THE DOCTRINE AND APPLICATION OF FLUXIONS.

CONTAINING

(BESIDES WHAT IS COMMON ON THE SUBJECT)

A NUMBER OF

NEW IMPROVEMENTS IN THE THEORY;

AND

THE SOLUTION OF A VARIETY OF NEW AND VERY INTERESTING

PROBLEMS,

IN DIFFERENT BRANCHES OF

THE MATHEMATICS.

By THOMAS SIMPSON, F.R.S.

IN TWO VOLUMES:—VOL. II.

A NEW EDITION,

CAREFULLY REVISED, AND ADAPTED, BY

COPIOUS APPENDIXES,

TO THE PRESENT ADVANCED STATE OF SCIENCE.

BY A GRADUATE OF THE UNIVERSITY OF CAMBRIDGE.

LONDON:

PRINTED FOR J. COLLINGWOOD, AND G. & W. B. WHITTAKER;

AND SOLD BY MESSRS. BELL & BRADFUTE, EDINBURGH.

1823.

Printed by T. C. Hansard, Peterborough-court, Fleet-street, London.

THE

DOCTRINE AND APPLICATION

OF

FLUXIONS.

PART THE SECOND.

SECTION I.

The Manner of Investigating the FLUXIONS *of* Exponentials, *with* Those *of the Sides and Angles of Spherical Triangles.*

250. THE method of deriving the fluxion of any power, x^v, of a flowing quantity, when the exponent (v) is given or invariable, has been already shown: but, if the exponent be variable, that method fails; in which case the quantity x^v is called an exponential; whose fluxion is thus determined.

Put $z = x^v$, and let the hyperbolic logarithm of x be denoted by y; then that of x^v (z) will, by the nature of logarithms, be $= vy$; and therefore its fluxion $= \dot{v}y + v\dot{y}$: but the fluxion of the logarithm of z $(=x^v)$

* Art. 196. is also expressed by $\frac{\dot{z}}{z}$;* whence we have $\frac{\dot{z}}{z} = v\dot{y} + y\dot{v}$, and consequently $\dot{z} = zv\dot{y} + zy\dot{v}$: which equation, by substi-

† Art. 126. tuting $\frac{\dot{x}}{x}$ for its equal $\dot{y}$,† becomes $\dot{z} = zy\dot{v} + \frac{zv\dot{x}}{x} = x^v y\dot{v} + x^v \times \frac{v\dot{x}}{x} = x^v y\dot{v} + vx^{v-1}\dot{x} = x^v\dot{v} \times$ hyp. log. $x + vx^{v-1}\dot{x}$.

The same otherwise, without introducing the Properties of Logarithms.

251. Let $1+z=x$, and $n+w=v$, supposing n constant and w variable: then $x^v = \overline{1+z}|^{n+w} = \overline{1+z}|^n \times \overline{1+z}|^w = \overline{1+z}|^n \times (1 + wz + \frac{w}{1} \times \frac{w-1}{2} \times z^2 +$

‡ Art. 99. $\frac{w}{1} \times \frac{w-1}{2} \times \frac{w-2}{3} \times z^3 +$ &c.)‡ $= \overline{1+z}|^n \times \overline{1 + wz + \tfrac{1}{2}w^2 - \tfrac{1}{2}w \times z^2 + \tfrac{1}{6}w^3 - \tfrac{1}{2}w^2 + \tfrac{1}{3}w \times z^3 + \&c.}$ whose fluxion, found the common way, is $n\dot{z} \times \overline{1+z}|^{n-1} \times (1 + wz + \overline{\tfrac{1}{2}w^2 - \tfrac{1}{2}w} \times z^2 + \overline{\tfrac{1}{6}w^3 - \tfrac{1}{2}w^2 + \tfrac{1}{3}w} \times z^3$ &c.) $+ \overline{1+z}|^n \times (\dot{w}z + w\dot{z} + \overline{w\dot{w} - \tfrac{1}{2}\dot{w}} \times z^2 + \overline{\tfrac{1}{2}w^2 - \tfrac{1}{2}w} \times 2z\dot{z} + \overline{\tfrac{1}{2}w^2\dot{w} - w\dot{w} + \tfrac{1}{3}\dot{w}} \times z^3 + \overline{\tfrac{1}{6}w^3 - \tfrac{1}{2}w^2 + \tfrac{1}{3}w} \times 3z^2\dot{z}$ &c.) which, by substituting $\dot{x}$ and $\dot{v}$ for their equals $\dot{z}$ and $\dot{w}$, becomes $n\dot{x} \times \overline{1+z}|^{n-1} \times (1 + wz + \overline{\tfrac{1}{2}w^2 - \tfrac{1}{2}w} \times z^2 +$ &c.) $+ \overline{1+z}|^n \times \overline{\dot{v}z + w\dot{x} + \overline{w\dot{v} - \tfrac{1}{2}\dot{v}} \times z^2 + \&c.}$

But, if w be, now, supposed to vanish, we shall have the true value of the fluxion when $v = n$; which, in that circumstance, appears to be $= n\dot{x} \times \overline{1+z}|^{n-1}$

$+\overline{1+z}|^{v} \times \overline{z\dot{v} - \frac{1}{2}z^2\dot{v} + \frac{1}{3}z^3\dot{v} - \frac{1}{4}z^4\dot{v}}$ &c. $= v\dot{x} \times x^{v-1}$

$+ \dot{v}x^{v} \times \overline{z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4}$ &c. *Q. E. I.*

It is plain, because the series, $z - \frac{1}{2}z^2 + \frac{1}{3}z^3$ &c. here brought out, is known to express the fluent of $\frac{\dot{z}}{1+z}$, or the hyperbolic logarithm of $1+z$,* that the two conclusions agree exactly with each other: from either of which the following *Rule*, for the fluxions of exponentials, is deduced.

* Art. 126.

252. *To the Fluxion found by the common Rule* (Art. 14) *considering the Exponent as constant, add the Quantity arising by multiplying the Fluxion of the Exponent, the hyperbolic Logarithm of the Root, and the proposed Quantity itself, continually, together: the sum will be the Fluxion when the Exponent is variable.*

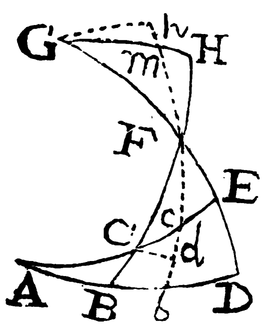
Thus, for example, let the quantity proposed be $\overline{a^2+z^2}|^{z}$, then the fluxion thereof will be $z \times 2z\dot{z} \times \overline{a^2+z^2}|^{z-1} + \dot{z} \times \overline{a^2+z^2}|^{z} \times$ hyp. log. (a^2+z^2).

But, if the root is constant, and only the exponent variable, the *exponential* will be more simple; and its fluxion will then be had by *barely multiplying the quantity itself by the product under the logarithm of the root and the fluxion of the exponent.*

Thus, the fluxion of a^x will be expressed by $a^x \times \dot{x} \times$ hyp. log. a; and that of $\overline{a^2+b^2}|^{nx}$ by $\overline{a^2+b^2}|^{nx} \times n\dot{x} \times$ hyp. log. $\overline{a^2+b^2}$. These kind of *exponentials* oftner occur, in practice, than any other; but, as it is very rare that we meet with *any*, I shall therefore proceed now to the other consideration proposed in the head of this section; namely, the method of determining the fluxions of the sides and angles of spherical triangles (a thing very useful in Practical Astronomy) which I shall deliver in the following Propositions.

PROPOSITION I.

253. *To determine the Ratio of the Fluxions of the several Parts of a right-angled spherical Triangle; supposing the Hypothenuse, one Leg, or one Angle, to remain constant, while the other Parts vary.*



Let A, F, and G be the poles of the three great-circles D E F G, ABD, and A C E; whereof the position of each is supposed to continue invariable, while another great-circle H F C B is conceived to revolve about the pole F: whence, if G H be supposed perpendicular to F H, three variable right-angled triangles, F G H, F C E, and A B C, will be formed; in the first whereof, the hypothenuse F G will remain constant; in the second, the leg E F; and in the third the angle A.

Let B *b* (*q*) be the fluxion (or indefinitely small increment)* of the base A B, or the angle F; and let C *d* meet the great circle *b* F *h*, at right-angles, in *d*; then it will be (*per Spherics*) as *sin.* F B (*rad.*) : *sin.*

$$\text{F C} :: \text{B}\,b\,(q) : \text{C}\,d = \frac{sin.\ \text{FC}}{rad.} \times q = \frac{co\text{-}s.\ \text{BC}}{rad.} \times q:$$

and, *tang.* C : *rad.* :: C *d* $\left(\frac{co\text{-}s.\ \text{BC}}{rad.} \times q\right) : \frac{co\text{-}s.\ \text{BC}}{tang.\ \text{C}}$ $\times q =$ the fluxion of B C.

Moreover, *sin.* C : *rad.* :: C *d* $\left(\frac{co\text{-}s.\ \text{BC}}{rad.} \times q\right)$:

$\frac{co\text{-}s.\ \text{BC}}{sin.\ \text{C}} \times q =$ the fluxion of A C.

* Art. 134.

Lastly, *sine* of FB *(rad.)* : *sin.* FH (BC) : : Bb(q) : $\frac{sin.\ BC}{rad.} \times q$ (= Hm) = the fluxion of G H, or its complement C.

Now, if the several quantities, in these three equations for the triangle A B C, be expounded by their respective equals in the other two triangles C E F and F G H, we shall also have

$$\frac{sin.\ CF}{tang.\ C} \times q = -flux.\ C\,F.$$

$$\frac{sin.\ CF}{sin.\ C} \times q = -\text{flux. } C\,E.$$

$$\frac{co\text{-}s.\ C\,F}{rad.} \times q = \text{flux. } C.$$

And

$$\frac{co\text{-}s.\ FH}{co\text{-}tang.\ GH} \times q = \text{flux. } F\,H.$$

$$\frac{co\text{-}s.\ F\,H}{co\text{-}s.\ G\,H} \times q = \text{flux. } G.$$

$$\frac{sin.\ F\,H}{rad.} \times q = -\text{flux. } G\,H.$$

Q. E. I.

COROLLARY 1.

254. Hence, if, in any right-angled spherical-triangle, the hypothenuse be denoted by h, the two legs by L and l, the angles, respectively, adjacent to them by A and a, we shall, by substituting above, have three equations for each of the three cases. From the comparison and composition of which, the three following tables are deduced; exhibiting all the different varieties that can possibly happen, whether an angle, a leg, or the hypothenuse be supposed invariable.

TABLE I.

When one Angle A is invariable,

$$\dot{L}=\frac{tang.\ a}{co\text{-}s.\ l}\times\dot{l}=\frac{sin.\ a}{co\text{-}s.\ l}\times\dot{h}=\frac{rad.}{sin.\ l}\times\dot{a}$$

$$\dot{l}=\frac{co\text{-}s.\ l}{tang.\ a}\times\dot{L}=\frac{co\text{-}s.\ a}{R}\times\dot{h}=\frac{co.\ tang.\ l}{tang.\ a}\times\dot{a}$$

$$\dot{h}=\frac{co\text{-}s.\ l}{sin.\ a}\times\dot{L}=\frac{R}{co\text{-}s.\ a}\times\dot{l}=\frac{co\text{-}tang.\ l}{sin.\ a}\times\dot{a}$$

$$\dot{a}=\frac{sin.\ l.}{R}\times\dot{L}=\frac{tang.\ a}{co\text{-}tang.\ l}\times\dot{l}=\frac{sin.\ a}{co\cdot tang.\ l}\times\dot{h}$$

TABLE II.

When one Leg L is invariable,

$$\dot{A}=\frac{tang.\ a}{sin.\ h}\times\dot{h}=\frac{sin.\ a}{sin.\ h}\times\dot{l}=-\frac{R}{co\text{-}s.\ h}\times\dot{a}$$

$$\dot{a}=-\frac{co\text{-}s.\ h}{R}\times\dot{A}=-\frac{sin.\ a}{tang.\ h}\times\dot{l}=-\frac{tang.\ a}{tang.\ h}\times\dot{h}$$

$$\dot{h}=\frac{sin.\ h}{tang.\ a}\times\dot{A}=\frac{co\text{-}s.\ a}{R}\times\dot{l}=-\frac{tang.\ h}{tang.\ a}\times\dot{a}$$

$$\dot{l}=\frac{sin.\ h}{sin.\ a}\times\dot{A}=\frac{R}{co\text{-}s.\ a}\times\dot{h}=-\frac{tang.\ h}{sin.\ a}\times\dot{a}$$

TABLE III.

When the Hyp. is invariable,

$$\dot{A}=-\frac{co\text{-}tang.\ l}{co\text{-}s.\ L}\times\dot{L}=-\frac{co\text{-}s.\ l}{co\text{-}s.\ L}\times\dot{a}=\frac{R}{sin.\ L}\times\dot{l}$$

$$\dot{L}=-\frac{co\text{-}s.\ L}{co\text{-}tang.\ l}\times\dot{A}=\frac{sin.\ l}{R}\times\dot{a}=-\frac{tang.\ l}{tang.\ L}\times\dot{l}$$

Where, and also in the two preceding tables, the leg L is adjacent to the angle A, and the leg l to the angle a.

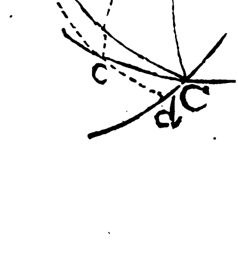
Corollary II.

255. From the third original equation, expressing the fluxion of the angle *C* (*Vide* Art. 253) it appears that the superficies of any spherical-triangle A B C, is proportional to the excess of its three angles above two right-angles. For (B C *d b*) the fluxion of the *triangle* ABC, is = *sine* BC × B*b* (by Art. 161) which being to, $\frac{sin.\ BC}{rad.} \times Bb$, the fluxion of the angle *C*, above specified, in the constant ratio of radius to unity, the fluents themselves (properly corrected) must therefore be in that ratio; that is, the superficies of the triangle A B C will always be proportional to the increase of the angle *C*, from its coinciding with *A*, or as the excess of *A* and *C* above two right-angles.

PROPOSITION II.

256. *To determine the Ratio of the Fluxions, or the indefinitely small Increments, of the different Parts of an oblique Spherical-Triangle* A B C; *two Sides thereof* A B, A C *being invariable, in Length.*

Let C *c* be an indefinitely small part of the parallel described by the extreme C of the given side A C, in its motion about the given point A; moreover, let C *d* be part of another parallel, whose pole is the given point B; let the great-circle B *c* meet C *d* in *d*; and let the three sides, AB, A C, and B C, of the triangle be denoted by *D*, *E*, and *F* respectively.



Then *(per Spherics)* we shall have

$$R : S.E :: \text{CA}c\ (\dot{A}) : \text{C}c = \frac{S.E.}{R} \times \dot{A};$$

and, $R : S.\ F :: \text{CB}d\ (\dot{B}) : \text{C}d = \frac{S.F}{R} \times \dot{B}$.

Also, $R : S.d\text{C}c\,(\text{ACB}) :: \text{C}c : \dot{F} = \frac{S.E \times S.C}{R^2} \times \dot{A}$:

But $S.C : S.D :: S.\text{B} : \text{S.E}$; therefore $S.E \times S.C = S.D \times S.B$, and consequently $\dot{F}$, also, $= \frac{S.D \times S.B}{R^2} \times \dot{A}$

Again, $R : \textit{co-s.}\ d\text{C}c\,(\text{ACB}) :: \text{C}c\left(\frac{S.E}{R} \times \dot{A}\right) : \frac{S.E. \times \textit{co-s.}\ C}{R^2} \times \dot{A}\ (= \text{C}d) = \frac{S.F}{R} \times \dot{B}$;

whence $\dot{B} = \frac{S.E \times \textit{co-s.}\ C}{R \times S.F} \times \dot{A}$.

Lastly, $\textit{co-t.}\ c\text{C}d\ (C) : R :: \text{C}d\left(\frac{S.F}{R} \times \dot{B}\right) : \dot{F} = \frac{S.F}{\textit{co-t.}\ C} \times \dot{B}$.

Whence, by the very same argument (substituting D for E, and C for B in the two last equations) we likewise have $\dot{C} = \frac{S.D \times \textit{co-s.}\ B}{R \times S.F} \times \dot{A}$, and $\dot{F}$ $\left(= \frac{S.F}{\textit{co-t.}\ C} \times \dot{B}\right) = \frac{S.F}{\textit{co-t.}\ B} \times \dot{C}$.

Now, from the equations thus found, it is manifest,

1°. $\dot{A} : \dot{F} :: R^2 : S.D \times S.B$ ($::$ *co-seca.* $D : S.B$)

2°. $\dot{A} : \dot{B} :: R \times S.F : S.E \times \textit{co-s.}\ C$

3°. $\dot{A} : \dot{C} :: R \times S.F :: S.D \times \textit{co-s.}\ B$

4°. $\dot{B} : \dot{F} :: \textit{co-t.}\ C : S.F$

5°. $\dot{C} : \dot{F} :: \textit{co-t.}\ B : S.F$

6°. $\dot{B} : \dot{C} :: \textit{co-t.}\ C : \textit{co-t.}\ B$ ($:: T.B : T.C$) Q.E.I.

257. These proportions, for the fluxions of the parts of a spherical triangle, are very useful in various cases in *Practical Astronomy*; whereof I shall here put down one or two instances.

The first is: to determine the annual alteration of the declination and right-ascension of a fixed star, through the precession of the equinox.

Here *A* must denote the pole of the ecliptic, *B* that of the equinoctial, and *C* the place of the star; and then (by the first and fourth proportions) we have

Co-seca. D : *sin.* $B :: \dot{A} : \dot{F}$; and

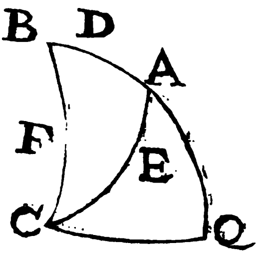
$S. F.$: *co-t.* $C :: \dot{F} : \dot{B}$;

That is 1°, *As the co-secant of the obliquity of the ecliptic is to the sine of the star's right ascension from the* solstitial colure, *so is the precession of the equinox, or alteration of longitude, to the alteration of declination.*

2°. *As the co-sine of the star's declination is to the co-tangent of its angle of position, so is the alteration of declination (found as above) to the alteration of right ascension corresponding.*

The second example is to find how much the amplitude, and the time of the apparent rising and setting of the sun, or a star, are affected by refraction.

In this case *A* must denote the pole of the equator, and *B* the *zenith*, and the side B C must be an arch of 90 degrees, so that the star *C* may coincide with the horizon Q C: then, from the very same proportion, we have



Sin. B : *co-seca.* $D :: \dot{F} : \dot{A}$,

And, R : *co-t.* $C :: \dot{F} : \dot{B}$

But, R : *co-t.* C $(T.QCA)$ :: *sin.* B (CQ) : *co.-tang.* D (*tang.* QA).

Hence it appears,

1°. *That, as the co-sine of the true amplitude (considered independent of refraction) is to the tangent of the pole's elevation, so is the given horizontal refraction to the difference of amplitudes thence arising.*

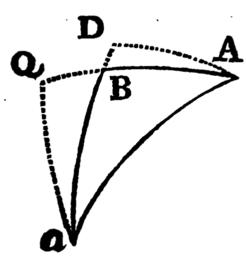
2°. And that, *as the co-sine of the true amplitude is to the secant of the pole's elevation, so is the said horizontal refraction to the effect thereof in the time of rising, or setting of the sun, or star.*

But this last proportion may be otherwise expressed, without the amplitude: thus,

$S.AB \times S.AC \times S.A : R^3 ::$ the horizontal refraction, to the same effect.

PROPOSITION III.

258. *To determine the same as in the preceding Problem; supposing one side* A B *and one of its adjacent Angles,* B, *to continue invariable.*



If from the end of the given side, opposite to the given angle, a perpendicular A D be let fall, that perpendicular, as well as the segment B D cut off thereby, will be a constant quantity, while the other parts of the triangle A*a* D vary, by the motion of *a* along the arch *a* B D. Therefore the problem is resolved by Case 2 of right angled triangles. *Vide* Art. 254.

259. It may not be amiss to give one example of the use of this last proposition: which shall be, in finding the parallax of a planet in longitude and latitude; that of altitude being given.

Here *A* must stand for the pole of the ecliptic, *B* the zenith, and *a* the planet: then, if the hypothenuse A *a* be denoted by *h*, the leg. D *a* by *l*, and the given parallax in altitude by $\dot{l}$, it will appear, from the place above quoted, that $\dot{A}$ (the parallax in long.) will be $= \frac{sin.\ a}{sin.\ h} \times \dot{l} = \frac{sin.\ BaA}{sin.\ Aa} \times \dot{l}$, and $\dot{h}$ (the parallax in lat.) $= \frac{co\text{-}s.\ a}{rad.} \times \dot{l} = \frac{co\text{-}s.\ BaA}{rad.} \times \dot{l}$.

If the planet be in (or very near) the ecliptic, and aQ be supposed a portion of the ecliptic, meeting A B, at right-angles, in Q, then *(per Spherics)* $\frac{sin.\ BaA}{sin.\ Aa} \left(\frac{co\text{-}s.\ BaQ}{radius}\right) = \frac{tang.\ Qa}{tang.\ Ba}$; also $\frac{co\text{-}s.\ BaA}{rad.} \left(\frac{sin.\ BaQ}{rad.}\right) = \frac{sin.\ QB}{sin.\ Ba}$; whence, by substituting these values above, we shall, *in this case*, have $\dot{A} = \frac{tang.\ Qa}{tang.\ Ba} \times \dot{l}$, and $\dot{h} = \frac{sin.\ QB}{sin.\ Ba} \times \dot{l}$; that is, in words,

As the tangent of the planet's zenith distance, is to the tangent of its longitude from the nonagesimal degree of the ecliptic, so is the parallax in altitude to the parallax in longitude.

And, as the sine of the zenith distance to the co-sine of the altitude of the nonagesimal degree, so is the parallax in altitude to the parallax in latitude.

Because the parallax in altitude, the horizontal parallax *(M)* being given, is nearly $= \frac{sin.\ Ba}{rad.} \times M$, if this value be substituted for $\dot{l}$, in the two last equations, we shall get $\dot{h} = \frac{sin.\ QB}{rad.} \times M$, and $\dot{A} = \frac{tang. Qa \times sin. Ba}{rad. \times tang.\ Ba} \times M = \frac{sin.\ AB \times sin.\ BAa}{rad.^2} \times M$.

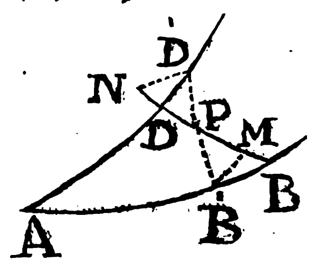
Whence, we have these two other theorems, for finding the required parallaxes immediately from the horizontal parallax, without either the altitude or its *parallax*.

1. *As radius to the co-sine of the altitude of the* nonagesimal degree *of the ecliptic, so is the horizontal parallax to the parallax in latitude.*

2. *And as the square of the radius to the rectangle under the sines of the altitude of the* nonagesimal degree *and the planet's longitude from thence, so is the horizontal parallax to the parallax in longitude.*

PROPOSITION IV.

260. *Still, to determine the same Thing; supposing one Angle* A, *and the Length of its opposite Side* B D *(or)* B́ D́) *to remain constant.*



Let B́ D́ (equal to B D) intersect B D in an indefinitely small angle at *P*, and meet AB and AD in *B́* and *D́*; also in B D produced, let there be taken P N = P D́ and P M = P B́, and let *N*, *D́*, and *M*, *B́* be joined.

Since, by hypothesis, D B = D́ B́ = M N, if from the first and last of these equal quantities D M, common, be taken away, there will remain B M = D N.

Moreover, since the triangles B M B́ and D N D́, in their ultimate state, may be considered as rectilineal, and right-angled at *M* and *N*,* it will therefore be, as

* Art. 134.

B M : B B́ :: *co-s. B* : *radius.*

And D N : D D́ :: *co-s. D* : *radius.*

From whence, the extremes in both proportions being the same, we have $BB : D\dot{D} :: co\text{-}s.\ D : co\text{-}s.\ B$: and therefore, if A B be denoted by H, and A D by K, it appears that $\dot{H} : \dot{K} :: co\text{-}s.\ D : co\text{-}s.\ B$.

Again, *per Spherics*, $sin.\ A : sin.$ BD $(G) :: sin.\ D : sin.\ H :: flux.\ sin.\ D : flux.\ sin.\ H$; because, the sines themselves being in a constant ratio, their fluxions must be in the same ratio: but the fluxion of the sine of any arc, or angle, is to the fluxion of the arc or angle itself, as the co-sine to radius :* * Art. 142. therefore the *flux. sin.* D being $= \frac{co.\text{-}s.\ D}{rad.} \times \dot{D}$, and $flux.\ sin.\ H = \frac{co\text{-}s.\ H}{rad.} \times \dot{H}$, it follows that, $sin.\ A : sin.\ G :: co\text{-}s.\ D \times \dot{D} : co\text{-}s.\ H \times \dot{H}$; or $\dot{D} : \dot{H} :: sin.\ A \times co\text{-}s.\ H : sin.\ G \times co\text{-}s.\ D$: and, by the very same argument, $\dot{B} : \dot{K} :: sin.\ A \times co\text{-}s.\ K : sin.\ G \times co\text{-}s.\ B$. Now, by compounding the former of these two proportions with the first above given, we get $\dot{D} : \dot{K} :: sin.\ A \times co\text{-}s.\ H : sin.\ G \times co\text{-}s.\ B$. And, by compounding this last with $\dot{K} : \dot{B} :: sin.\ G \times co\text{-}s.\ B : sin.\ A \times co\text{-}s.\ K$ (that immediately preceding it) we also obtain $\dot{D} : \dot{B} :: co\text{-}s.\ H : co\text{-}s.\ K$.

Whence, by collecting these several proportions together, we have the following Table for all the different cases.

$\dot{H} : \dot{K} :: co\text{-}s.\ D : co\text{-}s.\ B$

$\dot{D} : \dot{B} :: co\text{-}s.\ H : co\text{-}s.\ K$

$\dot{D} : \dot{H} :: tang.\ D : tang.\ H$

$\dot{B} : \dot{K} :: tang.\ B : tang.\ K$

$\dot{K} : \dot{D} :: sin.\ G \times co\text{-}s.\ B : sin.\ A \times co\text{-}s.\ H$

$\dot{H} : \dot{B} :: sin.\ G \times co\text{-}s.\ D : sin.\ A \times co\text{-}s.\ K$

It may be observed, that the fourth and the last are no new cases, but only the third and fifth repeated: and that, though the former of the two last named differs from that found above, yet it is very easily deduced from it: for, since it appears that $\dot{D} : \dot{H} :: \frac{sin. A}{co\text{-}s. D} : \frac{sin. G}{co\text{-}s. H}$, and because $sin. A : sin. G :: sin. D : sin. H$, it follows that $\dot{D} : \dot{H} :: \frac{sin. D}{co\text{-}s. D} : \frac{sin\ H}{co\text{-}s. H} :: tang. D : tang. H$. Q. E. I.

There is yet another problem, when two angles remain constant; but this, by taking the triangle formed by the poles of the three given circles, is reduced to Problem 2.

SECTION II.

Of the Resolution of fluxional Equations, or the manner of finding the Relation of the flowing Quantities from that of the Fluxions.

261. WHEN an equation, expressing the relation of the fluxions of the two variable quantities, contains *only* one of those fluxions with its respective flowing quantity in each term, the relation of the quantities will be obtained by finding the fluent of every term: as has been already taught, in Sect. VI, Part I.

Thus, if $ax^2\dot{x} = y^3\dot{y}$, then will $\frac{ax^3}{3} = \frac{y^4}{4}$.

And, if $x^n y^m \dot{x} = a\dot{y}$; by reducing it first to $x^n \dot{x} = ay^{-m}\dot{y}$ (so that its variable quantities may be separated) we have $\frac{x^{n+1}}{n+1} = \frac{ay^{1-m}}{1-m}$.

But, if the given equation has its indeterminate quantities and their fluxions so complicated together, that it cannot be brought under the form there prescribed, the task will become much more difficult; nor is there any general method to be given for such kinds of equations, whereof there are an infinite variety.

The method of infinite series (in some measure explained already, and more fully considered hereafter) is indeed very comprehensive, and may be applied to good purpose in various cases; but, being tedious and attended with a number of inconveniencies, it is a method we ought never to have recourse to till we have tried what may be, otherways, effected, by help of such particular rules and observations as we have been able to collect.

Accordingly, I shall, here, first point out some of the most proper ways to be tried, in order, if possible, to bring out the solution without an infinite series.

262. *The first method is, by multiplying, or dividing, the given equation into some power or product of the quantities concerned; so as to bring it, if possible, under the form of such fluxions, as, we know, do arise, if not from the first, yet from the second or third, of the three general rules in the direct method.*

Thus, if the given equation be $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}=\frac{x^m\dot{x}}{ay^n}$; then, the whole being multiplied by xy, so that the two first terms, $y\dot{x}+x\dot{y}$, may become the (known) fluxion of the rectangle xy,* there arises $y\dot{x}+x\dot{y}=\frac{x^{m+1}\dot{x}}{ay^{n-1}}$: but still we are at a loss for the fluent of the last term, unless n be taken $=1$ (so that y may vanish). In that case we have $xy=\frac{x^{m+2}}{\overline{m+2}\times a}$; expressing the relation of the fluents when that of the fluxions is $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}=\frac{x^m\dot{x}}{ay}$: which appears to be the only case, of the given equation, where this method is of use.

* Art. 10.

Again, let the equation $\frac{p\dot{x}}{x}+\frac{r\dot{y}}{y}=\frac{x^m\dot{x}}{ay^n}$ be proposed.

Here, multiplying by $x^p y^r$ (where the exponents are the same as the co-efficients of $\frac{\dot{x}}{x}$ and $\frac{\dot{y}}{y}$) we get $px^{p-1}\dot{x} \times y^r + x^p \times ry^{r-1}\dot{y} = \frac{x^{m+p}\dot{x}}{ay^{n-r}}$; in which the former part of the equation is known to express the fluxion of $x^p y^r$.* Therefore, when $n=r$, the relation of the fluents may be found, and will be expressed by $x^p y^r = \frac{x^{m+p+1}}{\overline{m+p+1} \times a}$: which, if no correction by a constant quantity be necessary, may be reduced to $y^r = \frac{x^{m+1}}{\overline{m+p+1} \times a}$.

* Art. 15.

The same method may also be extended to fluxions of the higher orders: let $\ddot{x}-x\dot{z}^2=f\dot{z}^2$ (which equation occurs hereafter, in the resolution of a problem of some difficulty). Then, multiplying by $\dot{x}$, it becomes $\dot{x}\ddot{x}-x\dot{x}\dot{z}^2=f\dot{z}^2\dot{x}$; where, $\dot{z}$ being constant, each term admits, now, of a perfect fluent, and we therefore have $\frac{\dot{x}^2}{2}-\frac{x^2\dot{z}^2}{2}=fx\dot{z}^2$: from whence, supposing no correction necessary, $\dot{z}=\frac{\dot{x}}{\sqrt{2fx+x^2}}$, and $z=$ hyp. log. $f+x+\sqrt{2fx+x^2}$ (by Art. 126).

268. *It may happen that the solution of an equation will become more easy by first taking the fluxion thereof; when, by that means, some of the terms destroy each other.*

The following is an instance of it (which, also, occurs hereafter). Let $y+\frac{\dot{y}\times\overline{a-x}}{\dot{x}}=x-\frac{y\dot{x}}{\dot{y}}$: whose fluxion, making $\dot{x}$ constant, is $\dot{y} + \frac{\ddot{y} \times \overline{a-x} - \dot{x}\dot{y}}{\dot{x}} = \dot{x} - \frac{\dot{y}\dot{x}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}^2}$: which, by reason of the terms destroying one another, is reduced to $\frac{\ddot{y} \times \overline{a-x}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}^2}$: therefore, by expunging $\ddot{y}$, &c. we get $\dot{y}y^{-\frac{1}{2}} = \dot{x} \times \overline{a-x}|^{-\frac{1}{2}}$, and consequently $2y^{\frac{1}{2}} = -2 \times \overline{a-x}|^{\frac{1}{2}} + $ *some constant quantity.*

264. *Another method, applicable to equations, of the first order of fluxions, wherein only one of the two variable quantities (x or y) enters, is, to substitute for the ratio of the two fluxions ($\dot{x}$ and $\dot{y}$): from whence the value of that quantity will be had, immediately, in terms of the said assumed ratio: and then, by taking its fluxion, that of the other quantity (and from thence the quantity itself) will become known.*

Thus, let $a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}^2 + \dot{y}^2}|^2$ (being the equation of the curve that generates the solid of the *least resistance*, when the bulk and greatest diameter are given). Then, by putting $\frac{\dot{x}}{\dot{y}} = v$, and substituting above, we get $av\dot{y}^4 = y \times \overline{v^2\dot{y}^2 + \dot{y}^2}|^2 = y\dot{y}^4 \times \overline{v^2+1}|^2$; and consequently $y = \frac{av}{\overline{v^2+1}|^2}$: therefore $\dot{y} = \frac{a\dot{v} - 3av^2\dot{v}}{\overline{v^2+1}|^3}$; and consequently $\dot{x}$ $(= v\,\dot{y}) = \frac{av\dot{v} - 3av^3\dot{v}}{\overline{v^2+1}|}$: whose fluent may be found, from Art. 84, or, otherwise, thus: put $w^2 = v^2 + 1$; then $v^2 = w^2 - 1$, and $w\dot{w} = v\dot{v}$; by substituting which values there arises $\dot{x} = \frac{aw\dot{w} - 3aw\dot{w} \times \overline{w^2-1}}{w^6} = 4a\dot{w}w^{-5} - 3a\dot{w}w^{-3}$; and therefore $x = \frac{4aw^{-4}}{-4} - \frac{3aw^{-2}}{-2} = -\frac{a}{w^4} + \frac{3a}{2w^2} = \frac{3aw^2 - 2a}{2w^4}$

$= \frac{3a \times \overline{v^2+1} - 2a}{2 \times \overline{v^2+1}|^2} = \frac{a \times \overline{3v^2+1}}{2 \times \overline{v^2+1}|^2}$; which, corrected (by taking y, or $v=0$) becomes $x = \frac{a \times \overline{3v^2+1}}{2 \times \overline{v^2+1}|^2} - \frac{a}{2}$. From this equation, by completing the square, &c. v may be found in terms of x; whence the corresponding value of y $\left(= \frac{av}{\overline{v^2+1}|^2}\right)$ will also be known.

265. *The fourth method*, which chiefly obtains when one of the indeterminate quantities and its fluxion, arise but to a single dimension each, *may be thus:*

Let the value of that quantity, which is least involved, be first sought, from the fictitious equation arising by neglecting all the terms in the given equation, where neither that quantity, nor its fluxion, are found: then, to that value, let some power, or powers, of the other quantity, with unknown co-efficients, be added (according to the dimensions of the terms neglected) and let the sum be substituted in the given equation, as the true value of the first mentioned quantity: by which means a new equation will result; from whence the assumed co-efficients may, sometimes, be determined.

Ex. Let the given equation be $cx^2\dot{x} + y\dot{x} = a\dot{y}$.

By neglecting $cx^2\dot{x}$, or feigning $y\dot{x} = a\dot{y}$, we get $\frac{\dot{x}}{a} = \frac{\dot{y}}{y}$: and consequently $\frac{x}{a}$ = hyp. log. y − hyp. log. d* = hyp log. $\frac{y}{d}$; d being any constant quantity, which the nature of the problem may require. Hence $\frac{y}{d}$ = the number whose hyperbolical logarithm is $\frac{x}{a}$: which number, if M be put for (2,71828 &c.)

* Art. 126 & 78.

the number whose hyp. log. is unity, will be expressed by $\overline{M}|^{\frac{x}{a}}$ (since it is evident that the hyp. log. hereof is $\frac{x}{a} \times \log. M = \frac{x}{a}$): therefore $\frac{y}{d} = \overline{M}|^{\frac{x}{a}}$ and $y = d \times \overline{M}|^{\frac{x}{a}}$. Now, to the value thus found, let there be added $Ax^2 + Bx + C$, in order to get the true value; and then, $\dot{y}$ being $= 2Ax\dot{x} + B\dot{x} + \frac{d\dot{x}}{a} \times \overline{M}|^{\frac{x}{a}}$,* we shall, by substituting in the given equation, have $cx^2\dot{x} + Ax^2\dot{x} + Bx\dot{x} + C\dot{x} + d\dot{x}M^{\frac{x}{a}} = 2Aax\dot{x} + Ba\dot{x} + d\dot{x}M^{\frac{x}{a}}$, and consequently $\overline{c + A} \times x^2\dot{x} + \overline{B - 2Aa} \times x\dot{x} + \overline{C - Ba} \times \dot{x} = 0$. Whence $A = -c$,† $B = -2ac$, $C = -2a^2c$; and consequently $y = -c \times \overline{x^2 + 2ax + 2a^2} + dM^{\frac{x}{a}}$. By the very same way, the value of y, in the equation $cx^n\dot{x} + y\dot{x} = a\dot{y}$, will come out $= -c \times nx^n + (ax^{n-1} + n.\overline{n-1}.a^2x^{n-2} + n.\overline{n-1}.\overline{n-2}.a^3x^{n-3} + \&c.) + dM^{\frac{x}{a}}$.

* Art. 14.

† Art. 84.

266. But, what is a little remarkable, in these equations, is, that the *Exponential* $dM^{\frac{x}{a}}$, though a variable quantity, should only serve, as it were, to correct the fluent, or perform the office of a constant quantity. What I here mean will plainly appear, if it be considered, that the equation $y = -c \times \overline{x^2 + 2ax + 2a^2}$, where the said *Exponential* is wanting, answers all the conditions of the fluxional equation first proposed; which, upon trial, will be found; and must needs be

the case, seeing d may be, either, taken nothing at all, or any quantity at pleasure.

But the equation $y = -c \times \overline{x^2 + 2ax + 2a^2}$ (when $dM^{\frac{x}{a}}$ is wanting) cannot be corrected, in the usual way, so as to give $y=0$, when $x=0$; since, if any other constant quantity, besides $-2a^2c$ be introduced, the first conditions will not be answered: the correction must, therefore, be by the exponential $dM^{\frac{x}{a}}$; and is thus:

Since $y = -cx^2 - 2cax - 2ca^2 + dM^{\frac{x}{a}}$, if y be taken $=0$ and $x=0$, then $-2ca^2 + dM^0 = 0$, or $d = 2ca^2$; and so the equation, truly corrected, is $y = -c \times \overline{x^2 + 2ax + 2a^2} + 2a^2cM^{\frac{x}{a}}$.

267. We come now to the last method; namely, that of infinite series; which, though less accurate, is vastly more comprehensive, than any yet explained: the manner of it is thus:

For the quantity whose value you would find, let an infinite series, consisting of the powers of the other quantity with unknown co-efficients, be assumed; which series, together with its fluxion, or fluxions, must be substituted instead of their equals in the given equation; when a new equation will arise, from which, by comparing the homologous terms, the assumed co-efficients, and consequently the value sought, will be determined.

Thus, let the equation $\frac{\dot{x}}{1+x} = \dot{y}$ (reducible to $\dot{x} - \dot{y} - x\dot{y} = 0$) be proposed; to find x in terms of y.

Then, assuming $x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5$ &c.

We have $\dot{x} = A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + 5Ey^4\dot{y} +$ &c.

Which values being substituted in $\dot{x} - \dot{y} - x\dot{y} = 0$, we get

$$\left.\begin{array}{l} A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + \&c. \\ -\dot{y} - Ay\dot{y} - By^2\dot{y} - Cy^3\dot{y} - \&c. \end{array}\right\} = 0.$$

Therefore $A-1=0$, or $A=1$; $2B-A=0$, or $B=\frac{A}{2}=\frac{1}{2}$; $3C-B=0$, or $C=\frac{B}{3}=\frac{1}{2.3}$; $4D-C=0$, or $D=\frac{C}{4}=\frac{1}{2.3.4}$ &c.

And consequently $x\,(Ay+By^2+Cy^3\ \&c.)=y+\frac{y^2}{2}+\frac{y^3}{2.3}+\frac{y^4}{2.3.4}+\frac{y^5}{2.3.4.5}+\&c.$

Again, let it be required to find the value of y, in the equation $cx^2\dot{x}+y\dot{x}=a\dot{y}$, or $a\dot{y}-y\dot{x}-cx^2\dot{x}=0$. Here, assuming $y=Ax+Bx^2+Cx^3+Dx^4+Ex^5+Fx^6$ &c. and proceeding as before, we shall have

$$\left.\begin{array}{llllll} aA\dot{x}+2aBx\dot{x}+3aCx^2\dot{x}+4aDx^3\dot{x}+5aEx^4\dot{x}+\&c. \\ 0\quad-\quad Ax\dot{x}-\quad Bx^2\dot{x}-\quad Cx^3\dot{x}-\quad Dx^4\dot{x}-\&c. \\ 0\qquad 0\quad-\quad cx^2\dot{x} \end{array}\right\}=0$$

Whence $A=0$; $2aB-A=0$; $3aC=B+c=c$, or $C=\frac{c}{3a}$; $4aD=C=\frac{c}{3a}$, or $D=\frac{c}{3.4a^2}$; $5aE=D=\frac{c}{3.4a^2}$, or $E=\frac{c}{3.4.5a^3}$ &c. and consequently y

$$(Ax+Bx^2+Cx^3+\&c.)=\frac{cx^3}{3a}+\frac{cx^4}{3.4a^2}+\frac{cx^5}{3.4.5a^3}+\frac{cx^6}{3.4.5.6a^4}+\&c.$$

268. It appears from this example, that the quantity to be found, will not always require all the terms of the series $Ax+Bx^2+Cx^3$ &c. And it may happen, in innumerable cases, that the series to be assumed will demand a very different law from *that* where the exponents proceed according to the terms of an arithmetical progression having unity for the common difference. And, indeed, the greatest difficulty we have here to encounter is, to know what kind of series, with regard to its exponents, ought to be assumed, so as to answer the conditions of the equation, without introducing more terms than are actually necessary.

The following rules will be found very useful upon this occasion: which, though they may become impracticable in certain particular cases, never take in any superfluous terms.

1°. *Having (if necessary) freed your equation from fractions and surds, let the quantity, whose value is sought, be supposed equal to some power of the other quantity with an unknown exponent (n); and let that power, together with its fluxion, or fluxions, be substituted for their (supposed) equals in the given equation.*

2°. *Let the least exponents of the variable, or indeterminate, quantity, in the new equation, thence arising, be put equal to each other: whence the value of the unknown exponent n will be found.*

3°. *Substitute the value of n, so found, in all the exponents where n is concerned; and then take the difference between one of the equal* ones, *above mentioned, and every* other *exponent, of the variable quantity, in the whole equation.*

4°. *To these differences, write down all the least numbers that can be composed out of them, by continual addition, either to themselves, or to one another; till you have, by that means, got, in the whole, as many different terms, as you would have the required series continued to.*

5°. *Lastly, let each of those terms be increased by the value of n (found by Rule 2) and you will then have the exponents of the series to be assumed.*

EXAMPLE I.

269. *Let the Value of x, in the Equation* $a^2\dot{x}^2 + x^2\dot{z}^2 - a^2\dot{z}^2 = 0$, *be required.*

First, by writing z^n for x, and $nz^{n-1}\dot{z}$ for $\dot{x}$, the indices of z will be $2n-2$, $2n$, and 0 (which are determined by inspection, without regarding the co-efficients) whereof the two least ($2n-2$ and 0) being put equal to each other, we here find $n=1$: therefore, the exponents being 0, 2, 0, the differences (according to *Rule* 3) are also 0, 2; from whence, by adding 2 continually, we get 0, 2, 4, 6, 8 &c. which (being each increased by the value of n) give 1, 3, 5, 7, 9 &c. for the exponents in this case.

Let, therefore, $x = Az + Bz^3 + Cz^5 + Dz^7 + \&c.$ Then, putting $\dot{z}=1$, in order to facilitate the operation, we shall have $\dot{x} = A + 3Bz^2 + 5Cz^4 + 7Dz^6 + \&c.$ which two values being squared, and substituted in the given equation, it will become

$$\left.\begin{array}{llll} a^2A^2 + 6a^2ABz^2 + & 10a^2ACz^4 + & 14a^2ADz^6 & + \&c. \\ & + 9a^2B^2z^4 + & 30a^2BCz^6 & + \&c. \\ * \quad + A^2z^2 & + 2ABz^4 + & 2ACz^6 & + \&c. \\ -a^2 & & + B^2z^6 & + \&c. \end{array}\right\} = 0$$

Whence, $a^2A^2=a^2$, and therefore $A=1$; $6a^2B=-A$, and therefore $B=-\frac{1}{6a^2}=-\frac{1}{2.3a^2}$; $10a^2AC = -9a^2B^2 - 2AB = -B \times \overline{9a^2B + 2A} = -B \times \overline{-\frac{3}{2}+2} = -\frac{B}{2}=\frac{1}{2.3.2a^2}$, and therefore $C = \frac{1}{2.3.4.5a^4}$; $14AD = -30a^2 \times -\frac{1}{6a^2} \times \frac{1}{120a^4} - 2 \times \frac{1}{120a^4} - \frac{1}{36a^4} = \frac{1}{24a^4} - \frac{1}{60a^4} - \frac{1}{36a^4} = -\frac{1}{360a^4}$, and therefore $D = -\frac{1}{14.360a^6} = -\frac{1}{2.3.4.5.6.7.a^6}$; and, consequently, $x=z-\frac{z^3}{2.3a^2}+\frac{z^5}{2.3.4.5a^4}-\frac{z^7}{2.3.4.5.6.7a^6}$ &c.

EXAMPLE II.

270. *Let the given Equation be* $a^2x\ddot{y}-2a^2\dot{x}\dot{y}+ax\dot{x}^2+x^3\ddot{y}=0$; *to find* y.

Here, substituting x^n for y, the exponents will be $n-1$, $n-1$, 1, and $n+1$; where, making $n-1=1$, we get $n=2$: whence, the differences being 0, 2, the series to be assumed for y will be $Ax^2+Bx^4+Cx^6+Dx^8+Ex^{10}+$ &c. From which, making $\dot{x}=1$, we have $\dot{y}=2Ax+4Bx^3+6Cx^5+8Dx^7$ &c. and

$\ddot{y}=2A+12Bx^2+30Cx^4+56Dx^6$ &c.

And, these values being substituted, the equation becomes

$$\left.\begin{array}{llllll} 2a^2Ax & +12a^2Bx^3 & +30a^2Cx^5 & +56a^2Dx^7 & + \&c. \\ -4a^2Ax & -8a^2Bx^3 & -12a^2Cx^5 & -16a^2Dx^7 & + \&c. \\ +ax & +2Ax^3 & +12Bx^5 & +30Cx^7 & + \&c. \end{array}\right\}=0$$

Therefore $A=-\frac{1}{2a}$; $B=-\frac{2A}{4a^2}=-\frac{1}{4a^3}$; $C=-\frac{12B}{18a^2}=\frac{1}{6a^5}$; $D=-\frac{30C}{40a^2}=-\frac{1}{8a^7}$ &c.

and so $y=\frac{x^2}{2a}-\frac{x^4}{4a^3}+\frac{x^6}{6a^5}-\frac{x^8}{8a^7}+\frac{x^{10}}{10a^9}-$ &c.

Which series is known to express the fluent of $\frac{ax\dot{x}}{a^2+x^2}$, or, $\frac{1}{2}a\times$ hyp. log. $\frac{a^2+x^2}{a^2}$: consequently y is also $=\frac{1}{2}a\times$ hyp. log. $\frac{a^2+x^2}{a^2}$. In this manner, it comes to pass, *that*, though we are obliged, in very complicated cases, to have recourse to infinite series, we are sometimes able, at last, to give the solution in finite terms, or at least, by help of logarithms, sines and tangents: which will always happen when the series can be summed, or is found to agree with that arising from some known quantity.

271. Sometimes it happens, in equations involving the higher orders of fluxions, that the exponents, mentioned in *Rule* 2, whereof the least ought to be made equal to each other, are so expressed, as to render such an equality impossible. When this is the case, the value of n, and the first term of the required series, can *only* be determined from the nature of the problem to which the equation belongs. We know,

indeed, from the equation itself, that n must be either equal to nothing, or to some positive integer, less than that expressing the order of the highest fluxion in the equation: because the term that has the least exponent, and which therefore cannot be compared with any other (being always affected by two or more of the factors n, $n-1$, $n-2$, &c.) will then (one of those factors being $=0$) vanish entirely out of the equation; which, thereby, is rendered possible.

When n and A are known, the rest of the terms will be found in the common way, as in

EXAMPLE III.

Where the Equation proposed is $y\dot{x}^2 + a\dot{x}\dot{y} - a^2\ddot{y} = 0$; to find y.

By supposing $\dot{x} = 1$, and writing x^n for y, nx^{n-1} for $\dot{y}$, and $n \times \overline{n-1} \times x^{n-2}$ for $\ddot{y}$, we get $x^n + nax^{n-1} - n \times \overline{n-1} \times a^2x^{n-2}$: but it is plain that no two of the indices of x can, *here*, be equal: the value of n must therefore be either $=0$, or unity (in both which cases the term $-n \times \overline{n-1} \times a^2x^{n-2}$ vanishes) but I shall take the latter value, and suppose the first term of the series to be Ax; then, the differences of the foresaid exponents being 1 and 2, the law of the series will be expressed by 1, 2, 3, 4, &c. Whence, assuming $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. and proceeding as in the former examples, y will be found $=$ A *into* $x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{8a^3} + \frac{x^5}{24a^4} + \frac{x^6}{90a^5}$ &c. or $=$ A *into* $x + \frac{x^2}{2a} + \frac{2x^3}{2.3\,a^2} + \frac{3x^4}{2.3.4\,a^3} + \frac{5x^5}{2.3.4.5\,a^4} + \frac{8x^6}{2.3.4.5.6\,a^5} +$ &c. where the law of continuation is manifest, the co-efficient of every numerator being composed by the addition of the two preceding ones.

272. It will be proper to observe here, that, in equations like the two last proposed, where the higher orders of fluxions are concerned, the series expressing the relation of the two quantities must always be found in terms of the quantity flowing uniformly. And, that, if the number of dimensions of the fluxion of the said quantity, after substitution, be not the same in every term, the equation itself, put down to be resolved, is absurd and impossible, and such as never can arise in the solution of any problem. In all proper equations the number of fluxional points (supposing the powers of the fluxions to be wrote without indices) will be the same in every term.

EXAMPLE IV.

273. *Where let the given Equation be* $a^3\dot{y} - ay^2\dot{x} + x^2y\dot{y} = x^3\dot{x}$; *to find* y.

By proceeding as usual the indices will here be $n-1$, $2n$, $2n+1$ and 3; whereof the least (which can be no other than $n-1$ and 3) being compared, n will be given $=4$: and the differences will therefore be 0, 5, 6; to which the double of the second and the sum of the second and third, &c. being put down, and then every term increased by 4, there arises 4, 9, 10, 14, 15, 16, 19, &c. for the exponents of the series to be assumed for y.

Let therefore $y = Ax^4 + Bx^9 + Cx^{10} + Dx^{14}$ &c. then, making $\dot{x} = 1$, $\dot{y}$ is $= 4Ax^3 + 9Bx^8 + 10Cx^9 + 14Dx^{13} + \&c.$

And, by substituting these values above, we have

$$\left.\begin{array}{l} 4a^3Ax^3 + 9a^3Bx^8 + 10a^3Cx^9 + 14a^3Dx^{13} + \&c. \\ -\dot{x}^3 \quad - aA^2x^8 \quad + 4A^2x^9 \quad - 2aABx^{13} + \&c. \end{array}\right\} = 0$$

Whence $A = \frac{1}{4a^3}$, $B = \frac{1}{144a^8}$, &c.

And* $y = \frac{x^4}{4a^3} - \frac{x^9}{144a^8} - \frac{x^{10}}{40a^9} + \frac{x^{14}}{4032a^{13}}$ &c.

* If, for y, the series $Ax^4 + Bx^5 + Cx^6 + Dx^7$ &c. whose exponents are in arithmetical progression, had been assumed, according to the method of some very good authors, no less than seven superfluous terms must have been introduced to obtain the four above given.

274. **Before I quit this subject, it may not be amiss to subjoin the following remarks.**

1°. If the indeterminate quantities are great in respect to the given ones, a descending series will, in most cases (where it is practicable) converge better than an ascending one. To obtain such a series, compare the greatest exponents, mentioned in Rule 2, instead of the least, and proceed according to the third and fourth Rules,* whence a series of numbers will be found; which, being successively subtracted from the value of n, you will have the exponents of a descending series.

* Art. 268.

Thus, let the common algebraic equation $a^3x+ax^3-a^3y-y^4=0$ be propounded; to find y, when x is great in comparison of a.

Then, proceeding as usual, the exponents of the four terms of the equation will be 1, 3, n, $4n$; whereof the two greatest ($4n$ and 3) being made equal, we get $n=\frac{3}{4}$; therefore the differences are 0, 2 and $2\frac{1}{4}$; and the numbers to be subtracted from n, are 0, 2, $\frac{9}{4}$, 4, $\frac{17}{4}$, &c. Consequently the series to be assumed for y is

$$Ax^{\frac{3}{4}}+Bx^{-\frac{5}{4}}+Cx^{-\frac{6}{4}}+Dx^{-\frac{13}{4}}+\text{\&c.}$$ From whence

$$y \text{ will be found}=a^{\frac{1}{4}}x^{\frac{3}{4}}+\frac{a^{\frac{9}{4}}}{4x^{\frac{5}{4}}}-\frac{a^{\frac{10}{4}}}{4x^{\frac{6}{4}}}-\frac{3a^{\frac{17}{4}}}{32x^{\frac{13}{4}}}\text{ \&c.}$$

2°. But, if the quantity (x) in whose terms the other is to be expressed, be neither much greater nor much smaller than the given quantity (a), it will be proper to substitute for the excess, or defect, of the said quantity (x) above, or below, some given quantity; so that, having, by this means, exterminated x, the series arising from the new equation (wherein the said excess, or defect, is the converging quantity) will have a due rate of convergency.

The use of this is so obvious that it needs no example, or farther explanation.

3°. Lastly, it will be proper to observe, that, if the equation for the value of A, arising from the first column of homologous terms, admits of two or more, equal roots (which is a case that may, perhaps, never happen in practice) all the foregoing precepts will be insufficient; unless the equation also admits of some other root, besides the equal ones, whereby A may be more commodiously expressed. To determine the exponents, in that particular case, divide each of the differences, mentioned in Rule 3, by the number of the equal roots; and then proceed as usual. The reasons of which, as well as of the rules themselves, I have long ago given elsewhere, and have not room to repeat them here.

SCHOLIUM.

275. Although the business of reverting series is not a branch of the doctrine of fluxions, but, more properly, belongs to common *algebra*; yet, as it is often useful where fluxions are concerned, and falls under the general rules illustrated in the foregoing pages, I shall here add an example or two on that head.

Let, then, $ax+bx^2+cx^3+dx^4+ex^5$ &c. $=y$; to revert the series, or, to find x in an infinite series expressed in the powers of y.

Here, by writing y^n for x, the indices of the powers of y, in the equation, will be n, $2n$, $3n$, &c. and 1; therefore $n=1$, and the differences are 0, 1, 2, 3, 4, 5, &c. and so the series to be assumed, in this case, is $Ay+By^2+Cy^3+Dy^4$ &c. Which being involved and substituted for the respective powers of x (neglecting, every where, all such powers of x and y as exceed the highest you would have the series carried to) there arises

$$\left.\begin{array}{lllll} aAy & +aBy^2 & +aCy^3 & +aDy^4 & \&c. \\ * & +bA^2y^2 & +2bABy^3 & \left.\begin{array}{l}+2bACy^4 \\ +bB^2y^4\end{array}\right\} & \begin{array}{l}\&c.\\ \&c.\end{array} \\ * & * & +cA^3y^3 & +3cA^2By^4 & \&c. \\ * & * & * & +dA^4y^4 & \&c. \end{array}\right\} = y$$

Whence, by comparing the homologous terms, $A = \frac{1}{a}$; $B = -\frac{b}{a^3}$; $C \left(= -\frac{2bAB + cA^3}{a}\right) = \frac{2b^2 - ac}{a^5}$; $D \left(= -\frac{2bAC + bB^2 + 3cA^2B + dA^4}{a}\right) = \frac{5abc - 5b^3 - a^2d}{a^7}$, &c. and consequently $x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2b^2 - ac}{a^5} \times y^3 - \frac{5b^3 - 5abc + a^2d}{a^7} \times y^4$ &c.

For an instance of the use of this conclusion, let $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ &c. $= y$: then, a being, in this case, $=1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, &c. we shall, by substituting these values, have $x = y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}$ &c. From whence, when y is given, x will also be given; provided the value of y be sufficiently small,*

* Art. 92.

Example 2. Let there be given $ax + by + cx^2 + dxy + ey^2 + fx^3 + gx^2y + hxy^2 + iy^3 + kx^4 + lx^3y$ &c. $=0$; to find y.

By assuming $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. and proceeding as above, A will be found $= -\frac{a}{b}$, $B = -\frac{c + dA + eA^2}{b}$, $C = -\frac{dB + 2eAB + f + gA + hA^2 + iA^3}{b}$, $D = -\frac{dC + 2eAC + eB^2 + gB + 2hAB + 3iA^2B + k + lA}{b} + \frac{mA^2 + nA^3 + pA^4}{b}$ &c.

Example 3. Lastly, let $x^m + bx^{m+p} + cx^{m+2p} + dx^{m+3p} +$ &c. $= z$.

Here, in order to determine the form of the series to be assumed, let z^n be wrote for x in the given equation, according to the usual method; and then the exponents, supposing z transposed, will be 1, nm, $nm + np$, $nm + 2np$, $nm + 3np$, &c. respectively; whereof the two least (1 and nm) being made equal to each other, n is found $= \frac{1}{m}$; and the differences are $\frac{p}{m}$, $\frac{2p}{m}$, $\frac{3p}{m}$, &c. Whence the series to be assumed for x is $z^{\frac{1}{m}} + Bz^{\frac{1+p}{m}} + Cz^{\frac{1+2p}{m}} + Dz^{\frac{1+3p}{m}} +$ &c. (for it is evident, by inspection, that the co-efficient (A) of the first term must here be an unit). This series being therefore raised to the several powers of x, in the given equation, by Art. 108, and the co-efficients of the homologous terms in the new equation compared together, it will be found that, $B = -\frac{b}{m}$, $C = \frac{\overline{1+m+2p} \times b^2 - 2mc}{2m^2}$,

$$D = -\frac{\overline{2m^2 + 9mp + 9p^2 + 3m + 6p + 1} \times b^3}{6m^3} + \frac{\overline{1+m+3p} \times bc}{m^2} - \frac{d}{m}, \text{ \&c.}$$

From the general value of x, found above, innumerable theorems, for reverting particular forms of series, may be deduced.

Thus, if $x + bx^2 + cx^3 + dx^4$, &c. $= z$; then (m being $= 1$ and $n = 1$) x is $= z - bz^2 + \overline{2b^2 - c} \times z^3 - \overline{5b^3 - 5bc + d} \times z^4$ &c.

And, if $x+bx^3+cx^5+dx^7+$ &c. $=z$; (m being $=1$, and $p=2$) $x=z-bz^3+\overline{3b^2-c}\times z^5-\overline{12b^3-8cb+d}\times z^7$ &c.

Also, if $x^{\frac{1}{2}}+bx^{\frac{3}{2}}+cx^{\frac{5}{2}}+dx^{\frac{7}{2}}$ &c. $=z$; then (m being $=\frac{1}{2}$ and $p=1$) $x=z^2-2bz^4+\overline{7b^2-2c}\times z^6-\overline{30b^3-18bc+2d}\times z^8$ &c. &c.

276. It may be observed that, in all these forms of series, the first term is without a co-efficient (which renders the conclusion much more simple). Therefore, when the series to be reverted has a co-efficient in its first term, the whole equation must be first of all divided thereby: thus, if the equation was $3x-6x^2+8x^3-13x^4$ &c. $=y$; by dividing the whole by 3 it will become $x-2x^2+\frac{8x^3}{3}-\frac{13x^4}{3}$ &c. $=\frac{1}{3}y$: where, putting $z=\frac{1}{3}y$, we have, by Form. 1, $x=z+2z^2+\frac{16}{3}z^3$ &c. $=\frac{y}{3}+\frac{2y^2}{9}+\frac{16y^3}{81}$ &c.

SECTION III.

Of the Comparison of Fluents, or the Manner of finding one Fluent from another.

277. WE have, already, pointed out the most remarkable forms of fluxions whose fluents are explicable in finite terms;* and also shown the use of infinite series in approximating the values of such fluents as do not come under any of those forms:† but this last method (as is before hinted) being troublesome, and attended with many obstacles; mathematicians have therefore invented, and shown, the way of deriving one fluent from another: which is of good

* Art. 77, 78, 83, 84, & 85.

† Art. 99.

advantage when the fluent sought can be referred to one, like those in Art. 126, and 142, expressing the logarithm of a number, or the arch of a circle; since the trouble of an infinite series is, then, avoided.

As the subject here proposed is of such a nature, that it would be very tedious and difficult, if not altogether impracticable, to lay down rules and precepts for all the various cases; I shall deliver, what I have to offer thereon, by way of *Problems;* beginning with some very easy ones, for the sake of the *Young Proficient.*

PROB. I.

278. *The Fluent of* $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ *being given (by Art.* 126) *it is proposed to find, from thence, the Fluent of* $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$.

Let both the numerator and denominator of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, be multiplied by x, so that the quantity without the vinculum, in the fluxion, $\frac{x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, thus transformed, may become some constant part of the fluxion of the highest term under the vinculum: which part, in this case, being $\frac{1}{4}$, let $\frac{1}{4}$ of the fluxion of the first term under the vinculum (or $\frac{1}{2}a^2x\dot{x}$) be therefore added to the numerator, in order to have the whole, $\frac{\frac{1}{2}a^2x\dot{x}+x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, a *complete* fluxion; and then the fluent thereof, by the common rule,* will be $\frac{1}{2}\sqrt{a^2x^2+x^4} = \frac{1}{2}x\sqrt{a^2+x^2}$: but, from *this*, we are now to deduct the fluent of the quantity $\frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2+x^4}}$ $\left(=\frac{\frac{1}{2}a^2\dot{x}}{\sqrt{a^2+x^2}}\right)$ that was added: which fluent, as

* Art. 77.

that of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ is given $=$ *hyp. log.* $(x + \sqrt{a^2 + x^2})$,* will be $= \frac{1}{2} a^2 \times$ *hyp. log.* $(x + \sqrt{a^2 + x^2})$; and consequently the fluent sought $= \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \times$ *hyp. log.* $\overline{x + \sqrt{a^2 + x^2}}$. *Q. E. I.*

* Art. 126.

PROBLEM II.

279. *Let it be proposed to find the Fluent of* $\frac{x^2 \dot{x}}{\sqrt{a^2 - x^2}}$, *from that of* $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$; *given by Art.* 142.

By proceeding as above, and adding $-\frac{1}{2} a^2 x \dot{x}$ to the numerator, we have $-\frac{\frac{1}{2} a^2 x \dot{x} - x^3 \dot{x}}{\sqrt{a^2 x^2 - x^4}}$; whereof the fluent, by the *common rule*, is $-\frac{1}{2} \sqrt{a^2 x^2 - x^4}$ $(= -\frac{1}{2} x \sqrt{a^2 - x^2})$: from which, deducting the fluent of $-\frac{\frac{1}{2} a^2 x \dot{x}}{\sqrt{a^2 x^2 - x^4}}$, or $-\frac{\frac{1}{2} a^2 \dot{x}}{\sqrt{a^2 - x^2}}$ (given $= -\frac{1}{2} a^2 \times$ arc (A) whose radius is unity, and sine $= \frac{x}{a}$ †) there comes out $\frac{1}{2} a^2 A - \frac{1}{2} x \sqrt{a^2 - x^2}$. *Q. E. I.*

† Art. 142.

280. In the same manner, if the power without the vinculum, in the expression whose fluent is sought, exceeds *that* in the other expression given, by the exponent under the vinculum, or by any *multiple* of it, the required fluent may be determined, by one, or by several operations, according to the value of the said *multiple*.

Thus, if the fluent of $\frac{x^4 \dot{x}}{\sqrt{a^2 - x^2}}$ was sought; then, because the index of x, without the *vinculum*, exceeds that in $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$ by twice the exponent under the *vinculum*, the required fluent may be had from that of $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$, at two operations; by the first whereof, we have already found the fluent of $\frac{x^2\dot{x}}{\sqrt{a^2 - x^2}}$ to be $=$ $\frac{1}{2} a^2 A - \frac{1}{2} x \sqrt{a^2 - x^2}$: whence, putting this value $= B$, and proceeding as before, we also get $- \frac{1}{4} \sqrt{a^2x^6 - x^8}$ $+ \frac{3}{4} a^2 B = - \frac{1}{4} x^3 \sqrt{a^2 - x^2} - \frac{3a^2x}{8} \sqrt{a^2 - x^2} +$ $\frac{3a^4 A}{8} = \frac{3a^4 A - \overline{2x^2 + 3a^2} \times x\sqrt{a^2 - x^2}}{8}$ = the true fluent of $\frac{x^4\dot{x}}{\sqrt{a^2 - x^2}}$.

PROBLEM III.

281. *Supposing the Fluent of* $\overline{a + cz^n}|^m \times z^{pn-1} \dot{z}$ *to be given* $= A$, *to find the Fluent of* $\overline{a + cz^n}|^m \times z^{pn+n-1} \dot{z}$ $= \dot{B}$ *(where the Exponent of* z, *without the* Vinculum *is increased by the Exponent under the* Vinculum).

Let the part affected by the *vinculum* be multiplied by z^{mq}, and the part without be divided by the same quantity; then our fluxion will be transformed to $\overline{az^q + cz^{n+q}}|^m \times z^{pn+n-mq-1} \dot{z} = \dot{B}$: where let q be now so taken that the exponent $(n + q)$ of the highest power of z under the *vinculum* may be equal to $(pn + n - mq)$ that of the power without the *vinculum* $+ 1$; that is, let $q = \frac{pn}{m+1}$: then (by Art. 77, if the first term under the *vinculum* was constant, the fluent of the said expression, or its equal $\overline{az^q + cz^{n+q}}|^m \times z^{n+q-1}\dot{z}$, would be had $= \frac{\overline{az^q + cz^{n+q}}|^{m+1}}{\overline{m+1} \times \overline{n+q} \times c}$. But the fluxion hereof, supposing both terms to be variable (as they actually are) is $\overline{az^q + cz^{n+q}}|^m \times z^{n+q-1}\dot{z} + \frac{qa}{\overline{n+q} \times c} \times \overline{az^q + cz^{n+q}}|^m \times z^{q-1}\dot{z}$ (by the common rule). Therefore $\frac{\overline{az^q + cz^{n+q}}|^{m+1}}{\overline{m+1} \times \overline{n+q} \times c} - \frac{qa}{\overline{n+q} \times c} \times$ *flu. of* $\overline{az^q + cz^{n+q}}|^m \times z^{q-1}\dot{z} = B$; that is, $\frac{\overline{a+cz^n}|^{m+1} \times z^{qm+q}}{\overline{m+1} \times \overline{n+q} \times c} - \frac{qa}{\overline{n+q} \times c} \times$ *flu.* $\overline{a+cz^n}|^m \times z^{mq+q-1}\dot{z} = B$; or, by substituting for q, $\frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times nc} - \frac{pa}{\overline{m+p+1} \times c} \times$ *flu.* $\overline{a + cz^n}|^m \times z^{pn-1}\dot{z} = B$: but the *flu. of* $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ is given $= A$; therefore, lastly, $\frac{\overline{a + cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times nc} - \frac{paA}{\overline{m+p+1} \times c} = B$. *Q. E. I.*

282. If the quantity under the *vinculum* be a multinomial, $a + cz^n + dz^{2n} + ez^{3n}$, &c. Then, since the fluxion of $\overline{a + cz^n + dz^{2n} + ez^{3n} \&c.}|^{m+1} \times z^{pn}$ is $\overline{m+1} \times \overline{ncz^{n-1}\dot{z} + 2ndz^{2n-1}\dot{z} + 3nez^{3n-1}\dot{z} \&c.} \times$

$\overline{a + cz^n + dz^{2n}\ \&c.}|^m \times z^{pn} + \overline{a + cz^n + dz^{2n}\ \&c.}|^{m+1}$
$\times\, pnz^{pn-1}\dot{z} =$

$$\left\{\begin{array}{l} * \qquad\quad \overline{m+1} \times ncz^{pn+n-1}\dot{z} + \overline{m+1} \times 2ndz^{pn+2n-1}\dot{z}\&c. \\ pnaz^{pn-1}\dot{z} + pncz^{pn+n-1}\dot{z} + pndz^{pn+2n-1}\dot{z}\ \&c. \end{array}\right\} \times$$

$\overline{a + cz^n + dz^{2n}\ \&c.}|^m$, it is evident, that if the fluents of $z^{pn-1}\dot{z}$, $z^{pn+n-1}\dot{z}$, $z^{pn+2n-1}\dot{z}$ &c. drawn into the general multiplicator $\overline{a + cz^n + dz^{2n}\ \&c.}|^m$, be denoted by A, B, C, D, &c. the fluent of the whole quantity exhibited above (which fluent is $\overline{a + cz^n + dz^{2n} + ez^{3n}\ \&c.}|^{m+1} \times z^{pn}$) will also be expressed by $pnaA + \overline{p+m+1} \times ncB + \overline{p+2m+2} \times ndC + \overline{p+3m+3} \times neD$ &c. Therefore, if there be given as many of the fluents A, B, C, D, &c. as there are terms in $a + cz^n + dz^{2n} + ez^{3n}$ &c. *minus one*, that other fluent, be it which it will, will also be given from hence. Thus, if $d=0$, $e=0$, &c. and the value of A be given, we shall have $\overline{a + cz^n}|^{m+1} \times z^{pn} = pnaA + \overline{p+m+1} \times ncB$; and consequently $B = \dfrac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{p+m+1} \times nc} - \dfrac{paA}{\overline{p+m+1} \times c}$, the very same as before.

PROBLEM IV.

283. *The Fluent of* $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ *being given (as in the preceding Problem) to determine, from thence, the Fluent of* $\overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$; *supposing* v *to denote a whole positive Number.*

Let $\overline{a+cz^n}|^{m+1}$ be denoted by M; also put $p+1=\acute{p}$, $\acute{p}+1\ (p+2)=\acute{\acute{p}}$, $\acute{\acute{p}}+1\ (p+3)=\acute{\acute{\acute{p}}}$ &c. and let the fluents of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{\acute{p}n-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{\acute{\acute{p}}n-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{\acute{\acute{\acute{p}}}n-1}\dot{z}$, &c. be represented by A, B, C, D, &c. respectively. Then, since

$$\frac{Mz^{pn}}{m+p+1\times nc} - \frac{paA}{m+p+1\times c} = B$$ *(by the preceding Prob.)* it follows, from the very same argument, that

$$\frac{Mz^{\acute{p}n}}{m+\acute{p}+1\times nc} - \frac{\acute{p}aB}{m+\acute{p}+1\times c} = C$$

$$\frac{Mz^{\acute{\acute{p}}n}}{m+\acute{\acute{p}}+1\times nc} - \frac{\acute{\acute{p}}aC}{m+\acute{\acute{p}}+1\times c} = D$$

&c. &c.

Hence, by writing the value of B in the second equation, we have $\frac{Mz^{\acute{p}n}}{m+\acute{p}+1\times nc} - \frac{\acute{p}aMz^{pn}}{m+\acute{p}+1\times m+p+1\times nc^2}$ $+\frac{p\acute{p}a^2A}{m+p+1\times m+\acute{p}+1\times c^2}=C$. In the same manner, by substituting this value for C in the 3d equation, we get

$$\frac{Mz^{\acute{\acute{p}}n}}{m+\acute{\acute{p}}+1\times nc} - \frac{\acute{\acute{p}}aMz^{\acute{p}n}}{m+\acute{\acute{p}}+1\times m+\acute{p}+1\times nc^2}+$$

$$\frac{\overset{\prime\prime}{p}\overset{\prime}{p}a^2Mz^{pn}}{\overline{m+\overset{\prime\prime}{p}+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+p+1}\times nc^3} - \frac{\overset{\prime}{p}\overset{\prime\prime}{p}pa^3A}{\overline{m+p+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+\overset{\prime\prime}{p}+1}\times c^3} = D.$$

Where the law of continuation is manifest; and from whence it appears that the value of any of the quantities *B*, *C*, *D*, *E*, &c. or the fluent expressed in a general manner, will be $\frac{Mz^{vn}}{\overline{m+q+1}\times nc} -$

$$\frac{qaMz^{\overline{q-1}\times n}}{\overline{m+q+1}\times\overline{m+q}\times ncc} + \frac{q\times\overline{q-1}\times a^2Mz^{\overline{q-2}\times n}}{\overline{m+q+1}\times\overline{m+q}\times\overline{m+q-1}\times nc^3}$$

$$(v) \pm \frac{p\times\overline{p+1}\times\overline{p+2}\times\overline{p+3}\ (v)\times a^vA}{\overline{m+p+1}\times\overline{m+p+2}\times\overline{m+p+3}\ (v)\times c^v};\ \text{or,}$$

$$\frac{\overline{a+cz^n}|^{m+1}\times z^{pn}}{\overline{s+1}\times nc}\times\left(z^{vn-n} - \frac{qaz^{vn-2n}}{sc} + \frac{q.\overline{q-1}\times a^2z^{vn-3n}}{s.\overline{s-1}\times c^2}\right.$$

$$\left. - \frac{q.\overline{q-1}.\overline{q-2}\times a^3z^{vn-4n}}{s.\overline{s-1}.\overline{s-2}\times c^3}\ (v)\right) \pm \frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}$$

$$\times\frac{p+3}{t+3}\ (v)\times\frac{a^vA}{c^v}:$$ where, A = fluent of $\overline{a+cz^n}|^m$

$\times z^{pn-1}\dot{z}$, $q=p+v-1$, $s=q+m$, $t=p+m+1$; and where the sign of the last term (in which *A* is found) must be taken + or − according as *v* is an even or odd number: note, also, that the parenthesis (*v*) is put to express the number of terms, or factors, to which the series, or product, preceding it, is to be continued. The like notation is to be understood in other cases of the same kind, when they hereafter occur.

The same otherwise.

284. Let $q = p + v - 1$, and let $\overline{a+cz^n}|^{m+1} \times \overline{Rz^{qn} + Sz^{qn-n} + Tz^{qn-2n} \ldots\ldots + \Delta z^{pn}} + \beta A$, be assumed for the fluent sought: then, by taking the fluxion thereof, you will have $\overline{m+1} \times ncz^{n-1}\dot{z} \times \overline{a+cz^n}|^m \times \overline{Rz^{qn} + Sz^{qn-n} \ldots\ldots + \Delta z^{pn}} + \overline{a+cz^n}|^{m+1} \times \overline{qn\dot{z}Rz^{qn-1} + \overline{qn-n} \times \dot{z}Sz^{qn-n-1} \ldots\ldots + pn\dot{z}\Delta z^{pn-1}} + \beta \times \overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$; which must be $= \overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$ (or $\overline{a+cz^n}|^m \times z^{qn+n-1}\dot{z}$) the fluxion proposed: whence, dividing the whole equation by $\overline{a+cz^n}|^m \times z^{n-1}\dot{z}$, and transposing, there comes out

$$\left.\begin{array}{l} \overline{m+1} \times nc \times \overline{Rz^{qn} + Sz^{qn-n} + Tz^{qn-2n} \ldots\ldots + \Delta z^{pn}} \\ \overline{a+cz^n} \times \overline{qnRz^{qn-n} + \overline{qn-n} \times Sz^{qn-2n} \ldots\ldots \times pn\Delta z^{pn-n}} \\ \quad -z^{qn} \qquad * \qquad * \qquad\qquad +\beta z^{pn-n} \end{array}\right\} = 0$$

Which, reduced, and the homologous terms united, becomes

$$\left.\begin{array}{l}\overline{m+q+1} \times ncR \\ \quad -1\end{array}\right\} \times z^{qn} + \left.\begin{array}{l}\overline{m+q} \times ncS \\ \quad +qnaR\end{array}\right\} \times z^{qn-n} + \left.\begin{array}{l}\overline{m+q-1} \times ncT \\ \quad +\overline{qn-n} \times aS\end{array}\right\} \times z^{qn-2n} \ldots\ldots \left.\begin{array}{l}+pna\Delta \\ \quad +\beta\end{array}\right\} \times z^{pn-n}$$

$=0$: where, by making $\overline{m+q+1} \times ncR - 1 = 0$, $\overline{m+q} \times ncS + qnaR = 0$, &c. we have $R = \frac{1}{\overline{m+q+1} \times cn}$, $S = -\frac{qaR}{\overline{m+q} \times c}$, $T = -\frac{\overline{q-1} \times aS}{\overline{m+q-1} \times c}$; or (putting

$m+q=s$) $R=\frac{1}{\overline{s+1}\times nc}$, $S=-\frac{qaR}{sc}=-\frac{qa}{\overline{s+1}\times snc^2}$

$T=-\frac{\overline{q-1}\times aS}{\overline{s-1}\times c}=\frac{q\times\overline{q-1}\times a^2}{\overline{s+1}\times s\times\overline{s-1}\times nc^3}$, &c.

Where, because the exponent of the first term of the equation is qn $(pn+vn-n)$ and that of the last term (in which Δ and β are concerned) $=pn$, it follows that the number of coefficients to be taken as above (whereof Δ is the last) is expressed by v: from which last, the value of β is given $=-pna\,\Delta$.

But, from the law of the said coefficients, R, S, ... Δ, it appears that the value of Δ (whose place from the beginning is denoted by v) will be $=\pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{q-v+2}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{s-v+2}}\times\frac{a^{v-1}}{nc^v}=\pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{p+m+1}}\times\frac{a^{v-1}}{nc^v}$: and therefore β $(=-pna\Delta)=\pm\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}.p}{(s+1).s.(s-1)\ldots\ldots p+m+1}$ $\times\frac{a^v}{c^v}=\pm\frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}\,(v)}{t.\overline{t+1}.\overline{t+2}.\overline{t+3}\,(v)}\times\frac{a^v}{c^v}$ (putting $p+m+1=t$, as before). Now, if the several values of R, S, T...... and β, thus found, be substituted in the assumed expression, you will have the very same conclusion as in the preceding article.

Corollary I.

285. Since q is $=p+v-1$, the fluent $\overline{a+cz^n}|^{m+1}\times$ $\overline{Rz^{qn}+Sz^{qn-n}\ldots\ldots+\Delta z^{pn}}+\beta A$, given above, may be expressed by $N\times\overline{Rz^{vn-n}+Sz^{vn-2n}+Tz^{vn-3n}}$ $(v)+\beta A$; where $N=\overline{a+cz^n}|^{m+1}\times z^{pn}$, $R=$

$\frac{1}{m+p+v \times nc}$, $S = -\frac{\overline{p+v-1}.aR}{\overline{m+p+v-1}.c}$, $T = -\frac{\overline{p+v-2}.aS}{\overline{m+p+v-2}.c}$: and, where the co-efficient (β) of the given fluent (A) will always be expressed by the last of the quantities R, S, T . . . $\triangle$, multiplied by $-pna$: this is evident, because it is found that $\beta = -pna\triangle$. And the same thing will also appear from the several particular cases (in Art. 283) for the values of B, C, and D: in each of which the co-efficient of the last term (where A is concerned) is to *that* of the term immediately preceding it, in the constant ratio of pa to $\frac{1}{n}$, or of pna to unity.

COROLLARY II.

286. If the value of c be negative, the general fluent (in Art. 283) when $a+cz^n=0$ (provided $m+1$, n, and p be positive) will become barely $= \pm \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{a^v A}{c^v}$; because, in this circumstance, all the terms multiplied by $\overline{a+cz^n}|^{m+1}$ entirely vanish. If, therefore, b be written for $-c$ (to render the expression more commodious) we shall have $\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{a^v A}{b^v}$ for the true fluent of $\overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, generated while bz^n, from nothing, becomes $= a$: where A denotes the fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, generated in the same time; and where

$t = p + m + 1$. Hence it follows that the fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z} \times \overline{e + fz^n + gz^{2n} + hz^{3n}\ \&c.}$ (where e, f, g are any given quantities) will be $= A \times \overline{e + \frac{paf}{tb} + \frac{p.\overline{p+1}.a^2g}{t.\overline{t+1}.b^2} + \frac{p.\overline{p+1}.\overline{p+2}.a^3h}{t.\overline{t+1}.\overline{t+2}.b^3}} + \&c.$ in the forementioned circumstance.

PROBLEM. V.

287. *The Fluent* (A) *of* $\overline{a+cz^n}|^m \times z^{pn+1}\dot{z}$ *being given, to find the Fluent of* $\overline{a+cz^n}|^{m+r} \times z^{pn-1}\dot{z}$; *supposing* r *to denote a whole positive Number.*

Since $\overline{a+cz^n}|^{m+1} = \overline{a+cz^n}|^m \times \overline{a+cz^n}$, it is evident that $\overline{a+cz^n}|^{m+1} \times z^{pn-1}\dot{z} = \overline{a+cz^n}|^m \times az^{pn-1}\dot{z} + \overline{a+cz^n}|^m \times cz^{pn+n-1}\dot{z}$: whose fluent (by Prob. 3) is $aA + \frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times n} - \frac{paA}{m+p+1} = \frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{p+m+1} \times n} + \frac{\overline{m+1} \times aA}{p+m+1}$. In like manner, if this fluent, of $\overline{a+cz^n}|^{m+1} \times z^{pn-1}\dot{z}$, be denoted by B, that of $\overline{a+cz^n}|^{m+2} \times z^{pn-1}\dot{z}$ by C, &c. it will appear that $\frac{\overline{a+cz^n}|^{m+2} \times z^{pn}}{\overline{p+m+2} \times n} + \frac{\overline{m+2} \times aB}{p+m+2} = C$; $\frac{\overline{a+cz^n}|^{m+3} \times z^{pn}}{\overline{p+m+3} \times n} + \frac{\overline{m+3} \times aC}{p+m+3} = D$, &c. Whence, by substituting these values, one by one, as in the pre-

ceding problem, and putting $Q = a + cz^n$, we get
$$C = \frac{Q^{m+2} z^{pn}}{p + m + 2.n} + \frac{\overline{m+2} \times aQ^{m+1} z^{pn}}{\overline{p + m + 2}.\overline{p + m + 1}.n} + \frac{\overline{m + 2}.\overline{m + 1} \times a^2 A}{\overline{p + m + 2}.\overline{p + m + 1}}; \; D = \frac{Q^{m+3} z^{pn}}{p + m + 3.n} +$$
$$\frac{\overline{m + 3} \times aQ^{m+2} z^{pn}}{\overline{p+m+3}.\overline{p+m+2}.n} + \frac{\overline{m+3}.\overline{m+2} \times a^2 Q^{m+1} z^{pn}}{\overline{p+m+3}.\overline{p+m+2}.\overline{p+m+1}.n}$$
$$+ \frac{\overline{m+3}.\overline{m+2}.\overline{m+1}.a^3 A}{\overline{p+m+3}.\overline{p+m+2}.\overline{p + m + 1}},$$
&c. Whence it is evident, by inspection, that the fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn-1}\dot{z}$, expressed in a general manner, will be
$$\frac{Q^{m+r} z^{pn}}{p+m+r.n} + \frac{\overline{m+r} \times aQ^{m+r-1} z^{pn}}{\overline{p+m+r} \times \overline{p+m+r-1}.n}$$
&c. Which, by putting $m+r=f$, $p+m+r=g$, and making $Q^{m+1} \times z^{pn}$ a general multiplicator, will be reduced to $Q^{m+1} \times$
$$z^{pn} \times \overline{\frac{Q^{r-1}}{gn} + \frac{f \times aQ^{r-2}}{g.\overline{g-1}.n} + \frac{f.\overline{f-1} \times a^2 Q^{r-3}}{g.\overline{g-1}.\overline{g-2}.n} (r)} +$$
$$\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \times \frac{m+3}{p+m+3} \; (r) \; a^r A;$$
where it appears (from the foregoing values of B, C, and D) that the co-efficient of A is always equal to the last term of the preceding series, multiplied by $\overline{m+1} \times na$ (instead of $Q^{m+1} z^{pn}$). *Q. E. I.*

COROLLARY.

288. If c be negative, so that Q, or its equal, $a+cz^n$, may become equal to nothing, the fluent will,

in that circumstance, be barely $= \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \times \frac{m+3}{p+m+3} (r) \times a^r A$; provided the values of $m+1$, p, and n are positive: or, if c, p and n be positive, and $m+r+p$ negative, the same expression will exhibit the true value of the whole fluent, generated while z, from nothing, becomes infinite.

PROBLEM VI.

289. *The same being given as in the preceding Problems; it is proposed to find the Fluent of* $\overline{a+ez^n}|^{m-r} \times z^{pn-1} \dot{z}$.

If $-r$ be written instead of r, in the last article, we shall have $m-r=f$, $p+m-r=g$, and $Q^{m+1} z^{pn} \times \overline{\frac{Q^{-r-1}}{gn} + \frac{f \times . a Q^{-r-2}}{g \cdot g-1 . n}} (-r) + \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} (-r) \times a^{-r} A$, expressing the required fluent in this case.

But $\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}$, &c. continued to $-r$ factors, signifies the same thing as the product continued downwards, or the contrary way, to r factors, according to the same law: and therefore is $= \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2} (r)$. After the same manner we have $\frac{Q^{-r-1}}{gn} + \frac{f \times a Q^{-r-2}}{g \cdot \overline{g-1} . n} (-r) =$

$$-\frac{Q^{-r}}{\overline{f+1} . na} - \frac{\overline{g+1} . Q^{-r+1}}{\overline{f+1} . \overline{f+2} . na^2} - \frac{\overline{g+1} . \overline{g+2} . Q^{-r+2}}{\overline{f+1} . \overline{f+2} . \overline{f+3} . na^3},$$

(r) and consequently the fluent itself $= Q^{m+1} z^{tn} \times$

$$\frac{-Q^{-r}}{\overline{f+1}.na} - \frac{\overline{g+1}.Q^{1-r}}{\overline{f+1}.\overline{f+2}.na^2} - \frac{\overline{g+1}.\overline{g+2}.Q^{2-r}}{\overline{f+1}.\overline{f+2}.\overline{f+3}.na^3}\ (r)$$

$$+\frac{p+m}{m}\times\frac{p+m-1}{m+1}\times\frac{p+m-2}{m-2}(r)\times\frac{A}{a^r}. \qquad Q.\ E.\ I.$$

Corollary.

290. It appears from hence that the co-efficient of A, the given fluent, will always be equal to *that* of the last term of the preceding series, multiplied by $\overline{p+m}\times n$: for, seeing the co-efficient of the said last term (whose distance from the first, inclusive, is denoted by r) must be $\frac{\overline{g+1}.\overline{g+2}.\overline{g+3}\ldots\overline{g+r-1}}{\overline{f+1}.\overline{f+2}.\overline{f+3}\ldots\overline{f+r}}\times\frac{1}{na^r}$ (by the law of the series) where $f+r=m$ and $g+r-1=p+m-1$ (as appears from above), it follows, by inverting the order of both progressions, that $\frac{\overline{p+m-1}.\overline{p+m-2}.(r-1)}{m.\overline{m-1}.\overline{m-2}.(r)}\times\frac{1}{na^r}$ will also express the same co-efficient: which, multiplied by $\overline{p+m}\times n$, gives $\frac{\overline{p+m}.\overline{p+m-1}.\overline{p+m-2}\,(r)}{m.\overline{m-1}.\overline{m-2}\,(r)}\times\frac{1}{a^r}$, the very co-efficient of A, above determined. The use of this conclusion will be seen in what follows.

PROBLEM VII.

291. *The same being, still, given; to find the Fluent of*
$$\overline{a+cz^n}|^m \times z^{pn-vn-1}\dot{z}.$$

By proceeding as in the last Problem, the required fluent of $\overline{a+cz^n}|^m \times z^{pn-vn-1}$ is derived from *that* of $\overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$ (*given by Prob.* 4) and comes out

$$= Q^{m+1} z^{pn} \times \left(\frac{z^{-vn}}{\overline{q+1}.na} - \frac{\overline{s+2}.cz^{n-vn}}{\overline{q+1}.\overline{q+2}.na^2} + \frac{\overline{s+2}.\overline{s+3}.c^2z^{2n-vn}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.na^3} (v) \right) \pm \frac{t-1}{p-1} \times \frac{t-2}{p-2} \times \frac{t-3}{p-3} (v)$$
$$\times \frac{c^v A}{a^v}:$$ where, $Q = a + cz^n$, $q = p - v - 1$, $s = m + q$, $t = p + m + 1$: and where the co-efficient of A is equal to that of the last of the preceding terms, multiplied by $-\overline{m+p} \times nc$. If the manner of deducing the required fluent, in this, and the last, problem, should not appear sufficiently plain and satisfactory to the beginner; the same conclusions may be, otherwise, brought out; by finding A, in terms of B, C, or D, from the several particular equations in Art. 283, or, by assuming a descending series, instead of an ascending one. *Vide* Art. 284.

PROBLEM VIII.

292. *The same being, still, given; to find the Fluent of*
$$\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}.$$

Let the fluent of $\overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$ (*given by Prob.* 4) be denoted by B, and that required, by F:

then, if $p+v$ be put $= \acute{p}$, the value of F (the fluent of $\overline{a+cz^n}|^{m+r} \times z^{\acute{p}n-1}\dot{z}$) will be given from that of B (the fluent of $\overline{a+cz^n}|^m \times z^{\acute{p}n-1}\dot{z}$) by writing B for A and $\acute{p}$ for p, in Art. 287. Whence we get $F=Q^{m+1}$

$$z^{\acute{p}n} \times \overline{\frac{Q^{r-1}}{gn} + \frac{fa\,Q^{r-2}}{g.\overline{g-1}.n} + \frac{f.\overline{f-1}.a^2Q^{r-3}}{g.\overline{g-1}.\overline{g-2}.n}\ (r)} + \frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2} + \frac{m+3}{\acute{p}+m+3}\ (r) \times a^r B:$$

where $\acute{p}=p+v$, $f=m+r$, $g\,(=\acute{p}+m+r)=p+m+v+r$, and $Q=a+cz^n$.

Which fluent, by substituting the value of B (in Prob. 4) becomes

$$F = Q^{m+1} z^{\acute{p}n} \times \overline{\frac{Q^{r-1}}{gn} + \frac{fa\,Q^{r-2}}{g.\overline{g-1}.n} + \frac{f.\overline{f-1}.a^2Q^{r-3}}{g.\overline{g-1}.\overline{g-2}.n}\ (r)} + \frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2}\ (r)$$

$$\times a^r \times Q^{m+1} z^{pn} \times \overline{\frac{z^{vn-n}}{\overline{s+1}.nc} - \frac{qaz^{vn-2n}}{\overline{s+1}.snc^2} + \frac{q.\overline{q-1}a^2z^{vn-3n}}{\overline{s+1}.s.\overline{s-1}.nc^3}}$$

$$(v) \pm \frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2}\ (r) \times a^r \times \frac{p}{t} \times \frac{p+1}{t+1}$$

$$(v) \times \frac{a^v A}{c^v}:$$

where $q=p+v-1$, $s=m+q=m+p+v-1$, and $t=p+m+1$; and where the sign of the last term is $+$ or $-$ according as v is an even or odd number. Q. E. I.

292. If the last term of the first series, exclusive of the general multiplicator $Q^{m+1} z^{pn}$, be denoted by β, the multiplicator, $\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}$ $(r) \times a^r$, to the second series will be $= \overline{m+1} \times n\,a\,\beta$;* and therefore the first term of this series, including its multiplicators, is $= \frac{\overline{m+1}.\,a\beta Q^{m+1}\,z^{pn+vn}}{\overline{s+1}.\,cz^n}$: which, if R be put to denote the last term $\beta\,Q^{m+1}\,z^{pn+vn}$ of the first series (with its multiplicator) will be expounded by $\frac{\overline{m+1}.\,a\,R}{\overline{s+1}.\,cz^n}$. Hence it follows, that the fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$, given above, will also be truly expressed by $\frac{Q^{m+r} \times z^{pn+vn}}{gn} + \frac{f}{g-1} \times \frac{aH}{Q} + \frac{f-1}{g-2} \times \frac{aI}{Q} + \frac{f-2}{g-3} \times \frac{aK}{Q}\,(r) + \frac{\overline{m+1}}{\overline{s+1}} \times \frac{aR}{cz^n} - \frac{q}{s} \times \frac{aS}{cz^n} - \frac{q-1}{s-1} \times \frac{aT}{cz^n} - \frac{q-2}{s-2} \times \frac{aV}{cz^n}\,(v) \pm \frac{\overline{m+1}.\,\overline{m+2}.\,\overline{m+3}\,(r) \times p.\,\overline{p+1}.\,\overline{p+2}\,(v)}{\overline{p+m+1}.\,\overline{p+m+2}\,(r) \times t.\,\overline{t+1}.\,\overline{t+2}\,(v)} \times \frac{a^{v+r}A}{c^v}$; where H, I, K, L ..... R, S, T, V, &c. represent the terms immediately preceding those where they stand, under their proper signs: R being the last term of the first series; also $f = m+r$, $g = m+r+p+v$, $q = p+v-1$, $s = m+q$, $t = m+p+1$, and $Q = a+cz^n$.

* Art. 287.

COROLLARY II.

293. Since the divisor, $\overline{p+m+1}.\ \overline{p+m+2}\ (r) \times t.(t+1).(t+2)\ (v)$, of the last term of the fluent (by substituting for t and p &c.) is $= \overline{p+m+1}.\ \overline{p+m+2}\ (v) \times \overline{p+v+m+1}.\ \overline{p+v+m+2}\ (r)$: where, the last factor $(p+m+v)$ of the first progression, is less by unity than the first factor of the second; it is evident that the said second progression is only a continuation of the first to r *more* factors: and so, the last term of the fluent, where A is found, is truly expressed by $\pm$

$$\frac{p.\overline{p+1}.\overline{p+2}\ (v) \times \overline{m+1}.\overline{m+2}.\overline{m+3}\ (r)}{\overline{m+p+1}.\overline{m+p+2}.\overline{m+p+3}\ (v+r)} \times \frac{a^{v+r}A}{c^v}.$$

Hence it follows, that the fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$, or that of $\overline{a-bz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$ (making $c = -b$) will, when $a-bz^n$ becomes equal to nothing, be barely $=$

$$\frac{p.\ \overline{p+1}.\ \overline{p+2}\ (v) \times \overline{m+1}.\ \overline{m+2}.\ \overline{m+3}\ (r)}{\overline{m+p+1}.\ \overline{m+p+2}.\ \overline{m+p+3}\ (v+r)} \times \frac{a^{v+r}A}{b^v}$$

A being the fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, in that circumstance, v and r whole positive numbers, and p and $m+1$ any positive numbers, either whole or broken.

SCHOLIUM.

294. If the fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn-1}\dot{z}$ (given by Prob. 5) be denoted by C; then (F) the fluent of $\overline{a+cz^n}|^{m'} \times z^{pn+vn-1}\dot{z}$ (where $m'=m+r$) will be had, from C (by Prob. 4) according to a new form, dif-

ferent from those already given. And, by following the same method, the fluents of $\overline{a+cz^n}|^{m-r} \times z^{pn+vn-1}\dot{z}$, $\overline{a+cz^n}|^{m+r} \times z^{pn-vn-1}\dot{z}$, and $\overline{a+cz^n}|^{m-r} \times z^{pn-vn-1}\dot{z}$ may also be found, each, according to two different forms, from a combination of the corresponding cases in the foregoing Problems.

But as it is extremely tiresome to repeat the same thing again and again, where such a number of symbols are necessarily concerned, I shall here put down one solution to each case (because of their use) leaving the process and the other forms (which contain no new difficulty) to *those* who will be at the trouble to set about them.

1°. The fluent of $\overline{a+cz^n}|^{m-r} \times z^{pn+vn-1}\dot{z}$ is =

$$-\frac{Q^{m-r+1} \times z^{pn+vn}}{\overline{f+1}.\,na}+\frac{g+1}{f+2}+\frac{QH}{a}+\frac{g+2}{f+3}+\frac{QI}{a}\ (r)$$

$$-\frac{QR}{cz^n}-\frac{q}{s}\times\frac{aS}{cz^n}-\frac{q-1}{s-1}\times\frac{aT}{cz^n}-\frac{q-2}{s-2}\times\frac{aV}{cz^n}\ (v)$$

$$+\frac{s+1}{m}\times\frac{s}{m-1}\times\frac{s-1}{m-2}(r)\times\frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}(v)\times\frac{a^{-r}A}{\overline{-c}|^{r}}.$$

Where *H*, *I*, *K*, *L*....*R*, *S*, *T*, &c. denote the terms immediately preceding those where they stand, under their proper signs; *R* being the last term of the first series, also $Q=a+cz^n$, $f=m-r$, $g=p+m+v-r$, $q=p+v-1$, $s=m+p+v-1$, $t=p+m+1$, and $A=$ the given fluent of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$.

2°. The fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn-vn-1}\dot{z}$ is $=$

$$\frac{Q^{m+r+1} \times z^{pn-vn}}{\overline{q+1}.na} - \frac{s+2}{q+2} \times \frac{Hcz^n}{a} - \frac{s+3}{q+3} \times \frac{Icz^n}{a} \; (v)$$

$$-\frac{Rcz^n}{Q} + \frac{f}{g-1} \times \frac{aS}{Q} + \frac{f-1}{g-2} \times \frac{aT}{Q} + \frac{f-2}{g-3} \times \frac{aV}{Q} \quad (r)$$

$$+\frac{s+2}{q+1} \times \frac{s+3}{q+2} \times \frac{s+4}{q+3}(v) \times \frac{m+1}{t} \times \frac{m+2}{t+1} \times \frac{m+3}{t+2}(r) \times \frac{\overline{-c}|^{v}A}{a^{v-r}} :$$

where $q=p-v-1$, $s=m+r+q$, $f=m+r$, $g=p+m+r$, and the rest as in the preceding case.

3°. The fluent of $\overline{a+cz^n}|^{m-r} \times z^{pn-vn-1}\dot{z}$ is $=$

$$-\frac{Q^{m-r+1} \times z^{pn-vn}}{\overline{f+1}.na} + \frac{g+1}{f+2} \times \frac{QH}{a} + \frac{g+2}{f+3} \times \frac{QI}{a} \; (r)$$

$$-\frac{s+1}{q+1} \times \frac{QR}{a} - \frac{s+2}{q+2} \times \frac{Scz^n}{a} - \frac{s+3}{q+3} \times \frac{Tcz^n}{a} \; (v)$$

$$+\frac{\overline{t-1}.\overline{t-2}.\overline{t-3}.\overline{t-4}.\overline{t-5}\,(r+v)}{m.\overline{m-1}.\overline{m-2}(r) \times \overline{p-1}.\overline{p-2}.\overline{p-3}(v)} \times \frac{\overline{-c}|^{v}A}{a^{r+v}}.$$

In which $f=m-r$, $g=m+p-r-v$, $q=p-v-1$, $s=q+m$, and the rest as before.

295. From what has been delivered in this section, the fluents of various forms of fluxions may be exhibited by means of circular arcs and logarithms. For, since the fluents of $\overline{a+cz^n}|^{-1} \times z^{\frac{1}{2}n-1}\dot{z}$, $\overline{a+cz^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}$, and $\overline{a+cz^n}|^{-\frac{1}{2}} \times z^{-1}\dot{z}$ (which I call original ones) are all of them explicable by one or the other of these two kinds of quantities (as will appear farther on) *those* of $\overline{a+cz^n}|^{-1\pm r} \times z^{\frac{1}{2}n\pm vn-1}\dot{z}$, $\overline{a+cz^n}|^{-\frac{1}{2}\pm r} \times z^{\frac{1}{2}n\pm vn-1}\dot{z}$, and $\overline{a+cz^n}|^{-\frac{1}{2}\pm r} \times z^{\pm vn-1}\dot{z}$ will also be given from thence, by the fore-

going theorems. Whence the most useful forms of fluents in *Cotes's Harmonia Mensurarum* will be obtained, besides some others, more general than any, of the same kind, put down by that sagacious author.

Here follow a few examples of some of the most useful cases.

EXAMPLE I.

296. *Let the Fluxion given be* $\frac{z^{2v}\dot{z}}{\sqrt{d^2 \pm z^2}}$ (or $\overline{d^2+z^2}|^{-\frac{1}{2}} \times z^{2v}\dot{z}$) *v being any whole positive Number.*

Then, the fluent of $\overline{d^2 \pm z^2}|^{-\frac{1}{2}} \times \dot{z}$, or $\frac{\dot{z}}{\sqrt{d^2 \pm z^2}}$ being $=$ *hyp. log.* $\frac{z + \sqrt{d^2 + z^2}}{d}$; or, equal to the

* Art. 126 & 142. arch whose sine is $\frac{z}{d}$ and radius unity;* according as the second term, in $d^2 \pm x^2$, is positive or negative; let A be, therefore, taken to denote the said arch, or logarithm; and let $\overline{d^2 \pm z^2}|^{-\frac{1}{2}} \times \dot{z}$ be compared with $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ (whose fluent is, all along, supposed to be given $= A$) and you will have $a=d^2$, $c=\pm 1$, $n=2$, $m=-\frac{1}{2}$, $2p-1=0$, and therefore $p=\frac{1}{2}$: whence, by substituting those values in Art. 283, we likewise get $q\ (p+v-1) = \frac{2v-1}{2}$, $s\ (m+q) = v-1$, $t\ (m+p+1) = 1$; and, consequently, the fluent sought $= \overline{d^2 \pm z^2}|^{\frac{1}{2}} \times (\pm \frac{z^{2v-1}}{2v} - \frac{\overline{2v-1}.\ d^2 z^{2v-3}}{2v.\ \overline{2v-2}} \pm \frac{\overline{2v-1}.\ \overline{2v-3}.\ d^4 z^{2v-5}}{2v.\ \overline{2v-2}.\ \overline{2v-4}} - \frac{\overline{2v-1}.\ \overline{2v-3}.\ \overline{2v-5}.\ d^6 z^{2v-7}}{2v.\ \overline{2v-2}.\ \overline{2v-4}.\ \overline{2v-6}})$

$(v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} (v) \times d^{2v} A$: in which the last term is negative, when the given fluxion is $\frac{z^{2v}\dot{z}}{\sqrt{d^2 + z^2}}$, and v at the same time, an odd number; but in all other cases, affirmative.

EXAMPLE II.

297. *Let* $z^{2v}\dot{z}\sqrt{d^2 \pm z^2}$ (or $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$) *be propounded.*

Here, denoting the fluent of $\overline{d^2 \pm z^2}|^{-\frac{1}{2}}\dot{z}$ by A (as above) and comparing $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$, with $\overline{a + cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$ (*vide* Prob. 8) we have $r=1$, and the rest as in the last example: whence also $p\ (p+v) = v+\frac{1}{2}$, $f\ (m+r) = \frac{1}{2}$, $g=v+1$, $Q=d^2 \pm z^2$, and the fluent itself $= \frac{z^{2v+1}\sqrt{d^2 \pm z^2}}{2v+2} \pm \frac{d^2 R}{2vz^2} \mp \frac{\overline{2v-1}.\,d^2 S}{\overline{2v-2}.\,z^2} \mp \frac{\overline{2v-3}.\,d^2 T}{\overline{2v-4}.\,z^2} (1+v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} (v) \times \frac{d^{2v+2}A}{2v+2}$* ($R$, S, T, &c. being the preceding terms with their signs) $= \frac{\sqrt{d^2 \pm z^2}}{2v+2} \times (z^{2v+1} \pm \frac{d^2 z^{2v-1}}{2v} - \frac{\overline{2v-1}.\,d^4 z^{2v-3}}{2v.\,\overline{2v-2}} \pm \frac{\overline{2v-1}.\,\overline{2v-3}.\,d^6 z^{2v-5}}{2v.\,\overline{2v-2}.\,\overline{2v-4}})$ $(v+1) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} (v) \times \frac{d^{2v+2}A}{2v-2}$: where the sign of the last term must be regulated as in the

* Art. 292.

preceding example. If the fluent of $\frac{z^{-m}\dot{z}}{\sqrt{d^2 \pm z^2}}$, or of $z^{-m}\dot{z}\sqrt{d^2 \pm z^2}$ (in which the exponent is negative) be required; the answer will be had in finite terms, independent of A, by Art. 85.

EXAMPLE III.

298. *Wherein the Fluxion proposed is* $\overline{d^n - z^n}|^{-\frac{1}{2}+r} \times z^{\frac{1}{2}n+vn-1}\dot{z}$; *r and v being any whole positive Numbers.*

Since the fluent of $\overline{d^n - z^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}$ (as will appear hereafter) is truly expressed by $\frac{2}{n} \times$ *arch*, whose sine is $\frac{z^{\frac{1}{2}n}}{d^{\frac{1}{2}n}}$ and radius unity, let this value be denoted by A; and then, by writing d^n for a, -1 for c, $-\frac{1}{2}$ for m, and $\frac{1}{2}$ for p, in Art. 292, we shall have f $(m+r) = \frac{2r-1}{2}$, $g\,(m+p+r+v) = r+v$, $q\,(p+v-1) = \frac{2v-1}{2}$, $s\,(m+q) = v-1$, $t\,(p+m+1) = 1$, Q $(a+cz^n) = d^n - z^n$, and the fluent itself equal to

$$\frac{Q^{-\frac{1}{2}}z^{vn+\frac{1}{2}n}}{\overline{r+v}.n} + \frac{\overline{2r-1}}{r+v-1} \times \frac{\frac{1}{2}d^nH}{Q} + \frac{2r-3}{r+v-2} \times \frac{\frac{1}{2}d^nI}{Q} + \frac{2r-5}{r+v-3} \times \frac{\frac{1}{2}d^nK}{Q}\,(r) - \frac{\frac{1}{2}d^nR}{vz^n} + \frac{2v-1}{v-1} \times \frac{\frac{1}{2}d^nS}{z^n} + \frac{2v-3}{v-2} \times \frac{\frac{1}{2}d^nT}{z^n}\,(v) + \frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\,(r+v)}$$

* Art. 293. $^* \times d^{n+vn}A$: in which H, I, K . . . R, S, T, &c. denote the preceding terms, with their signs; R being

the last term of the first series. Hence, because all the terms but the last vanish, when $Q=0$, it follows that the whole fluent of $\overline{d^n - z^n}|^{r-\frac{1}{2}} \times z^{vn+\frac{1}{2}n-1}\dot{z}$, generated while z, from nothing, becomes equal to d, is truly expressed by $\frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\,(r+v)} \times d^{rn+vn}A$, or by $\frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\,(r+v)} \times \frac{d^{rn+vn}G}{n}$; G being the semi-periphery of the circle whose radius is unity.

EXAMPLE IV.

299. *Let it be required to find the whole Fluent of* $\frac{\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}}{\overline{d+kz^n}|^\beta}$, *generated while* bz^n, *from Nothing, becomes* $=a$; *that of* $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$ *being given* $(=A)$.

Here, by expanding $\overline{d+kz^n}|^{-\beta}$, our given fluxion becomes $= \overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$ into $d^{-\beta} \times (1 - \frac{\beta k z^n}{d} + \frac{\beta.\overline{\beta+1}.k^2z^{2n}}{1.2.d^2} - \frac{\beta.\overline{\beta+1}.\overline{\beta+2}.k^3z^{3n}}{1.2.3.d^3}$ &c.) Which series being compared with $e+fz^n+gz^{2n}$ &c. (*vide* Art. 286) we have $e=1$, $f=-\frac{\beta k}{d}$, $g=\frac{\beta.\overline{\beta+1}.k^2}{1.2.d^2}$, &c. and consequently the fluent sought (by substituting these values) equal to $\frac{A}{a^\beta}$ into $1 -$

$\frac{p}{t} \times \frac{\beta}{1} \times \frac{ak}{bd} + \frac{p}{t} \cdot \frac{p+1}{t+1} \times \frac{\beta}{1} \cdot \frac{\beta+1}{2} \times \overline{\frac{ak}{bd}}|^2 -$

$\frac{p}{t} \cdot \frac{p+1}{t+1} \cdot \frac{p+2}{t+2} \times \frac{\beta}{1} \cdot \frac{\beta+1}{2} \cdot \frac{\beta+2}{3} \times \overline{\frac{ak}{bd}}\Big|^3$ + &c. (t being $= p+m+1$).

Here the values of $m+1$, n and p are supposed positive;* and it is requisite that $1 + \frac{ak}{bd}$ should also be positive; otherwise the fluent will fail. Although the series brought out above runs on to infinity, yet it may be summed, in many cases: thus, if the given fluxion be $\frac{\overline{a-bz^n}\big|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}}{d+kz^n}$; then, the foresaid series becoming $1 - \frac{1}{2} \times \frac{ak}{bd} + \frac{1}{2} \times \frac{3}{4} \times \overline{\frac{ak}{bd}}\Big|^2$ − &c. its sum will be $\overline{1+\frac{ak}{bd}}\Big|^{-\frac{1}{2}}$: and consequently $\frac{A}{d} \times \overline{1+\frac{ak}{bd}}\Big|^{-\frac{1}{2}}$ = the fluent sought: where A (the *whole* fluent of $\overline{a-bz^n}\big|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}$) being $= \frac{1}{n\sqrt{b}} \times$ semi-periphery of the circle whose radius is unity, the fluent given above will, therefore, be $= \frac{1}{n\sqrt{bd^2 + adk}}$ × by the same semi-periphery. If the reader is desirous to see a further application of the summation of series, to the finding of fluents, I must refer him to my *Dissertations* (where it is handled in a general manner) having neither room nor inclination to treat of it here.

* Art. 286.

SECTION IV.

Of the Transformation of Fluxions.

301. By the Transformation of Fluxions may be understood, the reducing *any* fluxional quantity to a different, or more commodious, form; according to which sense, a great part of the second section would properly fall under this head. But, *what* is here proposed, and *what* is commonly meant by the transformation of fluxions, *is*, the method of ordering those kinds of expressions which involve one variable quantity *only* with its fluxion; which, yet, are so affected by radical signs, that the fluent, without an infinite series, would be impracticable, were it not for a new substitution, or some other kind of transformation, whereby the given fluxion is rendered more manageable.

Something of this sort has been already touched upon in Art. 83. And in what follows I shall farther point out and exemplify the principal cases wherein such a *procedure* will be of service.

302. *If the number of dimensions of the variable quantity, without the* vinculum, *increased by unity, be some aliquot part, or parts, of the dimensions of the same quantity, under the* vinculum, *the fluxion will be reduced to a better form by substituting for that power of the variable quantity, which arises by dividing its exponent, under the* vinculum, *by the denominator of the fraction expressing the said* aliquot *part, or parts.*

Thus, if the fluxion propounded be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n \pm z^n}}$; by substituting $x = z^{\frac{1}{2}n}$, and taking the fluxion of both sides of the equation, we have $\dot{x} = \frac{1}{2} n z^{\frac{1}{2}n-1}\dot{z}$; and therefore $z^{\frac{1}{2}n-1}\dot{z} = \frac{\dot{x}}{\frac{1}{2}n}$: which value, with that of z^n, being wrote for their equals, in the given fluxion, it will be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{c^n \pm x^2}}$: which, putting $a = c^{\frac{1}{2}n}$ (to make the terms homologous), is also expressed by $\frac{\dot{x}}{\frac{1}{2}n\sqrt{a^2 \pm x^2}}$: whereof the fluent will be given by Art. 126, or Art. 142, according as the sign of x^2 is positive or negative.

303. If the power of the variable quantity under the *vinculum* has a co-efficient, it will be best to bring that co-efficient without the *vinculum*.

Ex. 2. Where let the fluxion given be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a+cz^n}}$: which, by bringing c without the *vinculum*, becomes $\frac{z^{\frac{1}{2}n-1}\dot{z}}{c^{\frac{1}{2}}\sqrt{\frac{a}{c}+z^n}}$: from whence, by putting $x = z^{\frac{1}{2}n}$ and proceeding as above, we get $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^2}}$: whose fluent, by Art. 126, is $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}} \times$ hyp. log. $(x + \sqrt{\frac{a}{c}+x^2})$. This, by restoring z, becomes $\frac{2}{nc^{\frac{1}{2}}} \times$ hyp. log $(z^{\frac{1}{2}n} + \sqrt{\frac{a}{c}+z^n})$. Which, corrected (by supposing it $= 0$ when $z = 0$) gives, at length, $\frac{2}{nc^{\frac{1}{2}}} \times \overline{\text{hyp. log. } (z^{\frac{1}{2}n} + \sqrt{\frac{a}{c}+z^n}) - \text{hyp. log. } \sqrt{\frac{a}{c}}} = \frac{2}{nc^{\frac{1}{2}}} \times$ hyp. log. $\left(\sqrt{\frac{cz^n}{a}} + \sqrt{1+\frac{cz^n}{a}}\right)$ for the true fluent of the quantity proposed.

But, when c is a negative quantity, this fluent fails, because the square root of c is to be extracted. In this case $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^2}}$ must be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{-c}\times\sqrt{\frac{a}{-c}-x^2}}$: and then its fluent (by Art. 142) will be had $=\frac{1}{\frac{1}{2}n\sqrt{-c}}\times$ the arch of a circle whose radius is unity, and right-sine $=\frac{x}{\sqrt{\frac{a}{-c}}}=\sqrt{\frac{-cz^n}{a}}$.

Ex. 3. Let the given fluxion be $\frac{\dot{z}}{z\sqrt{a+cz^n}}$. Which, by bringing c without the *vinculum*, and putting $x=z^{\frac{1}{2}n}$, is transformed to $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}x\sqrt{\frac{a}{c}+x^2}}$: whereof the fluent, by Art. 126, is $\frac{1}{n\sqrt{a}}\times$ hyp. log. $\frac{\sqrt{\frac{a}{c}}-\sqrt{\frac{a}{c}+x^2}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{a}{c}+x^2}}=\frac{1}{n\sqrt{a}}\times$ hyp. log. $\frac{\sqrt{a}-\sqrt{a+cz^n}}{\sqrt{a}+\sqrt{a+cz^n}}$. But here, when c is positive, the numerator will be negative; in which case it will be proper to change its signs, and express the fluent by $\frac{1}{n\sqrt{a}}\times$ hyp. log. $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{a+cz^n}+\sqrt{a}}$. That, such an alteration of the signs can make no difference in the fluxion, is evident from the nature of logarithms; because the fluxion of the log. of $-x$ $\left(=\frac{-\dot{x}}{-x}=\frac{\dot{x}}{x}\right)$ is the same with *that* of the hyp. log. of x. It will be proper to observe farther, that, instead of the logarithm above derived, any one of the following equal quantities may be taken; *viz.* hyp. log. $\frac{\overline{\sqrt{a+cz^n}-\sqrt{a}}^2}{cz^n}$ (found by multiplying both the numerator and denominator of the foresaid logarithm by $\sqrt{a+cz^n}-\sqrt{a}$) $=2\times$ hyp. log. $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{cz^n}}$ (by the nature of logarithms) $=2\times$ hyp. log. $\frac{\sqrt{cz^n}}{\sqrt{a+cz^n}+\sqrt{a}}$ (by multiplying, equally, by $\sqrt{a+cz^n}+\sqrt{a}$) But, take which of these forms you will, the fluent fails when a is negative; because the general multiplicator $\frac{1}{n\sqrt{a}}$ is then impossible. In this case the fluent of $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\times x\sqrt{\frac{a}{c}+x^2}}$, or its equal $\frac{\dot{z}}{z\sqrt{a+cz^n}}$, will be given by Art. 142, and is expounded by $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{-a}{c}}}\times A=\frac{2}{n\sqrt{-a}}$; where A denotes the *arch* whose radius is unity, and secant $\frac{x}{\sqrt{\frac{-a}{c}}}$ $\left(=\sqrt{\frac{cz^n}{-a}}\right)$.

In the same manner the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a + cz^n}$, is found $= \frac{1}{n\sqrt{ac}} \times$ *arch*, whose radius is unity and tangent $\sqrt{\frac{cz^n}{a}}$, or equal to $\frac{1}{n\sqrt{-ca}} \times$ hyp. log. $\frac{\sqrt{a} + \sqrt{-cz^n}}{\sqrt{a} - \sqrt{-cz^n}}$, according as the value of c is affirmative or negative; a being supposed affirmative.

304. *When the power, or powers, of the variable quantity without the vinculum, or radical sign, fall, mostly, in the denominator, it may be of use to substitute for the reciprocal of the said quantity, or for the quotient which arises by dividing some known quantity, either, by it, or by some compound of it in the denominator.*

Ex. 1. *Let the proposed Fluxion be* $\frac{a^3\dot{z}}{z^2\sqrt{a^2 + z^2}}$;

then, putting $x = \frac{a^2}{z}$, we have $z = \frac{a^2}{x}$, and $\dot{z} = -\frac{a^2\dot{x}}{x^2}$; and consequently $\frac{a^3\dot{z}}{z^2\sqrt{a^2 + z^2}} = \frac{-x\dot{x}}{\sqrt{x^2 + a^2}}$: whereof the fluent is $-\sqrt{x^2 + a^2} = -\sqrt{\frac{a^4}{z^2} + a^2}$.

Ex. 2. *Let the given Fluxion be* $\frac{z\dot{z}}{\overline{a+z}|^3 \times \sqrt{a^2 + az + z^2}}$.

Here, putting $x = \frac{a^2}{a+z}$, we have $z = \frac{a^2 - ax}{x} = a \times \frac{a - x}{x}$, $\dot{z} = -\frac{a^2\dot{x}}{x^2}$, $z\dot{z} = -\frac{a^3\dot{x} \times \overline{a - x}}{x^3}$, $\sqrt{a^2 + az + z^2} = \frac{a}{x}\sqrt{a^2 - ax + x^2}$; and therefore the quantity proposed is transformed to $\frac{x^2\dot{x} - ax\dot{x}}{a^4\sqrt{a^2 - ax + x^2}}$: whose fluent may be found from a table of logarithms; as will appear farther on.

305. *If the Fluxion given is affected by two different surds, and the rational factor, or the quantity without the* vinculum, *be in a constant ratio to the fluxion of the quantity under the* vinculum *of either surd, or be related to it as in* Art. 83, *the given fluxion will be reduced to a more simple form, by substituting for that surd.*

Ex. 1. *Let* $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}$ *be propounded.*

Then putting $x = \sqrt{b^2+z^2}$, we have $z^2 = x^2 - b^2$, $z\dot{z} = x\dot{x}$, and $\sqrt{c^2 - z^2} = \sqrt{c^2 + b^2 - x^2} = \sqrt{a^2 - x^2}$ (by making $a = \sqrt{c^2 + b^2}$) Whence $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}} =$

* Art. 279. $\frac{x^2\dot{x}}{\sqrt{a^2 - x^2}}$.*

Or, if x be put $= \sqrt{c^2 - z^2}$ (instead of $\sqrt{b^2+z^2}$); then $z^2 = c^2 - x^2$, $z\dot{z} = -x\dot{x}$, $\sqrt{b^2 + z^2} = \sqrt{b^2 + c^2 - x^2} = \sqrt{a^2 - x^2}$; and consequently $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2 - z^2}} = -\dot{x}\sqrt{a^2 - x^2}$: whose fluent is given by Art. 297, or 131.

Ex. 2. *Let the given Fluxion be* $\overline{a+cz^n}|^m \times \overline{e+fz^n}|^r \times$ † Art. 83. $z^{pn-1}\dot{z}$; *supposing p to denote any whole positive Number.*†

In this case, let that of the two quantities, $a + cz^n$ and $e + fz^n$, whose index (m or r) is the most complex (which we will suppose the latter) be put $= x$; then we shall have $z^n = \frac{x - e}{f}$; $z^{n-1}\dot{z} = \frac{\dot{x}}{nf}$;

$z^{pn-1}\dot{z}$ ($= z^{pn-n} \times z^{n-1}\dot{z}$) $= \frac{\overline{x-e}|^{p-1}}{f^{p-1}} \times \frac{\dot{x}}{nf}$;

$a + cz^n = a + \frac{cx - ce}{f} = d + \frac{cx}{f}$ (by putting $d = a - \frac{ce}{f}$) and consequently $\overline{d + \frac{cx}{f}}|^m \times \frac{\overline{x-e}|^{p-1} \times x^r \dot{x}}{nf^p}$

= the fluxion proposed: where, $p - 1$ being a whole positive number, the value of $\overline{x-e}|^{p-1}$ will therefore be expressed in finite terms; whence, if m be also a whole positive number, the fluent itself will be had in finite terms: but, if m and r be the halves of odd numbers, then the fluent will be found (from Art. 293 or 294) by means of circular arcs and logarithms.

306. *If the given expression be affected by two surds wherein the powers of the variable quantity are the same, and the rational quantity, without the* vinculums, *be related to the fluxion of either surd, as in Art. 83, it may be of use to substitute for the quotient, or ratio, of the two quantities under the radical signs; especially, if the sum of the said radical signs, or exponents (supposing both surds to be reduced to the denominator) is a whole number.*

Ex. 1. *Let the given Fluxion be* $\frac{z^2\dot{z}}{\overline{b^3+z^3}|^{\frac{2}{3}} \times \overline{c^3-z^3}|^{\frac{4}{3}}}$.

Then, writing $x = \frac{b^3+z^3}{c^3-z^3}$, we have $z^3 = \frac{c^3x - b^3}{1+x}$;

$3z^2\dot{z} = \frac{\overline{c^3+b^3} \times \dot{x}}{\overline{1+x}|^2}$; $\overline{b^3+z^3}|^{\frac{2}{3}} \times \overline{c^3-z^3}|^{\frac{4}{3}}$ ($= \overline{\frac{b^3+z^3}{c^3-z^3}}|^{\frac{2}{3}}$

$\times \overline{c^3-z^3}|^2 = x^{\frac{2}{3}} \times \overline{c^3 - \frac{c^3x-b^3}{1+x}}|^2$) $= \frac{\overline{b^3+c^3}|^2 \times x^{\frac{2}{3}}}{\overline{1+x}|^2}$;

and consequently $\frac{z^2\dot{z}}{\overline{b^3+z^3}|^{\frac{2}{3}} \times \overline{c^3-z^3}|^{\frac{4}{3}}} = \frac{x^{-\frac{2}{3}}\dot{x}}{3\times(b^3+c^3)}$:

whose fluent is $\frac{x^{\frac{1}{3}}}{b^3+c^3} = \frac{1}{b^3+c^3}\times\sqrt[3]{\frac{b^3+z^3}{c^3-z^3}}$.

Ex. 2. *Let there be given* $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^m \times \overline{e+fz^n}|^r}$.

Here, putting $x = \frac{e+fz^n}{a+cz^n}$, you will have $z^n = \frac{ax-e}{f-cx}$; $nz^{n-1}\dot{z} = \frac{\overline{af-ce}\times\dot{x}}{\overline{f-cx}|^2}$; $z^{pn-1}\dot{z}$ ($= z^{pn-n} \times z^{n-1}\dot{z}$) $= \overline{\frac{ax-e}{f-cx}}|^{p-1} \times \frac{\overline{af-ce}\times\dot{x}}{n\times\overline{f-cx}|^2}$; $\overline{a+cz^n}|^m \times \overline{e+fz^n}|^r$ ($=\overline{a+cz^n}|^{m+r} \times \frac{\overline{e+fz^n}|^r}{\overline{a+cz^n}|^r} = \overline{a+c\times\frac{ax-e}{f-cx}}|^{m+r} \times x^r$) $= \overline{\frac{af-ce}{f-cx}}|^{m+r} \times x^r$; and consequently the fluxion given $= \frac{\overline{ax-e}|^{p-1}\times\overline{f-cx}|^{m+r-p-1}\times x^{-r}\dot{x}}{n\times\overline{af-ce}|^{m+r-1}}$.

Where, if $m+r$ be a whole positive number, greater than p (also a whole positive number) the fluent will be truly had in finite terms; because both the series for the values of $\overline{ax-e}|^{p-1}$ and $\overline{f-cx}|^{m+r-p-1}$ do in that case terminate.* But, if r and $m+r-p-1$ be the halves of whole numbers, positive or negative, then the fluent will be given by the last section.

* Art. 99.

207. *A trinomial is reduced to a binomial by taking away its middle term; that is, by substituting for the sum or difference of the power of the variable quantity*

in that term and half its coefficient; according as the signs of the two terms, where the said quantity is found, are alike or unlike.

Ex. 1. *Let the given fluxion be* $\frac{\dot{z}}{\sqrt{b^2+cz+z^2}}$;

then, putting $x=z+\frac{1}{2}c$, or $z=x-\frac{1}{2}c$, we have $\dot{z}=\dot{x}$, and $\sqrt{b^2+cz+z^2}$ $(=\sqrt{b^2+cx-\frac{1}{2}c^2+x^2-cx+\frac{1}{4}c^2})$ $=\sqrt{b^2-\frac{1}{4}c^2+x^2}$; whence (making $a^2=b^2-\frac{1}{4}c^2$; there results $\frac{\dot{z}}{\sqrt{b^2+cz+z^2}}=\frac{\dot{x}}{\sqrt{a^2+x^2}}$: whose fluent is given by Art. 126.

Ex. 2. *Let the fluxion given be* $\frac{fz^{n-1}\dot{z}}{\sqrt{a+bz^n+cz^{2n}}}$.

First, by bringing c without the *vinculum*, according to Art. 303, we have $\sqrt{a+bz^n+cz^{2n}}=\sqrt{c}\times\sqrt{\frac{a}{c}+\frac{bz^n}{c}+z^{2n}}$: and, by putting $x=z^n+\frac{b}{2c}$, or $z^n=x-\frac{b}{2c}$, we also get $z^{n-1}\dot{z}=\frac{\dot{x}}{n}$, and $\sqrt{\frac{a}{c}+\frac{bz^n}{c}+z^{2n}}$ $(=\sqrt{\frac{a}{c}+\frac{bx}{c}-\frac{b^2}{2c^2}+x^2-\frac{bx}{c}+\frac{b^2}{4c^2}})=\sqrt{\frac{a}{c}-\frac{b^2}{4c^2}+x^2}$: therefore the fluxion transformed is $\frac{f\dot{x}}{n\sqrt{c}\times\sqrt{\frac{a}{c}-\frac{b^2}{4c^2}+x^2}}$: whose fluent is given by Art. 126, when c is a positive quantity: but when c is negative, the fluxion must be expressed thus, $\frac{f\dot{x}}{n\sqrt{-c}\times\sqrt{\frac{a}{-c}+\frac{b^2}{4c^2}-x^2}}$: answering to Form 2, Art. 142.

Ex. 3. *Let* $\frac{fz^{n-1}\dot{z} + gz^{2n-1}\dot{z} + hz^{3n-1}\dot{z} + kz^{4n-1}\dot{z}}{\overline{a+bz^n+cz^{2n}}|^m}$ *be proposed.*

Then, following the steps of the last example, $\overline{a + bz^n + cz^{2n}}|^m$ ($= c^m \times \overline{\frac{a}{c} + \frac{bz^n}{c} + z^{2n}}|^m$) will be transformed to $c^m \times \overline{\frac{a}{c} - \frac{b^2}{4c^2} + x^2}|^m$: moreover, z^n being $= x - \frac{b}{2c} = x - d$ (by putting $d = \frac{b}{2c}$) and $z^{n-1}\dot{z} = \frac{\dot{x}}{n}$, we also have $z^{2n-1}\dot{z}$ ($= z^n \times z^{n-1}\dot{z} = \overline{x-d} \times \frac{\dot{x}}{n}$) $= \frac{x\dot{x} - d\dot{x}}{n}$; $z^{3n-1}\dot{z}$ ($= z^{2n} \times z^{n-1}\dot{z}$) $= \overline{x-d}|^2 \times \frac{\dot{x}}{n} = \frac{x^2\dot{x} - 2dx\dot{x} + d^2\dot{x}}{n}$; &c. &c. From whence, by substituting these several values in the given fluxion, and putting $\frac{a}{c} - \frac{b^2}{4c^2} = e^2$, there comes out

$$\frac{f\dot{x} + g \times \overline{x\dot{x} - d\dot{x}} + h \times \overline{x^2\dot{x} - 2dx\dot{x} + d^2\dot{x}} + \&c.}{nc^m \times \overline{e^2+x^2}|^m}:$$

whose fluent, when the exponent m is the half of any *integer*, positive or negative, will be found, by means of circular arcs and logarithms, from Art. 295.

308. *When the Denominator is a rational trinomial, or multinomial (that is, when it is without a vinculum) the best way of proceeding, for the general part, is, to resolve the given fraction into binomial ones. In order to this, let its denominator be feigned* $= 0$; *by means*

of which equation, whose roots must be found, you will, by subtracting each root from the indeterminate quantity (x), have the binomial denominators of the required fractions into which the given one may be resolved: whose corresponding numerators, let be denoted $A\dot{x}$, $B\dot{x}$, $C\dot{x}$, *&c.; then, by putting the sum of the fractions, thus arising, equal to the given fraction, and reducing the whole equation to the same denominator, the assumed quantities, A, B, C, &c. by comparing the homologous terms, will be determined.*

Ex. 1. *Let the given fraction be* $\frac{\dot{x}}{x^2+ax+b}$; then, feigning $x^2+ax+b=0$, the two roots of the equation will be $-\frac{1}{2}a-\sqrt{\frac{1}{4}a^2-b}$, and $-\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-b}$: which being denoted by p and q, we have $x-p$ and $x-q$ for the two binomial factors, whereby x^2+ax+b may be resolved, or by whose multiplication $(\overline{x-p}\times\overline{x-q})$ the said quantity is produced.

Let therefore $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}$ be now assumed $\left(=\frac{\dot{x}}{x^2+ax+b}\right)=\frac{\dot{x}}{\overline{x-p}\times\overline{x-q}}$; then, by reducing the whole equation to one denomination, &c. we get $\overline{A+B}\times x\dot{x}-\overline{qA+pB+1}\times\dot{x}=0$: whence A is found $=\frac{1}{p-q}$, $B=\frac{1}{q-p}$; and, consequently, $\frac{\dot{x}}{\overline{p-q}\times\overline{x-p}}+\frac{\dot{x}}{\overline{q-p}\times\overline{x-q}}=\frac{\dot{x}}{x^2+ax+b}$.

Ex. 2. *Let the quantity proposed be* $\frac{x^2\dot{x}}{x^3+ax^2+bx+c}$.

Here, if the binomial factors whereby x^3+ax^2+bx+c is produced be represented by $x-p$, $x-q$, and $x-r$, and there be assumed $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}$

$\left(=\frac{x^2\dot{x}}{x^3+ax^2+bx+c}\right)=\frac{x^2\dot{x}}{\overline{x-p}\times\overline{x-q}\times\overline{x-r}}$; then, in this case, we shall have $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2=0$; that is, by reduction,

$$\left.\begin{matrix}A\\B\\C\\-1\end{matrix}\right\}\times x^2-\left.\begin{matrix}\overline{q+r}\times A\\\overline{p+r}\times B\\\overline{p+q}\times C\end{matrix}\right\}\times x+\left.\begin{matrix}qrA\\prB\\pqC\end{matrix}\right\}=0.$$

Whence $A+B+C=1$, $A\times\overline{q+r}+B\times\overline{p+r}+C\times\overline{p+q}=0$, and $Aqr+Bpr+Cpq=0$. Now, from the first of these equations, multiplied by $p+q$, subtract the second, and you will have $A\times\overline{p-r}+B\times\overline{q-r}=p+q$: also, from the first, multiplied by pq, subtract the third; then $A\times\overline{pq-rq}+B\times\overline{pq-pr}=pq$: lastly, from the former of the two equations thus arising, multiplied by p, subtract the latter, then $A\times\overline{p^2-pr-pq+qr}=p^2$, that is, $A\times\overline{p-q}\times\overline{p-r}=p^2$; and consequently $A=\frac{p^2}{\overline{p-q}\times\overline{p-r}}$: whence, by the very same argument, $B=\frac{q^2}{\overline{q-p}\times\overline{q-r}}$, and $C=\frac{r^2}{\overline{r-p}\times\overline{r-q}}$.

309. After the same manner, you may proceed in other cases: but there is an artifice, or compendium, for more readily determining the assumed quantities A, B, C, &c. by which a great deal of trouble is avoided: and that is, by considering the equation in such circumstances of the indeterminate quantity x, when it becomes most simple, or when most of its terms vanish.

Thus, in the preceding example, because $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2$ is $=0$ (in all circumstances of x whatever) let x be taken $=p$; then, all the terms vanishing, except the first and last, we have $A \times \overline{p-q} \times \overline{p-r} - p^2 = 0$; and consequently $A = \frac{p^2}{\overline{p-q} \times \overline{p-r}}$; *the very same as before.*

More universally, let the given fraction be $\frac{x^m \dot{x}}{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ \&c.}} = \frac{x^m \dot{x}}{\overline{x-p} \times \overline{x-q} \times \overline{x-r} \times \overline{x-s} \text{ \&c.}}$ (where m and n may represent any whole positive numbers whatever, provided the latter be greater than the former). Then, assuming $\frac{A\dot{x}}{x-p} + \frac{B\dot{x}}{x-q} + \frac{C\dot{x}}{x-r} + \frac{D\dot{x}}{x-s}$ &c. $= \frac{x^m \dot{x}}{x^n + ax^{n-1} + bx^{n-2} \text{ \&c.}}$ &c. we shall have $A \times \overline{x-q} \times \overline{x-r} \times \overline{x-s}$ &c. $+ B \times \overline{x-p} \times \overline{x-r} \times \overline{x-s}$ &c. $+ C \times \overline{x-p} \times \overline{x-q} \times \overline{x-s}$ &c. &c. $- x^m = 0$: from whence, by expounding x by p, q, r, &c. successively, we obtain $A = \frac{p^m}{\overline{p-q}.\ \overline{p-r}.\ \overline{p-s} \text{ \&c.}}$, $B = \frac{q^m}{\overline{q-p}.\ \overline{q-r}.\ \overline{q-s} \text{ \&c.}}$, $C = \frac{r^m}{\overline{r-p}.\ \overline{r-q}.\ \overline{r-s} \text{ \&c.}}$, &c. Whence the fractions themselves, whereof these quantities are the coefficients, or numerators, will likewise be given.

But the numerators thus found may sometimes be more commodiously expressed by help of the given coefficients a, b, c, d, &c. so as to involve only one of the roots p, q, r, &c. in each fraction. For, since $\overline{x-p} \times \overline{x-q} \times \overline{x-r}$ &c. is supposed, *universally,* $= x^n + ax^{n-1} + bx^{n-2} + cx^{n-3}$ &c. if both sides of the equation be divided by $x-p$, we shall have $\overline{x-q}\times\overline{x-r}\times\overline{x-s}$ &c. $=\dfrac{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ \&c.}}{x-p}$

Which last expression, when x is $=p$, that is, when both the numerator and the denominator become equal to nothing, will manifestly be equal to $(\overline{p-q}\times\overline{p-r}\times\overline{p-s}$ &c.) the divisor of A. Therefore, if the fluxion of the numerator be taken and divided by that of the denominator, and p be wrote instead of x (*vide page* 155.) we shall have $np^{n-1}+\overline{n-1}\times ap^{n-2}+\overline{n-2}\times bp^{n-3}$ &c. $=\overline{p-q}\times\overline{p-r}\times\overline{p-s}$ &c. and therefore A $\left(=\dfrac{p^m}{\overline{p-q}.\ \overline{p-r}.\ \overline{p-s} \text{ \&c.}}\right)=\dfrac{p^m}{np^{n-1}+\overline{n-1}.\,ap^{n-2}+\overline{n-2}.\ bp^{n-3} \text{ \&c.}}$. By the very same reasoning $B=$

$$\frac{q^m}{nq^{n-1}+\overline{n-1}.\ aq^{n-2}+\overline{n-2}.\ bq^{n-3}\ \text{\&c.}}\quad C=$$

$$\frac{r^m}{nr^{n-1}+\overline{n-1}.\ ar^{n-2}+\overline{n-2}.\ br^{n-3}\ \text{\&c.}}\ \text{\&c.}$$

Hence it appears, that, if the numerator of the given fraction be divided by the fluxion of the denominator (neglecting $\dot{x}$*) and the several roots,* p, q, r, *&c. (found by feigning the denominator* $=0$*) be, successively, substituted in the quotient, instead of* x; *I say, it is evident, that the quantities so resulting, divided by* $x-p$, $x-q$, $x-r$ *&c. will be the required, binomial, fractions into which the proposed multinomial one may be resolved.*

310. If some of the roots p, q, r, &c. are impossible, which is often the case, the fractions thus found, where the impossible roots are concerned, must

be united in pairs, and so reduced to trinomial ones, in order to take away the *imaginary terms.*

Thus, let the fraction proposed be $\frac{x\dot{x}}{x^3+ax^2+bx+c}$, and let two of the roots, p and q, of the equation $x^3+ax^2+bx+c=0$ be impossible: then, $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}$ being $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$, we shall, by uniting the *imaginary terms*, have $\frac{\overline{A+B}\times x\dot{x}-\overline{Aq+Bp}\times\dot{x}}{x^2-\overline{p+q}\times x+pq}+\frac{C\dot{x}}{x-r}$, also $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$; where the impossible quantities destroy one another. But, to render this more obvious, let a be taken $=0$, $b=0$, and $c=-1$, so that the given fraction may become $\frac{x\dot{x}}{x^3-1}$ then the three roots (p, q, r) of the equation, $x^3-1=0$, will here be $-\frac{1}{2}+\sqrt{\frac{-3}{4}}$, $-\frac{1}{2}-\sqrt{\frac{-3}{4}}$, and 1; whereof the two former are impossible. Moreover, by dividing the numerator (x) by the fluxion of the denominator $(3x^2)$ (according to the *prescript*) we have $\frac{1}{3x}$; which, by writing p, q, r, successively, instead of x, becomes $\frac{1}{3p}$, $\frac{1}{3q}$ and $\frac{1}{3r}$ for the values of A, B, and C, respectively. Whence $\frac{\overline{A+B}\times x-Aq-Bp}{x^2-\overline{p+q}\times x+pq}+\frac{C}{x-r}\left(=\frac{x}{x^3-1}\right)$ is $=\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}+\frac{\frac{1}{3}}{x-1}=\frac{1-x}{3x^2+3x+3}+\frac{1}{3x-3}$. But the same may be, otherwise, investigated in a more general manner; by assuming $\frac{Px+Q}{x^2+x+1}+\frac{R}{x-1}=\frac{x}{x^3-1}$, and proceeding as in the first and second examples; whence the very same conclusion will be derived.

If the fraction proposed be of this form, viz. $\frac{z^{pn-1}\dot{z}}{z^{mn}+az^{mn-n}+bz^{mn-2n}\ \&c.}$, the method of resolution will still be the same: since, by putting $x=z^n$, the given expression is reduced to

$$\frac{\frac{1}{n}\times x^{p-1}\dot{x}}{x^m+ax^{m-1}+bx^{m-2}\ \&c.}$$

It may also be proper to observe, *that*, in very complicated cases, the application of two, or more, of the six foregoing rules, may become necessary. Thus, for example, if the fluxion given be $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{e+fz^n+gz^{2n}}}$; by resolving $\frac{1}{e+fz^n+gz^{2n}}$ into two binomial fractions, $\frac{A}{h+z^n}+\frac{B}{k+z^n}$ (*according to Art.* 308) we shall have $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{e+fz^n+gz^{2n}}}$ $=\frac{Az^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{h+z^n}}+\frac{Bz^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{k+z^n}}$: where, if m be a whole positive number, greater than p, the fluent will be had in finite terms (*by Art.* 306, *Ex.* 2).

SECTION V.

The Investigation of Fluents of rational Fractions, of several Dimensions, according to the Forms *in* COTES'S HARMONIA MENSURARUM.

311. AS the subject here proposed is a matter of considerable difficulty, and has exercised the attention of some of the most celebrated mathematicians (who, yet, seem to have condescended very little to the information of their less experienced readers) I shall endeavour to set it in the clearest light possible: in order to which, it will be requisite to premise the following Lemmas.

LEMMA I.

If the Sine of the Mean of three equi-different Arcs, supposing Radius Unity, be multiplied by the Double of the Co-sine of the common Difference, and from the Product, the Sine of the lesser Extreme be subtracted, the Remainder will be the sine of the greater Extreme.

LEMMA II.

312. *If G be taken to denote the greater, and L the lesser, of two unequal Arcs, and their Difference be expressed by D; then will,*

1. $\frac{sin. G. \times co\text{-}s. D - sin. L. \times rad.}{sin. D} = co\text{-}s. G$

2. $\frac{co\text{-}s. L \times rad. - co\text{-}s. G \times co\text{-}s. D}{sin. D} = sin. G$

3. $\frac{sin. G. \times rad. - sin. L \times co\text{-}s. D}{sin. D} = co\text{-}s. L.$

The former of these two Lemmas may be met with in most authors upon Trigonometry; and the latter is nothing more than a Corollary to the *common* theorems for finding the sine and co-sine of the sum and difference of two given arcs; for which reasons I shall not stop here to give their demonstration.

COROLLARY.

313. If any arch of the circle, whose radius is unity, be denoted by Q, its sine by s, and its co-sine by a; and there be taken $A=2a$, $B=2aA-1$, $C=2aB-A$, $D=2aC-B$, $E=2aD-C$, $F=2aE-D$, &c. it follows (*from Lemma* 1), that,

Sin. $2Q$ (*sin.* $Q\times 2a-$*sin.* $0)=2sa-0=sA$
Sin. $3Q$ (*sin.* $2Q\times 2a-$*sin.* $Q)=2sAa-s=sB$
Sin. $4Q$ (*sin.* $3Q\times 2a-$*sin.* $2Q)=2sBa-sA=sC$
Sin. $5Q$ (*sin.* $4Q\times 2a-$*sin.* $3Q)=2sCa-sB=sD$
Sin. $6Q$ (*sin.* $5Q\times 2a-$*sin.* $4Q)=2sDa-sC=sE$
&c. &c.

LEMMA III.

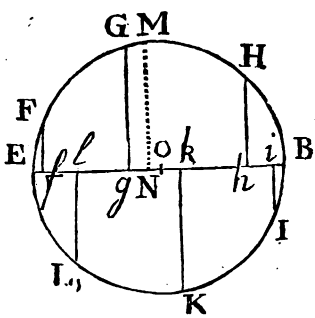
314. *To resolve the Trinomial* $r^{2n}-2kr^nx^n+x^{2n}$, *where n is any whole Number, into simple trinomial Factors.*

Since the first term of the given quantity $r^{2n}-2kr^nx^n+x^{2n}$ is divisible, only by the powers of r, and the last, only, by those of x; and it appears that r and x are concerned, exactly, alike; let therefore $r^2-2arx+x^2$ (where r and x are, *also*, alike concerned) be assumed for one of the required trinomial factors, whereby $r^{2n}-2kr^nx^n+x^{2n}$ may be resolved: and let $\overline{r^2-2arx+x^2}\times(r^8+Ar^7x+Br^6x^2+Cr^5x^3+Dr^4x^4+Cr^3x^5+Br^2x^6+Arx^7+x^8)$ (where r and x are, *still*, affected alike) be assumed $=r^{10}-2kr^5x^5+x^{10}$ (the value of n, to render the operation more perspicuous, being first expressed by 5).

Then, by multiplication and transposition, we shall have

$$\left.\begin{array}{l} r^{10}+Ar^9x+Br^8x^2+Cr^7x^3+Dr^6x^4+Cr^5x^5+Br^4x^6+Ar^3x^7+r^2x^8\ldots * \ldots\ldots * \\ * -2ar^9x-2aAr^8x^2-2aBr^7x^3-2aCr^6x^4-2aDr^5x^5-2aCr^4x^6-2aBr^3x^7-2aAr^2x^8-2arx^9\ldots * \\ * \quad * \quad +r^8x^2+Ar^7x^3+Br^6x^4+Cr^5x^5+Dr^4x^6+Cr^3x^7+Br^2x^8+Arx^9+x^{10} \\ -r^{10}\ldots\ldots\ldots\ldots\ldots\ldots+2kr^5x^5\ldots\ldots\ldots\ldots\ldots\ldots-x^{10} \end{array}\right\} = 0$$

Whence, $A=2a$, $B=2Aa-1$, $C=2aB-A$, $D=2aC-B$, and $2C-2aD+2k=0$. But, if Q be taken to denote the arch (E F) of a circle E H K, whose radius E O is unity, and co-sine (Of) $= a$; and s be put for (Ff) the sine of the same arch; then (*by Corol. to Lem.* 1) $s\,A = sin.\,2\,Q$, $s\,B = sin.\,3\,Q$, $s\,C = sin.\,4\,Q$ &c. and consequently $A = \frac{sin.\,2\,Q}{s}$, $B = \frac{sin.\,3Q}{s}$, $C = \frac{sin.\,4\,Q}{s}$, $D = \frac{sin.\,5\,Q}{s}$ (or, $\frac{sin.\,n\,Q}{s}$). Moreover, because, $2C-2aD+2k=0$, or $D \times a - C \times 1 = k$, where (as appears from above) $D \times a - C \times 1 = \frac{sin.\,5Q \times co\text{-}s.\,Q - sin.\,4Q \times rad.}{s} = co\text{-}s.\,5\,Q$ (*by Case* 1, *Lem.* 2) we therefore have $co\text{-}s.\,5\,Q\ (n\,Q) = k$. Whence this construction.



Take R to denote the arch (E M) whose co-sine (O N) is the given co-efficient k, and let Q (E F) be taken to EM as 1 to n; then the co-sine ($O\,f$) of this last arch will be the true value of d. But this is only one of the values that a will admit of: for it is well known that the co-sine of any arch, is also the co-sine of the same arch increased by any number of times the whole periphery (P). Therefore, seeing the co-sine of n Q (= co-sine of R) is likewise = co-sine $\overline{P+R}$ = co-s. $\overline{2P+R}$ = cos. $\overline{3P+R}$ &c. it follows that Q (whose co-sine is a) will be expressed by any one of the arcs, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$, $\frac{3P+R}{n}$ &c. (or by EF, EG, EH, EI,

&c. supposing the whole periphery to be divided into n equal parts, from the point F). Hence, if the co-sines of these several arcs, expressing all the different values of a, be represented by b, c and d, &c. respectively, we shall have $r^2-2brx+x^2$, $r^2-2crx+x^2$, $r^2-2drx+x^2$, &c. for the several required factors, by which $r^{2n}-2kr^n x^n+x^{2n}$ may be resolved: and consequently $\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\times\overline{r^2-2drx+x^2}\ (n) = r^{2n}-2kr^n x^n+x^{2n}$. Q. E. I.

Note. If the sign of the middle term $2kr^n x^n$ be positive, the distance (or co-sine) O N must be taken on the contrary side of the center: but when k is greater than unity, this method of solution fails; since no co-sine can be greater than the radius.

Corollary I.

315. If $k=1$, the arch R (whose co-sine is k) being $=0$, the values of b, c, d, &c. will be expressed by the co-sines of the arcs $\frac{o}{n}$, $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ &c. respectively: and our general equation will here become $r^{2n}-2r^n x^n+x^{2n}=\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\times\overline{r^2-2drx+x^2}\ (n)$. From whence, by extracting the square-root, on both sides, we also have $r^n \backsim x^n = \overline{r^2-2brx+x^2}|^{\frac{1}{2}}\times\overline{r^2-2crx+x^2}|^{\frac{1}{2}}\ (n)$.

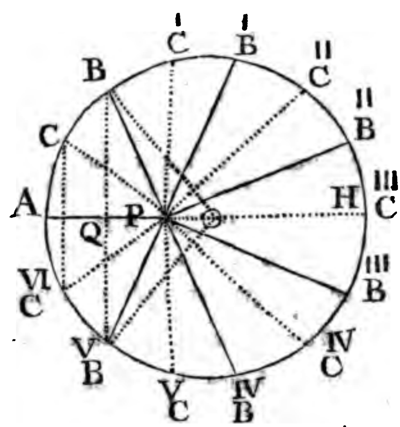
Corollary II.

316. But, if $k=-1$ (or the middle term be $+2r^n x^n$) then the arch R being $=\frac{P}{2}$, the values of b, c, d, &c. will, *here*, be defined by the co-sines of

the arcs $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$, &c. and our equation, by taking the root, as above, will become $r^n + x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}}$ (n).

SCHOLIUM.

317. From the two preceding corollaries, the demonstration of that remarkable property of the circle given, and applied to finding a vast number of fluents, in *Cotes's Harmonia Mensurarum*, is very easily, and naturally, deduced.



For, let the periphery of the circle ABB́ &c. whose radius is expressed by r, be divided into as many equal parts AB, BB́, B́B̋, &c. as there are units in the given integer n; so that AB, AB́, AB̋, &c. may respectively exhibit the values of the foresaid arcs $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ &c. (*vide* Corol. 1). Moreover, let O Q be the co-sine of the first of them; and, in the radius OA (produced if necessary) let there be taken O P $= x$; and let O B, Q B, P B, &c. &c. be drawn: then, the co-sine of the angle A O B ($= \frac{P}{n}$) to the radius 1, being expressed by c (*vide* Corol. 1) it will be $1 : c :: r$ (O B) : O Q $= cr$: whence P B^2 ($=$ OB^2 + OP^2 $-$ 2OQ $\times$ OP) $= r^2 + x^2 - 2crx = r^2 - 2crx + x^2$.

By the very same argument P$\acute{B}^2$ is $= r^2 - 2drx + x^2$, &c. &c. Therefore, because $r^n \backsim x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2drx + x^2}|^{\frac{1}{2}}$ (n), by Corol. 1 it follows that their equals, AO$^n \backsim$ OP^n and PA $\times$ PB $\times$ P$\acute{B}$ $\times$ P$\acute{\acute{B}}$ &c. must be equal likewise: *which is the first part of the theorem above hinted at.*

After the same manner, if the arcs AC, A$\acute{C}$, A$\acute{\acute{C}}$, A$\acute{\acute{\acute{C}}}$ be taken respectively equal to $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$ &c. it will appear (from Corol. 2) that A O^n + P O^n is $=$ PC $\times$ P$\acute{C}$ $\times$ P$\acute{\acute{C}}$ (n) *which is the latter part of the same theorem.*

Hence (by the bye) all the roots of the equation $x^n = r^n$ are very readily found: for, since AO$^n \backsim$ PO^n = PA $\times$ PB $\times$ P$\acute{B}$ &c. where the second factor and the last, the third and the last but one, &c. are respectively equal to each other, it is evident that AO$^n \backsim$ PO^n ($r^n \backsim x^n$) is also $=$ PA $\times$ PB^2 $\times$ P$\acute{B}^2$ $\times$ P$\acute{\acute{B}}^2$ $= \overline{r \backsim x} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2}$ &c. Whence, $x^n \backsim r^n$ being $= 0$, it follows that $\overline{r \backsim x} \times \overline{r^2 - 2crx + x^2}$ &c. is $= 0$: from which, by extracting the roots out of the equations $r \backsim x = 0$, $r^2 - 2crx + x^2 = 0$, $r^2 - 2drx + x^2 = 0$ &c. we get r, $r \times \overline{c + \sqrt{c^2 - 1}}$, $r \times \overline{c - \sqrt{c^2 - 1}}$, $r \times \overline{d + \sqrt{d^2 - 1}}$,

&c. for the several roots of the equation $x^n = r^n$; whereof the first, only, is possible, when n is odd; and the first and last when n is even.

By the same way of proceeding all the roots of the equation, $x^n + r^n = 0$, will also be found: for, seeing

$$x^n + r^n \text{ is } = \overline{r^2 - 2brx + x^2}\Big|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}\Big|^{\frac{1}{2}} \text{ \&c.}$$

$(= \text{PC} \times \text{P}\acute{\text{C}} \times \text{P}\acute{\acute{\text{C}}}$ &c.) where the first factor and the last, the second and the last but one, &c. are respectively equal to each other, it is plain that $x^n + r^n$ is likewise $= \overline{r^2 - 2brx + x^2} \times \overline{r^2 - 2crx + x^2}$ &c. and consequently $x = r \times \overline{b \pm \sqrt{b^2 - 1}}$ &c. &c. Where the roots are all impossible; except the last, when their number (n) is odd.

LEMMA IV.

318. *Supposing every thing to remain as in the preceding* Lemma, *and that* $\acute{k}$, $\acute{b}$, $\acute{c}$, $\acute{d}$, &c. *denote the Sines of the Arcs* R, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ &c. (*whose Co-sines are* k, b, c, d, &c.) *then, I say, the Fraction*

$$\frac{n\acute{k}r^n x^n}{r^{2n} - 2kr^n x^n + x^{2n}} \text{ is equal to } \frac{\acute{b}rx}{r^2 - 2brx + x^2} + \frac{\acute{c}rx}{r^2 - 2crx + x^2} + \frac{\acute{d}rx}{r^2 - 2drx + x^2} \text{ \&c.}$$

For, since $r^{10} - 2kr^5 x^5 + x^{10}$ $(r^{2n} - 2kr^n x^n + x^{2n})$ is $= \overline{r^2 - 2arx \times x^2} \times (r^8 + Ar^7 x + Br^6 x^2 + Cr^5 x^3 + Dr^4 x^4 + Cr^3 x^5 + Br^2 x^6 + Arx^7 + x^8)$ (*by the foresaid* Lemma) and it is also proved that $A = \frac{\sin. 2Q}{s}$, $B = \frac{\sin. 3Q}{s}$,

$C = \frac{sin.\ 4\ Q}{s}$ &c. it is evident, therefore, that $\frac{r^{10} - 2kr^5x^5 + x^{10}}{r^2 - 2arx + x^2}$ ($= r^8 + Ar^7x + Br^6x^2$ &c.) is $= r^8 +$ $\frac{sin.\ 2\ Q}{s} \times r^7x$ &c. and consequently $\frac{s \times \overline{r^{10} - 2kr^5x^5 + x^{10}}}{r^2 - 2arx + x^2}$ $= sin.\ Q \times r^8 + sin.\ 2\ Q \times r^7x + sin.\ 3\ Q \times r^6x^2 +$ $sin.\ 4\ Q \times r^5x^3 + sin\ 5\ Q \times r^4x^4 + sin.\ 4\ Q \times r^3x^5$ &c. In which equation, for a and s, let their several respective values b, c, d, &c. and b', c', d', &c. be, successively, substituted; and let the corresponding arcs $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ &c. be represented by Q, Q', Q'', &c. then we shall have

$$\frac{b' \times \overline{r^{10} - 2kr^5x^5 + x^{10}}}{r^2 - 2brx + x^2} = sin.\ Q \times r^8 + sin.\ 2\ Q \times r^7x \text{ \&c.}$$

$$\frac{c' \times \overline{r^{10} - 2kr^5x^5 + x^{10}}}{r^2 - 2crx + x^2} = sin.\ Q' \times r^8 + sin.\ 2\ Q' \times r^7x \text{ \&c.}$$

&c. &c.

Which equations, added all together, give

$$\overline{r^{10} - 2kr^5x^5 + x^{10}} \times \left(\frac{b'}{r^2 - 2brx + x^2} + \frac{c'}{r^2 - 2crx + x^2} + \frac{d'}{r^2 - 2drx + x^2} \text{ \&c.}\right)$$

$$= \left\{\begin{array}{l} \sin. Q \\ \sin. Q' \\ \sin. Q'' \\ \sin. Q''' \\ \sin. Q'''' \end{array}\right\} \times r^8 + \left\{\begin{array}{l} \sin. 2Q \\ \sin. 2Q' \\ \sin. 2Q'' \\ \sin. 2Q''' \\ \sin. 2Q'''' \end{array}\right\} \times r^7 x + \ldots\ldots + \left\{\begin{array}{l} \sin. nQ \\ \sin. nQ' \\ \sin. nQ'' \\ \sin. nQ''' \\ \sin. nQ'''' \end{array}\right\} \times r^4 x^4 + \&c.$$

But the sines of the first column, being those of an arithmetical progression (whose common difference is $\frac{P}{n}$) by which the whole periphery is divided into n (5) equal parts, their sum will therefore, it is well known, be equal to nothing; or all the negative ones equal to all the positive ones.

The same is also true with regard to the sines of the second column; whose arcs $\frac{2R}{n}$, $\frac{2P+2R}{n}$, $\frac{4P+2R}{n}$ &c. (having $\frac{2P}{n}$ for their common difference) divide the periphery (twice taken) into the same number (n) of equal parts. But the sines of the middle column (which is the last above exhibited) will not vanish, as all the rest do: for, $n\,Q$ being $=R$, $n\,\acute{Q}=P+R$, $n\acute{\acute{Q}}=2\,P+R$, &c. the common difference will here be equal to (P) the whole periphery; and therefore, every arch terminating in the same point with the first, the circle will, in this case, remain undivided, and the sine of each be equal to $(\acute{k})$ the sine of the first.

Hence, our equation is reduced to $\overline{r^{10}-2kr^5x^5+x^{10}}\times$

$$\overline{\frac{\acute{b}}{r^2-2\acute{b}rx+x^2}+\frac{\acute{c}}{r^2-2\acute{c}rx+x^2}\ \&c.}=5\,\acute{k}\,r^4\,x^4;$$ which divided by $r^{10}-2kr^5x^5+x^{10}$, and multiplied by rx, gives

$$\frac{\acute{b}rx}{r^2-2\acute{b}rx+x^2}+\frac{\acute{c}rx}{r^2-2\acute{c}rx+x^2}+\frac{\acute{d}rx}{r^2-2\acute{d}rx+x^2}\ \&c.=$$

$$\frac{5\acute{k}r^5x^5}{r^{10}-2kr^5x^5+x^{10}}=\frac{n\acute{k}r^n\,x^n}{r^{2n}-2kr^n\,x^n+x^{2n}}.\qquad Q.\ E.\ D.$$

The same otherwise.

319. Since $r^{2n}-2kr^n\,x^n+x^{2n}$ is $=\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\times\overline{r^2-2drx+x^2}\ (n)$ *by Lemma* 3, it is evident that, $\log.\ \overline{r^{2n}-2k\,r^n\,x^n+x^{2n}}=\log.\overline{r^2-2brx+x^2}+\log.\overline{r^2-2crx+x^2}+\log.\overline{r^2-2drx+x^2}$ (n). And, as this equation holds universally, let k and x be what they will (which two quantities may be supposed to flow independently of each other) let the

fluxion of the *whole* equation be taken, making k variable (and x constant); which gives $\frac{-2\dot{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}}$

$$= \frac{-2\dot{b}rx}{r^2-2brx+x^2} - \frac{2\dot{c}rx}{r^2-2crx+x^2} - \frac{2\dot{d}rx}{r^2-2drx+x^2}$$

* Art. 126. (n).* But, k, b, c, d, &c. are the co-sines of the arcs R, $\frac{R}{n}$, $\frac{R+P}{n}$, $\frac{R+2P}{n}$ &c. (whereof the corrresponding sines are k', b', c', &c.) therefore, the fluxion of the first of these arcs being denoted by $\dot{R}$, the fluxion of each of the rest will be expressed by $\frac{\dot{R}}{n}$: and so (the fluxion of the co-sine of an arch being equal to the fluxion of the arch itself drawn into its sine, applied to

† Art. 142. radius)† it follows that $\dot{k} = \dot{R}k'$, $\dot{b} = \frac{\dot{R}}{n} \times b'$, $\dot{c} = \frac{\dot{R}}{n} \times c'$, &c. Which values being substituted in the foregoing equation, and the whole divided by $\frac{-2\dot{R}}{n}$, we have $\frac{nk'r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{b'rx}{r^2-2brx+x^2} + \frac{c'rx}{r^2-2crx+x^2} + \frac{d'rx}{r^2-2drx+x^2}$ (n).

Lemma V.

320. *To determine the Series arising from the Division of Unity by a Trinomial, $x^2-2arx+r^2$; and to exhibit the Remainder after any given Number (v) of Terms in the Quotient.*

Let $x^{-2}+Arx^{-3}+Br^2x^{-4}+Cr^3x^{-5}+Dr^4x^{-6}$ represent the required quotient continued to 5 terms

(v, to render the process the more obvious, being first expounded by that number) and let $Er^5x^{-5}+Fr^6x^{-6}$ be the remainder. Then, because $\frac{1}{x^2-2arx+r^2}$ is $=$ $x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + Dr^4x^{-6} + \frac{Er^5x^{-5}+Fr^6x^{-6}}{x^2-2arx+r^2}$, we shall, by reducing the whole equation to one denomination, have

$$\left.\begin{array}{l} 1+Arx^{-1}+Br^2x^{-2}+Cr^3x^{-3}+Dr^4x^{-4}+Er^5x^{-5}+Fr^6x^{-6} \\ * -2arx^{-1}-2aAr^2x^{-2}-2aBr^3x^{-3}-2aCr^4x^{-4}-2aDr^5x^{-5} \; * \\ 1 \quad * \quad +r^2x^{-2}+Ar^3x^{-3}+Brx^{4-4}+Crx^{5-5}+Dr^6x^{-6} \end{array}\right\} = 0$$

Whence $A=2a$, $B=2aA-1$, $C=2aB-A$, $D=2aC-B$, $E=2aD-C$, and $F=-D$.

Therefore, if Q be now put for the arch whose radius is 1 and co-sine a, and there be taken $S = \sin. Q$, $\acute{S} = \sin. 2Q$, $\acute{\acute{S}} = \sin. 3Q$, &c. we shall also have $A\ (2a) = \frac{\acute{S}}{S}$, $B = \frac{\acute{\acute{S}}}{S}$, $C = \frac{\acute{\acute{\acute{S}}}}{S}$, $D = \frac{\acute{\acute{\acute{\acute{S}}}}}{S}$, $E = \frac{\overset{v}{S}}{S} = \frac{\sin. 6Q}{S}$, $F\ (-D) = -\frac{\sin. 5Q}{S}$ (*by Corol. to Lem.* 1). And consequently $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \acute{S}rx^{-3} + \acute{\acute{S}}r^2x^{-4} + \acute{\acute{\acute{S}}}r^3x^{-5} + \acute{\acute{\acute{\acute{S}}}}r^4x^{-6}}{S} +$$

$$\frac{\sin. 6Q \times r^5x^{-5} - \sin. 5Q \times r^6x^{-6}}{S \times \overline{x^2 - 2arx + r^2}}.$$ Whence, *universally*, $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \acute{S}rx^{-3} + \acute{\acute{S}}r^2x^{-4} + \acute{\acute{\acute{S}}}r^3x^{-5} \text{ \&c. (to } v \text{ terms)}}{S} +$$

$$\frac{\sin. \overline{v+1}. Q \times r^vx^{-v} - \sin. vQ \times r^{v+1}x^{-v-1}}{S \times \overline{x^2 - 2arx + r^2}}.$$ Which last equation (though obvious enough from the preceding one) may be investigated in a general manner (if required) by assuming $x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + \ldots\ldots dr^{v-2}x^{-v} + er^{v-1}x^{-v-1} + \frac{fr^v x^{-v} + gr^{v+1}x^{-v-1}}{x^2 - 2arx + r^2} = \frac{1}{x^2 - 2arx + r^2}$, and proceeding as above: by which means you will find $A = 2a$, $B = 2aA - 1$, &c. $f = 2ae - d = \frac{\sin. \overline{v+1} \times Q}{S}$, and $g\ (= -e) = -\frac{\sin. vQ}{S}$. And thus may the third

Lemma be made out, if any objection or difficulty should arise about its being general.

COROLLARY.

321. If, in the given trinomial $x^2 - 2arx + r^2$, we suppose r^2, instead of x^2, to be the leading term whereby the quotient is produced; then, since r and x are affected exactly alike, we shall, by writing r for x, and x for r, have $\frac{1}{r^2 - 2axr + x^2} =$

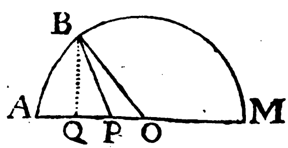
$$\frac{Sr^{-2} + \acute{S}xr^{-3} + \ddot{S}x^2r^{-4} \ (v)}{S} +$$

$$\frac{\overline{\sin. v + 1} \times Q \times x^v r^{-v} - \overline{\sin. vQ} \times x^{v+1} \times r^{-v-1}}{S \times \overline{r^2 - 2axr + x^2}}.$$

PROBLEM I.

322. *To find the Fluent of* $\frac{\dot{x}}{r^2 - 2arx + x^2}$, *together with that of* $\frac{x\dot{x}}{r^2 - 2arx + x^2}$.

Let A B M &c. be a circle whose radius O A (or O M) is r, and let the angle A O B be such, that its co-sine, to the radius 1, may be equal to a; or so, that O Q (supposing BQ perpendicular to OA) may be $= ar$: moreover let s denote the sine of the said angle A O B, corresponding to the co-sine a, and let O P (considered as variable by the motion of P along O A) express the value of x: then, PB^2 $(OB^2 + OP^2 - 2OQ \times OP)$ $= r^2 - 2arx + x^2$: and the fluxion of the measure of the angle Q B P (radius being unity) will be repre-



sented by $\frac{BQ \times flux.\, QP}{BP^2}$ (*vide Art.* 142) or by $\frac{rs \times -\dot{x}}{r^2 - 2arx + x^2}$; and consequently *that* of OBP, by $\frac{rs\dot{x}}{r^2 - 2arx + x^2}$: whence it is evident that the fluent of $\frac{\dot{x}}{r^2 - 2arx + x^2}$ (contemporaneous with x) is truly expressed by $\frac{1}{rs} \times OBP$.

Again, since $\frac{x\dot{x}}{r^2 - 2arx + x^2}$ may be transformed to $\frac{-ar\dot{x} + x\dot{x}}{r^2 - 2arx + x^2} + \frac{ar\dot{x}}{r^2 - 2arx + x^2}$; where the fluent of the former part is $= \frac{1}{2}$ hyp. log. $\frac{r^2 - 2arx + x^2}{r^2}$ * $=$ $\frac{1}{2}$ *hyp. log.* $\frac{PB^2}{OB^2}$ $=$ *hyp. log.* $\frac{PB}{OB}$; and that of the latter part $= \frac{a}{s} \times OBP$; it appears that the fluent of $\frac{x\dot{x}}{r^2 - 2arx + x^2}$ is truly expounded by *hyp. log.* $\frac{PB}{OB}$ + $\frac{a}{s} \times OBP$. Q. E. I.

* Art. 126.

COROLLARY.

323. Since, $PB : PO :: sin.\ BOP$ (s) : $sin.\ OBP =$ $\frac{sx}{\sqrt{r^2 - 2arx + x^2}}$; it follows, if the hyperbolical logarithm of $\frac{\sqrt{r^2 - 2arx + x^2}}{r}$, be represented by M, and the arch, whose sine is $\frac{sx}{\sqrt{r^2 - 2arx + x^2}}$ and radius

unity by N, that the fluents of $\frac{\dot{x}}{r^2-2arx+x^2}$ and $\frac{x\dot{x}}{r^2-2arx+x^2}$ will be expressed by $\frac{N}{rs}$ and $M+\frac{aN}{s}$ respectively.

PROBLEM II.

324. *To determine the Fluent of* $\frac{x^m\dot{x}}{x^2-2arx+r^2}$, *supposing* m *any whole positive Number, and* a *less than Unity.*

Let every thing remain as in Lemma 5, and then, if the equation there brought out be multiplied by $x^m\dot{x}$, and v at the same time be expounded by $m-1$, we shall get

$$\frac{x^m\dot{x}}{x^2-2arx+r^2}=\frac{Sx^{m-2}\dot{x}+\acute{S}rx^{m-3}\dot{x}+\acute{\acute{S}}r^2x^{m-4}\dot{x}}{S}$$

$$(m-1)+\frac{\overline{\sin.\ mQ}\times r^{m-1}x\dot{x}-\sin.\ \overline{\overline{m-1}\times Q}\times r^m\dot{x}}{S\times\overline{x^2-2arx+r^2}}$$

whose fluent will therefore be given by the preceding proposition: for, supposing the values of M, and N to be as there specified, the fluent of the last term

$$\left(\frac{\overline{\sin.\ mQ}\times r^{m-1}x\dot{x}-\sin.\ \overline{\overline{m-1}\times Q}\times r^m\dot{x}}{S\times\overline{x^2-2arx+r^2}}\right)$$ will, it

is manifest,* be expressed by $\frac{1}{S}$ into $\overline{\sin.\ mQ}\times r^{m-1}\times$

$$\overline{M+\frac{aN}{S}}-\sin.\ \overline{\overline{m-1}\times Q}\times r^m\times\frac{N}{rS}=\frac{r^{m-1}}{S}\text{ into}$$

$$\sin.\ mQ\times M+\frac{\overline{\sin.\ mQ\times a-\sin.\ \overline{m-1}\times Q}}{S}\times N$$

$$=\frac{r^{m-1}}{S}\text{ into }\overline{\sin.\ mQ}\times M+\overline{\text{co-s.}\ mQ}\times N\ (\textit{by Lem. 2,}$$

* Art. 323.

Case 1.). To which adding the fluent of the preceding series, there results $\frac{1}{S} \times \overline{\frac{Sx^{m-1}}{m-1} + \frac{\acute{S}rx^{m-2}}{m-2} + \frac{\acute{\acute{S}}r^2x^{m-3}}{m-3} (m-1)}$ $+ \frac{r^{m-1}}{S} \times \overline{\sin.\ mQ \times M + \text{co-}s.\ mQ \times N}$. *Q. E. I.*

Corollary.

325. Hence, the fluent of $\frac{-ax^m\dot{x} + rx^{m-1}\dot{x}}{x^2 - 2arx + r^2}$ may be deduced: for, by writing $m-1$, instead of m, the fluent of $\frac{x^{m-1}\dot{x}}{x^2 - 2arx + r^2}$ will be found $= \frac{1}{S} \times \overline{\frac{Sx^{m-2}}{m-2} + \frac{\acute{S}rx^{m-3}}{m-3} + \frac{\acute{\acute{S}}r^2x^{m-4}}{m-4} (m-2)} + \frac{r^{m-2}}{S} \times \overline{\sin.\ \overline{m-1} \times Q \times M + \text{co-}s.\ \overline{m-1} \times Q \times N}$: which fluent being multiplied by r, and *that* of $\frac{x^m\dot{x}}{x^2 - 2arx + r^2}$ (given above) by $-a$, we shall, when the homologous terms are united, have $\frac{1}{S} \times (-aS \times \frac{x^{m-1}}{m-1} \overline{-a\acute{S} - S} \times \frac{rx^{m-2}}{m-2} - \overline{a\acute{\acute{S}} - \acute{S}} \times \frac{r^2x^{m-3}}{m-3} (m-1)) + \frac{r^{m-1}}{S}$ into $-\overline{\sin.\ mQ \times a - \sin.\ \overline{m-1} \times Q} \times M - (\text{co-}s.\ mQ \times a - \text{co-}s.\ \overline{m-1} \times Q) \times N$, for the true fluent of the quantity propounded.

But (*by Case* 1, *Lem.* 2) $\frac{a\acute{S} - S}{S}$ $(= \frac{\sin.\ 2Q \times a - \sin.\ Q \times rad.}{\sin.\ Q}) = \text{co-}s.\ 2Q$: also

$\frac{a\dot{S}-\dot{S}}{S}\left(=\frac{\text{sin. } 3\,Q\times a-\text{sin. } 2\,Q\times \text{rad.}}{S}\right)=\text{co-s. } 3Q$ &c. And, by *Case* 2. of the same *Lemma*, $\frac{\text{co-s. } \overline{m-1}\times Q-\overline{\text{co-s. } mQ}\times a}{S}=\text{sin. } m\,Q:$ whence, by substituting these values, our fluent is reduced to

$$-\text{co-s. } Q\times\frac{x^{m-1}}{m-1}-\text{co-s. } 2Q\times\frac{rx^{m-2}}{m-2}-\text{co-s. } 3Q\times\frac{r^2x^{m-3}}{m-3}-\text{co-s. } 4\,Q\times\frac{r^3x^{m-4}}{m-4}\,(m-1)-r^{m-1}\times\overline{\text{co-s. } mQ\times M-\text{sin. } MQ\times N}.$$

PROBLEM III.

326. *To determine the Fluent of* $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}$; *under the Restrictions specified in the preceding Problem.*

If the equation in Art. 321 be multiplied by $x^{-m}\dot{x}$, and v at the same time be expounded by m, we shall have $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}=$

$$\frac{Sr^{-2}x^{-m}\dot{x}+\dot{S}r^{-3}x^{1-m}\dot{x}+\ddot{S}r^{-4}x^{2-m}\dot{x}}{S}\,(m)+\frac{r^{-m-1}}{S}\times\frac{\text{sin. } \overline{m+1}\times Q\times r\dot{x}-\overline{\text{sin. } mQ}\times x\dot{x}}{r^2-2arx+x^2}:$$

where, the fluent of the last term being $\frac{r^{-m-1}}{S}\times$

$$\overline{\text{sin. } \overline{m+1}\times Q\times\frac{N}{S}-\overline{\text{sin. } m\,Q}\times\overline{M+\frac{aN}{S}}}\,{}^{*}=$$

$\frac{r^{-m-1}}{S}$ into $-\overline{\text{sin. } mQ}\times M+$

* Art. 323.

$\frac{\sin.\ \overline{m+1} \times Q - \sin.\ mQ \times a}{S} \times N = \frac{r^{-m-1}}{S} \times$

$\overline{-\sin.\ mQ \times M + \text{co-s.}\ mQ \times N}$ (by Case 3, Lem. 2) it follows that the fluent of the whole expression, or the quantity sought, will be truly expressed by

$\frac{1}{S} \times \left(\frac{Sr^{-2} x^{1-m}}{1-m} + \frac{Sr^{-3} x^{2-m}}{2-m} + \&c.\right)$ or its equal

$\frac{-1}{S} \times \left(\frac{Sx^{1-m}}{m-1 . r^2} + \frac{Sx^{2-m}}{m-2 . r^3} + \frac{Sx^{3-m}}{m-3 . r^4} (m)\right) +$

$\frac{1}{Sr^{m+1}} \times \overline{\text{co-sin.}\ mQ \times N - \sin.\ mQ \times M}.$

PROBLEM IV.

327. *To find the fluent of* $\frac{x^{m-1} \dot{x}}{r^n + x^n}$; *m and n being any whole positive Numbers, whereof the former does not exceed the latter.*

Let b, c, d, &c. denote the co-sines of the arcs $\frac{360°}{2n}$, $\frac{3 \times 360°}{2n}$, $\frac{5 \times 360°}{2n}$, &c. (radius being unity). Then (*by Corol.* 2, *Lem.* 3) we shall have $r^n + x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2drx + x^2}|^{\frac{1}{2}}$ (n). Whence $\log. \overline{r^n + x^n} = \frac{1}{2} \log. \overline{r^2 - 2brx + x^2} + \frac{1}{2} \log. \overline{r^2 - 2crx + x^2} + \frac{1}{2} \log. \overline{r^2 - 2drx + x^2}$ (n) and, consequently, by taking the fluxion on both sides, $\frac{nx^{n-1}\dot{x}}{r^n + x^n} = \frac{x\dot{x} - br\dot{x}}{x^2 - 2brx + r^2} + \frac{x\dot{x} - cr\dot{x}}{x^2 - 2crx + r^2} + \frac{x\dot{x} - dr\dot{x}}{x^2 - 2drx + r^2}$* (n); which last equation, multiplied by $\frac{x}{\dot{x}}$, gives $\frac{nx^n}{r^n + x^n} = \frac{x^2 - brx}{x^2 - 2brx + r^2} + \frac{x^2 - crx}{x^2 - 2crx + r^2}$

* Art. 126.

$+ \frac{x^2 - drx}{x^2 - 2drx + r^2}$ (n). Let each side hereof be now subtracted from n, (or, which comes to the same thing, let $\frac{nx^n}{r^n + x^n}$ be taken from n, and each of the (n) terms on the other side, from unity); then we shall have $\frac{nr^n}{r^n + x^n} = \frac{-brx + r^2}{x^2 - 2brx + r^2} + \frac{-crx + r^2}{x^2 - 2crx + r^2}$ $+ \frac{-drx + r^2}{x^2 - 2drx + r^2}$ (n): which multiplied by $\frac{x^{m-1}\dot{x}}{r}$, gives $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n + x^n} = \frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{x^2 - 2brx + r^2} +$ $\frac{-cx^m\dot{x} + rx^{m-1}\dot{x}}{x^2 - 2crx + r^2}$ &c.

But now, to determine the fluent hereof, let the several arcs $\left(\frac{180°}{n}, \frac{3 \times 180°}{n}, \frac{5 \times 180°}{n}\right.$ &c.) above specified, be denoted by Q, $\acute{Q}$, $\acute{\acute{Q}}$, $\acute{\acute{\acute{Q}}}$, &c. respectively; also let N, $\acute{N}$, $\acute{\acute{N}}$, &c. express the measures of the angles whose sines are $\frac{x \times sin.\, Q}{\sqrt{r^2 - 2brx + r^2}}$, $\frac{x \times sin.\, \acute{Q}}{\sqrt{x^2 - 2crx + r^2}}$, $\frac{x \times sin.\, \acute{\acute{Q}}}{\sqrt{x^2 - 2drx + r^2}}$ &c. and M, $\acute{M}$, $\acute{\acute{M}}$, &c. the hyperbolic logarithms of $\frac{\sqrt{x^2 - 2brx + r^2}}{r}$ $\frac{\sqrt{x^2 - 2crx + r^2}}{r}$, $\frac{\sqrt{x^2 - 2drx + r^2}}{r}$ &c. Then (*by Corol. to Prob.* 2) the fluent of the first term, $\frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{x^2 - 2brx + r^2}$ (expounding a by b) comes out

$$-\text{co-s.}\ Q \times \frac{x^{m-1}}{m-1} - \text{co-s.}\ 2Q \times \frac{rx^{m-2}}{m-2} - \text{co-s.}3\ Q \times \frac{r^2x^{m-3}}{m-3}\ (m-1) + r^{m-1} \text{ into } \overline{\text{sin.}\ mQ} \times N - \overline{\text{co-s.}\ mQ} \times M.$$

In the same manner, by writing c for a, $\acute{Q}$ for Q, $\acute{M}$ for M, and $\acute{N}$ for N) the fluent of the second term, $\frac{-cx^m\dot{x}+rx^{m-1}\dot{x}}{x^2-2crx+r^2}$, is found $= -$ co-s. $\acute{Q} \times \frac{x^{m-1}}{m-1} -$ co-s. $2\acute{Q} \times \frac{rx^{m-2}}{m-2}$ &c. &c.

Therefore the fluent of the whole expression, by collecting the homologous terms, appears to be

$$-\left\{\begin{array}{l}\text{co-s. } Q\\ \text{co-s. } Q'\\ \text{co-s. } Q''\\ \text{co-s. } Q'''\\ \text{\&c.}\end{array}\right\} \times \frac{x^{m-1}}{m-1} - \left\{\begin{array}{l}\text{co-s. } 2Q\\ \text{co-s. } 2Q'\\ \text{co-s. } 2Q''\\ \text{co-s. } 2Q'''\\ \text{\&c.}\end{array}\right\} \times \frac{rx^{m-2}}{m-2} - \left\{\begin{array}{l}\text{co-s. } 3Q\\ \text{co-s. } 3Q'\\ \text{co-s. } 3Q''\\ \text{co-s. } 3Q'''\\ \text{\&c.}\end{array}\right\} \times \frac{r^2x^{m-3}}{m-3}\ \text{\&c.}$$

$$+ r^{m-1} \times \begin{cases} \overline{sin.\ mQ} \times N - \overline{co\text{-}s.\ mQ} \times M \\ \overline{sin.\ mQ'} \times N' - \overline{co\text{-}s.\ mQ'} \times M' \\ \overline{sin.\ mQ''} \times N'' - \overline{co\text{-}s.\ mQ''} \times M'' \\ \overline{sin.\ mQ'''} \times N''' - \overline{co\text{-}s.\ mQ'''} \times M''' \\ \qquad \&c. \qquad\qquad \&c. \end{cases}$$

But the co-sines of the first column being those of an arithmetical progression $(\frac{180^\circ}{n}, \frac{3 \times 180^\circ}{n}, \frac{5 \times 180^\circ}{n},$ &c. whose common difference is $\frac{360^\circ}{n}$, whereby the whole periphery is divided into n equal parts (*vide* Art. 317.) they will therefore destroy one another; since it is well known that if the periphery of any circle be divided into any number (n) of equal parts, the negative sines and co-sines will be equal to the positive ones; which is self-evident when their number is even.

Hence the co-sines in the second and third columns, &c. will also destroy one another (*vide* Art. 318.). But *those* of the last column of all, as well as the sines, having unequal multiplicators, must remain as above, and that column, *alone*, (drawn into r^{m-1}) will be the true fluent of $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n + x^n}$. Whence, putting mQ $(= m \times \frac{180^\circ}{n}) = R$, and dividing by nr^{n-1}, we shall (because $Q' = 3Q$, $Q'' = 5Q$, $Q''' = 7Q$ &c.) have

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{c} \overline{\sin. R} \times N - \overline{\text{co-s}. R} \times M \\ \overline{\sin. 3R} \times N' - \overline{\text{co-s}. 3R} \times M' \\ \overline{\sin. 5R} \times N'' - \overline{\text{co-s}. 5R} \times M'' \\ \overline{\sin. 7R} \times N''' - \overline{\text{co-s}. 7R} \times M''' \\ \&c. \text{ (to } n \text{ lines).} \end{array} \right\} = \text{fluent of } \frac{x^{m-1}\dot{x}}{r^n + x^n}$$

Q. E. I.

Corollary.

328. Since the first and the last, the second and the last but one, &c. of the foregoing quantities $x^2 - 2brx + r^2$, $x^2 - 2crx + r^2$, $x^2 - 2drx + r^2$, &c. are respectively equal to each other (*vide* Art. 317), the corresponding fluents, found above, will likewise be equal: and therefore the fluent of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ will *also* be expressed by

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{c} \overline{\sin. R} \times 2N - \overline{\text{co-s}. R} \times 2M \\ \overline{\sin. 3R} \times 2N' - \overline{\text{co-s}. 3R} \times 2M' \\ \overline{\sin. 5R} \times 2N'' - \overline{\text{co-s}. 5R} \times 2M'' \\ \&c. \qquad \&c. \end{array} \right.$$

The number of lines to be thus taken being $= \frac{1}{2} n$, when n is even; but, otherwise, $= \frac{n+1}{2}$; in which last case, the logarithm, &c. in the last line, must be taken only once, instead of twice; being that of $\frac{r+x}{r}$ (*vide* Art. 317).

PROBLEM V.

329. *To find the Fluent of* $\frac{x^{m-1}\dot{x}}{r^n - x^n}$; m *and* n *being as in the preceding Problem.*

If b, c, d, &c. be taken to denote the co-sines of the arcs $\frac{0}{n}$, $\frac{360°}{n}$, $\frac{2 \times 360°}{n}$ &c. to n terms, it will appear (*from Corol.* 1 *to Lem.* 3) that $r^n - x^n$ is $=$ $\overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2arx + x^2}|^{\frac{1}{2}}$ (n). From whence, by following the method of the last problem, we also have $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n - x^n} =$ $\frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{x^2 - 2brx + r^2} + \frac{-cx^m\dot{x} + rx^{m-1}\dot{x}}{x^2 - 2crx + r^2}$ &c.

Which fluxion having exactly the same *form* with *that* in the preceding Problem, its fluent will also be expressed in the very same manner, that is, by

$$r^{m-1} \times \begin{cases} \sin.\ mQ \times N - \text{co-}s.\ mQ \times M \\ \sin.\ m\acute{Q} \times \acute{N} - \text{co-}s.\ m\acute{Q} \times \acute{M} \\ \sin.\ m\acute{\acute{Q}} \times \acute{\acute{N}} - \text{co-}s.\ m\acute{\acute{Q}} \times \acute{\acute{M}} \\ (\&c.\ to\ n\ Lines). \end{cases}$$

Only Q, $\acute{Q}$, $\acute{\acute{Q}}$ &c. must here stand for $\frac{0}{n}$, $\frac{360°}{n}$, $\frac{2 \times 360°}{n}$, $\frac{3 \times 360°}{n}$ &c. (instead of $\frac{180°}{n}$, $\frac{3 \times 180°}{n}$, $\frac{5 \times 180°}{n}$ &c.)

Therefore, since the multiple arcs mQ, $m\acute{Q}$, $m\acute{\acute{Q}}$ &c. are, in this case, equal to 0, $m \times \frac{360^\circ}{n}$, $2m \times \frac{360^\circ}{n}$, $3m \times \frac{360^\circ}{n}$ &c. (whereof the sine of the first is $= 0$, and its co-sine = unity) we shall, by putting $R = m \times \frac{360^\circ}{n}$, and dividing the foresaid fluent by $n r^{n-1}$, have

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} * \ldots\ldots - \ldots\ldots M \\ sin.\ R \times \acute{N} - co\text{-}s.\ R \times \acute{M} \\ sin.\ 2R \times \acute{\acute{N}} - co\text{-}s.\ 2R \times \acute{\acute{M}} \\ sin.\ 3R \times \acute{\acute{\acute{N}}} - co\text{-}s.\ 3R \times \acute{\acute{\acute{M}}} \\ (\&c.\ to\ n\ Lines). \end{array} \right\} = \text{fluent of } \frac{x^{m-1}\dot{x}}{r^n - x^n}$$

Q. E. I.

Corollary.

330. Since in the fluent here given, the second line and the last, the third and the last but one, &c. are respectively equal (*vide* Art. 317) the same may also be exhibited, thus:

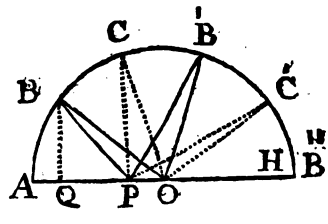
$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} * \ldots\ldots\ldots - \ldots\ldots \quad M \\ sin.\ R \times 2\acute{N} - co\text{-}s.\ R \times 2\acute{M} \\ sin.\ 2R \times 2\acute{\acute{N}} - co\text{-}s.\ 2R \times 2\acute{\acute{M}} \end{array} \right.$$

(&c. to $\frac{n+1}{2}$ lines).

Scholium.

331. If the semi-periphery A B C H of the circle whose diameter A H is $2r$, be divided into as many

equal parts A B, B C, C B', B' C' &c. as there are units in n (so that $AB = \frac{180°}{n} = Q$, $AB' = 3 \times \frac{180°}{n} = Q'$



&c. (*vide* Art. 317, and 327), and in the radius O A (produced, if necessary) there be taken $OP = x$, and P B, O B &c. be drawn, it will appear (from the said articles, and from Prop. 1.) that the quantities $\sqrt{r^2-2brx+x^2}$, $\sqrt{r^2-2crx+x^2}$ &c. in the former of the two preceding Problems, will here be expounded by P B, P B' &c. respectively: from whence it is also plain, that the measures N, N' &c. of the angles whose sines are $\frac{x \times \sin. Q}{\sqrt{r^2-2brx+x^2}}$, $\frac{x \times \sin. Q'}{\sqrt{r^2-2crx+x^2}}$ &c.* will here be expounded by OBP, OB'P, &c. &c.

Therefore the fluent of $\frac{x^{m-1}\dot{x}}{r^n+x^n}$, given in the corollary to the foresaid proposition, may be thus exhibited,

$$\frac{r^{m-n}}{n} \times \begin{cases} \overline{\sin. R} \times 2(OBP) - \overline{\cos. R} \times 2(OA : PB) \\ \overline{\sin. 3R} \times 2(OB'P) - \overline{\cos. 3R} \times 2(OA : PB') \\ \&c. \qquad\qquad\qquad \&c. \end{cases}$$

Where the arch R is $\left(= m \times \frac{180°}{n}\right) = m \times AB$, and where $(OA : PB)$ is put (after the manner of *Cotes*) to express the hyperbolical logarithm of $\frac{PB}{OA}$. It is also to be observed, *that*, when the last of the points B,

* Art. 322. & 323.

$\acute{B}$, $\acute{\acute{B}}$ &c. falls upon H (which will always happen when n is an odd number) the angle, in the last line of the fluent, will vanish, and the corresponding logarithm (which is that of $\frac{PH}{AO}$) must then be taken, instead of twice, only once.

In the very same manner it will appear, *that*, the arcs $\acute{Q}$, $\acute{\acute{Q}}$ &c. in the second case, where the fluent of $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ is sought, will be, respectively, expounded by AC, $A\acute{C}$ &c. also the corresponding angles $\acute{N}$, $\acute{\acute{N}}$ &c. by OCP, $O\acute{C}P$ &c. and the fluent itself by

$$\frac{r^{m-n}}{n} \times \begin{cases} * \quad \ldots\ldots\ldots \quad - \quad \ldots\ldots \quad (OA:PC) \\ \overline{\sin.\ R} \times 2\,(OCP) - \overline{\cos.\ R} \times 2\,(OA:PC) \\ \overline{\sin.\ 2R} \times 2\,(O\acute{C}P) - \overline{\cos.\ 2R} \times 2\,(OA:P\acute{C}) \\ \quad \&c. \qquad\qquad\qquad\qquad \&c. \end{cases}$$

where the arch R $\left(= m \times \frac{360^\circ}{n}\right) = m \times AC$; and where, as well as in the preceding case, all the arcs, sines and cosines are supposed to have unity for their radius.

332. From the fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$, thus given, *those* of $\frac{x^{vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{-vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{vn+m-1}\dot{x}}{r^n - x^n}$, and $\frac{x^{-vn+m-1}\dot{x}}{r^n - x^n}$, where v denotes any whole number, may be very easily deduced; either from Art. 283, and 291, or (more readily) by dividing the numerator by the denominator, and continuing the

quotient to as many terms as there are units in v.* By which means, if p be put $= vn + m$, $q = vn - m$, and the fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ be denoted by V and W respectively, the fluents, in the four cases specified above, will be expressed by

* Art. 150.

$$\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} + \frac{r^{2n} x^{p-3n}}{p-3n} (v) \pm r^{vn} V,$$

$$\frac{x^{-q}}{-qr^n} - \frac{x^{n-q}}{\overline{n-q}.\, r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}.\, r^{3n}} (v) \pm \frac{V}{r^{vn}},$$

$$-\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} - \frac{r^{2n} x^{p-3n}}{p-3n} (v) + r^{vn} W,$$

and, $\frac{x^{-q}}{qr^n} + \frac{x^{n-q}}{\overline{n-q}.\, r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}.\, r^{3n}} (v) + \frac{W}{r^{vn}}$,

respectively.

Moreover, from the same fluents, *those* of $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e + fz^q}$, and $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e - fz^q}$ will likewise become known:

For (having transformed the fluxions here proposed to $\frac{1}{e} \times \frac{z^{\frac{m}{n}q-1}\dot{z}}{1 + \frac{fz^q}{e}}$, &c.) let $\frac{fz^q}{e}$ be put $= x^n$, or $x = \overline{\frac{fz^q}{e}}\Big|^{\frac{1}{n}}$; then will $z^{\frac{m}{n}q} = \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times x^m$, and consequently $\frac{mq}{n} \times z^{\frac{m}{n}q-1}\dot{z} = \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times m x^{m-1}\dot{x}$.

Whence $z^{\frac{m}{n}q-1}\dot{z} = \frac{n}{q} \times \left(\frac{e}{f}\right)^{\frac{m}{n}} \times x^{m-1}\dot{x}$, and $1 \pm \frac{fz^q}{e} = 1 \pm x^n$; and therefore $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e \pm fz^q}$ $\left(= \frac{1}{e} \times \frac{n}{q} \times \left(\frac{e}{f}\right)^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n}\right) = \frac{n}{qe} \times \left(\frac{e}{f}\right)^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n}$: whose fluent is given, by Prob. 4 or 5. But, r being here $= 1$, the general multiplicator $\frac{r^{m-n}}{n}$, there given, will be barely $= \frac{1}{n}$: which, drawn into $\frac{n}{qe} \times \left(\frac{e}{f}\right)^{\frac{m}{n}}$, gives $\frac{1}{qe} \times \left(\frac{e}{f}\right)^{\frac{m}{n}}$, for the general multiplicator in this case.

One thing more, though well known to Mathematicians, it may be proper here to take notice of; and that relates to the sines and co-sines of the fore-mentioned arcs, R, $2R$, $3R$, &c. &c. (multiplying the several angles and ratios) some of which arcs do frequently exceed the whole periphery: when this happens to be the case, the periphery, or 360°, must be subtracted as often as possible, and the sine and co-sine of the remainder be taken. If the remainder be greater than 180°, the sine, falling in the lower semi-circle, will be negative; if, between 90° and 270°, the co-sine, falling beyond the center, will be negative.

PROBLEM VI.

333. *To find the Fluent of* $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^nx^n + x^{2n}}$; *where* n *and* m *denote any whole positive Numbers, and where the given Expression cannot be resolved into two Binomials* (k *being less than Unity.* Art. 308 and 310).

Let R be the arch whose co-sine is k, and radius unity, and let k' be the sine of the same arch; moreover, let the arcs $\frac{R}{n}$, $\frac{R+360°}{n}$, $\frac{R+2\times 360°}{n}$, $\frac{R+3\times 360°}{n}$ &c. be denoted by Q, $\dot{Q}$, $\ddot{Q}$ $\dddot{Q}$, $\ddddot{Q}$, &c. and let b', c', d' &c. and b, c, d &c. express the sines, and the co-sines of the same arcs respectively.

Then will $\frac{nk'r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{b'rx}{r^2-2brx+x^2} + \frac{c'rx}{r^2-2crx+x^2} + \frac{d'rx}{r^2-2drx+x^2}$ &c. (n) (*by Lemma* 4) From whence, multiplying the whole equation by $\frac{x^{m-1}\dot{x}}{nk'r^n}$ we have $\frac{x^{n+m-1}\dot{x}}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{1}{nk'r^{n-1}}$ into $\frac{b'x^m\dot{x}}{r^2-2brx+x^2} + \frac{c'x^m\dot{x}}{r^2-2crx+x^2} + \frac{d'x^m\dot{x}}{r^2-2drx+x^2}$ &c.

Now, the fluent of the first term hereof $\frac{b'x_m\dot{x}}{r^2-2brx+x^2}$ (if M be put for the hyp. log. of $\frac{\sqrt{x^2-2brx+x^2}}{r}$, and N for the arch whose radius is unity, and sine $\frac{x\times\overline{\sin. Q}}{\sqrt{r^2-2brx+x^2}}$) will appear (*from Prop.* 2) to be $= \overline{\sin. Q}\times\frac{x^{m-1}}{m-1} + \overline{\sin. 2Q}\times\frac{rx^{m-2}}{m-2} + \overline{\sin. 3Q}\times\frac{r^2x^{m-3}}{m-3}$ &c. $(m-1) + r^{m-1}\times(\sin. mQ\times M + \text{co-sin.}\ mQ\times N)$.

From whence, if the arcs whose sines are

$\frac{x \times sin. \acute{Q}}{\sqrt{r^2 - 2crx + x^2}}, \frac{x \times sin. \acute{\acute{Q}}}{\sqrt{r^2 - 2drx + x^2}}$ &c. be represented by $\acute{M}$, $\acute{\acute{M}}$ &c. and the logarithms whose numbers are $\frac{\sqrt{r^2-2crx + x^2}}{r}$, $\frac{\sqrt{r^2-2drx + x^2}}{r}$ &c. by $\acute{N}$, $\acute{\acute{N}}$ &c. respectively, the fluent of the whole expression, omitting the general multiplicator $\left(\frac{1}{nkr^{n-1}}\right)$ will be

$$\left\{\begin{matrix} sin.\ Q \\ sin.\ \acute{Q} \\ sin.\ \acute{\acute{Q}} \\ sin.\ \acute{\acute{\acute{Q}}} \\ \&c. \end{matrix}\right\} \times \frac{x^{m-1}}{m-1} + \left\{\begin{matrix} sin.\ 2Q \\ sin.\ 2\acute{Q} \\ sin.\ 2\acute{\acute{Q}} \\ sin.\ 2\acute{\acute{\acute{Q}}} \\ \&c. \end{matrix}\right\} \times \frac{rx^{m-2}}{m-2} + \left\{\begin{matrix} sin.\ 3Q \\ sin.\ 3\acute{Q} \\ sin.\ 3\acute{\acute{Q}} \\ sin.\ 3\acute{\acute{\acute{Q}}} \\ \&c. \end{matrix}\right\}$$

$$\times \frac{r^2x^{m-3}}{m-3} \text{ (\&c. to } m-1 \text{ terms)}$$

$$+ r^{m-1} \times \left\{\begin{matrix} \overline{sin.\ mQ} \times M + \overline{co\text{-}s.\ mQ} \times N \\ \overline{sin.\ m\acute{Q}} \times \acute{M} + \overline{co\text{-}s.\ m\acute{Q}} \times \acute{N} \\ \overline{sin.\ m\acute{\acute{Q}}} \times \acute{\acute{M}} + \overline{co\text{-}s.\ m\acute{\acute{Q}}} \times \acute{\acute{N}} \\ \overline{sin.\ m\acute{\acute{\acute{Q}}}} \times \acute{\acute{\acute{M}}} + \overline{co\text{-}s.\ m\acute{\acute{\acute{Q}}}} \times \acute{\acute{\acute{N}}} \\ \&c. \qquad \&c. \end{matrix}\right\}$$

But, the sines of the first column being those of an arithmetical progression (whose common difference is

$\frac{360^\circ}{n}$) which arises by dividing the whole periphery into n equal parts, their sum will, therefore, be equal to nothing.

Moreover, the sines of the second column, having $\frac{2 \times 360^\circ}{n}$ for the common difference of their respective arcs do also divide the whole periphery (twice taken) into n equal parts, and therefore destroy each other.

The same is likewise true with regard to the sines of every other column (except the last of all) when $m-1$ is less than n. But, if m be greater than n, the arcs in the column, whose place from the first, inclusive, is denoted by n, being expressed by nQ, $n\acute{Q}$, $n\ddot{Q}$, &c. (or R, $R+360^\circ$, $R+2\times360^\circ$ &c.) whereof the common difference is the whole periphery; the sines of that column do not destroy one another, but each is equal to that of the first arc R (*vide* Art. 314 and 318) and consequently their sum equal to $n \times sin. R$.

In like manner, if m be greater than $2n$, the series, continued to $m-1$ terms, will take in the column, where the arcs are $2nQ$, $2n\acute{Q}$, $2n\ddot{Q}$ &c. (or $2R$, $2R + 2 \times 360^\circ$, $2R+4\times360^\circ$ &c.) whereof the sine of each is also equal to the sine of the first ($2R$) and therefore their sum $= n \times sin.\ 2R$.

Thus also it will appear that the sines of the column whose distance from the first, inclusive, is $3n$ (when m is greater than $3n$) will be each equal to $sin.\ 3R$; &c. &c.

Therefore, seeing all the columns do actually vanish, except those above specified, whose places from the beginning are denoted by n, $2n$, $3n$ &c. and whose corresponding terms, or multiplicators, are therefore represented by $\frac{r^{n-1}x^{m-n}}{m-n}$, $\frac{r^{2n-1}x^{m-2n}}{m-2n}$, $\frac{r^{3n-1}x^{m-3n}}{m-3n}$ &c. it is evident that the whole expression will be reduced to

$$\overline{sin.\ R} \times \frac{nr^{n-1}\,x^{m-n}}{m-n} + \overline{sin.\ 2R} \times \frac{nr^{2n-1}\,x^{m-2n}}{m-2n}$$

$$+ \overline{sin.\ 3R} \times \frac{nr^{3n-1}\,x^{m-3n}}{m-3n} \ \&c.$$

$$+ r^{n-1} \text{ into } \begin{cases} sin.\ mQ \times M + co\text{-}s.\ mQ \times N \\ sin.\ m\acute{Q} \times \acute{M} + co\text{-}s.\ m\acute{Q} \times \acute{N} \\ sin.\ m\ddot{Q} \times \ddot{M} + co\text{-}s.\ m\ddot{Q} \times \ddot{N} \\ sin.\ m\dddot{Q} \times \dddot{M} + co\text{-}s.\ m\dddot{Q} \times \dddot{N} \\ \&c. \qquad \&c. \end{cases}$$

Which, multiplied by $\frac{1}{nkr^{n-1}}$, the foresaid general multiplicator, gives $\overline{sin.\ R} \times \frac{x^{m-n}}{\overline{m-n}\,.\,k} + \overline{sin.\ 2R} \times \frac{r^n x^{m-2n}}{\overline{m-2n}\,.\,k} + \overline{sin.\ 3R} \times \frac{r^{2n} x^{m-3n}}{\overline{m-3n}\,.\,k}$ &c.

$$+ \frac{r^{m-n}}{nk} \times \begin{cases} sin.\ mQ \times M + co\text{-}s.\ mQ \times N \\ sin.\ m\acute{Q} \times \acute{M} + co\text{-}s.\ m\acute{Q} \times \acute{N} \\ sin.\ m\ddot{Q} \times \ddot{M} + co\text{-}s.\ m\ddot{Q} \times \ddot{N} \\ sin.\ m\dddot{Q} \times \dddot{M} + co\text{-}s.\ m\dddot{Q} \times \dddot{N} \\ \&c. \qquad \&c. \end{cases}$$

for the true fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$: where the former part of the expression must be continued to as many terms as there are units in $\frac{m-1}{n}$ (the remainder, if any, being neglected). *Q. E. I.*

Corollary

334. If the quotient arising from the division of m by n (when the former exceeds) be denoted by v, and the remainder by t; or, which is the same, if $vn + t = m$, it is evident the arcs $m\acute{Q}$, $m\acute{\acute{Q}}$, $m\acute{\acute{\acute{Q}}}$ &c. which are respectively equal to $mQ + m \times \frac{360°}{n}$, $mQ + 2m \times \frac{360°}{n}$, $mQ + 3m \times \frac{360°}{n}$, &c. (by construction) will also be equal to $mQ + v \times 360° + t \times \frac{360°}{n}$, $mQ + 2v \times 360° + 2t \times \frac{360°}{n}$ &c. whereof the sines and co-sines (omitting $v \times 360°$, $2v \times 360°$ &c. the multiples of the whole periphery) are the same with those of $mQ + t \times \frac{360°}{n}$, $mQ + 2t \times \frac{360°}{n}$ &c. respectively.

Therefore, if the arcs of the progression, whereof the first term is mQ, and the common difference $t \times \frac{360°}{n}$, be represented by T, $\acute{T}$, $\acute{\acute{T}}$ &c. respectively; it follows that the fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$ (or $\frac{x^{n+vn+t-1}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$) will, *also*, be truly expressed by

$$\overline{\sin. R} \times \frac{x^{m-n}}{\overline{m-n}\,.\,\acute{k}} + \overline{\sin. 2R} \times \frac{r^n x^{m-2n}}{\overline{m-2n}\,.\,\acute{k}} + \overline{\sin. 3R} \times \frac{r^{2n} x^{m-3n}}{\overline{m-3n}\,.\,\acute{k}} \text{ \&c. } \left(\frac{m-1}{n}\right)$$

$$+\frac{r^{m-n}}{n\acute{k}}\left\{\begin{array}{l} \sin. T\times M+\text{co-s}. T\times N \\ \sin. \acute{T}\times \acute{M}+\text{co-s}. \acute{T}\times \acute{N} \\ \sin. \acute{\acute{T}}\times \acute{\acute{M}}+\text{co-s}. \acute{\acute{T}}\times \acute{\acute{N}} \\ \sin. \acute{\acute{\acute{T}}}\times \acute{\acute{\acute{M}}}+\text{co-s}. \acute{\acute{\acute{T}}}\times \acute{\acute{\acute{N}}} \\ \&c. \qquad \&c. \end{array}\right.$$

In the very same manner the fluent of $\frac{x^{n+m-1}}{r^{2n}+2kr^{n}x^{n}+x^{2n}}$ (where the sine of the second term is positive) will be exhibited; if R be taken to denote the arch whose co-sine is $-k$; which will, in this case, be greater than a quadrant.

PROPOSITION VII.

335. *To find the Fluent of* $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}$; *under the Restrictions mentioned in the last Problem.*

Let every thing remain as before; then we shall have $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}=\frac{1}{n\acute{k}r^{n-1}}$ into $\frac{\acute{b}x^{-m}\dot{x}}{r^{2}-2brx+x^{2}}$

$+\frac{\acute{c}x^{-m}\dot{x}}{r^{2}-2crx+x^{2}}$ (n) whereof the fluent (*by Prob.* 3) appears to be $\frac{1}{n\acute{k}r^{n-1}}$ into

$$-\left\{\begin{array}{l}\sin. Q \\ \sin. \acute{Q} \\ \sin. \acute{\acute{Q}} \\ \&c.\end{array}\right\}\times\frac{x^{1-m}}{m-1 . r^{2}}-\left\{\begin{array}{l}\sin. 2Q \\ \sin. 2\acute{Q} \\ \sin. 2\acute{\acute{Q}} \\ \&c.\end{array}\right\}\times\frac{x^{2-m}}{m-2\times r^{3}}$$

$$- \left\{ \begin{matrix} \sin. 3Q \\ \sin. 3\acute{Q} \\ \sin. 3\acute{\acute{Q}} \\ \&c. \end{matrix} \right\} \times \frac{x^{3-m}}{\overline{m-3}.\ r^4}\ (m)\ +$$

$$\frac{1}{r^{m+1}} \times \left\{ \begin{matrix} -\sin. mQ \times M + \text{co-s}. mQ \times N \\ -\sin. m\acute{Q} \times \acute{M} + \text{co-s}. m\acute{Q} \times \acute{N} \\ -\sin. m\acute{\acute{Q}} \times \acute{\acute{M}} + \text{co-s}. m\acute{\acute{Q}} \times \acute{\acute{N}} \\ \&c. \qquad \&c. \end{matrix} \right.$$

Which, by reasoning as above, will be reduced to

$$- \overline{\sin. R} \times \frac{x^{n-m}}{\overline{m-n}.\ kr^{2n}} - \overline{\sin. 2R} \times \frac{x^{2n-m}}{\overline{m-2n}.\ kr^{3n}}$$

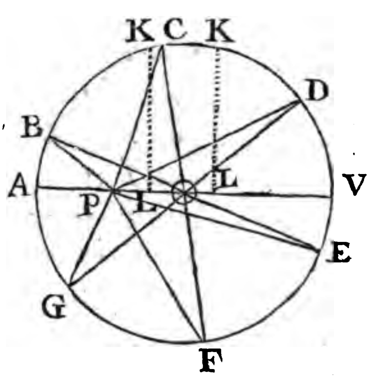
$$- \overline{\sin. 3R} \times \frac{x^{3n-m}}{\overline{m-3n}.\ kr^{4n}}\ (\text{to } \frac{m}{n} \text{ terms})$$

$$+ \frac{1}{knr^{n+m}} \times \left\{ \begin{matrix} -\sin. T \times M + \text{co-s}. T \times N \\ -\sin. \acute{T} \times \acute{M} + \text{co-s}. \acute{T} \times \acute{N} \\ -\sin. \acute{\acute{T}} \times \acute{\acute{M}} + \text{co-s}. \acute{\acute{T}} \times \acute{\acute{N}} \\ \&c. \qquad \&c. \end{matrix} \right.$$

Q. E. I.

SCHOLIUM.

336. If, from the center O, of the circle ABCD, whose radius OA, or OV, is r, there be taken O L equal to k and O P$=x$; and if the arch A B be to the arch A K, whose co-sine is $\pm k$, as 1 to n; and each



of the arcs B C, C D, D E, &c. be taken equal to $\frac{360^\circ}{n}$ &c. &c. Then the angles R, Q, $\acute{Q}$, &c. specified (in the two preceding Problems) being here expounded by AK, AB, AC &c. respectively, we have $PB = \sqrt{r^2 - 2brx + x^2}$, $PC = \sqrt{r^2 - 2crx + x^2}$ &c. (*vide* Art. 317 and 323.). Whence, also, the angles N, $\acute{N}$, $\acute{\acute{N}}$ &c. whose sines are $\frac{x \times sin.\, Q}{\sqrt{r^2 - 2brx + x^2}}$, $\frac{x \times sin.\, \acute{Q}}{\sqrt{r^2 - 2crx + x^2}}$, $\frac{x \times sin.\, \acute{\acute{Q}}}{\sqrt{r^2 - 2drx + x^2}}$ &c. will here be equal to B, C, D, &c. Therefore the fluents of $\frac{x^{n+m-1}\dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$, and $\frac{x^{n-m-1}\dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$ (there given) will, also, be truly defined by

$$\frac{x^{m-n}}{m-n} + \frac{sin.\, 2R}{sin.\, R} \times \frac{r^n x^{m-2n}}{m-2n} + \frac{sin.\, 3R}{sin.\, R} \times \frac{r^{2n} x^{m-3n}}{m-3n}$$

(to $\frac{m-1}{n}$ terms)

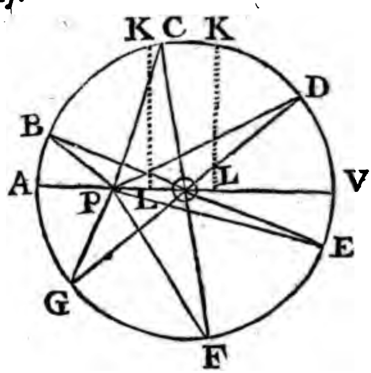
$$+ \frac{r^{m-n}}{n \times sin. R} \times \begin{cases} sin.\, T \times (OB : PB) + co\text{-}s.\, T \times (B) \\ sin.\, \acute{T} \times (OC : PC) + co\text{-}s.\, \acute{T} \times (C) \\ sin.\, \acute{\acute{T}} \times (OD : PD) + co\text{-}s.\, \acute{\acute{T}} \times (D) \\ sin.\, \acute{\acute{\acute{T}}} \times (OE : PE) + co\text{-}s.\, \acute{\acute{\acute{T}}} \times (-E) \\ sin.\, \acute{\acute{\acute{\acute{T}}}} \times (OF : PF) + co\text{-}s.\, \acute{\acute{\acute{\acute{T}}}} \times (-F) \\ \&c. \qquad\qquad \&c. \end{cases}$$

And by $-\dfrac{x^{n-m}}{m-n \,.\, r^{2n}} - \dfrac{\sin.\ 2R}{\sin.\ R} \times \dfrac{x^{2n-m}}{m-2n \,.\, r^{3n}} -$

$-\dfrac{\sin.\ 3R}{\sin.\ R} \times \dfrac{x^{3n-m}}{m-3n \,.\, r^{4n}} \left(\dfrac{m}{n}\right)$

$$+\frac{1}{nr^{n+m} \times \sin.\ R} \times \begin{cases} -\sin. T \times (OB : PB) + \text{co-s}. T \times (B) \\ -\sin. T' \times (OC : PC) + \text{co-s}. T' \times (C) \\ -\sin. T'' \times (OD : PD) + \text{co-s}. T'' \times (D) \\ -\sin. T''' \times (OE : PE) + \text{co-s}. T''' \times (-E) \\ -\sin. T'''' \times (OF : PF) + \text{co-s}. T'''' \times (-F) \\ \quad \&c. \end{cases}$$

respectively.



Where the arc AK (or R) will be greater than a quadrant when the sine of k is positive, but less when negative; and where the arcs T, T', T'' &c. denote an Arithmetical Progression, whose first term (T) is equal to $m \times AB$, and whereof the common difference is equal to $\frac{360°}{n}$ (or BC) multiplied by m, when m is less than n; but otherwise by the remainder of m divided by n.

337. Hence the fluent of $\frac{z^{q \pm \frac{m}{n}q-1}\dot{z}}{e \mp fz^q + gz^{2q}}$, where q is any number, either whole or broken, may be very easily deduced: for, having transformed the denominator to $g \times \overline{\frac{e}{g} \mp \frac{fz^q}{g} + z^{2q}}$, put $\frac{e}{g} = r^{2n}$, $\frac{f}{g} = 2kr^n$, and $z^q = x^n$; and then it will become $= g \times \overline{r^{2n} \mp 2kr^n x^n + x^{2n}}$: moreover, $z^{q \pm \frac{m}{n}q}$ being $= \overline{x^n}|^{1 \pm \frac{m}{n}} = x^{n \pm m}$, and $\overline{q \pm \frac{m}{n}q} \times z^{q \pm \frac{m}{n}q-1}\dot{z} = \overline{n \pm m} \times x^{n \pm m-1}\dot{x}$, the numerator will be reduced to $\frac{n}{q} \times x^{n \pm m-1}\dot{x}$: and so, we have $\frac{z^{q \pm \frac{m}{n}q-1}\dot{z}}{e \mp fz^q + gz^{2q}} = \frac{n}{qg} \times \frac{x^{n \pm m-1}\dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$: in which $x = z^{\frac{q}{n}}$, $r = \overline{\frac{e}{g}}|^{\frac{1}{2n}}$, and $k \left(= \frac{\frac{1}{2}f}{gr^n}\right) = \frac{\frac{1}{2}f}{\sqrt{eg}}$. But, it may be observed, that the fluent hereof is only given when $\frac{\frac{1}{2}f}{\sqrt{eg}}$ (or its equal k) is less than unity.* Therefore, if $\frac{1}{2}f$ be greater than $\sqrt{eg}$; or if the values of e and g are unlike, with regard to positive and negative, so that $\sqrt{eg}$ is impossible, the above solution fails. But here the given trinomial may be resolved into two binomials (*by Art.* 310) and, from thence, the fluent may be found at two operations (*by Prob.* 4 *and* 5).

* Art. 333.

For, by feigning $e \mp fy + gy^2 = 0$, in order to such a resolution, we get $\frac{\pm \frac{1}{2}f + \sqrt{\frac{1}{4}f^2 - eg}}{g}$, and $\frac{\pm \frac{1}{2}f - \sqrt{\frac{1}{4}f^2 - eg}}{g}$ for the roots of that equation, or the two first terms of the required binomials: which therefore are always possible when $\frac{1}{4}f^2 - eg$ is positive, or when the foregoing solution fails.

By denoting the said roots by H and K, the trinomial $e \mp fz^q + gz^{2q}$ is resolved into $g \times \overline{H - z^q} \times \overline{K - z^q}$, from whence $\frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{e \mp fz^q + gz^{2q}}$ is reduced to

$$\frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{g \times \overline{K - H} \times \overline{H - z^q}} + \frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{g \times \overline{H - K} \times \overline{K - z^q}},$$ whose fluent is given by Art. 332.

338. By proceeding the same way the fluent of $\frac{z^{\frac{m}{n}q - 1}\dot{z}}{e + fz^q + gz^{2q} + hz^{3q}}$ may likewise be found: for, since one, at least, of the three roots of the equation $e + fy + gy^2 + hy^3 = 0$, must be possible, the proposed fluxion, if it cannot be resolved into three binomials, may, however, be reduced to one binomial and one trinomial; and so, be brought under the foregoing forms: but this being a speculation too much out of the way of common use to be farther pursued, I shall here conclude this section, with observing, that, when k, in the original trinomial, above specified, is neither less, nor greater than unity, the fluent cannot then be had directly, from either of the preceding methods; but must be found by comparison from the fluent of $\frac{x^{n \pm m - 1}\dot{x}}{r^n \pm x^n}$. *Vide* Art. 289.

SECTION VI.

The Manner of investigating Fluents, when Quantities, and their Logarithms; Arcs and their Sines, &c. are involved together: with other cases of the like Nature.

PROBLEM I.

339. *SUPPOSING Q and n to denote given Quantities; it is proposed to find the Fluent of* $x^n \dot{x} Q^x$.

Let $Q^x \times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}}$ &c. be assumed for the fluent required: then the fluxion thereof, which is

* Art. 252. $Q^x\dot{x} \times$ hyp. log. Q * $\times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}}$ &c. +

$Q^x \times \overline{n\dot{x}Ax^{n-1} + \overline{n-1}.B\dot{x}x^{n-2} + \overline{n-2}.C\dot{x}x^{n-3}}$ &c.

must consequently be $= x^n \dot{x} Q^x$: and therefore, by putting m for the hyp. log. of Q, we have

$$\left.\begin{array}{l} mAx^n + mBx^{n-1} + \quad mCx^{n-2} + mDx^{n-3} \text{ \&c.} \\ -x^n + nAx^{n-1} + \overline{n-1}.Bx^{n-2} + \overline{n-2}.Cx^{n-3} \text{ \&c.} \end{array}\right\} = 0$$

Whence comparing the co-efficients of the homologous terms, we get $A = \frac{1}{m}$, $B = -\frac{nA}{m} = -\frac{n}{m^2}$, $C = -\frac{\overline{n-1}.B}{m} = \frac{n.\overline{n-1}}{m^3}$ &c. and consequently $Q^x \times$

$\overline{Ax + Bx^{n-1} + Cx^{n-2} + \text{\&c.}} = \frac{Q^x}{m} \times (x^n - \frac{nx^{n-1}}{m} + \frac{n.\overline{n-1}.x^{n-2}}{m^2} - \frac{n.\overline{n-1}.\overline{n-2}.x^{n-3}}{m^3}$ &c.) Which series, it is plain, will always terminate when n is a whole positive number. *Q. E. I.*

340. In the preceding problem the co-efficients A, B, C, &c. of the assumed series were taken, in the common way, as constant quantities; which, because of the general multiplicator Q^r, was sufficient.

But in other cases, where a proper multiplicator, to express the Mechanical, or Logarithmic, &c. part of the required fluent, cannot readily be known, it will be convenient to assume a series for the *whole* (independent of any general multiplicator) wherein the quantities A, B, C, D, &c. must be considered as variable.

PROBLEM II.

341. *To find the Fluent of $z^m x^{n-1}\dot{x}$; z being the Hyperbolic Logarithm of x; and m and n any given Numbers.*

Let there be assumed $Az^m + Bz^{m-1} + Cz^{m-2} + Dz^{m-3}$ &c. $=$ the fluent of $z^m x^{n-1}\dot{x}$: then, in fluxions, we shall have

$$\left.\begin{array}{l}\dot{A}z^m + \dot{B}z^{m-1} + \dot{C}z^{m-2} + \dot{D}z^{m-3}\text{ \&c.} \\ + mAz^{m-1}\dot{z} + \overline{m-1}.\,Bz^{m-2}\dot{z} + \overline{m-2}.\,Cz^{m-3}\dot{z}\text{ \&c.}\end{array}\right\} = z^m x^{n-1}\dot{x}$$

But $\dot{z} = \frac{\dot{x}}{x}$; whence, by ordering the equation, there arises

$$\left.\begin{array}{c}\dot{A} \\ -x^{n-1}\dot{x}\end{array}\right\} \times z^m \left.\begin{array}{c}+\dot{B} \\ \frac{mA\dot{x}}{x}\end{array}\right\} \times z^{m-1} \left.\begin{array}{c}+\dot{C} \\ \frac{\overline{m-1}.\,B\dot{x}}{x}\end{array}\right\} z^{m-2},\text{ \&c.} = 0$$

Now, by making the co-efficients of the like powers of z, equal to nothing, we have $\dot{A} = x^{n-1}\dot{x}$, $A = \frac{x^n}{n}$; $\dot{B} = \left(-\frac{mA\dot{x}}{x}\right) = -\frac{mx^{n-1}\dot{x}}{n}$, $B = -\frac{mx^n}{n^2}$;

$\dot{C}$ $\left(= -\frac{\overline{m-1}.B\dot{x}}{x} =\right)$ $\frac{m.\overline{m-1}.x^{m-1}\dot{x}}{n^2}$ $C =$ $\frac{m.\overline{m-1}.x^m}{n^3}$ &c. and consequently the fluent sought $= \frac{x^n}{n}$ into $z^m - \frac{mz^{m-1}}{n} + \frac{m.\overline{m-1}.z^{m-2}}{n^2} - \frac{m.\overline{m-1}.\overline{m-2}.z^{m-3}}{n^3} + \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}.z^{m-4}}{n^4}$ &c. Which, when m is a whole positive number, will terminate in $m+1$ terms. *Q. E. I.*

PROBLEM III.

342. *To find the Fluent of $z^n\dot{y}$; z being the Arch of a given Circle, and y the Sine corresponding.*

Let there be assumed $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3}$ = fluent of $z^n\dot{y}$; then, by taking the fluxion, we shall have

$$\left.\begin{array}{l}\dot{A}z^n + \dot{B}z^{n-1} + \dot{C}z^{n-2} + \dot{D}z^{n-3} \text{ \&c.} \\ -z^n\dot{y} + nAz^{n-1}\dot{z} + \overline{n-1}.Bz^{n-2}\dot{z} \text{ \&c.}\end{array}\right\} = 0$$

Whence, putting $\dot{A} - \dot{y} = 0$, $\dot{B} + nA\dot{z} = 0$, $\dot{C} + \overline{n-1}.B\dot{z} = 0$, $\dot{D} + \overline{n-2}.C\dot{z} = 0$, &c. we get $A = y$; $\dot{B} = -ny\dot{z}$, $\dot{C} = -\overline{n-1}.B\dot{z}$ &c.

But, if a and x be taken to denote the radius and co-sine of the arch z, it will appear, from Art. 142, that $y\dot{z} = -a\dot{x}$ and $x\dot{z} = a\dot{y}$: therefore $\dot{B} = na\dot{x}$, and $B = nax$; also $\dot{C}$ $(= -\overline{n-1}.B\dot{z}) = -n.\overline{n-1}.ax\dot{z} = -n.\overline{n-1}.a^2\dot{y}$, and $C = -n.\overline{n-1}.a^2y$; likewise $\dot{D}$ $(= -\overline{n-2}.C\dot{z}) = n.\overline{n-1}.\overline{n-2}.a^2y\dot{z}$ $= -n.\overline{n-1}.\overline{n-2}.a^3\dot{x}$, and $D = -n.\overline{n-1}.\overline{n-2}.a^3x$

&c. &c. and consequently $Az^n + Bz^{n-1} + Cz^{n-2}$ &c. $= yz^n + naxz^{n-1} - n.\overline{n-1}.a^2yz^{n-2} - n.\overline{n-1}\times\overline{n-2}.a^3xz^{n-3} + \&c.$

$$=\left\{\begin{array}{l} y\times\overline{z^n - n.\overline{n-1}.a^2z^{n-2} + n.\overline{n-1}.\overline{n-2}.\overline{n-3}.a^4z^{n-4}\ \&c.} \\ x\times\overline{naz^{n-1} - n.\overline{n-1}.\overline{n-2}.a^3z^{n-3} + n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.a^5z^{n-5}\ \&c.} \end{array}\right.$$

Q. E. I.

In the very same manner the fluent of $z^n\dot{w}$, or $z^n \times -\dot{x}$ (*w* being the versed-sine of the arch *z*) will be found $= -xz^n + nyaz^{n-1} + n.\overline{n-1}.xa^2z^{n-2} - n.\overline{n-1}.\overline{n-2}.ya^3z^{n-3} - n.\overline{n-1}.\overline{n-2}.\overline{n-3}.xa^4z^{n-4} + \&c.$

PROBLEM IV.

343. *The Quantities, x, y and z being the same as in the preceding Problem; to find the Fluent of $z^n x^r y^m \dot{y}$.*

By assuming $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3}$ &c. and proceeding as above, we have $\dot{A} = x^r y^m \dot{y}$, $\dot{B} = -nA\dot{z}$, $\dot{C} = -\overline{n-1}.B\dot{z}$, $\dot{D} = -\overline{n-2}.C\dot{z}$ &c. or (because $\dot{z} = \frac{a\dot{y}}{x}$) $\dot{B} = -\frac{naA\dot{y}}{x}$, $\dot{C} = -\frac{\overline{n-1}.aB\dot{y}}{x}$, $\dot{D} = -\frac{\overline{n-2}.aC\dot{y}}{x}$ &c. Therefore, if the fluent of $x^r y^m \dot{y}$ (found from Art. 142 and 291) be denoted by Q; *that* of $\frac{Q\dot{y}}{x}$, by R; *that* of $\frac{R\dot{y}}{x}$, by S; *that* of $\frac{S\dot{y}}{x}$, by T &c. it follows that the fluent of $z^n x^r y^m \dot{y}$ will be truly represented by $Qz^n - naRz^{n-1} + n.\overline{n-1} \times a^2Sz^{n-2} - n.\overline{n-1}.\overline{n-2}.a^3Tz^{n-3}$ &c.

COROLLARY.

344. Since $\dot{y} = -\frac{x\dot{x}}{y} = \frac{x\dot{z}}{a}$ (*Vide* Art. 142) it follows that $z^n x^r y^m \dot{y}$ is $= -z^n x^{r+1} y^{m-1} \dot{x} = \frac{z^n x^{r+1} y^m \dot{z}}{a}$: therefore the fluents of these two last expressions are, *also*, exhibited in the foregoing series.

345. As the values of Q, R, S &c. in the preceding articles, are too complex to be pursued in a general manner, it may not be amiss to illustrate the method of proceeding by an example or two.

Let, then the fluxion proposed be $\frac{zy^2\dot{y}}{x}$: where n being $= 1$, $m=2$, and $r=-1$, we have $\dot{Q}=\frac{y^2\dot{y}}{x}=\frac{y^2\dot{y}}{\sqrt{a^2-y^2}}$ (because $\sqrt{a^2-y^2}=x$). Whence $Q=-\frac{1}{2}y\sqrt{a^2-y^2}+\frac{1}{2}az=-\frac{1}{2}yx+\frac{1}{2}az$,* and therefore $\dot{R}$ $\left(=\frac{Q\dot{y}}{x}\right)=-\frac{1}{2}y\dot{y}+\frac{\frac{1}{2}az\dot{y}}{x}=-\frac{1}{2}y\dot{y}+\frac{1}{2}z\dot{z}$ (because $\frac{a\dot{y}}{x}=\dot{z}$) and consequently $R=-\frac{1}{4}y^2+\frac{1}{4}z^2$; and so $\frac{az-yx}{2}\times z+a\times\frac{y^2-z^2}{4}$, or $\frac{az^2-2xyz+ay^2}{4}$, is the true fluent of $\frac{zy^2\dot{y}}{x}$ $\left(=-zy\dot{x}=\frac{y^2z\dot{z}}{a}\right)$.†

* Art. 279.

† Art. 344.

Again, let the fluent of $-p\dot{x}\times\overline{z+y}|^2$ (expressing the content of the solid generated by the revolution of the *cycloid*) be required.

Here, the given expression, in simple terms, will become $-pz^2\dot{x}-2pzy\dot{x}-py^2\dot{x}$: whereof the fluent of the first term $-pz^2\dot{x}$, will be had, by making $n=2$, $m-1=0$, and $r+1=0$ (*Vide Form.* 2, *in Corol.*)

Where, we therefore, have $\dot{Q}=\frac{y\dot{y}}{x}=-\dot{x}$; whence $Q=-x$; also $\dot{R}$ $\left(\frac{Q\dot{y}}{x}\right)=-\dot{y}$, and $R=-y$; likewise $\dot{S}$ $\left(=\frac{R\dot{y}}{x}\right)=-\frac{y\dot{y}}{x}=\dot{x}$, $S=x$; and consequently the fluent of $-z^2\dot{x}$ $(Qz^n-naRz^{n-1}+n.\overline{n-1}.a^2Sz^{n-2}$ &c.$)=-xz^2+2ayz+2a^2x$: to which, adding the fluent $\left(\frac{az^2-2xyz+ay^2}{2}\right)$ of the second term $-2zy\dot{x}$ (found in the preceding example) and also that of $-y^2\dot{x}$ (or $-a^2\dot{x}+x^2\dot{x}$, found the common way) we get, in the whole, $\frac{1}{2}\overline{a-x}\times z^2 + \overline{2ay-yx}\times z + \frac{1}{2}ay^2 + a^2x + \frac{1}{3}x^3$; which, multiplied by p, and corrected, gives, p *into* $\frac{1}{2}\overline{a-x}\times z^2 + \overline{2ay-yx}\times z + \frac{1}{2}ay^2 + a^2x + \frac{1}{3}x^3 - \frac{4}{3}a^4$, for the true fluent that was to be determined.

PROBLEM V.

346. *Supposing H to denote the Fluent of $\overline{k+lz^n}|^r \times z^{vn-1}\dot{z}$; to find the whole Fluent of $H\times\overline{a-bz^n}|^m\times z^{qn-1}\dot{z}$, (when $a-bz^n$ becomes equal to Nothing).*

By resolving $\overline{k+lz^n}|^r \times z^{vn-1}\dot{z}$ into simple terms, and taking the fluent, the ordinary way, we get $H = \frac{k^r z^{vn}}{n}\times\overline{\frac{1}{v}+\frac{rlz^n}{v+1.k}+\frac{r.\overline{r-1}.l^2z^{2n}}{2.\overline{v+2}.k^2}\ \&c.}$ Which value being substituted above, and p wrote instead of $q+v$, we shall have $H\times\overline{a-bz^n}|^m\times z^{qn-1}\dot{z} = \frac{k^r}{n}\times\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$ *into* $\frac{1}{v}+\frac{rlz^n}{v+1.k}+\frac{r.\overline{r-1}.l^2z^{2n}}{2.\overline{v+2}.k^2}+\frac{r.\overline{r-1}.\overline{r-2}.l^3z^{3n}}{2.3.\overline{v+3}.k^3}$ &c.

Let, now, the fluent of $\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$ (in the proposed circumstance) be denoted by A, and put $t = p+m+1$; then it follows, from Art. 286 (by writing $\frac{1}{v}$ for e, $\frac{rl}{v+1.k}$ for f, &c.) that $\frac{k^r}{n}\times A$ *into* $\frac{1}{v}+$

$$\frac{p\,.\,r}{t\,.\,\overline{v+1}}\times\frac{al}{bk}+\frac{p.\,\overline{p+1}.\,r.\,\overline{r-1}}{t.\,\overline{t+1}.\,2.\,\overline{v+2}}\times\overline{\frac{al}{bk}}\Big|^{2}+$$

$$\frac{p.\,\overline{p+1}.\,\overline{p+2}.\,r.\,\overline{r-1}.\,\overline{r-2}}{t.\,\overline{t+1}\,\overline{t+2}.\,2.\,3.\,\overline{v+3}}\times\overline{\frac{al}{bk}}\Big|^{3}+\ \&c.$$ will be the true value of the fluent. *Q. E. I.*

Note, p and $m+1$ must here be positive quantities;* and it is also requisite that $\frac{l}{k}$ should be greater than $-\frac{b}{a}$; otherwise the fluent will fail.

* Art. 286.

Ex. 1. *Let* $\dot{H}=\overline{1-y^2}\Big|^{-\frac{1}{2}}\times\dot{y}$; *and let the whole Fluent of* $H\times\overline{1-y^2}\Big|^{-\frac{1}{2}}\dot{y}$, *be demanded.*

Then, k being $=1$, $l=-1$, $z=y$, $n=2$, $r=-\frac{1}{2}$, $v=\frac{1}{2}$; also $a=1$, $b=1$, $m=-\frac{1}{2}$, $q=\frac{1}{2}$; $p\ (=q+v)\ =1$, $t\ (=p+m+1)=\frac{3}{2}$, and A (=the *whole* fluent of $\overline{1-y^2}\Big|^{-\frac{1}{2}}y\dot{y}$) $=1$; we shall, by substituting these several values above, get $1+\frac{1}{3.3}+\frac{1}{5.5}+\frac{1}{7.7}+\frac{1}{9.9}+\frac{1}{11.11}$ &c. $=$ fluent of $H\times\overline{1-y^2}\Big|^{-\frac{1}{2}}\times\dot{y}$ (or $H\dot{H}$) when $y=1$. Which fluent being also expressed by $\frac{H^2}{2}$, it follows that $\frac{H^2}{2}=\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}$ &c. Where H is $\frac{1}{4}$ of the periphery of the circle whose radius is unity.

Ex. 2. *Let* $\dot{H} = \overline{c^2+z^2}|^{-1} \times \dot{z}$; *to find the fluent of* $H \times \overline{h^2-z^2}|^{-\frac{1}{2}} \times c^2\dot{z}$.

Here, $k=c^2$, $l=1$, $n=2$, $r=-\frac{1}{2}$, $v=\frac{1}{2}$; also $a=h^2$, $b=1$, $m=-\frac{1}{2}$, $q=\frac{1}{2}$, $p\,(q+v)=1$, $t\,(p+m+1)=\frac{3}{2}$, and A (= *whole* fluent of $\overline{h^2-z^2}|^{-\frac{1}{2}} \times z\dot{z}$) $= h$: whence, by substitution, we have $c^{-2} \times$

$$h \times \overline{1 - \frac{1}{3} \times \frac{h^2}{c^2} + \frac{1}{5} \times \frac{h^4}{c^4} - \frac{1}{7} \times \frac{h^6}{c^6} \text{ \&c.}}$$ which

multiplied by c^2 (the co-efficient of $\dot{z}$) gives $\frac{1}{c} \times$

$$\overline{h - \frac{h^3}{3c^2} + \frac{h^5}{5c^4} - \frac{h^7}{7c^6}} \text{ \&c.}$$ for the true fluent in this

case: where the series is *that* expressing the arch of the circle whose tangent is h and radius c;* and is therefore equal to $c \times$ arch, whose radius is unity and tangent $= \frac{h}{c}$: whence this last arch (taken without the multiplicator c) is the true value of the fluent.

* Art. 142.

SECTION VII.

Showing how Fluents, found by means of Infinite Series, are made to converge.

347. It is found, *in Art.* 85, that the fluent of $\overline{a+cz^n}|^m \times dz^{qn-1}\dot{z}$, in an infinite series, (making $m+q=s$) is expressed by $\frac{\overline{a+cz^n}|^{m+1} \times dz^{qn}}{qna} \times$

$\overline{1 - \frac{s+1.cz^n}{q+1.a} + \frac{s+1.\ s+2.\ c^2z^{2n}}{q+1.\ q+2.\ a^2} - \&c.}$ Whence it follows (and is evident by bare inspection) that the fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ (where the second term under the *vinculum* is negative) will be truly defined by

$$\frac{\overline{a-cy^n}|^{r+1} \times y^{qn}}{qna} \text{ into } 1 + \frac{\overline{s+1}.\ cy^n}{q+1.\ a} + \frac{\overline{s+1}.\overline{s+2}.\ c^2y^{2n}}{q+1.\ q+2.\ a^2}$$

$+$ &c. supposing $s=r+q$.

But, besides the series here given, and those, in Art. 83, 84, expressing the same value, the fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ will yet admit of another form, different from all of them; by means whereof and *that* above, we shall be enabled to draw out some very useful conclusions.

348. Put $z^n = \frac{ay^n}{a-cy^n}$; then $y^n = \frac{az^n}{a+cz^n}$, and therefore $ny^{n-1}\dot{y} = \frac{na^2z^{n-1}\dot{z}}{\overline{a+cz^n}|^2}$; also $a-cy^n = \frac{a^2}{a+cz^n}$, and $y^{qn-1}\dot{y}\ (= y^{qn-n} \times y^{n-1}\dot{y}) = \frac{a^{q+1}z^{qn-1}\dot{z}}{\overline{a+cz^n}|^{q+1}}$; and consequently $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y} = a^{2r+q+1} \times \overline{a+cz^n}|^{-r-q-1} \times z^{qn-1}\dot{z}$: which fluxion, so transformed, being compared with $\overline{a+cz^n}|^m \times dz^{qn-1}\dot{z}$; we have $m=-r-q-1$, $d=a^{2r+q+1}$, and $s\ (q+m) = -r-1$; whence, by substituting these values in the first series, above given, the fluent sought will be had $= \frac{\overline{a+cz}|^{-r-q} \times a^{2r+q} \times z^{qn}}{qn} \times (1 + \frac{rcz^n}{q+1.a}$

$$+ \frac{r.\overline{r-1}.\ c^2z^{2n}}{q+1.q+2.a^2} + \frac{r.\overline{r-1}.\overline{r-2}.\ c^3z^{3n}}{q+1.\ q+2.\ q+3.\ a^3}\ \&c.)$$

Which, by restoring y (or writing $\frac{a^2}{a-cy^n}$ and $\frac{ay^n}{a-cy^n}$ for their equals $a + cz^n$, and z^n) becomes

$$\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn} \times (1 + \frac{r}{q+1} \times \frac{cy^n}{a-cy^n} + \frac{r \cdot \overline{r-1}}{\overline{q+1} \cdot \overline{q+2}} \times$$

$$\frac{c^2 y^{2n}}{\overline{a-cy^n}|^2} \text{ \&c) the true fluent, of } \overline{a-cy^n}|^r \times y^{qn-1}\dot{y}.$$

349. This fluent may be otherwise found, independent of *that* above, in the following manner:

It is evident, by taking the fluxion of $\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn}$ (which quantity would be the fluent sought, if $\overline{a-cy^n}|^r$ was constant) that $\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn}$ is = the fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ — fluent of $\frac{rc}{q} \times \overline{a-cy^n}|^{r-1} \times y^{qn+n-1}\dot{y}$: this equation, by transposing the last term, and writing x in the room of $a-cy^n$ (for the sake of brevity) will become *flu.* $x^r y^{qn-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rc}{q} \times$ *flu.* $x^{r-1} y^{qn+n-1}\dot{y}$. From the very same argument (if, instead of r, we substitute $r-1$, $r-2$, &c. successively; and, for q, write $q+1$, $q+2$, $q+3$, &c. respectively) we shall, also, have

$$flu.\ x^{r-1} y^{qn+n-1}\dot{y} = \frac{x^{r-1} y^{qn+n}}{\overline{q+1} \cdot n} + \frac{\overline{r-1} \cdot c}{\overline{q+1}} \times$$
$$flu.\ x^{r-2} y^{qn+2n-1}\dot{y};$$

$$flu.\ x^{r-2} y^{qn+2n-1}\dot{y} = \frac{x^{r-2} y^{qn+2n}}{\overline{q+2} \cdot n} + \frac{\overline{r-2} \cdot c}{\overline{q+2}} \times$$
$$flu.\ x^{r-3} y^{qn+3n-1}\dot{y};$$

&c. &c.

Whence, by substituting these values, one by one, in that of, *flu.* $x^r y^{qn-1} \dot{y}$, we get

$$flu.\ x^r y^{qn-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rc}{q} \times \frac{x^{r-1} y^{qn+n}}{\overline{q+1}.n} + \frac{r.\overline{r-1}.c^2}{q.\overline{q+1}}$$

$$\times\ flu.\ x^{r-2} y^{qn+2n-1} \dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q.\overline{q+1}.n} +$$

$$\frac{r.\overline{r-1}.c^2}{q.\overline{q+1}} \times \frac{x^{r-2} y^{qn+2n}}{\overline{q+2}.n} + \frac{r.\overline{r-1}.\overline{r-2}.c^3}{q.\overline{q+1}.\overline{q+2}} \times$$

$$flu.\ x^{r-3} y^{qn+3n-1} \dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q.\overline{q+1}.n} +$$

$$\frac{r.\overline{r-1}.c^2 x^{r-2} y^{qn+2n}}{q.\overline{q+1}.\overline{q+2}.n} + \frac{r.\overline{r-1}.\overline{r-2}.c^3 x^{r-3} y^{qn+3n}}{q.\overline{q+1}.\overline{q+2}.\overline{q+3}.n}$$

&c. Where the law of continuation is manifest; and where, by making $\frac{x^r y^{qn}}{qn}$ a general multiplicator, we shall have the very series above exhibited.

350. From the equality of the two foregoing expressions, for the fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$, (or $x^r y^{qn-1}\dot{y}$) the business of finding fluents, by infinite series, will, in many cases, be very much facilitated.

For, in the first place, it follows (by dividing both by $\frac{\overline{a-c^n}|^{r+1} \times y^{qn}}{qna}$, or $\frac{x^{r+1} y^{qn}}{qna}$) that the series $1 + \frac{\overline{s+1}.cy^n}{\overline{q+1}.a} + \frac{\overline{s+1}.\overline{s+2}.c^2 y^{2n}}{\overline{q+1}.\overline{q+2}.a^2}$ &c. and $\frac{a}{x} \times$

$$\overline{1 + \frac{rcy^n}{\overline{q+1}.x} + \frac{r.\overline{r-1}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.x^2} + \frac{r.\overline{r-1}.\overline{r-2}.c^3y^{3n}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.x^3} +}$$

&c. must also be equal to each other, let the several quantities, therein concerned, be what they will (which may be otherwise proved, independent of fluxions). Therefore, if in the room of q and s we write any other Quantities p and t, the equation will, *still*, hold, and will then become $1+\frac{\overline{t+1}.cy^n}{\overline{p+1}.a}+\frac{\overline{t+1}.\overline{t+2}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.a^2}$ $+\ \&c. = \frac{a}{x}\times\overline{1+\frac{rcy^n}{\overline{p+1}.x}+\frac{r.\overline{r-1}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2}\ \&c.}$ (t being $=p+r$).

Moreover, if as many terms of the first series $1+\frac{\overline{s+1}.cy^n}{\overline{q+1}.a}+\frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.a^2}+\frac{\overline{s+1}.\overline{s+2}.\overline{s+3}.c^3y^{3n}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.a^3}$ &c. be taken as are denoted by any given number, v, and the last of them be represented by Q, it is evident, from the law of the series, that the first of the remaining terms will be expressed by $Q\times\frac{s+v}{q+v}\times\frac{cy^n}{a}$; the second, of them, by $Q\times\frac{s+v}{q+v}\times\frac{s+v+1}{q+v+1}\times\frac{c^2y^{2n}}{a^2}$ &c. and therefore the sum of all of them (putting $q+v=p$ and $s+v$ $(=r+q+v=t)$ will be $= Q\times\frac{t}{p}\times\frac{cy^n}{a}+Q\times\frac{t}{p}\times\frac{t+1}{p+1}\times\frac{c^2y^{2n}}{a^2}+\&c. =$ $\frac{tQcy^n}{pa}\times\overline{1+\frac{\overline{t+1}.cy^n}{\overline{p+1}.a}+\frac{\overline{t+1}.\overline{t+2}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.a^2}+\&c.}$ $=\frac{tQcy^n}{px}\times\overline{1+\frac{rcy^n}{\overline{p+1}.x}+\frac{r.\overline{r-1}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2}\ \&c.}$ (by writing the series found above in the room of its equal) and consequently the whole series (including the v first terms) $=1+\frac{\overline{s+1}.cy^n}{\overline{q+1}.a}+$

$$\frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.a^2}\ (v)+\frac{tQey^n}{px}\times(1+\frac{rcy^n}{\overline{p+1}.x}+$$

$$\frac{r.\overline{r-1}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2}+\frac{r.\overline{r-1}.\overline{r-2}.c^3y^{3n}}{\overline{p+1}.\overline{p+2}.\overline{p+3}.x^3}+\&c.)$$

Which drawn into the general multiplicator $\frac{x^{r+1}\times y^{qn}}{qna}$ (*vide* Art. 347) will give the fluent of $\overline{a-cy^n}|^r\times y^{qn-1}\dot{y}$ (or $x^r y^{qn-1}\dot{y}$) according to a new form, compounded out of the two preceding ones; where the second series (the value of p being large in respect of r) will always converge much faster than the remaining part of the first, for which it is substituted: but this will, more fully, appear from what follows hereafter. It will be proper to take notice here that the fluent of $\overline{a+cz^n}|^m\times z^{qn-1}\dot{z}$ (the fluxion first proposed, where the second term under the vinculum is positive) will also be had from hence (by writing z for y, m for r, and $-c$ for c) and is therefore equal to $\frac{x^{m+1}z^{qn}}{qna}$ drawn into the sum of the two following series,

$$1-\frac{\overline{s+1}.cz^n}{\overline{q+1}.a}+\frac{\overline{s+1}.\overline{s+2}.c^2z^{2n}}{\overline{q+1}.\overline{q+2}.a^2}-\frac{\overline{s+1}.\overline{s+2}.\overline{s+3}.c^3z^{3n}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.a^3}\ (v)$$

$$-\frac{tQcz^n}{px}\times(1-\frac{mcz^n}{\overline{p+1}.x}+\frac{m.\overline{m-1}.c^2z^{2n}}{\overline{p+1}.\overline{p+2}.x^2}-$$

$$\frac{m.\overline{m-1}.\overline{m-2}.c^3z^{3n}}{\overline{p+1}.\overline{p+2}.\overline{p+3}.x^3}+\&c.)$$

Where, $s=m+q$, $p=v+q$, $t=s+v$, $x=a+cz^n$, and $Q=$ the last term of the first series continued to v terms, v being any whole number, at pleasure. A few examples will show the use of what is above delivered.

351. Ex. 1. *Let* $\frac{\dot{z}}{1+z}$, or $\overline{1+z}|^{-1}\dot{z}$, *be propounded.*

Which being compared with $\overline{a+cz^n}|^m \times z^{qn-1}\dot{z}$, we have $a=1$, $c=1$, $n=1$, $x=1+z$, $m=-1$, $qn-1=0$, or $q=1$; whence also $s\ (m+q)=0$, $p\ (v+q)=v+1$, $t\ (s+v)=v$, and consequently the fluent itself (by substituting these several values in the last general theorem) $= z$ into $1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4}\ (v) - \frac{vzQ}{v+1.x}$ $\times \overline{1+\frac{z}{v+2.x}+\frac{2.z^2}{v+2.v+3.x^2}+\frac{2.3.z^3}{v+2.v+3.v+4.x^3}}$ &c. Where (Q) the last term of the first series being $\pm\frac{z^{v-1}}{v}$, the multiplicator $\left(\frac{vzQ}{v+1.x}\right)$ to the second, will be $= \mp\frac{z^v}{v+1.x}$; and so the fluent itself will be reduced to $z - \frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}\ (v) \mp\frac{z^{v+1}}{v+1.x}\times$ $\overline{1+\frac{z}{v+2.x}+\frac{2.z^2}{v+2.v+3.x^2}}+$ &c. In which the signs $-$ and $+$, before z^{v+1}, obtain alternately, according as v is an odd or even number. But, to show the advantage of expressing the fluent in this manner, by two different series, let $z=1$, and let v be taken $=8$; then the value of the first series (continued to 8 terms) being $= 0{,}6345238$ &c. and *that* of the second series $= \frac{1}{18}+\frac{A}{20}+\frac{2B}{22}+\frac{3C}{24}+\frac{4D}{26}$ $+\frac{5E}{28}$ &c. (where A, B, C, D &c. denote the terms preceding those where they stand) $= 0{,}0555555 + 0{,}0027778 + 0{,}0002525 + 0{,}0000316 + 0{,}0000048 + 0{,}0000009 + 0{,}0000002 = 0{,}0586233$; it is evident that the fluent of $\frac{\dot{z}}{1+z}$, when z becomes $= 1$, will be $= 0,6345238 + 0,0586233 = 0,6931471$: which is true to the very last decimal place; and would have required, at least, 100000 terms of the first, or common, series.

352. Ex. 2. *Let the fluent of* $\frac{\dot{z}}{1+z^2}$ *(expressing the Arch whose Radius is* 1 *and Tangent* z*) be required.*

In this case we have $a=1$, $c=1$, $n=2$, $x=1+z^2$, $m=-1$, $qn-1=0$, or $q=\frac{1}{2}$, $s=-\frac{1}{2}$, $p=v+\frac{1}{2}$, and the fluent itself $= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7}\ (v) \pm$

$$\frac{z^{2v+1}}{\overline{2v+1}\,.\,x} \times (1 + \frac{2\,.\,z^2}{\overline{2v+3}\,.\,x} + \frac{2\,.\,4\,.\,z^4}{\overline{2v+3}\;\overline{2v+5}\,.\,x^2} + \frac{2\,.\,4\,.\,6\,.\,z^6}{\overline{2v+3}\,.\,\overline{2v+5}\,.\,\overline{2v+7}\,.\,x^3}$$

&c.) Where, if z be taken $=1$, and $v=6$, we shall have $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{26} \times \overline{1 + \frac{1}{15} + \frac{1}{15} \times \frac{2}{17} + \frac{1}{15} \times \frac{2}{17} \times \frac{3}{19}}$ &c. $= 0,785398 =$ the fluent of $\frac{\dot{z}}{1+z^2}$ when $z = 1$ ($= \frac{1}{4}$ of the periphery of the foresaid circle). Which number, brought out by taking only 8 terms of the second series, is more exact than if 100000 terms of the common series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ &c. had been used. And, if z be taken $= \sqrt{\frac{1}{3}}$ (= tangent of 30°) and $v=6$, as before, the same number of terms will be sufficient to give the answer, true to twice the decimal places above exhibited.

353. Ex. 3. *Let the Fluxion proposed be* $\overline{c^4+y^4}|^{\frac{1}{2}} \times \dot{y}$.

Here we have $a=c^4$, $c=1$, $z=y$, $n=4$, $x=c^4+y^4$, $m=\frac{1}{2}$, $q=\frac{1}{4}$, $s\ (m+q) = \frac{3}{4}$, $p\ (v+q) = v+\frac{1}{4}$; $t\ (s+v) = v+\frac{3}{4}$; and therefore the fluent sought (by substitution) is $= \frac{x^{\frac{3}{4}}y}{c^4}$ into $1 - \frac{7y^4}{5c^4} + \frac{7.11y^8}{5.9c^8} - \frac{7.11.15y^{12}}{5.9.13c^{12}}\ (v) - \frac{\overline{4v+3}.Qy^4}{\overline{4v+1}.x} \times (1 - \frac{2y^4}{\overline{4v+5}.x} - \frac{2.2y^8}{\overline{4v+5}.\overline{4v+9}.x^2} - \frac{2.6.2y^{12}}{\overline{4v+5}.\overline{4v+9}.\overline{4v+13}.x^3} -$ &c.) In which (as in all other cases) Q denotes the last term of the first series. This fluent approximates equally fast with those in the foregoing examples: and it may be observed farther, that the fluent will always converge, however great the value of z is taken, if both a and c, in the general fluxion $\overline{a+cz^n}|^m \times z^{\theta n-1}\dot{z}$, are positive quantities. But, if the second term under the vinculum be negative, the case will be otherwise, when that term becomes greater than half the first; since the powers of $\frac{cz^n}{x}$, in the latter part of the fluent, will then form an increasing geometrical progression. It may, therefore, be of use to show how the theorem may be varied so as to answer in this case. In order thereto, if in the equations $s=r+q$, and $1 + \frac{\overline{s+1}.cy^n}{\overline{q+1}.a} + \frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.a^2}$ &c. $= \frac{a}{x} \times \overline{1 + \frac{rcy^n}{\overline{q+1}.x} + \frac{r.\overline{r-1}.c^2y^{2n}}{\overline{q+1}.\overline{q+2}.x^2}}$ &c. (*given in Art.* 350) you write k for r, and p for q, and multiply by $\frac{x}{a}$, you will have $s=k+p$, and $1 + \dfrac{kcy^n}{\overline{p+1}\,.\,x} + \dfrac{k.\overline{k-1}.\,c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2}$ &c. $= \dfrac{x}{a} \times (1 + \dfrac{\overline{s+1}.\,cy^n}{\overline{p+1}.\,a} + \dfrac{\overline{s+1}.\,\overline{s+2}.\,c^2y^{2n}}{\overline{p+1}.\,\overline{p+2}.\,a^2}$ &c.)

Moreover, if the v first terms of the above series $1 + \dfrac{rcy^n}{\overline{q+1}\,.\,x} + \dfrac{r.\overline{r-1}.\,c^2y^{2n}}{\overline{q+1}.\,\overline{q+2}.\,x^2}$ &c. be taken, and the last of them be denoted by Q, it is plain the first of the remaining terms will be $= Q \times \dfrac{r-v+1}{q+v} \times \dfrac{cy^n}{x}$, the second $= Q \times \dfrac{r-v+1}{q+v} \times \dfrac{r-v}{q+v+1} \times \dfrac{c^2y^{2n}}{x^2}$ &c. and the sum of them all (putting $q+v=p$, and $r-v=k$) equal to $\dfrac{\overline{k+1}.\,Qcy^n}{px} \times (1 + \dfrac{kcy^n}{\overline{p+1}.\,x} + \dfrac{k.\overline{k-1}.\,c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2}$ &c.) $= \dfrac{\overline{k+1}.\,Qcy^n}{px} \times \dfrac{x}{a} \times (1 + \dfrac{\overline{s+1}.\,cy^n}{\overline{p+1}.\,a} + \dfrac{\overline{s+1}.\,\overline{s+2}.\,c^2y^{2n}}{\overline{p+1}.\,\overline{p+2}.\,a^2}$ &c.) (by the equation above) and consequently the sum of the whole series $(1 + \dfrac{rcy^n}{\overline{q+1}.\,x}$ &c.) $= 1 + \dfrac{rcy^n}{\overline{q+1}.\,x} + \dfrac{r.\overline{r-1}.\,c^2y^{2n}}{\overline{q+1}.\,\overline{q+2}.\,x^2} + \dfrac{r.\overline{r-1}.\,\overline{r-2}.\,c^3y^{3n}}{\overline{q+1}.\,\overline{q+2}.\,\overline{q+3}.\,x^3}\ (v) + \dfrac{\overline{k+1} \times cy^nQ}{pa} \times$

$$\overline{1 + \frac{\overline{s+1}.\,cy^n}{\overline{p+1}.\,a} + \frac{\overline{s+1}.\,\overline{s+2}.\,c^2y^{2n}}{\overline{p+1}.\,\overline{p+2}.\,a^2} + \&c.}$$

Which,

multiplied by $\frac{x^r y^m}{qn}$, gives the fluent of $\overline{a-cy^n}|^r$ × $y^{qn-1}\dot{y}$ (* or $x^r y^{qn-1}\dot{y}$) where $k = r - v$, $p = v + q$, s $(= k + p) = r + q$ and $x = a - cy^n$. I shall put down one example of the use of this last general expression; where we will take $\dot{y}\sqrt{2y - y^2}$, or $\overline{2-y}|^{\frac{1}{2}} \times y^{\frac{1}{2}}\dot{y}$ (being the fluxion of the area of the circle whose radius is unity and versed sine y). In which case $a = 2$, $c = 1$, $n = 1$, $r = \frac{1}{2}$, $qn - 1 = \frac{1}{2}$, or $q = \frac{3}{2}$, $k = -v + \frac{1}{2}$, $p = v + \frac{3}{2}$, $s = 2$, $x = 2 - y$; and therefore the

* Art. 348, 349.

fluent sought $= \frac{2x^{\frac{1}{2}}y^{\frac{3}{2}}}{3}$ into $1 + \frac{y}{5x} - \frac{y^2}{5.7x^2} + \frac{3y^3}{5.7.9x^3} - \frac{3y^4}{7.9.11x^4} + \frac{3y^5}{9.11.13x^5}\ (v) \mp \frac{\overline{2v-3}.yQ}{\overline{2v+3}.2} \times (1 + \frac{3y}{2v+5} + \frac{3.4y^2}{\overline{2v+5}.\overline{2v+7}} + \frac{3.4.5y^3}{\overline{2v+5}.\overline{2v+7}.\overline{2v+9}}$ &c.) Which, if y be taken $= 1$, and $v = 5$, will become $= \frac{2}{3} + \frac{A}{5} - \frac{B}{7} + \frac{C}{3} - \frac{5D}{11} + \frac{7E}{2\times 13} + \frac{3F}{15} + \frac{4G}{17} + \frac{5H}{19}$ &c. $= 0{,}785398$ (where A, B, C &c. denote the several terms, respectively, without their signs). In bringing out which conclusion, six terms of the second series are required: but if y be taken $= \frac{1}{2}$ the radius of the foresaid circle, then four terms of each series will be more than sufficient to give the same number of decimal places. And it may likewise be observed, that, although no general rule can be laid down for assigning the value of v, so as to answer the best in all cases, yet the conclusion will, for the general part, require the fewest terms, when the number of those, taken in each series, is nearly the same.

354. But, after all, another theorem or series still seems wanting, to express the value of the whole fluent, when the quantity under the *vinculum* becomes equal to nothing (which, in the resolution of problems, is commonly what is required). For it is plain the last, above given, answers no better here than that preceding it; because (the divisor (x) being nothing) the former part of it fails.

In order, therefore, to determine a proper *form*, to obtain in this circumstance, it will be requisite to observe, first of all, *from Article* 286, that the whole fluent of $\overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, supposing *that* of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$ to be denoted by A, will be truly expressed by $\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}(v) \times \frac{a^v A}{b^v}$: in which $t=m+p+1$; and where it is requisite that the values of $m+1$ and p should be positive, otherwise, A being infinite, the fluent (or comparison) fails. Hence, because the whole fluent of $\overline{a-bz^n}|^m \times z^{n-1}\dot{z}$ (when $a-bz^n=0$) is found $=\frac{a^{m+1}}{\overline{m+1}\times nb}$, by the common way,* it follows, by writing this value in the room of A, and expounding p by 1, that the *whole fluent* of $\overline{a-bz^n}|^m \times z^{n+vn-1}\dot{z}$ is rightly expressed by $\frac{1}{m+2} \times$ Art. 77 & 78.

$\frac{2}{m+3} \times \frac{3}{m+4}(v) \times \frac{a^{m+v+1}}{\overline{m+1}\times nb^{v+1}}$, or by $\frac{1}{m+1} \times$

$\frac{2}{m+2} \times \frac{3}{m+3}(v+1) \times \frac{a^{m+v+1}}{\overline{v+1}\times nb^{v+1}}$: whence that of $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, by substituting r instead of $v+1$, will consequently be equal to $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\,(r) \times \frac{a^{m+r}}{rnb^r}$. Let this quantity be denoted by B; then, *by the same Article*, the fluents of the several terms of the series 1, $\frac{bz^n}{a}$, $\frac{b^2z^{2n}}{a^2}$, $\frac{b^3z^{3n}}{a^3}$ &c. drawn into the general multiplicator $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, will be respectively expounded by those of the series 1, $\frac{r}{t}$, $\frac{r.\overline{r+1}}{t.\overline{t+1}}$, $\frac{r.\overline{r+1}.\overline{r+2}}{t.\overline{t+1}.\overline{t+2}}$ &c. drawn into B; t being $=m+r+1$.

If now the differences of the quantities 1, $\frac{r}{t}$, $\frac{r.\overline{r+1}}{t.\overline{t+1}}$ &c. be continually taken;* and for $r-t$ its equal $-m-1$ be substituted, the value of any term of the series, whose distance from the first, exclusive, is denoted by s, or whose corresponding term, in the preceding series, is $\frac{b^s z^m}{a^s}$, will be universally expressed by $1-\frac{s.\overline{m+1}}{1.t}+\frac{s.\overline{s-1}.\overline{m+1}.\overline{m+2}}{1.2\times t.\overline{t+1}}-\frac{s.\overline{s-1}.\overline{s-2}.\overline{m+1}.\overline{m+2}.\overline{m+3}}{1.2.3\times t.\overline{t+1}.\overline{t+2}}+$ &c. Where, if s be interpreted by 0, 1, 2, 3 &c. successively, you will have the values 1, $\frac{r}{t}$, $\frac{r.\overline{r+1}}{t.\overline{t+1}}$ &c. above exhibited: but, if s be taken as a fraction, then the value of such an intermediate term will be found as will give

* See my Mathematical Essays, p. 94.

the fluent of $\frac{b^s z^{sn}}{a^s} \times \overline{a - bz^n}|^m \times z^{rn-1}\dot{z}$, in any proposed circumstance of s; which fluent, it is evident, will therefore be expressed by $B^! \times (1 - \frac{s.\ m+1}{1.\ t} + \frac{s.\ \overline{s-1}.\ \overline{m+1}.\ \overline{m+2}}{1.\ 2.\ t.\ \overline{t+1}}$ &c.) or its equal $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\ (r) \times \frac{a^{m+r}}{rnb^r}$ into $1 - \frac{s.\ \overline{m+1}}{1.\ t} - \frac{\overline{s-1}.\ \overline{m+2}}{2.\ \overline{t+1}} \times E - \frac{\overline{s-2}.\ \overline{m+3}}{3.\ \overline{t+2}} \times F - \frac{\overline{s-3}.\ \overline{m+4}}{4.\ \overline{t+3}} \times G$ &c. (where E, F, G &c. denote the terms immediately preceding those where they stand, under their proper signs). Whence, dividing by $\frac{b^s}{a^s}$, we have $\frac{1}{m+1} \times \frac{2}{m+2}\ (r) \times \frac{a^{m+r}}{rnb^{r+s}} \times (1 - \frac{s.\ \overline{m+1}}{t} - \frac{\overline{s-1}.\ \overline{m+2}}{2.\ \overline{t+1}} \times E$ &c.) for the true fluent of $\overline{a - bz^n}|^m \times z^{rn+sn-1}\dot{z}$.

From the last fluent *that* of $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z}$ (in which p denotes any positive fraction, proper or improper) is very readily obtained: for, if the same (when $a - bz^n = 0$) be denoted by A; then the fluent of $\overline{a - bz^n}|^m \times z^{pn+vn-1}\dot{z}$ will (according to the article above quoted) be expressed by $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2} \times \frac{p+2}{p+m+3}\ (v) \times \frac{a^v A}{b^v}$; supposing v any positive integer. Therefore, by making $\overline{a-bz^n}|^m \times z^{pn+m-1}\dot{z} = \overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, or $r+s=p+v$, the corresponding fluents must, also, be equal; that is, $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2}(v) \times \frac{a^v A}{b^v} = \frac{1}{m+1} = \frac{2}{m+2} \times \frac{3}{m+3}(r) \times \frac{a^{m+p+v}}{rnb^{p+v}} \times \overline{1 - \frac{s.\overline{m+1}}{t} - \&c.}$ And consequently A (the whole fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}) = \frac{p+m+1}{p} \times \frac{p+m+2}{p+1} \times \frac{p+m+3}{p+2}(v) \times \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}(r) \times \frac{a^{m+p}}{nrb^p} \times$ into the series $1 - \frac{s.\overline{m+1}}{1.t} - \frac{\overline{s-1}.\overline{m+2}}{2.\overline{t+1}}E - \frac{\overline{s-2}.\overline{m+3}}{3.\overline{t+2}}F - \frac{\overline{s-3}.\overline{m+4}}{4.\overline{t+3}}G$ &c. where $t=r+m+1$ and $s=p+v-r$; v and r being any whole positive numbers at pleasure.

355. An Example or two of the use of this conclusion, may be proper.

1°. Let the *whole fluent* of $\overline{1-x^2}|^{-\frac{1}{2}}\dot{x}$ (expressing the length of $\frac{1}{4}$ of the periphery of the circle whose radius is unity) be demanded. In which case, a being $=1$, $b=1$, $m=-\frac{1}{2}$, $n=2$, $p=\frac{1}{2}$, $t=r+\frac{1}{2}=\frac{2r+1}{2}$, and $s=v-r+\frac{1}{2}=\frac{2v-2r+1}{2}$, the fluent sought will, therefore (by substituting these values) be had $= \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5}(v) \times \frac{2}{1} \times \frac{4}{3} \times \frac{5}{6}(r) \times$

$\frac{1}{2r}$ into $1 - \frac{1 . \overline{2v - 2r + 1}}{2 . \overline{2r+1}} - \frac{3 . \overline{2v - 2r - 1}}{4 . \overline{2r+3}} E -$

$\frac{5 . \overline{2v - 2r - 3}}{6 . \overline{2r+5}} F - \frac{7 . \overline{2v - 2r - 5}}{8 . \overline{2r+7}} G$ &c. Which, by expounding v by 5 and r by 3, will become $=$ $2,16719$ &c. into $1 - \frac{1 . 5}{2 . 7} - \frac{3 . 3}{4 . 9} E - \frac{5 . 1}{6 . 11} F +$ $\frac{7 . 1}{8 . 13} G + \frac{9 . 3}{10 . 15} H + \frac{11 . 5}{12 . 17} I +$ &c. $= 1,5708$. In the bringing out of which value, all the terms above exhibited are requisite: but, of the common series, $1 +$ $\frac{1}{2 . 3} + \frac{1 . 3}{2 . 4 . 5} + \frac{1 . 3 . 5}{2 . 4 . 6 . 7} +$ &c. more than 10 times that number of terms would be necessary to answer with the same degree of exactness.

Ex. 2°. *Let the Fluxion proposed be* $\frac{d\dot{x}}{x^{\frac{1}{2}} \sqrt{d^2 - x^2}}$ (whose whole fluent, when $x = d$, expresses the time of descent of a heavy body in half the arch of a semi-circle, whose radius is d).*

* Art. 207.

Here, by comparing $\overline{d^2 - x^2}|^{-\frac{1}{2}} \times x^{-\frac{1}{2}} \dot{x}$ with $\overline{a - bx^n}|^m \times x^{pn-1} \dot{x}$, we have $a = d^2$, $b = 1$, $n = 2$, $pn - 1 = -\frac{1}{2}$, or $p = \frac{1}{4}$; also $s\ (p + v - r) = v - r + \frac{1}{4}$, $t\ (r + m + 1) = r + \frac{1}{2}$: whence, by taking r and v, each, equal to 4, the fluent, itself, comes out $=$ $\frac{3}{1} \times \frac{7}{5} \times \frac{11}{9} \times \frac{15}{13}$ into $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ $\frac{d^{\frac{1}{2}}}{8}$ into $1 - \frac{1 . 1}{4 . 9} + \frac{3 . 3}{8 . 11} E + \frac{7 . 5}{12 . 13} F + \frac{11 . 7}{16 . 15} G$ $+$ &c. $= \frac{11 . 16 d^{\frac{1}{2}}}{13 . 5}$ into $\overline{1 - \frac{1}{36} + \frac{9E}{88} + \text{\&c.}}$

$=2.6215d^{\frac{1}{2}}$: which is to $2\sqrt{2d}$, the time of descent along the vertical diameter of the foresaid circle, as 2.6215 to 2.8284, or as 100 to 108, nearly.

After the same manner the fluent will be found in other cases: but, with regard to the assigning of the values of r and v, it may be observed, that the answer will, commonly, be brought out with the least trouble when v is taken greater by an unit or two than r; which last quantity must be greater or less, according as a greater or less degree of exactness is necessary.—From the foregoing expressions, by varying the values of v and r, a great number of theorems, for the summation of series, may be deduced. But this being foreign to my present purpose, I am not at leisure to pursue it here.

256. Hitherto regard has been had to fluxions of the binomial-kind: but, from thence, the fluents of trinomials may also be found; when these last can be reduced to binomials (by Art. 307) without introducing new radical quantities.—Besides which method, I shall, here, give another, which will answer where that fails, and is also applicable to *multinomials*.

In order thereto, let the fluent of $\overline{a + cz^n}|^m \times z^{pn-1}\dot{z}$, be denoted by A; and let it be required to find, from thence, the fluent of the radical *multinomial*, or infinite series, $\overline{a + cx^n + dx^{2n} + ex^{3n} + fx^{4n} \&c.}|^m \times x^{pn-1}\dot{x}$.

Make $cz^n = cx^n + dx^{2n} + ex^{3n} + \&c.$ and $y = x^{pn}$; then, x^n being $= y^{\frac{1}{p}}$, if this value be substituted for x^n, in the first equation, it will become $cz^n = cy^{\frac{1}{p}} + dy^{\frac{2}{p}} + ey^{\frac{3}{p}}$ &c. Whence, by reverting the series (by

Art. 275.) y (x^{pn}) is found $= z^{pn} + Rz^{pn+n} + Sz^{pn+2n} + Tz^{pn+3n} +$ &c.

Where $R = -\frac{pd}{c}$, $S = \frac{p\,.\,\overline{p+3}}{2} \times \frac{d^2}{c^2} - \frac{pe}{c}$, $T =$

$$-\frac{p\,.\,\overline{p+4}\,.\,\overline{p+5}}{6} \times \frac{d^3}{c^3} + p\,.\,\overline{p+4} \times \frac{de}{c^2} - \frac{pf}{c} \text{ \&c.}$$

Moreover, by taking the fluxion of the equation thus brought out, and dividing by pn, we have $x^{pn-1}\dot{x}$

$$= z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1}\dot{z}$$

$$+\frac{p+3}{p} \times Tz^{pn+3n-1}\dot{z} + \text{\&c.}$$

Now let this value, with that of $cx^n + dx^{2n} + ex^{3n}$ + &c. (given above) be substituted in the proposed fluxion, and it will become $\overline{a + cz^n}|^m \times (z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1}\dot{z} +$ &c.)

Also, let v denote the place, or distance, of any term of this series from the first, exclusive; then the term itself, drawn into the general multiplicator, will be expressed by $\overline{a + cz^n}|^m \times \frac{p+v}{p}\,\Delta\, z^{pn+vn-1}\dot{z}$ (Δ being the corresponding co-efficient R, S, T, &c.) and the fluent thereof by $\frac{p+v}{p}\,\Delta \times \overline{a+cz^n}|^{m+1} \times z^{pn} \times$

$$\overline{\frac{z^{vn-n}}{\overline{s+1}\,.\,nc} - \frac{qaz^{vn-2n}}{\overline{s+1}\,.\,snc^2} + \frac{q\,.\,\overline{q-1}\,.\,a^2z^{vn-3n}}{\overline{s+1}\,.\,s\,.\,\overline{s-1}\,.\,nc^3}}\ (v) \pm$$

$$\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\ (v) \times \frac{p+v}{p} \times \Delta \times \frac{a^vA}{c^v} \text{ (Art. 283)}$$

Where, $q=p+v-1$, $s=n+q$, $t=p+n+1$, and the sign of the last term is $+$ or $-$, according as v is an even or odd number. Now, if in the fluent thus given, v be expounded by 1, 2, 3, 4 &c. successively, it is evident the fluent of the whole expression will, in all circumstances of z, be obtained. But, if the coefficient c be negative, so that $a+cz^n$ may (by increasing z) become equal to nothing; then, in that circumstance, the fluent of the foresaid general term $\overline{a+cz^n}|^m \times \frac{p+v}{p} \Delta\, z^{pn+vn-1}\dot{z}$ (or $\overline{a-bz^n}|^m \times \frac{p+v}{p} \Delta \times z^{pn+vn-1}\dot{z}$, making $-c=b$) being, *barely*, $= \frac{p}{t} \times$

Art. 286. $\frac{p+1}{t+1} \times \frac{p+2}{t+2}\,(v) \times \frac{p+v}{p} \times \frac{\Delta\, a^v A}{b^v}$, * it follows that the whole fluent of the given expression, or its equal, $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z} + \frac{p+1}{p} R z^{pn+n-1}\dot{z}$ &c. will be truly represented by $A \times \left(1 + \frac{\overline{p+1}\,.\,Ra}{tb} + \frac{\overline{p+1}\,.\,\overline{p+2}\,.\,Sa^2}{t\,.\,\overline{t+1}\,.\,b^2} + \frac{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,Ta^3}{t\,.\,\overline{t+1}\,.\,\overline{t+2}\,.\,b^3}\right.$ &c.$)$ In which, $R = \frac{pd}{b}$, $S = \frac{p\,.\,\overline{p+3}}{2} \times \frac{d^2}{b^2} + \frac{pe}{b}$, $T = \frac{p\,.\,\overline{p+4}\,.\,\overline{p+5}}{6} \times \frac{d^3}{b^3} + \frac{p\,.\,\overline{p+4}}{1} \times \frac{de}{b^2} + \frac{pf}{b}$, &c. and $A =$ the fluent $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, when $a-bz^n=0$.

357. Hence, if the fluxion given be of the trinomial kind (then, e, f, &c. vanishing) the *whole fluent*

of $\overline{a-bx^n+dx^{2n}}|^p \times x^{pn-1}\dot{x}$ (when $a-bx^n+dx^{2n}=0$) will, by substituting for R, S, T, &c. be $=A\times$

$$\left(1+\frac{p.\overline{p+1}}{1.t}\times\frac{ad}{b^2}+\frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}}{1.2.t.\overline{t+1}}\times\frac{ad}{b^2}\Big|^2+\right.$$

$$\left.\frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}.\overline{p+4}.\overline{p+5}}{1.2.3.t.\overline{t+1}.\overline{t+2}}\times\frac{ad}{b^2}\Big|^3+\text{\&c.}\right)$$

358. If $m+1$ and p are the halves of any odd affirmative numbers, the *fluent* of $\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$, when $a-bz^n=0$, will be equal to

$$\frac{1.3.5.7\,(m+\frac{1}{2})\times1.3.5.7\,(p-\frac{1}{2})}{2.4.6.8.10.12\,(m+p)}\times\frac{a^{m+p}G}{nb^p};\;*$$

* Art. 293 & 298.

G being the periphery of the circle whose diameter is unity. Therefore the fluent of $\overline{a-bx^n+dx^{2n}+ex^{3n}\text{ \&c.}}|^p$ $\times x^{pn-1}\dot{x}$, or its equal, $\overline{a-bz^n}|\times(z^{pn-1}\dot{z}+\frac{p+1}{p}$ $\times Rz^{pn+n-1}\dot{z}$ &c.) is found, in this case, by multiplying the expression here given, into the foregoing series, $1+\frac{\overline{p+1}.Ra}{tb}+$ &c.

359. An example or two will help to show the use of what is above delivered.

First, let the *fluent* of $\dfrac{\dot{x}}{\sqrt{a^2-x^2-\frac{x^4}{ra^2}}}$ (when the divisor becomes equal to nothing) be required.

Then, by comparing $\overline{a^2-x^2-\frac{x^4}{ra^2}}|^{-\frac{1}{2}}$ with the general *trinomial* $\overline{a - bx^n + dx^{2n}}|^p \times x^{pn-1}\dot{x}$, it appears that a^2 must be, here, wrote in the room of a, and that n, m, p, b and d, will be interpreted by 2, $-\frac{1}{2}, \frac{1}{2}$, 1, and $-\frac{1}{ra^2}$ respectively: whence we have $t\ (p+m+1) = 1$, $\frac{1.3.5\ (m+\frac{1}{2}) \times 1.3.5\ (p-\frac{1}{2})}{2\ .\ 4\ .\ 6\ (m+p)}$ $\times \frac{a^{m+p}G}{nb^p} = \frac{G}{2}$, and the fluent sought $= \frac{G}{2} \times$

$$\overline{1 - \frac{1.3}{2.2r} + \frac{1.3.5.7}{2.2.4.4r^2} - \frac{1.3.5.7.9.11}{2.2.4.4.6.6r^3} + \&c.}$$

360. The second example shall be, to find the fluent expressing the *apside-angle* in an orbit described by means of a centripetal force varying according to any power of the distance.

In which case the given fluxion being

$$\frac{\pm p\dot{x}}{x\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2 + \frac{2x^{n+3}}{n+1}}}$$ (*vide* Art. 242, where A is supposed the higher apse, and C A and consequently Cb, equal to unity) we shall, by putting $1 - p^2 = \beta$, $\frac{n+3}{2} = v$, and $1 - x^2 = y$, reduce it to

$$\frac{\frac{1}{2}\sqrt{1-\beta} \times \dot{y}}{\overline{1-y} \times \sqrt{\beta y + \frac{1 - vy - \overline{1-y}|^v}{1-v}}} = \frac{1}{2}\sqrt{1-\beta} \times$$

$$\overline{\beta - \frac{vy}{2} + \frac{v\ .\ \overline{v-2}}{2\ .\ 3}\ .\ y^2 - \frac{v\ .\ \overline{v-2}\ .\ \overline{v-3}}{2\ .\ 3\ .\ 4}\ .\ y^3 + \&c.}\Big|^{-\frac{1}{2}}$$

$\times\ \overline{y^{-\frac{1}{2}}\dot{y} + y^{\frac{1}{2}}\dot{y} + y^{\frac{3}{2}}\dot{y} + y^{\frac{5}{2}}\dot{y} + \&c.}$ Where the quantity under the radical sign (now answering to the form above prescribed) being compared with $\overline{a - bx^n + dx^{2n} + ex^{3n}\ \&c.}|^m$, we have $m = -\frac{1}{2}$, $n = 1$, $b = \frac{v}{2}$, $\frac{d}{b} = \frac{v-2}{3}$, $\frac{e}{b} = -\frac{\overline{v-2}.\overline{v-3}}{3.4}$ &c. Also the value of p with regard to the first term $(y^{-\frac{1}{2}}\dot{y})$ will be $= \frac{1}{2}$ (because $pn - 1 = -\frac{1}{2}$) likewise its value in the second term $(y^{\frac{1}{2}}\dot{y})$ is $= \frac{3}{2}$; in the third $= \frac{5}{2}$ &c. In the first of these cases we, therefore, have $t\ (m + p + 1) = 1$, $R\ (p \times \frac{d}{b}) = \frac{v-2}{6}$, $S = \frac{\overline{v-2}.\overline{4v-5}}{72}$, $T = \frac{\overline{v-2}.\overline{16v^2-37v+22}}{16 \times 45}$. Whence it follows, that the fluent of the first term $(\overline{\beta - \frac{vy}{2} + \frac{v.\overline{v-2}}{2.3}.y^2\ \&c.}|^{-\frac{1}{2}} \times y^{-\frac{1}{2}}\dot{y})$ when the quantity under the radical sign becomes equal to nothing (or the body arrives at its lower apse) will be truly expressed by $\frac{G}{\sqrt{\frac{1}{2}v}}$ into $1 + \frac{v-2}{2v}.\beta + \frac{5.\overline{v-2}.\overline{4v-5}}{48v^2}.\beta^2 + \frac{7.\overline{v-2}.\overline{16v^2-37v+22}}{6 \times 48v^3}.\beta^3$ + &c.

In the same manner it will appear, that the fluent of the second term, in that circumstance, is $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{1}{v}.\beta + \frac{5.\overline{v-2}}{4v^2}.\beta^2 + \frac{35.\overline{v-2}.\overline{2v-3}}{48v^3}.\beta^3}$

&c. that of the third $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \left(\frac{8}{2v^2} \cdot \beta^2 + \right.$ $\frac{35 \cdot \overline{v-2}}{12v^3} \cdot \beta^3$ &c.) that of the fourth $= \frac{5}{2v^3} \cdot \beta^3$ &c. &c.

Whence the fluent of the whole series, by collecting these several values together, will come out $=$

$\frac{G}{\sqrt{\frac{1}{2}v}} \times (1 + \frac{1}{4}\beta + \frac{20v^2 - 5v + 2}{48v^2} \cdot \beta^2 +$

$\frac{112v^3 - 63v^2 - 42v - 8}{6 \times 48v^3} \beta^3$ + &c.) Which, drawn into

$\frac{1}{2} \times \overline{1 - \frac{1}{2}\beta - \frac{1}{8}\beta^2 - \frac{1}{16}\beta^3 -}$ &c. (the value of the general multiplicator $\frac{1}{2}\sqrt{1-\beta}$) gives $\frac{G}{\sqrt{2v}} \times$

$(1 * + \frac{\overline{v-2} \cdot \overline{2v-1}}{48} \cdot \frac{\beta^2}{v^2} + \frac{\overline{v-2} \cdot \overline{2v-1} \cdot \overline{2v-1}}{72}$

$\times \frac{\beta^3}{v^3}$ &c.) for the true measure of the angle required, in parts of the radius, or unity: from whence, by writing 180 instead of G, we shall have the same in degrees: which, last of all, by restoring n, becomes

$$\frac{180^\circ}{\sqrt{n+3}} \times \left(1 * + \frac{\overline{n-1} \cdot \overline{n+2}}{24} \times \left|\frac{\beta}{n+3}\right|^2 + \right.$$

$$\left.\frac{\overline{n-1} \cdot \overline{n+2} \cdot \overline{n+2}}{18} \times \left|\frac{\beta}{n+3}\right|^3 \text{ \&c.}\right)$$

Where n is the exponent of the law of the force, whereby the orbit is described; and β, the defect of the square of the measure of the celerity, at the higher *apse*, below that which the body ought to have to revolve in a circle, this last being denoted by unity.

The same conclusion may be otherwise derived, by bringing $1-y$, in the transformed fluxion, under the *vinculum*; but this way of going to work, though we have but one series to manage, will prove rather more troublesome than the foregoing.

It will appear from the two preceding examples, especially the first of them, that this last method of finding fluents is, chiefly, useful when all the terms of the given expression, after the two first, in respect of these, are but small. Which is a circumstance that frequently occurs in the resolution of physical problems; such as determining the effect of the atmosphere's resistance upon the vibration of pendulums; and the *inequalities* of the planets arising from their action on each other.—In short, wherever the fluent, or the quantity it expresses, would belong to the circle, or some other of the conic-sections, were it not for the interposition of some small perturbating force (whereby new terms, small in comparison of the two first, are introduced) the said method will be found of very great service.

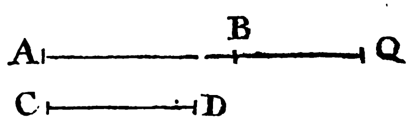
SECTION VIII.

The Use of Fluxions in determining the Motion of Bodies in resisting Mediums.

PROBLEM I.

361. *Supposing that a Body, let go from a given Point A, with a given Celerity, in a Right-line A Q, is resisted by a Medium (or any force) acting according to a given Power of the Velocity: to determine the Velocity, and also the Space run over, at the End of a given Time.*

LET the given celerity at A (measured by the space which would be uniformly described in any proposed time r) be put $= c$, and that at any other point, B, $= v$; moreover put A B $= x$, and the time



of its description $= z$; and let the resistance, or force, acting upon the body at A, be such, that, if the same was to be uniformly continued, the body would have all its motion destroyed thereby, in the time wherein it might move, uniformly, over a given distance d (C D) with its first velocity c: which time, let be denoted, by t.

Then, since the whole celerity c would be destroyed in the time t, that part of it which would be uniformly taken away in the time r, above proposed, will be truly represented by $\frac{r}{t} \times c$; or by $\frac{c^2}{d}$; which is equal to it, because the spaces (c and d) described with the same celerity are always as the times (r and t) of their description; and therefore $\frac{r}{t} = \frac{c}{d}$.

Hence, the resistance at B being to that at A (*by hypothesis*) as v^n to c^n, it follows that the velocity which might be destroyed in the given time r, by a force equal to the resistance at B, will be expressed by $\frac{c^2}{d} \times \frac{v^n}{c^n}$, or its equal $\frac{v^n}{dc^{n-2}}$: which expression is, therefore, the true measure of the force of the said resistance.

Now, it appears, from Art. 218, that, if the force with which the body is acted on (or the velocity it would generate in the given time r) be represented by F, the relation of the measures of the velocity and space gone over, will be expressed by the equation $\pm v\dot{v} = F\dot{x}$: from whence, by writing $\frac{v^n}{dc^{n-2}}$ instead of F, we have $-v\dot{v} = \frac{v^n\dot{x}}{dc^{n-2}}$ (the sign of $v\dot{v}$ being negative, because v decreases while x increases).* From this equation, we get $\dot{x} = -dc^{n-2}v^{1-n}\dot{v}$; whose fluent is $x = -\frac{dc^{n-2} \times v^{2-n}}{2-n} + \text{cor.}$; which, corrected (by taking $x = 0$, and $v = c$) becomes $x = \frac{-dc^{n-2} \times v^{2-n} + d}{2-n} = \frac{d}{n-2} \times \overline{\frac{c}{v}}\Big|^{n-2} - 1$.

* Art. 5.

Moreover, since the time ($\dot{z}$) is to the time r, as the distance $\dot{x}$ to the distance v, we also have $\dot{z}$ ($= \frac{r\dot{x}}{v}$) $= -rdc^{n-2}v^{-n}\dot{v}$; and consequently $z =$

$\frac{rd}{\overline{n-1}\times c}\times\left.\overline{\frac{c}{v}}\right|^{n-1}-1=\frac{t}{n-1}\times\left.\overline{\frac{c}{v}}\right|^{n-1}-1$ (by writing t for its equal $\frac{rd}{c}$): from which equation we get $\frac{c}{v}=\left.\overline{1+\overline{n-1}\times\frac{z}{t}}\right|^{\frac{1}{n-1}}$: likewise, from the preceding equation, we get $\frac{c}{v}=$ $\left.\overline{1+\overline{n-2}\times\frac{x}{d}}\right|^{\frac{1}{n-2}}$: which two equal values being compared together, there, at length, results $x=$ $\frac{d}{n-2}$ into $\left.\overline{1+\overline{n-1}\times\frac{z}{t}}\right|^{\frac{n-2}{n-1}}-1$, for the required relation of x and z. *Q. E. I.*

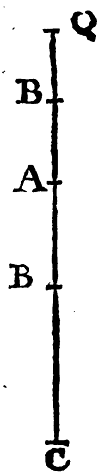
Corollary.

362. If $n=2$, or, the resistance be in the duplicate ratio of the velocity, the equation exhibiting the relation of z and v, will be $\frac{c}{v}=1+\frac{z}{t}$, or $v=\frac{c}{1+\frac{z}{t}}$: but the other equation (the fluent failing) becomes impracticable. Here x, the fluent of $-\frac{d\dot{v}}{v}$, will be explicable by $d\times$ hyp. log. $\frac{c}{v}$,* or by $d\times$ hyp. log. $\left(1+\frac{z}{t}\right)$; because $v=\frac{c}{1+\frac{z}{t}}$.

* Art. 126.

In the like manner, when $n=1$, or the resistance is as the velocity, the relation of v, x and z, will be exhibited by the equations $v = c \times \frac{d-x}{d}$, and $z = t \times$ hyp. log. $\frac{c}{v} = t \times$ hyp. log. $\frac{d}{d-x}$. Which case, and that above, are the only two wherein the general solution fails.

PROBLEM II.



363. *If a Body, let go from a given Point A with a given Celerity, in a vertical Line C A Q, is acted on by an uniform Gravity, and also by a Medium, resisting according to any given Power of the Velocity; it is proposed to determine the Relation of the Times, the Velocities, and the Spaces gone over.*

Let the notation in the preceding problem be retained; and let the force of gravity, in the given medium (measured by the velocity it might generate in the proposed time r)* be represented by b. Then, this value being added to, or subtracted from $\left(\frac{v^n}{dc^{n-2}}\right)$ the measure of the resistance,† according as the body is in its ascent, or descent, we thence get $\frac{v^n}{dc^{n-2}} \pm b$ for the whole force (F) whereby the motion, at B, is affected: whence (by Art. 218) $\dot{x} = \left(\frac{-v\dot{v}}{F}\right) = \frac{-dc^{n-2}v\dot{v}}{v^n \pm bdc^{n-2}}$; and $\dot{z}\ \left(= \frac{r\dot{x}}{v}\right)$‡ $= \frac{-rdc^{n-2}\dot{v}}{v^n \pm bdc^{n-2}}$: whose fluents may

* Art. 361.

† Art. 361

‡ Art. 361

be had, by the means of circular arcs, and logarithms, from Art. 331. *Q. E. I.*

Corollary I.

364. It appears that the force $\left(\frac{v^n}{dc^{n-2}}\right)$ of the resistance is to (b) *that* of gravity, in the given medium, as v^n to bdc^{n-2}: therefore, if this ratio be expounded by that of v^n to a^n, or a^n be put $= bdc^{n-2}$, it follows that a will express the celerity with which the resistance would be equal to the gravity (since, when $v=a$, the said ratio becomes that of equality). Hence, also, by substituting $\frac{a^n}{b}$ for its equal dc^{n-2}, we get

$$\dot{x} = \frac{-a^n v\dot{v}}{b \times \overline{v^n \pm a^n}}, \text{ and } \dot{z} = \frac{-ra^n \dot{v}}{b \times \overline{v^n \pm a^n}}$$

Corollary II.

365. If the resistance be in the duplicate ratio of the celerity, our two last equations will become $\dot{x} = \frac{-a^2 v\dot{v}}{b \times \overline{v^2 \pm a^2}}$, and $\dot{z} = \frac{-ra^2 \dot{v}}{b \times \overline{v^2 \pm a^2}}$: from the former

* Art. 126. whereof we get $x = -\frac{a^2}{2b} \times \text{hyp. log.} \frac{v^2 \pm a^2}{c^2 \pm a^2}$ *

$$= \frac{a^2}{2b} \times \text{hyp. log.} \frac{c^2 \pm a^2}{v^2 \pm a^2} = \frac{d}{2} \times \text{hyp. log.} \frac{c^2 \pm bd}{v^2 \pm bd}$$

(because, here, $a^2 = bd$.). From whence, when $v = 0$, (supposing the body to ascend) there comes out $x = \frac{d}{2} \times \text{hyp. log.} \left(1 + \frac{c^2}{a^2}\right)$, for the height ($AQ$) of the whole ascent. But, if c be taken $= 0$, or the body be supposed to descend from rest, we shall then have

$$-\frac{d}{2}\times\text{hyp. log.}\ \overline{1-\frac{v^2}{a^2}}=\text{the distance } AB \text{ descended.}$$

Whence, if N be put for the number whose hyperbolical logarithms is $\frac{2x}{d}$, it follows (because, log. $(1 - \frac{v^2}{a^2}) = -\frac{2x}{d} = -\log. N$) that $1-\frac{v^2}{a^2} = \frac{1}{N}$, and consequently $v=a\sqrt{\frac{N-1}{N}}$. From which, the distance AB being given, the velocity acquired in the fall will be determined. But, if the body, first, ascends from a given point A, with a given celerity c, and the celerity, acquired in falling, when it arrives, again, at that point, be required; the same may be exhibited in a more commodious form, independent of logarithms, and will be equal to $\frac{c}{\sqrt{1+\frac{c^2}{a^2}}}$; because N, in this case, is found above to be $= 1 + \frac{c^2}{a^2}$. Furthermore, with regard to the time (z), we have already found that $\dot{z}$ is $= \frac{-ra^2\dot{v}}{b\times\overline{v^2+a^2}}$, or $= \frac{-ra^2\dot{v}}{b\times\overline{v^2-a^2}}$ $(= \frac{ra^2\dot{v}}{b\times\overline{a^2-v^2}})$ according as the motion of the body is from, or towards the center of force. Therefore the time itself, in the former case, will be $= \frac{ra}{b}$ drawn into the difference of the two circular arcs whose tangents are $\frac{c}{a}$ and $\frac{v}{a}$, and whereof the common radius is unity:* whence it follows that the

* Art. 142.

time of the whole ascent will be denoted by $\frac{ra}{b}$ multiplied into the former of the said arcs.

But, in the other case, the fluent exhibiting the time of descent, is not explicable by the arcs of a circle, but by the difference of the hyperbolical logarithms of $\frac{a+v}{a-v}$ and $\frac{a+c}{a-c}$ drawn into $\frac{ra}{2b}$.* Therefore, when $c = 0$, or the body falls from rest, the time z will be barely $= \frac{ra}{2b} \times$ hyp. log. $\frac{a+v}{a-v} = \frac{ra}{b}$ $\times$ hyp. log. $(N^{\frac{1}{2}} + \overline{N-1}|^{\frac{1}{2}})$ (by substituting the value of v found above, and ordering the logarithm as in Art. 303). This equation, in the forementioned circumstance, where $N = 1 + \frac{c^2}{a^2}$, and $v = \frac{c}{\sqrt{1+\frac{c^2}{a^2}}}$, becomes $z = \frac{ra}{b} \times$ hyp. log. $\sqrt{1+\frac{c^2}{a^2}} + \frac{c}{a}$.

* Art. 126.

SCHOLIUM.

366. If, according to *Sir Isaac Newton*, we suppose the resistance of the air, to bodies moving in it, to be in the duplicate ratio of the celerities:* and that

* That the resistance is as the square of the celerity, the learner may, in some measure, conceive, by considering that the same body, with a double velocity, not only puts twice the number of resisting particles in motion, in the same time, but also acts upon each with a double force ; and therefore must suffer a four-fold resistance, or a resistance proportional to the square of the velocity. This would be strictly true, were it not that the particles so put in motion impel others lying before them, and thereby prevent, as it were, the action of the body. What deviation from the foregoing law may hence arise, is not easy to determine. This, however, seems plain, that the resistance at the beginning of any very swift motion (till the air in the way of the body comes duly to participate of that motion) will be greater than that sustained by another equal body, moving with the same celerity, that has been in motion some time.

a ball, in the time it might move uniformly over a space (d) which is to $\frac{2}{3}$ of its diameter as the density of the ball to that of the medium, would have all its motion taken away by a force equal to that of the resistance, uniformly continued: then, from these *data*, applied to the theorems in the preceding article, we shall be able to determine the velocities, and the times of the perpendicular ascent and descent of bodies near the earth's surface; allowing for the resistance of the atmosphere.

Thus, for instance, let a cannon ball of four inches diameter (whereof the density or specific gravity, is to that of air as 6000 to 1, nearly) be supposed to be projected, perpendicular to the horizon, with a velocity sufficient to cause it to ascend to the height of half a mile, or 2640 feet, *in vacuo;* which velocity (by Art. 203) will be found to answer to the rate of about 412 feet *per second:* then, according to the proportion just now mentioned, it will be as $1 : 6000 :: \frac{2}{3} \times 4 : 64000$ inches, or 5333 feet; which is the value of d in this case. Therefore, if the time r, in the preceding article (which may be assumed at pleasure) be here interpreted by *one second*, the corresponding values of d, c, and b will be expounded by 5333 f. 412 f. and $32\frac{1}{6}$ f.* respectively. Which values being substituted in the several equations in the last article, we shall get

* Art. 202.

1°. $a\ (= \sqrt{bd}) = 414$ f. the velocity, *per second*, wherewith the resistance would be equal to the gravity, or weight, of the ball.

2°. $\frac{d}{2} \times hyp.\ log.\ (1 + \frac{c^2}{a^2}) = 1835$ feet, the whole height of the ascent.

3°. $\frac{ra}{b} \times arch$, whose tang. is $\frac{c}{a} = 10{,}08$ *seconds*, the whole time of the ascent (which is less than the time *in vacuo*, by $2{,}73^{s}$).

4°. $v \left(= \frac{c}{\sqrt{1 + \frac{c^2}{a^2}}}\right) = 292$ f. the velocity, *per second*, acquired in the descent.

5°. Lastly, $\frac{ra}{b} \times hyp.\ log.\ \sqrt{1 + \frac{c^2}{a^2}} + \frac{c}{a} =$ 11,30 seconds, the time of the descent.

Note. In this example, the measure of the absolute gravity of the body, *in vacuo*, is taken, instead of its gravity in air (the difference *there* being too inconsiderable to be regarded). But, in cases where the specific gravity of the medium bears a sensible proportion to that of the body, the force of gravity (b) must be expounded by $32\frac{1}{12} \times \frac{B - M}{B}$ (instead of $32\frac{1}{12}$) where B is to M as the specific gravity of the body to that of the medium.

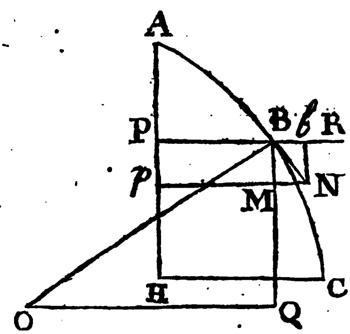
PROBLEM III.

367. *To determine the Resistance, by means whereof a Body, gravitating uniformly in the Direction of parallel Lines, may describe a given Curve.*

Let A B C be the given curve, and let B Q, parallel to the axis (or any given line) A H, be the direction of gravitation at any point B: make P B R perpendicular to AH and BQ; and let AP $= x$, PB $= y$, AB $= z$, B M (N b) $= \dot{x}$, M N (B b) $= \dot{y}$, B N $= \dot{z}$, and the velocity of the body at B in the direction P B R $= v$. Then, the decrease of velocity in the said direction, which is wholly owing to the resistance,* being represented by $- \dot{v}$, it follows that the corresponding decrease of motion in the direction B N, arising from the same cause, will be expressed by $\frac{\dot{z}}{\dot{y}} \times - \dot{v} = - \frac{\dot{v}\dot{z}}{\dot{y}}$; and, that in the direction B M, by $- \frac{\dot{v}\dot{x}}{\dot{y}}$. But, the

* Art. 209.

celerity in this last direction being every where represented by $v \times \frac{\dot{x}}{\dot{y}}$, its fluxion $\frac{v\ddot{x} + \dot{v}\dot{x}}{\dot{y}}$ will be the



whole alteration of motion in the said direction, arising from the resistance and the force of gravity, conjunctly: from which deducting the part owing to the resistance, found above to be $\frac{\dot{v}\dot{x}}{\dot{y}}$, the remainder $\frac{v\ddot{x}}{\dot{y}}$ will be the effect of the gravity. Which being to $(-\frac{\dot{v}\dot{z}}{\dot{y}})$ the effect of the absolute resistance in the direction B N, as 1 to $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$, the force of gravity, must therefore be to that of the required resistance, in the same ratio of 1 to $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$.

Moreover, the force of gravity, measured by the velocity it would generate in a given part of time (1), being denoted by unity, the velocity generated thereby, in the time $(\frac{\dot{y}}{v})$ of describing B*b*, with the celerity v, will likewise be truly expressed by $\frac{\dot{y}}{v}$, the measure of the said time: which being put = to $\left(\frac{v\ddot{x}}{\dot{y}}\right)$ the value of the same quantity, given above, we thence have $v^2 = \frac{\dot{y}^2}{\ddot{x}}$: from whence, not only the velocity, but the resistance will be found. But, if you would have the resistance expressed independent of v, then let the fluxion $\left(2v\dot{v} = -\frac{\dot{y}^2 \dddot{x}}{\ddot{x}^2}\right)$ of the last equation be divided by the fluent, which will give $\frac{\dot{v}}{v} = -\frac{\frac{1}{2}\dddot{x}}{\ddot{x}}$: and then, by substituting this value in $-\frac{\dot{v}\dot{s}}{v\ddot{x}}$, you will get $\frac{\dot{s}\dddot{x}}{2\ddot{x}^2}$ for the true force of the resistance, *that* of gravity (or the weight of the body) being expounded by unity.

The same otherwise.

Let B O be the radius of curvature at B, and let O Q be parallel to P B, meeting B M, produced, in Q: then, if the absolute gravity, acting in the direction B Q, be denoted by unity, its force in the direction B O, whereby the body is retained in the curve, will be represented by $\frac{BQ}{BO}$. Therefore, since the velocities in circles are known to be in the subduplicate ratio of the radii and of the forces conjunctly,* the velocity at B will be rightly expressed by $\sqrt{BO \times \frac{BQ}{BO}}$, or its equal $\sqrt{BQ}$. (For the curve at, and indefinitely near B, may be taken as an arch of a circle whose radius is B O: and it is evident that the resistance has nothing to do in forcing the body from the

* Art. 212.

tangent, but only serves to retard its motion, so that it may, every where, bear a due proportion to the given force of gravity acting in the direction B O). Hence, putting B Q = s, the increase of the celerity in the time $\left(\frac{\dot{z}}{\sqrt{s}}\right)$ of describing B N, will be expressed by the fluxion of $\sqrt{s}$, or $\frac{\dot{s}}{2\sqrt{s}}$. Moreover, the celerity that might be generated by gravity in the said time $\frac{\dot{z}}{\sqrt{s}}$ being measured thereby, the increase, in B N, arising from the same cause, will therefore be $= \frac{\dot{z}}{\sqrt{s}} \times \frac{\dot{x}}{\dot{z}} = \frac{\dot{x}}{\sqrt{s}}$: which, being taken from $\left(\frac{\dot{s}}{2\sqrt{s}}\right)$ the whole increase, found above, the remainder, $\frac{\dot{s}-2\dot{x}}{2\sqrt{s}}$, will be the effect of the resistance: which is to the effect, $\frac{\dot{z}}{\sqrt{s}}$, of the absolute gravity as $\frac{\dot{s}-2\dot{x}}{2\dot{z}}$ to 1. Therefore the resistance is to the gravity (or weight of the body) as $\frac{2\dot{x}-\dot{s}}{2\dot{z}}$ to unity: where the signs are changed, because the two forces act in contrary directions.

Because B O $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$,* therefore s $\left(\text{B O} \times \frac{\dot{y}}{\dot{z}}\right) = \frac{\dot{z}^2}{\ddot{x}} = \frac{\dot{y}^2+\dot{x}^2}{\ddot{x}}$ (= the square of the celerity) whence $\dot{s} = \frac{2\dot{x}\ddot{x}^2 - \overline{\dot{y}^2+\dot{x}^2} \times \dot{\ddot{x}}}{\ddot{x}^2}$, and consequently the resist-

* Art. 68.

sistance $\frac{2\dot{x}-\dot{s}}{2\dot{s}}=\frac{\overline{\dot{y}^2+\dot{x}^2}\times\ddot{x}}{2\dot{x}\dot{x}^2}=\frac{\dot{s}\ddot{x}}{2\dot{x}^3}$, *the very same as before.*

COROLLARY.

368. If the resistance be supposed as any given power of the velocity drawn into (D) the density of the medium; then, from hence, the density of the medium, at every point of the curve, may be determined: for, the absolute celerity at B being represented by $\frac{v\dot{z}}{\dot{y}}$, the resistance at that point will, according to the said hypothesis, be as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^n \times D$; and therefore the velocity that would be destroyed thereby, in the time $\left(\frac{\dot{y}}{v}\right)$ of describing BN, as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^n \times \frac{D\dot{y}}{v}$: which being put $=\left(-\frac{\dot{v}\dot{z}}{\dot{y}}\right)$ the effect of the same resistance, found above, we thence get $D=\frac{-\dot{v}\dot{y}^{n-2}}{\overline{v\dot{z}}\big|^{n-1}}$: which, by substituting for v and $\dot{v}$, becomes $D=\frac{\ddot{x}}{2\dot{z}^{n-1}\times\ddot{x}^{2-\frac{1}{2}n}}$.

In this corollary, and what elsewhere relates to unequal densities, the gravity of the body in the medium is supposed to continue every where the same, or, that the attraction increases with the density, so that the difference between the specific gravities of the body and medium may, at every point, be a constant quantity.

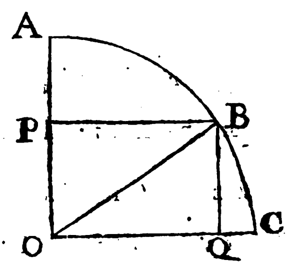
EXAMPLE I.

369. *Let the proposed Curve* A B C *be the common Parabola.*

Then, x being here $= \frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, $\ddot{x} = \frac{2\dot{y}^2}{a}$ and $\dddot{x} = 0$; and therefore $\frac{\dot{z}\dddot{x}}{2\ddot{x}^2}$ * is also $= 0$: whence it appears that a body, to describe this curve, must move in spaces entirely void of resistance. * Art. 367.

EXAMPLE II.

370. *Let the Curve* A B C *be taken as a Quadrant of a Circle, whose Radius* B O *is* $=a$.



In this case we have s (B Q)† $= a - x$ (= A O − A P) whence $\dot{s} = -\dot{x}$, and therefore $\frac{2\dot{x}-\dot{s}}{2\dot{z}} = \frac{3\dot{x}}{2\dot{z}} = \frac{3PB}{2AO}$.‡ From which it is evident, that the velocity is, every where, as $\sqrt{BQ}$, and the resistance to the gravity (or weight of the body) as 3 P B to 2 O B.

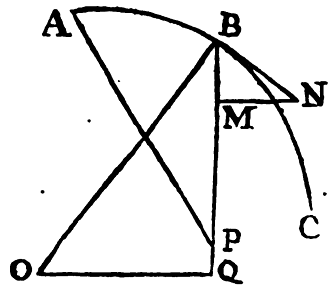
† Art. 367.

‡ Art. 142.

PROBLEM IV.

371. *The Centripetal Force (F) being given; to find the Resistance and Velocity whereby a Body may describe a given Spiral (or any other, possible, Curve) about the Center of Force.*

Let P be the center of force, and B O the radius of curvature at any point B in the proposed curve,



and let O Q be perpendicular to B P Q; also let B P $= y$, B Q $= s$, A B $= z$, B M
* Art. 5. $= -\dot{y}$,* and B N $= \dot{z}$. Then, it is evident from Art. 367, that the velocity at B will be expressed by $\sqrt{BO \times \frac{BQ}{BO} \times F}$; or, its equal, $\sqrt{sF}$: and therefore its increase in the time $\left(\frac{\dot{z}}{\sqrt{sF}}\right)$ of describing B N will be $\frac{s\dot{F}+F\dot{s}}{2\sqrt{sF}}$: from which, deducting $\left(F \times \frac{\dot{z}}{\sqrt{sF}} \times \frac{-\dot{y}}{\dot{z}}\right)$ the effect of the centripetal force, in the same time and direction, the remainder, $\frac{s\dot{F}+F\dot{s}+2F\dot{y}}{2\sqrt{sF}}$, is the effect of the resistance. Therefore the resistance is to the centripetal force as $\frac{s\dot{F}+F\dot{s}+2F\dot{y}}{2\sqrt{sF}}$ to $\frac{F\dot{z}}{\sqrt{sF}}$, or as $\frac{s\dot{F}+F\dot{s}+2F\dot{y}}{2F\dot{z}}$ to unity. Q. E. I.

EXAMPLE.

372. Let the measure (F) of the centripetal force be expounded by any power y^n of the distance; and let the curve be the logarithmic spiral; putting the
† Art. 61. co-sine of the given angle P B N† (to the radius r)
‡ Art. 74. $= c$. Then, s being here $= y$‡, and $\dot{F} = n y^{n-1} \dot{y}$,

we have $\frac{s\dot{F}+F\dot{s}+2F\dot{y}}{2F\dot{z}}=\frac{ny^n\dot{y}+y^n\dot{y}+2y^n\dot{y}}{2y^n\dot{z}}=\frac{n+3}{2}$ $\times\frac{\dot{y}}{\dot{z}}=\frac{n+3}{2}\times\frac{c}{r}$.

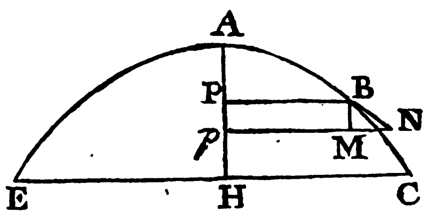
Hence it appears that the velocity must be, every where, as $y^{\frac{n+1}{2}}$; and the resistance to the centripetal force, as $\frac{n+3}{2}\times\frac{c}{r}$ to unity. But, when $n=-3$, $\frac{n+3}{2}\times\frac{c}{r}$ becomes $=0$; therefore the body, in this case, must move in spaces entirely void of resistance; agreeable to Art. 233. And if $n+3$ be negative, an accelerating, instead of a resisting force, will be required.

Scholium.

373. If the density of a medium, wherein a body moves, be either uniform, or varies according to a given law, the nature of the curve, or trajectory, may be determined from what is delivered in the preceding pages.

Thus, for example, let the density be supposed every where the same, and the resistance as the square of the celerity; then, from Art. 368, we have $\frac{\dot{x}}{\dot{z}\ddot{x}}=D$; which, in order to exterminate $\dot{z}$, may be transformed to $\dot{x}^2=\overline{\dot{y}^2+\dot{x}^2}\times D^2\ddot{x}^2$: where, D being a constant quantity (depending upon the given density of the medium) the value of x will be found, as is taught in Sect. 2, Art. 268, 271, and comes out $=\frac{y^2}{p}+\frac{Dy^3}{3p}$ $+\frac{D^2y^4}{12p}$ &c. In which p is put to denote the parameter of the curve at the vertex, or highest point A, (to be determined from the force of gravity and the given velocity of the body at that point). This solution answers near enough when the resistance is but small in proportion to the gravity: in other circumstances, the series not converging, it becomes useless: for which reason, and because the case above specified is that supposed to obtain, in respect to the air near the earth's surface, and its resistance to bodies moving therein, I shall show, by a different method, how the nature of the curve may be investigated.

In order thereto, let the celerity at the highest point A, above the plane of the horizon E C, be denoted by *c*; and let *a* be the celerity with which the resistance is equal to the gravity (*vide* Art. 365 and 366).



Moreover, let *d* be put for the distance over which the ball might uniformly move in the time that the medium would destroy all its motion, was the resistance to continue the same, all along, as at the first instant. (Which distance, according to Sir Isaac Newton, is always in proportion to $\frac{2}{3}$ of the ball's diameter, as the density of the ball is to that of the medium).

Then it will be, as $d : \dot{z}$ (B N) $:: \frac{v\dot{z}^2}{\dot{y}}$, the absolute celerity at B, to $\left(\frac{v\dot{z}^2}{d\dot{y}}\right)$ the part thereof that would be uniformly destroyed by the resistance in the time of describing B N, with the velocity at B: which value being also expressed by $\frac{-\dot{v}\dot{z}}{\dot{y}}$ (*vide* Art. 367) we therefore have $\frac{v\dot{z}^2}{d\dot{y}} = -\frac{\dot{v}\dot{z}}{\dot{y}}$; whence $\frac{\dot{z}}{d} = \frac{-\dot{v}}{v}$, and consequently, by taking the fluent $\frac{z}{d} = -$ hyp. log. v; which corrected (by putting $z=0$, and $v=c$) gives $\frac{z}{d}$ ($=$hyp. log. $c -$ hyp. log. v) $=$hyp. log. $\frac{c}{v}$.

Furthermore, since (by hypothesis) the resistance with the celerity $\frac{v\dot{z}}{\dot{y}}$ (at B) is to the force of gravity, or the resistance with the celerity a, as $\frac{v^2\dot{z}^2}{\dot{y}^2}$ to a^2; and it appears, from the aforesaid article, that the same ratio is also universally expressed by *that* of $\frac{-\dot{v}\dot{z}}{v\ddot{x}}$ to 1, it follows, from the equality of these ratios, that $\frac{\dot{z}\ddot{x}}{\dot{y}^2}$ is $= -\frac{a^2\dot{v}}{v^3}$. But, in order to the resolution of the equation thus given, let the tangent of the angle P B A (or N) which the ordinate P B makes with the curve (supposing radius unity) be, every where, represented by w: then, because $\dot{x}=w\dot{y}$, $\dot{z}$ ($\sqrt{\dot{y}^2 + \dot{x}^2}$) $= \dot{y}\sqrt{1+w^2}$, and $\ddot{x} = \dot{w}\dot{y}$ ($\dot{y}$ being constant) we shall, by substituting these values in the aforesaid equation, get $-\frac{a^2\dot{v}}{v^3} = \dot{w}\sqrt{1+w^2}$; whereof the fluent will be given, $\frac{\frac{1}{2}a^2}{v^2} = \frac{1}{2}w\sqrt{1+w^2} + \frac{1}{2}$ hyp. log. $w + \sqrt{1+w^2}$:* which corrected (by taking* $v=c$ and $w=0$) becomes $\frac{\frac{1}{2}a^2}{v^2} - \frac{\frac{1}{2}a^2}{c^2} = \frac{1}{2}w\sqrt{1+w^2}$ $+ \frac{1}{2}$ hyp. log. $w + \sqrt{1+w^2}$. But, to shorten the

* Art. 126 & 281.

remaining part of the process, let the latter part of the equation, or the fluent of $\dot{w}\sqrt{1+w^2}$ be denoted by Q; then $\frac{a^2}{2v^2}$ being $= \frac{a^2}{2c} + Q$, we have $v = \frac{ac}{\sqrt{a^2 + 2c^2Q}}$; and consequently $\frac{z}{d}$ $(=\text{hyp. log.} \frac{c}{v})$ $= \text{hyp log.} \frac{\sqrt{a^2+2c^2Q}}{a} = \frac{1}{2} \text{ hyp. log. } \overline{1 + \frac{2c^2Q}{a^2}}$

From which two equations, the velocity of the ball, and the distance it has moved, when its direction makes any given angle with the horizon, may be computed, let the medium be as dense as it will: also, from hence, if the celerity answering to any one given angle of direction be known, the celerity corresponding to any other given direction may be found, together with the distance described between the two positions. For v (in the descent of the body) being *universally*, equal to $\frac{ac}{\sqrt{a^2+2c^2Q}}$, the value of c, expressing the celerity at the vertex *A*, will be had from that equation, and comes out $= \frac{av}{\sqrt{a^2 - 2v^2Q}}$; whence also z $(= d \times \text{hyp. log. } \frac{c}{v}) = d \times \text{hyp. log.} \frac{a}{\sqrt{a^2 - 2v^2Q}} = -\frac{1}{2} d \times \text{hyp. log } \overline{1 - \frac{2v^2Q}{a^2}}$.

From which, the celerity at A being known, the rest is obvious.

But, in the ascending part of the curve E A, both z and Q must be considered as negative, or wrote with contrary signs: and then, from the foregoing equations, we shall also get $v = \frac{ac}{\sqrt{a^2 - 2c^2Q}}$, $c = \frac{av}{\sqrt{a^2 + 2v^2Q}}$, and $-z = \frac{1}{2}d \times$ hyp. log. $\overline{1 - \frac{2c^2Q}{a^2}} = -\frac{1}{2}d \times$ hyp. log. $\overline{1 + \frac{2v^2Q}{a^2}}$; and, consequently, $z = -\frac{1}{2}d \times$ hyp. log. $\overline{1 - \frac{2c^2Q}{a^2}} = \frac{1}{2}d \times$ hyp. log. $\overline{1 + \frac{2v^2Q}{a^2}}$ $= d \times$ hyp. log. $\frac{v}{c}$: answering in this case.

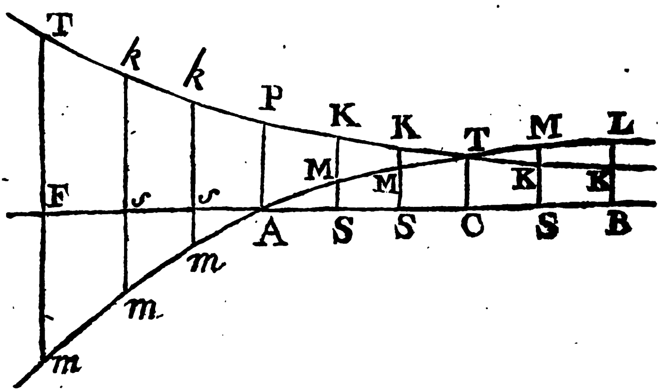
It still remains to take some notice of the values of x and y (in order to have the form, as well as the length of the curve). These, indeed, are not so easy to bring out as that of z, given above; nor can they be exhibited in a general manner, either by circular arcs, or logarithms (that I have been able to discover) but may, however, be approximated to any required degree of exactness, as will appear from what follows.

Since z $(= AB)$ is found $= \frac{1}{2}d \times$ hyp. log. $\frac{a^2+2c^2Q}{a^2}$, by taking the fluxion thereof, we get $\dot{z} = \frac{c^2d\dot{Q}}{a^2 + 2c^2Q} = \frac{c^2d\dot{w}\sqrt{1+w^2}}{a^2+2c^2Q}$ (because $\dot{Q} = \dot{w}\sqrt{1+w^2}$) therefore $\dot{y}$ $\left(= \frac{\dot{z}}{\sqrt{1+w^2}}\right) = \frac{c^2d\dot{w}}{a^2 + 2c^2Q}$; and $\dot{x}$ $(= w\dot{y}) = \frac{c^2dw\dot{w}}{a^2 + 2c^2Q}$: which equations, by taking r to 1, as a^2 to c^2 (or as the square of the force of gravity to the square of the resistance at A) are reduced to $\dot{y} = \frac{d\dot{w}}{r+2Q}$, and $\dot{x} = \frac{dw\dot{w}}{r + 2Q}$: whence we get $y = d$ into $\frac{w}{r + 2Q} + \frac{\frac{2}{3} \times \overline{1 + w^2}|^{\frac{3}{2}} - \frac{2}{3}}{\overline{r+2Q}|^2} +$

$\frac{8}{5} w \times \overline{1 + \frac{2}{3} w^2 + \frac{1}{5} w^4} - \frac{8}{5} Q \over \overline{r+2Q}|^3$ &c. And $x = d$ into

$\frac{\frac{1}{2} w^3}{r + 2Q} + \frac{\frac{1}{4} w \times \overline{1 + w^2}|^{\frac{3}{2}} - \frac{1}{4} Q}{\overline{r+2Q}|^2} +$

$\frac{\frac{1}{6} \times \overline{1 + w^2}|^3 - \frac{1}{6} - \frac{1}{2} Q^2}{\overline{r+2Q}|^3}$ &c. These expressions (brought out by assuming $\frac{A}{r + 2Q} + \frac{B}{\overline{r+2Q}|^2} +$ &c. for the fluent sought, and proceeding as in Art. 340) converge very fast when r is large in comparison to Q; but in other cases the required values will be had, with less trouble, from the following method.



Let P K T K and A M T M be two curves, whereof the ordinates S K and S M, to the common *abscissa* w (= A S) are expressed by $\frac{1}{r + 2Q}$ and $\frac{w}{r + 2Q}$ respectively: then it is plain, from the foregoing equations, that the measures of the areas of the said curves, multiplied by d, will truly exhibit the values of y and x;

answering to any given value of w (or A S) the tangent of the angle of direction; or, to speak more geometrically, a square upon A C (supposing A C = radius = unity) will be to either of the said areas A S K P, or A S M as the given distance d, to the value of y or x required. But now, as to a way for computing these areas (without which what has been said about them would be to very little purpose) the method of *equi-distant ordinates* may here be applied to very good advantage (when the foregoing series do not converge). By means whereof the required quantities may, with a little trouble, be brought out to a sufficient degree of exactness, let the resistance be as great as it will.

According to the same way of proceeding, the values of x and y, in the ascent of the ball, will also be found, if the ordinates sk and sm, generating the required areas, be taken, every where, equal to $\frac{1}{r-2Q}$ and $\frac{w}{r-2Q}$ instead of $\frac{1}{r+2Q}$ and $\frac{w}{r+2Q}$.

From what has been thus far delivered, it will not be very difficult to calculate (according to the foregoing hypothesis) all the principal requisites concerning the motion and track of a ball in the air, projected with a given velocity, at a given elevation; as will be more clearly seen by the example subjoined.

Suppose a cannon ball of 4 inches diameter (whereof the weight is nearly 9 pounds) to be discharged at an elevation of 45 degrees, with a velocity sufficient to carry it to the distance of one mile, on the plane of the horizon, were it not for the resistance of the air. Then that velocity, being the same as might be freely acquired in a perpendicular descent of half a mile,* will be found to answer to the rate of 412 feet *per second*, according to Art. 202 and 366. From whence it is also plain, that the distance d (so often mentioned above) will here be expounded by 5333 feet; and that the celerity (a) with which the resistance would be equal to the gravity (or weight of the ball) answers to the rate of about 414 feet *per second*.

* Art. 366.

Moreover, since the tangent of the angle of elevation, or the first value of w, is given *equal to* unity, (or radius) we have Q $(\frac{1}{2} w \sqrt{w^2+1} + \frac{1}{2}$ hyp. log. $\overline{w + \sqrt{w^2+1}}) = 1.1478$: from which, and v $(= 412 \sqrt{\frac{1}{2}})$, we get z $(= \frac{1}{2} d \times$ hyp. log. $\overline{1 + \frac{2v^2 Q}{a^2}})$ $= 2025$ feet $=$ the arch described in the whole ascent. Also c $\left(= \frac{v}{\sqrt{1 + \frac{2v^2 Q}{a^2}}}\right) = 199\frac{1}{3}$ feet, for the rate of the velocity, *per second*, at the highest point: whence r $(= \frac{a^2}{c^2}) = 4314$; by means whereof the greatest altitude of the ball, and the horizontal distance corresponding thereto will likewise be found: for let A F, *in the preceding figure*, be taken $=1$ (the given value of w) and let the same be divided into three parts by equi-distant ordinates (which number will answer sufficiently exact) then the successive values of w, for the ordinates A P, ks, ks, and TF, being 0, $\frac{1}{3}$, $\frac{2}{3}$, and 1, *those* of Q will be 0, 0.3394, 0.713, and 1.1478, and the ordinates themselves (or the corresponding values of $\frac{1}{r - 2Q}$ = to 0.2318, 0.2751, 0.3463 and 0.4953, respectively. From whence, by adding the two extremes to three times the sum of the two middle terms, and dividing the whole by 8, we get 0.3239 for the value of a mean ordinate:* which, as A F is here equal to unity, is also the measure of the required area A F T P: which, therefore, being multiplied by 5333 (d) gives 1727 feet for the horizontal distance made good in the whole ascent. In

* See p. 117 of my Mathematical Dissertations.

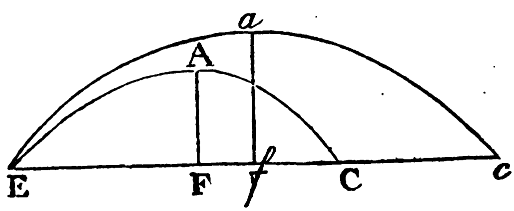
the same way the area A F m is found = 0. 1828. Whence the greatest height of the ball appears to be (0. 1828 × 5333) = 975 feet.

By taking AC = 1, and repeating the operation (only changing $r-2Q$ to $r+2Q$) the area A C T P will come out = 0. 1883, and A T C = 0. 0875; which multiplied by 5333 (as above) give 1004 F. and 467 F. for the amplitude, and the distance descended, from the highest point, when the direction of the ball makes an angle with the horizon equal to that in which it was projected.

But, to have the direction when the ball strikes the ground, and the whole amplitude of the projection, we must find the value of the tangent A B, when the area ABL is equal to (0. 1828) the area AFm (so that the descent, from the highest point, may become equal to the whole ascent). In order thereto, let 0. 0875 (ATC) be deducted from 0. 1828 (A F m) and the remainder 0. 0953 will be = C T B L; this, divided by T C (0. 1513) quotes 0. 63; which would be the value of CB, if all the ordinates C T, S M, &c. were equal: but, as it is obvious from the nature of the problem, and from the law of the ordinates already computed, that B L will be something greater than CT, and consequently C B less than 0. 63—I therefore suppose the value of C B may be about 0. 56; and, accordingly, proceed to compute the area of CBLT answering to this number; by means of CT (0. 1513) and BL (0. 1852) and one intermediate ordinate S M (0. 1715) and find it (from the approximation $\frac{CT+BL+4SM}{6} \times CB$) to come out = 0. 0955: which is so near the required value 0. 0953, that it will be altogether needless to repeat the operation. It is evident from hence, that the tangent (A B) of the angle of direction, when the ball strikes the ground, is 1. 56; answering to 57° : 20′: from whence, C B K T being found = 0. 0752, the whole area A B K P will be had = 0. 2635, and consequently 0. 2635 × 5333 = 1405 F. = the amplitude in the whole descent.

Furthermore, from the said value of w and that of c $(=199\frac{1}{3})$ given above, we get z $(=\frac{1}{2}d \times$ hyp. log. $\overline{1+\frac{2c^2Q}{a^2}}) = 1788$ feet, for the arch described in the descent; and also $v = 142\frac{1}{2}$ F. which multiplied by 1.8527, the secant of $57^\circ : 20'$, gives 264 F. for the celerity of the ball, *per second*, at the end of its flight.

Now, by collecting the principal of the foregoing conclusions, it appears,



1°. That the velocity at the highest point A of the trajectory will be at the rate of $199\frac{1}{3}$ feet, *per second:* which is to the velocity at the highest point a of the parabola (E a c) that would be described, were it not for the resistance, as 2 to 3, nearly.

2°. E A = 2025 and E a = 3030 ⎫
3°. E F = 1727 and E f = 2640 ⎪
4°. A F = 975 and $a f$ = 1320 ⎬ Feet.
5°. A C = 1788 and $a c$ = 3030 ⎪
6°. F C = 1405 and $f c$ = 2640 ⎭

7°. Angle C = $57^\circ : 20'$ and c (=E) = 45°.

8°. Velocity at C to that at E, as 264 to 412, or as 2 to 3, nearly.

These proportions, between the distances, in air and *in vacuo*, hold at an elevation of 45°, when the resistance, at going off, is nearly equal to the gravity, or weight, of the ball. If the velocity be greater than that above specified, or the body, projected, be, either, less, or less dense, the curve will differ, *still*, more from a parabola.

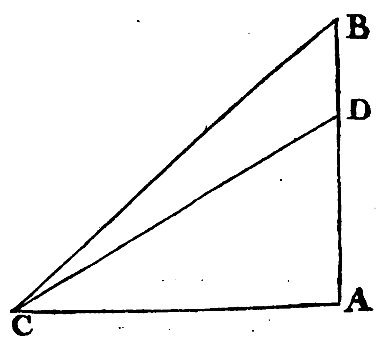
Hence it evidently appears, that the effect of the air's resistance upon very swift motions, is too considerable to be entirely disregarded in the art of gunnery.—It is true the method given above is, by much, too intricate for common practice; but when the law of the resistance to very swift motions is once sufficiently established (which, according to some late experiments, seems to be in a ratio greater than that of the square of the celerity) it will be no very difficult matter to find out proper approximations to correct the proportions in common use.

SECTION IX.

The Use of Fluxions in determining the Attraction of Bodies under different Forms.

PROBLEM I.

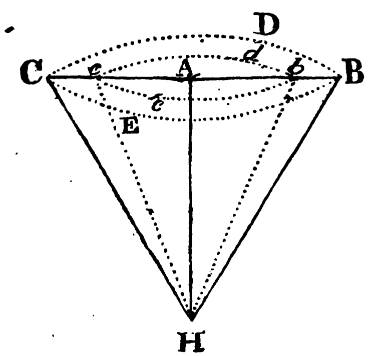
374. SUPPOSING AC *perpendicular to* AB, *and that a Corpuscle at* C *is attracted towards every Point or Particle of the Line* A B, *by Forces in the reciprocal duplicate Ratio of the distances; to determine the Ratio of the whole Force whereby the Corpuscle is urged in the Direction* C A.



Put AC $=a$, and let AD (considered as variable by the motion of D towards B) be denoted by x: then, the force of a particle at D being as $\frac{1}{CD^2}$ (by hypothesis) its efficacy in the proposed direction AC (will by the resolution of forces) be as $\frac{AC}{CD} \times \frac{1}{CD^2} = \frac{AC}{CD^3} = \frac{a}{\overline{a^2 + x^2}|^{\frac{3}{2}}}$: therefore $\frac{a\dot{x}}{\overline{a^2 + x^2}|^{\frac{3}{2}}}$ is the fluxion of the whole force; whose fluent, which (by Art. 85) is $= \frac{x}{a \times \overline{a^2 + x^2}|^{\frac{1}{2}}}$ $= \frac{AD}{CA \times CD}$, will, when AD = AB, be as the force itself. *Q. E. I.*

PROBLEM II.

375. *Supposing* BDÇE *to represent a circular Plane, and that a Corpuscle* H, *in the Axis thereof* AH, *is attracted by every Point or Particle of the Plane by Forces in the reciprocal duplicate Ratio of the Distances; to find the whole Force by which the Corpuscle is urged towards the Plane.*



Let $AH = a$, and $Hb = x$; then $Ab^2 = x^2 - a^2$; which multiplied by p ($= 3.14159$ &c.) the area of the circle whose radius is unity, gives $p \times \overline{x^2 - a^2}$ for the area of the circle $Acdbe$: whose fluxion is $= 2px\dot{x}$. But the force of a single particle at b, in the direction HA, is as $\frac{AH}{Hb^3}$, or $\frac{a}{x^3}$ (see the last Problem) therefore the fluxion of the whole force is truly defined by $2px\dot{x} \times \frac{a}{x^3}$ or its equal $\frac{2pa\dot{x}}{x^2}$, and the force itself by the fluent of $\frac{2pa\dot{x}}{x^2}$; which (properly corrected) is $-\frac{2pa}{x} + \frac{2pa}{a} = 2p \times \overline{1 - \frac{a}{x}} = 2p \times \overline{1 - \frac{\text{AH}}{\text{BH}}}$, when $x = \text{HB}$. *Q. E. I.*

376. In the preceding problems, we have supposed the attraction of each particle, to be as the square of the distance inversely; that being the law which is found to obtain in nature: but if the force, according to any other law of attraction, be required, the process will be very little different.

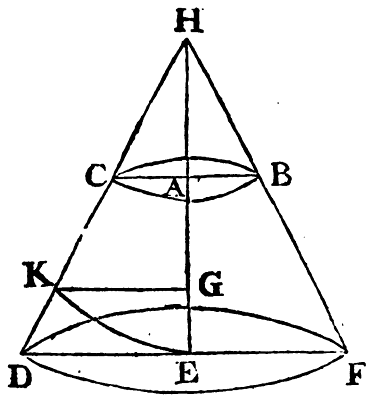
Thus, let the attraction be as any power (n) of the distance: then (in the last Prob.) the force of a particle at b (upon H) being as x^n, its force in the direction H A will be as $\frac{a}{x} \times x^n$ or ax^{n-1}; which multiplied by $2px\dot{x}$ (as before) gives $2pax^n\dot{x}$: whereof the fluent $\frac{2pax^{n+1} - 2pa^{n+2}}{n+1}$ $\left(= \frac{2p}{n+1} \times \overline{\text{AH} \times \text{BH}^{n+1} - \text{AH}^{n+2}}\right)$ will be as the force required.

PROBLEM III.

377. *To determine the Attraction of a Cone* DHF *at its Vertex; the Attraction of each Particle being as the Square of the Distance inversely.*

Put the axis E H $= a$, the length of the slant-side H D (or H F) $= b$, and A H (considered as variable) $= x$: then *(by sim. triangles)* a (H E) : b (H F)

$\because x$ (H A) : H B $= \frac{bx}{a}$ But, by the last problem,



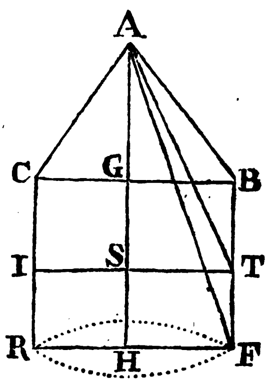
the attraction of all the particles in the circle B C will be measured by $2p \times \overline{1 - \frac{AH}{BH}} = 2p \times \overline{1 - \frac{a}{b}}$ (because $HB = \frac{bx}{a}$): which therefore being multiplied by $\dot{x}$, and the fluent taken, we thence have $x - \frac{ax}{b}$ for the attraction of A C H B: and this, when $x = a$, will be $2p \times \overline{EH - \frac{EH^2}{DH}}$, the force of the whole cone D E H F: which, if H K be made $=$ H E, and KG perpendicular to HE, will likewise be truly defined by $2p \times EG$ (because $HG = \frac{EH^2}{DH}$). *Q. E. I.*

Corollary.

378. Seeing the attraction of A C H B is, every where, as $x - \frac{ax}{b}$, or $\frac{b-a}{b} \times x$, it follows that the forces of similar cones, at their vertexes, are directly as their altitudes.

PROBLEM IV.

379. *To find the Force of a Cylinder* CBFR, *at any Point* A *in the Produced Axis; the Law of Attraction being still as in the preceding Problems.*

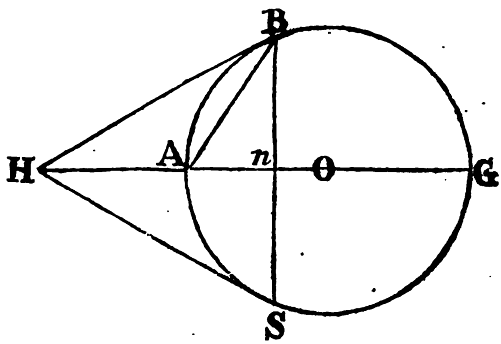


Put BG ($=$ CG $=$ RH) $= b$; and let AS (taken as variable) $= x$: therefore $AT = \sqrt{b^2+x^2}$, and $2p \times \overline{1 - \frac{AS}{AT}} = 2p \times \overline{1 - \frac{x}{\sqrt{b^2+x^2}}}$: which (by Prob. 2) expresses the force of all the particles in the circular surface IST.

Therefore $2p \times \overline{\dot{x} - \frac{x\dot{x}}{\sqrt{b^2+x^2}}}$ is the fluxion of the required force: whose fluent $(2p \times \overline{x - \sqrt{b^2+x^2}})$ when $x =$ AG, will be $= 2p \times \overline{AG - AB}$; but when $x =$ AH, it will be $= 2p \times \overline{AH - AF}$: hence, by taking the former of these values from the latter, we have $2p \times \overline{AB + BF - AF}$ for the measure of the true force by which a corpuscle at A is urged towards the cylinder.

PROBLEM V.

380. *The Law of the Force being still supposed the same; to determine the Attraction of a Sphere* OABGS, *at any given Point* H *above its Surface.*



Let B S be perpendicular to H G, and let H B be drawn; also put the radius AO $=a$, OH $=b$, AH $(b-a)$ $=c$, H$n=y$, and H B $=c+x$; then A$n=y-c$, Gn $=2a-y+c$, and consequently $\overline{y-c}\times\overline{2a-y+c}$ ($=$ A$n\times$G$n=$B$n^2=$BH$^2-$Hn^2)$=\overline{c+x}|^2-y^2$: from which equation we get $y=\frac{2ac+2c^2+2cx+x^2}{2a+2c}=$ $\frac{2bc+2cx+x^2}{2b}$ (because $a+c=b$). Whence also $2p\times$

$$\overline{1-\frac{\text{H}n}{\text{HB}}}^{*}=2p\times\overline{1-\frac{2bc+2cx+x^2}{2b\times\overline{c+x}}}=\frac{2p\times\overline{2ax-x^2}}{2b\times\overline{c+x}}:$$

which multiplied by $\frac{c\dot{x}+x\dot{x}}{b}=\dot{y}$ gives $\frac{p\times\overline{2ax\dot{x}-x^2\dot{x}}}{b^2}$ for the fluxion of the required force; whereof the fluent

* Art. 375.

$\frac{p \times \overline{ax^2 - \frac{1}{3}x^3}}{b^2}$ will be the attraction of the segment ABS: which therefore, when B coincides with G and x is $= 2a$, becomes $\frac{4pa^3}{3b^2}$, for the measure of the attraction of the whole sphere. *Q. E. I.*

Corollary I.

381. Hence the attraction $\left(\frac{4pa^3}{3b^2}\right)$ at the surface of the sphere, where b is $= a$, will be $\frac{4pa}{3}$; and therefore is directly as the radius of the sphere.

Corollary II.

382. Since $\frac{4pa^3}{3}$ is known to express the content of a sphere whose radius is a*, it is evident that *the attraction* $\left(\frac{4pa^3}{3b^2}\right)$ *of any sphere is, universally, as its quantity of matter directly, and the square of the distance from its center inversely; and is,* moreover, *the very same as it would be, was all the matter in the sphere to be united in a point at the center.*

* Art. 148.

Corollary III.

383. If instead of a corpuscle, or a single particle of matter, at H, we suppose another sphere, having its center at H: then, since the two spheres, at O and H, act upon each other with the very same forces as if each mass was contracted into its center, it follows that the *absolute force with which two spherical bodies tend towards each other, is as the product of their masses directly, and the square of the distance of their*

centers inversely: and therefore, if the masses are given, will be barely as the square of the distance.

PROBLEM VI.

384. *To determine the same as in the last Problem, the Force of each Particle being as any Power* (n) *of the Distance.*

Let HB $= x$, and let every thing else remain as above; then we shall have $y = \frac{c^2+2ac+x^2}{2b} = d + \frac{x^2}{2b}$ (by putting $d = \frac{c^2+2ac}{2b}$) and consequently $\dot{y} = \frac{x\dot{x}}{b}$.

Now the attraction of all the particles in the circular surface BS, is as $\frac{2p}{n+1} \times \overline{Hn \times HB^{n+1} - Hn^{n+2}}$ (by Art. 376) $= \frac{2p}{n+1} \times \overline{yx^{n+1} - y^{n+2}}$: which, multiplied by $\dot{y}$, gives $\frac{2p}{n+1} \times \overline{x^{n+1}y\dot{y} - y^{n+2}\dot{y}}$ for the fluxion of the required force: which, because $y\dot{y}$ is $= \overline{d + \frac{x^2}{2b}} \times \frac{x\dot{x}}{b} = \frac{dx\dot{x}}{b} + \frac{x^3\dot{x}}{2b^2}$, will likewise be expressed by $\frac{2p}{n+1} \times \overline{\frac{dx^{n+2}\dot{x}}{b} + \frac{x^{n+4}\dot{x}}{2b^2} - y^{n+2}\dot{y}}$: whereof the fluent is $\frac{2p}{n+1} \times \overline{\frac{dx^{n+3}}{n+3 \times b} + \frac{x^{n+5}}{n+5 \times 2b^2} - \frac{y^{n+3}}{n+3}}$: which, when B coincides with A, or $x = y = c$, will be $= \frac{2p}{n+1} \times \overline{\frac{dc^{n+3}}{n+3 \times b} + \frac{c^{n+5}}{n+5 \times 2b^2} - \frac{c^{n+3}}{n+3}}$ but, when

B coincides with G, or $x = y = 2a + c$ ($= f$) it will become $= \frac{2p}{n+1} \times \overline{\frac{df^{n+3}}{\overline{n+3} \times b} + \frac{f^{n+5}}{\overline{n+5} \times 2b^2} - \frac{f^{n+3}}{n+3}}$; therefore the difference of these two, which is $=$

$$\frac{2pf^{n+3}}{n+1} \times \frac{\overline{n+5} \times \overline{2bd - 2b^2} + \overline{n+3} \times f^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} - \frac{2pc^{n+3}}{n+1} \times \frac{\overline{n+5} \times \overline{2bd - 2b^2} - \overline{n+3} \times c^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} =$$

$$\frac{\overline{\overline{1+n} \times ab - c^2} \times 2pf^{n+3} + \overline{\overline{5+n} \times ab + c^2} \times 2pc^{n+3}}{\overline{n+1} \times \overline{n+3} \times \overline{n+5} \times b^2}$$

(because $f = a + b$, and $2db = c^2 + 2ac$) will be the attraction of the whole sphere. *Q. E. I.*

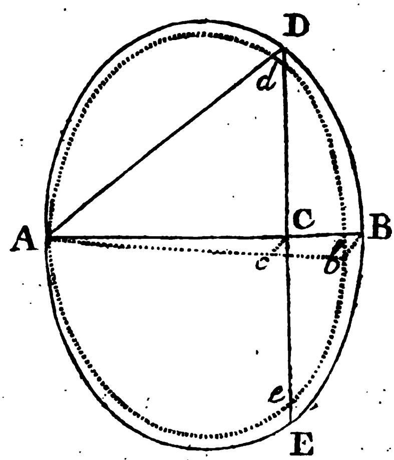
COROLLARY.

385. Hence, the attraction at the surface of the sphere (where $c = 0$) will be $\frac{2p}{n+1} \times \frac{\overline{1+n} \times \overline{2a}|^{n+3} + \overline{n+5} \times 0^{n+3}}{\overline{n+1}.\ \overline{n+3}.\ \overline{n+5}}$: which, if $n + 3$ be positive, will be $= \frac{2p \times \overline{2a}|^{n+3}}{\overline{n+3} \times \overline{n+5}}$; but, otherwise, infinite.

PROBLEM VII.

386. *Supposing* ADB*b*A *to be a* Cuneus *of uniformly dense Matter, comprized by two equal and similar elliptic Planes* ADBEA *and* A*dbe*A, *inclined to each other at the common Vertex* A, *of either their first or second Axes, in an indefinitely small angle* B A *b*; *to determine the Attraction thereof at the Point* A, *supposing the Force of each Particle of Matter to be as the Square of the Distance inversely.*

Let DE be any ordinate to the axis AB, and let AD be drawn; also put $AB=a$, $BC=x$, $CD=y$, and the sine of the angle BAb, formed by the two planes



(to the radius 1) $= d$; and let the equation of either curve be $y^2 = fx - x^2 - gx^2$: which will answer to the conjugate or transverse axis thereof, according as the value of g is positive or negative.

Now it will be, 1 (radius) : d :: $a-x$ (AC) : Cc $= d\times\overline{a-x}$, the thickness of the *cuneus* at the ordinate (or section) DE: moreover, because $AD^2 = AC^2 + CD^2$, we have $AD = \sqrt{\overline{a-x}|^2 + fx - x^2 - gx^2}$: whence, $\frac{DE \times Cc}{AC \times AD}$, expressing (by Art. 374) the attraction of the particles in the indefinitely narrow rectangle $DE \times Cc$, will be defined by $\frac{2d\sqrt{fx-x^2-gx^2}}{\sqrt{\overline{a-x}|^2 + fx - x^2 - gx^2}}$: which therefore, multiplied by $\dot{x}$, will give the fluxion of the force to be found. But when $fx - x^2 - gx^2$

becomes $= 0$, x will be $= \frac{f}{1+g} (= AB) = a$; therefore, by substituting $\overline{1+g} \times a$ for f, our fluxion will be transformed to $\frac{2d\dot{x}\sqrt{\overline{1+g} \times ax - \overline{1+g} \times x^2}}{\sqrt{\overline{a-x}|^2 + \overline{1+g} \times ax - \overline{1+g} \times x^2}} =$

$\frac{2d\dot{x}\sqrt{\overline{1+g} \times \overline{ax-x^2}}}{\sqrt{\overline{a-x}|^2 + \overline{1+g} \times \overline{ax-x^2}}} = \frac{2d\dot{x}\sqrt{\overline{1+g} \times x}}{\sqrt{a - x + \overline{1+g} \times x}} =$

$\frac{2d \times \overline{1+g}|^{\frac{1}{2}} \times x^{\frac{1}{2}}\dot{x}}{\overline{a+gx}|^{\frac{1}{2}}} = \frac{\overline{1+g}|^{\frac{1}{2}} \times 2dx^{\frac{1}{2}}\dot{x}}{a^{\frac{1}{2}}} \times$

$\overline{1 - \frac{gx}{2a} + \frac{3g^2x^2}{2.4a^2} - \frac{3.5g^3x^3}{2.4.6a^3} \text{ \&c.}}$ Whereof the fluent, when $x = a$, will be $\overline{1+g}|^{\frac{1}{2}} \times 2ad \times$

$\overline{\frac{2}{3} - \frac{2}{5} \times \frac{g}{2} + \frac{2}{7} \times \frac{3g^2}{2.4} - \frac{2}{9} \times \frac{3.5g^3}{2.4.6} \text{ \&c.}}$

Which, because $\overline{1+g}|^{\frac{1}{2}} \times a$ is $= f \times \overline{1+g}|^{-\frac{1}{2}} = f \times$

$\overline{1 - \frac{g}{2} + \frac{3g}{2.4} - \frac{3.5g^2}{2.4.6} \text{ \&c.}}$ will (by multiplying the two series together, &c.) be reduced to $2df \times$

$\overline{\frac{2}{3} - \frac{2.4g}{3.5} + \frac{2.4.6g^2}{3.5.7} - \frac{2.4.6.8g^3}{3.5.7.9} \text{ \&c.}}$

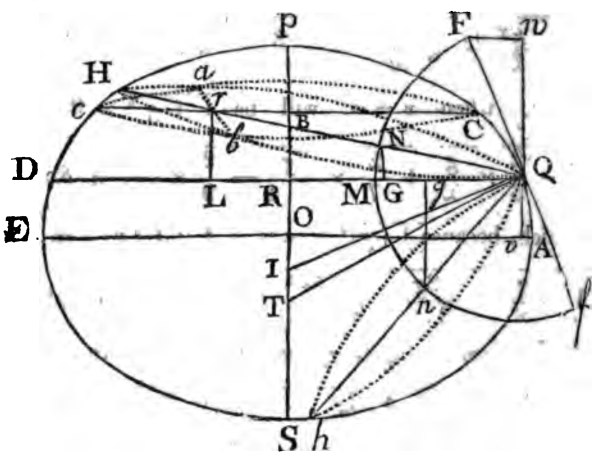
Q. E. I.

It may be observed, that the fluent given above may be brought out without an infinite series (by Art. 126 and 278). But the solution here exhibited is best adapted to what follows hereafter; to which the Proposition itself is premised as a *Lemma*.

PROBLEM VIII.

387. *To determine the Attraction at any Point Q in the Surface of a given Spheroid* O A P E S.

Let Q R L be perpendicular to the axis P S of the spheroid, and Q T perpendicular to the tangent F*f* of the generating ellipsis at Q, meeting P S in T: moreover, let Q *a* H *b* be a section of the spheroid by a plane perpendicular to that of the ellipsis APES, and through any point *r*, in the axis thereof, draw CB*c* and *r*L parallel to A E and PS: and make the abscissa Q*r* = *x*, its corresponding semi-ordinate *ra* (or *rb*) = *y*, Q R = *a*,



and R T = *b*; also let the sine (N G) of the angle H Q D (to the radius N Q = 1) = *p*, its co-sine Q G = *q*, and the ratio of O A² to O P², as any given quantity *h* to unity. Now, by reason of the similar triangles Q*rL* and Q N G, we have *r*L (B R) = *px*, and Q L = *qx*, and therefore B*r* (R L) = *qx* − *a*: also, from the nature of the ellipsis, A O² : P O² ($h : 1$) :: R T (b) : O R $= \frac{b}{h}$: likewise A O² : PO²

$(h:1)::QR^2:OP^2-OR^2$; and $PO^2:AO^2\,(1:h)::OP^2-OB^2:BC^2=h\times\overline{OP^2-OB^2}=h\times\overline{OP^2-\overline{OR+RB}|^2}=h\times\overline{OP^2-OR^2-2OR\times RB-RB^2}=QR^2+h\times\overline{-2OR\times RB-RB^2}$; because (by the former proportion) $QR^2=h\times\overline{OP^2-OR^2}$: whence, by the property of the circle, $Cacb$, we get $y^2\,(=BC^2-Br^2)=QR^2-Br^2-h\times\overline{2OR\times RB+RB^2}=a^2-\overline{qx-a}|^2-h\times\overline{\frac{2b}{h}\times px+p^2x^2}=\overline{aq-bp}\times 2x-\overline{q^2+hp^2}\times x^2$: which equation, by making $1+B=h$, becomes $y^2=\overline{aq-bp}\times 2x-\overline{q^2+p^2+Bp^2}\times x^2=\overline{aq-bp}\times 2x-x^2-Bp^2x^2$ (because $q^2+p^2=1=QN^2$: which being only of two dimensions, the curve $QaHb$, whereto it belongs, is an ellipsis.

The equation of the curve $QaHb$ being now obtained, let its axis QH be supposed to revolve about Q, as a center (the plane of the curve being always perpendicular to that of the ellipsis APES) and let the fluxion of the arch MN (expressing the angle described from the time the said axis begins its motion at the position ALD) be denoted by $\dot{A}$: then, it is evident from the preceding problem, that $\overline{2aq-2bp}\times 2\dot{A}\times\overline{\frac{2}{3}-\frac{2.4\,Bp^2}{3.5}+\frac{2.4.6\,B^2p^4}{3.5.7}\ \&c.}$ will be the fluxion of the attraction of the corresponding part DQH of the solid, upon a corpuscle at Q, considered as acting in the direction HQ (which expression is found, by barely writing $2aq-2bp$, $\dot{A}$, and Bp^2, in the said problem, for f, d, and g, respectively).

Hence, by the resolution of forces, the fluxion of the attraction, in the directions Q R and Q *w* (perpendicular to Q R) will be truly exhibited by $\overline{2aq - 2bp}$ $\times 2\dot{A}q \times \overline{\frac{2}{3} - \frac{2.4Bp^2}{3.5} + \frac{2.4.6B^2p^4}{3.5.7}}$ &c. and $\overline{2aq - 2bp} \times 2\dot{A}p \times \overline{\frac{2}{3} - \frac{2.4Bp^2}{3.5} + \frac{2.4.6B^2p^4}{3.5.7}}$ &c.

Let now another plane Q *h* be supposed to revolve about the point Q, the contrary way to the former, from Q D towards Q *f*; and let (*ng*) the sine of the angle R Q *h* be denoted by *P*, and its co-sine (Q *g*) by *Q*; then the fluxion of the attraction of the part D Q *h*, in the foresaid directions Q R and Q *w* (by writing $-P$ instead of *p*, and *Q* instead of *q*) will appear to be

$$\overline{2aQ + 2bP} \times 2\dot{A}Q \times \overline{\frac{2}{3} - \frac{2.4BP^2}{3.5} + \frac{2.4.6B^2P^4}{3.5.7}} \text{ \&c.}$$

and $\overline{2aQ + 2bP} \times -2\dot{A}P \times \left(\frac{2}{3} - \frac{2.4BP^2}{3.5} + \frac{2.4.6B^2P^4}{3.5.7}\right.$ &c.) Which being added to those of the former part, in the same directions, and $\frac{\dot{p}}{q}$ and $\frac{\dot{P}}{Q}$ respectively substituted instead of $\dot{A}$,* we have

$$4a \text{ into } \frac{2}{3} \times \overline{q\dot{p} + Q\dot{P}} - \frac{2.4B}{3.5} \times \overline{qp^2\dot{p} + QP^2\dot{P}} \text{ \&c.}$$

$$+ 4b \text{ into } \frac{2}{3} \times \overline{P\dot{P} - p\dot{p}} - \frac{2.4B}{3.5} \times \overline{P^3\dot{P} - p^3\dot{p}} \text{ \&c.}$$

And

$$4a \text{ into } \frac{2}{3} \times \overline{p\dot{p} - P\dot{P}} - \frac{2.4B}{3.5} \times \overline{p^3\dot{p} - P^3\dot{P}} \text{ \&c.}$$

$$-4b \text{ into } \frac{2}{3} \times \overline{\frac{p^2\dot{p}}{q} + \frac{P^2\dot{P}}{Q}} - \frac{2.4B}{3.5} \times \overline{\frac{p^4\dot{p}}{q} + \frac{P^4\dot{P}}{Q}} \text{ \&c.}$$

* Art. 142.

for the fluxion of the attraction of both parts together in the foresaid directions: whereof the fluents, when N coincides with F, and n with f, will be the attraction of the whole spheroid in those directions. But now in order to determine these fluents with as little trouble as possible, let m be assumed to denote any whole positive number; then the fluent of $\frac{p^{2m}\dot{p}}{q}$ or $\frac{p^{2m}\dot{p}}{\sqrt{1-p^2}}$, will be universally $= \frac{-q}{2m} \times (p^{2m-1} + \frac{2m-1}{2m-2} \times p^{2m-3} + \frac{\overline{2m-1}.\overline{2m-3}}{\overline{2m-2}.\overline{2m-4}} \times p^{2m-5} (m)) + \frac{1.3.5\ldots\overline{2m-1}}{2.4.6\ldots 2m} \times$ the arch (MN) whose sine is p:* and that of $\frac{P^{2m}\dot{P}}{Q}$, or $\frac{P^{2m}\dot{P}}{\sqrt{1-P^2}}$ (in the same manner) $= \frac{-Q}{2m} \times \overline{P^{2m-1} + \frac{2m-1}{2m-2} \times P^{2m-3}}$ &c. $+ \frac{1.2.3\ldots.\overline{2m-1}}{2.4.6\ldots 2m} \times$ arch (Mn) whose sine is P. But when N coincides with F, and n with f, the sines p and P, of the arches MF and Mf, becoming equal, and (the co-sine) $Q = -$ (co-sine) q, it is evident that the sum of the fluents of $\frac{p^{2m}\dot{p}}{q}$ and $\frac{P^{2m}\dot{P}}{Q}$, will, in that case, be truly exhibited by $\frac{1.3.5\ldots.\overline{2m-1}}{2.4.6\ldots 2m} \times \text{MF} + \frac{1.3.5\ldots.\overline{2m-1}}{2.4.6\ldots 2m} \times$ Mf, or its equal $\frac{1.3.5\ldots.\overline{2m-1}}{2.4.6\ldots 2m} \times \text{FM}f$; be-

* Art. 296.

cause, then all the rest of the terms (by reason of the equal quantities P, p and Q, $-q$) destroy one another. After the same manner the sum of the fluents of $q p^{2m} \dot{p}$ and $Q P^{2m} \dot{P}$, in the foresaid circumstance, will appear to be $= \frac{1.3.5.7 \ldots \overline{2m-1}}{2.4.6.8 \ldots \overline{2m+2}} \times \text{F M} f$.*

Now, to apply this to the matter in hand, let the exponent of B, in any term of either of the above found fluxions be, universally, expressed by n; then the numeral co-efficient (annexed to B) will be defined by $\frac{2.4.6 \ldots \overline{2n+2}}{1.3.5 \ldots \overline{2n+3}}$, and the variable quantities multiplied thereby, in the first line of the former fluxion, will be $q p^{2n} \dot{p} + Q P^{2n} \dot{P}$: therefore $\frac{2.4.6 \ldots \overline{2n+2}}{1.3.5 \ldots \overline{2n+3}} \times B^n \times \overline{q p^{2n} \dot{p} + Q P^{2n} \dot{P}}$ is a general term (from whence, if n be expounded by 1, 2, 3, &c. successively, that whole line will be produced). But the fluent of $q p^{2n} \dot{p} + Q P^{2n} \dot{P}$, in the circumstance above specified (putting $m = n$, and $\text{F M} f = k$), appears to be $= \frac{1.3.5.7 \ldots \overline{2n-1}}{2.4.6.8 \ldots \overline{2n+2}} \times k$: which therefore multiplied by $\frac{2.4.6 \ldots \overline{2n+2}}{3.5 \ldots \overline{2n+3}} \times B^n$, gives $\frac{1.3.5.7 \ldots \overline{2n-1}}{2.4.6.8 \ldots \overline{2n+2}} \times \frac{2.4.6 \ldots \overline{2n+2}}{3.5 \ldots \overline{2n+3}} \times B^n k = \frac{B^n k}{\overline{2n+1} \times \overline{2n+3}}$, for the true fluent of the said general term: which, if n be expounded by

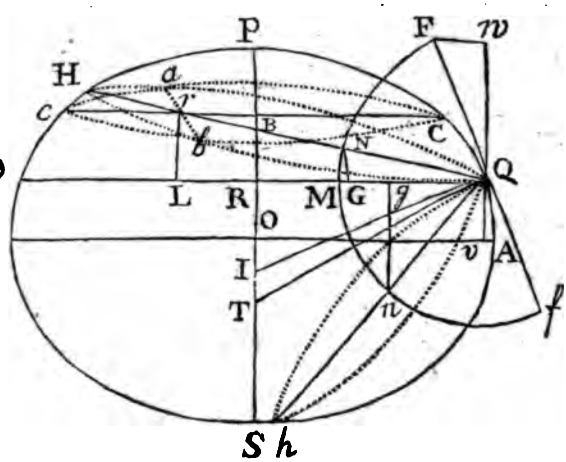
* Art. 297.

0, 1, 2, 3 &c. successively, will become equal to $\frac{k}{1.3}$, $\frac{Bk}{3.5}$, $\frac{B^2k}{5.7}$, $\frac{B^3k}{7.9}$ &c. respectively; and therefore the fluent of the whole line (drawn into the general multiplicator $4a$) is $= 4ak \times \left(\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} - \frac{B^3}{7.9}\right.$ &c.$\left.\right)$ But now, for the fluent of the second line: this, it is plain, will be $= 4b$ into $\frac{2}{3} \times \overline{\frac{P^2}{2} - \frac{p^2}{2} - \frac{2.4B}{3.5} \times \overline{\frac{P^4}{4} - \frac{p^4}{4}}}$ &c. Which, in the foresaid circumstance, when $P = p$, entirely vanishes. Therefore it appears, that the attraction of the whole Spheroid, in the direction Q R, is truly expressed by $4ak \times \overline{\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} - \frac{B^3}{7.9}}$, or its equal $4k \times \overline{\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}}$ &c. $\times QR$.

After the same manner the fluent of the first line, in the latter of our two fluxions, will be found to vanish: and $\frac{2.4.6\ldots\overline{2n+2}}{1.3.5\ldots\overline{2n+3}} \times B^n \times \left(\frac{p^{2n+2}\dot{p}}{q} + \frac{P^{2n+2}\dot{P}}{Q}\right)$ will be a general term to the second line. Whereof the fluent (by expounding $2m$ by $2n+2$) appears, from above, to be $= \frac{2.4.6\ldots\overline{2n+2}}{3.5.7\ldots\overline{2n+3}} \times B^n k \times \frac{1.3.5\ldots\overline{2n+1}}{2.4.6\ldots\overline{2n+2}} = \frac{B^n k}{2n+3}$: which, when

n is interpreted by 0, 1, 2, 3, &c. successively, comes out equal to $\frac{k}{3}$, $\frac{Bk}{5}$, $\frac{B^2k}{7}$ &c. respectively: therefore the attraction of the spheroid, in the direction Qw, is exhibited by $-4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}}$ &c. and consequently, that in the opposite direction Qv, by $4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}}$ &c. $= 4k \times \left(\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\right.$ &c.$\left.\right) \times RT = 4k \times \overline{1+B} \times \left(\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\right.$ &c.$\left.\right) \times OR$ (because $\overline{1+B} \times OR = RT$).

From which and the force in the direction QR (found above) not only the direction of the absolute



attraction, but that attraction itself will be known: for, let RI be taken to QR, as the force in the direction Qv to that in the direction QR; and then, by the composition of forces, Q I will be the direction of the attraction, or the line in which a corpuscle at Q tends to descend: and the attraction itself, in that direction (being to that in Q R, as Q I to Q R) will be defined by $4k \times \overline{\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}\ \&c.} \times QI$; which, since $4k$ is constant, will also be as $\left(\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}\ \&c.\right) \times QI$. *Q. E. I.*

Corollary.

388. Since, by construction, $RI : QR :: \overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}\ \&c.} \times OR : \left(\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}\ \&c.\right) \times QR$, it follows that $\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}\ \&c. : \overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}\ \&c.} :: RO : RI$; whence (by Division) $\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}\ \&c. : \frac{3B}{3.5} - \frac{3B^2}{5.7} + \frac{3B^3}{7.9}\ \&c. :: OR \left(: \frac{OT}{B}\right) : OI$; and consequently, $\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}\ \&c. : 3 \times \left(\frac{1}{3.5} - \frac{B}{5.7} + \frac{B^2}{7.9}\ \&c.\right) :: OT : OI$.

Hence it appears that the *direction* Q I, *of the absolute Attraction, divides the part of the axis* O T, *intercepted by the center and normal, in a given ratio:* and that *the attraction itself* (being de-found by $\overline{\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}}$ &c. × QI) *is every where as the said line of direction* Q I.

SCHOLIUM.

389. Although the foregoing conclusions are exhibited by infinite series, yet the sums of those series are explicable by means of the arch of a circle.

Thus, let the series $\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}$ &c. (which is one of the two original ones above found) be put $= S$, and let $B = t^2$; then by substitution, and multiplying the whole equation by t^3, we shall have $\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7}$ &c. $= t^3 S$, and consequently $t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7}$ &c. $= t - t^3 S$: where, the former part of the equation is known to express the arch of a circle, whose

* Art. 142. tangent is t $(B^{\frac{1}{2}})$ and radius unity:* wherefore, putting that arch $= A$, we have $A = t - t^3 S$, and consequently $S = \frac{t - A}{t^3} = \frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}$ &c.

Moreover, since it appears that

$$\left.\begin{array}{l} \frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7} \text{ &c.} \\ -\frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9} \text{ &c.} \end{array}\right\} \text{ is } = \frac{2B}{3.5} - \frac{2B^2}{5.7} + \frac{2B^3}{7.9} \text{ &c.}$$

(where the sum of $\frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7}$ &c. is already found $= \frac{t - A}{t^3} \times B = \frac{t - A}{t}$, and where that of $-\frac{B}{5}+\frac{B^2}{7}$ &c. by the same method will come out $=\frac{t-A-\frac{1}{3}t^3}{t^3}$) it is evident that $\frac{2B}{3.5}-\frac{2B^2}{5.7}$ $+\frac{2B^3}{7.9}$ &c. $=\frac{t-A}{t}+\frac{t-A-\frac{1}{3}t^3}{t^3}=$ $\frac{\frac{2}{3}t^3+t-A\times\overline{1+t^2}}{t^3}$; and consequently $\frac{1}{1.3}-$ $\frac{B}{3.5}+\frac{B^2}{5.7}$ &c. $\left(=\frac{1}{3}-\frac{\frac{2}{3}t^3+t-A\times\overline{1+t^2}}{2t^3}\right)$ $=\frac{A\times\overline{1+t^2}-t}{2t^3}$: which is the value of the other original series found above: from whence that of $\frac{3}{3.5}-\frac{3B}{5.7}+\frac{3B^2}{7.9}$ will also be had $=$ $\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t^5}$.

Hence, if

$$\frac{t-A}{t^3}\ \left(=\frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}-\frac{B^3}{9}\right) \text{ be put } = f;$$

$$\frac{A\times\overline{1+t^2}-t}{2t^3}\ \left(=\frac{1}{1.3}-\frac{B}{3.5}+\frac{B^2}{5.7}\ \text{\&c.}\right) = g;$$

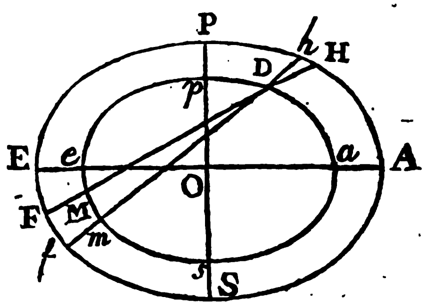
And

$$\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t^5}\ \left(=\frac{3}{3.5}-\frac{3B}{5.7}+\frac{3B^2}{7.9}\ \text{\&c.}\right)=h$$

it is evident that O T will be to O I, in the constant ratio of g to h; aud that the forces in the directions Q I, Q R, and Qv, will be as $g\times$QI, $g\times$QR, and $f\times$ $\overline{1+B}\times$OR respectively: where $1+B$ is $=\frac{AO^2}{PO^2}$.

PROBLEM IX.

390. *To determine the Attraction at any Point* D *within a given Spheroid* O A P E S.



Let O*apes* be another spheroid, concentric with, and similar to, the given one; whose surface D *e* M &c. passes through the given point D; also let F D *f* and HD*h* be taken as two opposite, indefinitely slender, Cones (or Pyramids) conceived to be formed by drawing innumerable lines H D F, *h* D *f* &c. through the common vertex D which Cones (or Pyramids) having the same angle, may be considered as similar; and so their forces, at D, will be as the altitudes D F and D H:* and, therefore, the excess of the former, above the latter, or the force whereby a corpuscle at D, tends towards F, through the contrary action of the two opposite cones, will be as DF—DH, or as DM; because *(by the property of the Ellipsis)* MF is, in all positions, equal to DH.

* Art. 378.

Hence it appears that the parts of matter F M *m f* and HD*h*, without the Spheroid *a p e s* (acting equally, in contrary directions) can have no effect at D: and this, being every where the case, the whole, efficacious, force at D must therefore be that of the Spheroid O *a p e s*.

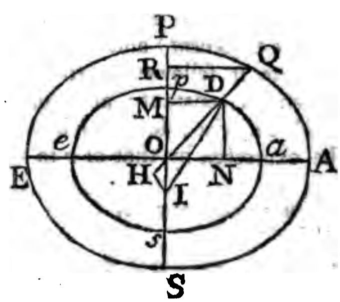
Hence, if the ratio of Oa^2 to Op^2 (or of OA^2 to OP^2) be denoted by that of $1 + B$ to 1, as in the last Problem), it follows, from thence, that the attraction at D, in the directions DM and DN (perpendicular to PS and AE; *see the next fig.*) will be expounded by $\overline{\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}}$ &c. $\times$ D M, and $\overline{1 + B} \times \overline{\frac{1}{3} - \frac{B}{.5} + \frac{B^2}{.7} - \frac{B^3}{.9}}$ &c. $\times$ DN respectively, or by their equals $g \times$ D M and $f \times \overline{1+B} \times$ DN: where the values of f and g are the same as given in the preceding article.

Corollary.

391. Hence the force wherewith a corpuscle, any where within a given spheroid, is attracted, either, towards the axis, or the plane of its equator, is directly as the distance therefrom.

PROBLEM. X.

392. *Supposing every Particle of Matter in a Spheroid to have a Tendency to recede, both, from the Axis* P S, *and from the Plane of the greatest Circle, by Means of Forces that are as the Distances from the said Axis, and Plane respectively; to find the Direction* D I *wherein a Corpuscle, at any Point* D, *tends to move through the Action of the said Forces and the Attraction conjunctly; and likewise the whole compound Force in that Direction.*



Let D M and D N be perpendicular to P S and A E, and let the given forces, in the direction of those lines (independent of the attraction) be expressed by $m \times$ DM and $n \times$ D N respectively.

Therefore, since (by the last Problem) the force of attraction in the said directions is defined by $g \times DM$ and $f \times \overline{1 + B} \times DN$, the whole resulting forces will be truly denoted by $\overline{g-m} \times DM$, and $f \times \overline{1+B-n} \times DN$: whence (by the composition of forces) it will be, $\overline{g-m} : f \times \overline{1+B-n} :: DN\ (OM) : MI$; whence the point I is given.

Also $DM : DI :: \overline{g-m} \times DM$ (the force in the direction DM) $: \overline{g-m} \times DI$, the force in DI. *Q.E.I.*

PROBLEM XI.

393. *Every thing being supposed as in the preceding Problems, it is required to determine the Force of all the Particles in the Line (or Column)* Q D O *tending to the Center* O *of the Spheroid.*

Let I H be perpendicular to Q O produced (*see the last Fig.*) then the absolute force, in the direction D I, being $\overline{g-m} \times DI$, that in the direction D H, whereby a corpuscle at D is urged towards the center, will be $\overline{g-m} \times DH$. Let now O D (considered as variable) be denoted by x; then because the ratio of OM to MI is given (being every where as $g-m$ to $f \times \overline{1+B-n}$, *by the precedent*) and the triangles ODM and IOH are similar, it follows that the ratio of O D to O H will be given, or constant; and consequently that of D H to OH, likewise: let therefore this ratio of D H to O H be expressed by that of r to s, and we shall have $DH = \frac{rx}{s}$, and consequently ($\overline{g-m} \times DH$) the force at D, equal to $\overline{g-m} \times \frac{rx}{s}$: which therefore being multiplied by $\dot{x}$, and the fluent taken, there comes out $\overline{g-m} \times \frac{rx^2}{2s} = \frac{g-m}{2} \times \text{D O} \times \text{D H}$, for the whole force of the line or column O D at the center.

Q. E. I.

Corollary.

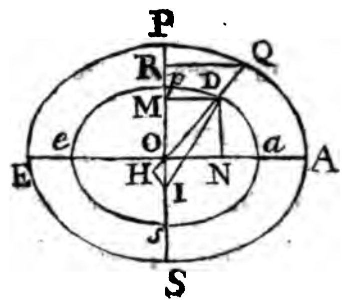
394. If the given forces *m* and *n* be such that the ratio of O M to M I (which is found to be universally as $g-m$ to $f \times \overline{1+\text{B}} - n$) may become as $1 : 1+\text{B}$ (or as $p\text{O}^2 : a\text{O}^2$) it is evident (from the property of the ellipsis) that the line of direction D I will be always perpendicular to the surface of the spheroid *Oapes*. In which case O D × D H is also (by the nature of the ellipsis) $= \text{O}a^2$: and therefore the force $\left(\frac{g-m}{2} \times \text{O D} \times \text{D H}\right)$ of O D is $= \frac{g-m}{2} \times \text{O}a^2$: which, when D coincides with Q, will become $\frac{g-m}{2} \times \text{AO}^2$; and is, therefore, a constant quantity.

Moreover, since in this case, $g-m : f \times \overline{1+\text{B}} - n :: 1 : 1+B$ (by hypothesis) we have $m - \frac{n}{1+B} = g - f$: which equation, if *n* be taken $= 0$, gives $m = g - f = \frac{2B}{3.5} - \frac{4B^2}{3.7} + \frac{6B^3}{7.9}$ &c. $= \frac{\overline{3+t^2} \times A - 3t}{2t^3}$;* but, if *m* be taken $= 0$, it will then give $n = -\overline{1+B} \times \overline{g-f} = -\overline{1+B} \times \overline{\frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9}}$ &c. Where, $t = B^{\frac{1}{2}}$, and $A =$ the arch whose tangent is *t*, and radius unity.

* Art. 389.

PROP. XII.

395. *If an oblate Spheroid* O A P E S, *whereof the Square of the Equatoreal Diameter* AE, *is to that of the Axis* P S, *in any given Ratio of* $1+B$ *to* 1, *revolves about its Axis, in such a Time, that the centrifugal Force, at the Equator* A, *is to the Attraction at the Surface of a Sphere whose Radius is* OA, *in the Ratio of* $\frac{2B}{3.5}-\frac{4B^2}{5.7}+\frac{6B^3}{7.9}$ &c. to $\frac{1}{3}$: *I say, in that Case, every Particle of the Spheroid will be* in Equilibrio; *so that, though the Cohesion of the Parts was to cease, the Figure itself would remain unchanged.*



For, the attraction of the spheroid, at A, being defined by $\overline{\frac{1}{1.3}-\frac{B}{3.5}+\frac{B^2}{5.7}\text{ \&c.}}\times AO$ (Art. 387) it is evident (by conceiving $B=0$) that $\frac{AO}{3}$ will represent the attraction at the surface of the sphere whose radius is A O: whence (by hypothesis) the centrifugal force at A (putting $m=\frac{2B}{3.5}-\frac{4B^2}{5.7}+\frac{6B^3}{7.9}$ &c.) will be truly defined by $m\times AO$; and consequently that, at any other point D, by $m \times DM$ (because the centrifugal forces of bodies describing unequal circles, in equal times, are known to be directly as the radii).* Hence, and from the Corollary to the last* Problem, it appears that the direction of gravitation D I is always perpendicular to the surface *apes*; and that the force of all the particles in the line (or canal) O D or O Q, towards the center O, will continue invariable, take the point Q in what part of the arch A P E you will: from which last consideration, it follows that the force, or pressure of every canal Q O, at the center O (considering the body in a fluid state) will be the same: whence (by the principles of hydrostatics) a corpuscle at D has no tendency to move, either way, in the line O Q: and therefore, as it hath no tendency to move in the direction at the surface D *p e* (the gravitation being perpendicular thereto) it is evident, *from Mechanics*, that no motion at all can ensue, in any direction. *Q. E. D.* Art. 218.

Corollary I.

396. Since m is $= \frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9}$ &c. the gravitation $(\overline{g - m} \times DI)$ at any point D in the spheroid will therefore be as $\overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \text{ \&c.}}$ $\times DI = \frac{t-A}{t^3} \times DI$ (see Art. 389).

Corollary II.

397. If the time of revolution be given $=p$, and q be put to denote the time wherein a (solid) sphere, of the same density with the spheroid, must revolve; so that the centrifugal force, at the equator thereof, may be equal to the gravity: then, as this last time is known to continue the same, whatever the magnitude of that sphere is;† and the centrifugal forces, in equal† Art. 213 & 381.

circles, are also known to be inversely as the squares of the periodic times—it follows, that $p^2 : q^2 :: \frac{1}{5}$ A O (the attraction, or centrifugal force, respecting the sphere O A, revolving in the time q) : $\left(\frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9}\right.$ &c.$\left.\right)$ $\times$ A O, the centrifugal force of the spheroid at A, revolving in the time p. From which proportion we get $\frac{q^2}{3p^2} = \frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9}$ &c. $= \frac{\overline{3+t^2} \times A - 3t}{2t^3}$ (Art. 394). Whence, by help of the trigonometrical-canon, the value of t ($= B^{\frac{1}{2}}$) and, consequently, the ratio of the two principal diameters, will be found; so that all the parts of the spheroid may remain *in equilibrio*. But, when $\frac{q^2}{3p^2}$ is small, the solution by an infinite series is preferable: for, then the series $\frac{2B}{3.5} - \frac{4B^2}{5.7}$ &c. $\left(= \frac{q^2}{3p^2}\right)$ converging sufficiently swift, we shall, by the reversion thereof, find $B = \frac{5q^2}{2p^2} + \frac{25 \times 6q^4}{4 \times 7p^4} + \frac{125 \times 37q^6}{8 \times 49p^6}$ &c. In which case the ratio of the equatoreal diameter to the axis, if we take only the first term of the series, will be, as

$$\sqrt{1 + \frac{5q^2}{2p^2}} : 1, \text{ or as } 1 + \frac{5q^2}{4p^2}, \text{ nearly.}$$

Which, if $\frac{p^2}{q^2} = 289$, or the centrifugal force at the equator be to the gravity as 1 to 289 (that being the proportion at the equator of the earth)* will come out as 231 to 230.

* Art. 217.

COROLLARY III.

398. Because, $\frac{\overline{3+t^2} \times A - 3t}{2t^3}$, the latter part of our foregoing equation will be equal to nothing, both when t is nothing and infinite, it is evident that the value thereof cannot, in any intermediate circumstance of t, exceed a certain assignable quantity.

Wherefore, to determine this limit of the value of $\frac{q^2}{3p^2}$ (beyond which the problem becomes impossible) let the fluxion of $\frac{\overline{3+t^2} \times A - 3t}{2t^3}$, or its double $\frac{\overline{3+t^2} \times A}{t^3} - \frac{3}{t^2}$ be taken and put $= 0$, and you will have $-\overline{9+t^2} \times A\dot{t} + \overline{3t+t^3} \times \dot{A} + 6t\dot{t} = 0$: which, because $\dot{A} = \frac{\dot{t}}{1+t^2}$ * will be reduced to $9t$ * Art. 142.
$+ 7t^3 - \overline{1+t^2} \times \overline{9+t^2} \times A = 0$; where t is found $= 2{,}5293$, from whence the corresponding values of $\sqrt{1+t^2}$, and $\frac{q}{p}$ come out $= 2{,}7198$, and $0{,}5805$ &c. respectively. Hence it appears that it is impossible for the parts of the spheroid, in a fluid state, to continue at rest among themselves, when the time of revolution is so great that $\frac{q}{p}$ exceeds $0{,}5805$ &c. And that, of all the spheroids which can be assumed by a fluid revolving about an axis, that whose equatoreal diameter is to its axis as $2{,}7198$ to unity, will perform its revolutions in the shortest time.

Thus, for example, if a (solid) sphere of the same common density with the earth was to revolve about its axis in the time of $84\frac{3}{4}$ minutes, the centrifugal

force at the equator thereof would, it is known, be
t. 217. equal to the gravity :* therefore, by taking $\frac{84\frac{1}{4}}{p}$ $(=\frac{q}{p})$ $= 0{,}5805$ &c. The time p will come out $=$ $\overset{M}{146}$ or $\overset{H}{2}\ \overset{M}{26}$. Which time is the least, possible, wherein a fluid, of the same common density with the earth, can revolve, so as to preserve its spheroidal figure. And this holds universally, let the magnitude of the body, or fluid, be what it will.

Corollary IV.

399. Hence also may be determined the spheroid, which a spherical body (of ice or any other matter) revolving in a given time s, will converge to, when reduced to a fluid state.

For, since the momenta of rotation, in equal spheres and spheroids, are to one another, in a ratio compounded of the direct ratio of their equatoreal diameters, and the inverse ratio of the times of their rotation, it follows, if d be put $=$ the diameter of the given sphere, and $E =$ the equatoreal diameter of the required spheroid, that $\frac{d}{s} = \frac{E}{p}$ (because the quantity of motion about the axis is not affected by the action of the particles one upon another, while the figure of the fluid is changing). Moreover, since the masses of the sphere and spheroid are also equal to each other (by hypothesis) we have d^3 $(= AE^2 \times PS) =$ $\frac{E^3}{\overline{1+t^2}|^{\frac{1}{2}}}$: from which two equations, exterminating d, there arises $p = \overline{1+t^2}|^{\frac{1}{6}} \times s$, for the time of revolution of the required spheroid: whence, by substituting this value of p in the general equation $\frac{q^2}{3p^2}$

$= \frac{\overline{3+t^2} \times A - 3t}{2t^3}$, we get $\frac{q^2}{3s^2} = \overline{1+t^2}|^{\frac{1}{3}} \times$ $\frac{\overline{3+t^2} \times A - 3t}{2t^2}$; from the solution of which the value of t, and the spheroid itself will be given.

But, since the value of the latter part of the equation can never exceed a certain assignable quantity, the matter proposed can therefore be only possible under certain limitations: in order to determine these limitations, let the fluxion of $\overline{1+t^2}|^{\frac{1}{3}} \times \frac{\overline{3+t^2} \times A - 3t}{2t^3}$ be taken and put $= 0$, and it will be found that $\overline{t^4 + 24t^2 + 27} \times A - 15t^3 - 27t = 0$: whence t comes out $= 7{,}5$, and the corresponding value of $\frac{q}{s} = 0{,}927$, nearly.

Hence the parts of the fluid cannot possibly come to an equilibrium among themselves, when the time s is less than $\frac{q}{0{,}927}$, but will continue to recede from the axis, *in infinitum*.

If q be taken $= 84\frac{1}{2}^m$ (as in the example to the preceding corollary, s will be equal $91^m = 1^h : 31^m$. From which it appears, that if the earth (or a spherical body of the same density) was to revolve about its axis in less than $1^h : 31^m$, and in the mean time be reduced to a state of fluidity, the parts thereof towards the equator would ascend, and continue to recede from the axis, *in infinitum*.

Corollary V.

400. Seeing the values of t and A are given when the spheroid is given, it follows that the gravitation

$\left(\frac{t-A}{t^3} \times QI\right)$ at any point in the surface of a spheroid, whereof the parts are kept *in equilibrio*, by their rotation about the axis, will be accurately as a perpendicular to the surface at that point, continued to the axis of the figure. Therefore the gravitation at the equator is to that at either of the poles, as the equatoreal diameter to the axis inversely.

Corollary VI.

401. But if the spheroid differs but little from a sphere, the excess of Q I above A O will (by the property of the ellipsis) be nearly as OR^2. Whence it appears that the increase of gravitation, in going from the equator to the pole, is as the square of the sine of latitude, nearly.

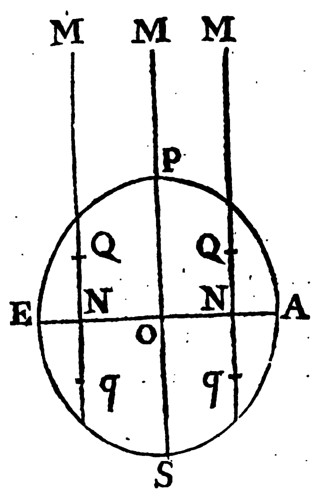
Corollary VII.

402. Moreover, since the ratio of the equatoreal diameter to the axis is found, in this case, to be that of $1 + \frac{5q^2}{4p^2}$ to 1,* the excess of that diameter above the axis will be to the axis as $\frac{5q^2}{4p^2}$ to unity; that is, as $\frac{5}{4}$ of the centrifugal force at the equator to the mean force of gravity. Whence, as the centrifugal forces, in unequal circles, are universally as the radii directly, and the squares of the periodic times inversely, it follows that the foresaid excess (in figures nearly spherical) will be as the radii directly, and as the density and the square of the time of rotation inversely: from which proportions, the ratios of the greatest and least diameters of the planets may be inferred from each other; supposing the times of their rotation about their axes to be known.

* Art. 397.

PROBLEM XIII.

403. *To determine the Figure which a Fluid will acquire when, besides the mutual Gravitation of the Parts thereof, it is attracted by another Body, so remote, that all Lines drawn from it to the Surface of the Fluid, may be taken as Parallels.*



Let OAPES be the proposed fluid, and let MPS and MQq be right-lines, drawn from the remote body *M*; whereof the former MPS passes through the center of gravity O: moreover, let the plane AE be perpendicular to the axis MOS; and put NQ$=a$ and OM (the distance of the remote body) $=d$; also put the semi-diameter of the body (at M) $=r$, and let its density be to that of the fluid APES, as any quantity v to unity. Then since, according to the foregoing calculations, the attraction at the surface of a sphere (of a given density) is expressed by $\frac{1}{3}$ of the radius, it follows that the attraction of the body *M*, at its surface, will be explicable by $\frac{vr}{3}$: and therefore, the force varying according to the square of the distance inversely,* it will be d^2 (M N^2) : r^2 :: $\frac{vr}{3} : \frac{vr^3}{3d^2}$, the attraction of *M*, at the distance MN: also $\overline{d-a}|^2$ (M Q^2) : r^2:: $\frac{vr}{3} : \frac{vr^3}{3\times\overline{d-a}|^2}$, its attraction at the

* Art. 382.

distance MQ. Whence the difference of these two, or $\frac{vr^3}{3 \times \overline{d-a}|^2} - \frac{vr^3}{3d^2} \left(= \frac{vr^3}{3d^3} \times \overline{2a + \frac{3a^2}{d} + \frac{4a^3}{d^2}} \right.$ &c.$\left.\right)$ will be as the force whereby a corpuscle at Q endeavours to recede from the plane AE: which because (by hypothesis) d is very great in respect of a, will (by rejecting all the terms after the first) be expressed by $\frac{2vr^3}{3d^3} \times a$, or its equal $\frac{2vr^3}{3d^3} \times$ NQ.

In the very same manner, the force whereby a corpuscle at q, below the plane AE, tends to recede therefrom, will be defined by $\frac{2vr^3}{3d^3} \times$ Nq.

Now, therefore, seeing these forces are, every where, as the distances NQ, Nq, from the plane AE, it appears (by Art. 393 and 394) that the figure OAPES will be a spheroid; whereof the equation, for the relation of its two principal diameters (putting $n = \frac{2vr^3}{3d^3}$) is $n = -\overline{1+B} \times \overline{\frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9}}$ &c. (In which the ratio of PS^2 to AE^2 is denoted by that of 1 to $1+B$). Hence, by reverting the series, we have $B = -\frac{15n}{2} - \frac{225n^2}{28}$ &c. and consequently PS : AE :: 1 : $\sqrt{1 - \frac{15n}{2} - \frac{225n^2}{28}}$ &c. :: 1 : $1 - \frac{15n}{4}$, nearly: which, by restoring the value of n, becomes PS : AE :: 1 : $1 - \frac{5vr^3}{2d^3}$. *Q. E. I.*

COROLLARY.

404. Because $\frac{r}{d}$ expresses the sine of the apparent semi-diameter of the body *M* (to the radius 1) seen at the distance O M, it follows, if the said sine be denoted by *c*, that $PS : AE :: 1 : 1 - \frac{5v}{2} \times c^3$; and consequently, by division, $PS : PS - AE :: 1 : \frac{5v}{2} \times c^3$.

Hence it appears, that the forces of the planets, to produce tides at the earth's surface, are to one another as their densities, and the cubes of their apparent diameters conjunctly. (For the sines of small arcs are nearly as the arcs themselves.)

EXAMPLE.

405. If *c* be taken = the sine of 16′ (expressing the mean apparent semi-diameter of the moon) and $v = \frac{5}{4}$ (the ratio of her density with respect to that of the earth) our last proportion will become $PS : PS - AE :: 1 : 0{,}000000315$: whence if PS be taken = 42000000 feet, (the measure of the earth's diameter) $PS - AE$ will come out = 13,23′.

SECTION X.

Of the Application of FLUXIONS *to the Resolution of such Kinds of Problems* DE MAXIMIS ET MINIMIS, *as depend upon a particular Curve, whose Nature is to be determined.*

I SHALL begin this section with premising the following useful

THEOREM.

406. *If the relation of two flowing quantities y and u be required; so that, when the fluent of $y^m \dot{u}$ becomes equal to a given value, that of $\frac{y^r \times \overline{\dot{u}^2 \pm \dot{y}^2}|^n}{\dot{y}^{2n-1}}$ may be a* maximum *or a* minimum; *I say, their relation must be such that $\frac{y^{r-m} \dot{u} \times \overline{\dot{u}^2 \pm \dot{y}^2}|^{n-1}}{\dot{y}^{2n-1}}$ may be every where the same, or equal to a constant quantity.*

The demonstration hereof depends upon the subsequent

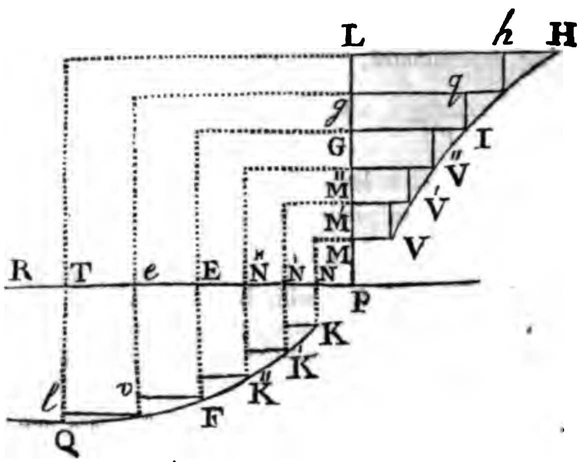
LEMMA.

407. If $a\alpha + b\beta = Q$, wherein α and β are indeterminate, the value of $A \times \overline{\alpha^2 \pm p^2}|^n + B \times \overline{\beta^2 + p^2}|^n$ will be a *maximum* or *minimum*, when $\frac{A\alpha}{a} \times \overline{\alpha^2 \pm p^2}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta^2 \pm p^2}|^{n-1}$ are equal to each

other. For, by taking the fluxions of both expressions we have $a\dot{\alpha} + b\dot{\beta} = 0$, and $2nA\alpha\dot{\alpha} \times \overline{\alpha^2 \pm p^2}|^{n-1}$ $+ 2nB\beta\dot{\beta} \times \overline{\beta^2 \pm p^2}|^{n-1} = 0$: from whence, $\dot{\alpha}$ and $\dot{\beta}$ being exterminated, there results $\frac{A\alpha}{a} \times \overline{\alpha^2 \pm p^2}|^{n-1}$ $= \frac{B\beta}{b} \times \overline{\beta^2 \pm p^2}|^{n-1}$. Q. E. I.

Hence, if $a\alpha + b\beta + c\gamma + d\delta$ &c. $= Q$ (where $\alpha, \beta, \gamma, \delta$, &c. are indeterminate) it follows that $A \times \overline{\alpha^2 \pm p^2}|^n + B \times \overline{\beta^2 \pm p^2}|^n + C \times \overline{\gamma^2 \pm p^2}|^n$ $+ D \times \overline{\delta^2 \pm p^2}|^n$ &c. will be a *maximum* or *minimum*, when all the quantities $\frac{A\alpha}{a} \times \overline{\alpha^2 \pm p^2}|^{n-1}$, $\frac{B\beta}{b} \times \overline{\beta^2 \pm p^2}|^{n-1}$, $\frac{C\gamma}{c} \times \overline{\gamma^2 \pm p^2}|^{n-1}$ &c. are equal to each other. For that expression is a *maximum* (or *minimum*) when it cannot be increased, or decreased, by altering the values of the indeterminate quantities involved therein; but it may be increased, or decreased, by altering only two of them (as α and β) whilst the rest remain unchanged; unless $\frac{A\alpha}{a} \times \overline{\alpha^2 \pm p^2}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta^2 \pm p^2}|^{n-1}$ are equal to each other. (This is proved above). Therefore, when $A \times \overline{\alpha^2 \pm p^2}|^n + B \times$ $\overline{\beta^2 \pm p^2}|^n + C \times \overline{\gamma^2 \pm p^2}|^{n-1}$ + &c. is a *maximum* or *minimum*, the quantities $\frac{A\alpha}{a} \times \overline{\alpha^2 \pm p^2}|^{n-1}$ and $\frac{B\beta}{b}$ $\times \overline{\beta^2 \pm p^2}|^{n-1}$ cannot be unequal: and, by the very same argument, no other two of the quantities above specified can be unequal.

If, in the right-line P R, there be now assumed $NN = \alpha$, $\acute{N}\acute{\acute{N}} = \beta$, &c. and upon these, as bases,



rectangles $\acute{N}K$, $\acute{\acute{N}}\acute{K}$ be supposed, whose altitudes NK, $\acute{N}\acute{K}$ &c. are denoted by a, b, c, d, &c. it is evident that $a\alpha + b\beta + c\gamma + d\delta$ &c. $(= Q)$ will be expressed by the sum of all the said rectangles, or the whole polygon Nl.

Moreover, if, in the right-line P L (perpendicular to P R) there be taken $M\acute{M}$, $\acute{M}\acute{\acute{M}}$ &c. each equal to p, and, upon these equal bases, rectangles $\acute{M}V$, $\acute{\acute{M}}\acute{V}$ &c. be constituted, whose altitudes are denoted by $A \times \frac{\overline{\alpha^2 \pm p^2}|^n}{p^{2n}}$, $B \times \frac{\overline{\beta^2 \pm p^2}|^n}{p^{2n}}$, &c. it is likewise plain that the value of $\frac{A \times \overline{\alpha^2 \pm p^2}|^n}{p^{2n-1}} + \frac{B \times \overline{\beta^2 \pm p^2}|^n}{p^{2n-1}} + \frac{C \times \overline{\gamma^2 \pm p^2}|^n}{p^{2n-1}}$ will be truly represented by the whole polygon $M h$. Which polygon (as p is constant) will be a *maximum* or *minimum*, when $A \times \overline{\alpha^2 \pm p^2}|^n + B \times \overline{\beta^2 \pm p^2}|^n +$ &c. is a *maximum* or *minimum*; that is, when all the quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha^2 \pm p^2}|^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta^2 \pm p^2}|^{n-1}}{p^{2n-1}}$ &c. are equal to each other (as has been proved above).

Let now, A, B, C, D, &c. be expounded by any powers (MP^r, $\acute{M}P^r$, $\acute{\acute{M}}P^r$, &c.) of the respective distances from a given point P; and let, at the same time, the corresponding values of a, b, c, d, &c. be interpreted by any other proposed powers MP^m, $\acute{M}P^m$, $\acute{\acute{M}}P^m$ &c. of the same given distances: then the area of the polygon $N l$ will be expressed by $MP^m \times \alpha + \acute{M}P^m \times \beta + \acute{\acute{M}}P^m \times \gamma$ &c. $(=Q)$; and that of the polygon $M h$, by $MP^r \times \frac{\overline{\alpha^2 \pm p^2}|^n}{p^{2n-1}} + \acute{M}P^r \times \frac{\overline{\beta^2 \pm p^2}|^n}{p^{2n-1}} + \acute{\acute{M}}P^r \times \frac{\overline{\gamma^2 \pm p^2}|^n}{p^{2n-1}} +$ &c. And the foresaid equal quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha^2 \pm p^2}|^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta^2 \pm p^2}|^{n-1}}{p^{2n-1}}$ &c. will become $MP^{r-m} \times \frac{\alpha \times \overline{\alpha^2 \pm p^2}|^{n-1}}{p^{2n-1}}$, $\acute{M}P^{r-m} \times \frac{\beta \times \overline{\beta^2 \pm p^2}|^{n-1}}{p^{2n-1}}$, &c. respectively.

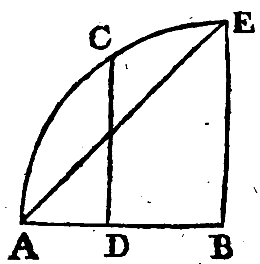
Now let the number of the rectangles be supposed indefinitely great, and their breadths indefinitely small, so that the area of each of the two polygons $N\,l$ and $M\,h$ may be taken for that of its circumscribing curve: moreover, let u and y be put to represent the distances of any two corresponding ordinates E F and G I from the given point P; and let $\dot{y}$ be every where expressed by p ($=\acute{M}M=\acute{M}\acute{\acute{M}}=$&c.). Then, $\dot{u}$ being a general value for any of the quantities $\alpha, \beta, \gamma, \delta$ &c. (or $N\acute{N}$, $\acute{N}\acute{\acute{N}}$ &c.) it follows; first, that the fluxion of the area of the curve N E F K (the ordinate being, every where, $=y^r$) will be truly defined by $y^r\dot{u}$; secondly, that the fluxion of the area M G I V (by substituting y, $\dot{u}$, and $\dot{y}$ instead of their equals) will be $\frac{y^r \times \overline{\dot{u}^2 \pm \dot{y}^2}|^n}{\dot{y}^{2n-1}}$; and, lastly, that the value of each of the equal quantities, $MP^{r-m} \times \frac{\alpha \times \overline{\alpha^2 \pm p^2}|^{n-1}}{p^{2n-1}}$ $\acute{M}P^{r-m} \times \frac{\beta \times \overline{\beta^2 \pm p^2}|^{n-1}}{p^{2n-1}}$, &c. above specified, will be expressed by $\frac{y^{r-m} \times \dot{u} \times \overline{\dot{u}^2 \pm \dot{y}^2}|^{n-1}}{\dot{y}^{2n-1}}$. *Whence the Theorem is manifest.*

408. If R and S be assumed to denote any functions of y (that is, any two quantities expressed in terms of y and given co-efficients); then, in order to have the fluent of $S \times \frac{\overline{\dot{u}^2 \pm \dot{y}^2}|^n}{\dot{y}^{2n-1}}$ a *maximum* or *minimum*, when that of $R\dot{u}$ becomes equal to a given value, it is requisite that $\frac{S\dot{u}}{R} \times \frac{\overline{\dot{u}^2 \pm \dot{y}^2}|^{n-1}}{\dot{y}^{2n-1}}$ should be a constant

quantity: this, also, is evident from the preceding demonstration; and may be of use when the above premised theorem is not sufficiently general.

PROBLEM I.

409. *To determine the Nature of the Curve* A C E; *so that the Length of the Arch* A E *being given, the Area* A B E *shall be a* maximum.



Calling (as usual) the abscissa A D, x; the ordinate D C, y; and the arch A C, z, we have $\dot{x} = \sqrt{\dot{z}^2 - \dot{y}^2}$;* and therefore $y\dot{x}$ † $= y \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ = the fluxion of the area A D C. Now, since, by the question, the fluent of $y \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ is to be a *maximum*, when that of $\dot{z}$ becomes equal to a given quantity (ACE) let these two fluxions be, respectively, compared with $\frac{y^r \times \overline{\dot{u}^2 - \dot{y}^2}|^n}{\dot{y}^{2n-1}}$ and $y^m\dot{u}$ (as given in the foregoing theorem).‡ By which means, $n = \frac{1}{2}$, $r = 1$, $\dot{u} = \dot{z}$, and $m = 0$; and consequently $\frac{y^{r-m}\dot{u} \times \overline{\dot{u}^2 - \dot{y}^2}|^{n-1}}{\dot{y}^{2n-1}}$ $= y\dot{z} \times \overline{\dot{z}^2 - \dot{y}^2}|^{-\frac{1}{2}}$: which (according to the said theorem) being, every where, equal to a constant quantity, we shall, by putting that quantity $= a$, and ordering the equation, get $\dot{z}^2 = \frac{a^2\dot{y}^2}{a^2 - y^2}$, and $\dot{x}$ $(\sqrt{\dot{z}^2 - \dot{y}^2}) = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}$; and consequently (by

* Art. 135

† Art. 112.

‡ Art. 406.

taking the fluent) $x=a-\sqrt{a^2-y^2}$, or $2ax-x^2=y^2$; which is the common equation of a circle.

Q. E. I.

COROLLARY.

410. It follows from hence, that the greatest area that can possibly be contained by a right line (A E) joining two given points, and any curve-line A C E of a given length, terminating in the same points, will be when the said curve-line is an arch of a circle.

PROBLEM II.

411. *The Length of the Arch* A E *(see the preceding Figure) being given, to determine the Nature of the Curve, so that the Solid generated by the Rotation thereof may be a* Maximum.

* Art. 145. Since the fluent of $y^2 \times \overline{\dot{z}^2-\dot{y}^2}|^{\frac{1}{2}}$ $(= y^2\dot{x}^*)$ is required to be a *maximum*, when that of $\dot{z}$ has a given value A C E, every thing will remain as in the last Problem; *only*, r must here be $= 2$: and therefore (*by the Theorem*) we have $y^2\dot{z} \times \overline{\dot{z}^2-\dot{y}^2}|^{-\frac{1}{2}} = a$. Whence $\dot{z} = \frac{a\dot{y}}{\sqrt{a^2-y^4}}$; and consequently $\dot{x}$ ($=$ $\sqrt{\dot{z}^2-\dot{y}^2}$) $= \frac{y^2\dot{y}}{\sqrt{a^2-y^4}}$: which values, if b^2 be put $= a$ (in order to have the powers homologous) will become $\dot{z} = \frac{b^2\dot{y}}{\sqrt{b^4-y^4}}$, and $\dot{x} = \frac{y^2\dot{y}}{\sqrt{b^4-y^4}}$: whence z and x will be known.

Q. E. I.

PROBLEM III.

412. *The Superficies generated by the Arch of a Curve, in its Rotation about its axis, being given; to determine the Curve, so that the Solid itself may be a* maximum.

† Art. 145. Because the fluent of $y^2 \times \overline{\dot{z}^2-\dot{y}^2}|^{\frac{1}{2}}$ † is to be a *maximum*, when that of $y\dot{z}$ becomes equal to a given

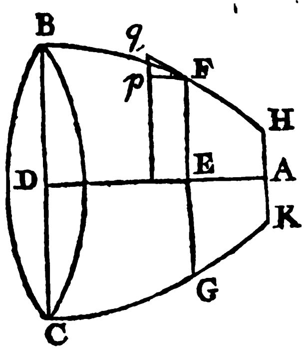
quantity; let the fluxions here exhibited be therefore compared with $\frac{y^r \times \overline{\dot{u}^2 \pm \dot{y}^2}|^n}{\dot{y}^{2n-1}}$ and $y^m\dot{u}$ (given in the theorem). By means whereof (r being $= 2$, $\dot{u} = \dot{z}$, $n = \frac{1}{2}$, and $m = 1$) we have $y\dot{z} \times \overline{\dot{z}^2 - \dot{y}^2}|^{-\frac{1}{2}} = a$ (a constant quantity;*) which is the very equation found in Prob. 1, belonging to a circle. *Art. 406.

If the solid be supposed given, and the superficies a *minimum*, we shall come at the very same conclusion: for, $y^2\dot{x}$ and $y \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (which are respectively as their fluxions) being compared with $y^m\dot{u}$ and $\frac{y^r \times \overline{\dot{u}^2 + \dot{y}^2}|^n}{\dot{y}^{2n-1}}$ we have $m = 2$, $\dot{u} = \dot{x}$, $r = 1$, and $n = \frac{1}{2}$; and therefore $\frac{\dot{x}}{y\sqrt{\dot{x}^2+\dot{y}^2}}$ equal to a constant quantity: which being denoted by $\frac{1}{a}$ (so that the terms may be homologous) there comes out $a\dot{x} = y\sqrt{\dot{x}^2 + \dot{y}^2}$; whence $2ax - x^2 = y^2$ (*as before*).

PROBLEM IV.

413. *To determine the Curve* H F B, *from whose Revolution a Solid* B K *shall be generated; which, moving forward, in a Medium, in the Direction of its Axis* D A, *will be less resisted than any other Solid of the same given Length* D A *and Base* B C.

If $AE = x$, $EF = y$, $Fp = \dot{x}$, &c. it is evident, from the principles of Mechanics, that the resisting force of a particle of the medium at F (being as the square of the sine of the angle of inclination pFq) will be truly represented by $\frac{\dot{y}^2}{\dot{x}^2+\dot{y}^2}$ $\left(\frac{\overline{pq}|^2}{\overline{Fq}|^2}\right)$. Moreover, since the



whole number of particles acting upon F H K G is proportional to the area of the circle F G, or as y^2; the fluxion hereof ($2y\dot{y}$) drawn into $\frac{\dot{y}^2}{\dot{x}^2 + \dot{y}^2}$, will therefore give $\frac{2y\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$ for the fluxion of the resistance upon F H K G.

Now, since it is required (by the question) to have the fluent of $\frac{y\dot{y}^3}{\dot{x}^2 + \dot{y}^2}$ (or $\frac{y \times \overline{\dot{x}^2 + \dot{y}^2}|^{-1}}{\dot{y}^{-3}}$) a *maximum*, when that of $\dot{x}$ becomes equal to a given quantity (A D), let these two fluxions be therefore compared with $\frac{y^r \times \overline{\dot{u}^2 + \dot{y}^2}|^n}{\dot{y}^{2n-1}}$ and $y^m \dot{u}$.* Whence (r being $= 1$, $\dot{u} = \dot{x}$, $n = -1$, and $m = 0$) we get $\frac{y\dot{x} \times \overline{\dot{x}^2 + \dot{y}^2}|^{-2}}{\dot{y}^{-3}}$ = (a constant quantity †); and consequently $y\dot{y}^3\dot{x} = a \times \overline{\dot{x}^2 + \dot{y}^2}|^2$: whereof the fluent will be found, by Art. 264. That the curve does not meet its axis in the extreme point A, but has an ordinate A H at that point (as represented in the figure) is evident from the foregoing equation. For $\overline{\dot{x}^2 + \dot{y}^2}|^2$ ($\overline{Fq}|^4$) being always greater than $\dot{y}^3\dot{x}$ ($\overline{pq}|^3 \times Fp$), y must therefore be greater than a, in the same proportion; and so, can never be equal to nothing.

Now, as it is demonstrable that the angle AHF must be $\frac{1}{2}$ of a right angle, A H (the least value of y) will therefore be $= 4a$ (since $\dot{x}$ and $\dot{y}$ are, in this circum-

* Art. 406.

† Art. 406.

stance, equal to each other). But, what a itself ought to be, must be determined from the given values of A D and B D, and the resolution of the foresaid equation.

PROBLEM V.

414. *To determine the Solid of the least Resistance, supposing the Area of the generating Plane* A H B D, *and its greatest Ordinate* D B *to be given (see the preceding Figure).*

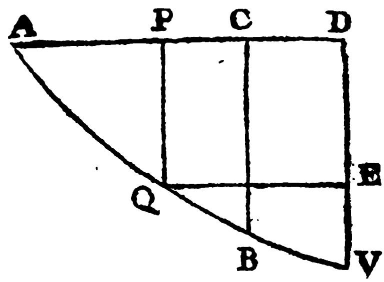
Since (by the last article) the fluxion of the resistance is expressed by $\frac{y \times \overline{\dot{x}^2 + \dot{y}^2}|^{-1}}{\dot{y}^{-3}}$, and that of the area A E F H by $y\dot{x}$, it is plain (from the premised theorem*) that $\frac{\dot{x} \times \overline{\dot{x}^2 + \dot{y}^2}|^{-2}}{\dot{y}^{-3}}$ is a constant quantity. * Art. 406.

Whence $\frac{\dot{y}^3\dot{x}}{\overline{\dot{x}^2 + \dot{y}^2}|^2}$, or its equal $\frac{\overline{pq}|^3 \times Fp}{\overline{qF}|^4}$, being every where the same, the angle pFq must also be invariable; and consequently HFB a right-line. Therefore the solid of the least resistance is *(in this case)* either a whole cone, or the frustrum of a greater cone. But it is easy to show, *that*, when the area of the generating plane A B is given so small, that the angle B may be taken equal to the half of a right-angle; I say, it is demonstrable in this case, that the frustrum so arising will be less resisted than a whole cone, or any other frustrum, whereof the base and the area of the generating plane are the same.

In like manner the solid of *least resistance*, when its bulk and greatest diameter are given, may be determined: the equation of the generating curve being $\frac{y^{-1}\dot{x} \times \overline{\dot{x}^2 + \dot{y}^2}|^{-2}}{\dot{y}^{-3}} = \frac{1}{a}$, or $a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}^2 + \dot{y}^2}|^2$: whereof the solution is given in Art. 264.

PROBLEM VI.

415. *To determine the Line along which a Body, by its own Gravity, will descend, from one given Point* A *to another* B, *in the shortest Time possible.*



Let A D be parallel, and B C perpendicular, to the horizon, intersecting each other in C; and let Q P be any ordinate to the curve parallel to B C: then (calling A P, x; P Q, y, &c.) the celerity at Q will be expressed by $y^{\frac{1}{2}}$; also the fluxion of the time of descent through

* Art. 204. A Q will be truly defined by $\frac{\dot{z}}{y^{\frac{1}{2}}}$,* or its equal $y^{-\frac{1}{2}} \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$. Here, therefore, the fluent of $y^{-\frac{1}{2}} \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ is to be a *minimum*, when that of $\dot{x}$ arrives to

† Art. 405. to a given value (A C). Whence, *by the theorem*,† $y^{-\frac{1}{2}}\dot{x} \times \overline{\dot{x}^2+\dot{y}^2}|^{-\frac{1}{2}}$ must be = a constant quantity: which (to have the term homologous) let be denoted by $a^{-\frac{1}{2}}$ (or $\frac{1}{\sqrt{a}}$). Then $a^{\frac{1}{2}}\dot{x} = y^{\frac{1}{2}} \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$; whence $\dot{x} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}} = \frac{y\dot{y}}{\sqrt{ay-y^2}}$; $\dot{z} = (\sqrt{\dot{x}^2+\dot{y}^2})$

$= \frac{a^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}}$; and consequently $z = 2a - 2a^{\frac{1}{2}}\sqrt{a-y}$.

Therefore when $y = a$, z is $= 2a$; which two corresponding values let be denoted by DV and AV; and let Q E, parallel to A D, meet D V in E; then V E (VD − ED) being $= a - y$, and VQ (AV − AQ) $= 2a^{\frac{1}{2}}\sqrt{a-y}$, it follows that

VD (a) : VE $(a-y)$: : VA² $(4a^2)$: VQ² $(4a \times \overline{a-y})$.

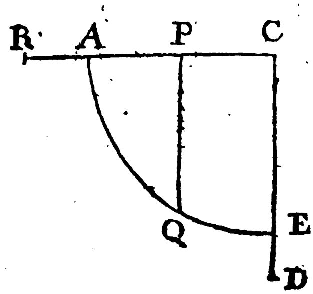
Which is one of the most remarkable properties of the cycloid; the curve which, therefore, answers the conditions of the problem.

If the celerity be supposed as any function (S) of the quantity y, the problem will be resolved in the same manner: the equation of the curve being

$$\frac{\dot{x} \times \overline{\dot{x}^2 + \dot{y}^2}|^{-\frac{1}{2}}}{S} = \frac{1}{a}.*$$ * Art. 408.

PROBLEM VII.

416. *To find the Nature of the Curve* AQE, *along which a heavy Body must descend from an horizontal Line* RC *to a vertical Line* CD, *so that the Area* CAE *may be given, and the Time of the Descent a* Minimum.

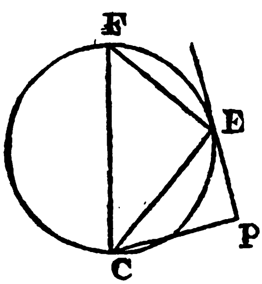


If the ordinate P Q (parallel to C D) be called y, and the velocity at Q be denoted by y^n; it is evident that the fluent of $y^{-n} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ $\left(= \frac{\dot{z}}{y^n}\right)$† must be a *minimum* when that of $y\dot{x}$ amounts to a given value. † Art. 204.

Therefore (by the theorem already mentioned so often) we have $y^{-n-1}\dot{x} \times \overline{\dot{x}^2 + \dot{y}^2}|^{-\frac{1}{2}} = a^{-n-1}$; and consequently $\dot{x} = \frac{y^{n+1}\dot{y}}{\sqrt{a^{2n+2} - y^{2n+2}}}$; which, by writing $\frac{1}{2}$ instead of n, becomes $\dot{x} = \frac{y^{\frac{3}{2}}\dot{y}}{\sqrt{a^3 - y^3}}$: whence x will be known. But, if the celerity was to be everywhere uniform, then (n being $= 0$) we should have $\dot{x} = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}$; and therefore $x = a - \sqrt{a^2 - y^2}$: which answers to a circle.

Lemma.

417. *If, upon a Tangent* E P, *from any Point* C *in the Circumference of a Circle* F E C, *a Perpendicular* CP *be let fall, the Chord* (CE) *joining that Point and the Point of Contact, will be a Mean-Proportional between the said Perpendicular* C P *and the Diameter* C F *of the Circle.*

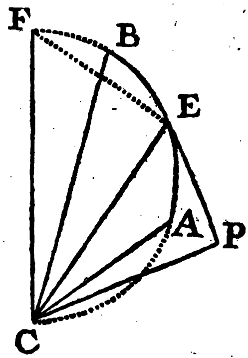


For, the angles P and C E F being both right; and also C E P = F, the triangles CPE and CEF are similar: and therefore CP : CE :: CE : CF.

Q. E. D.

PROBLEM VIII.

418. *In the mixt-lined Triangle* A C B, *the Lengths of all the sides (whereof* C A *and* C B *are Right-lines) are supposed given; it is required to find the Nature of the Curve-side* AEB, *so that the Area may be a* Maximum.



Put the arch $AE = z$, and the ordinate $CE = y$; then, the fluxion of the area ACE being $\frac{y}{2}\sqrt{\dot{z}^2 - \dot{y}^2}$,* the fluent of $y \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$, generated in the time wherein y, from C A, increases to C B, must be a *maximum*: therefore, *by the Theorem*,† we have $y\dot{z} \times \overline{\dot{z}^2 - \dot{y}^2}|^{-\frac{1}{2}} = a$,† or $\frac{\dot{z}}{\sqrt{\dot{z}^2 - \dot{y}^2}} = \frac{a}{y}$. But, if C P be supposed perpendicular to the tangent E P, then will $\frac{\dot{z}}{\sqrt{\dot{z}^2 - \dot{y}^2}}$ (Art. 35) $= \frac{CE}{CP} = \frac{y}{CP}$; and consequently $\frac{a}{y} = \frac{y}{CP}$; or, CP : CE (y) :: CE (y) : a: which proportion, *by the Lemma*, answers to a circle; whereof the quantity a is the diameter.

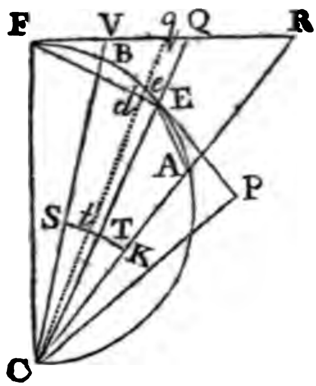
* Art. 113.

† Art. 406.

Now, that A E B must be an arch of a circle is also evident from Prob. 1; but, that the same arch, continued out, will pass through the angle C, does not appear from thence. This is known from above; and is requisite in finding the particular circle answering to any proposed *data*.

PROBLEM IX.

419. *To find the Path* AEB *which a Body must describe in moving uniformly from one given Point* A *to another* B; *so that, being every where acted on by a Force, or Virtue, which varies according to the Inverse-Duplicate-Ratio of the Distances from a given Center* C, *the whole Action upon the Body shall be a* Minimum.



Putting AE = z, CE = y, de (indefinitely small) = $\dot{y}$,* $Ee = \dot{z}$, and Ed $(\sqrt{\dot{z}^2 - \dot{y}^2})$ = $\dot{u}$; we have $\frac{\dot{z}}{y^2}$ $\left(= y^{-2} \times \overline{\dot{u}^2 + \dot{y}^2}\right|^{\frac{1}{2}})$ for the measure of the force which acts upon the body in describing the particle E e $(\dot{z})$: moreover, if from the center C, with any given radius (r) an arch KTtS of a circle be described, intersecting C E in T, we shall have Tt (the measure of the angle ECe) = $\frac{r\dot{u}}{y}$. Therefore, since the fluent of $y^{-2} \times \overline{\dot{u}^2 + \dot{y}^2}\right|^{\frac{1}{2}}$ is required to be a *minimum*, and the contemporary fluent of $y^{-1}\dot{u}$ (between C A and C B) a given quantity; it follows, from the theorem premised at the beginning of the section, that $y^{-2+1}\dot{u} \times \overline{\dot{u}^2 + \dot{y}^2}\right|^{-\frac{1}{2}}$ must be equal to a constant quantity $\left(\frac{1}{a}\right)$ and consequently $\frac{\dot{u}}{\sqrt{\dot{u}^2 + \dot{y}^2}}$ $\left(= \frac{\sqrt{\dot{z}^2 - \dot{y}^2}}{\dot{z}}\right) = \frac{y}{a}$: which is the very equation found in the preceding problem. Therefore, if through the three given points A, B, and C, the circumference of a circle be described, the arch thereof terminated by A and B will be the path of the body. *Q. E. I.*

* Art. 134.

COROLLARY.

420. If F R be a tangent to the circle, at the extremity of the diameter C F, and C A and C E be pro-

duced to meet it in R and Q, it follows that the whole action upon the body, in describing the arch AE, will be proportional to the corresponding part RQ of the said tangent. For, if Ce be also produced to meet FR in q, and EF be drawn, it is plain that the triangles CEF and CFQ, as also CEe and CqQ, are similar: whence it will be, CE (y) : CF (a) :: CF (a) : CQ (or Cq) $= \frac{a^2}{y}$; and CE (y) : Ee $(\dot{z})$ :: Cq $\left(\frac{a^2}{y}\right)$: Qq $= \frac{a^2\dot{z}}{y^2}$: which (a being constant) is as $\left(\frac{\dot{z}}{y^2}\right)$ the force that acts upon the body in describing Ee $(\dot{z})$. And, as this every where holds, the whole action in describing AE must therefore be proportional to RQ. Which force (it is easy to prove) will be to that exerted on the body in moving through the chord AE, as the chord to the arch.

PROBLEM X.

421. *To determine the Path in which a Body may move from one given Point* A *to another* B, *in the shortest Time possible; supposing the Velocity to be, every where, proportional to any Power* (y^p) *of the Distance from a given Center* C. (See the last Figure).

Here every thing will remain as in the preceding problem; only y^{-p} must be wrote instead of $y^{-\frac{1}{2}}$. Therefore we have $y^{-p+1} \times \dot{u} \times \overline{\dot{u}^2 + \dot{y}^2}|^{-\frac{1}{2}} =$ a constant quantity: which quantity (to have the terms homologous) let be denoted by $\frac{b}{a^p}$; then, by reduction,

$$\frac{by^{p-1}}{a^p} = \frac{\dot{u}}{\sqrt{\dot{u}^2 + \dot{y}^2}} \left(= \frac{Ed}{Ee}\right) = \frac{CP}{CE} = \frac{CP}{y}:$$

and consequently $CP = \frac{by^p}{a^p}$. Hence, if $p=0$, or the velocity be constant; then CP being every where $= b$, the body must, in this case, describe a right-line.

But, if $p = 1$, then CP being $= \frac{by}{a}$; the curve will be a logarithmic spiral, whose center is C: * except in that particular case, where CA = CB, when it degenerates to a circle.

* Art. 74.

Lastly, if $p = 2$, the curve will be a circle (*by the preceding Lemma*) whose diameter is $\frac{a^2}{b}$, and whose periphery passes through the given point C.

After the same manner, the value of CP (upon which the nature of the curve depends) may be determined, when the velocity is expounded by any given function (S) of the distance (y) from the center of force: and (by writing S in the room of y^p † &c.) will come out $CP = \frac{bS}{c}$; where b and c represent constant quantities.

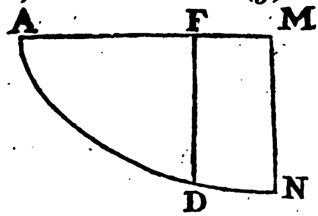
† Art. 407.

When the velocity is that which the body may acquire, in descending through BE, by a centripetal force expressed by y^p, then the value of S (the measure of that velocity) being interpreted by $\sqrt{d^{p+1} - y^{p+1}}$ ‡ (where $CB = d$) we therefore have $CP = \frac{b\sqrt{d^{p+1} - y^{p+1}}}{c}$ for the equation of the curve of the swiftest descent, according to this last *hypothesis* of a centripetal force varying as any power p of the distance.

‡ Art. 221 & 206.

422. Besides the Problems already resolved in this section, there are others of the same nature which are confined to more particular restrictions, and require a different method of solution.

Thus, if Q, R and S be supposed to denote any given powers, or functions, of the ordinate (y) of a curve ANM, and the nature of the curve be required, so that, when the fluent of $Q\dot{x}$ becomes equal to a given quantity, the fluent of $R\dot{z}$, may also become equal to another given quantity, and that of $S\dot{z}$, a *maximum* or *minimum:* then, because there is, in this case, a second equation, or new condition, beyond what is to be met with in any of the foregoing problems, the method of solution hitherto explained, will, therefore, be insufficient. But, by a process similar to that whereby the said method was demonstrated (assuming, here, three expressions, and three indeterminate quantities, instead of two)* a general answer to this problem (under all its restrictions) will be obtained: and is exhibited by the equation,



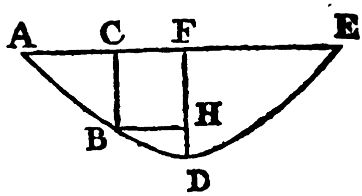
* Art. 407.

$\frac{\dot{z}}{\dot{x}} = \frac{pR \pm qS}{Q}$; wherein p and q denote constant quantities.

428. Though it seems unnecessary to put down the invention of this equation, after what has been hinted above, yet it may not be improper to observe, by way of corollary, that if $Q = 1$, $R = 1$, and $S = y^n$, the equation will then become $\frac{\dot{z}}{\dot{x}} = p \pm qy^n$; expressing the nature of the curve, when, the whole abscissa (AM) and corresponding arch (AN) being both given quantities, the fluent of $y^n\dot{z}$ is a *maximum* or *minimum*, according as the value of n is positive or negative: in both which cases, it is very easy to perceive, that the curve must be concave to A M, and that the value of $\frac{\dot{z}}{\dot{x}}$, or its

equal $p \pm qy^n$, must, therefore, decrease as y increases: whence we may infer that the sign of qy^n must be negative in the former case, and positive in the latter.

Ex. Let the curve A B D E, be the *catenaria*; formed by a slender chain, or perfectly flexible cord,



suspended by its two extremes in the horizontal line A E: then, since its center of gravity must be the lowest possible, the fluent of $y\dot{z}$, when AC=AE, must
* Art. 173. therefore be a *maximum*:* whence (n being here=1) our equation $\left(\frac{\dot{z}}{\dot{x}} = p \pm qy^n\right)$ becomes $\frac{\dot{z}}{\dot{x}} = p - qy$.

But, in order to reduce it to a more convenient form, let the distance (D F) of the lowest point of the curve from the horizontal-line A E be put $= b$; then, when y (B C) becomes $= b$, $\dot{x}$ will be $= \dot{z}$; and therefore the equation, in that circumstance, is $1 = p - qb$; whence $p = 1 + qb$, and consequently $\frac{\dot{z}}{\dot{x}} = 1 + qb - qy = 1 + q \times \overline{b - y}$: which, by putting $b - y$ (D H) $= s$ and $a = \frac{1}{q}$ is reduced to $\frac{\dot{z}}{\dot{x}} = 1 + \frac{s}{a}$: from whence $a^2\dot{z}^2 \left(= \overline{a+s}|^2 \times \dot{x}^2\right) = \overline{a+s}|^2 \times \overline{\dot{z}^2 - \dot{s}^2}$; and consequently B D $= \sqrt{2as + s^2}$.

For another example (wherein the exponent n will be negative) let the required curve be that along which a body may descend, by its own gravity, from one given point A to another B, in less time than through any other line of the same length. In which case, the fluent of $\dot{z}y^{-\frac{1}{2}}$ being a *minimum*, when x and z become equal to given quantities, our equation (by writing $-\frac{1}{2}$ for n) will here become $\frac{\dot{z}}{\dot{x}}=p+qy^{-\frac{1}{2}}$: from whence exterminating $\dot{x}$, or $\dot{z}$, by means of the equation $\dot{x}^2+\dot{y}^2=\dot{z}^2$, the fluent may also be determined.

SECTION XI.

The Resolution of Problems of various Kinds.

PROBLEM I.

424. *Any hyperbolical Logarithm* (y) *being given, it is proposed to find the natural Number answerng thereto.*

If the number sought be denoted by $1+x$, we shall (by Art. 126) have $\dot{y}=\frac{\dot{x}}{1+x}$, or $\dot{y}+x\dot{y}-\dot{x}=0$. Let $Ay+By^2+Cy^3$ &c. $=x$; then $A\dot{y}+2By\dot{y}+3Cy^2\dot{y}$ &c. $=\dot{x}$, and our equation will become

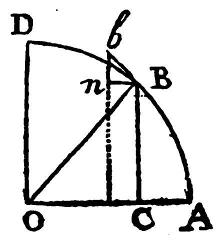
$$\left.\begin{array}{l}\dot{y}+Ay\dot{y}+By^2\dot{y}+Cy^3\dot{y}\text{ \&c.}\\ -A\dot{y}-2By\dot{y}-3Cy^2\dot{y}-4Dy^3\dot{y}\text{ \&c.}\end{array}\right\}=0.$$

Whence, by comparing the homologous terms, we get $A=1$, $B=\frac{A}{2}=\frac{1}{2}$, $C=\frac{B}{3}=\frac{1}{2.3}$, $D=\frac{C}{4}=\frac{1}{2.3.4}$ &c. Therefore $1+y+\frac{y^2}{2}+\frac{y^3}{2.3}+\frac{y^4}{2.3.4}+\frac{y^5}{2.3.4.5}$ &c. is $(=1+x)$ the number sought.

PROBLEM II.

425. *The Radius* AO *and any arch* AB *of a Circle* ABD *being given; to find the Sine* BC, *and Co-sine* OC *of that Arch.*

Let AO (BO) $= r$, AB $= z$, AC $= x$, BC $= y$,



B$b = \dot{z}$, B$n = \dot{x}$, and $bn = \dot{y}$; because of the similar triangles OBC and Bnb, it will be

OB (r) : BC (y) :: Bb $(\dot{z})$: Bn $(\dot{x})$

And OB (r) : OC $(r-x)$ :: Bb $(\dot{z})$: bn $(\dot{y})$

From which we have

$y\dot{z} = r\dot{x}$

And $r\dot{y} = r\dot{z} - x\dot{z}$.

Let $x = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5$ &c.

And $y = az + bz^2 + cz^3 + dz^4 + ez^5$ &c.

Then, by substitution and transposition, our two equations will become

$$\left.\begin{array}{l} * \quad + az\dot{z} + bz^2\dot{z} + cz^3\dot{z} + dz^4\dot{z} \text{ \&c.} \\ -rA\dot{z} - 2rBz\dot{z} - 3rCz^2\dot{z} - 4rDz^3\dot{z} - 5rEz^4\dot{z} \text{ \&c.} \end{array}\right\} = 0$$

And

$$\left.\begin{array}{l} ra\dot{z} + 2rbz\dot{z} + 3rcz^2\dot{z} + 4rdz^3\dot{z} + 5rez^4\dot{z} \text{ \&c.} \\ -r\dot{z} + Az\dot{z} + Bz^2\dot{z} + Cz^3\dot{z} + Dz^4\dot{z} \text{ \&c.} \end{array}\right\} = 0$$

From which, by equating the homologous terms, we get $A = 0$, $a = 2rB$, $b = 3rC$, $c = 4rD$, $d = 5rE$ &c.

Also $a = 1$, $b = -\dfrac{A}{2r}$, $c = -\dfrac{B}{3r}$, $d = -\dfrac{C}{4r}$ &c.

Therefore $2rB = 1$, $3rC = -\frac{A}{2r}$, $4rD = -\frac{B}{3r}$, $5rE = -\frac{C}{4r}$, &c. and consequently $B = \frac{1}{2r}$, $C = 0$, $D = -\frac{B}{3 \cdot 4r^2} = -\frac{1}{2 \cdot 3 \cdot 4r^3}$, $E = 0$, $F = -\frac{D}{5 \cdot 6r^2} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot r^5}$ &c.
Whence, also $b\ (= 3rC) = 0$, $c\ (= 4rD) = -\frac{1}{2 \cdot 3r^2}$ &c. &c.

Hence it is evident that $y\ (= az + bz^2 + cz^3$ &c.) $= z - \frac{z^3}{2 \cdot 3r^2} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5r^5} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7r^6}$ + &c. And that $x\ (= Az + Bz^2 + Cz^3$ &c.) $= \frac{z^2}{2r} - \frac{z^4}{2 \cdot 3 \cdot 4r^3} + \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6r^5} -$ &c.*

PROBLEM III.

426. *To find the Value of x, when x^x is a Minimum.*

The logarithm of x^x is $= x \times l \cdot x$; whose fluxion $\dot{x} \times l \cdot x + \dot{x}$ being $= 0$, we have $l \cdot x = -1$. But (by Prob. 1) the number whose hyp. log. is y will be $1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4}$ &c. Therefore, by writing -1 instead of y, we have $x = 1 - 1 +$

* The substance of this solution (being the most neat and artful I have seen to that useful Problem) I had from a Letter signed ——— *Needler*; which was put into my hands by a friend, who received it from the late Dr. *Halley*, to whom it was wrote.

$\frac{1}{2} - \frac{1}{2\,.\,3} + \frac{1}{2\,.\,3\,.\,4} - \frac{1}{2\,.\,3\,.\,4\,.\,5}$ &c. $= 0{,}367878$ &c.

PROBLEM IV.

427. *To divide a given Number (a) so that the continual Product of all its Parts may be a* Maximum.

It is evident (from Art. 23) that all the parts must be equal: if, therefore, any one of them be denoted by x, their number will be $\frac{a}{x}$, and we shall have $\overline{x}|^{\frac{a}{x}}$ a *maximum:* and therefore its logarithm $\frac{a}{x} \times$ $L\,.\,x$ a *maximum* also: and its fluxion $- \frac{a\dot{x}}{x^2} \times L\,.\,x$ $+ \frac{a\dot{x}}{x^2} = 0$:* whence H-$L\,.\,x = 1$, and consequently $x = 1 + 1 + \frac{1}{2} + \frac{1}{2\,.\,3} + \frac{1}{2\,.\,3\,.\,4}$ &c.$=2.71828$† &c. Therefore the next inferior, or superior, number to 2.71828 &c. that will exactly measure the given number a, is the required value of each part: thus, let $a = 10$; then because $\frac{10}{2\,.\,71828\text{ \&c.}} = 4$ nearly, the number of parts in this case, will be 4, and the value of each$=\frac{10}{4}=2\,.\,5$.

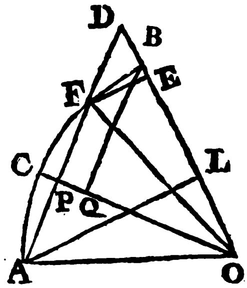
* Art. 22 & 126.

† Art. 424.

PROBLEM V.

428. *To divide a given Angle* A O B *into two Parts* A O C *and* B O C, *so that the Product of any given Powers,* AP^n, $\times$ BQ^m, *of their Sines* A P *and* B Q *may be a* Maximum.

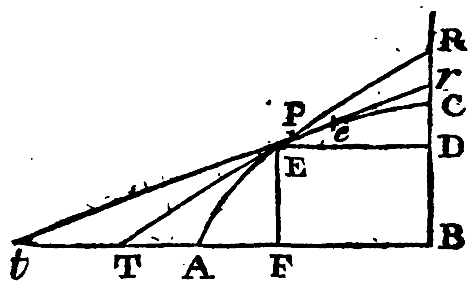
Let A P, produced, cut the radius O B in D, and the arch AB in F; likewise let FE and AL be perpendicular to OB, and join O, F: putting $AO=r$, $AP=x$ and $BQ=y$. Then, because $x^n y^m$ is to be a *maximum*, we have $nx^{n-1}\dot{x} \times y^m + x^n \times my^{m-1}\dot{y} = 0$; and consequently $ny\dot{x} = -mx\dot{y}$.



Moreover, since the fluxion of the arch AC is $= \frac{r\dot{x}}{\sqrt{r^2 - x^2}}$ and that of BC $= \frac{r\dot{y}}{\sqrt{r^2 - y^2}}$ (Art. 142) we also have $\frac{r\dot{y}}{\sqrt{r^2 - y^2}} + \frac{r\dot{x}}{\sqrt{r^2 - x^2}} = 0$, or $\frac{\dot{y}}{\sqrt{r^2 - y^2}} = \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; which multiplied by the former equation, &c. gives $\frac{ny}{\sqrt{r^2 - y^2}} = \frac{mx}{\sqrt{r^2 - x^2}}$, or $n \times \frac{y\sqrt{r^2 - x^2}}{\sqrt{r^2 - y^2}} = mx$: whence, because OQ $(\sqrt{r^2 - y^2})$: QB (y) :: OP $(\sqrt{r^2 - x^2})$: PD $= \frac{y\sqrt{r^2 - x^2}}{\sqrt{r^2 - y^2}}$, we have $n \times$ PD $(=mx) = m \times$ AP; and therefore PD : AP :: $m : n$; whence (by composition and division) AD : DF :: $m+n : m-n$: but (*by sim. Trian.*) AD : DF :: AL : FE; consequently $m+n : m-n$ :: AL : FE; that is, *as the sum of the indices of the two proposed powers is to their difference, so the sine of the whole given angle to the sine of the difference of its two required parts.* This proportion is given in words, at length, because it will be found of frequent use in the solution of Mechanical Problems.

PROBLEM VI.

429. *To show that the least Triangle that can be described about, and the greatest Parallelogram in, a given Curve* ABC, *concave to its Axis, will be when the Subtangent* FT *is equal to the Base* BF *of the Parallelogram, or half the Base* B T *of the Triangle.*



It appears from Art. 25, and is demonstrable by common geometry, that the greatest parallelogram that can be inscribed in the triangle BTR (supposing the position of T R to remain the same) will be *that* whose base BF is half the base BT of the triangle: therefore, as a greater figure cannot possibly be inscribed in the curve BAC than in the triangle BTR circumscribing it, the greatest parallelogram that can be inscribed, either in the triangle or the curve, must be that above specified.

But now, to make it also appear that the triangle BTR is a *minimum* when FT = BF; let B *t r* be any other circumscribing triangle, and let the two tangents T E R and *t e r* intersect each other in P. Then, ER being = ET, it is plain that R P is less than PT, and P*r* (less than PR less than PT) less than P*t*; therefore, the sides P R and P *r* of the triangle R P *r* being less than the sides, PT and P*t* of the triangle T P *t*, and the opposite angles R P *r* and T P *t* equal to each other, it follows that the triangle RP*r* is less than TP*t*; and consequently, by adding the trapezium B T P *r* to both, it appears that B T R is less than B *t r*.

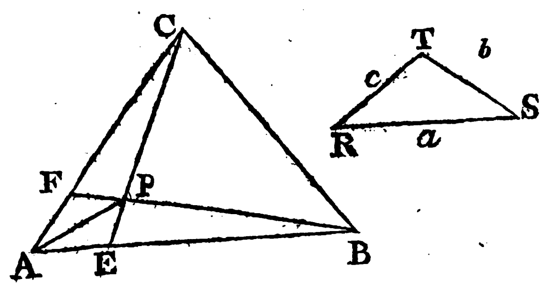
COROLLARY.

430. Hence *the greatest inscribed parallelogram is half the least circumscribing triangle.*

In the same way it may be proved, that *the greatest inscribed cylinder, and the least circumscribing cone, in, and about, the solid generated by revolution of a given curve, will be when the sub-tangent is equal to twice the altitude of the cylinder, or* $\frac{3}{2}$ *of the altitude of the cone: and that the two figures will be to each other in the ratio of* 4 *to* 9.

PROBLEM VII.

431. *Three Points* A, B, C *being given, to find the Position of a fourth Point* P, *so that, if Lines be drawn from thence to the three former, the Sum of the Products* $a \times AP$, $b \times BP$, *and* $c \times CP$ *(where a, b and c denote given Numbers) shall be a* Minimum.

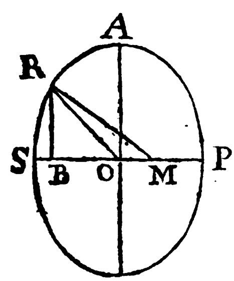


If CP and BP be produced to E and F, it will appear from Art. 35 and 36, that the sine of BPE must be to that of APE, as *a* to *b*; and the sine of CPF (BPE) to that of A P F, as *a* to *c*. Therefore, the sines of the three angles BPE, APE, and APF (which angles, taken all together, make two right-ones) being in the given ratio of *a*, *b* and *c*, it follows, that, if a triangle RST be constructed, whose sides RS, ST and RT are in the said ratio of *a*, *b* and *c*, the angles T, R and S opposite thereto, will be respectively equal to the fore-mentioned angles BPE, APE and APF. From whence, all the angles at the point P being given, the position of that point is given by common Geometry.

But it is observable, that, when one of the three given quantities a, b, c (suppose a) is equal to, or greater than, the sum of the other two, a triangle cannot then be formed whose sides are proportional to the said quantities: in that case the point P will fall in the point (A) corresponding to the greatest quantity (a). For, it is plain that $b \times AB$ is less than $b \times BP + b \times AP$; and that $c \times AC$ is less than $c \times CP + c \times AP$; whence, by adding the less to the less, and the greater to the greater, it also appears that $b \times AB + c \times AC$ must be less than $b \times BP + c \times CP + \overline{b+c} \times AP$ less than $b \times BP + c \times CP + a \times AP$; because a (by hypothesis) is equal to, or greater than, $b + c$.

PROBLEM VIII.

432. *To determine in what Latitude a Right-line perpendicular to the Surface of the Earth, and Another drawn, from the same Point, to the Center, make the greatest Angle, possible, with each other; the ratio of the Axis and the Equatoreal Diameter being supposed given.*



Let AE represent the Equatoreal diameter, and S P the axis of the earth (taken as an oblate spheroid) also let R O and R M represent the two lines specified in the problem, whereof let the latter (perpendicular to ARS) meet SP in M; and let RB be perpendicular to S P.

It is evident, from the property of the ellipsis, that $SP^2 : AE^2 :: BO : BM$. And (by Trigonometry) BO : BM :: tang. BRO : tang. BRM; whence, by equality, $SP^2 : AE^2 :: \text{tang. } BRO : \text{tang. } BRM$; therefore, by composition and division, $AE^2 + SP^2 : AE^2 - SP^2 :: \text{tang. } BRM + \text{tang. } BRO : \text{tang. } BRM - \text{tang. } BRO$. But, *the sum of the tangents of any two angles is to their difference, as the sine of the sum of those angles to the sine of their difference;** whence it follows that $AE^2 + SP^2 : AE^2 - SP^2 :: \text{sine } (BRM + BRO) : \text{sine } \overline{BRM - BRO}\ (ORM)$.

Now, since the ratio of the two first terms is constant, or in every part of the ellipsis the same, it is obvious that the angle ORM, or its sine, will be the greatest possible, when its antecedent (the sine of $\overline{BRM + BRO}$) is the greatest possible, that is when $BRM + BRO =$ a right-angle and its sine $=$ radius. Therefore, in the proposed circumstance, when ORM is *a maximum*, our last proportion will become $AE^2 + SP^2 : AE^2 - SP^2 :: \text{radius} : \text{sine of } ORM$: and half the angle, so found, added 45°, will give (BRM) the complement of the required latitude; because $BRM + BRO$ (or $2BRM - ORM$) being $= 90°$, it is evident that $2BRM = 90° + ORM$, and consequently $BRM = 45° + \frac{1}{2} ORM$.

PROBLEM IX.

433. *Of all the Semi-cubical Parabolas, to determine that whereof the Length of the Curve being given, the Area shall be a* Maximum.

The general equation is $ax^2 = y^3$: moreover, the area is universally $= \frac{3y^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$, and the length of the curve $= \frac{\overline{4a + 9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27}$ (see Art. 137). Let the last of these be put $= c$, and, by ordering the equation, you will

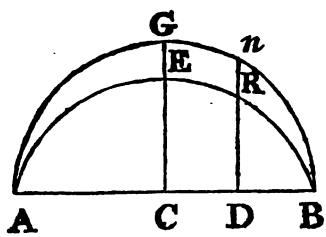
* Vide p. 56 of my Trigonometry.

get $y = \frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a}{9}$: whence $\frac{3y^{\frac{5}{3}}}{5a^{\frac{1}{3}}}$ (and consequently $\frac{y}{a^{\frac{1}{5}}}$) being a *maximum*, it is evident that $\frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a}{a^{\frac{1}{5}}}$, or its equal $a^{\frac{2}{15}} \times \overline{27c+8a}|^{\frac{2}{3}} - 4a^{\frac{4}{5}}$ must likewise be a *maximum*: which, put into fluxions, and reduced, gives $a = c \times \frac{9+3\sqrt{21}}{32}$: whence x and y will also be found.

PROBLEM X.

184. *To determine the Ratio of the Periphery of any given Ellipsis to that of its circumscribing Circle.*

Call the semi-transverse axis C B, a; the semi-conjugate C E, c; any ordinate D R, y; and its distance



CD from the center, x: then (by the nature of the curve) y being $= \frac{c}{a}\sqrt{a^2 - x^2}$, we have $\dot{y} = \frac{-cx\dot{x}}{a\sqrt{a^2 - x^2}}$; and consequently $\dot{z}$ $(\sqrt{\dot{x}^2 + \dot{y}^2}) = \frac{\dot{x}\sqrt{a^4 - \overline{a^2 - c^2} \times x^2}}{a\sqrt{a^2 - x^2}}$: which, by making $d =$

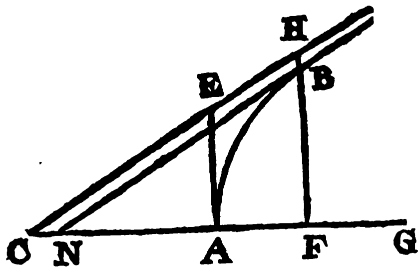
$\frac{a^2-c^2}{a^2}$ will be reduced to $\dot{z} = \frac{\dot{x}\sqrt{a^2-dx^2}}{\sqrt{a^2-x^2}} = \frac{a\dot{x}}{\sqrt{a^2-x^2}} \times \overline{1 - \frac{dx^2}{2a^2} - \frac{d^2x^4}{2.4a^4} - \frac{3d^3x^6}{2.4.6a^6}}$ &c. (by throwing the numerator into a series) whereof the *whole* fluent, when x becomes $= a$, will be z (E R B) $= A \times (1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6} - \frac{3.3.5.5.7d^4}{2.2.4.4.6.6.8.8}$ &c.) (by Art. 286) where A denotes the length of the arch G n B, or $\frac{1}{4}$ of the periphery of the circumscribing circle.

Hence it follows that the periphery of the ellipsis is to that of its circumscribing circle, as $1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$ &c. or as $1 - \frac{d}{2.2} \times A + \frac{1.3d}{4.4} \times B + \frac{3.5d}{6.6} \times C + \frac{5.7d}{8.8} \times D$ &c. to unity: where A, B, C, D &c. denote the preceding terms, under their proper signs.

PROBLEM XI.

435. *To determine the Difference between the Length of the Arch of a Semi-hyperbola infinitely produced, and its Asymptote.*

Call the semi-transverse axis (A C) a; the semi-conjugate (or its equal A F) b; the distance (CF) of any ordinate from the center, x; the ordinate itself, y; and the arch corresponding, z: then, from the nature of the curve we have $y = \frac{b\sqrt{x^2-a^2}}{a}$; whence



$\dot{y} = \frac{bx\dot{x}}{a\sqrt{x^2 - a^2}}$; and consequently $\dot{z}$ $(= \sqrt{\dot{x}^2 + \dot{y}^2}) =$

$$\frac{\dot{x}\sqrt{\frac{a^2x^2 + b^2x^2}{a^2} - a^2}}{\sqrt{x^2 - a^2}}$$

which, making $d^2 = \frac{a^2}{a^2 + b^2}$ $(= \frac{CA^2}{CE^2})$ and $u = \frac{a}{x}$ will be transformed to $\dot{z} = -\frac{a\dot{u}}{du^2} \times \frac{\overline{1 - d^2u^2}|^{\frac{1}{2}}}{\overline{1 - u^2}|^{\frac{1}{2}}}$; whereof the upper surd, expanded, is $= 1 - \frac{d^2u^2}{2} - \frac{d^4u^4}{8}$ &c. And therefore $\dot{z} =$

$$\frac{a}{d} \text{ into } \frac{-\dot{u}}{u^2\sqrt{1-u^2}} + \frac{d^2\dot{u}}{2\sqrt{1-u^2}} + \frac{d^4u^2\dot{u}}{8\sqrt{1-u^2}} + \frac{3d^6u^4\dot{u}}{8.6\sqrt{1-u^2}} + \frac{3.5d^8u^6\dot{u}}{8.6.8\sqrt{1-u^2}} \text{ \&c.}$$

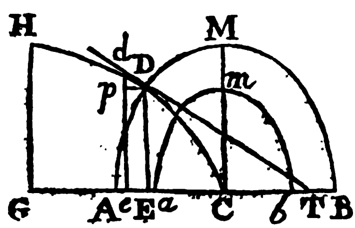
Now the fluent of the first term hereof, $\frac{a}{d}$ into $\frac{-\dot{u}}{u^2\sqrt{1-u^2}}$ $(= \frac{x\dot{x}}{d\sqrt{x^2 - a^2}})$ is universally expressed by $\frac{\sqrt{x^2 - a^2}}{d}$, or its equal $\frac{BF \times CE}{AE}$: which, if BN be parallel to the asymptote EC, will (because AE :

CE :: BF : BN) be also truly represented by BN: and this line BN, when x or z becomes infinite, will coincide with the asymptote. Therefore the fluent of the remaining terms is the difference sought; which fluent, when $u = 1$, or $y = 0$ (putting A for $\frac{1}{4}$ of the periphery of the circle whose radius is unity) will be $= aA \times \left(\frac{d}{2} + \frac{d^3}{2.2.4} + \frac{3.3d^5}{2.2.4.4.6} + \frac{3.3.5.5d^7}{2.2.4.4.6.6.8} + \frac{3.3.5.5.7.7d^9}{2.2.4.4.6.6.8.8.10} \text{ \&c.}\right)$ (by Art. 286) but $= 0$ when $u = 0$ (or y is infinite). Therefore the excess of the asymptote above the curve is truly exhibited by the preceding series. *Q. E. I.*

If a be taken $= 1$, and $b = 0$, then d will become $= 1$: and therefore, the curve in this case falling into its axis A G, we have $A \times \left(\frac{1}{2} + \frac{1}{2.2.4} + \frac{3.3}{2.2.4.4.6} + \frac{3.3.5.5}{2.2.4.4.6.6.8} \text{ \&c.}\right) = CA$, or unity. Whence it appears that the sum of the series $\frac{1}{2} + \frac{1}{2.2.4} + \frac{3.3}{2.2.4.4.6}$ is the reciprocal of $\frac{1}{4}$ of the periphery of the circle whose radius is unity. And, from the Problem preceding the last, it will likewise appear, that the sum of the series $1 - \frac{1}{2.2} - \frac{3}{2.2.4.4} - \frac{3.3.5}{2.2.4.4.6.6}$ &c. will be denoted by the same quantity; and consequently that these two series are equal to each other. From the addition and subtraction of which and their multiples, various other series may be produced, whose sums are explicable by means of the periphery of a circle.

PROBLEM XII.

436. *To determine the Nature of the Curve* C D H, *which will intersect any Number of similar and concentric Ellipsis* A M B *a m b &c. at Right-Angles.*



Let the tangent D T, which is a normal to the ellipsis A M B, meet the axis A B in T; and, supposing A C, C M, aC, Cm &c. to be the principal semi-diameters of their respective ellipsis, let the given ratio of A C^2 to C M^2 (or of aC^2 to Cm^2 &c.) be that of 1 to n: putting C E $=x$, E D $=y$, Dp (Ee) $=\dot{x}$, and $dp=\dot{y}$.

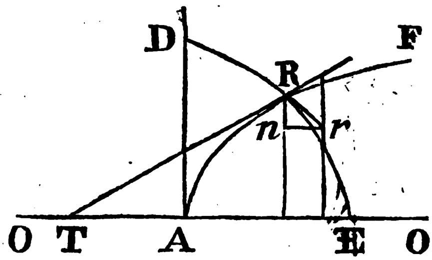
It is a known property of the ellipsis, that A C^2 : CM^2 :: CE : E T; therefore E T $=nx$: moreover E T (nx) : Dp $(\dot{x})$ :: ED (y) : pd $(\dot{y})$ by similar triangles, whence $\frac{\dot{x}}{nx}=\frac{\dot{y}}{y}$, or $\frac{\dot{x}}{x}=\frac{n\dot{y}}{y}$; whereof the fluent is L : $x-L$: $a=n$L : $y-n$L : a* (where a denotes any constant quantity at pleasure). Hence we also have L : $\frac{x}{a}=n\times$ L : $\frac{y}{a}=$ L : $\frac{y^n}{a^n}$, and consequently $\frac{x}{a}=\frac{y^n}{a^n}$, or $a^{n-1}x=y^n$.

* Art. 126.

PROBLEM XIII.

437. *To find the Equation of a Curve* E R D *that will cut any Number of Ellipses, or Hyperbolas, having the same Center* O *and Vertex* A, *at Right-Angles.*

Let R T be a tangent to any one of the proposed conic sections A R F, at the intersection R, meeting



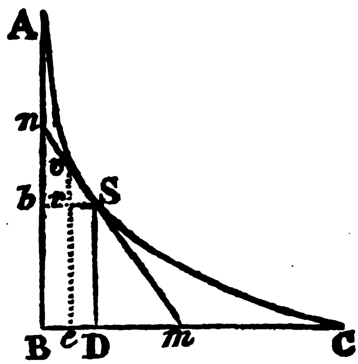
the axis AO in T; and put $AO = a$, $OB = x$, $BR = y$, $nr = \dot{x}$, $Rn = -\dot{y}$: then (*per Conics*) $BT = \frac{a^2 - x^2}{x}$, in the ellipsis, and $= \frac{x^2 - a^2}{x}$, in the hyperbola: whence, by reason of the similar triangles TBR, and R*rn*, it will be $\frac{a^2 \backsim x^2}{x}$ (BT) : y (BR) :: $-\dot{y}$ (R*n*) : $\mp \dot{x}$ (*rn*): therefore $+ y\dot{y} = \frac{a^2\dot{x} - x^2\dot{x}}{x} = \frac{a^2\dot{x}}{x} - x\dot{x}$, and consequently $+ \frac{y^2}{2} + d^2 = a^2 \times \mathrm{L} : \frac{x}{a} - \frac{1}{2}x^2$. Where d denotes a constant quantity, depending on the given value of A E.

PROBLEM XIV.

438. *Let two Points n and m move, at the same time, from two given Positions* B *and* C, *with equal Celerities, along two Right-Lines* BA *and* BC *perpendicular to each other: it is proposed to determine the Curve* ASC, *to which a Right-Line joining the said Points shall always be a Tangent.*

Let D S and *rv* be parallel to BA, and S*rb* perpendicular thereto: putting $BC = a$, $CD = x$, $SD = y$, $Sr = \dot{x}$, and $rv = \dot{y}$. Therefore (*by sim. Triangles*) $\dot{y} : \dot{x}$

$:: y : \frac{y\dot{x}}{\dot{y}} = \mathrm{D}m$, and $\dot{x} : \dot{y} :: a - x$ (Sb) : $\frac{\overline{a-x} \times \dot{y}}{\dot{x}} =$



bn: whence Cm (CD$-$Dm) $= x - \frac{y\dot{x}}{\dot{y}}$, and B$n$ (Bb $+ bn) = y + \frac{\overline{a-x} \times \dot{y}}{\dot{x}}$: which two last values, because the velocities of the bodies are equal, must also be equal to each other, that is, $x - \frac{y\dot{x}}{\dot{y}} = y + \frac{\overline{a-x} \times \dot{y}}{\dot{x}}$: hence, by making $\dot{x}$ constant, and taking the fluxion of the whole equation, we get $\dot{x} - \frac{\dot{x}\dot{y}^2 - y\dot{x}\ddot{y}}{\dot{y}^2} = \dot{y} - \frac{\dot{x}\dot{y} - \overline{a-x} \times \ddot{y}}{\dot{x}}$; or $\frac{\overline{a-x} \times \ddot{y}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}^2}$; from which there arises $\overline{a-x} \times \dot{y}^2 = y\dot{x}^2$, and $\frac{\dot{y}}{\sqrt{y}} = \frac{\dot{x}}{\sqrt{a-x}}$: where, the fluent on both sides being taken, we have $2\sqrt{y} = 2\sqrt{a} - 2\sqrt{a-x}$, and consequently $x = 2\sqrt{ay} - y$: which equation pertains to the common parabola.

Otherwise more universally, thus.

439. Put $Cm=v$ and $Bn=w$, and let these quantities (instead of being equal) have any given relation to each other. Then, since the absolute celerity of m is expressed by $\dot{v}$, its angular celerity, in a direction perpendicular to Sm, by which the line Sm tends to revolve about the point of contact S as a center, will be truly defined by $\frac{\text{sine of } Bmn}{\text{radius}} \times \dot{v}$ (Art. 35). In the same manner the angular celerity of n, about the point S, will be defined by $\frac{\sin. Bnm}{\text{rad.}} \times \dot{w}$. Now, as these celerities must be to each other as the distances Sm and Sn from the center S (or directly as the radii) we have $Sm : Sn$ $(:: DS : bn) :: \sin. Bmn \times \dot{v} : \sin. Bnm \times \dot{w}$; whence, because $\sin. Bmn : \sin. Bnm :: Bn\ (w) : Bm\ (a-v)$ we also have $DS : bn :: w \times \dot{v} : \overline{a-v} \times \dot{w}$: therefore, by composition, $DS : (DS + bn)\ w :: w\dot{v} : w\dot{v} + \overline{a-v} \times \dot{w}$, and consequently $DS = \frac{w^2\dot{v}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: whence $bn\ (w - SD) = \frac{\overline{a-v} \times w\dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$; and $BD \left(= Sb = \frac{bn \times Bm}{Bn}\right) = \frac{\overline{a-v}|^2 \times \dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: from whence the curve itself will be given.

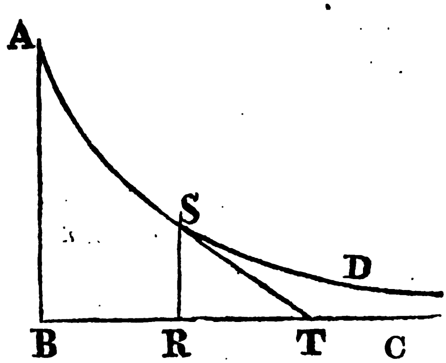
If v and w be taken equal to each other (as above) then SD (y) will become $= \frac{w^2}{a}$, and $BD = \frac{\overline{a-w}|^2}{a} = a - 2w + \frac{w^2}{a}$; in which last, if for w its equal

$\sqrt{ay}$ be substituted, we shall have B D $= a - 2\sqrt{ay} + y$; and consequently C D $(a - \text{B D}) = 2\sqrt{ay} - y$, the very same as before.

PROBLEM XV.

440. *Supposing a Body* T *to proceed, uniformly, along a Right-line* B C, *and another Body* S, *in pursuit of the same, always directly towards it, with a Celerity which is to that of* T, *in any given Ratio, of* 1 *to* n; *it is proposed to find the Equation of the Curve* A S D *described by the latter.*

Let the tangent A B, which makes right-angles with B C, be put $= a$, B R $= x$, R S $= y$, and A S $= z$:



then the sub-tangent R T being $= \frac{y\dot{x}}{-\dot{y}}$, we have BT $= x + \frac{y\dot{x}}{-\dot{y}}$: moreover, since the distances B T and A S gone over in the same time, are as the celerities n and 1, we also have B T $(= n \times \text{A S}) = nz = x + \frac{y\dot{x}}{-\dot{y}}$: whence, in fluxions (making $\dot{y}$ constant) $\frac{-y\ddot{x}}{\dot{y}} = n\dot{z}$; and consequently $\frac{-n\dot{y}}{y}$ $\left(= \frac{\ddot{x}}{\dot{z}} = \frac{\ddot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}}\right.$

the fluent of which (by Art. 126) is $-n \times \log. y = \log. \frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$: but when $y = a$, $\dot{x}$ is $= 0$; and then the equation becomes $-n \times \log. a = 0$; therefore the fluent, duly corrected, is $n \times \log. a - n \times \log. y = \log. \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$, or $\log. \frac{a^n}{y^n} = \log. \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$. Whence it is evident that $\frac{a^n}{y^n} = \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$, and $\frac{a^n\dot{y}}{y^n} - \dot{x} = \sqrt{\dot{y}^2 + \dot{x}^2}$; from which, by squaring both sides, $2\dot{x}$ is found $= \frac{a^n\dot{y}}{y^n} - \frac{y^n\dot{y}}{a^n}$; whose fluent is $2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1}$. But when $y = a$, x is $= 0$, and then, $0 = -\frac{a}{1-n} + \frac{a}{n+1} = -\frac{2na}{1-n^2}$; therefore the fluent corrected is $2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1} + \frac{2na}{1-n^2}$.

Q. E. I.

Otherwise (without second Fluxions).

441. Put $ST = P$ and $RT = Q$. Then since the absolute velocity of the body S is denoted by unity, that with which the ordinate S R is carried towards the body T will be denoted by $\frac{Q}{P} \times 1$ or $\frac{Q}{P}$ (by Art. 35) which subtracted from n the velocity of T, leaves $n - \frac{Q}{P}$ for the relative celerity with which T recedes from

it, after the same manner, if from $\frac{Q}{P}\times n$ the celerity of T in the direction ST produced, there be taken (1) the celerity of S in the same direction, the remainder, $\frac{nQ}{P}-1$ will be the celerity with which T recedes from S: therefore, the fluxions of quantities being as the celerities of their increase, we have $n-\frac{Q}{P}:\frac{nQ}{P}-1::\dot{Q}:\dot{P}$: and consequently $\overline{nQ-P}\times\dot{Q}=\overline{nP-Q}\times\dot{P}$. But, since the quantities P and Q are concerned exactly alike, the equation thus derived will, in all probability, become more simple, by substituting for their sum and difference: let therefore $P+Q=s$, and $P-Q=v$, or, which is the same, let $P=\frac{s+v}{2}$, and $Q=\frac{s-v}{2}$: then, by substitution, we shall have $\frac{ns-nv-s-v}{2}\times\frac{\dot{s}-\dot{v}}{2}=\frac{ns+nv-s+v}{2}\times\frac{\dot{s}+\dot{v}}{2}$; which contracted, &c. becomes $\overline{1+n}\times v\dot{s}=\overline{1-n}\times s\dot{v}$, or $\overline{1+n}\times\frac{\dot{s}}{s}=\overline{1-n}\times\frac{\dot{v}}{v}$; whose fluent (corrected) is $\overline{1+n}\times\log.\ s=\overline{1-n}\times\log.\ v+2n\times\log.\ a$, or $\log.\ s^{1+n}=\log.\ a^{2n}v^{1-n}$. Whence $s^{1+n}=a^{2n}v^{1-n}$, and consequently $s^{1+n}\times v^{1+n}=a^{2n}v^{2}$: but sv $(=\overline{ST+RT}\times\overline{ST-RT}=RS^{2})=y^{2}$; therefore $s^{1+n}\times v^{1+n}=y^{2n+2}=a^{2n}v^{2}$, and $v=\frac{y^{n+1}}{a^{n}}$; whence s $(=\frac{y^{2}}{v})$ $=\frac{a^{n}}{y^{n-1}}$ [illegible] ST $(=\frac{s+v}{2})$ $\frac{a^{n}}{2y^{n-1}}+\frac{y^{n+1}}{2a^{n}}$, RT $(=\frac{s-v}{2})$

$= \frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n}$. But RS (y) : RT $\left(\frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n}\right)$ $:: \dot{y} : \dot{x}$; whence $2\dot{x} = \frac{a^n \dot{y}}{y^n} - \frac{y^n \dot{y}}{a^n}$, and $2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1} + \frac{2na}{1-n^2}$, the very same as before.

COROLLARY.

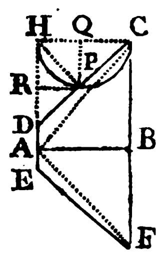
442. If the velocity of S be greater than that of T (or n be less than unity) the two bodies will concur when the latter has moved over a distanee expressed by $\frac{na}{1-n^2}$; because, when y becomes $= 0$, $2x$ is barely $= \frac{2na}{1-n^2}$. But if the velocity of S be less than that of T, it is plain that S can never come up with T: but its nearest approach will be when $y = \overline{\frac{n-1}{n+1}}\Big|^{\frac{1}{2n}} \times a$: for, since ST is universally $= \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}$, let the fluxion of this expression be taken and put equal to nothing; and y will be found as above exhibited.

If the celerities of S and T, instead of being uniform, vary according to a given law; then, denoting the former by A and the latter by B, the equation of the curve will be $\frac{\dot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}} = -\frac{B\dot{y}}{Ay}$: and if the fluent of $-\frac{B\dot{y}}{Ay}$ be explicable by a logarithm, as L. N; then, the fluent of $\frac{\dot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}}$ being L. $\frac{\dot{y} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$, ** Art. 126.

we shall have $N = \frac{\dot{y} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$; which ordered, gives $\dot{x} = \frac{N\dot{y}}{2} - \frac{\dot{y}}{2N}$: whence x will be found.

PROBLEM XVI.

443. *To determine the Frustum* CDEF *of a Triangular-Prism, of a given Base* C F *and Altitude* B A; *which, moving in a Medium, in the Direction of its Length* B A, *shall be resisted the least possible.*



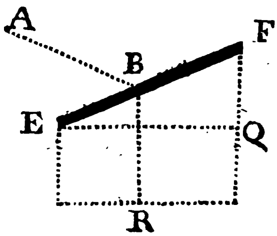
Draw C H parallel to B A meeting E D, produced, in H; moreover, let H P, P Q and P R be perpendicular to C D, C H and D H respectively.

Since the number of resisting particles acting upon D C is as D H, and the force of each as $\left(\frac{DR^2}{DP^2}\right)$ the square of the sine of the angle of incidence D P R, the whole resistance sustained by DC will therefore be expressed by $\frac{DH \times DR^2}{DP^2}$, or DR, which is equal to it (by the similarity of the triangles DHP and DPR). Whence the resistance upon ADC is truly expressed by AR (AD + DR) and is a *minimum* when its defect (PQ) below the given quantity AH (or BC) is a *maximum*: but PQ is a *maximum* when CQ and and HQ are equal; because, the angle CPH being right, a semi-circle described upon CH will always pass through the point P; and it is well known that the greatest ordinate in a semi-circle is that which divides the diameter into two equal parts.

Hence the angle D C H, when the resistance upon ADC is a *minimum*, will be just the half of a right-angle, provided BC be given greater than BA; otherwise, the whole prism C A F will be less resisted than any frustum C D E F of a greater prism.

PROBLEM XVII.

444. *To determine the Angle* R B E *which a Plane* E B F *must make with the Wind blowing in a given Direction* R B, *so that the Plane itself may be urged in another given Direction* B A *with the greatest Force possible.*



It is known, from the resolution of forces, that the force whereby the plane EF is urged in the given direction B A by a particle of air, acting in the direction R B, is directly as the rectangle of the sines of the angles (A B E, R B E) which the two given directions make with the plane: therefore, since the number of particles acting on E F is as the sine of R B E, it follows that the whole force, or effect, of the wind, in the direction B A, will be as $S.\ ABE \times (S.\ RBE)^2$; which being a *maximum*, we have (by Prob. 5) 3 : 1 :: sine of the whole given angle RBA : sine of $\overline{RBE - ABE}$. Whence the angles RBE and ABE are both given. *Q. E. I.*

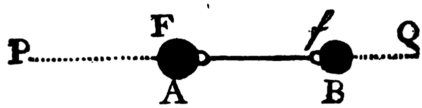
Corollary.

445. If the angle R B A be a right one (which is the case with regard to the sails of a windmill) then the sine of $\overline{RBE - ABE}$ being $= \frac{1}{3} =$,333 &c., we shall have $RBE - ABE = 19° : 28'$; and consequently $RBE \left(\frac{RBA + ABE}{2}\right) = 54° : 44'$.

PROBLEM XVIII.

446. *If two Bodies* A *and* B, *joined by a String, be urged in opposite Directions, towards* P *and* Q, *by any given Forces* F *and* f, *uniformly continued; it is proposed to find the Tension of the String, or the Force whereby the Bodies endeavour to recede from each other.*

Since $F - f$ is the absolute force by which the two bodies are, constantly, urged towards P, the *whole* motion, generated in both, in any time T, will therefore be expressed by $\overline{F-f} \times T$: whence, because both bodies (by reason of the string) acquire the same velocity, the motion generated in A, alone, will be $\frac{A}{A+B} \times \overline{F-f} \times T$, or that part of the *whole* defined by $\frac{A}{A+B}$. But the motion of A, had it not been retarded by the string (or B) would have been $F \times T$; therefore the loss of motion, by the action



upon the string, is $F \times T - \frac{A}{A+B} \times \overline{F-f} \times T$, $= \frac{fA+FB}{A+B} \times T$: which, divided by the time T (wherein that loss or effect is produced) gives $\frac{fA+FB}{A+B}$, for the tension of the thread, or the force sufficient to cause the said loss of motion.

The same otherwise.

447. Because the force F, was it to act alone, would communicate, by means of the string, the same velocity to B as to A; the part therefore of the force F employed upon B, by which the string is stretched, will be $\frac{B}{A+B} \times F$, or $\frac{BF}{A+B}$: and, from the very same argument, if the force f was to act alone, the tension of the thread would be $\frac{fA}{A+B}$: therefore, when both

the forces act together, the tension will be $\frac{fA+BF}{A+B}$: For it is very plain that their acting both at the same time, no way influences their respective effects on the thread. *Q. E. I.*

COROLLARY.

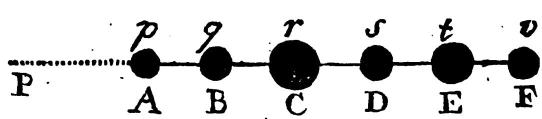
448. If the forces F and f be respectively expounded by the masses, or weights, of the bodies A and B; the tension of the thread will then become $\frac{2AB}{A+B}$. Whence it appears that the tension of a thread sliding over a pin or pulley, by means of two unequal weights A and B, suspended at the ends thereof, is equal to $\frac{2AB}{A+B}$: the double whereof, or $\frac{4AB}{A+B}$, is the weight which the pin or pulley sustains, while the bodies are in motion; because the thread hangs double, or on both sides the pulley.

If several bodies A, B, C, D, &c. communicating by means of a string or wire A F, be urged towards a point P, in the direction of the string or wire, by any given forces, p, q, r, s, &c. respectively, the tension of the part A B will be

$$= \frac{p \times \overline{B + C + D \text{ \&c.}} - A \times \overline{q + r + s \text{ \&c.}}}{A + B + C + D \text{ \&c.}};$$

of the part B C

$$= \frac{\overline{p+q} \times \overline{C+D+E \text{ \&c.}} - \overline{A+B} \times \overline{r+s+t \text{ \&c.}}}{A + B + C + D \text{ \&c.}};$$



of C D

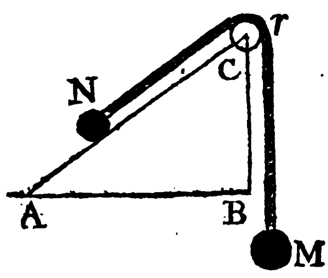
$$= \frac{\overline{p+q+r} \times \overline{D+E+F} - \overline{A+B+C} \times \overline{s+t+v}}{A+B+C+D \text{ \&c.}};$$

&c. &c.

All which easily follows from above; and will answer also in those cases where some of the forces are supposed to act in the contrary direction, if every such force be considered as a negative quantity.

PROBLEM XIX.

449. *Let it be required to raise a given Weight* N, *to a given Height* B C, *along an inclined Plane* A C, *by means of another given Weight* M, *connected to the former by a flexible Rope* N r M, *moving over a Pulley at* C; *to find the tension of the Rope; also the Inclination and Length of the Plane, so that the Time of the whole Ascent may be the least possible.*



It is well known that the force by which N tends to descend along the plane A C, or acts in opposition to M (supposing BC $= a$, and AC $= x$) will be $\frac{aN}{x}$, therefore $M - \frac{aN}{x}$, or $\frac{xM - aN}{x}$ is the efficacious force, by which the bodies are accelerated: but it is likewise demonstrable that the time of describing any line by means of a velocity uniformly accelerated, is in the subduplicate ratio of the length thereof, directly, and the subduplicate ratio of the accelerating force, inversely:* whence it follows that the time of describing A C will be represented by $\frac{x}{\sqrt{xM - aN}}$: whose fluxion (or that of its square) being made equal to nothing, x will be found $= \frac{2aN}{M}$, or $M : 2N :: a : x$. Hence the

* Art. 203.

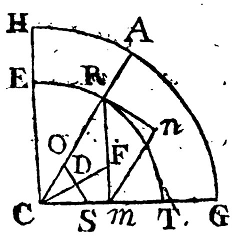
time of the ascent will be the least possible, when the sine of the plane's inclination is to the radius, as the power (M) is to twice the weight (N) to be raised.

The tension of the rope will be determined from the last Problem, (by writing N for A, $\frac{aN}{x}$ for F, M for B, and M for f) and comes out $= \frac{MN}{M+N} \times \frac{a+x}{x}$.

Q. E. I.

PROBLEM XX.

450. *Let* A C *represent a piece of Timber, moveable about a Center* C, *making any Angle* A C G *with the Plane of the Horizon* CG; *to determine the Position of a Prop or Supporter* O S, *of a given Length, which shall sustain it with the greatest Facility, in any given Position; and also what Inclination* AC *will have to the Horizon when the least Force that can sustain it, is greater than the least Force in any other Position.*



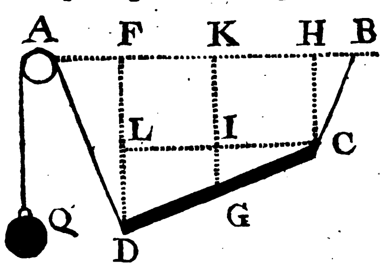
Let R be the center of gravity of the beam A C, and let Rn, Rm, and CD, be perpendicular to A C, C G, and O S respectively: putting SO $= a$, CR $= r$, C$m = x$, and the weight of the beam $= w$.

Then, *by the Principles of Mechanics*, we shall have, first, as Rm : Rn, or as $r : x :: w : \left(\frac{xw}{r}\right)$ the force, which acting at R, in the direction nR, is sufficient to sustain the beam AC; secondly, as C O : C R (r) :: $\frac{xw}{r}$ (the quantity last found) : $\frac{xw}{\text{CO}}$, the force able to support it at O, in a perpendicular direction; and lastly, as C D : C O :: $\frac{xw}{CO}$: $\frac{xw}{CD}$, the force or weight actually sustained by the given prop S O. Which force will therefore be the least possible when the perpendicular C D is the greatest possible, let the angle of inclination G C A be what it will: but of all triangles, having the same base (O S) and vertical angle (S C O) the isosceles one is known to have the greatest perpendicular: therefore the triangle C S O will be isosceles, and the angles S and O equal to each other, when the weight sustained by the prop O S is a *minimum*.

But now, to give a solution to the latter part of the Problem, or to find (supposing the angles S and O to be equal) when $\frac{x}{CD} \times w$ is a *maximum*, let C D produced meet mR in F; and then, because of the similar triangles C D S and Cm F, we shall have C D : x (Cm) :: S D ($\frac{1}{2}a$) : mF, or $\frac{x}{CD} = \frac{mF}{\frac{1}{2}a}$; and consequently $\frac{x}{CD} \times w = \frac{mF}{\frac{1}{2}a} \times w$: but, since C F bisects the angle m C R, we also have $r+x$ (C R + Cm) : x (Cm) :: $\sqrt{r^2 - x^2}$ (Rm) : Fm = $\frac{x\sqrt{r^2 - x^2}}{r+x}$ = $x\sqrt{\frac{r-x}{r+x}}$: whence the force $\frac{mF}{\frac{1}{2}a} \times w$, acting upon the supporter, is likewise truly expressed by $\frac{wx}{\frac{1}{2}a}\sqrt{\frac{r-x}{r+x}}$: whereof the fluxion being taken and put equal to nothing &c. we get $x = \frac{r\sqrt{5} - r}{2}$: therefore C R : C m ($::1 : \frac{\sqrt{5}-1}{2}$) :: radius : cosine of R C G ($=51^\circ : 50'$), the inclination required.

PROBLEM XXI.

451. *To determine the Position of a Beam* CD, *moveable about one End* C *as a Center, and sustained at the other End* D *by a given Weight* Q, *appended to a Cord* QAD *passing over a Pulley at a given Point* A.



Let G be the center of gravity of the beam; also let DF, GK and CH be perpendicular to the plane of the horizon, and CL and AH parallel to the same: putting $AH = a$, $CH = b$, $CD = c$, $CG = d$, $DL = x$, $CL = y$, and the weight of the beam $= w$; then $AF = a - y$, $DF = b + x$, and AD $(\sqrt{AF^2 + DF^2}) = \sqrt{a^2 - 2ay + y^2 + b^2 + 2bx + x^2}$; which (because $y^2 + x^2 = c^2$) will also be $= \sqrt{a^2 + b^2 + c^2 + 2bx - 2ay} = \sqrt{f^2 + 2bx - 2ay}$ (by putting $f^2 = a^2 + b^2 + c^2$) whose fluxion $\frac{b\dot{x} - a\dot{y}}{\sqrt{f^2 + 2bx - 2ay}}$, multiplied by Q, is the momentum of the weight Q, supposing the beam to be in motion. Moreover, because $DC : DL :: CG : GI$, we have $GI = \frac{dx}{c}$; whose fluxion, $\frac{d\dot{x}}{c}$, multiplied by w, is the momentum of the beam itself in a vertical direction.

Wherefore making these momenta equal to each other (according to the Principles of Mechanics) we get $\frac{b\dot{x} - a\dot{y}}{\sqrt{f^2 + 2bx - 2ay}} \times Q = \frac{d\dot{x}}{c} \times w$, and consequently $\overline{b\dot{x} - a\dot{y}} \times cQ = dw\dot{x}\sqrt{f^2 + 2bx - 2ay}$: but, since

$y^2 + x^2 = c^2$, we have $2y\dot{y} + 2x\dot{x} = 0$, or $-\dot{y} = \frac{x\dot{x}}{y}$: and therefore (by substitution) $\overline{b\dot{x} + \frac{ax\dot{x}}{y}} \times cQ = dw\dot{x}\sqrt{f^2 + 2bx - 2ay}$, or $\overline{by + ax} \times cQ = dyw \times \sqrt{f^2 + 2bx - 2ay}$: from whence, and the foregoing equation $x^2 + y^2 = c^2$, both x and y may be determined.

The same otherwise.

452. It is evident, from Mechanics, that the force which, acting in the direction D F, would sustain the end D, is to the whole weight w, as C G to C D; and therefore is $= \frac{CD}{CG} \times w$: it is likewise known that two forces acting in the different directions D F and D A, so as to have the same effect in sustaining DC, or causing it to move about the point C, must be to each other, inversely, as the sines of the angles of incidence FDC and ADC. Therefore we have $S.$ FDC : $S.$ ADC :: $Q : \frac{CD}{CG} \times w$; from which given ratio of the sines, the angles themselves will be found, by an algebraic process, independent of fluxions.

COROLLARY.

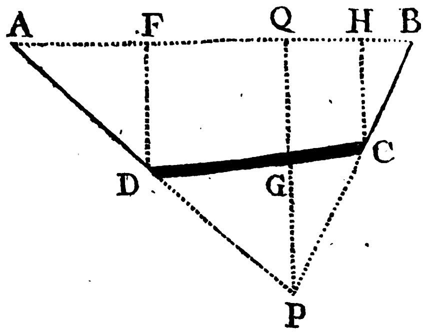
453. If the position of C D be supposed given, and the tension of A D (or the weight Q) be required: then, from the foregoing proportion, we shall have $Q = \frac{S.\,FDC}{S.\,ADC} \times \frac{CG}{CD} \times w$. Which will also express the tension of A D when the end C is sustained by a cord B C instead of a pin at C. whence it follows that the tensions of two cords A D and B C, sustaining a beam or rod C D, at its extremes D and C are expressed by $\frac{S.\,FDC}{S.\,ADC} \times \frac{CG}{CD} \times w$, and $\frac{S.\,HCD}{S.\,BCD} \times \frac{DG}{CD} \times w$; and

therefore are to each other as $\frac{CG}{S.ADC}$ to $\frac{DG}{S.BCD}$, or as $S.BCD \times CG$ to $S.ADC \times DG$ respectively; because the sine of F D C, and that of its supplement H C D are equal to each other.

PROBLEM XXII.

454. *To determine the Position of a Beam* DC, *suspended at its Extremes by two Cords* A D *and* B C *of given Lengths, from two given Points* A *and* B *in the same horizontal Line* A B.

Let G be the center of gravity of the beam, and let D F and C H be perpendicular to A B.



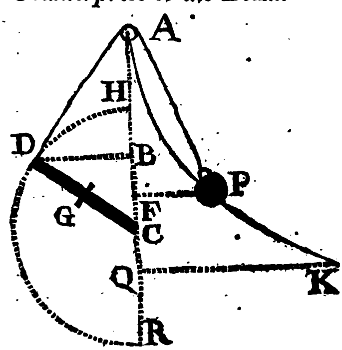
It appears, from the Corol. to the last Problem, that the tension of A D is to that of B C, as $\frac{CG}{S.ADC}$ to $\frac{DG}{S.BCD}$; whence (by the Resolution of Forces) the force of A D, in a direction parallel to the horizon, is to the force of BC, in the opposite direction, as $\frac{CG}{S.ADC} \times \frac{S.ADF}{rad.}$ to $\frac{DG}{S.BCD} \times \frac{S.BCH}{rad.}$. Which forces, that the beam may remain *in equilibrio*, must consequently be equal to each other; and therefore $\frac{S.\,BCD}{S.\,ADC}=\frac{S.\,BCH}{S.\,ADF}\times\frac{DG}{CG}$. But now, to determine the angles themselves, from this equation and the given lengths of A B, B C, &c. let A D and B C be produced to meet each other in P, and let P Q, perpendicular to A B, be drawn; putting $AB=a$, $AD=b$, $BC=c$, $DC=d$, $DG=f$, $CG=g$, $AP=x$, and $BP=y$.

Then, because $AB : AP+BP :: AP-BP : AQ-BQ = \frac{AP^2-BP^2}{AB}$, we have $AQ = \frac{1}{2}AB + \frac{AP^2-BP^2}{2AB} = \frac{AB^2+AP^2-BP^2}{2AB}$; and consequently the co-sine of A (=sine ADF) to the radius $1=\frac{AB^2+AP^2-BP^2}{2AB\times AP}$: whence, from the same argument, it is evident that the co-sine of B (= sine B C H) will be expressed by $\frac{AB^2+BP^2-AP^2}{2AB\times BP}$; and that of APB by $\frac{AP^2+BP^2-AB^2}{2AP\times BP}$ and also by $\frac{PD^2+PC^2-DC^2}{2PD\times PC}$; which two last quantities being equal to each other, we have $PD\times PC\times \overline{AP^2+BP^2-AB^2}=AP\times BP\times\overline{PD^2+PC^2-DC^2}$; that is $\overline{x-b}\times\overline{y-c}\times\overline{x^2+y^2-a^2}=xy\times\overline{\overline{x-b}|^2+\overline{y-c}|^2-d^2}$. Moreover, since $PC : PD :: S.ADC(\text{or } PDC) : S.BCD$ (or PCD) we also have $\frac{PD}{PC}=\frac{S.\,BCD}{S.\,ADC}=\frac{S.\,BCH}{S.\,ADF}\times\frac{DG}{CG}$ (by the first equation); whence $CG\times PD\times S.\,ADF=DG\times PC\times S.\,BCH$; that is $CG\times PD\times\frac{AB^2+AP^2-BP^2}{2AB\times AP}=DG\times PC\times\frac{AB^2+BP^2-AP^2}{2AB\times BP}$, or

$CG \times PD \times BP \times \overline{AB^2 + AP^2 - BP^2} = DG \times PC \times AP \times \overline{AB^2 + BP^2 - AP^2}$, which in algebraic terms, is $gy \times \overline{x-b} \times \overline{a^2 + x^2 - y^2} = fx \times \overline{y-c} \times \overline{a^2 + y^2 - x^2}$. From whence and the preceding equation the values of x and y will be known.

PROBLEM XXIII.

455. *Supposing a Beam* C D *moveable about one End* C, *as a Center, to be sustained at the other End* D *by means of a given Weight* P, *hanging at a Rope passing over a Pulley at a given Point* A, *vertical to* C; *it is proposed to find the Curve* A P K *along which the Weight must ascend, or descend, so as to be, every where, a just Counterprise to the Beam.*



From the center C, with the radius C D, let a semi-circle HDR be described, and let DB and PF be perpendicular to the vertical line AHCR; also let $CD = a$, $CA = b$, $AH = c$, $AF = x$, $PF = y$, $HB = z$, and the length of the rope $DAP = m$; likewise let HQ (h) be the given value of x (AF) when D coincides with H.

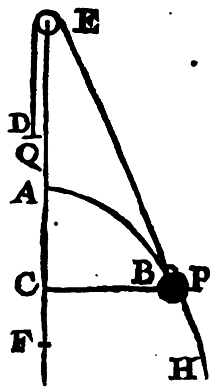
Because the weight and the beam are always *in equilibrio*, by hypothesis, their *momenta*, and consequently their velocities, in a vertical direction, must be every where in a constant ratio; and therefore the distance Q F $(h - x)$ ascended by the weight P, will be, to the distance H B descended by the end of the beam. D likewise in a constant ratio: let this ratio be that of b to any given quantity d, that is, let $h - x : z :: b : d$, and we shall have $dh - dx = bz$: moreover, we have AD^2 $(CD^2 + AC^2 - 2AC \times BC) = a^2 + b^2 - 2b \times \overline{a - z + b - a}^2 + 2bz = c^2 + 2bz = c^2 - 2dh + 2dx$: whence A P $(m - AD) = m -$

$\sqrt{c^2-2dh+2dx}$, and therefore, y^2 $(AP^2-AF^2)=$ $\overline{m-\sqrt{c^2-2dh+2dx}}|^2-x^2$. *Q. E. I.*

After the same manner a curve may be found, along which a weight descending, shall be every where in *equilibrio* with another weight ascending through the arch of a given curve.

PROBLEM XXIV.

456. *To find the Equation of a Curve* ABH, *along which a given Weight* P, *suspended by a String* PED *passing over a Pulley* E, *must descend, so that the Tension of the String may vary according to any given Law.*



Let E C be perpendicular, and C P parallel, to the plane of the horizon; also let $AE=a$, $AC=x$, $CB=y$, $EP=v$, and let the tension of the string (or the force acting at the end D) be denoted by any variable, or constant, quantity Q.

Therefore, because the celerity of the weight P, in a vertical direction, is to its celerity, in the direction E P produced, (or the celerity of the other end D) as $\dot{x}$ to $\dot{v}$, it is evident that the weight itself must be to the tending force Q, inversely in that ratio, and consequently $P\dot{x}=Q\dot{v}$.

Furthermore, because $EC=a+x$ and $BC^2=BE^2-EC^2$, we have $y^2=v^2-\overline{a+x}|^2$: from which equations, when the relation of P and Q is given, the curve itself will also be known.

Thus, for example, let the ratio of P to Q, be constant, or that of m to n, then $m\dot{x}$ being $=n\dot{v}$, we have (by taking the fluent) $mx+na=nv$; whence $v=a+\frac{mx}{n}$; and therefore y^2 $(=a^2+\frac{2max}{n}+\frac{m^2x^2}{n^2}$

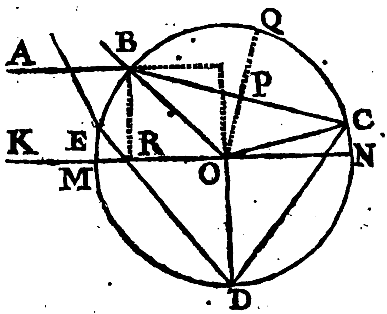
$- a^2 - 2ax - x^2) = \frac{m-n}{n} \times 2ax + \frac{m^2 - n^2}{n^2} \times x^2$; which is the equation of an hyperbola.

Again, for a second example, let the tending force Q be to the weight P, as DE^n to $AC^m \times c^{n-m}$, or as $\overline{b-v}|^n : x^m c^{n-m}$ (supposing $b =$ PED and $c =$ any given line AF). Therefore, since $Q = \frac{\overline{b-\dot{v}}|^n}{c^{n-m} x^m} \times P$, and $\frac{\overline{b-v}|^n}{c^{n-m} x^m} \times P\dot{v}$ $(= Q\dot{v}) = P\dot{x}$, we have $\overline{b-v}|^n \times \dot{v} = c^{n-m} x^m \dot{x}$, and so $\frac{\overline{b-a}|^{n+1} - \overline{b-v}|^{n+1}}{n+1} = \frac{c^{n-m} x^{m+1}}{m+1}$; whence $\overline{b-v}|^{n+1} = \overline{b-a}|^{n+1} - \frac{\overline{n+1} \times c^{n-m} x^{m+1}}{m+1}$, and v (EP) $= b - \overline{\overline{b-a}|^{n+1} - \frac{\overline{n+1} \times c^{n-m} x^{m+1}}{m+1}}|^{\frac{1}{n+1}}$. From which the relation of x and y, or the value of B C, is also known.

But if $m = 0$, and $n = 1$ (which will be the case when the force acting at D is equal to that by which a beam or rod is made to move about a center, as in the last Problem) v will then become, barely, $= b - \overline{\overline{b-a}|^2 - 2cx}|^{\frac{1}{2}}$, and therefore y^2 $(= v^2 - \overline{a+x}|^2)$ $= \overline{b - \sqrt{\overline{b-a}|^2 - 2cx}}|^2 - \overline{a+x}|^2$: therefore A B H is in this case, a line of the fourth order.

PROBLEM XXV.

457. *Supposing a Ray of Light* ABCD *to be refracted at the Surface of a given Sphere* MQND, *and afterwards reflected any given Number (n) of Times, within the Sphere; to determine the Distance of the Incident Ray* AB *from the Axis* MN, *so that the Arch* MBCDE, *intercepted by the given Point* M *and the emerging Ray at* E, *may be a* Minimum.



Let the radius $OB=1$, the sine of incidence $BR=x$ and the sine of refraction $OP=y$, and let the given ratio of the two last be that of p to q.

Since all the angles of incidence and reflexion BCO OCD, CDO &c. are equal, the arcs BC, CD and DE must also be equal; and consequently $MBCDE = MB + \overline{n+1} \times BC = MB + \overline{2n+2} \times BQ$: whose fluxion is to be equal to nothing.* Now the fluxion of the arch MB, whose sine is x and radius unity, will be $= \frac{\dot{x}}{\sqrt{1-x^2}}$;† and that of the arch BQ, whose co-sine (OP) is y, $= \frac{-\dot{y}}{\sqrt{1-y^2}}$. Hence we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times \dot{y}}{\sqrt{1-y^2}} = 0$: but, since $x : y :: p : q$, y is $= \frac{qx}{p}$ and $\dot{y} = \frac{q\dot{x}}{p}$; and so we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times q\dot{x}}{\sqrt{p^2 - q^2x^2}} = 0$; whence (putting

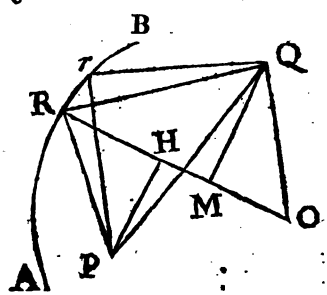
* Art. 22.

† Art. 142.

$m = 2n + 2$) x is found $= \frac{1}{q}\sqrt{\frac{m^2q^2-p^2}{m^2-1}}$: from which it is observable, that, when mq is less than p, or $2n+2$ less than $\frac{p}{q}$, the arch MBCD continually increases with B M; and therefore is the least possible, when B coincides with M. *Q. E. I.*

PROBLEM XXVI.

458. *If two Rays of Light* PR *and* Pr, *from a given Point* P, *making an indefinitely small Angle with each other, be reflected at a given Curve Surface* ARB; *it is proposed to determine the Concourse, or Focus,* Q *of the reflected Rays* R Q *and* r Q.



Let RO, perpendicular to the curve, be the radius of a circle having the same curvature with ARB at R; make PH and QM perpendicular to R O; join Q, O; and put RO $=r$, PR$=y$, RH$=v$, and R Q$=z$.

Then, because the angle of reflection ORQ is equal to the angle of incidence O R P, the triangles R Q M and RPH will be similar, and therefore $y : v :: z :$ RM $=\frac{vz}{y}$: whence OQ² (RO² + R Q² − 2RO × RM) $=r^2+z^2-\frac{2rvz}{y}$.

But, since this quantity O Q² continues the same (by hypothesis) whether we regard one ray or the other (that is, whether y stands for P R or P r) its fluxion must therefore be equal to nothing; that is $2z\dot{z} - \frac{2r\dot{v}zy + 2r\dot{v}\dot{z}y - 2rvz\dot{y}}{y^2} = 0$: whence $z = \frac{vy\dot{z}}{\frac{y^2\dot{z}}{r} + v\dot{y} - y\dot{v}}$: but (by Art. 35) $\dot{z} = -\dot{y}$; therefore $z = \frac{-vy\dot{y}}{-\frac{y^2\dot{y}}{r} + v\dot{y} - y\dot{v}}$: moreover (by Art. 73) $r = \frac{y\dot{y}}{\dot{v}}$; therefore

$$z = \frac{-vy\dot{y}}{-y\dot{v} + v\dot{y} - y\dot{v}} = \frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}. \qquad Q. E. I.$$

Example 1. Let ARB be an arch of the logarithmic spiral: whose equation is $av = by$:* and then, $\dot{v}$ being $= \frac{b\dot{y}}{a}$, we shall have $z \left(\frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}\right) = y$: therefore in this case the incident and reflected rays are equal to each other.

* Art. 74.

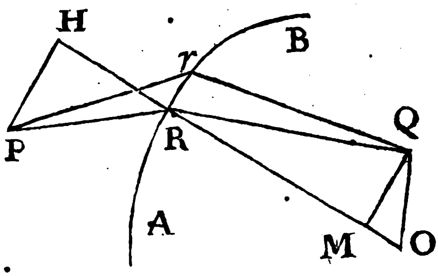
Ex. 2. Let ARB be supposed to degenerate into a right-line: in which case v being constant, its fluxion $\dot{v}$ is $= 0$; and therefore $z \left(= \frac{vy\dot{y}}{-v\dot{y}}\right) = -y$: which being negative, indicates that the rays do not converge after reflection, but, on the contrary, diverge from a point on the contrary side of ARB, at the distance y. Which is very easy to demonstrate by common geometry.

PROBLEM XXVII.

459. *Let two Rays of Light* PR *and* Pr, *from a given Point* P, *be refracted at a given Curve Surface* ARB; *to determine the Focus* Q *of the refracted Rays* RQ *and* rQ.

Let the lines RO, RH &c. be drawn, and denoted as in the preceding Problem: moreover, let the sine of incidence PRH (to the radius 1) be represented by s, and let it be to the sine of refraction ORQ, in the given ratio of 1 to n.

Then (*by Trigonometry*) 1 : ns (sine QRM) :: z (RQ) : QM $= nsz$; and therefore RM $= \sqrt{z^2-n^2s^2z^2}$



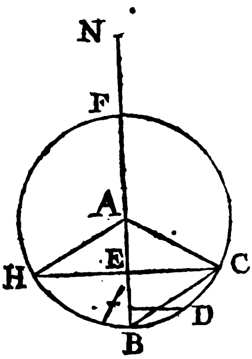
$= z\sqrt{1-n^2s^2}$. From whence, following the steps of the preceding Problem, we also get $OQ^2 = r^2+z^2 - 2rz\sqrt{1-n^2s^2}$; and its fluxion $2z\dot{z} - 2r\dot{z}\sqrt{1-n^2s^2} + \frac{2rzn^2s\dot{s}}{\sqrt{1-n^2s^2}} = 0$; or $z\dot{z}\sqrt{1-n^2s^2} - r\dot{z}\times\overline{1-n^2s^2} + n^2rzs\dot{s} = 0$. But (by Art. 35) $\dot{z} = -n\dot{y}$; therefore $-z\dot{y}\sqrt{1-n^2s^2} + r\dot{y}\times\overline{1-n^2s^2} + nrzs\dot{s} = 0$: moreover (by Trig.) 1 (radius) : s (sine of PRH) :: y (PR) : $\sqrt{y^2-v^2}$ (PH) whence we have $sy = \sqrt{y^2-v^2}$, $s^2 = 1-\frac{v^2}{y^2}$, and $s\dot{s} = \frac{-y^2v\dot{v}+v^2y\dot{y}}{y^4} = \frac{v^2\dot{y}-yv\dot{v}}{y^3}$; which values, of s^2 and $s\dot{s}$, being substituted in the foregoing equation, it becomes $-z\dot{y}\times\sqrt{1-n^2+\frac{n^2v^2}{y^2}} + r\dot{y}\times\overline{1-n^2+\frac{n^2v^2}{y^2}} + nrz\times\frac{v^2\dot{y}-yv\dot{v}}{y^3} = 0$, or $-zy^2\dot{y}\sqrt{\overline{1-n^2}\times y^2+n^2v^2} + ry\dot{y}\times\overline{\overline{1-n^2}\times y^2+n^2v^2} + nrz\times\overline{v^2\dot{y}-yv\dot{v}} = 0$; or (putting $\sqrt{\overline{1-n^2}\times y^2+n^2v^2} = w$) $\frac{-zy^2w\dot{y}}{r} + w^2y\dot{y} +$

$nzv^2\dot{y} - nyzv\dot{v} = 0$. But (by Art. 73) $r = \frac{y\dot{y}}{\dot{v}}$, therefore $- zyw\dot{v} + w^2y\dot{y} + nzv^2\dot{y} - nyzv\dot{v} = 0$, and consequently $z = \frac{w^2y\dot{y}}{\overline{wy + nvy} \times \dot{v} - nv^2\dot{y}}$ *Q. E. I.*

From this solution, that of the preceding Problem is easily derived: also from hence the caustic (or the curve which is the locus of all the points Q thus found) will likewise be given.

PROBLEM XXVIII.

460. *To find the Time of the Vibration of a Pendulum in the Arch of a Circle.*



Let AB denote the pendulum in a vertical position; and from any point D in the given arch CBH, wherein the vibrations are performed, draw Df parallel to CH; and let $AB = a$, $BE = c$, $Bf = x$, and $BD = z$: by the nature of the circle we have $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - x^2}}$: * whence the fluxion of the time, being as $\frac{\dot{z}}{\sqrt{Ef}}$, † will be defined by $\frac{a\dot{x}}{\sqrt{c - x} \times \sqrt{2ax - x^2}}$

$$= \frac{a\dot{x}}{\sqrt{cx - x^2} \times \sqrt{2a - x}} = \frac{\overline{\tfrac{1}{2}a}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - x^2}} \times \overline{1 - \frac{x}{2a}}|^{-\frac{1}{2}}$$

$$= \frac{\overline{\tfrac{1}{2}a}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - x^2}} \times \left(1 + \frac{x}{2 \cdot 2a} + \frac{3x^2}{2 \cdot 4 \cdot 4a^2} + \frac{3 \cdot 5x^3}{2 \cdot 4 \cdot 6 \cdot 8a^3}\right.$$

$$\left. + \frac{3 \cdot 5 \cdot 7x^4}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 16a^4} \text{ \&c.}\right)$$ Whereof the fluent,

* Art. 142.

† Art. 207.

when $x = c$, (or $\overline{cx - x^2}|^{\frac{1}{2}} = 0$) is, (by Art. 142 and 286) equal to $p\sqrt{\tfrac{1}{2}a} \times (1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot \overline{2a}|^2} + \frac{3 \cdot 3 \cdot 5 \cdot 5c^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot \overline{2a}|^3} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7c^4}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot \overline{2a}|^4}$ &c.) Which therefore is proportional to the time of half one vibration; where p stands for the semi-periphery of the circle whose radius is unity.

Corollary I.

461. Since the time of the perpendicular descent of a body through any given right-line u, computed according to the same method, is as the fluent of $\frac{\dot{u}}{\sqrt{u}}$ or $2\sqrt{u}$, it follows that the time of falling along the diameter B F ($2a$), or the chord C B,* will be truly defined by $2\sqrt{2a}$: which therefore is to the time of the descent through the arch CDB, as $\frac{4}{p}$ to $1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot \overline{2a}|^2}$ &c. From whence as the time of falling through the diameter B F, is absolutely given, by Art. 202, the true time of vibration will also be known.

* Art. 205.

Corollary II.

462. If the arch in which the pendulum vibrates be very small, the above proportion will become, nearly, as 4 to p: from which it appears, that *the time of descent through any very small arch* C B *is to that along the chord* C B, *as the periphery of any circle is to four times its diameter.*

Corollary III.

463. Hence, we have a method for determining how far a body freely descends in a given time; by knowing

the time of vibration of a given pendulum: for, if BN be assumed for the space through which a body would descend during the time of one whole vibration, in the very small arch C B H; then, the distances de-
* Art. 201. scended being as the squares of the * times we have, from the last Corollary, as $4^2 : \overline{2p}|^2 :: \text{BF} (2a) : \text{BN}$, or $1 : \frac{1}{4}p^2 : a : \text{BN}$; that is, *as the square of the diameter of a circle is to half the square of its periphery, so is the length of the pendulum, to the distance a body will freely descend, from rest, in the time of one oscillation.* Thus, for instance (because it is found from experiment that a pendulum 39,2 inches long vibrates seconds) it will be as $1 : 4{,}934\ (= \frac{1}{4}p^2) :: 39{,}2 : 193$ inches, the distance which a heavy body will fall in the first second of time.

Corollary IV.

464. Moreover, from the foregoing series, the time which a pendulum, vibrating in an exceeding small arch, will lose when made to vibrate in a greater arch of the same circle may also be deduced.

For let T be put to denote the number of seconds in 24 hours (or any other given time) then the number of vibrations performed in that time will be as

$$\frac{T}{1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot \overline{2a}|^2\ \&c.}}$$; which, therefore, in an exceeding small arch (where c may be taken as nothing) will be expressed by T: and so the time (t) or number of vibrations lost will be

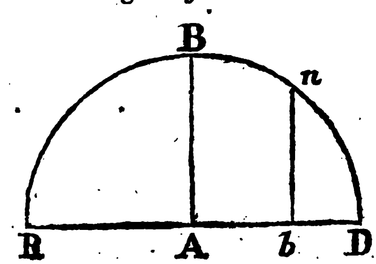
$$T - \frac{T}{1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2 .}{2 \cdot 2 \cdot 4 \cdot 4 \cdot \overline{2a}|^2\ \&c.}} = T \times$$

$$\overline{\frac{c}{8a} + \frac{5c^2}{256a^2}}$$ &c. (by dividing by the denominator). Now, if the number of degrees described on each side of the perpendicular be represented by D, the

arch itself, on each side, will be $= 3.14159$ &c. $\times a \times \frac{D}{180}$; which, if the value of D be not more than about 15 or 20 degrees, will be nearly equal to its chord, represented by $\sqrt{2ac}$ $(=\sqrt{\text{BF}\times\text{BE}})$. From which equation we get $\frac{c}{a} = \frac{D^2}{6560}$: this value, substituted above, gives $t = T \times \overline{\frac{D^2}{8\times 6560} + \frac{5D^4}{256\times \overline{6560}|^2}}$ &c. $= T \times \frac{D^2}{52480}$ nearly: which, when T is interpreted by 86400 seconds (or one whole day) becomes $= 1\frac{1}{2} \times D^2$, nearly: and so many are the seconds which will be lost *per diem* in the arch D. From whence we gather, that *if the pendulum measures true time in any small arch, whose degrees on each side the perpendicular are denoted by A, the number of seconds lost* per diem *in another arch whose degrees are B, will be nearly represented by* $\frac{3}{2} \times \overline{B^2 - A^2}$: thus, if a pendulum measures true time, in an arch of 3 degrees, it will lose $10\frac{1}{2}$ seconds a day in an arch of 4 degrees, and 24″ in an arch of 5 degrees.

PROBLEM XXIX.

465. *To determine the* Meridional Parts *answering to any proposed Latitude, according to* Wright's *Projection, applied to the true spheroidal Figure of the Earth.*



Let DAR be the axis, A B the semi-equatoreal diameter, and DBR a Meridian of the Earth; also let *bn* be an ordinate to the ellipsis DBR; putting AD(= AR)

$=1$, BA$=d$, A$b=x$, $bn=y$, B$n=z$, and the meridional distance (in parts of the semi-axis AD)$=u$.

Then, by the nature of the ellipsis, we have $y=d\times\sqrt{1-x^2}$; therefore $\dot{y}=\frac{-dx\dot{x}}{\sqrt{1-x^2}}$; and consequently $\dot{z}=\sqrt{\dot{x}^2+\frac{d^2x^2\dot{x}^2}{1-x^2}}$: which, by putting $b^2=d^2-1$, will be reduced to $\dot{z}=\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}$. Whence, by the nature of the Projection, it will be as bn $(d\sqrt{1-x^2})$: AB (d) :: $\dot{z}$ $\left(\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}\right)$: $\dot{u}=\frac{\dot{x}\sqrt{1+b^2x^2}}{1-x^2}$; which is the fluxion of the quantity required: but we are now to get the same thing expressed in terms of the Latitude of the Place n: in order thereto, putting the sine of that latitude$=s$, we have, by trigonometry, as $\dot{z}$ $\left(\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}\right)$: $-\dot{y}$ $\left(\frac{dx\dot{x}}{\sqrt{1-x^2}}\right)$ :: radius (1) : s; and consequently $s\sqrt{1+b^2x^2}=dx$; from which equation x is found$=\frac{s}{\sqrt{d^2-b^2s^2}}$: whence $\dot{x}=\frac{d^2\dot{s}}{\overline{d^2-b^2s^2}|^{\frac{3}{2}}}$; also $1-x^2=\frac{d^2-b^2s^2-s^2}{d^2-b^2s^2}=\frac{d^2-d^2s^2}{d^2-b^2s^2}$ (because $d^2=1+b^2$) and, lastly, $\sqrt{1+b^2x^2}\ (=\frac{dx}{s})=\frac{d}{\sqrt{d^2-b^2s^2}}$: which several values being substituted in that of $\dot{u}$, found above, it will become $\left(=\frac{d^2\dot{s}}{\overline{d^2-b^2s^2}|^{\frac{3}{2}}}\times\frac{d}{\sqrt{d^2-b^2s^2}}\times\frac{d^2-b^2s^2}{d^2\times\overline{1-s^2}}\right)=\frac{d\dot{s}}{\overline{d^2-b^2s^2}\times\overline{1-s^2}}$; which, resolved into two parts, for the more readily finding the fluent, gives $\dot{u} = \frac{\dot{s}}{1-s^2} - \frac{db^2\dot{s}}{d^2-b^2s^2}$: whereof the fluent being taken, we have

$$u = \begin{cases} 2.302585 \text{ \&c.} \times \frac{1}{2} d \times \log. \frac{1+s}{1-s} \\ -2.302585 \text{ \&c.} \times \frac{1}{2} b \times \log. \frac{d+bs}{d-bs} \end{cases}$$

But, as 3,14159 &c. × 2d (the measure of the whole periphery of the Earth at the Equator, in parts of the semi-axis A D) is to 21600 (the measure of the same periphery in geographical miles) so is the foresaid value of u to

$$\begin{cases} 3958 \times \log. \frac{1+s}{1-s} \\ -\frac{3958b}{d} \times \log. \frac{d+bs}{d-bs} \end{cases}$$ the corresponding value of u, in geographical miles, or the Meridional Parts required.

COROLLARY.

466. If the Earth be considered as differing but little from a sphere, d will be nearly $= 1$, and consequently ($\sqrt{d^2-1}$) the value of b, very small: therefore, in this case, the latter part of our fluent ($-\frac{3958b}{d} \times \log. \frac{d+bs}{d-bs}$) will become nearly $= 3440b^2s$ (because $\log. \frac{d+bs}{d-bs} = \frac{2bs}{d}) \times \frac{1}{2.3025 \text{ \&c.}}$.* But if the earth be taken as a perfect sphere, this last expression will vanish, and so the value of u will become barely $= 3958$

* There is a mistake in p. 43 and 44 of my Dissertations (by forgetting to divide by the Modulus 2.3025 &c.) which may from hence be rectified.

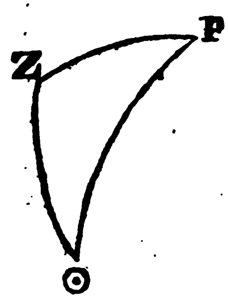
Lemma.

468. *In any Spherical Triangle, if radius be supposed Unity, the Product of the Sines of any two of the Sides drawn into the Co-sine of the Angle they include, added to the Product of their Co-sines, is equal to the Co-sine of the remaining side.*

This is demonstrated by the writers upon Spherics.

PROBLEM XXXI.

469. *The Elevation of the Pole and the Declination of the Sun being given, to find at what Time of the Day the Azimuth of the Sun increases the slowest.*



It is evident that the time sought will be when the fluxion of the hour-angle P, bears the greatest ratio possible to that of the azimuth Z.

Now the fluxion of the angle P is to that of Z, universally, as rad. $\times$ *S*. Z O : *S*. P O $\times$ *co-s*. O (by Art. 256, Case 2). Consequently $\frac{S.\,PO \times co\text{-}s.\,O}{rad. \times S.\,ZO}$, or $\frac{co\text{-}s.\,O}{S.\,ZO}$ is a *minimum*, in this case, because P O may be considered as constant.

Let now the sine of P O be put $= p$, its co-sine $= d$, the co-sine of P Z $= b$, that of Z O $= x$, and that of O $= y$; then, the sine of Z O being $= \sqrt{1-x^2}$, we have (*by the Lemma*) $p\sqrt{1-x^2} \times y + dx = b$; whence $y = \frac{b - dx}{p\sqrt{1-x^2}}$, and therefore $\frac{co\text{-}s.\,O}{S.\,ZO}\ (= \frac{y}{\sqrt{1-x^2}})$ $= \frac{b - dx}{p \times \overline{1 - x^2}}$: which put into fluxions, and reduced, gives $x = \dfrac{b - \sqrt{b^2 - d^2}}{d}$, for the sine of the sun's altitude at the time required: whence the time itself is given.

PROBLEM XXXII.

470. *To determine the Ratio of the Heat received from the Sun in different Latitudes, during the Time of one whole Day, or any Part thereof.*

Let p = the sine of the sun's polar-distance P☉ (*see the last Fig.*)

d = its co-sine, or the sine of the declination.

b = the sine of the pole's elevation.

c = its co-sine, or the sine of PZ.

z = the angle (P) expressing the time from noon.

x = its sine, and $\sqrt{1-x^2}$ = its cosine.

Then (by the foregoing Lemma) we shall have $pc\sqrt{1-x^2} + bd$ = co-sine Z☉ = sine of the sun's altitude.

Now, it is known that the number of rays falling in any given particle of time, upon a given horizontal plane, is as that time and the sine of the sun's altitude conjunctly: therefore the number of rays falling in the time $\dot{z}$, or $\dfrac{\dot{x}}{\sqrt{1-x^2}}$ (*vide* Art. 142) will be defined by $pc\dot{x} + bd\dot{z}$: whose fluent $pcx + bdz$ is, therefore, as the heat required.

Where it may be observed,

1. That when the latitude and declination are of different kinds, or P☉ is greater than 90 degrees, the value of d is to be considered as a negative quantity.

2. That, if the expression for the heat found above be divided by the square of the sun's distance from the earth, the quotient will exhibit the ratio of the heat, allowing for the excentricity of the earth's orbit.

Corollary I.

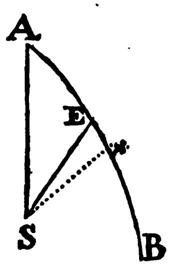
471. If the place proposed be at the equator, the heat, received in half one diurnal revolution, will be barely as p; because $b=0$, $c=1$, and $x=1$.

Corollary II.

472. But if the place be at the pole, then the heat will be as $d\times\overline{3.14159}$ &c. since, in this case, $c=0$, $b=1$, and z (=semi-circle)$=3.14159$ &c.

Lemma.

473. *The Number of Particles of light, ejected by the Sun, upon the Earth, in a given Time, is proportional to the Angle described about his Center in that Time.*

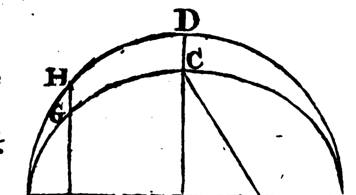


For, let S represent the center of the sun, AEB the orbit of the earth (or that of any other planet) and let E and r be two points therein as near as possible to each other: since the triangle E S r may be taken as rectilineal, its area, if the angle ESr be supposed given, or every where the same, will be as SE $\times$ Sr, or SE^2: and therefore the time of describing Er (being always as that area) is also explicable by SE^2: but the intensity of the light, or heat, at the distance of SE is as $\frac{1}{SE^2}$: therefore the intensity compounded with the time (or the whole number of particles received in that time) will consequently be as $\frac{1}{SE^2}\times SE^2$ $(=1)$ which being every where the same, the proposition is manifest.

PROBLEM XXXIII.

474. *To determine the Ratio of the Heat received from the Sun at the Equator and either of the Poles, during the Time of one whole Year, or any Part thereof.*

If the sine of the sun's declination be denoted by d and its co-sine by p, the heat received at the equator, and the pole, during half one diurnal revolution of the sun, will be as p and $d \times 3.14159$ &c. respectively *(by the Corollaries to the preceding Problem).*



Let the sun's longitude, considered as variable, be now denoted by z, and its sine by s; and let f be put for the sine of the obliquity of the ecliptic: then *(per Spherics)* we shall have $d = fs$, and consequently p $(= \sqrt{1-d^2}) = \sqrt{1-f^2s^2}$: wherefore, seeing the ratio of heat in the two places, for one half-day, is that of $\sqrt{1-f^2s^2}$ to $fs \times 3.14$ &c. let each of these terms be multiplied by $\frac{\dot{s}}{\sqrt{1-s^2}}$ $(=\dot{z})$ * expressing the quantity of heat falling upon the earth in the time of describing $\dot{z}$ (*see the foregoing Lemma*) then the products $\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-s^2}}$, and $3.14 f \times \frac{s\dot{s}}{\sqrt{1-s^2}}$ will be the fluxions of the required heat, answering to $\dot{z}$.

* Art. 142.

But now to exhibit the fluents hereof, let A C B be an ellipsis whose greater semi-axis A O is = unity, and its excentricity $FO = f$; and, supposing ADB to be a circle described about the ellipsis, let the arch D H express the sun's longitude from the equinoctial point; whose sine (OR) being $= s$, its co-sine R H will be $= \sqrt{1-s^2}$.

But, by the property of the ellipsis, O D (1) : O C : $(\sqrt{1-f^2})$ :: R H $(\sqrt{1-s^2})$: R G $= \sqrt{1-f^2} \times \sqrt{1-s^2}$: whose fluxion being =

$\frac{\sqrt{1-f^2} \times -s\dot{s}}{\sqrt{1-s^2}}$, we have $\sqrt{\dot{s}^2 + \frac{\overline{1-f^2} \times s^2 \dot{s}^2}{1-s^2}}$

$= \frac{\dot{s}\sqrt{1-f^2 s^2}}{\sqrt{1-s^2}}$ = the fluxion of C G. Whence it appears that the fluent of $\frac{\dot{s}\sqrt{1-f^2 s^2}}{\sqrt{1-s^2}}$ is truly defined by C G, or C G × A O².

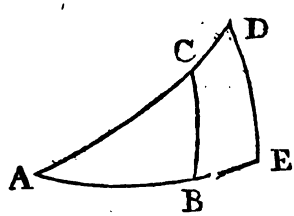
But the fluent of the other given fluxion, $3.14f \times \frac{s\dot{s}}{\sqrt{1-s^2}}$, will be $= 3.14f \times \overline{1 - \sqrt{1-s^2}}$ = A D B × F O × $\overline{\text{OD}-\text{RH}}$. Therefore the two fluents, when H and G coincide with A, will be to each other as C A × A O to A D B × F O: whereof the antecedent, multiplied by 4, will be as the heat received at the equator during one whole year; and the consequent, multiplied by 2, as the heat at the pole in the same time (because the sun shines at the pole only two quarters of the year). Hence the required ratio of the heat received at the equator and pole, in one whole year, will be that of C A × A O to D A × F O; or, in species, as $1 - \frac{f^2}{2.2} - \frac{3f^4}{2.2.4.4} - \frac{3.3.5f^6}{2.2.4.4.6.6}$ * &c. to f; which, in numbers, is as 959 to 396, or as 17 to 7, nearly.

* Art. 434.

PROBLEM XXXIV.

475. *To find when that Part of the Equation of Time, arising from the Obliquity of the Ecliptic to the Equinoctial, is a* Maximum.

In the right angled spherical triangle A B C let the angle A be that made by the ecliptic AC, and the equinoctial A B; then the problem will be, to find



when the difference between the base AB and the hypothenuse AC is the greatest possible (the angle A remaining invariable). Now, (by Art. 254) we have *co-s.* *BC* : *sin.* C :: fluxion of AC : fluxion of AB : also (*per Spherics*) *sin.* C : *co-s.* A :: *rad.* : *co-s.* BC $= \frac{co\text{-}s.\ A \times rad.}{sin.\ C}$: whence, by multiplying the two first terms of the former proportion by these equal quantities, respectively, we get this new proportion, *viz.* $\overline{co\text{-}s.\ BC}|^2$: *co-s.* A × *radius* :: so is the fluxion of AC to that of AB. But, when AC − AB is a *maximum*, these fluxions become equal; and consequently $\overline{co\text{-}s.\ BC}|^2 = co\text{-}s.\ A \times rad.$ From which equation BC, and from thence AC, will be known. *Q. E. I.*

The same, without Fluxions.

476. It will be (*per Spherics*) *rad.* *co-s.* A :: *tang.* AC : *tang.* AB; and therefore by composition and division, *rad.* + *co-s.* A : *rad.* − *co-s.* A :: *tang.* AC + *tang.* AB : *tang.* AC − *tang.* AB :: *sin.* $\overline{AC+AB}$: *sin.* $\overline{AC-AB}$, by the theorem mentioned in Problem 8th: from which, by following the steps there laid down, it appears that, radius + *co-s.* A : *radius* − *co-s.* A :: *radius* : sine of $\overline{AC-AB}$, when a *maximum*: whence (AC + AB being then = 90°) both AC and BC will be given.

COROLLARY.

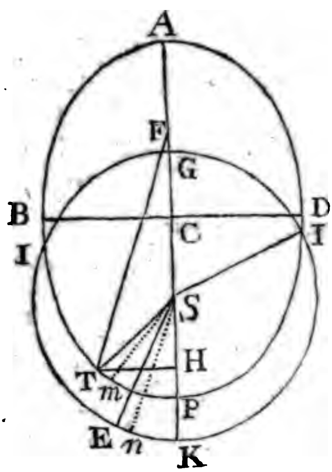
477. Since, *radius* + *co-s.* A : *radius* − *co-s.* A :: *co-tang.* ½ A : *tang.* ½ A * :: $\overline{radius}|^2$: $\overline{tang.\ \frac{1}{2} A}|^2$;

* Vide p. 70 and 71 of my Trigonometry.

therefore $\overline{radius}|^2 : \overline{tang. \frac{1}{2} A}|^2$ :: *radius* : *sine* of $\overline{AC - AB}$. Or, *radius* : *tang.* $\frac{1}{2}$ A :: *tang.* $\frac{1}{2}$ A : the *sine* of the greatest equation: which, supposing the angle A to be 23° 29′, comes out 2° . 28′ . 34″ . answering, in time, to 9 minutes 54 seconds.

PROBLEM XXXV.

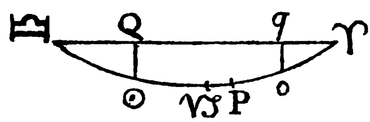
478. *To determine when the absolute Equation of Time, arising from the Inequality of the Sun's apparent Motion, and the Obliquity of the Ecliptic, conjunctly, is a* Maximum.



Let ABPD be the ellipsis in which the earth revolves about the sun, in the focus S; let F be the other focus, and T the place of the earth in its orbit at the time required. Moreover, about S, as a center, let a circle GEKI be described, whose diameter GK is a mean proportional between the two axes AP and BD of the ellipsis; so that the area thereof may be equal to that of the ellipsis: and, supposing S*m* to be indefinitely near to ST, let ES*n* be a sector of the said circle, equal to the area T S *m*.

Then, the time in which the earth moves through the arch T *m* being to the time of one intire revolution, as the area T S *m*, or E S *n*, is to the whole ellipsis, or the equal circle G E K F; and these areas ES*n*, and GEKI being in the ratio of the arch E *n* to the whole periphery GEKI; it is evident that E *n*, or the angles ESn, will express the increase of the *mean longitude*, in the foresaid time of describing the arch Tm: and that this angle or increase, by reason of the equality of the areas E S n and T S m, will be to the angle TSm, expressing the corresponding increase of the *true longitude*, as ST^2 to SE^2. Therefore, if the former be denoted by M, the latter will be represented by $\frac{SE^2}{ST^2} \times M$. But now to get a proper expression for the value of this increase of the *true longitude*, in algebraic terms; let F T be drawn, and also T H, perpendicular to AP: putting AC (=CP) = a, CB=b CS (=CF)=c, ST=z, and the co-sine of (T S P) the earth's distance from its *perihelion* (to the radius 1)=x: then F T being (= A P − S T) = $2a - z$ (by the property of the ellipsis) and SH=xz (*by Trig.*) we have $\overline{FT+ST} \times \overline{FT-ST}$ ($2a \times \overline{2a-2z}$)=FS $\times$ 2CH ($2c \times 2 \times \overline{c+xz}$) by a known property of triangles: from which equation z (S T) is found = $\frac{a^2-c^2}{a+cx} = \frac{b^2}{a+cx}$: and this value, with that of ES^2 (=ab) being substituted in the increase of the true longitude, found above, we thence get $\frac{a \times \overline{a+cx}|^2}{b^3} \times M$ for the measure of that increase; where M denotes the increment of the *mean motion* corresponding.

This being obtained, let ♎ ♑ ♈ (in the annexed figure) represent the southern semi-circle of the ecliptic, P the place of the perihelion, ♑ the tropic of capricorn, ☉ the apparent place of the sun in the ecliptic, and Q☉ his declination, at the time required: then it appears (from Art. 475) that the increase of the *true longitude* ♎ ☉, in an indefinitely small particle of time, will be to that of the *right-ascension* ♎ Q, in the same time, as the square of the co-sine of Q☉ is to a rectangle under the radius and the co-sine of the angle ♎: therefore, the former, being expressed by $\frac{a \times \overline{a+cx}|^2}{b^3} \times M$, the latter is truly represented by $\frac{a \times \overline{a+cx}|^2}{b^3} \times M \times \frac{rad. \times co\text{-}s. ♎}{\overline{co\text{-}s. ☉Q}|^2}$: which, in the required circumstance, when the proposed equation (or the difference between the sun's



mean motion and *right ascension*) is a *maximum*, must consequently be equal to (M) the corresponding increase of *mean motion*; and therefore $\frac{a \times \overline{a+cx}|^2}{b^3} = \frac{\overline{co\text{-}s. ☉Q}|^2}{rad. \times co\text{-}s. ♎}$.

But, to obtain the value of the latter part of this equation, also, in algebraic terms, let the sine and co-sine of (♑P) the distance of the *perihelion* from ♑, be denoted by m and n respectively; then, the co-sine of P☉ being (as above) expressed by x, and its sine by $\sqrt{1-x^2}$, we shall thence get $nx + m\sqrt{1-x^2}$ = co-sine of ☉ ♑ = sine of ♎ ☉ (*by the Elem. of Trig.*) But (putting the sine of the angle ♎ $= p$ and its co-sine $= q$) we have (*per Spherics*) radius (1) : sine ♎ ☉ $(nx + m\sqrt{1-x^2})$ :: p : $pnx + pm\sqrt{1-x^2}$ = sine of Q☉; from whence $\overline{co\text{-}s. Q☉}|^2 = 1 - \overline{pnx + pm\sqrt{1-x^2}}|^2$: which value, with that of the co-sine of the angle ♎, being substituted above, we, at length, get $\frac{a \times \overline{a+cx}|^2}{b^3} =$

$\frac{\overline{1-pnx+pm\sqrt{1-x^2}}|^2}{q}$; from which equation the value of x may be determined.

The foregoing equation, it may be observed, gives the time of the *maximum* which precedes the winter solstice; but if the *maximum* following that solstice be sought, it is but changing the sign of m, and then you will have $\frac{a\times\overline{a+cx}|^2}{b^3} = \frac{\overline{1-pnx-pm\sqrt{1-x^2}}|^2}{q}$, answering in this case. And from the negative roots of this and the preceding equation, the times of the other *maxima* after, and before, the summer solstice, will also be obtained. *Q. E. I.*

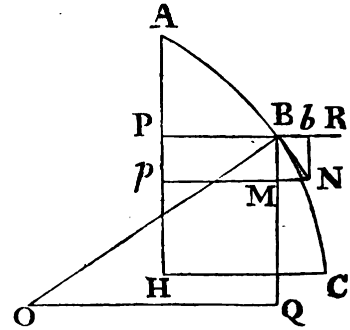
COROLLARY.

479. It is evident that the equation of the earth's orbit (or that part of the equation of time arising from the inequality of the sun's apparent motion) will be a *maximum*, when the center of the earth is in the intersection I of the ellipsis and the circle; where the *Mean Motion* and *True Longitude* increase with the same celerity.

PROBLEM XXXVI.

480. *To determine the Law of the Density of a Medium and the Curve described therein, by means of an uniform Gravity, so that the Projectile may, every where, move with the same Velocity.*

It appears, from Art. 367, that $\sqrt{\frac{\dot{y}^2}{\ddot{x}}}$ is a general expression for the celerity in the direction of the ordinate P B R; whence $\frac{\dot{z}}{\dot{y}}\times\sqrt{\frac{\dot{y}^2}{\ddot{x}}}$, or its equal, $\frac{\dot{z}}{\sqrt{\ddot{x}}}$, must be the true measure of the absolute celerity, in the direction B N: which being a constant quantity (by hypothesis) its square must also be con-



stant, and so we have $\frac{\dot{z}^2}{\ddot{x}} = a$; and consequently $\dot{x}^2 + \dot{y}^2 \, (= \dot{z}^2) = a\ddot{x}$.

But, in order to the solution of the equation thus given, make $u : 1 :: \dot{x} : \dot{y}$, or $\dot{x} = u\dot{y}$; then, $\ddot{x} = \dot{u}\dot{y}$, and, by substitution, $u^2\dot{y}^2 + \dot{y}^2 = a\dot{u}\dot{y}$: hence, $\dot{y}$ being $= \frac{a\dot{u}}{u^2+1}$, and $\dot{x} = \frac{au\dot{u}}{u^2+1}$, we get $y = a \times$ *arch*, whose

* Art. 142. tangent is u* (and secant $\sqrt{1+u^2}$); and $x = \frac{1}{2} a \times$

† Art. 126. hyp. log. $\overline{1 + u^2} = a \times$ hyp. log. $\sqrt{1 + u^2}$. †

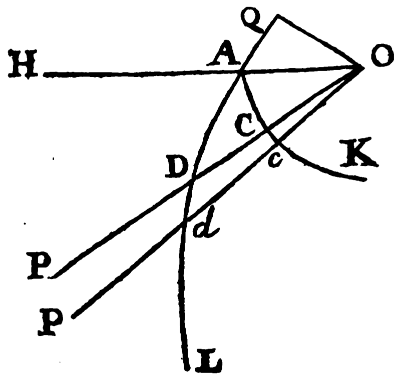
Therefore, as the hyp. log. of $\sqrt{1 + u^2}$ is $= \frac{x}{a}$, the common logarithm of $\sqrt{1 + u^2}$ will be $= \frac{0.4342944 \text{ \&c.} \times x}{a}$; and consequently $y = a \times$ *arch*, whose radius is unity, and log. secant $\frac{\overline{0.4342944 \text{ \&c.}} \times x}{a}$.

Moreover, with respect to the density of the medium; if the absolute force of gravity, in the direction Q B, be denoted by unity, its efficacy in the direction B N, whereby the body is accelerated, will be expressed by $\frac{\dot{x}}{\dot{z}}$, or its equal $\frac{u}{\sqrt{1+u^2}}$: which, as the velocity is supposed to remain every where the same, must also express the force of the resistance, in the opposite direction, or the true measure of the required density. This, therefore, if M be put for the absolute number whose hyperbolical logarithm is unity, may be had in terms of x, and will be $\overline{1 - \overline{M}|^{\frac{-2x}{a}}}|^{\frac{1}{2}}$: because hyp. log. $\overline{M}|^{\frac{x}{a}}$ $(= \frac{x}{a})$ being = hyp. log. $\sqrt{1+u^2}$, we have $\sqrt{1+u^2} = \overline{M}|^{\frac{x}{a}}$; whence $u = \overline{M^{\frac{2x}{a}} - 1}|^{\frac{1}{2}}$, and consequently $\frac{u}{\sqrt{1+u^2}} = \overline{1 - \overline{M}|^{\frac{-2x}{a}}}|^{\frac{1}{2}}$. *Q.E.I.*

PROBLEM XXXVII.

481. *Let a Line, or an inflexible Rod* OP *(considered without regard to Thickness) be supposed to revolve about one of its extremes O, as a Center, with a Motion regulated according to any given Law; whilst a Ring, or Ball, carried about with it, and tending to the Center* O *with any given Force, is suffered to move or slide freely along the said Line or Rod: it is proposed to determine the Velocity of the Ring, and its Pressure upon the Rod, in any proposed Position, together with the Nature of the Curve* A D L *described by means of that compound Motion.*

Let O D P be any position of the revolving line, and D the corresponding position of the body: moreover, supposing A C K to be the circumference of a circle described from the center O, through the given point A, let the measure of the angular celerity of that line, in the said circumference A C K, be repre-



sented by u; also let v denote the celerity of the ring at D in the direction DP; and w the true measure of the centripetal force: call O A, a; O D, x; and A C, z; and let the given values of u and v, at A, be denoted by b and c respectively. Then it will be, as a : x :: u : $\left(\frac{ux}{a}\right)$ the paracentric velocity of the body at D; whose square, divided by the distance O D, gives $\frac{u^2x}{a^2}$, for the true measure of the centrifugal force * arising from the revolution of the rod: from which the centripetal force w being deducted, the remainder, $\frac{xu^2}{a^2} - w$, is the true force whereby the velocity in the line O P is accelerated. Therefore (by Art. 218) we have $v\dot{v} = \overline{\frac{xu^2}{a^2} - w} \times \dot{x} = \frac{u^2x\dot{x}}{a^2} - w\dot{x}$.

* Art. 211.

Moreover, because the fluxion of the time is expressed either by $\frac{\dot{x}}{v}$, or by $\frac{\dot{z}}{u}$, these two values must, therefore, be equal to each other, and consequently $v = \frac{u\dot{x}}{\dot{z}}$: from which, and the preceding equation (when u and w are exhibited in terms of x or z) the required relation of v, x, and z will also become known. But now, in order to determine the action of the rod upon the ring, let O d P be indefinitely near to O D P, intersecting ADL and ACK in d and c; and put OD $= x + \dot{x}$. Then, because a body, acted on by no other force besides that tending to the center, about which it revolves, describes areas proportional to the times,* and the angular celerity of a ray revolving with the body, is, in that case, as the square of the distance of the body from the center, inversely (*vide Art.* 478) it follows that if the rod was to cease to act upon the ring, at the position O D P, the angular celerity at c, would then be $\frac{x^2}{\overline{x + \dot{x}}|^2} \times u$, instead of $u + \dot{u}$. Therefore the excess of $u + \dot{u}$ above $\frac{x^2}{\overline{x + \dot{x}}|^2} \times u$, which is $= \dot{u} + \frac{2u\dot{x}}{x} - \frac{3u\dot{x}^2}{x^2}$ &c. is the increase of the said angular celerity, at the distance O C, arising from the action of the rod. Therefore it will be, as O C (a) : OD (x) :: the said increase to $\left(\frac{x\dot{u}}{a} + \frac{2u\dot{x}}{a} - \frac{3u\dot{x}^2}{ax}\right.$ &c.$\left.\right)$ the alteration of the ring's paracentric velocity, arising from the same cause. Which, divided by $\left(\frac{\dot{x}}{v}\right)$ the time wherein it is produced, gives $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a} -$

* Art. 224.

$\frac{2uv\dot{x}}{ax}$ &c. for the measure of the force, by which *it* is produced. From whence, by substituting $\frac{\dot{u}}{x}$ in the room of $\frac{\dot{u}}{\dot{x}}$, and neglecting all the terms after the two first (in order to have the limiting ratio*) we get $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$. Therefore it will be, as $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$ to $\frac{b^2}{a}$,† or as $\frac{xv\dot{u}}{b^2\dot{x}} + \frac{2uv}{b^2}$ to unity, so is the action of the rod upon the ring, to the (given) centrifugal force at A (or the force that would retain a body in the circle A C K, with the velocity *b*). *Q. E. I.*

* Art. 134.

† Art. 211.

COROLLARY I.

482. If the angular motion be uniform, the equations found above, will become $v\dot{v} = \frac{b^2x\dot{x}}{a^2} - w\dot{x}$, and $v = \frac{b\dot{x}}{\dot{z}}$. From the latter of which, by taking the fluxion, we have $\dot{v} = \frac{b\ddot{x}}{\dot{z}}$; whence (by substitution) $\frac{b^2\dot{x}\ddot{x}}{\dot{z}^2} = \frac{b^2x\dot{x}}{a^2} - w\dot{x}$, and consequently $\ddot{x} - \frac{x\dot{z}^2}{a^2} = -\frac{w\dot{z}^2}{b^2}$; from the solution of which, the relation of x and z will be given. And then the value of v $\left(\frac{b\dot{x}}{\dot{z}}\right)$ being also known, the action upon the rod, which in this case is barely $= \frac{2bv}{a}$ $\left(= \frac{2b^2\dot{x}}{a\dot{z}}\right)$ will be given likewise,

being to $\left(\frac{b^2}{a}\right)$ the centrifugal force in the circle ACK, as $\frac{2\dot{x}}{\dot{z}}$ to unity.

COROLLARY II.

483. But if the angular celerity be proportional to any power (x^m) of the distance, and the centripetal force w be, also, supposed to vary according to some power (x^n) of the same distance: then, putting p to denote the centripetal, and q the centrifugal force, at the given point A, the value of w will here be expounded by $\frac{x^n}{a^n} \times p$, and that of u by $\frac{x^m}{a^m} \times b$: and therefore, the paracentric velocity of the ring at D being $= \frac{x^m}{a^m} \times b \times \frac{x}{a} \left(= \frac{bx^{m+1}}{a^{m+1}}\right)$ it will be as $\frac{b^2}{a} : \frac{b^2 x^{2m+2}}{xa^{2m+2}} :: q : \frac{x^{2m+1}}{a^{2m+1}} \times q$, the centrifugal force at D.* Hence $v\dot{v} = \frac{qx^{2m+1}\dot{x}}{a^{2m+1}} - \frac{px^n\dot{x}}{a^n}$; whereof the (corrected) fluent is $\frac{1}{2}v^2 - \frac{1}{2}c^2 = \frac{qx^{2m+2}}{2m+2 \times a^{2m+1}} - \frac{px^{n+1}}{n+1 \times a^n} - \frac{qa}{2m+2} + \frac{pa}{n+1}$: from whence v is found $=$

$$\sqrt{c^2 - \frac{qa}{m+1} + \frac{2pa}{n+1} + \frac{qx^{2m+2}}{m+1 \,.\, a^{2m+1}} - \frac{2px^{n+1}}{n+1 \,.\, a^n}}$$

and $\dot{z} \left(= \frac{u\dot{x}}{v} = \frac{bx^m\dot{x}}{a^m v}\right) =$

$$\frac{bx^m\dot{x}}{a^m\sqrt{c^2 - \frac{qa}{m+1} + \frac{2pa}{n+1} + \frac{qx^{2m+2}}{m+1 \,.\, a^{2m+1}} - \frac{2px^{n+1}}{n+1 \,.\, a^n}}}$$

* Art. 211.

moreover, by substituting for u, and its fluxion, we get $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a} = \overline{m+2} \times \frac{bx^m v}{a^{m+1}}$, expressing the action of the rod upon the ring: which, therefore, when m is expounded by -2, will entirely vanish: and, in that case, $\dot{z}$ will become =

$$\frac{a^2 b\dot{x}}{x\sqrt{\left(c^2+qa+\frac{2pa}{n+1}\right)\times x^2 - qa^3 - \frac{2px^{n+3}}{(n+1)\,.\,a^n}}}$$

expressing the nature of the trajectory described by means of a centripetal force, varying according to any power (x^n) of the distance. But this equation will be rendered somewhat more commodious, by substituting the values of b and c: for, if O Q (perpendicular to the tangent at A) be denoted by h, it will be, h : $\sqrt{a^2-h^2}$ (AQ) :: b (the celerity in the direction AC)

* Art. 35. to $c = \frac{b\sqrt{a^2-h^2}}{h}$ = the celerity in the direction AH.*

† Art. 211. Therefore, b being $= \sqrt{aq}$,† we have $c^2 = \frac{a^3 q}{h^2} - aq$;

$$\text{and } \dot{z} = \frac{a^2\dot{x}}{x\sqrt{\frac{a^2}{h^2}+\frac{2p}{\overline{n+1}.\,q}\times x^2 - a^2 - \frac{2px^{n+3}}{\overline{n+1}.\,qa^{n+1}}}}:$$

which equation is the same, in effect, with that given in Art. 242, by a different method.

Corollary III.

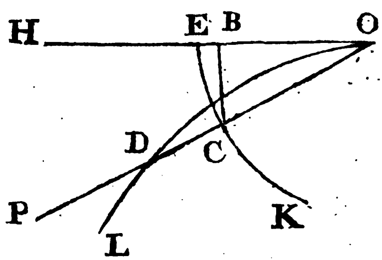
484. If the angular celerity be supposed uniform, and the ring to have no other motion along the rod than what it acquires from its centrifugal force; then c, m, and p being all of them equal to nothing, $\dot{z}$ will here become, barely $= \frac{b\dot{x}}{\sqrt{-qa+\frac{qx^2}{a}}} = \frac{a\dot{x}}{\sqrt{x^2-a^2}}$: and

therefore $z = a \times$ hyp. log. $\frac{x+\sqrt{x^2-a^2}}{a}$. Hence the number whose hyp. log. is $\frac{z}{a}$ be denoted by N, we shall have $\frac{x+\sqrt{x^2-a^2}}{a} = N$: from which x is found $= a \times \overline{\frac{N}{2}+\frac{1}{2N}}$; whence $\dot{x}$ is also had $=$ $\frac{a\dot{N}}{2} - \frac{a\dot{N}}{2N^2} = \frac{N\dot{z}}{2} - \frac{\dot{z}}{2N}$ (because $\frac{\dot{N}}{N} = \frac{\dot{z}}{a}$). Therefore, it will be (by Corol. 1) as unity is to $\frac{N}{2} - \frac{1}{2N}$, so is the angular velocity (b) in the arch A C K to the velocity with which the body recedes from the center of motion: and so, likewise, is the centrifugal force in that arch to half the pressure upon the rod. By taking $z =$ the whole periphery, or $\frac{z}{a} = 2 \times \overline{3.145}$ &c. N will come out $= 535.5$, and $x = 267.7 \times a$: from whence it appears that the distance of the ring from the center at the end of one entire revolution will be almost 268 times as great as at first.

COROLLARY IV.

485. If a body be supposed to descend from the point O, (*see the next fig.*) by the force of its own gravity, along an inclined plane OCP; whilst the plane itself moves uniformly about that point, from an horizontal position OEH; then the place, and the pressure of the body upon the plane, in any given position OCP, may also be derived from the equations in Corollary 1. For let CB (perpendicular to OH) be put $= y$; and let the ratio of the centrifugal force in the circle E C K, to the force of gravity (*given by Art.* 217) be as r to *unity*: then, as the measure of the former force is expressed by $\frac{b^2}{a}$,

that of the latter must be represented by $\frac{b^2}{ra}$; and, consequently, its efficacy in the direction P O, by $\frac{b^2y}{ra^2}$ $\left(=\left(\frac{b^2}{ra}\times\frac{CB}{OC}\right)\right)$: which value being substituted for $-w$, in the aforesaid Corollary, we have $\ddot{x}-\frac{x\dot{z}^2}{a^2}=\frac{y\dot{z}^2}{ra^2}$. But now, in order to the solution of this



equation, put the radius O C $(a) = 1$ (that the operation may be as simple as possible) also, instead of y,

* Art. 425. let its equal $z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}$* &c. be substituted and let x be assumed $= Az^3+Bz^5+Cz^7+Dz^9$ &c.

Then, by proceeding as is taught in Art. 267, the value of x will come out $=\frac{1}{r}$ into $\frac{z^3}{2.3}+\frac{z^7}{2.3.4.5.6.7}+\frac{z^{11}}{2.3.4.5.6.7.8.9.10.11}+$ &c. Whence the velocity $\left(\frac{b\dot{x}}{\dot{z}}\right)$ in the plane, is also found $=\frac{b}{r}$ into $\frac{z^2}{2}+\frac{z^6}{2.3.4.5.6}$ &c. Which, therefore, is to (*b*) the angular velocity of the plane, in the arch E C K, as $\frac{z^2}{2} + \frac{z^6}{2.3.4.5.6}$ + &c. to r. Moreover, the centrifugal force in the said arch being denoted by r (the force of gravity being unity) it will likewise be (by the above-mentioned Corol.) as $1 : \frac{2\dot{x}}{\dot{z}} :: r : \left(\frac{2r\dot{x}}{\dot{z}} =\right)$ $z^2 + \frac{z^6}{3.4.5.6} + \frac{z^{10}}{3.4.5.6.7.8.9.10}$ + &c. = the force sufficient to keep the body upon the plane. But the force of gravity in a direction perpendicular to the plane (the weight of the body being represented by *unity*) is $\frac{\text{OB}}{\text{OC}} = 1 - \frac{z^2}{2} + \frac{z^4}{2.3.4}$ * &c. From which deducting the quantity last found, there rests $1 - \frac{3z^2}{2} + \frac{z^4}{2.3.4} - \frac{3z^6}{2.3.4.5.6}$ &c. for the true pressure of the body upon the plane. By putting which equal to nothing, z^2 will be found = 0. 67715; answering to an angle (E O C) of 47° : 9′: which angle is therefore the inclination, when the force of gravity is no longer sufficient to keep the body upon the plane.

* Art. 425.

Though the value of x, given above, is found by an infinite series, yet the sum of that series is easily exhibited by the measures of angles and ratios. For, putting N to denote the number whose hyperbolical logarithm is z,

we have† $\begin{cases} 1 + z + \frac{z^2}{2} + \frac{z^3}{2.3} + \frac{z^4}{2.3.4} \text{ \&c.} = N \\ 1 - z + \frac{z^2}{2} - \frac{z^3}{2.3} + \frac{z^4}{2.3.4} \text{ \&c.} = \frac{1}{N}: \end{cases}$

† Art. 424.

half the difference of which two equations is $z + \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} + \frac{z^7}{2.3.4.5.6.7}$ &c. $= \frac{N}{2} - \frac{1}{2N}$:

from which taking $z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \frac{z^7}{2.3.4.5.6.7}$ &c. $= y$; and dividing the remainder by $2r$, there results $\left(\frac{1}{r} \times \overline{\frac{z^3}{2.3} + \frac{z^7}{2.3.4.5.6.7} \text{ \&c.}}\right)^{\frac{1}{2r}} \times \overline{\frac{N}{2} - \frac{1}{2N} - y}$, for the true value of x. Which, if required, may be expressed independent of r; by putting d for the distance through which a body freely descends in the first second of time, and taking b to denote the velocity of the plane (*per second*) in the arch E C K: for then, the ratio of the centrifugal force, in the said arch, to the force of gravity (or that of r to 1) being as $\frac{b^2}{1}$ $\left(= \frac{b^2}{\text{OC}}\right)$ to $2d$,* we shall have $r = \frac{b^2}{2d}$, and consequently $x = \frac{d}{b^2} \times \overline{\frac{N}{2} - \frac{1}{2N} - y}$.

By computations, not very unlike those above, the motion of the moon's *apogee*, and the principal equations of the lunar orbit may be exhibited, by means of proper approximations, derived from the general equations in Art. 481. But this is a consideration that would require a volume of itself, to treat it, from first principles, with all the attention and perspicuity suitable to the importance of the subject.

* Art. 211.

APPENDIX

TO

VOLUME THE SECOND.

SECT. I.—The method of investigating the indefinitely small increments of the sides and angles of triangles in this section, is by no means so elegant and comprehensive as the following.

The general proposition may be thus stated:—

Of the three Angles and three Sides of any Spherical Triangle, any two being constant, and the Fluxion of any one of the Variables being given, required the Fluxion of any one of the other Variables.

Let the angles of the triangle be denoted by A, B, C, and the sides respectively opposite, by a, b, c; and first let us suppose two sides, (as a, b) invariable, and the fluxion of the third side given, to find that of the angles A, B, C.

By Trigonometry (see Woodhouse, or Gregory) we have

$$\cos. A = \frac{\cos. a - \cos. b. \cos. c}{\sin. b. \sin. c} = \frac{\cos. a}{\sin. b}. \frac{1}{\sin. c} - \frac{\cos. b}{\sin. b}. \frac{\cos. c}{\sin. c}.$$

$$\cos. B = \frac{\cos. b}{\sin. a}. \frac{1}{\sin. c} - \frac{\cos. a}{\sin. a}. \frac{\cos. c}{\sin. c}.$$

$$\cos. C = \frac{\cos. c}{\sin. a \,.\, \sin. b} - \frac{\cos. a \,.\, \cos. b}{\sin. a \,.\, \sin. b}.$$

And taking the fluxions of these expressions we get (see vol. 1, p. 168).

$$-\dot{A} \,.\, \sin. A = -\frac{\cos. a}{\sin. b} \times \frac{\dot{c} \,.\, \cos. c}{\sin.^2 c} + \frac{\cos. b}{\sin. b} \times \frac{\dot{c}}{\sin.^2 c}$$

$$-\dot{B} \,.\, \sin. B = -\frac{\cos. b}{\sin. a} \times \frac{\dot{c} \,.\, \cos. c}{\sin.^2 c} + \frac{\cos. a}{\sin. a} \times \frac{\dot{c}}{\sin.^2 c}$$

$$-\dot{C} \,.\, \sin. C = -\frac{\dot{c} \,.\, \sin. c}{\sin. a \,.\, \sin. b}.$$

Hence $\frac{\dot{A}}{\dot{c}} = \frac{\cos. a. \cos. c - \cos. b}{\sin. A. \sin. b. \sin.^2 c} = \frac{-\sin. a}{\sin. A. \sin. b. \sin. c}$

$$\times \frac{\cos. b - \cos. a. \cos. c}{\sin. a \,.\, \sin. c} = -\frac{\sin. b}{\sin. B. \sin. b. \sin. c} \times \cos. B$$

$$= -\frac{\cos. B}{\sin. B \,.\, \sin. c}$$

Therefore $\dot{A} = -\frac{\cot. B}{\sin. c} \times \dot{c}$ 1

similarly $\dot{B} = -\frac{\cot. A}{\sin. c} \times \dot{c}$ 2

and $\dot{C} = \frac{\dot{c} \sin. c}{\sin. a. \sin. b. \sin. C} = \frac{\dot{c}}{\sin. b \times \sin. A}$ or $=$

$\frac{\dot{c}}{\sin. a \,.\, \sin. B}$ 3

Now, instead of a and b, let a and c be constant, and it will be found exactly in the same manner, that

$$\dot{A} = -\frac{\cot. C}{\sin. b} \times \dot{b}$$

$$\dot{B} = \frac{\dot{b}}{\sin. a \,.\, \sin. C} \text{ or } = \frac{\dot{b}}{\sin. c \,.\, \sin. A}$$

$$\dot{C} = -\frac{\cot. A}{\sin. b} \times \dot{b}$$

Again, suppose b and c constant, and we get

$$\dot{A} = \frac{\dot{a}}{sin.\ b\ .\ sin.\ C} \text{ or } = \frac{\dot{a}}{sin.\ c\ .\ sin.\ B}$$

$$\dot{B} = -\frac{cot.\ C}{sin.\ a} \times \dot{a}$$

$$\dot{C} = -\frac{cot.\ B}{sin.\ a} \times \dot{a}$$

The three first equations are sufficient to resolve, *Two sides being given and the fluxion of the third, of any spherical triangle; required the fluxion of each of the angles.*

Secondly. *Any two of the sides being constant, and the fluxion of an angle given; let it be required to find the fluxion of the third side, and of the remaining angles.*

From the above results we get

$$\dot{c} = -tan.\ B \times sin.\ c \times \dot{A} \ldots\ldots\ldots 4$$

$$\dot{B} = cot.\ A \times tan.\ B \times \dot{A} \ldots\ldots\ldots 5$$

$$\dot{C} = -\frac{sin.\ C}{sin.\ A\ .\ cos.\ B} \times \dot{A} \ldots 6$$

which three forms completely resolve the second case.

Thirdly. Let us suppose an angle, and its *opposite* side constant (as A, a) and the fluxion of a side ($\dot{c}$) given, to find $\dot{b}$, $\dot{B}$, and $\dot{C}$.

By Trigonometry we have

$$\frac{sin.\ A}{sin.\ a} = \frac{sin.\ C}{sin.\ c} = \frac{sin.\ B}{sin.\ b}.$$

Therefore, $sin.\ C = \dfrac{sin.\ A}{sin.\ a} \times sin.\ c$, and taking the fluxions

$$\dot{C} = \frac{sin.\ A}{sin.\ a} \times \frac{cos.\ c}{cos.\ C} \times \dot{c}$$

$$= \frac{\sin. C}{\sin. c} \times \frac{\cos. c}{\cos. C} \times \dot{c}$$

$$= \tan. C . \cot. c \times \dot{c}$$

Also $\sin. B = \frac{\sin. A}{\sin. a} \times \sin. b$

Therefore $\dot{B} = \frac{\sin. A}{\sin. a} \times \frac{\cos. b}{\cos. B} \times \dot{b} = \tan. B \times \cot. b \times \dot{b}$.

Now $\cos. A = \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c}$

Therefore $\sin. b . \sin. c \times \cos. A = \cos. a - \cos. b . \cos. c$, and $\cos. A \times (\dot{b} \cos. b . \sin. c + \dot{c} \cos. c . \sin. b) = \dot{b} \sin. b . \cos. c + \dot{c} \sin. c . \cos. b$.

Hence $\dot{b} = \frac{\dot{c} \times (\sin. c . \cos. b - \cos. c . \sin. b \times \cos. A)}{\cos. b . \sin. c . \cos. A - \sin. b . \cos. c}$

$$= \frac{\dot{c} \left(\sin. c . \cos. b - \cos. c . \sin. b \times \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c}\right)}{\cos. b . \sin. c \times \frac{\cos. a - \cos. b . \cos. c}{\sin. b . \sin. c} - \sin. b . \cos. c}$$

$$= \dot{c} \times \frac{(\sin.^2 c + \cos.^2 c) . \cos. b - \cos. a . \cos. c}{\cos. a . \cos. b - (\cos.^2 b + \sin.^2 b) . \cos. c} \times \frac{\sin. b}{\sin. c}$$

$$= -\dot{c} \times \frac{\frac{\cos. b - \cos. a . \cos. c}{\sin. a . \sin. c}}{\frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b}} = -\dot{c} \times \frac{\cos. B}{\cos. C}$$

Hence $\dot{b} = -\dot{c} \times \frac{\cos. B}{\cos. C}$ 7

$\dot{B} = \tan. B . \cot. b \times \dot{b} = -\dot{c} \times \frac{\sin. B}{\cos. C} \times \cot. b$...8

and $\dot{C} = \dot{c} \times \tan. C \times \cot. c$............ 9

Fourthly. Let A, a be constant as before, but the fluxion of an angle, $\dot{B}$, be given to find $\dot{b}$, $\dot{c}$, $\dot{C}$.

From equation (8) we get

$\dot{c} = -\frac{\cos. C \times \tan. b}{\sin. B} \times \dot{B}$............... 10

Also $\dot{b} = -\dot{c} \times \frac{cos.\ B}{cos.\ C} = \frac{tan.\ b \times cos.\ B}{sin.\ B} \times \dot{B} = tan.\ b \times$

$cot.\ B \times \dot{B}$ 11

and $\dot{C} = \dot{c} \times tan.\ C \times cot.\ c = -\frac{sin.\ C \times cot.\ c \times tan.\ b}{sin.\ B}$

$\times \dot{B} = -\frac{cos.\ c}{cos.\ b} \times \dot{B}$ 12

Fifthly. Let an angle and either of its *adjacent* sides (as A, b) be constant, and the fluxion of either of the other sides (as $\dot{c}$) be given; required to find $\dot{a}, \dot{B}, \dot{C}$.

As before, we have

$$cos.\ A\ .\ sin.\ b\ .\ sin.\ c = cos.\ a - cos.\ b\ .\ cos.\ c$$

Hence $\dot{a} = \dot{c} \times \frac{cos.\ b\ .\ sin.\ c - sin.\ b \times cos.\ c\ .\ cos.\ A}{sin.\ a}$

$$= \dot{c} \times \frac{cos.\ b\ .\ sin.\ c - sin.\ b \times cos.\ c \times \frac{cos.\ a - cos.\ b\ .\ cos.\ c}{sin.\ b\ .\ sin.\ c}}{sin.\ a}$$

$$= \dot{c} \times \frac{cos.\ b\ .\ (sin.\ ^2c + cos.\ ^2c) - cos.\ a\ .\ cos.\ c}{sin.\ a\ .\ sin.\ c}$$

$$= \dot{c} \times \frac{cos.\ b - cos.\ a\ .\ cos.\ c}{sin.\ a\ .\ sin.\ c} = \dot{c} \times cos.\ B$$

Again, $sin.\ B = sin.\ b \times \frac{sin.\ A}{sin.\ a}$, and taking the fluxions, &c. we obtain

$$\dot{B} = -\frac{sin.\ b.\ sin.\ A.\ cos.\ a}{cos.\ B.\ sin.\ ^2a} \times \dot{a} = \frac{-sin.\ b.\ sin.\ A.\ cos.\ a \times \dot{c}}{sin.\ ^2a}$$

$$= -\ sin.\ B \times cot.\ a \times \dot{c}$$

Again, $sin.\ C = \frac{sin.\ c \times sin.\ B}{sin.\ b}$, and taking the fluxions

we get $\dot{C} = \frac{\dot{c}\ .\ cos.\ c\ .\ sin.\ B + \dot{B}\ .\ cos.\ B\ .\ sin.\ c}{sin.\ b\ .\ cos.\ C}$

$$= \frac{\dot{c} \,.\, sin.\, B}{sin.\, a \,.\, sin.\, b \,.\, cos.\, C} \times (sin.\, a \,.\, cos.\, c - sin.\, c \,.\, cos.\, a \,.\, cos.\, B)$$

$$= \frac{\dot{c} \,.\, sin.\, B}{sin.\, a \times cos.\, C} \times \frac{cos.\, c - cos.\, a \,.\, cos.\, b}{sin.\, a \,.\, sin.\, b} \text{(by subst.)}$$

$$= \frac{\dot{c} \,.\, sin.\, B}{sin.\, a}$$

Hence, then, we collect

$$\dot{a} = \dot{c} \times cos.\, B \text{ .... } 13$$

$$\dot{B} = -sin.\, B \times cot.\, a \times \dot{c} \text{ ...... } 14$$

$$\dot{C} = \frac{sin.\, B}{sin.\, a} \times \dot{c} \text{ .... } 15$$

Sixthly. Let A, b be constant as before, and let the fluxion of either angle (as $\dot{B}$) be given; required to $\dot{a}$, $\dot{c}$, $\dot{C}$.

From (14) we get immediately

$$\dot{c} = -\frac{tan.\, a}{sin.\, B} \times \dot{B} \text{ .... } 16$$

Also $\dot{a} = \dot{c} \times cos.\, B = -tan.\, a \,.\, cot.\, B \times \dot{B}$ ... 17

and $\dot{C} = \frac{sin.\, B}{sin.\, a} \times \dot{c} = -\frac{\dot{B}}{cos.\, a}$ 18

Seventhly. Let any two angles (as A, B) be constant, and let the fluxion of a side opposite to one of them (as $\dot{a}$) be given; required to find $\dot{b}$, $\dot{c}$, and $\dot{C}$.

Since $sin.\, b = \frac{sin.\, B}{sin.\, A} \times sin.\, a$, we have

$$\dot{b} = \frac{sin.\, B \;\; cos.\, a}{cos.\, b \,.\, sin.\, A} \times \dot{a}$$

$$= \frac{sin.\, b}{cos\; b} \,.\, \frac{cos.\, a}{sin.\, a} \times \dot{a} = tan.\, b \times cot.\, a \times \dot{a}$$

Again, by Trigonometry, we have

$$tan.\ \frac{A}{2} \times tan.\ \frac{B}{2} = \frac{sin.\ \frac{a+b-c}{2}}{sin.\ \frac{a+b+c}{2}},$$ and taking the fluxions

$$\frac{\dot{a}+\dot{b}+\dot{c}}{2} \times cos.\ \frac{a+b+c}{2} \times tan.\ \frac{A}{2}\ tan.\ \frac{B}{2} = \frac{\dot{a}+\dot{b}-\dot{c}}{2}\ cos.\ \frac{a+b-c}{2}$$

Hence

$$\dot{c} = \frac{(\dot{a}+\dot{b}) \times \left(cos.\ \frac{a+b-c}{2} - cos.\ \frac{a+b+c}{2}\ tan.\ \frac{A}{2}\ tan.\ \frac{B}{2}\right)}{cos.\ \frac{a+b-c}{2} + cos.\ \frac{a+b+c}{2}\ tan.\ \frac{A}{2}\ tan.\ \frac{B}{2}}$$

$$= (\dot{a}+\dot{b}) \times \frac{sin.\ \frac{a+b+c}{2}\ .\ cos.\ \frac{a+b-c}{2} - cos.\ \frac{a+b+c}{2}\ sin.\ \frac{a+b-c}{2}}{sin.\ \frac{a+b+c}{2}\ .\ cos.\ \frac{a+b-c}{2} + cos.\ \frac{a+b+c}{2}\ .\ sin.\ \frac{a+b-c}{2}}$$

$$= (\dot{a}+\dot{b}) \times \frac{sin.\left(\frac{a+b+c}{2} - \frac{a+b-c}{2}\right)}{sin.\left(\frac{a+b+c}{2} + \frac{a+b-c}{2}\right)} = (\dot{a}+\dot{b})\ .\ \frac{sin.\ c}{sin.(a+b)}$$

But $\dot{b} = tan.\ b \times cot.\ a \times \dot{a}$

Therefore $\dot{c} = \dot{a} \times \left(1 + \frac{sin.\ b\ .\ cos.\ a}{cos.\ b\ .\ sin.\ a}\right) \times \frac{sin.\ c}{sin.\ (a+b)}$

$$= \frac{\dot{a}\ \ sin.\ c}{sin.\ a\ .\ cos.\ b}$$

Again, since $sin.\ C = sin.\ c \times \frac{sin.\ A}{sin.\ a}$

$$\dot{C} = \dot{c} \times \frac{\cos. c}{\cos. C} \times \frac{\sin. A}{\sin. a} - \dot{a} \times \frac{\cos. a}{\cos. C} \times \frac{\sin. c \times \sin. A}{\sin.^2 a}$$

$$= \frac{\dot{a} \sin. A}{\cos. C} \times \left(\frac{\sin. c . \cos. c}{\sin.^2 a . \cos. b} - \frac{\cos. a . \sin. c}{\sin.^2 a}\right).$$

$$= \frac{\dot{a} \sin. A . \sin. c \times \sin. b}{\sin. a . \cos. C . \cos. b} \times \left(\frac{\cos. c - \cos. a . \cos. b}{\sin. a . \sin. b}\right),$$

$$= \dot{a} \frac{\sin. A . \sin. c . \sin. b}{\sin. a . \cos. b} = \dot{a} \times \sin. C \times \tan. b$$

Hence, collecting results, we get

$$\dot{b} = \tan. b \times \cot. a \times \dot{a} \text{ ....} \quad 19$$

$$\dot{c} = \frac{\sin. c}{\sin. a . \cos. b} \times \dot{a} \text{ ....} \quad 20$$

$$\dot{C} = \sin. C . \tan. b \times \dot{a} \text{ ....} \quad 21$$

Eighthly. Let A, B be constant, and $\dot{C}$ given; to find $\dot{a}$, $\dot{b}$, $\dot{c}$.

From (21) we get

$$\dot{a} = \frac{\cot. b}{\sin. C} \times \dot{C} \text{ ....} \quad 22$$

Also $\dot{b} = \tan. b . \cot. a \times \dot{a} = \frac{\cot. a}{\sin. C} \times \dot{C} \text{...} \quad 23$

$$\dot{c} = \frac{\sin. c}{\sin. a . \cos. b} \times \dot{a} = \frac{\sin. c}{\sin. C . \sin. b . \sin. a} \times \dot{C} =$$

$$\frac{\dot{C}}{\sin. a . \sin. B} \text{ ....} \quad 24$$

Ninthly. Let A B be constant, and the fluxion of the side lying between them ($\dot{c}$) be given; to find $\dot{a}$, $\dot{b}$, and $\dot{c}$.

From (20) we get

$$\dot{a} = \frac{\sin. a . \cos. b}{\sin. c} \times \dot{c} \text{ ....} \quad 25$$

From (19) we have

$$\dot{b} = \tan. b \times \cot. a \times \dot{a} = \frac{\sin. b . \cos. a}{\sin. c} \times \dot{c} \text{ ...} \quad 26$$

And from (24)

$$\dot{C} = sin.\, a \,.\, sin.\, B \times \dot{c} \;\ldots\ldots\ldots\ldots\; 27$$

The last three cases we have resolved without the aid of the supplemental triangle.

This method has one great advantage over that in the text, in exhibiting whether the fluxion be positive or negative, or whether the side or angle be increasing or decreasing. See 1, 2, 4, 6, 7, 8, 10, 12, 14, 16, 17, 18, &c. The reader is advised to apply these observations in the astronomical examples.

The forms for right-angled triangles may evidently be deduced, as particular cases, from the above forms. Moreover, a similar process will apply in finding the fluxions of the sides and angles of *plane triangles* from corresponding data. See *Cotesii Estimatio Errorum in mixta Mathesi*.

SECT. II.—The subject of Fluxional Equations, from the time of Simpson, has been cultivated with very eminent success, and many important results, unknown at that period, have been obtained.

The limits of an appendix not permitting us to discuss the subject in its detail, we shall present the student merely with an outline of these improvements.

I. Let the variables be already separated.

$$1.\ \frac{\dot{x}}{\sqrt{(a+bx+cx^2+dx^3+ex^4)}} + \frac{\dot{y}}{\sqrt{(a+by+cy^2+dy^3+ey^4)}} = 0$$

$$\text{Make } \frac{\dot{x}}{\dot{t}} = \sqrt{a+bx+cx^2+dx^3+ex^4} = \sqrt{X}$$

$$\text{and } \frac{\dot{y}}{\dot{t}} = \sqrt{a+by+cy^2+dy^3+cy^4} = \sqrt{Y}$$

$$\text{Hence } \frac{\dot{x}^2}{\dot{t}^2} + \frac{\dot{y}^2}{\dot{t}^2} = X + Y$$

$$\text{and } \frac{\ddot{x}}{\dot{t}^2} + \frac{\ddot{y}}{\dot{t}^2} = \frac{\dot{X}}{2\dot{x}} + \frac{\dot{Y}}{2\dot{y}}$$

Put $x+y=u$, and $x-y=v$.

Then $x=\frac{u+v}{2}$, and $y=\frac{u-v}{2}$;

also $\frac{\ddot{u}}{\dot{t}^2}=\frac{\ddot{x}+\ddot{y}}{\dot{t}^2}=\frac{\dot{y}X'+\dot{x}Y'}{2\dot{x}\dot{y}}$

$$=b+cu+\tfrac{3}{4}d.(u^2+v^2)+\tfrac{1}{2}eu(u^2+3v^2)\ldots(1)$$

and $\frac{\dot{u}\dot{v}}{\dot{t}^2}=\frac{\dot{x}^2-\dot{y}^2}{\dot{t}^2}=X-Y$

$$=bv+cuv+\tfrac{1}{4}dv.(3u^2+v^2)+\tfrac{1}{2}e\,uv\,(u^2+v^2).(2)$$

Multiplying equation (1) by v, and subtracting, we get $\frac{v\ddot{u}}{\dot{t}^2}-\frac{\dot{u}\dot{v}}{\dot{t}^2}=\frac{d}{2}v^3+euv^3$, which being again multiplied by $\frac{2\dot{u}}{v^3}$, we have

$$\left(\frac{2\dot{u}\ddot{u}}{v^2}-\frac{2\dot{u}^2\dot{v}}{v^3}\right)\frac{1}{\dot{t}^2}=d\times\dot{u}+2eu\dot{u}$$

and taking the fluents, $\dot{t}$ being considered constant,

$$\frac{\dot{u}^2}{v^2}\times\frac{1}{\dot{t}^2}=const.+d\times u+eu^2.$$

Hence $\frac{\dot{u}}{\dot{t}}=\sqrt{X}+\sqrt{Y}=v\sqrt{C+du+eu^2}$

$$=(x-y)\sqrt{C+d.(x+y)+e.(x+y)^2}.$$

By the same process we resolve

$$\frac{\dot{x}}{\sqrt{X}}-\frac{\dot{y}}{\sqrt{Y}}=0,$$

and obtain

$$\sqrt{X}-\sqrt{Y}=(x-y)\sqrt{C+d.(x+y)+e.(x+y)^2}$$

Also

$$\sqrt{X}-\sqrt{Y}=\frac{X-Y}{\sqrt{X}+\sqrt{Y}}=\frac{bv+cvu+\ldots\ldots}{v\sqrt{C+d.(x+y)+e.(x+y)^2}}$$

$$=\frac{b+c.(x+y)+d.(x^2+xy+y^2)+e.(x^3+x^2y+xy^2+y^3)}{\sqrt{C+d.(x+y)+e.(x+y)^2}}.$$

By this method we find the fluent of

$$(2)\ldots\ldots \frac{\dot{\theta}}{\sqrt{1-e^2 \sin.^2\theta}} \pm \frac{\dot{\phi}}{\sqrt{1-e^2 \sin.^2\phi}} = 0$$

to be

$$\sqrt{1-e^2 \sin.^2\theta} \pm \sqrt{1-e^2 \sin.^2\phi} = C \times \sin.(\theta \pm \phi).$$

Also from

$$(3)\ldots\ldots \frac{\dot{x}}{\sqrt{a+bx+cx^2}} \pm \frac{\dot{y}}{\sqrt{a+by+cy^2}} = 0,$$

$$\sqrt{a+bx+cx^2} \pm \sqrt{a+by+cy^2} = \sqrt{C+2b.(x+y)+c.(x+y)^2}.$$

(4). Instead of resolving, by the above method, the equation

$$\frac{\dot{x}}{\sqrt{1-x^2}} \pm \frac{\dot{y}}{\sqrt{1-y^2}} = 0$$

we have ($\sin.^{-1}x = arc$ whose sine is x to radius 1, &c.)

$\sin.^{-1}x \pm \sin.^{-1}y = const. = \sin.^{-1}c$ (the constant being perfectly arbitrary).

Hence $c = \sin.(\sin.^{-1}x \pm \sin.^{-1}y)$

$= \sin.(\sin.^{-1}x)\cos.(\sin.^{-1}y) \pm \cos.(\sin.^{-1}x) \times \sin.(\sin.^{-1}y)$

$= x.\sqrt{1-y^2} \pm \sqrt{1-x^2} \times y$, as it will be readily perceived by attending to the meaning of the symbols $\sin.^{-1}x$, &c.

5. Required to resolve

$$\frac{\dot{x}}{1+x^2} \pm \frac{\dot{y}}{1+y^2} = 0.$$

We have $\tan.(\tan.^{-1}x \pm \tan.^{-1}y) = const. = \tan.^{-1}c$

Hence

$$c = \tan.(\tan.^{-1}x \pm \tan.^{-1}y) = \frac{\tan.(\tan.^{-1}x) \pm \tan.(\tan.^{-1}y)}{1 \mp \tan.(\tan.^{-1}x).\tan.(\tan.^{-1}y)}$$

$$= \frac{x \pm y}{1 \mp xy}$$

6. Required to resolve

$$\frac{\dot{x}}{\sqrt{1+x^2}} + \frac{\dot{y}}{\sqrt{1+y^2}} = 0.$$

Here we have

$$log.\,(x+\sqrt{1+x^2}) + log.\,(y+\sqrt{1+y^2}) = const. = log.\,c.$$

Therefore, by logarithms

$$c = (x+\sqrt{1+x^2})\,.\,(y+\sqrt{1+y^2})$$

which admits of further reductions.

The above methods show that the fluents, although logarithmic or circular, when taken separately, may frequently be rendered algebraic, by a proper assumption of the form of the arbitrary constant.

II. We will now proceed to solve *Homogeneous Equations*, or those whose terms are each of the same dimension, which comprehend an extensive class, and present little difficulty.

Every Homogeneous Equation may be reduced to this form

$$\frac{\dot{y}}{\dot{x}} = \text{a function of } \frac{y}{x} = f\left(\frac{y}{x}\right).$$

For the equation being $\frac{\dot{y}}{\dot{x}} = \frac{M}{N}$ (M and N are *homogeneous* functions of x and y), assume

$$\frac{y}{x} = u, \text{ or } y = xu$$

Then, substituting for y in M and N, x will rise to the same dimension in every term of them, and by division $\frac{\dot{y}}{\dot{x}} = \frac{M}{N}$ will become a function of the new variable only. That is

$$\frac{\dot{y}}{\dot{x}} = f.\,u = f.\left(\frac{y}{x}\right)$$

But $\dot{y}=\dot{x}u+\dot{u}x$.

Therefore $u+\frac{\dot{u}x}{\dot{x}}=fu$

and $\frac{\dot{x}}{x}=\frac{\dot{u}}{fu-u}$ in which the variables being separated we have

$$log.\ x=\int\frac{\dot{u}}{fu-u}$$

Ex. 1. $\frac{\dot{y}}{\dot{x}}=\frac{x^m+by^m}{ay^m}$.

Let $y=ux$

Then $\frac{\dot{u}x+\dot{x}u}{\dot{x}}=\frac{x^m+bu^mx^m}{au^mx^m}=\frac{1+bu^m}{au^m}$

and $\frac{\dot{x}}{x}=\frac{au^m\dot{u}}{1+bu^m-au^{m+1}}$ which is to be resolved as a rational fraction (Sect. V, Vol. II).

(2). As a more simple case take,

$$x\dot{y}-y\dot{x}=\dot{x}\,.\,\sqrt{x^2+y^2}.$$

Here we get

$$\frac{\dot{x}}{x}=\frac{\dot{u}}{\sqrt{1+u^2}}$$

and $log.\ x=log.\ (u+\sqrt{1+u^2})+log\ c$

Therefore $x=c\ (u+\sqrt{1+u^2})=c\,.\left(\frac{y}{x}+\frac{\sqrt{x^2+y^2}}{x}\right)$,

and by reduction

$$x^2=c^2-2cy.$$

(3). $\frac{\dot{x}}{\dot{y}}=\frac{x+y}{y}$.

Assume $x=yu$. Then $\frac{\dot{y}u+\dot{u}y}{\dot{y}}=\frac{yu+y}{y}=u+1$

Therefore $\frac{\dot{y}}{y} = \frac{\dot{u}}{u+1-u} = \dot{u}$,

and $log.\ y = u + const. = u - log\ c$,

or $u = log.\ cy$,

and $cy = e^u$ (e being the hyperbolic base)

Therefore $y = \frac{e^{\frac{x}{y}}}{c}$.

(4.) $\frac{\dot{x}}{\dot{y}} = -\frac{xy+x^2}{y^2}$. Make $x = yu$

and we get $y\sqrt{\frac{x}{2y+x}} = c$.

(5.) $(ax+by).\dot{y}+(fx+gy)\dot{x} = 0$. Make $y = xu$

and we get $\frac{\dot{x}}{x} = -\frac{(a+bu).du}{bu^2+(a+g).u+f}$ and taking the fluents we shall obtain, after the requisite reductions,

$$log\ C.(x+y) = -\frac{x}{x+y}$$

$$\text{or } C.(x+y) = e^{-\frac{x}{x+y}}.$$

This process is applicable even when such forms as $log.\left(\frac{y}{x}\right)$, $e^{\frac{y}{x}}$, $cos.\left(\frac{y}{x}\right)$ &c. and generally any functions of $\frac{y}{x}$, enter the equation.

III. (1). *Equations reducible to the form*

$$\frac{\dot{y}}{\dot{x}} + Ax^m y^n + A_1 x^{m_1} y^{n_1} + A_2 x^{m_2} y^{n_2} + \ldots\ldots = 0$$

may be rendered homogeneous by the assumption

$$y = u^r = u^{\frac{m+1}{1-n}},$$

whenever

$$\frac{m+1}{1-n} = \frac{m_1+1}{1-n_1} = \frac{m_2+1}{1-n_2} = \&c.$$

For we find, by taking the fluxion and substituting, that

$$ru^{r-1}.\frac{\dot{u}}{\dot{x}} + Ax^m y^{nr} + A_1 x^{m_1} y^{n_1 r} + \ldots\ldots\ldots\ldots = 0,$$

which will be *homogeneous* when

$$r - 1 = m + nr = m_1 + n_1 r = \&c.$$

or when

$$r = \frac{m+1}{1-n} = \frac{m_1+1}{1-n_1} = \&c.$$

Ex. 1. $\frac{\dot{y}}{\dot{x}} + Ax^m y^n + A_1 \frac{y}{x} = 0.$

Ex. 2. $\frac{\dot{y}}{\dot{x}} + Ax^m y^n + A_1 x^{\frac{m-n+2}{n-1}} y^2 + A_2 x^{\frac{2m-n+3}{n-1}} y^3 = 0,$

and so on to an indefinite extent.

Generally, to investigate the Conditions for rendering the above Equation homogeneous by assumptions of the form

$$y = u^r x^s.$$

Since $\frac{\dot{y}}{\dot{x}} = ru^{r-1}x^s.\frac{\dot{u}}{\dot{x}} + su^r x^{s-1}$, we have, by substitution,

$$ru^{r-1}x^s\frac{\dot{u}}{\dot{x}} + su^r x^{s-1} + Ax^{m+ns}u^{nr} + A_1 x^{m_1+n_1 s}u^{n_1 r} + \&c. = 0,$$

and it is evident that when this transformed equation is *homogeneous*, we must have

$$r + s - 1 = m + ns + nr = m_1 + n_1 s + n_1 r = \&c.$$

$$\therefore r = \frac{m+\overline{n-1}.s+1}{1-n} = \frac{m_1+\overline{n_1-1}.s+1}{1-n} = \&c.$$

the conditions required.

It appears hence that innumerable equations of the form

$$\frac{\dot{y}}{\dot{x}} + Ax^{m}y^{n} + A_1 x^{m_1} y^{n_1} + \&c. = 0,$$

by assumptions such as

$$y = x^{s} u^{m+ns-s+1}$$

may be rendered homogeneous, and therefore integrable.

In a similar manner may be found the conditions requisite to the applicability of the assumption

$$y = a u^{r} x^{s} + a_1 u^{r_1} x^{s_1} + a_2 u^{r_2} x^{s_2} + \ldots\ldots\ldots\ldots$$

in rendering an equation of the above form homogeneous, a, a_1, a_2, &c. r, r_1, r_2, &c. s, s_1, s_2, &c. being constants to be determined by convenient assumptions in the result, and u a new variable.

Ex. *Required the Conditions of Integrability of the Equation*

$$\frac{\dot{y}}{\dot{x}} + Ay^{2} + Bx^{m} = 0 \ldots\ldots\ldots (a)$$

by the Assumption

$$y = ax^{s} + ux^{s_1}.$$

By taking the fluxions, and substituting for y and $\dot{y}$ in the equation, we obtain

$$\frac{\dot{u}}{\dot{x}} x^{s_1} + (asx^{s-1} + Aa^{2}x^{2s}) + u\,(s_1 x^{s_1-1} + 2Aax^{s+s_1}) + Au^{2}x^{2s_1} + Bx^{m} = 0.$$

$$\text{Let } \left.\begin{array}{l} s-1 = 2s \\ s_1 - 1 = s + s_1 \end{array}\right\} \qquad \left.\begin{array}{r} as + Aa^{2} = 0 \\ \text{and } s_1 + 2Aa = 0 \end{array}\right\}$$

Hence $s = -1$, $a = \frac{1}{A}$ and $s_1 = -2Aa = -2$.

$$\therefore y = \frac{x^{-1}}{A} + ux^{-2}.$$

and $\frac{\dot{u}}{\dot{x}}x^{-2} + Au^2x^{-4} + Bx^m = 0,$

$$\text{or } \frac{\dot{u}}{\dot{x}} + Au^2x^{-2} + Bx^{m+2} = 0 \text{ .... } (b)$$

which is homogeneous when $m = -2$.

When m is -4, equation (b) is reducible to

$$\frac{\dot{u}}{Au^2+B} + \frac{\dot{x}}{x^2} = 0.$$

Hence, then, the given equation becomes integrable in the cases of $m=-2$, and $m=-4$, when transformed to equation (b) by the assumption

$$y = \frac{x^{-1}}{A} + ux^{-2}.$$

Again, let $u = y_1^{-1}$, and $x^{m+3} = x_1$.

Then, by substituting for u and $\dot{u}$, x and $\dot{x}$, we get, after proper reductions,

$$\frac{\dot{y}_1}{\dot{x}_1} - \frac{B}{m+3}y_1^2 - \frac{A}{m+3}\cdot x^{-\frac{m+4}{m+3}} = 0,$$

and putting

$$\frac{B}{m+3} = -A_1, \frac{A}{m+3} = -B_1 \text{ and } -\frac{m+4}{m+3} = m_1,$$

$$\frac{\dot{y}_1}{\dot{x}_1} + A_1y_1^2 + B_1x_1^{m_1} = 0 \text{ .... } (a_1)$$

which being similar to equation (a), may be treated in like manner.

Hence, by putting $y_1 = \frac{x_1^{-1}}{A_1} + u_1x_1^{-2}$, we obtain, as before,

$$\frac{\dot{u}_1}{\dot{x}_1} + A_1u_1^2x_1^{-2} + B_1x_1^{m_1+2} = 0 \text{............ } (b_1)$$

which is integrable when $m_1 = -2$, and when $m_1 = -4$.

Hence it is evident that by repeating the operation μ times and putting successively

$$\left.\begin{aligned} -\frac{B}{m+3}&=A_1\\ -\frac{B_1}{m_1+3}&=A_2\\ -\frac{B_2}{m_2+3}&=A_3\\ \&c.&=\&c.\end{aligned}\right\}\quad \left.\begin{aligned} -\frac{A}{m+3}&=B_1\\ -\frac{A_1}{m_1+3}&=B_2\\ -\frac{A_2}{m_2+3}&=B_3\\ \&c.&=\&c.\end{aligned}\right\}\quad \left.\begin{aligned} -\frac{m+4}{m+3}&=m_1\\ -\frac{m_1+4}{m_1+3}&=m_2\\ -\frac{m_2+4}{m_2+3}&=m_3\\ \&c.&=\&c.\end{aligned}\right\}$$

we shall at length arrive at an equation

$$\frac{\dot{u}_\mu}{\dot{x}_\mu} + A_\mu u_\mu^2 x_1^{-2} + B_\mu x_\mu^{m+2 \atop \mu} = 0 \;\ldots\ldots (b_\mu)$$

which is integrable when $m_\mu = -2$ or -4; and the equation from which this is immediately derived

$$\frac{\dot{y}_\mu}{\dot{x}_\mu} + A_\mu y_\mu^2 + B_\mu x^{m \atop \mu} = 0 \;\ldots\ldots\ldots (a_\mu)$$

is integrable when $m_\mu = 0$ (for then $\dfrac{\dot{y}_\mu}{A_\mu y_\mu^2 + B_\mu} + \dot{x}_\mu = 0$); hence the given equation (a) is integrable whenever $m_\mu = 0, -2,$ or -4.

$$\text{But } m_2 = -\frac{3m+8}{2m+5}$$

$$m_3 = -\frac{5m+12}{3m+7}$$

$$m_4 = -\frac{7m+12}{5m+9}$$

$$\&c. = \&c.$$

$$\therefore m_\mu = -\frac{(2\mu-1)\,m+4\mu}{\mu m+2\mu+1}.$$

Hence equation (a) is integrable when

$$-\frac{(2\mu-1)\,m+4\mu}{\mu m+2\mu+1} = 0 \text{ or } -2 \text{ or } -4,$$

$$\text{or when } m = -\frac{4\mu}{2\mu-1} \text{ or } -2, \text{ or } -\frac{4\mu}{2\mu+1}.$$

This example contains a complete discussion of *Count Riccati's* Equation.

Every equation of the forms

$$\left.\begin{array}{l}\dfrac{\dot{y}}{\dot{x}} + Ay^2x^p + Bx^m \\[2ex] \dfrac{\dot{y}}{\dot{x}} + Ay^2 + Bx^m + \dfrac{2y}{x} \\[2ex] \text{and } \dfrac{\dot{y}}{\dot{x}} + Ay^2x^n + Bx^m + Cyx^r \end{array}\right\} \text{ are reducible to the}$$

form of Riccati's equation, by the respective assumptions $x^{p+1} = u$, $y = u + \frac{1}{x}$; $x^{n+1} = u$, and then $y = v + \frac{1}{Au}$.

(2). *Equations of the form*

$$\frac{\dot{y}}{\dot{x}} + \frac{Ax^my^n + Bx^{m_1}y^{n_1} + \&c.}{ax^py^q + bx^{p_1}q^{q_1} + \&c.} = 0$$

may be rendered Homogeneous by the Assumption

$$y = u^r = u^{\frac{m-m_1}{n_1-n}} \text{ or } y = u^{\frac{m-p+1}{1-(n-q)}}$$

whenever

$$m + rn = m_1 + rn_1 = \&c. = p + rq + r - 1$$
$$= p_1 + rq_1 + r - 1 = \&c.$$

hold good simultaneously.

The truth of this becomes evident upon substituting in the given equation for y and $\dot{y}$, and equating the sums of the exponents in the results.

(3.) *Equations of the form*

$$\frac{\dot{y}}{\dot{x}} + \frac{A + Bx + Cy}{a + bx + cy} = 0 \ldots\ldots\ldots\ldots (a)$$

become Homogeneous by the Assumptions

$$\left.\begin{array}{r} A + Bx + Cy = u \\ \text{and } a + bx + cy = v \end{array}\right\}$$

For $\left.\begin{array}{r} B\dot{x} + C\dot{y} = \dot{u} \\ \text{and } b\dot{x} + c\dot{y} = \dot{v} \end{array}\right\}$. Hence, by elimination,

$$\dot{y} = \frac{b\dot{u} - B\dot{v}}{bC - cB}$$

$$\text{and } \dot{x} = \frac{C\dot{v} - c\dot{u}}{bC - cB}$$

∴ by substitution we have

$$\frac{b\dot{u} - B\dot{v}}{C\dot{v} - c\dot{u}} + \frac{u}{v} = 0,$$

whence is easily derived

$$\frac{\dot{u}}{\dot{v}} - \frac{Bv + Cu}{bv + cu} = 0 \text{ .... } (b)$$

which being *homogeneous*, is therefore integrable.

(4.) *Equations of the general form*

$$x_1^{m_1} \{a_1 \frac{\dot{x}_1}{x_1} + b_1 \frac{\dot{x}_2}{x_2} + \text{.....}\} + x_2^{m_2}\{a_2 \frac{\dot{x}_1}{x_1} + b_2 \frac{\dot{x}_2}{x_2} + \text{...}\}$$

$$+ x_3^{m_3} \{a_3 \frac{\dot{x}_3}{x_3} + b_3 . \frac{\dot{x}_2}{x_2} + \text{.........}\} + \text{\&c. \&c.} = 0,$$

the logarithmic forms $\frac{\dot{x}_1}{x_1}$ *&c. being taken in any order whatever, become homogeneous by assuming*

$$x_1^{m_1} = u_1$$

$$x_2^{m_2} = u_2$$

$$x_3^{m_3} = u_3$$

$$\text{\&c.} = \text{\&c.}$$

and making the proper substitutions.

$$\text{For } \frac{\dot{x}_1}{x_1} = \frac{1}{m_1} . \frac{\dot{u}_1}{u_1}$$

$$\frac{\dot{x}_2}{x_2} = \frac{1}{m_2} . \frac{\dot{u}_2}{u_1}$$

$$\text{\&c.} = \text{\&c.}$$

and ∴ we get, by substitution,

$$u_1\left\{\frac{a_1}{m_1}\cdot\frac{\dot{u}_1}{u_1}+\frac{b_1}{m_2}\cdot\frac{\dot{u}_2}{x_2}+\ldots\ldots\right\}+u_2\left\{\frac{a_2}{m_1}\cdot\frac{\dot{u}_1}{u_1}+\&c\right\}$$

+ &c. = 0, which is homogeneous, the sum of exponents in each term being Zero.

Ex. 1. $\frac{p\dot{x}}{x}+\frac{r\dot{y}}{y}=\frac{x^m\dot{x}}{y^n}$. (see p. 290).

First we reduce this to the above form by bringing it to

$$\left(\frac{p\dot{x}}{x}+\frac{r\dot{y}}{y}\right)y^n=x^{m+1}.\frac{\dot{x}}{x}.$$

Then, putting $y^n=u$, $x^{m+1}=v$, and substituting $\frac{1}{n}\cdot\frac{\dot{u}}{u}$ for $\frac{\dot{y}}{y}$, and $\frac{1}{m+1}\cdot\frac{\dot{v}}{v}$ for $\frac{\dot{x}}{x}$, we get

$$\left(\frac{p}{m+1}\cdot\frac{\dot{v}}{v}+\frac{r}{n}\cdot\frac{\dot{u}}{u}\right)u=\frac{1}{m+1}\dot{v}$$

which is manifestly homogeneous, and may therefore be integrated whatever be the value of n.

It appears from the text, that Simpson was not acquainted with any method of integrating the above equation, except in the case of $n=r$. It is easy even by the method of factors (that which he uses), to accomplish this object.

Equations of the form

$$(Ax^my^n+Bx^{m_1}y^{n_1}+\ldots\ldots\ldots\ldots)\left(a\frac{\dot{x}}{x}+b\frac{\dot{y}}{y}\right)+$$

$$(A^1x^py^q+B^1x^{p_1}y^{q_1}+\ldots\ldots\ldots)\left(a^1\frac{\dot{x}}{x}+b^1\frac{\dot{y}}{y}\right)=0$$

are reducible to homogeneity, whenever

$$\frac{m_1-m}{n-n_1}=\frac{m_2-m_1}{n_1-n_2}=\&c.=\frac{p_1-p}{q-q_1}=\frac{p_2-p_1}{q_1-q_2}=\&c.$$

subsist simultaneously.

For, putting $\left.\begin{array}{r} y = u^r \\ \text{and } x = v^s \end{array}\right\}$ and substituting for x, y, and $\dot{x}$, $\dot{y}$, the second factors preserve their form, and first factors are rendered homogeneous by making

$$nr + ms = n_1 r + m_1 s = \&c. = qr + ps = q_1 r + p_1 s = \&c.$$

whence

$$\frac{m_1 - m}{n - n_1} = \frac{m_2 - m_1}{n_1 - n_2} = \&c.$$

Generally $f.\,(x,\ y,\ z,\ \&c.)\,.\,(a\,\frac{\dot{x}}{x} + b\,\frac{\dot{y}}{y} + \&c.)$ $= f'.\,(x,\ y,\ z,\ \&c,)\,(a'\,\frac{\dot{x}}{x} + b'\,\frac{\dot{y}}{y} + \ldots.)$ *may always be rendered homogeneous when the functions denoted by* f, f', *are capable of such transformation by substitutions of the form*

$$x = u^r,\ y = u_1^{r_1}\ \&c.$$

For by substituting for x, y, &c. the powers of any other variables, we do not change the form of

$$(a\,.\,\frac{\dot{x}}{x} + b\,.\,\frac{\dot{y}}{y} + \ldots\ldots\ldots.)\ \&c.\ \&c.$$

which, therefore, remain homogeneous.

The foregoing considerations will enable the student to integrate a great number of curious and interesting equations. Many will, doubtless, occur, of a perfectly novel character.

The equation $\frac{\dot{y}}{\dot{x}} - \frac{x^2}{\sqrt{a + y}} \times \frac{\dot{y}}{\dot{x}} = x^3$ *becomes homogeneous by putting* $\left.\begin{array}{r} \sqrt{a+y} = z \\ x^2 = u \end{array}\right\}$, *and we find, after much labour, that*

$$\frac{a+y}{c} = \left\{\frac{x^2 + 2\,(1 + \sqrt{2})\,\sqrt{a+y}}{x^2 + 2\,(1 - \sqrt{2})\,\sqrt{a+y}}\right\}^{\frac{1}{\sqrt{2}}}$$

Also $\frac{\dot{y}}{\dot{x}} = \sqrt{2x^2+by}$ becomes so, by putting $y = z^2$.

Again, $\frac{\dot{y}}{\dot{x}} + \frac{Ay+Be^x}{ay+ye^{-x}} = 0$, becomes homogeneous by putting $u = e^x$.

IV. Having discussed at some length the subject of *Homogeneous Equations*, we now come to an extensive and highly important class, called *Linear Equations.*

To integrate the general Linear Equation of the first order

$$\frac{\dot{y}}{\dot{x}} + Xy + X' = 0$$

where X, X' denote any functions of x.

Let $y = uv$. Then

$$\frac{\dot{y}}{\dot{x}} = \frac{v \cdot \dot{u}}{\dot{x}} + \frac{u \cdot \dot{v}}{\dot{x}}$$

and by substitution we get

$$\frac{v \cdot \dot{u}}{\dot{x}} + \frac{v \cdot \dot{v}}{\dot{x}} + X \cdot uv + X' = 0;$$

and since we can make another assumption with regard to u, &c. let

$$\left.\begin{aligned} u \times \left(\frac{\dot{v}}{\dot{x}} + Xv\right) = 0, \\ \text{Then } v \cdot \frac{\dot{u}}{\dot{x}} + X' = 0. \end{aligned}\right\}$$

Hence $\frac{\dot{v}}{v} + X\dot{x} = 0$

$\therefore l \cdot v + \int X\dot{x} = \text{const.} = l \cdot c.$

$\therefore \frac{v}{c} = e^{-\int X\dot{x}}$

and $v = ce^{-\int X\dot{x}}$.

Hence $\dot{u} = -\frac{X'\dot{x}}{v} = -\frac{\dot{x}X'e^{\int X\dot{x}}}{c}$

$$\therefore u = -\int\frac{\dot{x}X'e^{\int X\dot{x}}}{c} + c'$$

$\therefore y = uv = -e^{-\int X\dot{x}} \times \{-\int\dot{x}X'e^{\int X\dot{x}} + C\}\ldots\ldots (a)$

C, the final arbitrary constant, being put $= cc'$.

Ex. 1. $\frac{\dot{y}}{\dot{x}} - \frac{y}{a} - \frac{c}{a} . x^n = 0,$

$$\text{Here } X = -\frac{1}{a}$$

$$X' = -\frac{c}{a}x^n$$

$$\therefore \int X\dot{x} = -\frac{x}{a}$$

and $\int\dot{x}X'e^{\int X\dot{x}} = -\frac{c}{a}\int x^n\dot{x}\, e^{-\frac{x}{a}}.$

But, integrating *by parts*, we have

$$\int x^n e^{-\frac{x}{a}}\dot{x} = -ax^n e^{-\frac{x}{a}} + an\int x^{n-1} e^{-\frac{x}{a}}\dot{x}$$

$$\int x^{n-1} e^{-\frac{x}{a}}\dot{x} = -ax^{n-1} e^{-\frac{x}{a}} + a.(n-1)\int x^{n-2} e^{-\frac{x}{a}}\dot{x}$$

$$\int x^{n-2} e^{-\frac{x}{a}}\dot{x} = -ax^{n-2} e^{-\frac{x}{a}} + a.(n-2)\int x^{n-3} e^{-\frac{x}{a}}\dot{x}$$

&c. = &c.

Hence, by successively substituting, we get

$\int\dot{x}X'e^{\int X\dot{x}} = ce^{-\frac{x}{a}}(x^n + nax^{n-1} + n.(n-1)a^2x^{n-2} + n.(n-1)(n-2)a^3x^{n-3}\ldots\ldots)$

$\therefore y = e^{\frac{x}{a}}\{-ce^{-\frac{x}{a}}(x^n + nax^{n-1} + n.(n-1)a^2x^{n-2} + \ldots) + C\}$

$$= -c(x^n + nax^{n-1} + n.(n-1)a^2x^{n-2} + \ldots) + Ce^{\frac{x}{a}}$$

which is exactly the same result as that in page 293, deduced by our author from principles less intelligible and conclusive. For the controversy which that equation led to, see *London Mag. for June* 1775, and *Turner's Exercises*, page 94. See also Irish Trans. VII, 351.

In his *Miscellaneous Tracts*, Simpson subsequently gave a general method of integrating *Linear Equations with constant coefficients of all orders.*

Ex. 2. $\frac{\dot{y}}{\dot{x}} + \frac{ny}{\sqrt{1+x^2}} - a = 0.$

Ans. $y = C\,(x+\sqrt{1+x^2})^{-n} \frac{a}{2(n-1)} \cdot (x+\sqrt{1+x^2})^{-1} + \frac{a}{2\,(n+1)}\,(x+\sqrt{1+x^2})$ ...... Euler Inst. Cal. Int. I.

Ex. 3. $\frac{\dot{y}}{\dot{x}} + (n-1)\frac{y}{x} - \frac{n-1}{x} + \frac{m}{x-1} - \frac{x}{(x-1)^2} = 0.$

Ans. $y = x^{1-n}\,\{C + x^{n+1} + m\int\frac{x^{n-1}\dot{x}}{1-x} + \int\frac{x^n\dot{x}}{(1-x)^2}\}$

which expresses the sum of the series

$$1 + \frac{m}{n}\,x + \frac{m+1}{n+1}\,x^2 + \frac{m+2}{n+2}\,x^3 + \ldots\ldots\ldots \infty.$$

See Lagrange. *Théorie des Fonctions Analytiques,* page 102.

Ex. 4. *To find the nature of a curve such that, a line being drawn from the vertex, making an angle of 45° with the axis, the ordinate may be to the corresponding subtangent, as (a) to that part of the ordinate produced, which is intercepted by the curve and the line drawn from the vertex.*

Let x = the abscissa of the curve measured from the vertex, and y the corresponding ordinate. Then since the $\angle$ is 45°, the part of the ordinate cut off by the line issuing from the vertex $= x$. $\therefore$ the part intercepted $= y - x$. Hence

$$y : \frac{y\dot{x}}{\dot{y}} :: a : y - x,$$

which gives

$$\frac{a\dot{x}}{\dot{y}} = y - x,$$

which is *linear*, with respect to x and its fluxion.

This may be integrated more expeditiously than by the general method, in putting

$$y - x = u,$$

which gives

$$\frac{\dot{y}}{a} = \frac{a\dot{u}}{a - u}.$$

Hence $\frac{y}{a} = lc - l\,(a - u) = l \,.\, \frac{c}{a - u}$

$$\therefore e^{\frac{y}{a}} = \frac{c}{a - y + x}$$

$$\therefore x = y + ae^{-\frac{y}{a}} + C = y + ae^{-\frac{y}{a}} - a,$$

which expresses the nature required.

The Equation

$$\frac{\dot{y}}{\dot{x}} + Xy + X'y^{n+1} = 0,$$

is reducible to the linear form, by putting $u = y^{-n}$.

For the equation becomes, by substituting for y and $\dot{y}$,

$$\frac{\dot{u}}{\dot{x}} - nXu - nX' = 0.$$

Hence

$$\frac{1}{y^n} = u = - e^{n\int X\dot{x}} \{\int n\dot{x}X'e^{\int n\dot{x}X} + C\}.$$

Ex. 1. $\frac{\dot{y}}{\dot{x}} + y - xy^3 = 0$, gives an integral,

$$\frac{1}{y^2} = u = x + \tfrac{1}{2} + ce^{2x}.$$

Ex. 2. $\frac{\dot{y}}{\dot{x}} + \frac{xy}{1 - x^2} - x\sqrt{y} = 0$, gives

$$\sqrt{y} = C \,.\, (1 - x^2)^{\frac{1}{4}} - \frac{1 - x^2}{3}.$$

V.—*Whenever the equation*

$$P\dot{y} + Q\dot{x} = 0 \ldots\ldots\ldots\ldots\ldots (a)$$

gives

$$\frac{P^{\cdot}}{\dot{x}} = \frac{Q^{\cdot}}{\dot{y}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b)$$

(P and Q denoting functions of x y and constants, and $\frac{P^{\cdot}}{\dot{x}}$, $\frac{Q^{\cdot}}{\dot{y}}$ *the partial fluxional coefficients of P and Q, relative to x and y respectively), it is integrable.*

For, let $P = \frac{\dot{u}}{\dot{y}}$ *(u* being a function of x, y, at present undetermined*)*.

Then $\frac{P^{\cdot}}{\dot{x}} = \frac{\ddot{u}}{\dot{y}\dot{x}} = \frac{Q^{\cdot}}{\dot{y}}$ (by equation b).

But $\frac{\ddot{u}}{\dot{y}\dot{x}} = \frac{\ddot{u}}{\dot{x}\dot{y}}$ (page 293, vol. 1)

$$\therefore \frac{Q^{\cdot}}{\dot{y}}\dot{y} = \frac{\ddot{u}}{\dot{x}\dot{y}}\dot{y},$$

and taking the integrals on the supposition that y is the only variable, we have $Q = \frac{\dot{u}}{\dot{x}}$.

Hence, by substitution, equation *(a)* becomes

$$\frac{\dot{u}}{\dot{y}} \cdot \dot{y} + \frac{\dot{u}}{\dot{x}}\dot{x} = 0 = \dot{u} \ldots\ldots\ldots\ldots\ldots (a^1)$$

which is a complete fluxion.

$$\left.\begin{array}{r}\text{Now, since } \frac{\dot u}{\dot y}\,\dot y = P\dot y \\ \text{and } \frac{\dot u}{\dot x}\,\dot x = Q\dot x\end{array}\right\}\text{ we have}$$

$$\left.\begin{array}{r}u = \int_x P\dot y + X \\ \text{or } u = \int_y Q\dot x + Y\end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ (c)$$

$\int_x$ or $\int_y$ denoting that the integral is taken on the supposition of x or y being constant, and X or Y a function of x or y respectively.

Hence

$$\int_x P\dot y - \int_y Q\dot x = Y - X \ldots\ldots\ldots\ldots\ (e),$$

from which equation we can evidently determine either Y or X, by putting $x = 0$, or $y = 0$.

The equations (e) and (c) are, consequently, sufficient to determine the integral of (a) whenever the equation of condition (b), called the Criterion of Integrability, *is found to subsist.*

Ex. 1. $\frac{\dot y}{y} - \frac{x\dot y - y\dot x}{y\sqrt{x^2+y^2}} = 0 = P\dot y + Q\dot x.$

$$\left.\begin{array}{r}\text{Here } P = \frac{1}{y} - \frac{x}{y\sqrt{x^2+y^2}} \\ Q = \frac{1}{\sqrt{x^2+y^2}}\end{array}\right\}$$

$$\therefore \frac{\dot P}{\dot x} = -\frac{y}{(x^2+y^2)^{\frac{3}{2}}} = \text{also } \frac{\dot Q}{\dot y}.$$

The equation is, therefore, integrable.

$$\text{Again } \int_x P\dot y = \int_x \left(\frac{\dot y}{y} - \frac{x\dot y}{y\sqrt{x^2+y^2}}\right)$$

$$= l\,.\,y - \tfrac{1}{2}\,l\,.\,\frac{\sqrt{x^2+y^2} - x}{\sqrt{x^2+y^2}+x} = ly - l\,.\,\frac{y}{\sqrt{x^2+y^2}+x}$$

(see Vol. I, pp. 141 and 342) $= l.(\sqrt{x^2+y^2}+x)$;

also $\int_y \dot{x} = \int_{\dot{y}} . \dfrac{\dot{x}}{\sqrt{x^2+y^2}} = l.(\sqrt{x^2+y^2}+x)$ (Vol. I, page 140.)

$$\therefore Y - X = 0.$$

$$\text{Let } x = 0$$

$$\text{Then } Y - const. = 0$$

$$\text{or } Y = C$$

$$\therefore u = \int_y Q\dot{x} + Y = l(\sqrt{x^2+y^2}+x) + C = 0.$$

$$\text{Let } C = -l.c.$$

$$\text{Then } l(\sqrt{x^2+y^2}+x) = lc$$

and we finally get

$$\sqrt{x^2+y^2}+x = c.$$

The equation, being *homogeneous*, might have been integrated by the process delivered in p. 305, Vol. II.

Ex. 2. $2axy\dot{x} + ax^2\dot{y} - y^3\dot{x} - 3xy^2\dot{y} = 0.$

$$\left.\begin{array}{l} \text{Here } P = ax^2 - 3xy^2 \\ Q = 2axy - y^3 \end{array}\right\}$$

$$\therefore \frac{P^{\cdot}}{\dot{x}} = 2ax - 3y^2 = \frac{Q^{\cdot}}{\dot{y}},$$

or the equation is integrable.

$$\text{Now } \int_x P\dot{y} = \int_x (ax^2\dot{y} - 3xy^2\dot{y}) = ax^2y - xy^3$$

$$\text{and } \int_y Q\dot{x} = \int_y (2axy\dot{x} - y^3\dot{x}) = ayx^2 - y^3x$$

$$\therefore Y - X = ax^2y - x^3y - ayx^2 + y^3x$$

$$= ax^2y - ayx.$$

$$\text{Let } x = 0.$$

$$\text{Then } Y - C = 0$$

$$u = \int_y Q\dot{x} + Y = ayx^2 - y^3x + C = 0$$

is the integral required.

Ex. 3. $\dot{x}\ \sin. y + x\dot{y}\cos. y + \dot{y}\sin. x + y\dot{x}\cos. x = 0.$

$$\left.\begin{array}{l}\text{Here } P = x\cos. y + \sin. x \\ \quad\ \ Q = y\cos. x + \sin. y\end{array}\right\}.$$

$$\therefore \frac{P^{\cdot}}{\dot{x}} = \cos. y + \cos. x = \frac{Q^{\cdot}}{\dot{y}},$$

or the equation is integrable.

Again,

$$\int_x P\dot{y} = \int_x (x\dot{y}\cos. y + \dot{y}\sin. x)$$
$$= x\sin. y + y\sin x \text{ (Vol. I, page 335)},$$
$$\text{and } \int_y Q\dot{x} = \int_y (y\dot{x}\cos. x + \dot{x}\sin. y)$$
$$= y\sin. x + x\sin y.$$

Hence $Y - X = 0$

Let $x = 0$

Then $Y - C = 0$, or $Y = C$.

$\therefore u = \int_y Q\dot{x} + Y = y\sin. x + x\sin. y + C = 0,$
is the integral required.

In the preceding examples, Y has merely been an arbitrary constant, or a function of y involving $\dot{y}^0$ only. Those that follow exemplify the Theory more fully.

Ex. 4. $\dot{x}(ax + by + g) + \dot{y}(bx + By + G) = 0.$

$$\left.\begin{array}{l}\text{Here } P = bx + By + G \\ \quad\ \ Q = ax + by + g\end{array}\right\}.$$

$$\therefore \frac{P^{\cdot}}{\dot{x}} = b = \frac{Q^{\cdot}}{\dot{y}},$$

or the equation is integrable.

Again,

$$\int_x P\dot{y} = bxy + \frac{B}{2}y^2 + Gy$$

$$\int_y Q\dot{x} = \frac{a}{2}x^2 + bxy + gx$$

$\therefore Y = X + \frac{B}{2}y^2 + Gy - \frac{a}{2}x^2 - yx$

Let $x = 0$.

Then $Y = C + \frac{B}{2}y^2 + Gy$.

$\therefore u = \frac{a}{2}x^2 + \frac{B}{2}y^2 + bxy + gx + Gy + C = 0$,

the integral required.

Ex. 5. Let $\dot{u} = \frac{a(x\dot{x}+y\dot{y})}{\sqrt{x^2+y^2}} + \frac{y\dot{x} - x\dot{y}}{x^2+y^2} + 3by^2\dot{y}$.

$$\left.\begin{aligned} \text{Then } P &= \frac{ay}{\sqrt{x^2+y^2}} - \frac{x}{x^2+y^2} + 3by^2 \\ Q &= \frac{ax}{\sqrt{x^2+y^2}} + \frac{y}{x^2+y^2} \end{aligned}\right\}$$

and $\frac{\dot{P}}{\dot{x}} = -\frac{axy}{(x^2+y^2)^{\frac{3}{2}}} - \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\dot{Q}}{\dot{y}}$,

or the equation is integrable.

Again,

$$\left.\begin{aligned} \textstyle\int_x P\dot{y} &= a\sqrt{x^2+y^2} - \tan.^{-1}\frac{y}{x} + by^3 \\ \text{and } \textstyle\int_y Q\dot{x} &= a\sqrt{x^2+y^2} - \tan.^{-1}\frac{x}{y} \end{aligned}\right\}$$

$\therefore Y = X + by^3 - \tan.^{-1}\frac{y}{x} - \tan.^{-1}\frac{x}{y}$.

Let $x = 0$.

Then $Y = C + by^3 - \frac{\pi}{2} - 0 = C' + by^3$.

Hence

$$u = a\sqrt{x^2+y^2} + by^3 + \tan.^{-1}\frac{x}{y} + C',$$

the integral required.

When $a = b = 0$, we have

$$u = \int \frac{y\dot{x} - x\dot{y}}{x^2 + y^2} = tan.^{-1} \frac{x}{y} + C',$$

which is employed by Laplace in the demonstration of the *parallelogram of forces.* (Mécan. Cél. t. I, p. 6).

Ex. 6. Let $\dot{u} = \dfrac{\dot{x}x + \dot{x}\sqrt{x^2 + y^2} + y\dot{y}}{(x + \sqrt{x^2 + y^2})\sqrt{x^2 + y^2}}$.

By proceeding as before, we get

$$u = C + l.\,(x + \sqrt{x^2 + y^2}).$$

Ex. 7. Let $\dot{u} = \dfrac{y\dot{x} - x\dot{y}}{(x+y)\sqrt{2y.(x-y)}} = 0.$

Then it will be easily found that

$$\frac{P'}{\dot{x}} = \frac{x^2 - xy + 2y^2}{2(x+y)^2.(x-y)\sqrt{2y(x-y)}} = \frac{Q'}{\dot{y}}.$$

Again,

$$\int_y Q\dot{x} = \int_y \frac{y\dot{x}}{(x+y)\sqrt{2y(x-y)}} = \sqrt{\frac{y}{2}} \int_y \frac{\dot{x}}{(x+y)\sqrt{x-y}}.$$

Let $\sqrt{\dfrac{x-y}{x+y}} = u$. Then $x = y \cdot \dfrac{1+u^2}{1-u^2}$.

$$\therefore \dot{x} = 4y \cdot \frac{u\dot{u}}{(1-u^2)^2},\quad x - y = 2y \cdot \frac{u^2}{1-u^2},$$

and $x + y = \dfrac{2y}{1-u^2}$. Hence by substitution we get

$$\int_y Q\dot{x} = \int_y \frac{\dot{u}}{\sqrt{1-u^2}} = sin.^{-1} u = sin.^{-1} \sqrt{\frac{x-y}{x+y}}$$

$$\therefore u = \int (P\dot{y} + Q\dot{x}) = sin.^{-1} \sqrt{\frac{x-y}{x+y}} + Y \ldots\ldots\ldots (1)$$

Now, instead of determining Y by finding the value of $\int_x P\dot{y}$, it will be easier to find its fluxion by taking that of equation (1.) on the supposition that x is constant.

Let $\sqrt{\frac{x-y}{x+y}} = v$

Then flux. of $\sin.^{-1} v = \frac{\dot{v}}{\sqrt{1-v^2}} = \frac{-\dot{y}x}{\sqrt{x-y}\,.\,(x+y)^{\frac{3}{2}}}$

$\times \frac{\sqrt{x+y}}{\sqrt{2y}} = \frac{-x\dot{y}}{(x+y)\sqrt{2y\,.\,(x-y)}} = P\dot{y}.$

Hence

$$\dot{Y} = P\dot{y} - P\dot{y} = 0.$$

and $Y = C$, an arbitrary constant.

$$\therefore u = \sin.^{-1} \sqrt{\frac{x-y}{x+y}} + C = 0$$

$$\therefore \sin.^{-1} \sqrt{\frac{x-y}{x+y}} = - C = \sin.^{-1} C'$$

$$\therefore \sqrt{\frac{x-y}{x+y}} = C'$$

Hence $\frac{x}{y} = \frac{1 + C'^2}{1 - C'^2} = C''$,

the integral in its most simple form.

The same result would have been more readily obtained by multiplying the proposed equation by its denominator, and dividing by y^2; for thereby we get at once,

$$\frac{y\dot{x}-x\dot{y}}{y^2} = \left(\frac{x}{y}\right)^{\cdot} = 0.$$

The above process, however, serves to show, that an equation $y\dot{x}-x\dot{y}=0$ may be rendered immediately integrable, either by multiplying each of its terms by

$\frac{1}{y^2}$ or by $\frac{1}{(x+y)\sqrt{2y.(x-y)}}$, a circumstance which induces us to investigate the integrability of fluxions, by means of the factor generally.

VI.—*There is a quantity, a function of x, y, which being multiplied into*

$$P\dot{y} + Q\dot{x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

will render it immediately integrable, P and Q being any functions whatever of x, y.

1. Supposing *eq.* (*a*) to have been derived from eliminating some constant (*m*) from its integral, $f(x, y, m) = 0$, and the immediate fluxion $\{f(x, y, m)\}^{\cdot} = 0$ (which may always be effected), let us put

$$m = \varphi(x, y)$$

Then $0 = m^{\cdot} = \frac{m^{\cdot}}{\dot{y}}\dot{y} + \frac{m^{\cdot}}{\dot{x}}\dot{x}$.

$$\left.\begin{array}{l} \therefore \dfrac{m^{\cdot}}{\dfrac{m^{\cdot}}{\dot{y}}} = \dot{y} + \dfrac{\dfrac{m^{\cdot}}{\dot{x}}}{\dfrac{m^{\cdot}}{\dot{y}}}\dot{x} = 0 \\ \text{also } \dot{y} + \dfrac{Q}{P}\dot{x} = 0 \end{array}\right\} \text{and since neither of these}$$

equations contains the constant (*m*), and are also prime to each other, they are identical.

$$\therefore m^{\cdot} = \frac{m^{\cdot}}{\dot{y}}.(\dot{y} + \frac{Q}{P}\dot{x}) = \frac{m^{\cdot}}{\dot{y}}.\frac{1}{P}(P\dot{y} + Q\dot{x}) \ldots\ldots\ldots (b)$$

and since $m^{\cdot}$ is a complete fluxion, $\frac{m^{\cdot}}{\dot{y}}.\frac{1}{P}$ must be the factor that renders $P\dot{y} + Q\dot{x}$ also a complete fluxion; that is, *whenever the equation (a) has arisen from the elimination of some constant (m) from its integral, by means of the immediate fluxion, that equation becomes immediately integrable by the factor* $\frac{m^{\cdot}}{\dot{y}}.\frac{1}{P}$.

The same may be proved of the factor $\frac{\dot{m}}{\dot{x}} \cdot \frac{1}{Q}$.

2. Let $P\dot{y}+Q\dot{x}=0$, be supposed to have been generated by the addition of $P_1\dot{y}+Q_1\dot{x}=0$, $P_2\dot{y}+Q_2\dot{x}=0$, $P_3\dot{y}+Q_3\dot{x}=0$, &c. which are each susceptible of immediate integration, by a factor of the form

$$\frac{\dot{m}_1}{\dot{y}} \cdot \frac{1}{P_1}, \frac{\dot{m}_2}{\dot{y}} \cdot \frac{1}{P_2}, \frac{\dot{m}_3}{\dot{y}} \cdot \frac{1}{P_3}, \text{ \&c. } m_1, m_2, \text{ \&c.}$$

being the eliminated constants; then, calling these factors, μ_1, μ_2, μ_3 &c. we have
$\dot{m}_1+\dot{m}_2+\dot{m}_3+\ldots=\mu_1.(P_1\dot{y}+Q_1\dot{x})+\mu_2.(P_2\dot{y}+Q_2\dot{x})$ $+\text{\&c.}=(\mu_1 P_1+\mu_2 P_2+\mu_3 P_3+\ldots)\,\dot{y}+(\mu_1 Q_1+\mu_2 Q_2+\ldots)\,\dot{x}$
and $\dot{m}_1+\dot{m}_2+\ldots\ldots$ being a complete fluxion, it will be shown as before, that $\frac{\mu_1 P_1+\mu_2 P_2+\ldots}{P}$ is a factor capable of completing the fluxion $P\dot{y}+Q\dot{x}$.

The proposition may be proved, in like manner for other cases. Those above, which, in a certain degree, coincide, are sufficient to show that a factor (μ) *exists* capable of rendering

$$(P\dot{y} + Q\dot{x})\,\mu = 0$$

a complete fluxion. Hence, it may easily be shown, that there are innumerable such factors;

For, putting $(P\dot{y}+Q\dot{x})\,\mu=\dot{m}$, we have

$$(P\dot{y}+Q\dot{x})\,\mu\,.\,\varphi\,(m) = \dot{m}\varphi m = \dot{m}',$$

φm denoting any function whatever of (m).

Ex. $x\dot{y} - \dot{y}x = 0$.

This equation we know has an integral $\frac{y}{x} = m$.

$$\therefore (x\dot{y} - y\dot{x})\,\mu = \dot{m} = \frac{x\dot{y} - y\dot{x}}{x^2}$$

$$\therefore \mu = \frac{1}{x^2} \text{ is one value of } \mu,$$

and $(x\dot{y}-y\dot{x})\frac{\varphi\cdot\frac{y}{x}}{x^2}=\left(\frac{y}{x}\right)^{\cdot}\varphi\cdot\frac{y}{x}=\left(\varphi'\frac{y}{x}\right)^{\cdot}=0$ being a complete fluxion, will give innumerable others.

The integral will be the same for every factor.

For, since $\left(\varphi'.\frac{y}{x}\right)^{\cdot}=0$,

$$\varphi'.\frac{y}{x}=c;$$

$$\therefore\frac{y}{x}=\varphi'^{-1}.c=c',$$

$\varphi^{-1}c$ denoting the inverse of φc.

Although, as we have just seen, there are innumerable factors capable of completing the fluxion

$$P\dot{y}+Q\dot{x}=0,\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (a)$$

they have been found but in very few cases, the principal of which are the following.

The Complementary Factor may be found, (1.) *when* $\frac{P^{\cdot}}{\dot{x}}=\frac{Q^{\cdot}}{\dot{y}}$; (2.) *when* $\frac{1}{P}.\left(\frac{Q^{\cdot}}{\dot{y}}-\frac{P^{\cdot}}{\dot{x}}\right)$ *is independent of* y, *or* $\frac{1}{Q}\left(\frac{Q^{\cdot}}{\dot{y}}-\frac{P^{\cdot}}{\dot{x}}\right)$ *of* x; (3) *when the integral is known;* (4) *when we can separate the variables;* (5) *when we can so decompose the equation into parts of the form* $P_1\dot{y}+Q_1\dot{x}$, $P_2\dot{y}+Q_2\dot{x}$ *&c. as to obtain out of the innumerable Complementary Factors of the parts, those which are identical.*

(1.) Let μ be the factor. Then since

$$\mu P\dot{y}+\mu Q\dot{x}=0,$$

is a complete fluxion, we have

$$\frac{(\mu P)^{\cdot}}{\dot{x}}=\frac{(\mu Q)^{\cdot}}{\dot{y}}\ \text{(p. 319.)}$$

$$\therefore P\,.\,\frac{\dot{\mu}}{\dot{x}} - Q\,\frac{\dot{\mu}}{\dot{y}} + \left(\frac{P^{\cdot}}{\dot{x}} - \frac{Q^{\cdot}}{\dot{y}}\right)\mu = 0 \;\ldots\ldots\ldots\; (b)$$

an equation which it generally is more difficult to integrate than the one proposed. In the present case, we have

$$P\,.\,\frac{\dot{\mu}}{\dot{x}} - Q\,\frac{\dot{\mu}}{\dot{y}} = 0.$$

But $Q = -\frac{\dot{y}}{\dot{x}}\,.\,P.$

$$\therefore \frac{\dot{\mu}}{\dot{x}}\,\dot{x} + \frac{\dot{\mu}}{\dot{y}}\,\dot{y} = 0,$$

or $\mu = const. = c.$

Equations of this kind satisfy the *Criterion of Integrability.*

(2.) Let $\frac{1}{P}\left(\frac{Q^{\cdot}}{\dot{y}} - \frac{P^{\cdot}}{\dot{x}}\right)$ be independent of y.

Then $\therefore \frac{\mu}{P}\,.\left(\frac{Q^{\cdot}}{\dot{y}} - \frac{P^{\cdot}}{\dot{x}}\right) = \frac{\dot{\mu}}{\dot{x}} - \frac{Q}{P}\,.\,\frac{\dot{\mu}}{\dot{y}}$ $(eq.\,6)$

$$= \frac{\dot{\mu}}{\dot{x}} + \frac{\dot{y}}{\dot{x}}\,.\,\frac{\dot{\mu}}{\dot{y}} = \frac{1}{\dot{x}}\left(\frac{\dot{\mu}}{\dot{x}}\,\dot{x} + \frac{\dot{\mu}}{\dot{y}}\,\dot{y}\right) = \frac{1}{\dot{x}}\,\dot{\mu}$$

$$\therefore \frac{\dot{\mu}}{\mu} = \frac{\dot{x}}{P}\left(\frac{Q^{\cdot}}{\dot{y}} - \frac{P^{\cdot}}{\dot{x}}\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (c)$$

whence we find μ in terms of x.

Similarly, if $\frac{1}{Q}\left(\frac{Q^{\cdot}}{\dot{y}} - \frac{P^{\cdot}}{\dot{x}}\right)$ be independent of x, it may be shown that

$$\frac{\dot{\mu}}{\mu} = \frac{\dot{y}}{Q}\left(\frac{Q^{\cdot}}{\dot{y}} - \frac{P^{\cdot}}{\dot{x}}\right) \;\ldots\ldots\ldots\ldots\; (e)$$

Ex. 1. *The Linear Equation is*

$$\dot{y} + Xy\dot{x} + X'\dot{x} = 0$$

Here $P = 1,\; Q = Xy + X'$

$\therefore \frac{1}{P} \cdot \left(\frac{\dot{Q}}{\dot{y}} - \frac{\dot{P}}{\dot{x}}\right) = X$ which being independent of y, we have

$$\frac{\dot{\mu}}{\mu} = \dot{x} X$$

$$\therefore \mu = e^{\int \dot{x} X}$$

$\therefore e^{\int \dot{x} X} \dot{y} + (Xy + X') e^{\int \dot{x} X} = 0$, is a complete fluxion, which may be integrated by the method in *p.* 319.

Ex. 2. $\dot{x} + (a\dot{x} + 2by\dot{y}) \sqrt{1+x^2} = 0$, gives $\frac{1}{P}\left(\frac{\dot{P}}{\dot{x}} - \frac{\dot{Q}}{\dot{y}}\right) = \frac{2byx}{\sqrt{1+x^2}} \times \frac{1}{2by\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$.
Hence by equation (*c*)

$$l.\,\mu = -l.\,\sqrt{1+x^2}$$

$$\therefore \mu = \frac{1}{\sqrt{1+x^2}}.$$

Ex. 3. $x^3\dot{y} + \left(4x^2 y - \frac{1}{\sqrt{1+x^2}}\right)\dot{x} = 0$, gives $\frac{\dot{\mu}}{\mu} = \frac{\dot{x}}{x}$, and multiplying by $\mu = x$, we get the integral

$$x^4 y + \sqrt{1-x^2} = c.$$

(3.) Let the known integral be $u = \int \mu (P\dot{y} + Q\dot{x})$. Then we have

$$\dot{u} = \frac{\dot{u}}{\dot{y}}\dot{y} + \frac{\dot{u}}{\dot{x}}\dot{x} = \mu P \dot{y} + \mu Q \dot{x},$$

which being identical

$$\mu P = \frac{\dot{u}}{\dot{y}} \text{ or } \mu Q = \frac{\dot{u}}{\dot{x}}$$

$$\therefore \mu = \frac{\dot{u}}{\dot{y}} \cdot \frac{1}{P} \text{ or } = \frac{\dot{u}}{\dot{x}} \cdot \frac{1}{Q}.$$

(4.) If x' and y' be the two new variables susceptible of being separated when substituted in the equation $P\dot{y}+Q\dot{x}=0$, the result will be of the form

$$X'\dot{x}' + Y'\dot{y}' = \mu \,.\, (P\dot{y} + Q\dot{x}),$$

X' being a function of x', and Y' of y.

Ex. *If* $P\dot{y} + Q\dot{x} = 0$, *be homogeneous, each term being supposed of* m *dimensions, the variables may be separated by putting* $y = x'x$, *and a Complementary Factor is* $\dfrac{1}{Py+Qx}$.

For substituting, we get

$$Px\dot{x}' + (Px' + Q)\,\dot{x} = P\dot{y} + Q\dot{x}$$

$$\therefore \dot{x}' + \left(x' + \frac{Q}{P}\right)\frac{\dot{x}}{x} = \frac{P\dot{y} + Q\dot{x}}{Px}$$

$$\text{and } \frac{\dot{x}'}{x' + \frac{Q}{P}} + \frac{\dot{x}}{x} = \frac{P\dot{y}+Q\dot{x}}{Px\left(x'+\frac{Q}{P}\right)} = \frac{P\dot{y}+Q\dot{x}}{Py+Qx}.$$

But $Q = x^m f.\,(x')$, and $P = x^m f'(x')$ since x will rise to the same dimension in every term of Q and P, by the substitution of $x'x$ for y.

$\therefore \dfrac{1}{x' + \frac{Q}{P}}$ is evidently a function of x' alone, or the first member of

$$\frac{\dot{x}'}{x' + \frac{Q}{P}} + \frac{\dot{x}}{x} = \frac{P\dot{y}+Q\dot{x}}{Py+Qx}$$

is immediately integrable.

$\therefore \dfrac{1}{Py+Qx}$ is a *complementary factor* of the *homogeneous equation*,

$$P\dot{y} + Q\dot{x} = 0.$$

This result may be verified by the *Criterion of Integrability*. It might have been deduced from Euler's Theorem (see page 295, Vol. I.)

The factor of

$$x\dot{y} - \dot{x}\,(y + \sqrt{x^2 + y^2}) = 0$$

being $xy - yx - x\sqrt{x^2 + y^2} = -x\sqrt{x^2 + y^2}$,

$$x^2\sqrt{x^2 + y^2}\,.\,\dot{y} - (xy\sqrt{x^2 + y^2} + x^3 + xy^2)\,\dot{x} = 0$$

becomes integrable by the method explained in p. 319, Vol. II. The integral is

$$y + \sqrt{x^2 + y^2} = cx^2.$$

(5.) Let $P\dot{y} + Q\dot{x} = P_1\dot{y} + Q_1\dot{x} + P_2\dot{y} + Q_2\dot{x}$

and suppose $\left.\begin{array}{l}(P_1\dot{y} + Q_1\dot{x})\,\mu_1 = \dot{u}\\ (P_2\dot{y} + Q_2\dot{x})\,\mu_2 = \dot{v}\end{array}\right\}$

Then $\mu_1\,\phi_1\,u$, $\mu_2\,\phi_2\,v$ will represent the general form of the *Complementary* Factors; and if, by any artifice, we can identify them, either of them will, evidently, be the Complementary Factor of $P\dot{y} + Q\dot{x} = 0$.

Ex. 1. $Ay\dot{x} + Bx\dot{y} + ay^m x^n\dot{x} + by^{m-1}x^{n+1}\dot{y} = 0$.

Here $\dot{u} = \dfrac{A\dot{x}}{x} + \dfrac{B\dot{y}}{y} = \dfrac{1}{xy}\,.\,(Ay\dot{x} + Bx\dot{y})$

and $\dot{v} = \dfrac{a\dot{x}}{x} + \dfrac{b\dot{y}}{y} = \dfrac{1}{x^{n+1}y^m}\,.\,(ay^m x^n\dot{x} + by^{m-1}x^{n+1}\dot{y})$

$$\therefore \mu_1 = \frac{1}{xy},\ \mu_2 = \frac{1}{x^{n+1}y^m}$$

$$u = Alx + Bly = l\,.\,x^A y^B$$

$$v = alx + bly = l\,.\,x^a y^b.$$

Hence the general Factors are

$$\frac{1}{xy}\,.\,\phi_1 u = \frac{1}{xy}\,\phi\,(l\,.\,x^A y^B) = \frac{1}{xy}\,.\,\phi_1\,(x^A y^B)$$

and $\frac{1}{x^{n+1}y^m}\cdot\phi_2 v = \frac{1}{x^{n+1}y^m}\cdot\phi_2(l.\,x^a y^b) = \frac{1}{x^{n+1}y^m}\,\phi_2(x^a y^b)$

since ϕ_1 and ϕ_2 represent arbitrary functions. Now to select such forms of these Factors as shall render them identical, we will put

$$\frac{1}{xy}\cdot\phi_1(x^A y^B) = \frac{1}{x^{n+1}y^m}\cdot\phi_2(x^a y^b)$$

and making $\phi_1(x^A y^B) = (x^A y^B)^G$

and $\phi_2(x^a y^b) = (x^a y^b)^g$

by substitution, &c. we get

$$x^{AG-1}y^{BG-1} = x^{ag-n-1}y^{bg-m}$$

which will be identical if

$$\left.\begin{array}{r} AG = ag - n \\ \text{and } BG - 1 = bg - m \end{array}\right\} \text{ or if}$$

$$G = \frac{a+bn-am}{aB-bA},$$

$$\text{and } g = \frac{A+Bn-Am}{aB-bA}.$$

$\therefore$ the Complementary Factor of the proposed equation is

$$x^{A\cdot\frac{a+bn-am}{aB-bA}-1} \times y^{B\cdot\frac{a+bn-am}{aB-bA}-1},$$

which will give an integral

$$\frac{1}{G}\cdot x^{AG}y^{BG} + \frac{1}{g}\cdot x^{ag}y^{bg} + C = 0.$$

Ex. 2. $(ax-y\sqrt{x^2+y^2-a^2})\,\dot{y}-(ay+x\sqrt{x^2+y^2-a^2})\dot{x}$
$= 0 = ax\dot{y} - ay\dot{x} - (y\dot{y} + x\dot{x})\sqrt{x^2+y^2-a^2}$
$= ax^2.\left(\frac{y}{x}\right)^{\cdot} - \frac{1}{3}.\left\{(x^2+y^2-a^2)^{\frac{3}{2}}\right\}^{\cdot}.$

$$\therefore \mu_1 = x^{-2},\ \mu_2 = 1$$

$$u = \frac{y}{x},\ v = (x^2 + y^2 - a^2)^{\frac{1}{2}}$$

$$\therefore \mu_1 \varphi_1 u = \frac{1}{x^2}\varphi_1 . \left(\frac{y}{x}\right) = \frac{1}{x^2}\varphi_1 . \frac{1}{1+\frac{y^2}{x^2}}$$

and $\mu_2 \varphi_2 v = \varphi_2 . (x^2+y^2-a^2)^{\frac{1}{2}} = \varphi_2 . (x^2+y^2)$

$$= \varphi_2 . \frac{1}{x^2+y^2} = \varphi_2 . \left(\frac{1}{x^2 e} - \frac{1}{1+\frac{y^2}{x^2}}\right)$$

Let $\frac{1}{x^2}\varphi_1 \frac{1}{1+\frac{y^2}{x^2}} = \frac{1}{x^2} = x\frac{1}{2}\frac{1}{x^2} . \frac{1}{1+\frac{y^2}{x^2}} = \frac{1}{x^2+y^2}$

and $\varphi_2 . \frac{1}{x^2} . \frac{1}{1+\frac{y^2}{x^2}} = \frac{1}{x^2} . \frac{1}{1+\frac{y^2}{x^2}} = \frac{1}{x^2+y^2}$,

which determining the identity of the factors of the parts, give that of the proposed equation $= \frac{1}{x^2+y^2}$, and the integral is

$$a\ tan.^{-1}\frac{y}{x} + a\ tan.^{-1}\frac{\sqrt{x^2+y^2-a^2}}{a} - \sqrt{x^2+y^2-a^2} = C$$

$$\text{or } a\ .\ tan.^{-1}\frac{ay+x\sqrt{x^2+y^2-a^2}}{ax-y\sqrt{x^2+y^2-a^2}} - \sqrt{x^2+y^2-a^2} = C.$$

In the preceding discussion we have seen that every equation of the form $P\dot{y}+Q\dot{x}=0$, is susceptible of being rendered immediately integrable by means of a Factor, although it is very difficult, if not impossible, in most cases, to discover the form of that factor. If this could be effected generally, the integration of equations between two variables, would be completely accomplished.

Euler and others, despairing of any great success in this pursuit, have abandoned it for the reverse one—that of finding the relation which ought to subsist between P and Q, given in form only, that the equation $P\dot{y}+Q\dot{x}=0$ may become integrable when multiplied by a factor also given in form. By this method an infinity of equations are integrable, but being such as scarcely ever are encountered in the Resolution of Problems in the other branches of science, it is sufficient to state that such a method exists. We should not, indeed, have insisted so much at length on the Direct Method, were it not of importance, as we shall see hereafter, to establish the fact of the existence of a Factor capable of rendering $P\dot{y}+Q\dot{x}=0$, a complete fluxion.

Simpson appears from pp. 289 and 290, Vol. II, to have been but little acquainted with this Theory.

We now pass on to *Singular Solutions of Fluxional Equations*, which are so denominated because of their not being comprised in the general form, with the arbitrary constant of the integral.

VII.—*Given $y = X$, (a function of x) an integral of $\frac{\dot{y}}{\dot{x}} = F(x, y)$, to find whether it be a Singular Solution, or merely a particular Integral deducible from the General Integral $y_1 = f(x, c)$ by assigning to the arbitrary constant c some particular value c'.*

If $y=X$ be a Particular Integral, since $y-y_1$ must $=0$ when $c = c'$, we have, by the Theory of Algebraic Equations,

$$y_1 - y = (c - c')^m \times Q \text{............} (1)$$

m being the highest index of $c - c'$, and Q such a function of x and $c-c'$ as shall neither $=0$ nor ∞ when $c = c'$.

Let $(c-c')^m \times Q = h$. Then

$y_1 = y + h$, and by the question $\frac{\dot{y}_1}{\dot{x}} = F(x, y_1)$

$= F(x, y+h) = \frac{\dot{y}}{\dot{x}} + \frac{\dot{h}}{\dot{x}} = F(x, y) + \frac{\dot{h}}{\dot{x}}$. But, by Taylor's Theorem, $F(x, y+h) = F(x, y) + \frac{F^{\cdot}(x, y)}{\dot{y}} \cdot h + \frac{F^{\cdot\cdot}(x, y)}{\dot{y}^2} \frac{h^2}{1 \cdot 2} + \&c.$

$$\therefore \frac{\dot{h}}{\dot{x}} = \frac{F^{\cdot}(x, y)}{\dot{y}} h + \frac{F^{\cdot\cdot}(x, y)}{\dot{y}^2} \cdot \frac{h^2}{1 \cdot 2} + \&c.$$

$$= \frac{\ddot{y}}{\dot{x}\dot{y}} h + \frac{\dddot{y}}{\dot{x}\dot{y}^2} \cdot \frac{h^2}{1 \cdot 2} + \&c. \ldots\ldots\ldots\ldots (2).$$

But $\frac{\dot{h}}{\dot{x}}$ is evidently (by the supposition with regard to Q) susceptible of developement, according to the ascending powers of h. Hence then, we infer that $y = X$, will be a Particular Integral or a Singular Solution, according as the members of equation (2) can or cannot be rendered *identical*, that is, according as $\frac{\ddot{y}}{\dot{x}\dot{y}}$ or $\frac{\dddot{y}}{\dot{x}\dot{y}^2}$ &c. is not ∞ or is ∞.

Hence *to find the Singular Solutions of an equation* $\frac{\dot{y}}{\dot{x}} = F(x, y)$, *put* $\frac{\ddot{y}}{\dot{x}\dot{y}} = \infty$, *and the resulting values between x and y which satisfy the proposed will be Singular Solutions. Others may be found by the like research with regard to* $\frac{\ddot{x}}{\dot{y}\dot{x}} = \infty$.

Ex. 1. $\frac{\dot{y}}{\dot{x}} = 1 + \frac{xy - y^2}{a^2}$.

Here $\frac{\ddot{y}}{\dot{x}\dot{y}} = \frac{x - 2y}{a^2} = \infty$ gives no result. But

$$\frac{\ddot{x}}{\dot{y}\dot{x}} = \frac{-a^2 y}{(a^2 + xy - y^2)^2} = \infty \text{ gives}$$

$$y^2 - xy + a^2 = 0$$

which does not satisfy the proposed, and is therefore no singular Solution.

Ex. 2. $\frac{\dot{y}}{\dot{x}} = 2x \pm 2\sqrt{x^2 - y}$.

Here $\frac{\ddot{y}}{\dot{x}\dot{y}} = \frac{1}{\sqrt{x^2 - y}} = \infty$

$\therefore y = x^2$ is a Singular Solution,

and $\frac{\ddot{x}}{\dot{y}\dot{x}} = \frac{-x - \sqrt{x^2 - y}}{2\sqrt{x^2 - y}\,(x + \sqrt{x^2 - y})^2} = \infty$

and $\therefore x + \sqrt{x^2 - y} = 0$, giving $y = 0$ a Singular Solution.

Ex. 3. $\frac{\dot{y}}{\dot{x}} = \frac{x}{\sqrt{x^2 + y^2 - c^2} - y}$, gives

$$x^2 + y^2 = c^2.$$

Ex. 4. $\frac{a\dot{y}}{\dot{x}} = \sqrt{y^2 - x^2} + a$, gives

$$y = x.$$

Other examples may be seen in a Collection of Examples of Applications of Integral Cal. by G. Peacock, A. M. &c.

Given the General Integral $u = f(x, y, c) = 0$ *of* $P\dot{y} + Q\dot{x} = 0$, *to find its Singular Solutions, if any.*

Since $\dot{u} = \frac{\dot{u}}{\dot{x}}\dot{x} + \frac{\dot{u}}{\dot{y}}\dot{y} + \frac{\dot{u}}{\dot{c}}\dot{c} = 0$, it is clear that c may either be constant, (when $\dot{c} = 0$), or such a function of x, y, that

$$\frac{\dot{u}}{\dot{c}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In the former case $c = const.$ being substituted in $f(x, y, c) = 0$, gives only a Particular Integral. In the latter, c may either be a particular constant, or a function of (x, y), which being substituted in $f(x, y, c)$

$=0$ will yield either a Particular Integral, or a Singular Solution respectively.

Hence the values of c *derivable from Eq*. (1) *which are variable, produce*, (except in extreme cases) *when substituted in the General Integral, Singular Solutions.*

Ex. 1. Given $u = x^2 - \sin. (2y + c) = 0$ the General Integral of $\dot{y} = \frac{x\dot{x}}{\sqrt{1 - x^4}}$, to find its Singular Solutions.

Here $\frac{\dot{u}}{\dot{c}} = \cos. (2y + c) = 0$; $\therefore 2y + c = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, &c. $\therefore 1 - x^2 = 0$, and $1 + x^2 = 0$ are both Singular Solutions.

Ex. 2. Given $(x^2 + y^2 - b)(y^2 - 2cy) + (x^2 - b)c^2 = 0$ to find the Singular Solutions.

By *Eq*. (1) we obtain

$c = \frac{y(x^2 + y^2 - b)}{x^2 - b}$, which gives, when substituted,

$$\frac{y^4(x^2 + y^2 - b)}{x^2 - b} = 0, \text{ and } \therefore x^2 + y^2 - b = 0$$

for a Singular Solution. This, however, is deducible from the General Integral in the extreme case of $c = 0$.

Ex. 3. $u = y^2 - 2cy + x^2 - c^2 = 0$, the Integral of $(x^2 - 2y^2)\left(\frac{\dot{y}}{\dot{x}}\right)^2 - 4xy\frac{\dot{y}}{\dot{x}} - x^2 = 0$, gives $c = -y$ and $x^2 + 2y^2 = 0$ for a Singular Solution.

Ex. 4. Required the nature of the curve to which a straight line cutting off the same area from an angular space in every position, is always a tangent.

Taking the line of abscissæ along that bisecting the given angle (C), and its origin at C, let x and y be the rectangular co-ordinates of the curve required. Also put $q^2 =$ the given area. Then T being the point of intersection of the axis CT with the cutting line AB, and PN the ordinate, at any point of contact P, we evidently have,

$$2q^2 = CT \times sin.\ T\ \{AT + BT\} = CT^2\ sin.\ T\ sin.\ \frac{C}{2} \cdot$$

$$\left\{ \frac{1}{sin.\ (T - \frac{C}{2})} + \frac{1}{sin.\ (T + \frac{C}{2})} \right\} = \frac{CT^2\ sin.^2\ T \times sin.\ C}{cos.^2\ \frac{C}{2} - cos.^2\ T}.$$

Hence, by proper reduction, we get

$$CT^2 + q^2\ tan.\ \frac{C}{2} \times cot.^2\ T = q^2\ cot.\ \frac{C}{2}.$$

But $CT = x - y\,\frac{\dot{x}}{\dot{y}}$, $cot.\ T = \frac{\dot{x}}{\dot{y}}$; substituting, therefore, and putting $q^2\ tan.\ \frac{C}{2} = b^2$, and $q^2\ cot.\ \frac{C}{2} = a^2$, and taking the root, we have,

$$x - y\,\frac{\dot{x}}{\dot{y}} = b \times \sqrt{\frac{a^2}{b^2} - \frac{\dot{x}^2}{\dot{y}^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now taking the fluxions on the supposition that $\dot{y}$ is constant, there results

$$\frac{\ddot{x}}{\dot{y}}\ \left\{ y \sqrt{\frac{a^2}{b^2} - \frac{\dot{x}^2}{\dot{y}^2}} - b\,\frac{\dot{x}}{\dot{y}} \right\} = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$\therefore \begin{cases} \frac{\ddot{x}}{\dot{y}} = 0 \text{ and } \frac{\dot{x}}{\dot{y}} = c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3) \\ \text{or } y \sqrt{\frac{a^2}{b^2} - \frac{\dot{x}^2}{\dot{y}^2}} = b\,\frac{\dot{x}}{\dot{y}}, \text{ and } \frac{\dot{x}}{\dot{y}} = \frac{\frac{a}{b}y}{\sqrt{b^2 + y^2}} \ \ldots \quad (4) \end{cases}$$

If $\frac{\dot{x}}{\dot{y}} = c$ be substituted in (1) there results

$$u = x - cy - b\sqrt{\frac{a^2}{b^2} - c^2} = 0 \ \ldots\ldots\ (g)$$

the *General Integral*, which is the equation to the cutting straight line.

If $\frac{a}{b} \cdot \frac{y}{\sqrt{b^2+y^2}}$ be substituted for $\frac{\dot{x}}{\dot{y}}$ in (1), we have

$$bx\sqrt{b^2+y^2} = a \,.\, (b^2+y^2), \text{ whence}$$

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2) \ \ldots\ldots\ldots\ldots\ldots\ (h)$$

a *Singular Solution*, which shews the locus of the intersections of all the straight lines represented by (g) taken two and two, or the curve to which they are tangents, to be an *hyperbola* whose semi-axes reckoned from the center C are b and a or $q\sqrt{\tan. \frac{C}{2}}$ and $q\sqrt{\cot. \frac{C}{2}}$ respectively.

We shall arrive at the same result by taking $\frac{\dot{u}}{\dot{x}} = 0$ in *eq*. (g), (which gives $c = \frac{a}{b} \cdot \frac{y}{\sqrt{b^2+y^2}}$), and substituting this value in *eq*. (g).

From this example, it appears, that if we can determine from given conditions, the equation of a curve $u = f(x, y, c) = 0$ *which is supposed to move according to a given law, then the locus of the intersections of this curve with itself, or the curve to which it is always a tangent, may be found, by substituting the value of c derived from*

$$\left.\begin{array}{l}\frac{\dot{u}}{\dot{c}} = 0 \\ \text{in} \\ f(x, y, c) = 0\end{array}\right\}$$

the equation to the locus being a Singular Solution of a certain fluxional equation, derivable from the consideration that the locus and curve, have in every position the same subtangent. By this process, which, in fact, obviates the integration of a Fluxional Equation, and is consequently attended with less labour than other methods, an infinite variety of elegant problems relating to the motion of curves may be resolved. Our limits will not permit more than two examples.

Ex. 1. *Supposing the center of a circle to move in the circumference of another, whose equation is* $\acute{x}^2+\acute{y}^2=r'^2$, *required the Locus of Intersections.*

Let the equation to the moving circle referred to same origin and line of abscissæ be

$$(x-\alpha)^2+(y-\beta)^2=r^2 \text{............} (1)$$

α and β being the co-ordinates of its center.

Then $\alpha^2+\beta^2=r'^2$, and substituting in (1)

$$(x-\alpha)^2+(y-\sqrt{r'^2-\alpha^2})^2-r^2=0=u \text{.........} (2)$$

$$\text{and } \frac{\dot{u}}{\dot{\alpha}} = - x + \alpha + \frac{\alpha}{\sqrt{r'^2-\alpha^2}} \cdot (y - \sqrt{r'^2 - \alpha^2})$$

$$= - x\sqrt{r'^2 - \alpha^2} + \alpha y = 0$$

$$\therefore \alpha = \frac{r'x}{\sqrt{x^2+y^2}}, \text{ and similarly } \beta = \frac{r'y}{\sqrt{x^2+y^2}}$$

which give from *eq*. (1)

$$x^2+y^2 = (r' \pm r)^2$$

the equation to a circle either exterior or interior with respect to the moving curve, which is the locus required.

Ex. 2. *Let a straight line move so as always to cut off from the lines* AX, AY, *forming an* $\angle = A$, *segments whose difference shall = a given quantity D; required the Locus of Intersections.*

Let $y = Mx + N$ ... (1) be the equation to the straight line in any position, referred to the lines AX, AY as co-ordinates originating in A, and let it be supposed to cut them in a, b, respectively. Then when $x=0$, we have $Ab=y=N$, and when $y=0$, $Aa=$ $x = -\frac{N}{M}$, and by the question $Ab - Aa = D$ $= N - \frac{N}{M} = \frac{N}{M} \cdot (M-1)$. Hence the equation (1) becomes

$$u = y - Mx - \frac{MD}{M-1} = 0 \;\ldots\ldots\ldots\ldots (2)$$

in which M varies with the position of the cutting lines. Hence

$$\frac{\dot{u}}{\dot{M}} = -x - \frac{D}{M-1} + \frac{MD}{(M-1)^2} = 0, \text{ gives}$$

$M = 1 + \sqrt{\frac{D}{x}}$, which being substituted in *eq*. (2), and necessary reductions made, we get

$$y^2 - 2yx + x^2 - 2D(y + x) + D^2 = 0$$

an equation to a *parabola*, since $4 \times 1 \times 1 = (-2)^2$. See Wood's Alg. page 291. The reader may transfer the co-ordinates to rectangular, and thence find the *latus rectum*, &c.

The above method will also conduct us to the equations of the Cycloid, Epicycloid, and generally of all curves generated by the revolution of one given curve upon another. It will, moreover, be useful in determining Caustic Curves, whether of Reflection or of Refraction.

VIII.—Hitherto we have confined ourselves to the Integration of Fluxional Equations of the First Order and Degree. In the resolution of Physical and other Problems, however, it frequently happens (from operations that may have been performed upon Fluxional expressions) that the equation, finally to be integrated, assumes the form

$$(a) \ldots P + Q \cdot \frac{\dot{y}}{\dot{x}} + R \cdot \frac{\dot{y}^2}{\dot{x}^2} + S \cdot \frac{\dot{y}^3}{\dot{x}^3} + \&c. = 0$$

in which P, Q, R, &c. are certain functions of x, y. Equations of this kind are evidently reducible to those of the First Order and First Degree, by the resolution of equation (a) algebraically with respect to $\frac{\dot{y}}{\dot{x}}$. In fact, supposing r, r_1, r_2, &c. functions of x, y, to represent the *real* roots of (a), we have

$$\frac{\dot{y}}{\dot{x}} - r = 0, \quad \frac{\dot{y}}{\dot{x}} - r_1 = 0, \ \&c.$$

which being integrated by the processes already explained, their integrals, supposed $R=0$, $R_1=0$, $R_2=0$, &c. will each satisfy the equation. Also since

$$\text{Flux.} (R \times R_1 \times \&c.) = \dot{R} \cdot R_1 \cdot R_2 \ \&c.$$
$$+ \dot{R}_1 \cdot R \cdot R_2 \cdot \&c. + \dot{R}_2 \cdot R \cdot R_1 \cdot \&c. + \&c.$$
$$\text{and } \dot{R} = 0, \ \dot{R}_1 = 0, \ \dot{R}_2 = 0 \ \&c.$$

it is evident that the product of any number of $R=0$, $R_1 = 0$, $R_2 = 0$, &c. will likewise satisfy the proposed equation.

Ex. 1. $y \cdot \frac{\dot{y}^2}{\dot{x}^2} + 2x \cdot \frac{\dot{y}}{\dot{x}} = y$.

Here $\frac{\dot{y}}{\dot{x}} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$

$$\therefore \frac{y\dot{y} + x\dot{x}}{\sqrt{x^2 + y^2}} = \pm \dot{x},$$

and $\sqrt{x^2 + y^2} = c + x$

or $y^2 = 2cx + c^2$.

Ex. 2. $\frac{\dot{y}^2}{\dot{x}^2} - ax = 0.$

Here $\frac{\dot{y}}{\dot{x}} = \pm \sqrt{ax}.$

$$\therefore y = \frac{2}{3} \sqrt{ax^3} + c, y = -\frac{2}{3}\sqrt{ax^3} + c,$$

and combining these values, we have

$$(y - c)^2 = \frac{4}{9} \cdot ax^3.$$

Although the above method is general in principle, yet it is frequently more commodious, in particular cases, to adopt the following processes.

(1.) *Let the Fluxional Equation involve* $\frac{\dot{y}}{\dot{x}}$, or $\frac{\dot{x}}{\dot{y}}$ *with one of the variables* (x, y) *only, or be of the form*

$$x = f \cdot \left(\frac{\dot{y}}{\dot{x}}\right) \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Then putting $\frac{\dot{y}}{\dot{x}} = p$, since

$$y = px - \int \dot{p} x$$

we have by substitution,

$$y = px - \int \dot{p} f \cdot (p) \ldots\ldots\ldots\ldots (2)$$

which being integrated, and p eliminated from equations (1) and (2), we shall have the integral required.

Ex. 1. Let $1 + \frac{\dot{y}^2}{\dot{x}^2} = \frac{1}{x}.$

Here $x = \frac{1}{1+p^2}.$

$$\therefore y = px - tan.^{-1} p + C.$$

But from $p = \sqrt{\frac{1-x}{x}}.$

$$y = \sqrt{x - x^2} - tan.^{-1} \sqrt{\frac{1-x}{x}} + C.$$

Ex. 2. Let $x^3 + \frac{\dot{y}^3}{\dot{x}^3} = ax \cdot \frac{\dot{y}}{\dot{x}}$.

Here (after making $p = ux$) &c. we get

$$y = \frac{a^2}{6} \cdot \frac{2u^3 - 1}{(1 + u^3)^2} + \frac{a^2}{3} \cdot \frac{1}{1 + u^3} + C.$$

and from the proposed equation determining the values of p and $\therefore$ of u, we easily obtain the required integral.

(2.) *All Equations of the Forms*

$$y = x \cdot \frac{\dot{y}}{\dot{x}} + f \cdot \left(\frac{\dot{y}}{\dot{x}}\right) \text{ ....} \quad (1)$$

$$x = y \cdot \frac{\dot{x}}{\dot{y}} + f \cdot \frac{\dot{x}}{\dot{y}} \text{ ....} \quad (2)$$

named after Clairaut, become integrable upon taking the fluxions.

For, putting $\frac{\dot{y}}{\dot{x}} = p$. Equation (1) becomes, after taking the fluxions,

$$\left\{x + \frac{(fp)^{\cdot}}{\dot{p}}\right\} \times \dot{p} = 0$$

$$\therefore \dot{p} = 0, \text{ or } p = c \text{ ....} \quad (3)$$

$$\text{and } x + \frac{(fp)^{\cdot}}{\dot{p}} = 0 \text{ ....} \quad (4)$$

and p, derived from (3), being substituted in (1) will give the *General Solution* involving an arbitrary constant (c). Also, by means of (1) and (4), we may eliminate p, and thus obtain an integral involving no arbitrary constant, which will be a *Particular Solution*.

Ex. 1. $y = x \cdot \frac{\dot{y}}{\dot{x}} + a\sqrt{1 + \frac{\dot{y}^2}{\dot{x}^2}}$.

Here, the general solution is

$$y = cx + a\sqrt{1 + c^2}.$$

And since

$$f(p) = a\sqrt{1+p^2}$$

$$\therefore \frac{(f'p)}{p'} = \frac{ap}{\sqrt{1+p^2}}$$

$$\therefore x + \frac{ap}{\sqrt{1+p^2}} = 0, \text{ or } p = \frac{\pm x}{\sqrt{a^2-x^2}},$$

which being substituted in the proposed equation, gives

$$\left.\begin{array}{l} y = \dfrac{a^2+x^2}{\sqrt{a^2-x^2}} \\ \text{and } y = \sqrt{a^2-x^2} \end{array}\right\} \text{ for the Particular Solutions.}$$

Ex. 2. $y = x \cdot \dfrac{\dot{y}}{\dot{x}} + a\sqrt[3]{1+\dfrac{\dot{y}^3}{\dot{x}^3}}$.

By the same process, we easily find

$$y = cx + \sqrt[3]{1+c^3}$$

$$\text{and } y = \frac{(a^3+x^3+y^3)^{\frac{2}{3}}}{a\sqrt[3]{4}}.$$

Ex. 3. *To find the nature of a curve, such that the locus of the extremity of a perpendicular let fall from a given point upon the tangent, may be a given straight line.*

Let a be the distance of the given straight line from the given point, and supposing x to be measured along this distance from the given point, we shall readily obtain

$$y = x \cdot \frac{\dot{y}}{\dot{x}} + a \cdot \frac{\dot{x}}{\dot{y}} \cdot \left(1 + \frac{\dot{y}^2}{\dot{x}^2}\right),$$

$$\text{or } y = px + \frac{a}{p} \cdot (1+p^2)$$

whose integration affords the *Particular Solution*

$$y^2 = 4 \cdot a\,(a+x)$$

the equation to a *parabola*, whose parameter is 4 a, and directrix the given straight line.

Another example is contained in Art. 438.

IX.—We now come to treat of the Integration of Fluxional Equations of Orders superior to the First; and more particularly of those which occur most frequently in Philosophical inquiries, viz. those of the Second Order.

The general equation of the second order is expressible by

$$f\cdot\left(y, x, \frac{\dot{y}}{\dot{x}}, \frac{\ddot{y}}{\dot{x}^2}\right) = 0$$

in which $\dot{x}$ is supposed constant, and f denotes any function whatever.

This equation may be subdivided into the particular cases

$$\left.\begin{array}{l} f\left(\frac{\ddot{y}}{\dot{x}^2}, x\right) = 0 \ldots 1 \\ f\left(\frac{\ddot{y}}{\dot{x}^2}, y\right) = 0 \ldots 2 \\ f\left(\frac{\ddot{y}}{\dot{x}^2}, \frac{\dot{y}}{\dot{x}}\right) = 0 \ldots 3 \end{array}\right\} \text{ and } \left\{\begin{array}{l} f\left(\frac{\ddot{y}}{\dot{x}^2}, \frac{\dot{y}}{\dot{x}}, x\right) = 0 \ldots 4 \\ f\left(\frac{\ddot{y}}{\dot{x}^2}, \frac{\dot{y}}{\dot{x}}, y\right) = 0 \ldots 5 \end{array}\right.$$

which we shall proceed to integrate separately.

(1.) Let $f\left(\frac{\ddot{y}}{\dot{x}^2}, x\right) = 0$.

Here putting the equation under the form

$$\frac{\ddot{y}}{\dot{x}^2} = \varphi x$$

we have

$$\frac{\dot{y}}{\dot{x}} = \int \dot{x}\varphi x = X + C$$

X denoting a function of X.

$$\therefore y = \int X\dot{x} + Cx + C_1$$
$$= X_1 + Cx + C_1 \;\ldots\ldots\ldots\ldots (a)$$

the integral required.

This method applies also to the form of the n^{th} order

$$\frac{\dot{y}^{(n)}}{\dot{x}^n} = \phi x$$

whose integral is of the form

$$y = X_{n-1} + \frac{Cx^{n-1}}{1.2.3\ldots n-1} + \frac{C_1 x^{n-2}}{1.2\ldots n-2} + \ldots C_{n-1},$$

or since C, C_1, C_2, &c. are arbitrary

$$y = X_{n-1} + Cx^{n-1} + C_1 x^{n-2} + \ldots\ldots\ldots C_{n-1} \ldots\ldots (b)$$

Ex. 1. Let $\frac{\ddot{y}}{\dot{x}^2} = ax^m$.

Here $\frac{\dot{y}}{\dot{x}} = a\int x^m \dot{x} = \frac{ax^{m+1}}{m+1} + C$

$$\therefore y = \frac{ax^{m+2}}{(m+1)(m+2)} + Cx + C_1.$$

Ex. 2. Let $\frac{\dot{y}^{(n)}}{\dot{x}^n} = x$.

Here $X_{n-1} = \int \dot{x} \int \dot{x} \int \dot{x} \int \dot{x} \ldots\ldots \int \dot{x} x$ to $n-1$ terms

$$= \frac{x^n}{1.2\ldots n}$$

$\therefore$ by *eq*. (*b*), we have

$$y = \frac{x^n}{1.2\ldots n} + Cx^{n-1} + C_1 x^{n-2} + \ldots\ldots C_{n-1}.$$

(2.) Let $\left(\frac{\ddot{y}}{\dot{x}^2}, y\right) = 0$, $\frac{\ddot{y}}{\dot{x}^2} = \varphi(y)$.

Here $\frac{2\dot{y}\ddot{y}}{\dot{x}^2} = 2\dot{y}\,\varphi y$, and supposing $\dot{x}$ constant we have

$$\frac{\dot{y}^2}{\dot{x}^2} = 2\int \dot{y}\, \phi y + c$$

$$\therefore \dot{y} = \dot{x}\sqrt{c + 2\int \dot{y}\, \phi y}$$

$$\therefore x = \int \frac{\dot{y}}{\sqrt{c + 2\int \dot{y}\, \phi y}} + C \text{ .... } (c)$$

the integral required.

This method applies also to all equations of the form

$$\frac{\dot{y}^{(n)}}{\dot{x}^n} = \phi \cdot \left(\frac{\dot{y}^{(n-2)}}{\dot{x}^{n-2}}\right)$$

For putting $\frac{\dot{y}^{(n-2)}}{\dot{x}^{n-2}} = u$, we have

$$\frac{\ddot{u}}{\dot{x}^2} = \frac{\dot{y}^{(n)}}{\dot{x}^n}$$

$$\therefore \frac{\ddot{u}}{\dot{x}^2} = \phi u,$$

and by the *eq*. (*c*) by integrating, we have

$$x = \int \frac{\dot{u}}{\sqrt{c + 2\int \dot{u}\, \phi u}} + C \text{ .... } (d)$$

Hence deducing u we have

$$u = \phi' x$$

$$\text{or } \frac{\dot{y}^{(n-2)}}{\dot{x}^{n-2}} = \phi' x$$

whose integral will be given by the form (*b*).

Ex. 1. $\frac{\ddot{y}}{\dot{x}^2} = -\frac{y}{a^2}$.

Here

$$x = \int \frac{\dot{y}}{\sqrt{c - 2\int \frac{y\dot{y}}{a^2}}} + C$$

$$= \int \frac{a\dot{y}}{\sqrt{c^2 - y^2}} + C \text{ (c being arbitrary)}$$

$$= a \sin.^{-1} \frac{y}{c} + C$$

$$\text{or } \frac{y}{c} = \sin. \frac{x - C}{a}$$

$$\therefore y = c \sin. \frac{x}{a} . \cos. \frac{C}{a} - c \sin. \frac{C}{a} \cos. \frac{x}{a}$$

$$= c' \sin. \frac{x}{a} + c'' \cos. \frac{x}{a},$$

since c and C are perfectly arbitrary.

Ex. 2. $\frac{\ddot{y}}{\dot{x}^2} = \frac{1}{\sqrt{ay}}$.

The integral is

$$x = \frac{2}{3} a^{\frac{1}{4}}(\sqrt{y} - 2c) \sqrt{\sqrt{y} + c} + c'.$$

Ex. 3. $\frac{\dddot{y}}{\dot{x}^4} = \frac{\ddot{y}}{\dot{x}^2}$.

Here $u = \frac{\ddot{y}}{\dot{x}^2}$.

$$\therefore \frac{\ddot{u}}{\dot{x}^2} = au$$

and equation (d) gives

$$x = \int \frac{\dot{u}}{\sqrt{c + 2\int \dot{u}u}} + C = \int \frac{\dot{u}}{\sqrt{c^2 + u^2}} + C$$

$$= l . (u + \sqrt{c^2 + u^2}) + C$$

$$\therefore u + \sqrt{c^2 + u^2} = e^{x-C} = c'e^x$$

$$\therefore u = \frac{c'^2 e^{2x} - c^2}{2c'e^x},$$

∴ the constants being arbitrary, we have

$$\frac{\ddot{y}}{\dot{x}^2} = u = c'e^x + ce^{-x}$$

$$\therefore \frac{\dot{y}}{\dot{x}} = c'e^x + ce^{-x} + C$$

$$y = c'e^x + ce^{-x} + Cx + C',$$

since $\int c'e^x = c'e^x$, and $\int ce^{-x} = -ce^{-x} = ce^{-x}$.

(3.) Let $f.\left(\frac{\ddot{y}}{\dot{x}^2}, \frac{\dot{y}}{\dot{x}}\right) = 0$, $\frac{\ddot{y}}{\dot{x}^2} = \varphi . \frac{\dot{y}}{\dot{x}}$.

Here putting $\frac{\dot{y}}{\dot{x}} = p$, we get

$$\dot{x} = \frac{\dot{p}}{\varphi(p)}$$

$$\therefore x = \int \frac{\dot{p}}{\varphi(p)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (e)$$

Also $\dot{y} = \frac{p\dot{p}}{\varphi(p)}$

$$\therefore y = \int \frac{p\dot{p}}{\varphi(p)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (f)$$

and eliminating p from equations (e) and (f), we get the integral required.

This process likewise applies to the form

$$\frac{\dot{y}^{(n)}}{\dot{x}^n} = \varphi . \left(\frac{\dot{y}^{(n-1)}}{\dot{x}^{n-1}}\right).$$

For making $\frac{\dot{y}^{(n-1)}}{\dot{x}^{n-1}} = u$, we have

$$\frac{\dot{u}}{\dot{x}} = \varphi u$$

$$\therefore x = \int \frac{\dot{u}}{\varphi u},$$

and we hence get

$$u = \phi' x \text{ .... } (1)$$

and by form (*b*), we finally get y in terms of x and constants.

Ex. 1. *Required the nature of the curve, whose radius of curvature is constant*, i. e. *let*

$$a = \frac{\left(1 + \frac{\dot{y}}{\dot{x}}\right)^{\frac{3}{2}}}{-\frac{\ddot{y}}{\dot{x}^2}}$$

Here $\frac{\ddot{y}}{\dot{x}^2} = -\frac{1}{a} \cdot \left(1 + \frac{\dot{y}}{\dot{x}}\right)^{\frac{3}{2}} = \phi\left(\frac{\dot{y}}{\dot{x}}\right)$,

and by equations (*e*) and (*f*)

$$x = \int \frac{-a\dot{p}}{(1+p^2)^{\frac{3}{2}}} = c - \frac{ap}{\sqrt{1+p^2}}$$

$$y = \int \frac{-ap\dot{p}}{(1+p^2)^{\frac{3}{2}}} = c' + \frac{a}{\sqrt{1+p^2}},$$

and eliminating p, we get after reduction

$$(c - x)^2 + (c' - y)^2 = a^2$$

the equation to a *circle* whose radius is (a).

(4.) *Let* $f\left(\frac{\ddot{y}}{\dot{x}^2}, \frac{\dot{y}}{\dot{x}}, x\right) = 0$, or $\frac{\ddot{y}}{\dot{x}^2} = \phi\left(\frac{\dot{y}}{\dot{x}}, x\right)$.

Making $\frac{\dot{y}}{\dot{x}} = p$, we have

$$\frac{\dot{p}}{\dot{x}} = \phi\,(p, x) \text{ .... } (g)$$

which being of the first order, may be integrated in certain cases by methods already explained.

Let the integral thus obtained be

$$p = \varphi'(x)$$

$$\therefore y = \int \dot{x}\varphi' x \text{ .... } (h)$$

N. B. It is, in some instances, more convenient to get the form

$$x = \varphi' p \text{ .... } (g')$$

in which case

$$y = \int p\dot{x} = px - \int x\dot{p} = px - \int \dot{p}\,\varphi' p \text{ ...... } (h')$$

whence by aid of equation (g') we eliminate p and thereby obtain the integral required.

Ex. (1.) *Required the curve whose radius of curvature is a given function X of its abscissa x.*

Here $X = \dfrac{\left(1+\dfrac{\dot{y}^2}{\dot{x}^2}\right)^{\frac{3}{2}}}{\dfrac{\ddot{y}}{\dot{x}^2}}$

whence

$$\frac{p}{\sqrt{1+p^2}} = \int \frac{\dot{x}}{X} = X'$$

$$\text{and } p = \frac{X'}{\sqrt{1-X'^2}}$$

$$\therefore y = \int \frac{X'\dot{x}}{\sqrt{1-X'^2}}$$

which expresses the nature of the curve required.

Ex. 2. Let $(1+p^2) + xp\dfrac{\dot{p}}{\dot{x}} = a\dfrac{\dot{p}}{\dot{x}}\sqrt{1+p^2}$.

This equation becomes *Linear*, when put under the form

$$a\dot{p} = \frac{xp}{\sqrt{1+p^2}} + \dot{x}\sqrt{1+p^2},$$

which being solved by the usual method, gives

$$x = \frac{ap+c}{\sqrt{1+p^2}},$$

whence, by eq. (h')

$$y = \frac{cp-a}{\sqrt{1+p^2}} - c\,l\,.\,\frac{p+\sqrt{1+p^2}}{c'}.$$

and substituting for (p), we finally get

$$y = \sqrt{a^2+c^2-x^2} - c\,l.\frac{a+x}{c'\,(c-\sqrt{a^2+c^2-x^2})}$$

(5.) Let $f\left(\frac{\ddot{y}}{\dot{x}^2}, \frac{\dot{y}}{\dot{x}}, y\right) = 0$, $\frac{\ddot{y}}{\dot{x}^2} = \varphi\left(\frac{\dot{y}}{\dot{x}}, y\right)$.

Making $\frac{\dot{y}}{\dot{x}} = p$, we have

$$\frac{\dot{p}}{\dot{x}} = \varphi\,(p,y)$$

$$\text{or } p\dot{p} = \dot{y}\,\varphi\,(p,y) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1)$$

whose integral, found by previous methods, let be

$$p = \varphi'(y) \;\ldots\; (2), \text{ or } y = \varphi'(p) \;\ldots\ldots\; (3)$$

as may be most convenient.

If $p = \varphi'(y)$, we have

$$x = \int \frac{\dot{y}}{\varphi' y} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (4)$$

If $y = \varphi'(p)$, we have

$$\dot{x} = \frac{\varphi'(p)}{p} + \int \frac{\varphi'(p)}{p^2}\,.\,\dot{p} \;\ldots\ldots\ldots\; (5).$$

Ex. 1. Let $\frac{\ddot{y}}{\dot{x}^2}\left(y\frac{\dot{y}}{\dot{x}} + a\right) = \frac{\dot{y}}{\dot{x}}\,.\,\left(1 + \frac{\dot{y}^2}{\dot{x}^2}\right)$.

Here $\dot{p}\,.\,(py + a) = \dot{y}\,.\,(1 + p^2)$

$$\text{or } \frac{\dot{y}}{\dot{p}} - y \cdot \frac{p}{1+p^2} = \frac{a}{1+p^2}$$

which being *Linear* with respect to y, by the form (a) in page 316, we have

$$y = ap + c\sqrt{1+p^2} \text{ .... } (6)$$

$$\text{and } x = \int \frac{\dot{y}}{p} = a\,l.\ (c'p) + c\,l.\ \{p + \sqrt{1+p^2}\}$$

and substituting for p its value $\frac{ay \pm c\sqrt{y^2 + a^2 - c^2}}{a^2 - c^2}$, derived from *eq* . 6, we have the integral required.

Ex. 2. $\frac{\ddot{y}}{\dot{x}^2} + \frac{A\dot{y}}{\dot{x}} + By = D$, *which is the Linear Equation of the Second Order with constant coefficients.*

Here $\frac{\dot{p}}{\dot{x}} + Ap + By = D$

or $\frac{p\dot{p}}{\dot{y}} + Ap + By = D.$

Let $By - D = Bu$. Then we easily get

$$\frac{\dot{p}}{\dot{u}} + B\frac{u}{p} + A = 0$$

which, being *homogeneous*, becomes, after making $p = uv$, and substituting, &c.

$$\frac{\dot{u}}{u} = \frac{-v\dot{v}}{v^2 + Av + B} = \frac{-v\dot{v}}{(v-a)\cdot(v-b)}$$

a and b being the roots of $v^2 + Av + B = 0$.

Hence

$$\dot{x} = \frac{\dot{y}}{p} = \frac{\dot{u}}{p} = \frac{\dot{u}}{uv}$$

$$= \frac{-\dot{v}}{(v-a)\cdot(v-b)}$$

and we $\therefore$ have

$$\frac{\dot{u}}{u} - a\dot{x} = \frac{-\dot{v}}{v-b}, \quad \frac{\dot{u}}{u} - b\dot{x} = \frac{-\dot{v}}{v-a}$$

$$\therefore l \,.\, cu \,.\, (v-b) = ax, \quad l \,.\, c'u\,(v-a) = bx$$

$$\therefore v - b = \frac{e^{ax}}{cu}, \quad v - a = \frac{e^{bx}}{c'u}$$

whence

$$u = \frac{1}{a-b} \cdot \left(\frac{e^{ax}}{c} - \frac{e^{bx}}{c'}\right) = ce^{ax} + c'e^{bx}$$

c, c' being arbitrary.

Hence

$$y = \frac{D}{B} + u = \frac{D}{B} + ce^{ax} + c'e^{bx}.$$

the integral required, which will require some slight modifications when a, b are either imaginary or equal. This will be noticed hereafter.

We will take two other particular cases of the general equation of the second degree, viz.

$$\left.\begin{aligned} \frac{\ddot{y}}{\dot{x}^2} + P\frac{\dot{y}}{\dot{x}} + Qy &= 0 \quad (6) \\ \text{and } \frac{\ddot{y}}{\dot{x}^2} + P\frac{\dot{y}}{\dot{x}} + Qy &= R \quad (7) \end{aligned}\right\}$$

P, Q, R being functions of x only.

(6.) *To integrate* $\frac{\ddot{y}}{\dot{x}^2} + P\frac{\dot{y}}{\dot{x}} + Qy = 0.$

Let $y = e^{\int u\dot{x}}$. Then $\frac{\dot{y}}{\dot{x}} = ue^{\int u\dot{x}}$, $\frac{\ddot{y}}{\dot{x}^2} = e^{\int u\dot{x}}.\left(\frac{\dot{u}}{\dot{x}} + u^2\right)$,

and substituting, &c. we get

$$\frac{\dot{u}}{\dot{x}} + u^2 + Pu + Q = 0 \;\ldots\ldots\ldots\ldots\ldots\; (A)$$

an equation of the *first order*, which being integrated will give u in terms of x and $\therefore y = e^{\int u\dot{x}}$ in terms of x.

(7.) *To integrate the general linear equation of the second order,*

$$\frac{\ddot{y}}{\dot{x}^2} + P.\frac{\dot{y}}{\dot{x}} + Qy = R.$$

This must first be reduced to the form (6) by assuming

$$y = y_1 v$$

which gives

$$\frac{\dot{y}}{\dot{x}} = \frac{\dot{y}_1}{\dot{x}} v + \frac{\dot{v}}{\dot{x}} y_1, \ \frac{\ddot{y}}{\dot{x}^2} = \frac{\ddot{y}_1}{\dot{x}^2} v + 2\dot{y}_1 \frac{\dot{v}}{\dot{x}^2} + \frac{\ddot{v}}{\dot{x}^2} y_1$$

and by substitution

$$\left(\frac{\ddot{y}_1}{\dot{x}^2} + P.\frac{\dot{y}_1}{\dot{x}} + Qy_1\right)v + \frac{\ddot{v}}{\dot{x}^2} y_1 + 2\dot{y}_1 . \frac{\dot{v}}{\dot{x}} + P.\frac{\dot{v}}{\dot{x}} y_1 = R.$$

Now assuming

$$\frac{\ddot{y}_1}{\dot{x}^2} + P.\frac{\dot{y}_1}{\dot{x}} + Qy_1 = 0 \ldots\ldots\ldots\ldots\ldots (a)$$

we have also

$$\left(\frac{\dot{v}}{\dot{x}}\right)^{\cdot} + \left(\frac{\dot{v}}{\dot{x}}\right) . \left(P + \frac{2 . \frac{\dot{y}_1}{\dot{x}}}{y_1}\right) \dot{x} = \frac{R\dot{x}}{y_1} \ldots\ldots\ldots (b)$$

But equation (a), the same in form as (6), may be integrated in like manner, producing

$$y_1 = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (c)$$

which being substituted in *eq*. b renders it a *Linear Equation of the First Order*, with respect to $\left(\frac{\dot{v}}{\dot{x}}\right)$. Hence we find

$$\frac{\dot{v}}{\dot{x}} = f'(x)$$

$$\text{and } v = \int \dot{x} f' x = \phi x \ \ldots\ldots\ldots (e)$$

Hence we finally determine

$$\dot{y} = y_1 \dot{v} = f\dot{x} \times \phi \dot{x}.$$

Ex. (1.) $\frac{\ddot{y}}{\dot{x}^2} + \frac{\dot{y}}{\dot{x}} \cdot \frac{1}{x} - \frac{y}{x^2} = \frac{a}{x^2-1}$.

Here $P = \frac{1}{x}$, $Q = -\frac{1}{x^2}$, $R = \frac{a}{x^2-1}$

eq. (*A*) by which (*a*) is integrated becomes

$$\frac{\dot{u}}{\dot{x}} + u^2 + \frac{u}{x} = \frac{1}{x^2} \text{ .... } (m)$$

Make $u_1 = \frac{1}{u}$; then we get

$$\frac{-\dot{u}_1}{\dot{x}\, u_1^2} + \frac{1}{u_1^2} + \frac{1}{xu_1} = \frac{1}{x^2} \text{ .... } (n)$$

which being *homogeneous*, put $x = x_1 u_1$, and we obtain

$$\frac{\dot{u}_1}{u} = -\frac{x_1^2 + x_1 - 1}{x_1 \,.\, (x_1^2 - 1)} \dot{x}_1$$

whence,

$$u_1 = \frac{1}{x_1} \cdot \sqrt{\frac{x_1 + 1}{x_1 - 1}}$$

and therefore, since $x_1 = ux$, we have

$$u = \frac{x^2+1}{x\,(x^2-1)}, \; \int u\dot{x} = l \,.\, \frac{x^2-1}{x}.$$

$$\therefore y_1 = e^{\int u\dot{x}} = \frac{x^2-1}{x} \text{ .... } (r)$$

Again, *eq*. (*b*) hence becomes

$$\left(\frac{\dot{v}}{\dot{x}}\right)^{\cdot} + \frac{\dot{v}}{\dot{x}} \cdot \frac{1+3x^2}{x\,(x^2-1)} \dot{x} = \frac{ax\dot{x}}{(x^2-1)^2}$$

whose integral, which is easily found (see p. 356) gives

$$v = \int \frac{(ax+c)\, x\dot{x}}{(x^2-1)^2}$$

$$= -2 \,.\, \frac{ax+c}{x^2-1} + a\,l \,.\, c' \,.\, \frac{x-1}{x+1}.$$

Hence, we finally have

$$y = -\frac{ax+c}{2x} + \frac{x^2-1}{4x}\, a\, l.\, c\, \frac{x-1}{x+1}.$$

Ex. (2.) $\frac{\ddot{y}}{\dot{x}^2} + y = -A\, cos.\, x.$

Here $P = 0$, $Q = 1$, $R = -A\, cos.\, x$. And the integral is

$$y = c\, sin.\, x + c'\, cos.\, x - \frac{Ax}{2}\, cos.\, x.$$

Ex. (3.) $\frac{\ddot{y}}{\dot{x}^2} - \frac{a^2-1}{4x^2}\, y = \frac{m}{x^{\frac{a+1}{2}}}.$

Here $y = \frac{1}{x^{\frac{a-1}{2}}} \cdot \left(c' x^a - \frac{c}{a} - \frac{mx}{a-1}\right).$

(8.) We shall conclude this division with exhibiting the integration of the *General Linear Equation*,

$$p_n + A p_{n-1} + B p_{n-2} + \ldots\ldots N p_1 + M y = X$$

where $p_1 = \frac{\dot{y}}{\dot{x}}$, $p_2 = \frac{\ddot{y}}{\dot{x}^2}$, &c. = &c. A, B, C ....

N, M are constant, and X is a function of x.

For this purpose, we will take the equation of the third order

$$p_3 + A p_2 + B p_1 + C y = X \ldots\ldots\ldots (A)$$

Assume

$$y = e^{mx} \int y_1\, \dot{x},\ y_1 = e^{m_1 x} \int y_2\, \dot{x},\ y_2 = e^{m_2 x} \int y_3\, \dot{x}.$$

Then since

$$p_1 = e^{mx}\{m \int y_1\, \dot{x} + y_1\}$$

$$p_2 = e^{mx}\left\{m^2 \int y_1\, \dot{x} + 2my_1 + \frac{\dot{y}_1}{\dot{x}}\right\}$$

$$p_3 = e^{mx}\left\{m^3 \int y_1 \dot{x} + 3m^2 y_1 + 3m\, \frac{\dot{y}_1}{\dot{x}} + \frac{\ddot{y}_1}{\dot{x}^2}\right\}.$$

By substituting &c. in the above equations, we get

$$\frac{\ddot{y}_1}{\dot{x}^2} + (3m^2 + 2Am + B)\frac{\dot{y}_1}{\dot{x}} + (3m + A)\, y_1 +$$

$$(m^3 + Am^2 + Bm + C) \int y_1 \dot{x} = \frac{X}{e^{mx}}.$$

Now assuming

$$m^3 + Am^2 + Bm + C = 0 \; \ldots\ldots \; (1)$$

we reduce the integration to that of

$$\frac{\ddot{y}_1}{\dot{x}^2} + A_1 \frac{\dot{y}_1}{\dot{x}} + B_1 y_1 = \frac{X}{e^{mx}} \; \ldots\ldots\ldots\ldots \; (a)$$

A_1, B_1 being put for $3m^2 + 2Am + B +$ and $3m + A$.

By the like assumptions with regard to y_1, we find

$$m_1^2 + A_1 m_1 + B_1 = 0 \; \ldots\ldots\ldots\ldots \; (2)$$

$$\text{and } \frac{\dot{y}_2}{\dot{x}} + (2m_1 + A_1)\, y_1 = \frac{X}{e^{mx} \,.\, e^{m_1 x}} \; \ldots\ldots \; (b)$$

And again (making $2m_1 + A_1 = A_2$)

$$m_2 + A_2 = 0 \; \ldots\ldots\ldots\ldots\ldots\ldots \; (3)$$

$$\text{and } y_3 = \frac{X}{e^{mx} \times e^{m_1 x} \times e^{m_2 x}} \; \ldots\ldots\ldots\ldots \; (c)$$

Hence then we get

$$y_2 = e^{m_2 x} \int y_3 \dot{x} = e^{m_2 x} \int \frac{X\dot{x}}{e^{x.(m+m_1+m_2)}}$$

$$y_1 = e^{m_1 x} \int y_2 \dot{x} = e^{m_1 x} \int e^{m_2 x} \dot{x} \int \frac{X\dot{x}}{e^{x(m+m_1+m_2)}}$$

$$\text{and } y = e^{mx} \int e^{m_1 x} \dot{x} \int e^{m_2 x} \dot{x} \int \frac{X\dot{x}}{e^{x.(m+m_1+m_2)}}$$

Again, let the roots of the equations (1), (2), (3), be denoted respectively by

$$a_1,\ a_2,\ a_3$$
$$b_1,\ b_2$$
$$c_1$$

and diminish the roots of (1) by any one of its roots (a_1 for instance) by putting

$$u + a_1 = m;$$

then the resulting equation will be

$$u^3 + (3a_1 + A)\,u^2 + (3a_1^2 + 2Aa_1 + B)\,u + a_1^3 + Aa_1^2 + Ba_1 + C = 0.$$

But by equation $a_1^3 + Aa_1^2 + Ba_1 + C = 0.$
we have

$$u^2 + (3a_1 + A)\,u + 3a_1^2 + 2Aa_1 + B = 0 \ \ldots\ldots\ldots\ (2')$$

whose co-efficients being $= A_1$ and B_1, (2) and (2′) are identical.

Hence $m_1 = u = m - a_1$

$$\therefore\ \left.\begin{aligned} b_1 &= a_2 - a_1 \\ b_2 &= a_3 - a_1 \end{aligned}\right\}$$

In the same manner it may be shown that $m_2 = m_1 - b_1$, or that

$$c_1 = a_3 - a_1 - (a_2 - a_1) = a_3 - a_2.$$

Hence, substituting a_1, $a_2 - a_1$, $a_3 - a_2$ for m, m_1, m_2, we finally obtain

$$y = e^{a_1 x} \int e^{\overline{a_2 - a_1}.x}\,\dot{x} \int e^{\overline{a_3 - a_2}.x}\,\dot{x} \int \frac{X\dot{x}}{e^{a_3 x}}$$

which will be the complete integral of the Linear Equation (A), because of the necessary introduction of three arbitrary constants by the triple integration.

By an extension of this process, the equation

$$p_n + Ap_{n-1} + \ldots Np_1 + My = X$$

may be integrated, and it will be found that if a_1, a_2 ... a_n be roots of the equation

$$m^n + Am^{n-1} + \dots \dots Nm + M = 0$$

the integral will be expressed by

$$y = e^{a_1 x} \int e^{(a_2 - a_1)x} \dot{x} \int e^{(a_3 - a_2)x} \dot{x} \int . e^{(a_4 - a_3)x} \dot{x} \int \dots \int \frac{X\dot{x}}{e^{a_n x}}.$$

If any of the roots a_1, a_2 &c. be equal, the above solution still holds good, containing as before (n) arbitrary constants.

If any of them be imaginary, the whole expression may be rendered real by means of the Theorem

$$\cos. \theta \pm \sqrt{-1} . \sin. \theta = e^{\pm \theta \sqrt{-1}}$$

and the circumstance of the entering by pairs.

Ex. 1. $\frac{\dddot{y}}{\dot{x}^3} - 3 . \frac{\ddot{y}}{\dot{x}^2} + 3 \frac{\dot{y}}{\dot{x}} - y = X.$

Here

$$m^3 - 3m^2 + 3m - 1 = 0 = (m - 1)^3$$

$$\text{or } a_1 = a_2 = a_3 = 1, \text{ and } \therefore$$

$$y = e^x \int \dot{x} \int \dot{x} \int \dot{x} \frac{X}{e^x}$$

which is easily integrated when X is known.

Let $X = 0$. Then $\int \dot{x} \times 0 = c$,

and $y = e^x \{c_1 x^2 + c_2 x + c_3\}$.

Ex. 2. $\frac{\ddot{y}}{\dot{x}^2} + By = X$, which is called the *Problem of the Three Bodies*, from the circumstance of its integration being required in the Theory of the Perturbations of the motions of Three Heavenly Bodies arising from their mutual attractions.

In this example, since $m^2 + B = 0$, we have

$a_1 = \sqrt{-B}$, $a_2 = -\sqrt{-B}$, and

$$y = e^{a_1 x} \int e^{(a_2 - a_1)x} \dot{x} \int \frac{\dot{x}X}{e^{a_2 x}} = e^{x\sqrt{-B}} \int e^{-2x\sqrt{-B}} \dot{x} \int \dot{x} X e^{x\sqrt{-B}}.$$

As a particular case, let $X = 0$.

Then $\int 0 \times \dot{x} = c$, $\therefore c \int e^{-2x\sqrt{-B}} \dot{x} = \frac{-c}{2\sqrt{-B}} . e^{-2x\sqrt{-B}} + c'$

$= ce^{-2x\sqrt{-B}} + c'$. Hence

$y = ce^{-x\sqrt{B}.\sqrt{-1}} + c'e^{x\sqrt{B}.\sqrt{-1}}$

$= c\,(cos.\ x\sqrt{B} - \sqrt{-1}\,.\,sin.\ x\sqrt{B}) + c'\,(cos.\ x\sqrt{B} + \sqrt{-1}\ sin.\ x\sqrt{B})$

$= (c + c')\,.\ cos.\ x\sqrt{B} - \sqrt{-1}\,.\,(c - c')\ sin.\ x\sqrt{B}$

$= c\ cos.\ x\sqrt{B} + c'\ sin\ x\sqrt{B}$.

since the constants are perfectly arbitrary.

For other particular cases, see Woodhouse's Astronomy, pp. 23, 97, 99, 100, 107, &c.

X. In the preceding divisions we have explained the principal methods of integrating Fluxional Equations between two variables. It yet remains for us to initiate the Student in the more refined and abstruse Theory which leads to the Integration of Equations involving *three variables*.

Equations of this kind may be represented generally by

$$P\dot{x} + Q\dot{y} + R\dot{z} = 0,$$

P, Q, R, being any functions of (x, y, z); and they are called *Total* or *Partial Fluxional Equations*, according as they involve *all* or only some one or two of $\dot{x}$, $\dot{y}$, $\dot{z}$.

The Total Equation

$$P\dot{x} + Q\dot{y} + R\dot{z} = 0 \ \dots\dots\dots \ (A)$$

being proposed, it is required to investigate the conditions of its integrability.

Transform the equation to

$$\dot{z} = -\frac{P}{R}\,\dot{x} - \frac{Q}{R}\,\dot{y}$$

$$= P_1\,\dot{x} + Q_1\,\dot{y} \text{ by hypoth.}$$

Then, supposing it a perfect fluxion, we have by page 319, Vol. II.

$$\frac{P^{\cdot}_1}{\dot{y}} = \frac{Q^{\cdot}_1}{\dot{x}}.$$

But since P_1 may be considered a function of (y, z) and, again, z a function of (x, y), we have

$$\frac{P^{\cdot}_1}{\dot{y}} = \frac{P^{\cdot}_1}{\dot{y}} + \frac{\dot{z}}{\dot{y}} \cdot \frac{P^{\cdot}_1}{\dot{z}} \text{ (see page 286, Vol. I)}$$

and by like reasoning we also get

$$\frac{Q^{\cdot}_1}{\dot{x}} = \frac{Q^{\cdot}_1}{\dot{x}} + \frac{\dot{z}}{\dot{x}} \cdot \frac{Q^{\cdot}_1}{\dot{z}}$$

$$\therefore \frac{P^{\cdot}_1}{\dot{y}} + \frac{\dot{z}}{\dot{y}} \cdot \frac{P^{\cdot}_1}{\dot{z}} = \frac{Q^{\cdot}_1}{\dot{x}} + \frac{\dot{z}}{\dot{x}} \cdot \frac{Q^{\cdot}_1}{\dot{z}}$$

and substituting for P_1, Q_1 their values we finally get after reduction

$$(a) \ldots P.\frac{R^{\cdot}}{\dot{y}} - R.\frac{P^{\cdot}}{\dot{y}} + R.\frac{Q^{\cdot}}{\dot{x}} - Q.\frac{R^{\cdot}}{\dot{x}} + Q.\frac{P^{\cdot}}{\dot{z}} - P.\frac{Q^{\cdot}}{\dot{z}} = 0$$

the condition required.

When this condition is fulfilled, we may proceed to integrate (A) as follows:

On the supposition that z (or x, or y) is constant, let the integral of $P\dot{x} + Q\dot{y} = 0$, found by previous methods, have the form of

$$f. \, (x, y, z, Z) = 0$$

Z being an arbitrary function of z; then taking the entire fluxion of this integral, and comparing its terms with those of the proposed *eq*. (A), we shall obtain $Z^{\cdot}$ in terms of Z, z, $\dot{z}$ alone, which will give Z, and therefore determine the integral required.

Ex. 1. Let $\dot{x}\,(y+z) + \dot{y}\,(x+z) + \dot{z}\,(x+y) = 0.$

Here, since $\dot{z} = 0$, we have

$$\dot{x} \,.\, (y + z) + \dot{y}\,(x + z) = 0$$

$$\therefore \frac{\dot{x}}{x+z} + \frac{\dot{y}}{y+z},$$

and considering z constant

$$l\,(x+z) + l\,.\,(y+z) = const. = l\,.\,Z$$

$$\therefore (x+z)\,.\,(y+z) = Z.$$

Again, taking the fluxions

$$\dot{Z} = (\dot{z}+\dot{x})\,(y+z) + (\dot{y}+\dot{z})\,(x+z)$$

$$= \dot{z}\,.\,(x+y+2z) + \dot{x}\,.\,(y+z) + \dot{y}\,(x+z)$$

$$= 2z\dot{z}$$

$$\therefore Z = z^2 + c.$$

Hence the required integral becomes

$$xz + yz + xy = c$$

Ex. 2. $z\dot{x} + x\dot{y} + y\dot{z} = 0$, does not satisfy the Criterion.

Ex. 3. $(x^2 + y^2)\,\dot{z} = (z - a)\,(x\dot{x} + y\dot{y})$.

Here $x^2 + y^2 = Z^2$,

and we easily find

$$Z = c\,.\,(z - a),$$

which gives

$$x^2 + y^2 = c^2\,.\,(z - a)^2.$$

Ex. 4. $(y^2 + yz + z^2)\,\dot{x} + (x^2 + xz + z^2)\,\dot{y} + (x^2 + xy + y^2)\,\dot{z} = 0.$

Here $\dfrac{\dot{x}}{x^2 + xz + z^2} + \dfrac{\dot{y}}{y^2 + yz + z^2} = 0,$

whose integral is

$$\frac{2}{z\sqrt{3}} \times \left\{ tan.^{-1} \frac{z+2x}{z\sqrt{3}} + tan.^{-1} \frac{z+2y}{z\sqrt{3}} \right\} = Z$$

or since Z is arbitrary,

$$tan.^{-1} \frac{z+2x}{z\sqrt{3}} + tan.^{-1} \frac{z+2y}{z\sqrt{3}} = tan.^{-1} Z$$

$$\therefore Z = \frac{\frac{z+2x}{z\sqrt{3}} + \frac{z+2y}{z\sqrt{3}}}{1 - \frac{(z+2x)(z+2y)}{3z^2}}$$

which reduces to

$$Z = \frac{x+y+z}{z^2 - zx - zy - 2xy}.$$

Hence taking the fluxions, &c. we finally get

$$xy + xz + yz = c\,(x+y+z).$$

Would our limits permit, much more might be said on the subject of Total Equations. It would be easy to show that the method of the Factor is not applicable to them, except when the condition (*a*) is fulfilled (and therefore that there are equations between three variables which it is impossible to integrate), and many other interesting circumstances; but we must proceed to Partial Fluxional Equations.

(1.) *Required to integrate*

$$f.\left(\frac{\dot z}{\dot x}, x, y, z\right) = 0.$$

Since $\dot y$ is not involved, y is supposed constant, and the integral containing an arbitrary function of y may be obtained, on that hypothesis, by the ordinary methods for equations between two variables. Similar reasoning will apply to the form

$$f.\left(\frac{\dot z}{\dot y}, x, y, z\right) = 0.$$

Ex. (1.) $\left(\frac{\dot z}{\dot x} - 3x^2\right)\dot x = 0.$

Here $\frac{\dot z}{\dot x} = 3x^2, \therefore z = x^3 + \phi y.$

Ex. (2.) $x - \frac{\dot z}{\dot x}\sqrt{x^2+y^2} = 0.$

Here $\dot{z} = \frac{x\dot{x}}{\sqrt{x^2 + y^2}}$

$$\therefore z = \sqrt{x^2 + y^2} + \phi y.$$

Ex. (3.) $\frac{\dot{z}}{\dot{x}} \sqrt{a^2 - y^2 - x^2} = a.$

Here $\dot{z} = \frac{a\dot{x}}{\sqrt{a^2 - y^2 - x^2}}$

$$\therefore z = a \sin.^{-1} \frac{x}{\sqrt{a^2 - y^2}} + \phi y.$$

Ex. (4.) $\frac{\dot{z}}{\dot{y}} xy + az = 0.$

Here x being constant, we have

$$l . z^x y^a = lc$$

$$\text{or } z^x y^a = c = \phi x.$$

Ex. (5.) $\frac{\dot{z}}{\dot{x}} . (y^2 + x^2) = y^2 + z^2$, which being homogeneous, we get

$$\tan.^{-1} \frac{z}{y} - \tan.^{-1} \frac{x}{y} = \tan.^{-1} c$$

$$\therefore \frac{\frac{z}{y} - \frac{x}{y}}{1 + \frac{zx}{y^2}} = c$$

$$\text{and } \frac{z - x}{y^2 + xz} = cy = \phi y.$$

(2.) *Required to integrate (when possible) the* PARTIAL LINEAR EQUATION

$$Pp + Qq = V \text{ ....} (A)$$

P, Q, V, being functions of (x, y, z) and p; q the partial fluxional co-efficients $\frac{\dot{z}}{\dot{x}}$, $\frac{\dot{z}}{\dot{y}}$, *of z relative to x, y respectively.*

Since z is supposed a function of (x, y), we have

$$\dot{z} = p\dot{x} + q\dot{y} \text{ .... } (B)$$

and substituting p, derived from (B), in *eq*. A, we get

$$(P\dot{z} - V\dot{x}) = q\ (P\dot{y} - Q\dot{x}) \text{ .... } (a)$$

the integration of which presents two cases.

Case 1. Let P and V contain only (x, z) and P, Q only (y, x).

Then, since q is indeterminate

$$P\dot{z} - V\dot{x} = 0, \text{ and } q\ (P\dot{y} - Q\dot{x}) = 0 \text{ .... } (b)$$

and there exist factors μ, μ' capable of rendering these equations perfect fluxions (see p. 319) $\dot{M}$, $\dot{M}'$.

Hence

$$\dot{M} = \mu\ .\ (P\dot{z} - V\dot{x}) = \mu q\ .\ (P\dot{y} - Q\dot{x})$$

$$= \frac{\mu}{\mu'} \cdot q \times \dot{M}',$$

which cannot be the case unless $\frac{\mu}{\mu'}\ q$ is a function of M',

$$\therefore M = \int \phi\ (M')\ \dot{M}' = \phi M'$$

ϕ denoting an arbitrary function.

Ex. (1.) Let $px + qy = nz$.

Here $P = x$, $Q = y$, and $V = nz$

$$\therefore x\dot{z} - nz\dot{x} = 0, \text{ and } x\dot{y} - y\dot{x} = 0$$

$$\therefore \frac{\dot{z}}{z} = n\frac{\dot{x}}{x}, \text{ and } \frac{\dot{y}}{y} = \frac{\dot{x}}{x}$$

$$M = \frac{z}{x^2}, \text{ and } M' = \frac{y}{x}$$

$$\therefore z = x^n \varphi \left(\frac{y}{x}\right).$$

which is an *homogeneous function* of $x\ y$.

Ex. (2.) $p x^2 + q y^2 = z^2.$

Here $P = x^2$, $Q = y^2$, and $V = z^2$

$$\therefore x^2\dot{z} - z^2\dot{x} = 0;\ x^2\dot{y} - y^2\dot{x} = 0$$

$$\therefore M = \frac{1}{x} - \frac{1}{z},\ M' = \frac{1}{x} - \frac{1}{y}$$

$$\therefore \frac{z-x}{xz} = \varphi.\left(\frac{y-x}{xy}\right)\ldots\ldots$$

Case 2. When x, y, z are intermixed in P, Q, V (eq. a), if we can determine two integrals of equations (b) viz. $M = \alpha$, $M' = \alpha'$, the integral of A will also in this case be

$$M = \varphi M'.$$

For let

$$\left.\begin{array}{l} \dot{M} = A\dot{x} + B\dot{y} + C\dot{z} = 0 \\ \dot{M}' = a\dot{x} + b\dot{y} + c\dot{z} = 0 \end{array}\right\} (e)$$

then, in order to find the conditions on which $M - \varphi M' = 0$ will satisfy $Pp + Qq = V$, we take the fluxions relatively to (z, x), and (z, y) thereby obtaining

$$(C - c\varphi_1(M'))\,p + A - a\varphi_1(M') = 0$$

$$(C - c\varphi_1(M))\,q + B - b\varphi_1(M') = 0.$$

Hence deriving p and q and substituting them in eq. (A) we get

$$AP + BQ + CV = \varphi_1(M').\ (aP + bQ + cV)\ \ldots (f)$$

the condition required.

But since, by hypothesis, M and M' satisfy equations (b), by substituting for $\dot{z}$ and $\dot{x}$ their values hence derived in eq. (e), we get

$$AP + BQ + CV = 0, \text{ and } aP + bQ + cV = 0.$$

which show that the condition (f) is fulfilled by

$M = \varphi M'$, and consequently that it is an integral of the proposed equation.

Now, since we obtain from equations (*b*) another $Q\dot{z} - V\dot{y} = 0$, the rule may be thus stated:

Find two integrals $M = \alpha$, $M' = \alpha'$ of any two of the equations

$$P\dot{z} - V\dot{x} = 0,\ P\dot{y} - Q\dot{x} = 0,\ Q\dot{z} - V\dot{y} = 0;$$

then the integral of $Pp + Qq = V$ will be

$$M = \varphi \,.\, M'.$$

Ex. 1. Let $qxy - px^2 = y^2$.

Here $x^2\dot{z} + y^2\dot{x} = 0$, $x^2\dot{y} + xy\dot{x} = 0$.

From the latter we get $xy = \alpha' = M'$, and substituting in the former for y,

$$\dot{z} + \alpha' \,.\, x^{-4}\dot{x} = 0, \text{ and } \therefore z - \frac{\alpha'}{3x^3} = z - \frac{y^2}{3x} = \alpha = M$$

and the integral is

$$z - \frac{y^2}{3x} = \varphi(xy).$$

Ex. 2. $px + qy = n\sqrt{x^2 + y^2}$.

Here $x\dot{z} - n\dot{x}\sqrt{x^2 + y^2} = 0$, $x\dot{y} - y\dot{x} = 0$.

The second gives $\alpha' = \dfrac{y}{x} = M'$

$$\text{and } \therefore x\dot{z} - n\dot{x}x\sqrt{1 + \alpha'^2} = 0$$

$$\therefore z - nx \,.\, \sqrt{1 + \alpha'^2} =$$

$$z - n\sqrt{x^2 + y^2} = \alpha = M$$

$$\therefore z = n\sqrt{x^2 + y^2} + \varphi\left(\frac{y}{x}\right).$$

For complete information on this subject, which is intimately connected with many of the most important problems in Geometry and Natural Philosophy (as he will discover on consulting the immortal works of D'Alembert, Euler, Monge, Lagrange, and Laplace), the Student is referred to Lacroix, 4to edition.

Having already exceeded the limits originally prescribed to ourselves in this Appendix, we shall forbear to comment upon the remaining Sections; and the rather, because the details by our author are of themselves sufficiently ample and diversified. We must not omit to notice, however, the oversight committed in Art. 399, which our author himself subsequently corrected in page 135 of his Miscellaneous Tracts.

THE END.

Printed by T. C. Hansard, Peterborough-court, Fleet-street, London.