}\left(|\epsilon_2\epsilon_3 \cdot \epsilon_1^2 + |\epsilon_3\epsilon_1 \cdot \epsilon_2^2 + |\epsilon_1\epsilon_2 \cdot \epsilon_3^2\right).$$

(15) The lines joining the corresponding vertices of a tetraedron and its complementary tetraedron are generators of the same system of a skew quadric. State the reciprocal proposition.

(16) The center of gravity of the *faces* of a tetraedron coincides with the center of the sphere inscribed within the tetraedron formed by joining by right lines the mean points of the faces of the first tetraedron.

(17) There are given six lines L_1, L_2, L_3, e_1e_1', e_2e_2', e_3e_3'; planes pass through L_1, L_2, L_3, and cut e_1e_1', e_2e_2', e_3e_3' respectively, in points which move along these lines uniformly at rates v_1, v_2, v_3; find the locus of the common point of these planes.

(18) If $P_1 = \Sigma_0^3 l|e$, $P_2 = \Sigma_0^3 m|e$, $P_3 = \Sigma_0^3 n|e$, show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ l_0 & l_1 & l_2 & l_3 \\ m_0 & m_1 & m_2 & m_3 \\ n_0 & n_1 & n_2 & n_3 \end{vmatrix} = 0$$

is the condition that they shall have a common point at ∞; that is, be all $\parallel$ to one line.

(19) Show that the condition that P_1 and P_2 of the last exercise shall be parallel, or have a common line at ∞, is

$$\begin{vmatrix} 1 & 1 & 1 \\ l_0 & l_1 & l_2 \\ m_0 & m_1 & m_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ l_2 & l_3 & l_0 \\ m_2 & m_3 & m_0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ l_3 & l_0 & l_1 \\ m_3 & m_0 & m_1 \end{vmatrix} = 0.$$

(20) Show that, if P_1 and P_2 be parallel, then $|P_1P_2$ is a line through the mean point of the reference tetraedron.

(21) If any plane be drawn through the middle points of two opposite edges of a tetraedron, it will divide the volume of the tetraedron into two equal parts.

(22) Show that $p_1L_1L_2L_3p_2 = -p_2L_3L_2L_1p_1$, and

$$L_1P_1L_2P_2L_3 = -L_3P_2L_2P_1L_1.$$

(23) Prove, by eq. (159), the relation

$$(l_1^2 + l_2^2 + l_3^2)(m_1^2 + m_2^2 + m_3^2) > (l_1 m_1 + l_2 m_2 + l_3 m_3)^2.$$

(24) Prove, by eq. (193), the formula of spherical trigonometry $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

(25) Show that ι, $|\iota\epsilon$, and $\iota\epsilon \cdot |\iota$ are three mutually perpendicular vectors, no matter what the directions of ι and ϵ may be.

(26) Show, by eq. (211), how to express the product of two determinants of the fourth order, as a determinant of the same order.

(27) Show that $SP = p_1 \cdot p_2 p_3 S + p_2 \cdot p_3 p_1 S + p_3 \cdot p_1 p_2 S$, if $P = p_1 p_2 p_3$; and, hence, that $Sp_1 \cdot Sp_2 \cdot Sp_3 = a\epsilon^2 \cdot S \cdot p_1 p_2 p_3$.

CHAPTER III.

APPLICATIONS TO PLANE GEOMETRY.

72. In the present chapter, since plane space is under consideration, we shall have constantly:

The product of two vectors a scalar quantity;
The product of three points a scalar quantity.

Also, if e_0, e_1, e_2 are reference points, and

$$\epsilon_1 = e_1 - e_0, \quad \epsilon_2 = e_2 - e_0,$$

we shall have the relation

$$e_0e_1e_2 = e_0(e_1 - e_0)(e_2 - e_0) = e_0\epsilon_1\epsilon_2 = \epsilon_1\epsilon_2 = e_1e_2 + e_2e_0 + e_0e_1 = 1;$$

and furthermore, if p be any unit point at a finite distance,

$$p|(e_0 + e_1 + e_2) = p(e_1e_2 + e_2e_0 + e_0e_1) = p\epsilon_1\epsilon_2 = \epsilon_1\epsilon_2 = 1. \quad (222)$$

In this equation of course $p\epsilon_1\epsilon_2$ is a *combinatory* product of the point and two vectors, and therefore not the same as p times the scalar $\epsilon_1\epsilon_2$. That $p|(e_0 + e_1 + e_2) = 1$ appears also from eq. (105), viz.:

$$p = e_0 \cdot p|e_0 + e_1 \cdot p|e_1 + e_2 \cdot p|e_2,$$

which requires the sum of the coefficients of the e's to be unity.

We shall have frequent occasion to use the mean point of the reference triangle, and shall designate it by $\bar{e}$, so that

$$3\bar{e} = e_0 + e_1 + e_2, \quad \ldots \quad (223)$$

and eq. (222) becomes

$$3p|\bar{e} = 1. \quad \ldots \quad (224)$$

The equations of curves in plane space may appear under any one of the six following forms: —

Non-scalar equations. { Expressed in points. / Expressed in lines. / Expressed in vectors.

Scalar equations. { Expressed in points. / Expressed in lines. / Expressed in vectors.

73. The non-scalar equation

$$p = ze_0 + xe_1 + ye_2 = e_0 + x(e_1 - e_0) + y(e_2 - e_0), \qquad (225)$$

the third member being obtained by the elimination of z, by the aid of the relation $x + y + z = 1$, which always exists because we use only *unit* points, may be called the *equation of our plane space;* for by giving suitable values to the scalar variables p may be moved to any point of this space. The corresponding scalar equation is

$$e_0e_1e_2p = 0, \quad \ldots \ldots \ldots \quad (226)$$

which is simply the condition that p shall lie in the plane $e_0e_1e_2$.

If a single condition be given between the scalar variables in eq. (225), such as $f(x, y, z) = 0$, or $f(x, y) = 0$, then p will vary according to a fixed law, and will therefore move on some curve.

Let $L = |p$; then

$$L = z|e_0 + x|e_1 + y|e_2 \quad \ldots \ldots \ldots \quad (227)$$

may be any line in the plane $e_0e_1e_2$; but if a relation exist, as above, between z, x, and y, then L will move according to some fixed law, and will envelope a curve.

Writing in (225), $p - e_0 = \rho$, $e_1 - e_0 = \epsilon_1$, $e_2 - e_0 = \epsilon_2$, we have

$$\rho = x\epsilon_1 + y\epsilon_2, \quad \ldots \ldots \ldots \quad (228)$$

a vector equation which will represent a curve when a relation exists between x and y, ρ being regarded as always drawn outwards from a fixed origin.

74. The equations

$$\left.\begin{array}{l} p = ze_0 + xe_1 + ye_2 \\ lx + my + nz = 0 \end{array}\right\}, \quad . \ . \ . \ . \ . \ . \ . \ . \qquad (229)$$

taken together, represent a right line; for, eliminating z, we have

$$p = \frac{1}{n}[x(ne_1 - le_0) + y(ne_2 - me_0)]; \quad . \qquad (230)$$

so that p lies on the right line through the two points $ne_1 - le_0$ and $ne_2 - me_0$. Multiplying by $(ne_1 - le_0)(ne_2 - me_0)$, we obtain the corresponding scalar equation

$$\left.\begin{array}{ll} & (ne_1 - le_0)(ne_2 - me_0)p = 0, \\ \text{or} & p(ne_1e_2 + le_2e_0 + me_0e_1) = 0, \\ \text{or} & p|(ne_0 + le_1 + me_2) = 0. \end{array}\right\} . \ . \ . \ . \qquad (231)$$

If $l = m = n$, this equation becomes that of the line at ∞, viz.:

$$p|\bar{e} = 0; \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (232)$$

for the points $ne_1 - le_0$ and $ne_2 - me_0$, in which the line cuts the reference lines e_0e_1 and e_0e_2, are in this case at ∞.

The equation of a line through any two points, p_1 and p_2, may be written

$$p = xp_1 + (1 - x)p_2 = p_2 + x(p_1 - p_2), \qquad (233)$$

and the corresponding scalar equation, found by multiplying by p_1p_2, is

$$pp_1p_2 = 0. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (234)$$

In general, the equation

$$pL = 0 \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \qquad (235)$$

is the scalar *point* equation of a *line*, if L be constant and p vary, and the scalar *line* equation of a *point*, if p be constant and L vary. Thus, the complementary equations to (231), (232), and (234) are

$$L(ne_0 + le_1 + me_2) = 0, \quad \ldots \quad (236)$$

$$L\bar{e} = 0, \quad \ldots \quad (237)$$

$$LL_1L_2 = 0, \quad \ldots \quad (238)$$

which are line equations of the points $ne_0 + le_1 + me_2$, $\bar{e}$, and L_1L_2, respectively.

If we have such an equation as $pp_1p_2 = C$, C being a scalar constant, it may always be rendered homogeneous in p; for, by eq. (224), $3p|\bar{e} = 1$, so that we may write

$$pp_1p_2 = 3\,Cp|\bar{e},$$

or

$$p(p_1p_2 - 3\,C \cdot |\bar{e}) = 0, \quad \ldots \quad (239)$$

which is a line through the common point of p_1p_2 and the line at ∞, and is therefore parallel to p_1p_2.

For vector equations of right lines we have

$$\left.\begin{array}{l} \rho = \epsilon_1 + x\epsilon_2 \\ (\rho - \epsilon_1)\epsilon_2 = 0 \end{array}\right\}, \quad \ldots \quad (240)$$

and

the scalar and non-scalar forms of the equation of a line through the end of ϵ_1 parallel to ϵ_2.

Also

$$\left.\begin{array}{l} \rho = \epsilon_1 + x(\epsilon_2 - \epsilon_1), \\ (\rho - \epsilon_1)(\epsilon_2 - \epsilon_1) = 0, \text{ or } \rho(\epsilon_2 - \epsilon_1) = \epsilon_1\epsilon_2, \end{array}\right\} \quad (241)$$

and

for the two forms of the vector equation of a line through the ends of ϵ_1 and ϵ_2 drawn outwards from the origin.

The equation

$$\rho\epsilon = C \quad \ldots \quad (242)$$

is that of some line *parallel* to ϵ, while

$$\rho|\epsilon = C \quad \ldots \quad (243)$$

is that of some line *perpendicular* to ϵ, as is easily seen from the meanings assigned to $\rho\epsilon$ and $\rho|\epsilon$ in Chap. II.

75. *Transformation of scalar equations from a point system to a vector system, and vice versâ.* Take e_0 for the origin of vectors, and write $p - e_0 = \rho$, $p_1 - e_0 = \epsilon_1$, etc., the difference between each fixed point and the origin being equal to some constant vector. Thus to transform (234) to a vector system, we have

$$pp_1p_2 = (e_0 + \rho)(e_0 + \epsilon_1)(e_0 + \epsilon_2) = e_0(\rho\epsilon_1 + \epsilon_1\epsilon_2 + \epsilon_2\rho) = 0.$$

The term $\rho\epsilon_1\epsilon_2$ vanishes by Art. 21, being the planimetric product of three vectors; also $e_0\rho\epsilon_1 = \rho\epsilon_1$, etc., hence the equation becomes $\rho(\epsilon_2 - \epsilon_1) = \epsilon_1\epsilon_2$, the same as (241). Since in changing from a point to a vector system we have dropped the point e_0 from each term, it follows that in the reverse change we must first multiply each term by some fixed point. Thus to change $(\rho - \epsilon_1)\epsilon_2 = 0$ to a point equation we have

$$(\rho - \epsilon_1)\epsilon_2 = e_0(p - p_1)(p_2 - e_0) = e_0(p - p_1)p_2 = 0.$$

76. EXERCISES. — (1) Find the equations of right lines satisfying the following conditions: —

Passing through p_1 and parallel to L_1;

Passing through p_1 and parallel to ϵ;

Passing through the common point of two right lines and having a given direction;

Passing through the common point of two right lines, and also through the common point of two other right lines;

Passing through the end of ϵ_1 perpendicular to ϵ_2.

Ans. $(p - p_1)L_1 = 0,\ pp_1\epsilon = 0,\ p\epsilon L_1L_2 = 0,$
$p(L_1L_2)(L_3L_4) = 0,\ (\rho - \epsilon_1)|\epsilon_2 = 0.$

(2) Interpret the equations obtained by putting lines for points, and points for lines, in the first four results of the previous exercise.

(3) Find the common points of the following pairs of right lines,

$$\begin{Bmatrix} p|p_1 = 0 \\ p|p_2 = 0 \end{Bmatrix}, \quad \begin{Bmatrix} pp_1p_2 = 0 \\ pq_1q_2 = 0 \end{Bmatrix}, \quad \begin{Bmatrix} \rho\epsilon_1 = C_1 \\ \rho\epsilon_2 = C_2 \end{Bmatrix}, \quad \begin{Bmatrix} \rho|\epsilon_1 = C_1 \\ \rho|\epsilon_2 = C_2 \end{Bmatrix}.$$

(4) Find the condition that the three lines $p|p_1 = 0$, $p|p_2 = 0$, $p|p_3 = 0$ shall have a common point; also the lines $\rho|\epsilon_1 = C_1$, $\rho|\epsilon_2 = C_2$, $\rho|\epsilon_3 = C_3$.

Ans. $p_1p_2p_3 = 0$, and $C_1\epsilon_2\epsilon_3 + C_2\epsilon_3\epsilon_1 + C_3\epsilon_1\epsilon_2 = 0.$

(5) Show that the common point of the two lines $pp_1p_2 = C_3$, $pp_2p_3 = C_1$ is $p_2 + \frac{1}{p_1p_2p_3}[C_3(p_3 - p_2) + C_1(p_1 - p_2)]$.

(6) Show that, if $p_1p_2p_3 = C_1 + C_2 + C_3$, then the three lines $pp_1p_2 = C_3$, $pp_2p_3 = C_1$, $pp_3p_1 = C_2$ have a common point.

(7) Show that if the equations of three lines, on being multiplied by any constants and added, *vanish identically*, that is, for all values of p or ρ, then the lines have a common point. Show also that the results in exercises (4) and (6) are in accordance with this.

(8) Find the condition that the three points whose line equations are $L|L_1 = 0$, $L|L_2 = 0$, $L|L_3 = 0$ shall be collinear.

(9) Show that the perpendiculars from the point e on the lines whose equations are $pL_1 = 0$, $pL_1 = C$, $(p - p_3)p_1p_2 = 0$, are respectively of the length $\frac{eL_1}{TL_1}$, $\frac{eL_1 - C}{TL_1}$, $\frac{p_1p_2(e - p_3)}{Tp_1p_2}$.

(10) Find the vector perpendiculars from the origin on the lines $\epsilon\rho = C$ and $\epsilon|\rho = C$. Also from the end of ϵ' on the same lines.

Ans. $\frac{C}{\epsilon^2} \cdot |\epsilon$, $\frac{C\epsilon}{\epsilon^2}$, $\frac{C - \epsilon\epsilon'}{\epsilon^2} \cdot |\epsilon$, $\frac{C - \epsilon|\epsilon'}{\epsilon^2} \cdot \epsilon$.

(11) If $L_1 = p_1\epsilon_1$, $L_2 = p_2\epsilon_2$, etc., show that $T\Sigma L = T\Sigma\epsilon$. We have, by Art. 61, since the lines are all in one plane, $\Sigma L = (e_0 + \alpha)\Sigma\epsilon$; but $e_0 + \alpha$ is a unit point, hence $T\Sigma L = T\Sigma\epsilon$.

(12) Show that $T(L + 3C|\bar{e}) = TL$.

77. If L_1 and L_2 are two straight lines, then the equation $L_1p \cdot L_2p = 0$ represents the two lines simultaneously, for it is satisfied whenever p lies on either of the lines. The equation

$$L_1p \cdot L_2p = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (244)$$

represents a locus that evidently differs less from being the two lines L_1 and L_2, the smaller C is; also, when p is indefinitely far from L_1, it is indefinitely near to L_2, and *vice versâ*. The locus is of the second order; *i.e.* it is cut in two points by a right line; for let $p = e + x\epsilon$ be the equation of some line; then, substituting this value of p in (244), we have a quadratic in x determining two points in which the line cuts the curve. The locus must therefore be a hyperbola. If C be *positive*, L_1p and L_2p must have like signs; hence p must be on the same side of L_1 and of L_2, *i.e.* in the *exterior* angle, while, if C be *negative*, p must be in the *interior* angle. Thus for the same numerical value opposite signs of C correspond to a primary and conjugate hyperbola.

The complementary equation

$$p_1L \cdot p_2L = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (245)$$

represents the reciprocal curve to (244). When $C = 0$, it represents the two points p_1 and p_2 and their connecting line; for it is satisfied when L passes through p_1 or p_2, or through both. When C is positive, p_1L and p_2L must have like signs, and hence L must not pass between p_1 and p_2; if C be negative, L must always pass between p_1 and p_2. The curve enveloped by L is of the second class, *i.e.* two tangents can be drawn from any point; it is therefore a conic.

78. It is easily seen, as in the last article, that the order of the curve represented by any scalar equation in terms of p, *i.e.* the number of points in which it can be cut by a right line, is simply the degree in p of the term of highest degree in the equation.

The equation

$$AL_2p \cdot L_3p + BL_3p \cdot L_1p + CL_1p \cdot L_2p = 0, \quad . \quad (246)$$

in which A, B, C are scalar coefficients, represents a curve of the second order passing through the points L_1L_2, L_2L_3, L_3L_1; for each term is of the second degree in p, and the equation is satisfied when p is on any two of the lines simultaneously. The complementary equation

$$Ap_2L \cdot p_3L + Bp_3L \cdot p_1L + Cp_1L \cdot p_2L = 0 \quad . \quad . \quad (247)$$

causes L to envelop a curve of the second class tangent to the three lines p_1p_2, p_2p_3, p_3p_1; for it is satisfied when L passes through any two of these points simultaneously. As an exercise let the student interpret the following equations, k being a scalar constant:

$$p_1p_2p \cdot p_3p_4p = kp_1p_4p \cdot p_2p_3p,$$

$$\frac{p_1p_2p \cdot p_3p_4p}{p_1p_2p_5 \cdot p_3p_4p_5} = \frac{p_1p_4p \cdot p_2p_3p}{p_1p_4p_5 \cdot p_2p_3p_5},$$

$$L_1L_2L \cdot L_3L_4L = kL_1L_4L \cdot L_2L_3L,$$

$$\frac{L_1L_2L \cdot L_3L_4L}{L_1L_2L_5 \cdot L_3L_4L_5} = \frac{L_1L_4L \cdot L_2L_3L}{L_1L_4L_5 \cdot L_2L_3L_5}.$$

79. *Differentiation.* Before proceeding to the general treatment of equations of the second degree in ρ and p, we will consider the question of differentiation as applied in this calculus.

Let

$$p = ze_0 + xe_1 + ye_2 = e_0 + x(e_1 - e_0) + y(e_2 - e_0) = e_0 + x\epsilon_1 + y\epsilon_2,$$

which implies that $x + y + z = 1$, as we always assume.

If p move from point to point, it is a function of the time, as are also x and y; hence

$$\frac{dp}{dt} = \epsilon_1\frac{dx}{dt} + \epsilon_2\frac{dy}{dt}. \quad . \quad . \quad . \quad . \quad (248)$$

Thus the differential coefficient of a *point* is a vector. Also, since $\rho = p - e_0$,

$$\frac{d\rho}{dt} = \frac{dp}{dt}. \quad . \quad . \quad . \quad . \quad (249)$$

If $T\epsilon_1 = T\epsilon_2 = 1$, $\epsilon_1|\epsilon_2 = 0$, and a relation subsist between x and y such as $f(x, y) = 0$, so that p moves along some curve,

then $$\left(\frac{dp}{dx}\right)^2 = \left(\epsilon_1 + \epsilon_2\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2,$$

or $$T\left(\frac{dp}{dx}\right) = \frac{ds}{dx}, \quad . \quad . \quad . \quad . \quad (250)$$

whence $$T\frac{dp}{ds} = 1, \quad . \quad . \quad . \quad . \quad (251)$$

Let $L = |p = |e_0 + xe_2(e_0 + e_1) - ye_1(e_2 + e_0)$, so that L is subjected to the same condition, $z + x + y = 1$, that p is, which, however, affects only its *length*, and not in any way its *position;*

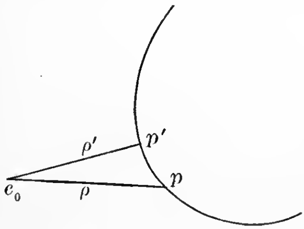
then $$\frac{dL}{dx} = e_2(e_0 + e_1) - \frac{dy}{dx}e_1(e_2 + e_0). \quad . \quad . \quad . \quad . \quad (252)$$

Multiply $\frac{dL}{dx}$ by $\bar{e}$, and we have

$$\bar{e}\frac{dL}{dx} = e_0e_2e_1 + e_1e_2e_0 - \frac{dy}{dx}(e_0e_1e_2 + e_2e_1e_0) = 0; \quad (253)$$

hence $\frac{dL}{dx}$ is a line through the mean point of the reference triangle.

By the figure it is evident that $p - p'$ or $\rho - \rho'$ is a chord of the curve which is the locus of p; as p' approaches p, $p - p'$ approaches the tangent at p in direction, and at the limit has this direction; hence



$$\text{limit of } \frac{p - p'}{T(p - p')} = \frac{dp}{Tdp} = \frac{dp}{ds} \quad . \quad . \quad . \quad . \quad (254)$$

is a *unit vector along the tangent at* p.

Similarly, if L envelops some curve, $\frac{dL}{TdL}$ is the limit of $\frac{L - L'}{T(L - L')}$ as L' approaches L. But $L - L'$ is always a line through the common point of L and L', which ultimately becomes the point of contact of L with the curve. Hence $\frac{dL}{TdL}$ is a unit line through the point of contact of L and the mean point $\bar{e}$.

If a *scalar* equation in p, L, or ρ be differentiated, it will necessarily become a homogeneous, linear function of dp, dL, or $d\rho$, and thus independent of the length of dp, dL, or $d\rho$; we may therefore, if we please, regard these *not* as infinitesimals, but as finite in length. Take for instance the equation

$$p|e_0 \cdot p|e_1 + pL_1 + pL_2 \cdot pL_3 \cdot pL_4 = 0;$$

differentiating, we have

$$dp|e_0 \cdot p|e_1 + p|e_0 \cdot dp|e_1 + dpL_1 + dpL_2 \cdot pL_3 \cdot pL_4 + pL_2 \cdot dpL_3 \cdot pL_4 + pL_2 \cdot pL_3 \cdot dpL_4 = 0;$$

and, as dp appears once, and only once, in each term, it is evident that its length may be taken as great or as small as we please without affecting in any way the meaning of the equation.

80. *Examples of differentiation.* As shown by the example just given, the process of differentiation does not differ in principle from that of ordinary algebraic equations; we have only to pay attention to the alternative law of multiplication.

$$\left.\begin{aligned} d(pp_1L_1p_2p) &= dpp_1L_1p_2p + pp_1L_1p_2dp \\ &= dpp_1L_1p_2p + p_2dp \cdot pp_1L_1 \\ &= dp(p_1L_1p_2p + p_2L_1pp_1) \end{aligned}\right\} \quad . \quad . \quad (255)$$

$$d(peq) = dpeq + pedq = e(qdp - pdq). \quad . \quad . \quad . \quad (256)$$

$$d(\rho^2) = d(\rho|\rho) = d\rho|\rho + \rho|d\rho = 2\rho|d\rho. \quad . \quad . \quad . \quad . \quad (257)$$

$$d(e|p)^n = n(e|p)^{n-1}e|dp. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (258)$$

$$dT^2\rho = 2T\rho dT\rho = d(\rho^2) = 2\rho|d\rho.$$

$$\therefore\ dT\rho = \frac{\rho|d\rho}{T\rho} = U\rho|d\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (259)$$

$$\left.\begin{aligned} d\rho &= d(T\rho U\rho) = U\rho dT\rho + T\rho dU\rho \\ &= U\rho \cdot U\rho|d\rho + T\rho dU\rho \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (260)$$

Also, by eqs. (189) and (260),

$$U\rho d\rho \cdot |U\rho = d\rho - U\rho \cdot d\rho|U\rho = T\rho dU\rho. \quad . \quad . \quad . \quad (261)$$

The student will find it interesting to examine the geometrical significance of the last three equations.

81. *Tangent and normal.* If the equation of a curve be given in the form

$$\rho = \phi(x) = x\epsilon_1 + f(x) \cdot \epsilon_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (262)$$

then, as $\frac{d\rho}{dx}$ is a vector along the tangent at the end of ρ, if we let σ be a vector to any point of the tangent, we have for the equation of the tangent at the end of ρ, u being a variable scalar,

$$\sigma = \rho + u\frac{d\rho}{dx} = \phi(x) + u\phi'(x). \quad . \quad . \quad . \quad . \quad . \qquad (263)$$

Multiplying by $\frac{d\rho}{dx}$, we have the scalar form

$$(\sigma - \rho)d\rho = 0 = (\sigma - \phi(x))\phi'(x). \quad . \quad . \quad . \quad . \qquad (264)$$

If ν be a vector $\|$ to the normal, *i.e.* $\perp$ to $d\rho$, the equation of the tangent may be written

$$(\sigma - \rho)|\nu = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (265)$$

and that of the normal,

$$(\sigma - \rho)\nu = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (266)$$

82. *The circle.* The equation

$$\rho = a(\iota_1 \cos\theta + \iota_2 \sin\theta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (267)$$

represents a circle whose radius is a; for, taking the co-square, we have

$$\rho^2 = a^2(\cos^2\theta + \sin^2\theta) = a^2, \text{ or } T\rho = a,$$

which is the scalar form of the equation, and evidently belongs to a circle of radius a, with the origin at the center.

If e_c be the center, and the origin be at e_0, let

$$e_c - e_0 = \epsilon \text{ and } p - e_0 = \rho;$$

then the equation of the circle may be written

$$T(p - e_c) = T(\rho - \epsilon) = a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (268)$$

or, squaring and transposing,

$$\left.\begin{aligned} &\rho^2 - 2\rho|\epsilon = a^2 - \epsilon^2 \\ \text{or, again,} \quad &\rho|(2\epsilon - \rho) = \epsilon^2 - a^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (269)$$

In the last form the equation gives an immediate proof of the proposition that the product of the segments of a secant line through a given point is constant, and equal to the square of the tangent from the point. This may be easily seen by drawing a diagram. If $a^2 = \epsilon^2$, the origin is at a point of the curve, and the equation becomes

$$\rho|(2\epsilon - \rho) = 0, \quad \ldots \ldots \ldots \quad (270)$$

which shows that the angle inscribed in a semicircle is a right angle.

If we have two circles whose equations are

$$(\rho - \epsilon_1)^2 - a_1^2 = 0 \text{ and } (\rho - \epsilon_2)^2 - a_2^2 = 0,$$

then the equation

$$(\rho - \epsilon_1)^2 - a_1^2 = (\rho - \epsilon_2)^2 - a_2^2$$

is that of some curve passing through the common points of the two circles. The first member of the equation is the square of the distance from the end of ρ to the point of contact of a tangent to the first circle drawn through the end of ρ, while the second member has a corresponding meaning for the other circle: hence the equation is *the locus of points from which equal tangents can be drawn to the two circles.* Expanding, it reduces to

$$2\rho_1|(\epsilon_2 - \epsilon_1) = \epsilon_2^2 - a_2^2 - \epsilon_1^2 + a_1^2, \quad \ldots \ldots \quad (271)$$

a straight line called the axis radical.

83. EXERCISES. — (1) Show that the equations

$$\rho^2 = k(a|\rho + C) \text{ and } \rho^2 = k'(a\rho + C')$$

represent circles, and find their radii, and the vectors to their centers. Also, if $C = C' = 0$, show that the two circles cut each other orthogonally. (a is some constant vector.)

(2) Show that the three axes radical of three circles have a common point.

(3) Show that if ϵ_1, ϵ_2, ϵ_3 are three vectors drawn outward from a common point, and they are connected by the relation

$$\epsilon_1\epsilon_2 \cdot \epsilon_3^2 + \epsilon_2\epsilon_3 \cdot \epsilon_1^2 + \epsilon_3\epsilon_1 \cdot \epsilon_2^2 = 0,$$

then their outer ends lie on a circle through their common point.

(4) By eq. (97) and the relation given in the last exercise show that the equations

$$\frac{\begin{vmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_1^2 & \epsilon_2^2 \end{vmatrix}}{\epsilon_1\epsilon_2} = \frac{\begin{vmatrix} \epsilon_2 & \epsilon_3 \\ \epsilon_2^2 & \epsilon_3^2 \end{vmatrix}}{\epsilon_2\epsilon_3} = \frac{\begin{vmatrix} \epsilon_3 & \epsilon_1 \\ \epsilon_3^2 & \epsilon_1^2 \end{vmatrix}}{\epsilon_3\epsilon_1}$$

also hold between three vectors which, being drawn outward from a point, terminate in a circle passing through this point.

(5) If perpendiculars be drawn from a point upon the three sides of a triangle, and the feet of these perpendiculars be collinear, then will the locus of the point be a circle circumscribed about the triangle.

(6) Show that the tangent line to the circle of eq. (267) has the equation

$$\sigma = a(\iota_1 \cos\theta + \iota_2 \sin\theta) + ua(\iota_2 \cos\theta - \iota_1 \sin\theta),$$

of which the scalar form is $\sigma|\rho = a^2$. Also the equations of the tangent and normal to (269) are respectively

$$(\sigma - \epsilon)|(\rho - \epsilon) = a^2 \text{ and } (\sigma - \epsilon)(\rho - \epsilon) = 0.$$

(7) Find what the equation $\sigma|\rho = a^2$ represents when ρ is not the vector to a point on the circle.

84. *The parabola.* The equations

$$\left.\begin{aligned} \rho &= x\iota_1 + y\iota_2 \\ y^2 &= 4ax \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots \quad (272)$$

represent a parabola; for, eliminating x, we have

$$\rho = \frac{y^2}{4a}\iota_1 + y\iota_2, \quad \ldots \ldots \ldots \ldots \quad (273)$$

which shows that the abscissa varies as the square of the ordinate, a property of the parabola.

Differentiating (273), we have

$$\frac{d\rho}{dy} = \frac{y}{2a}\iota_1 + \iota_2,$$

a vector parallel to the tangent at the end of ρ; hence the equation of the tangent may be written

$$\rho = \frac{y^2}{4a}\iota_1 + y\iota_2 + z\left(\frac{y}{2a}\iota_1 + \iota_2\right), \quad . \quad . \quad . \quad . \quad . \qquad (274)$$

in which y is to be taken as constant.

Eliminating y from (273), we have the scalar form of the equation, viz.:

$$(\rho|\iota_2)^2 = 4a \cdot \rho|\iota_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (275)$$

or, as it may be written,

$$\rho|(\iota_2 \cdot \rho|\iota_2 - 4a\iota_1) = 0.$$

In this latter form we see that the vector $\iota_2 \cdot \rho|\iota_2 - 4a\iota_1$ is always perpendicular to ρ. Let

$$\sigma = \iota_2 \cdot \rho|\iota_2 - 4a\iota_1; \text{ then } \sigma|\iota_1 = -4a,$$

and it appears that the locus of the end of σ drawn outward from the origin is a right line parallel to ι_2, at a distance of $4a$ to the left of the origin; also, $\sigma|\iota_2 = \rho|\iota_2$, so that the projections of ρ and σ on ι_2 are equal. The following proposition is a consequence, viz.: If a right-angle triangle have its rectangular vertex fixed, and one of the other vertices moves on a right line to which the hypotenuse remains perpendicular, then the third vertex generates a parabola.

To find the locus of the middle points of a system of parallel chords, *i.e.* a diameter. Let ϵ be parallel to the chords, and let the equation of some chord be $\rho = \rho_1 + x\epsilon$, in which ρ_1 satisfies eq. (275). Substitute this value of ρ in (275) to find the other end of the chord; therefore

$$((\rho_1 + x\epsilon)|\iota_2)^2 = 4a(\rho_1 + x\epsilon)|\iota_1,$$

or $$(\rho_1|\iota_2)^2 + 2x\epsilon|\iota_2 \cdot \rho|\iota_2 + x^2(\epsilon|\iota_2)^2 = 4a\rho_1|\iota_1 + 4ax\epsilon|\iota_1;$$
whence, by (275),
$$x = \frac{4a\epsilon|\iota_1 - 2\epsilon|\iota_2 \cdot \rho_1|\iota_2}{(\epsilon|\iota_2)^2}.$$
If σ be the vector to the middle point of the chord, then
$$\sigma = \rho_1 + \tfrac{1}{2}x\epsilon = \rho_1 + \frac{2a\epsilon|\iota_1 - \epsilon|\iota_2 \cdot \rho_1|\iota_2}{(\epsilon|\iota_2)^2} \cdot \epsilon;$$
whence $$\sigma|\iota_2 = 2a\frac{\epsilon|\iota_1}{\epsilon|\iota_2},$$
which is the equation of a right line parallel to ι_1. Hence the diameters of a parabola are all parallel. If $\epsilon = \iota_2$, $\sigma|\iota_2 = 0$, so that the line through the origin parallel to ι_1 bisects a system of chords perpendicular to it. This line is the *axis* of the parabola.

85. EXERCISES. — (1) Show that $T(\rho - a\iota_1) = a + \rho|\iota_1$, and interpret the equation.

We have, by (275),
$$(\rho|\iota_2)^2 \equiv \rho^2 - (\rho|\iota_1)^2 = 4a\rho|\iota_1 + a^2 - a^2,$$
or $$\rho^2 - 2a\rho|\iota_1 + a^2 = (\rho|\iota_1)^2 + 2a\rho|\iota_1 + a^2,$$
whence $$(\rho - a\iota_1)^2 = (a + \rho|\iota_1)^2.$$
The interpretation is easily obtained by a figure.

(2) Show that $d\rho(\iota_1 + U(\rho - a\iota_1)) = 0$, and interpret the equation.

(3) Show that the equations to the tangent and normal to (275) may be written respectively
$$\sigma|\iota_2 \cdot \rho|\iota_2 = 2a(\sigma + \rho)|\iota_1$$
and $$\sigma|\iota_1 \cdot \rho|\iota_2 = \rho|\iota_2 \cdot \rho|\iota_1 + 2a(\rho - \sigma)|\iota_2.$$

(4) Show that $\rho = \frac{1}{2}t^2\epsilon_1 + t\epsilon_2$ and $(\epsilon_1\rho)^2 + 2\epsilon_1\epsilon_2 \cdot \epsilon_2\rho = 0$, are respectively the vector and scalar forms of the equation of a parabola referred to a tangent whose direction is ϵ_2 and a

diameter through the point of contact whose direction is that of ϵ_1.

(5) Show that, with reference to the equations of Ex. 4, the lines $\sigma = \rho + x\frac{d\rho}{dt}$ and $\sigma = \rho + y\epsilon_2$ cut the line $\sigma = z\epsilon_1$ at equal distances on each side of the origin, and give the geometric interpretation.

86. *Ellipse and hyperbola.* The equations

$$\rho = a\iota_1 \cos\theta + b\iota_2 \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (276)$$

$$\rho = a\iota_1 \sec\theta + b\iota_2 \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (277)$$

represent respectively an ellipse and hyperbola, in which a and b are the semi-axes, and θ is the eccentric angle. This will be evident at once to any one familiar with the ordinary Cartesian equations, if we obtain the corresponding scalar forms. We find $\rho|\iota_1 = a\cos\theta$, $\rho|\iota_2 = b\sin\theta$, from (276), whence

$$\left(\frac{\rho|\iota_1}{a}\right)^2 + \left(\frac{\rho|\iota_2}{b}\right)^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (278)$$

From (277) we have $\rho|\iota_1 = a\sec\theta$, $\rho|\iota_2 = b\tan\theta$, whence

$$\left(\frac{\rho|\iota_1}{a}\right)^2 - \left(\frac{\rho|\iota_2}{b}\right)^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (279)$$

Since $\rho|\iota_1$ and $\rho|\iota_2$ are the Cartesian x and y, the equations are evidently those of the ellipse and hyperbola. The two equations may be combined by using the double sign; thus

$$\left(\frac{\rho|\iota_1}{a}\right)^2 \pm \left(\frac{\rho|\iota_2}{b}\right)^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (280)$$

EXERCISES. — Show that the equations

$$T(\rho + c\iota_1) = \frac{c}{a}\left(\frac{a^2}{c} + \rho|\iota_1\right) \text{ and } T(\rho - c'\iota_1) = \frac{c'}{a}\left(\rho|\iota_1 - \frac{a^2}{c'}\right),$$

in which $c = \sqrt{a^2 - b^2}$ and $c' = \sqrt{a^2 + b^2}$ are equivalent to (278) and (279) respectively, and interpret these forms of the equations.

Show that the equations

$$T(\rho + c\iota_1) + T(\rho - c\iota_1) = 2a$$

and

$$T(\rho + c'\iota_1) - T(\rho - c'\iota_1) = 2a$$

are also equivalent to (278) and (279) respectively, and interpret them.

Let us write the equation

$$\frac{\iota_1 \cdot \rho | \iota_1}{a^2} \pm \frac{\iota_2 \cdot \rho | \iota_2}{b^2} \equiv \phi\rho. \quad \ldots \ldots \quad (281)$$

This expression is a *linear* and *vector* function of ρ, and, by the aid of (281), equation (280) becomes

$$\rho | \phi\rho = 1. \quad \ldots \ldots \quad (282)$$

This remarkably simple equation may represent, as will appear hereafter, not merely (280), but *any* equation of the second degree in ρ, which contains no first-degree terms. It may thus represent not only any central conic with the origin at the center, but also any central quadric referred to its center as origin, when we are dealing with three-dimensional space, and the corresponding locus in n-dimensional space. Similarly, if ϕp be a linear, point function of a variable point p, we shall see that $p|\phi p = 0$ may represent any conic whatever in two-dimensional space, any quadric whatever in three-dimensional space, and any locus of the second order in n-dimensional space.

In the form given above the ϕ function will be found to possess the following properties, viz.:

$$\left.\begin{array}{ll} (\alpha) & \phi(\rho + \sigma) = \phi\rho + \phi\sigma \\ (\beta) & \phi(x\rho) = x\phi\rho \\ (\gamma) & d(\phi\rho) = \phi(d\rho) \\ (\delta) & \sigma|\phi\rho = \rho|\phi\sigma \end{array}\right\} \quad \ldots \ldots \quad (283)$$

The first three properties are possessed by *any* linear, vector function. When the last relation holds, the function is said to be *self-conjugate*, and, in dealing with curves and surfaces of the second order, ϕ may always be so taken that this relation exists, *i.e.* ϕ may be taken as self-conjugate.

87. *Tangent and normal.* Differentiate (282), having regard to (γ) and (δ) of (283); then

$$d\rho|\phi\rho + \rho|\phi d\rho = 2\,d\rho|\phi\rho = 0.$$

Hence $\phi\rho$ is a vector perpendicular to $d\rho$, *i.e.* parallel to the normal to (282) at the end of ρ. Therefore, if σ be a vector to any point of the tangent, and ρ the vector to the point of contact, so that $\sigma - \rho$ is parallel to $d\rho$, we have $(\sigma - \rho)|\phi\rho = 0$, or, by (282),

$$\sigma|\phi\rho = 1, \qquad (284)$$

as the equation of the tangent to the curve.

For the normal we have the equation

$$(\sigma - \rho)\,\phi\rho = 0. \qquad (285)$$

Since $\phi\rho$ is parallel to the normal at the end of ρ, the projection of ρ on $\phi\rho$ will be the perpendicular from the center on the tangent line. By Art. 46 this is

$$\frac{\phi\rho \cdot \rho|\phi\rho}{(\phi\rho)^2} = \frac{\phi\rho}{(\phi\rho)^2} = \frac{1}{T\phi\rho} \cdot U\phi\rho. \qquad (286)$$

Hence *the length of the vector $\phi\rho$ is the reciprocal of that of the perpendicular from the center on the tangent.*

88. *Diameter.* The diameter being the locus of the middle points of a system of parallel chords, we may find its equation as follows. Let the system of chords be parallel to ϵ, and let

$$\rho = \rho_1 + x\epsilon$$

be the equation of one of them, in which ρ_1 is a vector of the curve, *i.e.* $\rho_1|\phi\rho_1 = 1$. Substitute this value of ρ in (282), and we have, in order to find the other end of the chord,

$$(\rho_1 + x\epsilon)|\phi(\rho_1 + x\epsilon) = 1, \text{ or } \rho_1|\phi\rho_1 + 2\,x\epsilon|\phi\rho_1 + x^2\epsilon|\phi\epsilon = 1,$$

whence, because of the condition above,

$$x = 0, \text{ and } x = -\frac{2\,\epsilon|\phi\rho_1}{\epsilon|\phi\epsilon}.$$

Now, if σ is the vector to the middle point of the chord, we have

$$\sigma = \rho_1 + \tfrac{1}{2} x\epsilon = \rho_1 - \frac{\epsilon|\phi\rho_1}{\epsilon|\phi\epsilon} \cdot \epsilon.$$

Multiply into $|\phi\epsilon$, and we have

$$\sigma|\phi\epsilon = 0, \quad (287)$$

an equation independent of ρ_1, and depending only on the given direction ϵ and the function ϕ. (287) is therefore the equation of the required locus, which is a straight line perpendicular to $\phi\epsilon$, and consequently parallel to the tangents to the curve at the ends of a diameter parallel to ϵ. The direction of σ is said to be *conjugate* to that of ϵ, and the diameters parallel to ϵ and σ are conjugate diameters.

If α and β are any two conjugate vector semi-diameters, they must be subject, therefore, to the conditions

$$\left.\begin{aligned} \alpha|\phi\alpha &= \beta|\phi\beta = 1 \\ \alpha|\phi\beta &= \beta|\phi\alpha = 0 \end{aligned}\right\} \quad (288)$$

The results of Arts. 87 and 88 have been obtained with the functional symbol ϕ, without any reference to the *form* of the function, the only restriction being that it shall be subject to the conditions (283); hence these results are *general*, and hold for *any* form of the linear vector, self-conjugate function.

89. *Further development of the ϕ function.* Write

$$\phi\rho = g_1\iota_1 \cdot \rho|\iota_1 + g_2\iota_2 \cdot \rho|\iota_2, \quad (289)$$

so that, comparing with (281), we have

$$g_1 = \frac{1}{a^2}, \quad g_2 = \pm\frac{1}{b^2}. \quad (290)$$

Putting, in (289), successively ι_1 and ι_2 for ρ, we find

$$\phi\iota_1 = g_1\iota_1 \text{ and } \phi\iota_2 = g_2\iota_2. \quad (291)$$

Next substitute $\phi\rho$ for ρ in (289); therefore

$$\begin{aligned} \phi(\phi\rho) = \phi^2\rho &= g_1\iota_1 \cdot \phi\rho|\iota_1 + g_2\iota_2 \cdot \phi\rho|\iota_2 \\ &= g_1\iota_1 \cdot \rho|\phi\iota_1 + g_2\iota_2 \cdot \rho|\phi\iota_2 = g_1^2\iota_1 \cdot \rho|\iota_1 + g_2^2\iota_2 \cdot \rho|\iota_2. \end{aligned}$$

We have the fourth member from the third because

$$\phi\rho|\iota_1 = \iota_1|\phi\rho = \rho|\phi\iota_1,$$

by (79) and (283).

Similarly, $\phi(\phi^2\rho) = \phi^3\rho = g_1^3\iota_1 \cdot \rho|\iota_1 + g_2^3\iota_2 \cdot \rho|\iota_2$, etc.;

so that, if n is a positive whole number,

$$\phi^n\rho = g_1^n\iota_1 \cdot \rho|\iota_1 + g_2^n\iota_2 \cdot \rho|\iota_2. \quad . \quad . \quad . \quad . \quad . \qquad (292)$$

Let m be some other positive whole number; then also

$$\phi^m\rho = g_1^m\iota_1 \cdot \rho|\iota_1 + g_2^m\iota_2 \cdot \rho|\iota_2;$$

hence

$$\phi^n\phi^m\rho = \phi^{n+m}\rho = g_1^n\iota_1 \cdot \iota_1|\phi^m\rho + g_2^n\iota_2 \cdot \iota_2|\phi^m\rho$$
$$= g_1^{n+m}\iota_1 \cdot \rho|\iota_1 + g_2^{n+m}\iota_2 \cdot \rho|\iota_2. \quad . \quad . \qquad (293)$$

Suppose m to be negative and equal to $-n$; then by (293)

$$\phi^n(\phi^{-n}\rho) = \phi^0\rho = \iota_1 \cdot \rho|\iota_1 + \iota_2 \cdot \rho|\iota_2 = \rho,$$

so that the negative exponent gives a function, such that the operation indicated by ϕ with a positive exponent of the same numerical value being performed upon it, gives ρ as a result; *i.e.* the operations ϕ^n and ϕ^{-n} cancel each other. Hence (292) and (293) hold both for positive and negative values of the exponents.

Finally, suppose $n = \frac{m_1}{m_2}$; then we ought to have

$$\phi^{\frac{m_2 - m_1}{m_2}}(\phi^{\frac{m_1}{m_2}}\rho) = \phi\rho,$$

if the exponential law holds for this case, and the result is easily verified as before. Thus (292) holds for all real values of n.

If $$f(x) = Ax^n + Bx^{n-1} + \cdots N,$$

we may easily show that

$$\Phi\rho = (f(\phi))\rho = f(g_1) \cdot \iota_1 \cdot \rho|\iota_1 + f(g_2) \cdot \iota_2 \cdot \rho|\iota_2, \quad . \qquad (294)$$

in which $\Phi = f(\phi)$ is also a linear, vector, self-conjugate function.

Again, if f_1 and f_2 are functional symbols of the same form as f above, let $\Psi = \dfrac{f_1(\phi)}{f_2(\phi)} = \Phi_1\Phi_2^{-1}$; then

$$\Psi\rho = \Phi_1\Phi_2^{-1}\rho = \frac{f_1(g_1)}{f_2(g_1)} \cdot \iota_1 \cdot \rho|\iota_1 + \frac{f_1(g_2)}{f_2(g_2)} \cdot \iota_2 \cdot \rho|\iota_2, \quad . \quad (295)$$

so that Ψ is still a function of the same kind as ϕ.

Finally, let ϕ, ϕ', ϕ'' be three functions of the form (289) with corresponding g_1, g_1', g_1'', g_2, g_2', g_2''; then

$$\left.\begin{aligned} k\phi\rho + k'\phi'\rho + k''\phi''\rho &= (k\phi + k'\phi' + k''\phi'')\rho \\ &= (kg_1 + k'g_1' + k''g_1'')\iota_1 \cdot \rho|\iota_1 \\ &\quad + (kg_2 + k'g_2' + k''g_2'')\iota_2 \cdot \rho|\iota_2 = \psi\rho \end{aligned}\right\} . \quad (296)$$

All the results of this article hold equally well for n-dimensional space when $\phi\rho = \Sigma_1^n (g \cdot \iota \cdot \rho|\iota)$.

90. EXERCISES. — (1) Show that

$$\phi(\phi^2 + 1)\rho = \frac{1}{a^2}\left(\frac{1}{a^4} + 1\right)\iota_1 \cdot \rho|\iota_1 \pm \frac{1}{b^2}\left(\frac{1}{b^4} + 1\right)\iota_2 \cdot \rho|\iota_2.$$

(2) Show that $\iota_1|\phi(\phi^2 - 1)^{-1}\iota_1 = \dfrac{a^2}{1 - a^4}$.

(3) Show that $\left(\dfrac{1}{\phi + 1} + \dfrac{1}{\phi - 1}\right)\rho = 2\left(\dfrac{\phi}{\phi^2 - 1}\right)\rho$.

(4) Show that

$$\phi(\phi^2 - 1)^{-1}(\phi + 3)^{-1}\iota_1 = \tfrac{1}{4}(\phi + 1)^{-1}\iota_1 + \tfrac{1}{8}(\phi - 1)^{-1}\iota_1 - \tfrac{3}{8}(\phi + 3)^{-1}\iota_1$$
$$= \frac{a^4\iota_1}{(1 - a^4)(1 + 3a^2)}.$$

(5) Show that, if ϕ and ϕ' are of the form (289),

$$\phi\phi'\rho = \phi'\phi\rho.$$

(6) Show that

$$(\phi^2 - 2\phi + 1)\rho = (\phi - 1)^2\rho, \text{ and } (\phi^3 + 1)(\phi + 1)^{-1}\rho = (\phi^2 - \phi + 1)\rho.$$

91. *The function* $\phi^{\frac{1}{2}}$. Since $\phi = \phi^{\frac{1}{2}}\phi^{\frac{1}{2}}$, we have

$$\rho|\phi\rho = \rho|\phi^{\frac{1}{2}}(\phi^{\frac{1}{2}}\rho) = \phi^{\frac{1}{2}}\rho|\phi^{\frac{1}{2}}\rho = (\phi^{\frac{1}{2}}\rho)^2 = 1;$$

whence, $T\phi^{\frac{1}{2}}\rho = 1$, ... (297)

which is a form of the equation of the central conic analogous to that of the circle. We have

$$\phi^{\frac{1}{2}}\rho = g_1^{\frac{1}{2}}\iota_1 \cdot \rho|\iota_1 + g_2^{\frac{1}{2}}\iota_2 \cdot \rho|\iota_2 = \frac{1}{a} \cdot \iota_1 \cdot \rho|\iota_1 + \frac{1}{b\sqrt{\pm 1}} \cdot \iota_2 \cdot \rho|\iota_2,$$

so that, in the case of the hyperbola, $\phi^{\frac{1}{2}}$ is an imaginary function; nevertheless (297) is real and equivalent to (279), as is easily seen.

Let α and β be conjugate vector semi-diameters; then

$$\alpha|\phi\beta = 0 = \alpha|\phi^{\frac{1}{2}}\phi^{\frac{1}{2}}\beta = \phi^{\frac{1}{2}}\beta|\phi^{\frac{1}{2}}\alpha; \quad . \; . \; . \; . \; . \; . \; . \quad (298)$$

hence $\phi^{\frac{1}{2}}\alpha$ and $\phi^{\frac{1}{2}}\beta$ are *unit normal vectors*. We will determine what relation $\phi^{\frac{1}{2}}\rho$ bears to ρ in the ellipse.

Let $$\rho|\phi\rho = \left(\frac{\rho|\iota_1}{a}\right)^2 + \left(\frac{\rho|\iota_2}{b}\right)^2 = 1$$

and $$\rho'|\phi'\rho' = \left(\frac{\rho'|\iota_1}{a}\right)^2 + \left(\frac{\rho'|\iota_2}{b'}\right)^2 = 1$$

be the equations of two ellipses having the a axis in common, and let ρ and ρ' be so taken that they have the same projection on ι_1; that is, $\rho|\iota_1 = \rho'|\iota_1$.

Then $$\phi'^{\frac{1}{2}}\rho'\phi^{\frac{1}{2}}\rho = \left(\frac{\iota_1}{a} \cdot \rho'|\iota_1 + \frac{\iota_2}{b'} \cdot \rho'|\iota_2\right)\left(\frac{\iota_1}{a} \cdot \rho|\iota_1 + \frac{\iota_2}{b} \cdot \rho|\iota_2\right)$$

$$= \frac{\rho'|\iota_1 \cdot \rho|\iota_2}{ab} - \frac{\rho'|\iota_2 \cdot \rho|\iota_1}{ab'} = \frac{\rho|\iota_1}{a}\left(\frac{\rho|\iota_2}{b} - \frac{\rho'|\iota_2}{b'}\right).$$

But, from the equations of the curves,

$$\left(\frac{\rho|\iota_2}{b}\right)^2 = 1 - \left(\frac{\rho|\iota_1}{a}\right)^2 = 1 - \left(\frac{\rho'|\iota_1}{a}\right)^2 = \left(\frac{\rho'|\iota_2}{b'}\right)^2;$$

hence $$\phi'^{\frac{1}{2}}\rho'\phi^{\frac{1}{2}}\rho = 0.$$

Thus it appears that, if any two ellipses have a common axis, and ρ be taken in each so as to have the same projection on this axis, then $\phi^{\frac{1}{2}}\rho$ will be the *same unit vector* for each ellipse. Let one of the ellipses be a circle of radius a, *i.e.* let $b' = a$; then $a^2\phi'\rho' = a\phi'^{\frac{1}{2}}\rho' = \rho'$, and $\rho'\phi^{\frac{1}{2}}\rho = 0$, so that $\phi^{\frac{1}{2}}\rho$ in any ellipse is a *unit vector laid off along that radius of the circle, described on either axis, which has the same projection on this axis that* ρ *has.* By (298) it appears that the radii of this circle corresponding to a pair of conjugate semi-diameters are mutually perpendicular.

92. *Interpretation of the equation* $\sigma|\phi\epsilon = 1$. If ϵ satisfies the equation $\epsilon|\phi\epsilon = 1$, it is a vector of the curve, and the given equation is identical with (284), and therefore represents a tangent to (282).

If this is not the case, let ϵ be first a vector to some point from which a tangent can be drawn to the curve. Let ρ_1 and ρ_2 be vectors to the points of contact of the two tangents which pass through the end of ϵ. The equations of these tangents are

$$\sigma|\phi\rho_1 = 1 \text{ and } \sigma|\phi\rho_2 = 1;$$

and since they pass through the end of ϵ, the equations must be satisfied when ϵ is substituted for σ. Thus we have the relations

$$\epsilon|\phi\rho_1 = 1 = \epsilon|\phi\rho_2,$$

which must always hold between ϵ, ρ_1, and ρ_2.

If now, in the given equation $\sigma|\phi\epsilon = 1$, we make $\sigma = \rho_1$, or $\sigma = \rho_2$, it appears that the equation is satisfied, and hence the line represented by it *passes through the points of contact of the tangents through the end of* ϵ. Furthermore, the line is perpendicular to $\phi\epsilon$, and therefore parallel to the diameter conjugate to ϵ.

Next, write the equation $\epsilon|\phi\sigma = 1$; one value of σ will evidently coincide in direction with ϵ; when it has this direction, suppose it to become fixed, and ϵ to vary; the equation still represents a right line, which will evidently be parallel to the previous one; and since one value of ϵ will be its original value, this line must pass through the end of the original ϵ.

The line represented by the equation

$$\sigma|\phi\epsilon = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (299)$$

is the *polar* of the point at the end of ϵ.

Let ϵ_1 and ϵ_2 be the vectors from the origin to two points p_1 and p_2; then, *if the polar of p_1 passes through p_2, that of p_2 will also pass through p_1.* For the equations of the polars of p_1 and p_2 are respectively $\sigma|\phi\epsilon_1 = 1$ and $\sigma|\phi\epsilon_2 = 1$; if the first passes through p_2, we must have $\epsilon_2|\phi\epsilon_1 = 0$; but this is also the condition that the second shall pass through p_1. Since p_2 may be taken at any point of the polar of p_1, while *its* polar always passes through p_1, it appears that if a point move along a fixed right line, its polar passes through a fixed point; and reciprocally, if a line passes always through a fixed point, its pole moves on a fixed right line.

The semi-diameter along ϵ is a mean proportional between $T\epsilon$ and the distance along ϵ from the center to the polar of ϵ. Let the semi-diameter be $x\epsilon$, and the distance to the polar $y\epsilon$; then we must show that $T\epsilon \cdot yT\epsilon = (xT\epsilon)^2$, or $x^2 = y$. Put $x\epsilon$ for ρ in (282), therefore $x^2\epsilon|\phi\epsilon = 1$; also put $y\epsilon$ for σ in (299), and we have $y\epsilon|\phi\epsilon = 1$; therefore $x^2 = y$. Q.E.D.

93. EXERCISES. — (1) Show that the system of conics obtained by giving different values to k in the equation

$$\rho|(\phi^{-1} - k)^{-1}\rho = 1$$

have all the same foci. These are called *confocal* conics.

(2) Through any point in the plane of the confocal system there pass two conics of the system, which cut each other at right angles.

Let ϵ be the vector to any point; then, if the curve pass through it, we must have the equation of ex. (2) satisfied when ϵ is substituted for ρ. Hence

$$\epsilon|(\phi^{-1} - k)^{-1}\epsilon = 1 = \frac{(\epsilon|\iota_1)^2}{a^2 - k} + \frac{(\epsilon|\iota_2)^2}{b^2 - k},$$

a quadratic in k, giving two values corresponding to the two curves through the end of ϵ. Let the roots of this equation be k_1 and k_2; then the curves are

$$\rho|(\phi^{-1}-k_1)^{-1}\rho=1 \text{ and } \rho|(\phi^{-1}-k_2)^{-1}\rho=1,$$

and $(\phi^{-1}-k_1)^{-1}\epsilon$ and $(\phi^{-1}-k_2)^{-1}\epsilon$ are vectors parallel to the respective normals at ϵ, whose co-product must be zero if the normals are at right angles. Hence

$$(\phi^{-1}-k_1)^{-1}\epsilon|(\phi^{-1}-k_2)^{-1}\epsilon=\epsilon|(\phi^{-1}-k_2)^{-1}(\phi^{-1}-k_1)^{-1}\epsilon$$
$$=\frac{1}{k_2-k_1}\cdot\epsilon|[(\phi^{-1}-k_2)^{-1}-(\phi^{-1}-k_1)^{-1}]\epsilon=\frac{1-1}{k_2-k_1}=0.$$

(3) If ρ_1 and ρ_2 are any two vectors of the central conic, show that $\rho_2-\rho_1$ and $\rho_2+\rho_1$ (supplementary chords) are conjugate in direction.

(4) Show that the equations of the diagonals of the parallelogram formed by the tangents at the ends of ρ_1, ρ_2, $-\rho_1$, and $-\rho_2$ are $\sigma|\phi(\rho_2-\rho_1)=0$ and $\sigma|\phi(\rho_1+\rho_2)=0$, and that these diagonals are conjugate in direction.

(5) Find the condition that the line $\sigma|\epsilon=C$ shall be tangent to the curve $\rho|\phi\rho=1$.

The equation of the tangent is $\sigma|\phi\rho=1$; comparing this with the given equation, we have $\phi\rho=\frac{\epsilon}{C}$, or $\rho=\frac{\phi^{-1}\epsilon}{C}$. Substituting this value of ρ in the equation of the curve, we have $\frac{\phi^{-1}\epsilon|\epsilon}{C^2}=1$, or $C^2=\epsilon|\phi^{-1}\epsilon$, the required condition. Thus the line whose equation is

$$\sigma|\epsilon=\sqrt{\epsilon|\phi^{-1}\epsilon}=T\phi^{-\frac{1}{2}}\epsilon \quad \ldots\ldots\ldots \quad (300)$$

is always tangent to the central conic.

94. *The conic referred to conjugate diameters.*

The equation

$$(\rho\alpha)^2\pm(\rho\beta)^2=(\alpha\beta)^2 \quad \ldots\ldots\ldots \quad (301)$$

represents an ellipse or hyperbola referred to the conjugate semi-diameters α and β. The equation is satisfied when $\rho=\beta$, so that the curve passes through the end of β; also, the curve is satisfied when $\rho=\alpha\sqrt{\pm 1}$, so that the end of α is a real point of the ellipse, and an imaginary point of the hyperbola.

Write $$\phi\rho=\frac{|\alpha\cdot\alpha\rho}{(\alpha\beta)^2}\pm\frac{|\beta\cdot\beta\rho}{(\alpha\beta)^2},$$

so that (301) becomes $\rho|\phi\rho=1$; then we have

$$\alpha|\phi\beta=-\alpha\left(\frac{\alpha\cdot\alpha\beta}{(\alpha\beta)^2}\right)=0,$$

and α and β are conjugate in direction, by (288). Equation (301) shows that, in the hyperbola, if any diameter cuts the curve, its conjugate does not.

EXERCISE. — Show that the ellipse and hyperbola of (301) are also represented by

$$\rho=\alpha\cos\theta+\beta\sin\theta \text{ and } \rho=\alpha\sec\theta+\beta\tan\theta,$$

respectively.

95. *The area of the parallelogram formed by tangents at the ends of conjugate diameters of an ellipse is constant.*

If α and β are any pair of conjugate vector semi-diameters, then the area is $4\alpha\beta$. Now

$$\alpha=\phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}\alpha=a\iota_1\cdot\iota_1|\phi^{\frac{1}{2}}\alpha+b\iota_2\cdot\iota_2|\phi^{\frac{1}{2}}\alpha,$$

$$\beta=\phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}\beta=a\iota_1\cdot\iota_1|\phi^{\frac{1}{2}}\beta+b\iota_2\cdot\iota_2|\phi^{\frac{1}{2}}\beta;$$

hence $$4\alpha\beta=4ab\begin{vmatrix}\iota_1|\phi^{\frac{1}{2}}\alpha, & \iota_2|\phi^{\frac{1}{2}}\alpha\\ \iota_1|\phi^{\frac{1}{2}}\beta, & \iota_2|\phi^{\frac{1}{2}}\beta\end{vmatrix}=4ab\cdot\iota_1\iota_2\cdot\phi^{\frac{1}{2}}\alpha\phi^{\frac{1}{2}}\beta=4ab.$$

See (298) and (193).

96. EXERCISES. — (1) Find the locus of the intersection of tangents at the ends of conjugate diameters.

Taking α and β as usual, we have $\sigma=\alpha+\beta=$ vector of point whose locus is required. Therefore

$$\sigma|\phi\sigma=(\alpha+\beta)|\phi(\alpha+\beta)=\alpha|\phi\alpha+2\alpha|\phi\beta+\beta|\phi\beta=1+0+1=2,$$

a similar curve, which is, in this case, an ellipse. If the given curve be an hyperbola, the locus of $\sigma = \alpha + \beta$ becomes $\sigma|\phi\sigma = 0$, the asymptotes of the curve, because, in this case, if $\beta|\phi\beta = 1$, $\alpha|\phi\alpha = -1$.

(2) Find the locus of the extremity of $\phi\rho$; also of $\phi^n\rho$, noting particularly the case when $n = \frac{1}{2}$.

(3) Show that, when $T\phi\rho = c$, the locus of ρ is the curve whose equation is $\rho|\phi^2\rho = c^2$.

(4) Show that the locus of the foot of the perpendicular drawn from any point to the tangent to the conic, *i.e.* the pedal curve, has the equation $(\sigma - \epsilon)|\phi^{-1}(\sigma - \epsilon) = [\sigma|(\sigma - \epsilon)]^2$.

(5) If ϵ_1 and ϵ_2 are any two unit vectors at right angles to each other, show that $\epsilon_1|\phi^{-1}\epsilon_1 + \epsilon_2|\phi^{-1}\epsilon_2 = a^2 \pm b^2$.

The conditions give $\epsilon_2 = |\epsilon_1$. Thus we have

$$\epsilon_1|\phi^{-1}\epsilon_1 = a^2(\epsilon_1|\iota_1)^2 \pm b^2(\epsilon_1|\iota_2)^2,$$

and

$$\epsilon_2|\phi^{-1}\epsilon_2 = |\epsilon_1|\phi^{-1}|\epsilon_1 = a^2(\epsilon_1\iota_1)^2 \pm b^2(\epsilon_1\iota_2)^2;$$

whence, adding, we have the result.

(6) If α and β are vector conjugate semi-diameters, show, by the result of the last example, that $\alpha^2 + \beta^2 = a^2 \pm b^2$.

By Art. 91 $\phi^{\frac{1}{2}}\alpha$ and $\phi^{\frac{1}{2}}\beta$ are unit normal vectors; real, for the ellipse, and imaginary, for the hyperbola. Hence

$$\phi^{\frac{1}{2}}\alpha|\phi^{-1}\phi^{\frac{1}{2}}\alpha + \phi^{\frac{1}{2}}\beta|\phi^{-1}\phi^{\frac{1}{2}}\beta = a^2 \pm b^2 = \alpha^2 + \beta^2.$$

(7) Show that the locus of the common point of perpendicular tangents is $\sigma^2 = a^2 \pm b^2$.

Use eq. (300).

97. *The general linear, vector function in plane space.* The most general form of this function may be written

$$\phi\rho = \epsilon_1 \cdot \epsilon_1'|\rho + \epsilon_2 \cdot \epsilon_2'|\rho : \quad \ldots \ldots \ldots \quad (302)$$

for, suppose there were other terms such as $\epsilon_3 \cdot \epsilon_3'|\rho + k\rho$; write

$$\epsilon_3' = m_1\epsilon_1' + m_2\epsilon_2' \text{ and } \rho = \frac{1}{\epsilon_1'\epsilon_2'}(|\epsilon_1' \cdot \epsilon_2'|\rho - |\epsilon_2' \cdot \epsilon_1'|\rho),$$

and these terms become

$$\left(m_1\epsilon_3 - \frac{k}{\epsilon_1'\epsilon_2'}\cdot|\epsilon_2'\right)\cdot\rho|\epsilon_1' + \left(m_2\epsilon_3 + \frac{k}{\epsilon_1'\epsilon_2'}\cdot|\epsilon_1'\right)\cdot\rho|\epsilon_2',$$

which, on being combined with the terms composing (302), give an expression of the same *form* as before, merely having instead of ϵ_1 and ϵ_2 the vectors

$$\epsilon_1 + m_1\epsilon_3 - \frac{k}{\epsilon_1'\epsilon_2'}\cdot|\epsilon_2' \text{ and } \epsilon_2 + m_2\epsilon_3 + \frac{k}{\epsilon_1'\epsilon_2'}\cdot|\epsilon_1'.$$

This general form of ϕ possesses all the properties given in (283) except the last, or self-conjugate property. If, however, we write

$$\phi_c\rho = \epsilon_1'\cdot\epsilon_1|\rho + \epsilon_2'\cdot\epsilon_2|\rho, \quad \ldots \quad (303)$$

then we shall have

$$\rho|\phi\sigma = \sigma|\phi_c\rho \text{ and } \rho|\phi_c\sigma = \sigma|\phi\rho. \quad \ldots \quad (304)$$

ϕ_c is called the *conjugate* function to ϕ, and *vice versâ*. When $\phi_c = \phi$, the function is *self-conjugate,* and the condition (δ) of (283) is satisfied. In all geometrical applications it is possible to take ϕ self-conjugate, and we shall always so regard it, because it greatly facilitates necessary operations.

The sum of a linear function and its conjugate is always self-conjugate: for

$$\sigma|(\phi+\phi_c)\rho = \sigma|\phi\rho + \sigma|\phi_c\rho = \rho|\phi_c\sigma + \rho|\phi\sigma = \rho|(\phi+\phi_c)\sigma.$$

98. *Inversion of the ϕ function.* Suppose we wish to find the value of ρ from the equation

$$\phi\rho = \epsilon. \quad \ldots \quad (305)$$

This gives $\rho = \phi^{-1}\epsilon;$ . . . . (306)

and if we can find the form of the inverse function, we have the solution of (305). For convenience write $\epsilon = |\lambda$, so that $\phi\rho = |\lambda$ and $\rho = \phi^{-1}|\lambda$; then

$$\lambda|\phi\rho = 0 = \rho|\phi_c\lambda,$$

whence $\rho = x|\phi_c\lambda = \phi^{-1}|\lambda.$

To determine x, let μ be any vector whatever; then

$$\rho|\phi_c\mu = x|\phi_c\lambda|\phi_c\mu = x\phi_c\lambda\phi_c\mu = \phi^{-1}|\lambda|\phi_c\mu = \mu||\lambda = \lambda\mu.$$

$$\therefore\ x = \frac{\lambda\mu}{\phi_c\lambda\phi_c\mu},$$

and $$\rho = \phi^{-1}\epsilon = \phi^{-1}|\lambda = \frac{\lambda\mu\cdot|\phi_c\lambda}{\phi_c\lambda\phi_c\mu} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (307)$$

This equation affords a solution of (305), but we proceed to obtain a formula more convenient than this for most purposes. Let λ and μ be now any two constant vectors whatever; we have, from (305),

$$\lambda|\phi\rho = \rho|\phi_c\lambda = \lambda|\epsilon \text{ and } \mu|\phi\rho = \rho|\phi_c\mu = \mu|\epsilon;$$

also, writing ρ in terms of its projections on $|\phi_c\lambda$ and $|\phi_c\mu$, we have

$$\rho = \frac{1}{\phi_c\lambda\phi_c\mu}(|\phi_c\lambda\cdot\rho|\phi_c\mu - |\phi_c\mu\cdot\rho|\phi_c\lambda);$$

or, substituting from the preceding equations,

$$\phi^{-1}\epsilon = \rho = \frac{1}{\phi_c\lambda\phi_c\mu}(|\phi_c\lambda\cdot\mu|\epsilon - |\phi_c\mu\cdot\lambda|\epsilon), \quad . \qquad (308)$$

the proposed formula. If ϕ be self-conjugate, then $\phi_c = \phi$, and the suffix may be dropped in (307) and (308).

To invert such a function as $\phi_1 + k\phi_2 + k'\phi_3 + \text{etc.} = \Phi$, say, we have $\Phi_c = \phi_{1_c} + k\phi_{2_c} + k'\phi_{3_c} + \text{etc.}$, and Φ and Φ_c are to be substituted for ϕ and ϕ_c in (308). For example, suppose $\Phi = \phi + g$; then

$$(\phi + g)^{-1}\rho = \frac{|(\phi_c + g)\lambda\cdot\mu|\rho - |(\phi_c + g)\mu\cdot\lambda|\rho}{(\phi_c + g)\lambda(\phi_c + g)\mu}$$

$$= \frac{\phi^{-1}\rho\cdot\phi_c\lambda\phi_c\mu + \lambda\mu\cdot\rho\cdot g}{\phi_c\lambda\phi_c\mu + g(\lambda\phi_c\mu - \mu\phi_c\lambda) + \lambda\mu\cdot g^2}$$

$$= \frac{m_0\phi^{-1}\rho + g\rho}{m_0 + m_1 g + g_2},$$

if we write

$$m_0 = \frac{\phi_c\lambda\phi_c\mu}{\lambda\mu} \text{ and } m_1 = \frac{\lambda\phi_c\mu - \mu\phi_c\lambda}{\lambda\mu}. \quad . \quad . \qquad (309)$$

Hence $(m_0 + m_1 g + g^2)\rho = (\phi + g)(m_0\phi^{-1}\rho + g\rho)$
$$= m_0\rho + g\phi\rho + m_0 g\phi^{-1}\rho + g^2\rho.$$

$$\therefore\ m_1 g\rho = g\phi\rho + m_0 g\phi^{-1}\rho,$$

or
$$\phi^{-1}\rho = \frac{(m_1 - \phi)\rho}{m_0}. \qquad (310)$$

Thus we have still another inversion formula, which is, however, not generally so convenient as (308).

Operating by ϕ, (310) assumes the form

$$(\phi^2 - m_1\phi + m_0)\rho = 0, \qquad (311)$$

The quantities m_0 and m_1 are *invariants, i.e.* their value will be unchanged if other vectors be substituted for λ and μ. For, let $x_1\lambda + y_1\mu$ and $x_2\lambda + y_2\mu$ be any other two vectors in the plane space under consideration, and substitute them for λ and μ:

$$\therefore\ m_0 = \frac{\phi_c(x_1\lambda + y_1\mu)\,\phi_c(x_2\lambda + y_2\mu)}{(x_1\lambda + y_1\mu)(x_2\lambda + y_2\mu)}$$
$$= \frac{x_1y_2\phi_c\lambda\phi_c\mu + x_2y_1\phi_c\mu\phi_c\lambda}{x_1y_2\lambda\mu + x_2y_1\mu\lambda} = \frac{\phi_c\lambda\phi_c\mu}{\lambda\mu}.$$

Similarly,

$$m_1 = \frac{(x_1\lambda + y_1\mu)\,\phi_c(x_2\lambda + y_2\mu) - (x_2\lambda + y_2\mu)\,\phi_c(x_1\lambda + y_1\mu)}{(x_1\lambda + y_1\mu)(x_2\lambda + y_2\mu)}$$
$$= \frac{(x_1y_2 - x_2y_1)(\lambda\phi_c\mu - \mu\phi_c\lambda)}{(x_1y_2 - x_2y_1)\lambda\mu} = \frac{\lambda\phi_c\mu - \mu\phi_c\lambda}{\lambda\mu}.$$

From the above it follows that, in any given case, we may assume such values of λ and μ as may be most convenient. For example, take $\phi\rho = \epsilon_1 \cdot \epsilon_1'|\rho + \epsilon_2 \cdot \epsilon_2'|\rho$, as in (302); let $\lambda = |\epsilon_1$ and $\mu = |\epsilon_2$; then

$$\phi_c\lambda = \phi_c|\epsilon_1 = \epsilon_2' \cdot \epsilon_1\epsilon_2, \quad \phi_c\mu = \phi_c|\epsilon_2 = -\,\epsilon_1' \cdot \epsilon_1\epsilon_2.$$

$$\therefore\ \phi^{-1}\rho = \frac{|\epsilon_2' \cdot \epsilon_1\epsilon_2 \cdot \epsilon_2\rho + |\epsilon_1' \cdot \epsilon_1\epsilon_2 \cdot \epsilon_1\rho}{\epsilon_1'\epsilon_2' \cdot (\epsilon_1\epsilon_2)^2} = \frac{\epsilon_1\rho \cdot |\epsilon_1' + \epsilon_2\rho \cdot |\epsilon_2'}{\epsilon_1\epsilon_2 \cdot \epsilon_1'\epsilon_2'}.$$

99. EXERCISES. — (1) Invert

$$\phi\rho = \frac{1}{(\alpha\beta)^2}(\alpha \cdot \alpha|\rho + \beta \cdot \beta|\rho).$$

Find ρ as a function of ϵ in the following cases:

(2) $\alpha \cdot \alpha\rho + \beta \cdot \beta|\rho = \epsilon.$

(3) $A\rho + \alpha \cdot \alpha|\rho = \epsilon$, A being a scalar constant.

(4) $A\rho + \alpha \cdot \alpha\rho + \beta \cdot \beta\rho = \epsilon.$

100. *The general equation of the second degree in plane space.* This equation may be written

$$\rho|\phi\rho + 2\gamma|\rho = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (312)$$

For every term of the second degree in ρ may be expressed in the form $\rho|\epsilon \cdot \rho|\epsilon'$, so that the portion of the left-hand member which is of the second degree will consist of the sum of a series of such terms, *i.e.* $\Sigma_1^n(\rho|\epsilon \cdot \rho|\epsilon')$. This form evidently includes such forms as $(\rho|\epsilon)^2$, equivalent to $\rho|\epsilon \cdot \rho|\epsilon$, and $A\rho^2$, equivalent to $A((\rho|\iota_1)^2 + (\rho|\iota_2)^2 + (\rho|\iota_3)^2)$. If we write now $\Sigma_1^n(\epsilon \cdot \rho|\epsilon') = \phi\rho$, the second-degree terms assume the form $\rho|\phi\rho$, but ϕ will not be, in general, self-conjugate. If, however, we write

$$\tfrac{1}{2}\Sigma_1^n(\epsilon \cdot \rho|\epsilon' + \epsilon' \cdot \rho|\epsilon) = \phi\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (313)$$

then the second-degree terms will still assume the form $\rho|\phi\rho$, and ϕ *will* be self-conjugate because it is the sum of a linear vector function and its conjugate. See Art. 97.

The self-conjugate form of ϕ for any set of second-degree terms may be obtained by differentiating these terms, a method which may be convenient for the beginner. For suppose ϕ *not* self-conjugate in the expression $\rho|\phi\rho$; differentiating, we have

$$d\rho|\phi\rho + \rho|\phi d\rho = d\rho|\phi\rho + d\rho|\phi_c\rho = d\rho|(\phi + \phi_c)\rho,$$

and $\phi + \phi_c$ is self-conjugate. Also,

$$\rho|\phi_c\rho = \rho|\phi\rho, \text{ so that } \tfrac{1}{2}\rho|(\phi + \phi_c)\rho = \rho|\phi\rho.$$

For example, let

$$\rho|\phi\rho = \rho^2 + \rho\epsilon \cdot \rho|\epsilon'.$$

Then, $2\,d\rho|\rho + d\rho\epsilon \cdot \rho|\epsilon' + \rho\epsilon \cdot d\rho|\epsilon' = d\rho|(2\,\rho - |\epsilon \cdot \rho|\epsilon' + \epsilon' \cdot \rho\epsilon)$ is the differential; and if we put $\phi\rho = \rho + \frac{1}{2}(\epsilon' \cdot \rho\epsilon - |\epsilon \cdot \rho|\epsilon')$, we have $\rho|\phi\rho = \rho^2 + \rho\epsilon \cdot \rho|\epsilon'$. All terms of the first degree may evidently be combined in one; for example,

$$\epsilon_1\rho + \epsilon_2|\rho + \epsilon_3|\rho + \text{etc.} = \rho|(|\epsilon_1 + \epsilon_2 + \epsilon_3) = \rho|(2\,\gamma) = 2\,\gamma|\rho,$$

so that $\gamma = \frac{1}{2}(|\epsilon_1 + \epsilon_2 + \epsilon_3)$.

101. To determine the locus represented by eq. 312, we will first change the origin to the center, if the curve has such a point. Let δ be the vector to the center, and ρ' the new vector radius of any point; then, $\rho = \delta + \rho'$. Substituting in (312), we have

$$(\delta + \rho')|\phi(\delta + \rho') + 2\,\gamma|(\delta + \rho') = C,$$

or, expanding,

$$\delta|\phi\delta + 2\,\delta|\phi\rho' + \rho'|\phi\rho' + 2\,\gamma|\delta + 2\,\gamma|\rho' = C.$$

If the origin be now at the center, the terms of the first degree must disappear, because the equation must be unchanged when $-\rho'$ is substituted for $+\rho'$. Thus we must have

$$\rho'|(\phi\delta + \gamma) = 0.$$

But this is to be true for all values of ρ'; hence,

$$\phi\delta + \gamma = 0, \text{ or } \delta = -\phi^{-1}\gamma = \frac{\lambda|\gamma \cdot |\phi\mu - \mu|\gamma \cdot |\phi\lambda}{\phi\lambda\phi\mu}. \quad (314)$$

Eq. 314 gives the value of δ, the vector to the center. When $\phi\lambda\phi\mu$ differs from zero, the locus has a finite center; but when $\phi\lambda\phi\mu = 0$, the center is at ∞, unless the numerator is also zero. In this last case, δ becomes indeterminate; it is in fact the vector radius to a straight line, the locus of centers, whose equation is

$$-\phi\delta = \gamma, \quad \ldots \quad (315)$$

and (312) now represents two parallel or coincident right lines.

The equation

$$\phi\lambda\phi\mu = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (316)$$

requires that the function ϕ should consist of a single constant vector multiplied by a scalar factor, or should be reducible to that form. For, taking the general form of ϕ as given in Art. 97, we have

$$\phi\lambda\phi\mu = (\epsilon_1 \cdot \lambda|\epsilon_1' + \epsilon_2 \cdot \lambda|\epsilon_2')(\epsilon_1 \cdot \mu|\epsilon_1' + \epsilon_2 \cdot \mu|\epsilon_2')$$
$$= \epsilon_1\epsilon_2 \cdot (\lambda|\epsilon_1' \cdot \mu|\epsilon_2' - \lambda|\epsilon_2' \cdot \mu|\epsilon_1') = 0.$$

Since λ and μ are to have any values we choose to assign, this equation can only be satisfied by making

$$\epsilon_1\epsilon_2 = 0, \quad i.e. \quad \epsilon_2 = n\epsilon_1.$$

Thus we have

$$\phi\rho = \epsilon_1 \cdot \rho|(\epsilon_1' + n\epsilon_2').$$

But we are dealing only with *self-conjugate* functions; therefore the vector appearing in the scalar factor must be the same as the other; *i.e.* we must have the function of the form

$$\phi\rho = \epsilon \cdot \epsilon|\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (317)$$

Hence, when (316) is satisfied, and the numerator of the value of δ is not zero, eq. (312) takes the form

$$(\epsilon|\rho)^2 + 2\lambda|\rho = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (318)$$

Since we know now that the center is at ∞, change the origin to a point on the curve, by putting $\rho' + \delta'$ for ρ.

$$\therefore \; (\epsilon|(\rho' + \delta'))^2 + 2\gamma|(\rho' + \delta') = C = (\epsilon|\rho')^2 + 2\epsilon|\rho' \cdot \epsilon|\delta' + (\epsilon|\delta')^2$$
$$+ 2\gamma|\rho' + 2\gamma|\delta'.$$

If the origin be now on the curve, the equation must be satisfied when $\rho' = 0$; whence

$$(\epsilon|\delta')^2 + 2\gamma|\delta' = C,$$

an equation for determining δ'. The equation thus becomes

$$(\epsilon|\rho')^2 + 2(\epsilon \cdot \delta'|\epsilon + \gamma)|\rho' = 0, \quad . \quad . \quad . \quad . \quad (319)$$

which is of the same form as equations of Art. 84, which were shown to represent *parabolas*.

102. Resuming now the general equation with the origin at the center, it becomes by (314),

$$\rho|\phi\rho = C - \delta|\phi\delta - 2\gamma|\delta = C + \gamma|\phi^{-1}\gamma = C', \text{ say.} \quad . \quad . \quad (320)$$

Let us find the points at which the tangent is perpendicular to ρ; *i.e.* ρ is parallel to $\phi\rho$, or

$$\phi\rho = g\rho, \text{ or } (\phi - g)\rho = 0, \quad . \quad . \quad . \quad . \quad . \quad (321)$$

g being a scalar constant to be determined.

Eqs. (320) and (321) give

$$\rho|\phi\rho = C' = g\rho^2, \text{ or } \rho^2 = T^2\rho = \frac{C'}{g}. \quad . \quad . \quad (322)$$

Multiply the complement of (321) successively by any two vectors λ and μ; therefore

$$\lambda|(\phi - g)\rho = \rho|(\phi - g)\lambda = 0$$

and $$\mu|(\phi - g)\rho = \rho|(\phi - g)\mu = 0.$$

In order that ρ may be simultaneously perpendicular to $(\phi - g)\lambda$ and $(\phi - g)\mu$, these vectors must be parallel; hence we must have

$$(\phi - g)\lambda(\phi - g)\mu = 0$$

or $$g^2 - m_1 g + m_0 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (323)$$

m_0 and m_1 having the values given in (309).

Eqs. (322) and (323) show that $T\rho$ has two pairs of values, the values in each pair being numerically equal but of opposite sign. Thus there are four points at the opposite ends of two diameters at which $\rho\phi\rho = 0$.

Let g_1 and g_2 be the two roots of (323), and ρ_1 and ρ_2 the corresponding values of ρ. Then, by (321), we must have

$$\phi\rho_1 = g_1\rho_1 \text{ and } \phi\rho_2 = g_2\rho_2.$$

The most general form of a self-conjugate function may be written

$$2\phi\rho = g_1(\epsilon_1 \cdot \rho|\epsilon_1' + \epsilon_1' \cdot \rho|\epsilon_1) + g_2(\epsilon_2 \cdot \rho|\epsilon_2' + \epsilon_2' \cdot \rho|\epsilon_2).$$

whence, by the conditions above,

$$2\,\phi\rho_1 = g_1(\epsilon_1 \cdot \rho_1|\epsilon_1' + \epsilon_1' \cdot \rho_1|\epsilon_1) + g_2(\epsilon_2 \cdot \rho_1|\epsilon_2' + \epsilon_2' \cdot \rho_1|\epsilon_2) = 2\,g_1\rho_1,$$

and

$$2\,\phi\rho_2 = g_1(\epsilon_1 \cdot \rho_2|\epsilon_1' + \epsilon_1' \cdot \rho_2|\epsilon_1) + g_2(\epsilon_2 \cdot \rho_2|\epsilon_2' + \epsilon_2' \cdot \rho_2|\epsilon_2) = 2\,g_2\rho_2.$$

Hence $\rho_1|\epsilon_2' = \rho_1|\epsilon_2 = \rho_2|\epsilon_1' = \rho_2|\epsilon_1 = 0$; therefore ϵ_2 and ϵ_2' are both perpendicular to ρ_1, and hence parallel to each other, and ϵ_1 and ϵ_1' are both perpendicular to ρ_2, and hence parallel to each other. Therefore let $\epsilon_1' = \epsilon_1$ and $\epsilon_2' = \epsilon_2$; then we have

$$\phi\rho_1 = g_1 \cdot \epsilon_1 \cdot \rho_1|\epsilon_1 = g_1\rho_1 \text{ and } \phi\rho_2 = g_2 \cdot \epsilon_2 \cdot \rho_2|\epsilon_2 = g_2\rho_2.$$

Hence ϵ_1 is $\|$ to ρ_1 and ϵ_2 is $\|$ to ρ_2, also $\epsilon_1^2 = T^2\epsilon_1 = 1 = \epsilon_2^2 = T^2\epsilon_2$; thus ϵ_1 and ϵ_2 are *unit normal* vectors, and it appears that, whatever may be the original form of ϕ, it may always be reduced to the form

$$\phi\rho = g_1\iota_1 \cdot \rho|\iota_1 + g_2\iota_2 \cdot \rho|\iota_2,$$

ι_1 and ι_2 being unit normal vectors parallel to the values of ρ which satisfy the equation (321). Now, by (322),

$$g_1 = \frac{C'}{\rho_1^2} = \frac{C'}{a^2}, \text{ say, and } g_2 = \frac{C'}{\rho_2^2} = \frac{C'}{b^2}, \text{ say;}$$

so that ϕ has the form of (281), and (320) represents an ellipse or hyperbola, according as g_1 and g_2 are both positive, or one is positive and one negative, provided that C' be a positive quantity as it may always be taken. If g_1 and g_2 are both negative, the curve is imaginary. The vectors $(\phi - g_1)\lambda$ and $(\phi - g_2)\lambda$, or $(\phi - g_1)\mu$ and $(\phi - g_2)\mu$ are respectively perpendicular to the axes, which are thus completely determined.

In (323) we have $m_0 = g_1g_2$ and $m_1 = g_1 + g_2$; hence, when g_1 and g_2 are both $+$, m_0 is $+$; and if one of them be $-$, m_0 is $-$; consequently we have

$$m_0\left(=\frac{\phi\lambda\phi\mu}{\lambda\mu}\right)\left|\begin{array}{l} + \text{ for the ellipse} \\ 0 \text{ for the parabola} \\ - \text{ for the hyperbola}\end{array}\right| . \quad . \quad . \quad . \qquad (324)$$

103. EXERCISES. — (1) To discuss the equation

$$A\rho^2 + n(\epsilon|\rho)^2 + 2\gamma|\rho = C.$$

In this case $\phi\rho = A\rho + n\epsilon \cdot \rho|\epsilon$; suppose that $T\epsilon = 1$, and put $\lambda = \epsilon$ and $\mu = |\epsilon$; then $\phi\lambda = \phi\epsilon = (A+n)\epsilon$, $\phi\mu = \phi|\epsilon = A \cdot |\epsilon$.

$$\therefore\ \delta = -\frac{(A+n)\cdot|\epsilon\cdot\epsilon\gamma + A\epsilon\cdot\epsilon|\gamma}{A(A+n)} = -\frac{A\gamma + n\cdot|\epsilon\cdot\epsilon\gamma}{A(A+n)},$$

the last by (98).

The center is at ∞, and the curve a parabola, first, when $A = 0$, second, when $A + n = 0$, unless at the same time the numerator of the value of δ is zero.

When $A = 0$, $\delta = \frac{n\cdot|\epsilon\cdot\epsilon\gamma}{0}$, and $n\epsilon\gamma$ is zero when $n = 0$, when $\gamma = 0$, or when ϵ is parallel to γ. These cases correspond respectively to the line $\gamma|\rho = \frac{1}{2}C$, the parallel lines $\epsilon|\rho = \pm\sqrt{\frac{C}{n}}$, and the parallel lines $n\cdot\epsilon|\rho = -m \pm\sqrt{m^2 + nC}$, in which $m\epsilon$ has been substituted for γ.

When $A = -n$, $\delta = \frac{\epsilon\cdot\epsilon|\gamma}{0}$, and $\delta = \frac{0}{0}$, when $\gamma = 0$, or ϵ is perpendicular to γ. These cases correspond respectively to the pairs of parallel lines

$$\epsilon\rho = \pm\sqrt{\frac{C}{A}} \text{ and } A\cdot\epsilon\rho = -m \pm\sqrt{m^2 + AC}.$$

When $A(A+n)$ is not zero, the equation represents a central conic, of which we will find the axes.

We find $\qquad m_0 = A(A+n),\ m_1 = 2A + n;$

hence (323) becomes

$$g^2 - (2A+n)g + A(A+n) = 0 = (g-A)(g-A-n).$$

Thus $a^2 = \frac{C'}{A}$, $b^2 = \frac{C'}{A+n}$, in which $C' = C + \frac{A\gamma^2 + n(\epsilon\gamma)^2}{A(A+n)}$.

$(\phi - g_1)\lambda = (\phi - A)\epsilon = n\epsilon$ is $\perp$ to the a axis, so that the axes are now completely determined.

(2) Discuss the following equations:

(*a*) $(\alpha\rho)^2 \pm (\beta\rho)^2 = (\alpha\beta)^2$.

(*b*) $\rho^2 - \alpha|\rho \cdot \beta|\rho - \alpha\rho \cdot \beta\rho - (\alpha+\beta)|\rho = 1$, when $T\alpha = T\beta = 1$.

(*c*) $(\rho|(\alpha-\beta))^2 - (\rho|(\alpha+\beta))^2 = 4(\alpha\beta)^2$, when $T\alpha = T\beta = 1$.

(*d*) $\epsilon^2\rho^2 = e^2[\epsilon|(\rho - \epsilon)]^2$, e being a scalar constant.

In the last equation consider the cases when $e > 1$, $e < 1$, $e = 1$.

(3) Show that $C + \gamma|\phi^{-1}\gamma = 0$ is the condition that (312) shall represent two straight lines, real or imaginary.

(4) Find in how many ways a conic passing through the four common points of two given conics can be reduced to two right lines, using the condition of Exercise 3.

(5) Show that, if two conics have their axes parallel, any conic passing through the common points of these two will have its axes parallel to theirs.

(6) Hence show that, if a pair of right lines be drawn through the four common points of a circle and any conic, the bisectors of the angles between these two lines will be parallel to the axes of the conic.

CHAPTER IV.

SCALAR POINT EQUATIONS OF THE SECOND DEGREE IN PLANE SPACE.

104. We need only consider *homogeneous* equations; for, by eq. (224), $3p|\bar{e}=1$, so that a term of any degree in p may be raised to any other degree by multiplying by the proper power of $3p|\bar{e}$.

Any homogeneous equation of the second degree may be written in the form

$$p|\phi p=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (325)$$

in which ϕ is a linear self-conjugate function; for the left-hand member of such an equation can always be reduced to the sum of such terms as $A_1p|q_1\cdot p|q_1'$; that is, to the form

$$\Sigma(Ap|q\cdot p|q')=\tfrac{1}{2}\Sigma[p|(Aq\cdot p|q'+Aq'\cdot p|q)]$$
$$=\tfrac{1}{2}p|\Sigma[A(q\cdot p|q'+q'\cdot p|q)]=p|\phi p,$$

if we write

$$\phi p=\tfrac{1}{2}\Sigma[A(q\cdot p|q'+q'\cdot p|q)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (326)$$

Of course we may have $q=q'$ for some terms of the summation.

105. Eq. (325) represents a curve of the second order; that is, it is cut in two points by any right line. For, let $p=xq_1+yq_2$ be the equation of some right line, and substitute in (325):

$$\therefore\ (xq_1+yq_2)|\phi(xq_1+yq_2)$$
$$=x^2q_1|\phi q_1+2\,xyq_1|\phi q_2+y^2q_2|\phi q_2=0\,;$$

whence
$$\frac{y}{x}=\frac{-\,q_1|\phi q_2\pm\sqrt{(q_1|\phi q_2)^2-q_1|\phi q_1\cdot q_2|\phi q_2}}{q_2|\phi q_2}$$
$$=\frac{-\,q_1|\phi q_2\pm\sqrt{-\,q_1q_2|\phi q_1\phi q_2}}{q_2|\phi q_2}. \quad . \quad . \quad . \quad . \quad . \quad (327)$$

As $\frac{y}{x}$ has two values, it appears that the line must cut the curve at two points.

If the two values of $\frac{y}{x}$ are equal, the line must be tangent to the curve. The condition for this is

$$q_1 q_2 | \phi q_1 \phi q_2 = 0; \quad \ldots \quad (328)$$

and when this condition is satisfied, $\frac{y}{x} = -\frac{q_1 | \phi q_2}{q_2 | \phi q_2}$; so that the equation

$$p = \frac{q_1 \cdot q_2 | \phi q_2 - q_2 \cdot q_1 | \phi q_2}{q_2 | \phi q_2 - q_1 | \phi q_2} \quad \ldots \quad (329)$$

gives the point of contact of the line with the curve when (328) is satisfied.

In (328) suppose q_2 to vary, and replace it by p; then the equation

$$q_1 p | \phi q_1 \phi p = 0 \quad \ldots \quad (330)$$

causes p to be always on a straight line passing through the fixed point q_1 and tangent to the locus of (325). As (330) is of the second degree in p, there must be two such lines; *i.e.* two tangents to the locus through any point q_1. It will sometimes be more convenient to write the equation of the cutting line in the form

$$p = q_1 + y\epsilon;$$

when we must put in (327) $x = 1$, and ϵ for q_2, thus obtaining

$$y = \frac{-q_1 | \phi\epsilon \pm \sqrt{-q_1 \epsilon | \phi q_1 \phi \epsilon}}{\epsilon | \phi \epsilon}. \quad \ldots \quad (331)$$

106. *Diameters.* To find the locus of the middle points of a system of parallel chords of the locus of (325).

In (331) let q_1 be on the curve, so that we have $q_1 | \phi q_1 = 0$; then $y = \frac{-2 q_1 | \phi\epsilon}{\epsilon | \phi\epsilon}$. At the middle point of a chord having the direction ϵ, we have $p = q_1 + \frac{1}{2} y\epsilon = q_1 - \frac{q_1 \phi\epsilon}{\epsilon | \phi\epsilon} \cdot \epsilon$.

$$\therefore p | \phi\epsilon = q_1 | \phi\epsilon - q_1 | \phi\epsilon = 0. \quad \ldots \quad (332)$$

This is the equation of a diameter conjugate in direction to ϵ. Let p_1 and p_2 be any two points in this diameter, so that we have $p_1|\phi\epsilon=0$ and $p_2|\phi\epsilon=0$, and therefore $(p_2-p_1)|\phi\epsilon=0$. Now p_2-p_1, being a vector along the diameter, is conjugate to ϵ; hence two conjugate directions ϵ_1 and ϵ_2 must satisfy the condition

$$\epsilon_1|\phi\epsilon_2=0. \qquad (333)$$

107. *Tangent and polar.* Differentiating (325), we have

$$dp|\phi p=0.$$

Hence $|\phi p$ is parallel to the tangent at p; but (335) shows that the line $|\phi p$ passes *through* p; consequently $|\phi p$ *is* the tangent line to the locus at p. The equation of the tangent line is therefore

$$q|\phi p=0, \qquad (334)$$

q being a variable point, and p a point on the curve.

If e be some point *not* on the curve, let us determine what line $|\phi e$ is. Suppose tangents to be drawn from e to the curve, touching it at p_1 and p_2. Then these tangents will be $|\phi p_1$ and $|\phi p_2$. But as they pass through e, we must have

$$e|\phi p_1=0=p_1|\phi e$$

and

$$e|\phi p_2=0=p_2|\phi e.$$

These conditions show that $|\phi e$ passes through p_1 and p_2, the points of contact of the tangents drawn to the curve from e. $|\phi e$ is therefore the *polar* of e. Let q be any point on $|\phi e$; then we must have $q|\phi e=0=e|\phi q$; so that, wherever q be situated on the polar of e, *its* polar always passes through e.

Thus if a point move along a straight line, its polar passes through a fixed point, the pole of this line; and, reciprocally, if a revolving line pass through a fixed point, its pole moves along a fixed line, the polar of this point.

Equation (332) shows that a diameter is the polar of a point at ∞. Hence the polars of all points on a diameter have a common point at ∞; *i.e.* they are parallel to the diameter conjugate to this.

108. *Center of the locus.* The center is at the intersection of any two diameters; hence

$$q_c = m|\phi\epsilon_1\phi\epsilon_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (335)$$

is the center, m being a scalar factor so taken as to make q_c a unit point. To evaluate m, multiply both sides of the equation into $3|\bar{e}$; therefore

$$3q_c|\bar{e} = 1 = 3m|\phi\epsilon_1\phi\epsilon_2\bar{e} = 3m\bar{e}\phi\epsilon_1\phi\epsilon_2$$

so that (335) becomes

$$q_c = \frac{|\phi\epsilon_1\phi\epsilon_2}{3\,\bar{e}\phi\epsilon_1\phi\epsilon_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (336)$$

If we have $\quad \bar{e}\phi\epsilon_1\phi\epsilon_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (337)$

while $|\phi\epsilon_1\phi\epsilon_2$ is *not* zero, then the center of the locus is at ∞.

If q_c be substituted for q_1 in (330), we have the equation of the tangents passing through the center, that is, of the asymptotes, viz.:

$$pq_c|\phi p\phi q_c = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (338)$$

109. *Conjugate points.* Any set of three points which fulfil the conditions

$$q_1|\phi q_2 = q_2|\phi q_3 = q_3|\phi q_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (339)$$

is a set of *conjugate points.* These equations cause each point to be on the polar of each of the others; that is, the points are the vertices of a self-conjugate triangle, in which each side is the polar of the opposite vertex with reference to the curve $p|\phi p = 0$.

There is an infinite number of such sets of points; for take *any* point in the plane of the curve as q_1, then any point in the polar of q_1 as q_2, whose polar will pass through q_1, by Art. 107, and will cut the polar of q_1 in q_3. If one point, say q_3, is at ∞, the other two will be on a diameter; if q_2 be also at ∞, then q_1 will be at the intersection of two diameters, that is, it will be at the *center;* thus q_c, ϵ_1, ϵ_2 form a conjugate system if we have $q_c|\phi\epsilon_1 = \epsilon_1|\phi\epsilon_2 = \epsilon_2|\phi q_c$, and we see that conjugate directions are only a particular case of conjugate points.

110. *Normal system of conjugate points.* If a system of conjugate points, besides the conditions (339), satisfy also the conditions

$$q_1|q_2 = q_2|q_3 = q_3|q_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (340)$$

they may be called a *normal* system. We proceed to show, that with reference to any curve represented by the equation $p|\phi p = 0$, there is one, and only one, normal system of conjugate points.

111. *Solution of the equation $p\phi p = 0$.* This equation is equivalent to $\phi p = np$, or $(\phi - n)p = 0$. Multiply the complement of the first member by any three points q_1, q_2, q_3, and we have three scalar equations equivalent to the single non-scalar equation, viz.:

$$\left.\begin{array}{l} q_1|(\phi - n)p = p|(\phi - n)q_1 = 0 \\ q_2|(\phi - n)p = p|(\phi - n)q_2 = 0 \\ q_3|(\phi - n)p = p|(\phi - n)q_3 = 0 \end{array}\right\} . \quad . \quad . \quad . \quad . \quad (341)$$

Each of these equations must be satisfied by the same values of p which satisfy the given equation; *i.e.* they are simultaneous equations. The point p must therefore be simultaneously in each of the three lines $|(\phi - n)q_1$, $|(\phi - n)q_2$, $|(\phi - n)q_3$, which requires that these lines shall have a common point, the condition for which is

$$\left.\begin{array}{l} (\phi - n)q_1(\phi - n)q_2(\phi - n)q_3 = 0 \\ \text{or} \quad n^3 - k_2n^2 + k_1n - k_0 = 0 \end{array}\right\}, \quad . \quad . \quad . \quad . \quad (342)$$

$$\text{in which } \left.\begin{array}{l} k_0 = \phi q_1\phi q_2\phi q_3 \div q_1q_2q_3 \\ k_1 = (q_1\phi q_2\phi q_3 + q_2\phi q_3\phi q_1 + q_3\phi q_1\phi q_2) \div q_1q_2q_3 \\ k_2 = (q_1q_2\phi q_3 + q_2q_3\phi q_1 + q_3q_1\phi q_2) \div q_1q_2q_3 \end{array}\right\} . \quad (343)$$

The k's are *invariants; i.e.* they have the same values whatever position the points q_1, q_2, q_3 may occupy, which may be shown as in Art. 98 in the case of m_0 and m_1. The solution of (342) will give three values of n, which, substituted in (341), will give the required points at which $\phi p = np$. Let the roots of (342) be n_1, n_2, n_3, and let the corresponding values of p be p_1, p_2, p_3; then, by (341), the equations

$$\left.\begin{aligned} p_1|(\phi - n_1)q_1(\phi - n_1)q_2 = 0 \\ p_2|(\phi - n_2)q_1(\phi - n_2)q_2 = 0 \\ p_3|(\phi - n_3)q_1(\phi - n_3)q_2 = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (344)$$

give the points p_1, p_2, p_3.

112. *To show that the points just determined form a normal conjugate system.* We must have $\phi p_1 = n_1 p_1$, $\phi p_2 = n_2 p_2$, $\phi p_3 = n_3 p_3$. Now, as the function ϕ is self-conjugate, write it in the most general form of such a function in terms of the p's; that is,

$$2\phi p = n_1(p_1 \cdot p_2 p_3 p + |p_2 p_3 \cdot p_1|p) + n_2(p_2 \cdot p_3 p_1 p + |p_3 p_1 \cdot p_2|p) \\ + n_3(p_3 \cdot p_1 p_2 p + |p_1 p_2 \cdot p_3|p).$$

If this value of ϕ satisfies the conditions above, we must have $p_1|p_2 = p_2|p_3 = p_3|p_1 = 0$ and $p_1 p_2 p_3 = p_1^2 = p_2^2 = p_3^2 = 1$.

These conditions give at once

$$2\phi p_1 = n_1(p_1 + |p_2 p_3), \text{ etc.};$$

but they also cause p_1 to be on the lines $|p_2$ and $|p_3$ simultaneously, so that $p_1 = m|p_2 p_3$, m being some scalar constant. Hence $p_1|p_1 = m p_1 p_2 p_3$, or $m = 1$, so that the required condition $\phi p_1 = n_1 p_1$ is satisfied; and so for the others.

Finally $p_1|\phi p_2 = n_2 p_1|p_2 = 0$, etc., so that all the conditions of (339) and (340) are satisfied, and hence the points p_1, p_2, p_3 form a normal conjugate system. It appears then that, whatever be the original form of ϕ, one set of three points may always be found such that, if these be taken for reference points, ϕ is reduced to the form

$$\phi p = n_1 p_1 \cdot p|p_1 + n_2 p_2 \cdot p|p_2 + n_3 p_3 \cdot p|p_3. \quad . \quad (345)$$

Note that these will not in general be all *unit* points; for if we express p_1, p_2, p_3 in terms of the original reference points, we have *nine* constants, and have subjected the points to *seven* conditions, so that we cannot apply the *three* additional conditions necessary for unit points.

The three points above determined will always be *real, i.e.* the roots of (342) are always *real;* for, suppose one to be imaginary, and call it $n + n'i$, and the corresponding value of p, $p + p'i$, in which $i = \sqrt{-1}$; then

$$\phi(p + ip') = (n + in')(p + ip');$$

or, equating separately to zero real and imaginary parts,

$$\phi p = np - n'p', \quad \phi p' = n'p + np'.$$

$$\therefore\ p'|\phi p = np'|p - n'p'^2 = p|\phi p' = n'p^2 + np|p'.$$

$$\therefore\ n'(p^2 + p'^2) = 0,$$

which can only be satisfied by $n' = 0$, so that there can be no imaginary value of n or p.

113. *Canonical form of* $p|\phi p$. With the form of ϕ given in (345) we have

$$p|\phi p = n_1(p|p_1)^2 + n_2(p|p_2)^2 + n_3(p|p_3)^2, \quad . \quad (346)$$

which is the canonical form of the scalar quadratic in p, and we have shown that, whatever may have been the original form of $p|\phi p$, it can always be reduced to this canonical form by properly choosing the reference points. The first two terms of (346) may be written

$$p|(p_1\sqrt{n_1} + p_2\sqrt{-n_2}) \cdot p|(p_1\sqrt{n_1} - p_2\sqrt{-n_2}),$$

or $p|q_1 \cdot p|q_2$, if we put

$$q_1 = p_1\sqrt{n_1} + p_2\sqrt{-n_2} \text{ and } q_2 = p_1\sqrt{n_1} - p_2\sqrt{-n_2}.$$

Thus the equation of the locus becomes

$$p|q_1 \cdot p|q_2 + n_3(p|p_3)^2 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (347)$$

In this form the equation shows that the curve is tangent to $|q_1$ and $|q_2$ at the points where they are cut by $|p_3$. But, by the last article, $p_3|q_1 = 0$, and $p_3|q_2 = 0$, so that $|q_1$ and $|q_2$ pass through p_3, and touch the curve where it is cut by $p_1p_2 = |p_3$.

114. *Condition that $p|\phi p$ shall break up into two factors of the first degree.* If this be possible, it will take the form

$$p|\phi p = p|q_1 \cdot p|q_2,$$

whence $$\phi p = \tfrac{1}{2}(q_1 \cdot p|q_2 + q_2 \cdot p|q_1).$$

If we take any three values of p, as p_1, p_2, p_3, the points ϕp_1, ϕp_2, ϕp_3 will all be on the line $q_1 q_2$, so that their product will be zero. Conversely, if we have

$$\phi p_1 \phi p_2 \phi p_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (348)$$

for any three points whatever, p_1, p_2, p_3, then the function ϕ must have the above form, and $p|\phi p$ is factorable. The expression $\phi p_1 \phi p_2 \phi p_3$ is the *discriminant* of $p|\phi p$, and is the same as $p_1 p_2 p_3 \cdot k_0$ of eq. (343).

115. *Nature of the locus at ∞.* To determine this we will find the intersection of the line at ∞ with the locus. In (327) put ϵ_1 and ϵ_2 for q_1 and q_2, thus obtaining

$$\frac{y}{x} = \frac{-\epsilon_1|\phi\epsilon_2 \pm \sqrt{-\epsilon_1\epsilon_2|\phi\epsilon_1\phi\epsilon_2}}{\epsilon_2|\phi\epsilon_2}.$$

Now, as ϵ_1 and ϵ_2 may be any points at ∞ whatever, write $\epsilon_1 = e_1 - e_0$, $\epsilon_2 = e_2 - e_0$; then

$$\epsilon_1\epsilon_2 = e_1e_2 + e_2e_0 + e_0e_1 = |(e_0 + e_1 + e_2) = 3|\bar{e},$$

so that the quantity under the radical becomes $-3\bar{e}\phi\epsilon_1\phi\epsilon_2$. Thus the two vectors

$$\epsilon_2|\phi\epsilon_2 \cdot \epsilon_1 + (-\epsilon_1|\phi\epsilon_2 + \sqrt{-3\bar{e}\phi\epsilon_1\phi\epsilon_2})\epsilon_2$$

and $$\epsilon_2|\phi\epsilon_2 \cdot \epsilon_1 - (\epsilon_1|\phi\epsilon_2 + \sqrt{-3\bar{e}\phi\epsilon_1\phi\epsilon_2})\epsilon_2 \quad . \quad . \quad (349)$$

are respectively in the direction of the two points at ∞ of the locus. These vectors are imaginary, parallel, or real, according as $\bar{e}\phi\epsilon_1\phi\epsilon_2$ is $+$, 0, or $-$, corresponding respectively to *no* real points at ∞, two *coincident* points at ∞, and two *real* points at ∞. In the second case the vectors of (349) are parallel to the axis of the parabola; in the third case they are parallel to the asymptotes of the hyperbola. In the first case the curve is an ellipse. We have thus

$$\bar{e}\phi\epsilon_1\phi\epsilon_2 \left| \begin{array}{l} \text{positive for an ellipse} \\ \text{zero for a parabola} \\ \text{negative for an hyperbola} \end{array} \right| . \quad . \quad . \quad . \qquad (350)$$

116. The most general form of the homogeneous second-degree equation in p may evidently be written

$$A(p|e_0)^2 + B(p|e_1)^2 + C(p|e_2)^2 + 2A'p|e_1 \cdot p|e_2 \\ + 2B'p|e_2 \cdot p|e_0 + 2C'p|e_0 \cdot p|e_1 = 0 \quad . \quad (351)$$

if all the points involved except the variable point p are expressed in terms of the reference points; for no other combinations can be made of the three quantities $p|e_0$, $p|e_1$, $p|e_2$, which shall be of the second degree. As there are five arbitrary constants, the curve may be subjected to five arbitrary conditions. Write

$$\phi p = (Ae_0 + C'e_1 + B'e_2)p|e_0 + (C'e_0 + Be_1 + A'e_2)p|e_1 \\ + (B'e_0 + A'e_1 + Ce_2)p|e_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (352)$$

and (351) becomes $p|\phi p = 0$.

If we write

$$\left.\begin{array}{l} \mathfrak{A}e_0' = Ae_0 + C'e_1 + B'e_2 \\ \mathfrak{B}e_1' = C'e_0 + Be_1 + A'e_2 \\ \mathfrak{C}e_2' = B'e_0 + A'e_1 + Ce_2 \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (353)$$

we have $\phi p = \mathfrak{A}e_0' \cdot p|e_0 + \mathfrak{B}e_1' \cdot p|e_1 + \mathfrak{C}e_2' \cdot p|e_2, \quad . \quad . \quad (354)$

and (351) becomes

$$\mathfrak{A}p|e_0' \cdot p|e_0 + \mathfrak{B}p|e_1' \cdot p|e_1 + \mathfrak{C}p|e_2' \cdot p|e_2 = 0. \quad . \quad (355)$$

117. *Curve through the reference points.* Let $A = B = C = 0$ in (351); omit the primes and divide by 2, and the equation becomes

$$Ap|e_1 \cdot p|e_2 + Bp|e_2 \cdot p|e_0 + Cp|e_0 \cdot p|e_1 = 0. \quad . \quad (356)$$

As this equation is satisfied when p is replaced by e_0, e_1, or e_2, it follows that the curve passes through the reference points. It is the most general form of the equation of a conic through these points, because no other term can be formed which will vanish under each of these conditions. It is the same equation as (246) if L_1, L_2, L_3 be taken as sides of the reference triangle.

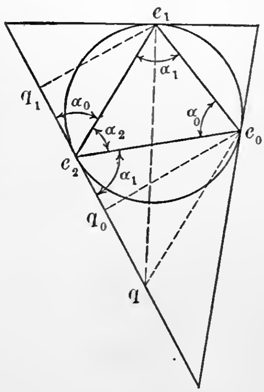
The equations of the tangent lines to (356) at the reference points are $p|\phi e_0 = 0$, etc.; and, by (353),

$$\phi e_0 = Ce_1 + Be_2,\quad \phi e_1 = Ae_2 + Ce_0,\quad \phi e_2 = Be_0 + Ae_1.$$

Hence these tangent lines may be written

$$\frac{p|e_1}{B} + \frac{p|e_2}{C} = \frac{p|e_2}{C} + \frac{p|e_0}{A} = \frac{p|e_0}{A} + \frac{p|e_1}{B} = 0. \quad . \quad . \quad (357)$$

118. *Circle through the reference points.* Let the circle be drawn through the reference points as in the figure, and also the tangents at these points. Let α_0, α_1, α_2 be the angles of the triangle as shown, and a_0, a_1, a_2 be equal respectively to $T(e_1e_2)$, $T(e_2e_0)$, and $T(e_0e_1)$. Let q be any point of the tangent at e_2, and let q_0, q_1, be the feet of the perpendiculars from e_0, e_1 on the tangent at e_2. Then we have



$$\frac{e_1e_2q}{e_0e_2q} \equiv -\frac{q|e_0}{q|e_1} = \frac{T(e_1q_1)}{T(e_0q_0)} = \frac{a_0 \sin \alpha_0}{a_1 \sin \alpha_1} = \frac{a_0^2}{a_1^2}.$$

Hence the equation of the tangent at e_2 is $\frac{q|e_0}{a_0^2} + \frac{q|e_1}{a_1^2} = 0$. And by symmetry the tangents at e_0 and e_1 are

$$\frac{q|e_1}{a_1^2} + \frac{q|e_2}{a_2^2} = \frac{q|e_2}{a_2^2} + \frac{q|e_0}{a_0^2} = 0.$$

Comparing these equations with (357), it appears that, if (356) is to represent a circle, we must have

$$\frac{A}{a_0^2}=\frac{B}{a_1^2}=\frac{C}{a_2^2}. \qquad (358)$$

119. Exercises. — (1) Verify the result of the last article by transforming (356), subject to the condition (358), to a vector system by Art. 75.

(2) Show in the same way that the equation

$$(a_1^2+a_2^2-a_0^2)(p|e_0)^2+(a_2^2+a_0^2-a_1^2)(p|e_1)^2 + (a_0^2+a_1^2-a_2^2)(p|e_2)^2=0$$

represents a circle, which is real or imaginary according as the reference triangle has an obtuse angle, or has all its angles acute.

(3) Show that the equation $p\epsilon|\phi p\phi\epsilon=0$ represents the two tangents to the conic parallel to ϵ.

(4) Hence find the tangents to (356) parallel to the sides of the reference triangle.

Ans. Tangents ‖ to e_0e_1 are $p|(e_0+e_1)=\pm p|e_2\sqrt{-\left(\frac{C}{A}+\frac{C}{B}\right)}$.

(5) If two conics through the reference points have respectively the coefficients A_1, B_1, C_1 and A_2, B_2, C_2, show that their fourth common point lies on the lines

$$\begin{vmatrix}B_1 & C_1\\ B_2 & C_2\end{vmatrix} p|e_0=\begin{vmatrix}C_1 & A_1\\ C_2 & A_2\end{vmatrix} p|e_1=\begin{vmatrix}A_1 & B_1\\ A_2 & B_2\end{vmatrix} p|e_2.$$

(6) Apply (336) to finding the center of the curves represented by (351) when $A'=B'=C'=0$, and when $A=B=C=0$.

Ans. $$\frac{BCe_0+CAe_1+ABe_2}{BC+CA+AB};$$

$$\frac{A'(B'+C'-A')e_0+B'(C'+A'-B')e_1+C'(A'+B'-C')e_2}{4A'B'-(A'+B'-C')^2}.$$

(7) Determine the nature of the curve represented by (351) under the following conditions:

(*a*) $A = B = C = 0,\ A' = B' = C' = 1.$

(*b*) $A = B = C = 0,\ A' = B' = -C' = 1.$

(*c*) $A = B = C = 0,\ A' = 4,\ B' = C' = 1.$

(*d*) $A = B = C = 0,\ A' = 3,\ B' = 4,\ C' = 5.$

(*e*) $A = B = -C = 1,\ A' = B' = C' = 0.$

(*f*) $A = -1,\ B = 2,\ C = 3,\ A' = B' = C' = 0.$

(*g*) $A = B = C = -1,\ A' = B' = C' = 1.$

In each of the above cases find the center, directions of the asymptotes when real, and the points in which the sides of the reference triangle cut the curve.

(8) Discuss the equations

$$(p|e_0)^2 - np|e_1 \cdot p|e_2 = 0$$

and
$$pe_0L_1e_1L_2e_2p = 0.$$

(9) If a triangle be inscribed in a conic, the tangents at the vertices cut the opposite sides in three collinear points.

(10) If a triangle circumscribe a conic, the lines joining the vertices with the points of contact of the opposite sides have a common point.

(11) If a line be tangent to a conic whose equation is $p|\phi_2 p = 0$, find the locus of its pole with reference to any other $p|\phi_1 p = 0$.

Ans. $q|\phi_2^{-1}\phi_1^2 q = 0.$

120. *The equations* $p|\phi p = C$ *and* $p|\phi p = 0$ *represent conics which are concentric, similar, and similarly placed.* For, since $3p|\bar{e} = 1$, we may write the first equation $p|\phi p = C(3p|\bar{e})^2$ without changing its meaning; but $p|\bar{e} = 0$ is the equation of the line at ∞; hence the equation in its present form is that of a conic tangent to $p|\phi p = 0$ at the points where it cuts the line at ∞; that is, the two curves have the *same asymptotes*, real or imaginary, which proves the proposition. It will also appear on examination that the expression for the center will be the same for the two curves, if we note that $\epsilon|\bar{e}$ is always zero, being the product of a point at ∞ into the line at ∞.

121. *Anti-polar of any point.* Let e be the point whose anti-polar is to be found, and let e' be so situated that $q_c = \frac{1}{2}(e+e')$; *i.e.* the center of the curve is midway between e and e' on the line joining them. Then the *polar* of e' will be the *anti-polar* of e; and since, by the given relation, $e' = 2q_c - e$, we have for the required equation

$$p|\phi(2q_c - e) = 0. \quad \text{. . . . .} \quad (359)$$

122. *Reciprocating ellipse.* We proceed now to find the equation of the ellipse referred to in Arts. 41–44, with reference to which each reference point is the anti-pole of the opposite reference line.

When the reference triangle is equilateral, the reciprocating curve is a circle. Now the equation of a circle through the reference points is, in this case,

$$p|e_1 \cdot p|e_2 + p|e_2 \cdot p|e_0 + p|e_0 \cdot p|e_1 = 0,$$

because $a_0 = a_1 = a_2$. Hence, by Art. 120, the reciprocating circle will be

$$p|e_1 \cdot p|e_2 + p|e_2 \cdot p|e_0 + p|e_0 \cdot p|e_1 = C(3p|\bar{e})^2 \quad . \quad (360)$$

if C be properly determined. We may infer that the equation of the reciprocating *ellipse* will have the same form when $e_0e_1e_2$ is *not* an equilateral triangle.

When $A = B = C = 0$ and $A' = B' = C'$ in (351), we have $q_c = \bar{e}$, so that (359) becomes $p|\phi(2\bar{e} - e) = 0$.

From (360) we have

$$\phi p = \tfrac{1}{2}(e_1 + e_2) \cdot p|e_0 + \tfrac{1}{2}(e_2 + e_0) \cdot p|e_1 + \tfrac{1}{2}(e_0 + e_1) \cdot p|e_2 - 9C\bar{e} \cdot p|\bar{e},$$

whence $\phi e_0 = \tfrac{1}{2}(e_1 + e_2) - 3C\bar{e}$, etc.

Now if any side of the reference triangle, as e_1e_2, is to be the anti-polar of the opposite vertex e_0, we must have

$$e_1e_2|\phi(2\bar{e} - e_0) = 0, \text{ or } |e_0\phi(2\bar{e} - e_0) = 0.$$

That is,

$$|e_0\phi(2e_1 + 2e_2 - e_0) = |e_0(e_2 + e_0 - 6C\bar{e} + e_0 + e_1 - 6C\bar{e} - \tfrac{1}{2}(e_1 + e_2) + 3C\bar{e})$$
$$= \tfrac{1}{2}(e_2 - e_1) - 3C(e_2 - e_1) = 0.$$

Whence $C = \tfrac{1}{6}$.

It is evident from the symmetry of the equations that the equations $|e_1\phi(2\bar{e} - e_1) = 0$ and $|e_2\phi(2\bar{e} - e_2) = 0$ will give the same value of C; hence the equation of the reciprocating ellipse is

$$6(p|e_1 \cdot p|e_2 + p|e_2 \cdot p|e_0 + p|e_0 \cdot p|e_1) - (3p|\bar{e})^2 = 0. \quad . \quad (361)$$

123. *Complement of any point.* We will give now the proof, referred to in Art. 44, that $|p$ is the anti-polar of p with reference to (361). Let

$$p = le_0 + me_1 + ne_2;$$

then its anti-polar is

$$|\phi(2\bar{e} - le_0 - me_1 - ne_2) = |\phi[(\tfrac{2}{3} - l)e_0 + (\tfrac{2}{3} - m)e_1 + (\tfrac{2}{3} - n)e_2],$$

ϕ being derived from (361). We find $\phi e_0 = 3(e_1 + e_2 - \bar{e})$, $\phi e_1 = 3(e_2 + e_0 - \bar{e})$, $\phi e_2 = 3(e_0 + e_1 - \bar{e})$. Now, if the proposition is true, we ought to have

$$|(le_0 + me_1 + ne_2)\phi[(2 - 3l)e_0 + (2 - 3m)e_1 + (2 - 3n)e_2] = 0.$$

$$\therefore\ |(le_0 + me_1 + ne_2)[(2 - 3l)(e_1 + e_2 - \bar{e}) + (2 - 3m)(e_2 + e_0 - \bar{e}) + (2 - 3n)(e_0 + e_1 - \bar{e})]$$
$$= |(le_0 + me_1 + ne_2)(3le_0 + 3me_1 + 3ne_2) = 0.$$

Hence the proposition is demonstrated.

124. *Line equations.* If, in eq. (356), we write L for $|p$, we have

$$ALe_1 \cdot Le_2 + BLe_2 \cdot Le_0 + CLe_0 \cdot Le_1 = 0, \quad . \quad . \quad (362)$$

the equation of a conic enveloped by L, and tangent to the sides of the reference triangle, because it is satisfied when $L = e_1e_2$, or e_2e_0, or e_0e_1.

Write

$$\psi L = |(Be_2 + Ce_1) \cdot Le_0 + |(Ce_0 + Ae_2) \cdot Le_1$$
$$+ |(Ae_1 + Be_0) \cdot Le_2, \quad . \quad (363)$$

and (362) becomes

$$L|\psi L = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (364)$$

Comparing with Arts. 116 and 117, it appears that

$$\psi L = |\phi p = |\phi\, L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (365)$$

We can thus pass at once from any point equation to its complementary line equation, or the reverse.

The function ψ is a linear line function of L, self-conjugate, and therefore possessing all the properties shown to belong to ϕ. Differentiating (364), we have

$$dL|\psi L + L|\psi dL = 2\, dL|\psi L = 0.$$

Hence, by Art. 79 and eq. (364), $|\psi L$ is a point on L, and also on dL, a line through $\bar{e}$ and the point of contact of L with the curve. Consequently $|\psi L$ must be itself the point of contact of L.

If L be *not* tangent to $L|\psi L = 0$, then $|\psi L$ is the *pole* of L.

125. *Center of $L|\psi L = 0$, ψ being any linear self-conjugate line function.* The center of any conic is the pole of the line at ∞. Hence

$$mq_c = |\psi|\bar{e} = \phi\bar{e},$$

m being a scalar constant. Multiply the complement of this equation by $3\,\bar{e}$; and we have

$$3\,m\bar{e}|q_c = m = 3\,\bar{e}|\phi\bar{e}.$$

$$\therefore\ q_c = \phi\bar{e} \div 3\,\bar{e}|\phi\bar{e} = |\psi|\bar{e} \div 3\,\bar{e}\psi|\bar{e}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (366)$$

If we have

$$\bar{e}|\phi\bar{e} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (367)$$

the center is at ∞, and the curve is a parabola. The condition (367) makes the curve $p|\phi p = 0$ pass through $\bar{e}$, the center of the reciprocating ellipse. Thus, when a curve passes through the mean point of the reference triangle, its reciprocal curve is a parabola.

126. *Determination of the curve $L|\psi L = 0$.* If *real* tangents can be drawn to the curve $p|\phi p = 0$ through $\bar{e}$, then the reciprocal curve (364) must have two *real* points at ∞, viz.: the antipoles of these two tangents, and must therefore be a hyperbola. If *no* real tangents can be drawn to $p|\phi p = 0$ through $\bar{e}$, then (364) has *no* real points at ∞, and is therefore an ellipse. If two coincident tangents can be drawn, the curve is a parabola, as was shown in the last article. Now (330) was shown to be the equation of the tangent lines to $p|\phi p = 0$ through q_1; hence, putting $\bar{e}$ for q_1, the tangents through the mean point are

$$p\bar{e}|\phi p\phi\bar{e} = 0 = \bar{e}|\phi\bar{e}\cdot p|\phi p - (p|\phi\bar{e})^2;$$

or, writing $\qquad \phi_1 p = \phi p\cdot\bar{e}|\phi\bar{e} - \phi\bar{e}\cdot p|\phi\bar{e}, \quad . \quad . \quad . \quad . \qquad (368)$

the equation becomes

$$p|\phi_1 p = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (369)$$

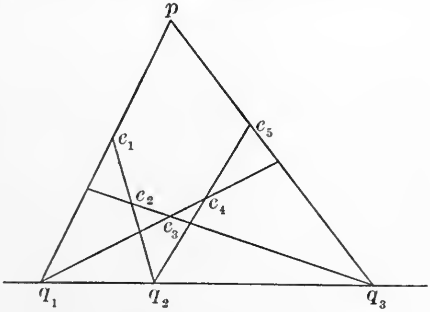
The two lines represented by (369) will be real, coincident, or imaginary, according as they cut the line at ∞ in real coincident or imaginary points; that is, by (350), according as $\bar{e}\phi_1\epsilon_1\phi_1\epsilon_2$ is negative, zero, or positive. Hence we have the criterion for eq. (364),

$$\bar{e}\phi_1\epsilon_1\phi_1\epsilon_2 \left|\begin{array}{l} +\text{ for an ellipse} \\ 0\text{ for a parabola} \\ -\text{ for a hyperbola} \end{array}\right| . \quad . \quad . \quad . \quad . \qquad (370)$$

127. *Pascal's theorem.* Let $e_1, \cdots e_5$, be any five fixed points; to find the locus of p when q_1, q_2, q_3 are in one straight line, as shown in the figure. The condition for this is

$$q_1q_2q_3 = 0,$$

and we have



$$q_1 = pe_1 \cdot e_3e_4,$$

$$q_2 = e_1e_2 \cdot e_4e_5,$$

$$q_3 = e_2e_3 \cdot e_5p.$$

Hence $$(pe_1 \cdot e_3e_4)(e_1e_2 \cdot e_4e_5)(e_2e_3 \cdot e_5p) = 0. \quad . \quad . \quad . \quad (371)$$

This is a scalar equation of the second degree in p. It therefore represents a conic. It is evidently satisfied when $p = e_1$, and when $p = e_5$. Let $p = e_2$; then q_1 and q_2 are each on the line e_1e_2, and q_3 is the point e_2; hence the equation is satisfied. Let $p = e_3$; then q_1 and q_3 each coincides with e_3, so that $q_1q_2q_3 = 0$. Finally, let $p = e_4$; then q_1 coincides with e_4, and q_2 and q_3 are both on the line e_4e_5, so that $q_1q_2q_3 = 0$ in this case also. Thus the conic passes through the five fixed points, and the hexagon $p, e_1, \cdots e_5$ is inscribed in a conic. We have, therefore, the following theorem:

If a hexagon be inscribed in a conic, the pairs of opposite sides intersect each other in three collinear points.

128. *Brianchon's theorem.* This theorem is the complementary, or reciprocal, of Pascal's, and will be obtained by writing lines for points in (371). It may be stated as follows:

If a hexagon be circumscribed about a conic, the three lines joining the opposite pairs of vertices will pass through a common point.

129. *Inversion of ϕ.* Let $\phi p = e$, so that we have also $p = \phi^{-1}e$, and let ϕ_c be the function conjugate to ϕ, so that

$q|\phi p = p|\phi_c q$. Also, let q_1, q_2, q_3 be any three non-collinear points. Then

$$q_1|\phi p = p|\phi_c q_1 = q_1|e, \quad q_2|\phi p = p|\phi_c q_2 = q_2|e, \quad q_3|\phi p = p|\phi_c q_3 = q_3|e.$$

In eq. (102) put $\phi_c q_1$ for p_0, $\phi_c q_2$ for p_1, and $\phi_c q_3$ for p_2; then, noting the equations just given, we have

$$\phi^{-1}e = p = \frac{1}{\phi_c q_1 \phi_c q_2 \phi_c q_3}\left[|\phi_c q_2 \phi_c q_3 \cdot q_1|e + |\phi_c q_3 \phi_c q_1 \cdot q_2|e + |\phi_c q_1 \phi_c q_2 \cdot q_3|e\right]. \quad (372)$$

When ϕ is *self*-conjugate, of course $\phi_c = \phi$. If $e = |q_2 q_3$, (372) becomes

$$\phi^{-1}|q_2 q_3 \cdot \phi_c q_1 \phi_c q_2 \phi_c q_3 = q_1 q_2 q_3 \cdot |\phi_c q_2 \phi_c q_3. \quad (373)$$

As the sum of any two linear functions is itself a linear function, and its conjugate is the sum of the conjugates of the two functions, we can invert such a function by (372). Take for instance $(\phi + n)p$, of which the conjugate is $(\phi_c + n)p$, and we have

$$(\phi+n)^{-1}p = \frac{|(\phi_c+n)q_2(\phi_c+n)q_3 \cdot p|q_1 + |(\phi_c+n)q_3(\phi_c+n)q_1 \cdot p|q_2 + |(\phi_c+n)q_1(\phi_c+n)q_2 \cdot p|q_3}{(q_c+n)q_1(\phi_c+n)q_2(\phi_c+n)q_3}$$

$$= \frac{k_0\phi^{-1}p + n\chi p + n_2 p}{k_0 + k_1 n + k_2 n^2 + n_3},$$

in which k_0, k_1, k_2 have the values given in (343) with ϕ_c substituted for ϕ, and χp is a linear function of p, the coefficient of n in the expansion of the numerator of the second member of the equation.

Clearing of fractions and operating by $\phi + n$, we have

$$(k_0 + k_1 n + k_2 n^2 + n^3)p = k_0 p + n(\phi\chi p + k_0\phi^{-1}p) + n^2(\phi + \chi)p + n^3 p.$$

This equation must be true for all values of n, and therefore the coefficients of different powers of n must vanish; hence

$$\phi\chi p + k_0\phi^{-1}p - k_1 p = 0 \text{ and } \phi p + \chi p - k_2 p = 0;$$

from which

$$\chi p = (k_2 - \phi)p,$$

and $$k_0\phi^{-1}p = (k_1 - k_2\phi + \phi^2)p. \quad (374)$$

Thus we have another inversion formula. If we substitute the value of χ just found in that of $(\phi + n)^{-1}$ as written above, we have a formula sometimes useful, viz.:

$$(\phi + n)^{-1}p = \frac{k_0\phi^{-1}p + n(k_2 - \phi)p + n^2p}{k_0 + k_1n + k_2n^2 + n^3}. \quad . \quad . \quad (375)$$

Finally, operating by ϕ, (374) may be written

$$(\phi^3 - k_2\phi^2 + k_1\phi - k_0)p = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (376)$$

130. EXERCISES. — (1) Invert ϕp as given in eq. (352) with $A = B = C = 0$. Also with $A' = B' = C' = 0$.

(2) Invert $\phi p = mp + e_1 \cdot p|e_2$

$$Ans. \ \phi^{-1}p = \frac{1}{m^2}[me_1 \cdot p|e_1 + (e_2 - e_1) \cdot p|e_2 + e_0m \cdot p|e_0].$$

(3) Show that if the sides of a triangle pass through three fixed points, and two of the vertices move on fixed right lines, then the third vertex describes a conic.

Ans. Equation of conic is $pp_1L_1p_2L_2p_3p = 0$.

(4) Show that if the vertices of a triangle move on three fixed right lines, and two of the sides pass through fixed points, then the third side envelops a conic.

Ans. Line equation of conic is $LL_1p_1L_2p_2L_3L = 0$.

Pascal's and Brianchon's theorems can be derived from (3) and (4) respectively.

(5) When the three points of Ex. (3) are collinear, show that the locus reduces to two straight lines.

(6) When the three lines of Ex. (4) have a common point, show that the envelope reduces to two points and the straight line joining them.

(7) Write the equation of a conic passing through four points and tangent at one of them to a given right line.

Ans. $(pe_1 \cdot e_3e_4)(e_1e_2 \cdot e_4\epsilon_4)(e_2e_3 \cdot e_4p) = 0$.

(8) Write the equation of a conic through three points and tangent at two of them to given lines.

Ans. $(pe_1 \cdot e_2e_3)(e_1\epsilon_1 \cdot e_3\epsilon_3)(e_2e_3 \cdot e_4p) = 0.$

(9) Write the line equation of a conic tangent to four right lines, the point of contact being given on one of them.

Ans. Lines are L_1, L_2, L_3, $e_4\epsilon_4$, and point e_4, and equation is

$$[LL_1 \cdot (L_3 \cdot e_4\epsilon_4)][L_1L_2 \cdot e_4][L_2L_3 \cdot (e_4\epsilon_4 \cdot L)] = 0.$$

(10) Write the equation of a conic tangent to three given lines, the points of contact being given on two of them.

Ans. If lines are $e_1\epsilon_1$, L_2, $e_3\epsilon_3$, and points e_1 and e_3, then the equation is $[(Le_1\epsilon_1)(L_2e_3\epsilon_3)]e_1e_3[(e_1\epsilon_1 \cdot L_2)(e_3\epsilon_3 \cdot L)] = 0.$

(11) If L, p, ϕ, and ψ are related as in Art. 124, show that $L|\psi L = 0$ and $p|\phi^{-1}p = 0$ are the line and point equations of the same curve, viz.: the anti-polar reciprocal of $p|\phi p = 0$. Also that $L|\psi^{-1}L = 0$ and $p|\phi p = 0$ are the line and point equations respectively of the anti-polar reciprocal of $L|\psi L = 0$.

(12) Show, by (328) and (373), that the condition that L_1 shall touch the conic $p|\phi p = 0$, is $L_1\phi^{-1}|L_1 = 0$.

(13) Interpret the complementary condition $p_1\psi^{-1}|p_1 = 0$.

(14) Find the locus of p under the following conditions: e_0, e_1, e_0', e_1' are fixed points; ϵ_0 and ϵ_1 are given vectors; $p' = e_0' + x\epsilon_0$, $p'' = e_1' + x\epsilon_1$, and $e_0p'p = e_1p''p = 0$.

Ans. $e_0e_0'p \cdot e_1\epsilon_1p = e_1e_1'p \cdot e_0\epsilon_0p.$

(15) Write in the result of the last exercise $e_0'' - e_0'$ for ϵ_0, and $e_1'' - e_1'$ for ϵ_1; take the complementary equation and interpret it.

(16) The sides of a triangle cut the corresponding sides of its polar triangle with reference to any conic in three collinear points. Reciprocally, the lines joining the corresponding vertices have a common point.

(17) Given four points; through them, two by two, draw six lines, cutting each other in three additional points, say q_1, q_2, q_3; then will any one of the q's be the pole of the line through the other two, with reference to any conic through the four given points.

(18) Derive the reciprocal proposition.

(19) Find the conditions that a triangle inscribed in a conic shall have maximum area.

Let p, p', p'' be the vertices of the triangle, subject to the conditions

$$p|\phi p = p'|\phi p' = p''|\phi p'' = 0.$$

The area is $u = pp'p''$, and for a maximum or minimum $du = 0$.

$$\therefore\ dpp'p'' + dp'p''p + dp''pp' = 0.$$

But the p's are independent of each other, so that we have

$$dpp'p'' = dp'p''p = dp''pp' = 0,$$

and also
$$dp|\phi p = dp'|\phi p' = dp''|\phi p'' = 0.$$

Hence $p'p''$ is $\parallel$ to $|\phi p$; *i.e.* to the tangent at p, and similarly for the other sides.

(20) Show that, if e and p are any two points, then e, $ep|e$, $|ep$ are a normal system of points; *i.e.* each point is the antipole of the line through the other two, with reference to the reciprocating ellipse.

CHAPTER V.

SOLID GEOMETRY.

131. As we are to deal in this chapter with space of three dimensions, the products of two vectors and of three points will no longer be scalar; but we shall have instead —

The product of *three* vectors a scalar quantity, and
The product of *four* points a scalar quantity.

Taking e_0, e_1, e_2, e_3 as reference points, and letting

$$e_1 - e_0 = \epsilon_1,\ e_2 - e_0 = \epsilon_2,\ e_3 - e_0 = \epsilon_3,$$

we shall assume always

$$\begin{aligned} 1 &= e_0e_1e_2e_3 = e_0\epsilon_1\epsilon_2\epsilon_3 = \epsilon_1\epsilon_2\epsilon_3 = (e_1 - e_0)(e_2 - e_0)(e_3 - e_0) \\ &= e_1e_2e_3 - e_2e_3e_0 + e_3e_0e_1 - e_0e_1e_2 = |(e_0 + e_1 + e_2 + e_3) \\ &= 4|\bar{e}, \text{ say.} \end{aligned}$$

Hence if p be any point at a finite distance, we shall have

$$p\epsilon_1\epsilon_2\epsilon_3 = p|(e_0 + e_1 + e_2 + e_3) = 4p|\bar{e} = 1. \quad . \quad . \quad . \quad . \qquad (377)$$

The equations of curves and surfaces in solid space may appear under the following forms:

Non-scalar equations.	Expressed in terms of points Expressed in terms of planes Expressed in terms of vectors
Scalar equations.	Expressed in terms of points Expressed in terms of planes Expressed in terms of vectors

132. The non-scalar equation

$$p = \Sigma_0^3(xe) = e_0 + \Sigma_1^3(x\epsilon), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (378)$$

$\epsilon_1, \epsilon_2, \epsilon_3$ having the values given in the last article, and x_0 being eliminated from the third member of the equation by the relation $x_0 + x_1 + x_2 + x_3 = 1$, may be called the *equation of solid space*, since, by giving proper values to x_0, x_1, etc., p may become any point whatever in space.

If a relation be given between the scalar quantities, such as

$$f_1(x_0, x_1, x_2, x_3) = 0, \text{ or } f_1(x_1, x_2, x_3) = 0,$$

then p will lose one degree of freedom of motion, and will lie on some surface. If another relation be given, such as

$$f_2(x_0, x_1, x_2, x_3) = 0, \text{ or } f_2(x_1, x_2, x_3) = 0,$$

then p will be compelled to move on a second surface simultaneously with the first, and hence along the curve of intersection of the two surfaces. It follows that the non-scalar equation of a surface has *two* independent scalar variables, while that of a curve has only *one*. Eq. (378) may be written in the form

$$p = e_0 \cdot p|e_0 + e_1 \cdot p|e_1 + e_2 \cdot p|e_2 + e_3 \cdot p|e_3, \quad . \quad . \quad (379)$$

from which it appears at once that the scalar coefficients are proportional to volumes of the tetraedra formed by joining p with the reference points. Since p is a unit point, the truth of (377) appears also from (379). It is easily seen likewise from (379) that when p passes through any face of the reference tetraedron, the coefficient of the opposite point changes sign. Thus when the coefficients are *all positive*, p is *inside* the tetraedron; when *one* is *negative*, it has passed through one face; when *two* are negative, it has passed through two faces, *i.e.* through one *edge;* and when *three* are negative, it has passed through three faces, *i.e.* through one *vertex.*

133. If $P = |p$, then the complementary equation to (378) or (379) is

$$P = \Sigma_0^3(x \cdot |e) = \Sigma_0^3(|e \cdot eP), \quad . \quad . \quad . \quad . \quad . \quad . \quad (380)$$

and P may be any plane whatever in space. But if a relation exists, as in the last article, between the scalar variables, then

P moves according to some definite law, and envelops a surface.

If a second relation exists between them, then P touches two surfaces simultaneously, or rolls on them, and therefore envelops a *developable surface,* which is reciprocal to a curve.

134. Writing in (378) $p - e_0 = \rho$, we have

$$\rho = \Sigma_1^3 (xe), \qquad (381)$$

a vector equation which, regarding ρ as always drawn outwards from a fixed origin, represents a surface or a curve under the same conditions as previously given.

135. *Equations of planes, lines, and points.* If, together with eq. (378), we have the linear relation

$$\Sigma_0^3 (mx) = 0, \qquad (382)$$

then (378) represents a *plane;* for, since $x_0 = p|e_0$, etc., we have

$$\Sigma_0^3 (mp|e) = p|\Sigma_0^3 (me) = 0, \qquad (383)$$

which is the condition that p shall be always on the plane

$$|(m_0 e_0 + m_1 e_1 + m_2 e_2 + m_3 e_3).$$

If we have also the relation

$$\Sigma_0^3 (nx) = 0, \qquad (384)$$

p must also lie on the plane whose equation is

$$p|\Sigma_0^3 (ne) = 0, \qquad (385)$$

and hence must lie on their common line.

The equation of a plane through any three points p_1, p_2, p_3 may be written

$$p = p_1 + x(p_2 - p_1) + y(p_3 - p_1), \qquad (386)$$

of which the scalar form is

$$pp_1p_2p_3 = 0, \qquad (387)$$

which may be derived from the non-scalar form by multiplying it by $p_1p_2p_3$.

The equation of a plane through p_1 and $\parallel$ to P_1 is

$$(p-p_1)P_1=0; \qquad (388)$$

for it is satisfied when $p=p_1$, and is the condition that the vector $p-p_1$ shall be $\parallel$ to P_1.

The complementary equations to (383), (385), and (387), viz.:

$$P\Sigma(me)=0,\ P\Sigma(ne)=0,\ PP_1P_2P_3=0,$$

are the conditions that the variable plane P shall always pass through the fixed points $\Sigma(me)$, $\Sigma(ne)$, and $P_1P_2P_3$, and hence are the plane equations of these points.

Eqs. (233), (234), and (235) are the equations of lines in *solid* space as well as in *plane* space, though in the former they are all non-scalar.

136. *Vector equations of planes and lines.* The equation

$$\Sigma_1^3(Ax)=C, \qquad (389)$$

taken in connection with (381), represents a plane, if ρ be always drawn outward from a fixed point; for, eliminating x_3 between the two equations, we have

$$\left.\begin{aligned}\rho&=\frac{C\epsilon_3}{A_3}+x_1\left(\epsilon_1-\frac{A_1\epsilon_3}{A_3}\right)+x_2\left(\epsilon_2-\frac{A_2\epsilon_3}{A_3}\right)\\ \text{or}\quad &(A_3\rho-C\epsilon_3)(A_3\epsilon_1-A_1\epsilon_3)(A_3\epsilon_2-A_2\epsilon_3)=0\end{aligned}\right\}, \qquad (390)$$

the non-scalar and scalar forms of the equation of a plane through the end of $C\epsilon_3\div A_3$ and parallel to the vectors $A_3\epsilon_1-A_1\epsilon_3$ and $A_3\epsilon_2-A_2\epsilon_3$. If a third equation,

$$\Sigma_1^3(Bx)=C', \qquad (391)$$

be given, we have, eliminating x_1 and x_2,

$$\begin{vmatrix}\epsilon_1, & \epsilon_2, & x_3\epsilon_3-\rho\\ A_1, & A_2, & x_3A_3-C\\ B_1, & B_2, & x_3B_3-C'\end{vmatrix}=0,\ \text{or}\ \begin{vmatrix}\rho & \epsilon_1 & \epsilon_2\\ C & A_1 & A_2\\ C' & B_1 & B_2\end{vmatrix}=\begin{vmatrix}\epsilon_1 & \epsilon_2 & \epsilon_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}x_3, \qquad (392)$$

an equation which evidently represents a right line.

Eqs. (390) and (392) are of the respective forms

$$\left.\begin{aligned} &\rho = \epsilon + x\epsilon' + y\epsilon'' \\ &(\rho - \epsilon)\epsilon'\epsilon'' = 0 \\ &\rho = \epsilon + z\epsilon' \end{aligned}\right\}, \quad \ldots \quad (393)$$

which are those practically used.

The equation of a plane through the ends of ϵ_1, ϵ_2, ϵ_3 drawn out from the origin may be written at once in either of the forms

$$\left.\begin{aligned} &\rho - \epsilon_1 = x(\epsilon_2 - \epsilon_1) + y(\epsilon_3 - \epsilon_1) \\ \text{or} \quad &(\rho - \epsilon_1)(\epsilon_2 - \epsilon_1)(\epsilon_3 - \epsilon_1) = 0 \end{aligned}\right\}, \quad \ldots \quad (394)$$

the last being equivalent to

$$\rho(\epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2) = \epsilon_1\epsilon_2\epsilon_3, \quad \ldots \quad (395)$$

as will be found on multiplying out.

137. EXERCISES. — (1) What is the meaning of the two equations $Pp_1 = 0$ and $Pp_2 = 0$ taken simultaneously?

(2) Interpret the following equations written in complementary pairs:

$$\begin{Bmatrix} pp_1\epsilon_1\epsilon_2 = 0 \\ PP_1p_1\bar{e} = 0 \end{Bmatrix}, \quad \begin{Bmatrix} pp_1p_2\epsilon_1 = 0 \\ PP_1P_2q_1q_2\bar{e} = 0 \end{Bmatrix}, \quad \begin{Bmatrix} pL_2L_1P_1 = 0 \\ PL_2L_1p_1 = 0 \end{Bmatrix},$$

$$\begin{Bmatrix} pP_1P_2 \cdot P_3P_4P_5 = 0 \\ Pp_1p_2 \cdot p_3p_4p_5 = 0 \end{Bmatrix}, \quad \begin{Bmatrix} p(k_1P_1 + k_2P_2) = 0 \\ P(k_1p_1 + k_2p_2) = 0 \end{Bmatrix},$$

$$\begin{Bmatrix} p(p_2 - p_1)P_1P_2 = 0 \\ P(P_2 - P_1)p_1p_2 = 0 \end{Bmatrix}, \quad \begin{Bmatrix} p(k_1P_1 + k_2P_2 + k_3P_3) = 0 \\ P(k_1p_1 + k_2p_2 + k_3p_3) = 0 \end{Bmatrix}.$$

If, in the fourth case, $P_5 = P_1$ and $p_5 = p_1$, what do the equations represent?

(3) Show that $p|(p_1 - 4C\bar{e}) = 0$ and $p|p_1 = C$ represent the same plane, parallel to $|p_1$ and distant from it by the amount $C \div T|p_1$.

(4) If ϵ be any point at ∞, show that we have always

$$\epsilon|\bar{e}=0. \qquad (396)$$

From this result, or otherwise, show that

$$p|\bar{e}=0 \qquad (397)$$

is the equation of the plane at ∞.

(5) Show that the common line of the two planes $pp_1p_2p_3=C$ and $pp_1p_2p_4=C'$ is

$$(p_2-p_1)(p_1p_2p_3p_4\cdot p_1+C(p_4-p_1)-C'(p_3-p_1)).$$

(6) Show that the common point of the three planes, $ep_1p_2p=C'''$, $ep_2p_3p=C'$, $ep_3p_1p=C''$ is

$$p_1p_2p_3e\cdot e+C'(e-p_1)+C''(e-p_2)+C'''(e-p_3).$$

(7) Show that $p_1p_2p_3=0$ is the condition that the three planes $p|p_1=p|p_2=p|p_3=0$ shall have a common line.

(8) Find the condition that the plane $p|p_4=0$, together with the three planes of the last exercise, shall have a common point.

(9) Show that if the equations of three planes, on being multiplied by any constants and added, vanish *identically*, *i.e.* for all values of p, then the three planes pass through a common line. Also that if the same holds for the equations of *four* planes, then the planes have a common point.

(10) Show that when $p_1p_2p_3p_4=C_1-C_2+C_3-C_4$, the four planes $pp_1p_2p_3=C_4$, $pp_2p_3p_4=C_1$, $pp_3p_4p_1=C_2$, $pp_4p_1p_2=C_3$ have a common point.

(11) Interpret the equations $\rho|\epsilon=C$, $\rho\epsilon\epsilon'=C$, $(\rho-\epsilon)|\epsilon=0$, $(\rho-\epsilon_1)(\epsilon_2-\epsilon_1)\epsilon_3=0$, $\rho=\epsilon+x|\epsilon\epsilon'+y\epsilon\epsilon'|\epsilon$, $\rho=\epsilon_1+x(\epsilon_2-\epsilon_1)+y\epsilon_3$.

(12) Find the vector perpendiculars from the origin on the planes of Ex. (11).

Ans. $\dfrac{C\epsilon}{\epsilon^{\underline{2}}}$, $\dfrac{C\cdot|\epsilon\epsilon'}{(\epsilon\epsilon')^{\underline{2}}}$, ϵ, $\dfrac{\epsilon_1\epsilon_2\epsilon_3\cdot|(\epsilon_2-\epsilon_1)\epsilon_3}{((\epsilon_2-\epsilon_1)\epsilon_3)^{\underline{2}}}$.

(13) Find the conditions of perpendicularity and parallelism of the two planes $\rho|\epsilon=C$ and $\rho|\epsilon'=C'$.

(14) Find the vector perpendicular from the end of the vector δ upon the line $(\rho-\epsilon)\epsilon'=0$.

Ans. $\epsilon'(\epsilon-\delta)\cdot|\epsilon'$.

(15) Find the conditions that the three lines $(\rho-\epsilon)\epsilon'=0$, $P_1P_2p=0$, $p_1p_2p=0$ shall lie respectively in the three planes $\rho|\epsilon''=C$, $P_3p=0$, $p|p_3=C$.

Ans. $\epsilon'|\epsilon''=0$ and $\epsilon|\epsilon''=C$, $P_1P_2P_3=0$, $(p_2-p_1)|p_3=0$ and $p_2|p_3=C$.

(16) Show that the shortest distance between the two right lines $(\rho-\epsilon_1)\epsilon_2=0$ and $(\rho-\epsilon_1')\epsilon_2'=0$ is $\dfrac{(\epsilon_1-\epsilon_1')\epsilon_2\epsilon_2'}{T\epsilon_2\epsilon_2'}$. (Use Art. 46.)

(17) Show that the equation of the common line of the two planes $(\rho-\epsilon)\epsilon'\epsilon''=0$ and $(\rho-\epsilon_1)\epsilon_1'\epsilon_1''=0$ is

$$\rho=\frac{\epsilon_1\epsilon_1'\epsilon_1''}{\epsilon'\epsilon_1'\epsilon_1''}\cdot\epsilon'+\frac{\epsilon\epsilon'\epsilon''}{\epsilon_1'\epsilon'\epsilon''}\cdot\epsilon_1'+x\epsilon'\epsilon''\cdot\epsilon_1'\epsilon_1''.$$

(18) Show that the equation of a plane through the line $(\rho-\epsilon_1)\epsilon_2=0$ parallel to the line $(\rho-\epsilon_1')\epsilon_2'=0$ may be written $(\rho-\epsilon_1)\epsilon_2|\epsilon_2\epsilon_2'=0$, and hence, by Ex. (17), that the line

$$\rho=\frac{\epsilon_1'\epsilon_2'|\epsilon_2\epsilon_2'}{(\epsilon_2\epsilon_2')^2}\cdot\epsilon_2-\frac{\epsilon_1\epsilon_2|\epsilon_2\epsilon_2'}{(\epsilon_2\epsilon_2')^2}\cdot\epsilon_2'+x|\epsilon_2\epsilon_2'$$

cuts these two lines at right angles.

(19) What are the conditions that the pairs of right lines $\begin{cases} q_1q_2(p-p_1)=0\\ q_1'q_2'(p-p_1')=0\end{cases}$ and $\begin{cases}(\rho-\epsilon_1)\epsilon_2=0\\ (\rho-\epsilon_1')\epsilon_2'=0\end{cases}$ shall intersect?

Ans. $p_1(q_2-q_1)p_1'(q_2'-q_1')=0$ and $(\epsilon_1-\epsilon_1')\epsilon_2\epsilon_2'=0$.

(20) Show that the common point of the three planes

$$\rho|\epsilon_1=C_1,\quad \rho|\epsilon_2=C_2,\quad \rho|\epsilon_3=C_3$$

is at the end of the vector

$$(\epsilon_1\epsilon_2\epsilon_3)^{-1}(C_1|\epsilon_2\epsilon_3+C_2|\epsilon_3\epsilon_1+C_3|\epsilon_1\epsilon_2).$$

(21) Show that the three planes of the last exercise will have a common line if we have

$$\epsilon_1\epsilon_2\epsilon_3 = 0 \text{ and } C_1\epsilon_2\epsilon_3 + C_2\epsilon_3\epsilon_1 + C_3\epsilon_1\epsilon_2 = 0.$$

(22) Show that, if

$$\epsilon_1\epsilon_2\epsilon_3 C_4 - \epsilon_2\epsilon_3\epsilon_4 C_1 + \epsilon_3\epsilon_4\epsilon_1 C_2 - \epsilon_4\epsilon_1\epsilon_2 C_3 = 0,$$

the plane $\rho|\epsilon_4 = C_4$, and the three planes of Ex. 20 have a common point.

(23) Interpret the equations

$$f(\rho|\epsilon) = 0, \quad f(T\rho) = 0, \quad f(U\rho) = 0;$$

if f is the symbol of a scalar function.

(24) Three planes pass through the three lines of intersection respectively of the three planes

$$(\rho - \epsilon_1)|\epsilon_1 = 0, \quad (\rho - \epsilon_2)|\epsilon_2 = 0, \text{ and } (\rho - \epsilon_3)|\epsilon_3 = 0,$$

each being perpendicular to the opposite plane (*i.e.* the plane through the common line of the first and second planes is perpendicular to the third, etc.); find the conditions that the three first mentioned planes may intersect in a common line through the origin.

Ans. $\epsilon_1|\epsilon_2 \cdot \epsilon_3^2 = \epsilon_2|\epsilon_3 \cdot \epsilon_1^2 = \epsilon_3|\epsilon_1 \cdot \epsilon_2^2.$

(25) Find the shortest distance between the diagonal of a cube and an edge that it does not meet.

(26) Find the equation of a line through p_1 cutting L_1 and L_2.

Ans. $pp_1L_1p_1L_2 = 0.$

(27) Derive (395) from (387) by transformation, as in Art. 75.

138. *The sphere.* The equation

$$\left.\begin{aligned}\rho &= a[(\iota_1\cos\theta' + \iota_2\sin\theta')\sin\theta + \iota_3\cos\theta] \\ &= a(\tau\sin\theta + \iota_3\cos\theta), \text{ say,}\end{aligned}\right\} \quad . \quad (398)$$

so that

$$\tau = \iota_1\cos\theta' + \iota_2\sin\theta', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (399)$$

represents a sphere. For, taking the co-square, we have

$$\rho^2 = a^2(\tau^2 \sin^2\theta + \iota_3^2 \cos^2\theta) = a^2,$$

since $\qquad \tau|\iota_3 = 0$ and $\iota_3^2 = \tau^2 = 1.$

Hence $T\rho = a =$ constant, which is a property of the sphere with center at the origin.

If the center be at the end of ϵ, we have for the scalar equation $T(\rho - \epsilon) = a$; whence

$$\rho^2 - 2\rho|\epsilon = a^2 - \epsilon^2 \text{ or } \rho|(2\epsilon - \rho) = \epsilon^2 - a^2. \quad . \quad (400)$$

This equation is identical with (269), so that it represents a circle or sphere according as it is interpreted in plane or solid space. The properties of the plane-radical can be proved precisely as, in Art. 82, those of the axis-radical were demonstrated.

139. EXERCISES. — (1) If α and β are any two non-parallel vectors, show that the equations

$$\rho^2 = k_1(\rho|\alpha + C_1), \ \rho^2 = k_2(\alpha\beta\rho + C_2), \ \rho^2 = k_3(\alpha\beta|\alpha\rho + C_3)$$

represent spheres; find their centers and radii, and their relative positions. If $C_1 = C_2 = C_3 = 0$, show that they cut each other orthogonally.

(2) If α, β, γ, δ be any four vectors drawn outwards from a point, and the relation

$$\alpha\beta\gamma \cdot \delta^2 - \beta\gamma\delta \cdot \alpha^2 + \gamma\delta\alpha \cdot \beta^2 - \delta\alpha\beta \cdot \gamma^2 = 0$$

exists between them, show that the extremities of the four vectors and their common point all lie on a sphere.

(3) Show that the equations of the tangent plane and normal line to (400) are respectively

$$(\sigma - \epsilon)|(\rho - \epsilon) = a^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (401)$$

$$(\sigma - \epsilon)(\rho - \epsilon) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (402)$$

(4) Show that $\sigma = \rho + x\frac{d\rho}{d\theta} + y\frac{d\rho}{d\theta'}$ is the vector *differential* equation of the tangent plane to (398), and from this find the equation of the tangent plane.

(5) Find the locus of the end of ρ when its distances from two fixed points have a constant ratio to each other.

(6) If ρ be the vector radius of the sphere of eq. (400), find the locus of the end of σ, when σ is subject to the conditions $U\sigma = U\rho$ and $T\sigma T\rho = k^2$.

140. *The paraboloids.* The equation

$$\rho = x\epsilon_1 + y\epsilon_2 + \frac{x^2 + my^2}{a}\epsilon_3 \quad \dots \dots \quad (403)$$

represents an elliptic or hyperbolic paraboloid, according as m is positive or negative, as will presently appear. Let

$$\alpha_1 = |\epsilon_2\epsilon_3, \ \alpha_2 = |\epsilon_3\epsilon_1, \ \alpha_3 = |\epsilon_1\epsilon_2, \text{ and } \epsilon_1\epsilon_2\epsilon_3 = 1;$$

then $$\alpha_1\alpha_2\alpha_3 = \epsilon_2\epsilon_3 \cdot \epsilon_3\epsilon_1 \cdot \epsilon_1\epsilon_2 = (\epsilon_1\epsilon_2\epsilon_3)^2 = 1,$$

$$\epsilon_1|\alpha_1 = 1, \ \epsilon_1|\alpha_2 = 0, \text{ etc., } \rho|\alpha_1 = x, \ \rho|\alpha_2 = y,$$

and $$\rho|\alpha_3 = \frac{x^2 + my^2}{a};$$

whence $$(\rho|\alpha_1)^2 + m(\rho|\alpha_2)^2 = a\rho|\alpha_3 = \rho|\phi\rho; \quad \dots \quad (404)$$

in which $$\phi\rho = \alpha_1 \cdot \rho|\alpha_1 + m\alpha_2 \cdot \rho|\alpha_2. \quad \dots \dots \quad (405)$$

Eq. (404) is the scalar form corresponding to (403).

Find the intersection of this locus with the plane

$$\rho|(n_1\alpha_1 + n_2\alpha_2) = 0. \quad \dots \dots \quad (406)$$

Eliminating $\rho|\alpha_1$, we obtain

$$(\rho|\alpha_2)^2 = \frac{an_1^2}{n_2^2 + mn_1^2} \cdot \rho|\alpha_3. \quad \dots \dots \quad (407)$$

Eqs. (407) and (406) taken simultaneously represent a parabola, as appears at once from the results of Chap. III.; thus any plane through $e_0\epsilon_3$ cuts a parabola from the surface, e_0 being the origin.

Again, intersect the surface by the plane

$$\rho|\alpha_3 = c\,; \quad \ldots \quad (408)$$

whence $$\rho|\phi\rho = (\rho|\alpha_1)^2 + m(\rho|\alpha_2)^2 = ac \quad \ldots \quad (409)$$

Eqs. (409) and (408) taken together represent, by Chap. III., an ellipse or hyperbola according as m is positive or negative; thus all sections parallel to $\epsilon_1\epsilon_2$ are ellipses, or else all are hyperbolas; hence the name of the surface as stated at the beginning of this article.

141. Eq. (404) may be written in the form

$$\left.\begin{array}{l} \rho|(\alpha_1 + \alpha_2\sqrt{-m})\cdot\rho|(\alpha_1 - \alpha_2\sqrt{-m}) = a\rho|\alpha_3 \\ \text{or} \quad \rho|\beta_1\cdot\rho|\beta_2 = a\rho|\alpha_3 \end{array}\right\}, \quad . \quad (410)$$

which shows that the surface passes through the two common lines of the plane $\rho|\alpha_3 = 0$, and the two planes $\rho|\beta_1 = 0$ and $\rho|\beta_2 = 0$. These are *real* when m is negative, and *imaginary* when m is positive.

Now (410) will be satisfied by any value of ρ which satisfies simultaneously either of the following pairs of equations, viz.

$$\left\{\begin{array}{l} \rho|\beta_1 = \frac{a}{n}\rho|\alpha_3 \\ \rho|\beta_2 = n \end{array}\right\}, \quad \left\{\begin{array}{l} \rho|\beta_2 = \frac{a}{n'}\rho|\alpha_3 \\ \rho|\beta_1 = n' \end{array}\right\}. \quad \ldots \quad (411)$$

Each of these pairs represents a *right line,* which will change its position as we change n or n': hence it appears that, when m is negative, there are two systems of right lines that lie wholly on the surface; these are called two systems of rectilinear generators of the surface. They are imaginary when m is positive. By Ex. (9) or (22) of Art. 137 it is easily shown that no two generators of the same system intersect, while every generator of one system cuts every one of the other.

142. EXERCISES. — (1) Find the common point of two generators belonging to the systems n and n' respectively.

Ans. $$\frac{1}{2\sqrt{-m}}[a\sqrt{-m}(n+n')\epsilon_1 + a(n'-n)\epsilon_2 + 2nn'\epsilon_3\sqrt{-m}].$$

(2) Show that all the generators of either system of a hyperbolic paraboloid are parallel to a single plane.

(3) Find the equations of the tangent planes to the loci of eqs. (403) and (404); and show that, when m is negative, the tangent plane contains a generator of each system.

(4) If $\alpha_1, \alpha_2, \alpha_3$ in (404) form a unit normal system, so that we may write $\alpha_1 = \epsilon_1 = \iota_1$, etc., find the locus of the points on the surface at which the two generators passing through each of such points are perpendicular to each other.

Ans.
$$\rho = \frac{a^2(1+m)}{8m}\left[-2\iota_3 - \frac{a}{n}\left(\iota_1 + \frac{\iota_2}{\sqrt{-m}}\right) + \frac{4mn}{a(1+m)}\left(\iota_1 - \frac{\iota_2}{\sqrt{-m}}\right)\right],$$
n being variable. This equation represents a hyperbola.

(5) In eq. (404) change the origin by putting $\rho' + \frac{c^2}{a}\epsilon_3$ for ρ, and then determine the nature of the surface when $a = 0$. Also when $a = c = 0$.

(6) Taking $\alpha_1, \alpha_2, \alpha_3$ as in Ex. (4), show that $\rho|\epsilon = C$ will represent a plane tangent to the locus of (404) if we have
$$C = -a\cdot\epsilon|\phi^{-1}\epsilon \div 4\epsilon|\epsilon_3.$$

143. *The central quadrics.* The surfaces represented by the three equations
$$\left.\begin{aligned} \rho &= \tau\sin\theta + c\iota_3\cos\theta \\ \rho &= \tau\operatorname{cosec}\theta + c\iota_3\cot\theta \\ \rho &= \tau'\operatorname{cosec}\theta + c\iota_3\cot\theta \end{aligned}\right\}, \qquad (412)$$
in which
$$\left.\begin{aligned} \tau &= a\iota_1\cos\theta' + b\iota_2\sin\theta' \\ \tau' &= a\iota_1\sec\theta' + b\iota_2\tan\theta' \end{aligned}\right\}, \qquad (413)$$
are called respectively the *ellipsoid*, the *hyperboloid of one sheet*, and the *hyperboloid of two sheets*, for reasons which will presently appear.

By comparison with eqs. (276) and (277) it will be seen that eqs. (413) represent respectively an ellipse whose semi-axes are a and b, and a hyperbola with the same semi-axes, each lying in the plane $\iota_1\iota_2$, the vector radius of one curve being τ, and that of the other τ'. In the first of eqs. (412) give to τ some particular value consistent with (413); then the equation represents an *ellipse* whose semi-axes are $T\tau$ and c; hence the first of (412) represents a surface generated by an ellipse revolving about ι_3 as an axis, c being the semi-axis along ι_3, while the other semi-axis is the radius-vector of the ellipse in $\iota_1\iota_2$ whose semi-axes are a and b.

Similarly, the second of (412) represents, for any given value of τ, a *hyperbola* whose semi-axes are $T\tau$ and c, and, when τ varies subject to (413), it represents the surface generated by this hyperbola revolving about ι_3, having c for its semi-axis along ι_3, while the other is the radius vector of the ellipse in $\iota_1\iota_2$.

Finally, the third of (412), for any given value of τ', represents a *hyperbola* with semi-axes $T\tau'$ and c, τ' being, as we saw above, the radius-vector of a hyperbola in $\iota_1\iota_2$. Thus the surface is generated in this case by a hyperbola revolving about ι_3, having its c-axis constant, while its other semi-axis is the radius-vector of a hyperbola in $\iota_1\iota_2$ whose semi-axes are a and b.

The methods of generation of these three surfaces show that the first is a *limited* surface, having no real points at ∞, and hence that no plane can cut from it a hyperbola or parabola; hence the name *ellipsoid:* that the second is generated by a hyperbola in such a way as to form one continuous surface, so that any two points whatever lying on it can be joined by a line also lying wholly on it; hence the name hyperboloid of *one sheet:* finally, that the third is generated by a hyperbola in such a way as to form two distinct portions, or sheets, so that points on these respective portions cannot be joined by a line lying wholly on the surface; whence the name as given above.

144. Eliminating θ and θ' from eqs. (412), we obtain

$$\left.\begin{aligned}\left(\frac{\rho|\iota_1}{a}\right)^2+\left(\frac{\rho|\iota_2}{b}\right)^2+\left(\frac{\rho|\iota_3}{c}\right)^2=1\\ \left(\frac{\rho|\iota_1}{a}\right)^2+\left(\frac{\rho|\iota_2}{b}\right)^2-\left(\frac{\rho|\iota_3}{c}\right)^2=1\\ \left(\frac{\rho|\iota_1}{a}\right)^2-\left(\frac{\rho|\iota_2}{b}\right)^2-\left(\frac{\rho|\iota_3}{c}\right)^2=1\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (414)$$

the scalar equations of the same surfaces.

As an exercise let the student determine the nature of the sections of the surfaces of (414) by the three planes $\rho|\iota_1=C_1$, $\rho|\iota_2=C_2$, $\rho|\iota_3=C_3$ when $C_1>a$, $C_2>b$, $C_3>c$, $C_1=a$, $C_2=b$, $C_3=c$, $C_1<a$, $C_2<b$, $C_3<c$.

145. Write

$$\phi\rho=\frac{\iota_1\cdot\rho|\iota_1}{a^2}\pm\frac{\iota_2\cdot\rho|\iota_2}{b^2}\pm\frac{\iota_3\cdot\rho|\iota_3}{c^2}; \quad . \quad . \quad . \quad . \quad . \quad (415)$$

then the equation

$$\rho|\phi\rho=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (416)$$

is equivalent to any one of the equations (414), if we select the signs properly in (415).

The function ϕ as given in (415) is evidently self-conjugate, and possesses all the properties proved in Arts. 86 and 89. Eq. (416) being of precisely the same form as (282), any operations performed on (282) which did not depend on the form of ϕ will give identical results when performed on (416), the *interpretation* of the results being, of course, different. *The discussions which follow hold for any linear, vector, self-conjugate form of ϕ, except when ϕ is specially restricted to the form* (415).

146. *Tangent plane and normal.* By differentiation we have, precisely as in Art. 87, that $\phi\rho$ is parallel to the normal at the end of ρ, and hence that the equations of the tangent plane and normal line at the end of ρ are respectively

$$\sigma|\phi\rho=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (417)$$

and

$$(\sigma-\rho)\phi\rho=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (418)$$

which are identical in form with (284) and (285). Also, as in the same article the perpendicular from the origin on the tangent plane is

$$\frac{1}{T\phi\rho}\cdot U\phi\rho. \qquad (419)$$

147. *Diametral plane.* Repeating exactly the operations of Art. 88, we obtain the same equation

$$\sigma|\phi\epsilon = 0, \qquad (420)$$

which now represents a *plane*, the locus of the middle points of a system of chords parallel to ϵ. This plane is perpendicular to $\phi\epsilon$, *i.e.* parallel to the tangent planes at the ends of the diameter parallel to ϵ, and is said to be *conjugate* in direction to ϵ. Also, any straight line in this plane is conjugate in direction to ϵ, so that (420) is the condition that two vectors σ and ϵ shall be conjugate. If vectors σ_1 and σ_2 be taken in (420) so that the tangent plane at the point where σ_2 drawn out from the origin pierces the surface is parallel to σ_1, then we shall have

$$\sigma_1|\phi\sigma_2 = 0 = \sigma_1|\phi\epsilon = \sigma_2|\phi\epsilon,$$

and σ_1, σ_2, and ϵ form a set of conjugate directions. If α, β, γ are three vectors which satisfy the conditions just found, and also the equation of the surface $\rho|\phi\rho = 1$; that is,

$$\left.\begin{array}{l} \alpha|\phi\beta = \beta|\phi\gamma = \gamma|\phi\alpha = 0 \\ \alpha|\phi\alpha = \beta|\phi\beta = \gamma|\phi\gamma = 1 \end{array}\right\}; \qquad (421)$$

then α, β, γ are a set of conjugate semi-diameters. From these relations we have $\alpha = m|\phi\beta\phi\gamma$; multiply into $|\phi\alpha$.

$$\therefore\ \alpha|\phi\alpha = 1 = m\phi\alpha\phi\beta\phi\gamma.$$

Substitute value of m, and we have

$$\text{and, similarly,}\quad \left.\begin{array}{l} \alpha\cdot\phi\alpha\phi\beta\phi\gamma = |\phi\beta\phi\gamma \\ \beta\cdot\phi\alpha\phi\beta\phi\gamma = |\phi\gamma\phi\alpha \\ \gamma\cdot\phi\alpha\phi\beta\phi\gamma = |\phi\alpha\phi\beta \end{array}\right\}. \qquad (422)$$

In the same way we find

$$\phi\alpha\cdot\alpha\beta\gamma = |\beta\gamma,\quad \phi\beta\cdot\alpha\beta\gamma = |\gamma\alpha,\quad \phi\gamma\cdot\alpha\beta\gamma = |\alpha\beta. \qquad (423)$$

148. *Interpretation of the equation* $\sigma|\phi\epsilon = 1$. In the first place it evidently represents some plane parallel to that of (420), *i.e.* conjugate in direction to ϵ.

If we have $\epsilon|\phi\epsilon = 1$, the equation is identical with (417) and hence represents the tangent plane at the end of ϵ. In general, let ρ be the vector to the point of contact of some plane, passing through the end of ϵ and tangent to the surface: then the equation of this tangent plane will be $\sigma|\phi\rho = 1$, with the condition $\epsilon|\phi\rho = 1 = \rho|\phi\epsilon$, which makes the plane pass through the end of ϵ. If ρ be *variable* in the last equation, we have the *locus* of the end of ρ, and the equation becomes identical with that at the head of the article when we put σ for ρ. Hence the equation

$$\sigma|\phi\epsilon = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (423\,a)$$

represents a plane containing the points of contact of all tangent planes to the surface which pass through the end of ϵ, or, in other words, the plane of the curve of contact of the circumscribed cone whose vertex is at the end of ϵ. Of course the end of ϵ may be so situated that this cone is imaginary. The plane $\sigma|\phi\epsilon = 1$ is called the *polar* plane of the point at the end of ϵ, while this point is the *pole* of that plane.

Now of the infinite number of directions of σ in (423 a) one will evidently coincide with that of ϵ; when σ has this direction suppose it to become fixed and ϵ to vary; we shall then have the plane $\epsilon|\phi\sigma = 1$ parallel to the plane which we had when ϵ was fixed and σ varied, and, since one value of ϵ must be its original value, the plane must now pass through the end of ϵ in its original position; thus the positions of the pole and polar plane have been exchanged as regards their distance from the center measured in the direction ϵ.

If a point p_1 be on the polar plane of p_2, then will p_2 be also on the polar plane of p_1.

Let p_1 and p_2 be at the ends of ϵ_1 and ϵ_2 respectively; the polar planes are then $\sigma|\phi\epsilon_1 = 1$ and $\sigma|\phi\epsilon_2 = 1$ respectively; if p_1 be on the polar plane of p_2, we must have the equation satisfied when $\sigma = \epsilon_1$; *i.e.* $\epsilon_1|\phi\epsilon_2 = 1 = \epsilon_2|\phi\epsilon_1$; but this is also the condition that p_2 shall be on the polar plane of p_1.

Finally, we have, precisely as in Art. 92, that the *semi-diameter along* ϵ *is a mean proportional between the distances along* ϵ *to the point and to its polar plane.*

149. As in Art. 91, we have

$$\rho|\phi\rho = \rho|\phi^{\frac{1}{2}}\phi^{\frac{1}{2}}\rho = \phi^{\frac{1}{2}}\rho|\phi^{\frac{1}{2}}\rho = (\phi^{\frac{1}{2}}\rho)^2 = 1,$$

or

$$T\phi^{\frac{1}{2}}\rho = 1; \qquad (424)$$

so that, when ρ is a vector of the surface, $\phi^{\frac{1}{2}}\rho$ is always a unit vector. Also, if α, β, γ are vector, conjugate semi-diameters,

$$\alpha|\phi\beta = \alpha|\phi^{\frac{1}{2}}\phi^{\frac{1}{2}}\beta = \phi^{\frac{1}{2}}\beta|\phi^{\frac{1}{2}}\alpha = 0,$$

and, similarly, $\phi^{\frac{1}{2}}\beta|\phi^{\frac{1}{2}}\gamma = \phi^{\frac{1}{2}}\gamma|\phi^{\frac{1}{2}}\alpha = 0,$

so that $\phi^{\frac{1}{2}}\alpha$, $\phi^{\frac{1}{2}}\beta$, $\phi^{\frac{1}{2}}\gamma$ form a *unit, normal* system of vectors.

150. *The volume of the parallelopiped formed by tangent planes at the ends of a set of conjugate diameters of an ellipsoid is constant.*

Taking α, β, γ as in the last article, the required volume is $8\alpha\beta\gamma$. Now with ϕ, as in eq. (415) with the upper signs, we have

$$\phi^{-\frac{1}{2}}\rho = a\iota_1 \cdot \rho|\iota_1 + b\iota_2 \cdot \rho|\iota_2 + c\iota_3 \cdot \rho|\iota_3,$$

$$\left.\begin{aligned} \text{and } \alpha &= \phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}\alpha = a\iota_1 \cdot \iota_1|\phi^{\frac{1}{2}}\alpha + b\iota_2 \cdot \iota_2|\phi^{\frac{1}{2}}\alpha + c\iota_3 \cdot \iota_3|\phi^{\frac{1}{2}}\alpha \\ \beta &= \phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}\beta = a\iota_1 \cdot \iota_1|\phi^{\frac{1}{2}}\beta + \text{etc.} \\ \gamma &= \phi^{-\frac{1}{2}}\phi^{\frac{1}{2}}\gamma = a\iota_1 \cdot \iota_1|\phi^{\frac{1}{2}}\gamma + \text{etc.} \end{aligned}\right\}. \qquad (425)$$

$$\therefore\ 8\alpha\beta\gamma = 8abc\begin{vmatrix} \iota_1|\phi^{\frac{1}{2}}\alpha, & \iota_2|\phi^{\frac{1}{2}}\alpha, & \iota_3|\phi^{\frac{1}{2}}\alpha \\ \iota_1|\phi^{\frac{1}{2}}\beta, & \iota_2|\phi^{\frac{1}{2}}\beta, & \iota_3|\phi^{\frac{1}{2}}\beta \\ \iota_1|\phi^{\frac{1}{2}}\gamma, & \iota_2|\phi^{\frac{1}{2}}\gamma, & \iota_3|\phi^{\frac{1}{2}}\gamma \end{vmatrix} = 8abc \cdot \iota_1\iota_2\iota_3 \cdot \phi^{\frac{1}{2}}\alpha\phi^{\frac{1}{2}}\beta\phi^{\frac{1}{2}}\gamma = 8abc,$$

by the last article and equation (195).

Take the co-square of each of equations (425) and add them, and we have

$$\alpha^2 + \beta^2 + \gamma^2 = a^2 + b^2 + c^2. \qquad (426)$$

151. The equation

$$(\beta\gamma\rho)^2 \pm (\gamma\alpha\rho)^2 \pm (\alpha\beta\rho)^2 = (\alpha\beta\gamma)^2 \quad . \quad . \quad . \quad . \quad (427)$$

represents a central quadric referred to the conjugate semi-diameters α, β, γ. The origin is at the center because the equation is unchanged by putting $-\rho$ for $+\rho$. If $\alpha = a\iota_1$, $\beta = b\iota_2$, $\gamma = c\iota_3$, the equation becomes identical with (414) Eq. (427) is satisfied when $\rho = \alpha$, or $\rho = \beta\sqrt{\pm 1}$, or $\rho = \gamma\sqrt{\pm 1}$, so that, when the signs are all positive, the ends of α, β, γ are real points on the surface; when one sign is negative, one point becomes imaginary; and when two signs are negative, two points become imaginary.

If we write

$$\phi\rho = \frac{1}{(\alpha\beta\gamma)^2}(|\beta\gamma \cdot \beta\gamma\rho + |\gamma\alpha \cdot \gamma\alpha\rho + |\alpha\beta \cdot \alpha\beta\rho), \quad (428)$$

eq. (427) becomes $\rho|\phi\rho = 1$, and we have at once

$$\alpha|\phi\beta = \beta|\phi\gamma = \gamma|\phi\alpha = 0,$$

the conditions for conjugate directions.

152. *Rectilinear generators.* Eq. (414) may be written in the form

$$\left.\begin{aligned} \rho\left|\left(\frac{\iota_2}{b\sqrt{\pm 1}} + \frac{\iota_3}{c\sqrt{\mp 1}}\right) \cdot \rho\right|\left(\frac{\iota_2}{b\sqrt{\pm 1}} - \frac{\iota_3}{c\sqrt{\mp 1}}\right) \\ = \left(1 + \frac{\rho|\iota_1}{a}\right)\left(1 - \frac{\rho|\iota_1}{a}\right) \\ \text{or} \quad a^2\rho|\alpha \cdot \rho|\alpha' = (a + \rho|\iota_1)(a - \rho|\iota_1) \end{aligned}\right\}, \quad . \quad (429)$$

which shows that the surface passes through the four lines of intersection of the pair of planes $\rho|\alpha = 0$ and $\rho|\alpha' = 0$ with the pair of planes $\rho|\iota_1 = -a$ and $\rho|\iota_1 = a$. Now if we take the *upper* signs throughout, α and α' are imaginary; hence for the ellipsoid these lines are imaginary. Again, taking the *lower* signs throughout, α and α' are imaginary, and the lines are therefore imaginary for the hyperboloid of two sheets. If, however, we take the upper sign for the second term of (415) and the lower sign for the third term, or *vice versâ*, then α and α' are *real*, and the four lines of intersection of the two pairs of planes are real lines on the surface, which is now a hyperboloid of one sheet. In this case we have

$$\alpha = b^{-1}\iota_2 + c^{-1}\iota_3 \text{ and } \alpha' = b^{-1}\iota_2 - c^{-1}\iota_3.$$

Either of the two pairs of planes,

$$\left\{\begin{array}{l} \rho|(a\alpha - n\iota_1) = na \\ \rho|(a\alpha' + \frac{1}{n}\iota_1) = \frac{a}{n} \end{array}\right\} \text{ and } \left\{\begin{array}{l} \rho|(a\alpha + m\iota_1) = ma \\ \rho|(a\alpha' - \frac{1}{m}\iota_1) = \frac{a}{m} \end{array}\right\}, \quad (429\,a)$$

taken simultaneously satisfies eq. (429). Hence the two planes of each pair intersect in a line lying wholly on the surface. By varying m and n we thus obtain two systems of rectilinear generators of the hyperboloid of one sheet.

153. EXERCISES. — (1) Show, by Ex. 22 or 9 of Art. 137, that no two generators belonging to the same system intersect, while every generator of one system cuts every generator of the other.

(2) Show that the vector to the common point of a generator of the system n and one of the system m is

$$\rho = (m+n)^{-1}[a\iota_1(m-n) + b\iota_2(mn+1) + c\iota_3(mn-1)].$$

(3) Find the condition between m and n in order that a generator of one system may be perpendicular to one of the other.

$$\textit{Ans.}\ 4a^2 - b^2\left(\frac{1}{m} - m\right)\left(\frac{1}{n} - n\right) - c^2\left(\frac{1}{m} + m\right)\left(\frac{1}{n} + n\right) = 0.$$

(4) Show that the projections of the generators on the reference planes are tangent to the principal sections of the surface; that is, that the projecting planes of the generators touch the surface at points lying in the reference planes.

(5) Show that if α, β, γ be substituted for $a\iota_1$, $b\iota_2$, $c\iota_3$ in (412) and (413), those equations will be equivalent to (427).

(6) Find the locus of the intersection of tangent planes at the ends of conjugate semi-diameters α, β, γ.

Let σ be the vector to the point of intersection; then

$$\sigma = \alpha + \beta + \gamma \text{ and } \sigma|\phi\sigma = 3$$

is the equation of the locus, a similar surface.

(7) Show that the locus of the extremity of the vector $\phi\rho$ is $\sigma|\phi^{-1}\sigma = 1$, if $\sigma = \phi\rho$.

(8) Show that, when $T\phi\rho = k =$ const., the locus of the end of ρ is $\rho|\phi^2\rho = k^2$.

(9) Find the equation of the pedal surface of the central quadric; *i.e.* the locus of the foot of the perpendicular from any point upon the tangent plane.

If σ be the vector to a point of the tangent plane, and ϵ the vector to the fixed point, we have

$$\sigma - \epsilon = x\phi\rho, \ \rho|\phi\rho = 1, \text{ and } \sigma|\phi\rho = 1;$$

therefore $\quad \sigma|(\sigma - \epsilon) = x$ and $(\sigma - \epsilon)|\phi^{-1}(\sigma - \epsilon) = x^2$,

whence $\quad (\sigma - \epsilon)|\phi^{-1}(\sigma - \epsilon) = [\sigma|(\sigma - \epsilon)]^2$, . . . (430)

a surface of the fourth order. If we change the origin to the end of ϵ, by writing ρ for $\sigma - \epsilon$, we have

$$\rho|\phi^{-1}\rho = [\rho|(\rho + \epsilon)]^2. \quad (431)$$

(10) Show that the vectors joining any point on the surface with the extremities of a diameter are conjugate in direction.

(11) Find the value of C so that $\rho|\epsilon = C$ may represent a plane tangent to $\rho|\phi\rho = 1$.

Ans. $C = \sqrt{\epsilon|\phi^{-1}\epsilon}$.

Compare with (417), and note that the equation

$$\rho|\epsilon = C = \sqrt{\epsilon|\phi^{-1}\epsilon}$$

is independent of $T\epsilon$.

(12) Taking ϕ as in (415), and ϵ_1, ϵ_2, ϵ_3, as unit normal vectors, show that

$$\epsilon_1|\phi\epsilon_1 + \epsilon_2|\phi\epsilon_2 + \epsilon_3|\phi\epsilon_3 = \frac{1}{a_2} \pm \frac{1}{b_2} \pm \frac{1}{c_2}.$$

(13) Find by Exs. (11) and (12) the locus of the common point of three perpendicular tangent planes to a central quadric.

Ans. $\sigma^2 = a^2 \pm b^2 \pm c^2$.

154. *Condition that $\rho|\phi\rho$ shall be factorable.* Proceeding precisely as in Art. 114, in fact simply putting vectors for points in that article, we have for the required condition

$$\phi\lambda\phi\mu\phi\nu = 0, \qquad (432)$$

λ, μ, ν being any three vectors whatever.

The function $\rho|\phi\rho - k\rho^2$ is *always* factorable; for, writing it in the form $\rho|(\phi - k)\rho$, and putting $\phi - k$ for ϕ in (432), we have

$$(\phi - k)\lambda(\phi - k)\mu(\phi - k)\nu = 0, \qquad (433)$$

a cubic in k which must have at least one real root: hence a value of k can always be found which will make $\rho|(\phi - k)\rho$ the product of two linear factors.

155. *The ϕ function in general.* Any linear vector function may be written in the form

$$\phi\rho = \epsilon_1 \cdot \epsilon_1'|\rho + \epsilon_2 \cdot \epsilon_2'|\rho + \epsilon_3 \cdot \epsilon_3'|\rho. \qquad (434)$$

This may be shown precisely as was done in Art. 97 for two-dimensional space. The function

$$\phi_c\rho = \epsilon_1' \cdot \epsilon_1|\rho + \epsilon_2' \cdot \epsilon_2|\rho + \epsilon_3' \cdot \epsilon_3|\rho \qquad (435)$$

is *conjugate* to ϕ; *i.e.* $\sigma|\phi\rho = \rho|\phi_c\sigma$, and $(\phi + \phi_c)\rho$ is always *self*-conjugate, as was shown in Art. 97. Again,

$$\rho|\phi\rho = \rho|\phi_c\rho \text{ or } \rho|(\phi - \phi_c)\rho = 0.$$

$\therefore$ $(\phi - \phi_c)\rho$ is perpendicular to ρ, and we may write

$$(\phi - \phi_c)\rho = |\epsilon\rho, \qquad (436)$$

ϵ being some real vector when ϕ is not self-conjugate. Then we have

$$\phi\rho = \tfrac{1}{2}(\phi + \phi_c)\rho + \tfrac{1}{2}(\phi - \phi_c)\rho = \tfrac{1}{2}(\phi + \phi_c)\rho + \tfrac{1}{2}|\epsilon\rho. \quad (437)$$

In Art. 129 replace p by ρ, e by ϵ, q_1, q_2, q_3 by λ, μ, ν, and we have the following inversion formulæ:

$$\left.\begin{array}{l}\rho = \phi^{-1}\epsilon = (\phi_c\lambda\phi_c\mu\phi_c\mu)^{-1} \\ \quad [|\phi_c\mu\phi_c\nu \cdot \lambda|\epsilon + |\phi_c\nu\phi_c\lambda \cdot \mu|\epsilon + |\phi_c\lambda\phi_c\mu \cdot \nu|\epsilon]\end{array}\right\} \quad . \quad (438)$$

$$\phi_c\lambda\phi_c\mu\phi_c\nu \cdot \phi^{-1}|\mu\nu = \lambda\mu\nu \cdot |\phi_c\mu\phi_c\nu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (439)$$

Putting also m's for k's and g for n, we have from (374), (375), and (376),

$$m_0\phi^{-1}\rho = (m_1 - m_2\phi + \phi^2)\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (440)$$

$$\left.\begin{array}{l}(m_0 + m_1g + m_2g^2 + g^3)(\phi + g)^{-1}\rho \\ \qquad = m_0\phi^{-1}\rho + g(m_2 - \phi)\rho + g^2\rho,\end{array}\right\} \quad . \quad . \quad . \quad . \quad (441)$$

$$(\phi^3 - m_2\phi^2 + m_1\phi - m_0)\rho = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (442)$$

The coefficients m_0, m_1, m_2 are *invariants; i.e.* their value is the same whatever λ, μ, ν may be, which may be shown, as in Art. 98.

156. *The general scalar equation of the second degree in terms of vectors.* This is identical in form with eq. (312) for plane space, viz.:

$$\rho|\phi\rho + 2\gamma|\rho = C; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (443)$$

for all second-degree terms may be included in $\rho|\phi\rho$, and all first-degree terms in $2\gamma|\rho$.

We will first show that *the surface represented by* (443) *has, in general, cyclic sections; i.e.* that certain planes will cut circles from the surface. Add and subtract $k\rho^2$ to and from eq. (443), which thus becomes

$$k\rho^2 + 2\gamma|\rho - C + \rho|(\phi - k)\rho = 0.$$

By Art. 154, k can be so determined that $\rho|(\phi - k)\rho$ shall be factorable; *i.e.* it may be written $\rho|\alpha \cdot \rho|\alpha'$, k being one of the roots of (443). Hence (443) may be written

$$k\rho^2 + 2\gamma|\rho - C + \rho|\alpha \cdot \rho|\alpha' = 0, \quad . \quad . \quad . \quad . \qquad (444)$$

which represents a surface passing through the curves of intersection of the sphere $k\rho^2 + 2\gamma|\rho - C = 0$ with the two planes $\rho|\alpha = 0$ and $\rho|\alpha' = 0$. As these curves are necessarily circles, these two planes cut circles from the surface.

157. Let us apply the results of the last article to eq. (414). Eq. (433) becomes in this case, if we put $\lambda = \iota_1$, $\mu = \iota_2$, $\nu = \iota_3$,

$$(a^{-2} - k)(\pm b^{-2} - k)(\pm c^{-2} - k) = 0,$$

which gives the three values of k. Using the value $k = b^{-2}$ for the ellipsoid first, we have

$$\left(\frac{\rho|\iota_1}{a}\right)^2 + \left(\frac{\rho|\iota_2}{b}\right)^2 + \left(\frac{\rho|\iota_3}{c}\right)^2 - \frac{(\rho|\iota_1)^2 + (\rho|\iota_2)^2 + (\rho|\iota_3)^2}{b^2} = 1 - \frac{\rho^2}{b^2},$$

or, writing $\frac{b^2}{c^2} - 1 = \mathfrak{e}_1^2$ and $1 - \frac{b^2}{a^2} = \mathfrak{e}_3^2$, and reducing,

$$\mathfrak{e}_1^2(\rho|\iota_3) - \mathfrak{e}_3^2(\rho|\iota_1)^2 = b^2 - \rho^2,$$

or again $\quad \rho|(\mathfrak{e}_1\iota_3 + \mathfrak{e}_3\iota_1) \cdot \rho|(\mathfrak{e}_1\iota_3 - \mathfrak{e}_3\iota_1) = b^2 - \rho^2, \quad . \quad . \quad . \qquad (445)$

an equation in the form of (444). The two planes

$$\rho|(\mathfrak{e}_1\iota_3 + \mathfrak{e}_3\iota_1) = 0 \text{ and } \rho|(\mathfrak{e}_1\iota_3 - \mathfrak{e}_3\iota_1) = 0$$

cut cyclic sections from the ellipsoid; they pass through the b axis of the ellipsoid, and are inclined to the a axis at the angles $\tan^{-1}\frac{\mathfrak{e}_3}{\mathfrak{e}_1}$ and $\tan^{-1}\left(-\frac{\mathfrak{e}_3}{\mathfrak{e}_1}\right)$; *i.e.*

$$\tan^{-1}\frac{c}{a}\sqrt{\frac{a^2 - b^2}{b^2 - c^2}} \text{ and } \tan^{-1}\left(-\frac{c}{a}\sqrt{\frac{a^2 - b^2}{b^2 - c^2}}\right).$$

The values of $\mathfrak{e}_1$ and $\mathfrak{e}_3$ are *real* only when b lies between a and c in value.

For the hyperboloid of two sheets b^2 and c^2 are negative, and therefore $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are real when b lies between a and c in numerical value, so that the cyclic planes of (445) pass through the greater axis which does not pierce the surface.

For the hyperboloid of one sheet let a^2 be negative and b^2 and c^2 positive; then ϵ_3 is real, and ϵ_1 is real when b is greater than c, so that the cyclic planes of (445) pass through the greater axis which pierces the surface.

158. We will next apply the results of Art. 156 to eq. (404), substituting ι_1 and ι_2 for α_1 and α_2.

We have then from (433)

$$(1-k)(m-k)k=0.$$

The root $k=0$ shows, as is evidently true, that $\rho|\phi\rho$ is, in this case, factorable without any addition of a multiple of ρ^2. This leads to the rectilinear generators of the hyperbolic paraboloid as in Art. 141, which are sections of the surface by infinite spheres. Taking next the root $k=m$, we have, after reduction, the equation

$$\left.\begin{array}{c}\rho|(\iota_1\sqrt{1-m}+\iota_3\sqrt{m})\cdot\rho|(\iota_1\sqrt{1-m}-\iota_3\sqrt{m})\\ =a\rho|\iota_3-m\rho^2\end{array}\right\}, \quad . \quad (446)$$

which gives real cyclic sections when m is positive and less than unity. Similarly, the root $k=1$ gives

$$\rho|(\iota_2\sqrt{m-1}+\iota_3)\cdot\rho|(\iota_2\sqrt{m-1}-\iota_3)=a\rho|\iota_3-\rho^2, \quad . \quad (447)$$

which gives real cyclic sections when m is positive and greater than unity. Eqs. (446) and (447) both represent *elliptic* paraboloids, so that the only cyclic sections of the *hyperbolic* paraboloid are the *generators* as mentioned above.

159. Exercises. — (1) Show that the two planes

$$\rho|(\epsilon_1\iota_3+\epsilon_3\iota_1)=C \text{ and } \rho|(\epsilon_1\iota_3-\epsilon_3\iota_1)=-C$$

cut the surface $\rho|\phi\rho=1$ in circles that lie also on the sphere

$$\rho^2+2C\epsilon_3\rho|\iota_1=b^2-C^2.$$

(2) Show that, when $C = \pm\sqrt{a^2 - c^2}$, the planes of the last exercise touch the surface in the points

$$\rho = (\epsilon_1 c^2 \iota_3 \pm \epsilon_3 a^2 \iota_1) \div (\pm\sqrt{a^2 - c^2}).$$

These points are called the umbilici.

(3) Show that the locus of the common point of tangent planes at the ends of three perpendicular radii vectores is $\sigma|\phi^2\sigma = a^{-2} + b^{-2} + c^{-2}$.

Let the vector radii be ρ_1, ρ_2, ρ_3; then

$$\rho_1|\phi\rho_1 = 1, \text{ etc.}, \quad \sigma|\phi\rho_1 = 1, \text{ etc.}, \quad \text{and} \quad \rho_1|\rho_2 = \rho_2|\rho_3 = \rho_3|\rho_1 = 0.$$

Hence,

$$|\rho_1\rho_2 = \frac{T\rho_1 T\rho_2}{T\rho_3} \cdot \rho_3, \text{ etc.}$$

Now, by (178), putting $\phi\rho_1$ for ϵ_1, etc., we have

$$\sigma = (\phi\rho_1\phi\rho_2\phi\rho_3)^{-1}(|\phi\rho_1\phi\rho_2 + |\phi\rho_2\phi\rho_3 + |\phi\rho_3\phi\rho_1),$$

or, by (439),
$$= (\rho_1\rho_2\rho_3)^{-1}\phi^{-1}(|\rho_1\rho_2 + |\rho_2\rho_3 + |\rho_3\rho_1)$$

$$= (T\rho_1 T\rho_2 T\rho_3)^{-1}\phi^{-1}\left(\frac{T\rho_1 T\rho_2}{T\rho_3} \cdot \rho_3 + \text{etc.}\right)$$

$$= \phi^{-1}\left(\frac{\rho_1}{\rho_1^2} + \frac{\rho_2}{\rho_2^2} + \frac{\rho_3}{\rho_3^2}\right).$$

$$\therefore \quad (\phi\sigma)^2 = \sigma|\phi^2\sigma = \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \frac{1}{\rho_3^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2},$$

by Ex. 12, Art. 153, since $\rho_1|\phi\rho_1 = 1$ gives $\dfrac{1}{\rho_1^2} = U\rho_1|\phi U\rho_1$, etc.

(4) Find the equation of a cone with vertex at the end of ϵ circumscribed about the central quadric.

Ans. With the origin at the vertex, the equation is
$$\rho|\phi\rho - \epsilon\rho|\phi\epsilon\phi\rho = 0.$$

(5) Show that the vector to the pole of the plane $\sigma|\alpha = C$ is $\phi^{-1}\alpha \div C$.

(6) A plane is tangent to a central quadric; find the locus of its pole with reference to another quadric concentric with the first.

Ans. If the equations of the quadrics are $\rho|\phi_2\rho = 1$ and $\rho|\phi_1\rho = 1$ respectively, the locus is $\rho|\phi_2^{-1}\phi_1^2\rho = 1$.

(7) Show that any sphere with its center at the origin cuts the surface $\rho|\iota_1 \cdot \rho|\iota_2 + \rho|\iota_2 \cdot \rho|\iota_3 + \rho|\iota_3 \cdot \rho|\iota_1 = 1$ in circles lying in two parallel planes.

(8) Show that the principal sections of the surfaces obtained by giving different values to k in the equation

$$\rho|(\phi^{-1} - k)^{-1}\rho = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (448)$$

have the same foci, ϕ being, as in (415), with the upper signs. These surfaces are said to be *confocal.*

(9) Show that through any point in space *three* confocal surfaces pass, corresponding to three values of k, and that these surfaces cut each other at right angles.

See Ex. 2, Art. 93.

(10) Show that the equation of the tangent plane to the locus of (443) is $\sigma|\phi\rho + \gamma|(\sigma + \rho) = C$. If ρ does not satisfy (443), the equation is that of the polar plane.

(11) In Ex. (6) substitute for the second quadric,

$$\rho|\phi_1\rho + 2\gamma|\rho = C,$$

not concentric with the first, and show that the locus is then

$$\rho|\phi_2^{-1}\phi_1^2\rho + 2\rho|\phi_2^{-1}\phi_1\gamma + \gamma|\phi_2^{-1}\gamma = (C - \rho|\gamma)^2.$$

160. *Center of the general quadric surface.* When the origin is at the center of a surface, terms of odd degrees must not appear in the equations, for it must be unchanged when $-\rho$ is put for $+\rho$. If then we change the origin in eq. (443) by putting $\rho + \delta$ for ρ, and cause the first degree terms to vanish, the origin will be at the center. Eq. (443) becomes thus

$$(\rho + \delta)|\phi(\rho + \delta) + 2(\rho + \delta)|\gamma = C,$$

or

$$\rho|\phi\rho + 2\rho|\phi\delta + \delta|\phi\delta + 2\rho|\gamma + 2\delta|\gamma = C.$$

If the first-degree terms are to vanish for *all* values of ρ, we must have $\phi\delta + \gamma = 0$, or

$$\left.\begin{aligned} \delta = -\phi^{-1}\gamma &= -\frac{|\phi\mu\phi\nu \cdot \lambda|\gamma + |\phi\nu\phi\lambda \cdot \mu|\gamma + |\phi\lambda\phi\mu \cdot \nu|\gamma}{\phi\lambda\phi\mu\phi\nu} \\ &= -\frac{\psi\gamma}{m_0} \end{aligned}\right\}, \quad (449)$$

say, so that

$$\left.\begin{aligned} m_0 &= (\lambda\mu\nu)^{-1}\phi\lambda\phi\mu\phi\nu \\ \text{and} \quad \psi\gamma &= (\lambda\mu\nu)^{-1}(|\phi\mu\phi\nu \cdot \lambda|\gamma + \text{etc.}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (450)$$

The quantity m_0 is an invariant, as we saw in Art. 155. The vector $\psi\gamma$ is also an invariant; that is, its length and direction are independent of the vectors λ, μ, ν, as may be shown by substituting $l\lambda + m\mu + n\nu$ for any one of the three. It also appears from the fact that, as (443) represents a definite surface, when ϕ and γ are given, its center must be a definite fixed point, and hence δ, the vector to this point, must have a definite fixed value, so that, m_0 being invariant, $\psi\gamma$ must be so likewise. If in (449) we have

$$m_0 = 0 \text{ and } T\psi\gamma \lessgtr 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (451)$$

the center is at ∞ in the direction of $\psi\gamma$, and the surface is said to be *non-central.* But, by Art. 154, $m_0 = 0$ is the condition that $\rho|\phi\rho$ shall be the product of two factors of the first degree; *i.e.* that it shall have the form $\rho|\beta_1 \cdot \rho|\beta_2$. Also, since the center is at ∞, take the origin on the surface by making the constant term disappear; *i.e.*

$$\delta|\phi\delta + 2\gamma|\delta - C = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (452)$$

an equation that gives two definite values for $T\delta$ when the *direction* of δ is assumed. Eq. (443) thus becomes

$$\rho|\beta_1 \cdot \rho|\beta_2 + 2(\phi\delta + \gamma)|\rho = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (453)$$

which is identical in form with (410), and must therefore represent an elliptic or hyperbolic paraboloid.

If we have

$$m_0 = 0 = T\psi\gamma, \qquad (454)$$

the value of δ is indeterminate, and $\phi\delta = -\gamma$ is then the equation of the *line* or *plane* of centers, the locus being in this case either a cylinder or two planes.

Resuming now the general value of δ, and substituting it, eq. (443) reduces to

$$\left.\begin{aligned} \rho|\phi\rho &= C - \delta|\phi\delta - 2\gamma|\delta = C + \gamma|\phi^{-1}\gamma \\ &= C + m_0^{-1}\gamma|\psi\gamma = C', \text{ say} \end{aligned}\right\} \qquad (455)$$

The sign of C' will be always taken as *positive*.

If we have

$$C' = C + \gamma|\phi^{-1}\gamma = 0;$$

that is, $$m_0 C + \gamma|\psi\gamma = 0, \qquad (456)$$

then (455) becomes

$$\rho|\phi\rho = 0, \qquad (457)$$

which, being independent of $T\rho$, must represent a *cone*, real or imaginary, which, when δ is indeterminate, breaks up into two real or imaginary planes.

161. *Maximum and minimum values of $T\rho$.* $T\rho$ will be a maximum or minimum when $dT\rho = 0$; *i.e.* when $\rho|d\rho = 0$. Eq. (455) gives $d\rho|\phi\rho = 0$; hence, at the maximum or minimum points, ρ and $\phi\rho$ are both $\perp$ to all values of $d\rho$, which requires them to be parallel to each other; *i.e.* we must have $\rho\phi\rho = 0$, an equation whose solution will give the required values of ρ.

This equation is equivalent to

$$\phi\rho = g\rho \text{ or } (\phi - g)\rho = 0, \qquad (458)$$

g being a scalar constant to be determined. Multiply the complement of (458) by ρ, and we have

$$\rho|\phi\rho = C' = g\rho^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (459)$$

which gives the relation between $(T\rho)_{\max.}$ and g.

By reference to Art. 111 it will be seen that the equation to be solved here is identical in form with the one there treated. Hence it will only be necessary to put vectors, say ρ, λ, μ, ν, in place of the points p, q_1, q_2, q_3, in Arts. 111 and 112, in order to obtain results for a vector system in solid space corresponding to those for a point system in plane space. We will also substitute g for n, and m for k. Thus we have, from (341),

$$\rho|(\phi - g)\lambda = \rho|(\phi - g)\mu = \rho|(\phi - g)\nu = 0; \quad . \quad (460)$$

from (342),

$$\left.\begin{array}{l}(\phi - g)\lambda(\phi - g)\mu(\phi - g)\nu = 0 \\ \text{or} \quad g^3 - m_2 g^2 + m_1 g - m_0 = 0\end{array}\right\}; \quad . \quad . \quad . \quad . \quad . \quad (461)$$

in which m_0 has the value given in (450),

$$\left.\begin{array}{l}m_1 = (\lambda\mu\nu)^{-1}(\lambda\phi\mu\phi\nu + \mu\phi\nu\phi\lambda + \nu\phi\lambda\phi\mu) \\ m_2 = (\lambda\mu\nu)^{-1}(\lambda\mu\phi\nu + \mu\nu\phi\lambda + \nu\lambda\phi\mu)\end{array}\right\} . \quad . \quad (462)$$

The m's are, of course, invariants like the k's of Art. 111. Eq. (461) is called the *discriminating cubic*, and plays an important part in the discussion of eq. (455). Eqs. (459), (460), and (461) give the complete solution of the problem in maxima and minima; for (461) gives three values of g, say g_1, g_2, g_3, which, substituted in (459), give the *lengths* of the maximum or minimum radii vectores, and substituted in (460) give the *directions* of the same. If we assume $g_1 < g_2 < g_3$, and let a, b, c be the corresponding values of $T\rho$, we have

$$a^2 = C'g_1^{-1}, \; b^2 = C'g_2^{-1}, \; c^2 = C'g_3^{-1}. \quad . \quad . \quad . \quad (463)$$

Proceeding precisely as in Art. 112, we find, if ρ_1, ρ_2, ρ_3 are the values of ρ corresponding to g_1, g_2, g_3,

$$\rho_1 = g_1 a_1, \; \rho_2 = g_2 a_2, \; \rho_3 = g_3 a_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (464)$$

in which α_1, α_2, α_3 are a *unit, normal* system of vectors, so that, whatever may be the original form of ϕ, it may always be transformed into

$$\phi\rho = g_1\alpha_1 \cdot \rho|\alpha_1 + g_2\alpha_2 \cdot \rho|\alpha_2 + g_3\alpha_3 \cdot \rho|\alpha_3, \quad . \quad . \quad . \quad . \quad (465)$$

in which $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1$, and $\alpha_1|\alpha_2 = \alpha_2|\alpha_3 = \alpha_3|\alpha_1 = 0$, as well as $\alpha_1|\phi\alpha_2 = \alpha_2|\phi\alpha_3 = \alpha_3|\phi\alpha_1 = 0$, so that we have obtained a system of mutually perpendicular conjugate diameters. It also appears, as in Art. 112, that the roots of (461) are always *real.*

162. Eq. (461) may also be written in the form

$$\left.\begin{array}{l}(g-g_1)(g-g_2)(g-g_3)=0 \\ \text{or } g^3-(g_1+g_2+g_3)g^2+(g_2g_3+g_3g_1+g_1g_2)g-g_1g_2g_3=0\end{array}\right\}. \quad (466)$$

From the form to which ϕ was shown in the last article to be reducible, and from (463), it appears that the general equation (455) can be written

$$\pm\left(\frac{\rho|\alpha_1}{a}\right)^2 \pm \left(\frac{\rho|\alpha_2}{b}\right)^2 \pm \left(\frac{\rho|\alpha_3}{c}\right)^2 = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (467)$$

the upper or lower signs to be taken according as the roots of (461) are positive or negative. Comparing this equation with (414), it appears that the general central equation represents one of the three surfaces considered in Arts. 143 and 144. By Descartes' rule of signs the positive roots of (461) are equal in number to the number of *variations* of sign, and the negative roots to the number of *permanences* of sign. Also, comparing (461) and (466), we see that the coefficient of g^3 is $+1$, and that m_0 is $+$ when all the roots are $+$, $-$ when one root is $-$, $+$ when two roots are $-$, and $-$ when three roots are $-$. Thus taking all the possible arrangements of sign, we have,

When all roots are positive, $\quad + - + -$

When one root is negative, $\quad \left\{\begin{array}{c} + - + + \\ + - - + \\ + + - + \end{array}\right\}$

When two roots are negative, $\left\{\begin{matrix} + & + & - & - \\ + & + & + & - \\ + & - & - & - \end{matrix}\right\}$

When all roots are negative, $+ \ + \ + \ +$

We may also have one root *zero*, when the cubic reduces to a quadratic, and $m_0 = 0$.

If *two* roots are zero, the equation is of the first degree, and m_1 is also zero.

These six cases correspond to the following surfaces:—

1st case. Ellipsoid; sphere; point (imaginary cone).

2d case. Hyperboloid of one sheet; cone.

3d case. Hyperboloid of two sheets; cone.

4th case. Imaginary ellipsoid; imaginary cone.

5th case. Cylinder, elliptic or hyperbolic; two real or imaginary intersecting planes:—because one axis is infinite.

6th case. Parallel planes:—because two axes are infinite.

If two roots of (461) are *equal*, the surface is one of *revolution*, for which the condition is

$$27 m_0^2 - 18 m_0 m_1 m_2 + 4 m_1^3 - m_1^2 m_2^2 + 4 m_0 m_2^3 = 0. \quad (467\,a)$$

163. *Further consideration of the case when* $m_0 = 0$.

Resuming the general equation $\rho|\phi\rho + 2\gamma|\rho = C$, which was shown to represent, in general, a paraboloid, when $m_0 = 0$, it may be seen at once that this paraboloid passes through the intersection of the central surface $\rho|\phi\rho = C$ with the plane $\rho|\gamma = 0$. But the condition $m_0 = 0$ causes the equation $\rho|\phi\rho = C$ to represent a *cylinder*, elliptic or hyperbolic, according as m_1 is $+$ or $-$, while, if m_1 is *also* zero, it represents two parallel planes. Hence the plane $\rho|\gamma = 0$ cuts from the locus of (455) an ellipse, hyperbola, or two parallel right lines, according as m_1 is $+$, $-$, or 0. The surface must therefore be in the corresponding cases an *elliptic paraboloid*, a *hyperbolic paraboloid*, or a *parabolic cylinder*.

164. *The quantity m_0C'.* The *sign* of this quantity determines whether (455) represents a *skew* or *convex* surface, and its *vanishing* shows that the surface is *developable*. We have seen in Art. 152 that the hyperboloid of one sheet is skew, while the ellipsoid and the other hyperboloid are not. Now C' is always to be taken positive, as stated in Art. 160, so that the sign of m_0C', for the central surfaces, depends only on m_0, which we have seen in Art. 162 to be $+$ for the ellipsoid and two-sheeted hyperboloid, and $-$ for the one-sheeted hyperboloid. Also, in Art. 160, we saw that when C' is zero, the central surface becomes a cone, a developable surface.

For the case when $m_0 = 0$, we have

$$m_0C' = m_0C + \gamma|\psi\gamma = \gamma|\psi\gamma.$$

Applying this to eq. (404), to which form the general equation was shown to be reducible in this case, we have

$$\gamma|\psi\gamma = \tfrac{1}{4}ma^2,$$

which is $+$ for the elliptic paraboloid, and $-$ for the hyperbolic paraboloid. If $m = 0$, the surface becomes a parabolic cylinder, and in this case $m_1 = 0$ also, which agrees with the last article. For the elliptic and hyperbolic cylinders $m_0C' = 0$ because $m_0 = 0$, while C' is finite.

We give here a table of the results which we have obtained in the treatment of the general equation of the second degree.

For the meaning of the symbols see as follows: for m_0, eq. (450); for m_1 and m_2, eq. (462); for δ, eq. (449); for C', eq. (455); for m_0C', the present article.

CLASSIFICATION OF THE QUADRIC.

	NAME OF SURFACE.	m_0C'	δ	C'	m_0	m_1	m_2
Central.	Ellipsoid.	+	Finite	+	+	+	+
	Imaginary cone (point).	0	"	0	"	"	"
	Hyperboloid, 1 sheet.	−	"	+	−	+ − −	+ + −
	Real cone.	0	"	0	"	"	"
	Hyperboloid, 2 sheets.	+	"	+	+	+ − −	− − +
	Real cone.	0	"	0	"	"	"
	Imaginary Ellipsoid.	−	"	+	−	+	−
	Imaginary cone (point).	0	"	0	"	"	"
Many centered.	Elliptic cylinder.	0	%	+	0	+	+
	Right line.	0	%	0	0	+	+ −
	Parallel planes.	0	%	+	0	0	+
	Coincident planes.	0	%	0	0	0	+ −
	Hyperbolic cylinder.	0	%	+	0	−	+ −
	Two intersecting planes.	0	%	0	0	−	+ −
Non-central.	Elliptic paraboloid.	+	∞	∞	0	+	+
	Parabolic cylinder.	0	∞	∞	0	0	+
	Hyperbolic paraboloid.	−	∞	∞	0	−	+ −

165. EXERCISES. — (1) Find what surface is represented by the equation $(\rho|\epsilon_1)^2 + 2\,\rho|\epsilon_1 \cdot \rho|\epsilon_2 + 2\,\gamma|\rho = C$, when $\epsilon_1\epsilon_2\gamma$ is not zero, and also when it vanishes.

(2) Discuss the equation $\rho^2 - n^2(\rho|\iota_1)^2 + 2\,\rho|\gamma = C$ when $n > 1$, when $n = 1$, and when $n < 1$.

(3) Discuss the equation

$$\rho|\alpha_1 \cdot \rho|\beta_1 + \rho|\alpha_2 \cdot \rho|\beta_2 + \rho|\alpha_3 \cdot \rho|\beta_3 + 2\,\rho|\gamma = C$$

in the following cases:

(*a*) $\alpha_1 = 5\iota_1 + 2\iota_2 + 4\iota_3$, $\alpha_2 = 2\iota_1 - 2\iota_2 - 3\iota_3$, $\alpha_3 = 4\iota_1 - 3\iota_2 - \iota_3$, $\beta_1 = \iota_1$, $\beta_2 = \iota_2$, $\beta_3 = \iota_3$, $\gamma = 0$, $C = 1$.

(*b*) $\beta_3 = \alpha_1$, $\beta_1 = \alpha_2$, $\beta_2 = \alpha_3$, $C = 2a^2$, $\gamma = 0$.

(*c*) The same as (*b*) except $\gamma = -\alpha_1 + 3\alpha_2 - 4\alpha_3$.

(*d*) The same case as (*b*) except add $2\rho^2$ to the first member.

(*e*) $\alpha_1 = \beta_1 = a\iota_2 + b\iota_1$, $\alpha_2 = \beta_2 = b\iota_3 + c\iota_2$, $\alpha_3 = \beta_3 = c\iota_1 + a\iota_3$, $\gamma = 0$, $C = 1$.

(4) Discuss the following equations:

(*a*) $(\epsilon_1\rho)^2 + (\epsilon_2\rho)^2 + (\epsilon_3\rho)^2 = a^2$.

(*b*) $\epsilon_1\rho \cdot \epsilon_2\epsilon_3 \cdot \epsilon_3\rho = c^2$.

(*c*) $(\epsilon_1\rho \cdot \epsilon_2\epsilon_3)\epsilon_3(\epsilon_2\rho \cdot \epsilon_3\epsilon_1) = c^2$.

(*d*) $\epsilon\rho\phi\rho = C$, ϕ being as in (415).

In the last case we may write

$$\epsilon\rho\phi\rho = \epsilon\rho\phi\rho \cdot \iota_1\iota_2\iota_3 = \begin{vmatrix} \iota_1|\epsilon, & \iota_1|\rho, & \iota_1|\phi\rho \\ \iota_2|\epsilon, & \iota_2|\rho, & \iota_2|\phi\rho \\ \iota_3|\epsilon, & \iota_3|\rho, & \iota_3|\phi\rho \end{vmatrix} = C.$$

(5) Show that, if $\rho|\phi_1\rho = 1$ and $\rho|\phi_2\rho = 1$ have parallel axes, then $\rho|(\phi_1 + k\phi_2)\rho = 1$ has its axes parallel to theirs.

(6) Find the condition that the surface of the second order passing through the common line of two quadrics shall be developable, and thus show that in general *four* different cones pass through this common line.

(7) Find the axes of a central plane section of a central quadric; also the area of the curve of section.

The equation of the surface is $\rho|\phi\rho = 1$, and that of the plane $\rho|\epsilon = 0$; hence, for a maximum or minimum value of $T\rho$, we have $d\rho|\rho = 0$; also, $d\rho|\epsilon = 0$ and $d\rho|\phi\rho = 0$. Therefore we have $\epsilon\rho\phi\rho = 0$, a cone cutting the curve of section at its maximum and minimum points. But $\epsilon\rho\phi\rho = 0$ is equivalent to

$$\phi\rho + k\rho + k'\epsilon = 0, \quad \text{whence } \rho|\phi\rho = 1 = -k\rho^2.$$

Also, $\rho = -k'(\phi + k)^{-1}\epsilon$, whence $\epsilon|\rho = 0 = \epsilon|(\phi + k)^{-1}\epsilon$.

By (441) this is equivalent to

$$m_0\epsilon|\phi^{-1}\epsilon + k\epsilon|(m_2 - \phi)\epsilon + k^2\epsilon^2 = 0.$$

$$\text{Area} = \pi a_1 b_1 = \frac{\pi}{\sqrt{k_1 k_2}} = \frac{\pi}{\sqrt{m_0\epsilon|\phi^{-1}\epsilon}}, \text{ if } T\epsilon = 1.$$

(8) Perpendiculars are drawn from p on the four faces of a tetraedron, the feet of the perpendiculars being coplanar; find the locus of p.

(9) Find the locus of a point, the sum of the squares of whose distances from n fixed points is constant.

(10) Find the locus of a point, the sum of the squares of whose distances from two fixed right lines is constant.

(11) Find the locus of a point, the sum of the squares of whose distances from three fixed right lines is constant.

Determine the nature of the locus in each case.

(12) Find the condition that the plane $\sigma|\epsilon = D$ shall be tangent to the surface of eq. (443).

Ans. $D + \gamma|\phi^{-1}\epsilon = \pm\sqrt{C''\epsilon|\phi^{-1}\epsilon}$.

CHAPTER VI.

SCALAR POINT EQUATIONS OF THE SECOND DEGREE IN SOLID SPACE.

166. *Differentiation.* As it will be necessary in this chapter to use dP, we proceed to determine its meaning. Reasoning precisely as in Art. 79, we see that, if p lie on some surface, dp must be a vector along some tangent to the surface at p; *i.e.* a point at ∞ in the tangent plane at p. Then, if $P = |p$, P will envelop some surface, and we have $dP = |dp$, so that dP is a plane through the mean point of the reference tetraedron. Also, as dP is the limit of $P - P'$ as these planes approach coincidence, it must always pass through the common line of P and P', and hence, at the limit, through the point of contact of P with the surface it envelops.

167. *The general homogeneous equation.* We shall deal *only* with homogeneous equations; for, by (377), $4p|\bar{e} = 1$, so that any term of lower degree than the highest can be raised to that degree by multiplying it by the proper power of $4p|\bar{e}$, without changing thereby the meaning of the equation.

Any homogeneous equation of the second degree in p may be written in the form

$$p|\phi p = 0, \qquad (468)$$

in which ϕ is self-conjugate; for such an equation can always be reduced to the sum of such terms as $A_1 p|q_1 \cdot p|q_1'$; that is, to the form $\Sigma(Ap|q \cdot p|q') = 0$. This is equivalent to

$$\Sigma[p|(Aq \cdot p|q' + Aq' \cdot p|q)] = p|\Sigma[A(q \cdot p|q' + q' \cdot p|q)]$$
$$= p|\phi p = 0,$$

if we write

$$\phi p = \Sigma[A(q \cdot p|q' + q' \cdot p|q)].$$

168. Eq. (468) represents a surface of the *second order; i.e.* it is pierced in *two* points by any right line. The demonstration is precisely as in Art. 105, and leads to the same value of $\frac{y}{x}$ given in (327). The equation

$$q_1p|\phi q_1\phi p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (469)$$

now represents a circumscribed *cone* with its vertex at q_1, while

$$\epsilon p|\phi\epsilon\phi p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (470)$$

represents a circumscribed *cylinder* ∥ to ϵ.

169. *Diametral planes.* The equation of the locus of the middle points of a system of chords ∥ to ϵ is found, as in Art. 106, to be

$$p|\phi\epsilon = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (471)$$

which represent now a diametral *plane*, conjugate in direction to ϵ. Every line in this plane is also conjugate in direction to ϵ. If p recede to ∞, it becomes a vector, say ϵ', and hence $\epsilon'|\phi\epsilon = 0$ is the condition for conjugate directions. If we take a second vector ϵ'' ∥ to the plane of (471), then we have also $\epsilon''|\phi\epsilon = 0$, and if we have also $\epsilon'|\phi\epsilon'' = 0$, then ϵ, ϵ', ϵ'' form a system of conjugate directions; *i.e.* they are ∥ to a set of conjugate diameters of the surface $p|\phi p = 0$.

170. *Significance of the quantity* $|\phi p$. Differentiating (468) we have $dp|\phi p = 0$; hence $|\phi p$, which, by (468), is a plane through p, is the locus of the tangents to the surface at p; *i.e.* the tangent plane. Its equation may be written

$$q|\phi p = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (472)$$

Suppose $|\phi p$ to pass through some fixed point e; then we shall have $e|\phi p = 0 = p|\phi e$. The point p in this equation is the point of contact of the tangent plane $|\phi p$; hence, if p *vary* subject to the above condition $p|\phi e = 0$, this equation is that of the *locus* of p, and is a *plane*, the *polar plane* of e. If e' be on the polar plane of e, we have $e'|\phi e = 0 = e|\phi e'$, so that e is also on the polar plane of e'. The points e and e' are *conjugate* to each other. We see, by (471), that a diametral plane is simply the polar plane of a point at ∞; hence the polar planes of all points in a diametral plane pass through the same point at ∞; *i.e.* they are parallel to the diameter conjugate to this plane. If a point move along a line, its polar plane turns about a line. For let $xp_1 + yp_2$ be any point on the line p_1p_2; then its polar plane is

$$|\phi(xp_1 + yp_2) = x|\phi p_1 + y|\phi p_2,$$

a plane through the common line of $|\phi p_1$ and $|\phi p_2$. Also, since the polar planes of points on $|\phi p_1$ and $|\phi p_2$ pass respectively through p_1 and p_2, it follows that the polar planes of points on $|\phi p_1 \phi p_2$ pass through p_1p_2. If we put ϵ_1 and ϵ_2 for p_1 and p_2, we see that the polar planes of points on a diameter pass through a common line at ∞, *i.e.* they are parallel.

171. *Center of the surface.* This is the common point of any three diametral planes; hence, if ϵ_1, ϵ_2, ϵ_3 are any points at ∞, we have for the center

$$q_c = n|\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3, \quad \ldots \quad (473)$$

n being a scalar factor so taken as to make q_c a unit point.

To evaluate n, multiply both sides of (473) into $4|\bar{e}$.

$$\therefore\ 4\,q_c|\bar{e} = 1 = 4\,n|\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3\bar{e} = -4\,n\bar{e}\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3.$$

$$\therefore\ q_c = -\frac{|\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3}{4\,\bar{e}\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3}. \quad \ldots \quad (474)$$

If we have

$$\bar{e}\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3 = 0, \quad \ldots \quad (475)$$

while the numerator of (474) is not zero, then the center is at ∞, and (468) represents a non-central surface. If we put q_c for q_1 in (469), we have the *asymptotic cone*

$$pq_c|\phi p\phi q_c = 0. \quad \ldots \quad (476)$$

172. *Sets of conjugate points.* Any four points q_1, q_2, q_3, q_4 which satisfy the six conditions, —

$$q_1|\phi q_2 = q_1|\phi q_3 = q_1|\phi q_4 = q_3|\phi q_4 = q_4|\phi q_2 = q_2|\phi q_3 = 0, \quad (477)$$

form a set of conjugate points. Each point evidently lies on the polar plane of each of the others; *i.e.* the four points form a tetraedron such that each vertex is the pole of the opposite face. There is an infinite number of such sets of points; for, take *any* point in space as q_1; then *any* point in $|\phi q_1$ as q_2; by Art. 170, $|\phi q_2$ also passes through q_1; next take *any* point in $|\phi q_1 \phi q_2$ as q_3; then $|\phi q_3$ passes through $q_1 q_2$ by Art. 170, and cuts $|\phi q_1 \phi q_2$ in q_4.

If q_4 be at ∞, $q_1 q_2 q_3$ is a diametral plane; if q_3 be also at ∞, then $q_1 q_2$ must be on a diameter, and, finally, if q_2 be at ∞, q_1 must be the *center* of the surface; thus q_c, ϵ, ϵ', ϵ'' are a set of conjugate points, three of which, being at ∞, reduce to directions, as in Art. 169.

173. *Normal set of conjugate points.* If four points, besides the conditions (477), satisfy also the following

$$q_1|q_2 = q_1|q_3 = q_1|q_4 = q_3|q_4 = q_4|q_2 = q_2|q_3 = 0, \quad . \quad . \quad (478)$$

they may be called a normal set of conjugate points. We proceed to show that, with reference to any given surface $p|\phi p = 0$, there is one and only one normal system of conjugate points.

174. *Solution of the equation $\phi p = np$.* Write the equation $(\phi - n)p = 0$, and multiply its complement successively by any four points e, e', e'', e''', thus obtaining

$$\left.\begin{aligned} e\ |(\phi - n)p &= p|(\phi - n)e = 0 \\ e'\ |(\phi - n)p &= p|(\phi - n)e' = 0 \\ e''\ |(\phi - n)p &= p|(\phi - n)e'' = 0 \\ e'''|(\phi - n)p &= p|(\phi - n)e''' = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (479)$$

Each of these equations must be satisfied by the same values of p that satisfy the given equation; therefore p must be simultaneously on the four planes $|(\phi - n)e$, etc.: hence these four planes must have a common point, the condition for which is

$$\left.\begin{aligned} &(\phi - n)e(\phi - n)e'(\phi - n)e''(\phi - n)e''' = 0, \\ \text{or} \quad &n^4 - k_3 n^3 + k_2 n^2 - k_1 n + k_0 = 0, \end{aligned}\right\} \quad . \quad (480)$$

in which the k's have the following values:

$$\left.\begin{aligned} k_0 &= (ee'e''e''')^{-1}\phi e\phi e'\phi e''\phi e''' \\ k_1 &= (ee'e''e''')^{-1}\Sigma(e\phi e'\phi e''\phi e''') \\ k_2 &= (ee'e''e''')^{-1}\Sigma(ee'\phi e''\phi e''') \\ k_3 &= (ee'e''e''')^{-1}\Sigma(ee'e''\phi e''') \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (481)$$

the summations being on the following plan, viz.: the ϕ's are to be placed before the different e's in succession, three by three, two by two, and one by one, for k_1, k_2, and k_3, respectively; thus $ee'\phi e''\phi e''' + \phi ee'e''\phi e''' +$ etc.

The solution of (480) will give four values of n, which, substituted in (479), will give the required points. Eq. (480) can have no imaginary root. This may be shown precisely as in Art. 112.

Let the roots be n_1, n_2, n_3, n_4, to which correspond the points p_1, p_2, p_3, p_4; then these points are given by the equations

$$\left.\begin{aligned} p_1|(\phi - n_1)e(\phi - n_1)e'(\phi - n_1)e'' &= 0 \\ p_2|(\phi - n_2)e(\phi - n_2)e'(\phi - n_2)e'' &= 0 \\ \text{etc., etc.} & \end{aligned}\right\}, \quad . \quad . \quad . \quad (482)$$

in accordance with (479).

175. *The points $p_1, \cdots p_4$, just determined, constitute a normal set of conjugate points.* Since $\phi p_1 = n_1 p_1$, $\phi p_2 = n_2 p_2$, etc., we have $p_1|\phi p_2 = n_2 p_1|p_2$, etc.; so that, if $p_1|p_2 = 0$, then $p_1|\phi p_2 = 0$ also. As ϕ is self-conjugate, write it in the most general form of such a function in terms of $n_1, \cdots n_4$ and $p_1, \cdots p_4$; *i.e.*

$$\begin{aligned} 2\phi p = &- n_1 p_1 \cdot p_2 p_3 p_4 p + n_2 p_2 \cdot p_3 p_4 p_1 p - n_3 p_3 \cdot p_4 p_1 p_2 p \\ &+ n_4 p_4 \cdot p_1 p_2 p_3 p - n_1 |p_2 p_3 p_4 \cdot p_1|p + n_2 |p_3 p_4 p_1 \cdot p_2|p \\ &- n_3 |p_4 p_1 p_2 \cdot p_3|p + n_4 |p_1 p_2 p_3 \cdot p_4|p. \end{aligned}$$

In order that this form of ϕ may satisfy the above conditions, we must have $p_1p_2p_3p_4 = 1$, $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 1$, and $p_1|p_2 = p_1|p_3 = p_1|p_4 = p_3|p_4 = p_4|p_2 = p_2|p_3 = 0$. These last conditions also imply $|p_2p_3p_4 = -p_1$, etc., for they make p_1 lie simultaneously on the three planes $|p_2, |p_3, |p_4$; therefore $p_1 = m|p_2p_3p_4$, m being a scalar constant. Multiply into $|p_1$.

$$\therefore\ p_1^2 = 1 = m|p_2p_3p_4p_1 = -mp_1p_2p_3p_4 = -m,$$

so that $$p_1 = -|p_2p_3p_4.$$

Thus, whatever the original form of ϕ may have been, by expressing it in terms of $p_1, \cdots p_4$ as determined in Art. 174, it will always be reduced to the form

$$\phi p = n_1p_1\cdot p|p_1 + n_2p_2\cdot p|p_2 + n_3p_3\cdot p|p_3 + n_4p_4\cdot p|p_4. \quad (483)$$

176. *Canonical form of $p|\phi p$.* With ϕ as in (483), we have

$$p|\phi p = n_1(p|p_1)^2 + n_2(p|p_2)^2 + n_3(p|p_3)^2 + n_4(p|p_4)^2 = 0, \quad (484)$$

as a form to which $p|\phi p = 0$ may always be reduced.

This equation may also be written

$$p|(p_1\sqrt{n_1} + p_2\sqrt{-n_2})\cdot p|(p_1\sqrt{n_1} - p_2\sqrt{-n_2}$$
$$+ p|(p_3\sqrt{n_3} + p_4\sqrt{-n_4})\cdot p|(p_3\sqrt{n_3} - p_4\sqrt{-n_4}) = 0,$$

or $$p|q_1\cdot p|q_1' + p|q_2\cdot p|q_2' = 0, \quad (485)$$

in which

$$q_1 = p_1\sqrt{n_1} + p_2\sqrt{-n_2},\quad q_1' = p_1\sqrt{n_1} - p_2\sqrt{-n_2},$$
$$q_2 = p_3\sqrt{n_3} + p_4\sqrt{-n_4},\quad q_2' = p_3\sqrt{n_3} - p_4\sqrt{-n_4}.$$

In this form it appears that the surface passes through the four common lines of two pairs of planes $|q_1, |q_1'$ and $|q_2, |q_2'$. If these planes are all *real*, then real right lines lie wholly on the surface; if any of them are imaginary, then there are *no* real right lines on the surface.

177. *Rectilinear generators.* Using q_1, q_1', etc., as in the last article, the pairs of planes

$$\left.\begin{cases} p|(q_1 + mq_2) = 0 \\ p|(q_1' - \frac{1}{m}q_2') = 0 \end{cases}\right\} \text{ and } \left.\begin{cases} p|(q_1 + nq_2') = 0 \\ p|(q_1' - \frac{1}{n}q_2) = 0 \end{cases}\right\}, \quad (486)$$

when different values are assigned to m and n, each intersect in a system of right lines on the surface of eq. (484). Thus the surface has two systems of rectilinear generators when the q's are all real.

Any two belonging to the systems m and n respectively intersect, for

$$(q_1+mq_2)(q_1'-\frac{1}{m}q_2')(q_1+nq_2')(q_1'-\frac{1}{n}q_2)=q_1q_1'q_2q_2'(1-1)=0.$$

No two of the same system intersect, for

$$(q_1+m_1q_2)\left(q_1'-\frac{1}{m_1}q_2'\right)(q_1+m_2q_2)\left(q_1'-\frac{1}{m_2}q_2'\right)$$
$$=q_1q_2q_1'q_2'(m_1m_2)^{-1}(m_1-m_2)^2,$$

which is not zero unless $m_1=m_2$, when the two generators coincide.

178. *The discriminant.* The expression k_0 as given in (481) is the discriminant of the quantity $p|\phi p$, and is the criterion for distinguishing whether (468) represents a skew, developable, or convex surface. k_0 is an *invariant*, as are also k_1, k_2, and k_3 of (481); *i.e.* they are unchanged if any other points be substituted for e, e', etc. Now we have seen that any form of ϕ may be transformed by changing the reference points into the form (483); hence, as this transformation will not affect k_0, it, k_0, must have the same meaning for this form as for the original one.

Eq. (483) gives

$$k_0=n_1n_2n_3n_4. \qquad (487)$$

Considering positive and negative values of k_0 as dependent on the signs of the n's, we have four cases:

1st. All the n's positive, k_0 positive;

2d. One of them zero, $k_0=0$;

3d. One of them negative, k_0 negative;

4th. Two of them negative, k_0 positive.

Three n's negative does not give a new case; for, by changing all the signs, we have only one negative.

In the first case all the q's of (485) are imaginary, and no real value of p will satisfy (484), which therefore represents an *imaginary* surface.

In the second case, suppose $n_4=0$; then (485) takes the form

$$p|q \cdot p|q_1' + n_3(p|p_3)^2 = 0,$$

which represents a *cone* touching $|q_1$ and $|q_1'$ along their intersections with $|p_3$, *imaginary* if n_1, n_2, n_3 are all positive, *real* if one of them is negative; thus we have a *developable* surface.

In the third case suppose n_4 negative; then q_2 and q_2' are real, while q_1 and q_1' are still imaginary; hence there are no real right lines on the surface, which is therefore *convex*.

In the fourth case let n_2 and n_4 be negative. [It manifestly makes no difference *which* two are supposed negative, for any other pair as well as n_2 and n_4 might just as well have had negative signs under the radicals in (485).] We now have all the q's *real*, so that the surface has two real systems of rectilinear generators, and is therefore skew.

If $n_3 = n_4 = 0$, the surface becomes simply two planes, real or imaginary, intersecting in a real right line.

We have thus

$$k_0 (= \phi e_0 \phi e_1 \phi e_2 \phi e_3) \left\{ \begin{array}{ll} \text{positive} & \text{for a skew surface} \\ \text{zero} & \text{for a developable surface} \\ \text{negative} & \text{for a convex surface} \end{array} \right\}. \quad (488)$$

In the value of k_0 we have put $e_0, \cdots e_3$ instead of $e, \cdots e'''$, because the value of k_0 is unchanged thereby, and the reference points are the ones generally most convenient.

179. *Nature of the surface at infinity.* To ascertain this, let the variable point p recede to ∞, by substituting for it

$$p_\infty = \rho = x\epsilon_1 + y\epsilon_2 + z\epsilon_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (489)$$

in the equation of the surface $p|\phi p = 0$.

$$\therefore \quad (x\epsilon_1 + y\epsilon_2 + z\epsilon_3)|\phi(x\epsilon_1 + y\epsilon_2 + z\epsilon_3) = 0,$$

or

$$\left. \begin{array}{l} x^2\epsilon_1|\phi\epsilon_1 + y^2\epsilon_2|\phi\epsilon_2 + z^2\epsilon_3|\phi\epsilon_3 + 2\,yz\epsilon_2|\phi\epsilon_3 \\ \quad + 2\,zx\epsilon_3|\phi\epsilon_1 + 2\,xy\epsilon_1|\phi\epsilon_2 = 0 \end{array} \right\} \quad . \quad . \quad . \quad (490)$$

Eqs. (489) and (490) taken together represent a cone passing through the intersection of $p|\phi p = 0$ with the plane at ∞. If this cone be *real,* the surface has a *real* curve at ∞; if it be *imaginary,* the surface has no real curve at ∞; if it break up into two real or imaginary *planes,* the surface has two real or imaginary right lines at ∞.

Take $\alpha_1, \alpha_2, \alpha_3$ as in Art. 140, so that $\alpha_1 = |\epsilon_2\epsilon_3$, etc., this complement being regarded as referring to a *vector* system.

Also write A, B, C, D, E, F for the coefficients in (490) taken in order, *i.e.* $\epsilon_1|\phi\epsilon_1 = A$, etc. From (489),

$$\rho|\alpha_1 = x, \quad \rho|\alpha_2 = y, \quad \rho|\alpha_3 = z,$$

so that (490) becomes

$$\left.\begin{aligned} A(\rho|\alpha_1)^2 + B(\rho|\alpha_2)^2 + C(\rho|\alpha_3)^2 + 2\,D\rho|\alpha_2 \cdot \rho|\alpha_3 \\ + 2\,E\rho|\alpha_3 \cdot \rho|\alpha_1 + 2\,F\rho|\alpha_1 \cdot \rho|\alpha_2 = 0 \end{aligned}\right\}, \quad . \quad (491)$$

and $$\phi'\rho = (A\alpha_1 + F\alpha_2 + E\alpha_3) \cdot \rho|\alpha_1 + (F\alpha_1 + B\alpha_2 + D\alpha_3) \cdot \rho|\alpha_2 + (E\alpha_1 + D\alpha_2 + C\alpha_3) \cdot \rho|\alpha_3.$$

We will next ascertain for this equation the values of m_0, m_1, m_2 appearing in the table in Art. 164. These will be sufficient to determine the cone completely.

We have, if we put $\lambda = \epsilon_1 = e_1 - e_0$, etc.,

$$\left.\begin{aligned} m_0 &= \phi'\epsilon_1\phi'\epsilon_2\phi'\epsilon_3 = (A\alpha_1 + F\alpha_2 + E\alpha_3)(F\alpha_1 + B\alpha_2 + D\alpha_3)(E\alpha_1 + D\alpha_2 + C\alpha_3) \\ &= \begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix} = \begin{vmatrix} \epsilon_1|\phi\epsilon_1 & \epsilon_2|\phi\epsilon_1 & \epsilon_3|\phi\epsilon_1 \\ \epsilon_1|\phi\epsilon_2 & \epsilon_2|\phi\epsilon_2 & \epsilon_3|\phi\epsilon_2 \\ \epsilon_1|\phi\epsilon_3 & \epsilon_2|\phi\epsilon_3 & \epsilon_3|\phi\epsilon_3 \end{vmatrix} = \epsilon_1\epsilon_2\epsilon_3|\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3 \\ &= (e_1 - e_0)(e_2 - e_0)(e_3 - e_0)|\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3 \\ &= 4\,|\bar{e} \cdot |\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3 = 4\,\bar{e}\phi\epsilon_1\phi\epsilon_2\phi\epsilon_3 \end{aligned}\right\}. \quad (492)$$

$$\left.\begin{aligned} m_1 &= \epsilon_1\phi'\epsilon_2\phi'\epsilon_3 + \epsilon_2\phi'\epsilon_3\phi'\epsilon_1 + \epsilon_3\phi'\epsilon_1\phi'\epsilon_2 \\ &= \begin{vmatrix} B & D \\ D & C \end{vmatrix} + \begin{vmatrix} C & E \\ E & A \end{vmatrix} + \begin{vmatrix} A & F \\ F & B \end{vmatrix} \\ &= \epsilon_2\epsilon_3|\phi\epsilon_2\phi\epsilon_3 + \epsilon_3\epsilon_1|\phi\epsilon_3\phi\epsilon_1 + \epsilon_1\epsilon_2|\phi\epsilon_1\phi\epsilon_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (493)$$

$$\left.\begin{aligned} m_2 &= \epsilon_1\epsilon_2\phi'\epsilon_3 + \epsilon_2\epsilon_3\phi'\epsilon_1 + \epsilon_3\epsilon_1\phi'\epsilon_2 = C + A + B \\ &= \epsilon_1|\phi\epsilon_1 + \epsilon_2|\phi\epsilon_2 + \epsilon_3|\phi\epsilon_3 \end{aligned}\right\} \quad . \quad . \quad . \quad (494)$$

Note that m_0 is the denominator of q_c in Art. 171, where we have already determined the meaning of $m_0 = 0$. By the aid of the values just found for the m's, k_0 of (488), and q_c of (474), together with the table in Art. 164, we construct the following table for the quadric in a point system.

NAME OF SURFACE.	q_c	k_0	m_0	m_1	m_2
Ellipsoid.	Finite.	−	+ −	+	+ −
Point (imaginary cone).	"	0	"	"	"
Imaginary ellipsoid.	"	+	"	"	"
Elliptic paraboloid.	∞	−	0	+	+ −
Elliptic cylinder.	%	0	0	+	+ −
Parabolic cylinder.	%	0	0	0	+
Hyperbolic cylinder.	%	0	0	−	+ −
Hyperbolic paraboloid.	∞	+	0	−	+ −
Hyperboloid of two sheets.	Finite.	−	+ - - - −	+ − − - - - + − −	− − + - - - + + −
Cone.	"	0	"	"	"
Hyperboloid of one sheet.	"	+	"	"	"

180. In (484) put $A_0, \cdots A_3$ for $n_1, \cdots n_4$ and $e_0, \cdots e_3$ for $p_1, \cdots p_4$, so that the equation becomes

$$\Sigma_0^3[A \cdot (p|e)^2] = 0, \quad \ldots \ldots \ldots \ldots \quad (495)$$

and (483) becomes $\Sigma[Ae \cdot p|e] = \phi p$. Then we have $\phi e_0 = A_0 e_0$, $\phi e_1 = A_1 e_1$, etc.; $\phi \epsilon_1 = \phi(e_1 - e_0) = A_1 e_1 - A_0 e_0$, etc. Hence we find

$$q_c = \frac{A_0^{-1}e_0 + A_1^{-1}e_1 + A_2^{-1}e_2 + A_3^{-1}e_3}{A_0^{-1} + A_1^{-1} + A_2^{-1} + A_3^{-1}},$$

$$k_0 = A_0A_1A_2A_3,$$

$$m_0 = A_1A_2A_3 + A_2A_3A_0 + A_3A_0A_1 + A_0A_1A_2,$$

$$m_1 = 2A_0(A_1 + A_2 + A_3) + A_2A_3 + A_3A_1 + A_1A_2,$$

$$m_2 = 3A_0 + A_1 + A_2 + A_3.$$

181. Consider next the equation

$$p|e_0(Ap|e_1 + Bp|e_2 + Cp|e_3) + p|e_1(C'p|e_2 + B'p|e_3) + A'p|e_2 \cdot p|e_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (496)$$

which represents a quadric surface passing through the four reference points, since it is satisfied when $p = e_0$, $p = e_1$, etc. It is the most general equation of the second degree in p representing a locus passing through these points; for it contains all the combinations of the quantities $p|e_0$, $p|e_1$, etc., taken two at a time, and there must be no term containing only *one* of these quantities, as $p|e_0$, because this term would not vanish when $p = e_0$.

Eq. (496) contains *five* arbitrary constants, and can therefore be made to fulfil five conditions, such as to pass through five given points; but the locus already passes through the four reference points; hence the general quadric can be subjected to nine conditions, and the general equation must contain nine arbitrary constants. We shall, in fact, obtain this general equation by adding together (495) and (496).

182. *Conditions in order that* (496) *shall represent a sphere.* If (496) represent a sphere, the sections of the surface by the reference planes must be circles. Take the section by the plane $p|e_0 = pe_1e_2e_3 = 0$, and the equation becomes

$$A'p|e_2 \cdot p|e_3 + B'p|e_3 \cdot p|e_1 + C'p|e_1 \cdot p|e_2 = 0.$$

Consider the plane space fixed by e_1, e_2, e_3; then, by (358), the condition that this equation shall represent a circle is

$\frac{A'}{a_{23}^2}=\frac{B'}{a_{31}^2}=\frac{C'}{a_{12}^2}$, in which $a_{23}=Te_2e_3$, $a_{31}=Te_3e_1$, $a_{12}=Te_1e_2$. Proceeding in the same way with the other reference planes, we find the required conditions to be

$$\frac{A}{a_{01}^2}=\frac{B}{a_{02}^2}=\frac{C}{a_{03}^2}=\frac{A'}{a_{23}^2}=\frac{B'}{a_{31}^2}=\frac{C'}{a_{12}^2}. \quad \cdots \cdots \quad (497)$$

183. EXERCISES. — (1) Show that for eq. (496) we have the following values, in which $2\mathfrak{A}=B+C-A'$, $2\mathfrak{B}=C+A-B'$, $2\mathfrak{C}=A+B-C'$, viz.:

$$k_0=\begin{vmatrix}0 & A & B & C\\ A & 0 & C' & B'\\ B & C' & 0 & A'\\ C & B' & A' & 0\end{vmatrix},\quad q_c=\frac{1}{m_0}\begin{vmatrix}e_0, & e_1, & e_2, & e_3\\ A, & -A, & C'-B, & B'-C\\ B, & C'-A, & -B, & A'-C\\ C, & B'-A, & A'-B, & -C\end{vmatrix},$$

$$m_0=-8\begin{vmatrix}A & \mathfrak{C} & \mathfrak{B}\\ \mathfrak{C} & B & \mathfrak{A}\\ \mathfrak{B} & \mathfrak{A} & C\end{vmatrix},\quad m_1=4\left\{\begin{vmatrix}A & \mathfrak{C}\\ \mathfrak{C} & B\end{vmatrix}+\begin{vmatrix}B & \mathfrak{A}\\ \mathfrak{A} & C\end{vmatrix}+\begin{vmatrix}C & \mathfrak{B}\\ \mathfrak{B} & A\end{vmatrix}\right\},$$

$$m_2=-2(A+B+C).$$

(2) Show that for the equation formed by adding (495) and (496), we have

$$k_0=\begin{vmatrix}A_0 & A & B & C\\ A & A_1 & C' & B'\\ B & C' & A_2 & A'\\ C & B' & A' & A_3\end{vmatrix},\quad q_c=\begin{vmatrix}e_0, & e_1, & e_2, & e_3\\ A-A_0, & A_1-A, & C'-B, & B'-C\\ B-A_0, & C'-A, & A_2-B, & A'-C\\ C-A_0, & B'-A, & A'-B, & A_3-C\end{vmatrix},$$

$$m_0=\begin{vmatrix}\mathfrak{A}' & \mathfrak{C}'' & \mathfrak{B}''\\ \mathfrak{C}'' & \mathfrak{B}' & \mathfrak{A}''\\ \mathfrak{B}'' & \mathfrak{A}'' & \mathfrak{C}'\end{vmatrix},\quad m_1=\begin{vmatrix}\mathfrak{A}' & \mathfrak{C}''\\ \mathfrak{C}'' & \mathfrak{B}'\end{vmatrix}+\begin{vmatrix}\mathfrak{B}' & \mathfrak{A}''\\ \mathfrak{A}'' & \mathfrak{C}'\end{vmatrix}+\begin{vmatrix}\mathfrak{C}' & \mathfrak{B}''\\ \mathfrak{B}'' & \mathfrak{A}'\end{vmatrix},$$

$$m_2=\mathfrak{A}'+\mathfrak{B}'+\mathfrak{C}',$$

in which

$$\mathfrak{A}'=A_0+A_1-2A,\quad \mathfrak{B}'=A_0+A_2-2B,$$

$$\mathfrak{C}'=A_0+A_3-2C,\quad \mathfrak{A}''=A_0+A'-B-C,$$

$$\mathfrak{B}''=A_0+B'-C-A,\quad \mathfrak{C}''=A_0+C'-A-B.$$

(3) Show that (496), with the conditions (497), represents a sphere, by transforming the equation to a vector system. See Art. 75.

(4) Show that the most general equation of the second degree in p, in solid space, may be written

$$p|e_0 \cdot p|e_0' + p|e_1 \cdot p|e_1' + p|e_2 \cdot p|e_2' + p|e_3 \cdot p|e_3' = 0; \quad . \quad (498)$$

write the corresponding self-conjugate form of ϕ, and determine $\phi e_0, \cdots \phi e_3$.

(5) Show that planes, tangent to (496) at the reference points, cut the opposite reference planes in four right lines, which are generators of the same system of a skew quadric.

(6) Show that the four lines of Ex. (5) are *coplanar* if

$$AA' = BB' = CC'.$$

(7) Prove the proposition — of which Ex. (5) is a particular case — that the corresponding faces of any tetraedron and its polar tetraedron with reference to any quadric, cut each other in four right lines which are generators of the same system of a skew quadric.

(8) Show that the conditions given in Ex. (6) make eq. (496) represent a convex surface.

(9) Find the condition that the plane $p|e = C$ shall be tangent to the quadric $p|\phi p = 0$, determining thus the equation of the tangent plane independent of the point of contact. By using the result, find the tangent plane conjugate in direction to ϵ.

Ans. $4\,C\bar{e}|\phi^{-1}\bar{e} = \bar{e}|\phi^{-1}e \pm \sqrt{-e\bar{e}|\phi^{-1}e\phi^{-1}\bar{e}}$, $4p|\phi\epsilon = \pm\sqrt{\dfrac{-\epsilon|\phi\epsilon}{\bar{e}|\phi^{-1}\bar{e}}}$.

(10) If a plane be tangent to any quadric $p|\phi_2 p = 0$, show that the locus of its pole, with reference to any other quadric $p|\phi_1 p = 0$, is $q|\phi_2^{-1}\phi_1^2 q = 0$.

(11) Find the nature of the surface in each of the following cases, the position of the center, and the asymptotic cone, if real:

(*a*) In (496) let $A = B = C = A' = B' = C' = 1$. Ellipsoid, $q_c = \bar{e}$.

(*b*) In (496) let $C' = 0$. Skew or developable surface.

(*c*) In (496) let $B' = C' = 0$. Skew surface.

(*d*) In (496) let $A = B = C = 0$. Cone, vertex at e_0.

(12) In cases (*b*), (*c*), (*d*) of the last exercise, show that one, two, and three edges respectively of the reference tetraedron lie on the surface.

(13) Show how the cone of Ex. (11), case (*d*), is related to the surface

$$A_0(p|e_0)^2 + A'p|e_2 \cdot p|e_3 + B'p|e_3 \cdot p|e_1 + C'p|e_1 \cdot p|e_2 = 0.$$

(14) Discuss, as in Ex. (11), the following cases:

(*a*) In (496) let $A = B = C = 1$, $A' = B' = C' = 2$. Ellipsoid.

(*b*) In (496) let $A = B = C = A' = B' = 1$, $C' = 3$. Elliptic paraboloid.

(*c*) In (496) let $A = -B = C = -A' = B' = -C' = 1$. Two-sheeted hyperboloid.

(*d*) In (496) let $A = B = C = A' = B' = 1$, $C' = 0$. Elliptic cylinder.

(*e*) In (496) let $A = -B = C = -A' = B'$, $C' = 0$. Cone, vertex at $\frac{1}{2}(e_2 + e_3)$.

(*f*) In (495) let $A_0 = A_1 = A_2 = -A_3 = 1$. Two-sheeted hyperboloid.

(*g*) In (495) let $A_0 = A_1 = -A_2 = -A_3$. Hyperbolic paraboloid.

(*h*) In (495) let $A_0 = 2$, $A_1 = A_2 = 4$, $A_3 = -1$. Elliptic paraboloid.

(i) In (498) let $e_0' = \frac{1}{3}(e_1 + e_2 + e_3)$, $e_1' = \frac{1}{3}(e_2 + e_3 + e_0)$, $e_2' = \frac{1}{3}(e_3 + e_0 + e_1)$, $e_3' = \frac{1}{3}(e_0 + e_1 + e_2)$.

(j) In (498) let $e_0' = e_1 + e_2 + e_3 - 2e_0$, $e_1' = e_2 + e_3 + e_0 - 2e_1$, $e_2' = e_3 + e_0 + e_1 - 2e_2$, $e_3' = e_0 + e_1 + e_2 - 2e_3$.

(k) In (498) let $e_0' = e_1' = \bar{e}$, $e_2' = e_2$, $e_3' = e_3$.

(l) In (498) let $e_0' = e_0$, $e_1' = \frac{1}{2}(e_0 + e_1)$, $e_1' = \frac{1}{2}(e_0 + e_2)$, $e_3' = \frac{1}{2}(e_0 + e_3)$.

(15) Discuss the equation $pL_1L_2L_3p = 0$, and show that it is the locus of a right line moving on three rectilinear directrices.

(16) A tetraedron has one variable vertex p, and three fixed vertices e_1, e_2, e_3; the variable faces pe_1e_2, etc., pass through the points in which L_3, L_1, L_2 pierce a variable plane which always passes through e_0: show that the equation of the locus is

$$(L_1 \cdot pe_2e_3)(L_2 \cdot pe_3e_1)(L_3 \cdot pe_1e_2)e_0 = 0.$$

(17) Three planes pass respectively through the lines L_1, L_2, L_3, and two of the common lines of these planes always cut L_1' and L_2' respectively; the locus of the third common line is a quadric whose equation is $pL_1L_1'L_3L_2'L_2p = 0$.

(18) There are given four lines L_1, $e_1\epsilon_1$, L_2, $e_2\epsilon_2$; two planes pass respectively through L_1 and L_2, cutting $e_1\epsilon_1$ and $e_2\epsilon_2$ in two points which move along these lines at rates bearing a constant ratio to each other; show that the locus of the common line of these planes is a quadric passing through L_1 and L_2, whose equation is $l \cdot p\epsilon_1L_1 \cdot pe_2L_2 = m \cdot p\epsilon_2L_2 \cdot pe_1L_1$. Consider the cases when l and m are of like, and unlike, signs, and when $L_1L_2 = 0$.

(19) Show that the two surfaces $p|\phi p = 0$ and $p|\phi p = C$ are similar, concentric, and similarly placed.

(20) Discuss the following equations:

$$(pe_0)^2 = c^2, \quad (pe_0)^2 \pm (pe_1)^2 = c^2,$$
$$(pe_0)^2 + (pe_1)^2 \pm (pe_2)^2 = c^2,$$

$$(pe_0)^2 + (pe_1)^2 \pm (pe_2)^2 \pm (pe_3)^2 = c^2,$$
$$p^2 = c^2, \quad (pe_0e_1)^2 = c^2,$$
$$(pe_0e_1)^2 \pm (pe_2e_3)^2 = c^2,$$
$$(pe_0e_1)^2 \pm (pe_0e_2)^2 \pm (pe_0e_3)^2 = c^2.$$

(21) Find the equation of the surface reciprocal to $p|\phi p = 0$; *i.e.* the locus of $q = \phi p$.

184. *Inversion of* ϕ. If we have given $\phi p = q$, whence we have $p = \phi^{-1}q$, we must be able to invert ϕ in order to find p as an explicit function of q. Taking, for generality, ϕ as not self-conjugate, let ϕ_c be the conjugate function, so that

$$q|\phi p = p|\phi_c q.$$

Let e, e', e'', e''' be any four points; then we have

$$e|\phi p = p|\phi_c e = e|q, \qquad e'|\phi p = p|\phi_c e' = e'|q,$$
$$e''|\phi p = p|\phi_c e'' = e''|q, \qquad e'''|\phi p = p|\phi_c e''' = e'''|q.$$

Now substitute in (201) $\phi_c e$ for p_0, $\phi_c e'$ for p_1, etc.

$$\therefore\ p = \phi^{-1}q = \frac{-|\phi_c e'\phi_c e''\phi_c e''' \cdot q|e + |\phi_c e''\phi_c e'''\phi_c e \cdot q|e' - \text{etc.}}{\phi_c e\phi_c e'\phi_c e''\phi_c e'''}. \quad (499)$$

If we put $q = |e'e''e'''$, which we can do whatever point q may be, because the e's are any points whatever, and hence three of them may always be so taken that the complement of their product shall be q, then (499) reduces to

$$k_0\phi^{-1}|e'e''e''' = |\phi_c e'\phi_c e''\phi_c e''', \quad . \quad . \quad . \quad . \quad . \quad (500)$$

k_0 having the value given in (481) except that ϕ_c must be put for ϕ.

Now substitute $\phi + n$ for ϕ, where n is any scalar.

$$\therefore\ k_0'(\phi + n)^{-1}q = k_0'(\phi + n)^{-1}|(e'e''e''')$$
$$= |(\phi_c + n)e'(\phi_c + n)e''(\phi_c + n)e'''.$$

On expanding the last member, the first term is $k_0\phi^{-1}q$, the last is n^3q, and, as the first member is a function of q, it appears that the coefficients of n and n^2 should also be functions of q. Call them χq and ψq respectively; then, expanding k_0', the equation becomes

$$(n^4+k_3n^3+k_2n^2+k_1n+k_0)(\phi+n)^{-1}q = k_0\phi^{-1}q+n\chi q+n^2\psi q+n^3q.$$

Operate on this equation by $\phi + n$.

$$\therefore\ (n^4 + k_3n^3 + k_2n^2 + k_1n + k_0)q$$
$$= k_0q+k_0n\phi^{-1}q+n\phi\chi q+n^2\chi q+n^2\phi\psi q+n^3\psi q+n^3\phi q+n^4q.$$

This equation must hold for all values of n; hence the coefficients of the same powers of n on each side of the sign of equality must be equal.

$$\therefore\ k_3 = \psi + \phi,\ \ k_2 = \chi + \phi\psi,\ \ k_1 = k_0\phi^{-1} + \phi\chi;$$
$$\therefore\ \psi = k_3 - \phi,\ \text{and}\ \chi = k_2 - k_3\phi + \phi^2.$$

$$\left.\begin{array}{ll}\text{Also,} & k_0\phi^{-1}q = k_1q - k_2\phi q + k_3\phi^2q - \phi^3q, \\ \text{or} & (\phi^4 - k_3\phi^3 + k_2\phi^2 - k_1\phi + k_0)q = 0.\end{array}\right\} \quad . \quad . \quad . \quad (501)$$

Substituting the values of ψ and χ in the value of $(\phi + n)^{-1}$, we obtain also

$$(\phi+n)^{-1}q = \frac{k_0\phi^{-1}q+n(k_2-k_3\phi+\phi^2)q+n^2(k_3-\phi)q+n^3q}{n^4 + k_3n^3 + k_2n^2 + k_1n + k_0}. \quad (502)$$

185. *Equation of the anti-polar plane.* This is found precisely as in Art. 121, and is of the same form, viz.:

$$p|\phi(2q_c - e) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (503)$$

e being the point of which this equation represents the anti-polar plane.

186. *Reciprocating ellipsoid.* To obtain the equation of this surface we proceed in the same manner as in Art. 122 for the reciprocating ellipse.

Assume the equation

$$p|\phi p = p|e_0 \cdot p|(e_1 + e_2 + e_3) + p|e_1 \cdot p|(e_2 + e_3) + p|e_2 \cdot p|e_3$$
$$+ m(4p|\bar{e})^2 = 0,$$

and determine m, so that, with reference to this surface, each reference point shall be the anti-pole of the opposite reference plane. The equation gives

$$\phi p = (8m\bar{e} + e_1 + e_2 + e_3) \cdot p|e_0 + (8m\bar{e} + e_2 + e_3 + e_0) \cdot p|e_1 + (8m\bar{e} + e_3 + e_0 + e_1) \cdot p|e_2 + (8m\bar{e} + e_0 + e_1 + e_2) \cdot p|e_3.$$

The center of the surface is at $\bar{e}$, so that (503) becomes $p|\phi(2\bar{e} - e) = 0$, or, putting e_0 for e, and its value for $\bar{e}$, $p|\phi(e_1 + e_2 + e_3 - e_0) = 0$. We must then have $|e_0$ coincident with $|\phi(e_1 + e_2 + e_3 - e_0)$, whence $e_0\phi(e_1 + e_2 + e_3 - e_0) = 0$. Now

$$\begin{aligned} e_0\phi(e_1 + e_2 + e_3 - e_0) &= e_0[8m\bar{e} + e_2 + e_3 + e_0 + \text{etc.}] \\ &= e_0[16m\bar{e} + e_1 + e_2 + e_3] \\ &= (4m + 1)e_0(e_1 + e_2 + e_3). \end{aligned}$$

This expression becomes zero if $m = -\frac{1}{4}$, which value, on substitution, gives for the required equation

$$\left. \begin{aligned} p|e_0 \cdot p|(e_1 + e_2 + e_3) + p|e_1 \cdot p|(e_2 + e_3) \\ + p|e_2 \cdot p|e_3 - 4(p|\bar{e})^2 = 0 \end{aligned} \right\} . \quad . \qquad (504)$$

Also, ϕp may be written

$$\left. \begin{aligned} \phi p = (e_1 + e_2 + e_3 - e_0) \cdot p|e_0 \\ + (e_2 + e_3 + e_0 - e_1) \cdot p|e_1 + \text{etc.} \end{aligned} \right\} . \qquad (505)$$

187. We will now show that the complement of any point is its anti-polar plane with reference to the ellipsoid of (504.) If this is true, we must have

$$|p \cdot |\phi(2\bar{e} - p) = 0 \text{ or } p\phi(2\bar{e} - p) = 0.$$

Let $p = \Sigma_0^3 ne$, with the condition $\Sigma_0^3 n = 1$; then

$$\begin{aligned} p\phi(2\bar{e} - p) &= \Sigma ne \cdot \phi[(\tfrac{1}{2} - n_0)e_0 + (\tfrac{1}{2} - n_1)e_1 + \text{etc.}] \\ &= \Sigma ne \cdot [(\tfrac{1}{2} - n_0)(e_1 + e_2 + e_3 - e_0) + \text{etc.}] \\ &= \Sigma ne \cdot [(1 - n_1 - n_2 - n_3 + n_0)e_0 + \text{etc.}] \\ &= 2p\Sigma ne = 2pp = 0. \quad \text{Q.E.D.} \end{aligned}$$

188. *Scalar plane equations.* In eq. (496) put P for $|p$, and we obtain the complementary or anti-polar reciprocal equation

$$\left.\begin{array}{r} Pe_0 \cdot P(Ae_1 + Be_2 + Ce_3) + Pe_1 \cdot P(C'e_2 + B'e_3) \\ + A'Pe_2 \cdot Pe_3 = 0 \end{array}\right\}, \quad (506)$$

which is that of a surface touching the four reference planes, since it is satisfied when $P = |e_0$, $P = |e_1$, etc. If we write

$$\left.\begin{array}{r} |\psi P = (Ae_1 + Be_2 + Ce_3) \cdot Pe_0 + (C'e_2 + B'e_3 + Ae_0) \cdot Pe_1 \\ + (A'e_3 + Be_0 + C'e_1) \cdot Pe_2 + (Ce_0 + B'e_1 + A'e_2) \cdot Pe_3 \end{array}\right\}, \quad (507)$$

eq. (506) becomes

$$P|\psi P = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (508)$$

and ψ is a linear, self-conjugate function of P.

In the same way *any* scalar point equation $p|\phi p = 0$ may be changed into its complementary plane equation $P|\psi P = 0$. Bearing in mind that, if $P = |p$, then $|P = |(|p) = -p$, we have the following relations between ϕ and ψ, viz.:

$$\psi P = \psi|p = |\phi p = -|\phi|P. \quad . \quad . \quad . \quad (509)$$

Suppose (508) to represent *any* homogeneous, plane equation of the second degree. The equation shows at once that the point $|\psi P$ lies on the plane P. Differentiating, we have

$$dP|\psi P + P|\psi dP = 2\,dP|\psi P = 0;$$

but, by Art. 166, dP is a plane through the point of contact of P with the surface. Hence $|\psi P$ is on the line PdP. Now dP is a *variable* plane, subject only to the conditions of always passing through $\bar{e}$ and the point of contact of P; hence $|\psi P$ must *coincide* with the point of contact of P. If Q be any plane not tangent to (508), then $|\psi Q$ is its *pole* with reference to that surface.

189. *Center of surface $P|\psi P = 0$.* The center is the pole of the plane at ∞, *i.e.* of $|\bar{e}$; hence, by (509),

$$q_c = m|\psi|\bar{e} = m|(|\phi\bar{e}) = -m\phi\bar{e}.$$

To evaluate m, multiply into $4|\bar{e}$.

$$\therefore\ 4q_c|\bar{e} = 1 = -m\phi\bar{e}\cdot 4|\bar{e} = -4m\bar{e}|\phi\bar{e};$$

whence
$$q_c = \frac{\phi\bar{e}}{4\,\bar{e}|\phi\bar{e}}. \qquad (510)$$

If we have $\bar{e}|\phi\bar{e} = 0$, ... (511)

the center is at ∞, and the surface is a paraboloid. This is also the condition that the reciprocal surface $p|\phi p = 0$ shall pass through the mean point of the reference tetraedron.

190. *Reciprocal surfaces. A skew surface is reciprocal to a skew surface.* For such a surface is generated by a right line whose consecutive positions are *not coplanar;* hence the reciprocal surface is generated by a right line whose consecutive positions *have no common point.*

A developable surface is reciprocal to a curve, and vice versâ. For such a surface may be regarded as the envelope of a plane rolling on some two given surfaces S_1 and S_2; hence the point reciprocal to this plane lies simultaneously on the two surfaces reciprocal to S_1 and S_2; *i.e.* it generates their common line. When the developable surface is a *cone,* the reciprocal curve is *plane;* for, since all tangent planes to the cone have a common point, their reciprocal points are coplanar.

A convex surface is reciprocal to a convex surface. This follows from the preceding, since *all* surfaces are included under these three heads.

191. *The discriminant k_0 as a criterion.* Since, by (509), $\psi|p = |\phi p$, we have, by the last article, and by (488), that when

$$k_0(=\psi|e_0\psi|e_1\psi|e_2\psi|e_3) \text{ is } \begin{Bmatrix} + \\ 0 \\ - \end{Bmatrix}, P|\psi P = 0 \text{ is a } \begin{Bmatrix} \text{skew surface} \\ \text{plane curve} \\ \text{convex surface} \end{Bmatrix}.$$

192. To determine still farther the surface represented by (508) we will make use of the cone circumscribed about the surface $p|\phi p = 0$ and having its vertex at $\bar{e}$, the center of reciprocation. If this cone is *real,* the surface cuts the plane at

∞ in a real curve, the reciprocal of this cone. If the cone is imaginary, the surface has no real points at ∞.

Substituting $\bar{e}$ for q_1 in (469), we have the equation of the above-mentioned cone, viz.:

$$p\bar{e}|\phi p\phi\bar{e} = 0. \qquad (512)$$

Write
$$\phi' p = \phi p \cdot \bar{e}|\phi\bar{e} - \phi\bar{e} \cdot p|\phi\bar{e}, \qquad (513)$$

and with this value of ϕ' compute m_0, m_1, m_2 according to Art. 179, and determine by the table in the same article whether (512) is real or imaginary. We obtain thus the following scheme for the determination of the equation $P|\psi P = 0$.

SURFACE.	k_0	CONE OF EQ. (512).
Ellipsoid.	$-$	Imaginary.
Ellipse.	0	"
Imaginary ellipsoid.	$+$	"
Elliptic paraboloid.	$-$	Two coincident
Parabola.	0	planes tangent at
Hyperbolic paraboloid.	$+$	$\bar{e}$. $(\bar{e}\|\phi\bar{e} = 0)$.
Hyperboloid of two sheets.	$-$	Real.
Hyperbola.	0	"
Hyperboloid of one sheet.	$+$	"

193. EXERCISE. — If ϕ and ψ, p and P, are related as in Art. 188, show that $p|\phi^{-1}p = 0$ and $P|\psi P = 0$ are respectively the point and plane equations of the same surface, reciprocal to that represented by $p|\phi p = 0$. Also that $P|\psi^{-1}P = 0$ and $p|\phi p = 0$ are respectively the plane and point equations of a surface reciprocal to that represented by $P|\psi P = 0$.

CHAPTER VII.

APPLICATIONS TO STATICS.

194. It is proposed, in this concluding chapter, to give a few applications of our calculus to mechanics, merely to serve as an introduction to this field, and to indicate how perfectly the methods that have been developed adapt themselves to mechanical conceptions and processes.

195. A *force* is that which is postulated as the cause of any change, or tendency to change, in the rate of motion of some particle, or rigid body, on which it acts.

In *statics* we consider *systems of forces* aside from any question of rest or motion of the bodies on which they act; and, especially, cases in which the total resultant effect of all the forces applied to a body is null.

196. The *space* qualities of a force are *magnitude, direction* and *position,* and these are the only qualities with which we have to do mathematically. The intrinsic character of a force, such as that of gravity or magnetism, we know little or nothing about; but our knowledge is complete for its mathematical treatment, when we know its magnitude, direction, and line of action, or position. Now these *space* qualities are identical with those of a *point-vector;* hence, for the purposes of mathematical discussion, a point-vector completely represents a force, and therefore all that has been demonstrated in Chapter II. regarding the former can be applied immediately to the latter.

197. *Notation.* The notation for points and vectors, in general, will be as in previous chapters. The vector representing the magnitude and direction of a *force* will be denoted by a

German letter as $\mathfrak{F}$, while the *magnitude* of the same force will be denoted by F, the corresponding English letter. Thus F will be the tensor of $\mathfrak{F}$, or $F = T\mathfrak{F}$. If e be a point on the line of action of the force, it will then be completely denoted by $e\mathfrak{F}$, a notation which is practically more convenient than the use of a single letter to represent the point-vector or force. The *complement* will be used in this chapter only with reference to a *vector system in solid space.*

198. *Forces acting on a particle.* The parallelogram and polygon of forces follow at once from the nature and properties of vectors and point-vectors, as shown in Chapters I. and II. Let a system of forces acting on the point e be denoted by $e\mathfrak{F}_1, e\mathfrak{F}_2, \cdots e\mathfrak{F}_n$; then the resultant effect of the system will be found in this, as in all cases, by *simply adding the forces.* Thus

$$\text{Resultant} = \Sigma e\mathfrak{F} = e\Sigma\mathfrak{F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (514)$$

For equilibrium we must have $e\Sigma\mathfrak{F} = 0$, or

$$\Sigma\mathfrak{F} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (515)$$

an equivalent equation.

199. *Equilibrium of a particle contained to remain on a smooth curve.* In this case the resultant force must have no component along the tangent to the curve, at the point where the particle is; hence the resultant must be $\perp$ to this tangent.

Let the equation of the curve be

$$p - e_0 = \rho = \phi t,$$

ϕ being a vector function of the scalar variable t; then

$$\frac{dp}{dt} = \frac{d\rho}{dt} = \phi' t$$

is a vector along the tangent. Hence, if p be the position of the particle on the curve, the condition for equilibrium is

$$\frac{dp}{dt}|\Sigma\mathfrak{F} = \frac{d\rho}{dt}|\Sigma\mathfrak{F} = \phi' t|\Sigma\mathfrak{F} = 0. \quad . \quad . \quad . \quad . \quad . \quad (516)$$

For example, if the curve become the right line whose equation is $\rho = \epsilon + \epsilon' t$, then $d\rho = \epsilon' dt$, and the condition becomes $\epsilon' | \Sigma \mathfrak{F} = 0$.

Again consider the case of a particle resting on a diagonal of a parallelopiped, and acted on by three forces represented by the edges of the parallelopiped which meet at a corner not on the diagonal. Let the three edges be ϵ_1, ϵ_2, ϵ_3, and the equation of the diagonal

$$\rho = \epsilon_1 + t(\epsilon_2 + \epsilon_3 - \epsilon_1);$$

thus $$\Sigma \mathfrak{F} = \epsilon_1 + \epsilon_2 + \epsilon_3, \text{ and } \frac{d\rho}{dt} = \epsilon_2 + \epsilon_3 - \epsilon_1,$$

so that we have

$$(\epsilon_2 + \epsilon_3 - \epsilon_1) | (\epsilon_1 + \epsilon_2 + \epsilon_3) = 0,$$

or $$(\epsilon_2 + \epsilon_3)^2 - \epsilon_1^2 = 0; \text{ i.e. } T\epsilon_1 = T(\epsilon_2 + \epsilon_3).$$

200. *Equilibrium of particle constrained to remain on a smooth surface.* If ν be a vector ∥ to the normal at p, then for equilibrium $\Sigma \mathfrak{F}$ must be ∥ to this vector.

$$\therefore \nu \Sigma \mathfrak{F} = 0 \quad \ldots \quad (517)$$

is the required condition. If the equation of the surface is a scalar one in terms of vectors, then it will be linear in $d\rho$ after differentiation and will have the form $\nu | d\rho = 0$, ν being some function of ρ. To illustrate, let the equation of a surface be

$$(\rho|\epsilon_1)^3 + (\rho|\epsilon_2)^2 - \rho|\epsilon_3 = 0;$$

then $$3(\rho|\epsilon_1)^2 \cdot d\rho|\epsilon_1 + 2\rho|\epsilon_2 \cdot d\rho|\epsilon_2 - d\rho|\epsilon_3 = 0,$$

or $$d\rho|(3\epsilon_1 \cdot (\rho|\epsilon_1)^2 + 2\epsilon_2 \cdot \rho|\epsilon_2 - \epsilon_3) = d\rho|\nu = 0.$$

If we have a vector equation, it will be in the form

$$\rho = \phi(x, y).$$

Then $\frac{d\rho}{dx}$ and $\frac{d\rho}{dy}$ will be vectors ∥ to tangents to the surface at the end of ρ, and hence we may write

$$\nu = \left| \frac{d\rho}{dx} \frac{d\rho}{dy} \right. . \quad \ldots \quad (518)$$

201. EXAMPLES. — (1) If e_1, e_2, e_3 are the vertices of a triangle, and p_1, p_2, p_3 the middle points of its sides, p_1 opposite e_1, etc.; then forces represented by e_1p_1, e_2p_2, e_3p_3 are in equilibrium.

We have

$$\Sigma e\mathfrak{F} = e_1p_1 + e_2p_2 + e_3p_3 = e_1\left(\frac{e_2+e_3}{2}\right) + e_2\left(\frac{e_3+e_1}{2}\right) + e_3\left(\frac{e_1+e_2}{2}\right)$$
$$= \tfrac{1}{2}(e_1e_2 + e_1e_3 + e_2e_3 + e_2e_1 + e_3e_1 + e_3e_2) = 0.$$

(2) Forces are represented by perpendiculars drawn from the vertices of a triangle on the opposite sides; to show that they cannot be in equilibrium unless the triangle is equilateral.

Let the triangle be $e_1e_2e_3$; $a = Te_2e_3$, $b = Te_3e_1$, $c = Te_1e_2$; p_1 foot of $\perp$ on e_2e_3, etc.; l, m, n cosines of angles at e_1, e_2, e_3. Then,

$$p_1 = \frac{1}{a}(bne_2 + cme_3), \quad p_2 = \frac{1}{b}(cle_3 + ane_1), \quad p_3 = \frac{1}{c}(ame_1 + ble_2),$$

and

$$\Sigma e\mathfrak{F} = e_1p_1 + e_2p_2 + e_3p_3 = n\left(\frac{b}{a} - \frac{a}{b}\right)e_1e_2 + l\left(\frac{c}{b} - \frac{b}{c}\right)e_2e_3 + m\left(\frac{a}{c} - \frac{c}{a}\right)e_3e_1,$$

which cannot be zero unless $a = b = c$.

(3) Let ϵ_1, ϵ_2, ϵ_3 be any three *unit* vectors, and $e\epsilon_1F_1$, $e\epsilon_2F_2$, $e\epsilon_3F_3$ three forces acting at e. Then for equilibrium

$$\Sigma\mathfrak{F} = \Sigma\epsilon F = \epsilon_1F_1 + \epsilon_2F_2 + \epsilon_3F_3 = 0.$$

This gives $\epsilon_1\epsilon_2\epsilon_3 = 0$, so that the forces must be coplanar. Also, $\epsilon_1\epsilon_2F_2 = \epsilon_3\epsilon_1F_3$, and $\epsilon_1\epsilon_2F_1 = \epsilon_2\epsilon_3F_3$; *i.e.* $\epsilon_1\epsilon_2F_3^{-1} = \epsilon_2\epsilon_3F_1^{-1} = \epsilon_3\epsilon_1F_2^{-1}$. But the three vectors, being coplanar, may be taken as in plane space, and, therefore, $\epsilon_1\epsilon_2$, etc., scalar; so that $\epsilon_1\epsilon_2 = \sin\angle{}^{\epsilon_2}_{\epsilon_1}$, etc., and we have

$$F_1 : \sin\angle{}^{\epsilon_3}_{\epsilon_2} = F_2 : \sin\angle{}^{\epsilon_1}_{\epsilon_3} = F_3 : \sin\angle{}^{\epsilon_2}_{\epsilon_1}.$$

(4) If ϵ_1, ϵ_2, ϵ_3, ϵ_4 be any four unit vectors, and $e\epsilon_1F_1$, $e\epsilon_2F_2$, etc., be four forces; show that for equilibrium we have

$$\epsilon_2\epsilon_3\epsilon_4F_1^{-1} = -\epsilon_3\epsilon_4\epsilon_1F_2^{-1} = \epsilon_4\epsilon_1\epsilon_2F_3^{-1} = -\epsilon_1\epsilon_2\epsilon_3F_4^{-1}.$$

(5) Show that the point on the smooth surface

$$\left(\frac{\rho|\iota_1}{a}\right)^3+\left(\frac{\rho|\iota_2}{b}\right)^3+\left(\frac{\rho|\iota_3}{c}\right)^3=1,$$

where a particle attracted toward the origin will remain at rest, is given by

$$\frac{\rho|\iota_1}{a^3}=\frac{\rho|\iota_2}{b^3}=\frac{\rho|\iota_3}{c^3}=\frac{1}{\sqrt[3]{a^6+b^6+c^6}}.$$

(6) Through a point at the end of ϵ three chords are drawn, parallel to a set of conjugate diameters of a central quadric; forces act on this point, represented by the portions of the chords intercepted between the point and the surface, and towards the surface: show that $\Sigma\mathfrak{F}=-2\epsilon$.

Let α, β, γ be conjugate semi-diameters, and equations of chords $\rho=\epsilon+x\alpha$, $\rho=\epsilon+y\beta$, $\rho=\epsilon+z\gamma$. Substitute the first value of ρ in the equation of the surface;

$$\therefore\ (\epsilon+x\alpha)|\phi(\epsilon+x\alpha)=1=\epsilon|\phi\epsilon+2x\alpha|\phi\epsilon+x^2,$$

because $\alpha|\phi\alpha=1$. Now the sum of two of the forces will be $\mathfrak{F}_1+\mathfrak{F}_2=(x_1+x_2)\alpha$, in which x_1 and x_2 are the roots of the preceding equation.

$$\therefore\ \mathfrak{F}_1+\mathfrak{F}_2=-2\alpha\cdot\alpha|\phi\epsilon.$$

Hence we have

$$\Sigma\mathfrak{F}=-2(\alpha\cdot\epsilon|\phi\alpha+\beta\cdot\epsilon|\phi\beta+\gamma\cdot\epsilon|\phi\gamma)$$
$$=-2\left(\frac{\alpha\cdot\epsilon\beta\gamma+\beta\cdot\epsilon\gamma\alpha+\gamma\cdot\epsilon\alpha\beta}{\alpha\beta\gamma}\right)=-2\epsilon.$$

The last results are by eqs. (423) and (177).

(7) Show that, if a system of forces acting on a point are represented by vectors drawn outward from the point, and are in equilibrium; then this point is the *mean* of the extremities of the vectors.

202. *Forces acting on a rigid body.* Let the forces be $e_1\mathfrak{F}_1$, $e_2\mathfrak{F}_2$, etc.; then we have for the total resultant effect

$$\mathfrak{W} = \Sigma e\mathfrak{F} = e_0\Sigma\mathfrak{F} + \Sigma(e - e_0)\mathfrak{F} = e_0\Sigma\mathfrak{F} + \Sigma\epsilon\mathfrak{F}. \qquad (519)$$

In the last member we have written ϵ_1 for $e_1 - e_0$, etc. Comparing with Art. 61, we see that the quantity $\mathfrak{W}$ has the same *space* qualities as the quantity there denoted by S, and called a *screw.* Following again the nomenclature of R. S. Ball, Astronomer Royal of Ireland, we shall call the quantity $\mathfrak{W}$ *a wrench,* and it is evident that it corresponds to a screw, just as a force does to a point-vector.

Before proceeding to a general discussion of eq. (519), we will consider some special cases.

203. *Parallel forces.* Let $\mathfrak{F}_1 = F_1\epsilon$, $\mathfrak{F}_2 = F_2\epsilon$, etc., and $T\epsilon = 1$; then

$$\mathfrak{W} = \Sigma e\mathfrak{F} = \Sigma eF\epsilon = -\epsilon\Sigma eF = -\epsilon\bar{e}\Sigma F, \quad . \quad . \quad (520)$$

in which $\bar{e}$ is the mean of all the points e_1, e_2, etc.

Now $\bar{e}$ depends *only* on e_1, e_2, etc., and will be the same whatever the direction of ϵ may be; hence it is a unique point, with reference to the system of $\parallel$ forces, possessing the property that through it the resultant always passes, no matter what may be the direction of the forces. It is called the *center* of $\parallel$ forces, and is determined by the equation

$$\bar{e} = \Sigma Fe \div \Sigma F. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (521)$$

In this case the wrench $\mathfrak{W}$ reduces to a *force* acting at $\bar{e}$.

204. *Couples.* Suppose that we have

$$\Sigma\mathfrak{F} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (522)$$

then (519) becomes

$$\mathfrak{W} = \Sigma\epsilon\mathfrak{F}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (523)$$

so that $\mathfrak{W}$ reduces to a *plane*-vector. Consider, first, one plane-vector only, of those that make up $\mathfrak{W}$, for instance

$$\epsilon_1\mathfrak{F}_1 = (e_1 - e_0)\mathfrak{F}_1 = e_1\mathfrak{F}_1 - e_0\mathfrak{F}_1;$$

it appears as the sum of two unlike parallel forces, of equal magnitude. This is denominated a *couple*, and its only effect on the body is to produce, or tend to produce, rotation. $\epsilon_1\mathfrak{F}_1 = T\epsilon_1\mathfrak{F}_1 \cdot U\epsilon_1\mathfrak{F}_1$ and the first factor evidently measures the magnitude of the rotational effect; *i.e.* the product of F_1 into the $\perp$ distance between $e_1\mathfrak{F}_1$ and $e_0\mathfrak{F}_1$, which is called the *moment* of the couple. All the properties of couples follow at once from those of plane-vectors, as demonstrated in Chapter II. Eq. (523) gives the resultant of all the couples acting on the body, and therefore the total tendency to produce rotation.

205. *Condition for a single resultant force.* By eq. (218) the condition that $\mathfrak{W}$ shall reduce to a single force is

$$\mathfrak{W}^2 = 0 = (e_0\Sigma\mathfrak{F} + \Sigma\epsilon\mathfrak{F})^2 = 2\, e_0\Sigma\mathfrak{F}\Sigma\epsilon\mathfrak{F}.$$

Hence the condition may be written either

$$\mathfrak{W}^2 = 0, \text{ or } \Sigma\mathfrak{F}\Sigma\epsilon\mathfrak{F} = 0. \quad \quad (524)$$

This shows that the plane of the resultant couple must be parallel to the resultant force.

Eq. (524) may also be satisfied by $\Sigma\epsilon\mathfrak{F} = 0$, which, by (519), reduces $\mathfrak{W}$ to a single force; and, by $\Sigma\mathfrak{F} = 0$, which reduces the wrench to a couple, or zero force at ∞. It follows that any system of forces *confined to one plane* will be equivalent either to a single force, or to a couple.

206. For *equilibrium* we must have

$$\mathfrak{W} = e_0\Sigma\mathfrak{F} + \Sigma\epsilon\mathfrak{F} = 0, \quad \quad (525)$$

which, because $\mathfrak{W}$ is the sum of a point-vector and a plane-vector, requires that we have

$$\left.\begin{array}{l} \Sigma\mathfrak{F} = 0 \\ \Sigma\epsilon\mathfrak{F} = 0 \end{array}\right\} \quad \quad (526)$$

207. *Normal form of a wrench.* It appears, by Art. 67, that, by properly choosing the point at which the resultant force acts, $\mathfrak{W}$ may always be reduced to the sum of a force at q, and a couple whose plane is *perpendicular* to the force.

Writing in (215) $\Sigma\mathfrak{F}$ for α, and $\Sigma\epsilon\mathfrak{F}$ for $|\beta$, we have

$$\mathfrak{W} = e_0\Sigma\mathfrak{F} + \Sigma\epsilon\mathfrak{F} = q\Sigma\mathfrak{F} + \frac{\Sigma\mathfrak{F}\Sigma\epsilon\mathfrak{F}}{(\Sigma\mathfrak{F})^2} \cdot |\Sigma\mathfrak{F}, \quad . \quad . \quad . \quad (527)$$

and, by (214), the vector perpendicular between $e_0\Sigma\mathfrak{F}$ and $q\Sigma\mathfrak{F}$ is

$$\frac{|\Sigma\mathfrak{F} \cdot \rho\Sigma\mathfrak{F}}{(\Sigma\mathfrak{F})^2} = -\frac{\Sigma\epsilon\mathfrak{F}|\Sigma\mathfrak{F}}{(\Sigma\mathfrak{F})^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (528)$$

in which $\rho = q - e_0$.

By Art. 46, the second term of the third member of (527) is the orthogonal projection of the resultant couple $\Sigma\epsilon\mathfrak{F}$ on a plane $\perp$ to $\Sigma\mathfrak{F}$, *i.e.* on $|\Sigma\mathfrak{F}$. Now the orthogonal projection of any plane-vector upon a plane is always *less* than the projected plane-vector in magnitude; hence the couple in the *normal form* of $\mathfrak{W}$ is the *minimum* of all the couples obtained, when e_0 occupies positions not in the line of $q\Sigma\mathfrak{F}$, which is called the *axis* of the wrench.

208. Recurring to eq. (213), let us consider the equation

$$T(|\beta - \rho\alpha) = C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (529)$$

i.e. the area of plane-vector part of S constant.

Taking the co-square, we have

$$\beta^2 - 2\rho\alpha\beta + (\rho\alpha)^2 = \beta^2 - 2\rho\alpha\beta + \rho^2\alpha^2 - (\rho|\alpha)^2 = C^2,$$

or $$\rho^2\alpha^2 - (\rho|\alpha)^2 - 2\rho\alpha\beta = C^2 - \beta^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (530)$$

Writing $\phi\rho = \alpha^2\rho - \alpha \cdot \rho|\alpha$, so that the first two terms become $\rho|\phi\rho$, and comparing with (443), we have

$$\gamma = -|\alpha\beta, \text{ and } (C \text{ of } (443)) = C^2 - \beta^2. \quad . \quad . \quad . \quad (531)$$

In (449) put

$$\lambda = \alpha, \ \mu = \beta, \ \nu = |\alpha\beta,$$

so that

$$\phi\alpha = 0, \ \phi\beta = \alpha^2\beta - \alpha \cdot \beta|\alpha, \ \phi|\alpha\beta = \alpha^2 \cdot |\alpha\beta,$$

and we see that $\delta = \frac{0}{0}$, so that the surface represented by (529) is a cylinder. We have, as in Art. 160, for the equation of the axis of this cylinder $\phi\delta + \gamma = 0$;

$$i.e.\ \alpha^2\delta - \alpha \cdot \delta|\alpha - |\alpha\beta = 0,$$

or, by (189), $\alpha\delta|\alpha = |\alpha\beta$, ... (532)

which is identical with (214), from which (528) was derived. *Thus the axis of the cylinder coincides with the axis of the screw, $q\alpha$.*

The discriminating cubic (461) becomes $(g - \alpha^2)^2 = 0$, so that the cylinder is *circular.* Now putting, as in the last Article, $\Sigma\mathfrak{F}$ for α and $\Sigma\epsilon\mathfrak{F}$ for $|\beta$, we see that the *locus of q, when the moment of the resultant couple is constant, is a circular cylinder whose axis is the axis of the wrench.*

209. *Any wrench can be reduced in an infinite number of ways to the sum of two forces.* Let us write $\mathfrak{W} = W(e\epsilon + a|\epsilon)$, to which form we have seen that any wrench is reducible. Now put $Wa|\epsilon = (e' - e)\mathfrak{F}$, which requires $e' - e$ and $\mathfrak{F}$ to be both $\perp$ to ϵ, but can be satisfied by an infinite number of values of e' and $\mathfrak{F}$, subject to these conditions. Substituting, we have

$$\mathfrak{W} = We\epsilon + (e' - e)\mathfrak{F} = e(W\epsilon - \mathfrak{F}) + e'\mathfrak{F},$$

which, by what we have just seen, proves the proposition.

The tetraedron of which the opposite edges are *any* two forces whose sum is $\mathfrak{W}$ is of *constant volume.* Let $p_1\mathfrak{P}_1$ and $p_2\mathfrak{P}_2$ be two such forces, so that $p_1\mathfrak{P}_1 + p_2\mathfrak{P}_2 = \mathfrak{W}$. Therefore, squaring,

$$2p_1\mathfrak{P}_1p_2\mathfrak{P}_2 = \mathfrak{W}^2 = 2\,e_0\Sigma\mathfrak{F}\Sigma\epsilon\mathfrak{F}.$$

The first member of this equation is 12 times the volume of the before-mentioned tetraedron, and the last member is a constant when $\mathfrak{W}$ is given, which proves the proposition.

210. EXERCISES. — (1) Find the conditions in order that the orthogonal projections on any direction whatever of a system of forces may be themselves a system of forces in equilibrium.

Let ι be a unit vector in any direction; then the projection of $\mathfrak{F}_1$ on ι is $\iota\cdot\iota|\mathfrak{F}_1$, and similarly for the other forces. By (526) we have for equilibrium, putting $\iota\cdot\iota|\mathfrak{F}_1$ for $\mathfrak{F}_1$, etc.,

$$\Sigma(\iota\cdot\iota|\mathfrak{F})=\iota\cdot\iota|\Sigma\mathfrak{F}=0.$$

As this is to be true for all directions of ι we must have $\Sigma\mathfrak{F}=0$. Again, writing $\iota\cdot\iota|\mathfrak{F}$ for $\mathfrak{F}$ in the second of (526), we have

$$\Sigma(\epsilon\iota\cdot\iota|\mathfrak{F})=-\iota\Sigma(\epsilon\cdot\iota|\mathfrak{F})=0$$

as the other condition, which requires $\Sigma(\epsilon\cdot\iota|\mathfrak{F})$ to be $\parallel$ to ι.

(2) Forces act at the vertices of a tetraedron, in directions respectively $\perp$ to the opposite faces, and proportional to the areas of these faces in magnitude; show that the forces have the property discussed in Ex. (1).

Let $e_0\epsilon_1$, $e_0\epsilon_2$, $e_0\epsilon_2$ be three edges of the tetraedron; then,

$$\mathfrak{F}_0=|(\epsilon_1-\epsilon_3)(\epsilon_1-\epsilon_2),\ \mathfrak{F}_1=|\epsilon_2\epsilon_3,\ \mathfrak{F}_2=|\epsilon_3\epsilon_1,\ \mathfrak{F}_3=|\epsilon_1\epsilon_2.$$

Hence we have at once $\Sigma\mathfrak{F}=0$. Also,

$$\Sigma(\epsilon\cdot\iota|\mathfrak{F})=\epsilon_1\cdot\iota\epsilon_2\epsilon_3+\epsilon_2\cdot\iota\epsilon_3\epsilon_1+\epsilon_3\cdot\iota\epsilon_1\epsilon_2=\epsilon_1\epsilon_2\epsilon_3\cdot\iota,$$

so that the second condition is fulfilled.

(3) A cube is acted on by four forces; one is in a diagonal, and the others in edges, no two of which are coplanar, and which do not meet the diagonal; find the condition that they may be equivalent to a single force.

Ans. If $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ are along the edges, and $\mathfrak{F}_4$ along the diagonal, the condition is

$$F_4(F_1+F_2+F_3)\sqrt{\tfrac{1}{3}}+F_2F_3+F_3F_1+F_1F_2=0.$$

(4) Six equal forces act along six successive edges of a cube which do not meet a given diagonal; find the resultant wrench.

Ans. If F be the magnitude of each force, and the edges of the cube be $e_0\iota_1$, $e_0\iota_2$, $e_0\iota_3$, we have

$$\mathfrak{W}=-2F|(\iota_1+\iota_2+\iota_3).$$

(5) If $\mathfrak{W}=e_1\mathfrak{F}_1+e_2\mathfrak{F}_2$, and e_1e_2 is $\perp$ to $\mathfrak{F}_1$ and $\mathfrak{F}_2$, find the normal form of $\mathfrak{W}$ and the position of its axis.

Let $e_0 = \frac{1}{2}(e_1 + e_2)$, and $\epsilon = e_2 - e_0 = -(e_1 - e_0)$. Then

$$\epsilon|\mathfrak{F}_1 = \epsilon|\mathfrak{F}_2 = 0 \text{ and } \Sigma\epsilon\mathfrak{F} = \epsilon\mathfrak{F}_2 - \epsilon\mathfrak{F}_1 = \epsilon(\mathfrak{F}_2 - \mathfrak{F}_1).$$

Hence

$$\mathfrak{W} = q(\mathfrak{F}_1 + \mathfrak{F}_2) + \frac{2\,\mathfrak{F}_1\epsilon\mathfrak{F}_2}{(\mathfrak{F}_1 + \mathfrak{F}_2)^2} \cdot |(\mathfrak{F}_1 + \mathfrak{F}_2).$$

By (528) the projection of $q - e_0$ on a plane $\perp$ to $\Sigma\mathfrak{F}$ is

$$\frac{F_2^2 - F_1^2}{(\mathfrak{F}_1 + \mathfrak{F}_2)^2} \cdot \epsilon,$$

so that, if we take $q - e_0 \perp$ to $\Sigma\mathfrak{F}$, we have

$$(\mathfrak{F}_1 + \mathfrak{F}_2)^2(q - e_0) = (F_2^2 - F_1^2)\epsilon.$$

(6) Three forces whose magnitudes are 1, 2, and 3 act along three successive edges of a unit cube which are not coplanar; show that the vector equation of the axis of the wrench is

$$\rho = \tfrac{13}{14}\iota_1 + \tfrac{1}{2}\iota_2 - \tfrac{9}{14}\iota_3 + x(\iota_1 + 2\iota_2 + 3\iota_3).$$

(7) Forces act at the mean points of the faces of a tetraedron, $\perp$ and proportional to the faces on which they act; show that they are in equilibrium.

(8) Let p be any point within a tetraedron, and let a system of $\parallel$ forces act at the vertices, each proportional in magnitude to the tetraedron formed by joining p with the other three vertices; find the centre of $\parallel$ forces.

(9) The sides of a rigid plane polygon are acted on by forces $\perp$ to the sides and proportional to them in magnitude, all the forces acting in the plane of the polygon, and being directed inwards; also the sides taken in the same order are severally divided by the points of application in the constant ratio of m to n; show that the system of forces is equivalent to a couple whose moment is $\frac{\mu(m-n)}{2(m+n)}\Sigma\epsilon^2$, in which μ is the ratio of the magnitude of any force to the length of the side it acts upon.

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