THE DOCTRINE AND APPLICATION OF FLUXIONS.

CONTAINING

(Besides what is common on the Subject)

A Number of New Improvements in the THEORY.

AND

The Solution of a Variety of New, and very Interesting, Problems in different Branches of the MATHEMATICKS.

PART I.

By THOMAS SIMPSON, F.R.S.

THE SECOND EDITION.
Revised and carefully corrected.

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MDCCLXXVI.

TO THE

RIGHT HONOURABLE

George Earl of *Macclesfield.*

MY LORD,

AS I eſteem it a very great Honour to be permitted to place the following Sheets under your Lordſhip's Protection, who are not only an Encourager of, but an Ornament to, Mathematical Learning; I have taken more than ordinary Pains, that, *What* is here uſhered into the World, with ſuch Advantage, may not be found altogether unworthy of ſo diſtinguiſhed a Patron.

I am not vain enough to imagine, that, to One ſo deeply read in *theſe* abſtruſe and curious Speculations, as your Lordſhip is univerſally

verſally allowed to be, this Work will appear without Faults: But then, I have the Satiſfaction to think, on the other hand, that, whatever is Here to be met with capable of bearing the Teſt of an exact and ſolid Judgment, will *alſo* have its due Weight, and not fail of receiving your Lordſhip's Approbation: And if, upon the Whole, there is Merit enough found to entitle me to a favourable Reception, it will gratify the higheſt Ambition of,

MY LORD,

Your LORDSHIP'S

Moſt Obedient Humble Servant,

Tho. Simpſon.

PREFACE.

HAVING, in the Year 1737, publiſhed a Piece, on this ſame Subject, under the Title of *A Treatiſe of Fluxions* (whereof the whole Impreſſion hath been long ſince ſold) it may be proper here, firſt of all, to aſſign the Reaſons why this Work is ſent abroad into the World as a New Book, rather than a Second Edition of the ſaid Treatiſe. Which, in ſhort, are theſe two: Firſt, becauſe the preſent Work is vaſtly more full and comprehenſive; and, ſecondly, becauſe the principal Matters in it which are alſo to be met with in that Treatiſe, are handled in a different Manner.

BESIDES the Preſs-Errors with which the ſaid Treatiſe abounds, there are ſeveral Obſcurities and Defects (which the Author's Want of Experience, and the many Diſadvantages he then laboured under, in his firſt Sally, may, it is hoped, in ſome meaſure excuſe.) But what is now offered to the Publick, being a Performance of more mature Consideration and Judgment, it will, I flatter myself, be found much more correct, and claim a favourable Reception; especially, as particular Care and Pains have been taken to put every Thing in a clear Light, and to oblige the lower, as well as the more experienced, Class of Readers.

THE Notion and Explication *Here* given of the first Principles of Fluxions, are not essentially different from what they are in the abovementioned Treatise, tho' expressed in other Terms. The Consideration of Time, which I have introduced into the General Definition, will, perhaps, be disliked by *Those* who would have Fluxions to be *meer Velocities*: But the Advantage of considering them *otherwise* (not as the Velocities Themselves, but the Magnitudes *They* would, uniformly, generate in a given finite Time) appear to me sufficient to obviate any Objection on that Head.

BY taking Fluxions as *meer Velocities*, the Imagination is confined, as it were, to a Point, and, without proper Care, insensibly involved in metaphysical Difficulties: But according to our Method of conceiving and explaining the Matter, less Caution in the Learner is necessary, and the higher Orders of Fluxions are rendered much more easy and intelligible—— Besides, tho' Sir

Isaac Newton defines Fluxions to be *the Velocities of Motions*, yet He hath Recourse to the Increments, or Moments, generated in equal Particles of Time, in order to determine those Velocities; which he afterwards teaches us to expound by finite Magnitudes of other Kinds: Without which (as is already hinted above) we could have but very obscure Ideas of the higher Orders of Fluxions: For if Motion in (or at) a Point be so difficult to conceive, that, *Some* have, even, gone so far as to dispute the very Existence of Motion, how much more perplexing must it be to form a Conception, not only, of the Velocity of a Motion, but also in infinite Changes and Affections of *It*, in one and the same Point, where all the Orders of Fluxions are to be considered?

SEEING the Notion of a Fluxion, according to our Manner of defining It, supposes an uniform Motion, it may, perhaps, seem a Matter of Difficulty, at first View, how the Fluxions of Quantities, generated by Means of accelerated and retarded Motions, can be rightly assigned; since not any, the least, Time can be taken during which the generating Celerity continues the same: Here, indeed, we cannot express the Fluxion by any Increment or Space, *actually*, generated in a given Time (as in uniform Motions.) But, then, we can easily determine, what the contemporary Increment, or generated Space *would be*, if the Acceleration, or Retardation, was to cease

at

at the propoſed Poſition in which the Fluxion is to be found: Whence the true Fluxion, itſelf, will be obtained, without the Aſſiſtance of infinitely ſmall Quantities, or any metaphyſical Conſiderations.

Thus, for Example, the Motion of a Ball, deſcending by the Force of its own Gravity, is continually accelerated; but to have the Fluxion of the Diſtance fall'n thro' at any given Poſition of the Ball, we muſt find how far the Ball *would*, uniformly, deſcend, from that Point, in a given Time, if the Gravity, or the Earth's Attraction, from thence, was to ceaſe acting. By which Means we ſhall have as clear an Idea of the Fluxion and the true Meaſure of the Velocity of the Ball, at any Point aſſigned, as in thoſe Caſes where the Motion is, *actually*, uniform.

Again, if a Right-line be ſuppoſed to move parallel to itſelf with an equable Motion, and to increaſe in Length, at the ſame Time; the Area generated thereby, will increaſe with an accelerated Velocity: But the Fluxion thereof, at any given Poſition of the Line, will be had by taking that Part of the Increment which *would*, uniformly; ariſe, was the Length (as well as the Velocity) of the Line to continue invariable from the propoſed Poſition. For, if the Length be ſuppoſed to increaſe, from the ſaid Poſition, the Area generated, from thence, will be, evidently, greater than That which would uniformly ariſe in the ſame Time; ſince the new Parts, produced each ſucceeding Moment, are greater and greater. Therefore the Fluxion muſt be leſs than any Space that can be deſcribed, in the given Time, when the Line increaſes. And, in the ſame Manner, the Fluxion will appear to be greater than any Space that can be deſcribed, in the ſame Time, when the Line decreaſes. *It* muſt, therefore, be equal to that Space, which will ariſe, when the Length of the generating Line, from the given Poſition, is ſuppoſed neither to increaſe nor decreaſe: Agreeable to *Art.* 4.

THUS much it ſeem'd proper to offer Here with regard to the Firſt Principles—I ſhall now proceed to ſay ſomething concerning the Order obſerv'd in treating, and putting together, the ſeveral Parts of the Work; wherein the Eaſe and Benefit of the younger Beginner have been particularly conſulted: To load ſuch an One with a Multitude of Rules and Precepts, before giving him any Taſte of their Uſe and Application, would, certainly, be very diſcouraging; and like obliging a Traveller to aſcend an high Mountain, without allowing him to ſtop by the Way, to take Breath, and refreſh his Spirits with a Proſpect of the agreeable and extenſive View he has to expect when he arrives at the Summit: I have therefore, after demonſtrating the Firſt Principles, proceeded immediately to exemplify their Utility in ſeveral entertaining Enquiries, before touching at all upon the Inverſe Method, or the more difficult

ficult Parts of the Direct. And, since that Branch of the Inverse Method which treats of the Comparison of Fluents is, naturally, somewhat difficult, it is referred to the Second Part of the Work, together with such other Matters in the Theory as might appear, either, too tedious or hard to a Learner at first setting out. The like Care has been taken in the Disposal of the rest of the Work —— As to the several Particulars whereof *It* is composed, I must refer to the Book itself, They being too many to be here enumerated: One Thing, however, I must not omit to take notice of, relating to that Part which treats of the aforesaid Business of Fluents: To which it may, perhaps, be objected, That, notwithstanding my having insisted so largely on the Subject, there are a Number of Forms of Fluxions and Fluents to be met with in Authors, that I have not so much as touch'd upon. This is granted; but then they are most of them such as, I dare pronounce, can never arise in any Inquiry into Nature: And it would, doubtless, be Time and Labour misapply'd, to swell the Work, and embarrass the Learner with a Number of unnecessary Difficulties, and empty Speculations; when what is, really, proper and useful, in the Subject, is sufficient (it is well known) to exercise his utmost Attention and Resolution.

I CANNOT put an End to this Preface without acknowledging my Obligations to a small Tract,

in-

intitled, *An Explanation of Fluxions in a Short Essay on the Theory*; printed for *W. Innys:* Wrote by a worthy Friend of mine (who was too modest to put his Name to that, his first, Attempt) whose Manner of determining the Fluxion of a Rectangle, and illustrating the higher Order of Fluxions, I have, in particular, follow'd, with little or no Variation.

The following BOOKS are all written by Mr. *Thomas Simpson*, F. R. S. and printed for *J. Nourse*.

1. THE Elements of Geometry; with their Application to the Menfuration of Superficies and Solids, to the Determination of the Maxima and Minima of Geometrical Quantities, and to the Conftruction of a great Variety of Geometrical Problems, 8vo. the third Edition, 5*s*.

2. Trigonometry Plain and Spherical, with the Conftruction and Application to Logarithms, the Second Edition, 8vo, 1*s*. 6*d*.

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THE

THE

DOCTRINE and APPLICATION

OF

FLUXIONS.

PART the First.

SECTION I.

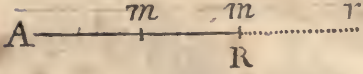
Of the Nature, and Investigation, of Fluxions.

1. IN order to form a proper Idea of the Nature of Fluxions, all Kinds of Magnitudes are to be considered as generated by the *continual* Motion of some of their Bounds or Extremes; as a Line by the Motion of a Point; a Surface by the Motion of a Line; and a Solid by the Motion of a Surface.

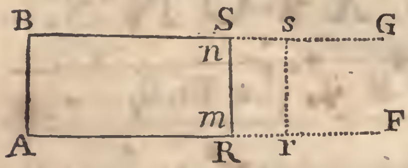
2. Every Quantity so generated is called a variable, or flowing Quantity: *And the Magnitude by which any flowing Quantity* WOULD BE *uniformly increased in a given Portion of Time, with the generating Celerity at any proposed Position, or Instant (was it from thence to continue invariable) is the Fluxion of the said Quantity at that Position, or Instant.*

 Thus,

Thus, let the Point *m* be conceived to move from A, and generate the variable Right-line A*m*, by a Motion any how regulated; and let the Celerity thereof, when it arrives at any proposed Position R, be such *as would*, was it to continue uniform from that Point, be sufficient to describe the Distance, or Line R*r*, in the given Time allotted for the Fluxion: Then will R*r* be the Fluxion of the variable Line A*m*, in that Position.



3. The Fluxion of a plane Surface is conceived in like Manner, by supposing a given Right-line *mn* to move parallel to itself, in the Plane of the parallel, and immoveable Lines AF and BG: For, if (as above) Rr be taken to express the Fluxion of the Line A*m*, and the Rectangle RrsS be completed; then that Rectangle, being the Space which *would be* uniformly described by the generating Line *mn*, in the Time that A*m would be* uniformly increased by *m*r, is therefore the Fluxion of the generated Rectangle B*m*, in that Position, according to the true Meaning of the Definition.



4. If the Length of the generating Line *mn* continually varies, the Fluxion of the Area will *still* be expounded by a Rectangle under that Line and the Fluxion of the Abscissa, or Base: For let the curvilineal Space A*mn* be generated by the continual, and parallel, Motion of the (now) variable Line *mn*, and let Rr be the Fluxion of the Base, or Abscissa, *Am* (as before); then the Rectangle RrsS will, here also, be the Fluxion of the generated Space A*mn*: Because, if the Length and Velocity of the generating Line *mn* were

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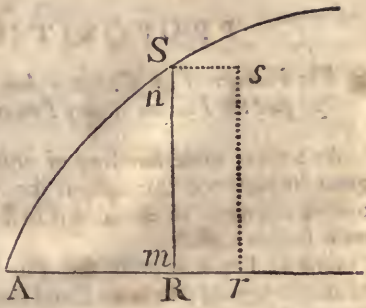
to continue invariable from the Position RS, the Rectangle RrsS would then be uniformly generated, with the very Celerity wherewith it begins to be generated, or with which the Space *Amn* is increased in that Position.



5. From what has been hitherto said it will appear, that *the Fluxions of Quantities are, always, as the Celerities by which the quantities themselves increase in Magnitude:* Whence it will not be difficult to form a Notion of the Fluxions of Quantities otherwise generated; as well such as arise from the Revolution of Right-lines and Planes, as those by parallel Motion: But of this hereafter. I come now to shew the Manner of determining the Fluxions of algebraic Quantities; by which all others, of what Kind soever, are explicable. But first of all it will be requisite to premise the following Observations.

I. *That the final Letters* u, w, x, y, z *of the Alphabet are commonly put for variable Quantities; and the initial Letters* a, b, c, d, &c. *for invariable ones:* Thus the Diameter of a given Circle may be denoted by a, and the Sine of any Arch thereof (considered as variable) by x.

II. *That the Fluxion of a Quantity represented by a single Letter, is usually expressed by the same Letter with a Dot or Full-point over it:* Thus the Fluxion of x is represented by $\dot{x}$, and that of y by $\dot{y}$.

III. *That the Fluxion of a Quantity which decreases, instead of increasing, is to be considered as negative.*

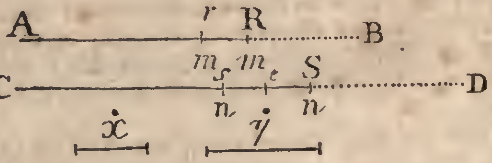
PROPOSITION I.

6. *The Fluxion of a Quantity being given, it is propoſed to find the Fluxion of any Power of that Quantity.*

As a clear underſtanding of this Problem will be of great Importance throughout the whole Work, it may not be improper to conſider it firſt in one or two of its moſt ſimple Caſes.

Caſe 1. Let $\dot{x}$ expreſs the Fluxion of x, (according to the foregoing Notation) and let the Fluxion of x^2 be required.

Conceive two Points m and n to proceed, at the ſame time, from two other Points A and C, along the Right-lines AB and CD, in ſuch ſort, that the Meaſure of the Diſtance CS *(y)*, deſcribed by the latter, may be, *always*, equal to the Square of that AR *(x)*, deſcribed by the former moving uniformly.



Furthermore, let r, s, and R, S, be any contemporary Poſitions of the generating Points, and let the Lines $\dot{x}$ and $\dot{y}$ repreſent the reſpective Diſtances that *would be* uniformly deſcribed, in the ſame time, with the Celerities of thoſe Points at R and S, then thoſe Lines will expreſs the Fluxions of Am and Cn in this Poſition, *(by the Definition*, Art. 2 and 5).

Moreover, ſince $Cs = Ar^2$ and $CS = AR^2$ *(by Hypotheſis)*, if Rr be denoted by v, we ſhall have CS $(y) = x^2$, and $Cs\ (= \overline{x - v}|^2) = x^2 - 2xv + v^2$, and conſequently $Ss\ (= CS - Cs) = 2xv - v^2$; from whence we gather, that, while the Point m moves over the Diſtance v, the Point n moves over the Diſtance

$2xv-v^2$. But this laſt Diſtance (ſince the Square of any Quantity is known to increaſe faſter, in Proportion, than the Root) is not deſcribed with an uniform Motion (like the former), but an accelerated one; and therefore is equal to, and may be taken to expreſs, the uniform Space that might be deſcribed with the mean Celerity at ſome intermediate Point e, in the ſame time. Therefore, ſeeing the Diſtances that might be deſcribed, in equal times, with the uniform Celerity of m, and the mean Celerity at e, are to each other as v to $2xv-v^2$, or as 1 to $2x-v$, or, laſtly, as $\dot{x}$ to $2x\dot{x}-v\dot{x}$, (all which are in the ſame Proportion) it is evident, that, in the time the Point m would move uniformly over the Diſtance $\dot{x}$, the other Point n, with its Celerity at e, *would* move uniformly over the Diſtance $2x\dot{x}-v\dot{x}$. This being the Caſe, let r, R, and s, S, be now ſuppoſed to coincide, by the Arrival of the generating Points at R and S, then e (being always between s and S, will likewiſe coincide with S; and the Diſtance, $2x\dot{x}-x\dot{x}$, which might be uniformly deſcribed in the aforeſaid time, with the Velocity at e, (now at S), will become barely equal to $2x\dot{x}$; which *(by the Defin.)* is equal to $(\dot{y})$, the true Fluxion of Cn or x^2 [a].

[a] *It may, perhaps, ſeem inaccurate, that the Fluxions of* x *and* x^2 *are compared together, and expreſſed both by Lines, when the flowing Quantities themſelves, conſidered as a Right Line and a Square, admit of no Compariſon.——This Objection would, indeed, be of force, were the Expreſſions reſtrained to a geometrical Signification; but here our Notions are more abſtracted and univerſal, not obliging us to regard what Kind of Extenſion, may be defined by this or that Expreſſion, but only the Values of the algebraic Quantities thereby ſignified; to which the Meaſures of all other Quantities whatever are ultimately referred.——And, though Quantities of different Kinds cannot be compared with each other, their Meaſures, in Numbers, may.——Thus, for Inſtance, though it would be wrong to affirm, that a Square whoſe Area is 9 Inches is equal to a Line of 9 Inches long, yet it is no Impropriety at all to ſay the Numbers expreſſing their Meaſures, in Inches, are equal.*

7. *Case* 2. Let the Fluxion of x^3 be required.

Suppose every Thing to remain as in the preceding Case; only let C*n* be here equal to the Cube of A*m* (instead of the Square).

Then, in the very same manner, we have S*s* ($=$CS$-$C$s = x^3 - \overline{x-v}|^3) = 3x^2v - 3xv^2 + v^3$: From whence it appears, that the Distances which *might be* described, in the same time, with the uniform Celerity of *m*, and the mean Celerity at *e*, will, in this Case, be to each other as v to $3x^2v - 3xv^2 + v^3$, or as $\dot{x}$ to $3x^2\dot{x} - 3xv\dot{x} + v^2\dot{x}$: Which last Expression, when *s*, *e*, and S coincide (as before) will become $3x^2\dot{x}$, the true Fluxion of x^3 required.

8. *Universally.* Let C*n* be, *always*, equal to $\overline{Am}|^n$; also let $\overline{x-v}|^n$ (or $x-v$ raised to the Power whose Exponent is n) be represented by $x^n - ax^{n-1}v + bx^{n-2}v^2 - cx^{n-3}v^3$, &c. and let every Thing else be supposed as above.

Then, since S*s* $(x^n - \overline{x-v}|^n)$ is $= ax^{n-1}v - bx^{n-2}v^2 + cx^{n-3}v^3$, &c. it is plain that the Spaces which might be described, in the same time, with the uniform Celerity of *m*, and the mean Celerity at *e*, will, here, be to each other as v to $ax^{n-1}v - bx^{n-2}v^2 + cx^{n-3}v^3$, &c. or as $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}v\dot{x} + cx^{n-3}v^2\dot{x}$, &c.

Therefore, all the Terms, wherein v is found, vanishing, when *s*, *e*, and S coincide, we have $ax^{n-1}\dot{x}$ for the required Fluxion of C*n*, or x^n; which Fluxion, because the numeral Co-efficient of the second Term of a Binomial involved is known to be, *universally*, equal to the Exponent of the Power, will also be truly expressed by $nx^{n-1}\dot{x}$. Q. E. I.

9. If the Quantity A*m* (or x) be generated with an accelerated, or a retarded Motion, instead of an uniform

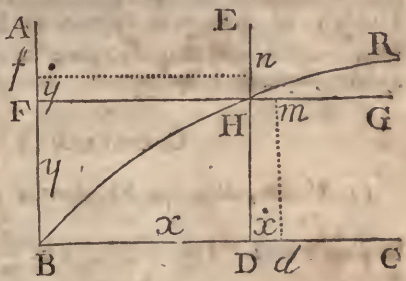
form one, the Fluxion of x^n (or Cn) will come out exactly the fame:

For the Spaces rR and sS, actually defcribed in the fame time, being always, to each other, in the Ratio of $\dot{x}$ to $ax^{n-1}\dot{x} - bx^{n-2}v\dot{x}$, *&c.* the mean Celerities, at certain intermediate Points between r, R and s, S muft, alfo, be in that Ratio: Which, when v vanifhes (as above) will become that of $\dot{x}$ to $ax^{n-1}\dot{x}$,(or $nx^{n-1}\dot{x}$) the very fame as before.

PROPOSITION II.

10. *To find the Fluxion of the Product or Rectangle of two variable Quantities.*

Conceive two Right-lines DE and FG, perpendicular to each other, to move, from two other Right-lines, BA and BC, continually parallel to themfelves, and thereby generate the Rectangle DF. Let the Path of their



Interfection, or the Loci of the Angle H, be the Line BHR; alfo let Dd ($\dot{x}$) and Ff ($\dot{y}$) be the Fluxions of the Sides BD (x) and BF (y), and let dm and fn, parallel to DH and FH, be drawn. Therefore, becaufe the Fluxion of the Space or Area BDH is truly expreffed by the Rectangle Dm ($= y\dot{x}$ *) and that of the Area, or Space BFH, by the Rectangle Fn, and equal Quantities have equal Fluxions, it follows that the Fluxion of the Rectangle xy = DF (= BDH + BFH) is truly expreffed by $\dot{y}x + \dot{x}y$. Q. E. I.

* Art. 4.

The same otherwise.

11. Let xy be the given Rectangle (as before); and put $z = x + y$, then z^2 being $= x^2 + 2xy + y^2$, we have $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 = xy$. But the Fluxion of $\frac{1}{2}z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, (and consequently that of its Equal xy) is $z\dot{z} - x\dot{x} - y\dot{y}$ (by *Art.* 6): Which, because $z = x + y$ and $\dot{z} = \dot{x} + \dot{y}$, is also equal to $\overline{x+y} \times \overline{\dot{x}+\dot{y}} - x\dot{x} - y\dot{y} = y\dot{x} + x\dot{y}$. Q. E. I.

COROLLARY 1.

12. Hence the Fluxion of the Product of three variable Quantities (yzu) may be derived: For, if x be put $= zu$; then yzu will become $= yx$, and its Fluxion $= y\dot{x} + x\dot{y}$ (*as above:*) But x being $= zu$, and, therefore, $\dot{x} = z\dot{u} + u\dot{z}$, if these Values be substituted in $y\dot{x} + x\dot{y}$, it will become $y \times \overline{z\dot{u} + u\dot{z}} + zu\dot{y} = yz\dot{u} + yu\dot{z} + zu\dot{y}$ the Fluxion of yzu required. In like Manner the Fluxion of $xyzu$ will appear to be $xyz\dot{u} + xy\dot{z}u + x\dot{y}zu + \dot{x}yzu$, and that of $xyzuw = xyzu\dot{w} + xyz\dot{u}w + xy\dot{z}uw + x\dot{y}zuw + \dot{x}yzuw$.

COROLLARY 2.

13. Hence, also, the Fluxion of a Fraction $\frac{u}{z}$ may be determined. For, putting $x = \frac{u}{z}$, we have $xz = u$, and therefore $x\dot{z} + z\dot{x} = \dot{u}$ (*as above*); whence, by Transposition and Division, $\dot{x} = \frac{\dot{u}}{z} - \frac{x\dot{z}}{z} = \frac{\dot{u}}{z} - \frac{u\dot{z}}{z^2}$ (by writing $\frac{u}{z}$ for x) $= \frac{z\dot{u} - u\dot{z}}{z^2}$; which is the true Fluxion of x, or its Equal $\frac{u}{z}$, the Fraction proposed.

14. Now, from the foregoing Propositions, and their subsequent Corollaries, the following practical Rules, for

for determining the Fluxions of algebraic Quantities, are obtained.

RULE I.

To find the Fluxion of any given Power of a variable Quantity.

Multiply the Fluxion of the Root by the Exponent of the Power, and the Product by that Power of the same Root whose Exponent is less by Unity than the given Exponent.

This Rule is investigated in Prop. 1, and is nothing more than $n x^{n-1} \dot{x}$ (the Fluxion of x^n) expressed in Words.

Hence the Fluxion of x^3 is $3x^2\dot{x}$; that of x^5 is $5x^4\dot{x}$; and that of $\overline{a+y}|^7$ is $7\dot{y} \times \overline{a+y}|^6$, (because, a being constant, $\dot{y}$ is the true Fluxion of the Root $a+y$, in this Case).

Moreover the Fluxion of $\overline{a^2+z^2}|^{\frac{3}{2}}$, will be $\frac{3}{2} \times 2z\dot{z} \times \overline{a^2+z^2}|^{\frac{1}{2}}$, or $3z\dot{z}\sqrt{a^2+z^2}$: For here, x being put $= a^2 + z^2$, we have $\dot{x} = 2z\dot{z}$, and therefore $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, the Fluxion of $x^{\frac{3}{2}}$ (or $\overline{a^2+z^2}|^{\frac{3}{2}}$) is $= 3z\dot{z}\sqrt{a^2+z^2}$, as above.

RULE II.

15. To find the Fluxion of the Product of several variable Quantities multiplied together.

Multiply the Fluxion of each, by the Product of the rest of the Quantities, and the Sum of the Products thus arising will be the Fluxion sought *.

Thus the Fluxion of xy, is $\dot{x}y + y\dot{x}$; that of xyz, is $xy\dot{z} + xz\dot{y} + yz\dot{x}$; and that of $xyzu$, is $xyz\dot{u} + xyu\dot{z} + xzu\dot{y} + yzu\dot{x}$. *Art. 12.

RULE

RULE III.

16. To find the Fluxion of a Fraction.

From the Fluxion of the Numerator drawn into the Denominator, subtract the Fluxion of the Denominator drawn into the Numerator, and divide the Remainder by the Square of the Denominator *.

*Art. 13.

Thus, the Fluxion of $\frac{x}{y}$ is $\frac{y\dot{x}-x\dot{y}}{y^2}$; that of $\frac{x}{x+y}$, is $\frac{\dot{x}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times x}{\overline{x+y}|^2}=\frac{\dot{x}y-\dot{y}x}{\overline{x+y}|^2}$; and that of $\frac{x+y+z}{x+y}$, or $1+\frac{z}{x+y}$, is $\frac{\dot{z}\times\overline{x+y}-\overline{\dot{x}+\dot{y}}\times z}{\overline{x+y}|^2}$; and so of others.

17. In the Examples hitherto given, each is resolved by its own particular Rule; but in those that follow, the Use of two, and sometimes of all the three, Rules is requisite.

Thus (*by Rule* 1. *and* 2.) the Fluxion of x^2y^2 is $2x^2y\dot{y}+2y^2x\dot{x}$; that of $\frac{x^2}{y^2}$ is $\frac{2y^2x\dot{x}-2x^2y\dot{y}}{y^4}$, (*by Rule* 1. *and* 3.) and that of $\frac{x^2y^2}{z}$ is $\frac{\overline{2x^2y\dot{y}+2y^2x\dot{x}}\times z-x^2y^2\dot{z}}{z^2}$; where all the three Rules are necessary.

When the proposed Quantity is affected by a Co-efficient, or constant Multiplicator, the Fluxion found as above, must be multiplied by that Co-efficient or Multiplicator.

Thus, the Fluxion of $5x^3$ is $15x^2\dot{x}$. For, the Fluxion of x^3 being $3x^2\dot{x}$, that of $5x^3$, which is 5 times as great, must consequently be $5\times 3x^2\dot{x}$, or $15x^2\dot{x}$.

And, in the very same Manner the Fluxion of ax^n will appear to be $nax^{n-1}\dot{x}$. Moreover, the Fluxion of $\frac{a}{\overline{x^2+y^2}|^{\frac{1}{2}}}$, or $a\times\overline{x^2+y^2}|^{-\frac{1}{2}}$, will be expressed by

$a\times$

$a \times -\frac{1}{2} \times \overline{2x\dot{x}+2y\dot{y}} \times \overline{x^2+y^2}|^{-\frac{3}{2}}$, or $-\frac{a \times \overline{x\dot{x}+y\dot{y}}}{\overline{x^2+y^2}|^{\frac{3}{2}}}$;

that of $\sqrt{x+y^{\frac{1}{2}}}$, or $\overline{x+y^{\frac{1}{2}}}|^{\frac{1}{2}}$, by $\overline{\frac{1}{2}\dot{x}+\frac{1}{2}\times\frac{1}{2}\dot{y}y^{-\frac{1}{2}}} \times \overline{x+y^{\frac{1}{2}}}|^{-\frac{1}{2}}$, (*Rule* 1.) or $\frac{\frac{1}{2}\dot{x}+\frac{1}{4}y\dot{y}^{-\frac{1}{2}}}{\sqrt{x+y^{\frac{1}{2}}}}$, or $\frac{\frac{1}{2}\dot{x}y^{\frac{1}{2}}+\frac{1}{4}\dot{y}}{\sqrt{xy+y^{\frac{3}{2}}}}$;

and that of $\frac{\overline{x+a}|^2}{\sqrt{x^2-a^2}}$, or $\frac{\overline{x+a}|^2}{\overline{x^2-a^2}|^{\frac{1}{2}}}$, by

$$\frac{2\dot{x} \times \overline{x+a} \times \overline{x^2-a^2}|^{\frac{1}{2}} - x\dot{x} \times \overline{x^2-a^2}|^{-\frac{1}{2}} \times \overline{x+a}|^2}{x^2-a^2}; \text{ which}$$

by Reduction, is $= \frac{2\dot{x} \times \overline{x^2-a^2}|^{\frac{1}{2}} - \dot{x}x \times \overline{x^2-a^2}|^{-\frac{1}{2}} \times \overline{x+a}}{x-a}$

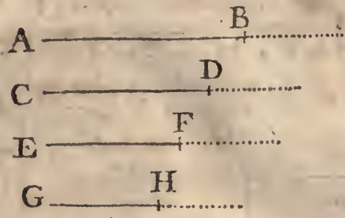
$$= \frac{2\dot{x} \times \overline{x^2-a^2} - x\dot{x} \times \overline{x+a}}{\overline{x-a} \times \overline{x^2-a^2}|^{\frac{1}{2}}} = \frac{2\dot{x} \times \overline{x-a} \times \overline{x+a} - \dot{x}x \times \overline{x+a}}{\overline{x-a} \times \sqrt{x^2-a^2}}$$

$$= \frac{\overline{x+a} \times \overline{x\dot{x}-2a\dot{x}}}{\overline{x-a} \times \sqrt{x^2-a^2}}.$$

Having explained the Manner of confidering and determining the firft Fluxions of variable or flowing Quantities, it will be proper to fay fomething, now, concerning the higher Orders, as Second, Third, Fourth, &c. Fluxions.

18. *The Second Fluxion of a Quantity is the Fluxion of the variable or algebraic Quantity expreffing the Firft Fluxion already defined* *. *By the Third Fluxion is* * Art. 2. *meant the Fluxion of the variable Quantity expreffing the Second: And by the Fourth, the Fluxion of the variable Quantity expreffing the Third Fluxion: And fo on.*

Thus, for Example, let the Line AB reprefent a variable Quantity, generated by the Motion of the Point B, and let the (firft) Fluxion thereof (or the Space that *might be* uniformly defcribed in a given Time, with the Celerity of B) be always expreffed by the Diftance of the Point D from a given, or fixed Point C: Then, if the Celerity of B be not every where the same; the Distance CD, expressing the Measure of that Celerity, must also vary, by the Motion of D, from, or towards C, according as the Celerity of B is an increasing or a decreasing one: And the Fluxion of the Line CD, so varying (or the Space (EF) that *might be* uniformly described in the aforesaid given Time, with the Celerity of D) is the second Fluxion of AB. Again, if the Motion of B be such that neither it, nor that of D, (which depends upon it) be equable, then EF, expressing the Celerity of D, will also have its Fluxion GH; which is the third Fluxion of AB, and the second Fluxion of CD.



And thus are the Fluxions of every other Order to be considered, *being the Measures of the Velocities by which their respective flowing Quantities, the Fluxions of the preceding Order, are generated* *.

*Art. 2.

19. Hence it appears, that a second Fluxion always shews the rate of the Increase, or Decrease, of the first Fluxion; and that Third, Fourth, *&c.* Fluxions, differ in Nothing (except their Order and Notation) from First Fluxions, being actually such to the Quantities from whence they are immediately derived; and therefore are also determinable, in the very same Manner, by the general Rules already delivered.

Thus, *by Rule* 3. the (first) Fluxion of x^3 is $3x^2\dot{x}$: And, if $\dot{x}$ be supposed constant, that is, if the Root x be generated with an equable Celerity, the Fluxion of $3x^2\dot{x}$ (or $3\dot{x} \times x^2$) again taken, by the same Rule, will be $3\dot{x} \times 2x\dot{x}$, or $6x\dot{x}^2$; which therefore is the second Fluxion of x^3: Whose Fluxion, found in like Sort, will be $6\dot{x}^3$, the third Fluxion of x^3. Farther than which

which we cannot go in this Caſe, becauſe the laſt Fluxion $6\dot{x}^3$ is here a conſtant Quantity.

20. In the preceding Example the Root x is ſuppoſed to be generated with an equable Celerity: But, if the Celerity be an increaſing or a decreaſing one, then $\dot{x}$, expreſſing the Meaſure thereof, being variable, will alſo have its Fluxion; which is uſually denoted by $\ddot{x}$: Whoſe Fluxion, according to the ſame Method of Notation, is again deſigned by $\dddot{x}$; and ſo on, with reſpect to the higher Orders.

21. Here follow a few Examples, wherein the Root x, (or y) is ſuppoſed to be generated with a variable Celerity.

Thus, the firſt Fluxion of x^3 is $3x^2\dot{x}$ (or $3x^2\times\dot{x}$). And, if the Fluxion of $3x^2\times\dot{x}$ (conſidered as a Rectangle) be, again, found *(by Rule* 2.) we ſhall have $6x\dot{x}\times\dot{x}+3x^2\times\ddot{x}=6x\dot{x}^2+3x^2\ddot{x}$, for the ſecond Fluxion of x^3.

Moreover, from the Fluxion laſt found we ſhall in like manner get $6\dot{x}\times\dot{x}^2 + 6x\times 2\dot{x}\ddot{x} + 6x\dot{x}\times\ddot{x} + 3x^2\times\dddot{x}$ (or $6\dot{x}^3+18x\dot{x}\ddot{x}+3x^2\dddot{x}$) for the third Fluxion of x^3.

Thus alſo, if $\dot{y}=nx^{n-1}\dot{x}$, then will $\ddot{y}=n\times\overline{n-1}\times x^{n-2}\dot{x}^2+n\ddot{x}x^{n-1}$; and if $\dot{z}^2=\dot{x}\dot{y}$, then will $2\dot{z}\ddot{z}=\dot{x}\ddot{y}+\dot{y}\ddot{x}$: And ſo of others. But, in the Solution of Problems, it will be convenient to make the firſt Fluxion of ſome one of the ſimple Quantities (x or y) invariable, not only to avoid Trouble, but that it may ſerve as a Standard to which the variable Fluxions of the other Quantities, depending thereon, may be always referred. The Reader is alſo deſired here (once for all) to take particular Notice, that *the Fluxions of all Kinds and Orders, whatever, are contemporaneous, or ſuch as may be generated together, with their reſpective Celerities, in one and the ſame Time.*

SECT.

SECTION II.

On the Application of Fluxions to the Solution of Problems DE MAXIMIS ET MINIMIS.

22. IF a Quantity, conceived to be generated by Motion, increaſes, or decreaſes, till it arrives at a certain Magnitude or Poſition, and then, on the contrary, grows leſſer or greater, and it be required to determine the ſaid Magnitude or Poſition, the Queſtion is called a Problem *de Maximis & Minimis.*

GENERAL ILLUSTRATION.

Let a Point *m* move uniformly in a Right Line, from A towards B, and let another Point *n* move after it, with a Velocity either increaſing, or decreaſing, but ſo that it may, at a certain Poſition, D, become equal to that of the former Point *m*, moving uniformly.

This being premiſed, let the Motion of *n* be firſt

conſidered as an increaſing one; in which Caſe the Diſtance of *n* behind *m* will continually increaſe, till the two Points arrive at the cotemporary Poſitions C and D; but afterwards it will, again, decreaſe; for the Motion of *n*, till then, being ſlower than at D, it is alſo ſlower than that of the preceding Point *m* (by Hypotheſis) but becoming quicker, afterwards, than that of *m*, the Diſtance *mn* (as has been already ſaid) will again decreaſe: And therefore is a *Maximum*, or the greateſt of all, when the Celerities of the two Points are equal to each other.

But, if *n* arrives at D with a decreaſing Celerity; then its Motion being firſt ſwifter, and afterwards ſlower, than that of *m*, the Diſtance *mn* will firſt decreaſe and then

then increase; and therefore is a *Minimum*, or the least of all, in the forementioned Circumstance.

Since then the Distance *mn* is a *Maximum* or a *Minimum*, when the Velocities of *m* and *n* are equal, or when that Distance increases as fast through the Motion of *m*, as it decreases by that of *n*, its Fluxion at that Instant is evidently equal to Nothing *. (* Art. 2 and 5.) Therefore, as the Motion of the Points *m* and *n* may be conceived such that their Distance *mn* may express the Measure of any variable Quantity whatever, it follows, that the Fluxion of any variable Quantity whatever, when a Maximum or Minimum, is equal to Nothing.

EXAMPLE I.

23. *To divide a given Right-line* AB *into two such Parts*, AC, BC, *that their Product, or Rectangle, may be the greatest possible.*

Put the given Line AB $= a$, and let the Part AC,

considered as variable (by the Motion of C from A towards B) be denoted by x: Then BC being $= a - x$, we have $AC \times BC = ax - x^2$: Whose Fluxion $a\dot{x} - 2\dot{x}x$ being put $= 0$, according to the prescript, we get $a\dot{x} = 2x\dot{x}$, and consequently $x = \frac{1}{2}a$. Therefore AC and BC, in the required Circumstance, are equal to each other: Which we also know from other Principles.

EXAMPLE II.

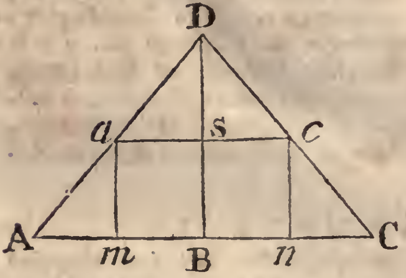
24. *To find the Fraction which shall exceed its Cube by the greatest Quantity possible.*

Let x denote a variable Quantity, expressing Number in general; then the Excess of x above x^3 being universally represented by $x - x^3$, if the Fluxion thereof be taken, &c. we shall have $\dot{x} - 3x^2\dot{x} = 0$; and therefore $x = \sqrt{\frac{1}{3}}$, the Fraction required.

E X-

EXAMPLE III.

25. *To determine the greatest Rectangle that can be inscribed in a given Triangle.*



Put the Base AC of the given Triangle $= b$, and its Altitude BD $= a$; and let the Altitude (BS) of the inscribed Rectangle *mc* (considered as variable) be denoted by x: Then, because of the parallel Lines AC, and *ac*, it will be as BD (a) : AC (b) :: DS $(a-x)$: $\frac{ba-bx}{a}$ $=ac$: Whence the Area of the Rectangle, or $ac \times$ BS will be $= \frac{bax-bx^2}{a}$: Whose Fluxion $\frac{ba\dot{x}-2bx\dot{x}}{a}$ being (as before) put $= 0$, we shall get $x = \frac{1}{2}a$. Whence the greatest inscribed Rectangle is that whose Altitude is just half the Altitude of the Triangle.

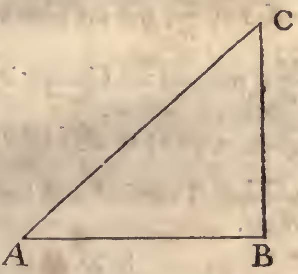
26. It will be proper to observe *here*, that the Value of a Quantity, when a *Maximum* or *Minimum*, may oftentimes be determined with more Facility by taking the Fluxion of some given Part, Multiple, or Power, thereof, than from the Fluxion of the Quantity itself. Thus, in the preceding Example, where the general Expression is $\frac{bax-bx^2}{a} = \frac{b}{a} \times \overline{ax-x^2}$, if the constant Multiplicator $\frac{b}{a}$ be rejected, we shall have $ax-x^2$; whose Fluxion $a\dot{x}-2x\dot{x}$ being put $= 0$, we get $x = \frac{1}{2}a$, *the very same as before.*

 The

The Reaſon of which is obvious; becauſe when the Quantity itſelf (be it of what Kind it will) is the greateſt, or leaſt poſſible, any *given* Part, Power, or Multiple of it is alſo the greateſt or leaſt poſſible.

EXAMPLE IV.

27. *Of all right-angled plain Triangles having the ſame given Hypothenuſe, to find that* (ABC) *whoſe Area is the greateſt.*



Let AC $=a$, AB $=x$, and BC $=y$: Then, x^2+y^2 being $=a^2$, we ſhall have $y=\sqrt{a^2-x^2}$, and conſequently $\frac{xy}{2}=\frac{x}{2}\cdot\sqrt{a^2-x^2}=$ the Area of the Triangle; whoſe Square $\frac{a^2x^2}{4}-\frac{x^4}{4}$ being, *alſo*, a *Maximum* *, the Fluxion thereof $\frac{a^2x\dot{x}}{2}-x^3\dot{x}$ muſt therefore be $=0$, †: Whence x is found $=a\sqrt{\frac{1}{2}}$, and y $(\sqrt{a^2-x^2})=a\sqrt{\frac{1}{2}}$.

*Art. 26.

†Art. 22.

The ſame otherwiſe.

Since $\frac{1}{2}xy$ is a *Maximum*, and $x^2+y^2=a^2$, let the Fluxions of both be taken, and you will have $\frac{1}{2}x\dot{y}+\frac{1}{2}y\dot{x}=0$, and $2x\dot{x}+2y\dot{y}=0$; from the former of which $\dot{y}$ will be $=-\frac{y\dot{x}}{x}$; and from the latter, it will be $=-\frac{x\dot{x}}{y}$: Therefore $\frac{y\dot{x}}{x}$ and $\frac{x\dot{x}}{y}$ are equal to each other, and conſequently $x=y$, (the ſame as before.)

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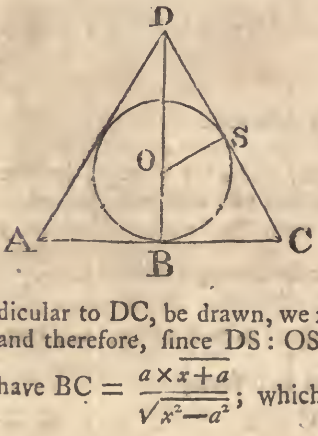
EXAMPLE V.

28. *Of all right-angled plain Triangles containing the ſame given Area, to find that whereof the Sum of the two Legs* AB+BC *is the leaſt poſſible.* (See the preceding Figure.)

Let one Leg, AB, be denoted by x, and the Area of the Triangle by a; then the other Leg will be denoted by $\frac{2a}{x}$, and the Sum of the two Legs will be $x + \frac{2a}{x}$; whereof the Fluxion is $\dot{x} - \frac{2a\dot{x}}{x^2}$; which, put $= 0$, gives x (AB) $= \sqrt{2a}$: Whence BC $\left(\frac{2a}{x}\right)$ is alſo $= \sqrt{2a}$. Therefore the two Legs are equal to each other.

EXAMPLE VI.

29. *To determine the Dimenſions of the leaſt Iſoſceles Triangle* ACD *that can circumſcribe a given Circle.*



Let the Diſtance (OD) of the Vertex of the Triangle from the Center of the Circle, be denoted by x, and let the remaining Part of the Perpendicular, which is the Radius of the Circle, be repreſented by a: Then, if OS, perpendicular to DC, be drawn, we ſhall have DS $= \sqrt{x^2 - a^2}$; and therefore, ſince DS : OS : : DB : BC, we likewiſe have BC $= \frac{a \times \overline{x+a}}{\sqrt{x^2-a^2}}$; which multiplied by $\overline{x+a}$ (BD) gives $\frac{a \times \overline{x+a}|^2}{\sqrt{x^2-a^2}}$ for the Area of the Triangle: Which being a *Minimum*, its Square must be a *Minimum*, and consequently $\frac{\overline{x+a}|^4}{x^2-a^2}$, or its Equal $\frac{\overline{x+a}|^3}{x-a}$, a *Minimum* also *. Whose Fluxion, therefore, which is *Art. 26. $\frac{3\dot{x} \times \overline{x+a}|^2 \times \overline{x-a} - \dot{x} \times \overline{x+a}|^3}{\overline{x-a}|^2}$, being put $= 0$, and the Whole divided by $\frac{\dot{x} \times \overline{x+a}|^2}{\overline{x-a}|^2}$, we also get $3 \times \overline{x-a} - \overline{x+a} = 0$; whence $x = 2a$: Therefore, OD being $= 2$OS, and the Triangles ODS and BDC equiangular, it is evident that DC is likewise $= 2$BC $=$ AC; and so the Triangle ACD, when the least possible, is equilateral.

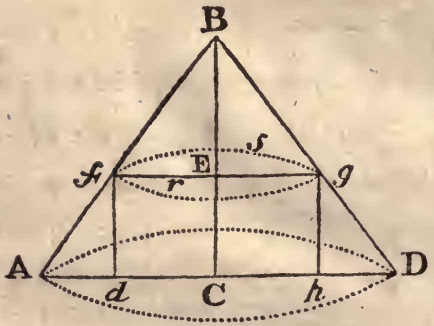
EXAMPLE VII.

30. *To determine the greatest Cylinder, dg, that can be inscribed in a given Cone* ADB.

Let $a =$ BC, the Altitude of the Cone;
$b =$ AD, the Diameter of its Base;
$x = fg$ (dh) the Diameter of the Cylinder, considered as variable;
$p = \left(\frac{3,14159, \&c.}{4}\right)$ the Area of the Circle whose Diameter is Unity.

Then, the Areas of Circles being to one another as the Squares of their Diameters, we have, $1^2 : x^2 :: p : (px^2)$ the Area of the Circle $fsgr$: Moreover, from the Similarity of the Triangles ABC and Adf, we have $\frac{1}{2}b$ (AC) : a (BC) :: $\frac{1}{2}b - \frac{1}{2}x$ (Ad) : $df = \frac{ab - ax}{b}$; which multiplied by the Area px^2 (found above) gives

 $pabx^2$

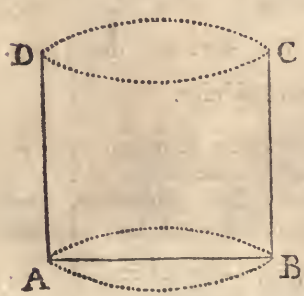


$\frac{pabx^2 - pax^3}{b}$ for the folid Content of the Cylinder: Which being a *Maximum*, its Fluxion $\frac{2pabx\dot{x}}{b} - \frac{3pax^2\dot{x}}{b}$ muft

*Art.22. be $= 0$ *, confequently $x = \frac{2b}{3}$ and $df = \frac{a}{3}$: From whence it appears, that the infcribed Cylinder will be the greateft poffible, when the Altitude thereof is juft $\frac{1}{3}$ of the Altitude of the whole Cone.

EXAMPLE VIII.

31. *To determine the Dimenfions of a cylindric Meafure* ABCD, *open at the Top, which fhall contain a given Quantity (of Liquor, Grain,* &c.) *under the leaft internal Superficies poffible.*

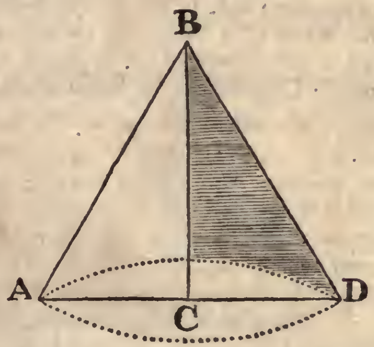


Let the Diameter AB $= x$, and the Altitude AD $= y$; moreover let p (3,14159, *&c.*) denote the Periphery of the Circle whofe Diameter is Unity, and let c be the given Content of the Cylinder. Then it will be $1 : p :: x : (px)$ the Circumference of the Bafe; which, multiplied by by the Altitude y, gives pxy for the concave Superficies of the Cylinder. In like Manner, the Area of the Base, by multiplying the same Expression into $\frac{1}{4}$ of the Diameter x, will be found $= \frac{px^2}{4}$; which drawn into the Altitude y, gives $\frac{px^2y}{4}$ for the solid Content of the Cylinder; which being made $= c$, the concave Surface pxy will be found $= \frac{4c}{x}$, and consequently the whole Surface $= \frac{4c}{x} + \frac{px^2}{4}$: Whereof the Fluxion, which is, $-\frac{4c\dot{x}}{x^2} + \frac{px\dot{x}}{2}$, being put $= 0$, we shall get $-8c + px^3 = 0$; and therefore $x = 2\sqrt[3]{\frac{c}{p}}$: Further, because $px^3 = 8c$, and $px^2y = 4c$, it follows, that $x = 2y$; whence y is also known, and from which it appears, that the Diameter of the Base must be just the Double of the Altitude.

EXAMPLE IX.

32. *Of all Cones under the same given Superficies* (s) *to find that* (ABD) *whose Solidity is the greatest.*

Let the Semidiameter of the Base, AC $= x$, and the Length of the slant Side AB $= y$; and let p (as in the preceding Examples) denote the Periphery of the Circle whose Diameter is Unity.



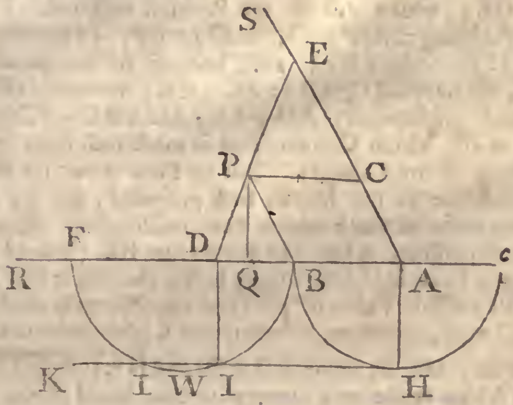
 Then

Then the Circumference of the Baſe will be $= 2px$, the Area of the Baſe $= px^2$, and the convex Superficies of the Cone $= pxy$, (which laſt is found by multiplying half the Periphery of the Baſe by the Length of the ſlant Side): Wherefore, ſince the whole Superficies is $= px^2 + pxy = s$, we have $y = \frac{s}{px} - x$; whence the Altitude CB ($\sqrt{AB^2 - AC^2}$) $= \sqrt{\frac{s^2}{p^2x^2} - \frac{2s}{p}}$; which multiplied by $\left(\frac{px^2}{3}\right)$ $\frac{1}{3}$ of the Area of the Baſe, gives $\frac{px^2}{3}\sqrt{\frac{s^2}{p^2x^2} - \frac{2s}{p}}$ for the ſolid Content of the Cone. Which being a *Maximum*, its Square $\frac{s^2x^2}{9} - \frac{2psx^4}{9}$ muſt alſo be a *Maximum*; and therefore $\frac{2s^2x\dot{x}}{9} - \frac{8psx^3\dot{x}}{9} = 0$; whence $s - 4px^2 = 0$, and conſequently $x = \sqrt{\frac{s}{4p}}$: From which $y \left(= \frac{s}{px} - x = \frac{s - px^2}{px} = \frac{3px^2}{px} = 3x\right)$ will likewiſe be known; and from whence it will appear that the greateſt Cone under a given Surface, (or a given Cone under the leaſt Surface) will be when the Length of the ſlant Side is to the Semi-diameter of the Baſe in the Ratio of 3 to 1, or, (which comes to the ſame) when the Square of the Altitude is to the Square of the whole Diameter in the Ratio of 2 to 1.

EX-

EXAMPLE X.

33. *To determine the Position of a Right-line* DE, *which, passing through a given Point* P, *shall cut two Right-lines* AR *and* AS, *given by Position, in such sort that the Sum of the Segments,* AD *and* AE, *made thereby, may be the least possible.*



Make PB, parallel to AS, $= a$, and PC, parallel to AR, $= b$; and let BD $= x$: Then, by reason of the parallel Lines, it will be, $x : a :: b : \text{CE} = \frac{ab}{x}$: Therefore $\text{AD} + \text{AE} = b + x + a + \frac{ab}{x}$, and its Fluxion, $\dot{x} - \frac{ab\dot{x}}{x^2}$, which, in the required Circumstance, being $= 0$, we have $x^2 - ab$ also $= 0$, and consequently $x = \sqrt{ab}$; whence the Position of DE is known. But the same Thing may be otherwise determined, independent of Fluxions, from the general Solution of the Problem for finding the Position of DE, when the Sum of the Segments AD and AE (instead of being a *Minimum*) shall be equal to a given Quantity. Of which Problem, the geometrical Construction may be as follows.

Compleat the Parallelogram ABPC (as before) and, in RA produced, take $Ac = AC$, and let cF be equal to the given Sum of the two Segments: Alſo let two Semi-circles be deſcribed upon Bc and BF, and let AH, perpendicular to Bc, interſect the former in H; likewiſe let HK, parallel to Fc, interſect the latter in I; draw ID perpendicular to Fc, and, through P and D draw DE; which will be the Poſition required. For $AB \times Ac$ being $= AH^2 = DI^2 = BD \times DF$, we have $BD : AB :: Ac\ (AC) : DF$; alſo, becauſe of the parallel Lines, we have $BD : AB :: AC : CE$; whence $DF = CE$, and conſequently $AD + AE\ (AD + AC + FD)$ is equal to cF, which Conſtruction is more neat than that in *p.* 155. of my *Geometry*. But to ſhew how far this may conduce to the Matter firſt propoſed; we are to obſerve, that, as the Problem here conſtructed appears to be impoſſible, when the Right-line HK (inſtead of cutting or touching) falls wholly below the Circle BWF, the leaſt poſſible Value of BF (and conſequently of $AD + AE$) muſt, therefore, be when that Right-line touches the Circle; that is, when $BD = DI = AH = \sqrt{AB \times AC}$; which Value is the very ſame with that found above.

The ſame Concluſion may alſo be deduced from the algebraic Solution of the foreſaid Problem: For, putting $b + x + a + \frac{ab}{x}\ (AD + AE) = s$, and ſolving the Equation, x will be found $= \frac{s-a-b}{2} \pm \sqrt{\frac{\overline{s-a-b}|^2}{4} - ab}$: Which Equation being no longer poſſible than till $\frac{\overline{s-a-b}|^2}{4} - ab$ is $= 0$, we have x, in that Circumſtance, $= \frac{s-a-b}{2} = \sqrt{ab}$; *ſtill as before*. In like Manner the *Maxima* and *Minima* may be determined in other Caſes, by finding the Poſition or Circumſtance wherein the general Problem begins to be impoſſible, (ſuppoſing the Quantity ſought to be given). But the Operation by

Fluxions

Fluxions is, for the general Part, much more ſhort and expeditious.

EXAMPLE XI.

34. *The ſame being given as in the preceding Example, to determine the Poſition, when the Line* DE, *itſelf, is the leaſt poſſible.*

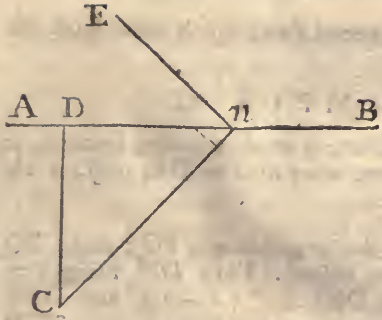
Upon AF let fall the perpendicular PQ; make BQ $=c$, and, the reſt, as before: Then DP^2 being $(= DB^2+BP^2-2BQ\times DB) = x^2+a^2-2cx$, and $DB^2 : DP^2 :: DA^2 : DE^2$, we have $x^2 : x^2+a^2-2cx :: \overline{b+x}|^2 : DE^2 = \frac{\overline{b+x}|^2\times\overline{x^2-2cx+a^2}}{x^2} = \overline{b+x}|^2\times\overline{1-\frac{2c}{x}+\frac{a^2}{x^2}}$; whoſe Fluxion, which is $2\dot{x}\times\overline{b+x}\times\overline{1-\frac{2c}{x}+\frac{a^2}{x^2}}+\overline{b+x}|^2\times\overline{\frac{2c\dot{x}}{x^2}-\frac{2a^2\dot{x}}{x^3}}$, being put $=0$, and the whole Equation divided by $2\dot{x}\times\overline{b+x}$, there will come out $1-\frac{2c}{x}+\frac{a^2}{x^2}+\overline{b+x}\times\overline{\frac{c}{x^2}-\frac{a^2}{x^3}}=0$; whence $x^3-2cx^2+a^2x+\overline{b+x}\times\overline{cx-a^2}=0$; that is, (by Reduction) $x^3-cx^2+bcx-a^2b=0$: From the Reſolution of which Equation, the Poſition of DE is determined.

LEMMA.

35. *If a Body or Point* (n) *be ſuppoſed to move in a Right-line* AB, *its abſolute Celerity, in the Direction of that Line, will be to the relative Celerity, whereby it tends to, or from, a given Point* C, *any where out of the Line, as the Diſtance* Cn, *is to the Diſtance* Dn, *intercepted by* n *and the Perpendicular* CD; *or, as Radius to the Co-ſine of the Angle of Inclination* DnC.

For, putting CD $=a$, D$n=x$, and C$n=y$, we have $a^2+x^2=y^2$, and conſequently $2x\dot{x}=2y\dot{y}$ *: Whence

* Art. 2 and 5.



Whence $\dot{x}$: $\dot{y}$ :: y (Cn) : x (Dn) :: Radius : Co-sine DnC : But, the Fluxions of Quantities are as the Celerities of their Increase *, therefore the Truth of the Proposition is manifest.

* Art. 2 and 5.

Corollary.

It follows from hence, that the relative Celerities in any two different Directions nE and nC, are directly as the Co-sines of the corresponding Angles DnE and DnC. Therefore, when nE is perpendicular to Cn, (and the Angle DnE therefore equal to C) the Celerity in the Direction nE, will be to that in the Direction nC, as the Sine of DnC is to its Co-sine. From whence it appears, that the Celerities in the Directions Dn, Cn, and En (perpendicular to nC) are to each other as Cn, Dn, and CD respectively.

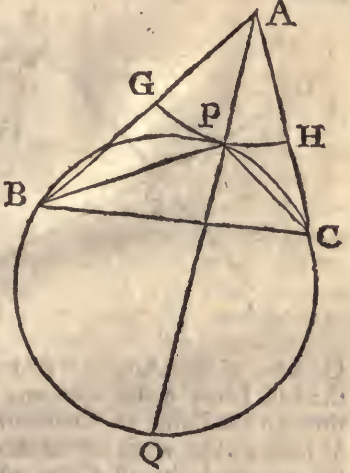
Example XII.

36. *To determine the Position of a Point, from whence, if three Right-lines be drawn to so many given Points* A, B, C, *their Sum shall be the least possible.*

Let HPG be the Periphery of a Circle described about the Point A, as a Center, at any Distance AG; in which let the Point P be conceived to move with an uniform Celerity, from G towards H. Then, because the relative Celerity thereof, in the Direction PC, is to that in the Direction BP produced, as the Co-sine of the Angle CPH to the Co-sine of the Angle BPG, *(by the preceding Lemma)*; and, since these Celerities, when the

the Sum of CP and BP is a *Minimum*, muſt be equal *, it follows, therefore, that the ſaid Angles CPH and BPG, as well as their Co-ſines, will in that Circumſtance become equal to each other; and conſequently APC alſo equal to APB. From whence it appears, that (take AG what you will) the Sum of the three Lines, AP, BP, and CP, cannot be the leaſt poſſible when the Angles APB and APC are unequal. And, by the ſame Argument, it alſo appears that their Sum cannot be the leaſt poſſible, when the Angles BPA and BPC are unequal: Therefore, their Sum muſt be the leaſt poſſible, when all the three Angles about the Point P are equal to one another; provided the Caſe will admit of ſuch an Equality, or that no one of the Angles of the Triangle ABC is equal to, or greater than $\frac{1}{3}$ of 4 Right Angles (for otherwiſe, the Point P will fall in the obtuſe Angle): Hence this

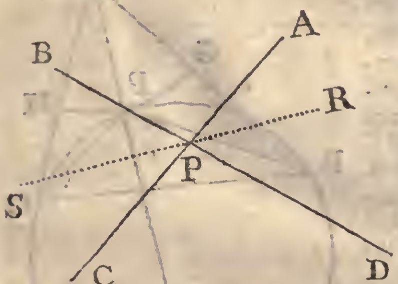
* Art. 2 and 22.



CONSTRUCTION.

Deſcribe, upon BC, a Segment of a Circle, to contain an Angle of 120°, and let the whole Circle BCQ be compleated; and from A, to the Middle (Q) of the Arch BQC, draw AQ interſecting the Circumference of the Circle in P; which will be the Point required. For, the Angles BPQ and CPQ, ſtanding upon the equal Arches BQ and CQ, have their Complements APB and APC equal to each other; and therefore, the Angle BPC being 120° (by Conſtruction) each of the ſaid

ſaid Angles APB, APC, will, likewiſe be 120 Degrees.

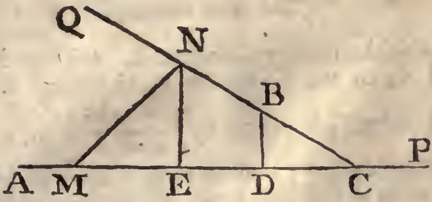


After the ſame Manner, it will appear that the Sum of all the Lines AP, BP, CP, &c. drawn from any Number of given Points A, B, C, &c. to meet in another Point P, will be the leaſt poſſible, when the Co-ſines of the Angles RPA, RPB, RPC, &c. that the ſaid Lines make with any other Line RS, paſſing through the Point of Concourſe, deſtroy each other: Which will be when all the Angles APB, BPC, CPD, &c. are equal, in all Caſes where the Poſition of the given Points will admit of ſuch an Equality. But, if the Number of given Points be four, the required Point will be in the Interſection of the two Right-lines joining the oppoſite Points: For, ſuppoſing APC and BPD to be continued Right-lines, the Co-ſine of RPA will be equal and contrary to that of RPC, and that of RPB equal and contrary to that of RPD.

EXAMPLE XIII.

37. *If two Bodies move at the ſame Time, from two given Places* A *and* B, *and proceed uniformly from thence in given Directions*, AP *and* BQ, *with Celerities in a given Ratio; it is propoſed to find their Poſition, and how far each has gone, when they are the neareſt poſſible to each other.*

Let M and N be any two cotemporary Poſitions of the Bodies, and upon AP let fall the Perpendiculars NE and BD; alſo let QB be produced to meet AP in



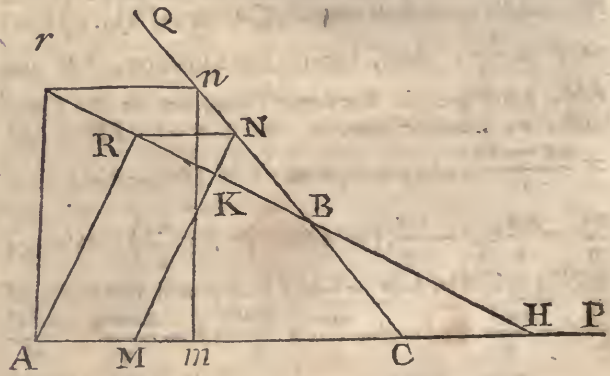
in C, and let MN be drawn: Moreover, let the given Celerity in BQ be to that in AP, as n to m, and let AC, BC, and CD, (which are alſo given) be denoted by a, b, and c reſpectively, and make the variable Diſtance CN $= x$: Then, by reaſon of the parallel Lines NE and BD, we ſhall have b (CB) : x (CN) :: c (CD) : CE $= \frac{cx}{b}$. Alſo, becauſe the Diſtances, BN and AM, gone over in the ſame Time, are as the Celerities, we likewiſe have, $n : m :: x - b$ (BN) : AM $= \frac{mx - mb}{n}$, and conſequently CM (AC—AM) $= a + \frac{mb}{n} - \frac{mx}{n} = d - \frac{mx}{n}$, (by writing $d = a + \frac{mb}{n}$). Whence MN^2 $(= CM^2 + CN^2 - CM \times 2CE)$ will alſo be found $= \overline{d - \frac{mx}{n}}|^2 + x^2 - \overline{d - \frac{mx}{n}} \times \frac{2cx}{b} = d^2 - \frac{2dmx}{n} + \frac{m^2x^2}{n^2} + x^2 - \frac{2cdx}{b} + \frac{2cmx^2}{nb}$; whoſe Fluxion $-\frac{2dm\dot{x}}{n} + \frac{2m^2x\dot{x}}{n^2} + 2x\dot{x} - \frac{2cd\dot{x}}{b} + \frac{4cmx\dot{x}}{nb}$ being made $= 0$ (becauſe MN is to be a *Minimum*) we get $-bdmn + m^2bx + n^2bx - n^2cd + 2mncx = 0$; and conſequently $x = \frac{mnbd + n^2cd}{m^2b + n^2b + 2mnc} = \frac{nd \times \overline{mb + nc}}{b \times \overline{m^2 + n^2 + 2mnc}}$; from whence BN, AM, and MN are alſo given.

The

The same otherwise.

Because the relative Celerities of the two Bodies, at M and N, in the Direction of the Line MN (produced) are truly expressed by $\frac{\textit{Co-sine}\ M}{\textit{Radius}} \times m$, and $\frac{\textit{Co-s.}N}{\textit{Rad.}}$
* Art.35. $\times n$, respectively *; and as these Celerities, when the Distance MN is a *Minimum*, do become equal to each
Art.22. other †, it follows that, in this Circumstance, $m : n ::$ Co-s. N. : Co-s. M : : Secant of M : Secant of N *(by plane Trig.)*

Whence this Construction. Take CH to CB in the given Ratio of m to n, and draw HB; upon which



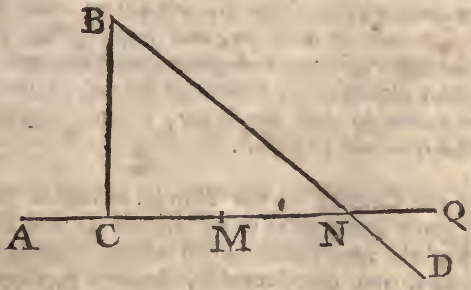
produced (if necessary) let fall the Perpendicular AR; draw RN parallel to AH, meeting CQ in N; lastly, draw NM parallel to AR, and it will give the Position required. For, first, it is plain, because AM (RN) : BN (: : CH : CB) : : $m : n$, that M and N are cotemporary Positions: It is likewise plain, that RN and BN will be Secants of the Angles KNR (CMN) and KNB (CNM) to the Radius NK; because the Angle NKR (=ARK) is a Right-one. Which Lines or Secants are in the proposed Ratio of m to n, as has been already shewn.

But

But the ſame Solution may be, yet, otherwiſe derived, independent of Fluxions, from Principles intirely geometrical. For, let m and n be any two cotemporary Poſitions at Pleaſure, and let CH (as before) be to CB, as the Celerity in AP to that in CQ; moreover, let nr, parallel to AP, be drawn, meeting HB produced in r, and let A, r be joined. Then, ſince CB : CH : : Bn : nr *(by ſim. Triangles)* and CB : CH : : Bn : Am, *(by Hyp.)* it follows, that nr and Am, (which are parallel) will alſo be equal to each other; and therefore Ar and mn, likewiſe equal and parallel. But Ar is the leaſt poſſible when perpendicular to Hr. Whence the Solution is manifeſt.

EXAMPLE XIV.

38. *Let the Body* M *move, uniformly, from* A *towards* Q, *with the Celerity m, and let another Body* N *proceed from* B, *at the ſame time, with the Celerity n. Now it is propoſed to find the Direction* (BD) *of the latter, ſo that the Diſtance* MN *of the two Bodies, when the latter arrives in the Way or Direction* AQ *of the former, may be the greateſt poſſible.*



Let BC be perpendicular to AQ, and make AC $= a$, BC$=b$, and BN$=x$. Therefore, if the Poſition M be ſuppoſed cotemporary with N, we ſhall have $n : m :: x : \text{AM} = \frac{mx}{n}$; whence CM $= \frac{mx}{n} - a$, and conſequently

ſequently MN (CN—CM) $= \sqrt{x^2-b^2} - \frac{mx}{n} + a$; whereof the Fluxion being taken, and made $= 0$, we get $\frac{x}{\sqrt{x^2-b^2}} = \frac{m}{n}$; therefore $x = \frac{mb}{\sqrt{m^2-n^2}}$, and CN $(\sqrt{x^2-b^2}) = \frac{nb}{\sqrt{m^2-n^2}}$: Whence, $m : n$ (:: BN : CN :: Radius : Co-ſine N. The ſame Concluſion is otherwiſe derived, thus,

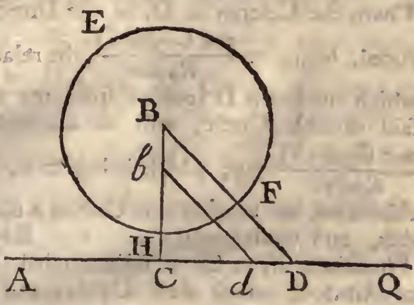
Let the Right-line BD be ſuppoſed to revolve about the Point B, as a Center, with a Motion ſo regulated, that the intercepted Part thereof BN may increaſe with the uniform Celerity n: Then, the Celerity with which
* Art. 35. CN is increaſed being $= \frac{n \times \textit{Radius}}{\textit{Co-ſine}\ N}$ *, this Expreſſion, when MN is a *Maximum*, muſt, conſequently, be equal
† Art. 22. to (m) the Velocity of the other Body † M; and therefore $m : n$ :: Radius : Co-ſine N, as before.

EXAMPLE XV.

39. *Suppoſing a Ship to ſail from a given Place* A, *in a given Direction* AQ, *at the ſame time that a Boat, from another given Place* B, *ſets out in order (if poſſible) to come up with her, and ſuppoſing the Rate at which each Veſſel runs to be given; it is required to find in what Direction the latter muſt proceed, ſo that, if it cannot come up with the former, it may, however, approach it as near as poſſible.*

Let the Celerity of the Ship be to that of the Boat in the given Ratio of m to n; alſo let D and F be the Places of the two Veſſels when neareſt poſſible to each other, and, from the Center B, through F, ſuppoſe the Circumference of a Circle to be deſcribed. Then (the Diſtance DF being the leaſt poſſible), the Point F muſt be in the Right-line (DB) joining the Point D and the Center

Center B; because no other Point in the whole Periphery, at which the Boat from B might arrive in the same time, is so near to D as that wherein the Line DB intersects the said



Periphery.—But now, to get an Expression for DF, in algebraic Terms, let BC be perpendicular to AQ, and make AC $=a$, BC $=b$, and CD $=x$; and then BD ($\sqrt{BC^2+CD^2}$) will be $=\sqrt{b^2+x^2}$; moreover, because $m:n::$ AD $(a+x)$: BF, you will have BF $=\frac{na+nx}{m}$, and consequently, DF $=\sqrt{b^2+x^2}-\frac{na+nx}{m}$; whose Fluxion, $\frac{x\dot{x}}{\sqrt{b^2+x^2}}-\frac{n\dot{x}}{m}$, being made $=0$, we find $x=\frac{nb}{\sqrt{m^2-n^2}}$; whence the Direction BD is known: And, if the Value of x, thus found, be substituted in that of DF, (found above) we shall have DF $=\frac{b\sqrt{m^2-n^2}-na}{m}$; whence the Position of F is known. And from which it is observable, that, as DF must be a *real*, *positive* Quantity (by the Question) this Method of Solution can only obtain when m is greater than n, and $b\sqrt{m^2-n^2}$, also greater than na: For in all other Cases the Boat will be able to come up with the Ship.

The same otherwise.

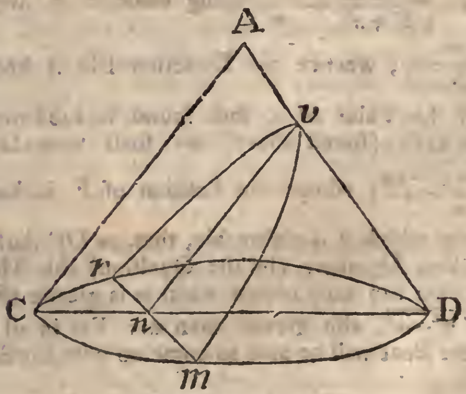
Let the Radius of the Circle EFH be conceived to increase uniformly, with the Celerity n, whilst the Point

 D moves

D moves uniformly along AQ, with the Celerity m: Then, the Celerity at D, in the Direction of BD produced, being $= \frac{m \times \textit{Co-fine}\ D}{\textit{Radius}}$, the relative Celerity with which the Point D recedes from the Periphery of the ſaid variable Circle, will be univerſally expreſſed by $\frac{m \times \textit{Co-fine}\ D}{\textit{Radius}} - n$; which being $= 0$, when DF is a *Minimum*, we have in this Caſe $m \times \textit{Co-fine}\ D = n \times \textit{Radius}$, and conſequently $m : n :: \textit{Radius} : \textit{Co-fine}\ D$. Therefore, if, at C, a right-angled Triangle Cbd be conſtituted, whoſe Baſe C$d = n$, and its Hypothenuſe $db = m$, and parallel to the latter you draw BD, it will be the Direction required: In which, if there be taken BF, a Fourth-proportional to m, n, and AD, you will alſo have the Poſition required.

EXAMPLE XVI.

40. *To determine the greateſt Parabola that can be formed by cutting a given Cone* ACD.



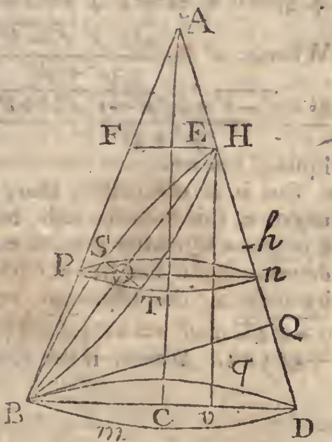
Let nv, parallel to CA, be the Axis of the Parabola rvm, and rm the Baſe (or Ordinate) thereof; putting DC

$DC = a$, $CA = b$, and $Dn = x$; then, becauſe of the parallel Lines, it will be, $a : b :: x : \frac{bx}{a} = nv$: Moreover, by the Property of the Circle, we have rn^2 $(= nm^2 = Dn \times Cn) = ax - x^2$, and conſequently rm $2\sqrt{ax - x^2}$; which multiplied by $\frac{2}{3} \times \frac{bx}{a}$ (becauſe every Parabola is $\frac{2}{3}$ of a Parallelogram of the ſame Baſe and Altitude) gives $\frac{4bx}{3a}\sqrt{ax - x^2}$ for the Content of the Parabola: Whoſe Fluxion, or that of $ax^3 - x^4$ * being put equal to Nothing; we find $x = \frac{3a}{4}$: Whence $nv = \frac{3}{4} \times AC$, $rm = CD \times \sqrt{\frac{3}{4}}$, and the Area of the greateſt, or required, Parabola $= AC \times CD \times \frac{\sqrt{3}}{4}$.

* Art. 26.

EXAMPLE XVII.

41. *To determine the greateſt Ellipſis* BTES *that can be formed by cutting a given Cone* ABD.



Let BE be the greater, and TS the leſſer, Axis of the Ellipſis BTES, conſidered as variable by the Motion of (the End of the Tranſverſe) E, along the Line AD; moreover let Ev be parallel to AC the Axis of the Cone, meeting the Diameter BD in v, and let the Diameters EF and np be parallel to BD; whereof the latter np is ſuppoſed to pass through O the Center of the Ellipsis: Then, putting $AC=a$, $CD=b$, and $Cv=x$, we shall have $Bv=b+x$; also, because of the parallel Lines we have CD (b) : CA (a) :: Dv $(b-x)$: $\frac{a\times\overline{b-x}}{b}=Ev$; whence

$$BE\ (\sqrt{Bv^2+Ev^2}) = \frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2}}{b}.$$

Furthermore, since the Triangles EOn, EBD, and BOp, BEF are equiangular, and EO $(=BO)=\frac{1}{2}$ BE, we likewise have $On=\frac{1}{2}BD=b$, and $Op=\frac{1}{2}EF=Cv=x$; and consequently $On\times Op$ ($=OT^2$, *by the Property of the Circle*) $=bx$; whence $ST=2\sqrt{bx}$, and

$$\text{therefore } BE\times ST=\frac{\sqrt{b^2\times\overline{b+x}|^2+a^2\times\overline{b-x}|^2\times 4bx}}{b}.$$

Now the Area of any Ellipsis being in a constant Ratio to the Rectangle of its greater and lesser Axes (namely as 3,14159, &c. to 4) the last general Expression must therefore be a *Maximum*, when the Area is so; and therefore its Fluxion, or that of $b^2x\times\overline{b+x}|^2+a^2x\times\overline{b-x}|^2$ $(=b^4x+2b^3x^2+b^2x^3+a^2b^2x$
Art.22. $-2a^2bx^2+a^2x^3)$ equal to Nothing; that is, $b^4\dot{x}+4b^3x\dot{x}+3b^2x^2\dot{x}+a^2b^2\dot{x}-4a^2bx\dot{x}+3a^2x^2\dot{x}=0$: Whence $x^2-\frac{4bx\times\overline{a^2-b^2}}{3a^2+3b^2}=-\frac{b^2}{3}$, and $x=\frac{2b\times\overline{a^2-b^2}\pm b\sqrt{a^4-14a^2b^2+b^4}}{3a^2+3b^2}$; from which the Ellipsis is known.

But it is observable, that, when $a^4-14a^2b^2+b^4$ is negative, this Solution fails, because the Square Root of a negative Quantity is to be extracted. Therefore, to determine the Limit, put $a^4-14a^2b^2+b^4=0$; then, by ordering the Equation, you will get $a^2=b^2\times\overline{7+\sqrt{48}}$, and $a=b\times\overline{2+\sqrt{3}}$; and therefore $a:b::2+\sqrt{3}:1$. Hence, if the Ratio of AC to CD be not greater

greater than that of $2+\sqrt{3}$ to 1, or (which comes to the ſame thing) if the Angle DAC be not leſs than 15 Degrees, the Fluxion of the Ellipſis can never become equal to Nothing; but the Ellipſis itſelf will increaſe continually, from the Vertex till it coincides with the Baſe of the Cone; and therefore is greater at the Baſe than in any other Poſition.

But it is further to be obſerved, that this Problem is confined to, yet, narrower Limits. For, either the Ellipſis will increaſe, continually, from the Vertex, to the Baſe, of the Cone, (which is ſhewn to be the Caſe when the Angle DAC is greater than 15°) or elſe it will increaſe till the Point E arrives at a certain Poſition H, and afterwards decreaſe to another certain Poſition *b*, and then increaſe again till it coincides with the Baſe of the Cone, (for it muſt always increaſe again before it coincides with the Baſe, becauſe, after the Point E is got below the Perpendicular BQ, both the Axes of the Ellipſis increaſe at the ſame time).

The ſame thing alſo appears from the foregoing Equation $x = \frac{2b \times \overline{a^2 - b^2} \pm b\sqrt{a^4 - 14a^2b^2 + b^4}}{3a^2 + 3b^2}$; whoſe two Roots expreſs the two Values of x (or Cv) at the Times of the *Maximum* (at H) and its ſucceeding *Minimum* (at *b*). Hence it is manifeſt, that the Ellipſis may admit of a *Maximum* between the Vertex of the Cone and the Perpendicular BQ, and yet, that *Maximum* be leſs than the Baſe of the Cone, unleſs the foreſaid Angle DAC be ſo much leſs than 15° (above found) that the Increaſe from *b* to D, be leſs than the Decreaſe from H to *b*. Now therefore, to determine the exact Limit, let the foreſaid Increment and Decrement be ſuppoſed equal to each other, or, which is the ſame in Effect, let the Ellipſis BTESB = the Circle BqDm, or BE × ST = BD², that is, let $\frac{\sqrt{b^2 \times \overline{b+x}|^2 + a^2 \times \overline{b-x}|^2 \times 4bx}}{b} = 4b^2$: From which

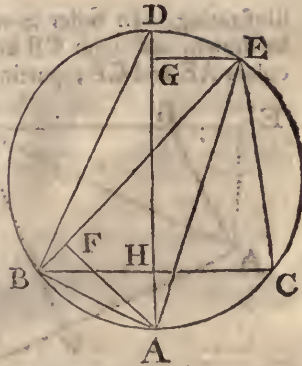
Equation you will get $a^2 = \frac{b^2}{x} \times \frac{4b^3 - b^2x - 2bx^2 - x^3}{\overline{b-x}|^2}$ $= \frac{b^2}{x} \times \frac{4b^2 + 3bx + x^2}{b-x}$: Moreover, from the Equation $b^4\dot{x} + 4b^3x\dot{x} + 3b^2x^2\dot{x} + a^2b^2\dot{x} - 4a^2bx\dot{x} + 3a^2x^2\dot{x} = 0$, (given above) you will, again, get $a^2 = \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{-b^2 + 4bx - 3x^2}$ $= \frac{b^2 \times \overline{b^2 + 4bx + 3x^2}}{\overline{b-x} \times \overline{3x-b}}$: Whence, by comparing these equal Values, there arises $\frac{4b^2 + 3bx + x^2}{x} = \frac{b^2 + 4bx + 3x^2}{3x - b}$ which, ordered, gives $x^2 + 2bx - b^2 = 0$, and therefore $x = b\sqrt{2} - b$.

Moreover, $\frac{a^2}{b^2}$ being $= \frac{4b^2 + 3bx + x^2}{bx - x^2}$, if $b^2 - 2bx$ be substituted herein for, its Equal, x^2; it will become $\frac{a^2}{b^2} = \frac{5b^2 + bx}{bx - x^2} = \frac{5b + x}{3x - b} = \frac{5b + b\sqrt{2} - b}{3b\sqrt{2} - 3b - b} = \frac{4 + \sqrt{2}}{-4 + 3\sqrt{2}}$ $= \frac{\overline{4 + \sqrt{2}} \times \overline{4 + 3\sqrt{2}}}{\overline{-4 + 3\sqrt{2}} \times \overline{4 + 3\sqrt{2}}} = \frac{22 + 16\sqrt{2}}{2} = 11 + 8\sqrt{2}$. Hence we have, $1 : \sqrt{11 + 8\sqrt{2}} :: b$ (DC) : a (AC) :: Radius to the Tangent of the Angle ADC $= 78° 3'$: Whose Complement DAC $= 11° 57'$, is the least Limit possible. Therefore, unless the Angle which the slant-Side makes with the Axis be less than $11° 57'$, the greatest Ellipsis will be less than the Base of the Cone.

EXAMPLE XVIII.

42. *Of all Triangles, having the same given Perimeter, and inscribed in the same given Circle; to determine the greatest.*

Let the Diameter DA bisect the Base BC of the required Triangle BEC in H, draw AE, AB and BD; also draw AF perpendicular to BE, and GE, parallel to BC,



BC, meeting AD in G: Then, putting AD $= a$, half the given Perimeter of the Triangle $= b$, and DH$=y$; we have BH $= \sqrt{ay-y^2}$, and therefore EF $= \overline{b-\sqrt{ay-y^2}}|^2$. Moreover DH (y) : AD (a) :: DB^2 : DA^2 :: EF^2 $\left(\overline{b-\sqrt{ay-y^2}}|^2\right)$: EA^2 $= \frac{a}{y} \times \overline{b-\sqrt{ay-y^2}}|^2$; therefore AG $\left(\frac{AE^2}{AD}\right) = \frac{\overline{b-\sqrt{ay-y^2}}|^2}{y}$, and HG $=$ (AG$-$AH$=$) $\frac{b^2-2b\sqrt{ay-y^2}}{y}$; whence the Area of the Triangle BEC (BH $\times$ HG) $= \frac{b^2\sqrt{ay-y^2}}{y} - 2ba + 2by$, whose Fluxion $2b\dot{y} - \frac{\frac{1}{2}ab^2\dot{y}}{y\sqrt{ay-yy}}$ being put $= 0$, gives $y\sqrt{ay-yy} = \frac{1}{4}ba$; whence y, and from thence the Sides of the Triangle may be determined.

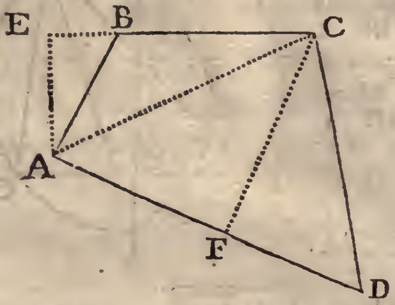
EXAMPLE XIX.

43. *To determine the greatest Area that can be contained under four given Right-lines.*

Though it is demonstrable from common Geometry that the Area will be a *Maximum*, when the Trapezium ABCD, formed by the given Lines, may be inscribed in a Circle [b], yet I shall here give the Solution from the Principles of Fluxions, (whose Uses I am now illustrating).

[a] By Prop. 13. Page 62. *Elem. Trig.*
[b] See Page 117. of *Elem. Geometry.*

illuſtrating. In order to which, let the Diagonal AC be drawn, and upon CB and AD let fall the Perpendiculars AE and CF; putting AB $=a$, BC $=b$, CD $=c$, DA $=d$, BE $=x$, and DF $=y$: Then AE being $=\sqrt{a^2-x^2}$, and CF $=\sqrt{c^2-y^2}$, the Area of the Trapezium ($\frac{1}{2}$BC $\times$ AE $+$ $\frac{1}{2}$AD $\times$ CF) will be $=\frac{1}{2}b\sqrt{a^2-x^2}+\frac{1}{2}d\sqrt{c^2-y^2}$;



*Art. 22. and its Fluxion $\frac{-\frac{1}{2}bx\dot{x}}{\sqrt{a^2-x^2}}-\frac{\frac{1}{2}dy\dot{y}}{\sqrt{c^2-y^2}}=0$ *;

and therefore $\frac{-dy\dot{y}}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$. Moreover, ſince b^2+a^2+2bx ($=$AC^2) $=d^2+c^2-2dy$, by taking the Fluxion thereof, we have $2b\dot{x}=-2d\dot{y}$, or $-d\dot{y}=b\dot{x}$; which, ſubſtituted for $-d\dot{y}$ in the foregoing Equation, gives $\frac{b\dot{x}y}{\sqrt{c^2-y^2}}=\frac{bx\dot{x}}{\sqrt{a^2-x^2}}$, and $\frac{y}{\sqrt{c^2-y^2}}=\frac{x}{\sqrt{a^2-x^2}}$; and conſequently, $\sqrt{c^2-y^2}$ (CF) : y (DF) :: $\sqrt{a^2-x^2}$ (AE) : x (BE) : From which it appears that the Triangles DCF and ABE are ſimilar, and that (D $+$ ABC being $=$ 2 Right-angles) the Trapezium may be inſcribed in a Circle; but this by the Bye. We are now to get an Expreſſion for the Area in known Terms, and in order thereto we have $b^2+a^2+2bx=dd+c^2-2dy$, $y=\frac{cx}{a}$, and CF $=\frac{c\sqrt{a^2-x^2}}{a}$ (becauſe AB : BE :: DC : DF, &c.) : Therefore, by Subſtitution, $b^2+a^2+2bx=d^2+c^2-\frac{2cdx}{a}$, and the Area ($\frac{1}{2}$BC $\times$ AE

$+\frac{1}{2}$AD

$+\frac{1}{2}AD\times CF) = \frac{1}{2}b\sqrt{a^2-x^2} + \frac{cd}{2a}\sqrt{a^2-x^2} = \frac{ab+cd}{2a}\sqrt{a^2-x^2}$; and therefore the Square thereof $= \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a^2-x^2} = \frac{\overline{ab+cd}|^2}{4a^2}\times\overline{a+x}\times\overline{a-x} = \frac{\overline{ab+cd}|^2}{4}\times\overline{1+\frac{x}{a}}\times\overline{1-\frac{x}{a}}$. But since $b^2+a^2+2bx = d^2+c^2-\frac{2cdx}{a}$, we have $\frac{x}{a} = \frac{d^2+c^2-b^2-a^2}{2ab+2cd}$, $1+\frac{x}{a} = 1+\frac{dd+c^2-b^2-a^2}{2ab+2cd} = \frac{2ab+2cd+dd+c^2-b^2-a^2}{2ab+2cd} = \frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}$; and $1-\frac{x}{a}=\frac{2ab+2cd-dd-c^2+b^2+a^2}{2ab+2cd} = \frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd}$; and consequently, the Square of the Area $= \frac{\overline{ab+cd}|^2}{4}\times\frac{\overline{d+c}|^2-\overline{b-a}|^2}{2ab+2cd}\times\frac{\overline{b+a}|^2-\overline{d-c}|^2}{2ab+2cd} = \frac{\overline{\overline{d+c}|^2-\overline{b-a}|^2}\times\overline{\overline{b+a}|^2-\overline{a-c}|^2}}{16}$ which (because the Difference of the Squares of any two Quantities is equal to a Rectangle under their Sum and Difference) will also be $=\frac{\overline{d+c+b-a}\times\overline{d+c-b+a}\times\overline{b+a+a-c}}{4}\times\frac{\overline{b+a-d+c}}{4} = \overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-a}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-b}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-c}\times\overline{\frac{1}{2}d+\frac{1}{2}c+\frac{1}{2}b+\frac{1}{2}a-d}$. Whence it appears, that, if from $\frac{1}{2}$ the Sum of all the four Sides each particular Side be subtracted, the continual Product of the Remainders will be the Square, or second Power, of the Area.

From this Theorem, the Rule in common Practice, for finding the Area of a Triangle, having the three Sides given, is deduced, as a Corollary: For, making $a=0$,

$a=0$, the Trapezium becomes a Triangle, and the second Power of its Area $=\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b}\times\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-b}\times\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-c}\times\overline{\tfrac{1}{2}d+\tfrac{1}{2}c+\tfrac{1}{2}b-d}$: Which, in Words, is the common Rule.

EXAMPLE XX.

44. *To find the greatest Value of y in the Equation* $a^4x^2=\overline{xx+yy}|^3$.

By putting the whole Equation into Fluxions, &c. we have $2a^4x\dot{x}=\overline{2x\dot{x}+2y\dot{y}}\times3\times\overline{x^2+y^2}|^2$; which in the required Circumstance, when $\dot{y}=0$ *, becomes $2a^4x\dot{x}=6x\dot{x}\times\overline{x^2+y^2}|^2$; whence $x^2+y^2=\frac{a^2}{\sqrt{3}}$, and $\overline{x^2+y^2}|^3=\frac{a^6}{3\sqrt{3}}$: But, by the given Equation $\overline{x^2+y^2}|^3=a^4x^2$; consequently $a^4x^2=\frac{a^6}{3\sqrt{3}}$, and therefore $x=a\sqrt{\frac{1}{3\sqrt{3}}}$; whence $y^2\left(=\frac{a^2}{\sqrt{3}}-x^2\right)=\frac{2a^2}{3\sqrt{3}}$, and $y=a\sqrt{\frac{2}{3\sqrt{3}}}$.

* Art. 22.

The same otherwise.

Since $\overline{xx+yy}|^3$ is given $=a^4x^2$, we have $x^2+y^2=a^{\frac{4}{3}}\times x^{\frac{2}{3}}$, and therefore $y^2=a^{\frac{4}{3}}\times x^{\frac{2}{3}}-x^2$; whose Fluxion, $\frac{2}{3}a^{\frac{4}{3}}\times x^{-\frac{1}{3}}\dot{x}-2x\dot{x}$, being put $=0$, we also get $\frac{a^{\frac{4}{3}}\times x^{-\frac{1}{3}}}{3}=x$; whose Cube is $\frac{a^4\times x^{-1}}{27}=x^3$, or $\frac{a^4}{27x}=x^3$; whence $27x^4=a^4$, and consequently $x=a\sqrt{\frac{1}{3\sqrt{3}}}$, the same as before.

45. When

45. When, in the general Expreſſion, whoſe *Maximum* or *Minimum* is ſought, there are two or more indeterminate Quantities, independent of each other, their reſpective Values, in the required Circumſtance, will be determined, by making them flow, one by one, while the others are ſuppoſed invariable; as in the following

EXAMPLE XXI.

Wherein it is propoſed to find three ſuch Values of x, y, and z, as ſhall make the Value of $\overline{b^3 - x^3} \times \overline{x^2z - z^3} \times \overline{xy - y^2}$ *the greateſt poſſible.*

Firſt, conſidering y as variable, and the reſt conſtant, we have $x\dot{y} - 2y\dot{y} = 0$ *; whence $y = \frac{1}{2}x$, and $xy - y^2 = \frac{1}{4}x^2$. By making z variable, we have $x^2\dot{z} - 3z^2\dot{z} = 0$; whence $z = \frac{x}{\sqrt{3}}$, and $x^2z - z^3 = \frac{2x^3}{3\sqrt{3}}$. Now let theſe Values of $xy - y^2$ and $x^2z - z^3$ be ſubſtituted in the given Expreſſion, and it will become $\frac{x^2}{4} \times \frac{2x^3}{3\sqrt{3}} \times \overline{b^3 - x^3} = \frac{b^3x^5 - x^8}{6\sqrt{3}}$; therefore $5b^3x^4\dot{x} - 8x^7\dot{x} = 0$: Whence $x = \frac{1}{2}b \times \sqrt[3]{5}$, $y\ (= \frac{1}{2}x) = \frac{1}{4}b \times \sqrt[3]{5}$, and $z \left(= \frac{x}{\sqrt{3}}\right) = \frac{1}{2}b \times \frac{\sqrt[3]{5}}{\sqrt{3}}$. *Art. 22.

The Reaſon of the foregoing Proceſs is obvious: For, if the Fluxion of the given Expreſſion, when any one of the indeterminate Quantities is made variable, be not equal to Nothing, that Expreſſion may become greater, without altering the Values of the reſt, which are conſidered as conſtant †: And therefore cannot be the greateſt poſſible, unleſs the ſaid Fluxion is equal to Nothing. †Art. 22.

EXAMPLE XXII.

46. *To determine the different Values of x, when that of* $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$ *becomes a* Maximum *or* Minimum.

The Fluxion of the given Expreſſion being *(as uſual)* put equal to Nothing, we have $12x^3-84ax^2+168a^2x-96a^3=0$, or $x^3-7ax^2+14a^2x-8a^3=0$: From whence (by the Method of Diviſors) we get $x-a=0$, $x-2a=0$, or $x-4a=0$: Therefore, the Roots of the Equation, or the three Values of x, are a, $2a$, and $4a$.

SCHOLIUM.

47. It appears, from the laſt Example, that a Quantity may admit of as many *Maxima* and *Minima* (according to the Meaning of the Definition *) as there are poſſible Roots in the Equation, ariſing from aſſuming its Fluxion equal to Nothing. Now to know which of thoſe Roots point out a *Maximum*, and which a *Minimum*; find whether the Value of the ſaid Fluxion, a little before it becomes equal to Nothing, be poſitive or negative; if *poſitive*, the ſucceeding Root gives a *Maximum*; but if *negative*, a *Minimum*: The Reaſon of which is extremely obvious; becauſe ſo long as any Quantity increaſes, its Fluxion is poſitive, but when it decreaſes the Fluxion is negative.

* Art. 22.

As an Example hereof, let the Quantity $3x^4-28ax^3+84a^2x^2-96a^3x+48b^4$, be again reſumed; whoſe Fluxion is $12\dot{x}\times\overline{x^3-7ax^2+14a^2x-8a^3}=12\dot{x}\times\overline{x-a}\times\overline{x-2a}\times\overline{x-3a}$: Whereof the Value, before it becomes equal to Nothing, the firſt time (or before $x=a$) being negative (becauſe the Product of three negative Factors is negative) its firſt Root (a) therefore indicates a *Minimum*: Whence we may conclude, without conſidering farther, that the ſecond Root $(2a)$ gives a *Maximum*, and the third $(4a)$ another *Minimum*. But, if you

you would know whether the firſt or third Root gives the leſſer Value of the two; it is but ſubſtituting in the given Quantity, which will come out $48b^4-37a^4$, and $48b^4-64a^4$ reſpectively; therefore the latter is the leſſer, and the very leaſt Value the propoſed Expreſſion can admit of.

When all the Roots prove impoſſible, the Quantity propoſed (as its Fluxion can never become $=0$) muſt either increaſe, or decreaſe, continually; and therefore can neither admit of a *Maximum nor a Minimum.*

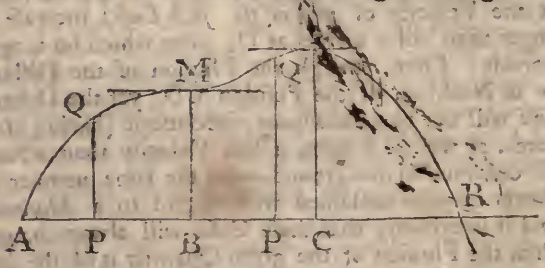
Moreover, it may ſo happen, that the Roots are poſſible, the Fluxion $=0$, and yet the Quantity itſelf be neither a *Maximum* nor a *Minimum* in that Circumſtance.

For let us, again, ſuppoſe the Point n to move after m, as in the general Illuſtration, (*vid. Art.* 22.) only let the Velocity of n (*in the firſt Caſe*) increaſe no longer than 'till it arrives at D; after which let it again decreaſe: Then, though the Fluxion of the Diſtance mn is Nothing, at the Poſition CD, yet the Diſtance itſelf will not be a *Maximum*; becauſe n (having afterwards, as well as before, a leſs Velocity than m) will ſtill continue to loſe ground.—In the ſame manner the Matter may be explained with regard to a *Minimum.* And it is evident, that theſe Caſes will always happen when the Fluxion of the given Quantity is of the ſame Denomination (with regard to poſitive and negative) both before and after, it becomes equal to Nothing: Which, by the Rules of common Algebra, is known to be when the Equation admits of an even Number of equal Roots.—An Example hereof, however, may not be improper.

Let then the Quantity propoſed be $24a^3x-30a^2x^2+16ax^3-3x^4$; whoſe Fluxion is $24a^3\dot{x}-60a^2x\dot{x}+48ax^2\dot{x}-12x^3\dot{x}=12\dot{x}\times\overline{a-x}\times\overline{a-x}\times\overline{2a-x}$: Which being made $=0$, it appears that the two leaſt Roots are equal. Therefore there is neither a *Maximum* nor *Minimum* when $x=a$ (becauſe whether x be taken a little leſs, or a little greater, than a, the Value of the Fluxion will ſtill be affirmative.) The greateſt Root, however, not being affected with another equal one, indicates a *Maximum*, according to the Rule above preſcribed.

To render what has been obſerved above ſtill more conſpicuous, let the given Expreſſion, $24a^3x - 30a^2x^2 + 16ax^3 - 3x^4$, be repreſented by the variable Ordinate PQ of the Curve AQMNR; whoſe Abſciſſa AP is (as uſual) denoted by x.

Then, whilſt $(12\dot{x} \times \overline{a-x} \times \overline{a-x} \times \overline{2a-x})$ the Fluxion of the Ordinate continues poſitive, (or till x becomes $= a =$ AB) the Ordinate itſelf will increaſe: But at the Poſition BM it becomes ſtationary (if I may be allowed the Expreſſion) the Fluxion being then $= 0$. After which, the Fluxion being again affirmative, the Ordinate will again increaſe, till x becomes $= 2a$ ($=$ AC); when, the Fluxion becoming Nothing, a ſe-

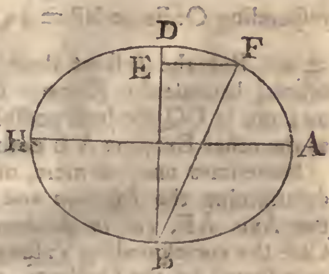


cond time,) and afterwards negative, CN will be a *Maximum*: Soon after which the Curve deſcends below its Axis, and continues to recede from it *in infinitum*.

Another Thing there is that ought to be regarded in the Solution of theſe Kinds of Problems, and that is, whether the *Maxima* or *Minima*, found by aſſuming the Fluxion $= 0$, fall within the Limits preſcribed by the Nature of the Queſtion or Figure; which is often reſtrained by Conditions that do not enter into the algebraic Computation.

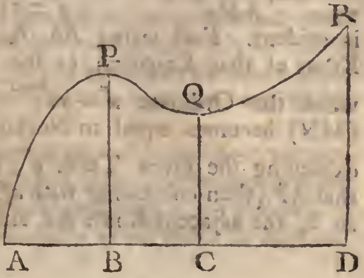
Thus, for Example; ſuppoſe it were required to find that Point (F) in a given Ellipſis ABHD which, of all others,

others, is the moſt remote from the Extreme B of the conjugate Axis BD.



Then, drawing FE parallel to the Tranſverſe AH, and putting AH $= a$, BD $= b$, and BE $= x$, we have, by the Property of the Curve BF^2 $(= BE^2 + EF^2)$ $x^2 + \overline{bx - x^2} \times \frac{a^2}{b^2}$; from whence x is found $= \frac{\frac{1}{2}a^2 b}{a^2 - b^2}$. But, from the Nature of the Figure, the greateſt Value that x $(= BE)$ can poſſibly admit of is b $(= BD)$, therefore if the Relation of a and b be ſuch, that $\frac{\frac{1}{2}a^2 b}{a^2 - b^2}$ is greater than b, this Solution is manifeſtly impoſſible. —— To determine the Limit, therefore, make $\frac{\frac{1}{2}a^2 b}{a^2 - b^2} = b$; then it will be found that $2b^2 = a^2$. Whence the foregoing Solution can only obtain when $2BD^2$ is equal to, or leſs than AH^2.

Again, it ought to be alſo conſidered whether the Value of x, found by the common Method, gives a leſs Quantity for the *Maximum*, and a greater for the *Minimum*, than will ariſe from the Extremes themſelves by which x is limited.

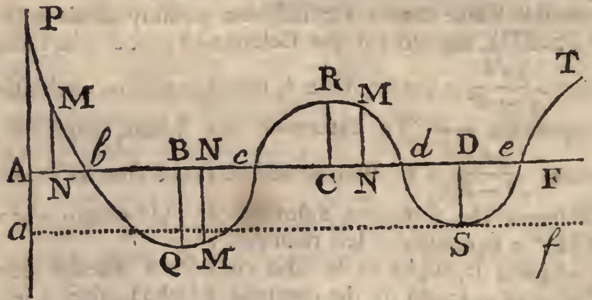


Thus, let it be required to determine the greateſt and leaſt Ordinates in a Curve, APR, whoſe Equation is $y^3 = 6a^2 x - 9ax^2 + 4x^3$, and whoſe greateſt Abſciſſa AD is given equal $2a$.

Here

Here we ſhall, by taking the Fluxion, &c. have $x = \frac{1}{2}a$, or $x = a$: The former of which Values gives the correſponding Ordinate BP $= a\sqrt[3]{\frac{5}{4}}$; and the latter, CQ $= a$: But the firſt of theſe is not the greateſt of all others, becauſe the Extreme DR exceeds it, being $= 2a$; nor is CQ the leaſt poſſible, becauſe the Ordinate at the other Extreme A is nothing at all.

Sometimes one, or more, of the Points Q, S, &c. determining the *Maxima* and *Minima*, will fall below the Axis AF, *(as in the annexed Figure)*. In which Caſe the correſponding Value of the general Expreſſion, repreſented by the Ordinate, will be negative: But at the Points *b*, *c*, *d*, &c. where the Curve interſects the



Axis, it will be equal to nothing: Whence (by the Bye) the Reaſon why the Roots of an Equation $(x^n - ax^{n-1} + b^2x^{n-2} \ldots + q^n = 0)$ are impoſſible by Pairs is evident. For, ſeeing A*b*, A*c*, A*d*, A*e*, &c. are the Roots of that Equation, or the different Values of x, when the Ordinate $x^n - ax^{n-1} + b^2x^{n-2} \ldots\ldots + q^n$ (MN) becomes equal to Nothing, it is plain, if PA, expreſſing the given Term q^n, be increaſed to P*a*, ſo that AF (then coinciding with *af*) may touch the Curve in S, the adjacent Roots A*d* and A*e* will then become equal;

equal; and if $\dot{q}^n$ be farther increased, so that the Axis may fall wholly below the Curve, not only those two, but also the other Roots, A*b* and A*c*, will become impossible.

Various other Observations might be made, relating to the Limits of Equations, determined by these *Maxima* and *Minima*; but this being foreign to the Matter in hand, I shall content myself with one Remark more, *viz.*

Any Expression which, being put equal to Nothing, admits of two or more equal Roots, has as many succeeding Orders of Fluxions equal to Nothing, at the same time, as are expressed by the Number of those Roots minus *one.*

Thus, an Equation, having three equal Roots, has both its first and second Fluxions equal to Nothing, when the Fluent itself is equal to Nothing.

Hence we have another Way (besides that given above) to know when a Quantity may have its Fluxion equal to Nothing, and yet neither admit of a *Maximum* nor a *Minimum:* For, since this Circumstance always takes place when the Equation admits of an *even* Number of equal Roots (as has been already shewn) the Number of Orders of Fluxions, equal to Nothing, at the same time (including the First) must also be even.

Hence, also, we have an easy Method for discovering when some of the Roots of an Equation are equal; and, if so, what they are.

Thus, let $x^3 - 3ax^2 + 4a^3 = 0$ be propounded; whereof the Fluxion $3x^2\dot{x} - 6ax\dot{x}$ being assumed equal to Nothing, we find $x = 2a$; which will also be a Root of the given Equation, if it admits of two equal ones: To try it, therefore, I substitute $2a$ for x, and find it answers.

Again, let $8x^4 - 28ax^3 + 18a^2x^2 + 27a^3x - 27a^4 = 0$; whereof the first and second Fluxions being $32x^3\dot{x} - 84ax^2\dot{x} + 36a^2x\dot{x} + 27a^3\dot{x}$ and $96x^2\dot{x}^2 - 168ax\dot{x}^2 + 36a^2\dot{x}^2$, if the latter of them be assumed $= 0$, x will be found $=\frac{7a}{8}\pm\sqrt{\frac{25a^2}{64}}=\frac{3a}{2}$, or $\frac{a}{4}$: One of which Quantities, if the Equation proposed admits of three equal Roots, will be the Value of each of them: By trying $\frac{3a}{2}$, it will be found to succeed. Whence, by a well known Rule, the fourth Root (being $=\frac{28a}{8}-\frac{3a}{2}\times 3=-a$) is also given.

The Reason of these Operations, as well as what is asserted above, may be thus demonstrated.

Let $\overline{r-x}\times\overline{r-x}$ &c. $\times\overline{A+Bx+Cx^2 \text{ \&c.}}=0$, be any Equation, having two or more equal Roots, represented, each, by r: Put $y=r-x$, and let the Number of the equal Roots be denoted by n; then, by Substitution, we have $y^n\times\overline{A+B\times\overline{r-y}+C\times\overline{r-y}|^2 \text{ \&c.}}=0$; which, by expanding the Powers of $r-y$, and putting $a=A+Br+Cr^2$ &c. $b=B+2Cr+3Dr^2$, &c. will be further transformed to $y^n\times\overline{a-by+cy^2-dy^3 \text{ \&c.}}=$: Whose Fluxion $na\dot{y}y^{n-1}-\overline{n+1}\cdot b\dot{y}y^n+\overline{n+2}\cdot c\dot{y}y^{n+1}$ &c. is evidently equal to Nothing, when y, or its Equal $r-x$, is Nothing (provided n be greater than Unity. It is equally plain, that the second Fluxion $n\cdot\overline{n-1}\cdot a\dot{y}^2y^{n-2}-\overline{n+1}\cdot nb\dot{y}^2y^{n-1}+\overline{n+2}\cdot\overline{n+1}\cdot c\dot{y}^2y^n$ &c. will also be equal to Nothing, in the same Circumstance, if n be greater than 2, &c. &c.

Hence, universally, let the Number (n) of equal Roots be what it will, that of the Orders of Fluxions equal to Nothing, at the same time, will be expressed by that Number *minus* one, as was to be shewn.

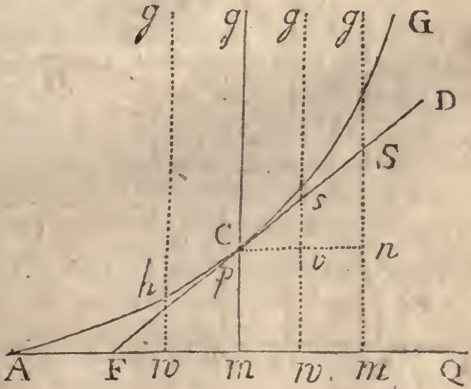
SECT.

SECTION III.

The Use of FLUXIONS *in drawing Tangents to Curves.*

ILLUSTRATION.

48. LET ACG be a Curve of any kind, and C the given Point from whence the Tangent is to be drawn.

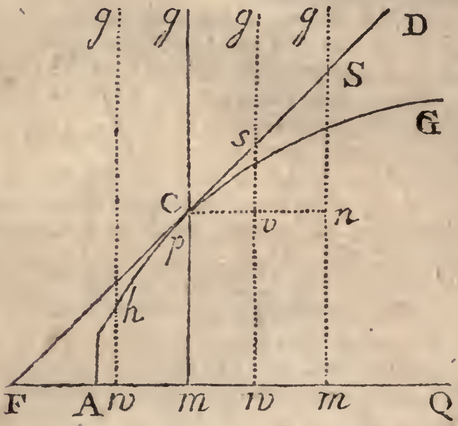


Conceive a Right-line *mg* to be carried along uniformly, parallel to itſelf, from A towards Q, and let, at the ſame time, a Point *p* ſo move in that Line, as to deſcribe, or trace out, the given Curve ACG: Alſo let *mm*, or C*n* (equal and parallel to *mm*) expreſs the Fluxion of A*m*, or the Celerity wherewith the Line *mg* is carried; and let *n*S expreſs the correſponding Fluxion of *mp*, in the Poſition *m*C*g*, or the Celerity of the Point *p*, in the Line *mg*. Moreover, through the Point C let the Right-line SF be drawn, meeting the Axis of the Curve (AQ) in F.

 Now,

Now, it is evident, if the Motion of *p*, along the Line *mg*, was to become equable at C, the Point *p* would be at S, when the Line itſelf had acquired the Poſition *m*S*g* (becauſe, by Hypotheſis, C*n* and *n*S expreſs the Diſtances that might be deſcribed by the two uniform Motions in the ſame time).

And, if *wsg* be aſſumed to repreſent any other Poſition of that Line, and *s* the contemporary Poſition of the Point *p* (ſtill ſuppoſing an equable Celerity of *p*); then the Diſtances C*v* and *vs*, gone over, in the ſame



time, by the two Motions, will, always, be to each other as the Celerities, or as C*n* to *n*S: Therefore, ſince C*v* : *vs* :: C*n* : *n*S (which is a known Property of ſimilar Triangles) the Point *s* will, always, fall in the Right-line FCS: Whence it appears, that, if the Motion of the Point *p* along the Line *mg* was to become uniform at C, that Point would then move in the Right-line CS, inſtead of the Curve-line CG.

Now, ſeeing the Motion of *p*, in the Deſcription of Curves, muſt, either, be an accelerated or a retarded one, let it be, firſt, conſidered as an accelerated one: In which Caſe the Arch CG will fall, *wholly*, above the Right-line CD (as in Fig. 1.) becauſe the Diſtance of of the Point p from the Axis AQ, at the End of any given Time, is greater than it would be if the Acceleration was to ceafe at C; and, if the Acceleration had ceafed at C, the Point p would (it is proved) have been always found in the faid Right-line FS.

But if the Motion of the Point p be a retarded one, it will appear, by reafoning in the fame manner, that the Arch CG will fall wholly below the Right-line CD (as in Fig. 2.)

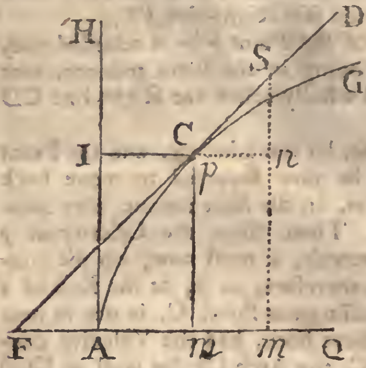
This being the Cafe, let the Line mg, and the Point p, along that Line, be now fuppofed to move back again, towards A and m, in the fame manner they proceeded from thence: Then, fince the Celerity of p (*Fig.* 1.) did before increafe, it muft now, on the contrary, decreafe; and, therefore, as p, at the End of a given Time, after repaffing the Point C, is not fo near to AQ, as it would have been, had the Velocity continued the fame as at C, the Arch Cb (as well as CG) muft fall wholly above the Right-line FCD. And, by the fame Method of arguing, the Arch Cb, *in the fecond Cafe*, will fall, *wholly*, below FCD: Therefore FCD, *in both Cafes*, is a Tangent to the Curve at the Point C: Whence, the Triangles FmC and CnS being fimilar, it appears, that the Sub-tangent mF is always a Fourth-proportional to (nS) the Fluxion of the ordinate (Cn), the Fluxion of the Abfciffa, and the Ordinate (Cm).

Otherwife.

49. Let ACG reprefent the propofed Curve, and let the Right-line FCD be a Tangent to it, at any Point C, meeting the Axis AQ (produced if neceffary) in F: Suppofe a Point p to move along the Curve, from A towards G, and let the abfolute Celerity thereof at C, in the Direction of the Tangent CD, or the Fluxion of the Line Ap fo generated *, be denoted by CS, any Part of the faid Tangent: Then, if AH, mp and mS be made perpendicular, and Ipn parallel, to AQ, the relative Celerities of that Point, in the Directions Cn and mC, wherewith Ip (= Am) and mp increafe in this Pofition,

* Art. 2 and 5.

*Art. 35. Position, will be truly expressed by Cn and nS *: But the Celerities by which Quantities increase are as the Fluxions of those Quantities: Therefore (CS being the Fluxion of the Curve-line Ap) Cn and nS are the corresponding Fluxions of the Abscissa Am and the Ordinate mp: And we have Sn : nC :: mC : mF, *the same as before.*

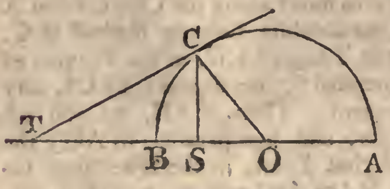


Hence, if the Abscissa Am be put $= x$, and the Ordinate $mp = y$, we shall have $mF = \frac{y\dot{x}}{\dot{y}}$: By means of which general Expression, and the Equation expressing the Relation between x and y, the Ratio of the Fluxions $\dot{x}$ and $\dot{y}$ will be found, and from thence the Length of the Sub-tangent (mF) as in the following Examples.

EXAMPLE I.

50. *To draw a Right-line* CT, *to touch a given Circle* BCA, *in a given Point* C.

Let CS be perpendicular to the Diameter AB, and put $AB = a$, $BS = x$ and $SC = y$: Then, by the Property of the Circle, y^2 (CS^2) $= BS \times AS$ $(= x \times \overline{a - x})$ $= ax - x^2$; whereof



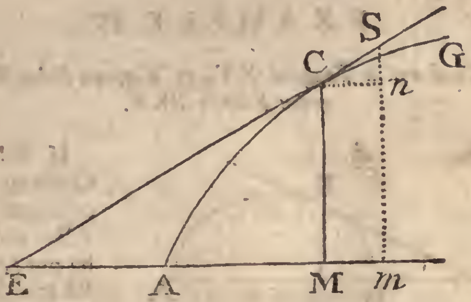
whereof the Fluxion being taken, in order to determine the Ratio of $\dot{x}$ and $\dot{y}$, we get $2y\dot{y} = a\dot{x} - 2x\dot{x}$; consequently $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a - 2x} = \frac{y}{\frac{1}{2}a - x}$; which, multiplied by y, gives $\frac{y\dot{x}}{\dot{y}} = \frac{y^2}{\frac{1}{2}a - x}$ = the Sub-tangent ST *. Whence (O being ſuppoſed the Center) we have OS $(\frac{1}{2}a - x)$: CS (y) :: CS (y) : ST; which we alſo know from other Principles.

* Art. 48 and 49.

EXAMPLE II.

51. *To draw a Tangent to any given Point* C *of the conical Parabola* ACG.

If the *Latus Rectum* of the Curve be denoted by a, the Ordinate MC by y, and its correſponding Abſciſſa



AM by x; then the known Equation, expreſſing the Relation of x and y, being $ax = y^2$, we have, in this Caſe, $a\dot{x} = 2y\dot{y}$; whence $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a}$, and conſequently $\frac{y\dot{x}}{\dot{y}}$† $= \frac{2y^2}{a} = \frac{2ax}{a} = 2x =$ MF. Therefore the Sub-tangent is juſt the double of its correſponding Abſciſſa AM: Which we likewiſe know from other Principles.

† Art. 48 and 49.

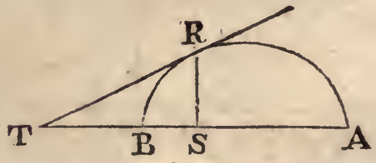
EXAMPLE III.

52. *To draw a Tangent to a Parabola of any kind.*

The general Equation of these sort of Curves being $a^m x^n = y^{m+n}$, we have $na^m x^{n-1}\dot{x} = \overline{m+n} \times y^{m+n-1}\dot{y}$, and therefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{m+n}\times y^{m+n-1}}{na^m x^{n-1}}$; whence $\frac{y\dot{x}}{\dot{y}} = \frac{\overline{m+n}\times y^{m+n}}{na^m x^{n-1}} = \frac{\overline{m+n}\times a^m x^n}{na^m x^{n-1}}$ (because $y^{m+n} = a^m x^n$) $= \frac{m+n}{n}\times x =$ the true Value of the Subtangent: Which, therefore, is to the Abscissa, in the constant Ratio of $m+n$ to n.

EXAMPLE IV.

53. *To draw a Tangent* RT, *to a given Point* R, *in a given Ellipsis* BRA.



If RS be an Ordinate to the principal Axis AB, and there be put *(as usual)* BS $= x$, RS $= y$, AB $= a$, and the lesser Axis $= b$; we shall, by the Property of the Curve, have $a^2 : b^2 :: ax - x^2$ (BS × AS) : y^2 (RS^2), and therefore $b^2 \times \overline{ax - x^2} = a^2y^2$: Whence $b^2 \times \overline{a\dot{x} - 2x\dot{x}} = 2a^2y\dot{y}$, and $\frac{\dot{x}}{\dot{y}} = \frac{2a^2y}{b^2\times\overline{a-2x}}$; and consequently the Sub-tangent

* Art. 49. ST $\left(\frac{y\dot{x}}{\dot{y}}\right)$* $= \frac{2a^2y^2}{b^2\times\overline{a-2x}} = \frac{a^2y^2}{b^2\times\overline{\frac{1}{2}a-x}} = \frac{b^2\times\overline{ax-x^2}}{b^2\times\overline{\frac{1}{2}a-x}} =$

$ax -$

$\frac{ax-x^2}{\frac{1}{2}a-x}$. Whence the Point T being given, through which the Tangent muſt paſs, the Tangent itſelf may be drawn.

But if you would derive an Expreſſion for the Subtangent, in any other kind of Ellipſes (beſides the conical) let the Equation $\overline{a-x}|^m \times x^n = \frac{e}{q} \times y^{m+n}$, exhibiting the Nature of all Kinds of Ellipſes, be aſſumed: Then, by taking the Fluxion thereof, you will have $-m\dot{x} \times \overline{a-x}|^{m-1} \times x^n + n\dot{x}x^{n-1} \times \overline{a-x}|^m = \frac{e}{q} \times \overline{m+n} \times y^{m+n-1}\dot{y}$; and therefore $\frac{y\dot{x}}{\dot{y}} =$

$$\frac{\frac{e}{q} \times \overline{m+n} \times y^{m+n}}{-m \times \overline{a-x}|^{m-1} \times x^n + nx^{n-1} \times \overline{a-x}|^m}$$

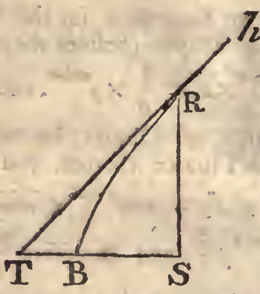
$= \frac{\overline{m+n} \times \overline{a-x}|^m \times x^n}{-mx^n \times \overline{a-x}|^{m-1} + nx^{n-1} \times \overline{a-x}|^m}$ (becauſe $\frac{e}{q} \times y^{m+n} = \overline{a-x}|^m \times x^n$) $= \frac{\overline{m+n} \times \overline{a-x} \times x}{-mx + n \times \overline{a-x}} = \frac{\overline{m+n} \times \overline{ax-x^2}}{na - \overline{n+m} \times x}$; which is the Sub-tangent required.

EXAMPLE V.

54. *To draw a Tangent, to any given Point* R, *in a given Hyperbola* BR*h*.

If a and c be put to denote the two principal Diameters of the Hyperbola, the Equation of the Curve will be $c^2 \times \overline{ax+x^2} = a^2y^2$: From whence we have $c^2 \times \overline{a\dot{x} +}$

$\overline{a\dot{x}+2x\dot{x}} = 2a^2y\dot{y}$, $\therefore \frac{\dot{x}}{\dot{y}} = \frac{a^2y}{c^2\times\overline{\frac{1}{2}a+x}}$, and consequently $\frac{y\dot{x}}{\dot{y}} = \frac{a^2y^2}{c^2\times\overline{\frac{1}{2}a+x}} = \frac{c^2\times\overline{ax+x^2}}{c^2\times\overline{\frac{1}{2}a+x}} = \frac{ax+x^2}{\frac{1}{2}a+x} =$ ST. Whence BT (ST — BS) $= \frac{\frac{1}{2}ax}{\frac{1}{2}a+x}$ is also known; and therefore the Point T being given the Tangent RT may be drawn.



The Manner of drawing Tangents to all Sorts of Hyperbolas, *universally*, will be the same as in the Ellipses, the Equations of the two Kinds of Curves differing in Nothing but their Signs.

EXAMPLE VI.

55. *Let the proposed Curve be that whose Equation is* $ax^2+xy^2+x^3-y^3=0$.

Then we shall have $2ax\dot{x}+y^2\dot{x}+2xy\dot{y}+3x^2\dot{x}-3y^2\dot{y} = 0$; therefore $2ax\dot{x}+y^2\dot{x}+3x^2\dot{x} = 3y^2\dot{y}-2xy\dot{y}$, $\frac{\dot{x}}{\dot{y}} = \frac{3y^2-2xy}{2ax+y^2+3x^2}$, and consequently $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3-2xy^2}{2ax+y^2+3x^2}$*.

*Art. 48 and 49.

EXAMPLE VII.

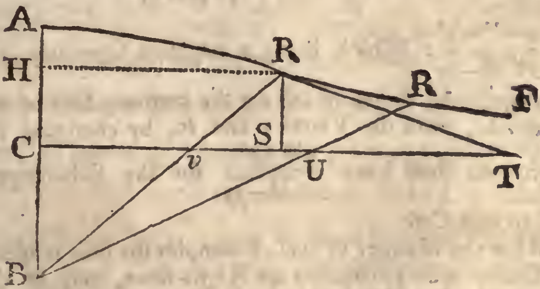
56. *Let the given Curve be the Ciſſoid of* Diocles, *whoſe Equation is* $y^2 = \frac{x^3}{a-x}$.

Here we have $2y\dot{y} = \frac{3x^2\dot{x} \times \overline{a-x} + \dot{x}x^3}{\overline{a-x}|^2} = \frac{3ax^2\dot{x} - 2x^3\dot{x}}{\overline{a-x}|^2}$:

Whence $\frac{\dot{x}}{\dot{y}} = \frac{2y \times \overline{a-x}|^2}{3ax^2 - 2x^3}$, and conſequently the Subtangent $\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2y^2 \times \overline{a-x}|^2}{3ax^2 - 2x^3} = \frac{2x^3}{a-x} \times \frac{\overline{a-x}|^2}{3ax^2 - 2x^3} = \frac{2x \times \overline{a-x}}{3a-2x}$.

EXAMPLE VIII.

57. *Let the Conchoid of* Nicomedes *be propoſed*; whereof the Nature is ſuch, that, if from a Point B, called



the Pole, any Number of Right-lines, BA, BR, BR, *&c.* be drawn, the Parts of thoſe Lines CA, *v*R, UR, *&c.* intercepted by the Curve and its Axis CT, ſhall be, all, equal to each other.

In

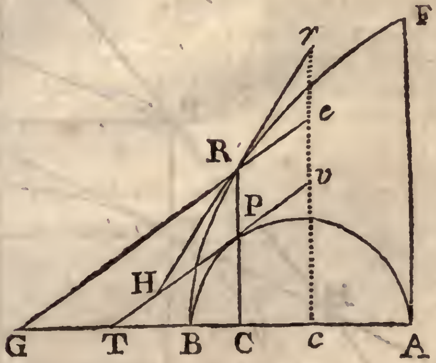
In this Case (supposing AB and RS perpendicular, and RH parallel, to CT; and putting BC $= a$, Rv (AC) $= b$, CS $= x$, and RS $= y$) we have, *per sim. Triang.* $a+y$ (BH) : x (RH) :: y (RS) : $\frac{xy}{a+y} =$ Sv: But Sv ($\sqrt{Rv^2 - RS^2}$) is also $= \sqrt{b^2-y^2}$; therefore $\frac{xy}{a+y} = \sqrt{b^2-y^2}$, or $x^2y^2 = \overline{a+y}|^2 \times \overline{b^2-y^2}$ is the general Equation of the Curve; which, in Fluxions, gives $2x^2y\dot{y} + 2y^2x\dot{x} = 2\dot{y} \times \overline{a+y} \times \overline{b^2-y^2} - 2y\dot{y} \times \overline{a+y}|^2 = 2\dot{y} \times \overline{a+y} \times \overline{b^2-ay-2y^2}$; and therefore $\frac{\dot{x}}{\dot{y}} = \frac{\overline{a+y} \times \overline{b^2-ay-2y^2} - x^2y}{xy^2}$, consequently $\frac{y\dot{x}}{\dot{y}} = \frac{\overline{a+y} \times y \times \overline{b^2-ay-2y^2} - x^2y^2}{y \times xy} = \frac{\overline{a+y} \times y \times \overline{b^2-ay-2y^2} - \overline{a+y}|^2 \times \overline{b^2-y^2}}{y \times \overline{a+y} \times \sqrt{b^2-y^2}}$ (because $x^2y^2 = \overline{a+y}|^2 \times \overline{b^2-yy}$) $= \frac{b^2y-ayy-2y^3-abb+ayy-bby+y^3}{y\sqrt{bb-yy}} = \frac{-ab^2-y^3}{y\sqrt{bb-yy}}$: Which being a negative Quantity, the Tangent will therefore fall on the contrary Side of the Ordinate, from the Vertex; and so, by changing the Signs we shall have $\frac{abb+y^3}{y\sqrt{bb-yy}}$ for the Sub-tangent ST in this Case.

After the Manner of these Examples the Sub-tangent, in Curves whose Abscissas are Right-lines, may be determined: But if the Abscissa, or Line terminating the Ordinate, on the lower Part, be another Curve, then the Tangent may be drawn as in the following

EX-

EXAMPLE IX.

58. Let the Curve BRF be a Cycloid; whose Abſciſſa is here ſuppoſed to be the Semicircle BPA, to which let the Tangent PT be drawn (as above). Moreover let rRH be a Tangent to the Cycloid, at the cor-

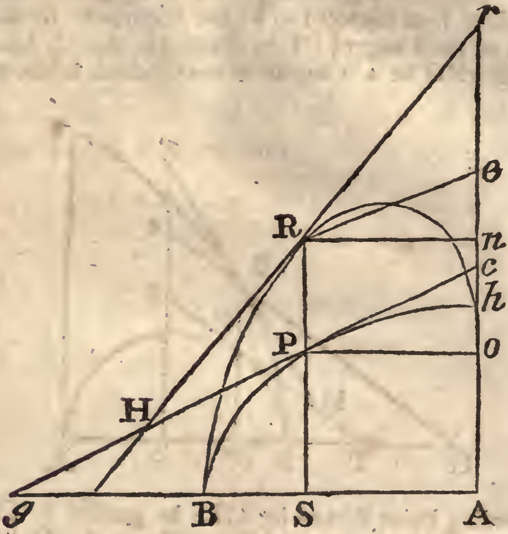


reſponding Point R, and let GRe be parallel to TPv; putting the Arch (or Abſciſſa) BP$=z$, its Ordinate PR$=y$, AF$=b$, and BPA$=c$: Then, by the Property of the Curve, we ſhall have c (BPA) : b (AF) :: z (BP) : y (PR): Therefore $y=\frac{bz}{c}$, and $\dot{y}=\frac{b\dot{z}}{c}=re$: But, by ſimilar Triangles, $re\ (\dot{y})$: Re ($=$ P$v=\dot{z}$) :: PR (y) : PH $=\frac{y\dot{z}}{\dot{y}}=z$ (becauſe $y=\frac{bz}{c}$). Therefore, if in the Right-line PT, there be taken PH, equal to the Arch PB, you will have a Point H, through which the Tangent of the Cycloid muſt paſs.

EXAMPLE X.

59. Let BPh be a Curve of any Kind, to which the Method of drawing the Tangent cPg is known; let BRh

BRh be another Curve of ſuch a Nature, that the Ordinate PR (y) ſhall always be a Mean-proportional be-



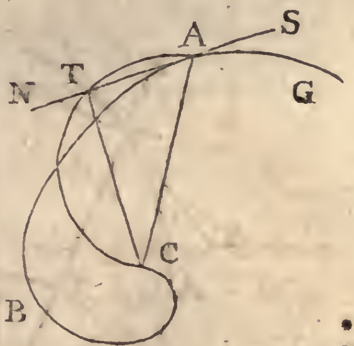
tween BS (x) and AS $(a-x)$ ſuppoſing RPS perpendi-
*Art.48 and 49. cular to AB: Put $Po = \dot{x}$, SP $= v$, $oc = \dot{v}$ *, and $er = \dot{y}$: Then, *(as above)* er $(\dot{y})$: Re ($=$ Pc $= \sqrt{\dot{x}^2+\dot{v}^2}$) :: RP (y) : PH $= \frac{y\sqrt{\dot{x}^2+\dot{v}^2}}{\dot{y}}$: But, by the Equation of the Curve $y^2 = ax - xx$; whence $2y\dot{y} = a\dot{x} - 2x\dot{x}$, and $\frac{y}{\dot{y}} = \frac{2ax-2x^2}{a\dot{x}-2x\dot{x}}$, and therefore PH $=$ $\frac{2ax-2x^2 \times \sqrt{\dot{x}^2+\dot{v}^2}}{a\dot{x}-2x\dot{x}}$: Which will be expreſſed independent of Fluxions, when the Property of the Curve BPh, or the Relation of x and v is given: Thus, let BPh be the common Parabola, and AB its *Latus Rectum*;

tum; then v being $=\sqrt{ax}$, $\dot{v}$ will be $=\frac{a\dot{x}}{2\sqrt{ax}}$, $\dot{x}^2+\dot{v}^2=\dot{x}^2+\frac{a\dot{x}\dot{x}}{4x}=\frac{\dot{x}\dot{x}\times\overline{4x+a}}{4x}$; and therefore PH $\left(\frac{\overline{2ax-2xx}\times\sqrt{\dot{x}^2+\dot{v}^2}}{a\dot{x}-2x\dot{x}}\right)=\frac{\overline{a-x}\times\sqrt{4x^2+ax}}{a-2x}$.

Thus far relates to Curves whoſe Ordinates are parallel to each other: We come now to Curves of the ſpiral Kind, whoſe Ordinates all iſſue from a Point: Such as the Spiral BAG, whoſe Ordinates CB, CA, CG, are all referred to the Point C, called the Center of the Spiral.

ILLUSTRATION.

60. Let SAN be a Tangent to the Spiral at any Point A, alſo let CT be perpendicular thereto, and let the Arch CBA (conſidered as variable by the Motion of A towards G) be denoted by z, and the Ordinate CA by y. Then $\dot{z}:\dot{y}::$ AC (y) : AT $=\frac{y\dot{y}}{\dot{z}}$.*



* Art. 5 and 35.

Hence, if upon CA, as a Diameter, a Semi-circle be deſcribed, and in it, from A, a Right-line AT equal to $\frac{y\dot{y}}{\dot{z}}$ be inſcribed, that Right-line will be a Tangent to the Spiral at the Point A.

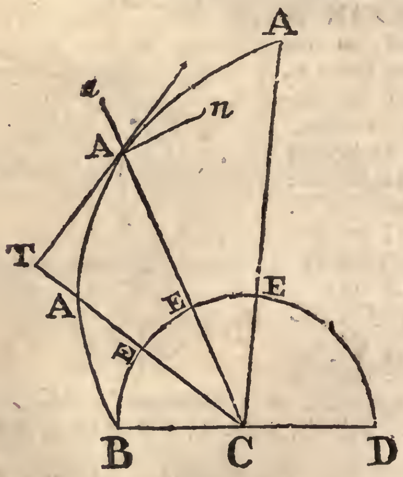
EXAMPLE I.

61. Let the Nature of the Curve CBA be ſuch that the Arch CBA may be, always, to its correſponding

responding Ordinate CA in a constant Ratio; namely as a to b: Then, because $z : y :: a : b$, we have $z = \frac{ay}{b}$, $\dot{z} = \frac{a\dot{y}}{b}$, and consequently AT $\left(\frac{y\dot{y}}{\dot{z}}\right) = \frac{by}{a} = \frac{b}{a} \times$ AC: Therefore, AC and AT being in a constant Ratio, the Angle CAT must also be invariable. Which is a known Property of the logarithmic Spiral.

EXAMPLE II.

62. Let BAA be the Spiral of *Archimedes*; whose Nature is such that the Part EA of the generating Ordinate, intercepted by the Spiral and a Circle BED described about the same Center C, is always in a constant Ratio to the corresponding Arch BE of that Circle.



Suppose An perpendicular to AC, &c.

Put BC $= c$, CA $= y$, and let the given Ratio of AE to BE, be that of b to c: Then $b : c :: y - c$ (AE) : $\frac{cy - cc}{b} =$ BE: whose Fluxion therefore is $= \frac{c\dot{y}}{b}$. Now

if

if the Right-line CEA*a* be ſuppoſed to revolve about the Center C, the angular Celerity of the generating Point A, in the perpendicular Direction A*n*, will be to that of E as AC to EC; therefore as the latter of theſe Celerities is expreſſed by $\frac{c\dot{y}}{b}$*, the former will be expreſſed by $\frac{y}{c} \times \frac{c\dot{y}}{b}$, or $\frac{y\dot{y}}{b}$: Which is to $(\dot{y})$ the Celerity of A, in the Direction A*a*, as $\frac{y}{b}$ to Unity, or as *y* to *b*. Therefore CT and AT are in the ſame Ratio, (by *Art.* 35) and conſequently AC : CT :: $\sqrt{yy+bb}$: *y*; and AC : AT :: $\sqrt{yy+bb}$: *b*; whence CT and AT are given equal to $\frac{y^2}{\sqrt{yy+bb}}$, and $\frac{by}{\sqrt{yy+bb}}$ reſpectively. From either of which (the Tangent AT) may be drawn by *Art.* 60. And, in the ſame manner may the Poſition of the Tangent of any other Spiral be determined.

*Art. 5.

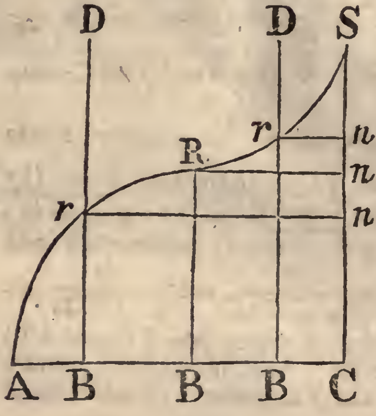
SECTION IV.

Of the Uſe of Fluxions in determining the Points of Retrogreſſion, or contrary Flexure in Curves.

63. WHEN a Curve ARS is, in one Part AR concave, and in the other Part RS convex, towards its Axis AC, the Point R limiting the two Parts is called a Point of Retrogreſſion, or contrary Flexure. The manner of determining which will appear from the following

ILLUSTRATION.

Suppose a Right-line BD to be carried along uniformly, parallel to itself, from A towards C; and let the Point r so move in that Line, at the same time, as to trace out, or describe, the given Curve-line ARS.



Then *(by Art.* 48.) while the Celerity of the Point r, in the Line BD, decreases, the Curve will be concave to its Axis AC; but when it increases, convex to the same: Therefore, as any Quantity is a *Minimum* at the End of its Decrease and the Beginning of its Increase *, it follows that the said Celerity, at the Point of Inflexion R, must be a *Minimum:* Whence, if the Fluxion of the Ordinate Br, expressing that Celerity †, be (as usual) denoted by $\dot{y}$; then will $\ddot{y}$ (the Fluxion of $\dot{y}$) be equal to Nothing in that Circumstance ‡.

*Art.22.
†Art. 5.
‡Art.22.

So far relates to Curves which are, in the former Part concave, and in the latter convex, to their Axes: But if (on the contrary) the Celerity of r first increases, and then decreases, that Celerity, at the required Point, between the Increase and Decrease, will be a *Maximum*; and therefore its Fluxion (or $\ddot{y}$) is *likewise* equal to Nothing in this Case §.

§Art.22.

Furthermore, if CS (perpendicular to AC) be now considered as an Axis, and the Abscissa Sn (or its Complement B$r = y$) be supposed to flow uniformly, (as AB was supposed before); then, by the same Argument, the second Fluxion $(-\ddot{x})$ of the Ordinate nr

(or

(or its Complement AB $= x$) will be equal to Nothing. Hence it is evident that, at the Point of contrary Flexure, the second Fluxion of the Ordinate will become equal to Nothing, if the Abſciſſa be made to flow uniformly; and *vice verſâ.*

EXAMPLE I.

64. Let the Nature of the Curve ARS (*ſee the preceding Figure*) be defined by the Equation $ay = a^{\frac{3}{2}}x^{\frac{1}{2}} + xx$ (the Abſciſſa AB and the Ordinate Br being, as uſual, repreſented by x and y reſpectively). Then $\dot{y}$, expreſſing the Celerity of the Point r, in the Line BD, will be equal to $\frac{\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}}\dot{x} + 2x\dot{x}}{a}$: Whoſe Fluxion, or that of $\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}} + 2x$ (becauſe a and $\dot{x}$ are conſtant) muſt be equal to Nothing *; that is, $-\frac{1}{4}a^{\frac{3}{2}}x^{-\frac{3}{2}}\dot{x} + 2\dot{x}$ $= 0$: Whence $a^{\frac{3}{2}}x^{-\frac{3}{2}} = 8$, $a^{\frac{3}{2}} = 8x^{\frac{3}{2}}$, $64x^3 = a^3$, and $x = \frac{1}{4}a = $ AB; therefore BR $\left(= \frac{a^{\frac{3}{2}}x^{\frac{1}{2}} + xx}{a}\right) = \frac{9}{16}a$: From which the Poſition of the Point R is given. *Art. 63.

EXAMPLE II.

65. Let the Nature of the propoſed Curve be defined by the Equation $ayy - aax - x^3 = 0$.

Then, by taking the firſt and ſecond Fluxions thereof (ſuppoſing $\dot{x}$ conſtant) we ſhall alſo have $2ay\dot{y} - aa\dot{x} - 3x^2\dot{x} = 0$, and $2a\dot{y}^2 + 2ay\ddot{y} - 6x\dot{x}\dot{x} = 0$; whereof the latter, when $\ddot{y}$ is $= 0$, becomes $2a\dot{y}^2 - 6x\dot{x}^2 = 0$, and therefore $\dot{y}^2 = \frac{3x\dot{x}^2}{a}$: But, by the former $\dot{y} = \frac{a^2\dot{x} + 3x^2\dot{x}}{2ay}$; whence $\frac{3x\dot{x}^2}{a} = \frac{\overline{a^2\dot{x} + 3x\dot{x}}|^2}{\overline{2ay}|^2}$, and conſequently $12axy^2$

 =

$=\overline{a^2+3x^2}|^2$; but, by the given Equation, $12axy^2 = 12a^2x^2+12x^4$, therefore $12a^2x^2+12x^4=\overline{a^2+3x^2}|^2$, or $3x^4+6a^2x^2-a^4=0$: Whence x will be found $= a\sqrt{:\sqrt{\tfrac{4}{3}}-1}$.

Otherwise.

Since $ay^2=a^2x+x^3$, we have $y=\dfrac{\overline{a^2x+x^3}|^{\frac{1}{2}}}{\sqrt{a}}$, and therefore $\dot{y} = \dfrac{\overline{\tfrac{1}{2}a^2\dot{x}+\tfrac{3}{2}x^2\dot{x}}\times\overline{a^2x+x^3}|^{-\frac{1}{2}}}{\sqrt{a}}$: Whose Fluxion, or that of $\overline{a^2+3x^2}\times\overline{a^2x+x^3}|^{-\frac{1}{2}}$ (because $\dot{x}$ is constant) being put $=0$, we get $6x\times\overline{a^2x+x^3}|^{-\frac{1}{2}}+\overline{a^2+3x^2}\times\overline{-\tfrac{1}{2}a^2-\tfrac{3}{2}x^2}\times\overline{a^2x+x^3}|^{-\frac{3}{2}}=0$, or $6x\times\overline{a^2x+x^3}+\overline{a^2+3x^2}\times-\dfrac{\overline{a^2+3x^2}}{2}$: Whence $3x^4+6a^2x^2-a^4=0$, and $x=a\sqrt{:\sqrt{\tfrac{4}{3}}-1}$, the same as before.

EXAMPLE III.

66. Let the proposed Curve be the Conchoid of *Nicomedes*, whereof the Equation is $x^2y^2=\overline{a+y}|^2\times$
Art. 57. $\overline{b^2-y^2}$ *, or $x^2=\dfrac{\overline{a+y}|^2\times\overline{b^2-y^2}}{y^2}$.

Here

Here we have $x\dot{x} = \dfrac{\overline{\dot{y}\times\overline{a+y}+\overline{b^2-y^2}-y\dot{y}\times\overline{a+y}|^2\times y^2}}{y^4}$

$\dfrac{-y\dot{y}\times\overline{a+y}|^2\times\overline{b^2-y^2}}{y^4} = -\dfrac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y} =$

$\overline{\dfrac{-a^2b^2}{y^3}-\dfrac{ab^2}{y^2}-a-y}\times\dot{y}$: Whence, making $\dot{y}$ invariable, we alſo have $\dot{x}^2+x\ddot{x} = \overline{\dfrac{3a^2b^2}{y^4}+\dfrac{2ab^2}{y^3}-1}\times\dot{y}^2$: Which, becauſe $\ddot{x}$ is $=0$ *, will be $\dot{x}^2 = \overline{\dfrac{3a^2b^2}{y^4}+\dfrac{2ab^4}{y^3}-1}$ *Art. 63.
$\times\dot{y}^2 = \dfrac{3a^2b^2+2ab^2y-y^4}{y^4}\times\dot{y}^2$. But ſince, by the former Equation, $x\dot{x} = -\dfrac{\overline{a+y}\times\overline{ab^2+y^3}}{y^3}\times\dot{y}$, we likewiſe get $\dot{x}^2 = \dfrac{\overline{a+y}|^2\times\overline{ab^2+y^3}|^2}{x^2y^6}\times\dot{y}^2$, and conſequently $\overline{3a^2b^2+2ab^2y-y^4}\times x^2y^2 = \overline{a+y}|^2\times\overline{ab^2+y^3}|^2$: But, by the Equation of the Curve x^2y^2 is $=\overline{a+y}|^2\times\overline{b^2-y^2}$; therefore $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{a+y}|^2\times\overline{b^2-y^2} = \overline{a+y}|^2\times\overline{ab^2+y^3}|^2$, and $\overline{3a^2b^2+2ab^2y-y^4}\times\overline{b^2-y^2} = \overline{ab^2+y^3}|^2$; whence $y^4+4ay^3+3a^2y^2-2ab^2y-2a^2b^2=0$; which divided by $y+a$, gives $y^3+3ay^2-2ab^2=0$; from whence y may be determined. But if $b=a$, the Equation will become more ſimple by dividing again by $y+a$; in which Caſe we get $y^2+2ay-2a^2=0$, and conſequently $y=a\sqrt{3}-a$.

EXAMPLE IV.

67. Let $a^4y = 180a^3x^2-110a^2x^3+30ax^4-3x^5$.

Then will $a^4\dot{y} = 360a^3x\dot{x}-330a^2x^2\dot{x}+120ax^3\dot{x}-15x^4\dot{x}$;

And $a^4\ddot{y} = 36ca^3\dot{x}^2 - 66ca^3x\dot{x}^2 + 36cax^2\dot{x}^2 - 6cx^3\dot{x}^2$

*Art. 63. Therefore, $6a^3 - 11a^2x + 6ax^2 - x^3 = 0$ *:

Which being divisible by any one of the three Quantities $a-x$, $2a-x$, or $3a-x$, the Root x must therefore have three Values, a, $2a$, and $3a$, and consequently the Curve, defined by the given Equation, as many Points of contrary Flexure.

But, if you would know whether the Part of the Curve lying between any two adjacent Points, thus found, be convex or concave towards the Axis; see whether the Value of the Expression for the second Fluxion of the Ordinate, between the two corresponding Roots, be positive or negative: For, in the former Case, the Curve is convex, and in the latter concave †, (provided the *whole* Curve lies on the same Side the Axis). Thus, in the Example before us; because the second Fluxion of the Ordinate is always as $6a^3 - 11aax + 6axx - x^3$ $(= \overline{a-x} \times \overline{2a-x} \times \overline{3a-x})$ and it appears that the Value of this Expression, while x is less than the first Root a, will be positive; the Curve, therefore, at the Beginning, will be convex to its Axis: But when x becomes greater than a, the said Expression being negative, the Curve will then be concave, and so continue 'till x is equal to the second Root $2a$; after which the Fluxion again becoming affirmative, the Curve will accordingly be convex till $x = 3a$; beyond which Limit the Curvature continually tends the same Way.

† Art. 5 and 48.

But it will be proper to observe, that there are Cases where the second Fluxion of the Ordinate may become equal to Nothing, without either changing its Value from positive to negative, or the contrary, (similar to those already taken Notice of in *Sect.* II. *p.* 45 and 46.) which Cases always happen when the Equation admits of an even Number of equal Roots: And then the Point found as above is not a Point of Inflexion; because the Curvature on either Side of it tends the same Way.

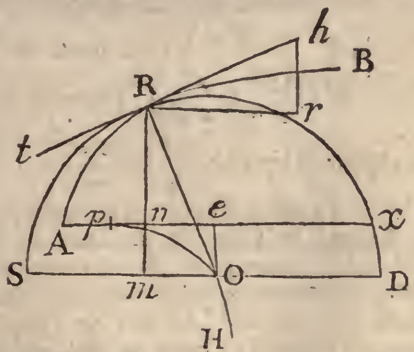
SECTION V.

The Uſe of Fluxions in determining the Radii of Curvature, and the Evolutes of Curves.

68. A Curve *p*OH is ſaid to be the Evolute of another Curve ARB, when it is of ſuch a Nature, that a Thread ROH, coinciding therewith (or wrapped upon the ſame) being unwound or diſengaged from it, by a Power acting at the End R, ſhall, by that End (the Thread continuing tight) deſcribe the given Curve ARB.

ILLUSTRATION.

From the Point O, where the Right-line RO (called the Radius of Curvature) touches the Evolute *p*OH,



let the Semi-circle SRD be deſcribed; which Semicircle, having the ſame Radius with the given Curve, at R, will conſequently have the ſame Degree of Curvature.——But the Curvature in two Curves is the ſame, when, the Fluxions of their Abſciſſas being the ſame, both the Firſt, and Second Fluxions of their

 cor-

corresponding Ordinates Rn and Rm are respectively equal to each other: For, the First Fluxions being equal, the two Curves will have, at the common Point
*Art.48. R, one and the same Tangent tRh *: And, if the Second Fluxions be likewise equal, the Curvature, or Deflection from that Tangent, will also be the same in both; because these last express the Increase or Decrease
†Art.19. of Motion in the Direction of the Ordinate †, upon
‡Art.48. which the Curvature intirely depends ‡.

This being premised, let the Abscissa Sm of the Semicircle (considered as variable) be put $= w$, its Ordinate R$m = v$, R$r = \dot{w}$, $rh = \dot{v}$, and R$h = \dot{z}$: Then, Rh being
‖Art.48. a Tangent to the Circle at R ‖, the Triangles Rhr and ROm will be equiangular, and therefore $\dot{w}$ (Rr) : $\dot{z}$ (RA) :: v (Rm) : RO $= \frac{v\dot{z}}{\dot{w}}$; which, because the Radius of every Circle is a constant Quantity, must be invariable, and consequently its Fluxion $\frac{\dot{v}\dot{z} + v\ddot{z}}{\dot{w}} = 0$: Whence v is found $= \frac{\dot{v}\dot{z}}{-\ddot{z}} = \frac{\dot{z}^2}{-\ddot{v}}$ (because, $\dot{w}$ being constant, and $\dot{w}^2 + \dot{v}^2 = \dot{z}^2$, we have, in Fluxions $2\dot{v}\ddot{v} = 2\dot{z}\ddot{z}$, and so $\frac{\dot{v}\dot{z}}{-\ddot{z}} = \frac{\dot{z}^2}{-\ddot{v}}$). Therefore since v is $= \frac{\dot{z}^2}{-\ddot{v}}$, we also get SO$=$RO $\left(\frac{v\dot{z}}{\dot{w}}\right) = \frac{\dot{z}^3}{-\dot{w}\ddot{v}} = \frac{\overline{\dot{v}^2 + \dot{w}^2}|^{\frac{3}{2}}}{-\dot{w}\ddot{v}}$: Which last is a general Expression for the Radius of any Circle, whatever, in Terms of the Fluxions of its Abscissa *(w)* and Ordinate *(v)*. But, by what is premised above, these Fluxions are respectively equal to those of the Abscissa An *(x)* and Ordinate Rn *(y)* of the proposed Curve ARB. Therefore, by writing $\dot{x}$, $\dot{y}$, and $\ddot{y}$, instead of $\dot{w}$, $\dot{v}$, and $\ddot{v}$, we have $\frac{\overline{\dot{y}^2 + \dot{x}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}} \left(= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right)$ for the general Value of the Radius of Curvature, RO.

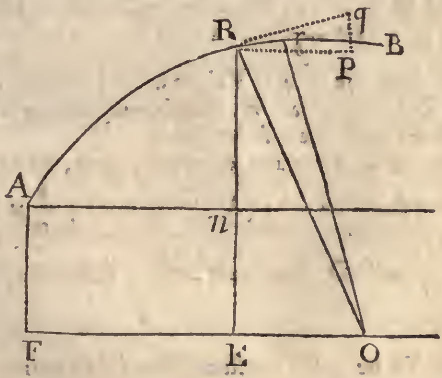
The same otherwise.

If the Radius of the Circle be put $= R$, and every Thing else be supposed as above; then (by the Property of the Circle) we shall have v^2 $(Rm^2) = 2\,Rw - w^2$ $(Sm \times Dm)$: Whence, in Fluxions (making $\dot{w}$ constant) we get $2v\dot{v} = 2R\dot{w} - 2w\dot{w}$, and $2\dot{v}^2 + 2v\ddot{v} = -2\dot{w}^2$: From the last of which Equations v is found $= \frac{\dot{v}^2 + \dot{w}^2}{-\ddot{v}}$ $= \frac{\dot{z}^2}{-\ddot{v}}$; and consequently RO $\left(\frac{v\dot{z}}{\dot{w}}\right) = \frac{\dot{z}^3}{-\dot{w}\ddot{v}} = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *the same as before.*

Otherwise without the Circle.

Let RO and rO be two Rays perpendicular to the Curve, indefinitely near to each other; and from their Intersection O, let OF be drawn parallel to An, cutting Rn and AF (parallel to Rn) in E and F.

Therefore, supposing $RE = v$, $An = x$, $Rn = y$, &c. (as before) we shall have, by similar Triangles, as RP



$(\dot{x})$: Pq $(\dot{y})$ :: RE (v) : EO $= \frac{v\dot{y}}{\dot{x}}$; and consequently FO $(An + EO) = x + \frac{v\dot{y}}{\dot{x}}$: Which Value (as well as

that

that of AF) continuing the ſame whether we regard the Radius RO, or the Radius rO, its Fluxion muſt therefore be equal to Nothing; that is, $\dot{x} + \frac{\overline{\dot{v}\dot{y} + v\ddot{y}} \times \dot{x} - v\dot{y}\ddot{x}}{\dot{x}^2}$ $= 0$; whence $v = \frac{\dot{x}^3 + \dot{x}\dot{v}\dot{y}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, and conſequently RO $\left(\frac{v\dot{z}}{\dot{x}}\right) = \frac{\dot{x}^2\dot{z} + \dot{v}\dot{y}\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{x}^2\dot{z} + \dot{y}^2\dot{z}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$: Which, if $\dot{x}$ is ſuppoſed conſtant, or $\ddot{x} = 0$, will become $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, *as above.*

But if $\dot{y}$ be ſuppoſed conſtant, it will be $\frac{\dot{z}^3}{\dot{x}\ddot{y}}$. And, if $\dot{z}$ be conſtant, it will then be $\frac{\dot{z}\dot{y}}{\ddot{x}}$: For, ſince $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$, by taking the Fluxion thereof, we have $2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} = 0$; whence $\ddot{y} = -\frac{\dot{x}\ddot{x}}{\dot{y}}$; and therefore RO $\left(= \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}\right) = \frac{\dot{z}^3}{\dot{y}\ddot{x} + \frac{\dot{x}^2\ddot{x}}{\dot{y}}} = \frac{\dot{y}\dot{z}^3}{\overline{\dot{y}^2 + \dot{x}^2} \times \ddot{x}} = \frac{\dot{y}\dot{z}}{\ddot{x}}$, *as before.*

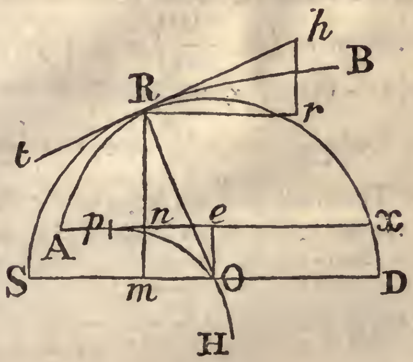
Now from the ſeveral Values of the Radius of Curvature RO, found above, the correſponding Values of Ae and eO will likewiſe be given.

Thus, if $\dot{x}$ be made conſtant; then, RO being $= \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, we ſhall have A$e$ (An+Om=An+ $\frac{\dot{y}}{\dot{z}}$ × RO) $= x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}$, and eO (Rm—nR$= \frac{\dot{x}}{\dot{z}}$ ×RO—Rn) $= \frac{\dot{z}^2}{-\ddot{y}} - y$.

But, if $\dot{y}$ be made conſtant, then, RO being $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$, we ſhall have AE $= x + \frac{\dot{z}^2}{\ddot{x}}$, and eO $= \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x}} - y$.

Laſtly,

Laſtly, if $\dot{z}$ be ſuppoſed conſtant; then RO being $= \frac{\dot{y}\dot{z}}{\ddot{x}}$, we ſhall have $Ae = x + \frac{\dot{y}^2}{\ddot{x}}$, and $eO = \frac{\dot{x}\dot{y}}{\ddot{x}} - y$.



Which ſeveral Expreſſions will ſerve as ſo many general Theorems for determining the Quantity of Curvature, and the Evolutes of given Curves: But, before we proceed to Examples, it will be proper to obſerve, that the Right-line Ap, denoting the Radius of Curvature at the Vertex A (to be found by making x, or y, $= 0$) muſt always be ſubtracted from RO and Ae, to have the true Length of the Arch pO, and its correſponding Abſciſſa pe.

EXAMPLE I.

69. Let the given Curve ARB be the common Parabola, whoſe Equation is $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$: Then will $\dot{y} = \frac{1}{2}a^{\frac{1}{2}}\dot{x}x^{-\frac{1}{2}}$ $= \frac{a^{\frac{1}{2}}\dot{x}}{2x^{\frac{1}{2}}}$, and (making $\dot{x}$ conſtant) $\ddot{y} = -\frac{1}{2} \times \frac{1}{2}a^{\frac{1}{2}}\dot{x}^2x^{-\frac{3}{2}}$ $= \frac{-a^{\frac{1}{2}}\dot{x}^2}{4x^{\frac{3}{2}}}$: Whence $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2}) = \frac{\dot{x}}{2}\sqrt{\frac{4x+a}{x}}$,

and

and the Radius of Curvature RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}}$: Which at the Vertex A, where $x=0$, will be $= \frac{1}{2}a =$ Ap. Moreover Ae $\left(x + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}}\right) = \frac{1}{2}a + 3x$, and therefore pe (Ae—Ap) $= 3x$, the Abſciſſa of the Evolute: Likewiſe Oe $\left(\frac{\dot{z}^2}{-\ddot{y}} - y\right) = \frac{4x^{\frac{3}{2}}}{\sqrt{a}}$ the Ordinate of the Evolute. Therefore, $\overline{Oe}|^2 \times a$ being in a conſtant Ratio to $\overline{pe}|^3$, namely as 16 to 27, the Curve is, in this Caſe, the Semi-cubical Parabola: Whoſe Arch pO (RO—Ap) is alſo given $= \frac{\overline{a+4x}|^{\frac{3}{2}}}{2\sqrt{a}} - \frac{1}{2}a$.

EXAMPLE II.

70. Let the Curve ARB denote a Parabola of any other Kind: Then, becauſe $y = ax^n$ is an Equation to all Kinds of Parabolas, we have $\dot{y} = nax^{n-1}\dot{x}$ and $\ddot{y} = n \times \overline{n-1} \times ax^{n-2}\dot{x}^2$: Therefore $\dot{z}$ $(\sqrt{\dot{x}^2+\dot{y}^2}) =$ $\dot{x}\sqrt{1+n^2a^2x^{2n-2}}$, RO $\left(\frac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \frac{\overline{1+n^2a^2x^{2n-2}}|^{\frac{3}{2}}}{-n\times\overline{n-1}\times ax^{n-2}}$, A$e$ $\left(x + \frac{\dot{y}\dot{z}^2}{-\ddot{y}}\right) = x - \frac{x+n^2a^2x^{2n-1}}{n-1}$, O$e$ $\left(\frac{\dot{z}^2}{-\ddot{y}} - y\right)$ $= \frac{1 + \overline{2n-1} \times na^2x^{2n-2}}{-\overline{n-1} \times nax^{n-2}}$, and A$p$ $= -\frac{n^2a^2 0^{2n-1}}{n-1}$: Which, if $n = \frac{1}{2}$, will become $= \frac{a^2}{2}$; but, if n be greater than $\frac{1}{2}$, it will be $= 0$; and, if n be leſs than $\frac{1}{2}$, it

it will be infinite: Whence it appears, that the Radius of Curvature at the Vertex will be a finite Quantity in Curves whoſe firſt (or leaſt) Ordinates are in the Subduplicate Ratio of their Abſciſſas, and in all other Caſes, either Nothing, or Infinite.

EXAMPLE III.

71. Suppoſe the given Curve to be an Ellipſis; whoſe Equation (putting a and c for the two principal Diameters) is $a^2y^2 = c^2 \times \overline{ax - x^2}$.

Here, by taking the Firſt and Second Fluxions of the given Equation, we have $2a^2y\dot{y} = c^2\dot{x} \times \overline{a - 2x}$, and $2a^2\dot{y}^2 + 2a^2y\ddot{y} = c^2\dot{x} \times -2\dot{x} = -2c^2\dot{x}^2$; whence $\dot{y} = \dfrac{c^2\dot{x} \times \overline{a-2x}}{2a^2y}$, and $-\ddot{y} = \dfrac{a^2\dot{y}^2 + c^2\dot{x}^2}{a^2y}$: Which, by ſubſtituting the Values of y and $\dot{y}$, will become $\dot{y} = \dfrac{c\dot{x} \times \overline{a-2x}}{2a\sqrt{ax-x^2}}$, and $-\ddot{y} = \dfrac{a^2c^2\dot{x}^2 \times \overline{a-2x}|^2}{4a^2 \times \overline{ax-xx} \times ac\sqrt{ax-x^2}}$ $+ \dfrac{c\dot{x}^2}{a\sqrt{ax-x^2}} = \dfrac{c\dot{x}^2}{a} \times \dfrac{\overline{a-2x}|^2 + 4 \times \overline{ax-x^2}}{4 \times \overline{ax-x^2}\sqrt{ax-x^2}} = \dfrac{ca\dot{x}^2}{4 \times \overline{ax-x^2}|^{\frac{3}{2}}}$:

Therefore $\dot{z}\ (\sqrt{\dot{y}^2 + \dot{x}^2}) = \sqrt{\dfrac{c^2\dot{x}^2 \times \overline{a-2x}|^2}{4a^2 \times \overline{ax-x^2}} + \dot{x}^2}$ $= \dfrac{\dot{x}}{2a}\sqrt{\dfrac{c^2a^2 + \overline{a^2-c^2} \times \overline{4ax-4x^2}}{ax-x^2}}$, and the Radius of Curvature $\left(\dfrac{\dot{z}^3}{-\dot{x}\ddot{y}}\right) = \dfrac{\overline{a^2c^2 + \overline{a^2-c^2} \times \overline{4ax-4x^2}}|^{\frac{3}{2}}}{2a^4c}$: Which when the Diameters a and c are equal, or the Ellipſis degenerates to a Circle, will be every where equal to $\dfrac{\overline{a^2c^2}|^{\frac{3}{2}}}{2a^4c}$, or $\frac{1}{2}a$; agreeable to the Definition of a Circle.

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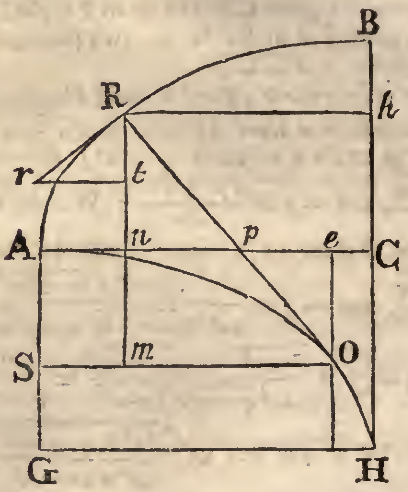
EXAMPLE IV.

72. *To find the Radius of Curvature, and the Evolute of the common Cycloid.*

Let ARB be the given Curve, and AOH its Evolute; alſo let Rh and OS be parallel to AC, and eO and Rm



perpendicular to AC; and put ARB ($=2$BC) $=a$, AR $=z$, A$n=x$, and R$n=y$: Then BR $=a-z$, Bh $=\frac{1}{2}a-y$; and, by the Property of the Curve, a^2 (AB²) : $\overline{a-z}|^2$ (BR²) : : $\frac{1}{2}a$ (BC) : $\frac{1}{2}a-y$ (Bh) whence $y=\frac{2az-z^2}{2a}$; therefore $\dot{y}=\frac{a\dot{z}-z\dot{z}}{a}$, $\dot{z}^2-\dot{y}^2$ $(\dot{x}^2)=\frac{\overline{2az-z^2}\times\dot{z}^2}{a^2}$, and $\dot{x}=\frac{\dot{z}\sqrt{2az-z^2}}{a}$. Whence (making $\dot{z}$ conſtant) $\ddot{x}=\frac{\dot{z}^2\times\overline{a-z}}{a\sqrt{2az-z^2}}$; from which

we

we get RO, or AO $\left(=\frac{\dot{y}\dot{z}}{\ddot{x}}{}^{*}\right)=\sqrt{2az-z^2}$, and *e*O, *Art. 68.

or AS $(=\frac{\dot{y}\dot{x}}{\ddot{x}}-y)=\frac{2az-z^2}{2a}$; which, when $z=a$, or ROH coincides with BH, become AOH (BH) $=a$, and CH (AG) $=\frac{1}{2}a$. Hence, becauſe it appears, that, $\overline{AH}|^2\ (a^2)$: AO^2 $(2az-z^2)$ :: AG $(\frac{1}{2}a)$: AS $\left(\frac{2az-z^2}{2a}\right)$ it follows that the Evolute AOH is alſo a Cycloid equal, and ſimilar, to the Involute ARB.

If the Evolute had been given, or ſuppoſed, a Cycloid, and the Involute required, the Proceſs would have been, more ſimple, as follows,

Let AH (2AG) $=a$, AO $(=\text{RO})=z$, AS $=x$, SO $=y$, BR $=v$, B*b* $=w$, R*r* $=\dot{v}$, R*t* $=\dot{w}$, &c. Then it will be †, †Art. 48.

$$\dot{y}:\dot{z}\ (::\text{O}m:\text{OR})::\text{R}t\ (\dot{w}):\text{R}r=\frac{\dot{w}\dot{z}}{\dot{y}}.$$

$$\dot{z}:\dot{y}::z\ (\text{RO}):\text{O}m=\frac{z\dot{y}}{\dot{z}},$$

$$\dot{z}:\dot{x}::z\ (\text{RO}):\text{R}m=\frac{z\dot{x}}{\dot{z}},$$

Whence we have $\dot{v}=\frac{\dot{w}\dot{z}}{\dot{y}}$, R*n* (R*m* — AS) $=\frac{z\dot{x}}{\dot{z}}-x$, and A*n* (OS — O*m*) $=y-\frac{z\dot{y}}{\dot{z}}$; which Expreſſions anſwer to any Curve whatever.

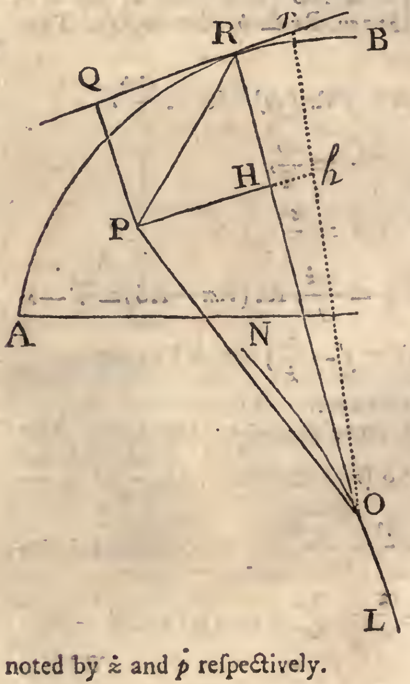
But, in the Caſe above propoſed, $\text{AH}^2\ (a^2)$: AO^2 (z^2) :: AG $(\frac{1}{2}a)$: AS (x); therefore $x=\frac{z^2}{2a}$, $\dot{x}=\frac{z\dot{z}}{a}$, and $\dot{y}\ (\sqrt{\dot{z}^2-\dot{x}^2})=\frac{\dot{z}\sqrt{a^2-z^2}}{a}$; and conſequently R*n* $\left(\frac{z\dot{x}}{\dot{z}}-x\right)=\frac{z^2}{a}-\frac{z^2}{2a}=\frac{z^2}{2a}=\frac{1}{2}a-w$ (or CB — B*b*):

Whence

Whence alſo $w = \frac{a^2 - z^2}{2a}$, and $\dot{v}\left(\frac{\dot{w}z}{y}\right) = \frac{a\dot{w}}{\sqrt{a^2 - z^2}}$ $= \frac{a\dot{w}}{\sqrt{2aw}}$: Therefore it will be $\dot{v} : \dot{w}$ ($: : a : \sqrt{2aw}$) $: : \sqrt{\frac{1}{2}a} : \sqrt{w}$; that is, as $Rr : Rt : : \sqrt{BC} : \sqrt{Bb}$: Which is a known Property of the Cycloid.

Hitherto regard has been had to Curves where the Ordinates are parallel to each other: But when the Ordinates are all referred to a given Point, as in Spirals, &c. other Theorems will become neceſſary; and may be thus derived.

73. Let ARB be the propoſed Curve, P the Point, or Center, to which its Ordinates are referred, NOL the Evolute, and RO the Ray of Curvature at R: Moreover, let PH be perpendicular to RO; and, ſuppoſing the Ordinate PR (y) to become variable by the Motion of the Point R along the Curve, let the Fluxions of AR and PH (p), expreſſing the Celerities of the Points R and H in Directions perpendicular to RO*, be denoted by $\dot{z}$ and $\dot{p}$ reſpectively.



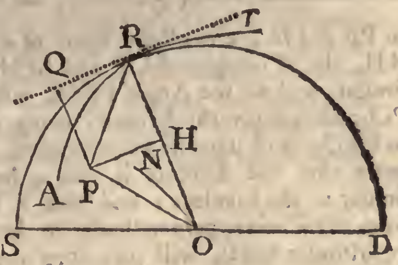
*Art. 5.

Therefore,

Therefore, the Celerities, of any two Points, in a Right-line revolving about a Center, being as the Distances from that Center, it follows that $\dot{p} : \dot{z} :: OH : OR$; whence by Division (putting $RH = v$) we have $\dot{z} - \dot{p} : \dot{z} :: v\ (RH) : RO = \frac{v\dot{z}}{\dot{z} - \dot{p}} = \frac{vp\dot{z}}{p\dot{z} - p\dot{p}}$: But $p\dot{z}$ $= y\dot{y}$ (*by Art.* 60.) and therefore $RO = \frac{vy\dot{y}}{y\dot{y} - p\dot{p}}$; which, because $y^2 - p^2$ is $= v^2$ (and therefore $y\dot{y} - p\dot{p} = v\dot{v}$) will also be $= \frac{vy\dot{y}}{v\dot{v}} = \frac{y\dot{y}}{\dot{v}}$.

The same otherwise.

Let SRD be a Circle described about the Point O, as a Center, and suppose the Distance PR to be variable by the Motion of the Point R along the Arch of the Circle (instead of the Curve): Then, drawing OP, and putting $OR = r$, $PR = y$, &c. as before, we shall get OP^2 $(OR^2 + PR^2 - 2OR \times RH) = r^2 + y^2 - 2rv$; which (as well as r) being a constant Quantity, its Fluxion $2y\dot{y} - 2r\dot{v}$ must be equal to nothing; and therefore $r = \frac{y\dot{y}}{\dot{v}}$, the very same as above. Nor is it of any Consequence whether $\dot{y}$ and $\dot{v}$ be here looked upon as respecting the Circle, or the Curve; since, at R, they must be the same in both Cases; otherwise the Curvature could not be the same *. Now from the Value of RO thus found, which (corrected, when necessary) will also express the Length of the Arch NO of the Evolute ‡, the Ordinate PO and the Tangent OH of the Evolute



* Art. 68.

‡ Art. 68.

may be easily deduced. For OH (RO—RH) $= \frac{y\dot{y}}{\dot{v}} - v = \frac{p\dot{p}}{\dot{v}}$, and PO ($= \sqrt{OH^2 + PH^2} = \frac{p\sqrt{\dot{p}^2 + \dot{v}^2}}{\dot{v}}$; whence the Nature of the Evolute is known.

EXAMPLE I.

74. Let the given Curve AR be the logarithmic Spiral, whose Nature is such, that the Angle PRQ (or RPH) which the Ordinate makes with the Curve is every where the same.

Then (denoting the Sine of that Angle by b, and the Radius of the Tables by a) we have RH $(v) = \frac{b\dot{y}}{a}$, and therefore RO $\left(\frac{y\dot{y}}{\dot{v}}\right) = \frac{ay\dot{y}}{b\dot{y}} = \frac{ay}{b}$; which being to PR (y) in the constant Ratio of a to b, or of PR to RH, the Triangles ROP and RPH must therefore be similar, and so the Angle POH, which the Ordinate PO makes with the Evolute, being every where equal to PRQ, will likewise be invariable. Whence it appears that the Evolute is also a logarithmic Spiral, similar to the Involute; and that a Right-line drawn from the Center, perpendicular to the Ordinate, of any logarithmic Spiral, will pass thro' the Center of Curvature.

EXAMPLE II.

75. Let the Curve proposed be the Spiral of *Archimedes*; where we have $p = \frac{by}{\sqrt{y^2+b^2}}$, and $v = \frac{y^2}{\sqrt{y^2+b^2}}$ (see *Art.* 62.) Therefore $\dot{v} = 2y\dot{y} \times \overline{y^2+b^2}|^{-\frac{1}{2}} + yy \times$ $-\frac{1}{2}$

$-\frac{1}{2} \times 2y\dot{y} \times \overline{y^2+b^2}|^{-\frac{3}{2}} \pm \frac{2y\dot{y}}{\overline{y^2+b^2}|^{\frac{1}{2}}} - \frac{y^3\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}} =$
$\frac{2y\dot{y} \times \overline{y^2+b^2} - y^3\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}} = \frac{y^3\dot{y}+2b^2y\dot{y}}{\overline{y^2+b^2}|^{\frac{3}{2}}}$; whence the Radius of
* Curvature $\frac{y\dot{y}}{\dot{v}}$ is here $= \frac{\overline{yy+bb}|^{\frac{3}{2}}}{y^2+2b^2}$; which being $= \frac{b}{2}$, *Art. 73.
when $y = 0$, the Arch of the Evolute †, reckoned from †Art. 68
the Vertex, is therefore $= \frac{\overline{yy+bb}|^{\frac{3}{2}}}{y^2+2b^2} - \frac{b}{2}$.

After the very ſame Manner you may proceed in other Caſes: But if the Value of $\dot{v}$ (or $\frac{y\dot{y}}{\dot{v}}$) changes, in any Caſe, from Poſitive to Negative, the Radius of Curvature (RO) after becoming infinite, will fall on the other Side of the Tangent, and the correſponding Point of the Curve, when $\dot{v} = 0$, will be a Point of *Contrary-Flexure.* Whence it may be obſerved that the Point of Inflection, in a Curve whoſe Ordinates are referred to a Center, may be found by making the Fluxion of the Perpendicular, drawn from the Center to the Tangent, equal to Nothing, which Caſe is not taken Notice of in the preceding Section.

SECTION VI.

Of the Inverse Method, or the Manner of determining the Fluents of given Fluxions.

76. IN the *Inverſe Method*, which teaches the Manner of finding the reſpective flowing Quantities of given Fluxions, there will be no great Difficulty in conceiving the Reaſons, if what is already delivered in *Sect.* 1. *on the direct Method*, has been duly conſidered: Though the Difficulties that occur in this Part, upon another Account, are indeed vaſtly ſuperior.

It is an eaſy Matter, or not impoſſible at moſt, to find the Fluxion of any flowing Quantity whatever; but in the *Inverſe Method* the Caſe is quite different: For, as there is no Method for deducing the Fluent from the Fluxion *a priori*, by a direct Inveſtigation, ſo it is impoſſible to lay down Rules for any other Forms of Fluxions, than thoſe particular ones which we know, from the direct Method, belong to ſuch and ſuch kinds of flowing Quantities. Thus, for Example, the Fluent of $2x\dot{x}$ is known to be x^2, becauſe it is found in *Art.* 6. and 14. that $2x\dot{x}$ is the Fluxion of x^2: But the Fluent of $y\dot{x}$ is unknown, ſince no Expreſſion has been diſcovered that produces $y\dot{x}$ for its Fluxion.

77. Now, as the principal Rule in the *direct Method* is that for the Fluxions of Powers, derived in *Art.* 8. (where it is proved that the Fluxion of x^n is, *univerſally*, expreſſed by $nx^{n-1}\dot{x}$); ſo the moſt general Rule, that can be given in the *Inverſe Method*, muſt be that ariſing from the converſe thereof; which *ſhews how to aſſign the Fluent of any Power of a variable Quantity drawn into the Fluxion of the Root*; and which, expreſſed in Words, will be as follows.

Divide by the Fluxion of the Root, add Unity to the Exponent of the Power, and divide by the Exponent ſo increaſed.

For,

For, dividing the Fluxion $n x^{n-1} \dot{x}$ by $\dot{x}$ (the Fluxion of the Root x) it becomes $n x^{n-1}$; and, adding 1 to the Exponent $(n-1)$ we have $n x^{n}$; which, divided by n, gives x^{n}, the true Fluent of $n x^{n-1}$, by *Art.* 8.

Hence (by the same Rule) the

Fluent of $3x^2\dot{x}$ will be $= x^3$;

That of $8x^2\dot{x} = \frac{8x^3}{3}$;

That of $2x^5\dot{x} = \frac{x^6}{3}$

That of $y^{\frac{1}{2}}\dot{y} = \frac{2}{3}y^{\frac{3}{2}}$;

That of $ay^{\frac{5}{3}}\dot{y} = \frac{3ay^{\frac{8}{3}}}{8}$;

That of $y^{\frac{m}{n}}\dot{y} = \frac{y^{\frac{m}{n}+1}}{\frac{m}{n}+1} = \frac{ny^{\frac{m+n}{n}}}{m+n}$;

That of $\frac{a\dot{x}}{x^n}$, or $a\dot{x}x^{-n}$, $= \frac{ax^{1-n}}{1-n}$;

That of $\overline{a+z}|^3 \times \dot{z} = \frac{\overline{a+z}|^4}{4}$;

And that of $\overline{a^m + z^m}|^n \times z^{m-1}\dot{z} = \frac{\overline{a^m + z^m}|^{n+1}}{m \times \overline{n+1}}$:

For *here* the Root, or the Quantity under the general Index n, being $a^m + z^m$, and its Fluxion $= mz^{m-1}\dot{z}$ (*Art.* 14.) we shall, by dividing by the last of these Quantities, have $\frac{\overline{a^m + z^m}|^n}{m}$; whence, increasing the

 Index

Index by Unity, and dividing by $(n+1)$ the Index so increased, there comes out $\frac{\overline{a^m+z^m}^{\,n+1}}{m\times \overline{n+1}}$.

After the very same Manner the Fluents of other Expressions may be deduced, when the Quantity, or Multiplicator, without the Vinculum is either equal, or in a constant Ratio, to the Fluxion of the Quantity, under the Vinculum: As in the Expression $\overline{a+cz^n}^{\,m}\times dz^{n-1}\dot{z}$; where the Number of Dimensions of z under the *Vinculum* (or general Index) being equal to those of z without the *Vinculum* $+\ 1$, the Fluent may therefore be had, as in the preceding Examples; and will come out $\frac{\overline{a+cz^n}^{\,m+1}\times d}{nc\times \overline{m+1}}$: And, that this (or any other Expression derived in like Manner) is the true Fluent will evidently appear, by supposing x equal to $a+cz^n$ the Quantity under the Vinculum; for then (equal Quantities having equal Fluxions) $\dot{x}$ will be

* Art. 8. $=ncz^{n-1}\dot{z}$ *; and consequently $\overline{a+cz^n}^{\,m}\times dz^{n-1}\dot{z}$ $\left(=x^m\times\frac{d\dot{x}}{nc}\right)=\frac{dx^m\dot{x}}{nc}$; whose Fluent is therefore

‡ Art. 77. $\frac{dx^{m+1}}{nc\times \overline{m+1}}$ ‡ $=\frac{d\times \overline{a+cz^n}^{\,m+1}}{nc\times \overline{m+1}}$, *as before.*

78. In assigning the Fluents of given Fluxions there is another Particular that ought to be attended to, not yet taken notice of; and that is, whether the flowing Quantity, found by the common Rule, above delivered, does not require the Addition or Subtraction of some constant Quantity to render it complete. This

indeed can, only, be known from the Nature of the Problem under Confideration; but that fuch an Addition or Subtraction may, in fome Cafes, become neceffary is evident from the Subject itfelf; fince a flowing Quantity increafed, or decreafed, by a conftant Quantity, has ftill the fame Fluxion; and therefore the Fluent of that Fluxion is as properly expreffed by the whole compound Expreffion, as by the variable Part of it, alone: Thus, for Inftance, the Fluent of $n x^{n-1} \dot{x}$ may be either reprefented by x^n or by $x^n \pm a$, becaufe (a being conftant) the Fluxion of $x^n \pm a$, as well as of x^n, is $n x^{n-1} \dot{x}$.

79. Hence it appears that it is the variable Part of a Fluent only which is affignable by the common Method; the conftant Part (when fuch becomes neceffary) being to be afcertained from the particular Nature of the Problem. Now to do this, the beft Way is to confider how much the variable Part of the Fluent, firft found, differs from the Truth, in that particular Circumftance when the required Quantity which the whole Fluent ought to exprefs, is equal to Nothing; then that Difference, added to, or fubtracted from, the faid variable Part, as occafion requires, will give the Fluent truly corrected: For, fince the Difference of two Quantities flowing with the fame Celerity (or having equal Fluxions) is either, Nothing at all, or *conftantly* the *fame*, the Difference in that Circumftance will likewife be the Difference in all other Circumftances: And therefore being added to the leffer Quantity, or fubtracted from the greater, both become equal.

80. To render what is above delivered as familiar as may be, I fhall put down a few Examples; in which the variable Quantities reprefented by x and y are fuppofed to begin their Exiftence together, or to be generated, at the fame time.

 1. Let

1. Let $\dot{y} = a^2x\dot{x}$; then the Fluent, found as uſual, will be $y = \frac{a^2x^2}{2}$; where taking $y = 0$, $\frac{a^2x^2}{2}$ alſo vaniſhes, (becauſe then $x=0$ by Hypotheſis): Therefore the Fluent requires no Correction in this Caſe.

2. Let $\dot{y} = \overline{a+x}|^3 \times \dot{x}$: Here we firſt have $y = \frac{\overline{a+x}|^4}{4}$; but when $y=0$, then $\frac{\overline{a+x}|^4}{4}$ becomes $= \frac{a^4}{4}$ (ſince x by Hypotheſis is then $= 0$:) Therefore $\frac{\overline{a+x}|^4}{4}$ *always* exceeds y by $\frac{a^4}{4}$; and ſo the Fluent properly corrected will be $y = \frac{\overline{a+x}|^4 - a^4}{4} = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$.

But the very ſame Fluent may be otherwiſe found, without needing any Correction: For the given Equation $(\dot{y} = \overline{a+x}|^3 \times \dot{x})$, by expanding $\overline{a+x}|^3$, is tranſformed to $\dot{y} = a^3\dot{x} + 3a^2x\dot{x} + 3ax^2\dot{x} + x^3\dot{x}$; whence $y = a^3x + \frac{3a^2x^2}{2} + ax^3 + \frac{x^4}{4}$; the ſame *as above.*

Hence it appears that the Fluent of an Expreſſion, found according to one Form, may require a very different Correction from the Fluent of the ſame Fluxion found according to another Form.

3. Let $\dot{y} = \overline{a^2 - x^2}|^{\frac{1}{2}} \times x\dot{x}$; then, firſt, $y = -\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$; where taking $y=0$, $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3}$ becomes

=

$= -\frac{a^3}{3}$; therefore $-\frac{\overline{a^2 - x^2}|^{\frac{3}{2}}}{3}$ is too little by $\frac{a^3}{3}$; and so the Fluent corrected will be $y = \frac{a^3}{3} - \frac{\overline{a^2 - x^2}|^{\frac{3}{2}}}{3}$.

4. Let $\dot{y} = \overline{a^m + x^m}|^n \times x^{m-1}\dot{x}$: Here we first have $y = \frac{\overline{a^m + x^m}|^{n+1}}{m \times \overline{n+1}}$; and making $y = 0$, the latter Part of the Equation becomes $\frac{\overline{a^m}|^{n+1}}{m \times \overline{n+1}} = \frac{a^{mn+m}}{m \times \overline{n+1}}$; whence the Equation, or Fluent, truly corrected is $y = \frac{\overline{a^m + x^m}|^{n+1} - a^{mn+m}}{m \times \overline{n+1}}$.

5. Lastly, let $\dot{y} = \overline{a + bx^m + cx^n}|^p \times \overline{mbx^{m-1}\dot{x} + ncx^{n-1}\dot{x}}$; then, in the first Place, we have $y = \frac{\overline{a + bx^m + cx^n}|^{p+1}}{p+1}$; which corrected, as above, becomes $y = \frac{\overline{a + bx^m + cx^n}|^{p+1} - a^{p+1}}{p+1}$.

81. Hitherto x and y are both supposed equal to Nothing at the same time; but that will not always be the Case in the Solution of Problems. Thus, for Instance, though the Sine and Tangent of an Arch are both equal to Nothing when the Arch itself is equal to Nothing, yet the

the Secant is then equal to the Radius: It will be proper therefore to add an Example or two wherein the Value of y is equal to Nothing, when that of x is equal to any given Quantity a.

Let, then, the Equation $\dot{y} = x^2\dot{x}$ be first proposed; whereof the Fluent (first taken) is $y = \frac{x^3}{3}$; but when $y = 0$, then $\frac{x^3}{3} = \frac{a^3}{3}$, by Hypothesis; therefore the Fluent, corrected, is $y = \frac{x^3 - a^3}{3}$.

Again, let the proposed Equation be $\dot{y} = -x^n\dot{x}$; then will $y = -\frac{x^{n+1}}{n+1}$; which corrected becomes $y = \frac{a^{n+1} - x^{n+1}}{n+1}$.

Lastly, let $\dot{y} = \overline{c^3 + bx^2}|^{\frac{1}{2}} \times x\dot{x}$; then, first, $y = \frac{\overline{c^3 + bx^2}|^{\frac{3}{2}}}{3b}$; and, when $y = 0$ and $x = a$, $\frac{\overline{c^3 + bx^2}|^{\frac{3}{2}}}{3b}$ becomes $= \frac{\overline{c^3 + ba^2}|^{\frac{3}{2}}}{3b}$: therefore the Fluent corrected is $y = \frac{\overline{c^3 + bx^2}|^{\frac{3}{2}} - \overline{c^3 + ba^2}|^{\frac{3}{2}}}{3b}$.

82. All the Examples hitherto given relate to such Fluxions as involve one variable Quantity only in each Term, whose Fluents are assignable from the Converse of the first General Rule, *in Section* I. But, besides these, various other Forms of Fluxions may be proposed, involving two or more variable Quantities, whose Fluents may also be found by Help of the other two General Rules delivered in the same Section.

Thus

Thus the Fluent of $y\dot{x}+x\dot{y}$ is expreſſed by xy *; that of $\frac{y\dot{x}-x\dot{y}}{y^2}$ by $\frac{x}{y}$ †; that of $a\dot{x}+x\dot{y}+y\dot{x}$ by $ax+xy$ ‡; and that of $\overline{nx\dot{y}y^{n-1}+y^n\dot{x}-nax^{n-1}\dot{x}} \times \overline{y^n x - ax^n}|^{\frac{p}{m}}$ by $\frac{m \times \overline{y^n x - ax^n}|^{\frac{p+m}{m}}}{p+m}$: For, dividing (in the laſt Caſe) by the Fluxion of the Root $y^n x - ax^n$ *, which (by *Art.* 14 and 15) is $nxy^{n-1}\dot{y}+y^n\dot{x}-nax^{n-1}\dot{x}$, we firſt have $\overline{y^n x - ax^n}|^{\frac{p}{m}}$; whence, adding Unity to the Exponent $\frac{p}{m}$, and dividing by the Exponent ſo increaſed, we get $\frac{\overline{y^n x - ax^n}|^{\frac{p}{m}+1}}{\frac{p}{m}+1} = \frac{m \times \overline{y^n x - ax^n}|^{\frac{p+m}{m}}}{p+m}$ for the true Fluent of the Quantity propoſed. But it ſeldom happens that theſe Kinds of Fluxions which involve two different variable Quantities in one Term, and yet admit of known, or perfect, Fluents, are to be met with in Practice: I ſhall therefore take no further Notice of them in this Place (but refer the Reader to the ſecond Part of the Work) my Deſign here being to inſiſt only upon what is moſt general and uſeful in the Subject; which brings me to further conſider thoſe Forms of Fluxions, involving one variable Quantity only, that frequently occur in the Solution of Problems, whoſe Fluents may (after proper Transformation) be found, by the Rule already delivered in *Art.* 77.

* Art. 10.
† Art. 13.
‡ Art. 10.
* Art. 77.

83. It

83. It has been already hinted, that if a Fluxion of the Binomial Kind, as $\overline{a+cz^{n}}^{m} \times dz^{n-1}\dot{z}$, has the Index $(n-1)$ of the variable Quantity (z) without the *Vinculum* + 1, equal to (n) the Index of the ſame Quantity under the *Vinculum*, the Fluent thereof may be then truly found by the forementioned Rule. But the ſame Obſervation may be farther extended to *thoſe Caſes where the Index without the Vinculum increaſed by Unity is equal to any Multiple of that under the Vinculum*; as in the Expreſſions, $\overline{a+cz^{n}}^{m} \times dz^{2n-1}\dot{z}$, $\overline{a+cz^{n}}^{m} \times dz^{3n-1}\dot{z}$, $\overline{a+cz^{n}}^{m} \times dz^{4n-1}\dot{z}$, &c. Whoſe Fluents are thus determined.

Put $a+cz^{n}=x$, then will $z^{n}=\frac{x-a}{c}$, and $nz^{n-1}\dot{z}$ $=\frac{\dot{x}}{c}$*; and therefore $z^{2n-1}\dot{z}=\frac{x-a}{c}\times\frac{\dot{x}}{nc}=$ $\frac{x\dot{x}-a\dot{x}}{ncc}$; whence by Subſtitution we get $\overline{a+cz^{n}}^{m} \times$ $dz^{2n-1}\dot{z}=\frac{x^{m}\times d\times\overline{x\dot{x}-a\dot{x}}}{nc^{2}}=d\times\frac{x^{m+1}\dot{x}-ax^{m}\dot{x}}{nc^{2}}$:

* Art. 8.

Whoſe Fluent (by *Art.* 77.) is therefore $=\frac{d}{nc^{2}}\times$ $\overline{\frac{x^{m+2}}{m+2}-\frac{ax^{m+1}}{m+1}}$; which, by reſtoring the Value of x, becomes $\frac{d}{nc^{2}}\times\overline{\frac{\overline{a+cz^{n}}^{m+2}}{m+2}-\frac{a\times\overline{a+cz^{n}}^{m+1}}{m+1}}=$

$d\times$

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2}\times\overline{\frac{a+cz^n}{m+2}-\frac{a}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2}\times\overline{\frac{cz^n}{m+2}-\frac{a}{\overline{m+2}\times\overline{m+1}}}$$

; the true Fluent of $\overline{a+cz^n}|^m\times dz^{2n-1}\dot{z}$.

Again; for the Fluent of $\overline{a+cz^n}|^m\times dz^{3n-1}\dot{z}$, because $z^{n-1}\dot{z}=\frac{\dot{x}}{nc}$, and $z^n=\frac{x-a}{c}$, we have $z^{3n-1}\dot{z}$ $\left(=z^{2n}\times z^{n-1}\dot{z}\right)=\frac{\overline{x-a}|^2}{c^2}\times\frac{\dot{x}}{nc}=\frac{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}{nc^3}$:

Whence, $\overline{a+cz^n}|^m$ being $=x^m$, we get $\overline{a+cz^n}|^m\times dz^{3n-1}\dot{z}=dx^m\times\frac{x^2\dot{x}-2ax\dot{x}+a^2\dot{x}}{nc^3}=\frac{d}{nc^3}\times\overline{x^{m+2}\dot{x}-2ax^{m+1}\dot{x}+a^2x^m\dot{x}}$; whose Fluent is therefore $=\frac{d}{nc^3}\times\overline{\frac{x^{m+3}}{m+3}-\frac{2ax^{m+2}}{m+2}+\frac{a^2x^{m+1}}{m+1}}=$

$$\frac{dx^{m+1}}{nc^3}\times\overline{\frac{x^2}{m+3}-\frac{2ax}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^3}\times$$

$$\overline{\frac{\overline{a+cz^n}|^2}{m+3}-\frac{2aa+2acz^n}{m+2}+\frac{a^2}{m+1}}=\frac{d\times\overline{a+cz^n}|^{m+1}}{nc^3}\times$$

$$\overline{\frac{c^2z^{2n}}{m+3}-\frac{2acz^n}{\overline{m+3}\times\overline{m+2}}+\frac{2a^2}{\overline{m+3}\times\overline{m+2}\times\overline{m+1}}}.$$

Uni-

Universally, let r denote any whole positive Number whatever, and let the Fluent of $\overline{a+cz^n}|^m \times dz^{mn-1}\dot{z}$ be required; then, by putting $a+cz^n=x$, and proceeding as above, our proposed Fluxion is transformed to $\frac{dx^m\dot{x}}{nc^r}\times\overline{x-a}|^{r-1}$; which, expanding $\overline{x-a}|^{r-1}$ (by the Binomial Theorem) becomes $\frac{d}{nc^r}\times$

$$\overline{x^{m+r-1}\dot{x}-\overline{r-1}\times ax^{m+r-2}\dot{x}+\overline{r-1}\times\frac{r-2}{2}\times a^2x^{m+r-3}\dot{x}}$$

&c. whose Fluent is therefore $=\frac{d}{nc^r}\times\overline{\frac{x^{m+r}}{m+r}-}$

$$\overline{\frac{\overline{r-1}\times ax^{m+r-1}}{m+r-1}+\frac{\overline{r-1}\times\overline{r-2}\times a^2x^{m+r-2}}{2\times\overline{m+r-2}}}\ \&c. =$$

$$\frac{dx^{m+1}}{nc^r}\times\overline{\frac{x^{r-1}}{m+r}-\frac{\overline{r-1}\times ax^{r-2}}{m+r-1}+\frac{\overline{r-1}\times\overline{r-2}\times a^2x^{r-3}}{2\times\overline{m+r-2}}}$$

&c.

Where, r being a whole positive Number, the Multiplicators $1, \overline{r-1}, \overline{r-1}\times\overline{r-2}, \overline{r-1}\times\overline{r-2}\times\overline{r-3}$, &c. will therefore become equal to Nothing, after the r first terms; and so, the Series terminating, the Fluent itself will be truly exhibited in that Number of Terms: Except when $m+r$ is likewise a whole positive Number, less than r; in which Circumstance the Divisors $m+r$, $m+r-1$, $m+r-2$, &c. becoming equal to Nothing, before the Multiplicators, the corresponding Terms of the Series will be infinite. And in that Case the Fluent is said to fail, since Nothing can then be determined from it.

84. Be-

84. Besides the foregoing, there is another Way of deriving the Fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, in Terms of the original flowing Quantity z; which will afford a Theorem more commodious for Practice than that above given: The Method of Investigation is thus.

Let $d\times\overline{a+cz^n}|^{m+1} \times Az^p + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}$ &c. (where p, v, A, B, C, &c. denote unknown, but determinate, Quantities) be assumed for the Fluent sought: Then by taking the Fluxion of the Quantity so assumed we shall have

$dcn\times\overline{m+1}\times z^{n-1}\dot{z}\times\overline{a+cz^n}|^m \times \overline{Az^p + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}\ \&c.} + d\times\overline{a+cz^n}|^{m+1} \times \overline{pAz^{p-1}\dot{z} + \overline{p-v}\times Bz^{p-v-1}\dot{z} + \overline{p-2v}\times Cz^{p-v-1}\dot{z}\ \&c.}$ * which being put equal to the given Fluxion, $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, and the whole Equation divided by $\overline{a+cz^n}|^m \times dz^{-1}\dot{z}$, there comes out

*Art. 8. 10.

$$\left.\begin{array}{l} +cn\times\overline{m+1}\times z^n \times \overline{Az^p + Bz^{p-v} + Cz^{p-2v} + Dz^{p-3v}\ \&c.} \\ +\overline{a+cz^n}\times\overline{pAz^p + \overline{p-v}\times Bz^{p-v} + \overline{p-2v}\times Cz^{p-2v}\ \&c.} \end{array}\right\} = z^{rn}$$

Whence, by collecting the Coefficients of the like Powers of z, we have

$$\left.\begin{array}{l} \left.\begin{array}{r} n\times\overline{m+1} \\ +p \end{array}\right\}\times cAz^{p+n} + \left.\begin{array}{r} n\times\overline{m+1} \\ +p-v \end{array}\right\}\times cBz^{p+n-v} + \left.\begin{array}{r} n\times\overline{m+1} \\ +p-2v \end{array}\right\}\times cCz^{p+n-2v}\ \&c. \\ -z^{rn} \qquad + \quad paAz^p \qquad\qquad + \overline{p-v}\times aBz^{p-v}\ \&c. \end{array}\right\} = 0$$

Where, comparing $p+n$ and rn, the two greatest Exponents of z, we find $p = rn - n = \overline{r-1}\times n$; and by comparing the two next inferior Exponents $p+n-v$, and p, we likewise

likewise get $v=n$; which Values being substituted above, our Equation is reduced to

$$\left.\begin{array}{l}\overline{m+r}\times ncAz^{rn}+\overline{m+r-1}\times ncBz^{rn-n}+\overline{m+r-2}\times ncCz^{rn-2n}\ \&c.\\ -z^{rn}+\quad\overline{r-1}\times naAz^{rn-n}+\overline{r-2}\times naBz^{rn-2n}\ \&c.\end{array}\right\}=0$$

Where, putting $m+r=s$, and comparing the Coefficients of the homologous Terms *, we have $A=\frac{1}{snc}$, $B=-\frac{\overline{r-1}\times aA}{\overline{s-1}\times c}=-\frac{\overline{r-1}\times a}{s\times\overline{s-1}\times nc^2}$, $C=-\frac{\overline{r-2}\times aB}{\overline{s-2}\times c}=\frac{\overline{r-1}\times\overline{r-2}\times a^2}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}$, $D=-\frac{\overline{r-3}\times aC}{\overline{s-3}\times c}=-\frac{\overline{r-1}\times\overline{r-2}\times\overline{r-3}\times a^3}{s\times\overline{s-1}\times\overline{s-2}\times\overline{s-3}\times nc^4}$, &c. &c. which Values, with those of p and v, being substituted in the assumed Fluent, it becomes $d\times\overline{a+cz^n}\big|^{m+1}\times$

$$\overline{\frac{z^{rn-n}}{snc}-\frac{\overline{r-1}\times az^{rn-2n}}{s\times\overline{s-1}\times nc^2}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{s\times\overline{s-1}\times\overline{s-2}\times nc^3}}$$

$$\&c.=\frac{d\times\overline{a+cz^n}\big|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2}}$$

&c. the true Fluent of $\overline{a+cz^n}\big|^{m+1}\times dz^{rn-1}\dot{z}$, which was to be determined: Which Fluent therefore, when r is a whole positive Number, will always terminate in as many Terms as are expressed by that Number; except in that particular Case, specified in the last Article. Thus, if $r=2$, or the

* *Vid. p.* 181. *of my Treatise of Algebra.*

the given Fluxion be $\overline{a+cz^n}|^m \times dz^{2n-1}\dot{z}$; then, s $(m+r)$ being $= m+2$, the Fluent itself will become

$$\frac{d\times\overline{a+cz^n}|^{m+1}}{nc\times m+2} \times \overline{\frac{z^n}{1} - \frac{a}{m+1\times c}} = \frac{d\times\overline{a+cz^n}|^{m+1}}{nc^2} \times$$

$$\overline{\frac{cz^n}{m+2} - \frac{a}{m+2\times m+1}};$$ which is exactly the same with the first of those found in *Art.* 83. by a different Method.

The like Agreement will likewise be found, when r is $= 3$: But when r, either denotes a broken, or a negative, Number, the Series for the Fluent will then run on to Infinity; because no one of the Multiplicators $r-1$, $r-2$, $r-3$, $r-4$, &c. can in that Case be equal to Nothing.

85. The foregoing Fluent, it may be observed, was found by assuming $d\times\overline{a+cz^n}|^{m+1} \times \overline{Az^p + Bz^{p-v} + Cz^{p-2v}}$ &c. and comparing the two greatest Exponents, of the Equation thence resulting: But if, instead of $Az^p + Bz^{p-v} + Cz^{p-2v}$ &c. an ascending Series, as $Az^p + Bz^{p+v} + Cz^{p+2v}$ &c. (where the Exponents of z continually increase) be taken, and the two least Indices of z in the Equation (in like Manner resulting) be compared together, the same Fluent will be had according to a different Form, which will be of good Use in many Cases, when the foregoing fails, or runs out into an Infinite Series.

Thus, if $p+v$, $p+2v$, &c. be wrote in the Room of $p-v$, $p-2v$, &c. respectively, in the first Equation of the last Article, it will appear that

$$\left.\begin{array}{l} +cn\times\overline{m+1}\times z^{n}\times\overline{Az^{p}+Bz^{p+v}+Cz^{p+2v}\ \&c.} \\ +\overline{a+cz^{n}}\times pAz^{p}+\overline{p+v}\times Bz^{p+v}+\overline{p+2v}\times Cz^{p+2v}\ \&c. \end{array}\right\}=z^{rn}$$

Which Equation may be reduced to

$$\left.\begin{array}{l} paAz^{p}+\overline{p+v}\times aBz^{p+v}+\overline{p+2v}\times aCz^{p+2v}\ \&c. \\ -z^{rn}+\left.\begin{array}{l} n\times\overline{m+1} \\ +p \end{array}\right\}\times cAz^{p+n}+\left.\begin{array}{l} n\times\overline{m+1} \\ +p+v \end{array}\right\}\times cBz^{p+n+v}\ \&c. \end{array}\right\}=0$$

Where, by comparing the two least Exponents, &c. p will be found $= rn$, $v = n$; $A = \frac{1}{pa} = \frac{1}{rna}$; $B = -\frac{\overline{p+n}\times\overline{m+1}\times cA}{\overline{p+v}\times a} = -\frac{\overline{r+m+1}\times ncA}{\overline{r+1}\times na} = -\frac{\overline{r+m+1}\times c}{r\times\overline{r+1}\times na^{2}}$; $C = -\frac{\overline{p+v+n}\times\overline{m+1}\times cB}{\overline{p+2v}\times a} = -\frac{\overline{r+m+2}\times ncB}{\overline{r+2}\times na} = \frac{\overline{r+m+1}\times\overline{r+m+2}\times c^{2}}{r\times\overline{r+1}\times\overline{r+2}\times na^{3}}$ &c. &c.

Therefore, denoting $r+m$ by s (as above) the Fluent of $\overline{a+cz^{n}}|^{m}\times dz^{rn-1}\dot{z}$, will *(also)* be truly represented by

$$\overline{d\times a+cz^{n}}|^{m+1}\times\overline{\frac{z^{rn}}{rna}-\frac{\overline{s+1}\times cz^{rn+n}}{r\times\overline{r+1}\times na^{2}}+\frac{\overline{s+1}\times\overline{s+2}\times c^{2}z^{rn+2n}}{r\times\overline{r+1}\times\overline{r+2}\times na^{3}}}\ \&c.$$

or its Equal

$$\frac{\overline{a+cz^{n}}|^{m+1}\times dz^{rn}}{rna}\times\overline{1-\frac{\overline{s+1}\times cz^{n}}{\overline{r+1}\times a}+\frac{\overline{s+1}\times\overline{s+2}\times c^{2}z^{2n}}{\overline{r+1}\times\overline{r+2}\times a^{2}}}\ \&c.$$

Which Series will terminate when s (or $r+m$) is a whole negative Number; and therefore in all such Cases the

the Fluent is exactly determined; provided r be not also a negative Integer less than s; for in this particular Circumstance the Fluent fails, the Divisor first becoming equal to Nothing. *Vid. Art.* 83.

The Use of the two foregoing general Expressions, for the Fluent of $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$, will appear from the following Examples.

EXAMPLE I.

86. *Let it be required to find the Fluent of* $\frac{bx\dot{x}}{\overline{a+x}|^{\frac{1}{2}}}$, or

$$\overline{a+x}|^{-\frac{1}{2}}\times bx\dot{x}.$$

By comparing the Fluxion here proposed with $\overline{a+cz^n}|^m\times dz^{rn-1}\dot{z}$, we have $a=a$, $c=1$, $z=x$, $n=1$, $m=-\frac{1}{2}$, $d=b$, $rn-1$ (or $r-1$) $=1$; whence $r=2$, and s $(r+m)$ $=\frac{3}{2}$; whereof the former being a whole positive Number, let these Values be therefore substituted in

$$\left(\frac{d\times\overline{a+cz^n}|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\frac{\overline{r-1}\times\overline{r-2}\times a^2z^{rn-3n}}{\overline{s-1}\times\overline{s-2}\times c^2},\ \&c.}\right)$$

the first of the two general Expressions for the Fluent, and it will become $\frac{b\times\overline{a+x}|^{\frac{3}{2}}}{\frac{3}{2}}\times\overline{x-\frac{a}{\frac{1}{2}}}=\frac{b\times\overline{a+x}|^{\frac{3}{2}}\times\overline{2x-4a}}{3}$, the Quantity sought in this Case.

EXAMPLE II.

87. *Let the Fluxion proposed be* $\frac{b\dot{x}x^{3n-1}}{\sqrt{a+fx^n}}$, or

$$\overline{a+fx^n}|^{-\frac{1}{2}} \times bx^{3n-1}\dot{x}.$$

Here, by proceeding as above, we have $a=a$, $c=f$, $z=x$, $n=n$, $m=-\frac{1}{2}$, $d=b$, $r=3$, and $s\ (r+m)=\frac{5}{2}$: Whence, by substituting these several Values in the same general Expression, we get $\frac{b\times\overline{a+fx^n}|^{\frac{1}{2}}}{\frac{5}{2}nf}\times$

$$\overline{x^{2n}-\frac{2ax^n}{\frac{3}{2}f}+\frac{2a^2}{\frac{3}{2}\times\frac{1}{2}f^2}} = \frac{b\times\overline{a+fx^n}|^{\frac{1}{2}}}{nf^3}\times \frac{6f^2x^{2n}-8afx^n+16a^2}{15}.$$

EXAMPLE III.

88. *Wherein the Quantity proposed is* $\frac{\dot{y}\sqrt{g^2+y^2}}{y^6}$, or

$$\overline{g^2+y^2}|^{\frac{1}{2}}+y^{-6}\dot{y}.$$

Here we have $a=g^2$, $c=1$, $z=y$, $n=2$, $m=\frac{1}{2}$, $d=1$, $rn-1$ (or $2r-1$) $=-6$; whence $r\left(=\frac{-6+1}{2}\right)=-\frac{5}{2}$, and $s\ (r+m)=-2$; whereof the latter being a whole Negative Number, let the several Values here exhibited be therefore substituted in

a+

$\left(\frac{\overline{a+cz^n}|^{m+1} \times dz^{rn}}{rna} \times \overline{1 - \frac{s+1 \times cz^n}{r+1 \times a} + \frac{s+1 \times s+2 \times c^2 z^{2n}}{r+1 \times r+2 \times a^2}}\right.$ &c.) the latter of the two general Expressions above derived, and it will become $\frac{\overline{g^2+y^2}|^{\frac{3}{2}} \times y^{-5}}{-5g^2} \times \overline{1 - \frac{-1 \times y^2}{-\frac{3}{2} \times g^2}} = \frac{\overline{g^2+y^2}|^{\frac{3}{2}} \times \overline{2y^2 - 3g^2}}{15g^4y^5}$; the true Fluent required.

EXAMPLE IV.

89. *Lastly, let the given Fluxion be* $\overline{a - fz^n}|^{\frac{1}{2}} \times z^{-\frac{7}{2}n-1}\dot{z}$.

Then, a being $= a$, $c = -f$, $m = \frac{1}{2}$, $d = 1$, $r = -\frac{7}{2}$, and the rest as in the general Fluxion $\overline{a+cz^n}|^m \times dz^{rn-1}\dot{z}$; we shall, by substituting in the second Form (because s is here equal to (-3) a whole negative Number) have $\frac{\overline{a-fz^n}|^{\frac{3}{2}} \times z^{-\frac{7}{2}n}}{-\frac{7}{2}na} \times \overline{1 - \frac{-2 \times -fz^n}{-\frac{5}{2}a} + \frac{-2 \times -1 \times f^2z^{2n}}{-\frac{5}{2} \times -\frac{3}{2}a^2}} = \frac{\overline{a-fz^n}|^{\frac{3}{2}}}{-\frac{7}{2}naz^{\frac{7}{2}n}} \times \overline{1 + \frac{4fz^n}{5a} + \frac{8f^2z^{2n}}{15a^2}}$

$= -\frac{\overline{a-fz^n}|^{\frac{3}{2}} \times \overline{30a^2 + 24afz^n + 16f^2z^{2n}}}{105na^3z^{\frac{7}{2}n}}$.

90. Having insisted largely on the Manner of finding such Fluents as can be truly exhibited in Algebraic Terms; it remains now to say something with regard

to those other Forms of Expressions, involving one variable Quantity only, which, *yet*, are so affected by compound Divisors and radical Quantities, that their Fluents cannot be *accurately* determined by any Method whatsoever; of which there are innumerable Kinds: But there is one general Method whereby the Fluents of such Expressions are approximated, to any assigned Degree of Exactness; namely, the Method of *Infinite Series*; which it will, therefore, be necessary to explain; so far as relates to the Manner of expounding the Value of any compound Fraction, or surd Quantity, by Help of such a Series.

EXAMPLE I.

91. *Let, then, the Fraction* $\frac{ax}{a-x}$ *be, first, given; to be converted into an Infinite Series.*

Divide the Numerator ax by the Denominator $a-x$, as is taught in Compound Division of common Algebra; then the Operation will stand as follows;

$$a-x)\,ax\quad\left(x+\frac{x^2}{a}+\frac{x^3}{a^2}+\frac{x^4}{a^3}+\text{\&c.}\right.$$

$$\begin{array}{l}
ax-xx \\ \hline
+xx \\
+xx-\frac{x^3}{a} \\ \hline
+\frac{x^3}{a} \\
+\frac{x^3}{a}-\frac{x^4}{a^2} \\ \hline
+\frac{x^4}{a^2} \\
\text{\&c.}
\end{array}$$

Where

Where the Quotient, or Series $x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} + \frac{x^5}{a^4} + \frac{x^6}{a^5}$ &c. infinitely continued, is taken to expound the Value of the proposed Fraction $\frac{ax}{a-x}$.

92. But, though the Series thus arising ought to be carried on to an Infinity of Terms, to have the true Value of the Quantity first proposed; or, though the Quotient, continued to ever so great a Number of Terms, will be *still* something defective of the Truth; yet, if the Value of the Quantity (x) in the Numerator be but small in Comparison of the Quantity (a) in the Denominator, the Remainder, after a few Terms in the Quotient, will become so exceeding small, as to be neglected without any considerable Error; and then the Value of the *Whole*, or of the Quantity first proposed, will be, very nearly, exhibited, by taking a small Number of the leading Terms only.

Thus, for Instance, let the Value of a be expounded by 10, and that of x by Unity; then the Remainder $\left(\frac{x^3}{a}\right)$ after the two first Terms of the Quotient, being $= \frac{1}{10}$, this Value, divided by the given Divisor ($a - x =$) 9, will therefore give $\frac{1}{90} = 0{,}01111111$, &c. for the Defect, by taking the two first Terms only: But, if the three first Terms be taken, the Defect will be *still* less considerable; amounting to no more than $\frac{1}{900}$, or 0,00111111, &c.

This may likewise be made to appear, without any regard to the Remainder, by collecting into one Sum, the Values of all the Terms to be taken: For, if only the first two $\left(x + \frac{x^2}{a}\right)$ be proposed, their Sum will be

= 1,1; which, deducted from the true Value of the given Fraction $\frac{ax}{a-x}\left(=\frac{10}{9}\right)$ = 1,1111111 &c. the Difference will come out 0.01111, *the very same as before.*

Thus, also, by collecting the Sum of the three, four and five, &c. first Terms of the Series, you will have 1,11; 1,111; and 1,1111 &c. which, being successively deducted from 1,111111111 &c. (as above) there will remain 0,001111 &c. 0,0001111 &c. 0,00001111 &c. for the Errors or Defects in those Cases respectively.

93. From what has been said in the preceding Article it appears, that Infinite Serieses, in Algebra (according to a common Observation) are similar to, or correspond with, Decimal Fractions in common Arithmetick: For, as a Decimal Fraction may be carry'd on to any proposed Number of Places, however great, and yet never amount to a Quantity, which but a very little exceeds the Value of the three or four first Places; so a Series may be infinite with regard to the Number of its Terms, and yet a few of the leading Terms only, may be sufficient to express the Value of the *Whole*, very nearly: Provided, always, that the Series has a sufficient Rate of Convergency, or that its Terms decrease in a pretty large Proportion: For, otherwise, *even*, a great Number of Terms may be used to little Purpose: Thus, in the foregoing Series, $x + \frac{x^2}{a} + \frac{x^3}{a^2}$ &c. if x be taken $= a$, no Number of Terms will be sufficient to exhibit the Value of the corresponding Fraction $\frac{ax}{a-x}$, it being infinite in that Circumstance.

94. Having endeavoured to shew, that the true Value of an infinite Series may be nearly obtained by adding together a few of the first Terms only, I shall now proceed to give other Examples of the Manner of con-

converting fractional, and surd, Quantities into such Kinds of Serieses, in order to the Approximation of the Fluents of Expressions affected by them.

EXAMPLE II.

Let the Quantity proposed be the Fraction $\frac{c^2}{c^2+2cy+y^2}$; then, by proceeding as in the first Example, you will have

$$c^2+2cy+y^2)\ c^2 \ldots\ldots \left(1-\frac{2y}{c}+\frac{3y^2}{c^2}-\frac{4y^3}{c^3}\right. \&c.$$

$$\frac{c^2+2cy+y^2}{-2cy-y^2}$$

$$\frac{-2cy-4y^2-\frac{2y^3}{c}}{+3y^2+\frac{2y^3}{c}\ \&c.}$$

Where, from a few of the first Terms of the Quotient, the Law of Continuation is manifest; the Numerators being in Arithmetical Progression; and the Signs, + and —, alternately.

EXAMPLE III.

95. *Let the Quantity given be* $\frac{1+x^2-2x^4}{1-x-x^2}$.

Then the Quotient will be $1+x+3x^2+4x^3+5x^4+9x^5+14x^6$ &c. where the *Law of Continuation* is manifest; being such that the Coefficient of each succeeding Term is equal to the Sum of those of the two Terms immediately preceding it.

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EXAMPLE IV.

96. *Let the Radical Quantity* $\sqrt{a^2+x^2}$ *be proposed.*

Here, according to the common Method of extracting the Square Root, the Process will stand as follows:

$$2a+\frac{x^2}{a}-\frac{x^4}{4a^3}\Big)\, aa+xx \,\Big(\, a+\frac{x^2}{2a}-\frac{x^4}{8a^3} \text{ \&c.}$$

$$aa$$

$$+xx$$

$$+xx+\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$+\frac{x^6}{8a^4}-\frac{x^8}{64a^6} \text{ \&c. \&c.}$$

97. The Law of Continuation in Serieses, thus arising, from radical Quantities, is not easily discovered: But, if you would carry on the Series to any proposed Number of Terms, the Work will be a good deal shortned, by dividing the Remainder by the Divisor, when half that Number of Terms is found (as in common Division) and observing, at the same time, to neglect all such Terms whose Indices would exceed the greatest, or the greatest Plus the common Difference, in the said Remainder, according as the whole Number of Terms proposed to be found is odd, or even.

Thus, if it were proposed to continue the foregoing Series $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}$ to 6 Terms, then the Divisor

(or

(or double Quotient) being $2a + \frac{x^2}{a} - \frac{x^4}{4a^3}$, and the Remainder $\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ (as appears from the laſt Article) the reſt of the Operation will ſtand thus:

$$2a + \frac{x^2}{a} - \frac{x^4}{4a^3}\Big)\frac{x^6}{8a^4} - \frac{x^8}{64a^6} + 0\Big(\frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$$

$$\frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8}$$

$$\overline{\quad - \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8}\quad}$$

$$- \frac{5x^8}{64a^6} - \frac{5x^{10}}{128a^8}$$

$$\overline{\quad + \frac{7x^{10}}{128a^8}\quad}$$

Which three Terms thus found being added to thoſe found above, we have $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$, for the 6 firſt Terms of an infinite Series exhibiting the Value of $\sqrt{a^2 + x^2}$.

98. Another Way of reſolving any radical Quantity, is to aſſume a Series (with unknown Coefficients) for the Value thereof; and then the Series ſo aſſumed being raiſed to the ſecond, third, or fourth Power, *&c.* according as the Root to be extracted is a ſquare, cubic, or biquadratic one, *&c.* an Equation will be obtained (free from Surds) from whence, by comparing the homologous Terms, the aſſumed Coefficients, and conſequently the Series ſought, will be determined; as in

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EXAMPLE V.

Where it is proposed to extract the Square Root of $a^{2n}+x^{2n}$ in an Infinite Series.

In which Case, assuming $A+Bx^{2n}+Cx^{4n}+Dx^{6n}+$ $+Ex^{8n}$ &c. for the required Series, and taking the Square thereof, we have

$$\left.\begin{array}{l} A^2+2ABx^{2n}+2ACx^{4n}+2ADx^{6n}+2AEx^{8n}\ \&c. \\ \qquad\qquad\quad +B^2x^{4n}\ \ +2BCx^{6n}+2BDx^{8n}\ \&c. \\ \qquad\qquad\qquad\qquad\qquad\quad +C^2x^{8n}\ \ \&c. \end{array}\right\} = a^{2n}+x^{2n};$$

and consequently

$$\left.\begin{array}{l} A^2\ \ +2ABx^{2n}+2ACx^{4n}+2ADx^{6n}+2AEx^{8n}\ \&c. \\ -a^{2n}-\quad x^{2n}+\ \ B^2x^{4n}+2BCx^{6n}+2BDx^{8n}\ \&c. \\ \qquad\qquad\qquad\qquad\qquad\qquad +\ \ C^2x^{8n}\ \&c. \end{array}\right\} = 0$$

Therefore $A^2-a^{2n}=0$, $2AB-1=0$, $2AC+B^2=0$, $2AD+2BC=0$, $2AE+2BD+C^2=0$, * &c. From which we get $A=a^n$; $B\ (=\frac{1}{2A}) = \frac{1}{2a^n}$; $C\ (= -\frac{B^2}{2A}) = -\frac{1}{8a^{3n}}$; $D\ (= -\frac{BC}{A}) = \frac{1}{16a^{5n}}$; $E\ (= -\frac{2BD+C^2}{2A}) = -\frac{5}{128a^{7n}}$ &c. whence we have

$A+Bx^{2n}+Cx^{4n}+Dx^{6n}$ &c. $(=\sqrt{a^{2n}+x^{2n}}) = a^n$

* *Vid. p.* 181 *of my Treatise of Algebra.*

+

$+ \frac{x^{2n}}{2a^n} - \frac{x^{4n}}{8a^{3n}} + \frac{x^{6n}}{16a^{5n}} - \frac{5x^{8n}}{128a^{7n}}$ &c. Which Series, if n be expounded by Unity, will become $a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$ &c. the very fame with that in the preceding Article found by the common Method.

EXAMPLE VI.

99. *Let it be required to refolve* $\overline{a + bx^n}|^{\frac{1}{3}}$ *into an Infinite Series.*

Here, by affuming $A + Bx^n + Cx^{2n} + Dx^{3n}$ &c. and cubing the fame, &c. we have

$$\left.\begin{array}{l} A^3 + 3A^2Bx^n + 3A^2Cx^{2n} + 3A^2Dx^{3n} + \&c. \\ -a - bx^n \quad + 3AB^2x^{2n} + 6ABCx^{3n} + \&c. \\ \qquad\qquad\qquad + \quad B^3x^{3n} + \&c. \end{array}\right\} = 0$$

Therefore $A = a^{\frac{1}{3}}$; $B \left(= \frac{b}{3A^2}\right) = \frac{b}{3a^{\frac{2}{3}}}$; $C \left(= -\frac{B^2}{A}\right) = -\frac{b^2}{9a^{\frac{5}{3}}}$; $D \left(= -\frac{6ABC + B^3}{3A^2}\right) = \frac{5b^3}{81a^{\frac{8}{3}}}$ &c.

and confequently, $\overline{a + bx^n}|^{\frac{1}{3}}$ $(= A + Bx^n + Cx^{2n} + \&c.)$ $= a^{\frac{1}{3}} + \frac{bx^n}{3a^{\frac{2}{3}}} - \frac{b^2x^{2n}}{9a^{\frac{5}{3}}} + \frac{5b^3x^{3n}}{81a^{\frac{8}{3}}} + \&c.$

And, in the fame Manner, may the Root of any other Quantity be extracted: But as the celebrated Binomial Theorem, difcovered by the illuftrious Sir *Ifaac Newton*, is vaftly more eafy and expeditious, in raifing Powers and extracting Roots than that, or any other, Method, I fhall now explain the Ufes thereof; but, firft first of all, it may not be amiss to shew how the Theorem itself, from the Principles of Fluxions, may be derived.

Let, then, $1+y$ be a Binomial whose first Term is Unity, and its second Term any proposed Quantity y; and let the Quantity to be expanded or thrown into a Series be $\overline{1+y}|^{v}$; where the Exponent v is supposed to denote any Number whatever, whole or broken, positive or negative.

Now it is evident that the first Term of the required Series must be Unity; because when y is $=0$, the other Terms all vanish; and, in that Case, $\overline{1+y}|^{v}$ is equal to Unity. Let, therefore, $1+Ay^{m}+By^{n}+Cy^{p}+Dy^{q}$ &c. be assumed to express the true Value of the said Series, or, which is the same, let

$\overline{1+y}|^{v}=1+Ay^{m}+By^{n}+Cy^{p}+Dy^{q}$ &c. where A, B, C, D, &c. m, n, p, q, &c. denote unknown, but determinate Quantities:

Then, by taking the Fluxion of the whole Equation, (supposing y variable) we shall have $v\dot{y}\times\overline{1+y}|^{v-1}=$ $m\dot{y}Ay^{m-1}+n\dot{y}By^{n-1}+p\dot{y}Cy^{p-1}+q\dot{y}Dy^{q-1}$ &c. Whence, multiplying the Sides of the two Equations, cross-wise, and dividing by $\dot{y}\times\overline{1+y}|^{v-1}$, there comes out $\overline{1+y}\times\overline{mAy^{m-1}+nBy^{n-1}+pCy^{p-1}+qDy^{q-1}\text{ \&c.}}$ $=v+vAy^{m}+vBy^{n}+vCy^{p}+vDy^{q}$ &c. which, by Reduction, is

$$\left.\begin{array}{lllll} mAy^{m-1} & +nBy^{n-1} & +pCy^{p-1} & +qDy^{q-1} & \text{\&c.} \\ \quad * & +mAy^{m} & +nBy^{n} & +pCy^{p} & \text{\&c.} \\ -v & -vAy^{m} & -vBy^{n} & -vCy^{p} & \text{\&c.} \end{array}\right\}=0$$

Now,

Now, ſince we are at Liberty to take the Exponents of y what we will, ſo as to anſwer the Conditions of the Equation, or ſo that all the Terms here put down may mutually deſtroy each other; let them, therefore, be ſo taken that the Terms themſelves may be homologous, that is, let $m-1=0$, $n-1=m$, $p-1=n$, $q-1=p$, &c. Then, m being $=1, n=2, p=3, q=4$, &c. if theſe ſeveral Values be ſubſtituted above, the Equation itſelf will become

$$\left.\begin{array}{l} A+2By+3Cy^2+4Dy^3+ \text{ \&c.} \\ *\ +Ay\ +2By^2+3Cy^3 \quad \text{\&c.} \\ -v-vAy-vBy^2-vCy^3 \quad \text{\&c.} \end{array}\right\}=0$$

Where, taking $A-v=0$, $2B+A-vA=0$, $3C+2B-vB=0$, $4D+3C-vC=0$, &c. ſo that every Column of homologous Terms (and, conſequently, the whole Expreſſion) may vaniſh, we alſo get $A=v$; B $\left(=\frac{vA-A}{2}=\frac{A\times\overline{v-1}}{2}\right)=\frac{v\times\overline{v-1}}{2}$; C $\left(=\frac{vB-2B}{3}=B\times\frac{\overline{v-2}}{3}\right)=v\times\frac{v-1}{2}\times\frac{v-2}{3}$; D $\left(=\frac{vC-3C}{4}=C\times\frac{v-3}{4}\right)=v\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}$, &c. &c.

Whence, by writing theſe Values, with thoſe of m, n, p, q, &c. in the Series $1+Ay^m+By^n+Cy^p$ &c. firſt aſſumed, we, at length, find $\overline{1+y}|^v = 1+vy+\frac{v}{1}\times\frac{v-1}{2}\times y^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times y^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times y^4+$ &c. which was to be inveſtigated.

From the Series here brought out, any Power or Root, of any other compound Quantity, whether Binomial, Trinomial, &c. is eaſily deduced: For, if p be put to repreſent the firſt Term of any ſuch Quantity, and Q the Quotient of the reſt of the Terms divided by the first; then the Quantity itself will be expressed by $P+PQ$ or $P\times\overline{1+Q}$, and the v Power thereof by $P^v\times\overline{1+Q}|^v$ which therefore is equal to $P^v\times\overline{1+vQ+\frac{v}{1}\times\frac{v-1}{2}\times Q^2+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times Q^3+\frac{v}{1}\times\frac{v-1}{2}\times\frac{v-2}{3}\times\frac{v-3}{4}\times Q^4+\&c.}$, by what is just now determined.

But when v is a Fraction, as in the Notation of Roots, the Theorem here given will be render'd somewhat more commodious for Practice, if, instead of v, a Fraction as $\frac{m}{n}$ be substituted; by which means it will become $P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}=P^{\frac{m}{n}}\times\overline{1+\frac{m}{n}Q+\frac{m}{n}\times\frac{m-n}{2n}Q^2+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}Q^3+\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\times\frac{m-3n}{4n}Q^4+\&c.}$ whose Use, in converting radical Quantities into Infinite Serieses will appear from the following Examples.

EXAMPLE VII.

100. *Wherein it is proposed to extract the Square Root of a^2+x^2, in an Infinite Series.*

Here the Quantity to be expanded being $\overline{a^2+x^2}|^{\frac{1}{2}}$, or $\overline{aa}|^{\frac{1}{2}}\times\overline{1+\frac{xx}{aa}}|^{\frac{1}{2}}$, by comparing it with the general Form, $P^{\frac{m}{n}}\times\overline{1+Q}|^{\frac{m}{n}}$, we have $P=a^2$, $Q=\frac{x^2}{a^2}$, $m=1$,

and

and $n=2$: Whence, by ſubſtituting theſe Values in the laſt general Equation, we get

$$\overline{a^2+x^2}|^{\frac{1}{2}} = a \times \overline{1 + \tfrac{1}{2}\times\frac{x^2}{a^2} + \tfrac{1}{2}\times -\tfrac{1}{4}\times\frac{x^4}{a^4} + \tfrac{1}{2}\times -\tfrac{1}{4} \times -\tfrac{3}{6}\times\frac{x^6}{a^6} + \tfrac{1}{2}\times -\tfrac{1}{4}\times -\tfrac{3}{6}\times -\tfrac{5}{8}\times\frac{x^8}{a^8}} + \&c. = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$$

&c. Which Series agrees exactly with thoſe found in *Art.* 97. and 98. by different Methods.

EXAMPLE VIII.

101. *Let it be required to extract the Cube-Root of b^3-y^3, in an Infinite Series.*

Here by comparing $\overline{b^3}|^{\frac{1}{3}} \times \overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}$ $\left(= \overline{b^3-y^3}|^{\frac{1}{3}}\right)$ with $P^{\frac{m}{n}} \times \overline{1+Q}|^{\frac{m}{n}}$, it will be $P=b^3$, $Q=-\frac{y^3}{b^3}$, $m=1$, and $n=3$: Therefore, by Subſtitution, we get

$$\overline{b^3-y^3}|^{\frac{1}{3}} \left(=b\times\overline{1-\frac{y^3}{b^3}}|^{\frac{1}{3}}\right) = b\times\overline{1 + \tfrac{1}{3}\times -\frac{y^3}{b^3} + \tfrac{1}{3}\times -\tfrac{2}{6}\times\frac{y^6}{b^6} + \tfrac{1}{3}\times\tfrac{2}{6}\times -\tfrac{5}{9}\times -\frac{y^9}{b^9} + \tfrac{1}{3}\times -\tfrac{2}{6}\times -\tfrac{5}{9}\times -\tfrac{8}{12}\times\frac{y^{12}}{b^{12}}} + \&c. = b - \frac{y^3}{3b^2} - \frac{y^6}{9b^5} - \frac{5y^9}{81b^8} - \frac{10y^{12}}{243b^{11}}$$

&c.

EXAMPLE IX.

102. *Let the Quantity to be converted into an Infinite Series be* $\frac{a}{\sqrt{ax-xx}}$.

In this Case the given Quantity being first transformed to $\sqrt{\frac{a}{x}} \times \overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$. and $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ afterwards compared with $\overline{1+Q}|^{\frac{m}{n}}$, we have $Q=-\frac{x}{a}$, $m=-1$, and $n=2$; and therefore $\overline{1-\frac{x}{a}}|^{-\frac{1}{2}}$ $(=\overline{1+Q}|^{\frac{m}{n}}=1+\frac{m}{n}Q+\frac{m}{n}\times\frac{m-2n}{2n}Q^2+$ &c.) $1+-\frac{1}{2}\times\frac{-x}{a}+-\frac{1}{2}\times-\frac{3}{4}\times\frac{x^2}{a^2}+-\frac{1}{2}\times-\frac{3}{4}\times-\frac{5}{6}\times\frac{-x^3}{a^3}$ &c. $=1+\frac{x}{2a}+\frac{3x^2}{8a^2}+\frac{5x^3}{16a^3}+\frac{35x^4}{128a^4}+$ &c. Which therefore, multiplied by $\sqrt{\frac{a}{x}}$, gives $\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}+\frac{x^{\frac{1}{2}}}{2a^{\frac{1}{2}}}+\frac{3x^{\frac{3}{2}}}{8a^{\frac{3}{2}}}+\frac{5x^{\frac{5}{2}}}{16a^{\frac{5}{2}}}+\frac{35x^{\frac{7}{2}}}{128a^{\frac{7}{2}}}+$ &c. $=\frac{a}{\sqrt{ax-xx}}$, the Quantity proposed.

103. It may not be improper to observe here, that, when both the Terms of the proposed Quantity are affirmative, and its Exponent also affirmative and less than Unity, the two first Terms of the equal Series will be positive, and the rest negative and positive, alternately; but if only the first Term of the Binomial be affirmative, all the Terms of the Series, after the first, will be negative: Moreover, if the Exponent of the

the given Quantity be negative, and both the Terms affirmative, the Signs will change alternately; but if only the first be affirmative, all the Terms of the equal Series will be positive.

EXAMPLE X.

104. *Let the Quantity proposed be the Trinomial*

$$\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}}.$$

Here, by dividing the rest of the Terms by the first, &c. our given Quantity is reduced to $\overline{x^3}|^{\frac{1}{3}} \times \overline{1+2x+3x^2}|^{\frac{1}{3}}$. Therefore, in this Case $P=x^3$, $Q=2x+3x^2$, $m=1$, and $n=3$: Whence (by Substitution) $\overline{x^3+2x^4+3x^5}|^{\frac{1}{3}} = x \times 1 + \frac{1}{3} \times \overline{2x+3x^2}| + \frac{1}{3} \times -\frac{2}{6} \times \overline{2x+3x^2}|^2 + \frac{1}{3} \times -\frac{2}{6} \times -\frac{5}{9} \times \overline{2x+3x^2}|^3$ &c. $=$ $x \times 1 + \frac{2x+3x^2}{3} - \frac{\overline{2x+3x^2}|^2}{9} + \frac{5 \times \overline{2x+3x^2}|^3}{81}$ &c.

Which, reduced to simple Terms, is $= x + \frac{2x^2}{3} + \frac{5x^3}{9} - \frac{68x^4}{81}$ &c.

105. When the proposed Expression consists of a rational, multiply'd by an irrational, Quantity, the Series answering to the irrational one must be first found, and afterwards multiply'd by the rational Quantity: But, if two, or more, compound irrational Quantities are to be drawn into each other, then take the Series answering to each Quantity, separately, and multiply them together; observing, always, to neglect all such Terms whose Indices would exceed that of the last, or highest,

 Term,

Term, which the Series sought is proposed to be continued to.

EXAMPLE XI.

106. *Let the Quantity proposed be* $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}}$

First we have $\overline{1-x}|^{\frac{1}{10}} = 1 - \frac{x}{10} - \frac{9x^2}{10 \times 20} - \frac{9 \times 19x^3}{10 \times 20 \times 30} - \frac{9 \times 19 \times 29x^4}{10 \times 20 \times 30 \times 40}$ —&c. Which, multiply'd by $1+x$, produces $\overline{1+x} \times \overline{1-x}|^{\frac{1}{10}} = 1 + \frac{9x}{10} - \frac{29x^2}{10.20} - \frac{9.49x^3}{10.20.30} - \frac{9.19.69x^4}{10.20.30.40}$ &c. $= 1 + \frac{9x}{10} - \frac{29x^2}{200} - \frac{147x^3}{2000} - \frac{3933x^4}{80000}$ — &c.

EXAMPLE XII.

107. *Where the Quantity to be expressed in an Infinite Series is* $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, or $\overline{a^2-x^2}|^{\frac{1}{2}} \times \overline{c^2-x^2}|^{-\frac{1}{2}}$.

Here we have, $\overline{a^2-x^2}|^{\frac{1}{2}}$ $(a \times \overline{1 - \frac{xx}{aa}}|^{\frac{1}{2}} = a \times$

$\overline{1 + \frac{1}{2} \times -\frac{x^2}{a^2} + \frac{1}{2} \times -\frac{1}{4} \times \frac{x^4}{a^4} + \frac{1}{2} \times -\frac{1}{4} \times -\frac{3}{6} \times -\frac{x^6}{a^6}}$

$\overline{+ \text{ \&c.}} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$ &c.

And

And $\overline{c^2 - x^2}|^{-\frac{1}{2}}$ ($= c^{-1} \times \overline{1 - \frac{xx}{cc}}|^{-\frac{1}{2}} = c^{-1} \times \overline{1 + -\frac{1}{2} \times -\frac{x^2}{c^2} + -\frac{1}{2} \times -\frac{3}{4} \times \frac{x^4}{c^4} + \&c.} = \frac{1}{c} + \frac{x^2}{2c^3} + \frac{3x^4}{8c^5} + \frac{5x^6}{16c^7}$ *&c.* Whence, multiplying theſe two Values, one by the other, we get

$$\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 + \&c.$$

for the four firſt Terms of the Series ſought.

EXAMPLE XIII.

108. *Let the Quantity to be expanded be the Multinomial, or infinite Series,* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v$; *whoſe Exponent* v *denotes any Number whatever, whole or broken, poſitive or negative.*

Here, dividing by the firſt Term, the given Quantity is transformed to $x^{pv} \times \overline{1 + ax^n + bx^{2n} + cx^{3n} + dx^{4n} + \&c.}|^v$; which, if $ax^n + bx^{2n} + cx^{3n}$ *&c.* be put $= y$, will become $x^{pv} \times \overline{1+y}|^v$; which laſt Expreſſion (by *Art.* 99.) is $= x^{pv} \times \overline{1 + vy + \frac{v}{1} \times \frac{v-1}{2} \times y^2 + \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3} \times y^3 + \&c.}$ Whence (for Brevity ſake) putting $A = v$, $B = \frac{v}{1} \times \frac{v-1}{2}$, $C = \frac{v}{1} \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = \frac{v}{1} \times$

$\frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, &c. and substituting for y, there comes out $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v =$ $x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^2} \times x^{pv+2n} +$ $\overline{Ac + 2Bab + Ca^3} \times x^{pv+3n} + \overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4}$ $\times x^{pv+4n} + \overline{Ae + 2Bad + 2Bbc + 3Ca^2c + 3Cab^2 + 4Da^3b}$ $\overline{+ Ea^5} \times x^{pv+5n} + \&c.$

EXAMPLE XIV.

109. *To extract the Square Root of* $a^2 - x^2$, *and from thence to determine the Fluent of* $\dot{x}\sqrt{a^2 - x^2}$, *in an Infinite Series.*

By proceeding as in the foregoing Examples, the Value of $\sqrt{a^2 - x^2}$ in an Infinite Series will be found to be $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} -$ &c. Which multiplied by $\dot{x}$ gives $\dot{x}\sqrt{a^2 - x^2} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$ &c. Whose Fluent therefore (by *Art.* 77.) is $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} \frac{5x^9}{1152a^7} -$ &c. Which was to be determined.

EXAMPLE XV.

110. *Let it be required to approximate the Fluent of* $\frac{\overline{a^2-x^2}|^{\frac{1}{2}} \times x^n \dot{x}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$ *in an Infinite Series.*

It appears, from Example 12, that the Value of $\frac{\overline{a^2-x^2}|^{\frac{1}{2}}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$, expressed in a Series, is $\frac{a}{c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times x^2 + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times x^4 + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times x^6 +$ &c. Which Value being therefore multiplied by $x^n \dot{x}$, and the Fluent taken (by the common Method) we get $\frac{ax^{n+1}}{\overline{n+1} \times c} + \overline{\frac{a}{2c^3} - \frac{1}{2ac}} \times \frac{x^{n+3}}{n+3} + \overline{\frac{3a}{8c^5} - \frac{1}{4ac^3} - \frac{1}{8a^3c}} \times \frac{x^{n+5}}{n+5} + \overline{\frac{5a}{16c^7} - \frac{3}{16ac^5} - \frac{1}{16a^3c^3} - \frac{1}{16a^5c}} \times \frac{x^{n+7}}{n+7} +$ &c.

EXAMPLE XVI.

111. *Wherein it is proposed to approximate the Fluent of* $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n} + \&c.}|^v \times x^{m-1}\dot{x}$ *in a Series.*

Here, if A be put $= v$, $B = v \times \frac{v-1}{2}$, $C = v \times \frac{v-1}{2} \times \frac{v-2}{3}$, $D = v \times \frac{v-1}{2} \times \frac{v-2}{3} \times \frac{v-3}{4}$, &c. the Quantity $\overline{x^p + ax^{p+n} + bx^{p+2n} + cx^{p+3n}\ \&c.}|^v$ expanded, will be $= x^{pv} + Aax^{pv+n} + \overline{Ab + Ba^2} \times x^{pv+2n} + \overline{Ac + 2Bab + Ca^3} \times x^{pv+3n} + \overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4} \times x^{pv+4n} +$ &c. as appears from *Art.* 108. Therefore this Expression being multiplied by $x^{m-1}\dot{x}$, and the Fluent taken (as usual) we shall have $\frac{x^{pv+m}}{pv+m} +$

$$\frac{Aax^{pv+m+n}}{pv+m+n} + \frac{\overline{Ab+Ba^2} \times x^{pv+m+2n}}{pv+m+2n} + \frac{\overline{Ac + 2Bab + Ca^3} \times x^{pv+m+3n}}{pv+m+3n} + \frac{\overline{Ad + 2Bac + Bb^2 + 3Ca^2b + Da^4} \times x^{pv+m+4n}}{pv+m+4n} + \&c.$$

for the Quantity proposed to be found.

SEC-

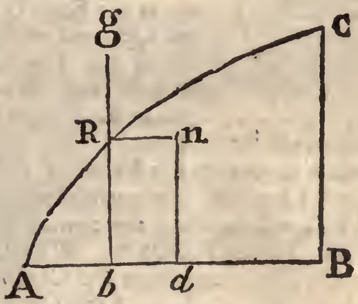
SECTION VII.

Of the Use of Fluxions in finding the Areas of Curves.

CASE I.

112. L*ET* ARC *be a Curve of any Kind whose Ordinates are perpendicular to an Axis* AB.

Imagine a Right-line *b*R*g* (perpendicular to AB) to move parallel to itself from A towards B; and let the Celerity thereof, or the Fluxion of the Absciſſa A*b*, in any propoſed Poſition of that Line, be denoted by *bd:*



Then it will appear, *from Art.* 4. that the Rectangle *(bn)* under *bd* and the Ordinate *b*R, will expreſs the correſponding Fluxion of the generated Area *ab*R: Which Fluxion, if $Ab = x$, and $bR = y$, will therefore be $= y\dot{x}$: From whence, by ſubſtituting for y or $\dot{x}$ (according to the Equation of the Curve) and taking the Fluent, the Area itſelf will become known.

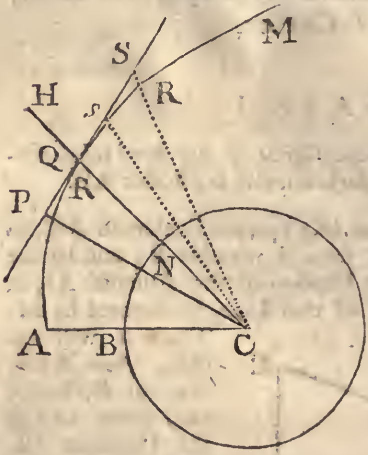
CASE II.

113. *Let* ARM *be any Curve whoſe Ordinates* CR, CR *are all referred to a Point or Center.*

Conceive a Right-line CRH to revolve about the given Center C, and let a Point R move along the ſaid

said Line, so as to trace out, or describe the proposed Curve Line ARM.

Now it is evident, that, if the Point R was to move from any Position Q, without changing its Direction and Velocity, it would proceed along the Tangent QS (instead of the Curve) and describe Areas QsC, QSC about the Center C, proportional to the Times of their Description; because those Areas, or Triangles, having the same Altitude (CP), are as the Bases Qs and QS, and these are as the Times, because the Motion in the Tangent (upon that Supposition) would be uniform.



Hence, if RS be taken to denote the Value of ($\dot{z}$) the Fluxion of the Curve Line AR, the corresponding Fluxion of the Area ARC, will be truly represented by the, uniformly generated, Triangle QCS *: Which, putting the Perpendicular (CP) drawn from the Center to the Tangent, $= s$, will therefore be ($= \frac{QS \times CP}{2} = \frac{s\dot{z}}{2}$; from whence the Area itself may be determined.

* Art. 2 and 5.

But, since in many Cases, the Value of $\dot{z}$ cannot be computed (from the Property of the Curve) without some Trouble, the two following Expressions, for the Fluxion of the Area, will commonly be found more commodious, viz. $\frac{sy\dot{y}}{2t}$ and $\frac{y^2\dot{x}}{2a}$; where $t = RP$ and $x =$ the Arch BN of a Circle, described about the Center C, at any

any Distance a ($=$ CB). These Expressions are derived from that above, in the following Manner; *viz.* $\dot{z} : \dot{y} :: y$ (CR) : t (RP) *; therefore $\dot{z} = \frac{y\dot{y}}{t}$; and *Art. 35.

consequently $\frac{s\dot{z}}{2} = \frac{sy\dot{y}}{2t}$; which is the first Expression.

Again, because the Celerity of R in the Direction of the Tangent is denoted by $\dot{z}$, that in a Direction perpendicular to CQ (whereby the Point R revolves about the Center C) will therefore be ($= \frac{CP}{CR} \times \dot{z}$) * $=$ *Art. 35.

$\frac{s\dot{z}}{y}$; which being to ($\dot{x}$) the Celerity of the Point N (about the same Center) as the Distance (or Radius) CR (y) to the Radius CN (a) we shall, by multiplying Extremes and Means, have $\frac{as\dot{z}}{y} = y\dot{x}$; and consequently $\frac{s\dot{z}}{2} = \frac{y^2\dot{x}}{2a}$; which is the other Expression.

The Method of applying this, together with the preceding Forms, will appear at large from the following Examples: Wherein x, y, z, and u are all along put to denote the Abscissa, Ordinate, Curve-line, and the Area respectively, unless where the contrary is expresly specified.

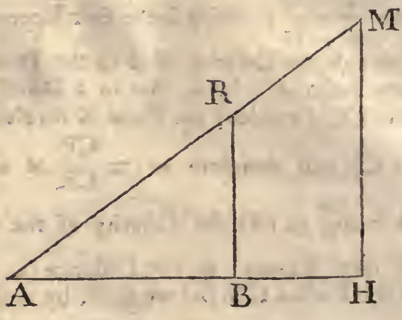
EXAMPLE I.

114. *Let it be proposed to determine the Area of a right-angled Triangle* AHM.

Put the Base AH$=a$, the Perpendicular HM$=b$; and let AB (x) be any Portion of the Base, considered as a flowing Quantity, and let BR (y) be the Ordinate, or Perpendicular, corresponding:

Then,

Then, because of the similar Triangles AHM and ABR, it will be, $a : b :: x : y = \frac{bx}{a}$: Whence $y\dot{x}$



Art. 112. (the Fluxion of the Area ABR) is, in this Case, $= \frac{bx\dot{x}}{a}$; and consequently the Fluent thereof, or the Area

†Art. 77. itself $= \frac{bx^2}{2a}$ †: Which therefore, when $x = a$, and BR coincides with HM, will become $\frac{ab}{2} = \frac{AH \times HM}{2} =$ the Area of the whole Triangle AHM; which we also know from other Principles.

EXAMPLE II.

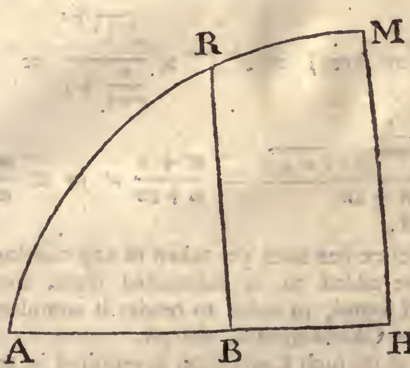
115. *Let the Curve* ARMH, *whose Area you would find, be the common Parabola.*

In which Case the Relation of AB (x) and BR (y) being expressed by $y^2 = ax$ (where a is the Parameter)

‡Art. 112. we thence get $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$; and therefore $\dot{u}$ $(= y\dot{x}$ ‡$)$ $= a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$: Whence $u = \frac{2}{3} \times a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}\, a^{\frac{1}{2}}x^{\frac{1}{2}} \times x = \frac{2}{3}yx$

(because

(because $a^{\frac{1}{2}}x^{\frac{1}{2}} = y) = \frac{2}{3} \times AB \times BR$: Hence *a Parabola is* $\frac{2}{3}$ *of a Rectangle of the same Base and Altitude.*



The Area is here found in Terms of x; but it will, many times, be more easily brought out in Terms of y (without radical Quantities) as in the very Case last proposed: Where x being $= \frac{y^2}{a}$, we therefore have $\dot{x} = \frac{2y\dot{y}}{a}$; and consequently $\dot{u}\ (y\dot{x}) = \frac{2y^2\dot{y}}{a}$: Whence $u = \frac{2y^3}{3a} = \frac{2y}{3} \times \frac{y^2}{a} = \frac{2y}{3} \times x = \frac{2}{3} \times AB \times BR$; *the same as before.*

EXAMPLE III.

116. *Let* ARM *(see the preceding Figure) be a Parabola of any Kind; whereof the general Equation is* $y^{m+n} = a^m x^n$.

Therefore, by extracting the Root, or dividing each Exponent by $m+n$, we have $y = a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}}$; whence

$\dot{u}\ (y\dot{x}) = a^{\frac{m}{m+n}} \times \dot{x}x^{\frac{n}{m+n}}$; and consequently u (the true Fluent, or Area) $= a^{\frac{m}{m+n}} \times \frac{x^{\frac{n}{m+n}+1}}{\frac{n}{m+n}+1} =$

$\frac{a^{\frac{m}{m+n}} \times x^{\frac{n}{m+n}} \times x \times \overline{m+n}}{m+2n} = \frac{m+n}{m+2n} \times yx = \frac{m+n}{m+2n} \times$ AB × BR.

No Notice has been yet taken of any constant Quantity to be added to, or subtracted from, the variable One, first found, in order to render it complete, agreeable to the Observation in *Art.* 78.

But that no such Correction is required in any of the preceding Examples, is evident from the Nature of the Figure; because, when x and y are nothing, the Area (u) ought also to be nothing, which it actually is according to the Equations above exhibited.

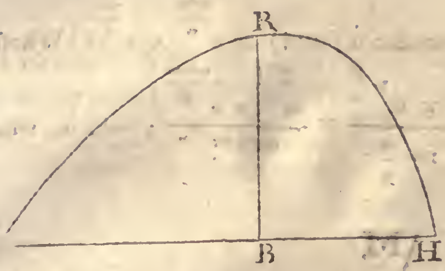
The Fluent found in the succeeding Example, will, however, stand in need of a Correction.

EXAMPLE IV.

117. *Where it is proposed to find the Area of the Curve* ARH, *whose Equation is* $x^4 - a^2x^2 + a^2y^2 = 0$.

Here, the given Equation is reduced to $y = \frac{x \times \overline{a^2 - x^2}|^{\frac{1}{2}}}{a}$; whence $\dot{u}\ (= y\dot{x}) = \frac{\overline{a^2 - x^2}|^{\frac{1}{2}} \times x\dot{x}}{a}$:

*Art. 77. Whereof the Fluent (by the common Rule *) is —



$\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$: Which, when $x=0$ and $u=0$, becomes $-\frac{a^2}{3}$; this therefore subtracted from $-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$, leaves $\frac{a^2}{3}-\frac{\overline{a^2-x^2}|^{\frac{3}{2}}}{3a}$ for the Fluent corrected, or the true Value of the Area ABR *. *Art. 78.

When the Ordinate BR $\left(\frac{x\sqrt{a^2-x^2}}{a}\right)$ becomes equal to Nothing, and B coincides with H, then x will become $=a=$ AH; and therefore the Area of the whole Curve ARH will be barely $=\frac{a^2}{3}=\frac{1}{3}\text{AH}^2$

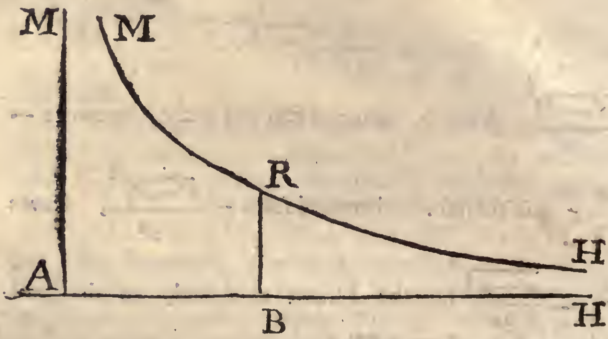
EXAMPLE V.

118. *Let it be required to determine the Area of the hyperbolical Curve whose Equation is* $x^m\ y^n = a^{m+n}$.

In this Case we have $y=\frac{a^{\frac{m+n}{n}}}{x^{\frac{m}{n}}}=a^{\frac{m+n}{n}}\times x^{\frac{-m}{n}}$;

and

and therefore $\dot{u}$ $(=y\dot{x}) = a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$: Whose Fluent is $\frac{a^{\frac{m+n}{n}} \times x^{1-\frac{m}{n}}}{1-\frac{m}{n}} = \frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}$; which, when x is



$=0$, will also be $=0$, if n be greater than m: Therefore, the Fluent requires no Correction in this Case; the Area AMRB, included between the Asymptote AM and the Ordinate BR, being truly defined by $\left(\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{n-m}\right)$ the Quantity above determined. But, if n be less than m, then the Fluent, when $x=0$, will be infinite (because the Index $\frac{n-m}{n}$ being negative, 0 becomes a Divisor to na^{m+n}:) Whence the Area AMRB will also be infinite.

But, here, the Area BRH comprehended between the Ordinate, the Curve, and the Part BH of the other Asymptote, is finite, and will be truly expounded by $\frac{na^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$ the same Quantity with its Signs changed. For the Fluxion

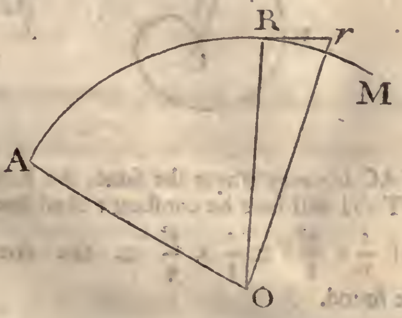
Fluxion of the Part AMRB being $a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$, that of its Supplement BRH must consequently be $-a^{\frac{m+n}{n}} \times x^{\frac{-m}{n}} \dot{x}$: Whereof the Fluent is $-\frac{a^{\frac{m+n}{n}} \times x^{\frac{1-m}{n}}}{1-\frac{m}{n}}$

$= \frac{a^{\frac{m+n}{n}} \times x^{\frac{n-m}{n}}}{m-n}$ = the Area BRH: Which wants no Correction; because, when x is infinite, and the Area BRH $= 0$, the said Fluent will also intirely vanish, seeing the Value of $x^{\frac{m-n}{n}}$ (which is a Divisor to $a^{\frac{m+n}{n}}$) is then infinite.

EXAMPLE VI.

119. *Where let it be required to determine the Area of the circular Sector* AOR.

Then, putting the Radius AO (or OR) $= a$, the



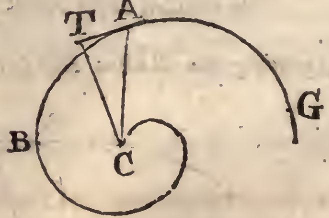
Arch AR (considered as variable by the Motion of R) $= z$, and R$r = \dot{z}$, the Fluxion of the Area will here

*Art. 113. be expressed by $\frac{a\dot{z}}{2}$ (= the Triangle ORr * :) Whence the Area itself is $= \frac{az}{2} = AO \times \frac{1}{2}AR$: From which it appears that the Area of any Circle is expressed by a Rectangle under half the Circumference and half the Diameter.

EXAMPLE VII.

120. *Wherein it is proposed to determine the Area* CBAC *of the logarithmic Spiral.*

Let the Right-line AT touch the Curve at A; upon which, from the Center C, let fall the Perpendicular CT: Then, since by the Nature of the Curve the



Angle TAC is every where the same, the Ratio of AT (t) to CT (s) will here be constant: And therefore the

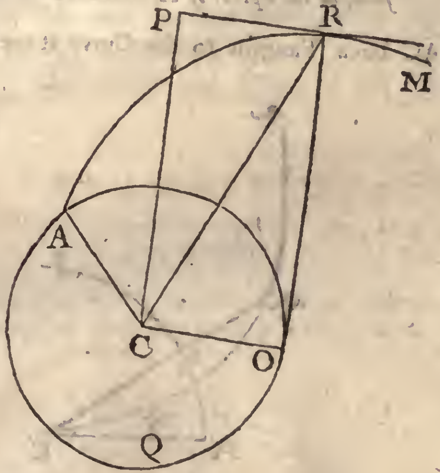
*Art. 113. Fluent of $\frac{s}{t} \times \frac{y\dot{y}}{2}$ * $= \frac{s}{t} \times \frac{y^2}{4}$ = the Area which was to be found.

EX-

EXAMPLE VIII.

121. *Let the Curve* ARM *be the Involute of a given Circle* AOQ.

In which Case the intercepted Part of the Tangent RP (t) being every where equal to the Radius CO (a)



of the generating Circle, we therefore have CP $(s) = \sqrt{CR^2 - RP^2}) = \sqrt{y^2 - a^2}$: Whence $\dot{u}$ $(= \frac{s y \dot{y}}{2t}$ * $)$ *Art. 113.

$= \frac{\sqrt{y^2 - a^2} \times y\dot{y}}{2a}$; and consequently $u = \frac{\overline{y^2 - a^2}|^{\frac{3}{2}}}{6a} = \frac{CP^3}{6CA}$ = the required Area ACR:

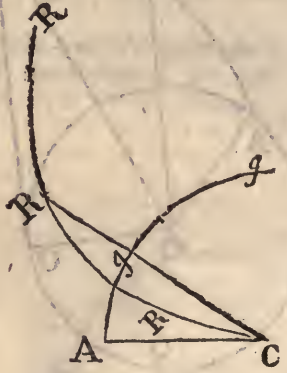
Which will also express the Area ARO generated by the Radius of Evolution RO; because, RO being = the

*Art. 119. the Arch AO, the Sector ACO ($\frac{1}{2}$AO × OC *) is equal to the Triangle CRO ($\frac{1}{2}$RO × OC) which equal Quantities being successively subtracted from CARO, there remains AOR = ACR.

EXAMPLE IX.

122. *Let the Curve* CRR, *whose Area* CRgC *you would find, be the Spiral of* Archimedes.

Let AC be a Tangent to the Curve at the Center



C, about which Center, with any Radius AC ($=a$) suppose a Circle Agg to be described; then the Arch (or Abscissa) Ag corresponding to any proposed Ordinate CR, being to that Ordinate in a given, or constant, Ratio (suppose as m to n) we have x (Ag) $= \frac{my}{n}$; therefore $\dot{u} = \frac{y^2\dot{x}}{2a}$* $= \frac{my^2\dot{y}}{2an}$, and consequently $u = \frac{my^3}{6an} =$ the Area CRRgC.

*Art. 113.

EX-

EXAMPLE X.

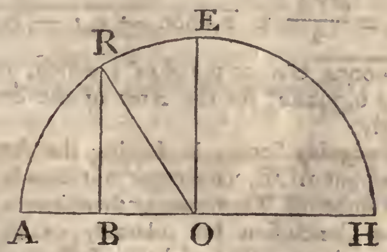
123. *Let the Equation of the Spiral* CRR *(see the last Figure) be* $x = by + cy^2 + dy^3 + ey^4 + fy^5 + \&c.$

Then, $\dot{x}$ being $= b\dot{y} + 2cy\dot{y} + 3dy^2\dot{y} + 4ey^3\dot{y} + \&c.$ we shall have $\dot{u}\ \left(= \frac{y^2\dot{x}}{2a}\right) = \frac{by^2\dot{y}}{2a} + \frac{2cy^3\dot{y}}{2a} + \frac{3dy^4\dot{y}}{2a} + \frac{4ey^5\dot{y}}{2a} + \&c.$ and therefore $u = \frac{by^3}{6a} + \frac{2cy^4}{8a} + \frac{3dy^5}{10a} + \frac{4ey^6}{12a}\ \&c.$ = the true Value of the Area in this Case.

EXAMPLE XI.

124. *Let it be proposed to find the Area of a Semicircle* AREH.

Here, putting the Diameter $AH = a$, $AB = x$, and $BR = y$ &c. *(as usual)* we have $y^2\ (BR^2) = ax - xx$



$(AB \times BH)$, and consequently $\dot{u}\ (y\dot{x}) = \dot{x}\sqrt{ax - xx} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} \times \overline{1 - \frac{x}{a}}\Big|^{\frac{1}{2}}$: Which Expression not being of the Kind described in *Art.* 83 and 85. that admit of Fluents in finite

finite Terms, let it therefore be resolved into an Infinite Series * and you will have $\dot{u} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} \times \overline{1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3} - \frac{5x^4}{128a^4}\ \&c.} = a^{\frac{1}{2}} \times \overline{x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{2a} - \frac{x^{\frac{5}{2}}\dot{x}}{8a^2} - \frac{x^{\frac{7}{2}}\dot{x}}{16a^3}\ \&c.}$ From whence, the Fluent of every Term being taken, according to the common Method, there will come out $u = a^{\frac{1}{2}} \times \overline{\frac{2x^{\frac{3}{2}}}{3} - \frac{x^{\frac{5}{2}}}{5a} - \frac{x^{\frac{7}{2}}}{28a^2} - \frac{x^{\frac{9}{2}}}{72a^3} - \frac{5x^{\frac{11}{2}}}{704a^4}\ \&c.} = x\sqrt{ax} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \frac{x^3}{72a^3} - \frac{5x^4}{704a^4}}$ — &c. = the Area ABR. Now, when, $x = \frac{1}{2}a$, the Ordinate BR will coincide with the Radius OE; in which Case the Area becomes $= \frac{1}{2}a\sqrt{\frac{1}{2}aa} \times \overline{\frac{2}{3} - \frac{1}{10} - \frac{1}{112} - \frac{1}{576} - \frac{5}{11264}\ \&c.} = \frac{a^2\sqrt{\frac{1}{2}}}{2} \times \overline{0{,}6666 - 0{,}1 - 0{,}0089 - 0{,}0017 - 0{,}0004\ \&c.} = 0{,}1964a^2$; which, multiply'd by 2, gives $0{,}3928a^2$ for the Area of the Semi-circle AEH, nearly.

*Art. 90 and 99.

As the foregoing Series, in finding the Area of the whole Quadrant AOE, converges but slowly, a considerable Number of Terms ought therefore to be taken to have the Conclusion but tolerably exact, the five first Terms above collected being sufficient to bring out no more than three Places of Figures that can be depended on. For which Reason it may be of Use to consider, whether, by computing the Area of some particular Portion (ABR) of the said Quadrant, that of the whole may not be deduced; where x being small in com-

comparison of a, the Series may have such a Rate of Convergency, that a smaller Number of Terms will be sufficient *. *Art.92.

Now, in order to this, it is well known that, if the Arch AR be taken $= \frac{1}{3}$AE (or 30 Degrees) the Sine BR will be $= \frac{1}{2}$ AO; and consequently AB $(x) =$ AO $-$ OB $=$ AO $- \sqrt{OR^2 - BR^2}$; which, if the Radius AO be expounded by Unity, (to facilitate the Operation) will be $= 0{,}1339746$ very nearly: This therefore, with the Value of a, being substituted in the forementioned Series, $\sqrt{ax^3} \times \overline{\frac{2}{3} - \frac{x}{5a} - \frac{x^2}{28a^2} - \&c.}$ we have $\overline{0{,}0693505} \times \overline{0{,}6666666 - 0{,}0133975 - 0{,}0001603 - 0{,}0000042 - \&c.} = 0{,}0693505 \times 0{,}6531046 = 0{,}0452931 =$ the Area ABR: Which added to the Area OBR ($=$ OB $\times \frac{1}{2}$BR $= \sqrt{\frac{3}{4}} \times \frac{1}{4} = 0{,}2165063$) gives $0{,}2617994$, for the Area of the Sector AOR; the treble whereof, or $0{,}7853982$ (because AR $= \frac{1}{3}$AE) will therefore be the Content of the whole Quadrant AOE: Which Number, found by taking four Terms of the Series only, is true to the last Decimal Place.

This Conclusion may be otherwise brought out, by finding a Series for the other Part of the Area, included between the Radius OE and the Ordinate BR; wherein the Co-sine OB (instead of the versed Sine AB) will be the converging (or variable) Quantity.

For, putting OB $= x$, and OR (OA) $= b$, we have y (BR $= \sqrt{OR^2 - OB^2} = \overline{b^2 - x^2}|^{\frac{1}{2}}$; and consequently $(y\dot{x})$ the Fluxion of the Area OBRE * $=$ *Art.112.
$\dot{x} \times \overline{b^2 - x^2}|^{\frac{1}{2}} = b\dot{x} - \frac{x^2\dot{x}}{2b} - \frac{x^4\dot{x}}{8b^3} - \frac{x^6\dot{x}}{16b^5} - \frac{5x^8\dot{x}}{128b^7} - \frac{7x^{10}\dot{x}}{256b^9}$ &c. Whence the Area itself is $= bx - \frac{x^3}{6b} - \frac{x^5}{40b^3} - \frac{x^7}{112b^5} - \frac{5x^9}{1152b^7} - \frac{7x^{11}}{2816b^9}$ &c.

 Now,

Now, if x (OB) be aſſumed $= \frac{1}{2}$ AO (ſo that the Arch ER may be $= \frac{1}{3}$ AE) and the Value of b (AO) be expounded by Unity, we ſhall have

$$x^3 \ \left(= x \times x^2 = ,5 \times \tfrac{1}{4} = \frac{,5}{4}\right) = ,125$$

$$x^5 \ \left(= x^3 \times x^2 = \frac{,125}{4}\right) = ,03125$$

$$x^7 \ \left(= x^5 \times x^2 = \frac{,03125}{4}\right) = ,0078125$$

$$x^9 \ \left(= x^7 \times x^2 = \frac{x^7}{4}\right) = ,0019531 +$$

$$x^{11} \ \left(= x^9 \times x^2 = \frac{x^9}{4}\right) = ,0004883 -$$

&c.

Which Values of the Powers of x being reſpectively divided by 6, 40, 112, 1152, 2816, &c. there will reſult 0,5000000 — 0,0208333 — 0,0007812 — 0,0000698 — 0,0000085 — 0,0000012 — 0,0000002 &c. = 0,4783057, for the Area OBRE in the forementioned Circumſtance, when OB $= \frac{1}{2}$ OA: From which, deducting the Triangle OBR $(= \sqrt{\frac{3}{4}} \times \frac{1}{4} = 0,2165063)$ the Remainder ,2617994 will conſequently be the Area of the Sector EOR; the treble whereof (becauſe ER is, *here*, $= \frac{1}{3}$ AE) will give the Area of the whole Quadrant, 0,7853982; *as before.*

EXAMPLE XII.

125. *Let the Curve, whoſe Area you would find, be the Ciſſoid of Diocles; whereof the Equation is* $y^2 = \dfrac{x^3}{a-x}$.

*Art. 112. Here we have $\dot{u}$ $(y\dot{x}$ *$) = \dfrac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a-x}} = \dfrac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}} \times \overline{1 - \frac{x}{a}}\big|^{\frac{1}{2}}}$

$= \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1 - \frac{x}{a}}\Big|^{-\frac{1}{2}}$: Which being none of the Kind that admit of Fluents in finite Terms *, let it therefore be resolved into an Infinite Series, and you will have $\dot{u} = \frac{x^{\frac{3}{2}}\dot{x}}{a^{\frac{1}{2}}} \times \overline{1 + \frac{x}{2a} + \frac{3x^2}{8a^2} + \frac{5x^3}{16a^3} + \frac{35x^4}{128a^4} + \&c.} = \frac{1}{a^{\frac{1}{2}}} \times \overline{x^{\frac{3}{2}}\dot{x} + \frac{x^{\frac{5}{2}}\dot{x}}{2a} + \frac{3x^{\frac{7}{2}}\dot{x}}{8a^2} + \frac{5x^{\frac{9}{2}}\dot{x}}{16a^3} + \&c.}$ Whence u (the Area itself) will come out $= \frac{1}{a^{\frac{1}{2}}} \times \overline{\frac{2x^{\frac{5}{2}}}{5} + \frac{x^{\frac{7}{2}}}{7a} + \frac{x^{\frac{9}{2}}}{12a^2} + \frac{5x^{\frac{11}{2}}}{88a^3} + \&c.} = x^2\sqrt{\frac{x}{a}} \times \overline{\frac{2}{5} + \frac{x}{7a} + \frac{x^2}{12a^2} + \frac{5x^3}{88a^3} + \&c.}$

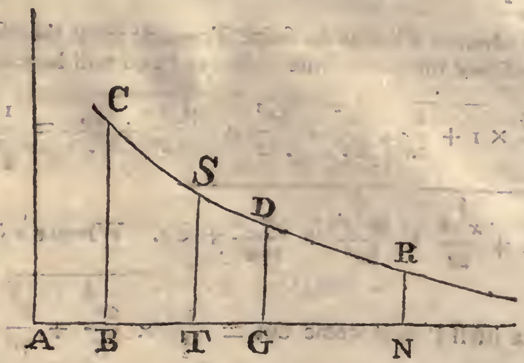
*Art. 83. and 85.

EXAMPLE XIII.

126. *Let the proposed Curve* CSDR *be of such a Nature, that (supposing* AB *Unity) the Sum of the Areas* CSTBC *and* CDGBC *answering to any two proposed Abscissas* AT *and* AG, *shall be equal to the Area* CRNBC *whose corresponding Abscissa* AN *drawn into* AB *is equal to,* AT × AG, *the Product of the Measures of the two former Abscissas.*

First, in order to determine the Equation of the Curve, (which must be known before the Area can be found) let the Ordinates GD and NR move parallel to themselves towards HF; and, then, having put GD $= y$, NR $= z$,

$NR = z$, $AT = a$, $AG = s$, and $AN = u$, the Fluxion of the Area CDGB will be repreſented by $y\dot{s}$, and that



*Art.112. of the Area CRNB by $z\dot{u}$ *: Which two Expreſſions muſt, by the Nature of the Problem, be equal to each other; becauſe the latter Area CRNB exceeds the former CDGB by the Area CSTB, which is here conſidered as a conſtant Quantity; and it is evident that two Expreſſions, that differ only by a conſtant Quantity, muſt always have equal Fluxions.

Since, therefore $y\dot{s}$ is $= z\dot{u}$, and $u = as$, *by Hypotheſis*, it follows that $\dot{u} = a\dot{s}$, and that the firſt Equation (by ſubſtituting for $\dot{u}$) will become $y\dot{s} = az\dot{s}$, or $y = az$, or laſtly $ys = zas$, that is, $GD \times AG = NR \times AN$: Therefore $GD : NR :: AN : AG$; whence it appears that every Ordinate of the Curve is reciprocally as its correſponding Abſciſſa.

Now, to find the Area of the Curve ſo determined, put $BC = b$, and $BG = x$: Then, ſince AG $(1 + x)$: AB (1) :: BC (b) : GD (y) we have $y = \frac{b}{1+x}$, and conſequently $\dot{u}\ (= y\dot{x}) = \frac{b\dot{x}}{1+x} = b \times \overline{\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x} - \&c.}$ Whence, BGDC, the Area itſelf

ſelf will be $= b \times x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$ &c. Which was to be found.

It may here be obſerved that the Areas of the Spaces above mentioned, are analogous to, and have the very ſame Properties as *Logarithms*; and that thoſe Spaces, or Logarithms, may be of different Forms or Values, according as you take the Value of the firſt Ordinate BC, which may be aſſumed at Pleaſure: Thus, if BC be taken $=$ AB $=$ Unity, the Curve will become an equilateral Hyperbola whoſe Center is A (becauſe then AG $\times$ GD $= AB^2$) and in that Caſe they are called hyperbolical Logarithms: But, if BC be taken $= 0{,}43429448$ (ſo that the Logarithm, or the Area of the Space CDGB, anſwering to the Abſciſſa AG, when expreſſed by the Number 10, may be expounded by Unity, or AB^2) we ſhall then have the common, or *Brigean* Form of Logarithms.

From theſe Logarithms (given by the Tables) the Buſineſs of finding Fluents, is in many Caſes, very much facilitated: For, if the Fluxion given appears to agree with the Fluxion of any known Logarithmic Expreſſion, its Fluent may, it is evident, be had by the Tables, ready calculated, without the Trouble of an Infinite Series.

But, now to know what Kinds of Fluents are explicable by Means of Logarithms, it will be neceſſary to obſerve that, *the Fluxion of any hyperbolic Logarithm is always expreſſed by the Fluxion of the correſponding Number divided by that Number*: This appears from above, where $(y\dot{x})$ the Fluxion of the Area (or Logarithm) BGDC, when BC $=$ AB $= 1$, is truly repreſented by $\frac{\dot{x}}{1+x}$; where $1 + x$ ($=$ AG) may ſtand for any Number whatever; and $\dot{x}$ for its Fluxion.

Hence

Hence the Fluent of $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$ will be expreſſed by the hyperbolical Logarithm of $x + \sqrt{x^2 \pm a^2}$: For the Fluxion of $(x + \sqrt{x^2 \pm a^2})$ the Number itſelf, being $\dot{x} + \frac{x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}\sqrt{x^2 \pm a^2} + x\dot{x}}{\sqrt{x^2 \pm a^2}} = \frac{\dot{x}}{\sqrt{x^2 \pm a^2}} \times \overline{\sqrt{x^2 \pm a^2} + x}$, this laſt Quantity, divided by that Number, gives $\frac{\dot{x}}{\sqrt{x^2 \pm a^2}}$, the very Fluxion firſt propoſed.

It alſo appears that the Fluent of $\frac{\dot{x}}{\sqrt{2ax + x^2}}$ will be truly expounded by the hyperbolical Logarithm of $a + x + \sqrt{2ax + x^2}$: Becauſe the Fluxion of the Number $(a + x + \sqrt{2ax + x^2})$ is here $= \dot{x} + \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}} = \frac{\dot{x}}{\sqrt{2ax + xx}} \times \overline{\sqrt{2ax + xx} + a + x}$; which divided by that Number produces $\frac{\dot{x}}{\sqrt{2ax + xx}}$.

Likewiſe the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ will be repreſented by the hyperbolical Logarithm of $\frac{a + x}{a - x}$: Becauſe, the Fluxion of $\frac{a + x}{a - x}$, being $\frac{\dot{x} \times \overline{a - x} + \dot{x} \times \overline{a + x}}{\overline{a - x}|^2} = \frac{2a\dot{x}}{\overline{a - x}|^2}$, if the ſame be therefore divided by $\frac{a + x}{a - x}$, we ſhall have $\frac{2a\dot{x}}{\overline{a - x}|^2} \times \frac{a - x}{a + x} = \frac{2a\dot{x}}{\overline{a - x} \times \overline{a + x}} = \frac{2a\dot{x}}{a^2 - x^2}$.

Laſtly,

Laſtly, the Fluent of $\frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$ will be denoted by the hyperbolical Logarithm of $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$; for here the Fluxion of the Number is $\frac{\mp x\dot{x}}{\sqrt{a^2\pm x^2}}\times$

$$\frac{a+\sqrt{a^2\pm x^2}}{\overline{a+\sqrt{a^2\pm x^2}}|^2}\mp\frac{x\dot{x}}{\sqrt{a^2\pm x^2}}\times\frac{a-\sqrt{a^2\pm x^2}}{\overline{a+\sqrt{a^2\pm x^2}}|^2}=$$

$\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2}$; which divided by $\frac{a-\sqrt{a^2\pm x^2}}{a+\sqrt{a^2\pm x^2}}$ gives $\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\overline{a+\sqrt{a^2\pm x^2}}|^2}\times$

$$\frac{a+\sqrt{a^2\pm x^2}}{a-\sqrt{a^2\pm x^2}}=\frac{\pm 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times a+\sqrt{a^2\pm x^2}\times a-\sqrt{a^2\pm x^2}}$$

$=\frac{\mp 2ax\dot{x}}{\sqrt{a^2\pm x^2}\times\mp x^2}=\frac{2a\dot{x}}{x\sqrt{a^2\pm x^2}}$, the Fluxion propoſed.

Theſe four are the principal Forms of Fluxions; whoſe Fluents may be found from a Table of Logarithms of the hyperbolic Kind: Which Table, upon Occaſion, may be eaſily ſupply'd by a Table of the common Form: For, ſince the hyperbolical Logarithm of any Number is to the common Logarithm of the ſame Number, in the conſtant Ratio of Unity to 0,43429448 (as appears from above) it follows that if any common Logarithm be, either, divided by 0,43429448, or multiply'd by its Reciprocal 2,30258509, you will thence obtain the hyperbolical Logarithm correſponding.

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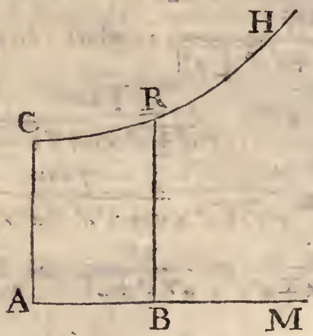
EXAMPLE XIV.

127. *Let it be required to determine the Area of the Curve; whose Equation is* $a^2y - x^2y - a^3 = 0$.

*Art. 112. In which Case y being $= \frac{a^3}{a^2 - x^2}$, we have $\dot{u}$ $(= y\dot{x})$ *

$$= \frac{a^3\dot{x}}{a^2 - x^2} = a\dot{x} + \frac{x^2\dot{x}}{a} + \frac{x^4\dot{x}}{a^3} + \frac{x^6\dot{x}}{a^5} + \frac{x^8\dot{x}}{a^7} + \&c.$$



Whence $u = ax + \frac{x^3}{3a} + \frac{x^5}{5a^3} + \frac{x^7}{7a^5} + \frac{x^9}{9a^7} + \&c.$ = the Area sought.

But the same Area (or Fluent) may be found without an Infinite Series, by Means of a Table of Logarithms, agreeable to the Observations in the last Article: For, since it there appears that the Fluent of $\frac{2a\dot{x}}{a^2 - x^2}$ is truly expressed by the hyperbolic Logarithm of $\frac{a+x}{a-x}$, it follows that that of $\frac{a^3\dot{x}}{a^2 - x^2}$ $\left(= \frac{2a\dot{x}}{a^2 - x^2} \times \frac{1}{2}a^2\right)$ will be expressed by the same Logarithm multiply'd by $\frac{1}{2}a^2$. Thus, for Example sake, let a (= AC) be taken

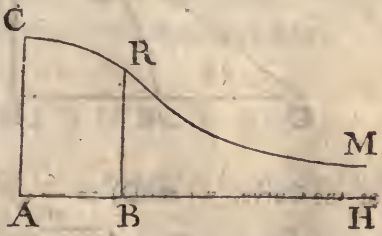
taken $= 10$, and x ($=$AB) $= 5$; then will $\frac{a+x}{a-x} = 3$; whose Logarithm taken from the common Tables is 0,4771213; which multiply'd by the *Modulus* 2,30258509 (see the last Article) gives 1,09861228 for the hyperbolical Logarithm of $\frac{a+x}{a-x}$; and this again multiply'd by 50 ($\frac{1}{2}a^2$) produces 54,930614 for the true Value of the Area ABRC, in the aforesaid Circumstance, when AC$=10$, and AB$=5$.

EXAMPLE XV.

128. *Where the proposed Curve is that whose Equation is* $a^2y^2 + x^2y^2 = a^4$.

Here, by reducing the given Equation, we get $y = \frac{a^2}{\sqrt{a^2+x^2}}$: Therefore $y\dot{x} = \frac{a^2\dot{x}}{\sqrt{a^2+x^2}} = *$. *Art. 11.

Whence, the Fluent of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ being $=$ hyperb.



Log. of $x + \sqrt{a^2+x^2}$ (by *Art.* 126. that of $\frac{a^2\dot{x}}{\sqrt{a^2+x^2}}$ will consequently be $=$ the same Logarithm multiply'd by a^2.

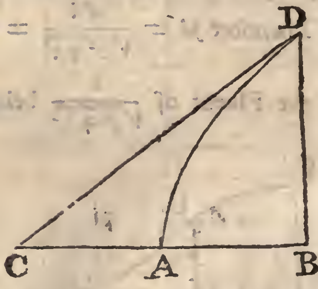
But to find whether the Fluent thus determined does not need a Correction ‡, let x be taken $= 0$; then the ‡Art. 78.
Fluent

Fluent will become = hyp. Log. $a : \times a^2$: Which, therefore, must be subtracted, to have the true Value of the
* Art. 78. Area ACRB *; and then there results $a^2 \times$ hyp. Log. $\overline{x + \sqrt{a^2 + x^2}} - a^2 \times$ hyp. Log. $a = a^2 \times$ hyp. Log. $\frac{x + \sqrt{a^2 + x^2}}{a} = u$.

EXAMPLE XVI.

129. *Let it be proposed to find the Area of the Hyperbola* ABD, *and also the Area of the hyperbolical Sector* CAD; *supposing* C *to be the Center, and* A *the principal Vertex of the Curve.*

Here, putting the Semi-transverse Axis CA $= a$, the Semi-conjugate $= c$, and CB $= x$; we have, by the



Property of the Curve, y (=BD) $= \frac{c}{a}\sqrt{xx - aa}$;

and therefore $\dot{u} = y\dot{x} = \frac{c\dot{x}}{a}\sqrt{x^2 - a^2}$ = the Fluxion

‡ Art. 112. of the Area ABD ‡

But to find the Fluxion of the Sector CAD, it is to be observed, that as the said Sector is = CBD − ABD $= \frac{xy}{2} - u$, its Fluxion will therefore be =

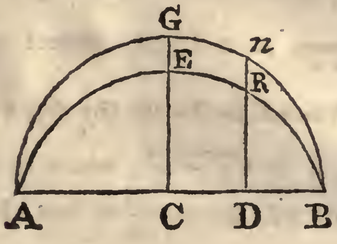
$\frac{x\dot{y}}{2} + \frac{y\dot{x}}{2} - \dot{u} = \frac{x\dot{y}}{2} - \frac{y\dot{x}}{2}$ (becauſe $\dot{u} = y\dot{x}$ *) which, *Art. 112.
by ſubſtituting for y and $\dot{y}$, their Equals $\frac{c}{a}\sqrt{x^2 - a^2}$ and $\frac{cx\dot{x}}{a\sqrt{x^2 - a^2}}$, is at length reduced to $\frac{ac}{2} \times \frac{\dot{x}}{\sqrt{x^2 - a^2}}$: Whereof the Fluent (by *Art.* 126.) is $\frac{ac}{2} \times$ hyp. Log. $x + \sqrt{x^2 - a^2}$; which corrected (by making $x = a$) will become $\frac{ac}{2} \times$ hyp. Log. $x + \sqrt{x^2 - a^2} - \frac{ac}{2} \times$ hyp. Log. $a = \frac{ac}{2} \times$ hyp. Log. $\frac{x + \sqrt{x^2 - a^2}}{a}$ = the Sector ADC: Which, ſubtracted from $\frac{cx\sqrt{x^2 - a^2}}{2a}$ ($= \frac{BC \times BD}{2}$ = the Triangle ABD) leaves $\frac{cx\sqrt{x^2 - a^2}}{2a} - \frac{ac}{2} \times$ hyp. Log. $\frac{x + \sqrt{x^2 - a^2}}{a}$ for the required Area of the Hyperbola ABD.

EXAMPLE XVII.

130. *Let the Curve propoſed be the Ellipſis* AEB.

Then, putting the tranſverſe Axis AB $= a$, and the Conjugate (2CE) $= c$; we ſhall, by the Property of the Curve, have y (DR) $= \frac{c}{a}\sqrt{ax - xx}$, and therefore $\dot{u}$ $(y\dot{x}) = \frac{c}{a} \times \dot{x}\sqrt{ax - xx}$ = the Fluxion of the Area ARD.

But $\dot{x}\sqrt{ax-xx}$ is known to expreſs the Fluxion of the correſponding Segment AD*n* of the circumſcribing

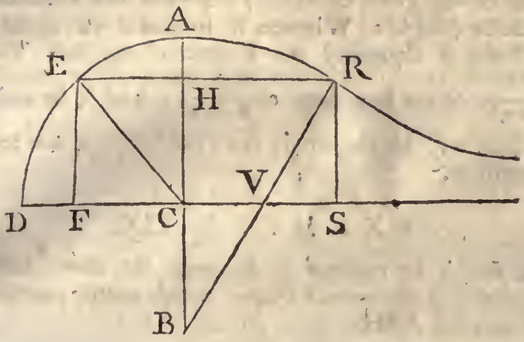


Semi-circle; whoſe Fluent is, therefore, given, by *Art.* 124; which being denoted by A, that of $\frac{c}{a}\times\dot{x}\sqrt{ax-x^2}$ will, conſequently, be $=\frac{c}{a}\times A$. Hence, the Area of the Segment of an Ellipſis, is to the Area of the correſponding Segment of its circumſcribing Circle, as the leſſer Axis of the Ellipſis is to the greater; whence, it follows that the whole Ellipſis muſt be to the whole Circle in the ſame Ratio.

EXAMPLE XVIII.

131. *Let the Curve* AR &c. *whoſe Area* CARS *you would find, be the Conchoid of* Nicomedes.

Whereof the Equation (putting BC $=a$, and RV (= AC) $=b$) is $x^2y^2=\overline{a+y}|^2\times\overline{b^2-y^2}$ (*Vid. Art.*57.) Which, by Reduction, becomes $x=\frac{a\sqrt{b^2-y^2}}{y}$ +



$\sqrt{b^2 - y^2}$: But, to bring it down to a, *still*, more simple Form, make $\sqrt{b^2 - y^2}$ ($=$ SV) $= z$; then $y = \sqrt{b^2 - z^2}$; whence, by Substitution, $x = \frac{az}{\sqrt{b^2 - z^2}} + z$; and consequently $\dot{x} =$

$$\frac{a\dot{z}\sqrt{b^2 - z^2} + \frac{z\dot{z}}{\sqrt{b^2 - z^2}} \times az}{b^2 - z^2} + \dot{z} =$$

$$\frac{a\dot{z} \times \overline{b^2 - z^2} + az^2\dot{z}}{\overline{b^2 - z^2} \times \sqrt{b^2 - z^2}} + \dot{z} = \frac{ab^2\dot{z}}{\overline{b^2 - z^2} \times \sqrt{b^2 - z^2}} + \dot{z};$$

and therefore $\dot{u}$ $(y\dot{x}) = \sqrt{b^2 - z^2} \times \overline{\frac{ab^2\dot{z}}{\overline{b^2 - z^2} \times \sqrt{b^2 - z^2}} + \dot{z}} = \frac{ab^2\dot{z}}{b^2 - z^2} + \dot{z}\sqrt{b^2 - z^2}$.

But now, to exhibit the Fluent hereof; upon C, as a Center, with the Radius AC (b) let a Quadrant of a Circle AED be described, and let RH, produced, meet the Periphery thereof in E, also let EF be parallel to AC, and let CE be drawn: It is evident (because CE (CA) $=$ VR and EF $=$ RS) that CF is also $=$ VS $= z$; and therefore, EF being ($= \sqrt{CE^2 - CF^2}$) $= \sqrt{b^2 - z^2}$, it appears that $\dot{z}\sqrt{b^2 - z^2}$ (the second Term

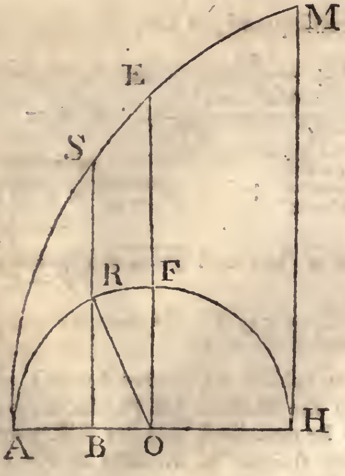
Term of our given Quantity) expresses the Fluxion of the Area AEFC: Whence, if to this Area (found *by the Table of Segments)* the Fluent of the first Term $\frac{ab^2\dot{z}}{b^2-z^2}$, or the hyp. Log. of $\frac{b+z}{b-z}$, $\times \frac{1}{2} ab$ *, be added, the Sum will be the whole Area ARCS, that was to be determined.

* Art. 126.

EXAMPLE XIX.

132. *Let it be required to determine the Area* ASRA *included by the common Cycloid* ASM *and its generating Semi-circle* ARH.

Put the Radius AO (or RO) $=a$, the Sine BR $=y$, the Co-sine OB $=x$, and the Arch AR ($=$ RS, by the Property of the Cycloid) $=z$: Then AB being $=a$



$-x$, its Fluxion will be $-\dot{x}$; whence ($\dot{u}$) that of the Area ARS is $=-z\dot{x}$ *. Now to find the Fluent thereof, make $w=-zx$ ($=$ the Fluent, if z was constant)

* Art. 112.

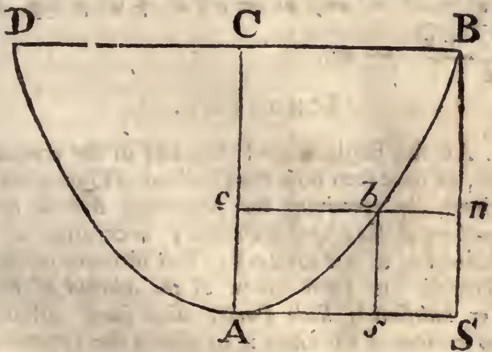
ftant) then $\dot{w}$ being $= - z\dot{x} - x\dot{z}$ *, we ſhall have $\dot{u}$ $(= - z\dot{x}) = \dot{w} + x\dot{z}$. But (*by Art.* 35.) $\dot{z}$ (AR Fluxion) : $\dot{y}$ (BR Fluxion) : : Radius : Co-ſine of the Angle ARB, or its Equal ROB : : OR (a) : OB (x): Therefore, by multiplying Extremes and Means, we get $x\dot{z} = a\dot{y}$: Whence, by Subſtitution $\dot{u}$ $(= \dot{w} + x\dot{z}) = \dot{w} + a\dot{y}$; and conſequently, by taking the Fluent, $u = w + ay = - zx + ay =$ AO × BR − BO × AR = the Area ARS. *Art. 10.

Hence it follows that the Area (AEFA) when RB coincides with the Radius FO, is barely = AO × FO = AO^2: And that the whole Area AMHFA is truly defined by − ARH × − OH, or by ARH × OH; that is by four times the Area of the generating Semi-circle.

EXAMPLE XX.

133. *Let the Curve propoſed be the* Catenaria DAB.

Then, drawing BS and *bs* parallel to the Axis AC, and AS and *cbn* perpendicular to the ſame; and making (as uſual) $Ac = x$, $cb = y$ and $Ab = z$, we ſhall have, by



the Property of the Curve, $2ax + x^2 = zz$: Whence $x = \sqrt{a^2 + z^2} - a$, and $\dot{x} = \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$: From which the

Value

* Art. 135. Value of $\dot{y}$ (which in all Curves is $= \sqrt{\dot{z}^2 - \dot{x}^2}$ *) will here be found $= \sqrt{\dot{z}^2 - \frac{z^2\dot{z}^2}{a^2+z^2}} = \sqrt{\frac{a^2\dot{z}^2}{a^2+z^2}}$ $= \frac{a\dot{z}}{\sqrt{a^2+z^2}}$; and this multiplied by $\sqrt{a^2+z^2} - a$ ($= bs$) gives $a\dot{z} - \frac{a^2\dot{z}}{\sqrt{a^2+z^2}}$ (= the Rectangle S*b*)

† Art. 112. = the Fluxion of the Area A*sb* †. From whence, by taking the Fluent, the Area itself is found $= az, - a^2$

‡ Art. 126. $\times$ *hyp. Log.* $\frac{z + \sqrt{a^2+z^2}}{a}$ ‡: Which therefore deducted from the Rectangle *sc* ($= yx = y\sqrt{a^2+z^2} - ay$), leaves $y\sqrt{a^2+z^2} - ay - az, + a^2 \times$ *hyp. Log.* $\frac{z+\sqrt{a^2+z^2}}{a}$ for the required Area A*bc*. But, since $\dot{y} = \frac{a\dot{z}}{\sqrt{a^2+z^2}}$ we have $y = a \times$ *hyp. Log.* $\frac{z + \sqrt{a^2+z^2}}{a}$; whence, by Substitution, the Area, at last comes out $= y\sqrt{a^2+z^2} - az$, or $= a\sqrt{a^2+z^2} \times$ *hyp. Log.* $\frac{z + \sqrt{a^2+z^2}}{a}, - az$.

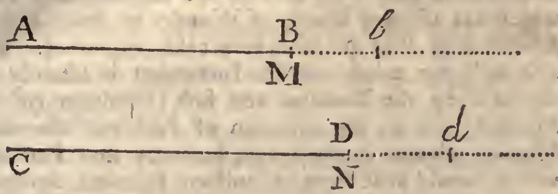
SCHOLIUM.

Read.

134. At the Beginning of this, and in the preceding Sections, we have seen how the Fluxions of Quantities are determined, *by conceiving the generating Motion to become uniform at the proposed Position*; according to the

§ Art. 2. true Definition of a Fluxion §: But hitherto no particular Notice has been taken of *the Method of Increments, or indefinitely little Parts*, used (and mistaken) by *many* for *that* of Fluxions: In which the Operations are, for the general Part, exactly the same; and which (tho' less accurate) may be applied to good Purpose in finding the Fluxions themselves, in many Cases. For which Reasons it may not be improper to add here a a few Lines on that Head, to ſhew the Beginner how the two Methods differ from each other; eſpecially as we ſhall be enabled, from thence, to draw out ſome Concluſions that will be of Uſe in the enſuing Part of the Work.

It hath been frequently inculcated in the foregoing Pages, that *the Fluxions of Quantities are always meaſured by how much the Quantities themſelves* would be *uniformly augmented in a given Time.* Therefore, if two



Quantities or Lines, AB and CD be generated together, by the uniform (or equable) Motion of two Points B and D, it follows, that any two Spaces B*b* and D*d* *actually* gone over (whereby AB and CD are augmented) in the ſame time, will truly expreſs the Fluxions of the generated Lines AB and CD: Whence it appears that the Increments (or Spaces actually gone over) and the Fluxions are the ſame in this Caſe, where the generating Velocities are equable.

But if, on the contrary, the Velocities of the two Points, in generating the Increments M*b* and N*d*, be ſuppoſed either to increaſe, or to decreaſe, the Lines or Increments ſo generated will, it is plain, no longer expreſs the Fluxions of AB and CD; being greater, or leſs than the Spaces that *might be uniformly* deſcribed, in the ſame Time, with the Velocities at M and N.

If, indeed, thoſe Increments, and the Time of their Deſcription, be taken ſo exceeding ſmall that the Motion of the Points during that Time may be conſidered as equable, the Ratio of the ſaid Increments, will then expreſs that of the Fluxions, or be as the Velocity at M to that at N, indefinitely near; but cannot be conceived to be *ſtrictly ſo*; unleſs, perhaps, in certain particular Caſes.

Hence we ſee that the *Differential Method*, which proceeds upon theſe indefinitely little Increments (actually generated) as we do upon Fluxions (or the Spaces that *might be uniformly* generated) differs little, or nothing, from the Method of Fluxions, except in the Manner of Conception, and in Point of Accuracy, wherein it appears defective: And yet it is very certain the Concluſions this Way derived are *mathematically* true; which has afforded Matter of Wonder to *ſome*: But the Reaſon why they are ſo is very eaſily explained. For, although the *whole complete* Increment is actually underſtood by the Notation and firſt Definition (of this Method) yet in the Solution of Problems the exact Meaſure thereof is not taken, but only that Part of it which would ariſe from an uniform Increaſe, agreeable to the Notion of a Fluxion; which admits of a ſtrict Demonſtration: But, after all, the *Differential Method* has one Advantage above that of Fluxions, which is, we are not there obliged to introduce the Properties of Motion. Since we reaſon upon the Increments themſelves, and not upon the Manner in which they may be generated.

It has been hinted above, that, though the Increments of Quantities are not, *ſtrictly*, as the Fluxions, yet from them the Ratio of the Fluxions may be deduced; and it appears that the ſmaller thoſe Increments are taken, the nearer their Ratio will approach to that of the Fluxions. Therefore, if we can, by any Means, find the Ratio to which the ſaid Increments, by conceiving them leſs and leſs, do perpetually converge, and which they may approach, before they vaniſh, nearer than any aſſignable Difference, that Ratio (called hereafter, for Diſtinction Sake, *the Ratio limiting that of the Increments*) will be, *ſtrictly*, that of the Fluxions.

This will more particularly appear from the following Inſtances; wherein the Manner of deriving the Ratio of the Fluxions, from that of the Increments, is ſhewn.

1°.

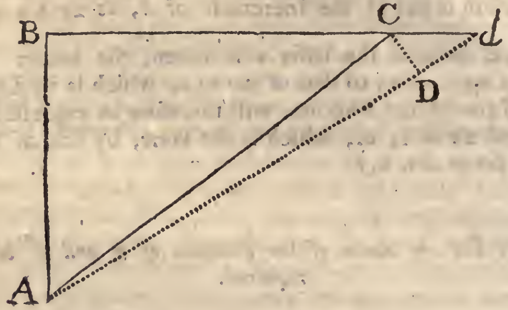
1°. *Let it be proposed to determine the Ratio of the Fluxions of x and x^2.*

Now, if x be ſuppoſed to be augmented by any (ſmall) Quantity $\acute{x}$, ſo as to become $x + \acute{x}$, its Square (x^2) will be augmented to $\overline{x+\acute{x}}|^2 = x^2 + 2x\acute{x} + \acute{x}\acute{x}$; whence the Increment of x^2 will be $2x\acute{x} + \acute{x}\acute{x}$; which therefore is to ($\acute{x}$) the Increment of x, as $2x + \acute{x}$ to 1. Hence, becauſe the leſſer $\acute{x}$ is taken, the nearer this Ratio approaches to that of $2x$ to 1, which is its *Limit*, the Ratio of the Fluxions will therefore be expreſſed by that of $2x$ to 1, or, which is the ſame, by that of $2x\dot{x}$ to $\dot{x}$ *(as in Art. 6.)*

2°. *Let the Ratio of the Fluxions of x and x^n be required.*

Then, if x be augmented to $x + \acute{x}$, x^n will be augmented to $\overline{x+\acute{x}}|^n = x^n + nx^{n-1}\acute{x} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\acute{x}^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\acute{x}^3$ &c. (*Vid. Art.* 99. Whence the Increments of x and x^n will be to each other as 1 to $nx^{n-1} + \frac{n}{1} \times \frac{n-1}{2} x^{n-2}\acute{x} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}\acute{x}\acute{x}$ &c. Where the ſmaller $\acute{x}$ is taken, the nearer the Ratio will approach to that of of 1 to $n\overline{x}^{n-1}$; which appears to be its Limit; Therefore this laſtRa tio, or that of $\dot{x}$ to $nx^{n-1}\dot{x}$, is the Ratio of the Fluxions required. *(Vid. Art.* 8.*)*

3°. *Let it be propoſed to determine the Proportion of the Fluxions of the Sides* AC *and* BC, *of a right-angled, plane Triangle* ABC; *ſuppoſing the Perpendicular* AB *to remain invariable.*



If C*d* be aſſumed to repreſent any Increment of BC and D*d*, the correſponding Increment of AC (=AD) the Ratio of thoſe Increments will be, univerſally, expreſſed by that of the Sine of the Angle CD*d* to the Sine of the Angle DC*d* *(by plane Trigonometry)* and the leſs the Increments are ſuppoſed to be, the nearer will the Angle CD*d* approach to a right one, or to an Equality with B; which is its Limit: And the nearer will DC*d* approach, at the ſame time, to an Equality with BAC. Therefore *the Ratio here limiting that of the Increments* is that of the Sine of B (or Radius) to the Sine of BAC: Which alſo expreſſes that of the required Fluxions. *(Vid. Art.* 35.*)*

In the ſame way the Proportion of the Fluxions of other Kinds of algebraical and geometrical Quantities may

may be investigated; but it will be unnecessary to dwell longer upon this Head: I shall therefore only add one other Observation from hence (which will be of use hereafter) relating to the Value of an algebraic Fraction, in that particular Circumstance when both its Numerator and Denominator become equal to Nothing, or vanish, at the same time. Which Value (it follows from above) will *be found by dividing the Fluxion of the Numerator by that of the Denominator.*

For, since the Value of any Fraction, in that Circumstance, is to be looked on as *the limiting Ratio* towards which its two Terms converge, before they vanish, and seeing the Fluxions are, always, expressed by that Ratio, the Truth of the Rule, or Position, is manifest.

An Example, however, may not be improper:

Let therefore the Fraction $\frac{x^2-a^2}{x-a}$ be propounded, to find the Value thereof when $x=a$. In which Case, the true Value sought, or the Fluxion of the Numerator divided by that of the Denominator, is $=\frac{2x\dot{x}}{\dot{x}}$ $=2x=2a$. And that this is the true Value, may be confirmed by common Division, whereby the Fraction proposed is reduced to $x+a$; whose Value when $x=a$, is therefore $=2a$, *the very same as before.*

SEC-

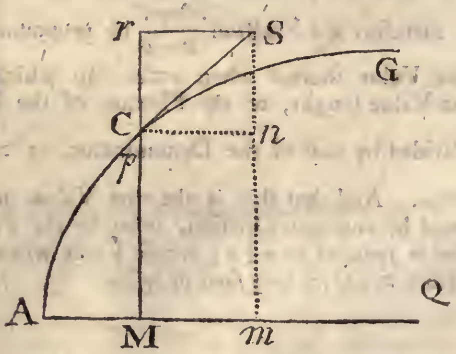
SECTION VIII.

The Use of Fluxions in the Rectification, or finding the Lengths, of Curves.

CASE I.

135. LET ACG *be a Curve of any Kind whose Ordinates are parallel to themselves and perpendicular to the Axis* AQ.

If the Fluxion of the Abscissa AM be denoted by M*m*, or by C*n* (equal and parallel to M*m*) and *n*S,



equal and parallel to C*r*, be taken to represent the corresponding Fluxion of the Ordinate MC; then will the
* Art. 48 and 49. Diagonal CS (touching the Curve in C *) be the Line which the generating point *(p)* would describe, was its Motion to become uniform at C *(Vid. Art.* 48 and 49.) which Line, therefore, truly expresses the Fluxion of
† Art. 2. the Space AC gone over, according to *the Definition* †.

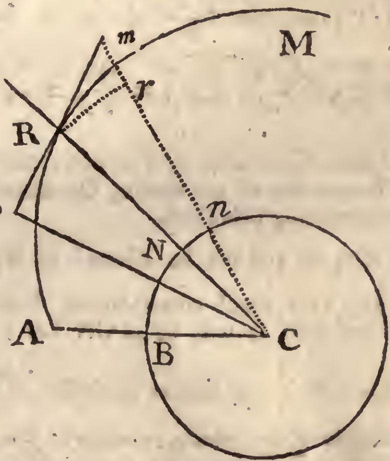
Hence, putting AM$=x$, CM$=y$, and AC$=z$, we have $\dot{z}$ $(=\text{CS}=\sqrt{\text{C}n^2+\text{S}n^2})=\sqrt{\dot{x}^2+\dot{y}^2}$; from which, and the Equation of the Curve, the Value of z may be determined.

CASE

CASE II.

136. *Let all the Ordinates of the propoſed Curve* ARM *be referred to a Center* C.

Then, putting the Tangent RP (intercepted by the Perpendicular CP) $= t$, the Arch BN, of a Circle deſcribed about the Center $C = x$; the Radius CN (or CB) $= a$, *&c.* (*Vid. Art.* 113.) we have $\dot{z} : \dot{y} :: y$ (CR)



$: t$ (RP*) and conſequently $\dot{z} = \frac{y\dot{y}}{t}$: From whence the Value of z will be found, if the Relation of y and t is given. *Art. 35.

But in other Caſes it will be better to work from the following Equation, *viz.* $\dot{z} = \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{a^2}}$. Which is thus derived.

Let the Right Line, CR, be conceived to revolve about the Center C; then ſince the Celerity of the generating

nerating Point R in a Direction perpendicular to CR is to $(\dot{x})$ the Celerity of the Point N, as CR (y) to CN (a) it will therefore be truly represented by $\frac{y\dot{x}}{a}$: Which being to $(\dot{y})$ the Celerity in the Direction of CR, pro-
*Art. 35. duced, as CP (s) : RP (t) * it follows that $\frac{y^2\dot{x}^2}{a^2} : \dot{y}^2 :: s^2 : t^2$: Whence, by Composition, $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 : \dot{y}^2 :: s^2 + t^2\ (y^2) : t^2$; therefore $\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2 = \frac{y^2\dot{y}^2}{t^2}$, and consequently $\sqrt{\frac{y^2\dot{x}^2}{a^2} + \dot{y}^2}\ \left(= \frac{y\dot{y}}{t}\right) = \dot{z}$; as was to be shewn.

But the same Conclusion may be more easily deduced from the Increments of the flowing Quantities, according to the preceding Scholium.

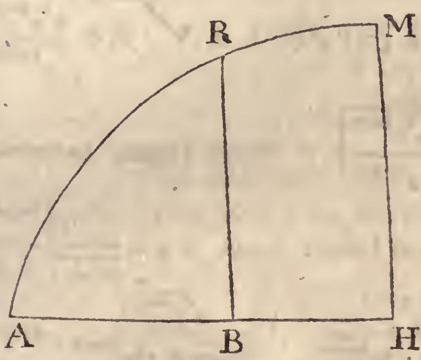
For, if R*m*, *rm* and N*n* be assumed to represent ($\acute{z}$, $\acute{y}$ and $\acute{x}$) any very small corresponding Increments of AR, CR and BN, it will be as CN (a) : CR (y) :: $\acute{x}$ (the Arch N*n*) : the similar Arch R*r* $= \frac{y\acute{x}}{a}$. And, if the Triangle R*rm* (which, while the Point *m* is returning back to R, approaches continually nearer and nearer to a Similitude with CRP) be considered as *rectilineal*, we shall also obtain $\acute{z}^2$ $(= Rm^2 = Rr^2 + rm^2)$ $= \frac{y^2\acute{x}^2}{a^2} + \acute{y}^2$: Whence, by writing $\dot{z}$, $\dot{x}$ and $\dot{y}$ for $\acute{z}$, $\acute{x}$ and $\acute{y}$ (according to the Scholium) there comes out $\dot{z}^2 = \frac{y^2\dot{x}^2}{a^2} + \dot{y}^2$, *as before.*

E X-

EXAMPLE I.

137. *Let the Curve* ARM *whose Length is sought, be the Semi-cubical Parabola.*

Whereof the Equation being $ax^2 = y^3$, or $x = \frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, we thence have $\dot{x} = \frac{3y^{\frac{1}{2}}\dot{y}}{2a^{\frac{1}{2}}}$: Whence $\dot{z}$ $(= \sqrt{\dot{y}^2 + \dot{x}^2}$*) *Art. 135.



$= \sqrt{\dot{y}^2 + \frac{9y\dot{y}^2}{4a}} = \frac{\dot{y} \times \overline{4a+9y}|^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$. Whose Fluent (found by the common Rule) is $\frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$; which, corrected (by making $y = 0$) becomes $\frac{\overline{4a+9y}|^{\frac{3}{2}}}{27a^{\frac{1}{2}}}$ $- \frac{8a}{27} = z$.

EX-

EXAMPLE II.

138. *Let the Curve proposed be a Parabola of any (other) Kind.*

Then $x = \frac{y^{n}}{a^{n-1}}$ being a general Equation to all Kinds of Parabolas, we here have $\dot{x} = \frac{ny^{n-1}\dot{y}}{a^{n-1}}$, and therefore $\dot{z}\ (=\sqrt{\dot{y}^2+\dot{x}^2}) = \sqrt{\dot{y}^2 + \frac{n^2y^{2n-2}\dot{y}^2}{a^{2n-2}}} = \dot{y} \times \overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}}$: Whose Fluent, universally expressed in an Infinite Series, is $y + \frac{n^2y^{2n-1}}{\overline{2n-1} \times 2a^{2n-2}} - \frac{n^4y^{4n-3}}{\overline{4n-3} \times 8a^{4n-4}} + \frac{n^6y^{6n-5}}{\overline{6n-5} \times 16a^{6n-6}}$, &c. $= z$.

But, when $2n-2$, the Index of y, in the given Fluxion, is either equal to Unity, or to any *aliquot Part* of it, the Fluent may be accurately had in finite Terms, *by Article* 84.

For, by putting $\frac{1}{2n-2} = v$, and $\frac{n^2}{a^{2n-2}} = c$, our Fluxion $\left(\overline{1 + \frac{n^2y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}} \times \dot{y}\right)$ is, in the first place, reduced to $\overline{1+cy^{\frac{1}{v}}}\Big|^{\frac{1}{2}} \times \dot{y}$: Which being compared with

with $\overline{a+cz^{n}}|^{m} \times dz^{rn-1}\dot{z}$, the general Expreſſion in the foreſaid *Article*, we have $a=1$, $z=y$, $n=\frac{1}{v}$, $m=\frac{1}{2}$, $d=1$, $\dot{z}=\dot{y}$, $rn-1=0$, or $\frac{r}{v}-1=0$; whence $r=v$, $s\ (r+m)=v+\frac{1}{2}$; and conſequently

$$\frac{d\times\overline{a+cz^{n}}|^{m+1}}{snc}\times\overline{\frac{z^{rn-n}}{1}-\frac{\overline{r-1}\times az^{rn-2n}}{\overline{s-1}\times c}+\&c.}$$ * * Art. 84.

$$\frac{\overline{1+cy^{\frac{1}{v}}}|^{\frac{3}{2}}}{c+\frac{c}{2v}}\times\overline{y^{\frac{v-1}{v}}-\frac{\overline{v-1}\times y^{\frac{v-2}{v}}}{\overline{v-\frac{1}{2}}\times c}+\frac{\overline{v-1}\times\overline{v-2}\times y^{\frac{v-3}{v}}}{\overline{v-\frac{1}{2}}\times\overline{v-\frac{3}{2}}\times c^{2}}-\&c.}$$ = the Fluent of

$\overline{1+cy^{\frac{1}{v}}}|^{\frac{1}{2}}\times\dot{y}$; which was to be determined, and which will (it is plain) always terminate in v Terms, when v, or its Equal $\frac{1}{2n-2}$, is a whole poſitive Number.

If $\frac{2v+1}{2v}$ (derived from $v=\frac{1}{2n-2}$) be ſubſtituted for its Equal n, the Equation of the Curve, will be changed to $ax^{2v}=y^{2v+1}$; which, if v be expounded by 1, 2, 3, 4, &c. ſucceſſively, will become $ax^{2}=y^{3}$, $ax^{4}=y^{5}$, $ax^{6}=y^{7}$, $ax^{8}=y^{9}$ &c. reſpectively: In all which Caſes the Length of the Curve may therefore be *accurately* had from the Fluent above exhibited.

Moreover, if n be aſſumed $= 2$ (or $v = \frac{1}{2}$) the general Equation, $x = \frac{y^n}{a^{n-1}}$, will then become $x = \frac{y^2}{a}$; anſwering to the common (or conical) Parabola.

And therefore in that Caſe $\dot{z}$ $(= \overline{1 + \frac{n^2 y^{2n-2}}{a^{2n-2}}}\Big|^{\frac{1}{2}} \times \dot{y})$ is $= \dot{y}\sqrt{1 + \frac{4y^2}{a^2}} = \frac{\dot{y}\sqrt{\frac{1}{4}a^2 + y^2}}{\frac{1}{2}a} = \frac{\dot{y}\sqrt{b^2 + y^2}}{b}$ (by putting $b = \frac{1}{2}a$) $= \frac{\dot{y} \times \overline{b^2 + y^2}}{b\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2\dot{y} + y^2\dot{y}}{\sqrt{b^2 + y^2}} = \frac{1}{b} \times \frac{b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2 y\dot{y}}{\sqrt{b^2y^2 + y^4}} = \frac{1}{b}$ into $\frac{\frac{1}{2}b^2 y\dot{y} + y^3\dot{y}}{\sqrt{b^2y^2 + y^4}} + \frac{\frac{1}{2}b^2\dot{y}}{\sqrt{b^2 + y^2}}$: Where, the Fluent of the firſt Term (of the Fluxion ſo transformed) being $= \frac{1}{2}\sqrt{b^2y^2 + y^4}$ (or $\frac{1}{2}y\sqrt{b^2 + y^2}$) by the common Rule; and that of the ſecond Term $= \frac{1}{2}b^2 \times$ hyp. Log. $\frac{y + \sqrt{b^2 + y^2}}{b}$, * it follows that the Length of the Curve will, in this Caſe, be $= \frac{\frac{1}{2}y\sqrt{b^2 + y^2}}{b} + \frac{1}{2}b \times$ hyp. Log. $\frac{y + \sqrt{b^2 + y^2}}{b}$.

* Art. 126.

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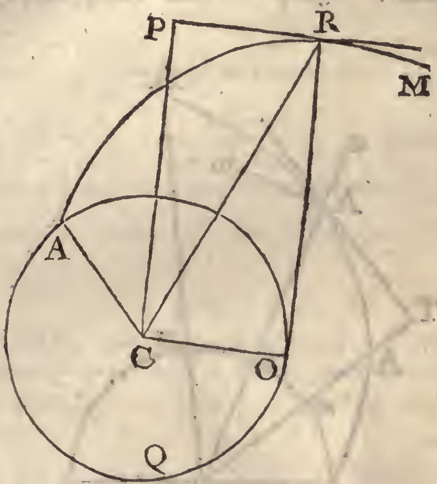
EXAMPLE III.

139. *Let the Curve proposed be the Involute of a Circle;* whose Nature is such, that the Part PR of the Tangent intercepted by the Point of Contact and the Perpendicular CP, is every where equal to the Radius CO of the ge-

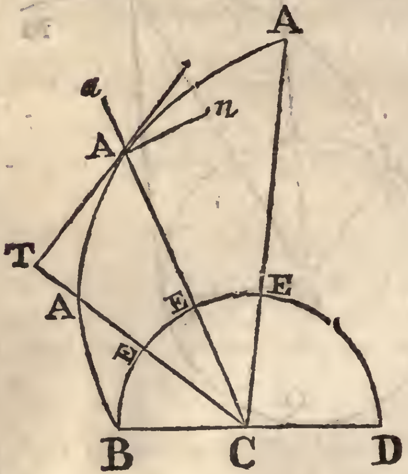


nerating Circle: Therefore $\dot{z}$ ($= \frac{y\dot{y}}{t}$ *) being here $= \frac{y\dot{y}}{a}$, we first get $z = \frac{y^2}{2a}$; which corrected, by making $y = a$ ($=$ AC) becomes $\frac{y^2 - a^2}{2a}$ $\left(\frac{CP^2}{2CA}\right)$ the true Measure of the required Arch AR. * Art. 136.

EXAMPLE IV.

140. *In which the Spiral of* Archimedes *is proposed.*

Where, the Value of t (AT) being denoted by $\frac{by}{\sqrt{b^2+y^2}}$ (*Vid. Art.* 62.) we get $\dot{z}$ $\left(=\frac{y\dot{y}}{t}\right)$ $=\frac{\dot{y}\sqrt{b^2+y^2}}{b}$: Which Fluxion being exactly the



same as that expressing the Arch of the common Parabola, found in *Article* 138. its Fluent will therefore be truly represented by the Measure of the said Arch, or by $\frac{\frac{1}{2}y\sqrt{b^2+y^2}}{b}+\frac{1}{2}b\times hyp.\ Log.\ \frac{y+\sqrt{b^2+y^2}}{b}$, the Value there exhibited.

EXAMPLE V.

141. *Let the Curve be a Spiral whose Equation is* $a^{m-1}x = y^m$ *(Vid. Art.* 136.)

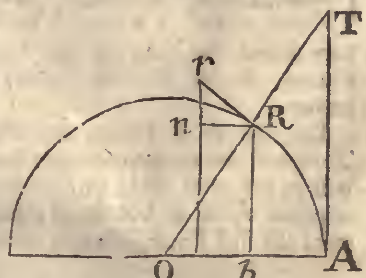
In which Case $\dot{x}$ being $= \frac{m\dot{y}y^{m-1}}{a^{m-1}}$, it is evident that $\dot{z}$ $\left(= \sqrt{\dot{y}^2 + \frac{y^2\dot{x}^2}{a^2}}\right.$*$\left.\right) = \sqrt{\dot{y}^2 + \frac{m^2y^{2m}\dot{y}^2}{a^{2m}}}$ *Art. 136.

$= \dot{y}\sqrt{1 + \frac{m^2y^{2m}}{a^{2m}}}$; and therefore $z = y + \frac{m^2y^{2m+1}}{\overline{2m+1} \times 2a^{2m}} - \frac{m^4y^{4m+1}}{\overline{4m+1} \times 8a^{4m}} + \frac{m^6y^{6m+1}}{\overline{6m+1} \times 16a^{6m}}$ *&c.* Which Value may be otherwise had, without an Infinite Series, when $\frac{1}{2m}$ is a whole positive Number, *Vid. Art.* 138.

EXAMPLE VI.

142. *Where, the Right-sine, Versed-sine, Tangent, or Secant of an Arch of a Circle, being given, it is required to find the Length of the Arch itself in Terms thereof.*

Put the Versed-sine A$b = x$, the Right-sine R$b = y$, the Tangent AT $= t$, the Secant OT $= s$, the Arch AR $= z$, and the Radius AO, or RO, $= a$; also let R$n = \dot{x}$, $nr = \dot{y}$ and R$r = \dot{z}$: Since the Angle rnR (= Right-angle) = ObR, and rRn (= Right-angle — nRO) = ORb, the Triangles rRn and ORb



 are

are therefore equi-angular; and it will be, Rb (y) : OR (a) :: Rn $(\dot{x})$: Rr $(\dot{z}) = \frac{a\dot{x}}{y} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ (because, by the Property of the Circle $\sqrt{2ax - xx} = y$.) Also, Ob $(\sqrt{a^2 - y^2})$: OR (a) :: nr $(\dot{y})$ Rr $(\dot{z}) = \frac{a\dot{y}}{\sqrt{a^2 - y^2}}$. These two Values exhibit the Fluxion of the Arch in Terms of the Versed-sine and Right-sine respectively: But, to get the same, in Terms of the Tangent and Secant, we have *(by sim. Triangles)* OT $(= s = \sqrt{a^2 + t^2})$: OA (a) :: OR (a) : Ob $= \frac{a^2}{s} = \frac{a^2}{\sqrt{a^2 + t^2}}$: Hence A$b = a - \frac{a^2}{s} = a - \frac{a^2}{\sqrt{a^2 + t^2}}$; whose Fluxion is therefore $= \frac{a^2\dot{s}}{s^2} = \frac{a^2 t\dot{t}}{\overline{a^2 + t^2}|^{\frac{3}{2}}}$: Whence (again by similar Triangles) AT $(= \sqrt{s^2 - a^2} = t)$: OT $(= s = \sqrt{a^2 + t^2})$ :: Rn : Rr $= \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}} = \frac{a^2\dot{t}}{a^2 + t^2} = \dot{z}$.

Now, from any one of the four Forms of Fluxions $\left(\frac{a\dot{x}}{\sqrt{2ax - xx}}, \frac{a\dot{y}}{\sqrt{a^2 - y^2}}, \frac{a^2\dot{t}}{a^2 + t^2}, \frac{a^2\dot{s}}{s\sqrt{s^2 - a^2}}\right)$ here found, the Value of the Arch itself (by taking the Fluent, in an Infinite Series) will likewise become known.

But the third Form, expressed in Terms of the Tangent, being intirely free from radical Quantities, will be the most ready in Practice, especially where the required Arch is but small; though the Series arising from the first Form, always, converges the fastest.

If,

If, therefore, $\frac{a^2\dot{t}}{a^2+t^2}$ be now converted to an Infinite Series, we shall have $\dot{z} = \dot{t} - \frac{t^2\dot{t}}{a^2} + \frac{t^4\dot{t}}{a^4} - \frac{t^6\dot{t}}{a^6}$ &c. and consequently $z = t - \frac{t^3}{3a^2} + \frac{t^5}{5a^4} - \frac{t^7}{7a^6} + \frac{t^9}{9a^8}$ &c. $=$ AR. Where, if *(for Example Sake)* AR be supposed an Arch of 30 Degrees, and AO (to render the Operation more easy) be put $=$ Unity, we shall have $t = \sqrt{\frac{1}{3}} = .5773502$ (because Ob $\sqrt{\frac{3}{4}}$: bR $(\frac{1}{2})$ :: OA (1) : AT $(t) = \sqrt{\frac{1}{3}}$)

Whence

$t^3 \; (= t \times t^2 = t \times \frac{1}{3}) = .1924500$

$t^5 \; \left(= t^3 \times t^2 = \frac{t^3}{3}\right) = .0641500$

$t^7 \; \left(= t^5 \times t^2 = \frac{t^5}{3}\right) = .0213833$

$t^9 \; \left(= t^7 \times t^2 = \frac{t^7}{3}\right) = .0071277$

$t^{11} \; \left(= t^9 \times t^2 = \frac{t^9}{3}\right) = .0023759$

$t^{13} \; \left(= t^{11} \times t^2 = \frac{t^{11}}{3}\right) = .0007919$

$t^{15} \; \left(= t^{13} \times t^2 = \frac{t^{13}}{3}\right) = .0002639$

&c.

And therefore AR $= .5773502 - \frac{.1924500}{3} + \frac{.0641500}{5} - \frac{.0213833}{7} + \frac{.0071277}{9} - \frac{.0023759}{11} +$

$$+\frac{.0007919}{13}-\frac{.0002639}{15}+\frac{.0000879}{17}-\frac{.0000293}{19}$$

$$+\frac{.0000097}{21}-\frac{.0000032}{23}=.5235987$$: Which multiplied by 6 gives 3.141592 + for the Length of the Semi-periphery of the Circle whoſe Radius is Unity.

At *Article* 126. certain Forms of Fluxions were pointed out, whoſe Fluents are explicable by means of hyperbolical Spaces, or a *Table of Logarithms*: Which Forms, it is obſervable, agree in every thing, but the Signs (and conſtant Quantities) with thoſe exhibited above, for the Arch of a Circle. And theſe laſt, like them, may ſerve as ſo many (other) Theorems for finding Fluents by means of a *Table of Sines, Tangents and Secants.* But, as ſuch a Table is uſually calculated to a Radius of 1,000000 *&c.* (or Unity) the following Equations, derived from thoſe above, being adapted to that Radius, will be rather more commodious.

$$\text{Thus, the Fluent of}\left\{\begin{array}{c}\dfrac{\dot{w}}{\sqrt{2aw-w^2}}\\[2ex]\dfrac{\dot{w}}{\sqrt{a^2-w^2}}\\[2ex]\dfrac{a\dot{w}}{a^2+w^2}\\[2ex]\dfrac{a\dot{w}}{w\sqrt{w^2-a^2}}\end{array}\right\}\text{is equal to the Arch whoſe}\left\{\begin{array}{l}\text{Verſed-ſine}\\[2ex]\text{Right-ſine}\\[2ex]\text{Tangent}\\[2ex]\text{Secant}\end{array}\right\}\text{is }\frac{w}{a}\text{, and Radius Unity.}$$

The way of deducing theſe Expreſſions, from the foregoing ones, is extremely eaſy: For, if A be put to denote the Arch whoſe Radius is Unity, and whoſe Verſed-ſine, Right-ſine, Tangent, or Secant is $\frac{w}{a}$ (according to the different Caſes here ſpecified). Then, becauſe ſimilar Arcs, of unequal Circles, are as their Radii,

Radii, it will be $1 : a :: A : (aA)$ the Length of the Arch AR (*see the Figure.*) Therefore, the Fluent of $\frac{a\dot{x}}{\sqrt{2ax - xx}}$ (or $\frac{a\dot{w}}{\sqrt{2aw - w^2}}$, putting $w = x$) being $= aA$ (AR), that of $\frac{\dot{w}}{\sqrt{2aw - w^2}}$ must necessarily be $= A$: And in the very same Manner the other Forms are made out.

EXAMPLE VII.

143. *Let the proposed Curve be the common Cycloid.*

Then, if the Radius AO of the generating Semi-circle* be denoted by a, we shall have $BR = \sqrt{2ax - x^2}$; and the Fluxion thereof $= \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - x^2}}$: Which being added to $\left(\frac{a\dot{x}}{\sqrt{2ax - xx}}\right)$ the Fluxion of AR or its Equal RS (given by the preceding Article) we thence get $\frac{2a\dot{x} - x\dot{x}}{\sqrt{2ax - x^2}} = \frac{\dot{x} \times \overline{2a - x}}{x^{\frac{1}{2}} \times \overline{2a - x}|^{\frac{1}{2}}} = \frac{\dot{x}}{x^{\frac{1}{2}}} \times \overline{2a - x}|^{\frac{1}{2}}$, for the true Fluxion of the Ordinate BS of the Cycloid.

* See Fig. Art. 132.

Hence $\dot{z}\ (\sqrt{\dot{x}^2 + \dot{y}^2}\ \dagger) = \sqrt{\dot{x}^2 + \frac{\dot{x}^2 \times \overline{2a - x}}{x}} = \dot{x}\sqrt{\frac{2a}{x}} = \overline{2a}|^{\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}$; and consequently, by taking the Fluent, $z = \overline{2a}|^{\frac{1}{2}} \times \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{2ax} =$ the Arch AS of the Cycloid.

† Art. 135.

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EXAMPLE VIII.

144. *Wherein it is required to determine the Length of the Arch of the common Hyperbola.*

In this Case (the Semi-transverse Axis being represented by b, and the Semi-conjugate by c) we have $\frac{b^2y^2}{c^2} = 2bx + x^2$; and therefore $x = \frac{b\sqrt{c^2 + y^2}}{c} - b$: Hence $\dot{x} = \frac{by\dot{y}}{c\sqrt{c^2+y^2}}$, and $\dot{z}$ $(= \sqrt{\dot{y}^2 + \dot{x}^2})$ $\sqrt{\dot{y}^2 + \frac{b^2y^2\dot{y}^2}{c^2 \times \overline{c^2+y^2}}} = \dot{y}\sqrt{1 + \frac{b^2y^2}{c^4 + c^2y^2}}$; which, by converting $\frac{b^2y^2}{c^4 + c^2y^2}$ into an *Infinite Series*, becomes $\dot{y}\sqrt{1 + \frac{b^2y^2}{c^4} - \frac{b^2y^4}{c^6} + \frac{b^2y^6}{c^8} - \frac{b^2y^8}{c^{10}}}$ &c. But still we have the Square Root to extract; In order thereto let it be assumed $= 1 + Ay^2 + By^4 + Cy^6 + Dy^8$ &c. Then, by squaring, and transposing (*Vid. Art.* 98.) there arises

$$\left.\begin{array}{l} 1 + 2Ay^2 + 2By^4 + 2Cy^6 + 2Dy^8 \text{ \&c.} \\ \qquad + A^2y^4 + 2ABy^6 + 2ACy^8 \text{ \&c.} \\ \qquad\qquad + B^2y^8 \text{ \&c.} \\ -1 - \frac{b^2}{c^4} \times y^2 + \frac{b^2}{c^6} \times y^4 - \frac{b^2}{c^8} \times y^6 + \frac{b^2}{c^{10}} \times y^8 \text{ \&c.} \end{array}\right\} = 0$$

Hence $A = \frac{b^2}{2c^4}$; $B = -\frac{b^2}{2c^6} - \frac{1}{2}A^2 = -\frac{b^2}{2c^6} - \frac{b^4}{8c^8}$; $C = \frac{b^2}{2c^8} - AB = \frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}$, &c. &c. Therefore $\dot{z}$ $(= \dot{y}\sqrt{1 + \frac{b^2y^2}{c^4} \text{ \&c.}} = \dot{y} \times \overline{1 + Ay^2 + By^4 \text{ \&c.}}) = \dot{y} + \frac{b^2}{2c^4} \times y^2\dot{y} - \overline{\frac{b^2}{2c^6} + \frac{b^4}{8c^8}} \times$ $y^4\dot{y}$

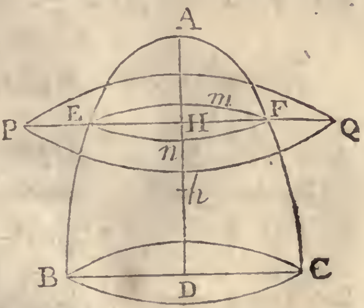
$y^4\dot{y} + \overline{\frac{b^2}{2c^8} + \frac{b^4}{4c^{10}} + \frac{b^6}{16c^{12}}} \times y^6\dot{y}$ &c. And consequently $z = y + \frac{b^2y^3}{6c^4} - \overline{\frac{b^2}{c^2} + \frac{b^4}{4c^4}} \times \frac{y^5}{10c^4} + \overline{\frac{b^2}{c^2} + \frac{b^4}{2c^4} + \frac{b^6}{8c^6}} \times \frac{y^7}{14c^6}$ &c.

By the very ſame way of proceeding the Arch of an Ellipſis may be found, the Equations of the two Curves differing in nothing but their Signs.

SECTION IX.

The Application of FLUXIONS *in inveſtigating the Contents of Solids.*

145. LET ABC repreſent any Solid; conceived to be generated (or deſcribed) by a Plane PQ paſſing over it, with a parallel Motion: Let Hh (perpendicular to PQ) be taken to expreſs the Fluxion of AH (x) or the Velocity with which the generating Plane is carry'd; alſo let the Area of the Part, EmFn, of the Plane intercepted by, or contained in, the Solid, be denoted by A: Then it follows, from *Art.* 2 and 5. that the Fluxion of the Solid AEF. will be expreſſed by $A\dot{x}$. From whence, by expounding A in Terms of x, (according to the Nature of the Figure) and then taking the Fluent, the Content of



of the Solid (which we shall, always, hereafter represent by s) will be given.

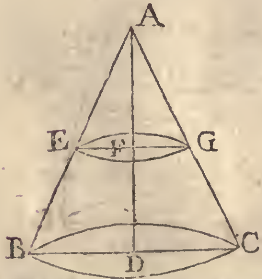
But, when the proposed Solid is that arising from the Revolution of any given Curve AEB about AHD, as an Axis, the Fluxion $(\dot{s})$ of the Solidity may be exhibited in a Manner more convenient for Practice: For, putting the Area (3,141592 &c.*) of the Circle, whose Radius is Unity, $= p$, and the Ordinate EH $= y$, it will be $1^2 : y^2 :: p : (py^2)$ the Area of the Circle EmFn, which being wrote above instead of A, we have $\dot{s} = py^2\dot{x}$. The Use of which will be sufficiently shewn in the following Examples.

* Art. 124.

EXAMPLE I.

146. *Let it be proposed to find the Content of a Cone* ABC.

Put the given Altitude (AD) of the Cone $= a$, and the Semi-diameter (BD of its Base $= b$: Then, the Distance (AF) of the Circle EG, from the Vertex A, being denoted by x, &c. we have, by similar Triangles, as $a : b :: x :$ EF $(y) = \frac{bx}{a}$. Whence, in this Case, $\dot{s}$



$(= py^2\dot{x}) = \frac{pb^2x^2\dot{x}}{a^2}$; and consequently $s = \frac{pb^2x^3}{3a^2}$; which, when $x = a$ (= AD) gives $\frac{pb^2a}{3}$ $(= p \times \text{BD}^2 \times \frac{1}{3}\text{AD})$ for the Content of the whole Cone ABC. Which appears, from hence, to be just $\frac{1}{3}$ of a Cylinder of the same Base and Altitude.

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EXAMPLE II.

147. *Where, let the Solid proposed be a parabolic Conoid, or that arising from the Revolution of any Kind of Parabola about its Axis.*

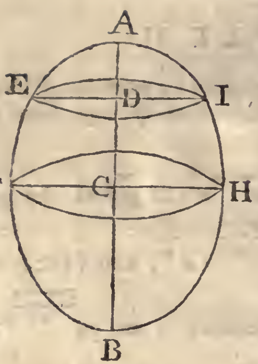
Then, from the Equation $a^{m-n} x^n = y^m$, of the generating Curve, we get $y = a^{\frac{m-n}{m}} \times x^{\frac{n}{m}}$, and $\dot{s}\ (= py^2\dot{x})$ $= pa^{\frac{2m-2n}{m}} \times \dot{x}x^{\frac{2n}{m}}$; and therefore $s = pa^{\frac{2m-2n}{m}} \times \frac{x^{\frac{2n}{m}+1}}{\frac{2n}{m}+1} = pa^{\frac{2m-2n}{m}} \times \frac{mx^{\frac{2n}{m}+1}}{2n+m} = pa^{\frac{2m-2n}{m}} \times x^{\frac{2n}{m}} \times \frac{mx}{2n+m} = py^2 \times \frac{mx}{2n+m} =$ the Content of the Solid; which therefore is to (py^2x) the Content of the circumscribing Cylinder, as m to $2n+m$. Whence the Solid generated by the conical Parabola (where $m=2$, and $n=1$) appears to be just $\frac{1}{2}$ of its circumscribing Cylinder.

EXAMPLE III.

148. *Let the proposed Solid* AFBH *be a Spheroid.*

In which Case, putting the Axis AB, about which the Solid is generated, $=a$, and the other Axis FH, of the generating Ellipsis $= b$, it follows, from the Property of the Ellipsis, that $a^2 : b^2 :: x \times \overline{a-x}$ (AD × BD) : y^2 $(DE)^2 = \frac{b^2}{a^2} \times \overline{ax-xx}$: Whence we have $\dot{s}\ (= py^2\dot{x}$ *$) = \frac{pb^2}{a^2} \times \overline{ax\dot{x} - x^2\dot{x}}$; and $s = \frac{pb^2}{a^2} \times \overline{\frac{1}{2}axx - \frac{1}{3}x^3} =$ the Segment AIE. Which, when

* Art. 145.



when AD (x) = AB (a), becomes $\left(\frac{pb^2}{a^2}\times\overline{\frac{1}{2}a^3-\frac{1}{3}a^3}\right)$ $\frac{1}{6}pab^2$ = the Content of the whole Spheroid. Where, if b (FH) be taken $= a$ (AB) we ſhall alſo get $\frac{1}{6}pa^3$ for the true Content of the Sphere whoſe Diameter is a. Hence a Sphere, or a Spheroid, is $\frac{2}{3}$ of its circumſcribing Cylinder; for the Area of the Circle FH being expreſſed by $\frac{pb^2}{4}$, the Content of the Cylinder whoſe Diameter is FH, and Altitude AB, will therefore be $\frac{pb^2a}{4}$; of which $\frac{1}{6}pab^2$, is, evidently, two third Parts.

EXAMPLE IV.

149. *Let the Solid, whoſe Content you would find, be the hyperbolical Conoid.*

Then, from the Equation, $y^2 = \frac{b^2}{a^2}\times\overline{ax+xx}$, of the generating Hyperbola, we have $\dot{s}\ (py^2\dot{x}) = \frac{pb^2}{a^2}\times\overline{ax\dot{x}+x^2\dot{x}}$, and conſequently $s = \frac{pb^2}{a^2}\times\overline{\frac{1}{2}ax^2+\frac{1}{3}x^3}$ = the Content of the Conoid; which therefore is to $\left(\frac{pb^2}{a^2}\times\overline{ax+x^2}\times x\right)$ that of a Cylinder of the ſame Baſe and Altitude, as $\frac{1}{2}a+\frac{1}{3}x$ to $a+x$. This Ratio, if x be extremely ſmall, will become as 1 to 2 very nearly; Whence it may be inferr'd, that the Content of of a very ſmall Part of any Solid, generated by a Curve, whoſe Ray of Curvature at the Vertex is a finite Quantity, is half that of a Cylinder of the ſame Baſe and Altitude, very nearly: Becauſe any ſuch Curve, for a ſmall Diſtance, will differ inſenſibly from an Hyperbola, whoſe Radius of Curvature, at the Vertex, is the ſame.

This might have been inferred, either, from the common parabolic Conoid, or the Spheroid, in the preceding Examples; but other Obſervations would not allow Room for it there.

EXAMPLE V.

150. *In which the propoſed Solid is that ariſing from the Rotation of the Ciſſoid of* Diocles, *about its Axis.*

Here, y^2 being $= \frac{x^3}{a-x}$, * we have $\dot{s}\ (py^2\dot{x}) \doteq$ * Art. 56.
$\frac{px^3\dot{x}}{a-x}$. But, in Caſes like this, (where the Denominator is rational and the variable Quantity in the Numerator of ſeveral Dimenſions) it will be neceſſary to divide the latter by the former, in order to obtain the Fluent, by leſſening the Number of Dimenſions: Thus, dividing $px^3\dot{x}$ by $-x+a$, according to the Manner of compound Quantities, the Work will ſtand thus:

$$
\begin{array}{lll}
-x+a) & px^3\dot{x}-0 & (-px^2\dot{x}-pax\dot{x}-pa^2\dot{x} \\
 & \underline{px^3\dot{x}-pax^2\dot{x}} & \\
 & \quad +pax^2\dot{x}-0 & \\
 & \quad \underline{+pax^2\dot{x}-pa^2x\dot{x}} & \\
 & \qquad +pa^2x\dot{x}-0 & \\
 & \qquad \underline{+pa^2x\dot{x}-pa^3\dot{x}} & \\
 & \qquad\quad +pa^3\dot{x} &
\end{array}
$$

Where, the Quotient being $-px^2\dot{x}-pax\dot{x}-pa^2\dot{x}$, and the Remainder $pa^3\dot{x}$, the Value of the given Fraction $\frac{px^3\dot{x}}{a-x}$,

will

will therefore be truly expressed by $-px^2\dot{x} - pax\dot{x} - pa^2\dot{x} + \frac{pa^3\dot{x}}{a-x}$: Whose Fluent, properly corrected, is $-\frac{1}{3}px^3 - \frac{1}{2}pax^2 - pa^2x + \overline{pa^3 \times hyp.\ Log. \frac{a}{a-x}}$ *Vid. Art.* 126.

EXAMPLE VI.

151. *Let the Solid be that arising from the Rotation of the Conchoid of* Nicomedes *about its Axis.*

The Sub-tangent $\frac{y\dot{x}}{\dot{y}}$ of this Curve being $= \frac{-ab^2-y^3}{y\sqrt{b^2-y^2}}$ (*Vid. Art.* 48 *and* 57.) we have $\dot{x} = \frac{-ab^2\dot{y}-y^3\dot{y}}{y^2\sqrt{b^2-y^2}}$, and

* Art. 145. therefore $\dot{s}\ (py^2\dot{x}\,*) = \frac{-pab^2\dot{y}-py^3\dot{y}}{\sqrt{b^2-y^2}} = -\frac{pab^2\dot{y}}{\sqrt{b^2-y^2}} - \frac{py^3\dot{y}}{\sqrt{b^2-y^2}}$. But, in order for the more easy finding the Fluent thereof, put $\sqrt{b^2-y^2} = u$; and then, y being $= \sqrt{b^2-u^2}$, and $\dot{y} = \frac{-u\dot{u}}{\sqrt{b^2-u^2}}$, we shall, by Substitution, get $\dot{s} = \frac{pab^2\dot{u}}{\sqrt{b^2-u^2}} + p \times \overline{b^2\dot{u} - u^2\dot{u}}$. Whence, the Fluent of $\frac{\dot{u}}{\sqrt{b^2-u^2}}$ being expressed by the Arch (A) of the Circle whose Radius is Unity and

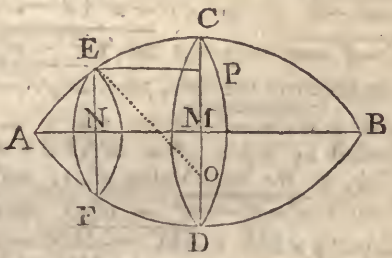
† Art. 142. Sine $\frac{u}{b}$ †, the Fluent of the whole Expression will be $pab^2 \times A + p \times \overline{b^2u - \frac{1}{3}u^3}$. Which, when $y=0$, or $u=b$, gives $(pab^2 \times \frac{1}{2}p + p \times \frac{2}{3}b^3)$ $pb^2 \times \overline{\frac{1}{2}pa + \frac{2}{3}b}$ for the Content of the whole Solid, when its Axis becomes infinite.

EX-

EXAMPLE VII.

152. *Where it is required to find the Content of a parabolic Spindle; generated by the Rotation of a given Parabola* ACB *about its Ordinate* AB.

Put CM (the Abſciſſa of the given Parabola) $= a$, and the Semi-ordinate AM (or BM) $= b$; and, ſuppoſing ENF to be any Section of the Solid parallel to DC, let its Diſtance MN (or EP) from DC, be denoted by w: Then, by the Property of the Curve, we ſhall



have AM^2 (b^2) : EP^2 (w^2) :: CM (a) : CP $= \frac{aw^2}{b^2}$: Therefore EN $(= CM - CP) = a - \frac{aw^2}{b^2} = \frac{a \times \overline{b^2 - w^2}}{b^2}$, and conſequently $p \times EN^2 = \frac{pa^2}{b^4} \times \overline{b^4 - 2b^2w^2 + w^4} =$ the Area of the Section EF: Which multiply'd by $(\dot{w})$ the Fluxion of MN, gives $\frac{pa^2}{b^4} \times \overline{b^4\dot{w} - 2b^2w^2\dot{w} + w^4\dot{w}}$ for the Fluxion of the Solidity, * whoſe Fluent, $\frac{pa^2}{b^4} \times \overline{b^4w - \frac{2}{3}b^2w^3 + \frac{1}{5}w^5}$, when w becomes $= b$, is $\left(\frac{8pa^2b}{15}\right)$ half the Content of the Solid.

* Art. 145.

EXAMPLE VIII.

153. *Let the Solid* ACBD *(see the last Figure) be a Spindle, generated by the Rotation of the Segment of a Circle,* ACB, *about its Chord, or Ordinate,* AB.

Then, if the Radius OE be put $= r$, OM $= d$, and EP $= w$ &c. (as before) we shall have OP $(= \sqrt{OE^2 - EP^2}) = \sqrt{r^2 - w^2}$, and EN $(= OP - OM)$ $= \sqrt{r^2 - w^2} - d$: Therefore $\dot{s}$, in this Case, is $= p\dot{w} \times \overline{\sqrt{r^2-w^2}-d}\,\vert^2 = p\dot{w} \times \overline{r^2 - w^2 + d^2 - 2d\sqrt{r^2-w^2}}$ $= p\dot{w} \times \overline{r^2 - d^2 - w^2} - p\dot{w} \times \overline{2d\sqrt{r^2-w^2} - 2d^2}$: Whence, the Fluent of the *Part*, $p\dot{w} \times \overline{2d\sqrt{r^2-w^2} - 2d^2}$ $(= 2dp \times \dot{w} \times \overline{\sqrt{r^2-w^2}-d} = 2dp \times \dot{w} \times EN)$ being expressed by $2dp \times$ *Area MNEC* * the Fluent of the *Whole*, or the true Value of s, will be expressed by $pw \times \overline{r^2 - d^2 - \frac{1}{3}w^2} - 2dp \times$ *Area MNEC*, or by its Equal $p \times MN \times \overline{AM^2 - \frac{1}{3}MN^2} - 2p \times OM \times$ *Area MNEC*: Which, when MN = MA, gives $p \times \frac{2}{3} AM^3 - 2p \times OM \times$ *Area ACM*, for the Content of half the Solid: Where the *Area ACM* may be found by *Art.* 124. or more easily by the common Table of the Areas of the Segments of a Circle; to be met with in most Books of *Gauging*.

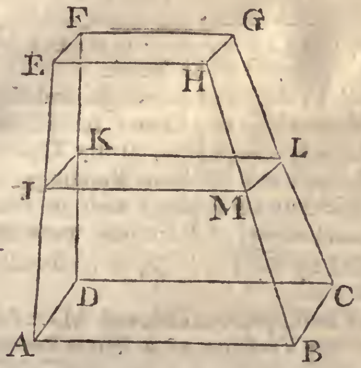
* Art. 112.

EXAMPLE IX.

154. *Let it be proposed to find the Content of the Solid* AEGB; whose four Sides AH, AF, CH, CF are plane Surfaces, and its Ends ADCB, EFGH given Rectangles, parallel to each other.

Let the Sides AB and AD, of the Base, be denoted by a and b; and those of the Top (EH and EF) by c and d respectively; moreover, let h express the perpendicular

dicular Height of the Solid; and let x (consider'd as variable) be the Distance of (IL) any Section thereof (parallel to the Base) from the Plane EG.



It is evident, from the Nature of the Figure, that the Section IL is a Rectangle; and that $b : x :: AB - EH : IM - EH :: BC - HG : ML - HG$.

From these Proportions we have $IM - EH = \frac{\overline{a-c} \times x}{b}$, and $ML - HG = \frac{\overline{b-d} \times x}{b}$: Hence $IM = \frac{\overline{a-c} \times x}{b} + c$, and $ML = \frac{\overline{b-d} \times x}{b} + d$; and consequently the Area of the Rectangle (IL) $= \frac{\overline{a-c} \times \overline{b-d}}{b^2} \times x^2 + \frac{ad - 2cd + cb}{b} \times x + cd$: Which being multiply'd by $\dot{x}$, and the Fluent taken, there results $\frac{\overline{a-c} \times \overline{b-d} \times x^3}{3b^2} + \frac{\overline{ad - 2cd + cb} \times x^2}{2b} + cdx$ for the Content of IFGL:

 Which,

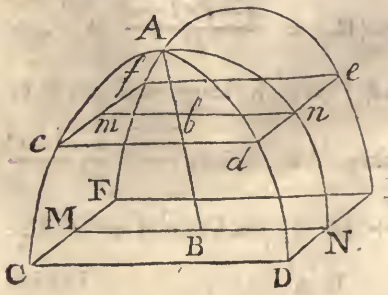
Which, when $x = b$, becomes $\left(\frac{\overline{a-c}\times\overline{b-d}\times b}{3}+\right.$ $\frac{\overline{ad-2cd+cb}\times b}{2}+cdb = \overline{2ab+ad+bc+2cd}\times\frac{1}{6}b =\left.\right)$ $\overline{AB\times AD+EH\times EF+\overline{AB+EH}\times\overline{AD+EF}}\times\frac{1}{6}b =$ the Quantity propoſed to be found.

If EF (d) be ſuppoſed to vaniſh, and the Lines EH and FG to coincide, the Planes AEHB and DFGC will form an Angle or Ridge, at the Top of the Solid (reſembling the Roofs of ſome Buildings, whoſe Ends as well as Sides run up ſloping) and, in this Caſe, the Content, found above, will become more ſimple, being then expreſſed by $\overline{2ab+bc}\times\frac{1}{6}b$, or its Equal $\overline{2AB+EH}\times AD\times\frac{1}{6}b$.

But, if EF be ſuppoſed $=$ EH, and AD $=$ AB, the Solid will then be the Fruſtrum of a ſquare Pyramid; and its Content $=\overline{a^2+ac+c^2}\times\frac{1}{3}b, = \overline{AB^2+AB\times EH+EH^2}\times\frac{1}{3}b$: From whence, by taking EH $=0$, the Content of the whole Pyramid whoſe Baſe is AB^2, and its Altitude b, will alſo be given, being $= AB^2\times\frac{1}{3}b$.

EXAMPLE X.

155. *Let the propoſed Solid be that, commonly known by the Name of a Groin*; whoſe Sections parallel to the Baſe are, all, Squares, and whereof the two Sections perpendicular to the Baſe, through the Middle of the oppoſite Sides, are Semi-circles.



Let $bcdef$ be any Section parallel to the Baſe; and let its Diſtance Ab from the Vertex of the Solid, be denoted by x; alſo let a repreſent the Radius AB (or BN) of the cir-

circular Section ABNA, perpendicular to the Base. Then, bn being (by the Property of the Circle) $= \sqrt{2ax - xx}$, the Side of the Square df, will be $= 2\sqrt{2ax - xx}$, and therefore the Area $= 4 \times \overline{2ax - xx}$; whence $\dot{s} = 4\dot{x} \times \overline{2ax - xx}$, and consequently $s = 4ax^2 - \frac{4x^3}{3}$: Which, when $x = a$, becomes $\frac{\overline{2a}|^3}{3} =$ the Content of the whole Solid.

If the Solid be a Groin of any other Kind, or such, that its two Sections perpendicular to the Base, through the Middle of the opposite Sides, are any other Curves than Semi-circles, the Content may, still, be found in the same Manner; and will be always in proportion to the Solid generated by the Revolution of the said Curve about its Axis, as a Square, is to its inscribed Circle. But, if the foresaid perpendicular Sections be Curves of different Kinds, the Sections parallel to the Base will no longer be Squares, but Rectangles; whose Sides are the corresponding (double) Ordinates of the respective Curves. Thus, for Instance, let one Section be a Circle and the other a Parabola, whose Ordinates, to the common Abscissa, x, are expressed by $\sqrt{dx - xx}$ and $\sqrt{ax}$, respectively; then the Sides of the rectangular Section, parallel to the Base of the Groin, will be $2\sqrt{dx - xx}$ and $2\sqrt{ax}$: Whence the Area of that Section is $= 4x\sqrt{ad - ax}$, and therefore $\dot{s} = 4x\dot{x}\sqrt{ad - ax}$: Where, by taking the Fluent, * $s =$

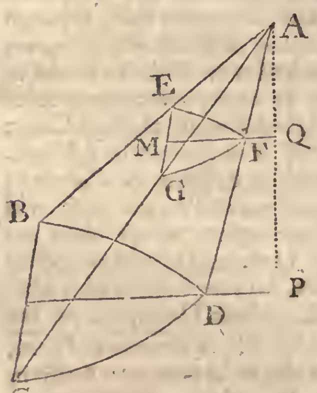
$$\frac{16d^2\sqrt{ad} - a^{\frac{1}{2}} \times \overline{d - x}|^{\frac{3}{2}} \times \overline{16d + 24x}}{15} = \text{the true}$$

Content of such a Solid.

* Art. 83.

EXAMPLE XI.

156. *Where the Solid* BACD *propofed is a kind of Cone, or Pyramid*; form'd by conceiving Right-lines to be drawn from every Point in the Perimeter of any given Plane BDC, to a given Point, or Vertex, A above that Plane.



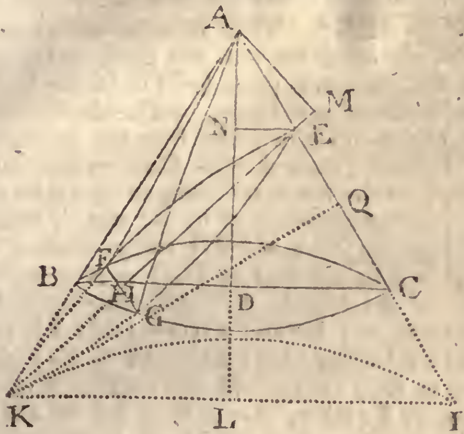
Let EFG be any Section parallel to BDC, whofe perpendicular Diftance (AQ) from the Vertex let be denoted by x; moreover, let the whole given Altitude (AP) of the Solid be put $= a$, and the Area of the Bafe BDC (which is alfo fuppofed given) $= b$.

In the firft place, it is eafy to conceive that the Planes BDC and EFG muft be fimilar: And therefore, fince fimilar Figures are to each other as the Squares of their like Sides, or Dimenfions, it follows that AP^2 (a^2) : AQ^2 (x^2) :: BDC (b) : EFG $= \frac{bx^2}{a^2}$.

Whence $\dot{s} = \frac{bx^2\dot{x}}{a^2}$, and confequently $s = \frac{bx^3}{3a^2} = \frac{ba}{3}$, when $x = a$. Therefore the Solidity of a Cone or Pyramid, let the Figure of its Bafe be what it will, is always had by multiplying the Area of the Bafe by $\frac{1}{3}$ of the Altitude.

EXAMPLE XII.

157. *Where it is proposed to find the Content of the Ungula* EFGC, cut off from a given Cone, ABC, by a Plane EFG passing through the Base thereof.



Let AD be the perpendicular Height of the Cone, also let AM be perpendicular to HE, the Axis of the Section FEG, and let FAG be another Section of the Cone, thro' FG and the Vertex A.

Since the Solids CAFG and EAFG, whose Bases are FCG, and FEG, come under the Form specified in the preceding Example, their Contents will therefore be expressed by $FCG \times \frac{1}{3}AD$ and $FEG \times \frac{1}{3}AM$ respectively: Whose Difference, $\frac{FCG \times AD - FEG \times AM}{3}$, is the Solidity of the *Ungula* CEFG: Where the Bases FCG and FEG being conic Sections, their Areas will be given by *Art.* 115. 124 and 129. from whence the whole will be known. Thus, if HE be supposed parallel to AB, the Section FEG, then being a Parabola, its Area will be $= \frac{2}{3} \times FG \times EH$ *: Whence the Solidity of the

* Art. 115.

 Segment

Segment EFGA is $= \frac{2}{9} \times FG \times EH \times AM$: Which being deducted from that of CFGA (found by Help of the common Table of circular Segments) the Remainder will be the Content of the *Ungula*. But, if the Axis EH produced, cuts AB, the Section FEG will be a Segment of an Ellipsis EFKG; whose conjugate Axis (supposing EN and KL perpendicular to AD) is

* Art. 41. $= 2\sqrt{EN \times KL}$ *. Now, in order to compute the Content, the easiest way, in this Case, let the Ratio of EH to EK (which is given by Trigonometry) be expressed by that of *m* to Unity, and let the Ratio of CH to CB, be as *n* to Unity: And from the common Table of *Segments* (adapted to the Circle whose Diameter is Unity) let the Areas answering to the versed Sines *m* and *n*, be taken and denoted by *M* and *N* respectively: Then, the Area of FEG being $= M \times EK \times$

† Art. 124 and 130. $2\sqrt{EN \times KL}$, and that of $FCG = N \times BC^2$ †, the Content of the *Ungula*, by substituting these Values, will become $= \frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times EK \times AM \times 2\sqrt{EN \times KL}$: But, since AM : AE :: KQ (perpendicular to AC) : KE; and AN : AE :: KQ : KI, it follows, by Equality, that $AM \times KE = AN \times KI$; whence the Content of the *Ungula* is also expressed by $\frac{1}{3} N \times BC^2 \times AD - \frac{1}{3} M \times AN \times KI \times 2\sqrt{EN \times KL}$. Which, if H be supposed to coincide with B, and KI with BC, will become $\frac{(0.78539}{3}$ &c. $\times BC^2 \times AD -$ $\frac{0.78539}{3}$ &c. $\times AN \times BC \times 2\sqrt{EN \times BD} = 0.26179$ &c. $\times BC \times \overline{BC \times AD - 2AN \times \sqrt{EN \times BD}}$.

When the Section EFG is an Hyperbola, its Area may be found by means of a Table of Logarithms (instead of a Table of Segments) whence the Content of the *Ungula* will likewise be had in that Case.

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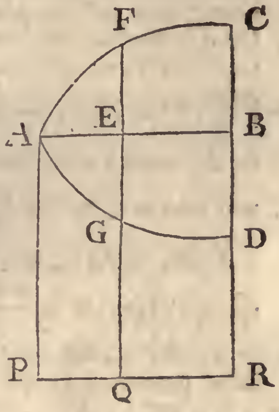
EXAMPLE XIII.

158. *Let* AFC, *or* AGD, *be a Curve of any Kind*; whofe Area, and the Content of the Solid arifing from its Rotation about its Axis, or Ordinate, AB, are both known; it is propofed to find, from thence, the Content of the Solid generated by the Revolution of that Curve about any other Line PR parallel to the faid Axis or Ordinate AB.



Let AP, FQ, and CR be all perpendicular to AB and to the Axis of Motion PQR; alfo let AP (or EQ) $= a$, AE, confidered as variable, $= w$, the Area AFE, or AEG $= M$, and the Solid, arifing from its Revolution about AB, $= N$. It is plain that the Area of the Circle generated by QF will be $= p \times FQ^2$ * $= p \times \overline{a + EF}|^2 = pa^2 + 2pa \times EF + p \times EF^2$; from which deducting the Area, pa^2, generated by QE, the Remainder, $2pa \times EF + p \times EF^2$, will be the Area of the Annulus generated by EF: Whence the Fluxion of the Solid generated by AEF is truly reprefented by $2pa \times EF \times \dot{w} + p\dot{w} \times EF^2$ †: And, in the fame manner, it will appear that the Fluxion of the Solid generated by AEG is $2pa \times EG \times \dot{w} - p\dot{w} \times EG^2$. But the Fluent of $EF \times \dot{w}$ (or $EG \times \dot{w}$) is $=$ the Area *(M)* of AEF (or AEG) ‡, and that of $p\dot{w} \times EF^2$ (or $p\dot{w} \times EG^2$) equal to *(N)* the given Solid arifing from that Area §; therefore the Fluent of the *Whole*, or the Solidity required, is $2paM + N$, in the former Cafe, and $2paM - N$ in the latter; where $2pa$, in

* Art. 145.

† Art. 145.

‡ Art. 112.

§ Art. 145.

in either Case, expresses the Periphery of the Cylinder described by AB, about the Axis of Rotation PR.

Hence, if ABC and ABD are equal and similar to each other, then the Value of M &c. being the same in both Cases, it follows that the Content of the Solid generated by AFG will be expressed by $2pa \times 2M$, or $2pa \times$ Area AFG.

Now, if (for Example sake) ACD be supposed a Circle, whose Semi-diameter is d, the Area of that Circle being $= pd^2$, the Solid generated by its Revolution (representing the Ring of an Anchor) will therefore be $= 2pa \times pd^2 = 2p^2ad^2$. But if you would know the Content of the Part generated by the upper Semicircle BAC, or the lower one BAD, let the Content

* Art. 148. $\left(\frac{4pd^3}{3}\right)$ * of a Sphere whose Semi-diameter is d, be wrote for N, in each of the two foregoing Expressions, and you will then get $p^2ad^2 + \frac{4pd^3}{3}$, and $p^2ad^2 - \frac{4pd^3}{3}$.

Again, if AFC, and AGD be taken as Right-lines, you will have $M = \frac{AB \times BC}{2}$ $\left(\text{or } \frac{AB \times BD}{2}\right)$ and N

† Art. 146. $= p \times BC^2 \times \frac{1}{3} AB$ (or $p \times BD^2 \times \frac{1}{3} AB$) †: Hence the Solid generated by the Triangle ABC is ($= 2pa \times \frac{AB \times BC}{2} + \frac{p}{3} \times BC^2 \times AB$) $= p \times AB \times BC \times \overline{RB + \frac{1}{3} BC}$; and that generated by ABD ($= 2pa \times \frac{AB \times BD}{2} - \frac{p}{3} \times BD^2 \times AB$) $= p \times AB \times BD \times \overline{RB - \frac{1}{3} BD}$.

Lastly, let ABC (or ABD) be considered as a Parabola, whose Ordinate is AB, and Axis CB (or DB): Then M being here $= \frac{2}{3} AB \times BC$ (or $\frac{2}{3} AB \times BD$) ‡

‡ Art. 115.

§ Art. 152. and $N = \frac{8p}{15} \times AB \times BC^2$ § (or $\frac{8p}{15} \times AB \times BD^2$)

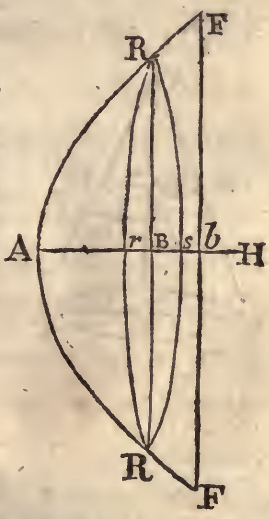
it

it follows that the Solid generated by ABC will be $(= 2pa \times \frac{2}{3} AB \times BC + \frac{8p}{15} \times AB \times BC^2) = 4p \times AB \times BC \times \frac{5BR + 2BC}{15}$, and that generated by ABD $= 4p \times AB \times BD \times \frac{5BR - 2BD}{15}$.

SECTION X.

The Use of Fluxions in finding the Superficies of solid Bodies.

159. LET FAF represent a Solid generated by the Revolution of any given Curve AF about its Axis AH; also let a Circle, whose Diameter is the variable Line (or Ordinate) RBR, be conceived to move uniformly from A towards FF, and to dilate itself so, on all Sides, at the same time, as to generate, by its Periphery, the proposed Superficies RAR: Then the Length of that Periphery, or the generating Line, being expressed by 3,141592 * &c. × RR ($= 2py$) and the Celerity with which it moves by $\dot{z}$ † the Fluxion of the Superficies RAR, or the Space that



* Art. 142.

† Art. 135.

 would

would be uniformly generated in the time of describing $\dot{z}$, will therefore be truly represented by $2py\dot{z}$.

Hence, if w be taken to represent the whole Surface RAR, generated from the beginning (according to the Method observed in the three last Sections) we shall

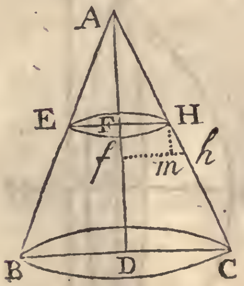
* Art. 135. have $\dot{w} = 2py\dot{z} = 2py\sqrt{\dot{x}^2 + \dot{y}^2}$ *; whence w itself may be found.

EXAMPLE I.

160. *Let it be proposed to determine the convex Superficies of a Cone* ABC.

Then, the Semi-diameter of the Base (BD, or CD) being put $= b$, the slanting Line, or Hypothenuse, AC $= c$, and FH (parallel to DC) $= y$ &c. we shall, from the Similarity of the Triangles ADC and Hmh,

† Art. 159. have $b : c :: \dot{y}\ (mh) : \dot{z}\ (\text{H}h) = \frac{c\dot{y}}{b}$: Whence $\dot{w}$ ($2py\dot{z}$ †) $= \frac{2pcy\dot{y}}{b}$; and consequently $w = \frac{pcy^2}{b}$. This, when



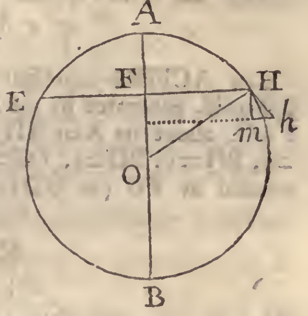
$y = b$, becomes $= pcb = p \times$ DC $\times$ AC $=$ the convex Superficies of the whole Cone ABC: Which therefore is equal to a Rectangle under half the Circumference of the Base and the slanting Line.

EX-

EXAMPLE II.

161. *Let the Solid, whose Surface you would find, be a Sphere* AEBH.

In which Case, putting the Radius $OH = a$, $AF = x$, $Hm = \dot{x}$, *&c.* we shall (by reason of the similar Triangles OHF and Hmh *) have y (FH) : a (OH) :: $\dot{x}$ (Hm) : $\dot{z}$ (Hh) $= \frac{a\dot{x}}{y}$: Therefore $\dot{w}$ $(2py\dot{z}) = 2pa\dot{x}$; and consequently the Superficies (w) itself $= 2pax = AF \times$ *Periph.* AEBH. Which, if the whole Sphere be taken, will become $AB \times$ *Periph.* AEBH = four times the Area BEAHO.



* Art. 68.

Hence the Superficies of a Sphere is equal to four times the Area of its greatest Circle: And the convex Superficies of any Segment thereof, is to that of the *Whole*, as the Axis (or Thickness) of the Segment to the Diameter of the Sphere.

EXAMPLE III.

162. *Wherein let the parabolic Conoid be proposed.*

The Equation of the generating Parabola being $ax = y^2$, or $x = \frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $\dot{z}\ (= \sqrt{\dot{y}^2 + \dot{x}^2}\ †) = \sqrt{\dot{y}^2 + \frac{4y^2\dot{y}^2}{a}} = \frac{\dot{y}\sqrt{a^2 + 4y^2}}{a}$: † Art. 135.

Hence $\dot{w}\ (2py\dot{z}) = \frac{2py\dot{y}}{a} \times \overline{a^2 + 4y^2}|^{\frac{1}{2}}$; whereof the

Fluent

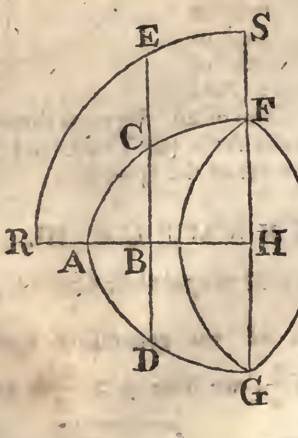
Fluent is $\frac{p \times \overline{a^2 + 4y^2}|^{\frac{3}{2}}}{6a}$; which corrected (by sup-

* Art. 79. posing $y = 0$ *) gives $\frac{p \times \overline{a^2 + 4yy}|^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$, for the Superficies sought.

EXAMPLE IV.

163. *Let it be required to determine the Superficies of a Spheroid.*

Let ACFHG represent one half of the proposed Spheroid, generated by the Rotation of the Semi-ellipsis FAG, about its Axis AH; put AH $= a$, FH (or HG) $= c$, BH $= x$, BC $= y$, FC $= z$, and the Superficies generated by FC (or GD) $= w$: Then, from the Na-



ture of the Ellipsis, we have $y = \frac{c}{a}\sqrt{a^2 - x^2}$; whence

† Art. 135. $\dot{y} = -\frac{cx\dot{x}}{a\sqrt{a^2 - x^2}}$, and consequently $\dot{z}$ ($= \sqrt{\dot{x}^2 + \dot{y}^2}$ †)

=

$= \sqrt{\dot{x}^2 + \frac{c^2x^2\dot{x}^2}{a^2 \times \overline{a^2 - x^2}}} = \frac{\dot{x}\sqrt{a^4 - \overline{aa - cc} \times xx}}{a\sqrt{aa - xx}} =$ $\frac{\dot{x}\sqrt{a^4 - b^2x^2}}{a\sqrt{a^2 - x^2}}$ = (by putting (the Excentricity) $\sqrt{a^2 - c^2} = b) = \frac{b\dot{x}\sqrt{\frac{a^4}{bb} - x^2}}{a\sqrt{a^2 - x^2}}$: Therefore, in this Caſe, $\dot{w}$ $(2py\dot{z}) = \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{bb} - x^2}$; whoſe Fluent, in an Infinite Series, is $2pcx \times \overline{1 - \frac{b^2x^2}{2.3a^4} - \frac{b^4x^4}{2.4.5a^8} - \frac{3b^6x^6}{2.4.6.7a^{12}}}$. But the ſame Fluent may be, *otherwiſe*, very eaſily exhibited by means of the Area of a Circle: For, if from the Center H, with a Radius equal to $\frac{aa}{b}$, a Circle SER be deſcribed, and the Ordinate BC be produced to interſect it in E, it is evident that BE $= \sqrt{\frac{a^4}{bb} - xx}$, and that the Fluxion of the Area ESHB will be expreſſed by $\dot{x}$ $\sqrt{\frac{a^4}{bb} - x^2}$; which being to $\frac{2pbc\dot{x}}{aa} \times \sqrt{\frac{a^4}{bb} - x^2}$, the Fluxion before found, in the conſtant Ratio of 1 to $\frac{2pbc}{a^2}$, their Fluents muſt therefore be in the ſame Ratio; and ſo the latter, expreſſing the Superficies CFGD, will conſequently be $= \frac{2pbc}{aa} \times$ BESFH $= 2p \times \frac{\text{FH}}{\text{HS}}$ $\times$ BESFH.

This Solution, it may be obſerved, obtains only in Caſe of an *oblong* Spheroid, generated by the Rotation of the Ellipſis about its greater Axis; for, in an *oblate* Spheroid,

Spheroid, generated about the lesser Axis, the Value of b $(\sqrt{a^2 - c^2})$ will be impossible; since, in this Case HF is greater than HA. But, if we, *here*, put $b = \sqrt{c^2 - a^2}$, and $d = \frac{a^2}{b}$, the Value of $\dot{w}$ (found above) will become $= \frac{2pbc\dot{x}}{a^2}\sqrt{\frac{a^4}{bb} + x^2} = \frac{2pc\dot{x}}{d}\sqrt{d^2 + x^2}$ $= \frac{2pc}{d} \times \dot{x}\sqrt{a^2 + x^2}$: Whose Fluent may be brought out by help of a Table of Logarithms: For, let the variable Part $\dot{x}\sqrt{d^2 + x^2}$ be transformed to $\left(\frac{\dot{x} \times \overline{d^2 + x^2}}{\sqrt{d^2 + x^2}} = \frac{d^2\dot{x} + x^2\dot{x}}{\sqrt{d^2 + x^2}} = \frac{d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}} = \right) \frac{\frac{1}{2}d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}} + \frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, so that the Numerator of the first Term $\frac{\frac{1}{2}d^2x\dot{x} + x^3\dot{x}}{\sqrt{d^2x^2 + x^4}}$ (now in a given Ratio to the Fluxion of the Quantity under the radical Sign) may be had by the common Rule *; by which means we get $\frac{1}{2}\sqrt{d^2x^2 + x^4}$, for the true Fluent of the said Term; to which adding the Fluent of the other Term $\frac{\frac{1}{2}d^2x\dot{x}}{\sqrt{d^2x^2 + x^4}}$, or $\frac{\frac{1}{2}d^2\dot{x}}{\sqrt{d^2 + x^2}}$ (given by *Art.* 126.) there arises $\frac{1}{2}x\sqrt{d^2 + x^2} + \frac{1}{2}d^2 \times$ hyp. Log. $x + \sqrt{d^2 + x^2}$, for the Fluent of $\dot{x}\sqrt{d^2 + x^2}$: And this, corrected † and multiplied by $\frac{2pc}{d}$, gives $\frac{pcx}{d}\sqrt{d^2 + x^2} + pcd \times$ hyp. Log. $\frac{x + \sqrt{dd + xx}}{d}$, for the Superficies in this Case, where the proposed Spheroid is an oblate One.

* Art. 77.

† Art. 78.

EXAMPLE V.

164. *Let the Solid, whose Superficies is sought, be the hyperbolical Conoid.*

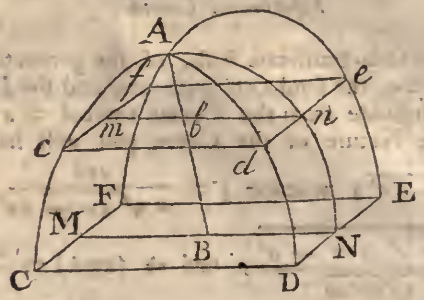
Let the semi-transverse Axis, of the generating Hyperbola, $=a$; the semi-conjugate $=c$, and the Distance of any Ordinate from the Center thereof $=x$; then from the Nature of the Curve you will have $y = \frac{c}{a}\sqrt{x^2 - a^2}$; whence $\dot{y} = \frac{cx\dot{x}}{a\sqrt{xx - aa}}$, $\dot{z} = \frac{\dot{x}\sqrt{\overline{a^2 + c^2} \times x^2 - a^4}}{a\sqrt{xx - aa}}$, and $\dot{s}$ $(2py\dot{z}) = \frac{2pc\dot{x}}{aa} \times \sqrt{\overline{aa + cc} \times xx - a^4}$; which last Value, if d^2 be put $= \frac{a^4}{a^2 + c^2}$, will be more commodiously expressed by $\frac{2pc\dot{x}}{d}\sqrt{x^2 - d^2}$: whereof the Fluent, by proceeding as in the latter Part of the foregoing Example, will come out $= \frac{pcx\sqrt{xx - dd}}{d} - pcd \times$ *hyp. Log.* $x + \sqrt{x^2 - d^2}$: Which corrected (by taking $x = a$) becomes $\frac{pcx}{d}\sqrt{xx - dd} - pc^2, - pcd \times$ *hyp. Log.* $\frac{x + \sqrt{x^2 - d^2}}{a + \frac{cd}{a}}$, the true Measure of the required Superficies.

EXAMPLE VI.

165. *Let it be proposed to find the Superficies of the Solid called a Grain.* (Vid. Art. 155.)

Let *bcdef* be any Section of the Solid parallel to the Base thereof, and let x denote its Distance from the Vertex

Vertex A, alſo put z equal to the correſponding Arch An of the ſemi-circular Section NnA *&c.* whoſe Radius AB or BN let be denoted by a.



It appears from *Art.* 161. that $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$:

* Art. 159. Which Value, multiplied by $(2\sqrt{2ax - xx})$ that of de $(= 2bn)$ gives $2a\dot{x}$ * for the Fluxion of one of the four equal convex Superficies by which the Solid is bounded. Hence the whole Superficies (excluding the Baſe) comes out $= 8a^2$: Which therefore is exactly equal to twice the Baſe.

If the Solid be ſuppoſed a Groin of any other Kind, ſuch that its two equal Sections, through the Middle of the oppoſite Sides, are other Curves than Circles, the Superficies may *ſtill* be had in the ſame manner; and will be always in proportion to the Superficies ariſing from the Revolution of either of the ſaid equal Curves about its Axis, as a Square is to its inſcribed Circle. Thus, the Superficies of a parabolic Conoid being $=$

$$\frac{p \times \overline{aa + 4yy}\,^{\frac{3}{2}}}{6a} - \frac{pa^2}{6}$$ (*by Art.* 162.) the convex Superficies of the Groin, ſuppoſing the generating Curve AnN to be a Parabola, will therefore be $=$

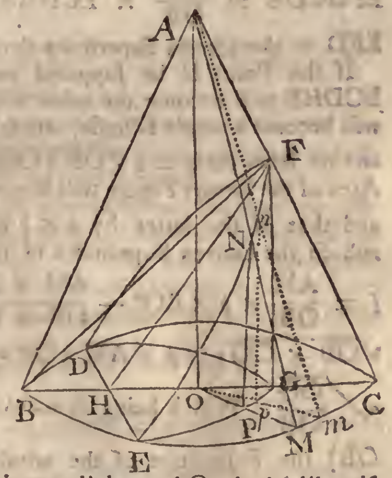
$$\frac{4 \times \overline{a^2 + 4yy}\,^{\frac{3}{2}}}{6a} - \frac{4a^2}{6}.$$

E X-

EXAMPLE VII.

166. *Wherein let it be required to find the convex Superficies of a conical* Ungula ECFD; formed by a Plane DFE paſſing through the Baſe of the Cone.

Let a right-angled Triangle AOM (whoſe Baſe OM is the Radius of the Circle BDCE) be ſuppoſed to revolve about the Axis AO; whilſt a Right-line NP, drawn perpendicular to OM from the interſection of AM and the Arch EFD, traces out, upon the Baſe of the Cone, the Curve-line EPGD.



If MPOAN and *mp*OA*n* be conſidered as two Poſitions of the generating Triangle indefinitely near to each other, it is evident that the Space MA*m*, generated by AM, will be to the Space MO*m*, generated by OM, as AM to OM, or OB. Whence, MN and MP being proportional Parts of AM and OM (becauſe NP is parallel to AO) it is likewiſe plain that the Spaces MN*nm* and MP*pm*, generated by thoſe Parts, will be to each other in the ſame Ratio of AM to OB. And ſince this every where holds, it follows that the whole Space (ENM) *&c.* generated by MN, will be to that (EPM) generated by PM, as AM to OB: And ſo the whole required Superficies (generated by AM) is truly repreſented by $\frac{AM}{OB} \times$ Area EPGDCE.

 But

But now, to find this Area, EPGDCE, it is observable that the Area of the Plane DFE (being the Segment of a Conic-section) is given, *by Art.* 115. 129 or 130. And it is very easy to apprehend and demonstrate that the Area so given will be to that of EGDH, as the Radius to the Co-sine of the Angle of the Inclination of the said Plane to the Base, or as HF to HG. Therefore, seeing EGDH is $= \frac{HG}{HF} \times EFD$, we have EPGDCE ($=$ ECDHE $-$ EGDH) $=$ ECDHE $- \frac{HG}{HF} \times EFD$; and consequently $\frac{AM}{OB} \times EPGCDE = \frac{AM}{OB} \times ECDHE - \frac{AM \times HG}{OB \times HF} \times EFD =$ the convex Superficies that was to be found.

If the Point H be supposed to coincide with B, ECDHE will become the *whole* Circle CB; and EDF will become a whole Ellipsis, whose greater Axis is BF, and its lesser Axis $= 2\sqrt{OB \times OG}$. * Therefore, the Area of the former Figure will be expressed by $p \times BO^2$ †, and that of the latter by $p \times \frac{1}{2} BF \times \sqrt{OB \times OG}$; and so the convex Superficies of the Part BFC will be $\left(= \frac{AM}{OB} \times p \times BO^2 - \frac{AM \times BG}{OB \times BF} \times p \times \frac{1}{2} BF \times \sqrt{OB \times OG}\right) = p \times AM \times OB - p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$: Which being deducted from ($p \times AM \times OB$) the Superficies of the whole Cone BAC, there rests $p \times AM \times \frac{1}{2} BG \times \sqrt{\frac{OG}{OB}}$, for the Superficies of the oblique Cone BAF; which from hence is also given.

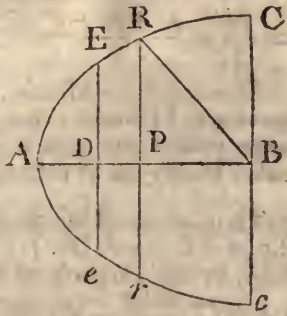
* Art. 41.

† Art. 124.

SCHO-

SCHOLIUM.

167. In most of the Examples, delivered in the four last Sections, the Part of the proposed *Figure* next the Vertex, whether, a Curve, Solid, or Superficies, is first found; from whence, by taking the Altitude (x) of that Part equal to (a) the Altitude given, the Content of the *Whole* is deduced: But, if the Content of the lower Segment (BCED) of any Figure (ABC) arising by taking away a Part (ADE) next the Vertex, be required; then the Difference between the *Whole* and the Part taken away (found as before explained) will be the Quantity sought.



Thus, for Example, let ABC be the common *Parabola*, and let it be proposed to find the Content of the Part, BCED, included between any two Ordinates BC (b) and DE (c) at a given Distance BD (d) from each other: Then, the Equation of the Curve being $ax = y^2$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $y\dot{x}$ * $= \frac{2y^2\dot{y}}{a}$, *Art. 112.

whose Fluent $\frac{2y^3}{3a}$ is a general Expression for the Area comprehended between the Vertex and the Ordinate y: Whence, expounding y, by b and c successively, we get $\frac{2b^3}{3a}$ and $\frac{2c^3}{3a}$ for the corresponding Values of ABC and ADE; whose Difference $\frac{2b^3 - 2c^3}{3a}$ is the required Area BCED: But, to express the same independent of a, it will be, by the Property of the Curve, $b^2 : c^2 :: AB : AD$;

 whence,

whence, by Division, $b^2 : b^2 - c^2 :: AB : BD$ (d) and consequently $\frac{b^2 - c^2}{d} = \frac{b^2}{AB} = a$; which first Value being wrote instead of a, there results $BCED = \frac{\overline{2b^3 - 2c^3} \times d}{3b^2 - 3c^2}$ $= \frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$.

After the same Manner, the Segments of other Figures may be found; but in many Cases they will be more readily had from a direct Investigation, without either finding the Whole or the Part taken away.

Thus, in the Case above, if the Excess of any Ordinate RP above DE (c) be denoted by w, we shall have, by the Property of the Curve, $b^2 - c^2$ $(BC^2 - DE^2) : \overline{c + w}|^2 - c^2$ $(RP^2 - DE^2) :: DB$ $(d) : DP = \frac{d \times \overline{2cw + w^2}}{b^2 - c^2}$; whose Fluxion $\left(d \times \frac{2c\dot{w} + 2w\dot{w}}{b^2 - c^2}\right)$ multiplied by $c + w$ $(= PR)$ gives $d \times \frac{2c^2\dot{w} + 4cw\dot{w} + 2w^2\dot{w}}{b^2 - c^2}$, for the Fluxion of the Area DPRE: Whereof the Fluent (which is $2dw \times \frac{c^2 + cw + \frac{1}{3}w^2}{b^2 - c^2}$) will, when $w = b - c$ (or $RP = BC$) be truly expounded by $\frac{2d \times \overline{b - c} \times \overline{\frac{1}{3}b^2 + \frac{1}{3}bc + \frac{1}{3}c^2}}{b^2 - c^2}$ or its Equal, $\frac{2d}{3} \times \frac{b^2 + bc + c^2}{b + c}$; *the same as before.*

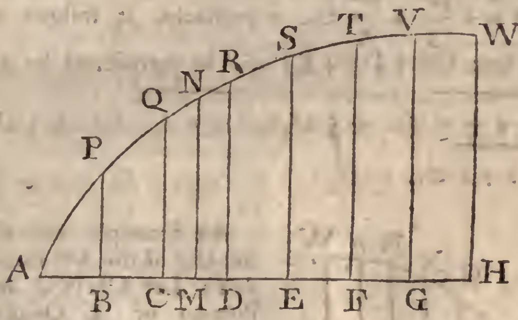
Again, for another Example, let CEDce be considered as the lower Frustrum of an Hemisphere, whose Center is the Point B: Then, BP being *here*, denoted by w, we shall have y^2 $(= BR^2 - BP^2) = b^2 - w^2$, and consequently $py^2\dot{w}$ * $= p \times \overline{b^2\dot{w} - w^2\dot{w}}$; whose Fluent

* Art. 145.

Fluent $(p \times \overline{b^2w - \frac{1}{3}w^3} = \frac{1}{3}pw \times \overline{3b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + b^2 - w^2} = \frac{1}{3}pw \times \overline{2b^2 + y^2} = \frac{1}{3}p \times BP \times \overline{2BC^2 + PR^2})$ is the true Content of the Part CED*ec*; which will alſo hold when the Figure is a Spheroid.

This laſt Method, of finding the Content of a Portion of a Figure, remote from the Vertex, will be of Service, when the general Value, for the *Whole*, cannot be expreſſed without an infinite Series; becauſe ſuch a Series, in that Caſe, not converging, becomes uſeleſs *. * Art. 93.

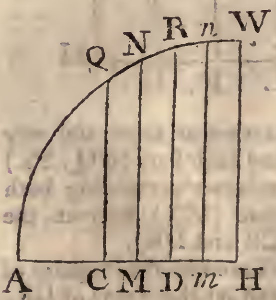
By dividing the whole propoſed Figure, AHW, into a Number of ſuch Portions, HV, GT, FS, &c. the Content thereof may be obtained, when to find it at once, by a Series, commencing from the Vertex, would be altogether impracticable.



But, to render ſuch an Operation as ſhort and eaſy as may be, it will be proper to find each Part (DQ, &c.) of the Figure, by means of a Series proceeding both Ways, from the middle Ordinate (MN) between the two correſponding Extremes (CR and DR).

Thus, let the Value of MN (found by the Property of the Curve) be denoted by a; and let the Value of DR, in a Series, be repreſented by $a + bx + cx^2 + dx^3 + ex^4 + fx^5 +$ &c. where $x =$ MD; then the Area MDRN will be repreſented by the Fluent of $\dot{x} \times \overline{a + bx + cx^2 + dx^3 +}$ &c.

&c. or by $x \times \overline{a + \frac{bx}{2} + \frac{cx^2}{3} + \frac{dx^3}{4}} + \&c.$ And by writing $-x$ instead of x, the Ordinate CQ will be expressed by $a - bx + cx^2 - dx^3$ &c. and the Area MCQN, by $x \times \overline{a - \frac{bx}{2} + \frac{cx^2}{3} - \frac{dx^3}{4} + \frac{ex^4}{5}}$ &c. whence the Area CDRQ is $= 2x \times \overline{a + \frac{cx^2}{3} + \frac{ex^4}{5} + \frac{gx^6}{7}} + \&c.$ Therefore, if DE, EF, FG, and GH be supposed, each, $=$ BC ($2x$) and the Areas DS, ET, &c. (found as above) be denoted by $2x \times \overline{a' + \frac{c'x^2}{3} + \frac{e'x^4}{5}}$ &c. and $2x \times \overline{a'' + \frac{c''x^2}{3} + \frac{e''x^4}{5}}$ &c. respectively, it follows that the Area CR + DS + ET will be represented by $2x \times \overline{a + a' + a''}$ &c. $+ \frac{2}{3} x^3 \times \overline{c + c' + c''}$ &c. $+ \frac{2}{3} x^5 \times \overline{e + e' + e''}$ &c.



An Example will shew the Use of this last Expression: Let CHWQ be a Portion of a Quadrant HAW of a Circle, whose Base HC (conceived to be divided into four equal Parts) is equal half the Radius AH, represented by *Unity*. Then, putting CM ($= DM = Dm = mH = \frac{1}{8}$) $= x$, HM ($= \frac{3}{8}$) $= p$, and Hm ($= \frac{1}{8}$) $= q$, we have, by the Property of the Circle, a (MN) $= \sqrt{\overline{HN}^2 - \overline{HM}^2} = \sqrt{1 - pp}$, and DR

DR $(=\sqrt{HR^2-HD^2}) = \sqrt{1-\overline{p-x}^2} = \sqrt{1-p^2+2px-x^2} = \sqrt{a^2+2px-x^2}$; which, in a Series, is $(=a+\frac{2px-x^2}{2a}-\frac{\overline{2px-x^2}^2}{8a^3}+$ &c.) $=a+\frac{px}{a}-\overline{\frac{1}{2a}+\frac{p^2}{2a^3}}\times x^2$ &c. Therefore, in this Case, $b=\frac{p}{a}$, $c=-\frac{1}{2a}+\frac{p^2}{2a^3}$, &c. Which Value of c, by writing $1-a^2$ for its Equal p^2, will be reduced to $-\frac{1}{2a^3}$. From whence it is also evident that $\acute{c}=-\frac{1}{2\acute{a}^3}$ (supposing $\acute{a}$ (mn) $=\sqrt{1-q^2}$)

Consequently $2x\times\overline{a+\acute{a}+\acute{\acute{a}}\text{ \&c.}}+\frac{2}{3}x^3\times\overline{c+\acute{c}+\acute{\acute{c}}\text{ \&c.}}+\frac{2}{5}x^5\times\overline{e+\acute{e}+\acute{\acute{e}}\text{ \&c.}}\ (=\overline{a+\acute{a}}\times 2x+\overline{c+\acute{c}}\times\frac{2x^3}{3}) = \overline{\sqrt{\frac{55}{64}}+\sqrt{\frac{63}{64}}}\times\frac{1}{4}-$

$$\overline{\frac{1}{\frac{2\times 55}{64}\sqrt{\frac{55}{64}}}+\frac{1}{\frac{2\times 63}{64}\sqrt{\frac{63}{64}}}}\times\frac{2}{64\times 8\times 3} =$$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{1}{3\times 55\sqrt{55}}-\frac{1}{3\times 63\sqrt{63}} =$$

$$\frac{\sqrt{55}+\sqrt{63}}{32}-\frac{\sqrt{55}}{3\times 55\times 55}-\frac{\sqrt{63}}{3\times 63\times 63} =$$

0,48730 = the Area, CHWQ, that was to be found.

This Example, chosen as an Illustration of the foregoing Method, may indeed be wrought the common Way; whence the very same Conclusion is brought out (*Vide*

(*Vide Art.* 124.) But that Method is alſo applicable to any other Caſe, whether the Part propoſed be near to the Vertex, or remote from it; and whether the Figure itſelf be a Curve, Solid or Superficies; ſince the Meaſure thereof may, always, be expreſſed by the Area of a Curve.

There is another Way, well known to Mathematicians, whereby the Area of a Curve may be determined, by means of a Number of equidiſtant Ordinates; which Method, derived from *that of Differences*, may, alſo, be uſed to good Purpoſe, in Caſes like thoſe above ſpecified: But, it having been treated of by ſeveral *others*, and alſo in my *Diſſertations*, the Reader will excuſe me, if no further Notice is taken of it here.

SECTION XI.

Of the Uſe of FLUXIONS *in finding the Centers of Gravity, Percuſſion, and Oſcillation of Bodies.*

168. THE Center of Gravity is that Point of a Body, by which, if it were ſuſpended, it would reſt *in Equilibrio*, in any Poſition.

LEMMA.

169. *Let p, q, r, s, &c. be any Number of given Weights, hanging at an inflexible Line (or Rod)* AM *ſuſpended* in Equilibrio, *in an horizontal Poſition, at the Point* O; *to determine the Poſition of that Point.*

Since *(by Mechanics)* the Force of any Weight *(p)* to raiſe the oppoſite End (M) of the Balance, is as that Weight drawn into its Diſtance (BO) from the Fulcrum,

crum, we ſhall, from the Equality of theſe Forces, have $p \times OB + q \times OC + r \times OD = s \times OE + t \times OF$,

that is $p \times \overline{AO-AB} + q \times \overline{AO-AC} + r \times \overline{AO-AD} = s \times \overline{AE-AO} + t \times \overline{AF-AO}$, and conſequently $AO = \dfrac{p \times AB + q \times AC + r \times AD + s \times AE + t \times AF}{p+q+r+s+t}$.

From which it appears, that, *if each Weight be multiply'd by its Diſtance from the End (or any given Point) of the Axis, the Sum of all the Products divided by the Sum of all the Weights, will give the Diſtance of the Center of Gravity from that End (or Point.)*

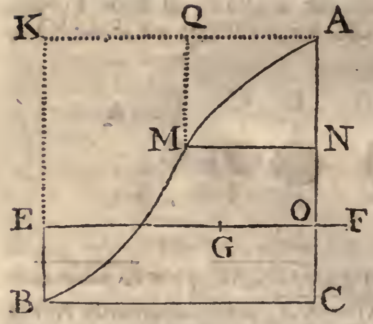
Note. The Products here mentioned are, uſually, call'd *the Forces*, of their reſpective Weights; not in reſpect to their Action at the Center O (which is expreſſed by a different Quantity) but with regard to the Effects they have in the Concluſion, or the Value of AO; which appear to be in that Ratio.

PROPOSITION I.

170. *To determine the Center of Gravity of a Line, Plane, Superficies, or Solid* (admitting the three former capable of being affected by Gravity.)

Let AMBC be the propoſed Figure, and G the Center of Gravity thereof; thro' which, parallel to the Horizon, let the Line EF be drawn, interſecting AC, at Right-angles, in O; alſo let AK and NM be perpendicular to AC, and parallel to EF.

171. Caſe



171. Case 1. *If the Figure* AMBC *be a Plane*; let it be suppos'd to rest *in Equilibrio* upon the Line EF; and then, if the Line MN be consider'd as a Weight, its Force *(defined above)* will be expressed by MN drawn into its Distance (AN) from the End of the Axis AC; that is by yx (supposing, as usual, AN$=x$ and MN$=y$.) This, therefore, multiply'd by the Fluxion of AN, gives $yx\dot{x}$ for the Fluxion of the Force of the Plane AMN; whose Fluent, when $x=$AC, expresses the Force of the whole Plane, or the Sum of all the Products of the Ordinates (or Weights) by their respective Distances from AK: Which Fluent being, therefore, divided by the Area ABC, or the Fluent of $y\dot{x}$ (according to the foregoing Lemma) the Quotient $\left(\frac{Flu.\ yx\dot{x}}{Flu.\ y\dot{x}}\right)$ will give (AO) the Distance of the Center of Gravity from the Line AK.

172. Case 2. *If the Figure be a Solid*; let MN be a Section thereof by a Plane perpendicular to the Horizon; then, the Area of that Section being denoted by A, the Force thereof (considered as above) will be expressed by Ax, and the Fluxion of the Force of the Solid AMN by $Ax\dot{x}$; whose Fluent, divided by the Content of the Body, or the Fluent of $A\dot{x}$, gives AO, in this Case. But, if the Solid be the half (or the whole) of that arising from the Rotation of a Curve AMB about its Axis AC; then (putting p for the Area of the Circle whose Radius is Unity) A will become $=\frac{1}{2}py^2$ *; and consequently AO $=\frac{Flu.\ \frac{1}{2}py^2x\dot{x}}{Flu.\ \frac{1}{2}py^2\dot{x}}=\frac{Flu.\ y^2x\dot{x}}{Flu.\ y^2\dot{x}}$.

* Art. 145.

173. Case

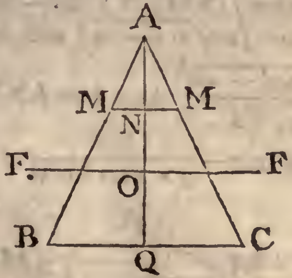
173. Case 3. *If the Figure proposed be the Curve-line* AMB; then, the Force of a Particle at M being expressed by AN or MQ (x) we shall (putting AM $= z$) have $\frac{Flu.\ x\dot{z}}{z} = \text{AO}.$

174. Case 4. *But if the Figure given be the Superficies generated by the Rotation of* AMB *about* AC.

Then, the Periphery of the Circle generated by the Point M being $= 2py$, it follows that $\frac{Flu.\ 2pyx\dot{z}}{Flu.\ 2py\dot{z}} = \frac{Flu.\ yx\dot{z}}{Flu.\ y\dot{z}} = \text{AO}.$

EXAMPLE I.

175. *Let the Figure proposed be the isosceles Triangle* ABC.



It is evident the Center of Gravity (O) will be somewhere in the Perpendicular AQ: And, if AQ$=a$, BC$=b$, AN $=x$, and MM$=y$; then y being $= \frac{bx}{a}$, we shall have, *by Case* 1, AO ($= \frac{Flu.\ yx\dot{x}}{Flu.\ y\dot{x}}$) $= \frac{Flu.\ x^2\dot{x}}{Flu.\ x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2} = \frac{2x}{3} = \frac{2}{3}$ AQ, when $x =$ AQ; and consequently OQ $= \frac{\text{AQ}}{3}$.

In the very same manner, the Center of Gravity of any other (plane) Triangle will appear to be at $\frac{1}{3}$ of the Altitude of the Triangle.

EX-

EXAMPLE II.

176. *Let the Figure proposed be a Parabola of any Kind;*

whereof the Equation is $y = \frac{x^{n}}{a^{n-1}}$.

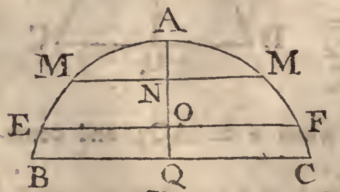
Here, $\frac{Flu.\ yx\dot{x}}{Flu.\ y\dot{x}} = \frac{Flu.\ x^{n+1}\dot{x}}{Flu.\ x^{n}\dot{x}} = \frac{n+1}{n+2} \times x =$ the Distance of the Center of Gravity from the Vertex of the Curve.

EXAMPLE III.

177. *Let* BAC *be a Segment of a Circle.*

Then, if the Radius thereof be put $= r$, we shall have y (NM) $= \sqrt{2rx - xx}$: Whence the Fluent of $yx\dot{x}$ $(x\dot{x}\sqrt{2rx - xx})$ will, by *Art.* 163. be found $= -\frac{\overline{2rx - xx}|^{\frac{3}{2}}}{3} + r \times$ Area ANM; which divided by ANM,

* Art. 171. gives $r - \frac{NM^3}{3 \times \text{Area ANM}} =$ AO*, This, therefore, when BAC is a Semi-circle, becomes $= \frac{576}{1000} \times r$, nearly.



But, with respect to the Center of Gravity of the Arch BAC; we have, *Flu.* $x\dot{z}$, (*by Case* 3.) $=$ Fluent of $\frac{rx\dot{x}}{\sqrt{2rx - xx}} = r \times \overline{AM - MN}$; and consequently AO *here* $= r - \frac{r \times MN}{AM}$.

EXAMPLE IV.

178. *Let* ABC (fee the preceding Figure) *reprefent a Segment of a Sphere, or Spheroid.*

In which Cafe, denoting the Axis of the Sphere, or Spheroid, by a, and the other Axis of the generating Curve, when an Ellipfis, by b, we have $y^2 = \frac{bb}{aa} \times \overline{ax - xx}$;

and therefore $\frac{Flu.\ y^2 x\dot{x}}{Flu.\ y^2 \dot{x}}$ * $= \frac{Flu.\ \overline{ax - xx} \times x\dot{x}}{Flu.\ \overline{ax - xx} \times \dot{x}} =$ * Art. 172.

$= \frac{\frac{1}{3} ax^3 - \frac{1}{4} x^4}{\frac{1}{2} ax^2 - \frac{1}{3} x^3} = \frac{\frac{1}{3} ax - \frac{1}{4} x^2}{\frac{1}{2} a - \frac{1}{3} x} = \frac{x \times \overline{4a - 3x}}{6a - 4x} = AO.$

If the Solid be an hyperbolical Conoid, the Diftance (AO) of its Center of Gravity from the Vertex, will alfo be exhibited by the Expreffion here brought out, when the negative Signs are changed to pofitive ones.

179. In thofe Cafes where the Figure cannot be divided into two Parts, equal and like to each other (as a Curve is by its Axis, &c.) the Pofition of two Lines EO, *eo* (*fee the enfuing Figure*) muft be determined, *as above*; in whofe Interfection (G) the Center of Gravity will be found.

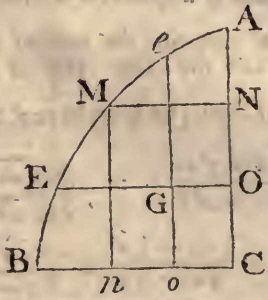
EXAMPLE V.

Let ABC *be a Semi-parabola of any Kind*; whereof the

Equation is $y = \frac{x^n}{a^{n-1}}$.

It appears, from *Ex.* 2. that (AO) the Diftance of EGO from the Vertex, is expreffed by $\frac{n+1}{n+2} \times$ AC: But to find the Pofition of *o*G*e* (perpendicular to EO) let M*n* be parallel to *eo*, or AC; then, AN being $= x$, and

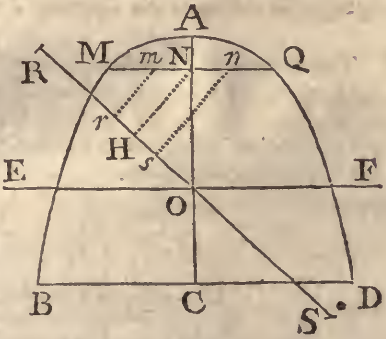
and NM $(y) = \frac{x^n}{a^{n-1}}$, if AC be denoted by b, we



ſhall have $Mn = b - x$, and $Mn \times NM \times \dot{y} = \overline{b-x} \times \frac{x^n}{a^{n-1}} \times \frac{nx^{n-1}\dot{x}}{a^{n-1}} = \frac{nbx^{2n-1}\dot{x} - nx^{2n}\dot{x}}{a^{2n-2}}$, for the Fluxion of the Sum of the Forces in this Caſe (*Vid. Art.* 171.) whoſe Fluent $\left(\frac{nbx^{2n}}{2na^{2n-2}} - \frac{nx^{2n+1}}{\overline{2n+1} \times a^{2n-2}}\right.$ $= \frac{x^{2n}}{a^{2n-2}} \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} = y^2 \times \overline{\frac{b}{2} - \frac{nx}{2n+1}} =$ $y^2 \times \frac{b}{4n+2}$, or $\frac{BC^2 \times AC}{4n+2}$, when $x = b$) divided by the Area ABC $\left(= \frac{BC \times AC}{n+1}\right)$ gives $\frac{n+1}{4n+2} \times$ BC for the true Value of Co, or OG. Which, in caſe of the common Parabola, where $n = \frac{1}{2}$, and where AO $\left(\frac{n+1}{n+2} \times AC\right) = \frac{3}{5}$AC, will become $= \frac{3}{8}$CB.

Before I leave this Subject it may not be improper to take notice, *that*, whatever Line you found your Calculations upon, by ſuppoſing the Figure to reſt, *in Equilibrio*,

Equilibrio, upon that Line, the very ſame Point, for the Place of the Center of Gravity, will be determined.



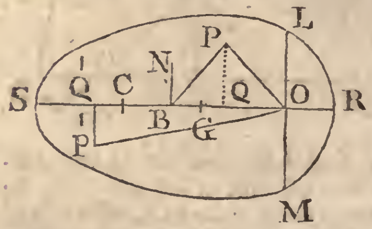
180. Thus, let O be the Point in the Axis AC, of a given Curve B A D, determined, as above, by ſuppoſing the Figure to reſt upon EF perpendicular to AC; and let RS be any other Line paſſing through the Point O; then I ſay the Sum of the *Momenta* of the Particles on each Side of RS will, *alſo*, be equal. For, if from two Points, in any Ordinate MQ, equally diſtant from the middle Point N, two Perpendiculars *mr* and *ns* be let fall upon RS, the Efficacy of thoſe two Points, in reſpect to RS, will be repreſented by $mr + ns$, or its Equal 2NH (ſuppoſing NH alſo perpendicular to RS.) Whence the Efficacy of all the Particles in MQ, will be expreſſed by their Number multiplied by NH, or by MQ × NH: Which is to their Efficacy (MQ × ON) when referred to the Line EF, in the conſtant Ratio of NH to ON, or of the Sine of the Angle RON to Radius. Whence it is evident that the Force of all the Ordinates (or the whole Curve) in the former Caſe, muſt be to that in the latter, in the ſame Ratio: But the ſaid Force, in the one Caſe, is equal to nothing by Hypotheſis, therefore it muſt be likewiſe ſo in the other: And conſequently the Sum of the *Momenta* of the Particles, on each Side of RS, equal to each other.

Much after the ſame manner the thing may be proved, in a Solid: Whence it will appear that there is actually ſuch a (fixed) Point in a Body as the Center of Gravity is defined to be: Which, however evident from mechanical Conſiderations, is not ſo eaſy to demonſtrate, *geometrically*, from the Reſolution of Forces.

PROPOSITION II.

181. *To determine the Center of Percussion of a Body.*

The Center of Percussion is that Point, in the Axis of Suspension of a vibrating (or revolving) Body, at which it may be stopt, by an immoveable Obstacle, so as to rest thereon *in Equilibrio* as it were, without acting upon the Center of Suspension.



Let O be the Point of Suspension, G the Center of Gravity, and SLM a Section of the Body, by the Plane wherein the Axis of Suspension OGS performs its Motion; to which Section let all the Particles of the Body be conceived to be transferred in such Parts thereof where they would be projected into *(orthographically)* by Lines parallel to the Axis of Motion; which Supposition will neither affect the Place of the Center of Gravity nor the angular Motion of the Body.

Since the angular Velocity of any Particle P is as the Distance, or Radius, OP, its Force in the Direction, PB, perpendicular to OP, will be expressed by $P \times OP$. Therefore the Efficacy of that Force upon the Axis, at B, in the perpendicular Direction BN (supposing the Axis stopt at C the Center of Percussion) will be $P \times OP \times \frac{OP}{OB}$, whose Power to turn the Body about the Point C is therefore as $P \times OP \times \frac{OP}{OB} \times BC = P \times \frac{OP^2 \times BC}{OB} = P \times \frac{OP^2 \times \overline{OC - OB}}{OB} = P \times \frac{OP^2 \times OC}{OB} - P \times OP^2$; which, if PQ be made Perpendicular to OS, OS, will at laſt (becauſe $\frac{OP^2}{OB} = OQ$) be reduced to $P \times OQ \times OC - P \times OP^2$. By the very ſame Argument, the Force of any other Particle $\acute{P}$ will be denoted by $\acute{P} \times O\acute{Q} \times OC - \acute{P} \times O\acute{P}^2$ *&c. &c.* But, as all theſe Forces muſt deſtroy one another (by the Nature of the Problem) the Sum of all the Quantities $P \times OQ \times OC$, $\acute{P} \times O\acute{Q} \times OC$, *&c.* muſt therefore be = the Sum of all the Quantities $P \times OP^2$, $\acute{P} \times O\acute{P}^2$ *&c.* and conſequently $OC = \frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.\ \&c.}{P \times OQ + \acute{P} \times O\acute{Q} + \&c.\ \&c.}$. But (by the preceding Propoſition) the Sum of all the Quantities $P \times OQ + \acute{P} \times O\acute{Q} +$ *&c.* is equal to $OG \times$ by the Content of the Body. Therefore OC is likewiſe = $\frac{P \times OP^2 + \acute{P} \times O\acute{P}^2 + \&c.\ \&c.}{OG \times Body}$.

The ſame otherwiſe.

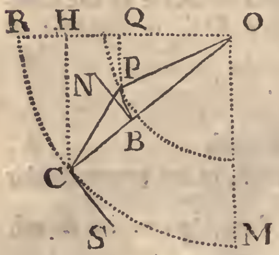
Since the Force of the Particle P, in the perpendicular Direction NB, is defined by $P \times \frac{OP^2}{OB}$, or its Equal, $P \times OQ$, the Sum of all the Quantities $P \times OQ$, $\acute{P} \times O\acute{Q}$, *&c. &c.* will expreſs the Force which, acting at C perpendicular to OS, is ſufficient to ſtop the Body, without the Center of Suſpenſion O being any way affected: This Sum, therefore, drawn into OC ($= OC \times \overline{P \times OQ + \acute{P} \times O\acute{Q} + \&c.\ \&c.}$) is as the Efficacy of the ſaid Force to turn the Body about the Point O. But the Force of the Particle P, in the Direction BN being $P \times \frac{OP^2}{OB}$, its Efficacy to turn the Body about the ſame

Point (the contrary way) is as $P \times OP^2$; and consequently the Efficacy of all the Particles as the Sum of all the Quantities $P \times OP^2$, $P' \times OP'^2$ *&c. &c.* Therefore (Action and Re-action being equal) we have $OC \times \overline{P \times OQ + P' \times OQ' + \&c.} = P \times OP^2 + P' \times OP'^2 + \&c.$ *the same as before.*

For the Center of Oscillation, it will be requisite to premise the following.

LEMMA.

182. *Suppose two exceeding small Weights* C *and* P, *acting on each other by means of an inflexible Line (or Wire* PC) *to vibrate in a vertical Plane* ROPCM, *about the Center* O; *it is required to determine how much the Motion of the one is affected by the other.*



Let CH and PQ be perpendicular to the horizontal Line OR; also let PB and CS be perpendicular to OP and OC respectively.

If the Force of Gravity be denoted by Unity, the Forces acting in the Directions CS and PB, whereby the Weights, in their Descent, are accelerated, will, according to the Resolution of Forces, be represented by $\frac{OH}{OC}$ and $\frac{OQ}{OP}$. Moreover, since the Velocities are always in the Ratio of the Radii OC and OP, if the foresaid Forces were to be in that Ratio, or that of P was to become $\frac{OH}{OC} \times \frac{OP}{OC}$, instead of $\frac{OQ}{OP}$. I say, in that Case, it is plain the Weights would continue their Motion without

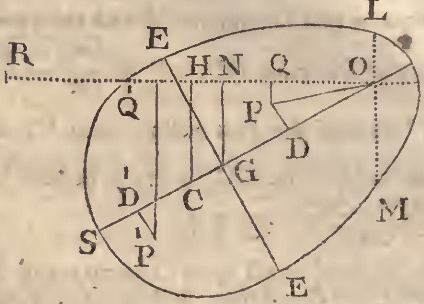
out affecting each other, or acting at all on the Line of Communication PC (or PB). Whence, the Excess of $\frac{OQ}{OP}$ above $\frac{OH}{OC} \times \frac{OP}{OC}$ must be the accelerative Force whereby the Weight P acts upon the Line (or Wire) OC, in the Direction PB; which multiply'd by the Weight P gives $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}$ for the absolute Force in that Direction: Which therefore, in the perpendicular Direction NB, is $P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}} \times \frac{OP}{OB}$; whereof the Part acting upon C, being to the Whole as OB to OC, is truly defined by $P \times \overline{\frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}}$. *Q. E. I.*

If P be supposed to act upon C by means of PC (instead of PB) the Conclusion will be no way different: For, let F (to shorten the Operation) be put to denote the Force $\left(P \times \overline{\frac{OQ}{OP} - \frac{OH \times OP}{OC^2}}\right)$ in the Direction PB, found above, then the Action thereof upon PC (according to the Principles of Mechanics) will be expressed by $F \times \frac{Radius}{Co\text{-}s.\ CPB}$; which therefore in the Direction SC, perpendicular to OC, is $F \times \frac{Radius}{Co\text{-}s.\ CPB} \times \frac{S.\ PCO}{Radius} = \frac{S.\ PCO}{Co\text{-}s.\ CPB} = \frac{S.\ PCO}{S.\ OPC} = F \times \frac{OP}{OC}$, *the very same as before.*

PROPOSITION III.

183. *To determine the Center of Oscillation of a Body.*

The Center of Oscillation is that Point, in the Axis (or Line) of Suspension of a vibrating Body, into which if the whole Body was contracted, the angular Velocity and the Time of Vibration would remain unaltered.



Let LMS be a Section of the Body by a Plane, perpendicular to the Horizon and the Axis of Motion, passing thro' the Center of Gravity G and the Point of Suspension O; and suppose all the Particles of the Body to be transferred to this Section, in such Places of it, as they would be projected into *(orthographically)* by Perpendiculars falling thereon. (Which Supposition will no way affect the Conclusion, the angular Motion continuing the same.) Moreover let C be the Center of Oscillation, or that Point in the Axis OS where a Particle of Matter (or a small Weight) may be placed so as to be neither accelerated nor retarded by the Action of the other Particles of Matter situate in the Plane. Then, if, from C and any other Point P in the Plane LMS, two Perpendiculars CH and PQ be let fall upon the horizontal Line OR, the Force of a Particle (or Weight) at P to accelerate the Weight at C, will *(according to the foregoing Lemma)* be represented by P ×

$\overline{\frac{OQ}{OC} - \frac{OH \times OP^2}{OC^3}}$: Which, ſuppoſing GN perpendicular to OR, will alſo be expreſſed by $P \times \overline{\frac{OQ}{OC} - \frac{ON}{OG} \times \frac{OP^2}{OC^2}}$, or its Equal $P \times \frac{OQ \times OG \times OC - ON \times OP^2}{OC^2 \times OG}$. In the very ſame manner the Force of any other Particle P' will be repreſented by $P' \times \frac{OQ' \times OG \times OC - ON \times OP'^2}{OC^2 \times OG}$ *&c. &c.*

Therefore the Forces of all the Particles deſtroying each other *(by Hypotheſis)* the Sum of all the Quantities $P \times \overline{OG \times OQ \times OC - ON \times OP^2} + P' \times \overline{OG \times OQ' \times OC - ON \times OP'^2}$ *&c. &c.* muſt be equal to nothing: Whence $P \times OG \times OQ \times OC + P' \times OG \times OQ' \times OC$ *&c. &c.* $= P \times ON \times OP^2 + P' \times ON \times OP'^2$ *&c. &c.* and conſequently $OC = \frac{ON}{OG} \times \frac{P \times OP^2 + P' \times OP'^2 + \&c.}{P \times OQ + P' \times OQ' + \&c.}$ But *(by Art.* 171. and 172.) the Sum of all the Quantities $P \times OQ + P' \times OQ'$ *&c.* is equal to the Content of the Body multiplied by the Diſtance (ON) of the Center of Gravity G from the Line LM (perpendicular to OC); whence OC is alſo $= \frac{ON}{OG} \times \frac{P \times OP^2 + P' \times OP'^2 \,\&c.\,\&c.}{ON \times Body} = \frac{P \times OP^2 + P' \times OP'^2 \,\&c.\,\&c.}{OG \times Body}$ Which Expreſſion continuing the ſame in all Inclina-

 tions

tions of the Axis OS, the Point C, thus determined is a fixed Point, agreeable to the Definition; and appears to be the ſame with the Center of Percuſſion; *ſee Art.* 181.

COROLLARY.

184. If PD, $\acute{P}\acute{D}$ &c. be perpendicular to OS, the Numerator of the Fraction found above, will become $P\times\overline{OG^2+GP^2-2OG\times GD}+\acute{P}\times\overline{OG^2+G\acute{P}^2+2OG\times G\acute{D}}$ + &c. &c. (ſince $OP^2=OG^2+GP^2-2OG\times GD$ &c.) Which, becauſe all the Quantities $P\times -2OG\times GD+\acute{P}\times 2OG\times G\acute{D}$ &c. or $P\times -GD+\acute{P}\times G\acute{D}$ &c. (by the Nature of the Center of Gravity) deſtroy one another, will be *barely* $=P\times\overline{OG^2+GP^2}+\acute{P}\times\overline{OG^2+G\acute{P}^2}$ + &c. &c. $=\overline{P+\acute{P}+\&c.}\times OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2+$ &c. $=$ *Maſs* $\times OG^2+P\times GP^2+\acute{P}\times G\acute{P}^2+$ &c. Whence it is evident that OC is, alſo,

$$\left(=\frac{Maſs\times OG^2,+P\times GP^2+\acute{P}\times G\acute{P}^2+\&c.\ \&c.}{Maſs\times OG}\right)$$

$$=OG+\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\&c.}{Maſs\times OG};$$ and conſequently

$$CG=\frac{P\times GP^2+\acute{P}\times G\acute{P}^2+\&c.\ \&c.}{Maſs\times OG}.$$

Whence it appears that, *if a Body be turned about its Center of Gravity, in a Direction, perpendicular to the Axis of the Motion, the Place of the Center of Oſcillation will remain unaltered*; becauſe the Quantities $P\times GP^2$, $\acute{P}\times G\acute{P}^2$ are no way affected by ſuch a Motion of the Body.

It

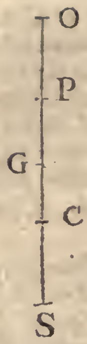
It alſo appears *that the Diſtance of the Center of Gravity from that of Oſcillation* (if the Plane of the Body's Motion remains unalter'd) *will be reciprocally as the Diſtance of the former from the Point of Suſpenſion. Therefore, if that Diſtance be found when the Point of Suſpenſion is in the Vertex, or ſo poſited, that the Operation may become the moſt ſimple, the Value thereof in any other propoſed Poſition of that Point will likewiſe be given, by one ſingle Proportion.*

185. But now, to ſhew how theſe Concluſions may be reduced to Practice, we muſt firſt of all obſerve, that *the Product of any Particle of the Body by the Square of its Diſtance from the Axis of Motion is* (here) *called the Force thereof* (its Efficacy to turn the Body about the ſaid Axis being in that Ratio.) According to which, and the firſt general Value of OC, it appears that, *if the Sum of all the Forces be divided by the Product of the Body into the Diſtance of the Center of Gravity from the Point of Suſpenſion, the Quotient thence ariſing will give the Diſtance of the Center of Percuſſion, or Oſcillation from the ſaid Point of Suſpenſion.*

The Manner of computing the Diviſor has been already explained; it remains therefore to ſhew how the Sum of all the Forces in the Numerator may be collected: Which will admit of ſeveral Caſes. Wherein, to avoid a Multiplicity of Words, I ſhall always expreſs the Diſtance of the Center of Gravity from the Point of Suſpenſion by g, and the Diſtance of the Center of Percuſſion, or Oſcillation, from the ſame Point, by C.

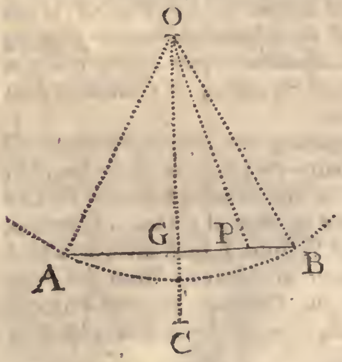
186. Caſe 1. *Let* OS *be a Line ſuſpended at one of its Extremes.*

Then, if the Part OP (conſidered as variable) be denoted by x, the Force of $\dot{x}$ Particles, at P, will *(as above)* be defined by $\dot{x} \times x^2$: Whoſe Fluent ($\frac{1}{3}x^3$) therefore expreſſes the Force of all the Particles in OP (or the Sum of all the Products, under each Particle, and the Square of its Diſtance from O the Point of Suſpenſion. This Quantity therefore (when x becomes



comes $= OS$) being divided by $OS \times \frac{1}{2}OS$ (according to the foregoing Rule or Observation) we get $\left(\frac{\frac{1}{3}OS^3}{\frac{1}{2}OS^2} =\right) \frac{2}{3}\,OS$ for the Value of *C*, the true Distance of the Center of Oscillation (or Percussion) from the Point of Suspension.

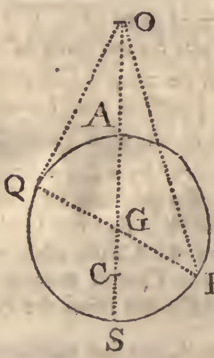
187. Case 2. *Let* AB *be a Line, vibrating in a vertical Plane, having its two Extremes* A *and* B *equally distant from the Point of Suspension* O.



If OG (perpendicular to AB) be put $=a$, and $GP = x$, the Force of $\dot{x}$ Particles at P, will be denoted by $\dot{x} \times \overline{a^2+x^2} = \dot{x} \times OP^2$ *: Whose Fluent, divided by ax (or $PG \times OG$) gives $\left(\frac{a^2x + \frac{1}{3}x^3}{ax}\right) a + \frac{x^2}{3a} = OG + \frac{BG^2}{3OG} = C$, when x becomes $= GB$.

* Art. 185.

188. Case 3. *Let* APSQ *be a Circle, vibrating in a vertical Plane.* Let PQ be any Diameter thereof; then $OP^2 + OQ^2$ being $= 2OG^2 + 2PG^2$, the Sum of the Forces of two Particles at P and Q, (putting $OG = a$, and $AG = r$) will be $= \overline{a^2+r^2} \times 2$; whence it is evident that the Sum of the Forces of all the Particles, in the whole Periphery, will be expressed by *their Number* $\times \overline{a^2+r^2}$, or by $\overline{a^2+r^2} \times$ *Periph.* APSQ: Which, if

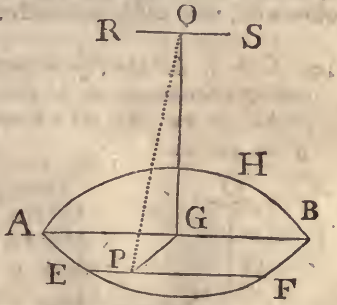


if p be put $= 3.141$ *&c.* will be $= \overline{a^2+r^2} \times 2pr = 2pa^2r + 2pr^3$. Hence the Force of the Circle itself is also given, being = Fluent of $\overline{2pa^2r+2pr^3} \times \dot{r} = pa^2r^2 + \frac{1}{2}pr^4 = \overline{a^2+\frac{1}{2}r^2} \times$ *Circle* APSQ. Now, if the two Expressions thus found be divided by $a \times$ *Periph.* APSQ, and $a \times$ *Circle* APSQ respectively *, we shall have

* Art. 185.

$a + \frac{r^2}{a}$ and $a + \frac{r^2}{2a}$, for the two corresponding Values of C.

189. Case 4. *Let* AHBE *be a Circle having its Plane (always) perpendicular to the Axis of Suspension* OG.

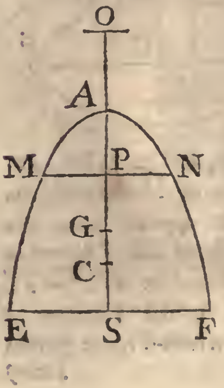


Let AGB be that Diameter of the Circle which is parallel to the Axis of Motion RS; and let EF be any Chord parallel to AB and RS; whose Distance, GP, from the Center of the Circle, let be denoted by x; putting OG $= a$, and AG $= r$: Then, by the Nature of the Circle, EF $= 2\sqrt{r^2-x^2}$; whose Distance OP, from the Axis of Motion RS, is also given $= \sqrt{a^2+x^2}$. Hence it appears that the Force of all the Particles in the Line EF *(defined in Art.* 185.) will be represented by $\overline{a^2+x^2} \times 2\sqrt{r^2-x^2}$. Therefore $\dot{x} \times \overline{a^2+x^2} \times 2\sqrt{r^2-x^2}$ is the Fluxion of the Force of the Plane ABFE; whose Fluent (when $x=r$)

$x = r$) is $= \overline{a^2 + \frac{1}{4}r^2} \times$ *Area AEFBG*; which, if p be put for the Area of the Circle whose Radius is Unity, will be $= \overline{a^2+\frac{1}{4}r^2} \times \frac{1}{2}pr^2$; whereof the Double $(pa^2r^2 + \frac{1}{4}pr^4)$ is the Force of the whole Circle AEFH: whose Fluxion $2par\dot{r} + pr^3\dot{r}$ (supposing r variable) being divided by $\dot{r}$, we likewise get $2pa^2r + pr^3$ $(= \overline{a^2+\frac{1}{2}r^2} \times$ *Periph. AEFH*) for the Force of the Periphery AEFH. But the Center of Gravity, whether we regard the Circle itself or its Periphery, is in the Center of the Circle; therefore the Distance of the Center of Oscillation from the Point of Suspension, will in these two Cases be exhibited by $a + \frac{r^2}{4a}$ and $a + \frac{r^2}{2a}$ respectively.

If the Circle, instead of being perpendicular to GO, either coincides, or makes a given Angle with it, the Value of C will come out exactly the same; provided the Diameter AB still continues parallel to the Axis of Motion RS: This appears from *Art.* 184. and may be, otherwise, very easily demonstrated.

190. Case 5. *Let the Figure proposed be a Curve* AEF, *moving (flat-ways, as it were) so that the Plane described by the Axis* OAS *may be perpendicular to that of the Curve.*



Here, putting AP $= x$, PN $= y$, AN $= z$, OA $= d$, OG $= g$, and AG $= a$, the Force of the Particles in MN will be defined by $2y \times \overline{a+x}|^2$. Therefore the Fluent of $2y\dot{x} \times \overline{d+x}|^2$ will be as the whole Force of the Plane NAM (or AEF, when $x =$ AS) and consequently $C = \frac{Flu.\ \overline{d+x}|^2 \times y\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}$: Which, therefore,

fore, when the Point of Suſpenſion is in the Vertex A, will become $C = \frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}$. Let this Value be denoted by v; then, the Diſtance of the Centers of Gravity and Oſcillation being $v-a$, we have (by *Art.* 184.) $g : a :: v-a : \left(\frac{a \times \overline{v-a}}{g}\right)$ the Diſtance of the ſame Centers, when the Point of Suſpenſion is at O, and conſequently C, in that Caſe, $= g + \frac{a \times \overline{v-a}}{g}$: Which Form will be found more commodious than the foregoing in moſt Caſes.

After the ſame Manner the Value of C, with reſpect of the Arch AEF, will appear to be $= \frac{Flu.\ \overline{d+x}|^2 \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}}$ $= g + \frac{a \times \overline{v-a}}{g}$, ſuppoſing $v = \frac{Flu.\ x^2\dot{z}}{Flu.\ x\dot{z}}$.

It may not be improper to give an Example or two of the Uſe of the foregoing Theorems.

191. Let therefore EAF be, firſt, conſider'd as an iſoſceles Triangle: In which Caſe AP (x) and PN (y) being in a conſtant Ratio, we have $y = \frac{bx}{c}$ (ſuppoſing SF$=b$ and AS$=c$.) Hence $C \left(= \frac{Flu.\ \overline{d+x}|^2 \times y\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}\right)$ $= \frac{Flu.\ d^2x\dot{x} + 2dx^2\dot{x} + x^3\dot{x}}{Flu.\ dx\dot{x} + x^2\dot{x}} = \frac{\frac{1}{2}d^2 + \frac{2}{3}dx + \frac{1}{4}x^2}{\frac{1}{2}d + \frac{1}{3}x} =$ $\frac{6d^2 + 8dx + 3x^2}{6d + 4x}$: Or (according to the ſecond Form) becauſe $v \left(\frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}\right) = \frac{3x}{4}$, and a is known to

be

* Art. 175. be $= \frac{2x}{3}$*, we have $C\left(=g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{x^2}{18g}$, where $g\ (=d+a) = d + \frac{2}{3}x$.

Again, becauſe $\dot{z}$ and $\dot{x}$ are in a conſtant Ratio, we alſo have $\frac{Flu.\ \overline{d+x}|^2 \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}} = \frac{Flu.\ \overline{d+x}|^2 \times \dot{x}}{Flu.\ \overline{d+x} \times \dot{x}} = \frac{d^2+dx+\frac{1}{3}x^2}{d+\frac{1}{2}x}$; whence the Center of Oſcillation of the Lines EH and AF is given.

192. For a ſecond Example, let EAF be ſuppoſed a Parabola of any Kind, whoſe Equation is $y = \frac{x^n}{c^{n-1}}$: Then *(according to Form 2.)* we ſhall firſt have $v\left(=\frac{Flu.\ yx^2\dot{x}}{Flu.\ yx\dot{x}}\right) = \frac{Flu.\ x^{n+2}\dot{x}}{Flu.\ x^{n+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3}$: Whence,

† Art. 176. a being $= \frac{\overline{n+1} \times x}{n+2}$†, we alſo get $C\left(=g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{\overline{n+1} \times x^2}{\overline{n+2}|^2 \times \overline{n+3} \times g}$; where $g = d + \frac{\overline{n+1} \times x}{n+2}$.

But, with reſpect to the Arch of the Curve, $v\left(=\frac{Flu.\ x^2\dot{z}}{Flu.\ x\dot{z}}\right)$ is $= \frac{Flu.\ x^2\dot{x}\sqrt{c^{2n-2}+nnx^{2n-2}}}{Flu.\ x\dot{x}\sqrt{c^{2n-2}+nnx^{2n-2}}}$: From which Value (found by infinite Series, and even with-

‡ Art. 138. out in ſome Caſes ‡) that of C will alſo be given.

193. Caſe. 6. *Let the propoſed Figure be a Curve vibrating (edge-ways) ſo that the Motion of the Axis may be in the Plane of the Curve.*

Then *(by Caſe 2.)* the Force of all the Particles in the Line PN *(ſee the preceding Figure)* being defined by $OP^2 \times PN + \frac{1}{3}PN^3$, or $\overline{d+x}|^2 \times y + \frac{1}{3}y^3$ (retaining the No-

Notation above) we have $C = \frac{Flu.\ \overline{d+x}|^2 \times y\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ \overline{d+x} \times y\dot{x}}$: Which, when the Point of Suspension is in the Vertex A, will become $\frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}}$: Let this (when found) be denoted by v; then, it appears from the preceding Case, that the general Value of C will, *also*, be represented by $g + \frac{a \times \overline{v-a}}{g}$.

In the same manner the Value of C, with respect to the Arch EAF, will be expounded by $\frac{Flu.\ \overline{d+x}|^2 + y^2 \times \dot{z}}{Flu.\ \overline{d+x} \times \dot{z}}$, or by $g + \frac{a \times \overline{v-a}}{g}$, supposing $v = \frac{Flu.\ \overline{x^2+y^2} \times \dot{z}}{Flu.\ x\dot{z}}$.

194. Example. *Let the Equation of the given Curve be* $y = \frac{x^n}{c^{n-1}}$: Then $v \left(= \frac{Flu.\ yx^2\dot{x} + \frac{1}{3}y^3\dot{x}}{Flu.\ yx\dot{x}}\right) =$

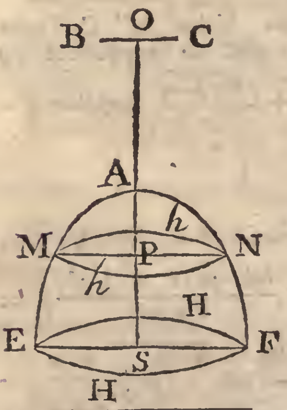
$$\frac{Flu.\ c^{1-n}x^{n+2}\dot{x} + \frac{1}{3}c^{3-3n}x^{3n}\dot{x}}{Flu.\ c^{1-n}x^{m+1}\dot{x}} = \frac{\overline{n+2} \times x}{n+3} +$$

$$\frac{\frac{1}{3}c^{2-2n}x^{3n+1} \times \overline{n+2}}{\overline{3n+1} \times x^{n+2}} = \frac{\overline{n+2} \times x}{x+3} + \frac{\overline{n+2} \times c^{2-2n} \times x^{2n-1}}{3 \times \overline{3n+1}}$$

$= \frac{n+2}{n+3} \times x + \frac{\overline{n+2}}{3 \times \overline{3n+1}} \times \frac{y^2}{x}$: From which the Value of C is also given; and from whence it appears, that if n be expounded by 0, v will become $= \frac{2x}{3} + \frac{2y^2}{3x} = \frac{2}{3} \times \frac{x^2+y^2}{y}$; in which Case the Figure will degenerate to a Rectangle: But, if n be interpreted by 1, the Figure EAF will then be an isosceles Triangle,

Triangle, and $v = \frac{3x}{4} + \frac{y^2}{4x}$: Laſtly, if n be taken $= \frac{1}{2}$, the Curve will be the common Parabola, and $v = \frac{5x}{7} + \frac{c}{3}$.

195. *Caſe* 7. *Let the Figure* AEFH *be a Solid generated by the Rotation of a Curve* EAF *about its Axis* AS ; *having its. Baſe* HH *parallel to the Axis of Motion* BOC.



It appears, *from Caſe* 4. that the Force of all the Particles in the circular Section *hh* (parallel to HH) will be expreſſed by $\overline{OP^2 + \frac{1}{4}PN^2} \times$ *Circle hh*, or $\overline{OP^2 \times PN^2 + \frac{1}{4}PN^4} \times p$ (p being $= 3.1415$ *&c.*) which, in algebraic Terms, is $\overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4}y^4} \times p$. Hence we have $C =$

$$\frac{Flu.\ \overline{\overline{d+x}|^2 \times y^2 + \frac{1}{4}y^4} \times p\dot{x}}{Flu.\ \overline{d+x} \times py^2\dot{x}} * = \frac{Flu\ \overline{d+x}|^2 \times y^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ \overline{d+x} \times y^2\dot{x}}.$$

* Art. 185.

Which, therefore, when the Point of Suſpenſion is in the Vertex A, becomes $\frac{Flu.\ y^2x^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ y^2x\dot{x}} = v$; and conſequently $C = g + \frac{a \times \overline{v-a}}{g}$, as in the preceding Caſes.

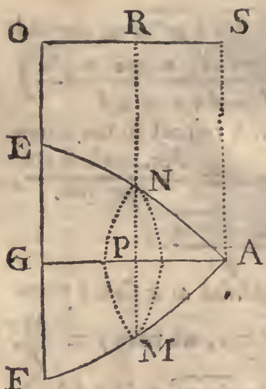
But, with regard to the Superficies of the Solid ; it is found, *in Caſe* 4. that the Force of the Particles in the Periphery M*h*N*h* is expreſſed by $\overline{OP^2 + \frac{1}{2}PN^2} \times$ *Periph. MhNh* $= \overline{d+x}|^2 \times 2py + py^3$.

5 Hence

Hence the Fluent of $\overline{d+x}|^2 \times \overline{2py+py^3} \times \dot{z}$, divided by that of $\overline{d+x} \times 2py\dot{z}$ $\left(= \frac{Flu.\ \overline{d+x}|^2 \times 2y\dot{z} + y^3\dot{z}}{Flu.\ \overline{d+x} \times 2y\dot{z}}\right)$ will give the true Value of C with respect to the curve Surface E*h*A*h*F. Which, putting $v = \frac{Flu.\ 2yx^2\dot{z} + y^3\dot{z}}{Flu.\ 2yx\dot{z}}$, is likewise expressed by $g + \frac{a \times \overline{v-a}}{g}$.

196. Ex. 1. *Let* EAF *be considered as a Cone*; then, putting AS $= f$, SF $= b$ and AF $= c$, we have $y = \frac{bx}{f}$, $z = \frac{cx}{f}$; and therefore C $\left(= \frac{Flu.\ \overline{d+x}|^2 \times y^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ \overline{d+x} \times y^2\dot{x}}\right)$ $= \frac{20d^2 + 30fd + 12f^2 + 3b^2}{20d + 15f}$, when $x = f$. But, with respect to the convex Superficies, C will be found $= \frac{12d^2 + 16df + 6f^2 + 3b^2}{12d + 8f}$.

197. Ex. 2. *Let* EAF *&c. be considered as a Sphere whose Center is* S, *and Radius* AS$=r$; in which Case, y^2 being $= 2rx - x^2$, we have v $\left(= \frac{Flu.\ y^2x^2\dot{x} + \frac{1}{4}y^4\dot{x}}{Flu.\ y^2x\dot{x}}\right)$ $= \frac{Flu.\ r^2x^2\dot{x} + rx^3\dot{x} - \frac{3}{4}x^4\dot{x}}{Flu.\ 2rx^2\dot{x} - x^3\dot{x}} = \frac{\frac{1}{3}r^2 + \frac{1}{4}rx - \frac{3}{20}x^2}{\frac{2}{3}r - \frac{1}{4}x}$; whence C is also given. But, when $x = 2r$ (or the whole Sphere is taken) $v = \frac{7r}{5}$: Therefore a being$=r$, and g = OS, in this Case, we have C $\left(= g + \frac{a \times \overline{v-a}}{g}\right) = g + \frac{r \times 2r}{5g} = g + \frac{2r^2}{5g}$.



198. Case 8. *Let the Figure proposed be a Solid, as in the preceding Case, but let its Axis* AG *be, here, parallel to the Axis of Motion* ORS.

Then, if RP (OG) be put $= g$, 3,1459 &c. $= p$, AP $= x$ &c. the Force of the Particles in the Circle NM (parallel to EF) will be exhibited by $\overline{g^2 + \frac{1}{2} y^2} \times py^2$, or $pg^2y^2 + \frac{1}{2}py^4$ (*Vid. Case* 3.) Hence we have $C =$

* Art. 185.
† Art. 145.

$$\frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Solid} * = \frac{Flu.\ pg^2y^2\dot{x} + \frac{1}{2}py^4\dot{x}}{g \times Flu.\ py^2\dot{x}} \dagger = g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}.$$

Moreover, with respect to the Superficies; the Force of the Particles in the Periphery of the said Circle MN being $\overline{2pg^2y + 2py^3}$ ‡, we have, in this Case, $C =$

‡ Art. 185.

$$\frac{Flu.\ \overline{2pg^2y + 2py^3} \times \dot{z}}{g \times Superficies.} = \frac{Flu.\ 2pg^2y\dot{z} + 2py^3\dot{z}}{g \times Flu.\ 2py\dot{z}} = g + \frac{Flu.\ y^3\dot{z}}{g \times Flu.\ y\dot{z}}.$$

199. Ex. 1. *Let* EAF *be a Segment of a Sphere, whose Radius is r*; then y^2 being $= 2rx - x^2$, we shall have

$$C\ \left(g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}}\right) = g + \frac{Flu.\ 2r^2x^2\dot{x} - 2rx^3\dot{x} + \frac{1}{2}x^4\dot{x}}{g \times Flu.\ \overline{2rx\dot{x} - x^2\dot{x}}}$$

$$= g + \frac{\frac{2}{3}r^2x - \frac{1}{2}rx^2 + \frac{1}{10}x^3}{g \times \overline{r - \frac{1}{3}x}} = g + \frac{\overline{20r^2 - 15rx + 3x^2} \times x}{\overline{30r - 10x} \times g}.$$

Which, when x is expounded, either, by r or $2r$, becomes $= g + \frac{2r^2}{5g}$, for the true Value of C, when

either

either the Hemisphere, or whole Sphere, is taken. But, with respect to the Center of Oscillation of the Superficies thereof, we have $\dot{z}$ in this Case $= \frac{r\dot{x}}{\sqrt{2rx-xx}}$ * $= \frac{r\dot{x}}{y}$: And therefore $g + \frac{Flu.\ y^3\dot{z}}{g \times Flu.\ y\dot{z}} = g + \frac{Flu.\ \overline{2rx-xx} \times r\dot{x}}{g \times Flu.\ r\dot{x}} = g + \frac{rx - \frac{1}{3}x^2}{g}$: Which, when $x = r$, or $x = 2r$, becomes $g + \frac{2r^2}{3g}$.

* Art. 142.

200. Ex. 2. *Let the Solid* EAF *be a Paraboloid, whose generating Curve is defined by the Equation* $y = \frac{x^n}{c^{n-1}}$:

Then $C = g + \frac{Flu.\ \frac{1}{2}y^4\dot{x}}{g \times Flu.\ y^2\dot{x}} = g + \frac{Flu.\ \frac{1}{2}x^{4n}\dot{x} \times c^{4-4n}}{g \times Flu.\ x^{2n}\dot{x} \times c^{2-2n}} = g + \frac{\overline{2n+1} \times x^{2n}}{\overline{4n+1} \times 2g \times c^{2n-2}} = g + \frac{\overline{2n+1} \times y^2}{\overline{4n+1} \times 2g}$. Where, if n be taken $= 0$, the Figure will become a Cylinder, and $C = g + \frac{y^2}{2g}$: But if n be expounded by 1, the Figure will be a Cone, and $C = g + \frac{3y^2}{10g}$. Lastly, if n be taken $= \frac{1}{2}$, the Figure will be the Solid generated from the common Parabola and $C = g + \frac{y^2}{3g}$.

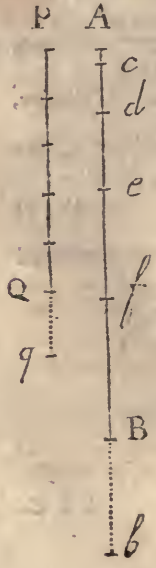
SECTION XII.

Of the Use of FLUXIONS *in determining the Motion of Bodies affected by centripetal Forces.*

PROPOSITION I.

201. THE *Motion, or Velocity, acquired by a Body freely descending from Rest, by the Force of an uniform Gravity, is proportional to the Time of its Descent; and the Space gone over, as the Square of that Time.*

The first Part of the Proposition is almost self-evident: For, since any Motion is proportional to the Force by which it is generated, that generated by the Force of an uniform Gravity must be as the Time of Descent; because the whole Effect of such a Force, acting equally every Instant, is as that Time.



Let, now, the Velocity acquired during a Descent of one Second of Time, be such as would carry the Body uniformly over any Distance b in one Second; and let AB (x) denote the Distance descended in any proposed Time t; which Time let be denoted by PQ; making $Bb = \dot{x}$ and $Qq = \dot{t}$: Then it will be, as $\overset{\prime}{1} : t :: b : (bt)$ the Distance that would be uniformly described in $\overset{\prime\prime}{1}$, with the Velocity at B: Also $\overset{\prime\prime}{1} : \dot{t} ::$ the said Distance (bt) to $bt\dot{t} = \dot{x}$ *. By taking the Fluent whereof we get

$\frac{1}{2}bt^2$

* Art. 2.

$\frac{1}{2}bt^2 = x$. Therefore the Diſtance deſcended ($\frac{1}{2}bt^2$) is as the Square of the Time. *Q. E. D.*

Otherwiſe, without Fluxions.

Conceive the Time (PQ) of falling thro' AB to be divided into an indefinite Number of very ſmall equal Particles, repreſented, each, by m; and let the Diſtance deſcended in the firſt of them be Ac, in the ſecond cd, in the third de, &c. &c. Then, the Velocity being always as the Time from the Beginning of the Deſcent, it will in the Middle of the firſt of the ſaid Particles be defined by $\frac{1}{2}m$; in the Middle of the ſecond by $1\frac{1}{2}m$; in the Middle of the third by $2\frac{1}{2}m$, &c. &c. But, ſince the Velocity at the Middle of any Particle of Time, is a Mean between thoſe at the two Extremes, or betwixt any other two equally remote from it, the correſponding Particle of the Diſtance AB may, therefore, be conſidered as deſcribed by that mean Velocity. And ſo, the Spaces Ac, cd, de, ef, &c. deſcribed in equal Times, being reſpectively as the ſaid mean Celerities $\frac{1}{2}m$, $1\frac{1}{2}m$, $2\frac{1}{2}m$, $3\frac{1}{2}m$, &c. it follows, by Addition, that the Diſtances, Ac, Ad, Ae, Af, &c. gone over from the Beginning, are to one another as $\frac{m}{2}$, $\frac{4m}{2}$, $\frac{9m}{2}$, $\frac{16m}{2}$, &c. or 1, 4, 9, 16, 25, &c. that is, as the Squares of the Times. *Q. E. D.*

Corollary I.

202. Since the Diſtance that might be uniformly run over in one Second, with the Velocity at B, is expreſſed by bt, the Diſtance that might be deſcribed with the ſame Velocity in the Time t will therefore be expreſſed by $bt \times t$, or bt^2: Whence it appears, that the Space AB ($\frac{1}{2}bt^2$) thro' which the Body falls in any given Time t, is juſt the half of that which would be uniformly deſcribed with the Celerity at B, in the ſame Time.

Therefore, ſince it is found from Experiment, that a Body near the Earth's Surface (where the Gravity may be taken as uniform) descends about $16\frac{1}{12}$ Feet in the first Second, it follows that the Value of b (is in this Case) $= 2 \times 16\frac{1}{12} = 32\frac{1}{6}$: And consequently the Number of Feet descended in t Seconds, equal to $16\frac{1}{12} \times t^2$.

COROLLARY 2.

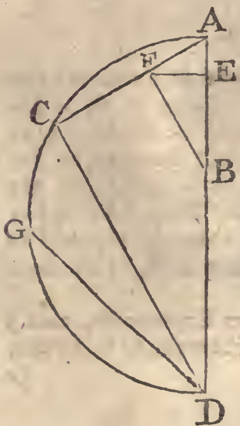
203. It is evident, whatever Force the Body descends by, the Value of b will always be as that Force; since a double Force, in the same time, generates a double Velocity; a treble Force, a treble Velocity, *&c.* Therefore, seeing our Equation $\frac{1}{2}bt^2 = x$, also gives $t = \sqrt{\frac{x}{\frac{1}{2}b}}$, and $b = \frac{x}{\frac{1}{2}t^2}$, it follows,

1. That the Distance descended is, universally, as the Force and the Square of the Time conjunctly.

2. That the Time is always as the Square-root of the Distance applied to the Force.

3. And that the Force is as the Distance apply'd to the Square of the Time——What is above demonstrated with respect to the Times, holds also in the Velocities, when the accelerating Forces are equal.

PROPOSITION II.



204. *To determine the Velocity, and Time of Descent, of a Body along an inclined Plane* AC.

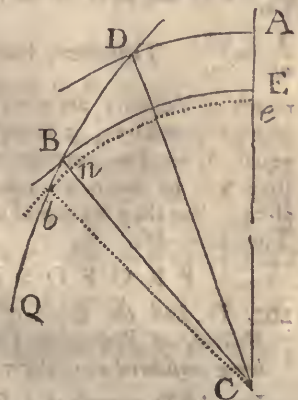
From any Point F, in AC, draw FE perpendicular to the vertical Line AD, and make FB and CD perpendicular to AC, meeting AD in B and D. Because (by the Principles of Mechanics) the Force of Gravity in the Direction CF, whereby the Body is made to descend along the Plane, is to the absolute Force thereof, as AF to AB, AB, or as AC to AD; and ſince *(by Caſe* 1. *Art.* 203.) the Diſtances deſcended in equal Times are as the Forces, it follows that the Time of Deſcent thro' AF will be equal to the Time of the perpendicular Deſcent thro' AB: And conſequently the Time of Deſcent thro' AC equal to that thro' AD; which is given by *Prop.* 1. Moreover, becauſe the Velocities at F and B, acquired in equal Times, are as the Forces, or as AF to AB; and it appears from *Prop.* 1. that the Velocity at E is to that at B, as $\sqrt{AE} : \sqrt{AB}$, or as $\sqrt{AE \times AB}$ (=AF) : $\sqrt{AB \times AB}$ (= AB) it follows, by Equality, that the Celerity at F is equal to that at E; which is therefore given, by the preceding Propoſition. *Q. E. I.*

COROLLARY.

205. Hence the Time of Deſcent along the Chord AC of a Semi-circle ACD is equal to the Time of Deſcent along the vertical Diameter AD: And, if the Chord DG be of the ſame Length with AC (its Inclination to the Horizon being alſo the ſame) the Time of Deſcent along it will alſo be equal to that along the vertical Diameter.

PROPOSITION III.

206. *If, from two Points* A *and* D, *equally remote from the Center of Attraction* C, *two Bodies move, with equal Celerities, the one along the Right-line* AC, *the other in a Curve-line* DBQ, *their Celerities at all other equal Diſtances from the Center, will be equal.*



For, let CB and CE be any two ſuch Diſtances; let the Arch BE be deſcribed,

ſcribed, from the Center C, and alſo *eb*, indefinitely near to it, cutting CB in *n*: Let the centripetal Force at the Diſtance of CB, or CE, be repreſented by *f*, and the Velocity at B, by *v*.

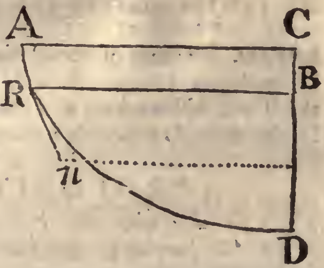
By the Reſolution of Forces, the Efficacy of the Force (*f*) in the Direction B*b*, whereby the Velocity of the Body is accelerated, will be $\frac{Bn}{Bb} \times f$: Alſo the Time of moving over B*b* (being as the Diſtance apply'd to the Velocity) is repreſented by $\frac{Bb}{v}$: Therefore the Increaſe of Velocity, in moving thro' B*b*, being as the Force and Time conjunctly, will be defined by $\frac{Bn}{Bb} \times f \times \frac{Bb}{v}$, or its Equal $\frac{Bn}{v} \times f$. In the ſame Manner, the Velocity at E being denoted by *w*, the Time of falling thro' E*e* will be repreſented by $\frac{Ee}{w}$, and the Velocity generated in that Time by $\frac{Ee}{w} \times f$: Which is to that $\left(\frac{Bn}{v} \times f\right)$ acquired in falling thro' the Arch B*b*, as $\frac{1}{w}$ to $\frac{1}{v}$. Therefore, ſeeing the correſponding Increments of Velocity are always, reciprocally, as the Velocities themſelves, it is manifeſt, if thoſe Velocities are equal, in any two correſponding Poſitions of the Bodies, they will be ſo in all others, being always increaſed alike. But they are equal at A and D by Suppoſition: Therefore, &c. *Q. E. D.*

PROPOSITION IV.

207. *To find the Ratio of the Velocities, and Times of Deſcent, of Bodies, in Curves; the Force of Gravity being conſidered as uniform.*

Let ARD repreſent a Curve of any Kind, along which a Body deſcends, by the Force of its own Gravity

vity from A; let AC, RB, *&c.* be parallel, and CD perpendicular, to the Horizon; moreover, let Rn touch the Curve at R; and let CB $= u$, AR $= w$, and R$n = \dot{w}$*. * Art. 135.



Since the Points B and R (as well as C and A) may be looked upon as equally remote from the Earth's Center (to which the Gravitation tends), the Velocity acquired in descending thro' the Arch AR will (*by the last Proposition*) be equal to that acquired by falling freely through the Right-line CB; which last Velocity (*by Prop.* 1.) is always as $\sqrt{CB}$ (or $u^{\frac{1}{2}}$). Therefore the Celerity, whether the Body moves in a Right-line, or a Curve, is always in the subduplicate Ratio of the perpendicular Descent; and so, the Time in which Rn ($\dot{w}$) would be uniformly described, with that Celerity, will be universally as $\frac{\dot{w}}{u^{\frac{1}{2}}}$; whose Fluent is as the Time of falling through AR. *Q. E. I.*

EXAMPLE.

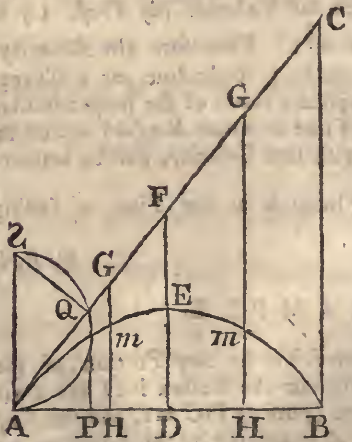
208. Let the Curve ARD be any Portion of the common Cycloid; whereof the Vertex is D and Axis DC; and whose Nature (putting DC $= c$, and the Ray of Curvature at D $= a$) is defined by the Equation $2a \times DB = DR^2$. Here, we have DR $(= \sqrt{2a} \times \sqrt{DB})$ $= \sqrt{2a} \times \overline{c-u}|^{\frac{1}{2}}$; whose Fluxion $-\sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\overline{c-u}|^{\frac{1}{2}}}$, with a contrary Sign, is the Value of Rn or $\dot{w}$;

and

therefore $\frac{\dot{w}}{u^{\frac{1}{2}}} = \sqrt{2a} \times \frac{\frac{1}{2}\dot{u}}{\sqrt{cu - uu}}$: Whose Fluent, at the lowest Point D, where u becomes $= c$, will (*by Art.* 142.) be equal to $\sqrt{2a}$ multiplied by $\left(\frac{3.14159 \&c.}{2}\right)$ half the Measure of the Periphery of the Circle whose Diameter is Unity. Which Fluent (and consequently the Time of Descent) will therefore continue the same, let the Arch DA be what it will.

PROPOSITION V.

209. *To determine the Paths of Projectiles near the Earth's Surface; (neglecting the Resistance of the Atmosphere.)*



Let a Body be projected from the Point A, in the Direction of the Line AC, with a Velocity sufficient to carry it uniformly over the Distance d in the Time t; and let the Space through which it would freely descend, by its own Gravity, in that time, be denoted by b; also let the Sine of the Angle of Elevation BAC (to the Radius r) be put $= s$, its Co-sine $= c$, and the Distance of the Point A from the Ordinate Hm (considered as moving parallel to itself along with the Body) $= x$; then, *by Trig.* HG (perpendicular to AB) will be $= \frac{sx}{c}$, and AG $= \frac{rx}{c}$.

Because the Projectile is turned aside, continually, from a rectilinear Path, by the Earth's Attraction, it must

muſt deſcribe a Curve-line $AmEmB$, to which AC is a Tangent at the Point A: But that Attraction, acting always in a Direction (Hm) perpendicular to the Horizon, can have no Effect upon that Part of the Velocity with which the Body approaches the Line BC, parallel to Hm; therefore the Right-line HG (in which the Body is always found) will continue to move uniformly towards BC, the ſame as if Gravity was not to act; and the Diſtance Gm deſcended from the Tangent AC, by means of the Attraction, will be the very ſame as if the Body was to deſcend from Reſt along the Line GH. This being premiſed, it is evident, that as d : AG $\left(\frac{rx}{c}\right) :: t : \left(\frac{rx}{cd} \times t\right)$ the Time of deſcribing Am; and, as $t^2 : \frac{r^2x^2}{c^2d^2} \times t^2 :: b : \left(\frac{br^2x^2}{c^2d^2}\right)$ the Space (Gm) through which a Body would freely deſcend in that Time (*by Prop.* 1.)

Hence $\frac{sx}{c} - \frac{br^2x^2}{c^2d^2}$, or $\frac{csd^2x - br^2x^2}{c^2d^2}$ is a general Value for the Ordinate Hm: By putting which $= 0$, we get $x = \frac{csd^2}{br^2} = AB =$ the Amplitude of the Projection. But, by putting its Fluxion equal to nothing, we have $x = \frac{csd^2}{2br^2}$; which ſubſtituted for x in the Value of Hm, gives $\frac{s^2d^2}{4br^2}$ for the Altitude DE of the Projection. *Q. E. I.*

Corollary.

210. If another Body be projected, with the ſame Celerity, in the vertical Direction AS; then, s becoming $= r$, the Altitude of that Projection $\left(\frac{s^2d^2}{4br^2}\right)$ will be-

come

come $\frac{d^2}{4b} =$ AS; which call b, and let this Value be substituted in those of AB and DE, and they will become $\frac{4bcs}{r^2}$ and $\frac{bs^2}{r^2}$ respectively.

Hence, if from the Point Q where the Line of Direction AC cuts a Semi-circle described upon AS, the Lines SQ and QP be drawn, the latter perpendicular to AB, the Triangles ASQ and AQP being similar, we shall have

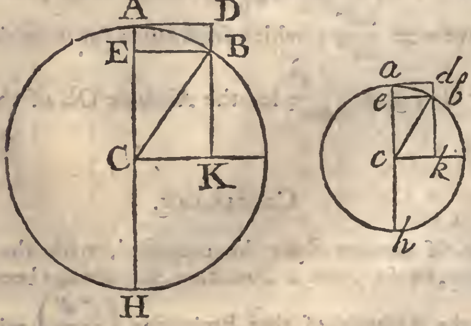
$$r : s :: b\ (AS) : \frac{sb}{r} = AQ$$

$$r : s :: \frac{sb}{r}\ (AQ) : \frac{s^2 b}{r^2} = PQ = DE$$

$$r : c :: \frac{sb}{r^2}\ (AQ) : \frac{scb}{r} = AP = \tfrac{1}{4}\ AB$$

PROPOSITION VI.

211. *To determine the Ratio of the Forces, whereby Bodies, tending to the Centers of given Circles, are made to revolve in the Peripheries thereof.*



Let ABH and *abh* be any two proposed Circles, whereof let AB and *ab* be similar Arcs; in which, let the

the Velocities of the revolving Bodies be reſpectively as V to v; make DBK and *dbk* parallel to the Radii AC and *ac*, putting AC $= R$, *ac* $= r$, and the Ratio of the centripetal Force in ABH to that in *abh*, as F to f.

It is plain, becauſe the Angles ABD and *abd* are equal, that the Velocities at B and *b*, in the Directions BK and *bk*, with which the Bodies recede from the Tangents AD and *ad*, are to each other as the abſolute Celerities V and v *. But thoſe Velocities, being the Effects of the centripetal Forces acting in correſponding, ſimilar, Directions during the Times of deſcribing AB and *ab*, will therefore be as the Forces themſelves when the Times are equal; but when unequal, as the Forces and Times conjunctly. Therefore, the Times being univerſally as $\frac{AB}{V}$ to $\frac{ab}{v}$, or as $\frac{R}{V}$ to $\frac{r}{v}$ (becauſe the Arcs AB and *ab* are ſimilar) we have, as $F \times \frac{R}{V} : f \times \frac{r}{v} :: V : v$; whence (multiplying the Antecedents by $\frac{V}{R}$ and the Conſequents by $\frac{v}{r}$) it will be, as $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$: Therefore the Forces are as the Squares of the Velocities directly, and as the Radii inverſely.

* Art. 35.

Otherwiſe.

Let the indefinitely little Arch AB be the Diſtance that the Body moves over in a given, or conſtant Particle of Time; and let the centripetal Force at B be meaſured by twice the Subtenſe or Space AE through which the Body is drawn from the Tangent AD in that Time †.

Then,

† *The Velocity which any Force, uniformly continued, is capable of generating, in a given Body, in a given Time, is the proper Meaſure of the Intenſity of that Force* *. *But this Velocity is itſelf meaſured by the Space the Body would move uniformly*

* Art. 203.

Then, by the Nature of the Circle, $AB^2 = AH \times AE = AC \times 2AE$, and consequently $2AE = \frac{AB^2}{AC}$: Therefore, the Force is as the Square of the Velocity applied to the Radius of the Circle *(as before)*.

COROLLARY I.

212. Because, $F : f :: \frac{V^2}{R} : \frac{v^2}{r}$, it follows that

$$V : v :: \sqrt{RF} : \sqrt{rf}, \text{ and}$$
$$R : r :: \frac{V^2}{F} : \frac{v^2}{f}.$$

COROLLARY II.

213. If the Ratio of the periodic Times be denoted by that of P to p; then the Ratio of the Velocities V, v being as $\frac{R}{P}$ to $\frac{r}{p}$, we shall have, by Equality $\sqrt{RF} : \sqrt{rf} :: \frac{R}{P} : \frac{r}{p}$; whence also

$$F : f :: \frac{R}{P^2} : \frac{r}{p^2}, \text{ and}$$
$$R : r :: FP^2 : fp^2.$$

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formly over in a given Time; which Space is always the double of that through which the Body would freely descend, from Rest, in the same time. Therefore* 2AE *is the proper Measure of the centripetal Force, according as we have assumed it.—— It is true, when the Forces to be compared are all computed in the same Manner, from the Nascent, or indefinitely small Subtenses of contemporaneous Arcs, it matters not whether we consider those Subtenses, or their Doubles, as the Measures of the Forces, the Ratio being the same in both Cases. But when the Forces so found are to be compared with others derived from a fluxional Calculus, it is absolutely necessary to take the double Subtense for the Measure of the Force.—— This Note is inserted, that the Learner may avoid the Errors, which some very considerable Mathematicians have fallen into by not properly attending to this Particular.*

* Art. 202.

Co-

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Corollary III.

214. If the Measure of the Force, or the Velocity that might be uniformly generated in a given Time (1) be expounded by any Power a^n of the Radius AC (a); then the Distance through which a Body would freely descend in the same Time, by that Force, uniformly continued, will be expressed by $\frac{1}{2}a^n$ *. Therefore, the Distances descended, by means of the same Force, uniformly continued, being as the Squares of the Times †, it is evident, if the Time of moving through AB be denoted by t, that the Distance AE descended in that Time, will be denoted by $\frac{t^2}{1^2} \times \frac{1}{2}a^n$: And so we shall have AB $(\sqrt{2AE \times AC}) = \frac{t}{1} \times a^{\frac{n+1}{2}}$; which being the Distance described by the revolving Body in the Time t, it follows that the Space gone over in the given Time (1) will be $a^{\frac{n+1}{2}}$: Which, therefore, is the true Measure of the Celerity in this Case. The same conclusion might have been derived in much fewer Words from *Corol.* 1. but, as a thorough understanding hereof is absolutely necessary in what follows hereafter, I have endeavoured to make it as plain as possible.

* Art. 202.

† Art. 201.

Corollary IV.

215. Hence the Time of Revolution is also derived; for it will be as $a^{\frac{n+1}{2}}$: 3.14159 &c. × $2a$ (the whole Periphery) :: 1 : $\frac{3.14 \text{ \&c.} \times 2a}{a^{\frac{n+1}{2}}}$ or 3.14159 &c. × $2a^{\frac{1-n}{2}}$, the true Measure of the periodic Time.

Co-

COROLLARY V.

216. Therefore, if n be expounded by 1, 0, — 1, — 2 and — 3 successively, then the Velocity corresponding will be as a, $a^{\frac{1}{2}}$, 1, $a^{-\frac{1}{2}}$, and a^{-1}; and the Time of Revolution, as 1, $a^{\frac{1}{2}}$, a, $a^{\frac{3}{2}}$ and a^2 respectively.

SCHOLIUM.

217. From the preceding Proposition, and its subsequent Corollaries, the Velocity and periodic Time of a Body revolving in a Circle at any given Distance from the Earth's Center, by means of its own Gravity, may be deduced: For let d be put for the Space thro' which a heavy Body, at the Surface of the Earth, descends in the first Second of Time, then $2d$ will be the Measure of the Force of Gravity at the Surface: And therefore, the Radius of the Earth being denoted by r, the Velocity, *per* Second, in a Circle at its Surface, will be $\sqrt{2rd}$; and the Time of Revolution $= \frac{3.14159 \&c. \times 2r}{\sqrt{2rd}}$ $= 3.14159 \&c. \times \sqrt{\frac{2r}{d}}$ *(Seconds)*; which two Expressions, because r is $= 21000000$ Feet and $d = 16\frac{1}{12}$ will therefore be nearly equal to 26000 Feet and 5075 *Seconds*, respectively. Let R be now put for the Radius of any other Circle described by a Projectile about the Earth's Center: Then, because the Force of Gravitation above the Surface is known to vary according to the Square of the Distance inversely, we have *(by Case* 4. *Corol.* 5.) $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} :: (26000^{F})$ the Velocity *(per* Second) at the Surface, to $26000 \times \sqrt{\frac{r}{R}}$, the Velocity

locity in the Circle whoſe Radius is *R*: And $r^{\frac{3}{2}} : R^{\frac{3}{2}}$ :: (5075 s.) the periodic Time at the Surface : to $5075 \times \sqrt{\frac{R^3}{r^3}}$, the Time of Revolution in the Circle *R*. Which, if *R* be aſſumed equal to (60*r*) the Diſtance of the Moon from the Earth, will give 2360000 s., or 27.3 D nearly, for the periodic Time at that Diſtance.

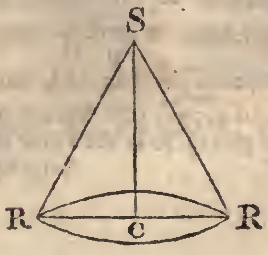
In like ſort the Ratio of the Forces of Gravitation of the Moon, towards the Sun and Earth, may be computed. For, the centrifugal Forces in Circles, being univerſally as the Radii apply'd to the Squares of the Times of Revolution, it will be as $\left(\frac{81000000}{1}\right)$ the Semi-diameter of the *Magnus Orbis* divided by the Square of one Year (the periodic Time of the Earth and Moon about the Sun) is to (240000×178) the Diſtance of the Moon from the Earth divided by $\frac{1}{178}$, the Square of the periodic Time of the Moon about the Earth, ſo is 1,9 to 1 nearly; and ſo is the Gravitation of the Moon towards the Sun to her Gravitation towards the Earth.

Alſo, after the ſame Manner, the centrifugal Force of a Body at the Equator, ariſing from the Earth's Rotation, is derived. For ſince it is found above, that 5075 Seconds is the Time of Revolution, when the centrifugal Force would become equal to the Gravity, and it appears (*by Caſe* 2. *Corol.* 2.) that the Forces, in Circles having the ſame Radii, are inverſely as the Squares of the periodic Times, we therefore have, as $\overline{86160}|^2$ (the Square of the Number of Seconds in (23 H 56 M) one whole Rotation of the Earth) to $\overline{5075}|^2$ (the Square of the Number of Seconds above given) ſo is the Force of

Gravity (which we will denote by Unity) to $\frac{1}{289}$, the centrifugal Force of a Body at the Equator arising from the Earth's Rotation.

But, to determine, in a more general Manner, the Ratio of the Force of a Body revolving in any given Circle, to its Gravity, we have already given 3.14 &c. $\times \sqrt{\frac{2r}{d}}$ for the Time of Revolution at the Surface of the Earth, when the Gravity and centrifugal Force are equal: Therefore, if the Time of Revolution in any Circle whose Radius is a, be denoted by t, it follows, from *Corol.* 2. *last Prop.* that, $\frac{r}{\overline{3.14}|^2 \text{ \&c.} \times \frac{2r}{d}} : \frac{a}{t^2}$ :: the Gravity of the Body : to its centrifugal Force in that Circle; which, therefore, is as Unity to $\frac{\overline{3.14}|^2 \text{ \&c.} \times 2a}{dt^2}$; or as 1 to $1.228 \times \frac{a}{t^2}$ very nearly: where a denotes the Number of Feet in the Radius of the proposed Circle, and t the Number of Seconds in one intire Revolution. So that, if the Length of a Sling, by which a Stone is whirled about, be two Feet, and the Time of Revolution $\frac{1}{2}$ of a Second, the Force by which the Stone endeavours to fly off, will be to its Weight as 9.824 to Unity.

From this general Proportion, the centrifugal Force and periodic Time of a Pendulum describing a conical Surface may likewise be deduced.



For let SR, the Length of the Pendulum, be denoted by g; the Altitude CS of the Cone, by c; the Semi-diameter CR of the Base by a; and the Time of Revolution by t: Then, the Force of Gravity being re-

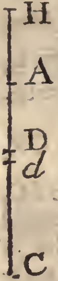
represented by Unity, the Force with which the revolving Body at R, the End of the Pendulum, tends to recede from the Center C, will be defined by $\frac{\overline{3.14\ \&c.}|^2 \times 2a}{dt^2}$, as has been already shewn. Therefore, because the Body is retained in the Circle RR by the Action of three different Powers, *i. e.* the centrifugal Force $\left(\frac{\overline{3.14\ \&c.}|^2 \times 2a}{dt^2}\right)$ in the Direction CR, the Force of Gravity (1) in a Direction parallel to SC, and the Force of the Thread or Wire RS, compounded of the former two; it follows, from the Principles of Mechanics, that as SC (c) to CR (g), so is the Weight of the Body at R, to the Force with which it acts upon the Thread or Wire RS; and as $1 : \frac{\overline{3.14\ \&c.}|^2 \times 2a}{dt^2}$ :: CS (c) : CR (a) : Whence $dt^2 = \overline{3.14\ \&c.}|^2 \times 2c$, and $t = 3.14\ \&c. \times \sqrt{\frac{2c}{d}} = 1,108\sqrt{c}$ nearly. Because dt^2, or its Equal $\overline{3.14\ \&c.}|^2 \times 2c$, expresses the Space a heavy Body will descend, by its own Gravity, in the Time t *, and since $1^2 : \overline{3.14\ \&c.}|^2 :: 2c : \overline{3.14\ \&c.}|^2 \times 2c\ (= dt^2)$ it therefore appears that, as the Square of the Diameter of any Circle, is to the Square of its Periphery, so is twice the perpendicular Altitude of the Cone, to the Distance a heavy Body will freely descend in the Time of one whole Gyration of the Pendulum, let the Base of the Cone and the Length of the Pendulum be what they will.

* Art. 202.

PROPOSITION VII.

218. *To determine the Ratio of the Velocities of Bodies descending, or ascending, in Right-lines, when accelerated, or retarded, by Forces, varying according to a given Law.*

Suppose a Body to move in the Right-line CH, and let the Force whereby it is urged towards C, or H, be as any variable Quantity F: Moreover, let the Velocity of the Body be represented by v; putting its Distance CD, from the Point $C = x$, and $Dd = \dot{x}$.



Then, since the Time wherein the Space D*d* $(\dot{x})$ would be uniformly described, with the Velocity at D, is known to be as $\frac{\dot{x}}{v}$, the Velocity that would be uniformly generated, or destroyed, in that Time by the Force F (being as the Time and Force conjunctly) will consequently be as $\frac{F\dot{x}}{v}$: Which therefore must be equal to, $\pm \dot{v}$, the uniform Increase or Decrease of Celerity in that Time; and consequently $\pm v\dot{v} = F\dot{x}$. From whence, when the Value of F is given in Terms of x, or v, the Value of v will likewise be known. *Q. E. I.*

COROLLARY I.

219. Hence, the Law of the Velocity being given, that of the Force is deduced: For, since $F\dot{x} = \pm v\dot{v}$, it is evident that $F = \pm \frac{v\dot{v}}{\dot{x}}$.

COROLLARY II.

220. Hence, also, the Ratio of the Velocity at D to that whereby a Body might revolve in the Periphery of a Circle about the Center C, at the Distance of CD, will be known: For, if this last Velocity be denoted by

* Art. 212. w, the Value of F will be rightly expressed by $\frac{w^2}{x}$ *:

Whence, by Substitution, we have $\pm v\dot{v} = \frac{w^2\dot{x}}{x}$, or

$\pm v^2$

$\pm v^2 \times \frac{\dot{v}}{v} = w^2 \times \frac{\dot{x}}{x}$: Whence $w^2 : v^2 : \pm \frac{\dot{v}}{v} : \frac{\dot{x}}{x}$, and consequently $w ; v :: \sqrt{\pm \frac{\dot{v}}{v}} : \sqrt{\frac{\dot{x}}{x}}$. Where, as well as above, the Sign of $\dot{v}$ must be taken + or — according as the Body is urged from, or towards the Center C.

PROPOSITION VIII.

221. *Suppoſing a Body, let go from a given Point* A *with a given Celerity (c) along a Right-line* CH, *to be urged, either way, in that Line, by a Force varying as any Power (n) of the Diſtance from a given Point* C; *to find, not only, the Relation of the Velocities, and Spaces gone over, but alſo the Times of Aſcent and Deſcent.*

The Conſtruction of the preceding Problem being retained, F will here be expounded by x^n, and we ſhall therefore have $\pm v\dot{v}$ $(=F\dot{x}) = x^n\dot{x}$; and conſequently, by taking the Fluent thereof, $\pm \frac{vv}{2} = \frac{x^{n+1}}{n+1}$; but to correct the Fluent thus found, let x be taken $=$ CA (which we will call a) then v being $= c$, the Fluent in that Circumſtance will become $\pm \frac{c^2}{2} = \frac{a^{n+1}}{n+1}$: Therefore the Fluent duly corrected is $\pm \frac{v^2}{2} \mp \frac{c^2}{2} = \frac{x^{n+1} - a^{n+1}}{n+1}$*, or $v^2 \backsim c^2 = \frac{2x^{n+1} \backsim 2a^{n+1}}{n+1}$: Whence v will come out $= \sqrt{c^2 + \frac{\mp 2a^{n+1} \pm 2x^{n+1}}{n+1}}$: Where the Signs of v and x^{n+1} muſt be alike, when both Quantities increaſe, or decreaſe, at the ſame time; that is, when

* Art. 78.

* Art. 220. when the Force, from C, is a repulsive one *; but, unlike, when one increases while the other decreases, or the Force, tending to C, is an attractive one. In the former Case we therefore have $v = \sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}$; and, in the latter, $v = \sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$.

The Value of v being thus obtained, let the required Time of moving over the Space AD be now denoted by T; then, since $\dot{T}$ is universally $= \frac{\dot{x}}{v}$, we have $\dot{T} = \frac{\dot{x}}{\sqrt{c^2 + \frac{2x^{n+1} - 2a^{n+1}}{n+1}}}$, or $\dot{T} = \frac{\dot{x}}{\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}}$ according to the two foresaid Cases respectively: From whence, by finding the Fluent, the Time itself will be known. *Q. E. I.*

COROLLARY.

222. If the Body proceeds from Rest at A, c will be $= 0$, and we shall have $\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2x^{n+1} - 2a^{n+1}}}$, or $\dot{T} = \frac{\overline{1+n}|^{\frac{1}{2}} \times \dot{x}}{\sqrt{2a^{n+1} - 2x^{n+1}}}$.

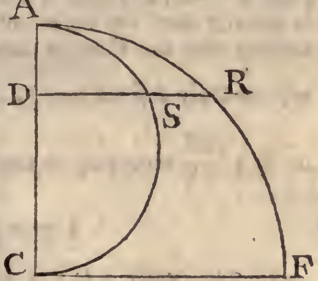
SCHOLIUM.

223. Although, the Fluents of the Expressions given above cannot be exhibited, in a general Manner, neither, in finite Terms, nor by means of circular Arcs and Logarithms; yet, in some of the most useful Cases

Caſes that occur in Nature, they may be obtained with great Facility.

Thus, if in $\frac{\overline{1+n}|^{\frac{1}{2}}\dot{x}}{\sqrt{2a^{n+1}-2x^{n+1}}}$ (expreſſing the Fluxion of the Time of Deſcent along AD) n be expounded by 1, 0, − 2, and − 3 ſucceſſively, the Fluxion itſelf will become equal to $\frac{\dot{x}}{\sqrt{a^2-x^2}}$, $\frac{\dot{x}}{\sqrt{2a-2x}}$, $\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, and $\frac{ax\dot{x}}{\sqrt{a^2-x^2}}$ reſpectively: Whence, if ARF be a Quadrant of a Circle whoſe Center is C, and ASC a Semi-circle whoſe Diameter is AC, and DSR be perpendicular to AC; then it will appear,

1°. That, when $n=1$, and $\dot{T}=\frac{\dot{x}}{\sqrt{a^2-x^2}}$, the Velocity ($\sqrt{a^2-x^2}$) at D will be repreſented by DR, and the Fluent ſought by $\frac{AR}{AC}$*.



* Art. 142.

2°. That, when $n=0$, and $\dot{T}=\frac{\dot{x}}{\sqrt{2a-2x}}$, the Velocity at D, and the Time of Deſcent thro' AD, will each be defined by $\sqrt{2AD}$.

3°. That, when $n=-2$, and $\dot{T}=\frac{\sqrt{\frac{1}{2}a}\times x\dot{x}}{\sqrt{ax-xx}}$, the Velocity $\left(\frac{\sqrt{ax-xx}}{x\sqrt{\frac{1}{2}a}}\right)$ will be as $\frac{DS}{CD\sqrt{\frac{1}{2}AC}}$, and the Time of Deſcent thro' AD, as $\sqrt{\frac{1}{2}AC}\times\overline{AS+DS}$.

 4°.

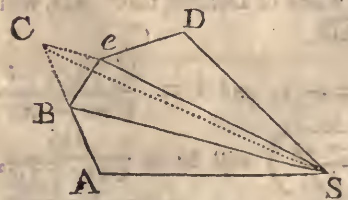
4°. And that, when $n = -3$, and $\dot{T} = \frac{ax\dot{x}}{\sqrt{a^2 - x^2}}$, the Velocity will be as $\frac{DR}{AC \times CD}$, and the Time as $AC \times DR$.

Hence the Time of the whole Descent thro' the Radius AC, appears to be as $\frac{AF}{AC}$, $\sqrt{2AC}$, $\sqrt{\frac{1}{2}AC} \times AF$, or AC^2. But the Time of one whole Revolution in

* Art. 215. the Periphery ARF &c. was found to be as $\frac{4AF}{AC^{\frac{n+1}{2}}}$*; which in the four Cases above specified is $\frac{4AF}{AC}$, $\frac{4AF}{\sqrt{AC}}$, $4AF \times \sqrt{AC}$, and $4AF \times AC$: Therefore, if the Time of moving over the Quadrant AF be denoted by $\mathfrak{Q}$, it follows that the Time of Descent thro' the Radius AC, will be truly defined by $\mathfrak{Q}$, $\mathfrak{Q} \times \frac{AC\sqrt{2}}{AF}$, $\mathfrak{Q} \times \sqrt{\frac{1}{2}}$, or $\mathfrak{Q} \times \frac{AC}{AF}$ according to the foresaid Cases respectively.

LEMMA.

224. *The Areas which a revolving Body describes, by Rays drawn to the Center of Force, are proportional to the Times of their Description.*



For, let a Body, in any given Time, describe the Right-line AB, with an uninterrupted uniform Motion; but upon its Arrival at B let it be impelled towards the Center S, so that, instead of proceeding along

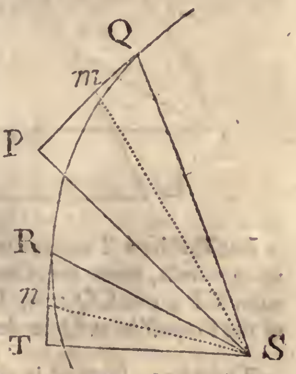
along ABC, it may, after the Impulse, describe the Right-line B*e*.

Because the Force, acting in the Line SB, can neither add to, nor take from, the Celerity which the Body has in a Direction perpendicular to that Line, the Distance of the Body from the said Line, at the end of a given Time, will therefore be the very same as if no Force had acted; and consequently the Area B*e*S equal to the Area BCS, which would have been described in the same time, had the Body proceeded uniformly along BC; because Triangles, having the same Base and Altitude, are equal.

Therefore seeing no Impulse, however great, can affect the Quantity of the Area described about the Center S, in a given Time, and because the Areas ASB, BSC, described about that Point, when no Force acts, are as the Bases AB, BC, or the Times of their Description, the Proposition is manifest.

COROLLARY.

225. Hence the Velocity of a revolving Body, at any Point Q or R, is inversely as the Perpendicular SP or ST, falling from the Center of Force upon the Tangent at that Point.



For, let two other Bodies *m* and *n* be supposed to move uniformly from Q and R, along the Tangents QP and RT, with Velocities respectively equal to those of the revolving Body at Q and R; then the Distances Q*m* and R*n*, gone over in the same Time, will be to each other as those Velocities; and the Areas QS*m* and RS*n* will be equal, being equal

to

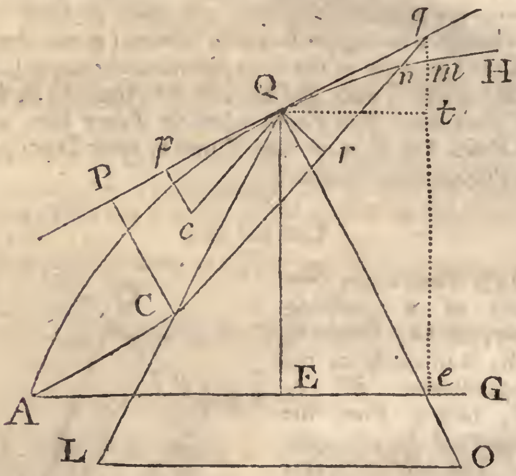
to those described by the revolving Body in the same time *: Whence $Qm \times SP$ being $= Rn \times ST$, it follows

* Art. 113.

that $Qm : Rn :: ST : SP :: \frac{1}{SP} : \frac{1}{ST}$.

PROPOSITION IX.

226. *To determine the Law of the centripetal Force, tending to a given Point* C, *whereby a Body may describe a given Curve* AQH.



Let QP be a Tangent to the Curve at any Point Q; upon which, from the Center C, let fall the Perpendicular CP; put $CQ = s$, $CP = u$; and let the Velocity of the Projectile at Q be denoted by v.

Therefore, since v^2 is always as $\frac{1}{u^2}$ (by the *Corol.* to *Lemma*) it is evident, by taking the Fluxions of both Quantities, that $v\dot{v}$ will also be as $\frac{-\dot{u}}{u^3}$: But the centripetal Force, whether the Body moves in a Right-line or or a Curve, is always as $-\frac{v\dot{v}}{\dot{s}}$ (by *Art.* 219. and 206.)

Therefore the centripetal Force is likewise as $\frac{\dot{u}}{u^3\dot{s}}$. *Q. E. I.*

The same otherwise.

227. Let the Ray of Curvature QO be denoted by *R*: Then, because the centripetal Forces in Circles are known to be as the Squares of the Velocities directly and the Radii inversely *, it follows that the Force, tending to the Point O, whereby the Body might be retained in its Orbit at Q, or in the Circle whose Radius is QO, will be defined by $\frac{1}{u^2}\times\frac{1}{R}$: Whence (by the Resolution of Forces) it will be CP (u) : CQ (s) :: $\frac{1}{u^2R}$ (the Force in the Direction QO) : $\frac{s}{u^3R}$, the Force in the Direction QC: Which, because $R=\frac{s\dot{s}}{\dot{u}}$ † will also be expressed by $\frac{\dot{u}}{u^3\dot{s}}$. *Q. E. I.*

* Art. 212.

† Art. 73.

Another Way.

228. Let *nq* be the indefinitely small Part of the Right-line C*q*, intercepted by the Curve and the Tangent Q*q*, expressing the Effect of the centripetal Force in the Time of describing the Area QC*n*. Now these Effects, or the Distances descended by means of Forces uniformly continued, are known to be in the duplicate Ratio of the Times ‡, or of the Areas denoting those Times §: Therefore, the centripetal Force at Q, or the Distance descended by means thereof in a given Time, will be as *nq* applied to the second Power of the Area QC*q*, or as $\frac{nq}{CP^2\times Qq^2}$. This Expression is the same with

‡ Art. 201.

§ Art. 224.

with that given by Sir *Isaac Newton*, in his *Principia*, *Book* 1. *Prop.* 6. But, to adapt it to a fluxional *Calculus*; let QE be an Ordinate to the principal Axis AG; and let (as usual) AE = x, EQ = y, AQ = z, Ee (or Qt) = $\dot{x}$, Qq = $\dot{z}$; supposing eq (parallel to EQ) to intersect the Curve and the Tangent in m and q.

Since Qq is conceived indefinitely small (or in its nascent State) the Triangle nmq may be taken as recti-

* Art. 136. lineal *; also the Angle n = CQP and the Angle m = Qqt: Whence, it will be (by Trigonometry) as *S.* CQP (n) : *S.* Qqt (m) :: mq : nq; that is, as $\frac{CP}{CQ} : \frac{Qt}{Qq}$ :: mq : $nq = \frac{CQ \times Qt \times mq}{CP \times Qq}$: Which substituted above gives $\frac{CQ \times Qt \times mq}{CP^3 \times \overline{Qq}^3}$ for the Measure of the centripetal Force at Q: But mq (supposing x to flow uniformly) is known to be as $-\ddot{y}$: Therefore the Force at Q, is as $\frac{CQ \times Qt \times -\ddot{y}}{CP^3 \times Qq^3}$, or its Equal $\frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$; where the Divisor ($u^3\dot{z}^3$) is as the Cube of (QC$q$) the Fluxion of the Area AQC.

The very same Theorem may likewise be deduced from that given by our second Method: For, since (R)

† Art. 68. the Ray of Curvature at Q is universally * = $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$, the Value of $\frac{s}{u^3 R}$ (there found) will here, by Substitution, become = $\frac{-s\dot{x}\ddot{y}}{u^3\dot{z}^3}$.

This Expression, tho' in appearance less simple than $\frac{\dot{u}}{u^3 \dot{s}}$, first found, is, for the general part, more commodious in Practice.

COROLLARY I.

229. If the Point C be ſo remote that all Right-lines drawn from thence to the Curve may be conſider'd as parallel to each other, the Force will then (making Qr perpendicular to Cq) be as $\frac{-s\dot{x}\ddot{y}}{Cq \times \overline{Qr}|^3}$, or barely as $\frac{-\dot{x}\ddot{y}}{\overline{Qr}|^3}$; ſince s (Cq) in this Caſe may be rejected.

From this Expreſſion, which is general, in all Caſes where the Force acts in the Direction of parallel Lines, it appears that the Force, which always acting in the Direction of the Ordinate QE, would retain the Body in its Orbit, is every where as $\frac{-\ddot{y}}{\dot{x}^2}$; becauſe QC here coincides with QE, and Qr becomes $= \dot{x}$.

COROLLARY II.

230. Becauſe the Force, tending to the Point C, is univerſally as $\frac{CQ}{CP^3 \times QO}$ $\left(\text{or } \frac{s}{u^3 R}\right)$ the Force to any other Point c, will, by the ſame Argument, be as $\frac{cQ}{\overline{cp}|^3 \times QO}$. Hence the Forces, to different Centers C and c (about which equal Areas are deſcribed in the ſame time) are to each other in the Ratio of $\frac{CP^3}{CQ}$ to $\frac{\overline{cp}|^3}{cQ}$ inverſely.

COROLLARY III.

231. Moreover, the Ratio of the Velocity at Q to the Velocity whereby the Body might revolve in a Circle about the Center at C, at the Diſtance CQ, is eaſily deduced from hence: For, ſince the Celerity at Q is that whereby

whereby the Body might revolve in a Circle about the Center O, and the Forces tending to the Centers O and C are to each other as u (CP) and s (CQ); it therefore follows, if the Ratio sought be aſſumed as v to w, that $\frac{v^2}{QO} : \frac{w^2}{QC} :: u : s$ (*by Art.* 212.) Whence alſo $v^2 : w^2 :: u \times QO\ (uR) : s \times QC\ (s^2)$ and conſequently $v : w :: \sqrt{\frac{uR}{ss}} : 1 :: \sqrt{\frac{u\dot{s}}{s\dot{u}}} : 1 :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{\frac{\dot{u}}{u}}$ (becauſe $R = \frac{s\dot{s}}{\dot{u}}$).

The ſame Proportion may alſo be derived from *Corol.* 2. *Prop.* 7. For it is there proved that $v : w :: \sqrt{\frac{\dot{s}}{s}} : \sqrt{-\frac{\dot{v}}{v}}$; and it appears from above, that $-\frac{\dot{v}}{v} = \frac{\dot{u}}{u}$: Whence the whole is manifeſt.

If OL be made perpendicular to QC, QL will be $\left(= \frac{CP \times QO}{CQ}\right) = \frac{uR}{s}$, and $\frac{QL}{CQ} = \frac{uR}{s^2}$; and therefore $v : w :: QL^{\frac{1}{2}} : CQ^{\frac{1}{2}}$: Which is another Proportion of the propoſed Celerities.

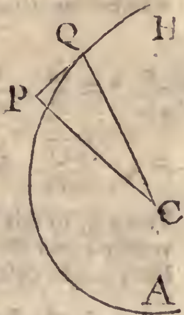
COROLLARY IV.

232. Laſtly, the Law of centripetal Force being given, the Nature of the Trajectory AQ may from hence be found; for ſince the Force (F) is univerſally defined by $\frac{\dot{u}}{u^3\dot{s}}$, it is evident that $\frac{-1}{2u^2}$ will be $=$ the Fluent of $F\dot{s}$; which, when F is given in Terms of s, will become known; and then, the Relation between u and s being given, the Curve itſelf is known.

EX-

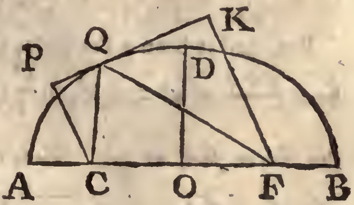
EXAMPLE I.

233. *Let the given Curve* AQH *be the logarithmic Spiral, and* C *the Center thereof:* Then u (CP) being in this Case $= \frac{b\dot{s}}{a}$ *, we have $\frac{\dot{u}}{u^3\dot{s}}$ † $\left(= \frac{b\dot{s}}{a\dot{s}} \times \frac{a^3}{b^3s^3}\right)$ * Art. 61. † Art. 227.
$= \frac{a^2}{b^2s^3}$, and $\sqrt{\frac{u\dot{s}}{\dot{s}\dot{u}}}$ ‡ $\left(= \sqrt{\frac{bs\dot{s}}{a} \times \frac{a}{bs\dot{s}}}\right)$ = Unity. Hence it appears that the Force is inversely as the Cube of the Distance; and the Velocity, every where, equal to that whereby the Body might revolve in a Circle at the same Distance. ‡ Art. 231.



EXAMPLE II.

234. *Let it be required to find the Law of the centripetal Force, whereby a Body, tending to the Focus* C, *is made to revolve in the Periphery of an Ellipsis* AQDB.



From the other Focus F draw FK parallel to CP meeting the Tangent PQ (at Right-angles) in K, join F, Q; putting the transverse Axis AB $= a$, the Semi-conjugate OD $= \frac{1}{2}b$, and the Parameter $\left(\frac{b^2}{a}\right)$ $= p$: Then, CQ and CP being denoted as above *, we have FQ ($=$AB$-$CQ) $= a - s$; whence, by reason of the similar Triangles CQP and FQK, it will be * Art. 231.

$s : u :: a - s : FK = \frac{\overline{a-s} \times u}{s}$. But FK $\times$ CP is $= OD^2$ (by the Nature of the Curve.) Hence we get $\frac{\overline{a-s} \times u^2}{s} = \frac{1}{4} b^2$; and consequently $\frac{1}{u^2} = \frac{4a}{b^2 s} - \frac{4}{b^2}$; whereof the Fluxion being $-\frac{2\dot{u}}{u^3} = -\frac{4a\dot{s}}{b^2 s^2}$, we obtain

* Art. 227. † Art. 231. $\frac{\ddot{u}}{u^3 \dot{s}}$ * $= \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, and $\sqrt{\frac{u\dot{s}}{s\dot{u}}}$ † $= \sqrt{\frac{2 \times \overline{a-s}}{a}}$ $= \sqrt{\frac{FQ}{AO}}$. Hence, it appears that the centripetal Force is, in this Case, as the Square of the Distance inversely; and the Velocity at Q is to that whereby the Body might describe a Circle at the Distance CQ, every where, in the Ratio of $FQ^{\frac{1}{2}}$ to $AO^{\frac{1}{2}}$.

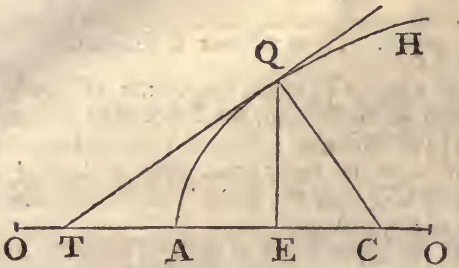
If the Curve had been an Hyperbola; then $\frac{a+s}{s} \times a^2$ (instead of $\frac{a-s}{s} \times u^2$) would have been $= \frac{1}{4} b^2$; and so $\frac{\ddot{u}}{u^3 \dot{s}} = \frac{2a}{b^2} \times \frac{1}{s^2} = \frac{2}{ps^2}$, the very same as before,

But, had it been a Parabola, the Equation would have been $\frac{a+0}{s} \times u^2 = \frac{1}{4} b^2$, or $\frac{u^2}{s} \left(= \frac{b^2}{4a}\right) = \frac{1}{4} p$; and the Force *still*, as $\frac{2}{ps^2}$. But, the Measure of the Velocity $\left(\sqrt{\frac{u\dot{s}}{s\dot{u}}} = \sqrt{\frac{2a-2s}{a}}\right)$ in this Case becoming barely $= \sqrt{2}$, it follows that the Velocity in a Parabola is to that whereby the Body might describe a Circle at the same Distance from the Center, in the constant Ratio of $\sqrt{2}$ to Unity.

EX-

EXAMPLE III.

235. *Let it be required to find the Law of the centripetal Force, by which a Body, tending to any given Point* C, *in the Axis, is made to describe a conic Section* AQH.



Put the ſemi-tranſverſe Axis (OA) $= a$, the ſemi-conjugate $= b$, and the given Diſtance of the Point C from the Vertex A $= c$: Put alſo the Abſciſſa AE, $= x$, the Ordinate EQ $= y$, and CQ $= s$ *(as before)*.

The Area of the Triangle ECQ being ($= \frac{1}{2}$EC×EQ) $= \frac{cy - xy}{2}$, its Fluxion is therefore $= \frac{c\dot{y} - x\dot{y} - y\dot{x}}{2}$; which added to $y\dot{x}$, the Fluxion of the Area AEQ, gives $\frac{c\dot{y} + y\dot{x} - x\dot{y}}{2}$ for the Fluxion of the whole Area ACQ deſcribed about the Center of Force. Whence (*by Art.* 228.) the required centripetal Force at Q will be as $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3}$. Which Expreſſion is general, let the Curve be of what Kind it will. But in the Caſe above, y being $= \frac{b}{a}\sqrt{2ax \pm x^2}$, we have $\dot{y} = \frac{b\dot{x} \times \overline{a \pm x}}{a\sqrt{2ax \pm x^2}}$, $\ddot{y} = \frac{-ab\dot{x}^2}{\overline{2ax \pm x^2}|^{\frac{3}{2}}}$, and $c\dot{y} + y\dot{x} - x\dot{y} =$

$\frac{b\dot{x} \times \overline{ca + ax \pm cx}}{a\sqrt{2ax \pm x^2}}$; and therefore, by ſubſtituting theſe Values, we get $\frac{-s\dot{x}\ddot{y}}{\overline{c\dot{y} + y\dot{x} - x\dot{y}}|^3} = \frac{a^4 s}{b^2 \times \overline{ca + ax \pm cx}|^3}$ Which, becauſe $\frac{a^4}{b^2}$ is conſtant, will alſo be as $\frac{s}{\overline{ca + ax \pm cx}|^3}$. From whence it follows,

1°. If c be $= \mp a$, or the Center of Force be in the Center of the Section, the Force itſelf will be barely as ($\pm s$) the Diſtance.

2°. If it be in the Focus, then $ac + ax \pm cx$ becoming $= CQ \times a$, the Force will be inverſely as the Square of the Diſtance.

3°. If the given Point be in the Vertex A, the Force will be as $\frac{s}{x^3}$: Which therefore in the Circle (where $x = \frac{s^2}{2a}$) will be as $\frac{1}{s^5}$, or the fifth Power of the Diſtance reciprocally.

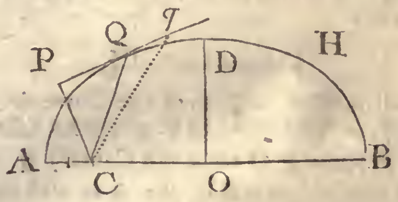
4°. Laſtly, if the Point C be at an indefinite Diſtance from the Vertex, or the Force be ſuppoſed to act in the Direction of Lines parallel to the Axis AO; then the Force will be as the Cube of OE inverſely.

PROPOSITION X.

236. *To determine the Ratio of the Velocities of Bodies revolving in different Orbits, about the ſame, or different, Centers; the Orbits themſelves, and the Forces whereby they are deſcribed, being given.*

Let AQH be any Orbit, deſcribed about the Center of Force C, and let the Force itſelf at the principal Vertex A be denoted by F; alſo let r ſtand for the Semi-parameter, or the Ray of Curvature at the Vertex, and let

let CP be perpendicular to the Tangent QP.



Then, the Celerity at A being, always, as $\sqrt{rF}$ (*by Art.* 212.) we have CP : CA :: $\sqrt{rF}$ (the Velocity at A) to $\frac{CA \times \sqrt{rF}}{CP}$, the Velocity at Q (*by Art.* 225.) Which anſwers in all Caſes, let the Values of AC, r and F be what they will. *Q. E. I.*

COROLLARY I.

237. If the centripetal Force be as the Square of the Diſtance inverſely, or F be expounded by $\frac{1}{AC^2}$, the Velocity at Q will become $\frac{AC}{CP} \times \sqrt{\frac{r}{AC^2}}$, or $\frac{\sqrt{r}}{CP}$: Whence the Velocities, in different Orbits, about the ſame Center, are in the ſubduplicate Ratio of the Parameters, and the inverſe Ratio of the Perpendiculars from the Center of Force to the Tangents, conjunctly.

COROLLARY II.

238. Hence, if the Celerity at Q be denoted by Qq, and Cq be drawn; then, Qq being as $\frac{\sqrt{r}}{CP}$, it follows that $\sqrt{r}$ is as CP × Qq, or as the Triangle QCq: Therefore

fore the Areas described about a common Center of Force in a given Time, are in the subduplicate Ratio of the Parameters.

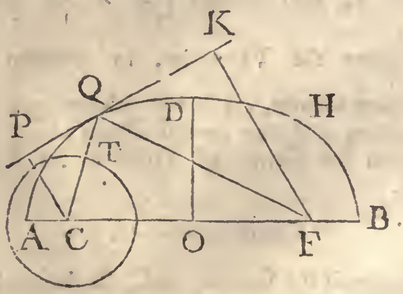
COROLLARY III.

239. Lastly, since the Area of the Curve AQHB *&c.* when an Ellipse*, is known to be as (AO × OD) AO × $\sqrt{r \times AO}$ (supposing O to be the Center) if the same be apply'd to $\sqrt{r}$, expressing the Area described in a given Part of Time (by the last *Corol.)* we shall thence have AO × $\sqrt{AO}$, or $AO^{\frac{3}{2}}$ for the Measure of the Time of one whole Revolution. From whence it appears, that the periodic Times, let the Species of the Ellipses be what they will, are in the sesquiplicate Ratio of their principal Axes.

* Art. 234.

PROPOSITION XI.

240. *The centripetal Force, tending to a given Point* C, *being as the Square of the Distances reciprocally, and the Direction and Velocity of a Body at any Point* Q *being given; to determine the Path in which the Body moves, and the periodic Time, in case it returns.*



It is evident from *Art.* 234. *and* 235. that the Trajectory AQB is a conic Section; whereof the Point C is one of the *Foci.*

Let

Let F be the other Focus, and upon the Tangent PQK let fall the Perpendiculars CP and FK, and let CQ and FQ be drawn: Alſo put the ſemi-tranſverſe Axis AO $= a$, the given focal Diſtance CQ $= d$, and the Sine of the Angle of Direction CQP (to the Radius 1) $= m$; and let the given Velocity at Q be to that whereby the Body might revolve in a Circle about the Center C, at that Diſtance, in any given Ratio of n to Unity: Then it will be $n : 1 :: FQ^{\frac{1}{2}} : AO^{\frac{1}{2}}$ (by *Art.* 234.) therefore $n^2 : 1^2 : FQ\ (2a-d) : AO\ (a)$; whence AO (a) is given $= \frac{d}{2-n^2}$. Moreover, ſince $CP = m \times CQ$, and $FK = m \times FQ$, we have OD^2 ($= CP \times FK = m^2 \times CQ \times FQ = \frac{m^2n^2d^2}{2-n^2}$; whence the ſemi-conjugate Axis (OD) is given likewiſe.

Laſtly, it will be (by *Art.* 239.) as $CT^{\frac{3}{2}} : AO^{\frac{3}{2}} ::$ *(P)* the periodic Time in any given Circle, whoſe Radius is CT, to $\left(\frac{AO^{\frac{3}{2}}}{CT^{\frac{3}{2}}} \times P\right)$ the required Time of one Revolution when the Orbit is an Ellipſis; that is, when n^2 is leſs than 2: For, if n^2 be $= 2$, the Curve (as its Axis $\frac{2d}{2-n^2}$ becomes infinite) will degenerate to a Parabola; and, if n^2 be greater than 2, the Axis being negative, it is then an Hyperbola; whoſe two principal Diameters are equal to $\frac{2d}{n^2-2}$ and $\frac{2mnd}{\sqrt{n^2-2}}$. *Q. E. I.*

Corollary.

241. Seeing neither the Value of AO, nor that of the periodic Time, is affected with m, it follows that the principal Axis, and the periodic Time, will remain

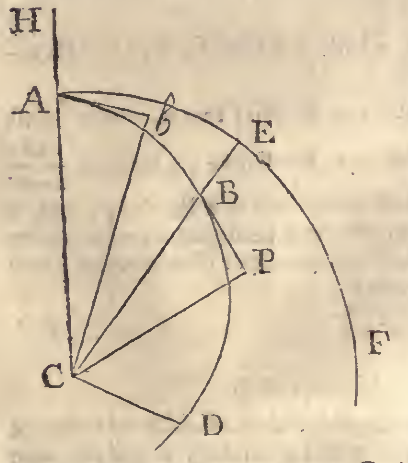
invariable, if the Velocity at Q be the same, let the Direction at that Point be what it will.

The same Solution may likewise be brought out, from *Art.* 238. by first finding the *principal Parameter:* For, it is evident that the Area described by the Body about the Center C, in any given Time, is to the Area described, in the same Time, by another Body revolving in a Circle at the Distance CQ, as mn to Unity: Hence, it will be $1^2 : m^2n^2 :: d : (m^2n^2d)$ the Semi-parameter *: From which (proceeding as above) we get $a \times m^2n^2d$ $(=OD^2) = m^2 \times \overline{2ad - d^2}$; and consequently $a = \frac{d}{2 - n^2}$, the same as before.

* Art. 238.

PROPOSITION XII.

242. *The centripetal Force being as any Power (n) of the Distance, and the Direction and Velocity of a Body at any Point A being given, to determine the Orbit or Trajectory.*



From the Center of Force, C, to any Point B in the required Trajectory ABD, let CB be drawn; join C, A, and let Ab be the given Direction of the Body at the Point A, and Cb perpendicular thereto; also let the Velocity at A be to that whereby a Body might describe a Circle AEF, about the Center C, in any given Ratio of p to Unity; putting CA$=a$, and CB$=x$: Then be because this last Velocity (the centripetal Force being as x^n (or a^n) is rightly defined by $a^{\frac{n+1}{2}}$ *, the Velocity of the Body at A will be truly expressed by $pa^{\frac{n+1}{2}}$:

* Art. 214.

Moreover, it is proved in *Art.* 221. *and* 206. that if the Celerity, at any given Distance a from the Center, be denoted by c, the Celerity at any other Distance x will be truly represented by $\sqrt{c^2 + \frac{2a^{n+1} - 2x^{n+1}}{n+1}}$:

Whence, $pa^{\frac{n+1}{2}}$ being substituted for c, we have $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}$ for the Celerity at B.

But now; to determine the Curve itself from hence, let BP be a Tangent to it at B, and CP perpendicular to BP; also let CB, produced, meet the Periphery of the Circle in E; putting the Arch AE $= z$, the foresaid Velocity at B (to shorten the Operation) $= v$, and C$b = b$: Then it will be (by *Art.* 225.) $v : c$ (the Velocity at A) :: b : CP $= \frac{bc}{v}$ Whence BP ($= \sqrt{CB^2 - CP^2}$) $= \frac{\sqrt{x^2v^2 - b^2c^2}}{v}$.

Moreover (by *Art.* 35.) we have, as CB : CP :: v : $\left(\frac{CP}{CB} \times v\right)$ the Velocity of the Body at B in a Direction perpendicular to CE; and consequently, as CB : CE :: $\frac{CP}{CB} \times v$ (the said Velocity) to $\frac{CP \times CE}{CB^2} \times v$ the angular Velocity of the Point E (revolving with the Body.) By the same *Article*, the Velocity at B in the

Direction CBE will be $\frac{BP}{CB} \times v$: Therefore, the Velocity of E being to the Velocity of B, in the said Direction, as $\frac{CP \times CE}{CB^2}$ to $\frac{BP}{CB}$, the Fluxions of AE (z) and CB (x) must consequently be in that Ratio; that is, $\frac{CP \times CE}{CB^2} : \frac{BP}{CB} :: \dot{z} : \dot{x}$; whence $\dot{z} = \frac{CP \times CE}{CB \times BP} \times \dot{x} = \frac{bc}{v} \times \frac{a}{x} \times \frac{v\dot{x}}{\sqrt{x^2v^2 - b^2c^2}} = \frac{abc\dot{x}}{x\sqrt{x^2v^2 - b^2c^2}} = \frac{ab\dot{x}}{x\sqrt{\frac{x^2v^2}{c^2} - b^2}}$. Which Equation is general, let the Law of the centripetal Force be what it will: But in the Case above proposed, v^2 being $= p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}$, and $c^2 = p^2a^{n+1}$; it becomes $\dot{z} = \frac{abp\dot{x}}{x\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2b^2 - \frac{2x^{n+3}}{n+1 \times a^{n+1}}}}$; whose Fluent is the Measure of the angular Motion; from which, when found, the Orbit may be constructed: Because, when AE, or the Angle ACE is given, as well as CB, the Position of the Point B is also given. But this Value of $\dot{z}$ is indeed too complex to admit of a Fluent in algebraic Terms, or even by circular Arcs and Logarithms, except in certain particular Cases; as when the Exponent n is equal to 1, -2, -3, or -5; besides some others wherein the Values of p and n are related in a particular Manner. *Q. E. I.*

Co-

Corollary I.

243. If the given Velocity at A be ſuch that $p^2 + \frac{2}{n+1} = 0$, or $p = \sqrt{\frac{-2}{n+1}}$ (which is always poſſible when the Value of $n+1$ is negative) our Equation will become $\dot{z} \times \frac{abp\dot{x}}{x\sqrt{-p^2b^2 + \frac{p^2x^{n+3}}{a^{n+1}}}}$: Which, by putting $n+3=m$, &c. is reduced to $\dot{z} = \frac{ab\dot{x}}{x\sqrt{-b^2+\frac{x^m}{a^{m-2}}}}$: Whereof the Fluent will be found (by the ſecond Part of this Work (equal to $\pm \frac{2a}{m}$ multiply'd by the Difference of the two circular Arcs, whoſe Secants are $\frac{x^{\frac{1}{2}m}}{ba^{\frac{1}{2}m-1}}$ and $\frac{a}{b}$ to the Radius Unity.) From this Value of the Arch AE the Poſition of the Point B, in the Orbit, is given.

But if the Angle of Direction CA*b* be a right one, the Fluent will become barely $= \pm \frac{2a}{m} \times$ Arch whoſe Secant is $\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}m}}$ (becauſe then $b=a$, and the Arch whoſe Secant is $\frac{a}{b}, = 0$) which therefore when $x^{\frac{m}{2}}$ becomes

in-

infinite, will be truly defined by $\pm \frac{1}{2m} \times$ whole Periphery AF, *&c.* Whence it is evident that the Body must either fly intirely off, or fall to the Center C, in a Number of Revolutions expressed by $\pm \frac{1}{2m}$; according as the Value of m is positive or negative.

Thus, if $n = -2$, and $m = 1$, the Body will fly intirely off in half a Revolution: And, if $n = -4$, and $m = -1$, it will fall to the Center in half a Revolution.

COROLLARY II.

244. Moreover, tho' the Fluent expressing the Angle at the Center cannot be exhibited in a general Manner yet there are certain Cases of the Exponent (n) where its respective Values may be derived from each other.

For let (as above) $n+3$ be put $= m$, and (to shorten the Operation) let CA (a) be taken as Unity: Then our Equation will be transformed to $\dot{z} = \frac{b\dot{x}}{x\sqrt{1+\frac{2}{m-2.p^2} \times x^2 - b^2 - \frac{2x^m}{m-2.p^2}}}$: Make $y = x^{\frac{m}{2}}$, and it will be farther transformed to $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{1+\frac{2}{m-2.p^2} \times y^{\frac{4}{m}} - b^2 - \frac{2y^2}{m-2.p^2}}}$: Put $r = \frac{4}{m}$, and it will become $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{\frac{ry^2}{r-2.p^2} - b^2 + 1 - \frac{r}{r-2.p^2} \times y^r}}$: Lastly, let $\frac{r}{r-2.p^2} = 1 + \frac{2}{r-2.q^2}$ (or $1 - \frac{r}{r-2.p^2} = -\frac{2}{r-2.q^2}$, or $q^2 = \frac{2p^2}{r-p^2 \times r-2}$) and then we shall have $\dot{z} = \frac{2}{m} \times \frac{b\dot{y}}{y\sqrt{1+\frac{2}{r-2.q^2} \times y^2 - b^2 - \frac{2y^r}{r-2.q^2}}}$.

Which Expression (excepting the general Multiplicator $\frac{2}{m}$) being exactly of the same Form with the first above given, must therefore be the Fluxion of the Angle at the Center, when the Index of the Force is $r-3$; for the very same Reasons that the former appears to be the Fluxion thereof when the Index is $m-3$ (or n.)

Hence, if the Fluent of

$\frac{b\dot{y}}{y\sqrt{1+\frac{2}{r-2.q^2} \times y^2 - b^2 - \frac{2y^r}{r-2.q^2}}}$, or the Angle at the Center, when the Exponent is $r-3$ (or $\frac{4}{m}-3 = \frac{4}{n+3}-3$) be denoted by w, the Value of z, (the Measure of the said Angle, when the Exponent is $m-3$ (or n) will be truly defined by $\frac{2w}{m}$.

From which we collect that, if the Indices of the Force, in any two Cases, be represented by n and $\frac{4}{n+3}-3$, and the respective Distances from the Center by x and $x^{\frac{n+3}{2}}$, then the Angles themselves corresponding to those Distances will be every where in the constant Ratio of 2 to $n+3$. Therefore, when the Orbit can be be conſtructed in the one Caſe, it alſo may in the other, provided the above Equation $q^2 \left(= \frac{2p^2}{r - p^2 \times \overline{r-2}}\right) = \frac{\overline{n+3} \cdot p^2}{2 + \overline{n+1} \cdot p^2}$, for the Relation of the Celerities at A, does not become impoſſible, as it will, ſometimes, when n is a negative Number.

COROLLARY III.

245. If the Body be ſuppoſed to move in a vertical Direction AH; then, putting the Velocity $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}} = 0$, we get x $(CH) = \overline{\tfrac{1}{2} p^2 \times \overline{n+1} + 1}^{\frac{1}{n+1}} \times a =$ the Height to which the Body will aſcend: Hence $\overline{\tfrac{1}{2} p^2 \times \overline{n+1} + 1}^{\frac{1}{n+1}}$ $\times a - a$ (= AH) is the Diſtance through which it muſt freely deſcend to acquire the given Celerity at A: This Diſtance, in caſe of an uniform Force, when $n = 0$, will become $= \tfrac{1}{2} p^2 a$: And, when the Force is inverſely as the Square of the Diſtance, it will then be $= \frac{p^2 a}{2 - p^2}$.

But, when $p = 1$, or the Velocity at A is juſt ſufficient to retain a Body in the Circle AEF, AH becomes $= \overline{\frac{3+n}{2}}^{\frac{1}{n+1}} \times a - a$: Which in the two Caſes aforeſaid will be equal to $\tfrac{1}{2}a$, and a reſpectively; but, infinite, when n is $= -3$.

Co-

Corollary IV.

246. When the Value of $n+1$ is positive, the Velocity at the Center, where $x=0$, will be barely $= \sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}}$; but if the Value of $n+1$ be negative, the Velocity at the Center will be infinite; becauſe, then 0^{n+1} is infinite.

Corollary V.

247. Moreover, when $n+1$ is negative and x infinite, the Velocity alſo becomes $= \sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}}$; becauſe then $x^{n+1} = 0$.

Hence, if the centripetal Force be inverſely as ſome Power of the Diſtance greater than the firſt, the Body may aſcend, *ad infinitum*, and have a Velocity always greater than $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}}$; which is to, $pa^{\frac{n+1}{2}}$, the given Velocity, at A, as $\sqrt{p^2 + \frac{2}{n+1}}$ to p. And this will actually be the Caſe when the Value of $p^2 + \frac{2}{n+1}$ is poſitive, or p^2 greater than $\frac{2}{-n-1}$, but not otherwiſe, the ſquare Root of a negative Quantity being impoſſible.

Thus, if $n = -2$, or the Force be inverſely as the Square of the Diſtance, and p^2, at the ſame time, greater than 2 $\left(\frac{2}{-n-1}\right)$ the Body will not only continue to aſcend *in infinitum*, but have a Velocity always greater than that defined by $\sqrt{p^2 - 2}$, which is its Limit.

Co-

COROLLARY VI.

248. Hence the leaſt Celerity ſufficient to cauſe the Body to aſcend for ever in a Right-line is given. For, putting $\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1}} = 0$, we have $p = \sqrt{\frac{2}{-n-1}}$. Therefore the leaſt Celerity by which the Body might aſcend for ever, is to that whereby it may revolve in a Circle AEF, as $\sqrt{\frac{2}{-n-1}}$ to Unity. From which it appears that, if the Force be inverſely as any Power of the Diſtance greater than the third, a leſs Velocity will cauſe a Body to aſcend *ad infinitum* than would retain it in a Circle.

SCHOLIUM.

249. From the Ratio of the Velocity $\left(\sqrt{p^2 + \frac{2}{n+1} \times a^{n+1} - \frac{2x^{n+1}}{n+1}}\right)$ wherewith the Body arrives at any Diſtance x from the Center, to that $\left(x^{\frac{n+1}{2}}\right)$ * which it ought to have to revolve in a Circle at the ſame Diſtance, it will not be difficult to determine in what Caſes the Body will be forced to the Center, and in what others it will continue to fly from it *ad infinitum*.

* Art. 214.

For, firſt, if the Angle CAb be acute, or the Body from A begins to deſcend, it will continue to do ſo till it actually arrives at the Center, if the former Velocity, during the Deſcent, be not ſomewhere greater than the latter, or the Quotient $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ greater than Unity; becauſe, if it ever begins to aſcend, it it muſt have an *Apſe*, as D (where a Right-line drawn from the Center cuts the Orbit at Right-angles) and there the Celerity muſt evidently be greater than that ſufficient to cauſe the Body to revolve in a Circle.

Secondly, but if the Quantity

$$\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}},$$ in the Acceſs of

the Body towards the Center, increaſes ſo as to become greater than Unity, or be every where ſo; then the Velocity at all inferior Diſtances being more than ſufficient to retain a Body in a Circle at any ſuch Diſtance, the Projectile cannot be forced to the Center.

After the ſame Manner, if the Angle CA*b* be obtuſe, or the Body from A begins to aſcend, it will continue to do ſo for ever, when the foreſaid Quantity is always greater than Unity, or, which is the ſame, when the Body, in its Receſs from the Center, has in every Place through which it paſſeth, a Velocity greater than ſufficient to retain it in a Circle at that Diſtance.

It therefore now remains to find in what Laws of the centripetal Force theſe different Caſes obtain: And, firſt, it is eaſy to perceive that when the Value of $n+1$ is poſitive, that of $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ will, by increaſing x, become equal to nothing. Therefore the Body cannot aſcend for ever in this Caſe: Neither can it deſcend to the Center (except in a Right-line) becauſe the foreſaid Quantity, by diminiſhing x, becomes greater than Unity (or any other aſſignable Magnitude.)

But, if the Value of n be betwixt -1, and -3, the ſaid general Expreſſion, taking x infinite, will alſo become infinite, provided the Value of $p^2 + \frac{2}{n+1}$ be poſitive (or p^2 greater than $\frac{2}{-n-1}$). Therefore the Body

Body, in this Cafe, may afcend *ad infinitum*, but cannot poffibly fall to the Center (except in a Right-line) fince, $\sqrt{-\frac{2}{n+1}}$, the Value of the general Expreffion, when $x = 0$, is greater than Unity.

Laftly, if n be expreffed by any negative Number greater than -3, or the Law of the Force be inverfely as any Power of the Diftance greater than the third, the two extreme Values of $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$ will, *ftill*, be denoted as in the preceeding Cafe; but here the latter of them, $\sqrt{\frac{-2}{n+1}}$, is lefs than Unity. Therefore the Body muft, in this Cafe, either afcend for ever, or be forced to the Center; except in one particular Circumftance, hereafter to be taken notice of.

Now, from thefe Obfervations we gather,

1°. That, when the centripetal Force is as any Power of the Diftance directly, or lefs than the firft Power thereof inverfely, the Orbit will always have an higher and a lower *Apfe*; beyond which the Body cannot afcend or defcend.

2°. That, when the centripetal Force is inverfely as any Power of the Diftance (whole or broken) betwixt the firft and third, the Orbit will alfo have two *Apfides*, if p be lefs than $\sqrt{-\frac{2}{n+1}}$; but otherwife, only one; in which laft Cafe the Body, after it has paffed its Apfe, will continue to recede from the Center *in infinitum*.

3°. That when the Force is inverfely as any Power greater than the third, the Orbit can, at moft, have but one *Apfe*; but, in fome Cafes, it will have none at all: And it may be worth while to inquire here, under what Reftrictions of the Velocity (p) this will happen; fince thereby, befides being able to know when the Body will

be

be forced to the Center, *&c.* we ſhall fall upon a Circumſtance ſomewhat remarkable and curious.

Now it appears, that, if the Body from A begins to deſcend, it muſt, when it comes to an *Apſe* at D, have a Velocity there greater than is ſufficient to retain it in a Circle; in which Caſe the general Expreſſion

$$\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$$ (ſo often mention'd above) muſt accordingly be greater than Unity. Let it be therefore made equal to Unity, which is the utmoſt Limit thereof, beyond which the Orbit cannot admit of an *Apſe*; putting at the ſame time $\dot{x}$, or its Diviſor

$$\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2b^2 - \frac{2x^{n+}}{\overline{n+1}.a^{n+1}}},$$ in the general Equation of the Orbit, equal to nothing (it being always ſo at the *Apſides.*) Then, from theſe two Equations, duly order'd, we ſhall get $x =$

$$\overline{\frac{2+\overline{n+1}.p^2}{n+3}}\Bigg|^{\frac{1}{n+1}} \times a, \text{ and } p^2 \left(= \frac{x^{n+3}}{a^{n+1}}\right) =$$

$$\overline{\frac{2+\overline{n+1}.p^2}{n+3}}\Bigg|^{\frac{n+3}{n+1}} \times \frac{a^2}{b^2}.$$ Now, it is evident, if the Value of p be greater than is given from the laſt Equation, the Orbit will have an Apſe; but if leſs, it can have none. In the former Caſe, the Body will therefore fly quite off; and in the latter, it will be forced to the Center. But we are now, naturally, led to inquire what will be the Conſequence when the Value of p is neither greater nor leſs, but exactly the ſame as given from the foreſaid Equation: This is the Caſe above hinted at; and here the Body will continue to deſcend for ever in a Spiral, yet never ſo low as to enter within the Circle

whoſe Radius CD is $= \overline{\frac{2+\overline{n+1}.p^2}{n+3}}\Bigg|^{\frac{1}{n+1}} \times a$. For, if the contrary were possible, the Body, at its Arrival to the Circumference of that Circle, would (because of the foresaid Equations) not only have a Direction, but also Velocity proper to retain it therein; which cannot be, because the Parts of the Orbit on either Side of an Apse are always similar to each other.

From the same Equation, the Value of the Limit will also be given when the Angle of Direction CAb is obtuse, or the Body is projected upwards:

For that Equation (as is easy to demonstrate *) admits of two different Roots, or Values of p; the one greater, the other less, than Unity: Whereof the former, giving CD (x) less than CA, is to be taken in the preceding Case, and the latter (making CD greater than CA) in the present. And the Body will, either, continue to ascend for ever, or come to an *Apse*, and from thence fall to the Center, according as the given Value of p is greater or less than that here specified. But if it be neither greater nor less, but exactly the same, then the Body, tho' it will still continue to ascend for ever in a Spiral, yet it can never rise so high as the Circumference of the Circle whose Radius CD is $=$

$$\left(\frac{2+\overline{n+1}\cdot p^2}{n+3}\right)^{\frac{1}{n+1}} \times a,$$ for Reasons similar to those already delivered, in respect to the preceding Case.

* *Mathematical Differt.* p. 167.

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THE

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FLUXIONS.

PART the Second.

SECTION I.

The Manner of investigating the FLUXIONS *of* Exponentials, *with* Thoſe *of the Sides and Angles of ſpherical Triangles.*

250. THE Method of deriving the Fluxion of any Power, x^v, of a flowing Quantity, when the Exponent (v) is given or invariable, has been already ſhewn: But, if the Exponent be variable, that Method fails; in which Caſe the Quantity x^v is called an Exponential; whoſe Fluxion is thus determined.

Put $z = x^v$, and let the hyperbolic Logarithm of x be denoted by y; then that of $x^v (z)$ will, by the Nature of Logarithms, be $= vy$; and therefore its Fluxion $= \dot{v}y + v\dot{y}$: But the Fluxion of the Logarithm of $z = x^v$) is

* Art. 126. is also expressed by $\frac{\dot{z}}{z}$ *; whence we have $\frac{\dot{z}}{z} = v\dot{y} + y\dot{v}$, and consequently $\dot{z} = zv\dot{y} + zy\dot{v}$: Which Equation, by substituting $\frac{\dot{x}}{x}$ for its Equal $\dot{y}$ †, becomes $\dot{z} = zy\dot{v} + \frac{zv\dot{x}}{x} = x^v y\dot{v} + x^v \times \frac{v\dot{x}}{x} = x^v y\dot{v} + vx^{v-1}\dot{x} = x^v \dot{v} \times \overline{hyp.\ Log.\ x} + vx^{v-1}\dot{x}$.

† Art. 126.

The same otherwise, without introducing the Properties of Logarithms.

251. Let $1 + z = x$, and $n + w = v$, supposing n constant and w variable: Then $x^v = \overline{1+z}|^{n+w} = \overline{1+z}|^n \times \overline{1+z}|^w = \overline{1+z}|^n \times \overline{1 + wz + \frac{w}{1} \times \frac{w-1}{2} \times z^2 + \frac{w}{1} \times \frac{w-1}{2} \times \frac{w-2}{3} \times z^3 + \&c.}$ ‡ $= \overline{1+z}|^n \times \overline{1 + wz + \overline{\tfrac{1}{2}w^2 - \tfrac{1}{2}w} \times z^2 + \overline{\tfrac{1}{6}w^3 - \tfrac{1}{2}w^2 + \tfrac{1}{3}w} \times z^3 + \&c.}$ whose Fluxion, found the common Way, is $n\dot{z} \times \overline{1+z}|^{n-1} \times \overline{1 + wz + \overline{\tfrac{1}{2}w^2 - \tfrac{1}{2}w} \times z^2 + \overline{\tfrac{1}{6}w^3 - \tfrac{1}{2}w^2 + \tfrac{1}{3}w} \times z^3\ \&c.} + \overline{1+z}|^n \times \overline{\dot{w}z + w\dot{z} + \overline{w\dot{w} - \tfrac{1}{2}\dot{w}} \times z^2 + \overline{\tfrac{1}{2}w^2 - \tfrac{1}{2}w} \times 2z\dot{z} + \overline{\tfrac{1}{2}w^2\dot{w} - w\dot{w} + \tfrac{1}{3}\dot{w}} \times z^3 + \overline{\tfrac{1}{6}w^3 - \tfrac{1}{2}w^2 + \tfrac{1}{3}w} \times 3z^2\dot{z}}$ &c. which, by substituting $\dot{x}$ and $\dot{v}$ for their Equals $\dot{z}$ and $\dot{w}$, becomes $n\dot{x} \times \overline{1+z}|^{n-1} \times \overline{1 + wz + \overline{\tfrac{1}{2}w^2 - \tfrac{1}{2}w} \times z^2 + \&c.} + \overline{1+z}|^n \times \overline{\dot{v}z + w\dot{x} + \overline{w\dot{v} - \tfrac{1}{2}\dot{v}} \times z^2 + \&c.}$

‡ Art. 99.

But, if w be, now, supposed to vanish, we shall have the true Value of the Fluxion when $v = n$; which, in that Circumstance, appears to be $= n\dot{x} \times \overline{1+z}|^{n-1}$.

$+ \overline{1+z}|^{v} \times \overline{z\dot{v} - \frac{1}{2}z^2\dot{v} + \frac{1}{3}z^3\dot{v} - \frac{1}{4}z^4\dot{v} \&c.} = v\dot{x} \times x^{v-1}$

$+ \dot{v}x^{v} \times \overline{z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 \&c.}$ *Q. E. I.*

It is plain, because the Series, $z - \frac{1}{2}z^2 + \frac{1}{3}z^3$ &c. here brought out, is known to expreſs the Fluent of $\frac{\dot{z}}{1+z}$, or the hyperbolic Logarithm of $1+z$ *, that the two Concluſions agree exactly with each other: From either of which the following *Rule*, for the Fluxions of Exponentials, is deduced. * Art. 126.

252. *To the Fluxion found by the common Rule* (Art. 14.) *conſidering the Exponent as conſtant, add the Quantity ariſing by multiplying the Fluxion of the Exponent, the hyperbolic Logarithm of the Root, and the propoſed Quantity itſelf, continually, together: The Sum will be the Fluxion when the Exponent is variable.*

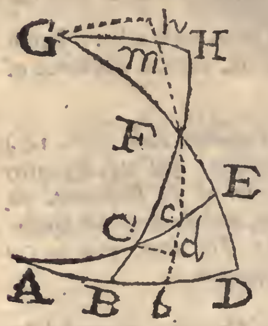
Thus, for Example, let the Quantity propoſed be $\overline{a^2+z^2}|^{z}$, then the Fluxion thereof will be $z \times 2z\dot{z} \times \overline{a^2+z^2}|^{z-1} + \dot{z} \times \overline{a^2+z^2}|^{z} \times \overline{hyp. Log. a^2+z^2}$.

But, if the Root is conſtant, and only the Exponent variable, the *Exponential* will be more ſimple; and its Fluxion will then be had by *barely multiplying the Quantity itſelf by the Product under the Logarithm of the Root and the Fluxion of the Exponent.*

Thus, the Fluxion of a^x will be expreſſed by $a^x \times \dot{x} \times hyp. Log. a$; and that of $\overline{a^2+b^2}|^{nx}$ by $\overline{a^2+b^2}|^{nx} \times n\dot{x} \times hyp. Log. \overline{a^2+b^2}$. Theſe Kind of *Exponentials* oftener occur, in Practice, than any other; but, as it is very rare that we meet with *any*, I ſhall therefore proceed now to the other Conſideration propoſed in the Head of this Section; namely, the Method of determining the Fluxions of the Sides and Angles of ſpherical Triangles (a Thing very uſeful in practical Aſtronomy) which I ſhall deliver in the following Propoſitions.

PROPOSITION I.

253. *To determine the Ratio of the Fluxions of the ſeveral Parts of a right-angled ſpherical Triangle; ſuppoſing the Hypothenuſe, one Leg, or one Angle, to remain conſtant, while the other Parts vary.*



Let A, F, and G be the Poles of the three Great-Circles DEFG, ABD, and ACE; whereof the Poſition of each is ſuppoſed to continue invariable, while another Great-Circle HFCB is conceived to revolve about the Pole F: Whence, if GH be ſuppoſed perpendicular to FH, three variable right-angled Triangles, FGH, FCE, and ABC, will be formed; in the firſt whereof, the Hypothenuſe FG will remain conſtant; in the ſecond, the Leg EF; and in the third, the Angle A.

Let B*b* (*q*) be the Fluxion (or indefinitely ſmall Increment *) of the Baſe AB, or the Angle F; and let C*d* meet the Great-Circle *b*F*h*, at Right-angles, in *d*; then it will be *(per Spherics)* as *Sin.* FB *(Rad.)* : *Sin.* FC :: B*b* (*q*) : $Cd = \frac{Sin.\,FC}{Rad.} \times q = \frac{Co\text{-}ſ.\,BC}{Rad.} \times q$:

* Art. 134.

And, *Tang.* C : *Rad.* :: C*d* $\left(\frac{Co\text{-}ſ.\,BC}{Rad.} \times q\right) : \frac{Co\text{-}ſ.\,BC}{Tang.\,C} \times q$ = the Fluxion of BC.

Moreover, *Sin.* C : *Rad.* :: C*d* $\left(\frac{Co\text{-}ſ.\,BC}{Rad.} \times q\right)$: $\frac{Co\text{-}ſ.\,BC}{Sin.\,C.} \times q$ = the Fluxion of AC.

Laſtly, *Sine* of FB *(Rad.)* : *Sin.* FH (BC) :: B*b* (*q*) : $\frac{Sin.\ BC}{Rad.} \times q$ (=H*m*) = the Fluxion of GH, or its Complement C.

Now, if the ſeveral Quantities, in theſe three Equations for the Triangle ABC, be expounded by their reſpective Equals in the other two Triangles CEF and FGH, we ſhall alſo have

$$\frac{Sin.\ CF.}{Tang.\ C} \times q = -\ Flux.\ CF.$$

$$\frac{Sin.\ CF}{Sin.\ C} \times q = -\ Flux.\ CE.$$

$$\frac{Co.\text{-}s.\ CF}{Rad.} \times q = Flux.\ C.$$

And

$$\frac{Co\text{-}s.\ FH}{Co\text{-}tang.\ GH} \times q = Flux.\ FH.$$

$$\frac{Co\text{-}s.\ FH}{Co\text{-}s.\ GH} \times q = Flux.\ G.$$

$$\frac{Sin.\ FH}{Rad.} \times q = -\ Flux.\ GH.$$

Q. E. I.

Corollary I.

254. Hence, if, in any right-angled Spherical-Triangle, the Hypothenuſe be denoted by *h*, the two Legs by *L* and *l*, the Angles, reſpectively, adjacent to them by *A* and *a*, we ſhall, by ſubſtituting above, have three Equations for each of the three Caſes. From the Compariſon and Compoſition of which, the three following Tables are deduced; exhibiting all the different Varieties that can poſſibly happen, whether an Angle, a Leg, or the Hypothenuſe be ſuppoſed invariable.

TABLE I.

When one Angle A is invariable,

$$\dot{L}=\frac{Tang.\ a}{Co\text{-}s.\ l}\times\dot{l}=\frac{Sin.\ a}{Co\text{-}s.\ l}\times\dot{h}=\frac{Rad.}{Sin.\ l}\times\dot{a}$$

$$\dot{l}=\frac{Co\text{-}s.\ l}{Tang.\ a}\times\dot{L}=\frac{Co\text{-}s.\ a}{R}\times\dot{h}=\frac{Co\text{-}tang.\ l}{Tang.\ a}\times\dot{a}$$

$$\dot{h}=\frac{Co\text{-}s.\ l}{Sin.\ a}\times\dot{L}=\frac{R}{Co\text{-}s.\ a}\times\dot{l}=\frac{Co\text{-}tang.\ l}{Sin.\ a}\times\dot{a}$$

$$\dot{a}=\frac{Sin.\ l}{R}\times\dot{L}=\frac{Tang.\ a}{Co\text{-}tang.\ l}\times\dot{l}=\frac{Sin.\ a}{Co\text{-}tang.\ l}\times\dot{h}$$

TABLE II.

When one Leg L is invariable,

$$\dot{A}=\frac{Tang.\ a}{Sin.\ h}\times\dot{h}=\frac{Sin.\ a}{Sin.\ h}\times\dot{l}=-\frac{R.}{Co\text{-}s.\ h}\times\dot{a}$$

$$\dot{a}=-\frac{Co\text{-}s.\ h}{R}\times\dot{A}=-\frac{Sin.\ a}{Tang.\ h}\times\dot{l}=-\frac{Tang.\ a}{Tang.\ h}\times\dot{h}$$

$$\dot{h}=\frac{Sin.\ h}{Tang.\ a}\times\dot{A}=\frac{Co\text{-}s.\ a}{R}\times\dot{l}=-\frac{Tang.\ h}{Tang.\ a}\times\dot{a}$$

$$\dot{l}=\frac{Sin.\ h}{Tang.\ a}\times\dot{A}=\frac{R.}{Co\text{-}s.\ a}\times\dot{h}=-\frac{Tang.\ h}{Sin.\ a}\times\dot{a}$$

TABLE III.

When the Hyp. is invariable,

$$\dot{A}=-\frac{Co\text{-}tang.\ l}{Co\text{-}s.\ L}\times\dot{L}=-\frac{Co\text{-}s.\ l}{Co\text{-}s.\ L}\times\dot{a}=\frac{R}{Sin.\ L}\times\dot{l}$$

$$\dot{L}=-\frac{Co\text{-}s.\ L}{Co\text{-}tang.\ l}\times\dot{A}=\frac{Sin.\ l}{R.}\times\dot{a}=-\frac{Tang.\ l}{Tang.\ L}\times\dot{l}$$

Where, and alſo in the two preceding Tables, the Leg L is adjacent to the Angle A, and the Leg l to the Angle a.

Co-

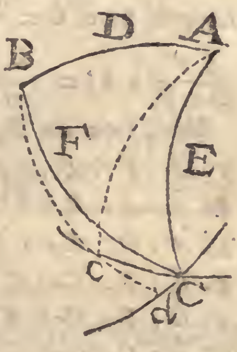
COROLLARY II.

255. From the third original Equation, expressing the Fluxion of the Angle *C* (*Vid. Art.* 253.) it appears that the Superficies of any Spherical-Triangle ABC, is proportional to the Excess of its three Angles above two Right-Angles. For (BC*db*) the Fluxion of the *Triangle* ABC, is $=$ *Sine* BC $\times$ B*b*, by *Art.* 161.) which being to, $\frac{Sin.\ BC}{Rad.} \times Bb$, the Fluxion of the Angle *C*, above specified, in the constant Ratio of Radius to Unity, the Fluents themselves (properly corrected) must therefore be in that Ratio; that is, the Superficies of the Triangle ABC will always be proportional to the Increase of the Angle *C*, from its coinciding with *A*, or as the Excess of *A* and *C* above two Right-Angles.

PROPOSITION II.

256. *To determine the Ratio of the Fluxions, or the indefinitely small Increments, of the different Parts of an oblique Spherical-Triangle* ABC; *two Sides thereof* AB, AC *being invariable, in Length.*

Let C*c* be an indefinitely small Part of the Parallel described by the Extreme C of the given Side AC, in its Motion about the given Point A; moreover, let C*d* be Part of another Parallel, whose Pole is the given Point B; let the Great-Circle B*c* meet C*d* in *d*; and let the three Sides, AB, AC, and BC, of the Triangle be denoted by *D*, *E*, and *F* respectively.



Then,

Then *(per Spherics)* we shall have

$$R : S.E :: CAc\ (\dot{A}) : Cc = \frac{S.E.}{R} \times \dot{A};$$

And, $R : S.F :: CBd\,(\dot{B}) : Cd = \frac{S.F}{R} \times \dot{B}.$

Also, $R : S.\ dCc\ (ACB) :: Cc : \dot{F} = \frac{S.E \times S.C}{R^2} \times \dot{A}:$ But $S.C : S:\dot{D} :: S.B : S.E$; therefore $S.E \times S.C = S.D \times S.B$, and consequently $\dot{F}$, also, $= \frac{S.D \times S.B}{R^2} \times \dot{A}$.

Again, $R : Co\text{-}f.\ dCc\ (ACB) :: Cc\ \left(\frac{S.E}{R} \times \dot{A}\right) : \frac{S.E. \times Co\text{-}f.\ C}{R^2} \times \dot{A}\ (= Cd) = \frac{S.F}{R} \times \dot{B}$;

Whence $\dot{B} = \frac{S.E \times Co\text{-}f.\ C}{R \times S.F} \times \dot{A}.$

Lastly, $Co\text{-}t.\ cCd : (C) : R :: Cd\ \left(\frac{S.F}{R} \times \dot{B}\right) : \dot{F} = \frac{S.F}{Co\text{-}t.\ C} \times \dot{B}.$

Whence, by the very same Argument (substituting D for E, and C for B in the two last Equations) we likewise have $\dot{C} = \frac{S.D \times Co\text{-}f.\ B}{R \times S.F} \times \dot{A}$, and $\dot{F}\ \left(= \frac{S.F}{Co\text{-}t.\ C} \times \dot{B}\right) = \frac{S.F}{Co\text{-}t.\ B} \times \dot{C}.$

Now, from the Equations thus found, it is manifest,

1°. $\dot{A} : \dot{F} :: R^2 : S.D \times S.B\ (:: Co\text{-}seca.\ D : S.B)$

2°. $\dot{A} : \dot{B} :: R \times S.F : S.E \times Co\text{-}f.\ C$

3°. $\dot{A} : \dot{C} :: R \times S.F : S.D \times Co\text{-}f.\ B$

4°. $\dot{B} : \dot{F} :: Co\text{-}t.\ C : S.F$

5°. $\dot{C} : \dot{F} :: Co\text{-}t.\ B : S.F$

6°. $\dot{B} : \dot{C} :: Co\text{-}t.\ C : Co\text{-}t.\ B\ (:: T.B : T.C)$ Q.E.I.

257. These Proportions, for the Fluxions of the Parts of a Spherical-Triangle, are very useful in various Cases in *Practical Astronomy*; whereof I shall here put down one or two Instances.

The first is; To determine the annual Alteration of the Declination and Right-Ascension of a fixt Star, through the Precession of the Equinox.

Here *A* must denote the Pole of the Ecliptic, *B* that of the Equinoctial, and *C* the Place of the Star; and then (by the first and fourth Proportions) we have

$Co\text{-}seca.\ D : Sin.\ B :: \dot{A} : \dot{F}$; and

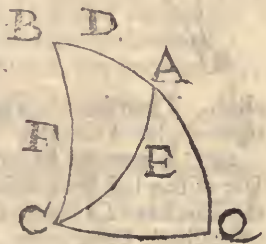
$S.\ F : Co\text{-}t.\ C :: \dot{F} : \dot{B}$;

That is, 1°, As the Co-secant of the Obliquity of the Ecliptic is to the Sine of the Star's Right-Ascension from the *solstitial Colure*, so is the *Precession* of the Equinox, or Alteration of Longitude, to the Alteration of Declination.

2°. As the Co-sine of the Star's Declination is to the Co-tangent of its Angle of Position, so is the Alteration of Declination (found as above) to the Alteration of Right-Ascension corresponding.

The second Example is to find how much the Amplitude, and the Time of the apparent Rising and Setting of the Sun, or a Star, are affected by Refraction.

In this Case *A* must denote the Pole of the Equator, and *B* the *Zenith*, and the Side BC must be an Arch of 90 Degrees, so that the Star *C* may coincide with the Horizon QC: Then, from the very same Proportion, we have,



$Sin.\ B : Co\text{-}seca.\ D :: \dot{F} : \dot{A}$,

And, $R : Co\text{-}t.\ C :: \dot{F} : \dot{B}$

But, $R : Co\text{-}t.\ C\ (T.\ QCA) :: Sin.\ B\ (CQ) : Co\text{-}tang.\ D\ (Tang.\ QA)$

Hence

Hence it apppears,

1°. That, as the Co-sine of the true Amplitude (considered independent of Refraction) is to the Tangent of the Pole's Elevation, so is the given horizontal Refraction to the Difference of Amplitudes thence arising.

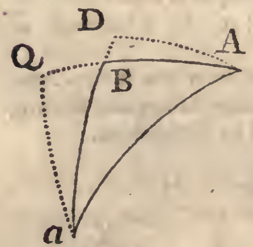
2°. And, that, as the Co-sine of the true Amplitude is to the Secant of the Pole's Elevation, so is the said horizontal Refraction to the Effect thereof in the Time of Rising, or Setting of the Sun, or Star.

But this last Proportion may be otherwise expressed, without the Amplitude: Thus,

$S.\,AB \times S.\,AC \times S.\,A : R^3 ::$ the horizontal Refraction, to the same Effect.

PROPOSITION III.

258. *To determine the same as in the preceding Problem; Supposing one Side* AB *and one of its adjacent Angles,* B, *to continue invariable.*



If from the End of the given Side, opposite to the given Angle, a Perpendicular AD be let fall, that Perpendicular, as well as the Segment BD cut off thereby, will be a constant Quantity, while the other Parts of the Triangle A*a*D vary, by the Motion of *a* along the Arch *a*BD. Therefore the Problem is resolved by *Case* 2. of right-angled Triangles. *Vid. Art.* 254.

259. It may not be amiss to give one Example of the Use of this last Proposition: Which shall be, in finding the Parallax of a Planet in Longitude and Latitude; that of Altitude being given.

Here *A* must stand for the Pole of the Ecliptic, *B* the Zenith, and *a* the Planet: Then, if the Hypothenuse A*a* be denoted by *h*, the Leg. D*a* by *l*, and the given Parallax, in Altitude, by *l*, it will appear, from the

the Place above quoted, that $\dot{A}$ (the Parallax in Long.) will be $= \frac{Sin.\ a}{Sin.\ b} \times \dot{l} = \frac{Sin.\ BaA}{Sin.\ Aa} \times \dot{l}$, and $\dot{b}$ (the Parallax in Lat.) $= \frac{Co\text{-}f.\ a}{Rad.} \times \dot{l} = \frac{Co\text{-}f.\ BaA}{Rad.} \times \dot{l}$.

If the Planet be in (or very near) the Ecliptic, and aQ be ſuppoſed a Portion of the Ecliptic, meeting AB, at Right-Angles, in Q, then *(per Spherics)* $\frac{Sin.\ BaA}{Sin.\ Aa} \left(\frac{Co\text{-}f.\ BaQ}{Radius}\right) = \frac{Tang.\ Qa}{Tang.\ Ba}$; alſo $\frac{Co\text{-}f.\ BaA}{Rad.} \left(\frac{Sin.\ BaQ}{Rad.}\right) = \frac{Sin.\ QB}{Sin.\ Ba}$; whence, by ſubſtituting theſe Values above, we ſhall, *in this Caſe*, have $\dot{A} = \frac{Tang.\ Qa}{Tang.\ Ba} \times \dot{l}$ and $\dot{b} = \frac{Sin.\ QB}{Sin.\ Ba} \times \dot{l}$; that is, in Words,

As, the Tangent of the Planet's Zenith Diſtance, is to the Tangent of its Longitude from the nonageſimal Degree of the Ecliptic, ſo is the Parallax in Altitude to the Parallax in Longitude.

And, as the Sine of the Zenith Diſtance to the Co-ſine of the Altitude of the nonageſimal Degree, ſo is the Parallax in Altitude to the Parallax in Latitude.

Becauſe the Parallax in Altitude, the horizontal Parallax *(M)* being given, is nearly $= \frac{Sin.\ Ba}{Rad.} \times M$, if this Value be ſubſtituted for $\dot{l}$, in the two laſt Equations, we ſhall get $\dot{b} = \frac{Sin.\ QB}{Rad.} \times M$, and $\dot{A} = \frac{Tang.\ Qa \times Sin.\ Ba}{Rad. \times Tang.\ Ba} \times M = \frac{Sin.\ AB \times Sin.\ BAa}{Rad.^2} \times M$.

Whence,

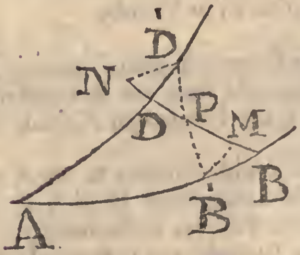
Whence, we have these two other Theorems, for finding the required Parallaxes immediately from the horizontal Parallax, without either the Altitude or its *Parallax.*

1. As Radius to the Co-sine of the Altitude of the *nonagesimal Degree* of the Ecliptic, so is the horizontal Parallax to the Parallax in Latitude.

2. And as the Square of Radius to the Rectangle under the Sines of the Altitude of the *nonagesimal Degree* and the Planet's Longitude from thence, so is the horizontal Parallax to the Parallax in Longitude.

PROPOSITION IV.

260. *Still, to determine the same Thing; supposing, one Angle* A, *and the Length of its opposite Side* BD (*or* $B'D'$) *to remain constant.*



Let $B'D'$ (equal to BD) intersect BD in an indefinitely small Angle at *P*, and meet AB and AD in B' and D'; also in BD produced let there be taken PN $= PD'$ and PM $= PB'$, and let *N*, D', and *M*, B' be joined.

Since, by Hypothesis, $DB = D'B' = MN$, if from the first and last of these equal Quantities DM, common, be taken away, there will remain $BM = DN$.

Moreover, since the Triangles BMB' and DND', in their ultimate State, may be considered as rectilineal,
* Art. 134. and right-angled at *M* and *N**, it will therefore be, as

BM : BB' :: *Co-s.* *B* : *Radius*

And DN : DD' :: *Co-s.* *D* :: *Radius.*

From

From whence, the Extremes in both Proportions being the same, we have $BB' : DD' :: Co\text{-}s.\ D : Co\text{-}s.\ B$: And therefore, if AB be denoted by H and AD by K, it appears that $\dot{H} : \dot{K} :: Co\text{-}s.\ D : Co\text{-}s.\ B$.

Again, *per Spherics*, $Sin.\ A : Sin.$ BD $(G) :: Sin.\ D : Sin.\ H :: Flux.\ Sin.\ D : Flux.\ Sin.\ H$; because, the Sines themselves being in a constant Ratio, their Fluxions must be in the same Ratio: But the Fluxion of the Sine of any Arc, or Angle, is to the Fluxion of the Arc or Angle itself, as the Co-sine to Radius *: *Art. 142. Therefore the *Flux. Sin.* D being $= \frac{Co\text{-}s.\ D}{Rad.} \times \dot{D}$, and $Flux.\ Sin.\ H = \frac{Co\text{-}s.\ H}{Rad.} \times \dot{H}$, it follows that, $Sin.\ A : Sin.\ G :: Co\text{-}s.\ D \times \dot{D} : Co\text{-}s.\ H \times \dot{H}$; or $\dot{D} : \dot{H} :: Sin.\ A \times Co\text{-}s.\ H : Sin.\ G \times Co\text{-}s.\ D$: And, by the very same Argument, $\dot{B} : \dot{K} :: Sin.\ A \times Co\text{-}s.\ K : Sin.\ G \times Co\text{-}s.\ B$. Now, by compounding the former of these two Proportions with the first above given, we get, $\dot{D} : \dot{K} :: Sin.\ A \times Co\text{-}s.\ H : Sin.\ G \times Co\text{-}s.\ B$. And, by compounding this last with $\dot{K} : \dot{B} :: Sin.\ G \times Co\text{-}s.\ B : Sin.\ A \times Co\text{-}s.\ K$ (that immediately preceding it) we also obtain $\dot{D} : \dot{B} :: Co\text{-}s.\ H : Co\text{-}s.\ K$.

Whence, by collecting these several Proportions together, we have the following Table, for all the different Cases.

$\dot{H} : \dot{K} :: Co\text{-}s.\ D : Co\text{-}s.\ B$

$\dot{D} : \dot{B} :: Co\text{-}s.\ H : Co\text{-}s.\ K$

$\dot{D} : \dot{H} :: Tang.\ D : Tang.\ H$

$\dot{B} : \dot{K} :: Tang.\ B : Tang.\ K$

$\dot{K} : \dot{D} :: Sin.\ G \times Co\text{-}s.\ B : Sin.\ A \times Co\text{-}s.\ H$

$\dot{H} : \dot{B} :: Sin.\ G \times Co\text{-}s.\ D : Sin.\ A \times Co\text{-}s.\ K$

It

It may be observed, that the fourth and the last are no new Cases, but only the third and fifth repeated: And that, though the former of the two, last named, differs from that found above; yet it is very easily deduced from it: For, since it appears that $\dot{D} : \dot{H} :: \frac{Sin.\ A}{Co\text{-}s.\ D} : \frac{Sin.\ G}{Co\text{-}s.\ H}$, and because $Sin.\ A : Sin.\ G :: Sin.\ D : Sin.\ H$, it follows that $\dot{D} : \dot{H} :: \frac{Sin.\ D}{Co\text{-}s.\ D} : \frac{Sin.\ H}{Co\text{-}s.\ H} ::$ $Tang.\ D : Tang.\ H$. Q. E. I.

There is yet another Problem, when two Angles remain constant; but this, by taking the Triangle formed by the Poles of the three given Circles, is reduced to *Problem* 2.

SECTION II.

Of the Resolution of fluxional Equations, or the Manner of finding the Relation of the flowing Quantities from that of the Fluxions.

261. WHEN an Equation, expressing the Relation of the Fluxions of the two variable Quantities, contains *only* one of those Fluxions with its respective flowing Quantity in each Term, the Relation of the Quantities will be obtained by finding the Fluent of every Term; as has been already taught, in *Sect.* VI. *Part* I.

Thus, if $ax^2\dot{x} = y^3\dot{y}$, then will $\frac{ax^3}{3} = \frac{y^4}{4}$.

And, if $x^n y^m \dot{x} = a\dot{y}$; by reducing it first to $x^n \dot{x} = ay^{-m}\dot{y}$ (so that its variable Quantities may be separated) we have $\frac{x^{n+1}}{n+1} = \frac{ay^{1-m}}{1-m}$.

 But,

But, if the given Equation has its indeterminate Quantities and their Fluxions ſo complicated together, that it cannot be brought under the Form there preſcribed, the Taſk will become much more difficult; nor is there any general Method to be given for ſuch Kinds of Equations, whereof there are an infinite Variety.

The Method of Infinite Serieſes (in ſome meaſure explained already, and more fully conſidered hereafter) is indeed very comprehenſive, and may be applied to good Purpoſe in various Caſes; but, being tedious and attended with a Number of Inconveniencies, it is a Method we ought never to have Recourſe to till we have tried what may be, otherways, effected, by help of ſuch particular Rules and Obſervations as we have been able to collect.

Accordingly, I ſhall, here, firſt point out ſome of the moſt proper Ways to be tried, in order, if poſſible, to bring out the Solution without an Infinite Series.

262. *The firſt Method is, by multiplying, or dividing, the given Equation by ſome Power or Product of the Quantities concerned; ſo as to bring it, if poſſible, under the Form of ſuch Fluxions, as, we know, do ariſe, if not from the firſt, yet from the ſecond, or third, of the three general Rules in the direct Method.*

Thus, if the given Equation be $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$; then, the whole being multiplied by xy, ſo that the two firſt Terms, $y\dot{x} + x\dot{y}$, may become the (known) Fluxion of the Rectangle xy *, there ariſes $y\dot{x} + x\dot{y} = \frac{x^{m+1}\dot{x}}{ay^{n-1}}$: But ſtill we are at a Loſs for the Fluent of the laſt Term, unleſs n be taken $= 1$ (ſo that y may vaniſh). In that Caſe we have $xy = \frac{x^{m+2}}{\overline{m+2} \times a}$; expreſſing the Relation of the Fluents when that of the Fluxions is $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay}$: Which appears to be the only Caſe, of the given Equation, where this Method is of Uſe.

* Art. 10.

 Again,

Again, let the Equation $\frac{p\dot{x}}{x} + \frac{r\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$ be proposed.

Here, multiplying by $x^p\ y^r$ (where the Exponents are the same as the Coefficients of $\frac{\dot{x}}{x}$ and $\frac{\dot{y}}{y}$) we get $p x^{p+1} \dot{x} \times y^r + x^p \times r y^{r-1}\ \dot{y} = \frac{x^{m+p}\ \dot{x}}{ay^{n-r}}$; in which the former Part of the Equation is known to express the Fluxion of $x^p\ y^r$ *. Therefore, when $n = r$, the Relation of the Fluxents may be found, and will be expressed by $x^p y^r = \frac{x^{m+p+1}}{m+p+1 \times a}$: Which, if no Correction by a constant Quantity be necessary, may be reduced to $y^r = \frac{x^{m+1}}{m+p+1 \times a}$.

* Art. 15.

The same Method may also be extended to Fluxions of the higher Orders: Let $\ddot{x} - x\dot{z}^2 = f\dot{z}^2$ (which Equation occurs hereafter, in the Resolution of a Problem of some Difficulty). Then, multiplying by $\dot{x}$, it becomes $\dot{x}\ddot{x} - x\dot{x}\dot{z}^2 = f\dot{z}^2\dot{x}$; where, $\dot{z}$ being constant, each Term admits, now, of a perfect Fluent, and we therefore have $\frac{\dot{x}^2}{2} - \frac{x^2\dot{z}^2}{2} = fx\dot{z}^2$: From whence, supposing no Correction necessary, $\dot{z} = \frac{\dot{x}}{\sqrt{2fx + xx}}$, and $z = hyp.$ $Log.\ f + x + \sqrt{2fx + x^2}$ (by Art. 126.)

263. *It may happen that the Solution of an Equation will become more easy by first taking the Fluxion thereof; when, by that means, some of the Terms destroy each other.*

The following is an Instance of it (which, also, occurs hereafter). Let $y + \frac{\dot{y} \times \overline{a - x}}{\dot{x}} = x - \frac{y\dot{x}}{\dot{y}}$: Whose Flux-

 ion,

ion, making $\dot{x}$ constant, is $\dot{y} + \frac{\ddot{y} \times \overline{a - x} - \dot{x}\dot{y}}{\dot{x}} = \dot{x} - \frac{\dot{y}\dot{x}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$: Which, by reason of the Terms destroying one another, is reduced to $\frac{\ddot{y} \times \overline{a - x}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$: Therefore, by expunging $\ddot{y}$, *&c.* we get $\dot{y}y^{-\frac{1}{2}} = \dot{x} \times \overline{a - x}|^{-\frac{1}{2}}$, and consequently $2y^{\frac{1}{2}} = -2 \times \overline{a - x}|^{\frac{1}{2}}$ + *some constant Quantity.*

264. *Another Method, chiefly applicable to Equations, of the first Order of Fluxions, wherein only one of the two variable Quantities* (x *or* y) *enters, is, to substitute for the Ratio of the two Fluxions* ($\dot{x}$ *and* $\dot{y}$) *: From whence the Value of that Quantity will be had, immediately, in Terms of the said assumed Ratio: And then, by taking its Fluxion, that of the other Quantity (and from thence the Quantity itself) will become known.*

Thus, let $a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^2$ (being the Equation of the Curve that generates the Solid of the *least Resistance,* when the Bulk and greatest Diameter are given). Then, by putting $\frac{\dot{x}}{\dot{y}} = v$, and substituting above, we get $av\dot{y}^4 = y \times \overline{v^2\dot{y}^2 + \dot{y}^2}|^2 = y\dot{y}^4 \times \overline{v^2 + 1}|^2$; and consequently $y = \frac{av}{\overline{v^2 + 1}|^2}$: Therefore $\dot{y} = \frac{a\dot{v} - 3av^2\dot{v}}{\overline{vv + 1}|^3}$; and consequently $\dot{x}\ (= v\dot{y}) = \frac{av\dot{v} - 3av^3\dot{v}}{\overline{vv + 1}|^3}$: Whose Fluent may be found, from *Art.* 84. or, otherwise, thus: Put $w^2 = v^2 + 1$; then $v^2 = w^2 - 1$, and $w\dot{w} = v\dot{v}$; by substituting which Values there arises $\dot{x} = \frac{aw\dot{w} - 3aw\dot{w} \times \overline{w^2 - 1}}{w^6} = 4a\dot{w}w^{-5} - 3a\dot{w}w^{-3}$; and

fore $x = \frac{4aw^{-4}}{-4} - \frac{3aw^{-2}}{-2} = -\frac{a}{w^4} + \frac{3a}{2w^2} = \frac{3aw^2 - 2a}{2w^4}$
$= \frac{3a \times \overline{v^2+1} - 2a}{2 \times \overline{vv+1}|^2} = \frac{a \times \overline{3vv+1}}{2 \times \overline{vv+1}|^2}$; which, corrected

(by taking y, or $v=0$) becomes $x = \frac{a \times \overline{3vv+1}}{2 \times \overline{vv+1}|^2} - \frac{a}{2}$.
From this Equation, by completing the Square, &c. v may be found in Terms of x; whence the correspond-

ing Value of y ($= \frac{av}{\overline{vv+1}|^2}$) will also be known.

265. *The fourth Method*, which chiefly obtains when one of the indeterminate Quantities and its Fluxion, arise but to a single Dimension each, *may be thus:*

Let the Value of that Quantity, which is least involved, be first sought, from the fictitious Equation arising by neglecting all the Terms in the given Equation, where neither that Quantity, nor its Fluxion, are found: Then, to that Value, let some Power, or Powers, of the other Quantity, with unknown Coefficients, be added (according to the Dimensions of the Terms neglected) and let the Sum be substituted in the given Equation, as the true Value of the first mentioned Quantity: By which means a new Equation will result; from whence the assumed Coefficients may, sometimes, be determined.

Ex. Let the given Equation be $cx^2\dot{x} + y\dot{x} = a\dot{y}$.

By neglecting $cx^2\dot{x}$, or feigning $y\dot{x} = a\dot{y}$, we get $\frac{\dot{x}}{a} = \frac{\dot{y}}{y}$: and consequently $\frac{x}{a}$; = hyp. Log. y — hyp. Log. d* = hyp. Log. $\frac{y}{d}$; d being any constant Quantity, which the Nature of the Problem may require.

* Art. 126, and 78.

Hence $\frac{y}{d}$ = the Number whose hyperbolical Logarithm is $\frac{x}{a}$: Which Number, if M be put for (2,71828 &c.) the

the Number whose hyp. Log. is Unity, will be expressed by $\overline{M}|^{\frac{x}{a}}$ (since it is evident that the hyp. Log. hereof is $\frac{x}{a} \times \text{Log.}\, M = \frac{x}{a}$): Therefore $\frac{y}{d} = \overline{M}|^{\frac{x}{a}}$ and $y = d \times \overline{M}|^{\frac{x}{a}}$. Now, to the Value thus found, let there be added $Ax^2 + Bx + C$, in order to get the true Value; and then, $\dot{y}$ being $= 2Ax\dot{x} + B\dot{x} + \frac{d\dot{x}}{a} \times \overline{M}|^{\frac{x}{a}}$ *, we shall, by substituting in the given Equation, have $cx^2\dot{x} + Ax^2\dot{x} + Bx\dot{x} + C\dot{x} + d\dot{x}M^{\frac{x}{a}} = 2Aax\dot{x} + Ba\dot{x} + d\dot{x}M^{\frac{x}{a}}$, and consequently $\overline{c+A} \times x^2\dot{x} + \overline{B-2Aa} \times x\dot{x} + \overline{C-Ba} \times \dot{x} = 0$. Whence $A = -c$†, $B = -2ac$, $C = -2aac$; and consequently $y = -c \times \overline{x^2 + 2ax + 2aa} + dM^{\frac{x}{a}}$. By the very same Way, the Value of y, in the Equation $cx^n\dot{x} + y\dot{x} = a\dot{y}$, will come out $= -c \times \overline{x^n + ax^{n-1} + n.\overline{n-1}.a^2x^{n-2} + n.\overline{n-1}.\overline{n-2}.a^3x^{n-3} + \&c.} + dM^{\frac{x}{a}}$.

* Art. 252.

† Art. 84.

266. But, what is a little remarkable, in these Equations, is, that the *Exponential* $dM^{\frac{x}{a}}$, tho' a variable Quantity, should only serve, as it were, to correct the Fluent, or perform the Office of a constant Quantity. What I here mean will plainly appear, if it be considered, that the Equation $y = -c \times \overline{x^2 + 2ax + 2aa}$, where the said *Exponential* is wanting, answers all the Conditions of the fluxional Equation first proposed; which, upon Trial, will be found; and must needs be

the Case, seeing d may be, either, taken Nothing at all, or any Quantity at Pleasure.

But the Equation $y = -c \times \overline{x^2 + 2ax + 2a^2}$ (when $dM^{\frac{x}{a}}$ is wanting) cannot be corrected, in the usual Way, so as to give $y=0$, when $x=0$; since, if any other constant Quantity, besides $-2a^2c$ be introduced, the first Conditions will not be answer'd: The Correction must, therefore, be by the Exponential $dM^{\frac{x}{a}}$; and is thus.

Since $y = -cx^2 - 2cax - 2ca^2 + dM^{\frac{x}{a}}$, if y be taken $= 0$ and $x = 0$, then $-2ca^2 + dM^0 = 0$, or $d = 2ca^2$; and so the Equation, truly corrected, is $y = -c \times \overline{x^2 + 2ax + 2a^2} + 2a^2cM^{\frac{x}{a}}$.

267. We come now to the last Method; namely, that of Infinite Serieses; which, tho' less accurate, is vastly more comprehensive, than *any* yet explained: The Manner of it is thus:

For the Quantity whose Value you would find, let an Infinite Series, consisting of the Powers of the other Quantity with unknown Coefficients, be assumed; which Series, together with its Fluxion, or Fluxions, must be substituted instead of their Equals in the given Equation; whence a new Equation will arise, from which, by comparing the homologous Terms, the assumed Coefficients, and consequently the Value sought, will be determined.

Thus, let the Equation $\frac{\dot{x}}{1+x} = \dot{y}$ (reducible to $\dot{x} - \dot{y} - x\dot{y} = 0$) be proposed; to find x in Terms of y.

Then, assuming $x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5$ &c.

We have $\dot{x} = A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + 5Ey^4\dot{y} +$ &c.

Which Values being substituted in $\dot{x} - \dot{y} - x\dot{y} = 0$, we get

$$\left.\begin{array}{l} A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y} + 4Dy^3\dot{y} + \text{\&c.} \\ -\dot{y} - Ay\dot{y} - By^2\dot{y} - Cy^3\dot{y} - \text{\&c.} \end{array}\right\} = 0.$$

There-

Therefore $A - 1 = 0$, or $A = 1$; $2B - A = c$, or $B = \frac{A}{2} = \frac{1}{2}$; $3C - B = 0$, or $C = \frac{B}{3} = \frac{1}{2.3}$; $4D - C = 0$, or $D = \frac{C}{4} = \frac{1}{2.3.4}$ &c.

And consequently $x\ (Ay + By^2 + Cy^3\ \&c\ ) = y + \frac{y^2}{2} + \frac{y^3}{2.3} + \frac{y^4}{2.3.4} + \frac{y^5}{2.3.4.5} + \&c.$

Again, let it be required to find the Value of y, in the Equation $cx^2\dot{x} + \dot{y}x = a\dot{y}$, or $a\dot{y} - y\dot{x} - cx^2\dot{x} = 0$. Here, assuming $y = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6$ &c. and proceeding as before, we shall have

$$\left.\begin{array}{llllll} aA\dot{x} + 2aBx\dot{x} + 3aCx^2\dot{x} + 4aDx^3\dot{x} + 5aEx^4\dot{x} + \&c. \\ 0 \quad - \quad Ax\dot{x} - \quad Bx^2\dot{x} - \quad Cx^3\dot{x} - \quad Dx^4\dot{x} - \&c. \\ 0 \qquad\quad 0 \quad - \quad cx^2\dot{x} \end{array}\right\} = 0$$

Whence $A = 0$; $2aB = A = 0$; $3aC = B + c = c$, or $C = \frac{c}{3a}$; $4aD = C = \frac{c}{3a}$, or $D = \frac{c}{3.4a^2}$; $5aE = D = \frac{c}{3.4a^2}$, or $E = \frac{c}{3.4.5a^3}$ &c. and consequently $y\ (Ax + Bx^2 + Cx^3 + \&c.) = \frac{cx^3}{3a} + \frac{cx^4}{3.4a^2} + \frac{cx^5}{3.4.5a^3} + \frac{cx^6}{3.4.5.6a^4} + \&c.$

268. It appears from this Example, that the Quantity to be found, will not always require all the Terms of the Series $Ax + Bx^2 + Cx^3$ &c. And it may happen, in innumerable Cases, that the Series to be assumed will demand a very different Law from *that* where the Exponents proceed according to the Terms of an arithmetical Progression having Unity for the common Difference. And, indeed, the greatest Difficulty we have here to encounter, is, to know what Kind of Series, with regard to its Exponents, ought to be assumed, so as to answer the Conditions of the Equation, without introducing more Terms than are actually necessary.

The following Rules will be found very useful upon this Occasion: Which, though they may become impracticable in certain particular Cases, never take in any superfluous Terms.

1°. *Having (if necessary) freed your Equation from Fractions and Surds, let the Quantity, whose Value is sought, be supposed equal to some Power of the other Quantity with an unknown Exponent (n); and let that Power, together with its Fluxion, or Fluxions, be substituted for their (supposed) Equals in the given Equation.*

2°. *Let the least Exponents of the variable, or indeterminate, Quantity, in the new Equation, thence arising, be put equal to each other: Whence the Value of the unknown Exponent n will be found.*

3°. *Substitute the Value of n, so found, in all the Exponents where n is concerned; and then take the Difference between one of the equal* ones, *above mentioned, and every other Exponent, of the variable Quantity, in the whole Equation.*

4°. *To these Differences, write down all the least Numbers that can be composed out of them, by continual Addition, either to themselves, or to one another; till you have, by that means, got, in the whole, as many different Terms, as you would have the required Series continued to.*

5°. *Lastly, let each of those Terms be increased by the Value of n (found by Rule 2.) and you will then have the Exponents of the Series to be assumed.*

EXAMPLE I.

269. *Let the Value of x, in the Equation* $a^2\dot{x}^2 + x^2\dot{z}^2 - a^2\dot{z}^2 = 0$, *be required.*

First, by writing z^n for x, and $nz^{n-1}\dot{z}$ for $\dot{x}$, the Indices of z will be $2n-2$, $2n$, and 0 (which are determined by Inspection, without regarding the Coefficients) whereof the two least ($2n-2$ and 0) being put equal to each other, we here find $n=1$: Therefore, the Exponents being $0, 2, 0$, the Differences (according to *Rule* 3.) are also $0, 2$; from whence, by adding 2 continually, we get $0, 2, 4, 6, 8$ &c. which (being each in-

increased by the Value of n) give 1, 3, 5, 7, 9 &c. for the Exponents in this Case.

Let, therefore, $x = Az + Bz^3 + Cz^5 + Dz^7 +$ &c. Then, putting $\dot{z}=1$, in order to facilitate the Operation, we shall have $\dot{x} = A + 3Bz^2 + 5Cz^4 + 7Dz^6 +$ &c. which two Values being squared, and substituted in the given Equation, it will become

$$\left.\begin{array}{llll} a^2A^2 + 6a^2ABz^2 + & 10a^2ACz^4 & + 14a^2ADz^6 & + \&c. \\ & +9a^2B^2z^4 & + 30a^2BCz^6 & + \&c. \\ * \quad + A^2z^2 & +2ABz^4 & + 2ACz^6 & + \&c. \\ -a^2 & & +B^2z^6 & + \&c. \end{array}\right\} = 0$$

Whence, $a^2A^2=a^2$, and therefore $A=1$; $6a^2B = -A$, and therefore $B = -\frac{1}{6a^2} = -\frac{1}{2.3a^2}$; $10a^2AC = -9a^2B^2 - 2AB = -B \times \overline{9a^2B + 2A} = -B \times -\frac{3}{2} + 2 = -\frac{B}{2} = \frac{1}{2.3.2a^2}$, and therefore $C = \frac{1}{2.3.4.5a^4}$; $14AD = -30a^2 \times -\frac{1}{6a^2} \times \frac{1}{120a^4} - 2 \times \frac{1}{120a^4} - \frac{1}{36a^4} = \frac{1}{24a^4} - \frac{1}{60a^4} - \frac{1}{36a^4} = -\frac{1}{360a^4}$, and therefore $D = -\frac{1}{14.360a^6} = -\frac{1}{2.3.4.5.6.7a^6}$; and, consequently, $x = z - \frac{z^3}{2.3a^2} + \frac{z^5}{2.3.4.5a^4} - \frac{z^7}{2.3.4.5.6.7a^6}$ &c.

EXAMPLE II.

270. *Let the given Equation be* $a^2x\ddot{y} - 2a^2\dot{x}\dot{y} + ax\dot{x}^2 + x^3\ddot{y} = 0$; *to find y.*

Here, substituting x^n for y, the Exponents will be $n-1$, $n-1$, 1, and $n+1$; where, making $n-1=1$, we we get $n = 2$: Whence, the Differences being 0, 2, the Series to be assumed for y will be $Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} +$ &c. From which, making $\dot{x} = 1$, we have $\dot{y} = 2Ax + 4Bx^3 + 6Cx^5 + 8Dx^7$ &c. and

$$\ddot{y} = 2A + 12Bx^2 + 30Cx^4 + 56Dx^6$$

And, these Values being substituted, the Equation becomes

$$\left.\begin{array}{l} 2a^2Ax + 12a^2Bx^3 + 30a^2Cx^5 + 56a^2Dx^7 + \&c. \\ -4a^2Ax - 8a^2Bx^3 - 12a^2Cx^5 - 16a^2Dx^7 + \&c. \\ +ax \quad + 2Ax^3 \quad + 12Bx^5 \quad + 30Cx^7 \quad + \&c. \end{array}\right\} = 0$$

Therefore $A = -\frac{1}{2a}$; $B = -\frac{2A}{4a^3} = -\frac{1}{4a^3}$;

$C = -\frac{12B}{18a^2} = \frac{1}{6a^5}$; $D = -\frac{30C}{40a^2} = -\frac{1}{8a^7}$ &c.

and so $y = \frac{x^2}{2a} - \frac{x^4}{4a^3} + \frac{x^6}{6a^5} - \frac{x^8}{8a^7} + \frac{x^{10}}{10a^9} -$ &c.

Which Series is known to express the Fluent of $\frac{ax\dot{x}}{a^2+x^2}$, or, $\frac{1}{2}a \times$ hyp. Log. $\frac{a^2+x^2}{aa}$: Consequently y is also $=$ $\frac{1}{2}a \times$ hyp. Log. $\frac{a^2+x^2}{a^2}$. In this manner, it comes to pass, *that*, though we are obliged, in very complicated Cases, to have recourse to Infinite Serieses, we are sometimes able, at last, to give the Solution in finite Terms, or, at least, by help of Logarithms, Sines and Tangents: Which will always happen when the Series can be summed, or is found to agree with that arising from some known Quantity.

271. Sometimes it happens, in Equations involving the higher Orders of Fluxions, that the Exponents, mention'd in *Rule* 2. whereof the least ought to be made equal to each other, are so express'd, as to render such an Equality impossible. When this is the Case, the Value of n, and the first Term of the required Series, can *only* be determined from the Nature of the Problem to which the Equation belongs. We know, in-

indeed, from the Equation itſelf, that n muſt be either equal to Nothing, or to ſome poſitive Integer, leſs than that expreſſing the Order of the higheſt Fluxion in the Equation: Becauſe the Term that has the leaſt Exponent, and which therefore cannot be compared with any other (being always affected by two or more of the Factors, n, $n-1$, $n-2$, &c. will then (one of thoſe Factors being $=0$) vaniſh intirely out of the Equation; which, thereby, is render'd poſſible.

When n and A are known, the reſt of the Terms will be found in the common Way, as in

EXAMPLE III.

Where the Equation propoſed is $y\dot{x}^2 + a\dot{x}\dot{y} - a^2\ddot{y} = 0$; *to find* y.

By ſuppoſing $\dot{x}=1$, and writing x^n for y, nx^{n-1} for $\dot{y}$, and $n \times \overline{n-1} \times x^{n-2}$ for $\ddot{y}$, we get $x^n + nax^{n-1} - n \times \overline{n-1} \times a^2x^{n-2}$: But it is plain that no two of the Indices of x can, *here*, be equal: The Value of n muſt therefore be either $=0$, or Unity (in both which Caſes the Term $-n \times \overline{n-1} \times a^2x^{n-2}$ vaniſhes) but I ſhall take the latter Value, and ſuppoſe the firſt Term of the Series to be Ax; then, the Differences of the foreſaid Exponents being 1 and 2, the Law of the Series will be expreſſed by 1, 2, 3, 4 &c. Whence, aſſuming $y =$ Ax + Bx^2 + Cx^3 + Dx^4 &c. and proceeding as in the former Examples, y will be found $=$ A *into* $x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{8a^3} + \frac{x^5}{24a^4} + \frac{x^6}{90a^5}$ &c. or $=$ A *into* $x + \frac{x^2}{2a} + \frac{2x^3}{2.3a^2} + \frac{3x^4}{2.3.4a^3} + \frac{5x^5}{2.3.4.5a^4} + \frac{8x^6}{2.3.4.5.6a^5} +$ &c. where the Law of Continuation is manifeſt, the Coefficient of every Numerator being compoſed by the Addition of the two preceding ones.

272. It

272. It will be proper to obſerve here, that, in Equations like the two laſt propoſed, where the higher Orders of Fluxions are concerned, the Series expreſſing the Relation of the two Quantities muſt always be found in Terms of the Quantity flowing uniformly. And, that, if the Number of Dimenſions of the Fluxion of the ſaid Quantity, after Subſtitution, be not the ſame in every Term, the Equation itſelf, put down to be reſolved, is abſurd and impoſſible, and ſuch as never can ariſe in the Solution of any Problem. In all proper Equations the Number of fluxional Points (ſuppoſing the Powers of the Fluxions to be wrote without Indices) will be the ſame in every Term.

EXAMPLE IV.

273. *Where let the given Equation be* $a^3\dot{y} - ay^2\dot{x} + x^2y\dot{y} = x^3\dot{x}$; *to find* y.

By proceeding as uſual the Indices will here be $n-1$, $2n$, $2n+1$ and 3; whereof the leaſt (which can be no other than $n-1$ and 3) being compared, n will be given $=4$: And the Differences will therefore be 0, 5, 6; to which the Double of the Second and the Sum of the ſecond and third, &c. being put down, and then every Term increaſed by 4, there ariſes 4, 9, 10, 14, 15, 16, 19 &c. for the Exponents of the Series to be aſſumed for y.

Let therefore $y = Ax^4 + Bx^9 + Cx^{10} + Dx^{14}$ &c. then, making $\dot{x} = 1$, $\dot{y}$ is $= 4Ax^3 + 9Bx^8 + 10Cx^9 + 14Dx^{13}$ + &c.

And, by ſubſtituting theſe Values above, we have

$$\left.\begin{matrix} 4a^3Ax^3 + 9a^3Bx^8 + 10a^3Cx^9 + 14a^3Dx^{13} + \&c. \\ -x^3 \quad - aA^2x^8 \quad + 4A^2x^9 \quad - 2aABx^{13} + \&c. \end{matrix}\right\} = 0$$

Whence $A = \frac{1}{4a^3}$, $B = \frac{1}{144a^8}$, &c.

And * $y = \frac{x^4}{4a^3} + \frac{x^9}{144a^8} - \frac{x^{10}}{40a^9} + \frac{x^{14}}{4032a^{13}}$ &c.

* *If for* y, *the Series* $Ax^4 + Bx^5 + Cx^6 + Dx^7$ &c. *whoſe Exponents are in arithmetical Progreſſion, had been aſſumed, according to the Method of ſome very good Authors, no leſs than ſeven ſuperfluous Terms muſt have been introduced to obtain the four above given.*

274. Before I quit this Subject, it may not be amiss to subjoin the following Remarks.

1°. If the indeterminate Quantities are great in respect to the given ones, a descending Series will, in most Cases (where it is practicable) converge better than an ascending one. To obtain such a Series, compare the greatest Exponents, mention'd in *Rule* 2 instead of the least, and proceed according to the third and fourth Rules *, whence a Series of Numbers will be found; which, being successively subtracted from the Value of n, you will have the Exponents of a descending Series. * Art. 268.

Thus, let the common-algebraic Equation $a^3x+ax^3-a^3y-y^4=0$ be propounded; to find y, when x is great in comparison of a.

Then, proceeding as usual, the Exponents of the four Terms of the Equation will be 1, 3, n, $4n$; whereof the two greatest ($4n$ and 3) being made equal, we get $n=\frac{3}{4}$; therefore the Differences are 0, 2 and $2\frac{1}{4}$; and $n=\frac{3}{4}$; therefore the Differences are 0, 2 and $2\frac{1}{4}$; and the Numbers to be subtracted from n, are 0, 2, $\frac{9}{4}$, 4, $\frac{17}{4}$, &c. Consequently the Series to be assumed for y is $Ax^{\frac{3}{4}}+Bx^{-\frac{5}{4}}+Cx^{-\frac{6}{4}}+Dx^{-\frac{13}{4}}+$ &c. From whence

$$y \text{ will be found } = a^{\frac{1}{4}}x^{\frac{3}{4}}+\frac{a^{\frac{9}{4}}}{4x^{\frac{5}{4}}}-\frac{a^{\frac{10}{4}}}{4x^{\frac{6}{4}}}-\frac{3a^{\frac{17}{4}}}{32x^{\frac{13}{4}}} \text{ \&c.}$$

2°. But, if the Quantity (x) in whose Terms the other is to be expressed, be neither much greater nor much smaller than the given Quantity (a), it will be proper to substitute for the Excess, or Defect, of the said Quantity (x) above, or below, some given Quantity; so that, having, by this means, exterminated x, the Series arising from the new Equation (wherein the said Excess, or Defect, is the converging Quantity) will have a due Rate of Convergency.

The Use of this is so obvious that it needs no Example, or farther Explanation.

3°. Lastly, it will be proper to observe, that, if the Equation for the Value of A, arising from the first Column of homologous Terms, admits of two or more, equal

equal Roots (which is a Case that may, perhaps, never happen in practice) all the foregoing Precepts will be insufficient; unless the Equation also admits of some other Root, besides the equal ones, whereby A may be more commodiously expressed. To determine the Exponents, in that particular Case, divide each of the Differences mention'd in *Rule* 3. by the Number of the equal Roots; and then proceed as usual. The Reasons of which, as well as of the Rules themselves, I have long ago given elsewhere, and have not Room to repeat them here.

SCHOLIUM.

275. Although the Business of reverting Serieses is not a Branch of the Doctrine of Fluxions, but, more properly, belongs to common *Algebra*; yet, as it is often useful where Fluxions are concerned, and falls under the general Rules illustrated in the foregoing Pages, I shall here add an Example or two on that Head.

Let, then, $ax + bx^2 + cx^3 + dx^4 + ex^5$ &c. $= y$; to revert the Series, or, to find x in an Infinite Series expressed in the Powers of y.

Here, by writing y^n for x, the Indices of the Powers of y, in the Equation, will be n, $2n$, $3n$, &c. and 1; therefore $n = 1$. and the Differences are 0, 1, 2, 3, 4, 5, &c. and so the Series to be assumed, in this Case, is $Ay + By^2 + Cy^3 + Dy^4$ &c. Which being involved and substituted for the respective Powers of x (neglecting, every where, all such Powers of x and y as exceed the highest you would have the Series carry'd to) there arises

$$\left.\begin{array}{lllll} aAy + aBy^2 & + aCy^3 & + aDy^4 & \&c. \\ * \quad + bA^2y^2 & + 2bABy^3 & + 2bACy^4 & \\ & & + bB^2y^4 & \&c. \\ * \quad\quad * & + cA^3y^3 & + 3cA^2By^4 & \&c. \\ * \quad\quad * & \quad * & + dA^4y^4 & \&c. \end{array}\right\} = y$$

Whence,

Whence, by comparing the homologous Terms, $A = \frac{1}{a}$; $B = -\frac{b}{a^3}$; $C\ (= -\frac{2bAB + cA^3}{a}) = \frac{2bb - ac}{a^5}$; $D\ (= -\frac{2bAC + bB^2 + 3cA^2B + dA^4}{a}) = \frac{5abc - 5b^3 - a^2d}{a^7}$, &c. and consequently $x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2bb - ac}{a^5} \times y^3 - \frac{5b^3 - 5abc + a^2d}{a^7} \times y^4$ &c.

For an Instance of the Use of this Conclusion, let $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ &c. $= y$: Then, a being, in this Case, $= 1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, &c. we shall, by substituting these Values, have $x = y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24}$ &c. From whence, when y is given, x will also be given; provided the Value of y be sufficiently small *.

* Art. 92.

Example 2. Let there be given $ax + by + cx^2 + dxy + ey^2 + fx^3 + gx^2y + hxy^2 + iy^3 + kx^4 + lx^3y$ &c. $= 0$; to find y.

By assuming $y = Ax + Bx^2 + Cx^3 + Dx^4$ &c. and proceeding as above, A will be found $= -\frac{a}{b}$, $B = -\frac{c + dA + eA^2}{b}$, $C = -\frac{dB + 2eAB + f + gA + hA^2 + iA^3}{b}$, $D = -\frac{dC + 2eAC + eB^2 + gB + 2hAB + 3iA^2B + k + lA +}{b}$ $\frac{mA^2 + nA^3 + pA^4}{b}$, &c.

Example

Example 3. Laftly, let $x^{m} + bx^{m+p} + cx^{m+2p} + dx^{m+3p} + \&c. = z$.

Here, in order to determine the Form of the Series to be affumed, let z^{n} be wrote for x in the given Equation, according to the ufual Method; and then the Exponents, fuppofing z tranfpofed, will be 1, nm, $nm + np$, $nm + 2np$, $nm + 3np$, &c. refpectively; whereof the two leaft (1 and nm) being made equal to each other, n is found $= \frac{1}{m}$; and the Differences are $\frac{p}{m}$, $\frac{2p}{m}$, $\frac{3p}{m}$, &c. Whence the Series to be affumed for x is $x^{\frac{1}{m}} + Bz^{\frac{1+p}{m}} + Cz^{\frac{1+2p}{m}} + Dz^{\frac{1+3p}{m}} + \&c.$ (for it is evident, by Infpection, that the Coefficient (A) of the firft Term muft here be an Unit.) This Series being therefore raifed to the feveral Powers of x, in the given Equation, *by Art.* 108. and the Coefficients of the homologous Terms in the new Equation compared together, it will be found that, $B = -\frac{b}{m}$, $C = \frac{\overline{1+m+2p} \times bb - 2mc}{2m^2}$,

$$D = -\frac{\overline{2m^2 + 9mp + 9p^2 + 3m + 6p + 1} \times b^3}{6m^3} + \frac{\overline{1+m+3p} \times bc}{m^2} - \frac{d}{m}, \&c.$$

From the general Value of x, found above, innumerable Theorems, for reverting particular Forms of Seriefes, may be deduced.

Thus, if $x + bx^2 + cx^3 + dx^4, \&c. = z$; then ($m$ being $= 1$ and $p = 1$) x is $= z - bz^2 + \overline{2bb - c} \times z^3 - \overline{5b^3 - 5bc + d} \times z^4$ &c.

And

And, if $x + bx^3 + cx^5 + dx^7 + \&c. = z$; ($m$ being $=1$, and $p=2$) $x = z - bz^3 + \overline{3bb - c} \times z^5 - \overline{12b^3 - 8cb + d} \times z^7$ &c.

Alſo, if $x^{\frac{1}{2}} + bx^{\frac{3}{2}} + cx^{\frac{5}{2}} + dx^{\frac{7}{2}}$ &c. $= z$; then (m being $= \frac{1}{2}$ and $p = 1$) $x = z^2 - 2bz^4 + \overline{7bb - 2c} \times z^6 - \overline{30b^3 - 18bc + 2d} \times z^8$ &c. &c.

276. It may be obſerved that, in all theſe Forms of Serieſes, the firſt Term is without a Coefficient (which renders the Concluſion much more ſimple.) Therefore, when the Series to be reverted has a Co-efficient in its firſt Term, the whole Equation muſt be firſt of all divided thereby: Thus, if the Equation was $3x - 6x^2 + 8x^3 - 13x^4$ &c. $= y$; by dividing the whole by 3 it will become $x - 2x^2 + \frac{8x^3}{3} - \frac{13x^4}{3}$ &c. $= \frac{1}{3}y$: Where, putting $z = \frac{1}{3}y$, we have, *by Form.* 1. $x = z + 2z^2 + \frac{16}{3}z^3$ &c. $= \frac{y}{3} + \frac{2y^2}{9} + \frac{16y^3}{81}$ &c.

SECTION III.

Of the Compariſon of Fluents, or the Manner of finding one Fluent from another.

277. WE have, already, pointed out the moſt remarkable Forms of Fluxions whoſe Fluents are explicable in finite Terms *; and alſo ſhewn the Uſe of Infinite Serieſes in approximating the Values of ſuch Fluents as do not come under any of thoſe Forms †: But this laſt Method (as is before hinted) being troubleſome, and attended with many Obſtacles; Mathematicians have therefore invented, and ſhewn, the Way of deriving one Fluent from another: Which is of good Advantage when the Fluent

* Art. 77. 78. 83. 84. and 85.

† Art. 99.

sought can be referred to one, like those in *Art.* 126 and 142. expressing the Logarithm of a Number, or the Arch of a Circle; since the Trouble of an infinite Series is, then, avoided.

As the Subject here proposed is of such a Nature, that it would be very tedious and difficult, if not altogether impracticable, to lay down Rules and Precepts for all the various Cases; I shall deliver, what I have to offer thereon, by way of *Problems*; beginning with some very easy ones, for the Sake of the *young Proficient*.

PROB. I.

278. *The Fluent of* $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ *being given (by Art.* 126.) *'tis proposed to find, from thence, the Fluent of* $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$.

Let both the Numerator and Denominator of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, be multiply'd by x, so that the Quantity without the Vinculum, in the Fluxion, $\frac{x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, thus transformed, may become some constant Part of the Fluxion of the highest Term under the Vinculum: Which Part, in this Case, being $\frac{1}{4}$, let $\frac{1}{4}$ of the Fluxion of the first Term under the Vinculum (or $\frac{1}{2}a^2x\dot{x}$) be therefore added to the Numerator, in order to have the *Whole*, $\frac{\frac{1}{2}a^2x\dot{x}+x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, a *complete* Fluxion; and then the

* Art. 77. Fluent thereof, by the common Rule *, will be $\frac{1}{2}\sqrt{a^2x^2+x^4} = \frac{1}{2}x\sqrt{a^2+x^2}$: But, from *this*, we are now to deduct the Fluent of the Quantity $\frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2+x^4}}$ $\left(= \frac{\frac{1}{2}a^2\dot{x}}{\sqrt{a^2+x^2}}\right)$ that was added: Which Fluent, as

that

that of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ is given $=$ *hyp. Log.* $x+\sqrt{a^2+x^2}$ *, * Art. 126.
will be $= \frac{1}{2}a^2 \times$ *hyp. Log.* $x + \sqrt{a^2+x^2}$; and consequently the Fluent fought $= \frac{1}{2}x\sqrt{a^2+x^2} - \frac{1}{2}a^2 \times$ *hyp. Log.* $x+\sqrt{a^2+x^2}$. *Q. E. I.*

PROB. II.

279. *Let it be propofed to find the Fluent of* $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$, *from that of* $\frac{\dot{x}}{\sqrt{a^2-x^2}}$; *given by* Art. 142.

By proceeding as above, and adding $-\frac{1}{2}a^2x\dot{x}$ to the Numerator, we have $-\frac{\frac{1}{2}a^2x\dot{x} - x^3\dot{x}}{\sqrt{a^2x^2-x^4}}$; whereof the Fluent, by the *common Rule*, is $-\frac{1}{3}\sqrt{a^2x^2-x^4}$ ($= -\frac{1}{2}x\sqrt{a^2-x^2}$: From which deducting the Fluent of $-\frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2-x^4}}$, or $-\frac{\frac{1}{2}a^2\dot{x}}{\sqrt{a^2-x^2}}$ (given $= -\frac{1}{2}a^2 \times$ Arc *(A)* whofe Radius is Unity and Sine $= \frac{x}{a}$ †) there comes out $\frac{1}{2}a^2A - \frac{1}{2}x\sqrt{a^2-x^2}$. † Art. 142.

Q. E. I.

280. In the fame Manner, if the Power without the Vinculum, in the Expreffion whofe Fluent is fought, exceeds *that* in the other Expreffion given, by the Exponent under the Vinculum, or by any *Multiple* of it, the required Fluent may be determined, by one, or by feveral Operations, according to the Value of the faid *Multiple.*

Thus, if the Fluent of $\frac{x^4\dot{x}}{\sqrt{a^2-x^2}}$ was fought; then, becaufe the Index of x, without the *Vinculum*, exceeds

 that

that in $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$ by twice the Exponent under the *Vinculum*, the required Fluent may be had from that of $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$, at two Operations; by the first whereof, we have already found the Fluent of $\frac{x^2\dot{x}}{\sqrt{a^2 - x^2}}$ to be $=$ $\frac{1}{2}a^2A - \frac{1}{2}x\sqrt{a^2 - x^2}$: Whence, putting this Value $= B$, and proceeding as before, we also get $-\frac{1}{4}\sqrt{a^2x^6 - x^8} + \frac{3}{4}a^2B = -\frac{1}{4}x^3\sqrt{a^2 - x^2} - \frac{3a^2x}{8}\sqrt{a^2 - x^2} + \frac{3a^4A}{8} = \frac{3a^4A - \overline{2xx + 3aa} \times x\sqrt{a^2 - x^2}}{8}$ = the true Fluent of $\frac{x^4\dot{x}}{\sqrt{a^2 - x^2}}$.

PROB. III.

281. *Supposing the Fluent of* $\overline{a + cz^n}|^m \times z^{pn-1}\dot{z}$ *to be given* $= A$, *to find the Fluent of* $\overline{a + cz^n}|^m \times z^{pn+n-1}\dot{z} = \dot{B}$ *(where the Exponent of z, without the* Vinculum *is increased by the Exponent under the* Vinculum*).*

Let the Part affected by the *Vinculum* be multiplied by z^{mq}, and the Part without be divided by the same Quantity; then our Fluxion will be transformed to $\overline{az^q + cz^{n+q}}|^m \times z^{pn+n-mq-1}\dot{z} = \dot{B}$: Where let q be, now, so taken that the Exponent $(n+q)$ of the highest Power of z under the *Vinculum* may be equal to $(pn+n-mq)$ that of the Power without the *Vinculum* $+ 1$; that is, let $q = \frac{pn}{m+1}$: Then *(by Art.* 77.*)* if the first Term under

under the *Vinculum* was conſtant, the Fluent of the ſaid Expreſſion, or its Equal $\overline{az^q + cz^{n+q}}|^m \times z^{n+q-1}\dot{z}$, would be had $= \frac{\overline{az^q + cz^{n+q}}|^{m+1}}{\overline{m+1} \times \overline{n+q} \times c}$. But the Fluxion hereof, ſuppoſing both Terms to be variable (as they actually are) is $\overline{az^q + cz^{n+q}}|^m \times z^{n+q-1}\dot{z} + \frac{qa}{\overline{n+q} \times c} \times \overline{az^q + cz^{n+q}}|^m \times z^{q-1}\dot{z}$ (by the common Rule.) Therefore $\frac{\overline{az^q + cz^{n+q}}|^{m+1}}{\overline{m+1} \times \overline{n+q} \times c} - \frac{qa}{\overline{n+q} \times c} \times$ *Flu. of* $\overline{az^q + cz^{n+q}}|^m \times z^{q-1}\dot{z} = B$; that is, $\frac{\overline{a+cz^n}|^{m+1} \times z^{qm+q}}{\overline{m+1} \times \overline{n+q} \times c} - \frac{qa}{\overline{n+q} \times c} \times$ *Flu.* $\overline{a+cz^n}|^m \times z^{mq+q-1}\dot{z} = B$; or, by ſubſtituting for q, $\frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times nc} - \frac{pa}{\overline{m+p+1} \times c} \times$ *Flu.* $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z} = B$: But the *Flu. of* $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ is given $= A$; therefore, laſtly, $\frac{\overline{a+cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times nc} - \frac{paA}{\overline{m+p+1} \times c} = B$. *Q. E. I.*

282. If the Quantity under the *Vinculum* be a Multinomial, $a + cz^n + dz^{2n} + ez^{3n}$ &c. Then, ſince the Fluxion of $\overline{a+cz^n + dz^{2n} + ez^{3n} \text{ \&c.}}|^{m+1} \times z^{pn}$ is $\overline{m+1} \times ncz^{n-1}\dot{z} + 2ndz^{2n-1}\dot{z} + 3nez^{3n-1}\dot{z}$ &c. $\times$

$\overline{a+cz^n+dz^{2n}\ \&c.}|^m \times z^{pn} + \overline{a+cz^n+dz^{2n}\ \&c.}|^{m+1}$
$\times pnz^{pn-1}\dot{z} =$
$$\left\{\begin{array}{l} * \quad \overline{m+1}\times ncz^{pn+n-1}\dot{z}+\overline{m+1}\times 2ndz^{pn+2n-1}\dot{z}\ \&c. \\ pnaz^{pn-1}\dot{z}+pncz^{pn+n-1}\dot{z}+pndz^{pn+2n-1}\dot{z}\ \&c. \end{array}\right\}$$
$\times\ \overline{a+cz^n+dz^{2n}\ \&c.}|^m$, it is evident, that, if the Fluents of $z^{pn-1}\dot{z}$, $z^{pn+n-1}\dot{z}$, $z^{pn+2n-1}\dot{z}$ &c. drawn into the general Multiplicator $\overline{a+cz^n+dz^{2n}\ \&c.}|^m$, be denoted by A, B, C, D, &c. the Fluent of the Whole-Quantity exhibited above (which Fluent is $\overline{a+cz^n+dz^{2n}+ez^{3n}\ \&c.}|^{m+1}\times z^{pn}$) will also be expressed by $pnaA+\overline{p+m+1}\times ncB+\overline{p+2m+2}\times ndC+\overline{p+3m+3}\times neD$ &c. Therefore, if there be given as many of the Fluents A, B, C, D &c. as there are Terms in $a+cz^n+dz^{2n}+ez^{3n}$ &c. *minus* one, that other Fluent, be it which it will, will also be given from hence. Thus if $d=0$, $e=0$, &c. and the Value of A be given, we shall have $\overline{a+cz^n}|^{m+1}\times z^{pn}=pnaA+\overline{p+m+1}\times ncB$; and consequently $B=\dfrac{\overline{a+cz^n}|^{m+1}\times z^{pn}}{\overline{p+m+1}\times nc}-\dfrac{paA}{\overline{p+m+1}\times c}$, the very same as before.

PROB. IV.

283. *The Fluent of* $\overline{a+cz^n}|^m\times z^{pn-1}\dot{z}$ *being given (as in the preceding Problem) to determine, from thence, the Fluent of* $\overline{a+cz^n}|^m\times z^{pn+vn-1}\dot{z}$; *supposing* v *to denote a whole positive Number.*

Let

Let $\overline{a+cz^n}|^{m+1}$ be denoted by M; also put $p+1 = p'$, $p'+1\ (p+2) = p''$, $p''+1\ (p+3) = p'''$ &c. and let the Fluents of $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{p'n-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{p''n-1}\dot{z}$, $\overline{a+cz^n}|^m \times z^{p'''n-1}\dot{z}$, &c. be represented by A, B, C, D, &c. respectively. Then, since $\frac{Mz^{pn}}{\overline{m+p+1} \times nc} - \frac{paA}{\overline{m+p+1} \times c} = B$ (*by the preceding Prob.*) it follows, from the very same Argument, that

$$\frac{Mz^{p'n}}{\overline{m+p'+1} \times nc} - \frac{p'aB}{\overline{m+p'+1} \times c} = C$$

$$\frac{Mz^{p''n}}{\overline{m+p''+1} \times nc} - \frac{p''aC}{\overline{m+p''+1} \times c} = D$$

&c. &c.

Hence, by writing the Value of B in the second Equation, we have $\frac{Mz^{p'n}}{\overline{m+p'+1} \times nc} - \frac{p'aMz^{pn}}{\overline{m+p'+1} \times \overline{m+p+1} \times ncc}$ $+ \frac{pp'a^2A}{\overline{m+p+1} \times \overline{m+p'+1} \times c^2} = C$. In the same Manner, by substituting this Value for C in the 3d Equation, we get

$$\frac{Mz^{p''n}}{\overline{m+p''+1} \times nc} - \frac{p''aMz^{p'n}}{\overline{m+p''+1} \times \overline{m+p'+1} \times nc^2} +$$

$$\frac{\overset{\prime\prime}{p}\,\overset{\prime}{p}a^2Mz^{pn}}{\overline{m+\overset{\prime\prime}{p}+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+p+1}\times nc^3} - \frac{\overset{\prime\prime\prime}{p}\,\overset{\prime\prime}{p}\,\overset{\prime}{p}\,pa^3A}{\overline{m+p+1}\times\overline{m+\overset{\prime}{p}+1}\times\overline{m+\overset{\prime\prime}{p}+1}\times c^3} = D.$$

Where the Law of Continuation is manifest; and from whence it appears that the Value of any of the Quantities B, C, D, E, &c. or the Fluent expressed in a general Manner, will be $\frac{Mz^{qn}}{\overline{m+q+1}\times nc} -$

$$\frac{qaMz^{\overline{q-1}\times n}}{\overline{m+q+1}\times\overline{m+q}\times ncc} + \frac{q\times\overline{q-1}\times a^2Mz^{\overline{q-2}\times n}}{\overline{m+q+1}\times\overline{m+q}\times\overline{m+q-1}\times nc^3} -$$

$$(v) \pm \frac{p\times\overline{p+1}\times\overline{p+2}\times\overline{p+3}\,(v)\times a^vA}{\overline{m+p+1}\times\overline{m+p+2}\times\overline{m+p+3}\,(v)\times c^v}\text{; or,}$$

$$\frac{\overline{a+cz^n}|^{m+1}\times z^{pn}}{\overline{s+1}\times nc}\times\overline{z^{vn-n} - \frac{qaz^{vn-2n}}{sc} + \frac{q.\overline{q-1}\times a^2z^{vn-3n}}{s.\overline{s-1}\times c^2} - \frac{q.\overline{q-1}.\overline{q-2}\times a^3z^{vn-4n}}{s.\overline{s-1}.\overline{s-2}\times c^3} - (v)} \pm \frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}\times$$

$\frac{p+3}{t+3}(v)\times\frac{a^vA}{c^v}$: Where, $A =$ Fluent of $\overline{a+cz^n}|^m$ $\times z^{pn-1}\dot{z}$, $q=p+v-1$, $s=q+m$, $t=p+m+1$; and where the Sign of the last Term (in which A is found) must be taken $+$ or $-$ according as v is an even or odd Number: Note, also, that the Parenthesis (v) is put to express the Number of Terms, or Factors, to which the Series, or Product, preceding it, is to be continued. The like Notation is to be understood in other Cases of the same Kind, when they hereafter occur.

The

The same otherwise.

284. Let $q = p + v - 1$, and let $\overline{a+cz^n}|^{m+1} \times \overline{Rz^{qn} + Sz^{qn-n} + Tz^{qn-2n} \ldots\ldots + \Delta z^{pn}} + \beta A$, be assumed for the Fluent sought: Then, by taking the Fluxion thereof, you will have $\overline{m+1} \times ncz^{n-1}\dot{z} \times \overline{a+cz^n}|^{m} \times \overline{Rz^{qn} + Sz^{qn-n} \ldots\ldots + \Delta z^{pn}} + \overline{a+cz^n}|^{m+1} \times \overline{qn\dot{z}Rz^{qn-1} + \overline{qn-n} \times \dot{z}Sz^{qn-n-1} \ldots\ldots + pn\dot{z}\Delta z^{pn-1}} + \beta \times \overline{a+cz^n}|^{m} \times z^{pn-1}\dot{z}$; which must be $= \overline{a+cz^n}|^{m} \times z^{pn+vn-1}\dot{z}$ (or $\overline{a+cz^n}|^{m} \times z^{qn+n-1}\dot{z}$) the Fluxion proposed: Whence, dividing the whole Equation by $\overline{a+cz^n}|^{m} \times z^{n-1}\dot{z}$, and transposing, there comes out

$$\left.\begin{array}{l} \overline{m+1} \times nc \times \overline{Rz^{qn} + Sz^{qn-n} + Tz^{qn-2n} \ldots\ldots + \Delta z^{pn}} \\ \overline{a+cz^n} \times \overline{qnRz^{qn-n} + \overline{qn-n} \times Sz^{qn-2n} \ldots\ldots + pn\Delta z^{pn-n}} \\ \quad -z^{qn} \quad * \quad * \quad + \beta z^{pn-n} \end{array}\right\} = 0$$

Which, reduced, and the homologous Terms united, becomes

$$\left.\begin{array}{r} \overline{m+q+1} \times ncR \\ -1 \end{array}\right\} \times z^{qn} + \left.\begin{array}{r} \overline{m+q} \times ncS \\ +qnaR \end{array}\right\} \times z^{qn-n} + \left.\begin{array}{r} \overline{m+q-1} \times ncT \\ +\overline{qn-n} \times aS \end{array}\right\} \times z^{qn-2n} \ldots\ldots\ldots + \left.\begin{array}{r} pna\Delta \\ +\beta \end{array}\right\} \times z^{pn-n}$$

$= 0$: Where, by making $\overline{m+q+1} \times ncR - 1 = 0$, $\overline{m+q} \times ncS + qnaR = 0$, &c. we have $R = \dfrac{1}{\overline{m+q+1} \times cn}$, $S = -\dfrac{qaR}{\overline{m+q} \times c}$, $T = -\dfrac{\overline{q-1} \times aS}{\overline{m+q-1} \times c}$; or (putting

$m+q$

$m+q=s$) $R=\frac{1}{\overline{s+1}\times nc}$, $S=-\frac{qaR}{sc}=-\frac{qa}{\overline{s+1}\times snc^2}$,

$T=-\frac{\overline{q-1}\times aS}{\overline{s-1}\times c}=\frac{q\times\overline{q-1}\times a^2}{\overline{s+1}\times s\times\overline{s-1}\times nc^3}$, &c.

Where, becauſe the Exponent of the firſt Term of the Equation is qn $(pn+vn-n)$ and that of the laſt Term (in which Δ and β are concerned) $=pn$, it follows that the Number of Coefficients to be taken as above (whereof Δ is the laſt) is expreſſed by v: From which laſt, the Value of β is given $=-pna\Delta$.

But, from the Law of the ſaid Coefficients, R, S, $\ldots\Delta$, it appears that the Value of Δ (whoſe Place from the Beginning is denoted by v) will be $=\pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{q-v+2}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{s-v+2}}\times\frac{a^{n-1}}{nc^n}=\pm$ $\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{p+m+1}}\times\frac{a^{n-1}}{nc^n}$: And therefore β $(=-pna\Delta)=\pm\frac{q.\overline{q-1}.\overline{q-2}\ldots\ldots\overline{p+1}.p}{\overline{s+1}.s.\overline{s-1}\ldots\ldots\overline{p+m+1}}$ $\times\frac{a^n}{c^n}=\pm\frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}\,(v)}{t.\overline{t+1}.\overline{t+2}.\overline{t+3}\,(v)}\times\frac{a^n}{c^n}$ (putting $p+m+1=t$, as before.) Now, if the ſeveral Values of R, S. $T\ldots\ldots$ and β, thus found, be ſubſtituted in the aſſumed Expreſſion, you will have the very ſame Concluſion as in the preceding Article.

Corollary I.

285. Since q is $=p+v-1$, the Fluent $\overline{a+cz^n}|^{m+1}\times$ $\overline{Rz^{qn}+Sz^{qn-n}\ldots\ldots+\Delta z^{pn}}+\beta A$, given above, may be expreſſed by $N\times\overline{Rz^{vn-n}+Sz^{vn-2n}+Tz^{vn-3n}}$ $(v)+\beta A$; where $N=\overline{a+cz^n}|^{m+1}\times z^{pn}$, $R=$

$\frac{1}{\overline{m+p+v} \times nc}$, $S = -\frac{\overline{p+v-1}.aR}{\overline{m+p+v-1}.c}$, $T = -\frac{\overline{p+v-2}.aS}{\overline{m+p+v-2}.c}$: And, where the Coefficient (β) of the given Fluent (A) will always be expreſſed by the laſt of the Quantities R, S, T . . . Δ, multiplied by $-pna$: This is evident, becauſe it is found that $\beta = -pna\,\Delta$. And the ſame thing will alſo appear from the ſeveral particular Caſes (*in Art.* 283.) for the Values of B, C and D: In each of which the Coefficient of the laſt Term (where A is concerned) is to *that* of the Term immediately preceding it, in the conſtant Ratio of pa to $\frac{1}{n}$, or of pna to Unity.

COROLLARY II.

286. If the Value of c be negative, the general Fluent (*in Art.* 283.) when $a + cz^n = 0$ (provided $m+1$, n, and p be poſitive) will become barely $= \pm \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\ (v) \times \frac{a^v A}{c^v}$; becauſe, in this Circumſtance, all the Terms multiplied by $\overline{a+cz^n}|^{m+1}$ intirely vaniſh. If, therefore, b be wrote for $-c$ (to render the Expreſſion more commodious) we ſhall have $\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\ (v) \times \frac{a^v A}{b^v}$ for the true Fluent of $\overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, generated while bz^n, from Nothing, becomes $= a$: Where A denotes the Fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, generated in the ſame time; and where

$t =$

$t = p + m + 1$. Hence it follows that the Fluent of $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z} \times \overline{e + fz^n + gz^{2n} + hz^{3n} \&c.}$ (where e, f, g, are any given Quantities) will be $= A \times$ $\overline{e + \frac{paf}{tb} + \frac{p.\overline{p+1}.a^2g}{t.\overline{t+1}.b^2} + \frac{p.\overline{p+1}.\overline{p+2}.a^3h}{t.\overline{t+1}.\overline{t+2}.b^3} + \&c.}$ in the forementioned Circumstance.

PROB. V.

287. *The Fluent (A) of* $\overline{a + cz^n}|^m \times z^{pn-1}\dot{z}$ *being given, to find the Fluent of* $\overline{a + cz^n}|^{m+r} \times z^{pn-1}\dot{z}$; *supposing r to denote a whole positive Number.*

Since $\overline{a + cz^n}|^{m+1} = \overline{a + cz^n}|^m \times \overline{a + cz^n}$, it is evident that $\overline{a + cz^n}|^{m+1} \times z^{pn-1}\dot{z} = \overline{a + cz^n}|^m \times az^{pn-1}\dot{z} + \overline{a + cz^n}|^m \times cz^{pn+n-1}\dot{z}$: Whose Fluent (*by Prop.* 3.) is $aA + \frac{\overline{a + cz^n}|^{m+1} \times z^{pn}}{\overline{m+p+1} \times n} - \frac{paA}{m + p + 1} = \frac{\overline{a + cz^n}|^{m+1} \times z^{pn}}{\overline{p+m+1} \times n} + \frac{\overline{m+1} \times aA}{p+m+1}$. In like Manner, if this Fluent, of $\overline{a + cz^n}|^{m+1} \times z^{pn-1}\dot{z}$, be denoted by B, that of $\overline{a + cz^n}|^{m+2} \times z^{pn-1}\dot{z}$ by C, &c. it will appear that $\frac{\overline{a + cz^n}|^{m+2} \times z^{pn}}{\overline{p+m+2} \times n} + \frac{\overline{m + 2} \times aB}{p+m+2} = C$; $\frac{\overline{a + cz^n}|^{m+3} \times z^{pn}}{\overline{p+m+3} \times n} + \frac{\overline{m+3} \times aC}{p+m+3} = D$, &c. Whence, by substituting these Values, one by one, as in the preceding

ceding Problem, and putting $Q = a + cz^n$, we get

$C = \frac{Q^{m+2}z^{pn}}{p+m+2 \cdot n} + \frac{\overline{m+2} \times aQ^{m+1}z^{pn}}{\overline{p+m+2}\,\overline{p+m+1} \cdot n} + \frac{\overline{m+2} \cdot \overline{m+1} \times a^2 A}{\overline{p+m+2} \cdot \overline{p+m+1}}$; $D = \frac{Q^{m+3}z^{pn}}{p+m+3 \cdot n} + \frac{\overline{m+3} \times aQ^{m+2}z^{pn}}{\overline{p+m+3} \cdot \overline{p+m+2} \cdot n} + \frac{\overline{m+3} \cdot \overline{m+2} \times a^2 Q^{m+1} z^{pn}}{\overline{p+m+3} \cdot \overline{p+m+2} \cdot \overline{p+m+1} \cdot n} + \frac{\overline{m+3} \cdot \overline{m+2} \cdot \overline{m+1} \cdot a^3 A}{\overline{p+m+3} \cdot \overline{p+m+2} \cdot \overline{p+m+1}}$, &c. Whence it is evident, by Inspection, that the Fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn-1}\dot{z}$, expressed in a general Manner, will be $\frac{Q^{m+r} z^{pn}}{p+m+r \cdot n} + \frac{\overline{m+r} \times aQ^{m+r-1} z^{pn}}{\overline{p+m+r} \times \overline{p+m+r-1} \cdot n}$ &c. Which, by putting $m+r=f$, $p+m+r=g$, and making $Q^{m+1} \times z^{pn}$ a general Multiplicator, will be reduced to $Q^{m+1} \times z^{pn} \times \overline{\frac{Q^{r-1}}{gn} + \frac{f \times aQ^{r-2}}{g \cdot \overline{g-1} \cdot n} + \frac{f \cdot \overline{f-1} \times a^2 Q^{r-3}}{g \cdot \overline{g-1} \cdot \overline{g-2} \cdot n} (r)} + \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \times \frac{m+3}{p+m+3} (r)\ a^r A$; where it appears (from the foregoing Values of B, C, and D) that the Coefficient of A is always equal to the last Term of the preceding Series, multiplied by $\overline{m+1} \times na$ (instead of $Q^{m+1} z^{pn}$). *Q. E. I.*

COROLLARY.

288. If c be negative, so that Q, or its Equal, $a + cz^n$, may become equal to Nothing, the Fluent will, in

in that Circumstance, be barely $= \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}$ $\times \frac{m+3}{p+m+3} (r) \times a^r A$; provided the Values of $m+1$, p, and n are positive: Or, if c, p, and n be positive, and $m+r+p$ negative, the same Expression will exhibit the true Value of the whole Fluent, generated while z, from Nothing, becomes infinite.

PROB. VI.

289. *The same being given as in the preceding Problems; it is proposed to find the Fluent of* $\overline{a+cz^n}|^{m-r} \times z^{pn-1}\dot{z}$.

If $-r$ be wrote instead of r, in the last Article, we shall have $m-r=f$, $p+m-r=g$, and $\overline{Q^{m+1} z^{pn} \times \frac{Q^{-r-1}}{gn} + \frac{f \times aQ^{-r-2}}{g.\overline{g-1}.n}} (-r) + \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} (-r) \times a^{-r} A$, expressing the required Fluent in this Case.

But $\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}$ &c. continued to $-r$ Factors, signifies the same thing as the Product continued downwards, or the contrary way, to r Factors, according to the same Law: And therefore is $= \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2} (r)$. After the same Manner we have $\frac{Q^{-r-1}}{gn} + \frac{f \times aQ^{-r-2}}{g.\overline{g-1}.n} (-r) =$ $-\frac{Q^{-r}}{\overline{f+1}.na} - \frac{\overline{g+1}.Q^{-r+1}}{\overline{f+1}.\overline{f+2}.na^2} - \frac{\overline{g+1}.\overline{g+2}.Q^{-r+2}}{\overline{f+1}.\overline{f+2}.\overline{f+3}.na^3}$ (r)

(r) and consequently the Fluent itself $= Q^{m+1} z^{pn} \times$

$$\overline{-\frac{Q^{-r}}{\overline{f+1}.na} - \frac{\overline{g+1}.Q^{1-r}}{\overline{f+1}.\overline{f+2}.na^2} - \frac{\overline{g+1}.\overline{g+2}.Q^{2-r}}{\overline{f+1}.\overline{f+2}.\overline{f+3}.na^3}\ (r)}$$

$$+\frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2}\ (r) \times \frac{A}{a^r}. \quad Q.E.I.$$

COROLLARY.

290. It appears from hence that the Coefficient of A, the given Fluent, will always be equal to *that* of the last Term of the preceding Series, multiplied by $\overline{p+m}\times n$: For, seeing the Coefficient of the said last Term (whose Distance from the first, inclusive, is denoted by r) must be $\frac{\overline{g+1}.\overline{g+2}.\overline{g+3}\ldots\overline{g+r-1}}{\overline{f+1}.\overline{f+2}.\overline{f+3}\ldots\overline{f+r}} \times \frac{1}{na^r}$ (by the Law of the Series) where $f+r=m$ and $g+r-1=p+m-1$ (as appears from above) it follows, by inverting the Order of both Progressions, that $\frac{\overline{p+m-1}.\overline{p+m-2}.(r-1)}{m.\overline{m-1}.\overline{m-2}\ (r)}$ $\times \frac{1}{na^r}$ will also express the same Coefficient: Which, multiplied by $\overline{p+m}\times n$, gives $\frac{\overline{p+m}.\overline{p+m-1}.\overline{p+m-2}\ (r)}{m.\overline{m-1}.\overline{m-2}\ (r)}$ $\frac{1}{a^r}$, the very Coefficient of A, above determined. The Use of this Conclusion will be seen in what follows.

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PROB. VII.

291. *The ſame being,* ſtill, *given; to find the Fluent of*

$$\overline{a+cz^n}|^m \times z^{pn-vn-1}\dot{z}.$$

By proceeding as in the laſt Problem, the required Fluent of $\overline{a+cz^n}|^m \times z^{pn-vn-1}$ is derived from *that* of $\overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$ (*given by Prob.* 4.) and comes out

$$= Q^{m+1} z^{pn} \times \overline{\frac{z^{-vn}}{\overline{q+1}.na} - \frac{\overline{s+2}.cz^{n-vn}}{\overline{q+1}.\overline{q+2}.na^2} + \frac{\overline{s+2}.\overline{s+3}.c^2z^{2n-vn}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.na^3}\ (v)} \pm \frac{t-1}{p-1} \times \frac{t-2}{p-2} \times \frac{t-3}{p-3}\ (v)$$

$\times \frac{c^v A}{a^v}$: Where, $Q = a+cz^n$, $q = p-v-1$, $s = m+q$, $t = p+m+1$: And where, the Coefficient of A is equal to that of the laſt of the preceding Terms, multiplied by $-\overline{m+p} \times nc$. If the Manner of deducing the required Fluent, in this, and the laſt, Problem, ſhould not appear ſufficiently plain and ſatisfactory to the Beginner; the ſame Concluſions may be, otherwiſe, brought out; by finding A, in Terms of B, C, or D, from the ſeveral particular Equations in *Art.* 283. or, by aſſuming a deſcending Series, inſtead of an aſcending one. *Vid. Art.* 284.

PROB. VIII.

292. *The ſame being,* ſtill, given; *to find the Fluent of*

$$\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}.$$

Let the Fluent of $\overline{a+cz^n}|^m \times z^{pn+vn-1}\dot{z}$ (*given by Prob.* 4.) be denoted by B, and that required by F: Then,

Then, if $p+v$ be put $=\acute{p}$, the Value of F (the Fluent of $\overline{a+cz^n}|^{m+r} \times z^{\acute{p}n-1}\dot{z}$) will be given from that of B (the Fluent of $\overline{a+cz^n}|^{m} \times z^{\acute{p}n-1}\dot{z}$) by writing B for A and $\acute{p}$ for p, in *Art.* 287. Whence we get $F=Q^{m+1}$

$$z^{\acute{p}n} \times \frac{Q^{r-1}}{gn} + \frac{faQ^{r-2}}{g\cdot\overline{g-1}.n} + \frac{f\cdot\overline{f-1}.a^2Q^{r-3}}{g\cdot\overline{g-1}\cdot\overline{g-2}.n}\ (r) + \frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2} \times \frac{m+3}{\acute{p}+m+3}\ (r) \times a^r B$$

: Where $\acute{p}=p+v$, $f=m+r$, $g\ (=\acute{p}+m+r)=p+m+v+r$, and $Q=a+cz^n$.

Which Fluent, by substituting the Value of B (*in Prob.* 4.) becomes

$$F=Q^{m+1}z^{\acute{p}n} \times \frac{Q^{r-1}}{gn} + \frac{faQ^{r-2}}{g\cdot\overline{g-1}.n} + \frac{f\cdot\overline{f-1}\cdot a^2Q^{r-3}}{g\cdot\overline{g-1}\cdot\overline{g-2}.n}\ (r) + \frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2}\ (r)$$

$$\times a^r \times Q^{m+1}z^{pn} \times \frac{z^{vn-n}}{\overline{s+1}.nc} - \frac{qaz^{vn-2n}}{\overline{s+1}.snc^2} + \frac{q\cdot\overline{q-1}.a^2z^{vn-3n}}{\overline{s+1}.s.\overline{s-1}.nc^3}.$$

$$(v) \pm \frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2}\ (r) \times a^r \times \frac{p}{t} \times \frac{\overline{p+1}}{t+1}.$$

$$(v) \times \frac{a^v A}{c^v}$$

: Where $q=p+v-1$, $s=m+q=m+p+v-1$, and $t=p+m+1$; and where the Sign of the last Term is + or — according as v is an even or odd Number. *Q. E. I.*

292. If the laſt Term of the firſt Series, excluſive of the general Multiplicator $\mathcal{Q}^{m+1}z^{\acute{p}n}$, be denoted by β, the Multiplicator, $\frac{m+1}{\acute{p}+m+1} \times \frac{m+2}{\acute{p}+m+2}(r) \times \acute{a}^{r}$, to the ſecond Series will be $= \overline{m+1} \times na\beta$ *; and therefore the firſt Term of this Series, including its Multiplicators, is $= \frac{\overline{m+1}.a\beta\mathcal{Q}^{m+1}z^{pn+vn}}{\overline{s+1}.cz^{n}}$: Which, if R be put to denote the laſt Term $\beta\mathcal{Q}^{m+1}z^{pn+vn}$ of the firſt Series (with its Multiplicator) will be expounded by $\frac{\overline{m+1}.aR}{\overline{s+1}.cz^{n}}$. Hence it follows, that the Fluent of $\overline{a+cz^{n}}|^{m+r} \times z^{pn+vn-1}\dot{z}$, given above, will alſo be truly expreſſed by $\frac{\mathcal{Q}^{m+r} \times z^{pn+vn}}{gn} + \frac{f}{g-1} \times \frac{aH}{\mathcal{Q}} + \frac{f-1}{g-2} \times \frac{aI}{\mathcal{Q}} + \frac{f-2}{g-3} \times \frac{aK}{\mathcal{Q}}(r) + \frac{m+1}{s+1} \times \frac{aR}{cz^{n}} - \frac{q}{s} \times \frac{aS}{cz^{n}} - \frac{q-1}{s-1} \times \frac{aT}{cz^{n}} - \frac{q-2}{s-2} \times \frac{aV}{cz^{n}}(v) \pm \frac{\overline{m+1}.\overline{m+2}.\overline{m+3}(r) \times p.\overline{p+1}.\overline{p+2}(v)}{\overline{\acute{p}+m+1}.\overline{\acute{p}+m+2}(r) \times t.\overline{t+1}.\overline{t+2}(v)} \times \frac{a^{v+r}A}{c^{v}}$: Where $H, I, K, L \ldots\ldots R, S, T, V,$ &c. repreſent the Terms immediately preceding thoſe where they ſtand, under their proper Signs: R being the laſt Term of the firſt Series; alſo $f = m+r$, $g = m+r+p+v$, $q = p+v-1$, $s = m+q$, $t = m+p+1$, and $\mathcal{Q} = a+cz^{n}$.

* Art. 287.

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COROLLARY II.

293. Since the Divisor, $\overline{\acute{p}+m+1}.\overline{\acute{p}+m+2}\ (r) \times \overline{t.t+1.t}\ (v)$, of the last Term of the Fluent (by substituting for t and $\acute{p}$ &c.) is $= \overline{p+m+1}.\overline{p+m+2}\ (v) \times \overline{p+v+m+1}.\overline{p+v+m+2}\ (r)$: Where, the last Factor $(p+m+v)$ of the first Progression, is less by Unity than the first Factor of the Second; it is evident that the said second Progression is only a Continuation of the first to r *more* Factors: And so, the last Term of the Fluent, where A is found, is truly expressed by $\pm$

$$\frac{p.\overline{p+1}.\overline{p+2}\ (v) \times \overline{m+1}.\overline{m+2}.\overline{m+3}\ (r)}{\overline{m+p+1}.\overline{m+p+2}.\overline{m+p+3}\ (v+r)} \times \frac{a^{v+r}A}{c^v}.$$

Hence it follows, that the Fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$, or that of $\overline{a-bz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$ (making $c=-b$) will, when $a-bz^n$ becomes equal to Nothing, be barely $=$

$$\frac{p.\overline{p+1}.\overline{p+2}\ (v) \times \overline{m+1}.\overline{m+2}.\overline{m+3}\ (r)}{\overline{m+p+1}.\overline{m+p+2}.\overline{m+p+3}\ (v+r)} \times \frac{a^{v+r}A}{b^v}:$$

A being the Fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, in that Circumstance, v and r whole positive Numbers, and p and $m+1$ any positive Numbers, either whole or broken.

SCHOLIUM.

294. If the Fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn-1}\dot{z}$ (given by *Prob.* 5.) be denoted by C; then (F) the Fluent of $\overline{a+cz^n}|^{\acute{m}} \times z^{pn+vn-1}\dot{z}$ (where $\acute{m}=m+r$) will be had, from C (by *Prob.* 4.) according to a new Form, different

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ferent from thofe already given. And, by following the fame Method, the Fluents of $\overline{a+cz^n}|^{m-r} \times z^{pn+vn-1}\dot{z}$, $\overline{a+cz^n}|^{m+r} \times z^{pn-vn-1}\dot{z}$, and $\overline{a+cz^n}|^{m-r} \times z^{pn-vn-1}\dot{z}$ may alfo be found, each, according to two different Forms, from a Combination of the correfponding Cafes in the foregoing Problems.

But, as it is extremely tirefome to repeat the fame thing, again and again, where fuch a Number of Symbols are neceffarily concerned, I fhall here put down one Solution to each Cafe (becaufe of their Ufe) leaving the Procefs and the other Forms (which contain no new Difficulty) to *Thofe* who will be at the Trouble to fet about them.

1°. The Fluent of $\overline{a+cz^n}|^{m-r} \times z^{pn+vn-1}\dot{z}$ is $=$

$$-\frac{Q^{m-r+1}\times z^{pn+vn}}{\overline{f+1}.na}+\frac{g+1}{f+2}\times\frac{QH}{a}+\frac{g+2}{f+3}\times\frac{QI}{a}\ (r)$$

$$-\frac{QR}{cz^n}-\frac{q}{s}\times\frac{aS}{cz^n}-\frac{q-1}{s-1}\times\frac{aT}{cz^n}-\frac{q-2}{s-2}\times\frac{aV}{cz^n}\ (v)$$

$$+\frac{s+1}{m}\times\frac{s}{m-1}\times\frac{s-1}{m-2}\ (r)\times\frac{p}{t}\times\frac{p+1}{t+1}\times\frac{p+2}{t+2}\ (v)\times\frac{a^{v-r}A}{\overline{-c}|^{v}}.$$

Where H, I, K, L, ... R, S, T, &c. denote the Terms immediately preceding thofe where they ftand, under their proper Signs; R being the laft Term of the firft Series, alfo $Q=a+cz^n$, $f=m-r$, $g=p+m+v-r$, $q=p+v-1$, $s=m+p+v-1$, $t=p+m+1$, and $A=$ the given Fluent of $\overline{a+cz^n}|^{m} \times z^{pn-1}\dot{z}$.

2°. The

2°. The Fluent of $\overline{a+cz^n}|^{m+r} \times z^{pn-vn-1}\dot{z}$ is $=$

$$\frac{Q^{m+r+1} \times z^{pn-vn}}{\overline{q+1}.na} - \frac{s+2}{q+2} \times \frac{Hcz^n}{a} - \frac{s+3}{q+3} \times \frac{Icz^n}{a}\ (v)$$

$$-\frac{Rcz^n}{Q} + \frac{f}{g-1} \times \frac{aS}{Q} + \frac{f-1}{g-2} \times \frac{aT}{Q} + \frac{f-2}{g-3} \times \frac{aV}{Q}\ (r)$$

$$+\frac{s+2}{q+1} \times \frac{s+3}{q+2} \times \frac{s+4}{q+3}(v) \times \frac{m+1}{t} \times \frac{m+2}{t+1} \times \frac{m+3}{t+2}(r) \times \frac{\overline{-c}|^{v} A}{a^{w-r}}:$$

Where $q=p-v-1$, $s=m+r+q$, $f=m+r$, $g=p+m+r$, and the rest as in the preceding Case.

3°. The Fluent of $\overline{a+cz^n}|^{m-r} \times z^{pn-vn-1}\dot{z}$ is $=$

$$-\frac{Q^{m-r+1} \times z^{pn-vn}}{\overline{f+1}.na} + \frac{g+1}{f+2} \times \frac{QH}{a} + \frac{g+2}{f+3} \times \frac{QI}{a}\ (r)$$

$$-\frac{s+1}{q+1} \times \frac{QR}{a} - \frac{s+2}{q+2} \times \frac{Scz^n}{a} - \frac{s+3}{q+3} \times \frac{Tcz^n}{a}\ (v)$$

$$+\frac{\overline{t-1}.\overline{t-2}.\overline{t-3}.\overline{t-4}.\overline{t-5}\ (r+v)}{mm-1.m-2(r) \times p-1\,p-2.p-3\ (v)} \times \frac{\overline{-c}|^{v} A}{a^{r+v}}.$$

In which $f=m-r$, $g=m+p-r-v$, $q=p-v-1$, $s=q+m$, and the rest as before.

295. From what has been delivered in this Section, the Fluents of various Forms of Fluxions may be exhibited, by means of circular Arcs and Logarithms. For, since the Fluents of $\overline{a+cz^n}|^{-1} \times z^{\frac{1}{2}n-1}\dot{z}$, $\overline{a+cz^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}$, and $\overline{a+cz^n}|^{-\frac{1}{2}} \times z^{-1}\dot{z}$ (which I call original Ones) are all of them explicable by one, or the other, of these two Kinds of Quantities (as will appear farther on) *those* of $\overline{a+cz^n}|^{-1\pm r} \times z^{\frac{1}{2}n\pm vn-1}\dot{z}$, $\overline{a+cz^n}|^{-\frac{1}{2}\pm r} \times z^{\frac{1}{2}n\pm vn-1}\dot{z}$, and $\overline{a+cz^n}|^{-\frac{1}{2}\pm r} \times z^{\pm vn-1}\dot{z}$ will also be given from thence, by the fore-

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going Theorems. Whence the most useful Forms of Fluents in *Cotes*'s *Harmonia Mensurarum* will be obtained, besides some others, more general than any, of the same Kind, put down by that sagacious Author.

Here follow a few Examples of some of the most useful Cases.

EXAMPLE I.

296. *Let the Fluxion given be* $\frac{z^{2v}\dot{z}}{\sqrt{d^2 \pm z^2}}$ (or $\overline{d^2+z^2}|^{-\frac{1}{2}} \times z^{2v}\dot{z}$) *v being any whole positive Number.*

Then, the Fluent of $\overline{d^2 \pm z^2}|^{-\frac{1}{2}} \times \dot{z}$, or $\frac{\dot{z}}{\sqrt{d^2 \pm z^2}}$ being $=$ *hyp. Log.* $\frac{z+\sqrt{d^2+z^2}}{d}$; or, equal to the

* Art. 126. 742. Arch whose Sine is $\frac{z}{d}$ and Radius Unity *; according as the second Term, in $d^2 \pm x^2$, is positive or negative; let A be, therefore, taken to denote the said Arch, or Logarithm; and let $\overline{d^2 \pm z^2}|^{-\frac{1}{2}} \times \dot{z}$ be compared with $\overline{a+cz^n}|^m \times z^{pn-1}\dot{z}$ (whose Fluent is, all along, supposed to be given $= A$) and you will have $a = d^2$, $c = \pm 1$, $n = 2$, $m = -\frac{1}{2}$, $2p - 1 = 0$, and therefore $p = \frac{1}{2}$: Whence, by substituting those Values in *Art.* 283. we likewise get $q\ (p+v-1) = \frac{2v-1}{2}$, $s\ (m+q) = v - 1$, $t\ (m+p+1) = 1$; and, consequently, the Fluent sought $= \overline{d^2 \pm z^2}|^{\frac{1}{2}} \times \overline{\pm \frac{z^{2v-1}}{2v} - \frac{\overline{2v-1}.d^2z^{2v-3}}{2v.\overline{2v-2}} \pm}$

$\overline{\frac{\overline{2v-1}.\overline{2v-3}.d^4z^{2v-5}}{2v.\overline{2v-2}.\overline{2v-4}} - \frac{\overline{2v-1}.\overline{2v-3}.\overline{2v-5}.d^6z^{2v-7}}{2v.\overline{2v-2}.\overline{2v-4}.\overline{2v-6}}}$

(v)

$(v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} (v) \times d^{2v}$: In which the laſt Term is negative, when the given Fluxion is $\frac{z^{2v}\dot{z}}{\sqrt{d^2+z^2}}$, and v, at the ſame time, an odd Number; but in all other Caſes, affirmative.

EXAMPLE II.

297. *Let* $z^{2v}\dot{z}\sqrt{d^2 \pm z^2}$ (or $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$) *be propounded.*

Here, denoting the Fluent of $\overline{d^2 \pm z^2}|^{-\frac{1}{2}}\dot{z}$ by A (as above) and comparing $\overline{d^2 \pm z^2}|^{-\frac{1}{2}+1} \times z^{2v}\dot{z}$, with $\overline{a+cz^n}|^{m+r} \times z^{pn+vn-1}\dot{z}$ (*Vid. Prob.* 8.) we have $r=1$, and the reſt as in the laſt Example: Whence alſo $p\ (p+v) = v+\frac{1}{2}$, $f\ (m+r) = \frac{1}{2}$, $g=v+1$, $Q=d^2 \pm z^2$, and the Fluent itſelf $= \frac{z^{2v+1}\sqrt{d^2 \pm z^2}}{2v+2} \pm \frac{d^2R}{2vz^2} \mp \frac{\overline{2v-1}.d^2S}{\overline{2v-2}.z^2} \mp \frac{\overline{2v-3}.d^2T}{\overline{2v-4}.z^2}\ (1+v) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6}(v) \times \frac{d^{2v+2}A}{2v+2}$ * (*R*, *S*, *T*, &c. being the preceding Terms with their Signs) $= \frac{\sqrt{d^2 \pm z^2}}{2v+2} \times \overline{z^{2v+1} \pm \frac{d^2z^{2v-1}}{2v} - \frac{\overline{2v-1}.d^4z^{2v-3}}{2v.\overline{2v-2}} \pm \frac{\overline{2v-1}.\overline{2v-3}.d^6z^{2v-5}}{2v.\overline{2v-2}.\overline{2v-4}}}\ (v+1) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8}(v) \times \frac{d^{2v+2}A}{2v+2}$: Where the Sign of the laſt Term muſt be regulated as in the pre-

* Art. 292.

preceding Example—If the Fluent of $\frac{z^{-vn}\dot{z}}{\sqrt{d^2 \pm z^2}}$, or of $z^{-vn}\dot{z}\sqrt{d^2 \pm z^2}$ (in which the Exponent is negative) be required; the Anſwer will be had in finite Terms, independent of *A*, *by Art.* 85.

EXAMPLE III.

298. *Wherein the Fluxion propoſed is* $\overline{d^n - z^n}|^{-\frac{1}{2}+r} \times z^{\frac{1}{2}n+vn-1}\dot{z}$; *r and v being any whole poſitive Numbers.*

Since the Fluent of $\overline{d^n - z^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1}\dot{z}$ (as will appear hereafter) is truly expreſſed by $\frac{2}{n} \times$ *Arch*, whoſe Sine is $\frac{z^{\frac{1}{2}n}}{d^{\frac{1}{2}n}}$ and Radius Unity, let this Value be denoted by *A*; and then, by writing d^n for *a*, -1 for *c*, $-\frac{1}{2}$ for *m*, and $\frac{1}{2}$ for *p*, *in Art.* 292. we ſhall have $f\ (m+r) = \frac{2r-1}{2}$, $g\ (m+p+r+v) = r+v$, $q\ (p+v-1) = \frac{2v-1}{2}$, $s\ (m+q) = v-1$, $t\ (p+m+1) = 1$, $Q\ (a+cz^n) = a^n - z^n$, and the Fluent, itſelf, equal to

$$\frac{Q^{r-\frac{1}{2}} z^{vn+\frac{1}{2}n}}{\overline{r+v}.n} + \frac{2r-1}{r+v-1} \times \frac{\frac{1}{2}d^n H}{Q} + \frac{2r-3}{r+v-2} \times \frac{\frac{1}{2}d^n I}{Q} + \frac{2r-5}{r+v-3} \times \frac{\frac{1}{2}d^n K}{Q}\ (r) - \frac{\frac{1}{2}d^n R}{vz^n} + \frac{2v-1}{v-1} \times \frac{\frac{1}{2}d^n S}{z^n} + \frac{2v-3}{v-2} \times \frac{\frac{1}{2}d^n T}{z^n}\ (v) + \frac{1.3.5.7\ (r) \times 1.3.5.7\ (v)}{2.4.6.8.10.12\ (r+v)}$$

* Art. 293. * $\times\ d^{rn+vn}A$: In which *H*, *I*, *K* . . . *R*, *S*, *T*, &c. denote the preceding Terms with their Signs; *R* being the

the laſt Term of the firſt Series. Hence, becauſe all the Terms, but the laſt, vaniſh, when $Q = 0$, it follows that the whole Fluent of $\overline{d^n - z^r}|^{r - \frac{1}{2}} \times z^{vn + \frac{1}{2}n - 1}\dot{z}$, generated while z, from Nothing, becomes equal to d, is truly expreſſed by $\frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\,(r+v)} \times d^{rn+vn} A$, or by $\frac{1.3.5.7\,(r) \times 1.3.5.7\,(v)}{2.4.6.8.10.12\,(r+v)} \times \frac{d^{rn+vn}G}{n}$; G being the Semi-Periphery of the Circle whoſe Radius is Unity.

EXAMPLE IV.

299. *Let it be required to find the* whole *Fluent of* $\frac{\overline{a - bz^n}|^m \times z^{pn-1}\dot{z}}{\overline{d + kz^n}|^\beta}$, *generated while* bz^n, *from Nothing, becomes* $= a$; *that of* $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z}$ *being given* ($= A$.)

Here, by expanding $\overline{d + kz^n}|^{-\beta}$, our given Fluxion becomes $= \overline{a - bz^n}|^m \times z^{pn-1}\dot{z}$ into $d^{-\beta} \times \overline{1 - \frac{\beta kz^n}{d} + \frac{\beta.\beta+1.k^2z^{2n}}{1.2.d^2} - \frac{\beta.\beta+1.\beta+2.k^3z^{3n}}{1.2.3.d^3}}$ &c. Which Series being compared with $e + fz^n + gz^{2n}$ &c. (*Vid. Art.* 286.) we have $e = 1$, $f = -\frac{\beta k}{d}$, $g = \frac{\beta.\beta+1.k^2}{1.2.d^2}$, &c. and conſequently the Fluent ſought (by ſubſtituting theſe Values) equal to $\frac{A}{d^\beta}$ into $1 - \frac{p}{t} \times \frac{\beta}{1} \times \frac{ak}{bd} + \frac{p}{t} \cdot \frac{p+1}{t+1} \times \frac{\beta}{1} \cdot \frac{\beta+1}{2} \times \overline{\frac{ak}{bd}}|^2 -$

$\frac{p}{t} \cdot \frac{p+1}{t+1} \cdot \frac{p+2}{t+2} \times \frac{\beta}{1} \cdot \frac{\beta+1}{2} \cdot \frac{\beta+2}{3} \times \overline{\frac{ak}{bd}}|^3 + \&c.$ (t being $= p+m+1$,)

Here the Values of $m+1$, n and p are supposed positive; * and it is requisite that $1 + \frac{ak}{bd}$ should also be positive; otherwise the Fluent will fail. Although the Series brought out above runs on to Infinity, yet it may be sum'd, in many Cases: Thus, if the given Fluxion be $\frac{\overline{a-bz^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1} \dot{z}}{d+kz^n}$; then, the foresaid Series becoming $1 - \frac{1}{2} \times \frac{ak}{bd} + \frac{1}{2} \times \frac{3}{4} \times \overline{\frac{ak}{bd}}|^2 - \&c.$ its Sum will be $\overline{1 + \frac{ak}{bd}}|^{-\frac{1}{2}}$: And consequently $\frac{A}{d} \times \overline{1 + \frac{ak}{bd}}|^{-\frac{1}{2}}$ = the Fluent sought; Where, A (the *whole* Fluent of $\overline{a-bz^n}|^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1} \dot{z}$ being $= \frac{1}{n\sqrt{b}} \times$ Semi-Periphery of the Circle whose Radius is Unity, the Fluent given above will, therefore, be $= \frac{1}{n\sqrt{bd^2 + adk}}$ $\times$ by the same Semi-Periphery. If the Reader is desirous to see a further Application of the Summation of Serieses, to the finding of Fluents, I must refer him to my *Dissertations* (where it is handled in a general Manner) having neither Room nor Inclination to treat of it here.

* Art. 286.

SECTION IV.

Of the Transformation of Fluxions.

301. BY the Transformation of Fluxions may be understood, the reducing *any* fluxional Quantity to a different, or more commodious, Form; according to which Sense, a great Part of the second Section would properly fall under this Head. But, *what* is here proposed, and *what* is commonly meant by the Transformation of Fluxions, *is*, the Method of ordering those Kinds of Expressions which involve one variable Quantity *only* with its Fluxion; which, yet, are so affected by radical Signs, that the Fluent, without an Infinite Series, would be impracticable, were it not for a new Substitution, or some other Kind of Transformation, whereby the given Fluxion is render'd more manageable.

Something of this Sort has been already touch'd upon in *Art.* 83. And in what follows I shall farther point out and exemplify the principal Cases wherein such a *Procedure* will be of Service.

302. *If the Number of Dimensions of the variable Quantity, without the* Vinculum, *increased by Unity, be some aliquot Part, or Parts, of the Dimensions of the same Quantity, under the* Vinculum, *the Fluxion will be reduced to a better Form by substituting for that Power of the variable Quantity, which arises by dividing its Exponent, under the* Vinculum, *by the Denominator of the Fraction expressing the said* aliquot *Part, or Parts.*

Thus, if the Fluxion propounded be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{c^n \pm z^n}}$; by substituting $x = z^{\frac{1}{2}n}$, and taking the Fluxion of both Sides of the Equation, we have $\dot{x} = \frac{1}{2}nz^{\frac{1}{2}n-1}\dot{z}$; and therefore $z^{\frac{1}{2}n-1}\dot{z} = \frac{\dot{x}}{\frac{1}{2}n}$: Which Value, with that of z^n, being wrote for their Equals, in the given Fluxion, it will be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{c^n \pm x^2}}$: Which, putting $a = c^{\frac{1}{2}n}$ (to make the Terms homologous), is alſo expreſſed by $\frac{\dot{x}}{\frac{1}{2}n\sqrt{a^2 \pm x^2}}$: Whereof the Fluent will be given by *Art.* 126. or *Art.* 142. according as the Sign of x^2 is poſitive or negative.

303. If the Power of the variable Quantity under the *Vinculum* has a Coefficient, it will be beſt to bring that Coefficient without the *Vinculum.*

Ex. 2. Where let the Fluxion given be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a+cz^n}}$: Which, by bringing c without the *Vinculum*, becomes $\frac{z^{\frac{1}{2}n-1}\dot{z}}{c^{\frac{1}{2}}\sqrt{\frac{a}{c}+z^n}}$: From whence, by putting $x = z^{\frac{1}{2}n}$ and proceeding as above, we get $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^2}}$: Whoſe Fluent, *by Art.* 126. is $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}} \times$ *hyp. Log.* $x + \sqrt{\frac{a}{c}+x^2}$. This, by reſtoring z, becomes $\frac{2}{nc^{\frac{1}{2}}} \times$ *hyp. Log.* $z^{\frac{1}{2}n} + \sqrt{\frac{a}{c}+z^n}$. Which, corrected (by ſuppoſing it $= 0$ when $z = 0$) gives, at length, $\frac{2}{nc^{\frac{1}{2}}} \times \overline{\textit{hyp. Log. } z^{\frac{1}{2}n} + \sqrt{\frac{a}{c}+z^n} - \textit{hyp. Log. } \sqrt{\frac{a}{c}}} = \frac{2}{nc^{\frac{1}{2}}} \times$ *hyp. Log.* $\sqrt{\frac{cz^n}{a}} + \sqrt{1+\frac{cz^n}{a}}$ for the true Fluent of the Quantity propoſed.

But,

But, when c is a negative Quantity, this Fluent fails, becauſe the ſquare Root of c is to be extracted. In this Caſe $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^2}}$ muſt be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{-c}\times\sqrt{\frac{a}{-c}-x^2}}$: And then its Fluent (*by Art.* 142.) will be had $=\frac{1}{\frac{1}{2}n\sqrt{-c}}\times$ the *Arch* of a Circle whoſe Radius is Unity, and Right-Sine $=$ $\frac{x}{\sqrt{\frac{a}{-c}}}=\sqrt{\frac{-cz^n}{a}}$.

Ex. 3. Let the given Fluxion be $\frac{\dot{z}}{z\sqrt{a+cz^n}}$. Which, by bringing c without the *Vinculum*, and putting $x=z^{\frac{1}{2}n}$, is transformed to $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}x\sqrt{\frac{a}{c}+x^2}}$: Whereof the Fluent, *by Art.* 126. is $\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{\frac{a}{c}}-\sqrt{\frac{a}{c}+x^2}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{a}{c}+x^2}}=\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{a}-\sqrt{a+cz^n}}{\sqrt{a}+\sqrt{a+cz^n}}$. But here, when c is poſitive, the Numerator will be negative; in which Caſe it will be proper to change its Signs, and expreſs the Fluent by $\frac{1}{n\sqrt{a}}\times$ *hyp. Log.* $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{a+cz^n}+\sqrt{a}}$. That, ſuch

an

an Alteration of the Signs can make no Difference in the Fluxion, is evident from the Nature of Logarithms; because the Fluxion of the *Log.* of $-x$ $\left(=\frac{-\dot{x}}{-x}=\frac{\dot{x}}{x}\right)$ is the same with *that* of the *hyp. Log.* of x: It will be proper to observe farther, that, instead of the Logarithm above derived, any one of the following, equal, Quantities may be taken; *viz. hyp. Log.* $\frac{\overline{\sqrt{a+cz^n}-\sqrt{a}}|^2}{cz^n}$ (found by multiplying both the Numerator and Denominator of the foresaid Logarithm by $\sqrt{a+cz^n}-\sqrt{a}$) $=2\times$ *hyp. Log.* $\frac{\sqrt{a+cz^n}-\sqrt{a}}{\sqrt{cz^n}}$ (by the Nature of Logarithms) $=2\times$ *hyp. Log.* $\frac{\sqrt{cz^n}}{\sqrt{a+cz^n}+\sqrt{a}}$ (by multiplying, equally, by $\sqrt{a+cz^n}+\sqrt{a}$) But, take which of these Forms you will, the Fluent fails when a is negative; because the general Multiplicator $\frac{1}{n\sqrt{a}}$ is then impossible. In this Case the Fluent of $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\times x\sqrt{\frac{a}{c}+x^2}}$, or its Equal $\frac{\dot{z}}{z\sqrt{a-cz^n}}$, will be given *by Art.* 142. and is expounded by $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{-a}{c}}}$ $\times A=\frac{2}{n\sqrt{-a}}$; where A denotes the *Arch* whose Radius is Unity, and Secant $\frac{x}{\sqrt{\frac{-a}{c}}}\left(=\sqrt{\frac{cz^n}{-a}}\right)$.

In

In the same Manner the Fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{a+cz^n}$, is found $= \frac{1}{n\sqrt{ac}} \times$ *Arch*, whose Radius is Unity and Tangent $\sqrt{\frac{cz^n}{a}}$, or equal to $\frac{1}{n\sqrt{-ca}} \times$ *hyp. Log.* $\frac{\sqrt{a}+\sqrt{-cz^n}}{\sqrt{a}-\sqrt{-cz^n}}$, according as the Value of c is affirmative or negative; a being supposed affirmative.

304. *When the Power, or Powers, of the variable Quantity without the* Vinculum, *or radical Sign, fall, mostly, in the Denominator, it may be of Use to substitute for the Reciprocal of the said Quantity, or for the Quotient which arises by dividing some known Quantity, either, by it, or by some Compound of it in the Denominator.*

Ex. 1. *Let the proposed Fluxion be* $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}}$; then, putting $x = \frac{a^2}{z}$, we have $z = \frac{a^2}{x}$, and $\dot{z} = -\frac{a^2\dot{x}}{x^2}$; and consequently $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}} = \frac{-x\dot{x}}{\sqrt{x^2+a^2}}$: Whereof the Fluent is $-\sqrt{x^2+a^2} = -\sqrt{\frac{a^4}{z^2}+a^2}$.

Ex. 2. *Let the given Fluxion be* $\frac{z\dot{z}}{\overline{a+z}|^3 \times \sqrt{a^2+az+z^2}}$.

Here, putting $x = \frac{a^2}{a+z}$, we have $z = \frac{aa-ax}{x} = a \times \frac{a-x}{x}$, $\dot{z} = -\frac{a^2\dot{x}}{x^2}$, $z\dot{z} = -\frac{a^3\dot{x} \times \overline{a-x}}{x^3}$, $\sqrt{a^2+az+z^2} = \frac{a}{x}\sqrt{a^2-ax+x^2}$; and therefore the

Quantity

Quantity propofed is transformed to $\frac{x^2\dot{x}-ax\dot{x}}{a^4\sqrt{a^2-ax+x^2}}$: Whofe Fluent may be found from a Table of Logarithms; as will appear farther on.

305. *If the Fluxion given is affected by two different Surds, and the rational Factor, or the Quantity without the* Vinculum, *be in a conftant Ratio to the Fluxion of the Quantity under the* Vinculum *of either Surd, or be related to it as in* Art. 83. *the given Fluxion will be reduced to a more fimple Form, by fubftituting for that Surd.*

Ex. 1. *Let* $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}$ *be propounded.*

Then, putting $x=\sqrt{b^2+z^2}$, we have $z^2=x^2-b^2$, $z\dot{z}=x\dot{x}$, and $\sqrt{c^2-z^2}=\sqrt{c^2+b^2-x^2}=\sqrt{a^2-x^2}$ (by making $a=\sqrt{c^2+b^2}$) Whence $\frac{z\dot{z}\sqrt{b^2+z^2}}{\sqrt{c^2-z^2}}=$

* Art. 279. $\frac{x^2\dot{x}}{\sqrt{a^2-x^2}}$ *:

Or, if x be put $=\sqrt{c^2-z^2}$ (inftead of $\sqrt{b^2+z^2}$); then $z^2=c^2-x^2$, $z\dot{z}=-x\dot{x}$, $\sqrt{b^2+z^2}=\sqrt{b^2+c^2-x^2}=\sqrt{a^2-x^2}$; and confequently $\frac{z\dot{z}\sqrt{bb+zz}}{\sqrt{cc-zz}}=-\dot{x}\sqrt{a^2-x^2}$: Whofe Fluent is given by *Art.* 297. or 131.

Ex. 2. *Let the given Fluxion be* $\overline{a+cz^n}|^m\times\overline{e+fz^n}|^r\times$

† Art. 83. $z^{pn-1}\dot{z}$; *fuppofing p to denote any whole pofitive Number* †:

In this Cafe, let that of the two Quantities, $a+cz^n$ and $e+fz^n$, whofe Index (m or r) is the moft complex (which we will fuppofe the latter) be put $=x$; then we fhall have $z^n=\frac{x-e}{f}$; $z^{n-1}\dot{z}=\frac{\dot{x}}{nf}$;

$z^{pn-1}\dot{z} \left(= z^{pn-n} \times z^{n-1}\dot{z}\right) = \frac{\overline{x-e}|^{p-1}}{f^{p-1}} \times \frac{\dot{x}}{nf}$;

$a + cz^n = a + \frac{cx - ce}{f} = d + \frac{cx}{f}$ by putting $d = a -$ $\left.\frac{ce}{f}\right)$ and consequently $\overline{d + \frac{cx}{f}}|^m \times \frac{\overline{x-e}|^{p-1} \times x^r \dot{x}}{nf^p}$ = the Fluxion proposed: Where, $p-1$ being a whole positive Number, the Value of $\overline{x-e}|^{p-1}$ will therefore be expressed in finite Terms; whence, if m be also a whole positive Number, the Fluent itself will be had in finite Terms: But, if m and r be the Halves of odd Numbers, then the Fluent will be found (*from Art.* 298 or 294.) by means of circular Arcs and Logarithms.

306. *If the given Expression be affected by two Surds wherein the Powers of the variable Quantity are the same, and the rational Quantity, without the* Vinculums, *be related to the Fluxion of either Surd, as in Art.* 83, *it may be of Use to substitute for the Quotient, or Ratio, of the two Quantities under the radical Signs; especially, if the Sum of the said radical Signs, or Exponents (supposing both Surds to be reduced to the Denominator) is a whole Number.*

Ex. 1. *Let the given Fluxion be* $\frac{z^2\dot{z}}{\overline{b^3+z^3}|^{\frac{2}{3}} \times \overline{c^3-z^3}|^{\frac{4}{3}}}$.

Then, writing $x = \frac{b^3+z^3}{c^3-z^3}$, we have $z^3 = \frac{c^3x - b^3}{1+x}$;

$3z^2\dot{z} = \frac{\overline{c^3+b^3} \times \dot{x}}{\overline{1+x}|^2}$; $\overline{b^3+z^3}|^{\frac{2}{3}} \times \overline{c^3-z^3}|^{\frac{4}{3}} \left(= \overline{\frac{b^3+z^3}{c^3-z^3}}|^{\frac{2}{3}}\right.$ $\left.\times \overline{c^3-z^3}|^2 = x^{\frac{2}{3}} \times \overline{c^3 - \frac{c^3x-b^3}{1+x}}|^2\right) = \frac{\overline{b^3+c^3}|^2 \times x^{\frac{2}{3}}}{\overline{1+x}|^2}$;

 and

and consequently $\frac{z^2\dot{z}}{\overline{b^3+z^3}|^{\frac{2}{3}}\times\overline{c^3-z^3}|^{\frac{4}{3}}} = \frac{x^{-\frac{2}{3}}\dot{x}}{3\times b^3+c^3}$;

Whose Fluent is $\frac{x^{\frac{1}{3}}}{b^3+c^3} = \frac{1}{b^3+c^3}\times\sqrt[3]{\frac{b^3+z^3}{c^3-z^3}}$.

Ex. 2. *Let there be given* $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^{m}\times\overline{e+fz^n}|^{r}}$

Here, putting $x = \frac{e+fz^n}{a+cz^n}$, you will have $z^n = \frac{ax-e}{f-cx}$; $nz^{n-1}\dot{z} = \frac{\overline{af-ce}\times\dot{x}}{\overline{f-cx}|^2}$; $z^{pn-1}\dot{z}$ $(= z^{pn-n}\times z^{n-1}\dot{z}) = \overline{\frac{ax-e}{f-cx}}|^{p-1}\times\frac{\overline{af-ce}\times\dot{x}}{n\times\overline{f-cx}|^2}$; $\overline{a+cz^n}|^{m}\times\overline{e+fz^n}|^{r}$ $(= \overline{a+cz^n}|^{m+r}\times\frac{\overline{e+fz^n}|^{r}}{\overline{a+cz^n}|^{r}} = \overline{a+c\times\frac{ax-e}{f-cx}}|^{r+m}\times x^r) = \overline{\frac{af-ce}{f-cx}}|^{m+r}\times x^r$; and consequently the

Fluxion given $= \frac{\overline{ax-e}|^{p-1}\times\overline{f-cx}|^{m+r-p-1}\times x^{-r}\dot{x}}{n\times\overline{af-ce}|^{m+r-1}}$.

Where, if $m+r$ be a whole positive Number, greater than p (also a whole positive Number) the Fluent will be truly had in finite Terms; because both the Serieses for the Values of $\overline{ax-e}|^{p-1}$ and $\overline{f-cx}|^{m+r-p-1}$ do in that Case terminate *. But, if r and $m+r-p-1$ be the Halves of whole Numbers, positive or negative, then the Fluent will be given by the last Section.

* Art. 99.

307. *A Trinomial is reduced to a Binomial by taking away its middle Term; that is, by substituting for the Sum or Difference of the Power of the variable Quantity in*

in that Term and half its Coefficient; according as the Signs of the two Terms, where the said Quantity is found, are like, or unlike.

Ex. 1. *Let the given Fluxion be* $\frac{\dot{z}}{\sqrt{b^2+cz+z^2}}$; then, putting $x=z+\frac{1}{2}c$, or $z=x-\frac{1}{2}c$, we have $\dot{z}=\dot{x}$, and $\sqrt{b^2+cz+z^2}$ $(=\sqrt{b^2+cx-\frac{1}{2}c^2+x^2-cx+\frac{1}{4}c^2})$ $=\sqrt{b^2-\frac{1}{4}c^2+x^2}$; whence (making $a^2=b^2-\frac{1}{4}c^2$) there results $\frac{\dot{z}}{\sqrt{b^2+cz+z^2}}=\frac{\dot{x}}{\sqrt{a^2+x^2}}$: Whose Fluent is given, *by Art.* 126.

Ex. 2. *Let the Fluxion given be* $\frac{fz^{n-1}\dot{z}}{\sqrt{a+bz^n+cz^{2n}}}$.

First, by bringing c without the *Vinculum*, according to *Art.* 303. we have $\sqrt{a+bz^n+cz^{2n}}=\sqrt{c}\times\sqrt{\frac{a}{c}+\frac{bz^n}{c}+z^{2n}}$: And, by putting $x=z^n+\frac{b}{2c}$, or $z^n=x-\frac{b}{2c}$, we also get $z^{n-1}\dot{z}=\frac{\dot{x}}{n}$, and $\sqrt{\frac{a}{c}+\frac{bz^n}{c}+z^{2n}}$ $(=\sqrt{\frac{a}{c}+\frac{bx}{c}-\frac{bb}{2cc}+x^2-\frac{bx}{c}+\frac{bb}{4cc}})$ $=\sqrt{\frac{a}{c}-\frac{bb}{4cc}+x^2}$: Therefore the Fluxion, transformed, is $\frac{f\dot{x}}{n\sqrt{c}\times\sqrt{\frac{a}{c}-\frac{bb}{4cc}+x^2}}$: Whose Fluent is given by *Art.* 126. when c is a positive Quantity: But, when c is negative, the Fluxion must be expressed thus, $\frac{f\dot{x}}{n\sqrt{-c}\times\sqrt{\frac{a}{-c}+\frac{bb}{4cc}-x^2}}$: Answering to *Form* 2. *Art.* 142.

Ex. 3. *Let* $\frac{fz^{n-1}\dot{z}+gz^{2n-1}\dot{z}+hz^{3n-1}\dot{z}+kz^{4n-1}\dot{z}.}{\overline{a+bz^n+cz^{2n}}|^m}$

be proposed.

Then, following the Steps of the last Example, $\overline{a+bz^n+cz^{2n}}|^m$ $(=c^m\times\overline{\frac{a}{c}+\frac{bz}{c}+z^{2n}}|^m)$ will be transformed to $c^m\times\overline{\frac{a}{c}-\frac{bb}{4cc}+x^2}|^m$: Moreover, z^n being $= x-\frac{b}{2c}=x-d$ (by putting $d=\frac{b}{2c}$) and $z^{n-1}\dot{z}=\frac{\dot{x}}{n}$, we also have z^{2n-1} $(= z^n\times z^{n-1}\dot{z}=\overline{x-d}\times\frac{\dot{x}}{n})=\frac{x\dot{x}-d\dot{x}}{n}$; $z^{3n-1}\dot{z}$ $(= z^{2n}\times z^{n-1}\dot{z})=\overline{x-d}|^2\times\frac{\dot{x}}{n}=\frac{x^2\dot{x}-2dx\dot{x}+d^2\dot{x}}{n}$; &c. &c. From whence, by substituting these several Values in the given Fluxion, and putting $\frac{a}{c}-\frac{bb}{4cc}=e^2$, there comes out

$$\frac{f\dot{x}+g\times\overline{x\dot{x}-d\dot{x}}+h\times\overline{x^2\dot{x}-2dx\dot{x}+a^2\dot{x}}+\&c.}{nc^m\times\overline{ee+xx}|^m}:$$

Whose Fluent, when the Exponent m is the Half of any *Integer*, positive or negative, will be found, by means of circular Arcs and Logarithms, from *Art.* 295.

308. *When the Denominator is a rational Trinomial, or Multinomial (that is, when it is without a* Vinculum) *the best Way of proceeding, for the general Part, is, to resolve the given Fraction into binomial Ones. In order to this, let its Denominator be feigned* = 0; *by means*

means of which Equation, whose Roots must be found, you will, by subtracting each Root from the indeterminate Quantity (x), *have the binomial Denominators, of the required Fractions into which the given* One *may be resolved: Whose corresponding Numerators, let be denoted* $A\dot{x}$, $B\dot{x}$, $C\dot{x}$ &c. *then, by putting the Sum of the Fractions, thus arising, equal to the given Fraction, and reducing the whole Equation to the same Denominator, the assumed Quantities* A, B, C *&c. by comparing the homologous Terms, will be determined.*

Ex. 1. *Let the given Fraction be* $\frac{\dot{x}}{x^2+ax+b}$; then, feigning $x^2+ax+b=0$, the two Roots of the Equation will be $-\frac{1}{2}a-\sqrt{\frac{1}{4}a^2-b}$, and $-\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-b}$: Which being denoted by p and q, we have $x-p$ and $x-q$ for the two binomial Factors whereby x^2+ax+b may be resolved, or by whose Multiplication $(\overline{x-p} \times \overline{x-q})$ the said Quantity is produced.

Let therefore $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}$ be now assumed $\left(=\frac{\dot{x}}{x^2+ax+b}\right)=\frac{\dot{x}}{\overline{x-p}\times\overline{x-p}}$; then, by reducing the whole Equation to one Denomination &c. we get $\overline{A+B}\times x\dot{x}-\overline{qA+pB+1}\times\dot{x}=0$: Whence A is found $=\frac{1}{p-q}$, $B=\frac{1}{q-p}$; and, consequently,

$$\frac{\dot{x}}{\overline{p-q}\times\overline{x-p}}+\frac{\dot{x}}{\overline{q-p}\times\overline{x-q}}=\frac{\dot{x}}{x^2+ax+b}.$$

Ex. 2. *Let the Quantity proposed be* $\frac{x^2\dot{x}}{x^3+ax^2+bx+c}$.

Here, if the binomial Factors whereby x^3+ax^2+bx+c is produced be represented by $x-p$, $x-q$, and $x-r$, and there be assumed $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}$

$\left(=\frac{x^2\dot{x}}{x^3+ax^2+bx+c}\right)=\frac{x^2\dot{x}}{\overline{x-p}\times\overline{x-q}\times\overline{x-r}}$; then, in this Case, we shall have $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2=0$; that is, by Reduction,

$$\left.\begin{matrix}A\\B\\C\\-1\end{matrix}\right\}\times x^2-\left.\begin{matrix}\overline{q+r}\times A\\\overline{p+r}\times B\\\overline{p+q}\times C\end{matrix}\right\}\times x+\left.\begin{matrix}qrA\\prB\\pqC\end{matrix}\right\}=0.$$

Whence $A+B+C=1$, $A\times\overline{q+r}+B\times\overline{p+r}+C\times\overline{p+q}=0$, and $Aqr+Bpr+Cpq=0$. Now, from the first of these Equations, multiply'd by $p+q$, subtract the second, and you will have $A\times\overline{p-r}+B\times\overline{q-r}=p+q$: Also, from the first, multiply'd by pq, subtract the third; then $A\times\overline{pq-rq}+B\times\overline{pq-pr}=pq$: Lastly, from the former of the two Equations thus arising, multiply'd by p, subtract the latter, then $A\times\overline{pp-pr-pq+qr}=pp$, that is, $A\times\overline{p-q}\times\overline{p-r}=p^2$; and consequently $A=\frac{p^2}{\overline{p-q}\times\overline{p-r}}$: Whence, by the very same Argument,

$$B=\frac{q^2}{\overline{q-p}\times\overline{q-r}},\text{ and }C=\frac{r^2}{\overline{r-p}\times\overline{r-q}}.$$

309. After the same Manner you may proceed in other Cases: But there is an Artifice, or Compendium, for more readily determining the assumed Quantities A, B, C &c. by which a great deal of Trouble is avoided: And that is, by considering the Equation in such Circumstances of the indeterminate Quantity x, when it becomes most simple, or when most of its Terms vanish.

Thus, in the preceding Example, because $A\times\overline{x-q}\times\overline{x-r}+B\times\overline{x-p}\times\overline{x-r}+C\times\overline{x-p}\times\overline{x-q}-x^2$ is $=0$ (in all Circumstances of x whatever) let x be taken $=p$; then, all the Terms vanishing, except the first and last,

we

we have $A \times \overline{p-q} \times \overline{p-r} - p^2 = 0$; and consequently $A = \frac{p^2}{\overline{p-q} \times \overline{p-r}}$; *the very same as before:*

More universally, let the given Fraction be

$$\frac{x^m \dot{x}}{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \&c.} = \frac{x^m \dot{x}}{\overline{x-p} \times \overline{x-q} \times \overline{x-r} \times \overline{x-s} \&c.}$$

(where m and n may represent any whole positive Numbers whatever, provided the latter be greater than the former.) Then, assuming $\frac{A\dot{x}}{x-p} + \frac{B\dot{x}}{x-q} + \frac{C\dot{x}}{x-r} + \frac{D\dot{x}}{x-s}$ &c. $= \frac{x^m \dot{x}}{x^n + ax^{n-1} + bx^{n-2} \&c.}$ &c. we shall have $A \times \overline{x-q} \times \overline{x-r} \times \overline{x-s}$ &c. $+ B \times \overline{x-p} \times \overline{x-r} \times \overline{x-s}$ &c. $+ C \times \overline{x-p} \times \overline{x-q} \times \overline{x-s}$ &c. &c. $- x^m = 0$: From whence, by expounding x by p, q, r &c. successively, we obtain $A = \frac{p^m}{\overline{p-q} \cdot \overline{p-r} \cdot \overline{p-s} \&c.}$, $B = \frac{q^m}{\overline{q-p} \cdot \overline{q-r} \cdot \overline{q-s} \&c.}$, $C = \frac{r^m}{\overline{r-p} \cdot \overline{r-q} \cdot \overline{r-s} \&c.}$, &c. Whence the Fractions themselves, whereof these Quantities are the Coefficient, or Numerators, will likewise be given.

But the Numerators thus found may, sometimes, be more commodiously expressed by Help of the given Coefficients, a, b, c, d &c. so as to involve only one of the Roots p, q, r &c. in each Fraction. For, since $\overline{x-p} \times \overline{x-q} \times \overline{x-r}$ &c. is supposed, *universally*, $= x^n + ax^{n-1} + bx^{n-2} + cx^{n-3}$ &c. if both Sides of the

Equation be divided by $x-p$, we shall have $\overline{x-q} \times \overline{x-r} \times \overline{x-s}$ &c. $= \frac{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ \&c.}}{x-p}$ Which last Expression, when x is $= p$, that is, when both the Numerator and the Denominator become equal to Nothing, will, manifestly, be equal to $(\overline{p-q} \times \overline{p-r} \times \overline{p-s}$ &c.) the Divisor of A. Therefore, if the Fluxion of the Numerator be taken and divided by that of the Denominator, and p be wrote instead of x (*vid.* *Page* 155.) we shall have $np^{n-1} + \overline{n-1} \times ap^{n-2} + \overline{n-2} \times bp^{n-3}$ &c. $= \overline{p-q} \times \overline{p-r} \times \overline{p-s}$ &c. and therefore $A \left(= \frac{p^m}{\overline{p-q} \cdot \overline{p-r} \cdot \overline{p-s} \text{ \&c.}}\right) =$

$$\frac{p^m}{np^{n-1} + \overline{n-1} \cdot ap^{n-2} + \overline{n-2} \cdot bp^{n-3} \text{ \&c.}}.$$ By the very same Reasoning $B =$

$$\frac{q^m}{nq^{n-1} + \overline{n-1} \cdot aq^{n-2} + \overline{n-2} \cdot bq^{n-3} \text{ \&c.}} \quad C =$$

$$\frac{r^m}{nr^{n-1} + \overline{n-1} \cdot ar^{n-2} + \overline{n-2} \cdot br^{n-3} \text{ \&c.}} \text{ \&c.}$$

Hence it appears, that, if the Numerator of the given Fraction be divided by the Fluxion of the Denominator (neglecting $\dot{x}$) and the several Roots p, q, r &c. (found by feigning the Denominator $= 0$) be, successively, substituted in the Quotient, instead of x; I say, it is evident, that the Quantities so resulting, divided by $x-p$, $x-q$, $x-r$ &c. will be the required, binomial, Fractions into which the proposed multinomial One may be resolved.

310. If some of the Roots p, q, r &c. are impossible, which is often the Case, the Fractions thus found, where the impossible Roots are concerned, must be

be united in Pairs, and ſo reduced to trinomial Ones, in order to take away the *imaginary Terms.*

Thus, let the Fraction propoſed be $\frac{x\dot{x}}{x^3+ax^2+bx+c}$, and let two of the Roots, p and q, of the Equation $x^3+ax^2+bx+c=0$ be impoſſible: Then, $\frac{A\dot{x}}{x-p}+\frac{B\dot{x}}{x-q}+\frac{C\dot{x}}{x-r}$ being $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$, we ſhall, by uniting the *imaginary Terms*, have $\frac{\overline{A+B}\times x\dot{x}-\overline{Aq+Bp}\times\dot{x}}{x^2-\overline{p+q}\times x+pq}+\frac{C\dot{x}}{x-r}$, alſo, $=\frac{x\dot{x}}{x^3+ax^2+bx+c}$; where the impoſſible Quantities deſtroy one another. But, to render this more obvious, let a be taken $=0$, $b=0$, and $c=-1$, ſo that the given Fraction may become $\frac{x\dot{x}}{x^3-1}$; then the three Roots (p, q, r) of the Equation, $x^3-1=0$, will here be $-\frac{1}{2}+\sqrt{\frac{-3}{4}}$, $-\frac{1}{2}-\sqrt{\frac{-3}{4}}$, and 1; whereof the two former are impoſſible. Moreover, by dividing the Numerator (x) by the Fluxion of the Denominator $(3x^2)$ (according to the *Preſcript)* we have $\frac{1}{3x}$; which, by writing p, q, r ſucceſſively, inſtead of x, becomes $\frac{1}{3p}$, $\frac{1}{3q}$ and $\frac{1}{3r}$ for the Values of A, B, and C, reſpectively. Whence $\frac{\overline{A+B}\times x-Aq-Bp}{x^2-\overline{p+q}\times x+pq}+\frac{C}{x-r}\left(=\frac{x}{x^3-1}\right)$ is $=\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}+\frac{\frac{1}{3}}{x-1}=\frac{1-x}{3x^2+3x+3}+\frac{1}{3x-3}$. But the ſame may be, other-

wiſe,

wise, investigated, in a more general Manner; by assuming $\frac{Px+Q}{x^2+x+1}+\frac{R}{x-1}=\frac{x}{x^3-1}$, and proceeding as in the first and second Examples; whence the very same Conclusion will be derived.

If the Fraction proposed be of this Form, *viz.* $\frac{z^{pn-1}\dot{z}}{z^{mn}+az^{mn-n}+bz^{mn-2n}\ \&c.}$, the Method of Resolution will, *still*, be the same: Since, by putting $x=z^n$, the given Expression is reduced to

$$\frac{\frac{1}{n}\times x^{p-1}\dot{x}}{x^m+ax^{m-1}+bx^{m-2}\ \&c.}$$

It may also be proper to observe, *that*, in very complicated Cases, the Application of two, or more, of the six foregoing Rules, may become necessary. Thus, for Example, if the Fluxion given be

$\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{e+fz^n+gz^{2n}}}$; by resolving $\frac{1}{e+fz^n+gz^{2n}}$ into two Binomial Fractions, $\frac{A}{h+z^n}+\frac{B}{k+z^n}$ (*according to Art.* 308.) we shall have $\frac{z^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{e+fz^n+gz^{2n}}}$

$=\frac{Az^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{h+z^n}}+\frac{Bz^{pn-1}\dot{z}}{\overline{a+cz^n}|^m\times\overline{k+z^n}}$: Where, if m be a whole positive Number, greater than p, the Fluent will be had in finite Terms (*by Art.* 306. *Ex.* 2.)

SEC-

SECTION V.

The Investigation of Fluents of Rational Fractions, of several Dimensions, according to the Forms *in* Cotes's HARMONIA MENSURARUM.

311. AS the Subject here proposed is a Matter of considerable Difficulty, and has exercised the Attention of some of the most celebrated Mathematicians (who, yet, seem to have condescended very little to the Information of their less experienced Readers) I shall endeavour to set it in the clearest Light possible: In order to which, it will be requisite to premise the following Lemmas.

LEMMA I.

If the Sine of the Mean of three equi-different Arcs, supposing Radius Unity, be multiplied by the Double of the Co-sine of the common Difference, and from the Product, the Sine of the lesser Extreme be subtracted, the Remainder will be the Sine of the greater Extreme.

LEMMA II.

312. *If G be taken to denote the greater, and L the lesser, of two unequal Arcs, and their Difference be expressed by D; then will,*

1. $\frac{Sin.\ G. \times Co\text{-}s.\ D - Sin.\ L. \times Rad.}{Sin.\ D} = Co\text{-}s.\ G$

2. $\frac{Co\text{-}s.\ L \times Rad. - Co\text{-}s.\ G \times Co\text{-}s.\ D}{Sin.\ D} = Sin.\ G$

3. $\frac{Sin.\ G. \times Rad. - Sin.\ L \times Co\text{-}s.\ D}{Sin.\ D} = Co\text{-}s.\ L.$

The former of these two Lemmas may be met with in most Authors upon Trigonometry; and the latter is nothing more than a Corollary to the *common* Theorems for finding the Sine and Co-sine of the Sum and Difference of two given Arcs; for which Reasons I shall not stop here to give their Demonstration.

COROLLARY.

313. If any Arch of the Circle, whose Radius is Unity, be denoted by $\mathcal{Q}$, its Sine by s, and its Co-sine by a; and there be taken $A=2a$, $B=2aA-1$, $C=2aB-A$, $=D=2aC-B$, $E=2aD-C$, $F=2aE-D$, *&c.* it follows (*from Lemma* 1.) that,

$Sin.\ 2\mathcal{Q}\ (Sin.\ \mathcal{Q}\times 2a-Sin.\ 0)=2sa-0=sA$

$Sin.\ 3\mathcal{Q}\ (Sin.\ 2\mathcal{Q}\times 2a-Sin.\ \mathcal{Q})=2sAa-s=sB$

$Sin.\ 4\mathcal{Q}\ (Sin.\ 3\mathcal{Q}\times 2a-Sin.\ 2\mathcal{Q})=2sBa-sA=sC$

$Sin.\ 5\mathcal{Q}\ (Sin.\ 4\mathcal{Q}\times 2a-Sin.\ 3\mathcal{Q})=2sCa-sB=sD$

$Sin.\ 6\mathcal{Q}\ (Sin.\ 5\mathcal{Q}\times 2a-Sin.\ 4\mathcal{Q})=2sDa-sC=sE$

&c. *&c.*

LEMMA III.

314. *To resolve the Trinomial* $r^{2n}-2kr^nx^n+x^{2n}$, *where* n *is any whole Number, into simple trinomial Factors.*

Since the first Term of the given Quantity $r^{2n}-2kr^nx^n+x^{2n}$ is divisible, only, by the Powers of r, and the last, only, by those of x; and it appears that r and x are concerned, exactly, alike; let therefore $r^2-2arx+x^2$ (where r and x are, *also*, alike concerned) be assumed for one of the required trinomial Factors, whereby $r^{2n}-2kr^nx^n+x^{2n}$ may be resolved: And let $\overline{r^2-2arx+x^2}\times\overline{r^8+Ar^7x+Br^6x^2+Cr^5x^3+Dr^4x^4+Cr^3x^5+Br^2x^6+Arx^7+x^8}$ (where r and x are, *still*, affected alike) be assumed $=r\quad-2kr^5x^5+x^{10}$ (the Value of n, to render the Operation more perspicuous, being first expressed by 5.)

Then,

Then, by Multiplication and Transposition, we shall have

$$\left.\begin{array}{l} r^{10} + Ar^9x + Br^8x^2 + Cr^7x^3 + Dr^6x^4 + Cr^5x^5 + Br^4x^6 + Ar^3x^7 + r^2x^8 \ldots * \ldots * \\ * - 2ar^9x - 2aAr^8x^2 - 2aBr^7x^3 - 2aCr^6x^4 - 2aDr^5x^5 - 2aCr^4x^6 - 2aBr^3x^7 - 2aAr^2x^8 - 2arx^9 \ldots * \\ * \quad * + r^8x^2 + Ar^7x^3 + Br^6x^4 + Cr^5x^5 + Dr^4x^6 + Cr^3x^7 + Br^2x^8 + Arx^9 + x^{10} \\ -r^{10} \ldots\ldots\ldots\ldots + 2kr^5x^5 \ldots\ldots\ldots\ldots - x^{10} \end{array}\right\} = 0$$

Whence

Whence, $A=2a$, $B=2Aa-1$, $C=2aB-A$, $D=2aC-B$, and $2C-2aD+2k=0$. But, if Q be taken to denote the Arch (EF) of a Circle EHK, whose Radius EO is Unity, and Co-sine $(Of)=a$; and s be put for (Ff) the Sine of the same Arch; then (*by Corol. to Lem.* 1.) $sA=Sin.\ 2Q$, $sB=Sin.\ 3Q$, $sC=Sin.\ 4Q$ &c. and consequently $A=\frac{Sin.\ 2Q}{s}$, $B=\frac{Sin.\ 3Q}{s}$, $C=\frac{Sin.\ 4Q}{s}$, $D=\frac{Sin.\ 5Q}{s}$ or, $\frac{Sin.\ nQ}{s}$). Moreover, because, $2C-2aD+2k=0$, or $D\times a-C\times 1=k$, where (as appears from above) $D\times a-C\times 1=\frac{Sin.\ 5Q\times Co\text{-}f.\ Q-Sin.4Q\times Rad.}{s}=Co\text{-}f.\ 5Q$: (*by Case* 1. *Lem.* 2.) we therefore have $Co\text{-}f.\ 5\ Q\ (n\ Q)=k$. Whence this Construction.

Take R to denote the Arch (EM) whose Co-sine (ON) is the given Co-efficient k, and let Q (EF) be taken to EM as 1 to n; then the Co-sine (Of) of this last Arch will be the true Value of a. But this is only one of the Values that a will admit of: For it is well known, that the Co-sine of any Arch, is also the Co-sine of the same Arch increased by any Number of Times the whole Periphery (P). Therefore, seeing the Co-sine of nQ (= Co-sine of R) is likewise = Co-sine $\overline{P+R}$ = Co-s. $\overline{2P+R}$ = Cos. $\overline{3P+R}$ &c. it follows that Q (whose Co-sine is a) will be expressed by any one of the Arcs, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$, $\frac{3P+R}{n}$ &c. (or by EF, EG, EH, EI,

&c.

&c. supposing the whole Periphery to be divided ino n equal Parts, from the Point F). Hence, if the Co-sines of these several Arcs, expressing all the different Values of a, be represented by b, c and d, *&c.* respectively, we shall have $r^2-2brx+x^2$, $r^2-2crx+x^2$, $r^2-2drx+x^2$, *&c.* for the several required Factors, by which $r^{2n}-2kr^nx^n+x^{2n}$ may be resolved; and consequently $\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\times\overline{r^2-2drx+x^2}\,(n)=r^{2n}-2kr^nx^n+x^{2n}$. *Q. E. I.*

Note, If the Sign of the middle Term $2kr^nx^n$ be positive, the Distance (or Co-sine) ON must be taken on the contrary Side of the Center: But when k is greater than Unity, this Method of Solution fails; since no Cosine can be greater than the Radius.

Corollary I.

315. If $k=1$, the Arch R (whose Co-sine is k) being $=0$, the Values of b, c, d, *&c.* will be expressed by the Co-sines of the Arcs $\frac{o}{n}$, $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ *&c.* respectively: And our general Equation will here become $r^{2n}-2r^nx^n+x^{2n}=\overline{r^2-2brx+x^2}\times\overline{r^2-2crx+x^2}\,\overline{r^2-2drx+x^2}\,(n)$. From whence, by extracting the Square-Root, on both Sides, we also have $r^n \backsim x^n=\overline{r^2-2brx+x^2}|^{\frac{1}{2}}\times\overline{r^2-2crx+x^2}|^{\frac{1}{2}}\,(n)$.

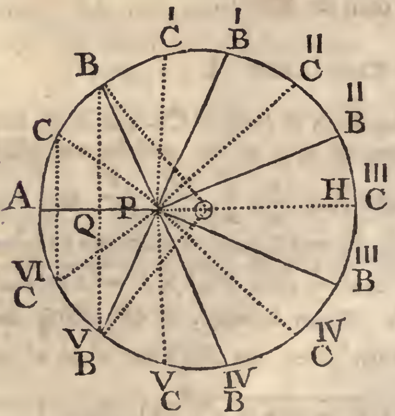
Corollary II.

316. But, if $k=-1$ (or the middle Term be $+2r^nx^n$) then the Arch R being $=\frac{P}{2}$, the Values of b, c, d, *&c.* will, *here*, be defined by the Cosines of the

the Arcs $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$, &c. and our Equation, by taking the Root, as above, will become $r^n + x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}}$ (n).

Scholium.

317. From the two preceding Corollaries, the Demonstration of that remarkable Property of the Circle given, and applied to finding a vast Number of Fluents, in *Cotes's Harmonia Mensurarum*, is very easily, and naturally, deduced.



For, let the Periphery of the Circle ABB' &c. whose Radius is expressed by r, be divided into as many equal Parts AB, BB', $B'B''$, &c. as there are Units in the given Integer n; so that AB, AB', AB'', &c. may respectively exhibit the Values of the foresaid Arcs $\frac{P}{n}$, $\frac{2P}{n}$, $\frac{3P}{n}$ &c. (*vid. Corol.* 1.) Moreover, let OQ be the Co-sine of the first of them; and, in the Radius OA (produced if necessary) let there be taken $OP = x$; and let OB, QB, PB, &c. &c. be drawn: Then, the Co-sine of the Angle AOB $(= \frac{P}{n})$ to the

the Radius 1, being expressed by c (*vid. Corol.* 1.) it will be $1 : c :: r$ (OB) : OQ $= cr$: Whence PB^2 ($= OB^2 + OP^2 - 2OQ \times OP$) $= r^2 + x^2 - 2crx = r^2 - 2crx + x^2$.

By the very same Argument $\overset{\prime}{P}B^2$ is $= r^2 - 2drx + x^2$, &c. &c. Therefore, because $r^n \backsim x^n = \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2drx + x^2}|^{\frac{1}{2}}$ (n), *by Corol.* 1. it follows that their Equals, $AO^n \backsim OP^n$ and $PA \times PB \times P\overset{\prime}{B} \times P\overset{\prime\prime}{B}$ &c. must be equal likewise: *Which is the first Part of the Theorem above hinted at.*

After the same Manner, if the Arcs AC, $A\overset{\prime}{C}$, $A\overset{\prime\prime}{C}$, $A\overset{\prime\prime\prime}{C}$ be taken respectively equal to $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$ &c. it will appear (*from Corol.* 2.) that $AO^n + PO^n$ is $= PC \times P\overset{\prime}{C} \times P\overset{\prime\prime}{C}$ (n) *Which is the latter Part of the same Theorem.*

Hence (by the Bye) all the Roots of the Equation $x^n = r^n$ are very readily found: For, since $AO^n \backsim PO^n = PA \times PB \times P\overset{\prime}{B}$ &c. where the second Factor and the last, the third and the last but one, &c. are respectively equal to each other, it is evident that $AO^n \backsim PO^n$ ($r^n \backsim x^n$) is also $= PA \times PB^2 \times P\overset{\prime}{B}{}^2 \times P\overset{\prime\prime}{B}{}^2 = \overline{r \backsim x} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2}$ &c. Whence, $x^n \backsim r^n$ being $= 0$, it follows that $\overline{r \backsim x} \times \overline{r^2 - 2crx + x^2}$ &c. is $= 0$: From which, by extracting the Roots out of the Equations $r \backsim x = 0$, $r^2 - 2crx + x^2 = 0$, $r^2 - 2drx + x^2 = 0$ &c. we get r, $r \times \overline{c + \sqrt{c^2 - 1}}$, $r \times \overline{c - \sqrt{c^2 - 1}}$, $r \times \overline{d + \sqrt{d^2 - 1}}$,

&c. for the several Roots of the Equation $x^n = r^n$; whereof the first, only, is possible when n is odd; and the first and last when n is even.

By the same Way of proceeding all the Roots of the Equation, $x^n + r^n = 0$, will also be found: For, seeing $x^n + r^n$ is $= \overline{r^2 - 2brx + x^2}|^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}|^{\frac{1}{2}}$ *&c.* $(= PC \times PC' \times PC''$ *&c.*) where the first Factor and the last, the second and the last but one, *&c.* are respectively equal to each other, it is plain that $x^n + r^n$ is likewise $= \overline{r^2 - 2brx + x^2} \times \overline{r^2 - 2crx + x^2}$ *&c.* and consequently $x = r \times \overline{b \pm \sqrt{b^2 - 1}}$ *&c.* *&c.* Where the Roots are all impossible; except the last, when their Number (n) is odd.

Lemma IV.

318. *Supposing every thing to remain as in the preceding* Lemma, *and that* k', b', c', d' &c. *denote the Sines of the Arcs* R, $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ *&c.* (*whose Co-sines are* k, b, c, d, &c.) *then, I say, the Fraction* $\frac{nk' r^n x^n}{r^{2n} - 2kr^n x^n + x^{2n}}$ *is equal to* $\frac{b' rx}{r^2 - 2brx + x^2} + \frac{c' rx}{r^2 - 2crx + x^2} + \frac{d' rx}{r^2 - 2drx + x^2}$ *&c.*

For, since $r^{10} - 2kr^5x^5 + x^{10}$ $(r^{2n} - 2kr^nx^n + x^{2n})$ is $= \overline{r^2 - 2arx + x^2} \times \overline{r^8 + Ar^7x + Br^6x^2 + Cr^5x^3 + Dr^4x^4 + Cr^3x^5 + Br^2x^6 + Arx^7 + x^8}$ (*by the foresaid* Lemma) and it is also proved that $A = \frac{Sin.\ 2Q}{s}$, $B = \frac{Sin.\ 3Q}{s}$, $C =$

$C = \frac{Sin.\ 4Q}{s}$ &c. it is evident, therefore, that $\frac{r^{10} - 2kr^5x^5 + x^{10}}{r^2 - 2arx + x^2}$ $(= r^8 + Ar^7x + Br^6x^2$ &c.) is $= r^8 + \frac{Sin.\ 2Q}{s} \times r^7x$ &c. and consequently $\frac{s \times \overline{r^{10} - 2kr^5x^5 + x^{10}}}{r^2 - 2arx + x^2}$ $= Sin.\ Q \times r^8 + Sin.\ 2Q \times r^7x + Sin.\ 3Q \times r^6x^2 + Sin.\ 4Q \times r^5x^3 + Sin.\ 5Q \times r^4x^4 + Sin.\ 4Q \times r^3x^5$ &c. In which Equation, for a and s, let their several respective Values b, c, d, &c. and $\acute{b}$, $\acute{c}$, $\acute{d}$, &c. be, successively, substituted; and let the corresponding Arcs $\frac{R}{n}$, $\frac{P+R}{n}$, $\frac{2P+R}{n}$ &c. be represented by Q, $\acute{Q}$, $\acute{\acute{Q}}$, &c. then we shall have

$$\acute{b} \times \frac{\overline{r^{10} - 2kr^5x^5 + x^{10}}}{r^2 - 2brx + x^2} = Sin.\ Q \times r^8 + Sin.\ 2Q \times r^7x \text{ &c.}$$

$$\frac{\acute{c} \times \overline{r^{10} - 2kr^5x^5 + x^{10}}}{r^2 - 2crx + x^2} = Sin.\ \acute{Q} \times r^8 + Sin.\ 2\acute{Q} \times r^7x \text{ &c.}$$

&c. &c.

Which Equations, added all together, give

$$\overline{r^{10} - 2kr^5x^5 + x^{10}} \times \overline{\frac{\acute{b}}{r^2 - 2brx + x^2} + \frac{\acute{c}}{r^2 - 2crx + x^2} + \frac{\acute{d}}{r^2 - 2drx + x^2}} \text{ &c.}$$

$$= \left\{\begin{array}{l} \text{Sin. ♌} \\ \text{Sin. ♌-} \\ \text{Sin. ♌=} \\ \text{Sin. ♌≡} \\ \text{Sin. ♌≣} \end{array}\right\} \times r^8 + \left\{\begin{array}{l} \text{Sin. 2♌} \\ \text{Sin. 2♌-} \\ \text{Sin. 2♌=} \\ \text{Sin. 2♌≡} \\ \text{Sin. 2♌≣} \end{array}\right\} \times r^7x + \ldots\ldots + \left\{\begin{array}{l} \text{Sin. }n\text{♌} \\ \text{Sin. }n\text{♌-} \\ \text{Sin. }n\text{♌=} \\ \text{Sin. }n\text{♌≡} \\ \text{Sin. }n\text{♌≣} \end{array}\right\} \times r^4x^4 + \&c.$$

But the Sines of the first Column, being those of an arithmetical Progression (whose common Difference is $\frac{P}{n}$) by which the whole Periphery is divided into n (5) equal Parts, their Sum will therefore, it is well known, be equal to Nothing; or all the negative ones equal to all the positive ones.

 The

The same is also true with regard to the Sines of the second Column; whose Arcs $\frac{2R}{n}$, $\frac{2P+2R}{n}$, $\frac{4P+2R}{n}$ &c. (having $\frac{2P}{n}$ for their common Difference) divide the Periphery (twice taken) into the same Number (n) of equal Parts. But the Sines of the middle Column (which is the last above exhibited) will not vanish, as all the rest do: For, $n\mathcal{Q}$ being $= R$, $n\mathcal{Q}' = P+R$, $n\mathcal{Q}'' = 2P + R$, &c. the common Difference will here be equal to (P) the whole Periphery; and therefore, every Arch terminating in the same Point with the first, the Circle will, in this Case, remain undivided, and the Sine of each be equal to (k') the Sine of the first.

Hence, our Equation is reduced to $\overline{r^{10}-2kr^5x^5+x^{10}} \times \overline{\frac{b}{r^2-2brx+x^2} + \frac{c'}{r^2-2crx+x^2}}$ &c. $= 5k'r^4x^4$; which divided by $r^{10}-2kr^5x^5+x^{10}$, and multiplied by rx, gives

$$\frac{b'rx}{r^2-2brx+x^2} + \frac{c'rx}{r^2-2crx+x} + \frac{d'rx}{r^2-2drx+x^2}\ \&c. =$$

$$\frac{5k'r^5x^5}{r^{10}-2kr^5x^5+x^{10}} = \frac{nkr^nx^n}{r^{2n}-2kr^nx^n+x^{2n}}.\quad \mathcal{Q}.\ E.\ D.$$

The same otherwise.

319. Since $r^{2n}-2kr^nx^n+x^{2n}$ is $= \overline{r^2-2brx+x^2} \times \overline{r^2-2crx+x^2} \times \overline{r^2-2drx+x^2}$ (n) *by Lemma* 3. it is evident that, Log. $\overline{r^{2n}-2kr^nx^n+x^{2n}} =$ Log. $\overline{r^2-2brx+x^2}$ + Log. $\overline{r^2-2crx+x^2}$ + Log. $\overline{r^2-2drx+x^2}$ (n). And, as this Equation holds universally, let k and x be what they will (which two Quantities may be supposed to flow independently of each other) let the

Fluxion of the *whole* Equation be taken, making k variable (and x constant); which gives $\frac{-2\dot{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}}$

$$= \frac{-2\dot{b}rx}{r^2-2brx+x^2} - \frac{2\dot{c}rx}{r^2-2crx+x^2} - \frac{2\dot{d}rx}{r^2-2drx+x^2}$$

* Art. 126. (n) *. But, $k, b, c, d,$ &c. are the Co-sines of the Arcs R, $\frac{R}{n}$, $\frac{R+P}{n}$, $\frac{R+2P}{n}$ &c. (whereof the corresponding Sines are $\acute{k}$, $\acute{b}$, $\acute{c}$, &c.) therefore, the Fluxion of the first of these Arcs being denoted by $\dot{R}$, the Fluxion of each of the rest will be expressed by $\frac{\dot{R}}{n}$: And so (the Fluxion of the Co-sine of an Arch being equal to the Fluxion of the Arch itself drawn into its Sine, applied to

† Art. 142. Radius †) it follows that $\dot{k} = \dot{R}\acute{k}$, $\dot{b} = \frac{\dot{R}}{n} \times \acute{b}$, $\dot{c} = \frac{\dot{R}}{n} \times \acute{c}$, &c. Which Values being substituted in the foregoing Equation, and the whole divided by $\frac{-2\dot{R}}{n}$, we have $\frac{n\acute{k}r^n x^n}{r^{2n}-2kr^n x^n+x^{2n}} = \frac{\acute{b}rx}{r^2-2brx+x^2} + \frac{\acute{c}rx}{r^2-2crx+x^2} + \frac{\acute{d}rx}{r^2-2drx+x^2}$ (n).

LEMMA V.

320. *To determine the Series, arising from the Division of Unity by a Trinomial,* $x^2-2arx+r^2$; *and to exhibit the Remainder after any given Number* (v) *of Terms in the Quotient.*

Let $x^{-2}+Arx^{-3}+Br^2x^{-4}+Cr^3x^{-5}+Dr^4x^{-6}$ represent the required Quotient continued to 5 Terms (v,

(v, to render the Process the more obvious, being first expounded by that Number) and let $Er^5x^{-5} + Fr^6x^{-6}$ be the Remainder. Then, because $\frac{1}{x^2 - 2arx + r^2}$ is $= x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + Dr^4x^{-6} + \frac{Er^5x^{-5} + Fr^6x^{-6}}{x^2 - 2arx + r^2}$, we shall, by reducing the whole Equation to one Denomination, have

$$\left.\begin{array}{llllllll}
1 & + Arx^{-1} & + Br^2x^{-2} & + Cr^3x^{-3} & + Dr^4x^{-4} & + Er^5x^{-5} & + Fr^6x^{-6} \\
* & -2arx^{-1} & -2aAr^2x^{-2} & -2aBr^3x^{-3} & -2aCr^4x^{-4} & -2aDr^5x^{-5} & * \\
1 & * & + r^2x^{-2} & + Ar^3x^{-3} & + Br^4x^{-4} & + Cr^5x^{-5} & + Dr^6x^{-6}
\end{array}\right\} = 0$$

Whence $A = 2a$, $B = 2aA - 1$, $C = 2aB - A$, $D = 2aC - B$, $E = 2aD - C$, and $F = -D$.

There-

Therefore, if $\mathcal{Q}$ be now put for the Arch whose Radius is 1 and Co-sine a, and there be taken $S = Sin.\ \mathcal{Q}$, $\overset{I}{S} = Sin.\ 2\mathcal{Q}$, $\overset{II}{S} = Sin.\ 3\mathcal{Q}$, &c. we shall, also, have $A\ (2a) = \frac{\overset{I}{S}}{S}$, $B = \frac{\overset{II}{S}}{S}$, $C = \frac{\overset{III}{S}}{S}$, $D = \frac{\overset{IIII}{S}}{S}$, $E = \frac{\overset{V}{S}}{S} = \frac{Sin.\ 6\mathcal{Q}}{S}$; $F\ (-D) = -\frac{Sin.\ 5\mathcal{Q}}{S}$ (*by Corol. to Lem.* 1.) And consequently $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \overset{I}{S}rx^{-3} + \overset{II}{S}r^2x^{-4} + \overset{III}{S}r^3x^{-5} + \overset{IIII}{S}r^4x^{-6}}{S} +$$

$$\frac{Sin.\ 6\mathcal{Q} \times r^5x^{-5} - Sin.\ 5\mathcal{Q} \times r^6x^{-6}}{S \times \overline{x^2 - 2arx + r^2}}.$$ Whence, *universally*, $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \overset{I}{S}rx^{-3} + \overset{II}{S}r^2x^{-4} + \overset{III}{S}r^3x^{-5}\ \&c.\ (to\ v\ Terms)}{S} +$$

$$\frac{Sin.\ \overline{v+1}\,.\,\mathcal{Q} \times r^v x^{-v} - Sin.\ v\mathcal{Q} \times r^{v+1}x^{-v-1}}{S \times \overline{x^2 - 2arx + r^2}}.$$ Which last Equation (though obvious enough from the preceding one) may be investigated in a general Manner (if required) by assuming $x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + \ldots\ldots dr^{v-2}x^{-v} + er^{v-1}x^{-v-1} + \frac{fr^v x^{-v} + gr^{v+1}x^{-v-1}}{x^2 - 2arx + r^2} = \frac{1}{x^2 - 2arx + r^2}$, and proceeding as above: By which Means you will find $A = 2a$, $B = 2aA - 1$, &c. $f = 2ae - d = \frac{Sin.\ \overline{v+1} \times \mathcal{Q}}{S}$, and $g\ (= -e) = -\frac{Sin.\ v\mathcal{Q}}{S}$. And thus may the third

Lemma

Lemma be made out, if any Objection, or Difficulty, should arise about its being general.

COROLLARY.

321. If, in the given Trinomial $x^2 - 2arx + r^2$, we suppose r^2, instead of x^2, to be the leading Term whereby the Quotient is produced; then, since r and x are affected exactly alike; we shall, by writing r for x, and x for r, have $\frac{1}{r^2 - 2axr + x^2} =$

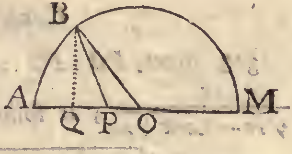
$$\frac{Sr^{-2} + \overset{\prime}{S}xr^{-3} + \overset{\prime\prime}{S}x^2r^{-4}\ (v)}{S} +$$

$$\frac{\overline{Sin.\ v+1} \times \mathfrak{Q} \times x^v r^{-v} - \overline{Sin.\ v\mathfrak{Q}} \times x^{v+1} \times r^{-v-1}}{S \times \overline{r^2 - 2axr + x^2}}.$$

P R O B. I.

322. *To find the Fluent of* $\frac{\dot{x}}{rr - 2arx + xx}$, *together with that of* $\frac{x\dot{x}}{rr - 2arx + xx}$.

Let ABM &c. be a Circle whose Radius OA (or OM) is r, and let the Angle AOB be such, that its Co-sine, to the Radius 1, may be equal to a; or so, that OQ (supposing BQ perpendicular to OA) may be $= ar$: Moreover let s denote the Sine of the said Angle AOB, corresponding to the Co-sine a, and let OP (considered as variable by the Motion of P along OA) express the Value of x: Then, PB^2 ($OB^2 + OP^2 - 2OQ \times OP$) $= rr - 2arx + xx$: And the Fluxion of the Measure of the Angle QBP (Radius being Unity) will be represented



sented by $\frac{BQ \times Flux.\ QP}{BP^2}$ (*vid. Art.* 142.) or by $\frac{rs \times -\dot{x}}{rr - 2arx + xx}$; and consequently *that* of *OBP*, by $\frac{rs\dot{x}}{rr - 2arx + xx}$: Whence it is evident that the Fluent of $\frac{\dot{x}}{rr - 2arx + xx}$ (contemporaneous with x) is truly expressed by $\frac{1}{rs} \times OBP$.

Again, since $\frac{x\dot{x}}{rr - 2arx + xx}$ may be transformed to $\frac{-ar\dot{x} + x\dot{x}}{rr - 2arx + xx} + \frac{ar\dot{x}}{rr - 2arx + xx}$; where the Fluent of

* Art. 326. the former Part is $= \frac{1}{2}$ hyp. Log. $\frac{rr - 2arx + xx}{rr}$ * $= \frac{1}{2}$ *hyp. Log.* $\frac{PB^2}{OB^2} =$ *hyp. Log.* $\frac{PB}{OB}$; and that of the latter Part $= \frac{a}{s} \times OBP$; it appears that the Fluent of $\frac{x\dot{x}}{rr - 2arx + xx}$ is truly expounded by *hyp. Log.* $\frac{PB}{OB}$, $+ \frac{a}{s} \times OBP$. *Q. E. I.*

Corollary.

323. Since, $PB : PO :: Sin.\ BOP\ (s) : Sin.\ OBP = \frac{sx}{\sqrt{rr - 2arx + xx}}$; it follows, if the hyperbolical Logarithm of $\frac{\sqrt{r^2 - 2arx + xx}}{r}$, be represented by *M*, and the Arch, whose Sine is $\frac{sx}{\sqrt{rr - 2arx + xx}}$ and Radius Unity,

Unity, by N, that the Fluents of $\frac{\dot{x}}{rr - 2arx + xx}$ and $\frac{x\dot{x}}{rr - 2arx + xx}$ will be expressed by $\frac{N}{rs}$ and $M + \frac{aN}{s}$ respectively.

PROB. II.

324. *To determine the Fluent of* $\frac{x^m \dot{x}}{x^2 - 2arx + r^2}$; *supposing* m *any whole positive Number, and* a *less than Unity.*

Let every thing remain as in *Lemma* 5. and then, if the Equation there brought out be multiplied by $x^m \dot{x}$, and v at the same time be expounded by $m-1$, we shall get $\frac{x^m \dot{x}}{x^2 - 2arx + r^2} = \frac{Sx^{m-2}\dot{x} + \overset{I}{S}rx^{m-3}\dot{x} + \overset{II}{S}r^2x^{m-4}\dot{x}}{S}$

$(m-1) + \frac{\overline{\text{Sin. } m\mathcal{Q}} \times r^{m-1}x\dot{x} - \overline{\text{Sin. } \overline{m-1} \times \mathcal{Q}} \times r^m\dot{x}}{S \times \overline{xx - 2arx + rr}}$

Whose Fluent will therefore be given by the preceding Proposition: For, supposing the Values of M and N to be as there specified, the Fluent of the last Term $\left(\frac{\overline{\text{Sin. } m\mathcal{Q}} \times r^{m-1}x\dot{x} - \overline{\text{Sin. } \overline{m-1} \times \mathcal{Q}} \times r^m\dot{x}}{S \times \overline{xx - 2arx + rr}}\right)$ will, it is manifest *, be expressed by $\frac{1}{S}$ *into* $\overline{\text{Sin. } m\mathcal{Q}} \times r^{m-1} \times$ * Art. 323.

$\overline{M + \frac{aN}{S}} - \overline{\text{Sin. } \overline{m-1} \times \mathcal{Q}} \times r^m \times \frac{N}{rS} = \frac{r^{m-1}}{S}$ into

$\overline{\text{Sin. } m\mathcal{Q}} \times M + \frac{\overline{\text{Sin. } m\mathcal{Q}} \times a - \overline{\text{Sin. } \overline{m-1} \times \mathcal{Q}}}{S} \times N$

$= \frac{r^{m-1}}{S}$ into $\overline{\text{Sin. } m\mathcal{Q}} \times M + \overline{\text{Co-s. } m\mathcal{Q}} \times N$ (*by Lem.* 2.

Case

Case 1.) To which adding the Fluent of the preceding Series, there results $\frac{1}{S} \times \overline{\frac{Sx^{m-1}}{m-1} + \frac{\overset{\prime}{S}rx^{m-2}}{m-2} + \frac{\overset{\prime\prime}{S}r^2x^{m-3}}{m-3} (m-1)}$ $+ \frac{r^{m-1}}{S} \times \overline{Sin.\ m\mathfrak{Q} \times M + Co\text{-}s.\ m\mathfrak{Q} \times N}$. *Q. E. I.*

COROLLARY.

325. Hence, the Fluent of $\frac{-ax^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2arx + rr}$ may be deduced: For, by writing $m-1$, instead of m, the Fluent of $\frac{x^{m-1}\dot{x}}{xx - 2arx + rr}$ will be found $= \frac{1}{S} \times$ $\overline{\frac{Sx^{m-2}}{m-2} + \frac{\overset{\prime}{S}rx^{m-3}}{m-3} + \frac{\overset{\prime\prime}{S}r^2x^{m-4}}{m-4} (m-2)} + \frac{r^{m-2}}{S} \times$ $\overline{Sin.\ \overline{m-1} \times \mathfrak{Q} \times M + Co\text{-}s.\ \overline{m-1} \times \mathfrak{Q} \times N}$: Which Fluent being multiplied by r, and *that* of $\frac{x^m\dot{x}}{xx - 2arx + rr}$ (given above) by $-a$, we shall, when the homologous Terms are united, have $\frac{1}{S} \times -aS \times \frac{x^{m-1}}{m-1} \overline{-a\overset{\prime}{S} - S} \times$ $\frac{rx^{m-2}}{m-2} \overline{- a\overset{\prime\prime}{S} - \overset{\prime}{S}} \times \frac{r^2x^{m-3}}{m-3} (m-1) + \frac{r^{m-1}}{S}$ into $-$ $\overline{Sin.\ m\mathfrak{Q} \times a - Sin.\ \overline{m-1} \times \mathfrak{Q} \times M} - \overline{Co\text{-}s.\ m\mathfrak{Q} \times a -}$ $\overline{Co\text{-}s.\ \overline{m-1} \times \mathfrak{Q} \times N}$, for the true Fluent of the Quantity propounded.

But (*by Case* 1. *Lem.* 2.) $\frac{a\overset{\prime}{S} - S}{S}$ $\left(= \frac{Sin.\ 2\mathfrak{Q} \times a - Sin.\ \mathfrak{Q} \times Rad.}{Sin.\ \mathfrak{Q}}\right) = Co\text{-}s.\ 2\mathfrak{Q}$: Also

$$\frac{a\overset{''}{S}-\overset{'}{S}}{S}\left(\frac{Sin.\ 3Q\times a - Sin.\ 2Q\times Rad.}{S}\right) = Co\text{-}s.\ 3Q$$

&c. And, by *Case* 2. of the same *Lemma*,

$$\frac{Co\text{-}s.\ \overline{m-1}\times Q - \overline{Co\text{-}s.\ mQ}\times a}{S} = Sin.\ mQ:$$ Whence,

by substituting these Values, our Fluent is reduced to

$$-Co\text{-}s.\ Q\times\frac{x^{m-1}}{m-1} - Co\text{-}s.\ 2Q\times\frac{rx^{m-2}}{m-2} - Co\text{-}s.\ 3Q\times\frac{r^2x^{m-3}}{m-3} - Co\text{-}s.\ 4Q\times\frac{r^3x^{m-4}}{m-4}\ (m-1) - r^{m-1}\times\overline{\overline{Co\text{-}s.\ mQ}\times M - \overline{Sin.\ MQ}\times N}.$$

PROB. III.

326. *To determine the Fluent of* $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}$; *under the Restrictions specified in the preceding Problem.*

If the Equation in *Art.* 321. be multiply'd by $x^{-m}\dot{x}$, and v at the same time be expounded by m, we shall have $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}=$

$$\frac{Sr^{-2}x^{-m}\dot{x}+\overset{'}{S}r^{-3}x^{1-m}\dot{x}+\overset{''}{S}r^{-4}x^{2-m}\dot{x}}{S}\ (m) +$$

$$\frac{r^{-m-1}}{S}\times\frac{\overline{Sin.\ \overline{m+1}\times Q}\times r\dot{x} - \overline{Sin.\ mQ}\times x\dot{x}}{r^2-2arx+x^2}:$$

Where, the Fluent of the last Term being $\frac{r^{-m-1}}{S}\times$

$$\overline{\overline{Sin.\ \overline{m+1}\times Q}\times\frac{N}{S} - \overline{Sin.\ mQ}\times\overline{M+\frac{aN}{S}}}\ ^{*} =$$ * Art. 323.

$$\frac{r^{-m-1}}{S}\ \text{into} - \overline{Sin.\ mQ}\times M +$$

$$\frac{\overline{Sin.\, m+1 \times Q} - Sin.\, mQ \times a}{S} \times N = \frac{r^{-m-1}}{S} \times$$
$$\overline{- \overline{Sin.\, mQ} \times M + \overline{Co\text{-}s.\, mQ} \times N}$$ (*by Case* 3. *Lem.* 2.) it follows that the Fluent of the whole Expression, or the Quantity sought, will be truly expressed by

$$\frac{1}{S} \times \frac{Sr^{-2}x^{1-m}}{1-m} + \frac{\acute{S}r^{-3}x^{2-m}}{2-m} + \&c.$$ or its Equal

$$\frac{-1}{S} \times \frac{Sx^{1-m}}{m-1.r^2} + \frac{\acute{S}x^{2-m}}{m-2.r^3} + \frac{\ddot{S}x^{3-m}}{m-3.r^4} \,(m) +$$
$$\frac{1}{Sr^{m+1}} \times \overline{Co\text{-}sin.\, mQ \times N - \overline{Sin.\, mQ} \times M}.$$

PROB IV.

327. *To find the Fluent of* $\frac{x^{m-1}\dot{x}}{r^n+x^n}$; *m and n being any whole positive Numbers, whereof the former does not exceed the latter.*

Let *b*, *c*, *d*, &c. denote the Co-sines of the Arcs $\frac{360^\circ}{2n}$, $\frac{3\times 360^\circ}{2n}$, $\frac{5\times 360^\circ}{2n}$, &c. (Radius being Unity) Then (*by Corol.* 2. *Lem.* 3.) we shall have $r^n + x^n =$ $\overline{rr-2brx+xx}|^{\frac{1}{2}} \times \overline{rr-2crx+xx}|^{\frac{1}{2}} \times \overline{rr-2drx+xx}|^{\frac{1}{2}}$ (*n*). Whence *Log.* $\overline{r^n+x^n} = \frac{1}{2}$ *Log.* $\overline{rr-2brx+xx}$ + $\frac{1}{2}$ *Log.* $\overline{rr-2crx+xx}$ + $\frac{1}{2}$ *Log.* $\overline{rr-2drx+xx}$ (*n*) and, consequently, by taking the Fluxion, on both Sides, $\frac{nx^{n-1}\dot{x}}{r^n+x^n} = \frac{x\dot{x}-br\dot{x}}{xx-2brx+rr} + \frac{x\dot{x}-cr\dot{x}}{xx-2crx+rr} +$

* Art. 126. $\frac{x\dot{x}-dr\dot{x}}{xx-2drx+rr}$ * (*n*); which last Equation, multiply'd by $\frac{x}{\dot{x}}$, gives $\frac{nx^n}{r^n+x^n} = \frac{xx-brx}{xx-2brx+rr} + \frac{xx-crx}{xx-2crx+rr}$

$+ \frac{xx - drx}{xx - 2drx + rr}$ (n). Let each Side hereof be now subtracted from n (or, which comes to the same thing, let $\frac{nx^n}{r^n + x^n}$ be taken from n, and each of the (n) Terms on the other Side, from Unity) then we shall have $\frac{nr^n}{r^n + x^n} = \frac{-brx + rr}{xx - 2brx + rr} + \frac{-crx + rr}{xx - 2crx + rr}$ $+ \frac{-drx + rr}{xx - 2drx + rr}$ (n): Which multiply'd by $\frac{x^{m-1}\dot{x}}{r}$, gives $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n + x^n} = \frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2brx + rr} +$ $\frac{-cx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2crx + rr}$ &c.

But now, to determine the Fluent hereof, let the several Arcs $\left(\frac{180°}{n}, \frac{3 \times 180°}{n}, \frac{5 \times 180°}{n}\right.$ &c.) above specified, be denoted by $\mathfrak{Q}, \overset{\prime}{\mathfrak{Q}}, \overset{\prime\prime}{\mathfrak{Q}}, \overset{\prime\prime\prime}{\mathfrak{Q}}$, &c. respectively; also let N, $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, &c. express the Measures of the Angles whose Sines are $\frac{x \times \text{Sin. } \mathfrak{Q}}{\sqrt{rr - 2brx + xx}}$, $\frac{x \times \text{Sin. } \overset{\prime}{\mathfrak{Q}}}{\sqrt{xx - 2crx + rr}}$, $\frac{x \times \text{Sin. } \overset{\prime\prime}{\mathfrak{Q}}}{\sqrt{xx - 2drx + rr}}$ &c. and M, $\overset{\prime}{M}$, $\overset{\prime\prime}{M}$, &c. the hyperbolic Logarithms of $\frac{\sqrt{xx - 2brx + rr}}{r}$, $\frac{\sqrt{xx - 2crx + rr}}{r}$, $\frac{\sqrt{xx - 2drx + rr}}{r}$ &c. Then (*by Corol. to Prob.* 2.) the Fluent of the first Term, $\frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2brx + rr}$ (expounding a by b) comes out

$$- \text{Co-ſ. } \mathfrak{Q} \times \frac{x^{m-1}}{m-1} - \text{Co-ſ. } 2\mathfrak{Q} \times \frac{rx^{m-2}}{m-2} - \text{Co-ſ. } 3\mathfrak{Q} \times \frac{r^2x^{m-3}}{m-3} \; (m-1) + r^{m-1} \text{ into } \overline{\text{Sin. } m\mathfrak{Q} \times N - \text{Co-ſ. } m\mathfrak{Q} \times M}.$$

In the ſame Manner, by writing c for a, $\acute{\mathfrak{Q}}$ for $\mathfrak{Q}$, M' for M, and N' for N) the Fluent of the ſecond Term, $\frac{-cx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2crx + rr}$, is found $= -\text{Co-ſ. } \mathfrak{Q}' \times \frac{x^{m-1}}{m-1} - \text{Co-ſ. } 2\mathfrak{Q}' \times \frac{rx^{m-2}}{m-2}$ &c. &c.

Therefore the Fluent of the whole Expreſſion, by collecting the homologous Terms, appears to be

$$- \left\{\begin{matrix} \text{Co-ſ. } \mathfrak{Q} - \\ \text{Co-ſ. } \mathfrak{Q}' - \\ \text{Co-ſ. } \mathfrak{Q}'' - \\ \text{Co-ſ. } \mathfrak{Q}''' \\ \text{\&c.} \end{matrix}\right\} \times \frac{x^{m-1}}{m-1} - \left\{\begin{matrix} \text{Co-ſ. } 2\mathfrak{Q} - \\ \text{Co-ſ. } 2\mathfrak{Q}' - \\ \text{Co-ſ. } 2\mathfrak{Q}'' - \\ \text{Co-ſ. } 2\mathfrak{Q}''' \\ \text{\&c.} \end{matrix}\right\} \times \frac{rx^{m-2}}{m-2} - \left\{\begin{matrix} \text{Co-ſ. } 3\mathfrak{Q} - \\ \text{Co-ſ. } 3\mathfrak{Q}' - \\ \text{Co-ſ. } 3\mathfrak{Q}'' - \\ \text{Co-ſ. } 3\mathfrak{Q}''' \\ \text{\&c.} \end{matrix}\right\} \times \frac{r^2x^{m-}}{m-}$$

$$+ r^{n-1} \times \begin{cases} \overline{Sin.\ m\mathcal{Q}} \times N - \overline{Co\text{-}s.\ m\mathcal{Q}} \times M \\ \overline{Sin.\ m\mathcal{Q}'} \times N' - \overline{Co\text{-}s.\ m\mathcal{Q}'} \times M' \\ \overline{Sin.\ m\mathcal{Q}''} \times N'' - \overline{Co\text{-}s.\ m\mathcal{Q}''} \times M'' \\ \overline{Sin.\ m\mathcal{Q}'''} \times N''' - \overline{Co\text{-}s.\ m\mathcal{Q}'''} \times M''' \\ \quad \&c. \qquad\qquad \&c. \end{cases}$$

But the Co-ſines of the firſt Column being thoſe of an arithmetical Progreſſion $\left(\frac{180^\circ}{n}, \frac{3 \times 180^\circ}{n}, \frac{5 \times 180^\circ}{n}\right.$ *&c.*) whoſe common Difference is $\frac{360^\circ}{n}$, whereby the whole Periphery is divided into n equal Parts (*vid. Art.* 317.) they will therefore deſtroy one another; ſince it is well known that, if the Periphery of any Circle be divided into any Number (n) of equal Parts, the negative Sines and Co-ſines will be equal to the poſitive ones; which is ſelf-evident when their Number is even.

Hence the Co-ſines in the ſecond and third Columns, *&c.* will alſo deſtroy one another (*vid. Art.* 318.) But *thoſe* of the laſt Column of all, as well as the Sines, having unequal Multiplicators, muſt remain as above, and that Column, *alone*, (drawn into r^{m-1}) will be the true Fluent of $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n + x^n}$. Whence, putting $m\mathcal{Q}$ $\left(= m \times \frac{180^\circ}{n}\right) = R$, and dividing by nr^{n-1}, we ſhall (becauſe $\mathcal{Q}' = 3\mathcal{Q}$, $\mathcal{Q}'' = 5\mathcal{Q}$, $\mathcal{Q}''' = 7\mathcal{Q}$ *&c.*) have

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{c} \overline{Sin.\ R} \times N - \overline{Co\text{-}f.\ R} \times M \\ \overline{Sin.\ 3R} \times N' - \overline{Co\text{-}f.\ 3R} \times M' \\ \overline{Sin.\ 5R} \times N'' - \overline{Co\text{-}f.\ 5R} \times M'' \\ \overline{Sin.\ 7R} \times N''' - \overline{Co\text{-}f.\ 7R} \times M''' \\ \text{\&c. (to } n \text{ Lines.)} \end{array} \right\} = \text{Fluent of } \frac{x^{m-1}\dot{x}}{r^n + x^n}.$$

Q. E. I.

COROLLARY.

328. Since the first and the last, the second and the last but one, &c. of the foregoing Quantities $x^2 - 2brx + rr$, $xx - 2crx + rr$, $xx - 2drx + rr$ &c. are respectively equal to each other (*vid. Art.* 317.) the corresponding Fluents, found above, will likewise be equal: And therefore the Fluent of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ will, *also*, be expressed by

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{c} \overline{Sin.\ R} \times 2N - \overline{Co\text{-}f.\ R} \times 2M \\ \overline{Sin.\ 3R} \times 2N' - \overline{Co\text{-}f.\ 3R} \times 2M' \\ \overline{Sin.\ 5R} \times 2N'' - \overline{Co\text{-}f.\ 5R} \times 2M'' \\ \text{\&c.} \qquad\qquad \text{\&c.} \end{array} \right.$$

The Number of Lines to be thus taken being $= \frac{1}{2}n$, when n is even; but, otherwise, $= \frac{n+1}{2}$; in which last Case, the Logarithm, &c. in the last Line, must be taken only once, instead of twice; being that of $\frac{r+x}{r}$ (*vid. Art.* 317.)

PROB.

PROB. V.

329. *To find the Fluent of* $\frac{x^{m-1}\dot{x}}{r^n - x^n}$; m *and* n *being as in the preceding Problem.*

If b, c, d, *&c.* be taken to denote the Co-sines of the Arcs $\frac{0}{n}$, $\frac{360°}{n}$, $\frac{2 \times 360°}{n}$ *&c.* to n Terms, it will appear (*from Corol.* 1. *to Lem.* 3.) that $r^n - x^n$ is $= \overline{rr - 2brx + xx}|^{\frac{1}{2}} \times \overline{rr - 2crx + xx}|^{\frac{1}{2}} \times \overline{rr - 2drx + xx}|^{\frac{1}{2}}$ (n). From whence, by following the Method of the last Problem, we also have $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n - x^n} =$
$\frac{-bx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2brx + rr} + \frac{-cx^m\dot{x} + rx^{m-1}\dot{x}}{xx - 2crx + rr}$ *&c.*
Which Fluxion having exactly the same *Form* with *that* in the preceding Problem, its Fluent will also be expressed in the very same Manner; that is, by

$$r^{m-1} \times \begin{cases} Sin.\ m\mathcal{Q} \times N - Co\text{-}s.\ m\mathcal{Q} \times M \\ Sin.\ m\acute{\mathcal{Q}} \times \acute{N} - Co\text{-}s.\ m\acute{\mathcal{Q}} \times \acute{M} \\ Sin.\ m\acute{\acute{\mathcal{Q}}} \times \acute{\acute{N}} - Co\text{-}s.\ m\acute{\acute{\mathcal{Q}}} \times \acute{\acute{M}} \\ (\&c.\ to\ n\ Lines.) \end{cases}$$

Only $\mathcal{Q}$, $\acute{\mathcal{Q}}$, $\acute{\acute{\mathcal{Q}}}$ *&c.* must here stand for $\frac{0}{n}$, $\frac{360°}{n}$, $\frac{2 \times 360°}{n}$, $\frac{3 \times 360°}{n}$ *&c.* (instead of $\frac{180°}{n}$, $\frac{3 \times 180°}{n}$, $\frac{5 \times 180°}{n}$ *&c.*)

There-

Therefore, since the multiple Arcs $m\mathcal{Q}$, $m\acute{\mathcal{Q}}$, $m\acute{\acute{\mathcal{Q}}}$ &c. are, in this Case, equal to 0, $m \times \frac{360^\circ}{n}$, $2m \times \frac{360^\circ}{n}$, $3m \times \frac{360^\circ}{n}$ &c. (whereof the Sine of the first is = 0, and its Co-sine = Unity) we shall, by putting $R = m \times \frac{360^\circ}{n}$, and dividing the 'foresaid Fluent by nr^{n-1}, have

$$\frac{r^{m-n}}{n} \times \left\{\begin{array}{l} * \ldots\ldots - \ldots\ldots M \\ Sin.\ R \times \acute{N} - Co\text{-}s.\ R \times \acute{M} \\ Sin.2R \times \acute{\acute{N}} - Co\text{-}s.2R \times \acute{\acute{M}} \\ Sin.3R \times \acute{\acute{\acute{N}}} - Co\text{-}s.3R \times \acute{\acute{\acute{M}}} \\ (\&c.\ to\ n\ Lines.) \end{array}\right\} = \text{Fluent of } \frac{x^{m-1}\dot{x}}{r^{n} - x^{n}}$$

Q. E. I.

Corollary.

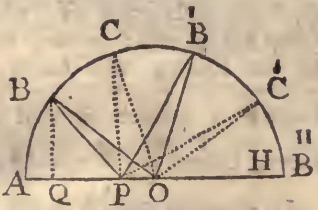
330. Since, in the Fluent here given, the second Line and the last, the third and the last but one, &c. are respectively equal (*vid. Art.* 317.) the same may also be exhibited, thus;

$$\frac{r^{m-n}}{n} \times \left\{\begin{array}{l} * \ldots\ldots - \ldots\ldots M \\ Sin.\ R \times 2\acute{N} - Co\text{-}s.\ R \times 2\acute{M} \\ Sin.\ 2R \times 2\acute{\acute{N}} - Co\text{-}s.\ 2R \times 2\acute{\acute{M}} \end{array}\right.$$

(&c. to $\frac{n+1}{2}$ Lines.)

Scholium.

331. If the Semi-Periphery ABCH of the Circle whose Diameter AH is $2r$, be divided into as many equal



equal Parts AB, BC, CB', $B'C'$ &c. as there are Units in n (so that $AB = \frac{180^\circ}{n} = Q$, $AB' = 3 \times \frac{180^\circ}{n} = Q'$ &c. *vid. Art.* 317. and 327.) and in the Radius OA (produced, if necessary) there be taken $OP = x$, and PB, OB &c. be drawn, it will appear (from the said Articles, and from *Prop.* 1.) that the Quantities $\sqrt{r^2 - 2brx + xx}$, $\sqrt{r^2 - 2crx + x^2}$ &c. in the former of the two preceding Problems, will here be expounded by PB, PB' &c. respectively: From whence it is also plain, that the Measures N, N' &c. of the Angles whose Sines are $\frac{x \times Sin.\, Q}{\sqrt{r^2 - 2brx + x^2}}$, $\frac{x \times Sin.\, Q'}{\sqrt{r^2 - 2crx + x^2}}$ &c. * will here be expounded by OBP, $OB'P$, &c. &c. * Art. 322, and 323.

Therefore the Fluent of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$, given in the Corollary to the foresaid Proposition, may be thus exhibited;

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} \overline{Sin.\ R} \times 2\,(OBP) - \overline{Co\text{-}s.\ R} \times 2\,(OA : PB) \\ \overline{Sin.\,3R} \times 2\,(OB'P) - \overline{Co\text{-}s.\ R} \times 2\,(OA : PB') \\ \quad \&c. \qquad\qquad\qquad\qquad \&c. \end{array} \right.$$

Where the Arch R is $\left(= m \times \frac{180^\circ}{n}\right) = m \times AB$, and where $(OA : PB)$ is put (after the Manner of *Cotes)* to express the hyperbolical Logarithm of $\frac{PB}{OA}$. It is also to be observed, *that,* when the last of the Points B,

$\overset{\prime}{B}$, $\overset{\prime\prime}{B}$ &c. falls upon H (which will always happen when n is an odd Number) the Angle, in the last Line of the Fluent, will vanish, and the corresponding Logarithm (which is that of $\frac{PH}{AO}$) must then be taken, instead of twice, only once.

In the very same Manner it will appear, *that*, the Arcs $\overset{\prime}{Q}$, $\overset{\prime\prime}{Q}$ &c. in the second Case, where the Fluent of $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ is sought, will be, respectively, expounded by AC, $\overset{\prime}{AC}$ &c. also the corresponding Angles $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$ &c. by OCP, $\overset{\prime}{OCP}$ &c. and the Fluent itself by

$$\frac{r^{m-n}}{n} \times \begin{cases} * \ldots\ldots\ldots - \ldots\ldots (OA : PC) \\ \overline{Sin.\ R} \times 2\,(OCP) - \overline{Co\text{-}f.\ R} \times 2\,(OA : PC) \\ \overline{Sin. 2R} \times 2\,(\overset{\prime}{OCP}) - \overline{Co\text{-}f. 2R} \times 2\,(OA : \overset{\prime}{PC}) \\ \&c. \qquad\qquad\qquad \&c. \end{cases}$$

Where the Arch R $\left(= m \times \frac{360^\circ}{n}\right) = m \times AC$; and where, as well as in the preceding Case, all the Arcs, Sines and Co-sines are supposed to have Unity for their Radius.

332. From the Fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$, thus given, *those* of $\frac{x^{vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{-vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{vn+m-1}\dot{x}}{r^n - x^n}$, and $\frac{x^{-vn+m-1}\dot{x}}{r^n - x^n}$, where v denotes any whole Number, may be very easily deduced; either from *Art.* 283. and 291. or (more readily) by dividing the Numerator by the Denominator, and continuing the Quo-

Quotient to as many Terms as there are Units in v*. By which means, if p be put $= vn + m$, $q = vn - m$, and the Fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ be denoted by V and W respectively, the Fluents, in the four Cases specified above, will be expressed by

* Art. 150.

$$\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} + \frac{r^{2n} x^{p-3n}}{p-3n} (v) \pm r^{vn} V,$$

$$\frac{x^{-q}}{-qr^n} - \frac{x^{n-q}}{\overline{n-q}.r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}.r^{3n}} (v) \pm \frac{V}{r^{vn}},$$

$$-\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} - \frac{r^{2n} x^{p-3n}}{p-3n} (v) + r^{vn} W,$$

and, $\frac{x^{-q}}{qr^n} + \frac{x^{n-q}}{\overline{n-q}\,.\,r^{2n}} + \frac{x^{2n-q}}{\overline{2n-q}\,.\,r^{3n}} (v) + \frac{W}{r^{vn}}$, respectively.

Moreover, from the same Fluents, *those* of $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e+fz^q}$, and $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e-fz^q}$ will likewise become known:

For (having transformed the Fluxions here proposed to $\frac{1}{e} \times \frac{z^{\frac{m}{n}q-1}\dot{z}}{1+\frac{fz^q}{e}}$, *&c.*) let $\frac{fz^q}{e}$ be put $= x^n$, or $x = \overline{\frac{fz^q}{e}}\Big|^{\frac{1}{n}}$; then will $z^{\frac{m}{n}q} = \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times x^m$, and consequently $\frac{mq}{n} \times z^{\frac{m}{n}q-1}\dot{z} = \overline{\frac{e}{f}}\Big|^{\frac{m}{n}} \times mx^{m-1}\dot{x}$.

 Whence

Whence $z^{\frac{m}{n}q-1}\dot{z} = \frac{n}{q} \times \overline{\frac{e}{f}}\big|^{\frac{m}{n}} \times x^{m-1}\dot{x}$, and $1 \pm \frac{fz^q}{e} = 1 \pm z^n$; and therefore $\frac{z^{\frac{m}{n}q-1}\dot{z}}{e \pm fz^q}$ $\left(= \frac{1}{e} \times \frac{n}{q} \times \overline{\frac{e}{f}}\big|^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n}\right) = \frac{n}{qe} \times \overline{\frac{e}{f}}\big|^{\frac{m}{n}} \times \frac{x^{m-1}\dot{x}}{1 \pm x^n}$: Whose Fluent is given, by *Prob.* 4. or 5. But, r being here $= 1$, the general Multiplicator $\frac{r^{m-n}}{n}$, there given, will be barely $= \frac{1}{n}$: Which, drawn into $\frac{n}{qe} \times \overline{\frac{e}{f}}\big|^{\frac{m}{n}}$, gives $\frac{1}{qe} \times \overline{\frac{e}{f}}\big|^{\frac{m}{n}}$, for the general Multiplicator in this Case.

One thing more, though well known to Mathematicians, it may be proper here to take notice of; and that relates to the Sines and Co-sines of the fore-mention'd Arcs, R, $2R$, $3R$, *&c. &c.* (multiplying the several Angles and Ratios) some of which Arcs do frequently exceed the whole Periphery: When this happens to be the Case, the Periphery, or 360°, must be subtracted as often as possible, and the Sine and Co-sine of the Remainder be taken. If the Remainder be greater than 180°, the Sine, falling in the lower Semi-Circle, will be negative; if, between 90° and 270°, the Co-sine, falling beyond the Center, will be negative.

PROB. VI.

333. *To find the Fluent of* $\frac{x^{n+m-1}\dot{x}}{r^{2n}-2kr^n x^n + x^{2n}}$; *where* n *and* m *denote any whole positive Numbers, and where the given Expression cannot be resolved into two Binomials,* (k *being less than Unity.* Art. 308. and 310.)

Let

Let R be the Arch whose Co-sine is k and Radius Unity, and let $\acute{k}$ be the Sine of the same Arch; moreover, let the Arcs $\frac{R}{n}$, $\frac{R + 360^\circ}{n}$, $\frac{R + 2 \times 360^\circ}{n}$, $\frac{R + 3 \times 360^\circ}{n}$ *&c.* be denoted by $\mathcal{Q}$, $\acute{\mathcal{Q}}$, $\mathcal{Q}''$, $\mathcal{Q}'''$, $\mathcal{Q}''''$ *&c.* and let $\acute{b}$, $\acute{c}$, $\acute{d}$ *&c.* and b, c, d *&c.* express the Sines, and the Co-sines of the same Arcs respectively.

Then will $\frac{n\acute{k}r^n x^n}{r^{2n} - 2kr^n x^n + x^{2n}} = \frac{\acute{b}rx}{r^2 - 2brx + x^2} + \frac{\acute{c}rx}{r^2 - 2crx + x^2} + \frac{\acute{d}rx}{r^2 - 2drx + x^2}$ *&c.* (n) *by Lemma* 4.) From whence, multiplying the whole Equation by $\frac{x^{m-1}\dot{x}}{n\acute{k}r^n}$ we have $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}} = \frac{1}{n\acute{k}r^{n-1}}$ into

$$\frac{\acute{b}x^m\dot{x}}{r^2 - 2brx + x^2} + \frac{\acute{c}x^m\dot{x}}{r^2 - 2crx + x^2} + \frac{\acute{d}x^m\dot{x}}{r^2 - 2drx + x^2} \ \&c.$$

Now, the Fluent of the first Term hereof $\frac{\acute{b}x^m\dot{x}}{r^2 - 2brx + x^2}$ (if M be put for the hyp. Log. of $\frac{\sqrt{x^2 - 2brx + x^2}}{r}$, and N for the Arch whose Radius is Unity, and Sine $\frac{x \times \overline{Sin.\ \mathcal{Q}}}{\sqrt{r^2 - 2brx + x^2}}$) will appear (*from Prop.* 2.) to be $=$

$$\overline{Sin.\ \mathcal{Q}} \times \frac{x^{m-1}}{m-1} + \overline{Sin.\ 2\mathcal{Q}} \times \frac{rx^{m-2}}{m-2} + \overline{Sin.\ 3\mathcal{Q}} \times \frac{r^2x^{m-3}}{m-3} \ \&c.\ (m-1) + r^{m-1} \times \overline{Sin.\ m\mathcal{Q} \times M + Co\text{-}sin.\ m\mathcal{Q} \times N}.$$

From

From whence, if the Arcs whose Sines are $\frac{x \times Sin.\ Q'}{\sqrt{r^2-2crx+x^2}}$, $\frac{x \times Sin.\ Q''}{\sqrt{r^2-2drx+x^2}}$ &c. be represented by M', M'' &c. and the Logarithms whose Numbers are $\frac{\sqrt{r^2-2crx+x^2}}{r}$, $\frac{\sqrt{r^2-2drx+x^2}}{r}$ &c. by N', N'' &c. respectively, the Fluent of the whole Expression, omitting the general Multiplicator $\left(\frac{1}{nk'r^{n-1}}\right)$ will be

$$\left.\begin{array}{l} Sin.\ Q \\ Sin.\ Q' \\ Sin.\ Q'' \\ Sin.\ Q''' \\ \&c. \end{array}\right\} \times \frac{x^{m-1}}{m-1} + \left\{\begin{array}{l} Sin.\ 2Q \\ Sin.\ 2Q' \\ Sin.\ 2Q'' \\ Sin.\ 2Q''' \\ \&c. \end{array}\right\} \times \frac{rx^{m-2}}{m-2} + \left\{\begin{array}{l} Sin.\ 3Q \\ Sin.\ 3Q' \\ Sin.\ 3Q'' \\ Sin.\ 3Q''' \\ \&c. \end{array}\right\}$$

$$\times \frac{r^2x^{m-3}}{m-3} \text{ (\&c. to } m-1 \text{ Terms)}$$

$$+ r^{m-1} \times \left\{\begin{array}{l} \overline{Sin.\ mQ} \times M + \overline{Co\text{-}s.\ mQ} \times N \\ \overline{Sin.\ mQ'} \times M' + \overline{Co\text{-}s.\ mQ'} \times N' \\ \overline{Sin.\ mQ''} \times M'' + \overline{Co\text{-}s.\ mQ''} \times N'' \\ \overline{Sin.\ mQ'''} \times M''' + \overline{Co\text{-}s.\ mQ'''} \times N''' \\ \quad \&c. \qquad\qquad \&c. \end{array}\right\}$$

But, the Sines of the first Column being those of an arithmetical Progression (whose common Difference is

$\frac{360^\circ}{n}$) which arises by dividing the whole Periphery into n equal Parts, their Sum will, therefore, be equal to Nothing.

Moreover, the Sines of the second Column, having $\frac{2 \times 360^\circ}{n}$ for the common Difference of their respective Arcs do, also, divide the whole Periphery (twice taken) into n equal Parts, and therefore destroy each other.

The same is likewise true, with regard to the Sines of every other Column (except the last of all) when $m-1$ is less than n. But, if m be greater than n, the Arcs, in the Column, whose Place from the first, inclusive, is denoted by n, being expressed by $n\mathfrak{Q}$, $n\acute{\mathfrak{Q}}$, $n\mathfrak{Q}''$ &c. (or R, $R+360^\circ$, $R+2\times360^\circ$ &c.) whereof the common Difference is the whole Periphery; the Sines of that Column do not destroy one another, but each is equal to that of the first Arc R (*Vid. Art.* 314. and 318.) and consequently their Sum equal to $n \times Sin. R$.

In like Manner, if m be greater than $2n$, the Series, continued to $m-1$ Terms, will take in the Column, where the Arcs are $2n\mathfrak{Q}$, $2n\mathfrak{Q}'$, $2n\mathfrak{Q}''$ &c. (or $2R$, $2R+2\times360^\circ$, $2R+4\times360^\circ$ &c.) whereof the Sine of each is, also, equal to the Sine of the first ($2R$) and therefore their Sum $= n \times Sin.\ 2\ R$.

Thus, also, it will appear that the Sines of the Column whose Distance from the first, inclusive, is $3n$ (when m is greater than $3n$) will be each equal to $Sin.\ 3\,R$; &c. &c.

Therefore, seeing all the Columns do actually vanish, except those above specified; whose Places from the Beginning are denoted by n, $2n$, $3n$ &c. and whose corresponding Terms, or Multiplicators are, therefore, represented by $\frac{r^{n-1}x^{m-n}}{m-n}$, $\frac{r^{2n-1}x^{m-2n}}{m-2n}$, $\frac{r^{3n-1}x^{m-3n}}{m-3n}$ &c. it is evident that the whole Expression will be reduced to

$$\overline{Sin. R} \times \frac{nr^{n-1}x^{m-n}}{m-n} + \overline{Sin. 2R} \times \frac{nr^{2n-1}x^{m-2n}}{m-2n}$$

$$+ \overline{Sin. 3R} \times \frac{nr^{3n-1}x^{m-3n}}{m-3n} \text{ \&c.}$$

$$+ r^{m-1} \text{ into } \begin{cases} Sin.\, m\mathcal{Q} \times M + Co\text{-}s.\, m\mathcal{Q} \times N \\ Sin.\, m\mathcal{Q}' \times M' + Co\text{-}s.\, m\mathcal{Q}' \times N' \\ Sin.\, m\mathcal{Q}'' \times M'' + Co\text{-}s.\, m\mathcal{Q}'' \times N'' \\ Sin.\, m\mathcal{Q}''' \times M''' + Co\text{-}s.\, m\mathcal{Q}''' \times N''' \\ \text{\&c.} \qquad\qquad \text{\&c.} \end{cases}$$

Which, multiply'd by $\frac{1}{nk'r^{n-1}}$, the foresaid, general, Multiplicator, gives $\overline{Sin. R} \times \frac{x^{m-n}}{\overline{m-n}\cdot k'} + \overline{Sin. 2R} \times \frac{r^{n}x^{m-2n}}{\overline{m-2n}\cdot k'} + \overline{Sin. 3R} \times \frac{r^{2n}x^{m-3n}}{\overline{m-3n}\cdot k'}$ &c.

$$+ \frac{r^{m-n}}{nk'} \times \begin{cases} Sin.\, m\mathcal{Q} \times M + Co\text{-}s.\, m\mathcal{Q} \times N \\ Sin.\, m\mathcal{Q}' \times M' + Co\text{-}s.\, m\mathcal{Q}' \times N' \\ Sin.\, m\mathcal{Q}'' \times M'' + Co\text{-}s.\, m\mathcal{Q}'' \times N'' \\ Sin.\, m\mathcal{Q}''' \times M''' + Co\text{-}s.\, m\mathcal{Q}''' \times N''' \\ \text{\&c.} \qquad\qquad \text{\&c.} \end{cases}$$

for the true Fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}$: Where the former Part of the Expression must be continued to as many Terms as there are Units in $\frac{m-1}{n}$ (the Remainder, if any, being neglected.) *Q. E. I.*

COROLLARY

Corollary.

334. If the Quotient arising from the Division of m by n (when the former exceeds) be denoted by v, and the Remainder by t; or, which is the same, if $vn + t = m$, it is evident the Arcs $m\mathcal{Q}'$, $m\mathcal{Q}''$, $m\mathcal{Q}'''$ &c. which are respectively equal to $m\mathcal{Q} + m \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 2m \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 3m \times \frac{360^\circ}{n}$, &c. (by Construction) will also be equal to $m\mathcal{Q} + v \times 360^\circ + t \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 2v \times 360^\circ + 2t \times \frac{360^\circ}{n}$ &c. whereof the Sines and Co-sines (omitting $v \times 360^\circ$, $2v \times 360^\circ$ &c. the Multiples of the whole Periphery) are the same with those of $m\mathcal{Q} + t \times \frac{360^\circ}{n}$, $m\mathcal{Q} + 2t \times \frac{360^\circ}{n}$ &c. respectively.

Therefore, if the Arcs of the Progression, whereof the first Term is $m\mathcal{Q}$, and the common Difference $t \times \frac{360^\circ}{n}$, be represented by T, T', T'' &c. respectively; it follows that the Fluent of $\frac{x^{n+m-1}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$ (or, $\frac{x^{n+vn+t-1}\dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$) will, *also*, be truly expressed by

$$\overline{Sin. R} \times \frac{x^{m-n}}{\overline{m-n} \cdot k'} + \overline{Sin. 2R} \times \frac{r^n x^{m-2n}}{\overline{m-2n} \cdot k'} + \overline{Sin. 3R} \times \frac{r^{2n} x^{m-3n}}{\overline{m-3n} \cdot k'} \text{ \&c. } \left(\frac{m-1}{n}\right)$$

$$+\frac{r^{m-n}}{nk^{\prime}}\left\{\begin{array}{l}\text{Sin. } T \times M+\text{Co-f. } T \times N \\ \text{Sin. } T' \times M'+\text{Co-f. } T' \times N' \\ \text{Sin. } T'' \times M''+\text{Co-f. } T'' \times N'' \\ \text{Sin. } T''' \times M'''+\text{Co-f. } T''' \times N''' \\ \quad \&c. \qquad\qquad \&c.\end{array}\right.$$

In the very ſame Manner the Fluent of $\frac{x^{n+m-1}}{r^{2n}+2kr^{n}x^{n}+x^{2n}}$ (where the Sign of the ſecond Term is poſitive) will be exhibited; if *R* be taken to denote the Arch whoſe Co-ſine is $-k$; which will, in this Caſe, be greater than a Quadrant.

PROPOSITION VII.

335. *To find the Fluent of* $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}$; *under the Reſtrictions mentioned in the laſt Problem.*

Let every thing remain as before: Then we ſhall have $\frac{x^{n-m-1}\dot{x}}{r^{2n}-2kr^{n}x^{n}+x^{2n}}=\frac{1}{nk^{\prime}r^{n-1}}$ into $\frac{b^{\prime}x^{-m}\dot{x}}{r^{2}-2brx+x^{2}}$ $+\frac{cx^{-m}\dot{x}}{r^{2}-2crx+x^{2}}$ (*n*) Whereof the Fluent (*by Prob.* 3.) appears to be $\frac{1}{nk^{\prime}r^{n-1}}$ into

$$-\left\{\begin{array}{l}\text{Sin. } Q \\ \text{Sin. } Q' \\ \text{Sin. } Q'' \\ \&c.\end{array}\right\}\times\frac{x^{1-m}}{m-1 \cdot r^{2}}-\left\{\begin{array}{l}\text{Sin. } 2Q \\ \text{Sin. } 2Q' \\ \text{Sin. } 2Q'' \\ \&c.\end{array}\right\}\times\frac{x^{2-m}}{m-2 \times r^{3}}$$

$$-\left\{\begin{matrix} \text{Sin. } 3\mathcal{Q} \\ \text{Sin. } 3\acute{\mathcal{Q}} \\ \text{Sin. } 3\acute{\acute{\mathcal{Q}}} \\ \&c. \end{matrix}\right\} \times \frac{x^{3-m}}{\overline{m-3} \cdot r^4}\,(m) +$$

$$\frac{1}{r^{m+1}} \times \left\{\begin{matrix} -\text{Sin. } m\mathcal{Q} \times M + \text{Co-s.} m\mathcal{Q} \times N \\ -\text{Sin. } m\acute{\mathcal{Q}} \times \acute{M} + \text{Co-s. } m\acute{\mathcal{Q}} \times \acute{N} \\ -\text{Sin. } m\acute{\acute{\mathcal{Q}}} \times \acute{\acute{M}} + \text{Co-s. } m\acute{\acute{\mathcal{Q}}} \times \acute{\acute{N}} \\ \&c. \qquad \&c. \end{matrix}\right.$$

Which, by Reasoning as above, will be reduced to

$$-\overline{\text{Sin. } R} \times \frac{x^{n-m}}{\overline{m-n} \cdot \acute{k}r^{2n}} - \overline{\text{Sin. } 2R} \times \frac{x^{2n-m}}{\overline{m-2n} \cdot \acute{k}r^{3n}}$$

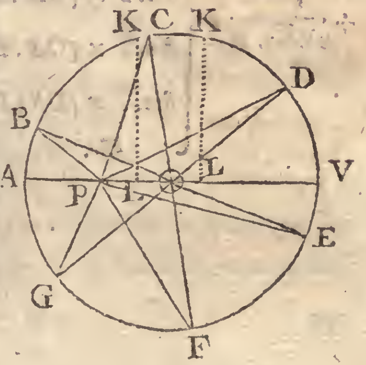
$$-\overline{\text{Sin. } 3R} \times \frac{x^{3n-m}}{\overline{m-3n} \cdot \acute{k}r^{4n}} \left(\text{to } \frac{m}{n} \text{ Terms}\right)$$

$$+\frac{1}{\acute{k}nr^{n+m}} \times \left\{\begin{matrix} -\text{Sin. } T \times M + \text{Co-s. } T \times N \\ -\text{Sin. } \acute{T} \times \acute{M} + \text{Co-s.} \acute{T} \times \acute{N} \\ -\text{Sin. } \acute{\acute{T}} \times \acute{\acute{M}} + \text{Co-s.} \acute{\acute{T}} \times \acute{\acute{N}} \\ \&c. \qquad \&c. \end{matrix}\right.$$

Q. E. I.

SCHOLIUM.

336. If, from the Center O, of the Circle ABCD, whose Radius OA, or OV, is r, there be taken OL equal to k and OP $= x$; and if the Arch AB be to the Arch AK, whose Co-sine is $\pm k$, as 1 to n; and each of



of the Arcs BC, CD, DE &c. be taken equal to $\frac{360^\circ}{n}$ &c. &c. Then the Angles R, Q, $\acute{Q}$ &c. specified (in the two preceding Problems) being here expounded by AK, AB, AC &c. respectively, we have $PB = \sqrt{r^2 - 2brx + x^2}$, $PC = \sqrt{r^2 - 2crx + x^2}$ &c. (*Vid. Art.* 317. and 323.) Whence, also, the Angles N, $\acute{N}$, $\acute{\acute{N}}$ &c. whose Sines are $\frac{x \times Sin.\, Q}{\sqrt{r^2 - 2brx + x^2}}$, $\frac{x \times Sin.\, \acute{Q}}{\sqrt{r^2 - 2crx + x^2}}$, $\frac{x \times Sin.\, \acute{\acute{Q}}}{\sqrt{r^2 - 2drx + x^2}}$ &c. will here be equal to B, C, D &c. Therefore the Fluents of $\frac{x^{n+m-1}\dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$, and $\frac{x^{n-m-1}\dot{x}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$ (there given) will, also, be truly defined by

$$\frac{x^{m-n}}{m-n} + \frac{Sin.\, 2R}{Sin.\, R} \times \frac{r^n x^{m-2n}}{m-2n} + \frac{Sin.\, 3R}{Sin.\, R} \times \frac{r_{2n} x^{m-3n}}{m-3n}$$

$\left(\text{to } \frac{m-1}{n} \text{ Terms}\right)$

$$+\frac{r^{m-n}}{n \times Sin.\, R} \times \begin{cases} Sin.\, T \times (OB : PB) + Co\text{-}s.\, T \times (B) \\ Sin.\, \acute{T} \times (OC : PC) + Co\text{-}s.\, \acute{T} \times (C) \\ Sin.\, \acute{\acute{T}} \times (OD : PD) + Co\text{-}s.\, \acute{\acute{T}} \times (D) \\ Sin.\, \acute{\acute{\acute{T}}} \times (OE : PE) + Co\text{-}s.\, \acute{\acute{\acute{T}}} \times (-E) \\ Sin.\, \acute{\acute{\acute{\acute{T}}}} \times (OF : PF) + Co\text{-}s.\, \acute{\acute{\acute{\acute{T}}}}\, (-F) \\ \quad \&c. \qquad\qquad \&c. \end{cases}$$

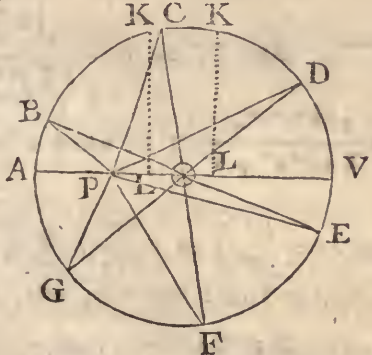
And

And by $-\frac{x^{n-m}}{\overline{m-n}\,.\,r^{2n}} - \frac{Sin.\ 2R}{Sin.\ R} \times \frac{x^{2n-m}}{\overline{m-2n}\,.\,r^{3n}} -$

$$-\frac{Sin.\ 3R}{Sin.\ R} \times \frac{x^{3n-m}}{\overline{m-3n}\,.\,r^{4n}} \left(\frac{m}{n}\right)$$

$$+\frac{1}{nr^{n+m} \times Sin.\ R} \times \left\{ \begin{array}{l} -Sin.\,T \times (OB:PB) + Co\text{-}s.\,T \times (B) \\ -Sin.\,T' \times (OC:PC) + Co\text{-}s.\,T' \times (C) \\ -Sin.\,T'' \times (OD:PD) + Co\text{-}s.\,T'' \times (D) \\ -Sin.\,T''' \times (OE:PE) + Co\text{-}s.\,T''' \times (-E) \\ -Sin.\,T'''' \times (OF:PF) + Co\text{-}s.\,T'''' \times (-F) \\ \quad \&c. \end{array} \right.$$

respectively.



Where the Arc *AK* (or *R*) will be greater than a Quadrant when the Sign of *k* is positive; but less, when negative; and where the Arcs T, T', T'' &c. denote an arithmetical Progression, whose first Term (T) is equal to $m \times AB$, and whereof the common Difference is equal to $\frac{360°}{n}$ (or *BC*) multiplied by *m*, when *m* is less than *n*; but otherwise by the Remainder, of *m* divided by *n*.

337. Hence the Fluent of $\frac{z^{q\pm\frac{m}{n}q-1}\dot{z}}{e\mp fz^{q}+gz^{2q}}$, where q is any Number, either whole or broken, may be very easily deduced: For, having transformed the Denominator to $g\times\overline{\frac{e}{g}\mp\frac{fz^{q}}{g}+z^{2q}}$, put $\frac{e}{g}=r^{2n}$, $\frac{f}{g}=2kr^{n}$, and $z^{q}=x^{n}$; and then it will become $=g\times\overline{r^{2n}\mp 2kr^{n}x^{n}+x^{2n}}$: Moreover, $z^{q\pm\frac{m}{n}q}$ being $=\overline{x^{n}}|^{1\pm\frac{m}{n}}=x^{n\pm m}$, and $\overline{q\pm\frac{m}{n}q}\times z^{q\pm\frac{m}{n}q-1}\dot{z}=\overline{n\pm m}\times x^{n\pm m-1}\dot{x}$, the Numerator will be reduced to $\frac{n}{q}\times x^{n\pm m-1}\dot{x}$: And so, we have $\frac{z^{q\pm\frac{m}{n}q-1}\dot{z}}{e\mp fz^{q}+gz^{2q}}=\frac{n}{qg}\times\frac{x^{n\pm m-1}\dot{x}}{r^{2n}\mp 2kr^{n}x^{n}+x^{2n}}$: In which $x=z^{\frac{q}{n}}$, $r=\overline{\frac{e}{g}}\Big|^{\frac{1}{2n}}$, and $k\left(=\frac{\frac{1}{2}f}{gr^{n}}\right)=\frac{\frac{1}{2}f}{\sqrt{eg}}$. But, it may be observed, that the Fluent hereof is, only, given when
* Art. 333. $\frac{\frac{1}{2}f}{\sqrt{eg}}$ (or its Equal k) is less than Unity *. Therefore, if $\frac{1}{2}f$ be greater than $\sqrt{eg}$; or if the Values of e and g are unlike, with regard to positive and negative, so that $\sqrt{eg}$ is impossible, the above Solution fails. But, here, the given Trinomial may be resolved into two Binomials (*by Art.* 310.) and, from thence, the Fluent may be found at two Operations (*by Prob.* 4. *and* 5.)

For,

For, by feigning $e \mp fy + gy^2 = 0$, in order to such a Resolution, we get $\frac{\pm\frac{1}{2}f + \sqrt{\frac{1}{4}f^2 - eg}}{g}$, and $\frac{\pm\frac{1}{2}f - \sqrt{\frac{1}{4}f^2 - eg}}{g}$ for the Roots of that Equation, or the two first Terms of the required Binomials: Which therefore are always possible when $\frac{1}{4}f^2 - eg$ is positive, or when the foregoing Solution fails.

By denoting the said Roots by H and K, the Trinomial $e \mp fz^q + gz^{2q}$ is resolved into $g \times \overline{H - z^q} \times \overline{K - z^q}$, from whence $\frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{e \mp fz^q + gz^{2q}}$ is reduced to

$$\frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{g \times \overline{K - H} \times \overline{H - z^q}} + \frac{z^{q \pm \frac{m}{n}q - 1}\dot{z}}{g \times \overline{H - K} \times \overline{K - z^q}},$$

whose Fluent is given by *Art.* 332.

338. By proceeding the same Way the Fluent of $\frac{z^{\frac{m}{n}q - 1}\dot{z}}{e + fz^q + gz^{2q} + hz^{3q}}$ may likewise be found: For, since one, at least, of the three Roots of the Equation $e + fy + gy^2 + hy^3 = 0$, must be possible, the proposed Fluxion, if it cannot be resolved into three Binomials, may, however, be reduced to one Binomial and one Trinomial; and so, be brought under the foregoing Forms: But this being a Speculation too much out of the Way of common Use to be farther pursued, I shall here conclude this Section, with observing, that, when k, in the original Trinomial, above specified, is neither less, nor greater than Unity, the Fluent cannot then be had directly, from either of the preceding Methods; but must be found by Comparison from the Fluent of $\frac{x^{n \pm m - 1}\dot{x}}{r^n \pm x^n}$. *Vid. Art.* 289.

SECTION VI.

The Manner of investigating Fluents, when Quantities, and their Logarithms; Arcs and their Sines, &c. are involved together: With other Cases of the like Nature.

PROB. I.

339. SUPPOSING Q and n to denote given Quantities; it is proposed to find the Fluent of $x^n\dot{x}Q^x$.

Let $Q^x \times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}}$ &c. be assumed for the Fluent required: Then the Fluxion thereof, which is

* Art. 252. $Q^x\dot{x} \times hyp.\ Log.\ Q$ * $\times \overline{Ax^n + Bx^{n-1} + Cx^{n-2}}$ &c. $+$ $Q^x \times \overline{n\dot{x}Ax^{n-1} + \overline{n-1}.B\dot{x}x^{n-2} + \overline{n-2}.C\dot{x}x^{n-3}}$ &c.

must consequently be $= x^n\dot{x}Q^x$: And therefore, by putting m for the *hyp. Log.* of Q, we have

$$\left.\begin{array}{l} mAx^n + mBx^{n-1} + \quad mCx^{n-2} + mDx^{n-3} \text{ \&c.} \\ -x^n + nAx^{n-1} + \overline{n-1}.Bx^{n-2} + \overline{n-2}.Cx^{n-3} \text{ \&c.} \end{array}\right\} = 0$$

Whence, comparing the Coefficients of the homologous Terms, we get $A = \frac{1}{m}$, $B = -\frac{nA}{m} = -\frac{n}{m^2}$, $C = -\frac{\overline{n-1}.B}{m} = \frac{n.\overline{n-1}}{m^3}$ &c. and consequently $Q^x \times \overline{Ax + Bx^{n-1} + Cx^{n-2} + \text{\&c.}} = \frac{Q^x}{m} \times \overline{x^n - \frac{nx^{n-1}}{m} +} \overline{\frac{n.\overline{n-1}.x^{n-2}}{m^2} - \frac{n.\overline{n-1}.\overline{n-2}.x^{n-3}}{m^3}}$ &c. Which

Series,

Series, it is plain, will always terminate when n is a whole positive Number. *Q. E. I.*

340. In the preceding Problem the Coefficients A, B, C, &c. of the assumed Series were taken, in the common Way, as constant Quantities; which, because of the general Multiplicator Qx, was sufficient.

But, in other Cases, where a proper Multiplicator, to express the mechanical, or logarithmic, &c. Part of the required Fluent, cannot readily be known, it will be convenient to assume a Series for the *Whole* (independent of any general Multiplicator) wherein the Quantities A, B, C, D, &c. must be considered as variable.

PROB. II.

341. *To find the Fluent of* $z^m x^{n-1}\dot{x}$; z *being the Hyperbolic-Logarithm of* x; *and* m *and* n *any given Numbers.*

Let there be assumed $Az^m + Bz^{m-1} + Cz^{m-2} + Dz^{m-3}$ &c. $=$ the Fluent of $z^m x^{n-1}\dot{x}$: Then, in Fluxions, we shall have

$$\left.\begin{matrix}\dot{A}z^m + \dot{B}z^{m-1} + \dot{C}z^{m-2} + \dot{D}z^{m-3} \text{ \&c.} \\ + mAz^{m-1}\dot{z} + \overline{m-1}.Bz^{m-2}\dot{z} + \overline{m-2}.Cz^{m-3}\dot{z} \text{ \&c.}\end{matrix}\right\} = z^m x^{n-1}\dot{x}.$$

But $\dot{z} = \frac{\dot{x}}{x}$; whence, by ordering the Equation, there arises

$$\left.\begin{matrix}\dot{A} \\ -x^{n-1}\dot{x}\end{matrix}\right\}\times z^m + \left.\begin{matrix}\dot{B} \\ \frac{mA\dot{x}}{x}\end{matrix}\right\}\times z^{m-1} + \left.\begin{matrix}\dot{C} \\ \frac{\overline{m-1}.B\dot{x}}{x}\end{matrix}\right\} z^{m-2}, \text{ \&c.} = 0$$

Now, by making the Coefficients of the like Powers of z, equal to Nothing, we have $\dot{A} = x^{n-1}\dot{x}$, $A = \frac{x^n}{n}$; $\dot{B} = \left(-\frac{mA\dot{x}}{x}\right) = -\frac{mx^{n-1}\dot{x}}{n}$, $B = -\frac{mx^n}{n^2}$;

$\dot{C}\,(= -\frac{\overline{m-1}\,.\,B\dot{x}}{x} =)\ \frac{m\,.\,\overline{m-1}\,.\,x^{n-1}\dot{x}}{n^2}\quad C =$ $\frac{m\,.\,\overline{m-1}\,.\,x^n}{n^3}$ *&c.* and consequently the Fluent sought $= \frac{x^n}{n}$ into $z^m - \frac{mz^{m-1}}{n} + \frac{m\,.\,\overline{m-1}\,.\,z^{m-2}}{n^2} - \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,z^{m-3}}{n^3} + \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}\,.\,z^{m-4}}{n^4}$ *&c.* Which, when m is a whole positive Number, will terminate in $m+1$ Terms. *Q. E. I.*

PROB. III.

342. *To find the Fluent of $z^n\dot{y}$; z being the Arch of a given Circle, and y the Sine corresponding.*

Let there be assumed $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3} =$ Fluent of $z^n\dot{y}$; then, by taking the Fluxion, we shall have

$$\left.\begin{array}{l} \dot{A}z^n + \dot{B}z^{n-1} + \dot{C}z^{n-2} + \dot{D}z^{n-3} \quad \&c. \\ -z^n\dot{y} + nAz^{n-1}\dot{z} + \overline{n-1}\,.\,Bz^{n-2}\dot{z} \quad \&c. \end{array}\right\} = 0$$

Whence, putting $\dot{A} - \dot{y} = 0$, $\dot{B} + nA\dot{z} = 0$, $\dot{C} + \overline{n-1}\,.\,B\dot{z} = 0$, $\dot{D} + \overline{n-2}\,.\,C\dot{z} = 0$, *&c.* we get $A = y$; $\dot{B} = -ny\dot{z}$, $\dot{C} = -\overline{n-1}\,.\,B\dot{z}$ *&c.*

But, if a and x be taken to denote the Radius and Co-sine of the Arch z, it will appear, from *Art.* 142. that $y\dot{z} = -a\dot{x}$ and $x\dot{z} = a\dot{y}$: Therefore $\dot{B} = na\dot{x}$, and $B = nax$; also $\dot{C}\ (= -\overline{n-1}\,.\,B\dot{z}) = -n\,.\,\overline{n-1}\,.\,ax\dot{z} = -n\,.\,\overline{n-1}\,.a^2\dot{y}$, and $C = -n.\overline{n-1}\,.a^2y$; likewise $\dot{D}\ (= -\overline{n-2}\,.\,C\dot{z}) = n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^2y\dot{z} = -n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^3\dot{x}$, and $D = -n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.a^3x$ *&c.*

&c. &c. and consequently $Az^n + Bz^{n-1} + Cz^{n-2}$ &c. $= yz^n + naxz^{n-1} - n.\overline{n-1}.a^2yz^{n-2} - n.\overline{n-1}.\overline{n-2}.a^3xz^{n-3}$ + &c.

$$= \begin{cases} y \times \overline{z^n - n.\overline{n-1}.a^2z^{n-2} + n.\overline{n-1}.\overline{n-2}.\overline{n-3}.a^4z^{n-4}} \text{ \&c.} \\ x \times \overline{naz^{n-1} - n.\overline{n-1}.\overline{n-2}.a^3z^{n-3} + n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.a^5z^{n-5}} \text{ \&c.} \end{cases}$$

Q. E. I.

In the very same Manner the Fluent of $z^n\dot{w}$, or $z^n \times -\dot{x}$ (w being the Versed-Sine of the Arch z) will be found $= -xz^n + nyaz^{n-1} + n.\overline{n-1}.xa^2z^{n-2} - n.\overline{n-1}.\overline{n-2}.ya^3z^{n-3} - n.\overline{n-1}.\overline{n-2}.\overline{n-3}.xa^4z^{n-4}$ + &c.

PROB. IV.

343. *The Quantities, x, y and z being the same as in the preceding Problem; to find the Fluent of $z^n x^r y^m \dot{y}$.*

By assuming $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3}$ &c. and proceeding as above, we have $\dot{A} = x^r y^m \dot{y}$, $\dot{B} = -nA\dot{z}$, $\dot{C} = -\overline{n-1}\,.\,B\dot{z}$, $\dot{D} = -\overline{n-2}\,.\,C\dot{z}$ &c. or (because $\dot{z} = \frac{a\dot{y}}{x}$) $\dot{B} = -\frac{naA\dot{y}}{x}$, $\dot{C} = -\frac{\overline{n-1}\,.\,aB\dot{y}}{x}$, $\dot{D} = -\frac{\overline{n-2}\,.\,aC\dot{y}}{x}$ &c. Therefore, if the Fluent of $x^r y^m \dot{y}$ (found from *Art.* 142. and 291.) be denoted by Q; *that* of $\frac{Q\dot{y}}{x}$, by R; *that* of $\frac{R\dot{y}}{x}$, by S; *that* of $\frac{S\dot{y}}{x}$, by T &c. it follows that the Fluent of $z^n x^r y^m \dot{y}$ will be truly represented by $Qz^n - naRz^{n-1} + n\,.\,\overline{n-1}\,.\,a^2Sz^{n-2} - n\,.\,\overline{n-1}\,.\,\overline{n-2}\,.\,a^3Tz^{n-3}$ &c.

COROLLARY.

344. Since $\dot{y} = -\frac{x\dot{x}}{y} = \frac{x\dot{z}}{a}$ (*Vid. Art.* 142.) it follows that $z^n x^r y^m \dot{y}$ is $= -z^n x^{r+1} y^{m-1} \dot{x} = \frac{z^n x^{r+1} y^m \dot{z}}{a}$: Therefore the Fluents of these two last Expressions are, *also*, exhibited in the foregoing Series.

345. As the Values of Q, R, S, &c. in the preceding Articles, are too complex to be pursued in a general Manner, it may not be amiss to illustrate the Method of proceeding by an Example or two.

Let, then, the Fluxion propofed be $\frac{zy^2\dot{y}}{x}$: Where n being $=1$, $m=2$, and $r=-1$, we have $\dot{Q}=\frac{y^2\dot{y}}{x}=\frac{y^2\dot{y}}{\sqrt{a^2-y^2}}$ (becaufe $\sqrt{a^2-y^2}=x$.) Whence $Q=-\frac{1}{2}y\sqrt{a^2-y^2}+\frac{1}{2}az=-\frac{1}{2}yx+\frac{1}{2}az$*, and therefore $\dot{R}$ $\left(=\frac{Q\dot{y}}{x}\right)=-\frac{1}{2}y\dot{y}+\frac{\frac{1}{2}az\dot{y}}{x}=-\frac{1}{2}y\dot{y}+\frac{1}{2}z\dot{z}$ (becaufe $\frac{a\dot{y}}{x}=\dot{z}$) and confequently $R=-\frac{1}{4}y^2+\frac{1}{4}z^2$; and fo, $\frac{az-yx}{2}\times z+a\times\frac{yy-zz}{4}$, or $\frac{az^2-2xyz+ay^2}{4}$, is the true Fluent of $\frac{zy^2\dot{y}}{x}$ $\left(=-zy\dot{x}=\frac{y^2z\dot{z}}{a}.\dagger\right)$

* Art. 279.

† Art. 344.

Again, let the Fluent of $-p\dot{x}\times\overline{z+y}|^2$ (expreffing the Content of the Solid generated by the Revolution of the *Cycloid*) be required.

Here, the given Expreffion, in fimple Terms, will become $-pz^2\dot{x}-2pzy\dot{x}-py^2\dot{x}$: Whereof the Fluent of the firft Term $-pz^2\dot{x}$, will be had, by making $n=2$, $m-1=0$, and $r+1=0$ (*Vid. Form.* 2. *in Corol.*) Where, we therefore, have $\dot{Q}=\frac{y\dot{y}}{x}=-\dot{x}$; whence $Q=-x$; alfo $\dot{R}$ $\left(\frac{Q\dot{y}}{x}\right)=-\dot{y}$, and $R=-y$; likewife $\dot{S}$ $\left(=\frac{R\dot{y}}{x}\right)=-\frac{y\dot{y}}{x}=\dot{x}$, $S=x$; and confequently the Fluent of $-z^2\dot{x}$ ($Qz^n-naRz^{n-1}+n.\overline{n-1}.a^2Sz^{n-2}$ &c.) $=-xz^2+2ayz+2a^2x$: To which, adding the Fluent $\left(\frac{az^2-2xyz+ay^2}{2}\right)$ of the

fecond

second Term $-2zy\dot{x}$ (found in the preceding Example) and also that of $-y^2\dot{x}$ (or $-a^2\dot{x}+x^2\dot{x}$, found the common Way) we get, in the Whole, $\frac{1}{2}\overline{a-x}\times z^2+\overline{2ay-yx}\times z+\frac{1}{2}ay^2+a^2x+\frac{1}{3}x^3$; which, multiply'd by p, and corrected, gives, p *into* $\frac{1}{2}\overline{a-x}\times z^2+\overline{2ay-yx}\times z+\frac{1}{2}ay^2+a^2x+\frac{1}{3}x^3-\frac{4}{3}a^4$, for the true Fluent that was to be determined.

PROB. V.

346. *Supposing H to denote the Fluent of $\overline{k+lz^n}|^r\times z^{vn-1}\dot{z}$; to find the whole Fluent of $H\times\overline{a-bz^n}|^m\times z^{qn-1}\dot{z}$, (when $a-bz^n$ becomes equal to Nothing.)*

By resolving $\overline{k+lz^n}|^r\times z^{vn-1}\dot{z}$ into simple Terms, and taking the Fluent, the ordinary Way, we get $H=$ $\frac{k^r z^{vn}}{n}\times\frac{1}{v}+\frac{rlz^n}{\overline{v+1}.k}+\frac{r.\overline{r-1}.l^2z^{2n}}{2.\overline{v+2}.k^2}$ &c. Which Value being substituted above, and p wrote instead of $q+v$, we shall have $H\times\overline{a-bz^n}|^m\times z^{qn-1}\dot{z}=\frac{k^r}{n}\times$ $\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$ *into* $\frac{1}{v}+\frac{rlz^n}{\overline{v+1}.k}+\frac{r.\overline{r-1}.l^2z^{2n}}{2.\overline{v+2}.k^2}$ $+\frac{r.\overline{r-1}.\overline{r-2}.l^3z^{3n}}{2.3.\overline{v+3}.k^3}$ &c.

Let, now, the Fluent of $\overline{a-bz^n}|^m\times z^{pn-1}\dot{z}$ (in the proposed Circumstance) be denoted by A, and put $t=p+m+1$; then it follows, from *Art.* 286. (by writing $\frac{1}{v}$ for e, $\frac{rl}{\overline{v+1}.k}$ for f, &c.) that $\frac{k^r}{n}\times A$ *into* $\frac{1}{v}+$

$\frac{p \cdot r}{t \cdot \overline{v+1}} \times \frac{al}{bk} + \frac{p \cdot \overline{p+1} \cdot r \cdot \overline{r-1}}{t \cdot \overline{t+1} \cdot 2 \cdot \overline{v+2}} \times \overline{\frac{al}{bk}}\Big|^2 +$

$\frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot r \cdot \overline{r-1} \cdot \overline{r-2}}{t \cdot \overline{t+1} \cdot \overline{t+2} \cdot 2.3 \, \overline{v+3}} \times \overline{\frac{al}{bk}}\Big|^3$ + &c. will be the true Value of the Fluent. *Q. E. I.*

Note, p and $m+1$ must here be positive Quantities *; and it is also requisite that $\frac{l}{k}$ should be greater than $-\frac{b}{a}$; otherwise the Fluent will fail.

* Art. 286.

Ex. 1. *Let* $\dot{H} = \overline{1-y^2}\big|^{-\frac{1}{2}} \times \dot{y}$; *and let the whole Fluent of* $H \times \overline{1-y^2}\big|^{-\frac{1}{2}} \dot{y}$, *be demanded.*

Then, k being $=1$, $l=-1$, $z=y$, $n=2$, $r=-\frac{1}{2}$, $v=\frac{1}{2}$; also $a=1$, $b=1$, $m=-\frac{1}{2}$, $q=\frac{1}{2}$; $p \; (=q+v) = 1$, $t \; (=p+m+1) = \frac{3}{2}$, and A (= the *whole* Fluent of $\overline{1-y^2}\big|^{-\frac{1}{2}} y\dot{y}) = 1$; we shall, by substituting these several Values above, get $1 + \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 5} + \frac{1}{7 \cdot 7} + \frac{1}{9 \cdot 9} + \frac{1}{11 \cdot 11}$ &c. = Fluent of $H \times \overline{1-y^2}\big|^{-\frac{1}{2}} \times \dot{y}$ (or $H\dot{H}$) when $y=1$. Which Fluent being also expressed by $\frac{H^2}{2}$, it follows that $\frac{H^2}{2} = \frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$ &c. Where H is $\frac{1}{4}$ of the Periphery of the Circle whose Radius is Unity.

Ex.

Ex. 2. *Let* $\dot{H} = \overline{c^2 + z^2}|^{-\frac{3}{2}} \times \dot{z}$; *to find the Fluent of* $H \times \overline{h^2 - z^2}|^{-\frac{1}{2}} \times c^2\dot{z}$.

Here, $k = c^2$, $l = 1$, $n = 2$, $r = -\frac{3}{2}$, $v = \frac{1}{2}$; alſo $a = h^2$, $b = 1$, $m = -\frac{1}{2}$, $q = \frac{1}{2}$, $p\,(q + v) = 1$, $t\,(p + m + 1) = \frac{3}{2}$, and A (= *whole* Fluent of $\overline{h^2 - z^2}|^{-\frac{1}{2}} \times z\dot{z}$) $= h$: Whence, by Subſtitution, we have $c^{-3}\,\overline{h \times 1 - \frac{1}{3} \times \frac{h^2}{c^2} + \frac{1}{5} \times \frac{h^4}{c^4} - \frac{1}{7} \times \frac{h^6}{c^6}}$ *&c.* which, multiplied by c^2 (the Coëfficient of $\dot{z}$) gives $\frac{1}{c} \times \overline{h - \frac{h^3}{3c^2} + \frac{h^5}{5c^4} - \frac{h^7}{7c^6}}$ *&c.* for the true Fluent in this Caſe: Where the Series is *that* expreſſing the Arch of the Circle whoſe Tangent is h and Radius c *; and is therefore equal to $c \times$ Arch, whoſe Radius is Unity and Tangent $= \frac{h}{c}$: Whence this laſt Arch (taken without the multiplicator c) is the true Value of the Fluent.

* Art. 142.

SECTION VII.

Shewing how Fluents, found by Means of Infinite Serieſes, are made to converge.

347. IT is found, *in Art.* 85. that the Fluent of $\overline{a + cz^n}|^m \times dz^{qn-1}\,\dot{z}$, in an infinite Series, (making $m + q = s$) is expreſſed by $\frac{\overline{a + cz^n}|^{m+1} \times dz^{qn}}{qna} \times$

$\overline{1 - \frac{\overline{s+1}\,.\,cz^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2z^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2} - \&c.}$ Whence it follows (and is evident by bare Inſpection) that the Fluent of $\overline{a - cy^n}|^r \times y^{qn-1}\dot{y}$ (where the ſecond Term under the *Vinculum* is negative) will be truly defined by $\frac{\overline{a - cy^n}|^{r+1} \times y^{qn}}{qna}$ into $1 + \frac{\overline{s+1}\,.\,cy^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2}$ + &c. ſuppoſing $s = r + q$.

But, beſides the Series here given, and Thoſe, *in Art.* 83. 84. expreſſing the ſame Value, the Fluent of $\overline{a - cy^n}|^r \times y^{qn-1}\dot{y}$ will, yet, admit of another Form, different from all of them; by means whereof and *that* above, we ſhall be enabled to draw out ſome very uſeful Concluſions.

348. Put $z^n = \frac{ay^n}{a - cy^n}$; then $y^n = \frac{az^n}{a + cz^n}$, and therefore $ny^{n-1}\dot{y} = \frac{na^2z^{n-1}\dot{z}}{\overline{a + cz^n}|^2}$; alſo $a - cy^n = \frac{a^2}{a + cz^n}$, and $y^{qn-1}\dot{y}$ $(= y^{qn-n} \times y^{n-1}\dot{y}) = \frac{a^{q+1}z^{qn-1}\dot{z}}{\overline{a + cz^n}|^{q+1}}$; and conſequently $\overline{a - cy^n}|^r \times y^{qn-1}\dot{y} = a^{2r+q+1} \times \overline{a + cz^n}|^{-r-q-1} \times z^{qn-1}\dot{z}$: Which Fluxion, ſo tranſformed, being compared with $\overline{a + cz^n}|^m \times dz^{qn-1}\dot{z}$; we have $m = -r - q - 1$, $d = a^{2r+q+1}$, and $s\ (q + m) = -r - 1$; whence, by ſubſtituting theſe Values in the firſt Series, above given, the Fluent ſought will be had $= \frac{\overline{a + cz}|^{-r-q} \times a^{2r+q} \times z^{qn}}{qn} \times \overline{1 + \frac{rcz^n}{\overline{q+1}\,.\,a}}$ $\overline{+ \frac{r\,.\,\overline{r-1}\,.\,c^2z^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3z^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,a^3}}$ &c.

Which, by restoring y (or writing $\frac{a^2}{a-cy^n}$ and $\frac{ay^n}{a-cy^n}$ for their Equals $a+cz^n$, and z^n) becomes

$$\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn} \times 1 + \frac{r}{q+1} \times \frac{cy^n}{a-cy^n} + \frac{r \cdot \overline{r-1}}{\overline{q+1} \cdot \overline{q+2}} \times \frac{c^2y^{2n}}{\overline{a-cy^n}|^2}$$ &c. the true Fluent, of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$.

349. This Fluent may be otherwise found, independent of *that* above, in the following Manner:

It is evident, by taking the Fluxion of $\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn}$ (which Quantity would be the Fluent sought, if $\overline{a-cy^n}|^r$ was constant) that $\frac{\overline{a-cy^n}|^r \times y^{qn}}{qn}$ is = the Fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ — Fluent of $\frac{rc}{q} \times \overline{a-cy^n}|^{r-1} \times y^{qn+n-1}\dot{y}$: This Equation, by transposing the last Term, and writing x in the room of $a-cy^n$ (for the Sake of Brevity) will become *Flu.* $x^r y^{qn-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rc}{q} \times$ *Flu.* $x^{r-1} y^{qn+n-1}\dot{y}$. From the very same Argument (if, instead of r, we substitute $r-1$, $r-2$ &c. successively; and, for q, write $q+1$, $q+2$, $q+3$, &c. respectively) we shall, also, have

Flu. $x^{r-1} y^{qn+n-1}\dot{y} = \frac{x^{r-1} y^{qn+n}}{\overline{q+1} \cdot n} + \frac{\overline{r-1} \cdot c}{q+1} \times$ *Flu.* $x^{r-2} y^{qn+2n-1}\dot{y}$;

Flu. $x^{r-2} y^{qn+2n-1}\dot{y} = \frac{x^{r-2} y^{qn+2n}}{\overline{q+2} \cdot n} + \frac{\overline{r-2} \cdot c}{q+2} \times$ *Flu.* $x^{r-3} y^{qn+3n-1}\dot{y}$;

&c. &c.

Whence, by ſubſtituting theſe Values, one by one, in that of, *Flu.* $x^r y^{qn-1}\dot{y}$, we get

$$Flu.\ x^r y^{qn-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rc}{q} \times \frac{x^{r-1} y^{qn+n}}{\overline{q+1}\,.\,n} + \frac{r\,.\,\overline{r-1}\,.\,c^2}{q\,.\,\overline{q+1}}$$

$$\times\ Flu.\ x^{r-2} y^{qn+2n-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q\,.\,\overline{q+1}\,.\,n} +$$

$$\frac{r\,.\,\overline{r-1}\,.\,c^2}{q\,.\,\overline{q+1}} \times \frac{x^{r-2} y^{qn+2n}}{\overline{q+2}\,.\,n} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3}{q\,.\,\overline{q+1}\,.\,\overline{q+2}} \times$$

$$Flu.\ x^{r-3} y^{qn+3n-1}\dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q\,.\,\overline{q+1}\,.\,n} +$$

$$\frac{r\,.\,\overline{r-1}\,.\,c^2 x^{r-2} y^{qn+2n}}{q\,.\,\overline{q+1}\,.\,\overline{q+2}\,.\,n} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3 x^{r-3} y^{qn+3n}}{q\,.\,\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,n}$$

&c. Where the Law of Continuation is manifeſt; and where, by making $\frac{x^r y^{qn}}{qn}$ a general Multiplicator, we ſhall have the very Series above exhibited.

350. From the Equality of the two foregoing Expreſſions, for the Fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$, (or $x^r y^{qn-1}\dot{y}$) the Buſineſs of finding Fluents, by infinite Serieſes, will, in many Caſes, be very much facilitated.

For, in the firſt Place, it follows (by dividing both by $\frac{\overline{a-cy^n}|^{r+1} \times y^{qn}}{qna}$, or $\frac{x^{r+1} y^{qn}}{qna}$) that the Serieſes $1 + \frac{\overline{s+1}\,.\,cy^n}{\overline{q+1}\,.\,a} + \frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2 y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,a^2}$ &c. and $\frac{a}{x} \times$

$$\overline{1 + \frac{rcy^n}{\overline{q+1}\,.\,x} + \frac{r\,.\,\overline{r-1}\,.\,c^2 y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,x^2} + \frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3 y^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,x^3} +}$$

&c. muſt alſo be equal to each other, let the ſeveral Quan-

Quantities, therein concerned, be what they will (which may be otherwise proved, independent of Fluxions.) Therefore, if in the room of q and s we write any other Quantities p and t, the Equation will, *still*, hold, and will then become $\overline{1 + \frac{\overline{t+1}\cdot cy^n}{\overline{p+1}\cdot a} + \frac{\overline{t+1}\cdot\overline{t+2}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot a^2}}$ $+$ &c. $= \frac{a}{x} \times \overline{1 + \frac{rcy^n}{\overline{p+1}\cdot x} + \frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot x^2}}$ &c. (t being $= p+r$.)

Moreover, if as many Terms of the first Series $1 + \frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a} + \frac{\overline{s+1}\cdot\overline{s+2}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot a^2} + \frac{\overline{s+1}\cdot\overline{s+2}\cdot\overline{s+3}\cdot c^3y^3}{\overline{q+1}\cdot\overline{q+2}\cdot\overline{q+3}\cdot a^3}$ &c. be taken as are denoted by any given Number v, and the last of them be represented by $\mathcal{Q}$, it is evident, from the Law of the Series, that the first of the remaining Terms will be expressed by $\mathcal{Q} \times \frac{s+v}{q+v} \times \frac{cy^n}{a}$; the second, of them, by $\mathcal{Q} \times \frac{\overline{s+v}}{q+v} \times \frac{\overline{s+v+1}}{q+v+1} \times \frac{c^2y^{2n}}{d^2}$ &c. and therefore the Sum of all of them (putting $q+v=p$ and $s+v$ $(=r+q+v) = t$) will be $= \mathcal{Q} \times \frac{t}{p} \times \frac{cy^n}{a} + \mathcal{Q} \times \frac{t}{p} \times \frac{t+1}{p+1} \times \frac{c^2y^{2n}}{a^2}$ + &c. $= \frac{t\mathcal{Q}cy^n}{pa} \times \overline{1 + \frac{\overline{t+1}\cdot cy^n}{\overline{p+1}\cdot a} + \frac{\overline{t+1}\cdot\overline{t+2}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot a^2}}$ + &c. $= \frac{t\mathcal{Q}cy^n}{px} \times \overline{1 + \frac{rcy^n}{\overline{p+1}\cdot x} + \frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{p+1}\cdot\overline{p+2}\cdot x^2}}$ &c. (by writing the Series found above in the room of its Equal) and consequently the whole Series (including the v first Terms) $= 1 + \frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a} +$

$$\frac{\overline{s+1}.\overline{s+2}.c^2y^{2n}}{\overline{q+1}\,\overline{q+2}.a^2}\ (v) + \frac{tQcy^n}{px} \times \overline{1 + \frac{rcy^n}{\overline{p+1}.x} +}$$
$$\overline{\frac{r.\overline{r-1}.c^2y^{2n}}{\overline{p+1}.\overline{p+2}.x^2} + \frac{r.\overline{r-1}.\overline{r-2}.c^3y^{3n}}{\overline{p+1}.\overline{p+2}.\overline{p+3}.x^3} + \&c.}$$

Which, drawn into the general Multiplicator $\frac{x^{r+1}\times y^{qn}}{qna}$ (*vid. Art.* 347.) will give the Fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ (or $x^r y^{qn-1}\dot{y}$) according to a new Form, compounded out of the two preceding ones; where the second Series (the Value of p being large in respect of r) will always converge much faster than the remaining Part of the first, for which it is substituted: But this will, more fully, appear from what follows hereafter. It will be proper to take notice here that the Fluent of $\overline{a+cz^n}|^m \times z^{qn-1}\dot{z}$ (the Fluxion first proposed, where the second Term under the Vinculum is positive) will also be had from hence (by writing z for y, m for r, and $-c$ for c) and is therefore equal to $\frac{x^{m+1}z^{qn}}{qna}$ drawn into the Sum of the two following Serieses,

$$1 - \frac{\overline{s+1}.cz^n}{\overline{q+1}.a} + \frac{\overline{s+1}.\overline{s+2}.c^2z^{2n}}{\overline{q+1}.\overline{q+2}.a^2} - \frac{\overline{s+1}.\overline{s+2}.\overline{s+3}.c^3z^{3n}}{\overline{q+1}.\overline{q+2}.\overline{q+3}.a^3}\ (v)$$
$$- \frac{tQcz^n}{px} \times \overline{1 - \frac{mcz^n}{\overline{p+1}.x} + \frac{m.\overline{m-1}.c^2z^{2n}}{\overline{p+1}.\overline{p+2}.x^2} -}$$
$$\overline{\frac{m.\overline{m-1}.\overline{m-2}.c^3z^{3n}}{\overline{p+1}.\overline{p+2}.\overline{p+3}.x^3}} + \&c.$$

Where, $s=m+q$, $p=v+q$, $t=s+v$, $x=a+cz^n$, and $Q=$ the last Term of the first Series continued to v Terms; v being any whole Number, at pleasure. A few Examples will shew the Use of what is above delivered.

351. Ex. 1. *Let* $\frac{\dot{z}}{1+z}$, or $\overline{1+z}|^{-1}\dot{z}$, *be propounded.* Which being compared with $\overline{a+cz^n}|^m \times z^{qn-1}\dot{z}$, we have $a=1$, $c=1$, $n=1$, $x=1+z$, $m=-1$, $qn-1=0$, or $q=1$; whence alſo $s(m+q)=0$, $p(v+q)=v+1$, $t(s+v)=v$, and conſequently the Fluent itſelf (by ſubſtituting theſe ſeveral Values in the laſt general Theorem) $= z$ into

$$1-\frac{z}{2}+\frac{z^2}{3}-\frac{z^3}{4}\,(v)-\frac{vQ}{\overline{v+1}\cdot x}\times\overline{1+\frac{z}{\overline{v+2}\cdot x}+\frac{2\cdot z^2}{\overline{v+2}\cdot\overline{v+3}\,x^2}+\frac{2\cdot 3\cdot z^3}{\overline{v+2}\cdot\overline{v+3}\cdot\overline{v+4}\cdot x^3}}$$

&c. Where (Q) the laſt Term of the firſt Series being $\pm\frac{z^{v-1}}{v}$, the Multiplicator $\left(\frac{vzQ}{\overline{v+1}\cdot x}\right)$ to the Second, will be $=\mp\frac{z^v}{\overline{v+1}\cdot x}$; and ſo the Fluent itſelf will be reduced to

$$z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}\,(v)\mp\frac{z^{v+1}}{\overline{v+1}\cdot x}\times\overline{1+\frac{z}{\overline{v+2}\cdot x}+\frac{2\cdot z^2}{\overline{v+2}\cdot\overline{v+3}\cdot x^2}}+\text{\&c.}$$

In which the Signs $-$ and $+$, before z^{v+1}, obtain alternately, according as v is an odd or even Number. But, to ſhew the Advantage of expreſſing the Fluent in this Manner, by two different Serieſes, let $z=1$, and let v be taken $=8$; then the Value of the firſt Series (continued to 8 Terms) being $=0{,}6345238$ &c. and *That* of the ſecond Series $=\frac{1}{18}+\frac{A}{20}+\frac{2B}{22}+\frac{3C}{24}+\frac{4D}{26}+\frac{5E}{28}$ &c. (where A, B, C, D &c. denote the Terms preceding thoſe where they ſtand) $=0{,}0555555+0{,}0027778+0{,}0002525+0{,}0000316+0{,}0000048+0{,}0000009+0{,}0000002=0{,}0586233$; it is evident that

that the Fluent of $\frac{\dot{z}}{1+z}$, when z becomes $= 1$, will be $= 0,6345238 + 0,0586233 = 0,6931471$: Which is true to the very laſt Decimal Place; and would have required, at leaſt, 100000 Terms of the firſt, or common, Series.

352. Ex. 2. *Let the Fluent of* $\frac{\dot{z}}{1+z^2}$ *(expreſſing the Arch whoſe Radius is* 1 *and Tangent* z*) be required.*

In this Caſe we have $a=1$, $c=1$, $n=2$, $x=1+zz$, $m=-1$, $qn-1=0$, or $q=\frac{1}{2}$, $s=-\frac{1}{2}$, $p=v+\frac{1}{2}$, and the Fluent itſelf $= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7}\ (v) \pm \frac{z^{2v+1}}{\overline{2v+1}\,.\,x} \times \overline{1 + \frac{2\,.\,z^2}{\overline{2v+3}\,.\,x} + \frac{2\,.\,4\,.\,z^4}{\overline{2v+3}\,.\,\overline{2v+5}\,.\,x^2} + \frac{2\,.\,4\,.\,6\,.\,z^6}{\overline{2v+3}\,.\,\overline{2v+5}\,.\,\overline{2v+7}\,.\,x^3}}$ &c. Where, if z be taken $=1$, and $v=6$, we ſhall have $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{26} \times \overline{1 + \frac{1}{15} + \frac{1}{15} \times \frac{2}{17} + \frac{1}{15} \times \frac{2}{17} \times \frac{3}{19}}$ &c. $= 0,785398 =$ the Fluent of $\frac{\dot{z}}{1+z^2}$ when $z = 1$ $(= \frac{1}{8}$ of the Periphery of the foreſaid Circle) Which Number, brought out by taking, only, 8 Terms of the ſecond Series, is more exact than if 100000 Terms of the common Series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ &c. had been uſed. And, if z be taken $= \sqrt{\frac{1}{3}}$ (= Tangent of 30°) and $v=6$, as before, the ſame Number of Terms will be ſufficient to give the Anſwer, true to twice the Decimal Places above exhibited.

353. Ex. 3. *Let the Fluxion proposed be* $\overline{e^4+y^4}|^{\frac{1}{2}}\times\dot{y}$.

Here we have, $a=e^4$, $c=1$, $z=y$, $n=4$, $x=e^4+y^4$, $m=\frac{1}{2}$, $q=\frac{1}{4}$, $s, (m+q) = \frac{3}{4}$, $p\ (v+q) = v+\frac{1}{4}$; $t\ (s+v) = v+\frac{3}{4}$; and therefore the Fluent sought (by Substitution) is $= \frac{x^{\frac{3}{2}}y}{e^4}$ into $1-\frac{7y^4}{5e^4}+\frac{7\cdot 11y^8}{5\cdot 9e^8}-\frac{7\cdot 11\cdot 15y^{12}}{5\cdot 9\cdot 13e^{12}}\ (v) - \frac{\overline{4v+3}\cdot Qy^4}{\overline{4v+1}\cdot x}\times\overline{1-\frac{2y^4}{\overline{4v+5}\cdot x}-\frac{2\cdot 2y^8}{\overline{4v+5}\cdot\overline{4v+9}\cdot x^2}-\frac{2\cdot 6\cdot 2y^{12}}{\overline{4v+5}\cdot\overline{4v+9}\cdot\overline{4v+13}\cdot x^3}-}$ &c. in which (as in all other Cases) Q denotes the last Term of the first Series. This Fluent approximates equally fast with those in the foregoing Examples: And it may be observed farther, that the Fluent will always converge, however great the Value of z is taken, if both a and c, in the general Fluxion $\overline{a+cz^n}|^m\times z^{qn-1}\dot{z}$, are positive Quantities. But, if the second Term under the Vinculum be negative, the Case will be otherwise, when that Term becomes greater than half the First; since the Powers of $\frac{cz^n}{x}$, in the latter Part of the Fluent, will then form an increasing Geometrical Progression. It may, therefore, be of use to shew how the Theorem may be varied so as to answer in this Case. In order thereto, if in the Equations $s=r+q$, and $1+\frac{\overline{s+1}\cdot cy^n}{\overline{q+1}\cdot a}+\frac{\overline{s+1}\cdot\overline{s+2}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot a^2}$ &c. $=\frac{a}{x}\times\overline{1+\frac{rcy^n}{\overline{q+1}\cdot x}+\frac{r\cdot\overline{r-1}\cdot c^2y^{2n}}{\overline{q+1}\cdot\overline{q+2}\cdot x^2}}$ &c. (*given in Art.* 350.) you write k for r, and p for q, and multiply by $\frac{x}{a}$, you

you will have $s=k+p$, and $\overline{1+\frac{kcy^n}{\overline{p+1}\,.\,x}+\frac{k\,.\,\overline{k-1}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,x^2}\ \&c.} = \frac{x}{a}\times\overline{1+\frac{\overline{s+1}\,.\,cy^n}{\overline{p+1}\,.\,a}+\frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,a^2}\ \&c.}$

Moreover, if the v first Terms of the above Series $1+\frac{rcy^n}{\overline{q+1}\,.\,x}+\frac{r\,.\,\overline{r-1}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,x^2}$ &c. be taken, and the last of them be denoted by Q, it is plain the first of the remaining Terms will be $= Q\times\frac{r-v+1}{q+v}\times\frac{cy^n}{x}$, the second $= Q\times\frac{r-v+1}{q+v}\times\frac{r-v}{q+v+1}\times\frac{c^2y^{2n}}{x^2}$, &c. and the Sum of them all (putting $q+v=p$, and $r-v=k$) equal to $\frac{\overline{k+1}\,.\,Qcy^n}{px}\times\overline{1+\frac{kcy^n}{\overline{p+1}\,.\,x}+\frac{k\,.\,\overline{k-1}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,x^2}\ \&c.} = \frac{\overline{k+1}\,.\,Qcy^n}{px}\times\frac{x}{a}\times\overline{1+\frac{\overline{s+1}\,.\,cy^n}{\overline{p+1}\,.\,a}+\frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,a^2}\ \&c.}$ (by the Equation above) and consequently the Sum of the whole Series ($1+\frac{rcy^n}{\overline{q+1}\,.\,x}$ &c.) $= 1+\frac{rcy^n}{\overline{q+1}\,.\,x}+\frac{r\,.\,\overline{r-1}\,.\,c^2y^{2n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,x^2}+\frac{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,.\,c^3y^{3n}}{\overline{q+1}\,.\,\overline{q+2}\,.\,\overline{q+3}\,.\,x^3}\ (v)+\frac{\overline{k+1}\times cy^n\,Q}{pa}\times\overline{1+\frac{\overline{s+1}\,.\,cy^n}{\overline{p+1}\,.\,a}+\frac{\overline{s+1}\,.\,\overline{s+2}\,.\,c^2y^{2n}}{\overline{p+1}\,.\,\overline{p+2}\,.\,a^2}+\&c.}$ Which,

 mul-

multiply'd by $\frac{x^r y^{qn}}{qn}$, gives the Fluent of $\overline{a-cy^n}|^r \times y^{qn-1}\dot{y}$ (* or $x^r y^{qn-1}\dot{y}$) where $k = r - v$, $p = v + q$, $s\ (=k+p) = r+q$ and $x = a - cy^n$. I ſhall put down one Example of the Uſe of this laſt general Expreſſion; where we will take $\dot{y}\sqrt{2y-y^2}$ or $\overline{2-y}|^{\frac{1}{2}} \times y^{\frac{1}{2}}\dot{y}$ (being the Fluxion of the Area of the Circle whoſe Radius is Unity and verſed Sine y) In which Caſe, $a=2$, $c=1$, $n=1$, $r=\frac{1}{2}$, $qn-1=\frac{1}{2}$, or $q=\frac{3}{2}$, $k=-v+\frac{1}{2}$, $p=v+\frac{3}{2}$, $s=2$, $x=2-y$; and therefore the Fluent ſought $= \frac{2x^{\frac{1}{2}}y^{\frac{3}{2}}}{3}$ into $1 + \frac{y}{5x} - \frac{y^2}{5\cdot 7x^2} + \frac{3y^3}{5\cdot 7\cdot 9x^3} - \frac{3y^4}{7\cdot 9\cdot 11x^4} + \frac{3y^5}{9\cdot 11\cdot 13x^5}\ (v) \mp \frac{\overline{2v-3}\cdot yQ}{\overline{2v+3}\cdot 2} \times \overline{1 + \frac{3y}{2v+5} + \frac{3\cdot 4y^2}{\overline{2v+5}\cdot\overline{2v+7}} + \frac{3\cdot 4\cdot 5y^3}{\overline{2v+5}\cdot\overline{2v+7}\cdot\overline{2v+9}}}$ &c. Which, if y be taken $=1$, and $v=5$, will become $= \frac{2}{3} + \frac{A}{5} - \frac{B}{7} + \frac{C}{3} - \frac{5D}{11} + \frac{7E}{2\times 13} + \frac{3F}{15} + \frac{4G}{17} + \frac{5H}{19}$ &c. $=0{,}785398$ (where A, B, C &c. denote the ſeveral Terms, reſpectively, without their Signs.) In bringing out which Concluſion, ſix Terms of the ſecond Series are required: But if y be taken $=\frac{1}{2}$ the Radius of the foreſaid Circle, then four Terms of each Series will be more than ſufficient to give the ſame Number of Decimal Places. And it may likewiſe be obſerved, that, although no general Rule can be laid down for aſſigning the Value of v, ſo as to anſwer the beſt in all Caſes, yet the Concluſion will,

* Art. 348, 349.

will, for the general Part, require the feweſt Terms, when the Number of thoſe, taken in each Series, is nearly the ſame.

354. But, after all, another Theorem or Series, ſtill, ſeems wanting, to expreſs the Value of the whole Fluent, when the Quantity under the *Vinculum* becomes equal to Nothing (which, in the Reſolution of Problems, is, commonly, what is required.) For, it is plain the laſt, above given, anſwers no better, *here*, than that preceding it; becauſe (the Diviſor (x) being Nothing) the former Part of it fails.

In order, therefore, to determine a proper *Form*, to obtain in this Circumſtance, it will be requiſite to obſerve, firſt of all, *from Article* 286. that the whole Fluent of $\overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, ſuppoſing *that* of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$ to be denoted by A, will be truly expreſſed by $\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\ (v) \times \frac{a^v A}{b^v}$: In which $t=m+p+1$; and where it is requiſite that the Values of $m+1$ and p ſhould be poſitive, otherwiſe, A being infinite, the Fluent (or Compariſon) fails. Hence, becauſe the whole Fluent of $\overline{a-bz^n}|^m \times z^{n-1}\dot{z}$, (when $a-bz^n=0$) is found $=\frac{a^{m+1}}{\overline{m+1}\times nb}$, by the common Way *, it follows, by writing this Value in the Room of A, and expounding p by 1, that the *whole Fluent* of $\overline{a-bz^n}|^m \times z^{n+vn-1}\dot{z}$ is rightly expreſſed by $\frac{1}{m+2} \times \frac{2}{m+3} \times \frac{3}{m+4}\ (v) \times \frac{a^{m+v+1}}{\overline{m+1}\times nb^{v+1}}$, or by $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\ (v+1) \times \frac{a^{m+v+1}}{\overline{v+1}\times nb^{v+1}}$: Whence

* Art. 77. and 78.

That of $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, by ſubſtituting r inſtead of $v+1$, will conſequently be equal to $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3} (r) \times \frac{a^{m+r}}{rnb^r}$. Let this Quantity be denoted by B; then, *by the ſame Article*, the Fluents of the ſeveral Terms of the Series $1, \frac{bz^n}{a}, \frac{b^2z^{2n}}{a^2}, \frac{b^3z^{3n}}{a^3}$ &c. drawn into the general Multiplicator $\overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, will be, reſpectively, expounded by thoſe of the Series 1, $\frac{r}{t}, \frac{r\cdot\overline{r+1}}{t\cdot\overline{t+1}}, \frac{r\cdot\overline{r+1}\cdot\overline{r+2}}{t\cdot\overline{t+1}\cdot\overline{t+2}}$ &c. drawn into B; t being $=m+r+1$.

If now the Differences of the Quantities 1, $\frac{r}{t}$, $\frac{r\cdot\overline{r+1}}{t\cdot\overline{t+1}}$ &c. be, continually, taken *; and for $r-t$ its Equal $-m-1$ be ſubſtituted, the Value of any Term of the Series, whoſe Diſtance from the firſt, excluſive, is denoted by s, or whoſe correſponding Term, in the preceding Series, is $\frac{b^sz^{sn}}{a^s}$, will be univerſally expreſſed by $1 - \frac{s\cdot\overline{m+1}}{1\cdot t} + \frac{s\cdot\overline{s-1}\cdot\overline{m+1}\cdot\overline{m+2}}{1\cdot 2 \times t\cdot\overline{t+1}} - \frac{s\cdot\overline{s-1}\cdot\overline{s-2}\cdot\overline{m+1}\cdot\overline{m+2}\cdot\overline{m+3}}{1\cdot 2\cdot 3 \times t\cdot\overline{t+1}\cdot\overline{t+2}}$ + &c. Where, if s be interpreted by 0, 1, 2, 3 &c. ſucceſſively, you will have the Values 1, $\frac{r}{t}, \frac{r\cdot\overline{r+1}}{t\cdot\overline{t+1}}$ &c. above exhibited: But, if s be taken as a Fraction, then the Value of ſuch an intermediate Term will be found as will give the

* *See my Mathematical Eſſays*, p. 94.

the Fluent of $\frac{b^s z^{sn}}{a^s} \times \overline{a-bz^n}|^m \times z^{rn-1}\dot{z}$, in any proposed Circumstance of s; which Fluent, it is evident, will therefore be expressed by $B \times \overline{1 - \frac{s.\overline{m+1}}{1.t} + \frac{s.\overline{s-1}.\overline{m+1}.\overline{m+2}}{1.2.t.\overline{t+1}}}$ &c. or its Equal $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3}\,(r) \times \frac{a^{m+r}}{rnb^r}$ into $1 - \frac{s.\overline{m+1}}{1.t} - \frac{\overline{s-1}.\overline{m+2}}{2.\overline{t+1}} \times E - \frac{\overline{s-2}.\overline{m+3}}{3.\overline{t+2}} \times F - \frac{\overline{s-3}.\overline{m+4}}{4.\overline{t+3}} \times G$ &c. (where E, F, G &c. denote the Terms immediately preceding those where they stand, under their proper Signs.) Whence, dividing by $\frac{b^s}{a^s}$, we have $\frac{1}{m+1} \times \frac{2}{m+2}\,(r) \times \frac{a^{m+r+s}}{rnb^{r+s}} \times \overline{1 - \frac{s.\overline{m+1}}{t} - \frac{\overline{s-1}.\overline{m+2}}{2.\overline{t+1}} \times E}$, &c. for the true Fluent of $\overline{a-bz^n}|^m \times z^{rn+sn-1}\dot{z}$.

From the last Fluent *that* of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$ (in which p denotes any positive Fraction, proper or improper) is very readily obtained: For, if the same (when $a-bz^n = 0$) be denoted *by* A; then the Fluent of $\overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$ will (according to the Article above quoted) be expressed by $\frac{p}{p+m+1} \times \frac{p+1}{p+m+2} \times \frac{p+2}{p+m+3}\,(v) \times \frac{a^v A}{b^v}$; supposing v any

positive

positive Integer. Therefore, by making $\overline{a-bz^n}|^m \times z^{rn+sn-1}\dot{z} = \overline{a-bz^n}|^m \times z^{pn+vn-1}\dot{z}$, or $r+s=p+v$, the corresponding Fluents must, also, be equal; that is,

$$\frac{p}{p+m+1} \times \frac{p+1}{p+m+2} (v) \times \frac{a^v A}{b^v} = \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3} (r) \times \frac{a^{m+p+v}}{rnb^{p+v}} \times \overline{1 - \frac{s \cdot \overline{m+1}}{t} - \&c.}$$

And consequently A (the whole Fluent of $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$)

$$= \frac{p+m+1}{p} \times \frac{p+m+2}{p+1} \times \frac{p+m+3}{p+2} (v) \times \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3} (r) \times \frac{a^{m+p}}{nrb^p} \times$$

into the Series

$$1 - \frac{s \cdot \overline{m+1}}{1 \cdot t} - \frac{\overline{s-1} \cdot \overline{m+2}}{2 \cdot \overline{t+1}} E - \frac{\overline{s-2} \cdot \overline{m+3}}{3 \cdot \overline{t+2}} F - \frac{\overline{s-3} \cdot \overline{m+4}}{4 \cdot \overline{t+3}} G \;\&c.$$

where $t=r+m+1$ and $s=p+v-r$; v and r being any whole positive Numbers at pleasure.

355. An Example, or two, of the Use of this Conclusion, may be proper.

1°. Let the *whole Fluent* of $\overline{1-x^2}|^{-\frac{1}{2}}\dot{x}$ (expressing the Length of $\frac{1}{4}$ of the Periphery of the Circle whose Radius is Unity) be demanded. In which Case, a being $+1$, $b=1$, $m=-\frac{1}{2}$, $n=2$, $p=\frac{1}{2}$, $t=r+\frac{1}{2}=\frac{2r+1}{2}$, and $s=v-r+\frac{1}{2}=\frac{2v-2r+1}{2}$, the Fluent sought will, therefore, (by substituting these Values) be had $= \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} (v) \times \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} (r) \times$

$\frac{1}{2r}$ into $1 - \frac{1 \cdot \overline{2v - 2r + 1}}{2 \cdot \overline{2r + 1}} - \frac{3 \cdot \overline{2v - 2r - 1}}{4 \cdot \overline{2r + 3}} E -$ $\frac{5 \cdot \overline{2v - 2r - 3}}{6 \cdot \overline{2r + 5}} F - \frac{7 \cdot \overline{2v - 2r - 5}}{8 \cdot \overline{2r + 7}} G$ &c. Which, by expounding v by 5 and r by 3, will become $=$ 2,16719 &c. into $1 - \frac{1 \cdot 5}{2 \cdot 7} - \frac{3 \cdot 3}{4 \cdot 9} E - \frac{5 \cdot 1}{6 \cdot 11} F +$ $\frac{7 \cdot 1}{8 \cdot 13} G + \frac{9 \cdot 3}{10 \cdot 15} H + \frac{11 \cdot 5}{12 \cdot 17} I +$ &c. $= 1,5703$. In the bringing out of which Value, all the Terms above exhibited are requiſite: But, of the common Series, $1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} +$ &c. more than 10 times that Number of Terms would be neceſſary to anſwer with the ſame Degree of Exactneſs.

Ex. 2°. *Let the Fluxion propoſed be* $\frac{d\dot{x}}{x^{\frac{1}{2}} \sqrt{d^2 - x^2}}$ (whoſe whole Fluent, when $x = d$, expreſſes the Time of Deſcent of a heavy Body in half the Arch of a Semicircle, whoſe Radius is d*.) * Art. 207.

Here, by comparing $\overline{d^2 - x^2}|^{-\frac{1}{2}} \times x^{-\frac{1}{2}} \dot{x}$ with $\overline{a - bz^n}|^m \times z^{pn-1} \dot{z}$, we have $a = d^2$, $b = 1$, $n = 2$, $pn - 1 = -\frac{1}{2}$, or $p = \frac{1}{4}$; alſo $s\ (p + v - r) = v - r + \frac{1}{4}$, $t\ (r + m + 1) = r + \frac{1}{2}$: Whence, by taking r and v, each, equal to 4, the Fluent, itſelf, comes out $=$ $\frac{3}{1} \times \frac{7}{5} \times \frac{11}{9} \times \frac{15}{13}$ into $\frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$ $\frac{d^{\frac{1}{2}}}{8}$ into $1 - \frac{1 \cdot 1}{4 \cdot 9} + \frac{3 \cdot 3}{8 \cdot 11} E + \frac{7 \cdot 5}{12 \cdot 13} F + \frac{11 \cdot 7}{16 \cdot 15}$ $G +$ &c. $= \frac{11 \cdot 16 d^{\frac{1}{2}}}{13 \cdot 5}$ into $\overline{1 - \frac{1}{36} + \frac{9E}{88} +}$ &c.

$= 2{,}6215d^{\frac{1}{2}}$: Which is to $2\sqrt{2d}$, the Time of Defcent along the vertical Diameter of the forefaid Circle, as 2,6215 to 2,8284, or as 100 to 108, nearly.

After the fame Manner the Fluent will be found in other Cafes: But, with regard to the affigning of the Values of r and v, it may be obferved, that the Anfwer will, commonly, be brought out with the leaft Trouble when v is taken greater by an Unit or two than r; which laft Quantity muft be greater or lefs, according as a greater or lefs Degree of Exactnefs is neceffary.——From the foregoing Expreffions, by varying the Values of v and r, a great Number of Theorems, for the Summation of Seriefes, may be deduced. But this being foreign to my prefent Purpofe, I am not at Leifure to purfue it here.

356. Hitherto Regard has been had to Fluxions of the Binomial-Kind: But, from thence, the Fluents of Trinomials may alfo be found; when thefe laft can be reduced to Binomials *(by Art.* 307.) without introducing new Radical Quantities.——Befides which Method, I fhall, here, give another, which will anfwer where that fails, and is alfo applicable to *Multinomials.*

In order thereto, let the Fluent of $\overline{a + cz^n}|^m \times z^{pn-1}\dot{z}$, be denoted by A; and let it be required to find, from thence, the Fluent of the Radical *Multinomial*, or Infinite Series, $\overline{a + cx^n + dx^{2n} + ex^{3n} + fx^{4n} \, \&c.}|^m \times x^{pn-1}\dot{x}$.

Make $cz^n = cx^n + dx^{2n} + ex^{3n} + \&c.$ and $y = x^{pn}$; then, x^n being $= y^{\frac{1}{p}}$, if this Value be fubftituted for x^n, in the firft Equation, it will become $cz^n = cy^{\frac{1}{p}} + dy^{\frac{2}{p}} + ey^{\frac{3}{p}} \, \&c.$ Whence, by reverting the Series, *(by Art.*

Art. 275.) $y\ (x^{pn})$ is found $= z^{pn} + Rz^{pn+n} + Sz^{pn+2n} + Tz^{pn+3n} + \&c.$

Where $R = -\frac{pd}{c}$, $S = \frac{p \cdot \overline{p+3}}{2} \times \frac{d^2}{c^2} - \frac{pe}{c}$, $T = -\frac{p \cdot \overline{p+4} \cdot \overline{p+5}}{6} \times \frac{d^3}{c^3} + p \cdot \overline{p+4} \times \frac{de}{c^2} - \frac{pf}{c}$ &c.

Moreover, by taking the Fluxion of the Equation thus brought out, and dividing by pn, we have $x^{pn-1}\dot{x} = z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1}\dot{z} + \frac{p+3}{p} \times Tz^{pn+3n-1}\dot{z} + \&c.$

Now let this Value, with that of $cx^n + dx^{2n} + ex^{3n} + \&c.$ (given above) be substituted in the proposed Fluxion, and it will become $\overline{a + cz^n}|^m \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1}\dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1}\dot{z} + \&c.}$

Also, let v denote the Place, or Distance, of any Term of this Series from the first, exclusive; then the Term itself, drawn into the general Multiplicator, will be expressed by $\overline{a + cz^n}|^m \times \frac{p+v}{p} \Delta\ z^{pn+vn-1}\dot{z}$ (Δ being the corresponding Coefficient R, S, T, &c.) and the Fluent thereof by $\frac{p+v}{p} \Delta \times \overline{a + cz^n}|^{m+1} \times z^{pn} \times \overline{\frac{z^{vn-n}}{s+1 \cdot nc} - \frac{qaz^{vn-2n}}{s+1 \cdot snc^2} + \frac{q \cdot \overline{q-1} \cdot a^2 z^{vn-3n}}{s+1 \cdot s \cdot \overline{s-1} \cdot nc^3}\ (v)} \pm \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2}\ (v) \times \frac{p+v}{p} \times \Delta \times \frac{a^v A}{c^v}$ (*Art.* 283.)

Where,

Where, $q=p+v-1$, $s=m+q$, $t=p+m+1$, and the Sign of the laſt Term is + or —, according as v is an even or odd Number. Now, if in the Fluent thus given, v be expounded by 1, 2, 3, 4, &c. ſucceſſively, it is evident the Fluent of the whole Expreſſion will, in all Circumſtances of z, be obtained. But, if the Coefficient c be negative, ſo that $a+cz^n$ may (by increaſing z) become equal to Nothing; then, in that Circumſtance, the Fluent of the foreſaid general Term $\overline{a+cz^n}|^m \times \frac{p+v}{p} \Delta z^{pn+vn-1}\dot{z}$ (or $\overline{a-bz^n}|^m \times \frac{p+v}{p} \Delta z^{pn+vn-1}\dot{z}$, making $-c=b$) being, *barely*, $= \frac{p}{t} \times$

* Art. 286. $\frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{p+v}{p} \times \frac{\Delta a^v A}{b^v}$ *, it follows that the whole Fluent of the given Expreſſion, or its Equal, $\overline{a-bz^n}|^m \times \overline{z^{pn-1}\dot{z} + \frac{p+1}{p} Rz^{pn+n-1}\dot{z}\ \&c.}$ will be truly repreſented by $A \times \overline{1 + \frac{\overline{p+1}\,.\,Ra}{tb} + \frac{\overline{p+1}\,.\,\overline{p+2}\,.\,Sa^2}{t\,.\,\overline{t+1}\,.\,b^2}}$ $+ \frac{\overline{p+1}\,.\,\overline{p+2}\,.\,\overline{p+3}\,.\,Ta^3}{t\,.\,\overline{t+1}\,.\,\overline{t+2}\,.\,b^3}$ &c. In which, $R=\frac{pd}{b}$, $S=\frac{p\,.\,\overline{p+3}}{2} \times \frac{d^2}{b^2} + \frac{pe}{b}$, $T=\frac{p\,.\,\overline{p+4}\,.\,\overline{p+5}}{6} \times \frac{d^3}{b^3} + \frac{p\,.\,\overline{p+4}}{1} \times \frac{de}{bb} + \frac{pf}{b}$, &c. and $A=$ the Fluent $\overline{a-bz^n}|^m \times z^{pn-1}\dot{z}$, when $a-bz^n=0$.

357. Hence, if the Fluxion given be of the Trinomial Kind (then, e, f, &c. vaniſhing the *whole Fluent* of

of $\overline{a - bx^n + dx^{2n}}|^m \times x^{pn-1}\dot{x}$ (when $a - bx^n + dx^{2n} = 0$) will, by substituting for *R*, *S*, *T*, &c. be $= A \times$

$$\overline{1 + \frac{p \cdot \overline{p+1}}{1 \cdot t} \times \frac{ad}{bb} + \frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot \overline{p+3}}{1 \cdot 2 \cdot t \cdot \overline{t+1}} \times \overline{\frac{ad}{bb}}|^2 +}$$

$$\overline{\frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot \overline{p+3} \cdot \overline{p+4} \cdot \overline{p+5}}{1 \cdot 2 \cdot 3 \cdot t \cdot \overline{t+1} \cdot \overline{t+2}} \times \overline{\frac{ad}{bb}}|^3} + \&c.$$

358. If $m+1$ and p are the Halves of any odd Affirmative-Numbers, the *Fluent* of $\overline{a - bz^n}|^m \times z^{pn-1}\dot{z}$, when $a - bz^n = 0$, will be equal to

$$\frac{1 \cdot 3 \cdot 5 \cdot 7\ (m+\frac{1}{2}) \times 1 \cdot 3 \cdot 5 \cdot 7\ (p-\frac{1}{2})}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12\ (m+p)} \times \frac{a^{m+p}G}{nb^p}.$$ * Art. 293, and 298.

G being the Periphery of the Circle whose Diameter is Unity. Therefore the Fluent of $\overline{a - bx^n + dx^{2n} + ex^{3n}\ \&c.}|^m \times x^{pn-1}\dot{x}$, or its Equal, $\overline{a - bz^n}| \times z^{pn-1}\dot{z} + \frac{\overline{p+1}}{p} \times Rz^{\overline{pn+n-1}}\dot{z}$ &c. is found, in this Case, by multiplying the Expression here given, into the foregoing Series, $1 + \frac{\overline{p+1} \cdot Ra}{tb} + \&c.$

359. An Example or two will help to shew the Use of what is above delivered.

First, let the *Fluent* of $\frac{\dot{x}}{\sqrt{a^2 - x^2 - \frac{x^4}{raa}}}$ (when the Divisor becomes equal to Nothing) be required.

Then, by comparing $\overline{a^2 - x^2 - \frac{x^4}{raa}}|^{-\frac{1}{2}}$ with the

the general *Trinomial* $\overline{a - bx^n + dx^{2n}}|^m \times x^{pn-1}\dot{x}$, it appears that a^2 must be, here, wrote in the room of a, and that n, m, p, b and d, will be interpreted by 2, $-\frac{1}{2}$, $\frac{1}{2}$, 1, and $-\frac{1}{raa}$ respectively: Whence we have t, $(p+m+1) = 1$, $\frac{1.3.5\,(m+\frac{1}{2}) \times 1.3.5\,(p-\frac{1}{2})}{2.4.6\,(m+p)}$ $\times \frac{a^{m+p}G}{nb^p} = \frac{G}{2}$, and the Fluent sought $= \frac{G}{2} \times$

$$\overline{1 - \frac{1.3}{2.2r} + \frac{1.3.5.7}{2.2.4.4r^2} - \frac{1.3.5.7.9.11}{2.2.4.4.6.6r^3} + \&c.}$$

360. The second Example shall be, to find the Fluent expressing the *Apside Angle* in an Orbit described by means of a centripetal Force varying according to any Power of the Distance.

In which Case the given Fluxion being

$$\frac{\pm p\dot{x}}{x\sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2 + \frac{2x^{n+3}}{n+1}}}$$ (*Vid. Art.* 242.

where A is supposed the higher Apse, and CA (and consequently Cb) equal to Unity) we shall, by putting $1 - p^2 = \beta$, $\frac{n+3}{2} = v$, and $1 - x^2 = y$, reduce it to

$$\frac{\frac{1}{2}\sqrt{1-\beta} \times \dot{y}}{\overline{1-y} \times \sqrt{\beta y + \frac{1 - vy - \overline{1-y}|^v}{1-v}}} = \frac{1}{2}\sqrt{1-\beta} \times$$

$$\overline{\beta - \frac{vy}{2} + \frac{v\,.\,\overline{v-2}}{2\,.\,3}\,.\,y^2 - \frac{v\,.\,\overline{v-2}\,.\,\overline{v-3}}{2\,.\,3\,.\,4}\,.\,y^3 + \&c.}|^{-\frac{1}{2}}$$

$\times y^{-\frac{1}{2}}\dot{y} + y^{\frac{1}{2}}\dot{y} + y^{\frac{3}{2}}\dot{y} + y^{\frac{5}{2}}\dot{y} + \&c.$ Where the Quantity

tity under the Radical Sign (now anſwering to the Form above preſcribed) being compared with $\overline{a-bx^n+dx^{2n}+ex^{3n}\ \&c.}\Big|^m$, we have $m=-\frac{1}{2}$, $n=1$, $b=\frac{v}{2}$, $\frac{d}{b}$, $=\frac{v-2}{3}$ $\frac{e}{b}=-\frac{\overline{v-2}\,.\,\overline{v-3}}{3\cdot 4}$ &c. Alſo the Value of p with regard to the firſt Term $(y^{-\frac{1}{2}}\dot{y})$ will be $=\frac{1}{2}$ (becauſe $pn-1=-\frac{1}{2}$) likewiſe its Value in the ſecond Term $(y^{\frac{1}{2}}\dot{y})$ is $=\frac{3}{2}$; in the third $=\frac{5}{2}$ &c. In the firſt of theſe Caſes we, therefore, have $t\ (m+p+1)=1$, $R\ \left(p\times\frac{d}{b}\right)=\frac{v-2}{6}$, $S=\frac{\overline{v-2}\,.\,\overline{4v-5}}{72}$, $T=\frac{\overline{v-2}\,.\,\overline{16v^2-37v+22}}{16\times 45}$. Whence it follows, that the Fluent of the firſt Term $\left(\overline{\beta-\frac{vy}{2}+\frac{v\,.\,\overline{v-2}}{2\cdot 3}\,.\,y^2\ \&c.}\Big|^{-\frac{1}{2}}\times y^{-\frac{1}{2}}\dot{y}\right)$ when the Quantity under the Radical Sign becomes equal to Nothing (or the Body arrives at its lower Apſe) will be truly expreſſed by $\frac{G}{\sqrt{\frac{1}{2}v}}$ into $1+\frac{v-2}{2v}\,.\,\beta+\frac{5\,.\,\overline{v-2}\,.\,\overline{4v-5}}{48v^2}\,.\,\beta^2+\frac{7\,.\,\overline{v-2}\,.\,\overline{16v^2-37v+22}}{6\times 48v^3}\,.\,\beta^3$ + &c.

In the ſame Manner it will appear, that the Fluent of the ſecond Term, in that Circumſtance, is $=\frac{G}{\sqrt{\frac{1}{2}v}}\times\overline{\frac{1}{v}\,.\,\beta+\frac{5\,.\,\overline{v-2}}{4v^2}\,.\,\beta^2+\frac{35\,.\,\overline{v-2}\,.\,\overline{2v-3}}{48v^3}\,.\,\beta^3}$ &c.

&c. that of the Third $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{\frac{3}{2v^2} \cdot \beta^2 + \frac{35 \cdot \overline{v-2}}{12v^3} \cdot \beta^3}$ &c. that of the Fourth $= \frac{5}{2v^3} \cdot \beta^3$ &c. &c.

Whence, the Fluent of the whole Series, by collecting these several Values together, will come out $= \frac{G}{\sqrt{\frac{1}{2}v}} \times \overline{1 + \frac{1}{2}\beta + \frac{20v^2 - 5v + 2}{48v^2} \cdot \beta^2 + \frac{112v^3 - 63v^2 - 42v - 8}{6 \times 48v^3} \cdot \beta^3 + \&c.}$ Which, drawn into $\frac{1}{2} \times \overline{1 - \frac{1}{2}\beta - \frac{1}{8}\beta^2 - \frac{1}{16}\beta^3 - \&c.}$ (the Value of the general Multiplicator $\frac{1}{2}\sqrt{1-\beta}$) gives $\frac{G}{\sqrt{2v}} \times \overline{1 \; * + \frac{\overline{v-2} \cdot \overline{2v-1}}{48} \cdot \frac{\beta^2}{v^2} + \frac{\overline{v-2} \cdot \overline{2v-1} \cdot \overline{2v-1}}{72} \times \frac{\beta^3}{v^3}}$ &c. for the true Measure of the Angle required, in Parts of the Radius, or Unity: From whence, by writing 180 instead of G, we shall have the same in Degrees: Which, last of all, by restoring n, becomes $\frac{180^\circ}{\sqrt{n+3}} \times \overline{1 \; * + \frac{\overline{n-1} \cdot \overline{n+2}}{24} \times \left.\frac{\beta}{n+3}\right| + \frac{\overline{n-1} \cdot \overline{n+2} \cdot \overline{n+2}}{18} \times \left.\frac{\beta}{n+3}\right|^3}$ &c.

Where n is the Exponent of the Law of the Force, whereby the Orbit is described; and β, the Defect of the Square of the Measure of the Celerity, at the higher *Apse*, below That which the Body ought to have to revolve in a Circle, this last being denoted by Unity. The

The ſame Concluſion may be otherwiſe derived, by bringing $1-y$, in the transformed Fluxion, under the *Vinculum*; but this Way of going to work, though we have but one Series to manage, will prove rather more troubleſome than the foregoing.

It will appear from the two preceding Examples, eſpecially the firſt of them, that this laſt Method of finding Fluents is, chiefly, uſeful when all the Terms of the given Expreſſion, after the two firſt, in reſpect of theſe, are but ſmall. Which is a Circumſtance that frequently occurs in the Reſolution of phyſical Problems; ſuch as determining the Effect of the Atmoſphere's Reſiſtance upon the Vibration of Pendulums; and the *Inequalities* of the Planets ariſing from their Action on each other.——In ſhort, wherever the Fluent, or the Quantity it expreſſes, would belong to the Circle, or ſome other of the Conic-Sections, were it not for the Interpoſition of ſome ſmall perturbating Force (whereby new Terms, ſmall in Compariſon of the two firſt, are introduced) the ſaid Method will be found of very great Service.

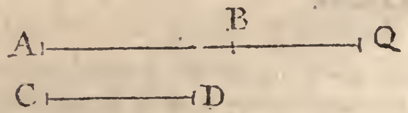
SECTION VIII.

The Use of Fluxions in determining the Motion of Bodies in resisting Mediums.

PROB. I.

361. *Supposing that a Body, let go from a given Point* A, *with a given Celerity, in a Right-line* AQ, *is resisted by a Medium (or any Force) acting according to a given Power of the Velocity: To determine the Velocity, and also the Space run over, at the End of a given Time.*

LET the given Celerity at A (measur'd by the Space which would be uniformly described in any proposed Time r) be put $= c$, and that at any other Point B, $= v$; moreover put AB $= x$, and the Time



of its Description $= z$; and let the Resistance, or Force, acting upon the Body at A, be such, that, if the same was to be uniformly continued, the Body would have all its Motion destroyed thereby, in the Time wherein it might move, uniformly, over a given Distance d (CD) with its first Velocity c: Which Time, let be denoted, by t.

Then, since the whole Celerity c would be destroy'd in the Time t, that Part of it which would be uniformly taken away in the Time r, above proposed, will be truly represented by $\frac{r}{t} \times c$; or by $\frac{cc}{d}$; which is equal to it, because the Spaces (c and d) described with the same Ce-

Celerity are always as the Times (*r* and *t*) of their Description; and therefore $\frac{r}{t} = \frac{c}{d}$.

Hence, the Resistance at B being to that at A (*by Hypothesis*) as v^n to c^n, it follows that the Velocity which might be destroyed in the given Time *r*, by a Force equal to the Resistance at B, will be expressed by $\frac{cc}{d} \times \frac{v^n}{c^n}$, or its Equal $\frac{v^n}{dc^{n-2}}$: Which Expression is, therefore, the true Measure of the Force of the said Resistance.

Now, it appears, from *Art.* 218. that, if the Force with which the Body is acted on (or the Velocity it would generate in the given Time *r*) be represented by *F*, the Relation of the Measures of the Velocity and Space gone over, will be expressed by the Equation $\pm v\dot{v} = F\dot{x}$: From whence, by writing $\frac{v^n}{dc^{n-2}}$ instead of *F*, we have $-v\dot{v} = \frac{v^n\dot{x}}{dc^{n-2}}$ (the Sign of $v\dot{v}$ being negative, because *v* decreases while *x* increases *.) From this Equation, we get $\dot{x} = -dc^{n-2}v^{1-n}\dot{v}$; whose Fluent is $x = -\frac{dc^{n-2} \times v^{2-n}}{2-n} +$; which, corrected (by taking $x = 0$, and $v = c$) becomes $x = \frac{-dc^{n-2} \times v^{2-n} + d}{2-n} = \frac{d}{n-2} \times \overline{\frac{c}{v}}\Big|^{n-2} - 1$.

* Art. 5.

Moreover, since the Time ($\dot{z}$) is to the Time *r*, as the Distance $\dot{x}$ to the Distance *v*, we also have $\dot{z}$ ($= \frac{r\dot{x}}{v}$) $= -rdc^{n-2}v^{-n}\dot{v}$; and consequently $z =$

$\frac{rd}{n-1\times c}\times\overline{\frac{c}{v}}\Big|^{n-1}-1=\frac{t}{n-1}\times\overline{\frac{c}{v}}\Big|^{n-1}-1$ (by writing t for its Equal $\frac{rd}{c}$): From which Equation we get $\frac{c}{v}=\overline{1+\overline{n-1}\times\frac{z}{t}}\Big|^{\frac{1}{n-1}}$: Likewise, from the preceding Equation, we get $\frac{c}{v}=$ $\overline{1+\overline{n-2}\times\frac{x}{d}}\Big|^{\frac{1}{n-2}}$: Which two equal Values being compared together, there, at length, results $x=$ $\frac{d}{n-2}$ into $\overline{1+\overline{n-1}\times\frac{z}{t}}\Big|^{\frac{n-2}{n-1}}-1$, for the required Relation of x and z. *Q. E. I.*

Corollary.

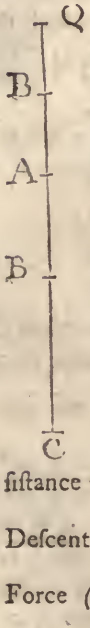
362. If $n=2$, or, the Resistance be in the Duplicate Ratio of the Velocity, the Equation exhibiting the Relation of z and v, will be $\frac{c}{v}=1+\frac{z}{t}$, or $v=\frac{c}{1+\frac{z}{t}}$: But the other Equation (the Fluent failing) becomes impracticable. Here x, the Fluent of $-\frac{\dot{v}}{v}$, will be explicable by $d\times$ *hyp. Log.* $\frac{c}{v}$ *, or by $d\times$ *hyp. Log.* $1+\frac{z}{t}$; because $v=\frac{c}{1+\frac{z}{t}}$.

* Art. 126.

In

In the like Manner, when $n=1$, or the Resistance is as the Velocity, the Relation of v, x and z, will be exhibited by the Equations $v = c\times\frac{\overline{d-x}}{d}$, and $z = t \times$ *hyp. Log.* $\frac{c}{v} = t \times$ *hyp. Log.* $\frac{d}{d-x}$. Which Case, and that above, are the only two wherein the general Solution fails.

PROB. II.



363. *If a Body, let go from a given Point A with a given Celerity, in a vertical Line CAQ, is acted on by an uniform Gravity, and also by a Medium, resisting according to any given Power of the Velocity; 'tis proposed to determine the Relation of the Times, the Velocities, and the Spaces gone over.*

Let the Notation in the preceding Problem be retained; and let the Force of Gravity, in the given Medium (measured by the Velocity it might generate in the proposed Time r *) be represented by b. Then, * Art. 361.
this Value being added to, or subtracted from $\left(\frac{v^n}{dc^{n-2}}\right)$ the Measure of the Resistance †, according as the Body is in its Ascent, or † Art. 361.
Descent, we thence get $\frac{v^n}{dc^{n-2}} \pm b$ for the whole Force (F) whereby the Motion, at B, is affected: Whence (*by Art.* 218) $\dot{x}\ \left(= \frac{-v\dot{v}}{F}\right) = \frac{-dc^{n-2}v\dot{v}}{v^n \pm bdc^{n-2}}$; and $\dot{z}\ \left(= \frac{r\dot{x}}{v}\ ‡\right) = \frac{-rdc^{n-2}\dot{v}}{v^2 \pm bdc^{n-2}}$: Whose Fluents ‡ Art. 362.

 may

may be had, by the Means of circular Arcs, and Logarithms, *from Art.* 331. *Q. E. I.*

COROLLARY I.

364. It appears that the Force $\left(\frac{v^n}{dc^{n-2}}\right)$ of the Resistance is to (*b*) *that* of Gravity, in the given Medium, as v^n to bdc^{n-2}: Therefore, if this Ratio be expounded by that of v^n to a^n, or a^n be put $= bdc^{n-2}$, it follows that a will express the Celerity with which the Resistance would be equal to the Gravity (since, when $v=a$, the said Ratio becomes that of Equality.) Hence, also, by substituting $\frac{a^n}{b}$ for its Equal dc^{n-2}, we get

$$\dot{x} = \frac{-a^n v\dot{v}}{b \times \overline{v^n \pm a^n}}, \text{ and } \dot{z} = \frac{-ra^n\dot{v}}{b \times \overline{v^n \pm a^n}}.$$

COROLLARY II.

365. If the Resistance be in the Duplicate Ratio of the Celerity, our two last Equations will become $\dot{x} = \frac{-a^2 v\dot{v}}{b \times \overline{vv \pm aa}}$, and $\dot{z} = \frac{-ra^2\dot{v}}{b \times \overline{vv \pm aa}}$: From the former whereof we get $x = -\frac{a^2}{2b} \times hyp.\ Log.\ \frac{vv \pm aa}{cc \pm aa}$ * $= \frac{a^2}{2b} \times hyp.\ Log.\ \frac{cc \pm aa}{vv \pm aa} = \frac{d}{2} \times hyp.\ Log.\ \frac{cc \pm bd}{vv \pm bd}$ (because, here, $a^2 = bd$.) From whence, when $v=0$, (supposing the Body to ascend) there comes out $x = \frac{d}{2} \times hyp.\ Log.\ \overline{1 + \frac{cc}{aa}}$, for the Height (*AQ*) of the whole Ascent. But, if c be taken $= 0$, or the Body be

* Art. 126.

be supposed to descend from Rest, we shall then have $-\frac{d}{2} \times hyp.\ Log.\ \overline{1 - \frac{vv}{aa}} =$ the Distance AB descended. Whence, if N be put for the Number whose Hyperbolical Logarithm is $\frac{2x}{d}$, it follows, (because, Log. $1 - \frac{vv}{aa} = -\frac{2x}{d} = -$ Log. N) that $1 - \frac{vv}{aa} = \frac{1}{N}$, and consequently $v = a\sqrt{\frac{N-1}{N}}$. From which, the Distance AB being given, the Velocity acquired in the Fall will be determined. But, if the Body, first, ascends from a given Point A, with a given Celerity c, and the Celerity, acquired in falling, when it arrives, again, at that Point, be required; the same may be exhibited in a more commodious Form, independent of Logarithms, and will be equal to $\frac{c}{\sqrt{1 + \frac{cc}{aa}}}$; because N, in this Case, is found above to be $= 1 + \frac{cc}{aa}$. Furthermore, with regard to the Time (z), we have already found that $\dot{z}$ is $= \frac{-ra^2\dot{v}}{b \times \overline{vv + aa}}$, or $= \frac{-ra^2\dot{v}}{b \times \overline{vv - aa}}$ $\left(= \frac{ra^2\dot{v}}{b \times \overline{aa - vv}}\right)$ according as the Motion of the Body is from, or towards the Center of Force. Therefore the Time itself, in the former Case, will be $= \frac{ra}{b}$ drawn into the Difference of the two circular Arcs whose Tangents are $\frac{c}{a}$ and $\frac{v}{a}$, and whereof the common Radius is Unity *: Whence it follows that the Time

* Art. 142.

Time of the whole Aſcent will be denoted by $\frac{ra}{b}$ multiplied into the former of the ſaid Arcs.

But, in the other Caſe, the Fluent, exhibiting the Time of Deſcent, is not explicable by the Arcs of a Circle, but by the Difference of the hyperbolical Logarithms of $\frac{a+v}{a-v}$ and $\frac{a+c}{a-c}$ drawn into $\frac{ra}{2b}$ *. Therefore, when $c=0$, or the Body falls from Reſt, the Time z will be barely $= \frac{ra}{2b} \times hyp.\ Log.\ \frac{a+v}{a-v} = \frac{ra}{b}$ $\times hyp.\ Log.\ N^{\frac{1}{2}} + \overline{N-1}|^{\frac{1}{2}}$ (by ſubſtituting the Value of v found above, and ordering the Logarithm as in *Art.* 303.) This Equation, in the forementioned Circumſtance, where $N = 1 + \frac{cc}{aa}$, and $v = \frac{c}{\sqrt{1+\frac{cc}{aa}}}$,

becomes $z = \frac{ra}{b} \times hyp.\ Log.\ \sqrt{1+\frac{cc}{aa}} + \frac{c}{a}$.

* Art. 126.

SCHOLIUM.

366. If, according to Sir *Iſaac Newton*, we ſuppoſe the Reſiſtance of the Air, to Bodies moving in it, to be in the Duplicate Ratio of the Celerities *; and that

* *That the Reſiſtance is as the Square of the Celerity, the Learner may, in ſome meaſure, conceive, by conſidering that the ſame Body, with a double Velocity, not only puts twice the Number of reſiſting Particles in Motion, in the ſame time, but alſo acts upon each with a double Force; and therefore muſt ſuffer a four-fold Reſiſtance, or a Reſiſtance proportional to the Square of the Velocity. This would be ſtrictly true, were it not that the Particles ſo put in Motion impel others lying before them, and thereby prevent, as it were, the Action of the Body. What Deviation from the foregoing Law may hence ariſe, is not eaſy to determine. This, however, ſeems plain, that the Reſiſtance at the Beginning of any very ſwift Motion (till the Air in the Way of the Body comes duly to participate of that Motion) will be greater than That ſuſtained by another equal Body, moving with the ſame Celerity, that has been in Motion ſome time.*

a Ball,

a Ball, in the Time it might move, uniformly, over a Space (d) which is to $\frac{8}{3}$ of its Diameter as the Density of the Ball to that of the Medium, would have all its Motion taken away by a Force equal to that of the Resistance, uniformly continued: Then, from these *Data*, applied to the Theorems in the preceding Article, we shall be able to determine the Velocities, and the Times of the perpendicular Ascent and Descent of Bodies near the Earth's Surface; allowing for the Resistance of the Atmosphere.

Thus, for Instance, let a Cannon Ball, of four Inches Diameter (whereof the Density, or specific Gravity, is to that of Air as 6000 to 1, nearly) be supposed to be projected, perpendicular to the Horizon, with a Velocity sufficient to cause it to ascend to the Height of half a Mile, or 2640 Feet, *in vacuo*; which Velocity (*by Art.* 203.) will be found to answer to the Rate of about 412 Feet *per Second:* Then, according to the Proportion just now mentioned, it will be as $1 : 6000 :: \frac{8}{3} \times 4 : 64000$ Inches, or 5333 Feet; which is the Value of d in this Case. Therefore, if the Time r, in the preceding Article (which may be assumed at pleasure) be here interpreted by *one Second*, the corresponding Values of d, c and b will be expounded by 5333 F. 412 F. and $32\frac{1}{12}$ F. * respectively. Which Values being substituted in the several Equations in the last Article, we shall get

* Art. 202.

1°. $a\ (=\sqrt{bd}) = 414$ F. the Velocity, *per Second*, wherewith the Resistance would be equal to the Gravity, or Weight, of the Ball.

2°. $\frac{d}{2} \times hyp.\ Log.\ 1 + \frac{cc}{aa} = 1835$ Feet, the whole Height of the Ascent.

3°. $\frac{ra}{b} \times Arch.$ whose Tang. is $\frac{c}{a} = 10{,}08$ *Seconds*, the whole Time of the Ascent (which is less than the Time, *in vacuo*, by 2,73.)

4°.

4°. $v \left(= \frac{c}{\sqrt{1 + \frac{cc}{aa}}}\right) = 292$, the Velocity, *per Second*, acquired in the Descent.

5°. Lastly, $\frac{ra}{b} \times$ *hyp. Log.* $\sqrt{1 + \frac{cc}{aa}} + \frac{c}{a} =$ 11,30 Seconds, the Time of the Descent.

Note, In this Example the Measure of the absolute Gravity of the Body, *in vacuo*, is taken, instead of its Gravity in Air (the Difference, *there*, being too inconsiderable to be regarded.) But, in Cases where the specific Gravity of the Medium bears a sensible Proportion to that of the Body, the Force of Gravity *(b)* must be expounded by $32\frac{1}{12} \times \frac{B - M}{B}$ (instead of $32\frac{1}{12}$) where B is to M as the specific Gravity of the Body to that of the Medium.

PROB. III.

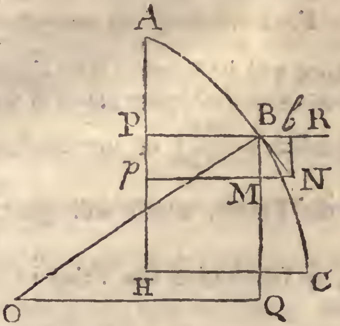
367. *To determine the Resistance, by means whereof a Body, gravitating uniformly in the Direction of parallel Lines, may describe a given Curve.*

Let ABC be the given Curve, and BQ, parallel to the Axis (or any given Line) AH, be the Direction of Gravitation at any Point B: Make PBR perpendicular to AH and BQ; and let $AP = x$, $PB = y$, $AB = z$, BM $(Nb) = \dot{x}$, MN $(Bb) = \dot{y}$, BN $= \dot{z}$, and the Velocity of the Body at B in the Direction PBR $= v$. Then, the Decrease of Velocity in the said Direction, which is wholly owing to the Resistance *, being represented by $-\dot{v}$, it follows that the corresponding Decrease of Motion in the Direction BN, arising from the same Cause, will be expressed by $\frac{\dot{z}}{\dot{y}} \times -\dot{v} = -\frac{\dot{v}\dot{z}}{\dot{y}}$; and, that in the Direction BM, by $-\frac{\dot{v}\dot{x}}{\dot{y}}$. But, the

* Art. 209.

Celerity

Celerity in this last Direction being, every where, represented by $v \times \frac{\dot{x}}{\dot{y}}$, its Fluxion $\frac{v\ddot{x} + \dot{v}\dot{x}}{\dot{y}}$ will be the



whole Alteration of Motion in the said Direction, arising from the Resistance and the Force of Gravity, conjunctly: From which deducting the Part owing to the Resistance, found above to be $\frac{\dot{v}\dot{x}}{\dot{y}}$, the Remainder $\frac{v\ddot{x}}{\dot{y}}$ will be the Effect of the Gravity. Which being to $\left(-\frac{\dot{v}\dot{z}}{\dot{y}}\right)$ the Effect of the absolute Resistance in the Direction BN, as 1 to $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$, the Force of Gravity, must therefore be to that of the required Resistance, in the same Ratio of 1 to $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$.

Moreover, the Force of Gravity, measured by the Velocity it would generate in a given Part of Time (1), being denoted by Unity, the Velocity generated thereby, in the Time $\left(\frac{\dot{y}}{v}\right)$ of describing B*b*, with the Celerity v, will likewise be truly expresed by, $\frac{\dot{y}}{v}$, the Measure of

the

the ſaid Time: Which being put = to $\left(\frac{v\ddot{x}}{\dot{y}}\right)$ the Value of the ſame Quantity, given above, we thence have $v^2 = \frac{\dot{y}\dot{y}}{\ddot{x}}$: From whence, not only the Velocity, but the Reſiſtance will be found. But, if you would have the Reſiſtance expreſſed independent of v; then let the Fluxion $\left(2v\dot{v} = -\frac{\dot{y}^2\dddot{x}}{\ddot{x}\ddot{x}}\right)$ of the laſt Equation be divided by the Fluent, which will give $\frac{\dot{v}}{v} = -\frac{\frac{1}{2}\dddot{x}}{\ddot{x}}$: And then, by ſubſtituting this Value in $-\frac{\dot{v}\dot{z}}{v\ddot{x}}$, you will get $\frac{\dot{z}\dddot{x}}{2\ddot{x}\ddot{x}}$, for the true Force of the Reſiſtance, *that* of Gravity (or the Weight of the Body) being expounded by Unity.

The ſame otherwiſe.

Let BO be the Radius of Curvature at B, and let OQ be parallel to PB, meeting BM, produced, in Q: Then, if the abſolute Gravity, acting in the Direction BQ, be denoted by Unity, its Force in the Direction BO, whereby the Body is retained in the Curve, will be repreſented by $\frac{BQ}{BO}$. Therefore, ſince the Velocities in Circles are known to be in the Subduplicate Ratio of the Radii and of the Forces conjunctly *, the Velocity at B will be rightly expreſſed by $\sqrt{BO \times \frac{BQ}{BO}}$, or its Equal $\sqrt{BQ}$. (For the Curve at, and indefinitely near, B may be taken as an Arch of a Circle whoſe Radius is BO: And it is evident that the Reſiſtance has nothing to do in forcing the Body from the Tangent,

* Art. 212.

Tangent, but only serves to retard its Motion so, that it may, every where, bear a due Proportion to the given Force of Gravity acting in the Direction BO.) Hence, putting BQ $= s$, the Increase of the Celerity in the Time $\left(\frac{\dot{z}}{\sqrt{s}}\right)$ of describing BN, will be expressed by the Fluxion of $\sqrt{s}$, or $\frac{\dot{s}}{2\sqrt{s}}$. Moreover, the Celerity that might be generated by Gravity in the said Time $\frac{\dot{z}}{\sqrt{s}}$ being measured thereby, the Increase, in BN, arising from the same Cause, will therefore be $= \frac{\dot{z}}{\sqrt{s}} \times \frac{\dot{x}}{\dot{z}} = \frac{\dot{x}}{\sqrt{s}}$: Which, being taken from $\left(\frac{\dot{s}}{2\sqrt{s}}\right)$ the whole Increase, found above, the Remainder, $\frac{\dot{s} - 2\dot{x}}{2\sqrt{s}}$, will be the Effect of the Resistance: Which is to the Effect, $\frac{\dot{z}}{\sqrt{s}}$, of the absolute Gravity as $\frac{\dot{s} - 2\dot{x}}{2\dot{z}}$ to 1. Therefore the Resistance is to the Gravity (or Weight of the Body) as $\frac{2\dot{x} - \dot{s}}{2\dot{z}}$ to Unity: Where the Signs are changed, because the two Forces act in contrary Directions.

Because BO $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ *, therefore s (BO $\times \frac{\dot{y}}{\dot{z}}$) $= \frac{\dot{z}^2}{\ddot{x}} = \frac{\dot{y}^2 + \dot{x}^2}{\ddot{x}}$ (= the Square of the Celerity) whence $\dot{s} = \frac{2\dot{x}\ddot{x}\ddot{x} - \overline{\dot{y}^2 + \dot{x}^2} \times \dddot{x}}{\ddot{x}\ddot{x}}$, and consequently the Re- * Art. 61.

siſtance

ſiſtance $\frac{2\dot{x} - \dot{s}}{2\dot{z}} = \frac{\overline{\dot{y}^2 + \dot{x}^2} \times \dddot{x}}{2\dot{z}\ddot{x}\ddot{x}} = \frac{\dot{z}\dddot{x}}{2\ddot{x}\ddot{x}}$, *the very ſame as before.*

Corollary.

368. If the Reſiſtance be ſuppoſed as any given Power of the Velocity drawn into (D) the Denſity of the Medium; then, from hence, the Denſity of the Medium, at every Point of the Curve, may be determined: For, the abſolute Celerity at B being repreſented by $\frac{v\dot{z}}{\dot{y}}$, the Reſiſtance at that Point will, according to the ſaid Hypotheſis, be as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^n \times D$; and therefore the Velocity that would be deſtroyed thereby, in the Time $\left(\frac{\dot{y}}{v}\right)$ of deſcribing BN, as $\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^n \times \frac{D\dot{y}}{v}$: Which being put $= \left(-\frac{\dot{v}\dot{z}}{\dot{y}}\right)$ the Effect of the ſame Reſiſtance, found above, we thence get $D = \frac{-\dot{v}\dot{y}^{n-2}}{\overline{v\dot{z}}|^{n-1}}$: Which, by ſubſtituting for v and $\dot{v}$, becomes $D = \frac{\dddot{x}}{2\dot{z}^{n-1} \times \ddot{x}^{2-\frac{1}{2}n}}$.

In this Corollary, and what, elſewhere, relates to unequal Denſities, the Gravity of the Body in the Medium is ſuppoſed to continue, every where, the ſame, or, that the Attraction increaſes with the Denſity, ſo that the Difference between the ſpecific Gravities of the Body and Medium may, at every Point, be a conſtant Quantity.

E X-

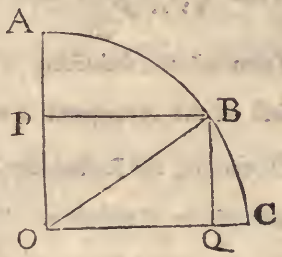
EXAMPLE I.

369. *Let the proposed Curve* ABC *be the common Parabola:*

Then, x being here $= \frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, $\ddot{x} = \frac{2\dot{y}\dot{y}}{a}$ and $\dddot{x} = 0$; and therefore $\frac{\dot{z}\dddot{x}}{2\dot{x}\ddot{x}}$ * is also $= 0$: * Art. 367.
Whence it appears that a Body, to describe this Curve, must move in Spaces intirely void of Resistance.

EXAMPLE II.

370. *Let the Curve* ABC *be taken as a Quadrant of a Circle, whose Radius* BO *is* $= a$.



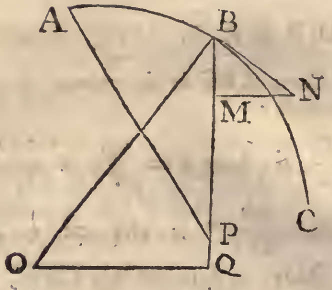
In this Case we have s (BQ) † $= a - x$ ($=$ AO $-$ AP) whence $\dot{s} = -\dot{x}$, † Art. 367.
and therefore $\frac{2\dot{x} - \dot{s}}{2\dot{z}} = \frac{3\dot{x}}{2\dot{z}} = \frac{3PB}{2AO}$ ‡. From which it is evident, that the ‡ Art. 142.
Velocity is, every where, as $\sqrt{BQ}$, and the Resistance to the Gravity (or Weight of the Body) as 3PB to 2OB.

PROB. IV.

371. *The Centripetal Force (F) being given; to find the Resistance and Velocity whereby a Body may describe a given Spiral (or any other, possible, Curve) about the Center of Force.*

Let P be the Center of Force, and BO the Radius of Curvature at any Point B in the proposed Curve,

 and



and let OQ be perpendicular to BPQ; alſo let BP $= y$, BQ $= s$, AB $= z$, BM $= -\dot{y}$ *, and BN $= \dot{z}$. Then, it is evident from *Art.* 367. that the Velocity at B will be expreſſed by $\sqrt{BO \times \frac{BQ}{BO} \times F}$;

* Art. 5.

or, its Equal, $\sqrt{sF}$: And therefore its increaſe in the Time $\left(\frac{\dot{z}}{\sqrt{sF}}\right)$ of deſcribing BN will be $\frac{s\dot{F} + F\dot{s}}{2\sqrt{sF}}$: From which, deducting $(F \times \frac{\dot{z}}{\sqrt{sF}} \times \frac{-\dot{y}}{\dot{z}})$ the Effect of the centripetal Force, in the ſame Time and Direction, the Remainder, $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2\sqrt{sF}}$, is the Effect of the Reſiſtance. Therefore the Reſiſtance is to the centripetal Force as $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2\sqrt{sF}}$ to $\frac{F\dot{z}}{\sqrt{sF}}$, or as $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2F\dot{z}}$ to Unity. *Q. E. I.*

EXAMPLE.

372. Let the Meaſure *(F)* of the centripetal Force be expounded by any Power y^n of the Diſtance; and let the Curve be the logarithmic Spiral; putting the Co-ſine of the given Angle PBN † (to the Radius r) $= c$. Then, s being here $= y$ ‡, and $\dot{F} = ny^{n-1}\dot{y}$, we

† Art. 61.
‡ Art. 74.

we have $\frac{s\dot{F} + F\dot{s} + 2F\dot{y}}{2F\dot{z}} = \frac{ny^n\dot{y} + y^n\dot{y} + 2y^n\dot{y}}{2y^n\dot{z}} = \frac{n+3}{2} \times \frac{\dot{y}}{\dot{z}} = \frac{n+3}{2} \times \frac{c}{r}$.

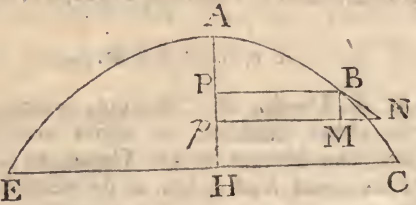
Hence it appears that the Velocity must be, every where, as $y^{\frac{n+1}{2}}$; and the Resistance, to the centripetal Force, as $\frac{n+3}{2} \times \frac{c}{r}$ to Unity. But, when $n = -3$, $\frac{n+3}{2} \times \frac{c}{r}$ becomes $= 0$; therefore the Body, in this Case, must move in Spaces intirely void of Resistance; agreeable to *Art.* 233. And, if $n+3$ be negative, an accelerating, instead of a resisting Force, will be required.

SCHOLIUM.

373. If the Density of a Medium, wherein a Body moves, be either uniform, or varies according to a given Law, the Nature of the Curve, or Trajectory may be determined from what is delivered in the preceding Pages.

Thus, for Example, let the Density be supposed every where the same, and the Resistance as the Square of the Celerity; then, from *Art.* 368. we have $\frac{\dddot{x}}{\dot{z}\ddot{x}} = D$; which, in order to exterminate $\dot{z}$, may be transformed to $\dddot{x}\dddot{x} = \overline{\dot{y}\dot{y} + \dot{x}\dot{x}} \times D^2\ddot{x}\ddot{x}$: Where, D being a constant Quantity (depending upon the given Density of the Medium) the Value of x will be found, as is taught in *Sect.* 2. *Art.* 268. 271. and comes out $= \frac{y^2}{p} + \frac{Dy^3}{3p} + \frac{D^2y^4}{12p}$ &c. In which p is put to denote the Parameter of the Curve at the Vertex, or higheſt Point A, (to be determin'd from the Force of Gravity and the given Velocity of the Body at that Point.) This Solution anſwers near enough when the Reſiſtance is but ſmall in Proportion to the Gravity; in other Circumſtances, the Series not converging, it becomes uſeleſs: For which Reaſon, and becauſe the Caſe above ſpecified is That ſuppoſed to obtain, in reſpect to the Air near the Earth's Surface, and its Reſiſtance to Bodies moving therein, I ſhall ſhew, by a different Method, how the Nature of the Curve may be inveſtigated.

In order thereto, let the Celerity at the higheſt Point, A, above the Plane of the Horizon EC, be denoted by c; and let a be the Celerity with which the Reſiſtance is equal to the Gravity (*vid. Art.* 365. *and* 366.)



Moreover, let d be put for the Diſtance over which the Ball might uniformly move in the Time that the Medium would deſtroy all its Motion, was the Reſiſtance to continue the ſame, all along, as at the firſt Inſtant (Which Diſtance, according to Sir *Iſaac Newton*, is, always, in Proportion to $\frac{8}{3}$ of the Ball's Diameter, as the Denſity of the Ball to that of the Medium.)

Then it will be, as $d : \dot{z}$ (BN) :: $\frac{v\dot{z}}{\dot{y}}$, the abſolute Celerity at B, to $\left(\frac{v\dot{z}^2}{d\dot{y}}\right)$ the Part thereof that would be uniformly deſtroyed by the Reſiſtance in the Time of deſcribing BN, with the Velocity at B: Which Value being alſo expreſſed by $\frac{-\dot{v}\dot{z}}{\dot{y}}$ (*vid. Art.* 367.) we

there-

therefore have $\frac{v\dot{z}^2}{d\dot{y}} = -\frac{\dot{v}\dot{z}}{\dot{y}}$; whence $\frac{\dot{z}}{d} = \frac{-\dot{v}}{v}$, and consequently, by taking the Fluent, $\frac{z}{d} = -$ hyp. Log. v; which corrected (by putting $z=0$, and $v=c$) gives $\frac{z}{d}$ ($=$ hyp. Log. c — hyp. Log. v) $=$ hyp. Log. $\frac{c}{v}$.

Furthermore, since (by Hypothesis) the Resistance with the Celerity $\frac{v\dot{z}}{\dot{y}}$ (at B) is to the Force of Gravity, or the Resistance with the Celerity a, as $\frac{vv\dot{z}\dot{z}}{\dot{y}\dot{y}}$ to a^2; and it appears, from the aforesaid Article, that the same Ratio is also universally expressed by *that* of $\frac{-\dot{v}\dot{z}}{v\ddot{x}}$ to 1, it follows, from the Equality of these Ratios, that $\frac{\dot{z}\ddot{x}}{\dot{y}\dot{y}}$ is $= -\frac{a^2\dot{v}}{v^3}$. But, in order to the Resolution of the Equation thus given, let the Tangent of the Angle PBA (or N) which the Ordinate, PB, makes with the Curve (supposing Radius Unity) be, every where, represented by w: Then, because $\dot{x}=w\dot{y}$, $\dot{z}$ ($\sqrt{\dot{y}^2+\dot{x}^2}$) $= \dot{y}\sqrt{1+w^2}$, and $\ddot{x} = \dot{w}\dot{y}$ ($\dot{y}$ being constant) we shall, by substituting these Values in the foresaid Equation, get $-\frac{a^2\dot{v}}{v^3} = \dot{w}\sqrt{1+w^2}$; whereof the Fluent will be given, $\frac{\frac{1}{2}a^2}{v^2} = \frac{1}{2}w\sqrt{1+w^2} + \frac{1}{2}$ hyp. Log. $w+\sqrt{1+w^2}$ *: Which corrected (by taking $v=c$ and $w=0$) becomes $\frac{\frac{1}{2}a^2}{v^2} - \frac{\frac{1}{2}a^2}{c^2} = \frac{1}{2}w\sqrt{1+w^2}$ $+ \frac{1}{2}$ hyp. Log. $w+\sqrt{1+w^2}$. But, to shorten the re-

* Art. 126, and 281.

remaining Part of the Procefs, let the latter Part of the Equation, or the Fluent of $\dot{w}\sqrt{1+w^2}$ be denoted by Q; then $\frac{aa}{2vv}$ being $= \frac{aa}{2cc} + Q$, we have $v = \frac{ac}{\sqrt{aa+2ccQ}}$; and confequently $\frac{z}{d}$ (= hyp. Log. $\frac{c}{v}$) = hyp. Log. $\frac{\sqrt{aa+2ccQ}}{a} = \frac{1}{2}$ hyp. Log. $\overline{1+\frac{2ccQ}{aa}}$. From which two Equations, the Velocity of the Ball, and the Diftance it has moved, when its Direction makes any given Angle with the Horizon, may be computed, let the Medium be as denfe as it will: Alfo, from hence, if the Celerity anfwering to any one given Angle of Direction be known, the Celerity correfponding to any other given Direction may be found, together with the Diftance defcribed between the two Pofitions. For v (in the Defcent of the Body) being, *univerfally*, equal to $\frac{ac}{\sqrt{aa+2c^2Q}}$, the Value of c, expreffing the Celerity at the Vertex *A*, will be had from that Equation, and comes out $= \frac{av}{\sqrt{aa-2v^2Q}}$; whence alfo z (= d × hyp. Log. $\frac{c}{v}$) = d × hyp. Log. $\frac{a}{\sqrt{aa-2v^2Q}} = -\frac{1}{2}d$ × hyp. Log. $\overline{1-\frac{2vvQ}{aa}}$. From which, the Celerity at A being known, the reft is obvious.

But, in the afcending Part of the Curve EA, both z and Q muft be confidered as negative, or wrote with contrary Signs: And then, from the foregoing Equations, we fhall alfo get $v = \frac{ac}{\sqrt{aa-2ccQ}}$, $c = \frac{av}{\sqrt{aa+2vvQ}}$, and

and $-z = \frac{1}{2}d \times$ hyp. Log. $\overline{1 - \frac{2cc\mathfrak{Q}}{aa}} = -\frac{1}{2}d \times$ hyp. Log. $\overline{1 + \frac{2vv\mathfrak{Q}}{aa}}$; and, consequently, $z = -\frac{1}{2}d \times$ hyp. Log. $\overline{1 - \frac{2cc\mathfrak{Q}}{aa}} = \frac{1}{2}d \times$ hyp. Log. $\overline{1 + \frac{2vv\mathfrak{Q}}{aa}}$ $= d \times$ hyp. Log. $\frac{v}{c}$: Answering in this Case.

It still remains to take some notice of the Values of x and y (in order to have the Form, as well as the Length of the Curve.) These, indeed, are not so easy to bring out as That of z, given above; nor can they be exhibited in a general Manner, either by circular Arcs, or Logarithms (that I have been able to discover) but may, however, be approximated to any required Degree of Exactness, as will appear from what follows.

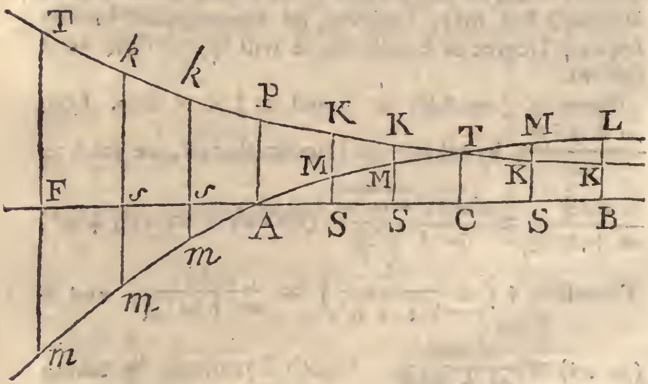
Since z ($=$ AB) is found $= \frac{1}{2}d \times$ hyp. Log. $\frac{aa + 2c^2\mathfrak{Q}}{aa}$, by taking the Fluxion thereof, we get $\dot{z} = \frac{ccd\dot{\mathfrak{Q}}}{aa + 2cc\mathfrak{Q}} = \frac{c^2d\dot{w}\sqrt{1 + w^2}}{aa + 2c^2\mathfrak{Q}}$ (because $\dot{\mathfrak{Q}} = \dot{w}\sqrt{1 + w^2}$) Therefore $\dot{y}$ ($= \frac{\dot{z}}{\sqrt{1 + w^2}}$) $= \frac{c^2d\dot{w}}{aa + 2c^2\mathfrak{Q}}$; and $\dot{x}$ ($= w\dot{y}$) $= \frac{c^2dw\dot{w}}{aa + 2c^2\mathfrak{Q}}$: Which Equations, by taking r to 1, as a^2 to c^2 (or as the Square of the Force of Gravity to the Square of the Resistance at A) are reduced to $\dot{y} = \frac{d\dot{w}}{r + 2\mathfrak{Q}}$, and $\dot{x} = \frac{dw\dot{w}}{r + 2\mathfrak{Q}}$: Whence we get $y = d$ into $\frac{w}{r + 2\mathfrak{Q}} + \frac{\frac{2}{3} \times \overline{1 + ww}|^{\frac{3}{2}} - \frac{2}{3}}{\overline{r + 2\mathfrak{Q}}|^2} +$

$\frac{\frac{8}{3}w \times \overline{1+\frac{2}{3}w^2+\frac{1}{5}w^4} - \frac{8}{3}\mathcal{Q}}{\overline{r+2\mathcal{Q}}|^3}$ &c. And $x = d$ into

$\frac{\frac{1}{2}w^2}{r+2\mathcal{Q}} + \frac{\frac{1}{4}w \times \overline{1+ww}|^{\frac{3}{2}} - \frac{1}{4}\mathcal{Q}}{\overline{r+2\mathcal{Q}}|^2} +$

$\frac{\frac{1}{6} \times \overline{1+ww}|^3 - \frac{1}{6} - \frac{1}{2}\mathcal{Q}^2}{\overline{r+2\mathcal{Q}}|^3}$ &c. These Expressions

(brought out by assuming $\frac{A}{r+2\mathcal{Q}} + \frac{B}{\overline{r+2\mathcal{Q}}|^2} +$ &c. for the Fluent sought, and proceeding as in *Art.* 340.) converge very fast when r is large in comparison to $\mathcal{Q}$; but in other Cases the required Values will be had, with less Trouble, from the following Method.



Let PKTK and AMTM be two Curves, whereof the Ordinates SK and SM, to the common *Abscissa* w (= AS) are expressed by $\frac{1}{r+2\mathcal{Q}}$ and $\frac{w}{r+2\mathcal{Q}}$ respectively: Then it is plain, from the foregoing Equations, that the Measures of the Areas of the said Curves, multiplied by d, will truly exhibit the Values of y and x; an-

answering to any given Value of w (or AS) the Tangent of the Angle of Direction; or, or speak more geometrically, a Square upon AC (supposing AC = Radius = Unity) will be to either of the said Areas ASKP, or ASM as the given Distance d, to the Value of y or x required——But now as to a Way for computing these Areas (without which what has been said about them would be to very little Purpose) the Method of *Equi-distant Ordinates* may here be applied to very good Advantage (when the foregoing Serieses do not converge) By means whereof the required Quantities may, with a little Trouble, be brought out to a sufficient Degree of Exactness, let the Resistance be as great as it will.

According to the same Way of proceeding, the Values of x and y, in the Ascent of the Ball, will also be found, if the Ordinates *sk* and *sm*, generating the required Areas, be taken, every where, equal to $\frac{1}{r - 2Q}$ and $\frac{w}{r - 2Q}$ (instead of $\frac{1}{r + 2Q}$ and $\frac{w}{r + 2Q}$).

From what has been thus far delivered, it will not be very difficult to calculate (according to the foregoing Hypothesis) all the principal Requisites concerning the Motion and Track of a Ball in the Air, projected with a given Velocity, at a given Elevation; as will be more clearly seen by the Example subjoined.

Suppose a Cannon Ball of 4 Inches Diameter (whereof the Weight is nearly 9 Pounds) to be discharged at an Elevation of 45 Degrees, with a Velocity sufficient to carry it to the Distance of one Mile, on the Plane of the Horizon, were it not for the Resistance of the Air.—— Then that Velocity, being the same as might be freely acquired in a perpendicular Descent of half a Mile *, will be found to answer to the Rate of 412 Feet, *per Second*, according to *Art.* 202. and 366. From whence it is also plain, that the Distance d (so often mentioned above) will here be expounded by 5333 Feet; and that the Celerity *(a)* with which the Resistance would be equal to the Gravity (or Weight of the Ball) answers to the Rate of about 414 Feet *per Second.*

* Art. 366.

More-

Moreover, since the Tangent of the Angle of Elevation, or the first Value of w, is given *equal to* Unity (or Radius) we have $\mathcal{Q}\left(\frac{1}{2}w\sqrt{w^2+1}+\frac{1}{2}\text{ hyp. Log. } \overline{w+\sqrt{w^2+1}}\right)=1.1478$: From which, and $v\left(=412\sqrt{\frac{1}{2}}\right)$, we get $z\left(=\frac{1}{2}d\times\text{hyp. Log. } \overline{1+\frac{2vv\mathcal{Q}}{aa}}\right)$ $= 2025$ Feet = the Arch described in the whole Ascent. Also $\left(c=\frac{v}{\sqrt{1+\frac{2vv\mathcal{Q}}{aa}}}\right)=199\frac{1}{3}$ Feet, for the Rate of the Velocity, *per Second*, at the highest Point: Whence $r\left(=\frac{aa}{cc}\right)=4,314$; by Means whereof the greatest Altitude of the Ball, and the horizontal Distance corresponding thereto will likewise be found: For let AF, *in the preceding Figure*, be taken = 1 (the given Value of w) and let the same be divided into three Parts by equi-distant Ordinates (which Number will answer sufficiently exact) then the successive Values of w, for the Ordinates AP, *ks*, *ks* and TF, being 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, *those* of $\mathcal{Q}$ will be 0, 0.3394, 0.713, and 1.1478, and the Ordinates themselves (or the corresponding Values of $\left.\frac{1}{r-2\mathcal{Q}}\right)$ = to 0.2318, 0.2751, 0.3463 and 0.4953, respectively. From whence, by adding the two Extremes to three times the Sum of the two middle Terms, and dividing the whole by 8, we get 0.3239 for the Value of a mean Ordinate *: Which, as AF is here equal to Unity, is also the Measure of the required Area AFTP: Which, therefore, being multiplied by 5333 *(d)* gives 1727 Feet, for the horizontal Distance made good in the whole Ascent. In

* *See* p. 117. *of my Mathematical Dissertations.*

the same Way the Area AFm is found $= 0.1828$. Whence the greatest Height of the Ball appears to be $(= 0.1828 \times 5333) = 975$ Feet.

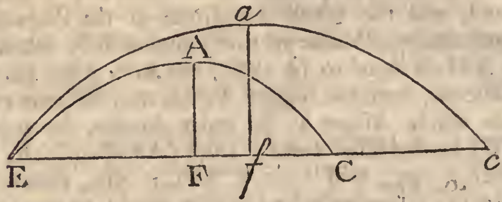
By taking AC$=1$, and repeating the Operation (only changing $r - 2\mathfrak{Q}$ to $r + 2\mathfrak{Q}$) the Area ACTP will come out $= 0.1883$, and ATC $= 0.0875$; which multiplied by 5333 (as above) give 1004 F. and 467 F. for the Amplitude, and the Distance descended, from the highest Point, when the Direction of the Ball makes an Angle with the Horizon equal to that in which it was projected.

But, to have the Direction when the Ball strikes the Ground, and the whole Amplitude of the Projection, we must find the Value of the Tangent AB, when the Area ABL is equal to (0.1828) the Area AFm (so that the Descent, from the highest Point, may become equal to the whole Ascent.) In order thereto, let 0.0875 (ATC) be deducted from 0.1828 (AFm) and the Remainder 0.0953 will be $=$ CTBL; this, divided by TC (0.1513) quotes 0.63; which would be the Value of CB, if all the Ordinates CT, SM, &c. were equal: But, as it is obvious from the Nature of the Problem, and from the Law of the Ordinates already computed, that BL will be something greater than CT, and consequently CB less than 0.63——I therefore suppose the Value of CB may be about 0.56; and, accordingly, proceed to compute the Area of CBLT answering to this Number; by means of CT (0.1513) and BL (0.1852) and one intermediate Ordinate SM (0.1715) and find it (from the Approximation $\frac{CT + BL + 4SM}{6} \times CB$) to come out $= 0.0955$: Which is so near the required Value 0.0953, that it will be altogether needless to repeat the Operation. It is evident from hence, that the Tangent (AB) of the Angle of Direction, when the Ball strikes the Ground, is 1.56; answering to $57° : 20'$: From whence, CBKT being found $= 0.0752$, the whole Area ABKP will be had $= 0.2635$, and consequently $0.2635 \times 5333 = 1405$ F. $=$ the Amplitude in the whole Descent.

Fur-

Furthermore, from the ſaid Value of w and that of c ($= 199\frac{1}{3}$) given above, we get z ($= \frac{1}{2} d \times$ hyp. Log. $\overline{1 + \frac{2cc\mathfrak{Q}}{aa}}$) $= 1788$ Feet, for the Arch deſcribed in the Deſcent; and alſo $v = 142\frac{1}{2}$ F. which multiplied by 1.8527, the Secant of 57°: 20′, gives 264 F. for the Celerity of the Ball, *per Second*, at the End of its Flight.

Now, by collecting the principal of the foregoing Concluſions, it appears,



1°. That the Velocity at the higheſt Point A of the Trajectory will be at the Rate of $199\frac{1}{3}$ Feet, *per Second*: Which is to the Velocity at the higheſt Point a of the Parabola (Eac) that would be deſcribed, were it not for the Reſiſtance, as 2 to 3, nearly.

2°.	EA = 2025 and Ea = 3030		Feet
3°.	EF = 1727 and Ef = 2640		
4°.	AF = 975 and af = 1320		
5°.	AC = 1788 and ac = 3030		
6°.	FC = 1405 and fc = 2640		

7°. Angle C = 57°: 20′ and c (= E) = 45°.
8°. Velocity at C to that at E, as 264 to 412, or as 2 to 3, nearly.

Theſe Proportions, between the Diſtances, in Air and *in vacuo*, hold at an Elevation of 45°, when the Reſiſtance, at going off, is nearly equal to the Gravity, or Weight, of the Ball. If the Velocity be greater than that above ſpecified, or the Body, projected, be, either,

either, less, or less dense, the Curve will differ, *still*, more from a Parabola.

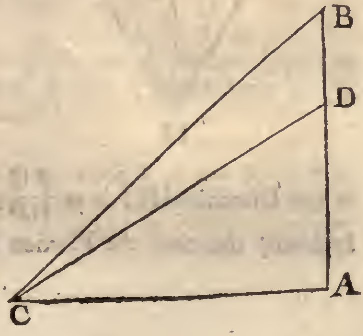
Hence it evidently appears, that the Effect of the Air's Resistance upon very swift Motions, is too considerable to be intirely disregarded in the Art of Gunnery.——'Tis true the Method given above is, by much, too intricate for common Practice; but when the Law of the Resistance to very swift Motions is once sufficiently established (which, according to some late Experiments, seems to be in a Ratio greater than that of the Square of the Celerity) it will be no very difficult Matter to find out proper Approximations to correct the Proportions in common Use.

SECTION IX.

The Use of Fluxions in determining the Attraction of Bodies under different Forms.

PROB I.

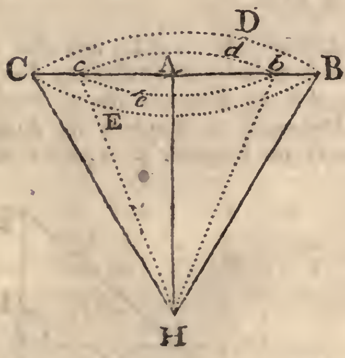
374. *Supposing* AC *perpendicular to* AB, *and that a Corpuscle at* C *is attracted towards every Point or Particle of the Line* AB, *by Forces in the reciprocal duplicate Ratio of the Distances; to determine the Ratio of the whole Force whereby the Corpuscle is urged in the Direction* CA.



Put $AC = a$, and let AD (considered as variable by the Motion of D towards B) be denoted by x: Then, the Force of a Particle at D being as $\frac{1}{CD^2}$ (by Hypothesis) its Efficacy in the propofed Direction AC will (by the Refolution of Forces) be as $\frac{AC}{CD} \times \frac{1}{CD^2} = \frac{AC}{CD^3} = \frac{a}{\overline{a^2+x^2}|^{\frac{3}{2}}}$: Therefore $\frac{a\dot{x}}{\overline{a^2+x^2}|^{\frac{3}{2}}}$ is the Fluxion of the whole Force; whofe Fluent, which (*by Art.* 85.) is $= \frac{x}{a \times \overline{a^2+x^2}|^{\frac{1}{2}}}$ $= \frac{AD}{CA \times CD}$, will, when AD=AB, be as the Force itfelf. *Q. E. I.*

PROB. II.

375. *Suppofing* BCDE *to reprefent a circular Plane, and that a Corpufcle* H, *in the Axis thereof* AH, *is attracted by every Point or Particle of the Plane by Forces in the reciprocal duplicate Ratio of the Diftances; to find the whole Force by which the Corpufcle is urged towards the Plane.*



Let AH $= a$, and H$b = x$; then A$b^2 = x^2 - a^2$; which multiply'd by ($p = 3{,}14159$ &c.) the Area of the Circle whofe Radius is Unity, gives $p \times \overline{x^2 - a^2}$ for the Area of the Circle A*cdbe*: whofe Fluxion is $= 2px\dot{x}$. But the Force of a fingle Particle at *b*, in the Direction HA, is as $\frac{AH}{Hb^3}$, or $\frac{a}{x^3}$ (fee the laft Problem) therefore the Fluxion of the whole Force is truly

truly defined by $2px\dot{x} \times \frac{a}{x^3}$ or its Equal $\frac{2p\dot{x}}{x^2}$ and the Force itself by the Fluent of $\frac{2pa\dot{x}}{x^2}$; which (properly corrected) is $-\frac{2pa}{x} + \frac{2pa}{a} = 2p \times \overline{1 - \frac{a}{x}} = 2p \times \overline{1 - \frac{AH}{BH}}$, when $x = HB$. *Q. E. I.*

376. In the preceding Problems, we have supposed the Attraction of each Particle, to be as the Square of the Distance inversely; that being the Law which is found to obtain in Nature: But if the Force, according to any other Law of Attraction, be required, the Process will be very little different.

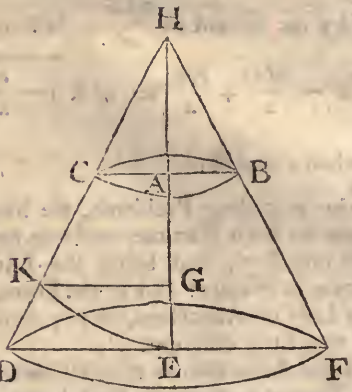
Thus, let the Attraction be as any Power *(n)* of the Distance: Then (in the last *Prob.)* the Force of a Particle at *b* (upon H) being as x^n, its Force in the Direction HA will be as $\frac{a}{x} \times x^n$ or ax^{n-1}; which multiply'd by $2px\dot{x}$ (as before) gives $2pax\,\dot{x}$: whereof the Fluent $\frac{2pax^{n+1} - 2pa^{n+2}}{n+1}$ $\left(= \frac{2p}{n+1} \times \overline{AH \times BH^{n+1} - AH^{n+2}}\right)$ will be as the Force required.

PROB. III.

377. *To determine the Attraction of a Cone* DHF *at its Vertex; the Attraction of each Particle being as the Square of the Distance inversely.*

Put the Axis $EH = a$, the Length of the Slant-Side HD (or HF) $= b$, and AH (considered as variable) $= x$: Then *(by sim. Triangles)* a (HE) : b (HF) :: x

$:: x$ (HA) : HB $= \frac{bx}{a}$. But, by the laſt Problem,



the Attraction of all the Particles in the Circle BC will be meaſured by $2p \times \overline{1 - \frac{AH}{BH}} = 2p \times \overline{1 - \frac{a}{b}}$ (becauſe HB $= \frac{bx}{a}$): Which therefore being multiply'd by $\dot{x}$, and the Fluent taken, we thence have $x - \frac{ax}{b}$ for the Attraction of ACHB: And this, when $x = a$, will be, $2p \times \overline{EH - \frac{EH^2}{DH}}$, the Force of the whole Cone DEHF: Which, if HK be made = HE, and KG perpendicular to HE, will likewiſe be truly defined by $2p \times EG$ (becauſe HG $= \frac{EH^2}{DH}$). *Q. E. I.*

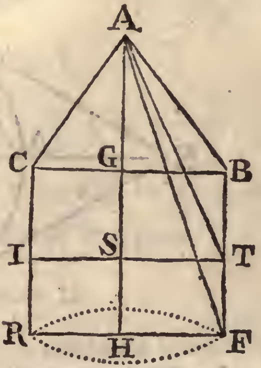
COROLLARY.

378. Seeing the Attraction of ACHB is, every where, as $x - \frac{ax}{b}$, or $\frac{b-a}{b} \times x$, it follows that the Forces of ſimilar Cones, at their Vertexes, are directly as their Altitudes.

PROB.

PROB. IV.

379. *To find the Force of a Cylinder* CBRF, *at any Point* A *in the produced Axis; the Law of Attraction being still as in the preceding Problems.*

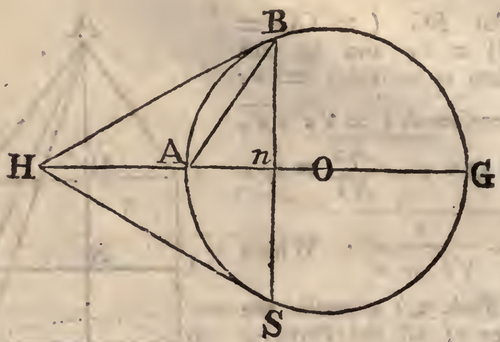


Put BG (= CG = RH) = b; and let AS (taken as variable) = x: Therefore AT $= \sqrt{b^2+x^2}$, and $2p \times \overline{1 - \frac{AS}{AT}} = 2p \times \overline{1 - \frac{x}{\sqrt{b^2+x^2}}}$: Which (*by Prob.* 2.) expresses the Force of all the Particles in the circular Surface IST.

Therefore $2p \times \overline{\dot{x} - \frac{x\dot{x}}{\sqrt{b^2+x^2}}}$ is the Fluxion of the required Force: Whose Fluent $(2p \times \overline{x - \sqrt{b^2+x^2}})$ when $x =$ AG, will be $= 2p \times \overline{AG - AB}$; but when $x =$ AH, it will be $= 2p \times \overline{AH - AF}$: Hence, by taking the former of these Values from the latter, we have $2p \times \overline{AB + BF - AF}$ for the Measure of the true Force by which a Corpuscle at A is urged towards the Cylinder.

PROB. V.

380. *The Law of the Force being* ſtill *ſuppoſed the ſame; to determine the Attraction of a Sphere* OABGS, *at any given Point* H *above its Surface.*



Let BS be perpendicular to HG, and let HB be drawn; alſo put the Radius AO $= a$, OH $= b$, AH $(b-a) = c$, H$n = y$, and HB $= c+x$; then A$n = y-c$, G$n = 2a-y+c$, and conſequently $\overline{y-c} \times \overline{2a-y+c}$ ($=$ A$n \times$ G$n =$ B$n^2 =$ BH$^2 -$ Hn^2) $= \overline{c+x}|^2 - y^2$: From which Equation we get $y = \frac{2ac+2c^2+2cx+x^2}{2a+2c} = \frac{2bc+2cx+x^2}{2b}$ (becauſe $a+c=b$.) Whence alſo $2p \times$

$$\overline{1 - \frac{Hn}{HB}} {}^{*} = 2p \times \overline{1 - \frac{2bc+2cx+x^2}{2b \times \overline{c+x}}} = \frac{2p \times \overline{2ax-x^2}}{2b \times \overline{c+x}}:$$

* Art. 375.

Which multiply'd by $\frac{c\dot{x}+x\dot{x}}{b} = \dot{y}$ gives $\frac{p \times \overline{2ax\dot{x} - x^2\dot{x}}}{b^2}$ for the Fluxion of the required Force; whereof the Fluent

$\frac{p \times \overline{ax^2 - \frac{1}{3}x^3}}{b^2}$ will be the Attraction of the Segment ABS: Which therefore, when B coincides with G and x is $= 2a$, becomes $\frac{4pa^3}{3b^2}$, for the Meaſure of the Attraction of the whole Sphere. *Q. E. I.*

COROLLARY I.

381. Hence the Attraction $\left(\frac{4pa^3}{3b^2}\right)$ at the Surface of the Sphere, where b is $= a$, will be $\frac{4pa}{3}$; and therefore is directly as the Radius of the Sphere.

COROLLARY II.

382. Since $\frac{4pa^3}{3}$ is known to expreſs the Content of Sphere whoſe Radius is a *, it is evident that the Attraction $\left(\frac{4pa^3}{3b^2}\right)$ of any Sphere is, univerſally, as its Quantity of Matter directly, and the Square of the Diſtance from its Center inverſely; and is, moreover, the very ſame as it would be, was all the Matter in the Sphere to be united in a Point at the Center.

* Art. 148.

COROLLARY III.

383. If inſtead of a Corpuſcle, or a ſingle Particle of Matter, at H, we ſuppoſe another Sphere, having its Center at H: Then, ſince the two Spheres, at O and H, act upon each other with the very ſame Forces, as if each Maſs was contracted into its Center, it follows that the abſolute Force, with which two ſpherical Bodies tend towards each other, is as the Product of their Maſſes directly, and the Square of the Diſtance of their

Centers inversely: And therefore, if the Masses are given, will be barely as the Square of the Distance.

PROB. VI.

384. *To determine the same as in the last Problem, the Force of each Particle being as any Power* (n) *of the Distance.*

Let HB $= x$, and let every thing else remain as above; then we shall have $y = \frac{c^2 + 2ac + x^2}{2b} = d + \frac{x^2}{2b}$ (by putting $d = \frac{c^2 + 2ac}{2b}$) and consequently $\dot{y} = \frac{x\dot{x}}{b}$.

Now the Attraction of all the Particles in the circular Surface BS, is as $\frac{2p}{n+1} \times \overline{Hn \times HB^{n+1} - Hn^{n+2}}$ (*by Art.* 376.) $= \frac{2p}{n+1} \times \overline{yx^{n+1} - y^{n+2}}$: Which, multiply'd by $\dot{y}$, gives $\frac{2p}{n+1} \times \overline{x^{n+1} y\dot{y} - y^{n+2}\dot{y}}$ for the Fluxion of the required Force: Which, because $y\dot{y}$ is $= \overline{d + \frac{x^2}{2b}} \times \frac{x\dot{x}}{b} = \frac{dx\dot{x}}{b} + \frac{x^3\dot{x}}{2b^2}$, will likewise be expressed by $\frac{2p}{n+1} \times \overline{\frac{dx^{n+2}\dot{x}}{b} + \frac{x^{n+4}\dot{x}}{2b^2} - y^{n+2}\dot{y}}$: Whereof the Fluent is $\frac{2p}{n+1} \times \overline{\frac{dx^{n+3}}{\overline{n+3} \times b} + \frac{x^{n+5}}{\overline{n+5} \times 2b^2} - \frac{y^{n+3}}{n+3}}$: Which, when B coincides with A, or $x = y = c$, will be $= \frac{2p}{n+1} \times \overline{\frac{dc^{n+3}}{\overline{n+3} \times b} + \frac{c^{n+5}}{\overline{n+5} \times 2b^2} - \frac{c^{n+3}}{n+3}}$: But, when

 B co-

B coincides with G, or $x = y = 2a + c\ (= f)$ it will become $= \frac{2p}{n+1} \times \overline{\frac{df^{n+3}}{\overline{n+3} \times b} + \frac{f^{n+5}}{\overline{n+5} \times 2b^2} - \frac{f^{n+3}}{n+3}}$: Therefore the Difference of these two, which is $= \frac{2pf^{n+3}}{n+1} \times \frac{\overline{n+5} \times \overline{2bd - 2b^2} + \overline{n+3} \times f^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} - \frac{2pc^{n+3}}{n+1} \times \frac{\overline{n+5} \times \overline{2bd - 2b^2} - \overline{n+3} \times c^2}{\overline{n+3} \times \overline{n+5} \times 2b^2} =$

$$\frac{\overline{\overline{1+n} \times ab - cc} \times 2pf^{n+3} + \overline{\overline{5+n} \times ab + cc} \times 2pc^{n+3}}{\overline{n+1} \times \overline{n+3} \times \overline{n+5} \times b^2}$$

(because $f = a+b$, and $2db = c^2 + 2ac$) will be the Attraction of the whole Sphere. *Q. E. I.*

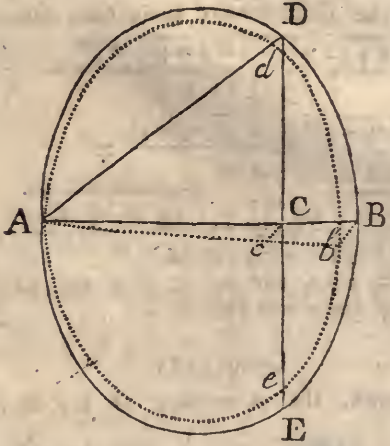
Corollary.

385. Hence, the Attraction at the Surface of the Sphere (where $c = 0$) will be $\frac{2p}{n+1} \times \frac{\overline{1+n} \times \overline{2a}|^{n+3} + \overline{n+5} \times 0^{+3}}{\overline{n+1} \cdot \overline{n+3} \cdot \overline{n+5}}$: Which, if $n+3$ be positive, will be $= \frac{2p \times \overline{2a}|^{n+3}}{\overline{n+3} \times \overline{n+5}}$; but, otherwise, infinite.

386. *Supposing* ADB*b*A *to be a* Cuneus *of uniformly dense Matter, compriz'd by two equal and similar elliptic Planes* ADBEA *and* Ad*b*eA, *inclin'd to each other, at the common Vertex* A, *of either their first or second Axes, in an indefinitely small Angle* BA*b*; *To determine the Attraction thereof at the Point* A, *supposing the Force of each Particle of Matter to be as the Square of the Distance inversely.*

 Let

Let DE be any Ordinate to the Axis AB, and let AD be drawn; alſo put $AB = a$, $BC = x$, $CD = y$, and the Sine of the Angle BAb, formed by the two Planes



(to the Radius 1) $= d$; and let the Equation of either Curve be $y^2 = fx - x^2 - gx^2$: Which will anſwer, to the Conjugate, or Tranſverſe Axis thereof, according as the Value of g is poſitive or negative.

Now it will be, 1 (Radius) : d :: $a - x$ (AC) : $Cc = d \times \overline{a-x}$, the Thickneſs of the *Cuneus* at the Ordinate (or Section) DE: Moreover, becauſe $AD^2 = AC^2 + CD^2$, we have $AD = \sqrt{\overline{a-x}|^2 + fx - x^2 - gx^2}$: Whence, $\frac{DE \times Cc}{AC \times AD}$, expreſſing (*by Art.* 374.) the Attraction of the Particles in the indefinitely narrow Rectangle $DE \times Cc$, will be defined by $\frac{2d\sqrt{fx - x^2 - gx^2}}{\sqrt{\overline{a-x}|^2 + fx - x^2 - gx^2}}$: Which therefore, multiply'd by $\dot{x}$, will give the Fluxion of the Force to be found. But when $fx - x^2 - gx^2$ be-

becomes $= 0$, x will be $= \frac{f}{1+g}$ ($=$AB) $= a$; therefore, by ſubſtituting for f, our Fluxion will be transformed to $\frac{2d\dot{x}\sqrt{\overline{1+g}\times ax - \overline{1+g}\times x^2}}{\sqrt{\overline{a-x}|^2 + \overline{1+g}\times ax - \overline{1+g}\times x^2}} =$ $\frac{2d\dot{x}\sqrt{\overline{1+g}\times \overline{ax-x^2}}}{\sqrt{\overline{a-x}|^2 + \overline{1+g}\times \overline{ax-x^2}}} = \frac{2d\dot{x}\sqrt{\overline{1+g}\times x}}{\sqrt{a-x+\overline{1+g}\times x}} =$ $\frac{2d\times \overline{1+g}|^{\frac{1}{2}}\times x^{\frac{1}{2}}\dot{x}}{\overline{a+gx}|^{\frac{1}{2}}} = \frac{\overline{1+g}|^{\frac{1}{2}}\times 2dx^{\frac{1}{2}}\dot{x}}{a^{\frac{1}{2}}} \times$ $\overline{1 - \frac{gx}{2a} + \frac{3g^2x^2}{2.4a^2} - \frac{3.5g^3x^3}{2.4.6a^3}\ \&c.}$ Whereof the Fluent, when $x = a$, will be $\overline{1+g}|^{\frac{1}{2}} \times 2ad \times$ $\overline{\frac{2}{3} - \frac{2}{5}\times\frac{g}{2} + \frac{2}{7}\times\frac{3g^2}{2.4} - \frac{2}{9}\times\frac{3.5g^3}{2.4.6}\ \&c.}$ Which, becauſe $\overline{1+g}|^{\frac{1}{2}}\times a$ is $= f\times \overline{1+g}|^{-\frac{1}{2}} = f \times$ $\overline{1 - \frac{g}{2} + \frac{3g}{2.4} - \frac{3.5g^2}{2.4.6}\ \&c.}$ will (by multiplying the two Serieſes together &c.) be reduced to $2df \times$ $\overline{\frac{2}{3} - \frac{2.4g}{3.5} + \frac{2.4.6g^3}{3.5.7} - \frac{2.4.6.8g^3}{3.5.7.9}\ \&c.}$

Q. E. I.

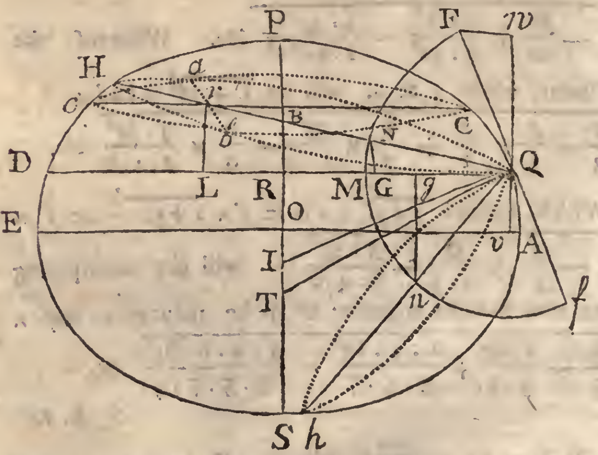
It may be obſerved, that the Fluent given above may be brought out without an Infinite Series (*by Art.* 126. *and* 278.) But the Solution here exhibited is beſt adapted to what follows hereafter; to which the Propoſition itſelf is premiſed as a *Lemma.*

 PROB.

PROB. VIII.

387. *To determine the Attraction at any Point Q in the Surface of a given Spheroid* OAPES.

Let QRL be perpendicular to the Axis PS of the Spheroid, and QT perpendicular to the Tangent Ff of the generating Ellipsis at Q, meeting PS in T: Moreover, let QaHb be a Section of the Spheroid by a Plane perpendicular to that of the Ellipsis APES, and thro' any Point r, in the Axis thereof, draw CBc and rL parallel to AE and PS: And make the Abscissa Qr $= x$, its corresponding Semi-Ordinate ra (or rb) $= y$, QR $= a$,



and RT $= b$; also let the Sine (NG) of the Angle HQD (to the Radius NQ$=1$) $= p$, its Co-sine QG $= q$, and the Ratio of OA^2 to OP^2, as any given Quantity h to Unity. Now, by reason of the similar Triangles QrL and QNG, we have rL (BR) $= px$, and QL $= qx$, and therefore Br (RL) $= qx - a$: Also, from the Nature of the Ellipsis, $AO^2 : PO^2$

$(h:1) :: \text{RT } (b) : \text{OR} = \frac{b}{h}$: Likewise $AO^2 : PO^2$

$(h:1)$

$(h:1)::QR^2:OP^2-OR^2$; and $PO^2:AO^2\ (1:h)$ $::OP^2-OB^2:BC^2=h\times\overline{OP^2-OB^2}=h\times$ $\overline{OP^2-\overline{OR+RB}^2}=h\times\overline{OP^2-OR^2-2OR\times RB-RB^2}$ $=QR^2+h\times\overline{-2OR\times RB-RB^2}$; becaufe (by the former Proportion) $QR^2=h\times\overline{OP^2-OR^2}$: Whence, by the Property of the Circle, $Cacb$, we get $y^2\ (BC^2-Br^2)=QR^2-Br^2-h\times\overline{2OR\times RB+RB^2}=a^2-\overline{qx-a}|^2-h\times$ $\overline{\frac{2b}{h}\times px+p^2x^2}=\overline{aq-bp}\times 2x-\overline{q^2+hp^2}\times x^2$: Which Equation, by making $1+B=h$, becomes $y^2=$ $\overline{aq-bp}\times 2x-\overline{q^2+p^2+Bp^2}\times x^2=\overline{aq-bp}\times 2x-x^2-Bp^2x^2$ (becaufe $q^2+p^2=1=QN^2$: Which being only of two Dimenfions, the Curve $QaHb$, whereto it belongs, is an Ellipfis.

The Equation of the Curve $QaHb$ being now obtained, let its Axis QH be fuppofed to revolve about Q, as a Center (the Plane of the Curve being always perpendicular to that of the Ellipfis APES) and let the Fluxion of the Arch MN (expreffing the Angle defcribed from the time the faid Axis begins its Motion at the Pofition ALD) be denoted by $\dot{A}$: Then, it is evident from the preceding Problem, that, $\overline{2aq-2bp}\times 2\dot{A}\times$ $\overline{\frac{2}{3}-\frac{2\cdot 4Bp^2}{3\cdot 5}+\frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}}$ &c. will be the Fluxion of the Attraction of the correfponding Part DQH of the Solid, upon a Corpufcle at Q, confidered as acting in the Direction HQ (which Expreffion is found, by, barely, writing $2aq-2bq$, $\dot{A}$, and Bp^2, in the faid Problem, for f, d, and g refpectively.)

Hence,

Hence, by the Reſolution of Forces, the Fluxion of the Attraction, in the Directions QR and Qw (perpendicular to QR) will be truly exhibited by $\overline{2aq-2bp} \times 2\dot{A}q \times \overline{\frac{2}{3} - \frac{2\cdot 4Bp^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}}$ &c. and $\overline{2aq-2bp} \times 2\dot{A}p \times \overline{\frac{2}{3} - \frac{2\cdot 4Bp^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2p^4}{3\cdot 5\cdot 7}}$ &c.

Let now another Plane Qb be ſuppoſed to revolve about the Point Q, the contrary Way to the former, from QD towards Qf; and let (ng) the Sine of the Angle RQb be denoted by P, and its Co-ſine (Qg) by $\mathcal{Q}$: Then the Fluxion of the Attraction of the Part DQb, in the foreſaid Directions QR and Qw (by writing $-P$ inſtead of p and $\mathcal{Q}$ inſtead of q) will appear to be $\overline{2a\mathcal{Q}+2bP} \times 2\dot{A}\mathcal{Q} \times \overline{\frac{2}{3} - \frac{2\cdot 4BP^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2P^4 \text{ \&c.}}{3\cdot 5\cdot 7}}$ and $\overline{2a\mathcal{Q}+2bP} \times -2\dot{A}P \times \overline{\frac{2}{3} - \frac{2\cdot 4BP^2}{3\cdot 5} + \frac{2\cdot 4\cdot 6B^2P^4}{3\cdot 5\cdot 7}}$ &c. Which being added to thoſe of the former Part, in the ſame Directions, and $\frac{\dot{p}}{q}$ and

* Art. 142. $\frac{\dot{P}}{\mathcal{Q}}$ reſpectively ſubſtituted inſtead of $\dot{A}$ *, we have

$$4a \text{ into } \frac{2}{3} \times \overline{q\dot{p}+\mathcal{Q}\dot{P}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{qp^2\dot{p}+\mathcal{Q}P^2\dot{P}} \text{ \&c.}$$

$$+\ 4b \text{ into } \frac{2}{3} \times \overline{P\dot{P}-p\dot{p}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{P^3\dot{P}-p^3\dot{p}} \text{ \&c.}$$

And

$$4a \text{ into } \frac{2}{3} \times \overline{p\dot{p}-P\dot{P}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{p^3\dot{p}-P^3\dot{P}} \text{ \&c.}$$

$$-4b \text{ into } \frac{2}{3} \times \overline{\frac{p^2\dot{p}}{q}+\frac{P^2\dot{P}}{\mathcal{Q}}} - \frac{2\cdot 4B}{3\cdot 5} \times \overline{\frac{p^4\dot{p}}{q}+\frac{P^4\dot{P}}{\mathcal{Q}}} \text{ \&c.}$$

for

for the Fluxion of the Attraction of both Parts together in the foresaid Directions: Whereof the Fluents, when N coincides with F, and n with f, will be the Attraction of the whole Spheroid in those Directions. But now, in order to determine these Fluents with as little Trouble as possible, let m be assumed to denote any whole positive Number; then the Fluent of $\frac{p^{2m}\dot{p}}{q}$, or $\frac{p^{2m}\dot{p}}{\sqrt{1-p^2}}$, will be universally $= \frac{-q}{2m} \times \overline{p^{2m-1} + \frac{2m-1}{2m-2} \times p^{2m-3} + \frac{2m-1 \cdot 2m-3}{2m-2 \cdot 2m-4} \times p^{2m-5}\ (m)}$ $+ \frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times$ the Arch (MN) whose Sine is p*: And that of $\frac{P^{2m}\dot{P}}{Q}$, or $\frac{P^{2m}\dot{P}}{\sqrt{1-P^2}}$ (in the same Manner) $= \frac{-Q}{2m} \times \overline{P^{2m-1} + \frac{2m-1}{2m-2} \times P^{2m-3}}$ &c. $+ \frac{1 \cdot 2 \cdot 3 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times$ Arch (Mn) whose Sine is P. But when N coincides with F, and n with f, the Sines p and P, of the Arches MF and Mf, becoming equal, and (the Co-sine) $Q = -$ (Co-sine) q, it is evident that the Sum of the Fluents of $\frac{p^{2m}\dot{p}}{q}$ and $\frac{P^{2m}\dot{P}}{Q}$, will, in that Case, be truly exhibited by $\frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times \text{MF} + \frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times$ Mf, or its Equal $\frac{1 \cdot 3 \cdot 5 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \ldots 2m} \times$ FMf; because,

* Art. 296.

cause, then all the rest of the Terms (by reason of the equal Quantities P, p and Q, $-q$) destroy one another. After the same Manner the Sum of the Fluents of $qp^{2m}\dot{p}$ and $QP^{2m}\dot{P}$, in the foresaid Circumstance, will appear to be $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots \overline{2m-1}}{2 \cdot 4 \cdot 6 \cdot 8 \ldots \overline{2m+3}} \times FMf$*.

* Art. 297.

Now, to apply this to the Matter in hand, let the Exponent of B, in any Term of either of the above found Fluxions be, universally, expressed by n; then the numeral Coefficient (annexed to B) will be defined by $\frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{1 \cdot 3 \cdot 5 \ldots \overline{2n+3}}$, and the variable Quantities multiplied thereby, in the first Line of the former Fluxion, will be $qp^{2n}\dot{p} + QP^{2n}\dot{P}$: Therefore $\frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{1 \cdot 3 \cdot 5 \ldots \overline{2n+3}} \times B^n \times \overline{qp^{2n}\dot{p} + QP^{2n}\dot{P}}$ is a General Term, (from whence, if n be expounded by 1, 2, 3 &c. successively, that whole Line will be produced.) But, the Fluent of $qp^{2n}\dot{p} + QP^{2n}\dot{P}$, in the Circumstance above specified, (putting $m = n$ and $FMf = k$) appears to be $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots \overline{2n-1}}{2 \cdot 4 \cdot 6 \cdot 8 \ldots \overline{2n+2}} \times k$: Which, therefore, multiplied by $\frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{3 \cdot 5 \ldots \overline{2n+3}} \times B^n$, gives $\frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots \overline{2n-1}}{2 \cdot 4 \cdot 6 \cdot 8 \ldots \overline{2n+2}} \times \frac{2 \cdot 4 \cdot 6 \ldots \overline{2n+2}}{3 \cdot 5 \ldots \overline{2n+3}} \times B^n k = \frac{B^n k}{\overline{2n+1} \times \overline{2n+3}}$, for the true Fluent of the said General Term: Which, if n be expounded by

0,

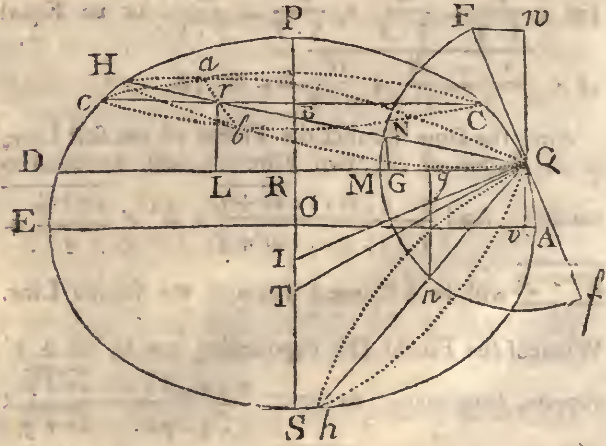
0, 1, 2, 3 &c. successively, will become equal to $\frac{k}{1\cdot3}$, $\frac{Bk}{3\cdot5}$, $\frac{B^2k}{5\cdot7}$, $\frac{B^3k}{7\cdot9}$ &c. respectively; and therefore the Fluent of the whole Line (drawn into the general Multiplicator $4a$) is $= 4ak \times \overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7} - \frac{B^3}{7\cdot9}}$ &c. But now, for the Fluent of the second Line: This, it is plain, will be $= 4b$ into $\frac{2}{3} \times \overline{\frac{P^2}{2} - \frac{p^2}{2}} - \frac{2\cdot4B}{3\cdot5} \times \overline{\frac{P^4}{4} - \frac{p^4}{4}}$ &c. Which, in the foresaid Circumstance, when $P = p$, intirely vanishes. Therefore it appears, that the Attraction of the whole Spheroid, in the Direction QR, is truly expressed by $4ak \times \overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7} - \frac{B^3}{7\cdot9}}$, or its Equal $4k \times \overline{\frac{1}{1\cdot3} - \frac{B}{3\cdot5} + \frac{B^2}{5\cdot7}}$ &c. $\times QR$.

After the same Manner the Fluent of the first Line, in the latter of our two Fluxions, will be found to vanish: And $\frac{2\cdot4\cdot6\ldots\overline{2n+2}}{1\cdot3\cdot5\ldots\overline{2n+3}} \times B^n \times \overline{\frac{p^{2n+2}\dot{p}}{q} + \frac{P^{2n+2}\dot{P}}{Q}}$ will be a General Term to the second Line. Whereof the Fluent (by expounding $2m$ by $2n+2$) appears, from above, to be $= \frac{2\cdot4\cdot6\ldots\overline{2n+2}}{3\cdot5\cdot7\ldots\overline{2n+3}} \times B^nk \times \frac{1\cdot3\cdot5\ldots\overline{2n+1}}{2\cdot4\cdot6\ldots\overline{2n+2}} = \frac{B^nk}{2n+3}$: Which, when

n is

n is interpreted by 0, 1, 2, 3 &c. successively, comes out equal to $\frac{k}{3}$, $\frac{Bk}{5}$, $\frac{B^2k}{7}$ &c. respectively: Therefore the Attraction of the Spheroid, in the Direction Qw, is exhibited by $-4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}}$ &c. and consequently, That in the opposite Direction Qv, by $4bk \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9}} = 4k \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}}$ &c. $\times$ RT $= 4k \times \overline{1+B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}}$ &c. $\times$ OR (because $\overline{1+B} \times \mathrm{OR} = \mathrm{RT}$.)

From which and the Force in the Direction QR (found above) not only the Direction of the absolute



Attraction, but that Attraction itself will be known: For, let RI be taken to QR, as the Force in the Direction Qv to that in the Direction QR; and then, by the

the Compoſition of Forces, QI will be the Direction of the Attraction, or the Line in which a Corpuſcle at Q tends to deſcend: And the Attraction itſelf, in that Direction, (being to that in QR, as QI to QR) will be defined by $4k \times \overline{\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ \&c.}} \times \text{QI}$; which, ſince $4k$ is conſtant, will alſo be as $\overline{\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ \&c.}} \times \text{QI}$. *Q.E.I.*

COROLLARY.

388. Since, by Conſtruction, RI : QR :: $\overline{1 + B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9} \text{ \&c.}} \times \text{OR} : \overline{\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ \&c.}} \times \text{QR}$, it follows that $\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ \&c.} : \overline{1 + B} \times \overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \text{ \&c.}} :: \text{RO} : \text{RI}$; whence (by Diviſion) $\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ \&c.} : \frac{3B}{3 \cdot 5} - \frac{3B^2}{5 \cdot 7} + \frac{3B^3}{7 \cdot 9} \text{ \&c.} :: \text{OR} \left(: \frac{\text{OT}}{B}\right) : \text{OI}$; and conſequently, $\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ \&c.} : 3 \times \overline{\frac{1}{3 \cdot 5} - \frac{B}{5 \cdot 7} + \frac{B^2}{7 \cdot 9} \text{ \&c.}} :: \text{OT} : \text{OI}$.

Hence it appears that the Direction QI, of the abſolute Attraction, divides the Part of the Axis OT, intercepted by the Center and Normal, in a given Ratio: And that the Attraction itſelf (being defined

fined by $\overline{\frac{1}{1\cdot3}-\frac{B}{3\cdot5}+\frac{B^2}{5\cdot7}\ \&c.\ \times QI}$) is every where as the said Line of Direction QI.

SCHOLIUM.

389. Although the foregoing Conclusions are exhibited by infinite Serieses, yet the Sums of those Serieses are explicable by means of the Arch of a Circle.

Thus, let the Series $\frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}$ &c. (which is one of the two original ones above found) be put $= S$, and let $B = t^2$; then by Substitution, and multiplying the whole Equation by t^3, we shall have $\frac{t^3}{3}-\frac{t^5}{5}+\frac{t^7}{7}$ &c. $= t^3S$, and consequently $t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}$ &c. $= t - t^3S$: Where, the former Part of the Equation is known to express the Arch of a Circle, whose Tangent is t $(B^{\frac{1}{2}})$ and Radius Unity *: Wherefore, putting that Arch $= A$, we have $A = t - t^3S$, and consequently $S = \frac{t-A}{t^3} = \frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}$ &c.

* Art. 142.

Moreover, since it appears that

$$\left.\begin{array}{l}\frac{B}{3}-\frac{B^2}{5}+\frac{B^3}{7}\ \&c.\\ -\frac{B}{5}+\frac{B^2}{7}-\frac{B^3}{9}\ \&c.\end{array}\right\}\text{is} = \frac{2B}{3\cdot5}-\frac{2B^2}{5\cdot7}+\frac{2B^3}{7\cdot9}\ \&c.$$

(where the Sum of $\frac{B}{3}-\frac{B^2}{5}+\frac{B^3}{7}$ &c. is already found $= \frac{t-A}{t^3}\times B = \frac{t-A}{t}$, and where That of of $-\frac{B}{5}+\frac{B^2}{7}$ &c. by the same Method will come out $=\frac{t-A-\frac{1}{3}t^3}{t^3}$) it is evident that $\frac{2B}{3\cdot5}-\frac{2B^2}{5\cdot7}+\frac{2B^3}{7\cdot9}$ &c. $=\frac{t-A}{t}+\frac{t-A-\frac{1}{3}t^3}{t^3}=\frac{\frac{2}{3}t^3+t-A\times\overline{1+t^2}}{t^3}$; and consequently $\frac{1}{1\cdot3}-\frac{B}{3\cdot5}+\frac{B^2}{5\cdot7}$ &c. $\left(=\frac{1}{3}-\frac{\frac{2}{3}t^3+t^2-A\times\overline{1+t^2}}{2t^3}\right)$ $=\frac{A\times\overline{1+t^2}-t}{2t^3}$: Which is the Value of the other original Series found above: From whence that of $\frac{3}{3\cdot5}-\frac{3B}{5\cdot7}+\frac{3B^2}{7\cdot9}$ will also be had $=\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t^5}$.

Hence, if

$$\frac{t-A}{t^3}\left(=\frac{1}{3}-\frac{B}{5}+\frac{B^2}{7}-\frac{B^3}{9}\right) \text{ be put } = f;$$

$$\frac{A\times\overline{1+t^2}-t}{2t^3}\left(=\frac{1}{1\cdot3}-\frac{B}{3\cdot5}+\frac{B^2}{5\cdot7}\text{ \&c.}\right)=g;$$

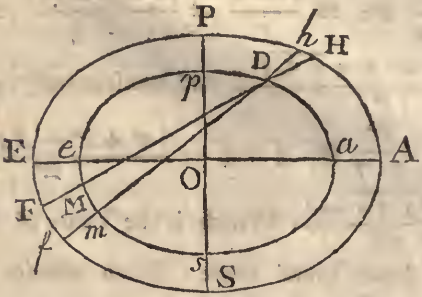
And

$$\frac{3t+2t^3-3A\times\overline{1+t^2}}{2t^5}\left(=\frac{3}{3\cdot5}-\frac{3B}{5\cdot7}+\frac{3B^2}{7\cdot9}\text{ \&c.}\right)=h$$

it is evident that OT will be to OI, in the constant Ratio of g to h; and that the Forces in the Directions QI, QR, and Qv, will be as $g\times$ QI, $g\times$ QR, and $f\times\overline{1+B}\times$ OR respectively: Where $1+B$ is $=\frac{AO^2}{PO^2}$.

PROB. IX.

390. *To determine the Attraction at any Point* D *within a given Spheroid* OAPES.



Let O*apes* be another Spheroid, concentric with, and ſimilar to, the given one; whoſe Surface D*e*M &c. paſſes through the given Point D; alſo let FD*f* and HD*h* be taken as two oppoſite, indefinitely ſlender, Cones (or Pyramids) conceived to be formed by drawing innumerable Lines HDF, *h*D*f* &c. through the common Vertex D) which Cones (or Pyramids) having the ſame Angle, may be conſidered as ſimilar; and ſo their

* Art. 378. Forces, at D, will be as the Altitudes DF and DH *: And, therefore, the Exceſs of the former, above the latter, or the Force whereby a Corpuſcle at D, tends towards F, through the, contrary, Action of the two oppoſite Cones, will be as DF — DH, or as DM; becauſe *(by the Property of the Ellipſis)* MF is, in all Poſitions, equal to DH.

Hence it appears that the Parts of Matter FM*mf* and HD*h*, without the Spheroid *apes* (acting equally, in contrary Directions) can have no Effect at D: And this, being every where the Caſe, the whole, efficacious, Force at D muſt therefore be that of the Spheroid O*apes*.

Hence, if the Ratio of Oa^2 to Op^2 (or of OA^2 to OP^2) be denoted by that of $1+B$ to 1, as in the laſt Problem, it follows, from thence, that the Attraction at D, in the Directions DM and DN (perpendicular to PS and AE; *see the next Fig.)* will be expounded by $\frac{1}{1\cdot3}-\frac{B}{3\cdot5}+\frac{B^2}{5\cdot7}$ &c. × DM, and $\overline{1+B}\times\frac{1}{3}-\frac{B}{\cdot5}+\frac{B^2}{\cdot7}-\frac{B^3}{9}$ &c. × DN respectively, or by their Equals $g\times$ DM and $f\times\overline{1+B}\times$ DN: Where the Values of f and g are the same as given in the preceding Article.

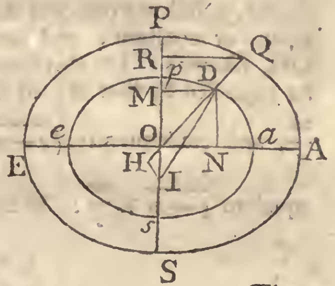
COROLLARY.

391. Hence the Force wherewith a Corpuscle, any where within a given Spheroid, is attracted, either, towards the Axis, or the Plane of its Equator, is directly as the Distance therefrom.

PROB. X.

392. *Supposing every Particle of Matter in a Spheroid to have a Tendency to recede, both, from the Axis* PS, *and from the Plane of the greatest Circle, by Means of Forces that are as the Distances from the said Axis, and Plane, respectively; to find the Direction* DI *wherein a Corpuscle, at any Point* D, *tends to move through the Action of the said Forces and the Attraction conjunctly; and likewise the whole compound Force in that Direction.*

Let DM and DN be perpendicular to PS and AE, and let the given Forces, in the Direction of those Lines (independent of the Attraction) be expressed by $m\times$DM and $n\times$DN respectively.



There-

Therefore, since (by the last Problem) the Force of Attraction in the said Directions is defined by $g \times$ DM and $f \times \overline{1 + B} \times$ DN, the whole resulting Forces will be truly denoted by $\overline{g - m} \times$ DM, and $f \times \overline{\overline{1 + B} - n} \times$ DN: Whence (by the Composition of Forces) it will be, $g - m : f \times \overline{1 + B} - n ::$ DN (OM) : MI; whence the Point I is given:

Also DM : DI :: $\overline{g - m} \times$ DM (the Force in the Direction DM) : $\overline{g - m} \times$ DI, the Force in DI. *Q. E. I.*

PROB. XI.

393. *Every thing being supposed as in the preceding Problems, it is required to determine the Force of all the Particles in the Line (or Column)* QDO *tending to the Center* O *of the Spheroid.*

Let IH be perpendicular to QO produced (*see the last Fig.*) then the absolute Force, in the Direction DI, being $\overline{g - m} \times$ DI, that in the Direction DH, whereby a Corpuscle at D is urged towards the Center, will be $\overline{g - m} \times$ DH. Let now OD (considered as variable) be denoted by x; then because the Ratio of OM to MI is given (being every where as $g - m$ to $f \times \overline{1 + B} - n$, *by the Precedent*) and the Triangles ODM and IOH are similar, it follows that the Ratio of OD to OH will be given, or constant; and consequently that of DH to OH, likewise: Let therefore this Ratio of DH to OH be expressed by that of r to s, and we shall have DH $= \frac{rx}{s}$, and consequently ($\overline{g - m} \times$ DH) the Force at D, equal to $\overline{g - m} \times \frac{rx}{s}$: Which therefore being multi-

plied

plied by $\dot{x}$, and the Fluent taken, there comes out $\frac{\overline{g-m} \times rx^2}{2s} = \frac{g-m}{2} \times DO \times DH$, for the whole Force of the Line or Column OD at the Center.

Q. E. I.

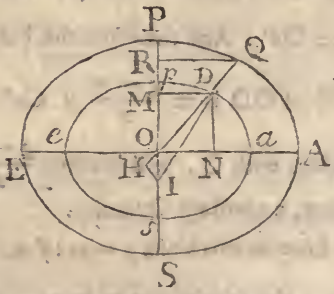
COROLLARY.

394. If the given Forces m and n be such that the Ratio of OM to MI, (which is found to be universally as $g-m$ to $f \times \overline{1+B-n})$ may become as $1 : 1+B$ (or as $pO^2 : aO^2$) it is evident (from the Property of the Ellipsis) that the Line of Direction DI will be always perpendicular to the Surface of the Spheroid *Oapes.* In which Case OD × DH is also (by the Nature of the Ellipsis) $= Oa^2$: And therefore the Force $\left(\frac{g-m}{2}\right.$ $\left.\times OD \times DH\right)$ of OD is $= \frac{g-m}{2} \times Oa^2$: Which, when D coincides with Q, will become $\frac{g-m}{2} \times AO^2$; and is, therefore, a constant Quantity.

Moreover, since in this Case, $g-m : f \times \overline{1+B}-n$ $:: 1 : 1+B$ (by Hypothesis) we have $m - \frac{n}{1+B} = g$ $-f$: Which Equation, if n be taken $= 0$, gives $m = g - f = \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}$ &c. $= \frac{\overline{3+t^3} \times A - 3t}{2t^3}$ *; * Art. 389.
But, if m be taken $= 0$, it will then give $n = -\overline{1+B}$ $\times \overline{g-f} = -\overline{1+B} \times \overline{\frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}}$ &c. Where, $t = B^{\frac{1}{2}}$, and $A =$ the Arch whose Tangent is t, and Radius Unity.

PROP. XII.

395. *If an oblate Spheroid* OAPES, *whereof the Square of the Equatoreal Diameter* AE, *is to that of the Axis* PS, *in any given Ratio of* $1+B$ *to* 1, *revolves about its Axis, in such a Time, that the centrifugal Force, at the Equator* A, *is to the Attraction at the Surface of a Sphere whose Radius is* OA, *in the Ratio of* $\frac{2B}{3\cdot5}-\frac{4B^2}{5\cdot7}+\frac{6B^3}{7\cdot9}$ *&c.* to $\frac{1}{3}$: *I say, in that Case, every Particle of the Spheroid will be* in Equilibrio; *so that, though the Cohesion of the Parts was to cease, the Figure itself would remain unchanged.*



For, the Attraction of the Spheroid, at A, being defined by $\overline{\frac{1}{1\cdot3}-\frac{B}{3\cdot5}+\frac{B^2}{5\cdot7}\ \&c.}\times AO$ (*Art.* 387.) it is evident (by conceiving $B=0$) that $\frac{AO}{3}$ will represent the Attraction at the Surface of the Sphere whose Radius is AO: Whence (by Hypothesis) the centrifugal Force at A (putting $m=\frac{2B}{3\cdot5}-\frac{4B^2}{5\cdot7}+\frac{6B^3}{7\cdot9}$ &c.) will be truly defined by $m\times AO$; and consequently

ſequently That, at any other Point D, by $m \times DM$ (becauſe the centrifugal Forces of Bodies deſcribing unequal Circles, in equal Times, are known to be directly as the Radii *.) Hence, and from the Corollary to the laſt Problem, it appears that the Direction of Gravitation DI is always perpendicular to the Surface *apes*; and that the Force of all the Particles in the Line (or Canal) OD or OQ, towards the Center O, will continue invariable, take the Point Q in what Part of the Arch APE you will: From which laſt Conſideration, it follows that the Force, or Preſſure of every Canal QO, at the Center O, (conſidering the Body in a fluid State) will be the ſame: Whence (by the Principles of Hydroſtatics) a Corpuſcle at D has no Tendency to move, either Way, in the Line OQ: And therefore, as it hath no Tendency to move in the Direction of the Surface D*pe* (the Gravitation being perpendicular thereto) it is evident, *from Mechanics*, that no Motion at all can enſue, in any Direction. *Q. E. D.* * Art. 213.

COROLLARY I.

396. Since m is $= \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}$ &c. the Gravitation $(\overline{g - m} \times DI)$ at any Point D in the Spheroid will therefore be as $\overline{\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \text{ \&c.}}$ $\times DI = \frac{t - A}{t^3} \times DI$ (*ſee Art.* 389.

COROLLARY II.

397. If the Time of Revolution be given $= p$, and q be put to denote the Time wherein a (ſolid) Sphere, of the ſame Denſity with the Spheroid, muſt revolve; ſo that the centrifugal Force, at the Equator thereof, may be equal to the Gravity: Then, as this laſt Time is known to continue the ſame, whatever the Magnitude of that Sphere is †; and the centrifugal Forces, in equal † Art. 213. and 381.

 Circles,

Circles, are alſo known to be inverſely as the Squares of the periodic Times——it follows, that $p^2 : q^2 :: \frac{1}{3}$AO (the Attraction, or centrifugal Force, reſpecting the Sphere OA, revolving in the Time q) : $\overline{\frac{2B}{3\cdot5} - \frac{4B^2}{5\cdot7} + \frac{6B^3}{7\cdot9}\text{ \&c.}} \times$ AO, the centrifugal Force of the Spheroid at A, revolving in the Time p. From which Proportion we get $\frac{q^2}{3p^2} = \frac{2B}{3\cdot5} - \frac{4B^2}{5\cdot7} + \frac{6B^3}{7\cdot9}$ &c. $= \frac{\overline{3+t^2} \times A - 3t}{2t^3}$ (*Art.* 394.) Whence, by Help of the Trigonometrical-Canon, the Value of t $(= B^{\frac{1}{2}})$ and, conſequently, the Ratio of the two principal Diameters, will be found; ſo that all the Parts of the Spheroid may remain *in Equilibrio*. But, when $\frac{q^2}{3p^2}$ is ſmall, the Solution by an Infinite Series is preferable: For, then the Series $\frac{2B}{3\cdot5} - \frac{4B^2}{5\cdot7}$ &c. $(= \frac{q^2}{3p^2})$ converging ſufficiently ſwift, we ſhall, by the Reverſion thereof, find $B = \frac{5q^2}{2p^2} + \frac{25 \times 6q^4}{4 \times 7p^4} + \frac{125 \times 37q^6}{8 \times 49p^6}$ &c. In which Caſe the Ratio of the Equatoreal Diameter to the Axis, if we take only the firſt Term of the Series, will be, as $\sqrt{1 + \frac{5q^2}{2p^2}} : 1$, or as $1 + \frac{5q^2}{4p^2}$, nearly.

Which, if $\frac{p^2}{q^2} = 289$, or the centrifugal Force at the Equator be to the Gravity as 1 to 289 (that being the Proportion at the Equator of the Earth *) will come out as 231 to 230.

* Art. 217.

Co-

COROLLARY III.

398. Becauſe, $\frac{\overline{3+t^2}\times A - 3t}{2t^3}$, the latter Part of our foregoing Equation will be equal to Nothing, both when t is Nothing and Infinite, it is evident that the Value thereof cannot, in any intermediate Circumſtance of t, exceed a certain aſſignable Quantity.

Wherefore, to determine this Limit of the Value of $\frac{q^2}{3p^2}$ (beyond which the Problem becomes impoſſible) let the Fluxion of $\frac{\overline{3+t^2}\times A - 3t}{2t^3}$, or its Double $\frac{\overline{3+t^2}\times A}{t^3} - \frac{3}{t^2}$ be taken and put $= 0$, and you will have $-\overline{9+t^2}\times A\dot{t} + \overline{3t+t^3}\times\dot{A} + 6t\dot{t} = 0$: Which, becauſe $\dot{A} = \frac{\dot{t}}{1+t^2}$ * will be reduced to $9t$ * Art. 142.
$+ 7t^3 - \overline{1+t^2}\times\overline{9+t^2}\times A = 0$; where t is found $= 2{,}5293$, from whence the correſponding Values of $\sqrt{1+t^2}$, and $\frac{q}{p}$ come out $= 2{,}7198$, and 0.5805 &c. reſpectively. Hence it appears that it is impoſſible for the Parts of the Spheroid, in a fluid State, to continue at Reſt among themſelves, when the Time of Revolution is ſo great that $\frac{q}{p}$ exceeds $0{,}5805$ &c. And that, of all the Spheroids which can be aſſumed by a Fluid revolving about an Axis, That whoſe Equatoreal Diameter is to its Axis as $2{,}7198$ to Unity, will perform its Revolutions in the ſhorteſt Time.

Thus, for Example, if a (ſolid) Sphere of the ſame common Denſity with the Earth was to revolve about its Axis in the Time of $84\frac{3}{4}$ Minutes, the centrifugal Force

Force at the Equator thereof would, it is known, be equal to the Gravity *: Therefore, by taking $\frac{84\frac{3}{4}}{p}\left(=\frac{q}{p}\right)$ $= 0,5805$ *&c.* the Time p will come out $= \overset{M}{146}$ or $\overset{H}{2}\ \overset{M}{26}$. Which Time is the least, possible, wherein a Fluid, of the same common Density with the Earth, can revolve, so as to preserve its spheroidal Figure. And this holds universally, let the Magnitude of the Body, or Fluid, be what it will.

* Art. 217.

COROLLARY IV.

399. Hence also may be determined the Spheroid, which a spherical Body (of Ice or any other Matter) revolving in a given Time s, will converge to, when reduced to a fluid State *:

For, since the Momenta of Rotation, in equal Spheres and Spheroids, are to one another, in a Ratio compounded of the direct Ratio of their Equatoreal Diameters, and the inverse Ratio of the Times of their Rotation, it follows, if d be put $=$ the Diameter of the given Sphere, and $E =$ the Equatoreal Diameter of the required Spheroid, that $\frac{d}{s} = \frac{E}{p}$ (because the Quantity of Motion about the Axis is not affected by the Action of the Particles one upon another, while the Figure of the Fluid is changing.) Moreover, since the Masses of the Sphere and Spheroid are also equal to each other (by Hypothesis) we have d^3 $(= AE^2 \times PS) = \frac{E^3}{\overline{1+t^2}|^{\frac{1}{2}}}$: From which two Equations, exterminating d, there arises $p = \overline{1+t^2}|^{\frac{1}{6}} \times s$, for the Time of Revolution of the required Spheroid: Whence, by substituting this Value of p in the general Equation $\frac{q^2}{3p^2}$

* The Author in a Note, page 135 of his *Miscellaneous Tracts in 4to*, has corrected an Oversight in this Corollary, by taking here $\frac{e}{d} \times s$ (instead of $\frac{e^2}{d^2} \times s$) whereby the remaining Part of this Article is rendered erroneous.

=

$= \frac{\overline{3+t^2} \times A - 3t}{2t^3}$, we get $\frac{q^2}{3s^2} = \overline{1+t^2}|^{\frac{1}{3}} \times$ $\frac{\overline{3+t^2} \times A - 3t}{2t^3}$; from the Solution of which the Value of t, and the Spheroid itself, will be given.

But, since the Value of the latter Part of the Equation can never exceed a certain assignable Quantity, the Matter proposed can therefore be only possible under certain Limitations: In order to determine these Limitations, let the Fluxion of $\overline{1+t^2}|^{\frac{1}{3}} \times \frac{\overline{3+t^2} \times A - 3t}{2t^3}$ be taken and put $= 0$, and it will be found that $\overline{t^4 + 24t^2 + 27} \times A - 15t^3 - 27t = 0$: Whence t comes out $= 7.5$, and the corresponding Value of $\frac{q}{s} = 0,927$, nearly.

Hence the Parts of the Fluid cannot possibly come to an Equilibrium among themselves, when the Time s is less than $\frac{q}{0,927}$, but will continue to recede from the Axis, *in Infinitum*.

If q be taken $= \overset{M}{84\tfrac{3}{4}}$ (as in the Example to the preceding Corollary) s will be equal $\overset{M}{91} = \overset{H}{1} : \overset{M}{31}$. From which it appears, that, if the Earth (or a spherical Body of the same Density) was to revolve about its Axis in less than $\overset{H}{1} : \overset{M}{31}$; and, in the mean time, be reduced to a State of Fluidity, the Parts thereof towards the Equator would ascend, and continue to recede from the Axis, *in Infinitum*.

Corollary V.

400. Seeing the Values of t and A are given when the Spheroid is given, it follows that the Gravitation

tation $\left(\frac{t-A}{t^3} \times QI\right)$ at any Point in the Surface of a Spheroid, whereof the Parts are kept *in Equilibrio*, by their Rotation about the Axis, will be accurately as a Perpendicular to the Surface at that Point, continued to the Axis of the Figure. Therefore the Gravitation at the Equator is to that at either of the Poles, as the Equatoreal Diameter to the Axis inversly.

COROLLARY VI.

401. But, if the Spheroid differs but little from a Sphere, the Excefs of QI above AO will (by the Property of the Ellipsis) be nearly as OR^2. Whence it appears that the Increafe of Gravitation, in going from the Equator to the Pole, is as the Square of the Sine of Latitude, nearly.

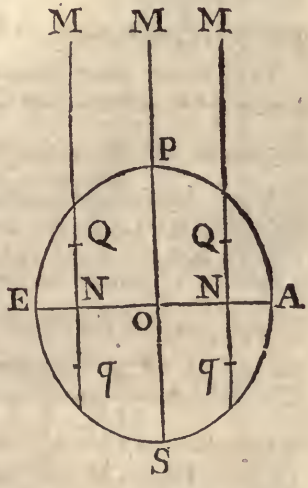
COROLLARY VII.

402. Moreover, fince the Ratio of the Equatoreal Diameter to the Axis is found, in this Cafe, to be that

† Art. 397. of $1 + \frac{5q^2}{4p^2}$ to 1 †, the Excefs of that Diameter above the Axis will be to the Axis as $\frac{5q^2}{4p^2}$ to Unity; that is, as $\frac{5}{4}$ of the centrifugal Force at the Equator to the mean Force of Gravity. Whence, as the centrifugal Forces, in unequal Circles, are univerfally as the Radii directly, and the Squares of the periodic Times inverfly, it follows that the forefaid Excefs (in Figures nearly fpherical) will be as the Radii directly, and as the Denfity and the Square of the Time of Rotation inverfly: From which Proportions, the Ratios of the greateft and leaft Diameters of the Planets may be inferred from each other; fuppofing the Times of their Rotation, about their Axes, to be known.

PROB. XIII.

403. *To determine the Figure which a Fluid will acquire, when, besides the mutual Gravitation of the Parts thereof, it is attracted by another Body, so remote, that all Lines drawn from it to the Surface of the Fluid, may be taken as Parallels.*



Let OAPES be the proposed Fluid, and let MPS and MQ*q* be Right-lines, drawn from the remote Body *M*; whereof the former MPS passes thro' the Center of Gravity O: Moreover, let the Plane AE be perpendicular to the Axis MOS; and put NQ = *a* and OM (the Distance of the remote Body) = *d*; also put the Semi-diameter of the Body (at M) = *r*, and let its Density be to that of the Fluid APES, as any Quantity *v* to Unity. Then, since, according to the foregoing Calculations, the Attraction at the Surface of a Sphere (of a given Density) is expressed by $\frac{1}{3}$ of the Radius, it follows that the Attraction of the Body *M*, at its Surface, will be explicable by $\frac{vr}{3}$: And therefore, the Force varying according to the Square of the Distance inversly *, it will be, d^2 (MN^2) : r^2 :: $\frac{vr}{3}$: $\frac{vr^3}{3d^2}$, the Attraction of *M*, at the Distance MN: Also $\overline{d-a}|^2$ (MQ^2) : r^2 :: $\frac{vr}{3}$: $\frac{vr^3}{3\times\overline{d-a}|^2}$, its Attraction at the

* Art. 382.

Distance

Distance MQ. Whence the Difference of these two, or $\frac{vr^3}{3 \times \overline{d-a}|^2} - \frac{vr^3}{3d^2}$ ($= \frac{vr^3}{3d^3} \times \overline{2a + \frac{3a^2}{d} + \frac{4a^3}{d^2}}$ *&c.*) will be as the Force whereby a Corpuscle at Q endeavours to recede from the Plane AE: Which because (by Hypothesis) d is very great in respect of a, will (by rejecting all the Terms after the first) be expressed by $\frac{2vr^3}{3d^3} \times a$, or its Equal $\frac{2vr^3}{3d^3} \times$ NQ.

In the very same Manner, the Force whereby a Corpuscle at q, below the Plane AE, tends to recede therefrom, will be defined by $\frac{2vr^3}{3d^3} \times Nq$.

Now, therefore, seeing these Forces are, every where, as the Distances NQ, Nq, from the Plane AE, it appears (*by Art.* 393. *and* 394.) that the Figure OAPES will be a Spheroid; whereof the Equation, for the Relation of its two principal Diameters (putting $n = \frac{2vr^3}{3dr^3}$) is $n = -\overline{1+B} \times \overline{\frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}}$ *&c.* (In which, the Ratio of PS² to AE² is denoted by that of 1 to $1+B$.) Hence, by reverting the Series, we have $B = -\frac{15n}{2} - \frac{225n^2}{28}$ *&c.* and consequently PS : AE :: 1 : $\sqrt{1 - \frac{15n}{2} - \frac{225n^2}{28} \, \&c.}$ :: 1 : $1 - \frac{15n}{4}$, nearly: Which, by restoring the Value of n, becomes PS : AE :: 1 : $1 - \frac{5vr^3}{2d^3}$.

Q. E. I.

Co-

COROLLARY.

404. Becauſe $\frac{r}{d}$ expreſſes the Sine of the apparent Semi-diameter of the Body *M*, to the Radius 1) ſeen at the Diſtance OM, it follows, if the ſaid Sine be denoted by *c*, that PS : AE :: 1 : $1 - \frac{5v}{2} \times c^3$; and conſequently, by Diviſion, PS : PS — AE :: 1 : $\frac{5v}{2} \times c^3$.

Hence it appears, that the Forces of the Planets, to produce Tides at the Earth's Surface, are to one another as their Denſities, and the Cubes of their apparent Diameters conjunctly. (For the Sines of ſmall Arcs are nearly as the Arcs themſelves.)

EXAMPLE.

405. If *c* be taken = the Sine of 16′ (expreſſing the mean Apparent Semi-diameter of the Moon) and $v = \frac{5}{4}$ (the Ratio of her Denſity with reſpect to that of the Earth) our laſt Proportion will become PS : PS — AE :: 1 : 0,000000315 : Whence, if PS be taken = 42000000 Feet (the Meaſure of the Earth's Diameter) PS — AE will come out = $\overset{F}{13,23}$.

SECTION X.

Of the Application of FLUXIONS *to the Resolution of such Kinds of Problems* DE MAXIMIS ET MINIMIS, *as depend upon a particular Curve, whose Nature is to be determined.*

I SHALL begin this Section with premising the following useful

THEOREM.

406. *If the Relation of two flowing Quantities y and u be required; so that, when the Fluent of* $y^m \dot{u}$ *becomes equal to a given Value, that of* $\dfrac{y^r \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ *may be a* Maximum *or a* Minimum; *I say, their Relation must be such that* $\dfrac{y^{r-m}\dot{u} \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ *may be, every where, the same, or equal to a constant Quantity.*

The Demonstration hereof depends upon the subsequent

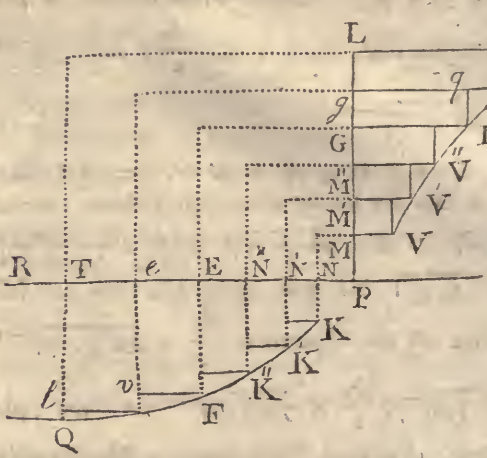
LEMMA.

407. If $a\alpha + b\beta = Q$, wherein α and β are indeterminate, the Value of $A \times \overline{\alpha\alpha \pm pp}|^n + B \times \overline{\beta\beta + pp}|^n$ will be a *Maximum* or *Minimum*, when $\dfrac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\dfrac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ are equal to each other

other. For, by taking the Fluxions of both Expreſsions we have $a\dot{\alpha} + b\dot{\beta} = 0$, and $2nA\alpha\dot{\alpha} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ $+ 2nB\beta\dot{\beta} \times \overline{\beta\beta \pm pp}|^{n-1} = 0$: From whence, $\dot{\alpha}$ and $\dot{\beta}$ being exterminated, there reſults $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ $= \frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$. *Q. E. D.*

Hence, if $a\alpha + b\beta + c\gamma + d\delta$ &c. $= Q$ (where α, β, γ, δ &c. are indeterminate) it follows that $A \times \overline{\alpha\alpha \pm pp}|^n + B \times \overline{\beta\beta \pm pp}|^n + C \times \overline{\gamma\gamma \pm pp}|^n + D \times \overline{\delta\delta \pm pp}|^n$ &c. will be a *Maximum* or *Minimum*, when all the Quantities $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$, $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$, $\frac{C\gamma}{c} \times \overline{\gamma\gamma \pm pp}|^{n-1}$ &c. are equal to each other. For that Expreſſion is a *Maximum* (or *Minimum)* when it cannot be increaſed (or decreaſed) by altering the Values of the indeterminate Quantities involved therein; but it may be increaſed (or decreaſed) by altering only two of them (as α and β) whilſt the reſt remain unchanged; unleſs $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ are equal to each other. (This is proved above.) Therefore, when $A \times \overline{\alpha\alpha \pm pp}|^n + B \times \overline{\beta\beta \pm pp}|^n + C \times \overline{\gamma\gamma \pm pp}|^n +$ &c. is a *Maximum* or *Minimum*, the Quantities $\frac{A\alpha}{a} \times \overline{\alpha\alpha \pm pp}|^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}|^{n-1}$ cannot be unequal: And, by the very ſame Argument, no other two of the Quantities above ſpecified can be unequal.

If, in the Right-line PR, there be now assumed $N\acute{N} = \alpha$, $\acute{N}\acute{\acute{N}} = \beta$, &c. and upon these, as Bases,



Rectangles $\acute{N}K$, $\acute{\acute{N}}\acute{K}$ be supposed, whose Altitudes NK, $\acute{N}\acute{K}$ &c. are denoted by a, b, c, d &c. it is evident that $a\alpha + b\beta + c\gamma + d\delta$ &c. $(= Q)$ will be expressed by the Sum of all the said Rectangles, or the whole Polygon Nl.

Moreover, if, in the Right-line PL (perpendicular to PR) there be taken $M\acute{M}$, $\acute{M}\acute{\acute{M}}$ &c. each equal to p, and, upon these equal Bases, Rectangles $\acute{M}V$, $\acute{\acute{M}}\acute{V}$ &c. be constituted, whose Altitudes are denoted by $A \times \frac{\overline{\alpha\alpha \pm pp}\,\rvert^{n}}{p^{2n}}$, $B \times \frac{\overline{\beta\beta \pm pp}\,\rvert^{n}}{p^{2n}}$, &c. it is likewise plain that the Value of $\frac{A \times \overline{\alpha\alpha \pm pp}\,\rvert^{n}}{p^{2n-1}} + \frac{B \times \overline{\beta\beta \pm pp}\,\rvert^{n}}{p^{2n-1}} + \frac{C \times \overline{\gamma\gamma \pm pp}\,\rvert^{n}}{p^{2n-1}}$ will be truly represented by the

whole

whole Polygon Mh. Which Polygon (as p is constant) will be a *Maximum* or *Minimum*, when $A \times \overline{\alpha\alpha \pm pp}|^{n} + B \times \overline{\beta\beta \pm pp}|^{n} +$ &c. is a *Maximum* or *Minimum*; that is, when all the Quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha\alpha \pm pp}|^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}|^{n-1}}{p^{2n-1}}$, &c. are equal to each other (as has been proved above.)

Let now, A, B, C, D &c. be expounded by any Powers, (MP^{r}, $\acute{M}P^{r}$, $\acute{\acute{M}}P^{r}$, &c.) of the respective Distances from a given Point P; and let, at the same time, the corresponding Values of a, b, c, d &c. be interpreted by any other proposed Powers MP^{m}, $\acute{M}P^{m}$, $\acute{\acute{M}}P^{m}$ &c. of the same given Distances: Then the Area of the Polygon Nl will be expressed by $MP^{m} \times \alpha + \acute{M}P^{m} \times \beta + \acute{\acute{\acute{M}}}P^{m} \times \gamma$ &c. $(= Q)$; and that of the Polygon Mh, by $MP^{r} \times \frac{\overline{\alpha\alpha \pm pp}|^{n}}{p^{2n-1}} + \acute{M}P^{r} \times \frac{\overline{\beta\beta \pm pp}|^{n}}{p^{2n-1}} + \acute{\acute{M}}P^{r} \times \frac{\overline{\gamma\gamma \pm pp}|^{n}}{p^{2n-1}} +$ &c. And the foresaid equal Quantities $\frac{A\alpha}{a} \times \frac{\overline{\alpha\alpha \pm pp}|^{n-1}}{p^{2n-1}}$, $\frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}|^{n-1}}{p^{2n-1}}$ &c. will become $MP^{r-m} \times \frac{\alpha \times \overline{\alpha\alpha \pm pp}|^{n-1}}{p^{2n-1}}$, $\acute{M}P^{r-m} \times \frac{\beta \times \overline{\beta\beta \pm pp}|^{n-1}}{p^{2n-1}}$, &c. respectively.

Now let the Number of the Rectangles be supposed indefinitely great, and their Breadths indefinitely small,

 so

ſo that the Area of each of the two Polygons Nl and Mb may be taken for that of its circumſcribing Curve: Moreover, let u and y be put to repreſent the Diſtances of any two correſponding Ordinates EF and GI from the given Point P; and let $\dot{y}$ be every where expreſſed by p ($=\overset{\prime}{M}M=\overset{\prime\,\prime\prime}{M}M=$ &c.) Then, $\dot{u}$ being a general Value for any of the Quantities α, β, γ, δ &c. (or NN, NN &c.) it follows; Firſt, that the Fluxion of the Area of the Curve NEFK (the Ordinate being, every where, $=y^m$) will be truly defined by $y^m\dot{u}$; Secondly, that the Fluxion of the Area MGIV (by ſubſtituting y, $\dot{u}$ and $\dot{y}$ inſtead of their Equals) will be $\frac{y^r\times\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$; and, laſtly, that the Value of each of the equal Quantities, $MP^{r-m}\times\frac{\alpha\times\overline{\alpha\alpha\pm pp}|^{n-1}}{p^{2n-1}}$, $\overset{\prime}{M}P^{r-m}\times\frac{\beta\times\overline{\beta\beta\pm pp}|^{n-1}}{p^{2n-1}}$, &c. above ſpecified, will be expreſſed by $\frac{y^{r-m}\times\dot{u}\times\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$. *Whence the Theorem is manifeſt.*

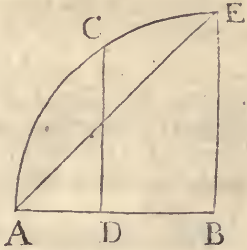
408. If R and S be aſſumed to denote any Functions of y (that is, any two Quantities expreſſed in Terms of y and given Coefficients; then, in order to have the Fluent of $S\times\frac{\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ a *Maximum* or *Minimum*, when that of $R\dot{u}$ becomes equal to a given Value, it is requiſite that $\frac{S\dot{u}}{R}\times\frac{\overline{\dot{u}\dot{u}\pm\dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ ſhould be a conſtant

Quan-

Quantity: This, alſo, is evident from the preceding Demonſtration; and may be of Uſe when the above premiſed Theorem is not ſufficiently general.

PROB. I.

409. *To determine the Nature of the Curve* ACE; *ſo that, the Length of the Arch* AE *being given, the Area* ABE *ſhall be a* Maximum.



Calling (as uſual) the Abſciſſa AD, x; the Ordinate DC, y; and the Arch AC, z, we have $\dot{x} = \sqrt{\dot{z}^2 - \dot{y}^2}$*; and therefore $y\dot{x}$ † $= y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{\frac{1}{2}}$ = the Fluxion of the Area ADC. Now, ſince, by the Queſtion, the Fluent of $y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{\frac{1}{2}}$ is to be a *Maximum*, when That of $\dot{z}$ becomes equal to a given Quantity (ACE) let theſe two Fluxions be, reſpectively, compared with $\frac{y^r \times \overline{\ddot{u}\ddot{u} - \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ and $y^m \dot{u}$ (as given in the foregoing Theorem ‡.) By which means, $n = \frac{1}{2}$, $r = 1$, $\dot{u} = \dot{z}$, and $m = 0$; and conſequently $\frac{y^{r-m} \dot{u} \times \overline{\ddot{u}\ddot{u} - \dot{y}\dot{y}}|^{n-1}}{\dot{y}^{2n-1}}$ $= y\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{-\frac{1}{2}}$: Which (according to the ſaid Theorem) being, every where, equal to a conſtant Quantity, we ſhall, by putting that Quantity $= a$, and ordering the Equation, get $\dot{z}^2 = \frac{a^2\dot{y}^2}{a^2 - y^2}$, and $\dot{x}$ $(\sqrt{\dot{z}^2 - \dot{y}^2}) = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}$; and, conſequently, (by taking

* Art. 135.

† Art. 112.

‡ Art. 406.

taking the Fluent) $x = a - \sqrt{a^2 - y^2}$, or $2ax - xx = y^2$; which is the common Equation of a Circle. *Q. E. I.*

COROLLARY.

410. It follows from hence, that the greatest Area that can possibly be contain'd by a Right-line (AE) joining two given Points, and any Curve-line ACE of a given Length, terminating in the same Points, will be when the said Curve-line is an Arch of a Circle.

PROB. II.

411. *The Length of the Arch* AE *(see the preceding Figure) being given, to determine the Nature of the Curve, so that the Solid generated by the Rotation thereof may be a* Maximum.

* Art. 145. Since the Fluent of $y^2 \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ ($= y^2\dot{x}$ *) is required to be a *Maximum*, when that of $\dot{z}$ has a given Value ACE, every thing will remain as in the last Problem; *only*, r must here be $= 2$: And therefore *(by the Theorem)* we have $y^2\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{-\frac{1}{2}} = a$. Whence $\dot{z} = \frac{a\dot{y}}{\sqrt{a^2 - y^4}}$; and consequently $\dot{x}$ ($= \sqrt{\dot{z}^2 - \dot{y}^2}$) $= \frac{y^2\dot{y}}{\sqrt{a^2 - y^4}}$: Which Values, if b^2 be put $= a$ (in order to have the Powers homologous) will become $\dot{z} = \frac{b^2\dot{y}}{\sqrt{b^4 - y^4}}$ and $\dot{x} = \frac{y^2\dot{y}}{\sqrt{b^4 - y^4}}$: Whence z and x will be known. *Q. E. I.*

PROB. III.

412. *The Superficies generated by the Arch of a Curve, in its Rotation, about its Axis, being given; to determine the Curve, so that the Solid, itself, may be a* Maximum.

† Art. 145. Because the Fluent of $y^2 \times \overline{\dot{z}^2 - \dot{y}^2}|^{\frac{1}{2}}$ † is to be a *Maximum*, when that of $y\dot{z}$ becomes equal to a given Quan-

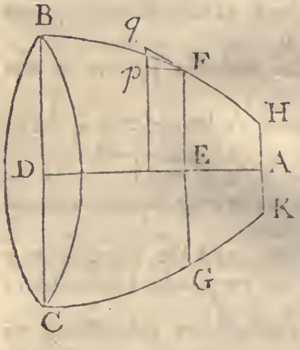
Quantity; let the Fluxions here exhibited be therefore compared with $\frac{y^r \times \overline{\dot{u}\dot{u} \pm \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ and $y^m\dot{u}$ (given in the Theorem.) By means whereof (r being $=2$, $\dot{u} = \dot{z}$, $n = \frac{1}{2}$, and $m = 1$) we have $y\dot{z} \times \overline{\dot{z}^2 - \dot{y}^2}|^{-\frac{1}{2}} = a$ (a constant Quantity *;) which is the very Equation found in *Prob.* 1. belonging to a Circle. * Art. 406.

If the Solid be supposed given, and the Superficies a *Minimum*, we shall come at the very same Conclusion: For, $y^2\dot{x}$ and $y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{1}{2}}$ (which are respectively as their Fluxions) being compared with $y^m\dot{u}$ and $\frac{y^r \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ we have $m = 2$, $\dot{u} = \dot{x}$, $r = 1$, and $n = \frac{1}{2}$; and therefore $\frac{\dot{x}}{y\sqrt{\dot{x}^2 + \dot{y}^2}}$ equal to a constant Quantity: Which being denoted by $\frac{1}{a}$ (so that the Terms may be homologous) there comes out $a\dot{x} = y\sqrt{\dot{x}^2 + \dot{y}^2}$; whence $2ax - x^2 = y^2$, *(as before.)*

PROB. IV.

413. *To determine the Curve* HFB, *from whose Revolution a Solid* BK *shall be generated; which, moving forward, in a Medium, in the Direction of its Axis* DA, *will be less resisted than any other Solid of the same given Length* DA *and Base* BC.

If AE $= x$, EF $= y$, F$p = \dot{x}$ *&c.* it is evident, from the Principles of Mechanics, that the resisting Force of a Particle of the Medium at F (being as the Square of the Sine of the Angle of Inclination pFq) will be truly represented by $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x} + \dot{y}\dot{y}}$ $\left(\frac{\overline{pq}|^2}{\overline{Fq}|^2}\right)$. Moreover, since the



the whole Number of Particles acting upon FHKG is proportional to the Area of the Circle FG, or as y^2; the Fluxion hereof $(2y\dot{y})$ drawn into $\frac{\dot{y}\dot{y}}{\dot{x}\dot{x}+\dot{y}\dot{y}}$, will therefore give $\frac{2y\dot{y}^3}{\dot{x}\dot{x}+\dot{y}\dot{y}}$ for the Fluxion of the Resistance upon FHKG.

Now, since it is required (by the Question) to have the Fluent of $\frac{y\dot{y}^3}{\dot{x}\dot{x}+\dot{y}\dot{y}}$ (or $\frac{y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-1}}{\dot{y}^{-3}}$) a *Maximum*, when That of $\dot{x}$ becomes equal to a given Quantity (AD), let these two Fluxions be therefore

* Art. 406. compared with $\frac{y^r \times \overline{\dot{u}\dot{u}+\dot{y}\dot{y}}|^n}{\dot{y}^{2n-1}}$ and $y^m\dot{u}$ *. Whence (r being $=1$, $\dot{u}=\dot{x}$, $n=-1$, and $m=0$) we get

† Art. 406. $\frac{y\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}} = a$ (a constant Quantity †); and consequently $y\dot{y}^3\dot{x} = a \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$: Whereof the Fluent will be found, by *Art.* 264. That the Curve does not meet its Axis in the extreme Point A, but has an Ordinate AH at that Point (as represented in the Figure) is evident from the foregoing Equation. For $\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$ $(\overline{Fq}|^4)$ being, always, greater than $\dot{y}^3\dot{x}$ $(\overline{pq}|^3 \times Fp)$, y must therefore be greater than a, in the same Proportion; and so, can never be equal to Nothing.

Now, as it is demonstrable that the Angle AHF must be $\frac{3}{2}$ of a Right-Angle, AH (the least Value of y) will therefore be $= 4a$ (since $\dot{x}$ and $\dot{y}$ are, in this Circumstance,

ſtance, equal to each other.) But, what a, itſelf, ought to be, muſt be determined from the given Values of AD and BD, and the Reſolution of the foreſaid Equation.

PROB. V.

414. *To determine the Solid of the leaſt Reſiſtance, ſuppoſing the Area of the generating Plane* AHBD, *and its greateſt Ordinate* DB *to be given; (ſee the preceding Figure.)*

Since (by the laſt Article) the Fluxion of the Reſiſtance is expreſſed by $\frac{y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-1}}{\dot{y}^{-3}}$, and that of the Area AEFH by $y\dot{x}$, it is plain (from the premiſed Theorem *) that $\frac{\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}}$ is a conſtant Quantity. * Art. 406.

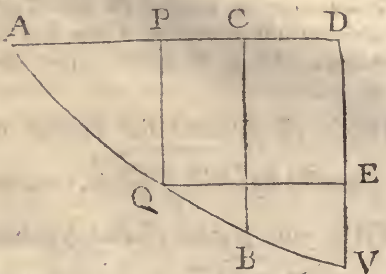
Whence, $\frac{\dot{y}^3\dot{x}}{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2}$, or its Equal $\frac{\overline{pq}|^3 \times Fp}{\overline{qF}|^4}$, being every where the ſame, the Angle pFq muſt alſo be invariable; and conſequently HFB a Right-line. Therefore the Solid of the leaſt Reſiſtance is *(in this Caſe)* either a whole Cone, or the Fruſtrum of a, greater, Cone. But it is eaſy to ſhew, *that*, when the Area of the generating Plane AB is given ſo ſmall, that the Angle B may be taken equal to the Half of a Right-angle; I ſay, it is demonſtrable, in this Caſe, that the Fruſtrum ſo ariſing will be leſs reſiſted than a whole Cone, or any other Fruſtrum, whereof the Baſe and the Area of the generating Plane are the ſame.

In like manner the Solid of *leaſt Reſiſtance*, when its Bulk and greateſt Diameter are given, may be determined: The Equation of the generating Curve being $\frac{y^{-1}\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-2}}{\dot{y}^{-3}} = \frac{1}{a}$, or $a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^2$: Whereof the Solution is given in *Art.* 264.

PROB.

PROB. VI.

415. *To determine the Line, along which a Body, by its own Gravity, will, descend, from one given Point* A *to another* B, *in the shortest Time possible.*



Let AD be parallel, and BC perpendicular, to the Horizon, interfecting each other in C; and let QP be any Ordinate to the Curve parallel to BC: Then (calling AP, x; PQ, y *&c.)* the Celerity at Q will be expressed by $y^{\frac{1}{2}}$; also the Fluxion of the Time of Descent thro' AQ will be truly defined by $\frac{\dot{z}}{y^{\frac{1}{2}}}$ *, or its Equal $y^{-\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$. Here, therefore, the Fluent of $y^{-\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$ is to be a *Minimum*, when that of $\dot{x}$ arrives to a given Value (AC). Whence, *by the Theorem* †, $y^{-\frac{1}{2}}\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-\frac{1}{2}}$ must be $=$ a constant Quantity: Which (to have the Terms homologous) let be denoted by $a^{-\frac{1}{2}}$ (or $\frac{1}{\sqrt{a}}$). Then $a^{\frac{1}{2}}\dot{x} = y^{\frac{1}{2}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$; whence $\dot{x} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}} = \frac{y\dot{y}}{\sqrt{ay-yy}}$; $\dot{z} = (\sqrt{\dot{x}^2+\dot{y}^2})$ $=$

* Art. 204.

† Art. 405.

$= \frac{a^{\frac{1}{2}}\dot{y}}{\sqrt{a-y}}$; and consequently $z = 2a - 2a^{\frac{1}{2}}\sqrt{a-y}$.

Therefore, when $y = a$, z is $= 2a$; which two corresponding Values let be denoted by DV and AV; and let QE, parallel to AD, meet DV in E; then VE (VD — ED) being $= a - y$, and VQ (AV — AQ) $= 2a^{\frac{1}{2}}\sqrt{a-y}$, it follows that

VD (a) : VE $(a-y)$ :: VA² $(4a^2)$: VQ² $(4a \times \overline{a-y})$

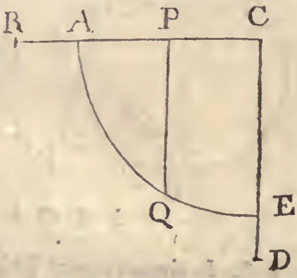
Which is one of the most remarkable Properties of the Cycloid; the Curve which, therefore, answers the Conditions of the Problem.

If the Celerity be supposed as any Function (S) of the Quantity y, the Problem will be resolved in the same manner: The Equation of the Curve being

$$\frac{\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-\frac{1}{2}}}{S} = \frac{1}{a} \text{*}.$$ * Art. 408.

PROB. VII.

416. *To find the Nature of the Curve* AQE, *along which a heavy Body must descend from an horizontal Line* RC *to a vertical Line* CD, *so that the Area* CAE *may be given, and the Time of the Descent a* Minimum.



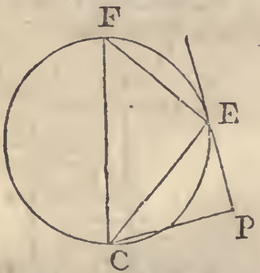
If the Ordinate PQ (parallel to CD) be called y, and the Velocity at Q be denoted by y^n; it is evident that the Fluent of $y^{-n} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{1}{2}}$ $\left(= \frac{\dot{z}}{y^n} \dagger\right)$ must be a *Minimum* when that of $y\dot{x}$ amounts to a given Value. † Art. 204.

Therefore

Therefore (by the Theorem already mention'd ſo often) we have $y^{-n-1}\dot{x} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{-\frac{1}{2}} = a^{-n-1}$; and conſequently $\dot{x} = \frac{y^{n+1}\dot{y}}{\sqrt{a^{2n+2} - y^{2n+2}}}$; which, by writing $\frac{1}{2}$ inſtead of n, becomes $\dot{x} = \frac{y^{\frac{3}{2}}\dot{y}}{\sqrt{a^3 - y^3}}$: Whence x will be known. But, if the Celerity was to be every where uniform, then (n being $= 0$) we ſhould have $\dot{x} = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}$; and therefore $x = a - \sqrt{a^2 - y^2}$: Which anſwers to a Circle.

Lemma.

417. *If, upon a Tangent* EP, *from any Point* C *in the Circumference of a Circle* FEC, *a Perpendicular* CP *be let fall, the Chord* (CE) *joining that Point and the Point of Contact, will be a Mean-Proportional between the ſaid Perpendicular* CP *and the Diameter* CF *of the Circle.*

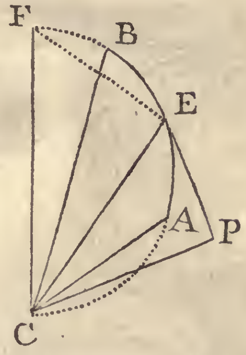


For, the Angles P and CEF being both Right; and alſo CEP = F, the Triangles CPE and CEF are ſimilar: And therefore CP : CE :: CE : CF. *Q. E. D.*

PROB. VIII.

418. *In the mixt-lin'd Triangle* ACB, *the Lengths of all the Sides (whereof* CA *and* CB *are Right-lines) are ſuppoſed given; 'tis required to find the Nature of the Curve-ſide* AEB, *ſo that the Area may be a* Maximum.

Put



Put the Arch AE = z, and the Ordinate CE = y; then, the Fluxion of the Area ACE being $\frac{\dot{y}}{2}\sqrt{\dot{z}^2 - \dot{y}^2}$, * the Fluent of $y \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{\frac{1}{2}}$, generated in the Time wherein y, from CA, increases to CB, must be a *Maximum*: Therefore, *by the Theorem* †, we have $y\dot{z} \times \overline{\dot{z}\dot{z} - \dot{y}\dot{y}}|^{-\frac{1}{2}} = a$ †, or $\frac{\dot{z}}{\sqrt{\dot{z}\dot{z} - \dot{y}\dot{y}}} = \frac{a}{y}$. But, if CP be supposed perpendicular to the Tangent EP, then will $\frac{\dot{z}}{\sqrt{\dot{z}\dot{z} - \dot{y}\dot{y}}}$ (*Art.* 35.) $= \frac{CE}{CP} = \frac{y}{CP}$; and consequently $\frac{a}{y} = \frac{y}{CP}$; or, CP : CE ($y$) :: CE ($y$) : a: Which Proportion, *by the Lemma*, answers to a Circle; whereof the Quantity a is the Diameter.

* Art. 113.

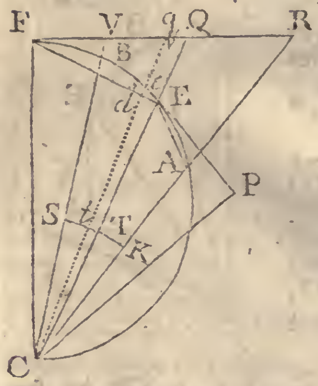
† Art. 406.

Now, that AEB must be an Arch of a Circle is also evident from *Prob.* 1. but, that the same Arch, continu'd out, will pass thro' the Angle C, does not appear from thence. This is known from above; and is requisite in finding the particular Circle answering to any proposed *Data.*

PROB. IX.

419. *To find the Path* AEB *which a Body must describe in moving uniformly from one given Point* A *to another* B; *so that, being every where acted on by a Force, or Virtue, which varies according to the Inverse-Duplicate-Ratio of the Distances from a given Center* C, *the whole Action upon the Body shall be a* Minimum.

Putting



Putting AE $= z$, CE $= y$, *de* (indefinitely small) $= \dot{y}$ *, E*e* $= \dot{z}$, and E*d* $(\sqrt{\dot{z}^2 - \dot{y}^2})$ $= \dot{u}$; we have $\frac{\dot{z}}{y^2}$ $(= y^{-2} \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}|^{\frac{1}{2}})$ for the Measure of the Force which acts upon the Body in describing the Particle E*e* $(\dot{z})$: Moreover, if from the Center C, with any given Radius (r) an Arch KT*t*S of a Circle be described, intersecting CE in T, we shall have T*t* (the Measure of the Angle EC*e*) $= \frac{r\dot{u}}{y}$. Therefore, since the Fluent of $y^{-2} \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}|^{\frac{1}{2}}$ is required to be a *Minimum*, and the cotemporary Fluent of $y^{-1}\dot{u}$ (between CA and CB) a given Quantity; it follows, from the Theorem premised at the Beginning of the Section, that $y^{-2+1}\dot{u} \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}|^{-\frac{1}{2}}$ must be equal to a constant Quantity $\left(\frac{1}{a}\right)$ and consequently $\frac{\dot{u}}{\overline{\dot{u}\dot{u} + \dot{y}\dot{y}}|}$ $\left(= \frac{\sqrt{\dot{z}^2 - \dot{y}^2}}{\dot{z}}\right) = \frac{y}{a}$: Which is the very Equation found in the preceding Problem. Therefore, if thro' the three given Points A, B, and C, the Circumference of a Circle be described, the Arch thereof terminated by A and B will be the Path of the Body. *Q. E. I.*

* Art. 134.

Corollary.

420. If FR be a Tangent to the Circle, at the Extremity of the Diameter CF, and CA and CE be produced

duced to meet it in R and Q, it follows that the whole Action upon the Body, in describing the Arch AE, will be proportional to the corresponding Part RQ of the said Tangent. For, if C*e* be, also, produced to meet FR in *q*, and EF be drawn, it is plain that the Triangles CEF and CFQ, as also CE*e* and C*q*Q, are similar: Whence it will be, CE (y) : CF (a) :: CF (a) : CQ (or C*q*) $= \frac{aa}{y}$; and CE (y) : E*e* $(\dot{z})$ :: C*q* $\left(\frac{aa}{y}\right)$: Q*q* $= \frac{aa\dot{z}}{yy}$: Which (a being constant) is as $\left(\frac{\dot{z}}{yy}\right)$ the Force that acts upon the Body in describing E*e* $(\dot{z})$. And, as this every where holds, the whole Action in describing AE must therefore be proportional to RQ. Which Force (it is easy to prove) will be to that exerted on the Body in moving through the Chord AE, as the Chord to the Arch.

PROB. X.

421. *To determine the Path in which a Body may move from one given Point* A *to another* B, *in the shortest Time possible; supposing the Velocity to be, every where, proportional to any Power* (y^p) *of the Distance from a given Center* C. (See the last Figure.)

Here every thing will remain as in the preceding Problem; *only* y^{-p} must be wrote instead of y^{-2}. Therefore we have $y^{-p+1} \times \dot{u} \times \overline{\dot{u}\dot{u} + \dot{y}\dot{y}}^{-\frac{1}{2}} =$ a constant Quantity: Which Quantity (to have the Terms homologous) let be denoted by $\frac{b}{a^p}$; then, by Reduction,

$$\frac{by^{p-1}}{a^p} = \frac{\dot{u}}{\sqrt{\dot{u}\dot{u} + \dot{y}\dot{y}}} \left(= \frac{\mathrm{E}d}{\mathrm{E}e}\right) = \frac{\mathrm{CP}}{\mathrm{CE}} = \frac{\mathrm{CP}}{y}:$$

And consequently $\mathrm{CP} = \frac{by^p}{a^p}$. Hence, if $p = 0$, or the

Ve-

Velocity be conſtant; then CP being every where $= b$, the Body muſt, in this Caſe, deſcribe a Right-line. But, if $p = 1$, then CP being $= \frac{by}{a}$; the Curve will be a logarithmic Spiral, whoſe Center is C *: Except in that particular Caſe, where CA = CB, when it degenerates to a Circle.

* Art. 74.

Laſtly, if $p = 2$, the Curve will be a Circle *(by the preceding Lemma)* whoſe Diameter is $\frac{aa}{b}$, and whoſe Periphery paſſes through the given Point C.

After the ſame manner, the Value of CP (upon which the Nature of the Curve depends) may be determined, when the Velocity is expounded by any given Function *(S)* of the Diſtance *(y)* from the Center of Force: And (by writing S in the room of y^n † *&c.)* will come out CP $= \frac{bS}{c}$; where b and c repreſent conſtant Quantities.

† Art. 407.

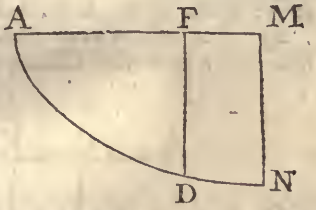
When the Velocity is That which the Body may acquire, in deſcending through BE, by a centripetal Force expreſſed by y^p, then the Value of S (the Meaſure of that Velocity) being interpreted by $\sqrt{d^{p+1} - y^{p+1}}$ ‡ (where CB $= d$) we therefore have CP $= \frac{b\sqrt{d^{p+1} - y^{p+1}}}{c}$ for the Equation of the Curve of the ſwifteſt Deſcent, according to this laſt *Hypotheſis* of a centripetal Force varying as any Power p of the Diſtance.

‡ Art. 221. and 206.

422. Beſides the Problems already reſolved in this Section, there are others of the ſame Nature which are confined to more particular Reſtrictions, and require a different Method of Solution.

Thus,

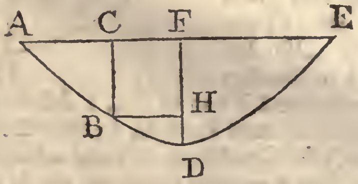
Thus, if $\mathcal{Q}$, R and S be ſuppoſed to denote any given Powers, or Functions, of the Ordinate (y) of a Curve ANM, and the Nature of the Curve be required, ſo that, when the Fluent of $\mathcal{Q}\dot{x}$ becomes equal to a given Quantity, the Fluent of $R\dot{z}$ may alſo become equal to another given Quantity, and That of $S\dot{z}$, a *Maximum* or *Minimum:* Then, becauſe there is, in this Caſe, a ſecond Equation, or new Condition, beyond what is to be met with in any of the foregoing Problems, the Method of Solution hitherto explained, will, therefore, be inſufficient. But, by a Proceſs ſimilar to that whereby the ſaid Method was demonſtrated (aſſuming, here, three Expreſſions, and three indeterminate Quantities, inſtead of two *) a general Anſwer to this Problem (under all its Reſtrictions) will be obtained: And is exhibited by the Equation, $\frac{\dot{z}}{\dot{x}} = \frac{pR \pm qS}{\mathcal{Q}}$; wherein p and q denote conſtant Quantities. * Art. 407.



423. Though it ſeems unneceſſary to put down the Invention of this Equation, after what has been hinted above, yet it may not be improper to obſerve, by way of Corollary, that, if $\mathcal{Q} = 1$, $R = 1$, and $S = y^n$, the Equation will then become $\frac{\dot{z}}{\dot{x}} = p \pm qy^n$; expreſſing the Nature of the Curve, when, the whole Abſciſſa (AM) and correſponding Arch (AN) being both given Quantities, the Fluent of $y^n\dot{z}$ is a *Maximum* or *Minimum*, according as the Value of n is poſitive or negative: In both which Caſes, it is very eaſy to perceive, that the Curve muſt be concave to AM, and that the Value of $\frac{\dot{z}}{\dot{x}}$, or its

Equal $p \pm qy^n$, must, therefore, decrease as y increases; whence we may infer that the Sign of qy^n must be negative in the former Case, and positive in the latter.

Ex. Let the Curve ABDE, be the *Catenaria*; formed by a slender Chain, or perfectly flexible Cord,



suspended by its two Extremes in the horizontal Line AE: Then, since its Center of Gravity must be the lowest possible, the Fluent of $y\dot{z}$, when AC=AE, must therefore be a *Maximum* *: Whence (n being here $=1$) our Equation $\left(\frac{\dot{z}}{\dot{x}} = p \pm qy^n\right)$ becomes $\frac{\dot{z}}{\dot{x}} = p - qy$.

* Art. 173.

But; in order to reduce it to a more convenient Form, let the Distance (DF) of the lowest Point of the Curve from the horizontal-Line AE be put $= b$; then, when y (BC) becomes $= b$, $\dot{x}$ will be $= \dot{z}$; and therefore the Equation, in that Circumstance, is $1 = p - qb$; whence $p = 1 + qb$, and consequently $\frac{\dot{z}}{\dot{x}} = 1 + qb - qy = 1 + q \times \overline{b - y}$: Which, by putting $b - y$ (DH) $= s$ and $a = \frac{1}{q}$ is reduced to $\frac{\dot{z}}{\dot{x}} = 1 + \frac{s}{a}$: From whence $a^2\dot{z}^2$ $(= \overline{a + s}|^2 \times \dot{x}^2) = \overline{a + s}|^2 \times \overline{\dot{z}^2 - \dot{s}^2}$; and consequently BD $= \sqrt{2as + ss}$.

For another Example (wherein the Exponent n will be negative) let the required Curve be That along which

which a Body may defcend, by its own Gravity, from one given Point A to another B, in lefs Time than through any other Line of the fame Length. In which Cafe, the Fluent of $\dot{z}y^{-\frac{1}{2}}$ being a *Minimum*, when x and z become equal to given Quantities, our Equation (by writing $-\frac{1}{2}$ for n) will here become $\frac{\dot{z}}{\dot{x}} = p + qy^{-\frac{1}{2}}$: From whence exterminating $\dot{x}$, or $\dot{z}$, by means of the Equation $\dot{x}^2 + \dot{y}^2 = \dot{z}^2$, the Fluent may alfo be determined.

SECTION XI.

The Refolution of Problems of various Kinds.

PROB. I.

424. *ANY hyperbolical Logarithm (y) being given, it is propofed to find the natural Number anfwering thereto.*

If the Number fought be denoted by $1 + x$, we fhall (by *Art.* 126.) have $\dot{y} = \frac{\dot{x}}{1 + x}$, or $\dot{y} + x\dot{y} - \dot{x} = 0$. Let $Ay + By^2 + Cy^3$ &c. $= x$; then $A\dot{y} + 2By\dot{y} + 3Cy^2\dot{y}$ &c. $= \dot{x}$, and our Equation will become

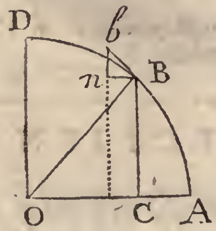
$$\left.\begin{array}{l} \dot{y} + Ay\dot{y} + By^2\dot{y} + Cy^3\dot{y} \text{ \&c.} \\ - A\dot{y} - 2By\dot{y} - 3Cy^2\dot{y} - 4Dy^3\dot{y} \text{ \&c.} \end{array}\right\} = 0.$$

Whence, by comparing the homologous Terms, we get $A = 1$, $B = \frac{A}{2} = \frac{1}{2}$, $C = \frac{B}{3} = \frac{1}{2 \cdot 3}$, $D = \frac{C}{4} = \frac{1}{2 \cdot 3 \cdot 4}$ &c. Therefore $1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} + \frac{y^5}{2 \cdot 3 \cdot 4 \cdot 5}$ &c. is $(= 1 + x)$ the Number fought.

PROB. II.

425. *The Radius* AO *and any Arch* AB *of a Circle* ABD *being given; to find the Sine* BC, *and Co-sine* OC *of that Arch.*

Let AO (BO) $= r$, AB $= z$, AC $= x$, BC $= y$,



B$b = \dot{z}$, B$n = \dot{x}$, and $bn = \dot{y}$: Because of the similar Triangles OBC and Bnb, it will be

OB (r) : BC (y) :: Bb $(\dot{z})$: Bn $(\dot{x})$

And OB (r) : OC $(r - x)$ :: Bb $(\dot{z})$: bn $(\dot{y})$

From which we have

$y\dot{z} = r\dot{x}$

And $r\dot{y} = r\dot{z} - x\dot{z}$.

Let $x = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5$ &c.

And $y = az + bz^2 + cz^3 + dz^4 + ez^5$ &c.

Then, by Substitution and Transposition, our two Equations will become

$$\left.\begin{array}{l} * \ + az\dot{z} + bz^2\dot{z} + cz^3\dot{z} + dz^4\dot{z} \text{ \&c.} \\ -rA\dot{z} - 2rBz\dot{z} - 3rCz^2\dot{z} - 4rDz^3\dot{z} - 5rEz^4\dot{z} \text{ \&c.} \end{array}\right\} = 0$$

And

$$\left.\begin{array}{l} ra\dot{z} + 2rbz\dot{z} + 3rcz^2\dot{z} + 4rdz^3\dot{z} + 5rez^4\dot{z} \text{ \&c.} \\ -r\dot{z} + Az\dot{z} + Bz^2\dot{z} + Cz^3\dot{z} + Dz^4\dot{z} \text{ \&c.} \end{array}\right\} = 0$$

From which, by equating the homologous Terms, we get $A = 0$, $a = 2rB$, $b = 3rC$, $b = 4rD$, $d = 5rE$ &c.

Also $a = 1$, $b = -\frac{A}{2r}$, $c = -\frac{B}{3r}$, $d = -\frac{C}{4r}$ &c.

There

Therefore $2rB = 1$, $3rC = -\frac{A}{2r}$, $4rD = -\frac{B}{3r}$, $5rE = -\frac{C}{4r}$, *&c.* and consequently $B = \frac{1}{2r}$, $C = 0$, $D = -\frac{B}{3 \cdot 4r^2} = -\frac{1}{2 \cdot 3 \cdot 4r^3}$, $E = 0$, $F = -\frac{D}{5 \cdot 6r^2} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot r^5}$ *&c.* Whence, also $b\ (=3rC) = 0$, $c\ (=4rD) = -\frac{1}{2 \cdot 3r^2}$ *&c. &c.*

Hence it is evident that $y\ (= az + bz^2 + cz^3$ *&c.*$)$ $= z - \frac{z^3}{2 \cdot 3r^2} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5r^5} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7r^6}$ + *&c.* And that $x\ (= Az + Bz^2 + Cz^3$ *&c.*$) = \frac{z^2}{2r} - \frac{z^4}{2 \cdot 3 \cdot 4r^3} + \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6r^5}$ — *&c.* ‡

PROB. III.

426. *To find the Value of x, when x^x is a* Minimum.

The Logarithm of x^x is $= x \times l.x$; whose Fluxion $\dot{x} \times l.x + \dot{x}$ being $= 0$, we have $l:x = -1$. But (*by Prob.* 1.) the Number whose hyp. Log. is y will be $1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4}$ *&c.* Therefore, by writing -1 instead of y, we have $x = 1 - 1 +$

‡ *The Substance of this Solution (being the most neat and artful I have seen to that useful Problem) I had from a Letter sign'd* ——— Needler; *which was put into my Hands by a Friend, who receiv'd it from the late Dr.* Halley, *to whom it was wrote.*

$\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}$ &c. $= 0,367878$ &c.

PROB. IV.

427. *To divide a given Number (a) so that the continual Product of all its Parts may be a* Maximum.

It is evident (*from Art.* 23.) that all the Parts must be equal: If, therefore, any one of them be denoted by x, their Number will be $\frac{a}{x}$, and we shall have $\overline{x}|^{\frac{a}{x}}$ a *Maximum*: And therefore its Logarithm $\frac{a}{x} \times L.x$ a *Maximum* also: And its Fluxion $-\frac{a\dot{x}}{x^2} \times L.x - \frac{a\dot{x}}{x^2} = 0$*: Whence H-$L.x = 1$, and consequently $x = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4}$ &c. $= 2.71828$ &c. Therefore the next inferior, or superior, Number to 2,71828 &c. that will exactly measure the given Number a, is the required Value of each Part: Thus, let $a = 10$; then because $\frac{10}{2.71828 \text{ \&c.}} = 4$ nearly, the Number of Parts, in this Case, will be 4, and the Value of each $= \frac{10}{4} = 2.5$.

* Art. 22. and 126.

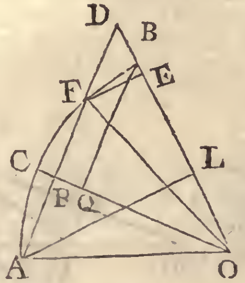
† Art. 424.

PROB. V.

428. *To divide a given Angle* AOB *into two Parts* AOC *and* BOC, *so that the Product of any given Powers,* $AP^n \times BQ^m$, *of their Sines* AP *and* BQ *may be a* Maximum.

Let

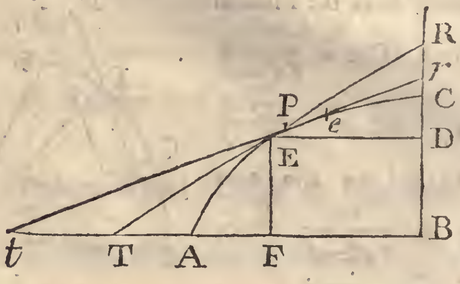
Let AP, produced, cut the Radius OB in D, and the Arch AB in F; likewise let FE and AL be perpendicular to OB, and join O, F: Putting $AO = r$, $AP = x$ and $BQ = y$. Then, because $x^n y^m$ is to be a *Maximum*, we have $nx^{n-1}\dot{x} \times y^m + x^n \times my^{m-1}\dot{y} = 0$; and consequently $ny\dot{x} = -mx\dot{y}$.



Moreover, since the Fluxion of the Arch AC is $= \frac{r\dot{x}}{\sqrt{r^2 - x^2}}$ and that of BC $= \frac{r\dot{y}}{\sqrt{r^2 - y^2}}$ (*Art.* 142.) we also have $\frac{r\dot{y}}{\sqrt{r^2 - y^2}} + \frac{r\dot{x}}{\sqrt{r^2 - x^2}} = 0$, or $\frac{\dot{y}}{\sqrt{r^2 - y^2}} = \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; which multiply'd by the former Equation, &c. gives $\frac{ny}{\sqrt{r^2 - y^2}} = \frac{mx}{\sqrt{r^2 - x^2}}$, or $n \times \frac{y\sqrt{r^2 - x^2}}{\sqrt{r^2 - y^2}} = mx$: Whence, because OQ $(\sqrt{r^2 - y^2})$: QB (y) :: OP $(\sqrt{r^2 - x^2})$: PD $= \frac{y\sqrt{r^2 - x^2}}{\sqrt{r^2 - y^2}}$, we have $n \times$ PD $(= mx) = m \times$ AP; and therefore PD : AP :: $m : n$; whence (by Composition and Division) AD : DF :: $m+n : m-n$: But (*by sim. Triang.*) AD : DF :: AL : EF; consequently $m+n : m-n$ :: AL : FE; that is, as the Sum of the Indices of the two proposed Powers is to their Difference, so the Sine of the whole given Angle to the Sine of the Difference of its two, required, Parts. This Proportion is given in Words, at length, because it will be found of frequent Use in the Solution of mechanical Problems.

PROB. VI.

429. *To shew that the least Triangle that can be described about, and the greatest Parallelogram in, a given Curve* ABC, *concave to its Axis, will be when the Subtangent* FT *is equal to the Base* BF *of the Parallelogram, or half the Base* BT *of the Triangle.*



It appears from *Art.* 25. and is demonstrable by common Geometry, that the greatest Parallelogram that can be inscrib'd in the Triangle BTR (supposing the Position of TR to remain the same) will be *that* whose Base BF is half the Base BT of the Triangle: Therefore, as a greater Figure cannot possibly be inscribed in the Curve BAC than in the Triangle BTR circumscribing it, the greatest Parallelogram that can be inscribed, either in the Triangle or the Curve, must be That above specified.

But now, to make it also appear that the Triangle BTR is a *Minimum* when FT=BF; let B*tr* be any other circumscribing Triangle, and let the two Tangents TER and *ter* intersect each other in P. Then, ER being = ET, it is plain that RP is less than PT, and P*r* (less than PR less than PT) less than P*t*: Therefore, the Sides PR and P*r* of the Triangle RP*r* being less than the Sides, PT and P*t* of the Triangle TP*t*, and the opposite Angles RP*r* and TP*t* equal to each other, it follows that the Triangle PR*r* is less than TP*t*; and consequently, by adding the Trapezium BTP*r* to both, it appears that BTR is less than B*tr*.

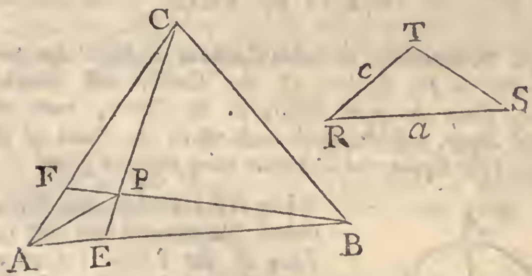
Corollary.

430. Hence the greateſt inſcribed Parallelogram is half the leaſt circumſcribing Triangle.

In the ſame Way it may be proved, that the greateſt inſcribed Cylinder, and the leaſt circumſcribing Cone, in, and about, the Solid generated by Revolution of a given Curve, will be when the Sub-tangent is equal to twice the Altitude of the Cylinder, or $\frac{2}{3}$ of the Altitude of the Cone: And that the two Figures will be to each other in the Ratio of 4 to 9.

PROB. VII.

431. *Three Points* A, B, C *being given, to find the Poſition of a fourth Point* P, *ſo that, if Lines be drawn from thence to the three former, the Sum of the Products* $a \times AP$, $b \times BP$, *and* $c \times CP$ *(where a, b and c denote given Numbers) ſhall be a* Minimum.

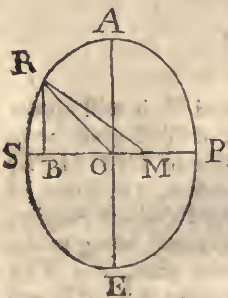


If CP and BP be produced to E and F, it will appear from *Art.* 35. and 36. that the Sine of BPE muſt be to that of APE, as *a* to *b*; and the Sine of CPF (BPE) to that of APF, as *a* to *c*. Therefore, the Sines of the three Angles BPE, APE, and APF (which Angles, taken all together, make two Right-ones) being in the given Ratio of *a*, *b* and *c*, it follows, that, if a Triangle RST be conſtructed, whoſe Sides RS, ST and RT are in the ſaid Ratio of *a*, *b* and *c*, the Angles T, R and S oppoſite thereto, will be reſpectively equal to to the fore-mention'd Angles BPE, APE, and APF. From whence, all the Angles at the Point P being given, the Position of that Point is given by common Geometry.

But it is observable, that, when one of the three given Quantities a, b, c (suppose a) is equal to, or greater than, the Sum of the other two, a Triangle cannot then be formed whose Sides are proportional to the said Quantities: In that Case the Point P will fall in the Point (A) corresponding to the greatest Quantity (a). For, it is plain that $b \times AB$ is less than $b \times BP + b \times AP$; and that $c \times AC$ is less than $c \times CP + c \times AP$; whence, by adding the Less to the Less, and the Greater to the Greater, it also appears that $b \times AB + c \times AC$ must be less than $b \times BP + c \times CP + \overline{b+c} \times AP$ less than $b \times BP + c \times CP + a \times AP$; because a (by Hypothesis) is equal to, or greater than, $b+c$.

PROB. VIII.

432. *To determine in what Latitude a Right-line perpendicular to the Surface of the Earth, and Another drawn, from the same Point, to the Center, make the greatest Angle, possible, with each other; the Ratio of the Axis and the Equatoreal Diameter being supposed given.*



Let AE represent the Equatoreal Diameter, and SP the Axis of the Earth (taken as an oblate Spheroid) also let RO and RM represent the two Lines specified in the Problem, whereof let the latter (perpendicular to ARS) meet SP in M; and let RB be perpendicular to SP.

It is evident, from the Property of the Ellipsis, that $SP^2 : AE^2 :: BO : BM$: And (by Trigonometry) BO : BM :: Tang. BRO : Tang. BRM; whence, by Equality,

lity, $SP^2 : AE^2 :: \text{Tang. } BRO : \text{Tang. } BRM$; therefore, by Composition and Division, $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Tang. } BRM + \text{Tang. } BRO : \text{Tang. } BRM - \text{Tang. } BRO$. But, *the Sum of the Tangents of any two Angles is to their Difference, as the Sine of the Sum of those Angles to the Sine of their Difference* *; whence it follows that $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Sine. } \overline{BRM + BRO} : \text{Sine. } \overline{BRM - BRO}$ (ORM).

Now, since the Ratio of the two first Terms is constant, or in every Part of the Ellipsis the same, it is obvious that the Angle ORM, or its Sine, will be the greatest possible, when its Antecedent (the Sine of $\overline{BRM + BRO}$) is the greatest possible, that is when $BRM + BRO =$ a Right-Angle and its Sine = Radius. Therefore, in the proposed Circumstance, when ORM is a *Maximum*, our last Proportion will become $AE^2 + SP^2 : AE^2 - SP^2 ::$ Radius : Sine of ORM : And half the Angle, so found, added 45°, will give (BRM) the Complement of the required Latitude; because $BRM + BRO$ (or $2BRM - ORM$) being $= 90^\circ$, it is evident that $2BRM = 90 + ORM$, and consequently $BRM = 45^\circ + \frac{1}{2} ORM$.

PROB. IX.

433. *Of all the Semi-cubical Parabolas, to determine that, whereof, the Length of the Curve being given, the Area shall be a* Maximum.

The general Equation is $ax^2 = y^3$: Moreover, the Area is universally $= \frac{3y^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$, and the Length of the Curve $= \frac{\overline{4a + 9y}^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27}$ (*see Art.* 137.). Let the last of these be put $= c$, and, by ordering the Equation, you will get

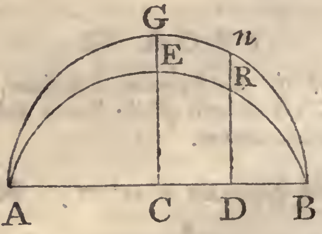
* *Vid. p.* 56. *of my Trigonometry.*

get $y = \frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a}{9}$: Whence, $\frac{3y^{\frac{5}{2}}}{5a^{\frac{1}{2}}}$ (and consequently $\frac{y}{a^{\frac{1}{5}}}$) being a *Maximum*, it is evident that $\frac{a^{\frac{1}{3}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a}{a^{\frac{1}{5}}}$, or its Equal $a^{\frac{2}{15}} \times \overline{27c + 8a}|^{\frac{2}{3}} - 4a^{\frac{4}{5}}$ must likewise be a *Maximum:* Which, put into Fluxions and reduced, gives $a = c \times \frac{9 + 3\sqrt{21}}{32}$: Whence x and y will also be found.

PROB. X.

434. *To determine the Ratio of the Periphery of any given Ellipsis to that of its circumscribing Circle.*

Call the Semi-transverse Axis CB, a; the Semi-conjugate CE, c; any Ordinate Dr, y; and its Distance



CD from the Center, x: Then (by the Nature of the Curve) y being $= \frac{c}{a}\sqrt{aa - xx}$, we have $\dot{y} = \frac{-cx\dot{x}}{a\sqrt{aa - xx}}$; and consequently $\dot{z}$ $(\sqrt{\dot{x}^2 + \dot{y}^2}) = \frac{\dot{x}\sqrt{a^4 - \overline{a^2 - c^2} \times x^2}}{a\sqrt{aa - xx}}$: Which by making $d =$

$\frac{aa - cc}{aa}$ will be reduced to $\dot{z} = \frac{\dot{x}\sqrt{aa - dxx}}{\sqrt{aa - xx}} =$
$\frac{a\dot{x}}{\sqrt{aa - xx}} \times \overline{1 - \frac{dx^2}{2a^2} - \frac{d^2x^4}{2 \cdot 4a^4} - \frac{3d^3x^6}{2 \cdot 4 \cdot 6a^6}}$ &c.
(by throwing the Numerator into a Series) whereof the *whole* Fluent, when x becomes $= a$, will be z (ERB)
$= A \times \overline{1 - \frac{d}{2 \cdot 2} - \frac{3d^2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5d^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} - \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7d^4}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}}$ &c. (*by Art.* 286.) where A denotes the Length of the Arch GnB, or $\frac{1}{4}$ of the Periphery of the circumſcribing Circle.

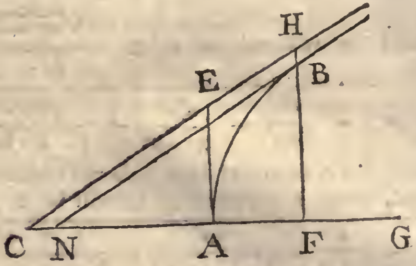
Hence it follows that the Periphery of the Ellipſis is to that of its circumſcribing Circle, as $1 - \frac{d}{2 \cdot 2} - \frac{3d^2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5d^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}$ &c. or as $1 - \frac{d}{2 \cdot 2} \times A + \frac{1 \cdot 3d}{4 \cdot 4} \times B + \frac{3 \cdot 5d}{6 \cdot 6} \times C + \frac{5 \cdot 7d}{8 \cdot 8} \times D$ &c. to Unity: Where A, B, C, D &c. denote the preceding Terms, under their proper Signs.

PROB. XI.

435. *To determine the Difference between the Length of the Arch of a Semi-hyperbola infinitely produced, and its Aſymptote.*

Call the Semi-tranſverſe Axis (AC) a; the Semi-conjugate (or its Equal AE); b the Diſtance (CF) of any Ordinate from the Center, x; the Ordinate itſelf, y; and the Arch correſponding, z: Then, from the Nature of the Curve we have $y = \frac{b\sqrt{x^2 - a^2}}{a}$; whence

$\dot{y} =$



$\dot{y} = \frac{bx\dot{x}}{a\sqrt{x^2 - a^2}}$; and consequently $\dot{z}\ (=\sqrt{\dot{x}^2 + \dot{y}^2}) =$

$\frac{\dot{x}\sqrt{\frac{aaxx + bbxx}{aa} - a^2}}{\sqrt{xx - aa}}$: Which, making $d^2 = \frac{a^2}{a^2 + b^2}$

$\left(= \frac{CA^2}{CE^2}\right)$ and $u = \frac{a}{x}$ will be transformed to $\dot{z} =$

$-\frac{a\dot{u}}{du^2} \times \frac{\overline{1 - dduu}|^{\frac{1}{2}}}{\overline{1 - uu}|^{\frac{1}{2}}}$; whereof the upper Surd, expanded, is $= 1 - \frac{d^2u^2}{2} - \frac{d^4u^4}{8}$ &c. And therefore $\dot{z} =$

$\frac{a}{d}$ into $\frac{-\dot{u}}{u^2\sqrt{1 - uu}} + \frac{d^2\dot{u}}{2\sqrt{1 - uu}} + \frac{d^4u^2\dot{u}}{8\sqrt{1 - uu}} +$

$\frac{3d^6u^4\dot{u}}{8 . 6\sqrt{1 - uu}} + \frac{3 . 5\ d^8u^6\dot{u}}{8 . 6 . 8\sqrt{1 - uu}}$ &c. Now the Fluent of the first Term hereof, $\frac{a}{d}$ into $\frac{-\dot{u}}{u^2\sqrt{1 - uu}}$

$\left(= \frac{x\dot{x}}{d\sqrt{x^2 - a^2}}\right)$ is universally expressed by

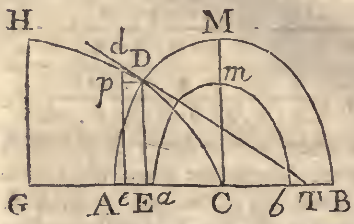
$\frac{\sqrt{x^2 - a^2}}{d}$, or its Equal $\frac{BF \times CE}{AE}$: Which, if BN be parallel to the Asymptote EC, will (because AE : CE ::

CE :: BF : BN) be alſo truly repreſented by BN: And this Line BN, when x or z becomes infinite, will coincide with the Aſymptote. Therefore the Fluent of the remaining Terms is the Difference ſought: Which Fluent, when $u = 1$, or $y = 0$ (putting A for $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity) will be $= aA \times \overline{\frac{d}{2} + \frac{d^3}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3d^5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} + \frac{3 \cdot 3 \cdot 5 \cdot 5d^7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7d^9}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10}}$ &c. (*by Art.* 286.) but $= 0$ when $u = 0$ (or y is infinite). Therefore the Exceſs of the Aſymptote above the Curve is truly exhibited by the preceding Series. *Q. E. I.*

If a be taken $= 1$, and $b = 0$, then d will become $= 1$: And therefore, the Curve in this Caſe falling into its Axis AG, we have $A \times \overline{\frac{1}{2} + \frac{1}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} + \frac{3 \cdot 3 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8}}$ &c. $=$ CA, or Unity. Whence it appears that the Sum of the Series $\frac{1}{2} + \frac{1}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6}$ is the Reciprocal of $\frac{1}{4}$ of the Periphery of the Circle whoſe Radius is Unity. And, from the Problem preceding the laſt, it will likewiſe appear, that the Sum of the Series $1 - \frac{1}{2 \cdot 2} - \frac{3}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5 \cdot}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}$ &c. will be denoted by the ſame Quantity; and conſequently that theſe two Serieſes are equal to each other. From the Addition and Subtraction of which and their Multiples, various other Serieſes may be produced, whoſe Sums are explicable by means of the Periphery of a Circle.

PROB. XII.

436. *To determine the Nature of the Curve* CDH, *which will intersect any Number of similar and concentric Ellipses* AMB, *amb* &c. *at Right-Angles.*



Let the Tangent DT, which is a Normal to the Ellipsis AMB, meet the Axis AB in T; and, supposing AC, CM, *aC*, C*m* &c. to be the principal Semi-diameters of their respective Ellipses, let the given Ratio of AC^2 to CM^2 (or of aC^2 to Cm^2 &c.) be that of 1 to *n*: Putting CE $= x$, ED $= y$, D*p* (E*e*) $= \dot{x}$, and $dp = \dot{y}$.

It is a known Property of the Ellipsis that $AC^2 : CM^2 :: CE : ET$; therefore ET $= nx$: Moreover ET (nx) : D*p* $(\dot{x})$ :: ED (y) : *pd* $(\dot{y})$ by similar Triangles) whence $\frac{\dot{x}}{nx} = \frac{\dot{y}}{y}$, or $\frac{\dot{x}}{x} = \frac{n\dot{y}}{y}$; whereof the Fluent is $L : x - L : a = nL : y - nL : a$ * (where *a* denotes any constant Quantity at Pleasure.) Hence we also have $L : \frac{x}{a} = n \times L : \frac{y}{a} = L : \frac{y^n}{a^n}$, and consequently

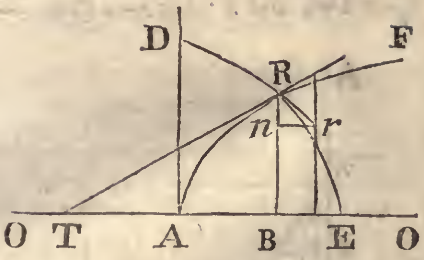
$$\frac{x}{a} = \frac{y^n}{a^n}, \text{ or } a^{n-1}x = y^n.$$

* Art. 126.

PROB. XIII.

437. *To find the Equation of a Curve* ERD *that will cut any Number of Ellipses, or Hyperbolas, having the same Center* O *and Vertex* A, *at Right-Angles.*

Let RT be a Tangent to any one of the proposed Conic Sections ARF, at the Intersection R, meeting the



the Axis AO in T; and put AO $=a$, OB $=x$, BR $=y$, $nr=\dot{x}$, R$n=-\dot{y}$: Then (*per Conics*) BT $=\frac{a^2-x^2}{x}$, in the Ellipsis, and $=\frac{x^2-a^2}{x}$, in the Hyperbola: Whence, by reason of the similar Triangles TBR, and Rrn, it will be $\frac{a^2 \backsim x^2}{x}$ (BT) : y (BR) :: $-\dot{y}$ (Rn) : $\mp \dot{x}$ (rn): Therefore $+y\dot{y}=\frac{a^2\dot{x}-x^2\dot{x}}{x}=\frac{a^2\dot{x}}{x}-x\dot{x}$, and consequently $+\frac{y^2}{2}+d^2=a^2 \times \text{L} : \frac{x}{a} - \frac{1}{2}x^2$. Where d denotes a constant Quantity, depending on the given Value of AE.

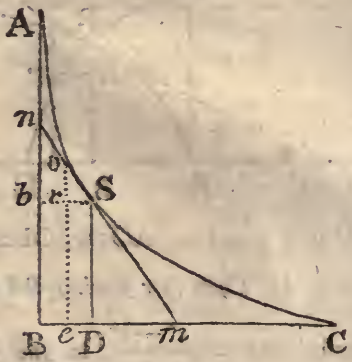
PROB. XIV.

438. *Let two Points* n *and* m *move, at the same time, from two given Positions* B *and* C, *with equal Celerities, along two Right-lines* BA *and* BC *perpendicular to each other: 'Tis proposed to determine the Curve* ASC, *to which a Right-line joining the said Points shall, always, be a Tangent.*

Let DS and ev be parallel to BA, and Srb perpendicular thereto: Putting BC $=a$, CD $=x$, SD $=y$, S$r=\dot{x}$, and $rv=\dot{y}$. Therefore (*by sim. Triangles*) $\dot{y}:\dot{x}$

 :: y

$:: y : \frac{y\dot{x}}{\dot{y}} = Dm$, and $\dot{x} : \dot{y} :: a-x$ (Sb) : $\frac{\overline{a-x} \times \dot{y}}{\dot{x}} =$



bn: Whence Cm (CD—Dm) $= x - \frac{y\dot{x}}{\dot{y}}$, and B$n$ (Bb $+ bn) = y + \frac{\overline{a-x} \times \dot{y}}{\dot{x}}$: Which two laſt Values, becauſe the Velocities of the Bodies are equal, muſt alſo be equal to each other, that is, $x - \frac{y\dot{x}}{\dot{y}} = y + \frac{\overline{a-x} \times \dot{y}}{\dot{x}}$: Hence, by making $\dot{x}$ conſtant, and taking the Fluxion of the whole Equation, we get $\dot{x} - \frac{\dot{y}\dot{x}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}^2} = \dot{y} - \frac{\dot{x}\dot{y} - \overline{a-x} \times \ddot{y}}{\dot{x}}$; or $\frac{\overline{a-x} \times \ddot{y}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}^2}$; from which there ariſes $\overline{a-x} \times \dot{y}^2 = y\dot{x}^2$, and $\frac{\dot{y}}{\sqrt{y}} = \frac{\dot{x}}{\sqrt{a-x}}$: Where, the Fluent on both Sides being taken, we have $2\sqrt{y} = 2\sqrt{a} - 2\sqrt{a-x}$, and conſequently $x = 2\sqrt{ay} - y$: Which Equation pertains to the common Parabola.

Other-

Otherwise more universally, thus.

439. Put C$m = v$ and B$n = w$, and let these Quantities (instead of being equal) have any given Relation to each other. Then, since the absolute Celerity of m is expressed by $\dot{v}$, its angular Celerity, in a Direction perpendicular to Sm, by which the Line Sm tends to revolve about the Point of Contact S as a Center, will be truly defined by $\frac{\textit{Sine of Bmn}}{\textit{Radius}} \times \dot{v}$ (*Art.* 35.) In the same manner the angular Celerity of n, about the Point S, will be defined by $\frac{\textit{Sin. Bnm}}{\textit{Rad.}} \times \dot{w}$. Now, as these Celerities must be to each other as the Distances Sm and Sn from the Center S (or directly as the Radii) we have Sm : Sn (:: DS : bn) :: *Sin. Bmn* $\times \dot{v}$: *Sin. Bnm* $\times \dot{w}$; whence, because *Sin. Bmn* : *Sin. Bnm* :: *Bn* (w) : *Bm* $(a - v)$ we also have DS : bn :: $w \times \dot{v} : \overline{a-v} \times \dot{w}$: Therefore, by Composition, DS : (DS $+ bn$) w :: $w\dot{v} : w\dot{v} + \overline{a-v} \times \dot{w}$, and consequently DS $= \frac{w^2\dot{v}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: Whence bn (w — SD) $= \frac{\overline{a-v} \times w\dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$; and BD $\left(= Sb = \frac{bn \times Bm}{Bn}\right)$ $= \frac{\overline{a-v}|^2 \times \dot{w}}{w\dot{v} + \overline{a-v} \times \dot{w}}$: From whence the Curve itself will be given.

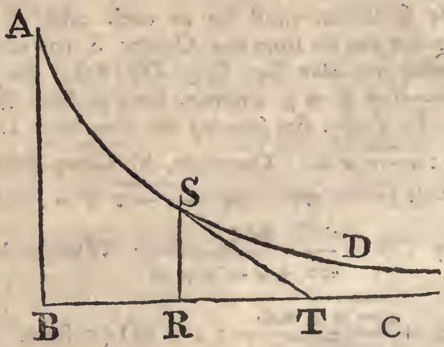
If $\dot{v}$ and w be taken equal to each other (as above) then SD (y) will become $= \frac{w^2}{a}$, and BD $= \frac{\overline{a-w}|^2}{a}$ $= a - 2w + \frac{w^2}{a}$; in which last, if for w its Equal

$\sqrt{ay}$ be substituted, we shall have BD $= a - 2\sqrt{ay} + y$; and consequently CD (a — BD) $= 2\sqrt{ay} - y$, the very same as before.

PROB. XV.

440. *Supposing a Body* T *to proceed, uniformly, along a Right-line* BC, *and another Body* S, *in pursuit of the same, always directly towards it, with a Celerity which is to that of* T, *in any given Ratio, of* 1 *to* n; *it is proposed to find the Equation of the Curve* ASD *described by the latter.*

Let the Tangent AB, which makes Right-Angles with BC, be put $= a$, BR $= x$, RS $= y$, and AS $= z$:



Then the Subtangent RT being $= \frac{y\dot{x}}{-\dot{y}}$, we have BT $= x + \frac{y\dot{x}}{-\dot{y}}$: Moreover, since the Distances BT and AS gone over in the same Time, are as the Celerities n and 1, we also have BT ($= n \times$ AS) $= nz = x + \frac{y\dot{x}}{-\dot{y}}$: Whence, in Fluxions (making $\dot{y}$ constant) $\frac{-y\ddot{x}}{\dot{y}} = n\dot{z}$; and consequently $\frac{-n\dot{y}}{y} \left(= \frac{\ddot{x}}{\dot{z}} \right) = \frac{\ddot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}}$:

The

The Fluent of which (*by Art.* 126.) is $-n\times$ Log. y $=$ Log. $\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$: But when $y=a$, $\dot{x}$ is $=0$, and then the Equation becomes $-n\times$ Log. $a=0$; therefore the Fluent, duly corrected, is $n\times$ Log. $a-n$ $\times$ Log. $y=$ Log. $\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$, or Log. $\frac{a^n}{y^n}=$ Log. $\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$. Whence it is evident that $\frac{a^n}{y^n}$ $=\frac{\dot{x}+\sqrt{\dot{y}^2+\dot{x}^2}}{\dot{y}}$, and $\frac{a^n\dot{y}}{y^n}-\dot{x}=\sqrt{\dot{y}^2+\dot{x}^2}$; from which, by ſquaring both Sides, $2\dot{x}$ is found $=$ $\frac{a^n\dot{y}}{y^n}-\frac{y^n\dot{y}}{a^n}$; whoſe Fluent is $2x=-\frac{a^n y^{1-n}}{1-n}+$ $\frac{a^{-n}y^{n+1}}{n+1}$. But when $y=a$, x is $=0$, and then, $0=-\frac{a}{1-n}+\frac{a}{n+1}=-\frac{2na}{1-nn}$; therefore the Fluent corrected is $2x=-\frac{a^n y^{1-n}}{1-n}+\frac{a^{-n}y^{n\times 1}}{n+1}+$ $\frac{2na}{1-n^2}$. *Q. E. I.*

Otherwiſe (without ſecond Fluxions.)

441. Put ST $=P$ and RT $=Q$. Then ſince the abſolute Velocity of the Body *S* is denoted by Unity, that with which the Ordinate SR is carry'd towards the Body *T* will be denoted by $\frac{Q}{P}\times 1$ or $\frac{Q}{P}$ (*by Art.* 35.) which ſubtracted from n the Velocity of *T*, leaves $n-\frac{Q}{P}$ for the relative Celerity with which *T* recedes from

R: After the same Manner, if from $\frac{Q}{P} \times n$ the Celerity of *T* in the Direction ST produced, there be taken (1) the Celerity of *S* in the same Direction, the Remainder, $\frac{nQ}{P} - 1$, will be the Celerity with which *T* recedes from *S*: Therefore, the Fluxions of Quantities being as the Celerities of their Increase, we have $n - \frac{Q}{P} : \frac{nQ}{P} - 1 :: \dot{Q} : \dot{P}$; and consequently $\overline{nQ - P} \times \dot{Q} = \overline{nP - Q} \times \dot{P}$. But, since the Quantities *P* and *Q* are concerned exactly alike, the Equation thus derived will, in all probability, become more simple, by substituting for their Sum and Difference: Let therefore $P + Q = s$, and $P - Q = v$, or, which is the same, let $P = \frac{s+v}{2}$, and $Q = \frac{s-v}{2}$: Then, by Substitution, we shall have $\frac{ns - nv - s - v}{2} \times \frac{\dot{s} - \dot{v}}{2} = \frac{ns + nv - s + v}{2} \times \frac{\dot{s} + \dot{v}}{2}$; which contracted, &c. becomes $\overline{1+n} \times v\dot{s} = \overline{1-n} \times s\dot{v}$, or $\overline{1+n} \times \frac{\dot{s}}{s} = \overline{1-n} \times \frac{\dot{v}}{v}$; whose Fluent (corrected) is $\overline{1+n} \times \text{Log. } s = \overline{1-n} \times \text{Log. } v + 2n \times \text{Log. } a$, or $\text{Log. } s^{1+n} = \text{Log. } a^{2n} v^{1-n}$. Whence $s^{1+n} = a^{2n} v^{1-n}$, and consequently $s^{1+n} \times v^{1+n} = a^{2n} v^2$: But sv $(= \overline{ST + RT} \times \overline{ST - RT} = RS^2) = y^2$ therefore $s^{1+n} \times v^{1+n} = y^{2n+2} = a^{2n} v^2$, and $v = \frac{y^{n+1}}{a^n}$; whence $s \left(= \frac{y^2}{v}\right) = \frac{a^n}{y^{n-1}}$, ST $\left(\frac{s+v}{2}\right) \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}$, RT $\left(\frac{s-v}{2}\right)$

=

$= \frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n}$. But RS (y) : RT $\left(\frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n}\right)$ :: $\dot{y} : \dot{x}$; whence $2\dot{x} = \frac{a^n\dot{y}}{y^n} - \frac{y^n\dot{y}}{a^n}$, and $2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1} + \frac{2na}{1-nn}$, the very ſame as before.

COROLLARY.

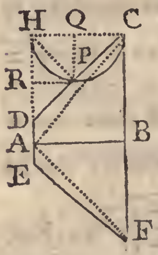
442. If the Velocity of S be greater than that of T (or n be leſs than Unity) the two Bodies will concur when the latter has moved over a Diſtance expreſſed by $\frac{na}{1-n^2}$; becauſe, when y becomes $= 0$, $2x$ is barely $= \frac{2na}{1-n^2}$. But if the Velocity of S be leſs than that of T, it is plain that S can never come up with T: But its neareſt Approach will be when $y = \overline{\frac{n-1}{n+1}}|^{\frac{1}{2n}} \times a$: For, ſince ST is univerſally $= \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}$, let the Fluxion of this Expreſſion be taken and put equal to Nothing; and y will be found as above exhibited.

If the Celerities of S and T, inſtead of being uniform, vary according to a given Law; then, denoting the former by A and the latter by B, the Equation of the Curve will be $\frac{\dot{x}}{\sqrt{\dot{y}^2 + \dot{x}^2}} = -\frac{B\dot{y}}{Ay}$: And if the Fluent of $-\frac{B\dot{y}}{Ay}$ be explicable by a Logarithm, as L. N; then, the Fluent of $\frac{\dot{x}}{\sqrt{\dot{y}^2+\dot{x}^2}}$ being L. $\frac{\dot{y} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$ *, *Art. 126.

we shall have $N = \frac{\dot{y} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}$; which, ordered, gives $\dot{x} = \frac{N\dot{y}}{2} - \frac{\dot{y}}{2N}$: Whence x will be found.

PROB. XVI.

443. *To determine the Frustum* CDEF *of a Triangular-Prism, of a given Base* CF *and Altitude* BA; *which, moving in a Medium, in the Direction of its Length* BA, *shall be resisted the least possible.*



Draw CH parallel to BA meeting ED, produced, in H: Moreover, let HP, PQ and PR be perpendicular to CD, CH and DH respectively.

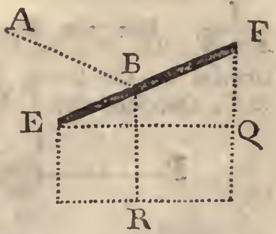
Since the Number of resisting Particles acting upon DC is as DH, and the Force of each as $\left(\frac{DR^2}{DP^2}\right)$ the Square of the Sine of the Angle of Incidence DPR, the whole Resistance sustained by DC will therefore be expressed by $\frac{DH \times DR^2}{DP^2}$, or DR, which is equal to it (by the Similarity of the Triangles DHP and DPR) Whence the Resistance upon ADC is truly expressed by AR (AD + DR) and is a *Minimum* when its Defect (PQ) below the given Quantity AH (or BC) is a *Maximum*: But PQ is a *Maximum* when CQ and HQ are equal; because, the Angle CPH being Right, a Semi-circle described upon CH will always pass through the Point P; and it is well known that the greatest Ordinate in a Semi-circle is That which divides the Diameter into two equal Parts.

Hence the Angle DCH, when the Resistance upon ADC is a *Minimum*, will be just the Half of a Right-Angle, provided BC be given greater than BA; otherwise,

wise, the whole Prism CAF will be less resisted than any Frustum CDEF of a greater Prism.

PROB. XVII.

444. *To determine the Angle* RBE *which a Plane* EBF *must make with the Wind blowing in a given Direction* RB, *so that the Plane itself may be urged in another given Direction* BA *with the greatest Force possible.*



It is known, from the Resolution of Forces, that the Force whereby the Plane EF is urged in the given Direction BA, by a Particle of Air, acting in the Direction RB, is directly as the Rectangle of the Sines of the Angles (ABE, RBE) which the two given Directions make with the Plane: Therefore, since the Number of Particles acting on EF is as the Sine of RBE, it follows that the whole Force, or Effect, of the Wind, in the Direction BA, will be as *S.* ABE × *Squ. S.* RBE; which being a *Maximum*, we have (*by Prob.* 5.) 3 : 1 :: Sine of the whole given Angle RBA : Sine of $\overline{\text{RBE}-\text{ABE}}$. Whence the Angles RBE and ABE are both given. *Q. E. I.*

Corollary.

445. If the Angle RBA be a Right one (which is the Case with regard to the Sails of a Windmill) then the Sine of $\overline{\text{RBE}-\text{ABE}}$ being $= \frac{1}{3} =$,333 &c. we shall have $\text{RBE} - \text{ABE} = 19° : 28'$; and consequently

$$\text{RBE} \left(\frac{\text{RBA} + \text{ABE}}{2}\right) = 54° : 44'.$$

PROB. XVIII.

446. *If two Bodies* A *and* B, *joined by a String, be urged in opposite Directions, towards* P *and* Q, *by any given Forces* F *and* f, *uniformly continued; it is proposed to find the Tension of the String, or the Force whereby the Bodies endeavour to recede from each other.*

Since

Since $F - f$ is the absolute Force by which the two Bodies are, constantly, urged towards P, the *whole* Motion, generated in Both, in any Time T, will therefore be expressed by $\overline{F-f} \times T$: Whence, because both Bodies (by reason of the String) acquire the same Velocity, the Motion generated in A, alone, will be $\frac{A}{A+B} \times \overline{F-f} \times T$, or that Part of the *Whole* defined by $\frac{A}{A+B}$. But the Motion of A, had it not been retarded by the String (or B) would have been $F \times T$; therefore the Loss of Motion, by the Action

upon the String, is $F \times T - \frac{A}{A+B} \times \overline{F-f} \times T$, $= \frac{fA+FB}{A+B} \times T$: Which, divided by the Time T, (wherein that Loss or Effect is produced) gives $\frac{fA+FB}{A+B}$, for the Tension of the Thread, or the Force sufficient to cause the said Loss or Motion.

The same otherwise.

447. Because the Force F, was it to act alone, would communicate, by means of the String, the same Velocity to B as to A, the Part therefore of the Force F employ'd upon B, by which the String is stretch'd, will be $\frac{B}{A+B} \times F$, or $\frac{BF}{A+B}$: And, from the very same Argument, if the Force f was to act alone, the Tension of the Thread would be $\frac{fA}{A+B}$: Therefore, when both

the

the Forces act together, the Tension will be $\frac{fA+BF}{A+B}$: For it is very plain that, their acting both at the same time, no way influences their respective Effects on the Thread. *Q. E. I.*

COROLLARY.

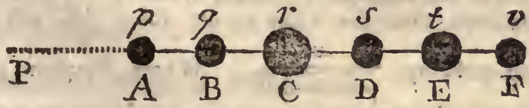
448. If the Forces F and f be respectively expounded by the Masses, or Weights, of the Bodies A and B; the Tension of the Thread will then become $\frac{2AB}{A+B}$. Whence it appears that the Tension of a Thread sliding over a Pin or Pulley, by means of two unequal Weights A and B, suspended at the Ends thereof, is equal to $\frac{2AB}{A+B}$: The Double whereof, or $\frac{4AB}{A+B}$, is the Weight which the Pin or Pulley sustains, while the Bodies are in Motion; because the Thread hangs double, or on both Sides the Pulley.

If several Bodies A, B, C, D &c. communicating by means of a String or Wire AF, be urged towards a Point P, in the Direction of the String or Wire, by any given Forces p, q, r, s &c. respectively, the Tension of the Part AB will be

$$= \frac{p \times \overline{B+C+D}\ \&c. - A \times \overline{q+r+s}\ \&c.}{A+B+C+D\ \&c.};$$

of the Part BC

$$= \frac{\overline{p+q} \times \overline{C+D+E}\ \&c. - \overline{A+B} \times \overline{r+s+t}\ \&c.}{A+B+C+D\ \&c.};$$



of CD

$$= \frac{\overline{p+q+r} \times \overline{D+E+F} - \overline{A+B+C} \times \overline{s+t+v}}{A+B+C+D\ \&c.};$$

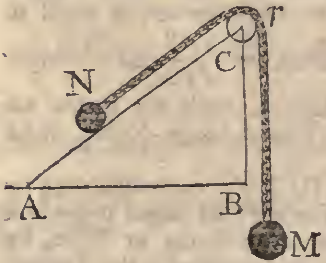
&c. &c.

All

All which eafily follows from above; and will anfwer alfo in thofe Cafes where fome of the Forces are fuppofed to act in the contrary Direction, if every fuch Force be confidered as a negative Quantity.

PROB. XIX.

449. *Let it be required to raife a given Weight* N, *to a given Height* BC, *along an inclin'd Plane* AC, *by means of another given Weight* M, *connected to the former by a flexible Rope* NrM, *moving over a Pulley at* C; *to find the Tenfion of the Rope; alfo the Inclination and Length of the Plane, fo that the Time of the whole Afcent may be the leaft poffible.*



It is well known that the Force by which N tends to defcend along the Plane AC, or acts in oppofition to M (fuppofing BC $= a$, and AC $= x$) will be $\frac{aN}{x}$: Therefore $M - \frac{aN}{x}$, or $\frac{xM - aN}{x}$ is the efficacious Force, by which the Bodies are accelerated: But it is likewife demonftrable that the Time of defcribing any Line by means of a Velocity uniformly accelerated, is in the fubduplicate Ratio of the Length thereof, directly, and the fubduplicate Ratio of the accelerating Force, inverfely *: Whence it follows that the Time of defcribing AC will be reprefented by $\frac{x}{\sqrt{xM - aN}}$: Whofe Fluxion (or that of its Square) being made equal to Nothing, x will be found $= \frac{2aN}{M}$, or $M : 2N :: a : x$. Hence the

* Art. 203.

Time

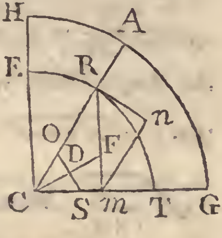
Time of the Aſcent will be the leaſt poſſible, when the Sine of the Plane's Inclination is to the Radius, as the Power (M) is to twice the Weight (N) to be raiſed.

The Tenſion of the Rope will be determined from the laſt Problem, (by writing N for A, $\frac{aN}{x}$ for F, M for B, and M for f) and comes out $= \frac{MN}{M+N} \times \frac{a+x}{x}$.

Q. E. I.

PROB. XX.

450. *Let* AC *repreſent a Piece of Timber, moveable about a Center* C, *making any Angle* ACG *with the Plane of the Horizon* CG; *to determine the Poſition of a Prop or Supporter* OS, *of a given Length, which ſhall ſuſtain it with the greateſt Facility, in any given Poſition; and alſo what Inclination* AC *will have to the Horizon when the leaſt Force that can ſuſtain it, is greater than the leaſt Force in any other Poſition.*



Let R be the Center of Gravity of the Beam AC, and let Rn, Rm and CD be perpendicular to AC, CG and OS reſpectively: Putting SO$=a$, CR $=r$, C$m = x$, and the Weight of the Beam $= w$.

Then, *by the Principles of Mechanics*, we ſhall have, firſt, as Rm : Rn, or as, $r : x :: w : \left(\frac{xw}{r}\right)$ the Force which acting at R, in the Direction Rn, is ſufficient to ſuſtain the Beam AC; ſecondly, as CO : CR $(r) :: \frac{xw}{r}$ (the Quantity laſt found) : $\frac{xw}{\text{CO}}$, the Force able to ſupport it, at O, in a perpendicular Direction; and, laſtly,

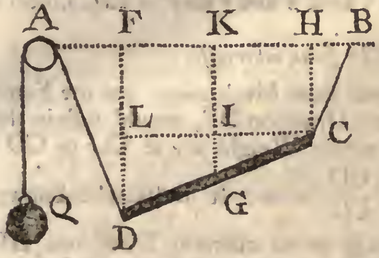
laſtly, as CD : CO :: $\frac{xw}{CO} : \frac{xw}{CD}$, the Force, or Weight, actually ſuſtained by the given Prop SO. Which Force will therefore be the leaſt poſſible when the Perpendicular CD is the greateſt poſſible, let the Angle of Inclination GCA be what it will: But of all Triangles, having the ſame Baſe (OS) and vertical Angle (SCO) the Iſoſceles one is known to have the greateſt Perpendicular: Therefore the Triangle CSO will be Iſoſceles, and the Angles S and O equal to each other, when the Weight ſuſtain'd by the Prop OS is a *Minimum*.

But, now, to give a Solution to the latter Part of the Problem, or to find (ſuppoſing the Angles S and O to be equal) when $\frac{x}{CD} \times w$ is a *Maximum*, let CD produced meet *m*R in F; and then, becauſe of the ſimilar Triangles CDS and C*m*F, we ſhall have CD : x (C*m*) :: SD ($\frac{1}{2}a$) : *m*F, or $\frac{x}{CD} = \frac{mF}{\frac{1}{2}a}$; and conſequently $\frac{x}{CD} \times w = \frac{mF}{\frac{1}{2}a} \times w$: But, ſince CF biſects the Angle *m*CR, we alſo have, $r+x$ (CR+C*m*) : x (C*m*) :: $\sqrt{r^2-x^2}$ (R*m*) : F*m* $= \frac{x\sqrt{r^2-x^2}}{r+x} = x\sqrt{\frac{r-x}{r+x}}$: Whence the Force $\frac{mF}{\frac{1}{2}a} \times w$, acting upon the Supporter, is likewiſe truly expreſſed by $\frac{wx}{\frac{1}{2}a}\sqrt{\frac{r-x}{r+x}}$: Whereof the Fluxion being taken and put equal to Nothing &c. we get $x = \frac{r\sqrt{5}-r}{2}$: Therefore CR : C*m* $\left(:: 1 : \frac{\sqrt{5}-1}{2}\right)$ :: Radius : Coſine of RCG $= 51^\circ : 50'$, the Inclination required.

PROB.

PROB. XXI.

451. *To determine the Position of a Beam* CD, *moveable about one End* C *as a Center, and sustained at the other End* D *by a given Weight* Q, *appended to a Cord* QAD *passing over a Pulley at a given Point* A.



Let G be the Center of Gravity of the Beam; also let DF, GK and CH be perpendicular to the Plane of the Horizon, and CL and AH parallel to the same: Putting AH $=a$, CH $=b$, CD $=c$, CG $=d$, DL $=x$, CL $=y$, and the Weight of the Beam $=w$. Then AF $=a-y$, DF $=b+x$, and AD $(\sqrt{AF^2+DF^2}) = \sqrt{a^2-2ay+y^2+b^2+2bx+x^2}$; which (because $y^2+x^2=c^2$) will also be $=\sqrt{a^2+b^2+c^2+2bx-2ay} = \sqrt{f^2+2bx-2ay}$ (by putting $f^2=a^2+b^2+c^2$) whose Fluxion, $\frac{b\dot{x}-a\dot{y}}{\sqrt{f^2+2bx-2ay}}$, multiply'd by $\mathcal{Q}$, is the Momentum of the Weight $\mathcal{Q}$, supposing the Beam to be in Motion. Moreover, because DC : DL :: CG : GI, we have GI $=\frac{dx}{c}$; whose Fluxion, $\frac{d\dot{x}}{c}$, multiply'd by w, is the Momentum of the Beam itself in a vertical Direction.

Wherefore making these *Momenta* equal to each other (according to the Principles of Mechanics) we get $\frac{b\dot{x}-a\dot{y}}{\sqrt{f^2+2bx-2ay}} \times \mathcal{Q} = \frac{d\dot{x}}{c} \times w$, and consequently $\overline{b\dot{x}-a\dot{y}} \times c\mathcal{Q} = dw\dot{x}\sqrt{f^2+2bx-2ay}$: But, since y^2+

$y^2 + x^2 = c^2$, we have $2y\dot{y} + 2x\dot{x} = 0$, or $-\dot{y} = \frac{x\dot{x}}{y}$: And therefore (by Substitution) $b\dot{x} + \frac{ax\dot{x}}{y} \times c\mathcal{Q} = dw\dot{x}\sqrt{f^2 + 2bx - 2ay}$, or $\overline{by + ax} \times c\mathcal{Q} = dwy \times \sqrt{f^2 + 2bx - 2ay}$: From whence, and the foregoing Equation $x^2 + y^2 = c^2$, both x and y may be determined.

The same otherwise.

452. It is evident, from Mechanics, that the Force which, acting in the Direction DF, would sustain the End D, is to the whole Weight w, as CG to CD; and therefore is $= \frac{CD}{CG} \times w$: It is likewise known that two Forces acting in the different Directions DF and DA, so as to have the same Effect in sustaining DC, or causing It to move about the Point C, must be to each other, inversely, as the Sines of the Angles of Incidence FDC and ADC. Therefore we have $S.\,FDC : S.\,ADC :: \mathcal{Q} : \frac{CD}{CG} \times w$; from which given Ratio of the Sines, the Angles themselves will be found, by an algebraic Process independent of Fluxions.

COROLLARY.

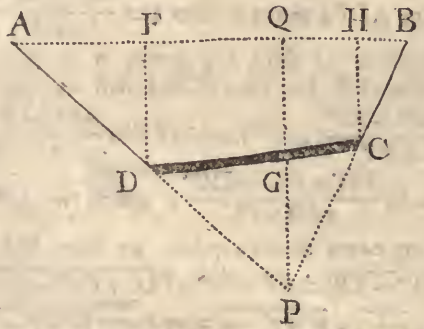
453. If the Position of CD be supposed given, and the Tension of AD (or the Weight $\mathcal{Q}$) be required: Then, from the foregoing Proportion, we shall have $\mathcal{Q} = \frac{S.\,FDC}{S.\,ADC} \times \frac{CG}{CD} \times w$. Which will also express the Tension of AD when the End C is sustained by a Cord BC instead of a Pin at C: Whence it follows that the Tensions of two Cords AD and BC, sustaining a Beam or Rod CD, at its Extremes D and C, are expressed by $\frac{S.\,FDC}{S.\,ADC} \times \frac{CG}{CD} \times w$, and $\frac{S.\,HCD}{S.\,BCD} \times \frac{DG}{CD} \times w$; and there-

therefore are to each other as $\frac{CG}{S.\ ADC}$ to $\frac{DG}{S.\ BCD}$, or as $S.\ BCD \times CG$ to $S.\ ADC \times DG$ respectively; because the Sine of FDC and that of its Supplement HCD are equal to each other.

PROB. XXII.

454. *To determine the Position of a Beam* DC, *suspended at its Extremes by two Cords* AD *and* BC *of given Lengths, from two given Points* A *and* B *in the same horizontal Line* AB.

Let G be the Center of Gravity of the Beam, and let DF and CH be perpendicular to AB.



It appears, from the Corol. to the last Problem, that the Tension of AD is to that of BC, as $\frac{CG}{S.\ ADC}$ to $\frac{DG}{S\ BCD}$; whence (by the Resolution of Forces) the Force of AD, in a Direction parallel to the Horizon, is to the Force of BC, in the opposite Direction, as $\frac{CG}{S.\ ADC} \times \frac{S.\ ADF}{Rad.}$ to $\frac{DG}{S.\ BCD} \times \frac{S.\ BCH}{Rad.}$. Which Forces, that the Beam may remain *in Equilibrio*, must

 con-

consequently be equal to each other; and therefore $\frac{S.\,BCD}{S.\,ADC}=\frac{S.\,BCH}{S.\,ADF}\times\frac{DG}{CG}$. But now, to determine the Angles themselves, from this Equation and the given Lengths of AB, BC *&c.* let AD and BC be produced to meet each other in P, and let PQ, perpendicular to AB, be drawn; putting $AB=a$, $AD=b$, $BC=c$, $DC=d$, $DG=f$, $CG=g$, $AP=x$, and $BP=y$.

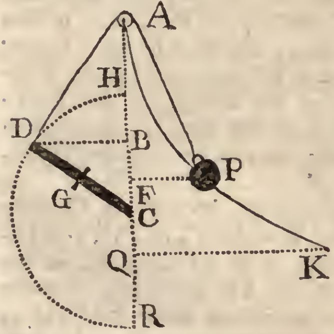
Then, because $AB : AP+BP :: AP-BP : AQ-BQ$ $=\frac{AP^2-BP^2}{AB}$, we have $AQ=\frac{1}{2}AB+\frac{AP^2-BP^2}{2AB}$ $=\frac{AB^2+AP^2-BP^2}{2AB}$; and consequently the Co-sine of A ($=$ Sine ADF) to the Radius $1=\frac{AB^2+AP^2-BP^2}{2AB\times AP}$: Whence, from the same Argument, it is evident that the Co-sine of B ($=$ Sine BCH) will be expressed by $\frac{AB^2+BP^2-AP^2}{2AB\times BP}$; and That of APB by $\frac{AP^2+BP^2-AB^2}{2AP\times BP}$; And also by $\frac{PD^2+PC^2-DC^2}{2PD\times PC}$; which two last Quantities being equal to each other, we have $PD\times PC\times\overline{AP^2+BP^2-AB^2}=AP\times BP\times\overline{PD^2+PC^2-DC^2}$; that is $\overline{x-b}\times\overline{y-c}\times\overline{x^2+y^2-a^2}=xy\times\overline{\overline{x-b}|^2+\overline{y-c}|^2-d^2}$. Moreover, since $PC : PD :: S.\,ADC$ (or PDC) $: S.\,BCD$ (or PCD) we also have $\frac{PD}{PC}=\frac{S.\,BCD}{S.\,ADC}=\frac{S.\,BCH}{S.\,ADF}\times\frac{DG}{CG}$ (by the first Equation); whence $CG\times PD\times S.\,ADF=DG\times PC\times S.\,BCH$; that is $CG\times PD\times\frac{AB^2+AP^2-BP^2}{2AB\times AP}=DG\times PC\times\frac{AB^2+BP^2-AP^2}{2AB\times BP}$, or

CG

$CG \times PD \times BP \times \overline{AB^2+AP^2-BP^2} = DG \times PC \times AP \times \overline{AB^2+BP^2-AP^2}$, which, in algebraic Terms, is $gy \times \overline{x-b} \times \overline{a^2+x^2-y^2} = fx \times \overline{y-c} \times \overline{a^2+y^2-x^2}$. From whence and the preceding Equation the Values of x and y will be known.

PROB. XXIII.

455. *Supposing a Beam* CD, *moveable about one End* C, *as a Center, to be sustained at the other End* D *by means of a given Weight* P, *hanging at a Rope passing over a Pulley at a given Point* A, *vertical to* C; *it is proposed to find the Curve* APK *along which the Weight must ascend, or descend, so as to be, every where, a just Counterpoise to the Beam.*



From the Center C, with the Radius CD, let a Semi-circle HDR be described, and let DB and PF be perpendicular to the vertical Line AHCR; also let $CD = a$, $CA = b$, $AH = c$, $AF = x$, $PF = y$, $HB = z$, and the Length of the Rope $DAP = m$; likewise let HQ (h) be the given Value of x (AF) when D coincides with H.

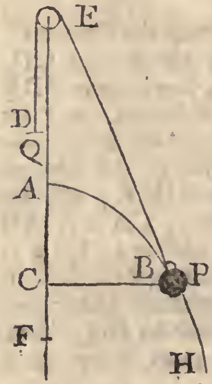
Because the Weight and the Beam are always in *Equilibrio*, by Hypothesis, their *Momenta*, and consequently their Velocities, in a vertical Direction, must be every where in a constant Ratio; and therefore the Distance QF ($h-x$) ascended by the Weight *P*, will be, to the Distance HB descended by the End of the Beam D likewise in a constant Ratio: Let this Ratio be that of b to any given Quantity d, that is, let $h-x : z :: b : d$, and we shall have $dh - dx = bz$: Moreover, we have AD^2 ($CD^2 + AC^2 - 2AC \times BC$) $= a^2 + b^2 - 2b \times \overline{a-z} = \overline{b-a}^2 + 2bz = c^2 + 2bz = c^2 - 2dh + 2dx$: Whence AP ($m - AD$) $= m -$

$\sqrt{cc - 2dh + 2dx}$, and therefore, y^2 ($AP^2 - AF^2$) $=$ $\overline{m - \sqrt{cc - 2dh + 2dx}}|^2 - x^2$. *Q. E. I.*

After the same manner a Curve may be found, along which a Weight descending, shall be every where in *Equilibrio* with another Weight ascending thro' the Arch of a given Curve.

PROB. XXIV.

456. *To find the Equation of a Curve* ABH, *along which a given Weight* P, *suspended by a String* PED *passing over a Pulley* E, *must descend, so that the Tension of the String may vary according to any given Law.*



Let EC be perpendicular, and CP parallel, to the Plane of the Horizon; also let AE $= a$, AC $= x$, CB $= y$, EP $= v$, and let the Tension of the String (or the Force acting at the End D) be denoted by any variable, or constant, Quantity Q.

Therefore, because the Celerity of the Weight P, in a vertical Direction, is to its Celerity, in the Direction EP produced, (or the Celerity of the other End D) as $\dot{x}$ to $\dot{v}$, it is evident that the Weight itself must be to the tending Force Q, inversely in that Ratio, and consequently $P\dot{x} = Q\dot{v}$.

Furthermore, because EC $= a + x$ and $BC^2 = BE^2 - EC^2$, we have $y^2 = v^2 - \overline{a+x}|^2$: From which Equations, when the Relation of P and Q is given, the Curve itself will also be known.

Thus, for Example, let the Ratio of P to Q, be constant, or that of m to n, then $m\dot{x}$ being $= n\dot{v}$, we have (by taking the Fluent) $mx + na = nv$; whence $v = a + \frac{mx}{n}$; and therefore y^2 ($= a^2 + \frac{2max}{n} + \frac{m^2x^2}{n^2}$

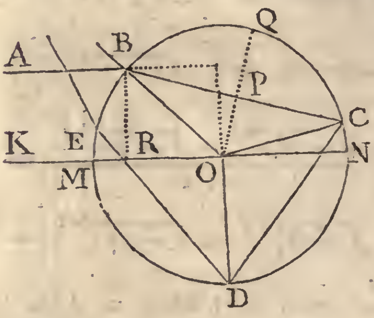
$-a^2-2ax-x^2) = \frac{m-n}{n}\times 2ax + \frac{m^2-n^2}{n^2}\times x^2$: Which is the Equation of an Hyperbola.

Again, for a second Example, let the tending Force $\mathcal{Q}$ be to the Weight P, as DE^n to $AC^m \times c^{n-m}$, or as $\overline{b-v}|^n : x^m c^{n-m}$ (supposing b = PED and c = any given Line AF.) Therefore, since $\mathcal{Q} = \frac{\overline{b-v}|^n}{c^{n-m} x^m}\times P$, and $\frac{\overline{b-v}|^n}{c^{n-m} x^m}\times P\dot{v}\ (=\mathcal{Q}\dot{v}) = P\dot{x}$, we have $\overline{b-v}|^n \times \dot{v} = c^{n-m} x^m \dot{x}$, and so $\frac{\overline{b-a}|^{n+1} - \overline{b-v}|^{n+1}}{n+1} = \frac{c^{n-m} x^{m+1}}{m+1}$; whence $\overline{b-v}|^{n+1} = \overline{b-a}|^{n+1} - \frac{\overline{n+1}\times c^{n-m} x^{m+1}}{m+1}$, and v (EP) $= b - \overline{\overline{b-a}|^{n+1} - \frac{\overline{n+1}\times c^{n-m} x^{m+1}}{m+1}}\Big|^{\frac{1}{n+1}}$. From which the Relation of x and y, or the Value of BC, is also known.

But if $m=0$, and $n=1$, (which will be the Case when the Force acting at D is equal to that by which a Beam or Rod is made to move about a Center, as in the last Problem) v will then become, barely, $= b - \overline{\overline{b-a}|^2 - 2cx}|^{\frac{1}{2}}$, and therefore $y^2\ (= v^2 - \overline{a+x}|^2)$ $= \overline{b - \sqrt{\overline{b-a}|^2 - 2cx}}|^2 - \overline{a+x}|^2$: Therefore ABH is, in this Case, a Line of the fourth Order.

PROB. XXV.

457. *Supposing a Ray of Light* ABCD *to be refracted at the Surface of a given Sphere* MQND, *and afterwards reflected any given Number (n) of Times, within the Sphere; to determine the Distance of the Incident Ray* AB *from the Axis* MN, *so that the Arch* MBCDE, *intercepted by the given Point* M *and the emerging Ray at* E, *may be a* Minimum.



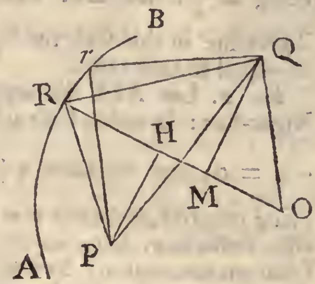
Let the Radius OB = 1, the Sine of Incidence BR = x, and the Sine of Refraction OP = y, and let the given Ratio of the two last be that of p to q. Since all the Angles of Incidence and Reflexion BCO OCD, CDO &c. are equal, the Arcs BC, CD and DE must also be equal; and consequently MBCDE = MB + $\overline{n+1}$ × BC = MB + $\overline{2n+2}$ × BQ:
* Art. 22. Whose Fluxion is to be equal to Nothing *. Now the Fluxion of the Arch MB, whose Sine is x and
† Art. 142. Radius Unity, will be $= \frac{\dot{x}}{\sqrt{1-x^2}}$ †; and that of the Arch BQ, whose Co-sine (OP) is y, $= \frac{-\dot{y}}{\sqrt{1-y^2}}$. Hence we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times \dot{y}}{\sqrt{1-y^2}} = 0$: But since $x : y :: p : q$, y is $= \frac{qx}{p}$ and $\dot{y} = \frac{q\dot{x}}{p}$; and so we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times q\dot{x}}{\sqrt{p^2 - q^2x^2}} = 0$; whence (putting

$m =$

$m = 2n+2$) x is found $= \frac{1}{q}\sqrt{\frac{m^2q^2-p^2}{m^2-1}}$: From which it is obſervable, that, when mq is leſs than p, or $2n+2$ leſs than $\frac{p}{q}$, the Arch MBCD continually increaſes with BM; and therefore is the leaſt poſſible, when B coincides with M. *Q. E. I.*

PROB. XXVI.

458. *If two Rays of Light* PR *and* Pr, *from a given Point* P, *making an indefinitely ſmall Angle with each other, be reflected at a given Curve Surface* ARB; *'tis propoſed to determine the Concourſe, or Focus,* Q *of the reflected Rays* RQ *and* rQ.



Let RO, perpendicular to the Curve, be the Radius of a Circle having the ſame Curvature with ARB at R; make PH and QM perpendicular to RO, join Q, O; and put RO $=r$, PR $=y$, RH $=v$, and RQ $=z$.

Then, becauſe the Angle of Reflection ORQ is equal to the Angle of Incidence ORP, the Triangles RQM and RPH will be ſimilar, and therefore $y : v :: z$: RM $= \frac{vz}{y}$: Whence OQ² (RO² + RQ² − 2RO × RM) $= r^2 + z^2 - \frac{2rvz}{y}$.

But, ſince this Quantity OQ² continues the ſame (by Hypotheſis) whether we regard one Ray or the other (that is, whether y ſtands for PR or Pr) its Fluxion muſt therefore be equal to Nothing; that

 is,

is, $2z\dot{z} - \frac{2r\dot{v}zy + 2rv\dot{z}y - 2rvz\dot{y}}{y^2} = 0$: Whence

$z = \frac{vy\dot{z}}{\frac{y^2\dot{z}}{r} + v\dot{y} - y\dot{v}}$: But (*by Art.* 35.) $\dot{z} = -\dot{y}$; therefore $z =$

$\frac{-vy\dot{y}}{-\frac{y^2\dot{y}}{r} + v\dot{y} - y\dot{v}}$: Moreover (*by Art.* 73.) $r = \frac{y\dot{y}}{\dot{v}}$; therefore

$z = \frac{-vy\dot{y}}{-y\dot{v} + v\dot{y} - y\dot{v}} = \frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}$. *Q. E. I.*

Example 1. Let ARB be an Arch of the Logarithmic Spiral: whose Equation is $av = by$ †: And then, $\dot{v}$ being $= \frac{b\dot{y}}{a}$, we shall have $z \left(\frac{vy\dot{y}}{2y\dot{v} - v\dot{y}}\right) = y$: Therefore in this Case the Incident and Reflected Rays are equal to each other.

Ex. 2. Let ARB be supposed to degenerate into a Right-line: In which Case v being constant, its Fluxion $\dot{v}$ is $= 0$; and therefore $z \left(= \frac{vy\dot{y}}{-v\dot{y}}\right) = -y$: Which being negative, indicates that the Rays do not converge after Reflection, but, on the contrary, diverge from a Point on the contrary Side of ARB, at the Distance y. Which is very easy to demonstrate by common Geometry.

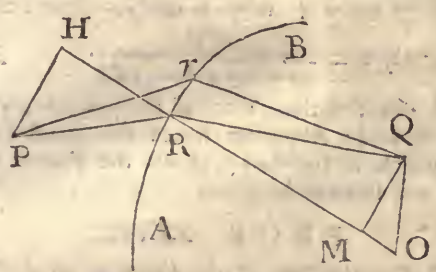
PROB. XXVII.

459. *Let two Rays of Light* PR *and* Pr, *from a given Point* P, *be refracted at a given Curve Surface* ARB; *to determine the Focus* Q *of the refracted Rays* RQ *and* rQ.

Let the Lines RO, RH &c. be drawn, and denoted as in the preceding Problem: Moreover, let the Sine of Incidence PRH (to the Radius 1) be represented by s, and let it be to the Sine of Refraction ORQ, in the given Ratio of 1 to n.

Then

Then *(by Trigonometry)* 1 : ns (Sine QRM) :: z (RQ) : QM $= nsz$; and therefore RM $= \sqrt{z^2 - n^2s^2z^2}$



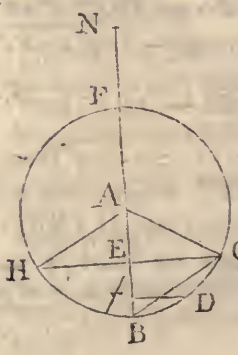
$= z\sqrt{1-n^2s^2}$. From whence, following the Steps of the preceding Problem, we also get $OQ^2 = r^2 + z^2 - 2rz\sqrt{1-n^2s^2}$; and its Fluxion $2z\dot{z} - 2r\dot{z}\sqrt{1-n^2s^2} + \frac{2rzn^2s\dot{s}}{\sqrt{1-n^2s^2}} = 0$; or $z\dot{z}\sqrt{1-n^2s^2} - r\dot{z} \times \overline{1-n^2s^2} + n^2rzs\dot{s} = 0$. But *(by Art.* 35.) $\dot{z} = -n\dot{y}$; therefore $-z\dot{y}\sqrt{1-n^2s^2} + r\dot{y} \times \overline{1-n^2s^2} + nrzs\dot{s} = 0$: Moreover (*by Trig.*) 1 (Radius) : s (Sine of PRH) :: y (PR) : $\sqrt{y^2-v^2}$ (PH) whence we have $sy = \sqrt{y^2-v^2}$, $s^2 = 1 - \frac{v^2}{y^2}$, and $s\dot{s} = \frac{-y^2v\dot{v} + v^2y\dot{y}}{y^4} = \frac{v^2\dot{y} - yv\dot{v}}{y^3}$; which Values, of s^2 and $s\dot{s}$, being substituted in the foregoing Equation, it becomes $-z\dot{y}\sqrt{1-n^2+\frac{n^2v^2}{y^2}} + r\dot{y} \times \overline{1-n^2+\frac{n^2v^2}{y^2}} + nrz \times \frac{v^2\dot{y}-yv\dot{v}}{y^3} = 0$, or $-zy^2\dot{y}\sqrt{\overline{1-nn}\times y^2 + n^2v^2} + ry\dot{y} \times \overline{\overline{1-nn}\times y^2 + n^2v^2} + nrz \times \overline{v^2\dot{y} - yv\dot{v}} = 0$; or (putting $\sqrt{\overline{1-n^2}\times y^2 + n^2v^2} = w$) $\frac{-zy^2w\dot{y}}{r} + w^2y\dot{y} +$

$nzv^2\dot{y} - nyzv\dot{v} = 0$. But (*by Art.* 73.) $r = \frac{y\dot{y}}{\dot{v}}$, therefore $- zyw\dot{v} + w^2y\dot{y} + nzv^2\dot{y} - nyzv\dot{v} = 0$, and consequently $z = \frac{w^2y\dot{y}}{\overline{wy + nvy} \times \dot{v} - nv^2\dot{y}}$. *Q. E. I.*

From this Solution, that of the preceding Problem is easily derived: Also from hence the Caustic (or the Curve which is the Locus of all the Points Q thus found) will likewise be given.

PROB. XXVIII.

460. *To find the Time of the Vibration of a Pendulum in the Arch of a Circle.*



Let AB denote the Pendulum in a vertical Position; and from any Point D in the given Arch CBH, wherein the Vibrations are perform'd, draw Df parallel to CH; and let AB $= a$, BE $= c$, B$f = x$, and BD $= z$: By the Nature of the Circle we have $\dot{z} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$ *: Whence the Fluxion of the Time, being as $\frac{\dot{z}}{\sqrt{Ef}}$ †, will be defined by $\frac{a\dot{x}}{\sqrt{c - x} \times \sqrt{2ax - xx}}$

* Art. 142.

† Art. 207.

$$= \frac{a\dot{x}}{\sqrt{cx - xx} \times \sqrt{2a - x}} = \frac{\overline{\tfrac{1}{2}a}^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - xx}} \times \overline{1 - \frac{x}{2a}}^{-\frac{1}{2}}$$

$$= \frac{\overline{\tfrac{1}{2}a}^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx - xx}} \times \overline{1 + \frac{x}{2 \cdot 2a} + \frac{3x^2}{2 \cdot 4 \cdot 4a^2} + \frac{3 \cdot 5x^3}{2 \cdot 4 \cdot 6 \cdot 8a^3} + \frac{3 \cdot 5 \cdot 7x^4}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 16a^4}}$$ &c. Whereof the Fluent,

when

when $x=c$, (or $\overline{cx-x^2}|^{\frac{1}{2}}=0$) is, (*by Art.* 142. *and* 286.) equal to $p\sqrt{\tfrac{1}{2}a}\times\overline{1+\frac{c}{2.2.2a}+\frac{3.3c^2}{2.2.4.4.\overline{2a}|^2}}$

$$+\frac{3.3.5.5c^3}{2.2.4.4.6.6.\overline{2a}|^3}+\frac{3.3.5.5.7.7c^4}{2.2.4.4.6.6.8.8.\overline{2a}|^4}$$

&c. Which therefore is proportional to the Time of half one Vibration; where p ſtands for the Semi-Periphery of the Circle whoſe Radius is Unity.

Corollary I.

461. Since the Time of the perpendicular Deſcent of a Body through any given Right-line u, computed according to the ſame Method, is as the Fluent of $\frac{\dot{u}}{\sqrt{u}}$ or $2\sqrt{u}$, it follows that the Time of falling along the Diameter BF (2a), or the Cord CB *, will be truly defined by $2\sqrt{2a}$: Which therefore is to the Time of the Deſcent thro' the Arch CDB, as $\frac{4}{p}$ to $1+\frac{c}{2.2.2a}+\frac{3.3c^2}{2.2.4.4.\overline{2a}|^2}$ &c. From whence, as the Time of falling thro' the Diameter BF, is abſolutely given, *by Art.* 202. the true Time of Vibration will alſo be known.

* Art. 205.

Corollary II.

462. If the Arch in which the Pendulum vibrates be very ſmall, the above Proportion will become, nearly, as 4 to p: From which it appears, that the Time of Deſcent thro' any very ſmall Arch CB is to that along the Chord CB, as the Periphery of any Circle is to four times its Diameter.

Corollary III.

463. Hence, we have a Method for determining how far a Body freely deſcends in a given Time; by knowing the

the Time of Vibration, of a given Pendulum: For, if BN be assumed for the Space thro' which a Body would descend during the Time of one whole Vibration, in the very small Arch CBH; then, the Distances de-
* Art. 201. scended being as the Squares of the * Times, we have, from the last Corollary, as $4^2 : \overline{2p}|^2 :: BF\ (2a) : BN$, or $1 : \frac{1}{2}p^2 : a : BN$; that is, as the Square of the Diameter of a Circle is to half the Square of its Periphery, so is the Length of the Pendulum, to the Distance a Body will freely descend, from Rest, in the Time of one Oscillation. Thus, for instance (because it is found from Experiment that a Pendulum 39,2 Inches long vibrates Seconds) it will be as $1 : 4{,}934\ (=\frac{1}{2}p^2) :: 39{,}2 : 193$ Inches, the Distance which a heavy Body will fall in the first Second of Time.

COROLLARY IV.

464. Moreover, from the foregoing Series, the Time which a Pendulum, vibrating in an exceeding small Arch, will lose when made to vibrate in a greater Arch of the same Circle may also be deduced:

For let *T* be put to denote the Number of Seconds in 24 Hours (or any other given Time) then the Number of Vibrations, performed in that Time will be as

$$\frac{T}{1 + \frac{c}{2\,.\,2\,.\,2a} + \frac{3\cdot 3c^2}{2\,.\,2\,.\,4\,.\,\overline{2a}|^2\ \&c.}};$$ which, therefore, in an exceeding small Arch (where *c* may be taken as Nothing) will be expressed by *T*: And so the Time (t) or Number of Vibrations lost will be $T -$

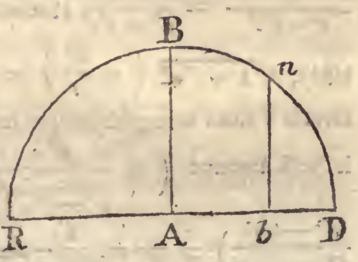
$$\frac{T}{1 + \frac{c}{2\,.\,2\,.\,2a} + \frac{3\cdot 3c^2}{2\,.\,2\,.\,4\,.\,4\,.\,\overline{2a}|^2\ \&c.}} = T \times \overline{\frac{c}{8a} + \frac{5c^2}{256a^2}\ \&c.}$$ (by dividing by the Denominator.)

Now, if the Number of Degrees described on each Side of the Perpendicular be represented by *D*, the Arch

Arch itſelf, on each Side, will be $= 3.14159$ &c. $\times a \times \frac{D}{180}$; which, if the Value of D be not more than about 15 or 20 Degrees, will be nearly equal to its Chord, repreſented by $\sqrt{2ac}$ $(=\sqrt{BF \times BE}.)$ From which Equation we get $\frac{c}{a} = \frac{D^2}{6560}$: This Value, ſubſtituted above, gives $t = T \times \overline{\frac{D^2}{8 \times 6560} + \frac{5D^4}{256 \times \overline{6560}^2}\ \&c.}$ $= T \times \frac{D^2}{52480}$ nearly: Which, when T is interpreted by 86400 Seconds (or one whole Day) becomes $= 1\frac{1}{2} \times D^2$, nearly: And ſo many are the Seconds which will be loſt *per Diem* in the Arch D. From whence we gather, that if the Pendulum meaſures true Time in any ſmall Arch, whoſe Degrees on each Side the Perpendicular are denoted by A, the Number of Seconds loſt *per Diem* in another Arch whoſe Degrees are B, will be nearly repreſented by $\frac{3}{2} \times \overline{B^2 - A^2}$: Thus, if a Pendulum meaſures true Time, in an Arch of 3 Degrees, it will loſe $10\frac{1}{2}$ Seconds a Day in an Arch of 4 Degrees, and 24″ in an Arch of 5 Degrees.

PROB. XXIX.

465. *To determine the* Meridional Parts *anſwering to any propoſed Latitude, according to* Wright's *Projection, applied to the true ſpheroidal Figure of the Earth.*



Let DAR be the Axis, AB the Semi-equatoreal Diameter, and DBR a Meridian, of the Earth; alſo let *bn* be an Ordinate to the Ellipſis DBR; puttingAD(=AR)

$=1$, BA$=d$, A$b=x$, $bn=y$, B$n=z$, and the Meridional Distance (in Parts of the Semi-Axis AD) $=u$.

Then, by the Nature of the Ellipsis, we have $y=d\sqrt{1-x^2}$; therefore $\dot{y}=\frac{-dx\dot{x}}{\sqrt{1-x^2}}$; and consequently $\dot{z}=\sqrt{\dot{x}^2+\frac{d^2x^2\dot{x}^2}{1-xx}}$: Which, by putting $b^2=d^2-1$, will be reduced to $\dot{z}=\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}$. Whence, by the Nature of the Projection, it will be as bn $(d\sqrt{1-x^2})$: AB (d) :: $\dot{z}$ $\left(\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-xx}}\right)$: $\dot{u}=\frac{\dot{x}\sqrt{1+b^2x^2}}{1-x^2}$; which is the Fluxion of the Quantity required: But we are now to get the same thing expressed in Terms of the Latitude of the Place n: In order thereto, putting the Sine of that Latitude $=s$, we have, by Trigonometry, as $\dot{z}$ $\left(\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}\right)$: $-\dot{y}$ $\left(\frac{dx\dot{x}}{\sqrt{1-x^2}}\right)$ :: Radius (1) : s; and consequently $s\sqrt{1+b^2x^2}=dx$; from which Equation x is found $=\frac{s}{\sqrt{d^2-b^2s^2}}$: Whence $\dot{x}=\frac{d^2\dot{s}}{\overline{d^2-b^2s^2}|^{\frac{3}{2}}}$; also $1-x^2=\frac{d^2-b^2s^2-s^2}{d^2-b^2s^2}=\frac{d^2-d^2s^2}{d^2-b^2s^2}$ (because $d^2=1+b^2$) and, lastly, $\sqrt{1+b^2x^2}\left(=\frac{dx}{s}\right)=\frac{d}{\sqrt{d^2-b^2s^2}}$: Which several Values being substituted in that of $\dot{u}$, found above, it will become $\left(=\frac{d^2\dot{s}}{\overline{d^2-b^2s^2}|^{\frac{3}{2}}}\times\frac{d}{\sqrt{d^2-b^2s^2}}\times\frac{d^2-b^2s^2}{d^2\times\overline{1-ss}}\right)=\frac{d\dot{s}}{\overline{d^2-b^2s^2}\times\overline{1-ss}}$; which resolved into two Parts, for the more readily finding the Fluent, gives $\dot{u} = \frac{d\dot{s}}{1 - s^2} - \frac{db^2\dot{s}}{d^2 - b^2s^2}$: Whereof the Fluent being taken, we have

$$u = \begin{cases} 2 \,.\, 302585 \text{ \&c.} \times \frac{1}{2}d \times \text{Log.} \frac{1+s}{1-s} \\ -2 \,.\, 302585 \text{ \&c.} \times \frac{1}{2}b \times \text{Log.} \frac{d+bs}{d-bs} \end{cases}$$

But, as 3,14159 &c. × 2d (the Meaſure of the whole Periphery of the Earth at the Equator, in Parts of the Semi-Axis AD) is to 21600 (the Meaſure of the ſame Periphery in Geographical Miles) ſo is the foreſaid Value of u to

$$\left\{ \begin{array}{l} 3958 \times \text{Log.} \frac{1+s}{1-s} \\ -\frac{3958b}{b} \times \text{Log.} \frac{d+bs}{d-bs} \end{array} \right\}$$ the correſponding Value of u, in Geographical Miles, or the Meridional Parts required.

COROLLARY.

466. If the Earth be conſidered as differing but little from a Sphere, d will be nearly $=1$, and conſequently $(\sqrt{d^2-1})$ the Value of b, very ſmall: Therefore, in this Caſe, the latter Part of our Fluent $\left(-\frac{3958b}{d} \times \text{Log.} \frac{d+bs}{d-bs}\right)$ will become nearly $= 3440b^2s$ (becauſe $\text{Log.} \frac{d+bs}{d-bs} = \frac{2bs}{d}$) $\times \frac{1}{2 \,.\, 3025 \text{ \&c.}}$ *. But if the Earth be taken as a perfect Sphere, this laſt Expreſſion will vaniſh, and ſo the Value of u will become barely $=3958$

* *There is a Miſtake in* p. 43. *and* 44. *of my Diſſertations (by forgetting to divide by the Modulus* 2.3025 &c.) *which may from hence be rectify'd.*

$\times$ Log. $\frac{1+s}{1-s}$. Which Logarithm, it is easy to prove, expresses twice the artificial Tangent of half the given Latitude increased by 45 Degrees (Radius being Unity.) Wherefore, if the Meridional Parts answering to any given Latitude, thus found (from a Table of logarithmic Tangents) when the Earth is considered as a perfect Sphere, be denoted by M, it follows that the Meridional Parts answering to the same Latitude, when the Earth is taken as a Spheroid, will be nearly equal to $M -$ $3440b^2s$: Which, because AD (1) : AB ($\sqrt{1+bb}$) ::
* Art. 397. 230 : 231 *, will (by substituting the Value of b hence arising) be reduced to $M-30s$. Whence the following Rule.

As Radius, to the Sine of the given Latitude, so is 30 to a Fourth-Proportional; which subtracted from the Meridional Parts when the Earth is taken as a Sphere (found as above) gives the Meridional Parts answering to the same Latitude, when it is considered as an oblate Spheroid.

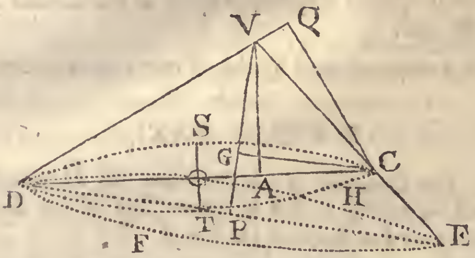
Thus, for Example, let the given Latitude be 50°: Then, first, for the Meridional Parts in the *Sphere*; we must, according to the foregoing Prescript, take the Logarithmic Tangent of 25°+45°, or 70°: Which, by the Table, is found = 0,43893 &c. This multiply'd by the constant Multiplicator 7916 (=2 × 3958) produces 3475 for the Meridional Parts in the Sphere: Then by the Rule above, it will be as Radius to the Sine of 50°, so is 30 to 23; which subtracted from 3475, leaves 3452 for the Meridional Parts answering to 50° Latitude, in the Spheroid.

PROB. XXX.

467. *To determine the Paths which Shadows of Objects describe, upon the Plane of the Horizon, during the Sun's apparent diurnal Revolution.*

Let CSODT be the Plane of the Horizon, and AV the perpendicular Height of the Object: Then, since the Rays, intercepted by the highest Point V, would, in the Sun's diurnal Revolution, form a conical Surface

face VDFEH about that Point as a Vertex; whoſe Axis PV produced paſſes thro' the Pole of the World; it is evident that the Path of the Shadow, being the Interſection of the Plane of the Horizon with that Surface, muſt be a Conic Section.



Let its two principal Diameters therefore (when an Ellipſis, that is, when the Sun never deſcends below the Horizon) be CD and ST; alſo let DPE and CG be perpendicular to VP the Axis of the Cone, and CQ perpendicular to DV: Putting the Sine of (QVC) twice the Sun's Declination VEP $=f$; the Sine of (DCV) his greater Meridional Altitude $= g$, and that of the leſſer (CDV) $= h$: Then (*by plane Trig.*) $g : 1$ (AV) :: 1 (Radius) : CV $= \frac{1}{g}$; and h (Sine of CDV : $\frac{1}{g}$ (CV) :: f (Sine of DVC) : DC $= \frac{f}{gh}$: Moreover, 1 (Radius) : $\frac{1}{g}$ (CV) :: p (the Sine of the Comp. Decl. GVC) : GC $= \frac{p}{g}$: And in the very ſame Manner it will be found that DP $= \frac{p}{h}$: But GC $\times$ DP $=$ OS^2 (*vid. Art.* 41.) whence we have ST (2OS) $= \frac{2p}{\sqrt{gh}}$: From which, and the Tranſverſe Axis (DC $= \frac{f}{gh}$) the Curve itſelf is given. *Q. E. I.*

 LEMMA.

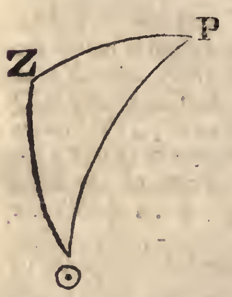
Lemma.

468. *In any fpherical Triangle, if Radius be fuppofed Unity, the Product of the Sines of any two of the Sides drawn into the Co-fine of the Angle they include, added to the Product of their Co-fines, is equal to the Co-fine of the remaining Side.*

This is demonftrated by the Writers upon Spherics.

PROB. XXXI.

469. *The Elevation of the Pole and the Declination of the Sun being given, to find at what Time of the Day the Azimuth of the Sun increafes the floweft.*



It is evident that the Time fought will be when the Fluxion of the Hour Angle P, bears the greateft Ratio poffible to That of the Azimuth Z.

Now the Fluxion of the Angle P is to that of Z, univerfally, as Rad. × *S.* ZO : *S.* PO × *Co-f.* O (*by Art.* 256. *Cafe* 2.) Confequently $\frac{S.\ PO \times Co\text{-}f.\ O}{Rad. \times S.\ ZO}$, or $\frac{Co\text{-}f.\ O}{S.\ ZO}$ is a *Minimum*, in this Cafe, becaufe PO may be confidered as conftant.

Let now the Sine of PO be put $= p$, its Co-fine $= d$, the Co-fine of PZ $= b$, that of ZO $= x$, and that of O $= y$; then, the Sine of ZO being $= \sqrt{1-x^2}$, we have (*by the Lemma*) $p\sqrt{1-x^2} \times y + dx = b$; whence $y = \frac{b-dx}{p\sqrt{1-x^2}}$ and therefore $\frac{Co\text{-}f.\ O}{S.\ ZO}$ $\left(= \frac{y}{\sqrt{1-x^2}}\right)$ $= \frac{h-dx}{p \times \overline{1-x^2}}$: Which put into Fluxions, and reduced, gives $x = \frac{b - \sqrt{b^2 - d^2}}{d}$, for the Sine of the Sun's Altitude at the Time required: Whence the Time itſelf is given.

PROB. XXXII.

470. *To determine the Ratio of the Heat received from the Sun in different Latitudes, during the Time of one whole Day, or any Part thereof.*

Let $p =$ the Sine of the Sun's Polar-Diſtance P☉ (*ſee the laſt Fig.*)
$d =$ its Co-ſine, or the Sine of the Declination.
$b =$ the Sine of the Pole's Elevation.
$c =$ its Co-ſine, or the Sine of PZ.
$z =$ the Angle (P) expreſſing the Time from Noon.
$x =$ its Sine, and $\sqrt{1 - x^2} =$ its Co-ſine.

Then (by the foregoing Lemma) we ſhall have $pc\sqrt{1 - x^2} + bd =$ Co-ſine Z☉ $=$ Sine of the Sun's Altitude.

Now, it is known that the Number of Rays falling in any given Particle of Time, upon a given horizontal Plane, is as that Time and the Sine of the Sun's Altitude conjunctly: Therefore the Number of Rays falling in the Time $\dot{z}$, or $\frac{\dot{x}}{\sqrt{1 - xx}}$ (*vid. Art.* 142.) will be defined by $pc\dot{x} + bd\dot{z}$: Whoſe Fluent $pcx + bdz$ is, therefore, as the Heat required.

Where it may be obſerved,

1. That when the Latitude and Declination are of different Kinds, or P☉ is greater than 90 Degrees, the Value of d is to be conſidered as a negative Quantity.

2. That, if the Expreſſion for the Heat found above be divided by the Square of the Sun's Diſtance from the Earth, the Quotient will exhibit the Ratio of the Heat, allowing for the Excentricity of the Earth's Orbit.

Corollary I.

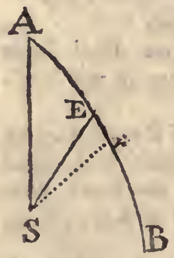
471. If the Place proposed be at the Equator, the Heat, received in half one diurnal Revolution, will be barely as p; because $b=0$, $c=1$, and $x=1$.

Corollary II.

472. But if the Place be at the Pole, then the Heat will be as $d \times \overline{3,14159}$ &c. since, in this Case, $c=0$, $b=1$, and z ($=$ Semi-Circle) $=3,14159$ &c.

Lemma.

473. *The Number of Particles of Light, ejected by the Sun, upon the Earth, in a given Time, is proportional to the Angle described about his Center in that Time.*



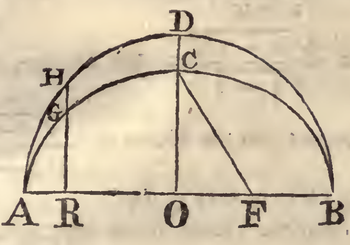
For, let S represent the Center of the Sun, AEB the Orbit of the Earth (or That of any other Planet) and let E and r be two Points therein as near as possible to each other: Since the Triangle ESr may be taken as rectilineal, its Area, if the Angle ESr be supposed given, or every where the same, will be as SE $\times$ Sr, or SE^2: And therefore the Time of describing Er (being always as that Area) is also explicable by SE^2: But the Intensity of the Light, or Heat, at the Distance of SE is as $\frac{1}{SE^2}$: Therefore the Intensity compounded with the Time (or the whole Number of Particles received in that Time) will consequently be as $\frac{1}{SE^2} \times SE^2$ ($=1$): Which being every where the same, the Proposition is manifest.

PROB. XXXIII.

474. *To determine the Ratio of the Heat received from the Sun at the Equator and either of the Poles, during the Time of one whole Year, or any Part thereof.*

If

If the Sine of the Sun's Declination be denoted by d and its Co-fine by p, the Heat received at the Equator, and the Pole, during half one diurnal Revolution of the Sun, will be as p and $d \times 3{,}14159$ &c. refpectively (*by the Corollaries to the preceding Problem*).



Let the Sun's Longitude, confidered as variable, be now denoted by z, and its Sine by s; and let f be put for the Sine of the Obliquity of the Ecliptic: Then (*per Spherics*) we fhall have $d = fs$, and confequently p $(= \sqrt{1 - d^2}) = \sqrt{1 - f^2s^2}$: Wherefore, feeing the Ratio of Heat in the two Places, for one Half-Day, is that of $\sqrt{1 - f^2s^2}$ to $fs \times 3{,}14$ &c. let each of thefe Terms be multiplied by $\frac{\dot{s}}{\sqrt{1 - ss}}$ $(= \dot{z})$ * expreffing the Quantity of Heat falling upon the Earth in the Time of defcribing $\dot{z}$ (*fee the foregoing Lemma*) then the Products $\frac{\dot{s}\sqrt{1 - f^2s^2}}{\sqrt{1 - s^2}}$, and $3.14f \times \frac{s\dot{s}}{\sqrt{1 - s^2}}$ will be the Fluxions of the required Heat, anfwering to $\dot{z}$.

* Art. 142.

But now to exhibit the Fluents hereof, let ACB be an Ellipfis whofe greater Semi-Axis AO is = Unity, and its Excentricity FO $= f$; and, fuppofing ADB to be a Circle defcribed about the Ellipfis, let the Arch DH exprefs the Sun's Longitude from the Equinoctial Point; whofe Sine (OR) being $= s$, its Co-fine RH will be $= \sqrt{1 - ss}$.

But, by the Property of the Ellipfis, OD (1) OC : $(\sqrt{1 - f^2})$ :: RH $(\sqrt{1 - ss})$: RG $= \sqrt{1 - ff} \times \sqrt{1 - ss}$: Whofe Fluxion being =

$\frac{\sqrt{1-ff}\times -\dot{ss}}{\sqrt{1-ss}}$, we have $\sqrt{\dot{s}^2+\frac{\overline{1-ff}\times s^2\dot{s}^2}{1-ss}}$ $=\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-ss}}=$ the Fluxion of CG. Whence it appears that the Fluent of $\frac{\dot{s}\sqrt{1-f^2s^2}}{\sqrt{1-ss}}$ is truly defined by CG; or CG $\times$ AO^2.

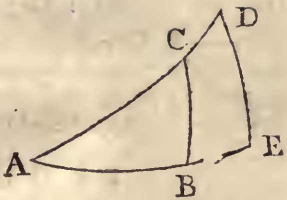
But the Fluent of the other given Fluxion, $3{,}14f\times\frac{\dot{ss}}{\sqrt{1-ss}}$, will be $=3{,}14f\times\overline{1-\sqrt{1-ss}}=$ ADB $\times$ FO $\times\overline{OD-RH}$. Therefore the two Fluents, when H and G coincide with A, will be to each other as CA$\times$AO to ADB$\times$FO: Whereof the Antecedent, multiplied by 4, will be as the Heat received at the Equator during one whole Year; and the Consequent, multiplied by 2, as the Heat at the Pole in the same Time (because the Sun shines at the Pole only two Quarters of the Year.) Hence the required Ratio, of the Heat received at the Equator and Pole, in one whole Year, will be That of CA $\times$ AO to DA$\times$FO; or, in Species, as $1-\frac{f^2}{2.2}-\frac{3f^4}{2.2.4.4}-\frac{3\cdot3\cdot5f^6}{2.2.4.4.6.6}$ * &c. to f; which, in Numbers, is as 959 to 396, or as 17 to 7, nearly.

* Art. 434.

PROB. XXXIV.

475. *To find when that Part of the Equation of Time, arising from the Obliquity of the Ecliptic to the Equinoctial, is a* Maximum.

In the right-angled spherical Triangle ABC let the Angle A be that made by the Ecliptic AC, and the Equinoctial AB; then the Problem will be, to find when



when the Difference between the Bafe AB and the Hypothenufe AC is the greateft poffible (the Angle A remaining invariable.) Now *(by Art.* 254.) we have *Co-f. BC* : *Sin. C* :: Fluxion of AC Fluxion of AB : Alfo *(per Spherics) Sin.* C : *Co-f.* A :: *Rad.* : *Cof.* BC $= \frac{Co\text{-}f.\ A \times Rad.}{Sin.\ C}$: Whence, by multiplying the two firft Terms of the former Proportion by thefe equal Quantities, refpectively, we get this new Proportion, *viz.* $\overline{Co\text{-}f.\ BC}|^2$: *Co-f.* A × *Radius* :: fo is the Fluxion of AC to That of AB. But, when AC—AB is a *Maximum*, thefe Fluxions become equal; and confequently $\overline{Co\text{-}f.\ BC}|^2 = Co\text{-}f.\ A \times Rad.$ From which Equation BC, and from thence AC, will be known. *Q.E.I.*

The fame, without Fluxions.

476. It will be *(per Spherics) Rad.* : *Co-f.* A :: *Tang.* AC : *Tang.* AB; and therefore by Compofition and Divifion, *Rad.* + *Co-f.* A : *Rad.* — *Co-f.* A :: *Tang.* AC + *Tang.* AB : *Tang.* AC — *Tang.* AB :: *Sin.* $\overline{AC+AB}$: *Sin.* $\overline{AC-AB}$, by the Theorem mentioned in Problem 8th : From which, by following the Steps there laid down, it appears that, *Radius* + *Co-f.* A : *Radius* — *Co-f.* A :: *Radius* : Sine of $\overline{AC-AB}$, when a *Maximum* : Whence (AC + AB being then = 90°) both AC and BC will be given.

COROLLARY.

477. Since, *Radius* + *Co-f.* A : *Radius* — *Co-f.* A :: *Co-tang.* ½ A : *Tang.* ½ A * :: $\overline{Radius}|^2$: $\overline{Tang.\ \frac{1}{2} A}|^2$;

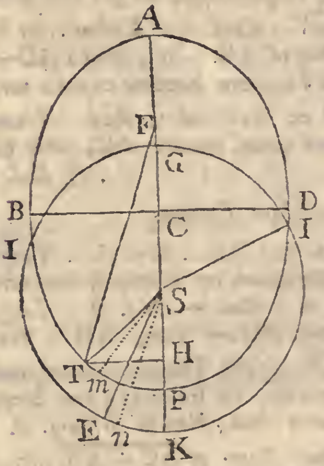
* *Vid. p.* 70. *and* 71. *of my Trigonometry.*

 There-

therefore $\overline{Radius}|^2 : \overline{Tang. \frac{1}{2} A}|^2 :: Radius : Sine$ of $\overline{AC - AB}$, Or, *Radius* : *Tang.* $\frac{1}{2}$ A :: *Tang.* $\frac{1}{2}$ A : the *Sine* of the greateſt Equation: Which, ſuppoſing the Angle A to be 23° 29′, comes out 2° : 28′ : 34″ : anſwering, in Time, to 9 Minutes : 54 Seconds.

PROB. XXXV.

478. *To determine when the abſolute Equation of Time, ariſing from the Inequality of the Sun's apparent Motion, and the Obliquity of the Ecliptic, conjunctly, is a* Maximum.

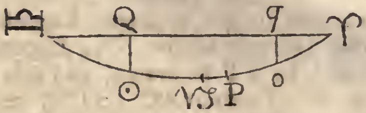


Let ABPD be the Ellipſis in which the Earth revolves about the Sun in the Focus S; let F be the other Focus, and T the Place of the Earth in its Orbit at the Time required. Moreover, about S, as a Center, let a Circle GEKI be deſcribed, whoſe Diameter GK is a Mean Proportional between the two Axes AP and BD of the Ellipſis; ſo that the Area thereof may be equal to That of the Ellipſis: And, ſuppoſing S*m* to be indefinitely near to ST, let ES*n* be a Sector of the ſaid Circle, equal to the Area TS*m*.

Then, the Time in which the Earth moves through the Arch T*m* being to the Time of one intire Revolution, as the Area TS*m*, or ES*n*, is to the whole Ellipſis, or the equal Circle GEKF; and theſe Areas ES*n*, and GEKI being in the Ratio of the Arch E*n* to the whole Periphery GEKI; it is evident that E*n*, or the Angle ESn, will expreſs the Increaſe of the *Mean Longitude*, in the foreſaid Time of deſcribing the Arch Tm: And that this Angle or Increaſe, by reaſon of the Equality of the Areas ESn and TSm, will be to the Angle TSm, expreſſing the correſponding Increaſe of the *True Longitude*, as ST^2 to SE^2. Therefore, if the former be denoted by M, the latter will be repreſented by $\frac{SE^2}{ST^2} \times M$. But now to get a proper Expreſſion for the Value of this Increaſe of the *True Longitude*, in Algebraic Terms; let FT be drawn, and alſo TH, perpendicular to AP: Putting AC ($=$CP) $=a$, CB$=b$, CS ($=$CF) $=c$, ST$=z$, and the Co-ſine of (TSP) the Earth's Diſtance from its *Perihelion* (to the Radius 1) $=x$: Then FT being ($=$AP $-$ ST) $=2a-z$ (by the Property of the Ellipſis) and SH$=xz$ (*by Trig.*) we have $\overline{FT+ST}\times\overline{FT-ST}$ ($2a\times\overline{2a-2z}$) $=$ FS $\times$ 2CH ($2c\times2\times\overline{c+xz}$) by a known Property of Triangles: From which Equation z (ST) is found $= \frac{a^2-c^2}{a+cx} = \frac{b^2}{a+cx}$: And this Value, with that of ES^2 ($=ab$) being ſubſtituted in the Increaſe of the True Longitude, found above, we thence get $\frac{a\times\overline{a+cx}^2}{b^3}\times M$ for the Meaſure of that Increaſe; where M denotes the Increment of the *Mean Motion* correſponding.

This being obtained, let ♎ ♑ ♈ (in the annexed Figure) repreſent the Southern Semi-Circle of the Ecliptic, P the Place of the Perihelion, ♑ the Tropic of Capricorn, ☉ the apparent Place of the Sun in the Ecliptic, and Q☉ his Declination, at the Time required: Then it appears, (*from Art.* 475.) that the Increaſe of the *True Longitude* ♎☉, in an indefinitely ſmall Particle of Time, will be to That of the *Right-Aſcenſion* ♎Q, in the ſame Time, as the Square of the Co-ſine of Q☉ is to a Rectangle under the Radius and the Co-ſine of the Angle ♎: Therefore, the former, being

being expressed by $\frac{a \times \overline{a+cx}|^2}{b^3} \times M$, the latter is truly represented by $\frac{a \times \overline{a+cx}|^2}{b^3} \times M \times \frac{Rad. \times Cos. ♎}{\overline{Co\text{-}s. ☉Q}|^2}$ Which, in the required Circumstance, when the proposed Equation (or the Difference between the Sun's



Mean Motion and *Right Ascension*) is a *Maximum*, must consequently be equal to (M) the corresponding Increase of *Mean Motion*; and therefore $\frac{a \times \overline{a+cx}|^2}{b^3}$

$= \frac{\overline{Co\text{-}s. ☉Q}|^2}{Rad. \times Co\text{-}s. ♎}$.

But, to obtain the Value of the latter Part of this Equation, also, in Algebraic Terms, let the Sine and Co-sine of (♑ P) the Distance of the *Perihelion* from ♑, be denoted by m and n respectively; then, the Co-sine of P ☉ being (as above) expressed by x, and its Sine by $\sqrt{1 - xx}$, we shall thence get $nx + m\sqrt{1 - xx}$ = Co-sine of ☉ ♑ = Sine of ♎ ☉ (*by the Elem. of Trig.*) But (putting the Sine of the Angle ♎ $= p$ and its Co-sine $= q$) we have (*per Spherics*) Radius (1) : Sine ♎☉ ($nx + m\sqrt{1 - xx}$ :: p : $pnx + pm\sqrt{1 - xx}$ = Sine of Q☉; from whence $\overline{Co\text{-}s. Q☉}|^2$ $= 1 - \overline{pnx + pm\sqrt{1 - xx}}|^2$: Which Value, with That of the Co-sine of the Angle ♎, being substituted above, we, at length, get $\frac{a \times \overline{a+cx}|^2}{b^3} =$

$\frac{\overline{1 - pnx + pm\sqrt{1 - xx}}|^2}{q}$; from which Equation the Value of x may be determined.

The foregoing Equation, it may be obſerved, gives the Time of the *Maximum* which precedes the Winter Solſtice; but if the *Maximum* following that Solſtice be ſought; it is but changing the Sign of m, and then you will have $\frac{a \times \overline{a + cx}|^2}{b^3} = \frac{\overline{1 - pnx - pm\sqrt{1 - xx}}|^2}{q}$, anſwering in this Caſe. And from the negative Roots of this, and the preceding, Equation, the Times of the other *Maxima* after, and before, the Summer Solſtice will alſo be obtained. *Q. E. I.*

Corollary.

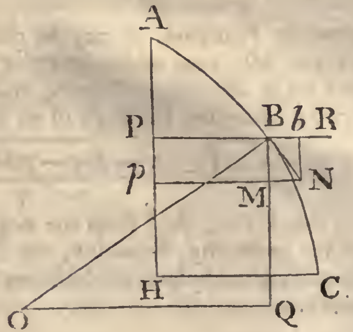
479. It is evident that the Equation of the Earth's Orbit (or that Part of the Equation of Time ariſing from the Inequality of the Sun's apparent Motion) will be a *Maximum*, when the Center of the Earth is in the Interſection I of the Ellipſis and the Circle; where the *Mean Motion* and *True Longitude* increaſe with the ſame Celerity.

PROB. XXXVI.

480. *To determine the Law of the Denſity of a Medium and the Curve deſcribed therein, by Means of an uniform Gravity, ſo that the Projectile may, every where, move with the ſame Velocity.*

It appears, from Art. 367. that $\sqrt{\frac{\dot{y}\dot{y}}{\ddot{x}}}$ is a general Expreſſion for the Celerity in the Direction of the Ordinate PBR; whence $\frac{\dot{z}}{\dot{y}} \times \sqrt{\frac{\dot{y}\dot{y}}{\ddot{x}}}$, or its Equal, $\frac{\dot{z}}{\sqrt{\ddot{x}}}$, muſt be the true Meaſure of the abſolute Celerity,

lerity, in the Direction BN: Which being a conſtant Quantity (by Hypotheſis) its Square muſt alſo be con-



ſtant; and ſo, we have $\frac{\dot{z}\dot{z}}{\ddot{x}} = a$, and conſequently $\dot{x}\dot{x} + \dot{y}\dot{y}\ (= \dot{z}\dot{z}) = a\ddot{x}$.

But, in order to the Solution of the Equation thus given, make $u : 1 :: \dot{x} : \dot{y}$, or $\dot{x} = u\dot{y}$; then $\ddot{x} = \dot{u}\dot{y}$, and, by Subſtitution, $u^2\dot{y}^2 + \dot{y}^2 = a\dot{u}\dot{y}$: Hence, $\dot{y}$ being $= \frac{a\dot{u}}{uu + 1}$, and $\dot{x} = \frac{au\dot{u}}{uu + 1}$, we get $y = a \times$ *Arch*, whoſe
* Art. 142. Tangent is u * (and Secant $\sqrt{1 + uu}$); and $x = \frac{1}{2} a \times$
† Art. 126. Hyp. Log. $\overline{1 + uu} = a \times$ Hyp. Log. $\sqrt{1 + uu}$ †.
Therefore, as the Hyp. Log. of $\sqrt{1 + uu}$ is $= \frac{x}{a}$, the Common Logarithm of $\sqrt{1 + uu}$ will be $= \frac{0{,}4342944 \text{ \&c.} \times x}{a}$; and conſequently $y = a \times$ *Arch*, whoſe Radius is Unity, and Log. Secant $\overline{\frac{0.4342944 \text{ \&c.} \times x}{a}}$.
Moreover, with reſpect to the Denſity of the Medium; if the abſolute Force of Gravity, in the Direction QB, be

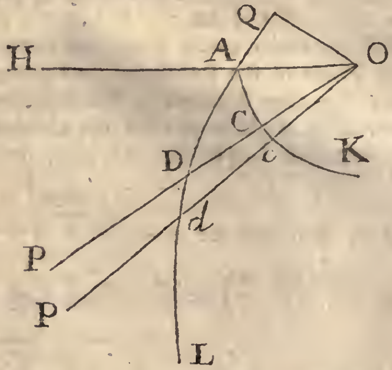
be denoted by Unity, its Efficacy in the Direction BN, whereby the Body is accelerated, will be expressed by $\frac{\dot{x}}{\dot{z}}$, or its Equal $\frac{u}{\sqrt{1+uu}}$: Which, as the Velocity is supposed to remain every where the same, must also express the Force of the Resistance, in the opposite Direction, or the true Measure of the required Density. This, therefore, if M be put for the absolute Number whose Hyperbolical Logarithm is Unity, may be had in Terms of x, and will be $\left|1-\overline{M}\right|^{\frac{-2x}{a}}\Big|^{\frac{1}{2}}$: Because Hyp. Log. $\overline{M}|^{\frac{x}{a}}$ $\left(=\frac{x}{a}\right)$ being $=$ Hyp. Log. $\sqrt{1+uu}$, we have $\sqrt{1+uu}=\overline{M}|^{\frac{x}{a}}$; whence $u=\overline{M^{\frac{2x}{a}}-1}|^{\frac{1}{2}}$, and consequently $\frac{u}{\sqrt{1+uu}}=\overline{1-\overline{M}|^{\frac{-2x}{a}}}|^{\frac{1}{2}}$. *Q. E. I.*

PROB. XXXVII.

481. *Let a Line, or an inflexible Rod* OP *(considered without regard to Thickness) be supposed to revolve about one of its Extremes* O, *as a Center, with a Motion regulated according to any given Law; whilst a Ring, or Ball, carried about with it, and tending to the Center* O *with any given Force, is suffered to move or slide freely along the said Line or Rod: It is proposed to determine the Velocity of the Ring, and its Pressure upon the Rod, in any proposed Position, together with the Nature of the Curve* ADL *described by means of that compound Motion.*

Le ODP be any Position of the revolving Line, and D the corresponding Position of the Body: Moreover, supposing ACK to be the Circumference of a Circle

Circle described from the Center O, through the given Point A, let the Measure of the angular Celerity of that Line, in the said Circumference ACK, be repre-



sented by u; also let v denote the Celerity of the Ring at D in the Direction DP; and w the true Measure of the centripetal Force: Call OA, a; OD, x; and AC, z; and let the given Values of u and v, at A, be denoted by b and c respectively. Then it will be, as $a : x :: u : \left(\frac{ux}{a}\right)$ the paracentric Velocity of the Body, at D; whose Square, divided by the Distance OD, gives

* Art. 211. $\frac{u^2x}{a^2}$, for the true Measure of the Centrifugal Force * arising from the Revolution of the Rod: From which the centripetal Force w being deducted, the Remainder, $\frac{xu^2}{a^2} - w$, is the true Force whereby the Velocity in the Line OP is accelerated. Therefore (*by Art.* 218.) we have $v\dot{v} = \overline{\frac{xu^2}{a^2} - w} \times \dot{x} = \frac{u^2x\dot{x}}{a^2} - w\dot{x}$.

Moreover, because the Fluxion of the Time is expressed either by $\frac{\dot{x}}{v}$ or by $\frac{\dot{z}}{u}$, these two Values must, therefore,

therefore, be equal to each other, and consequently $v = \frac{u\dot{x}}{\dot{z}}$: From which, and the preceding Equation (when u and w are exhibited in Terms of x or z) the required Relation of v, x and z will also become known— But now, in order to determine the Action of the Rod upon the Ring; let O*d*P be indefinitely near to ODP, intersecting ADL and ACK in *d* and *c*; and put O*d* = $x + \acute{x}$. Then, because a Body, acted on by no other Force besides That tending to the Center, about which it revolves, describes Areas proportional to the Times *, and the angular Celerity of a Ray revolving with the Body, is, in that Case, as the Square of the Distance of the Body from the Center, inversely (*vid. Art.* 478.) it follows, that, if the Rod was to cease to act upon the Ring, at the Position ODP, the angular Celerity at *c*, would then be $\frac{x^2}{(x+\acute{x})^2} \times u$, instead of $u + \acute{u}$. Therefore the Excess of $u + \acute{u}$ above $\frac{x^2}{(x+\acute{x})^2} \times u$, which is $= \acute{u} + \frac{2u\acute{x}}{x} - \frac{3u\acute{x}^2}{x^2}$ &c. is the Increase of the said angular Celerity, at the Distance OC, arising from the Action of the Rod. Therefore it will be, as OC (a) : OD (x) :: the said Increase to $\left(\frac{xu}{a} + \frac{2u\acute{x}}{a} - \frac{3u\acute{x}^2}{ax} \text{ \&c.}\right)$ the Alteration of the Ring's paracentric Velocity, arising from the same Cause. Which, divided by $\left(\frac{\acute{x}}{v}\right)$ the Time wherein *It* is produced, gives $\frac{xv\acute{u}}{a\acute{x}} + \frac{2uv}{a} -$

* Art. 224.

$\frac{3uvx'}{a\dot{x}}$ *&c.* for the Measure of the Force, by which *It* is produced. From whence, by substituting $\frac{\dot{u}}{\dot{x}}$ in the Room of $\frac{u}{x'}$, and neglecting all the Terms after the two

* Art. 134. first (in order to have the limiting Ratio *) we get $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$. Therefore it will be, as $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a}$ to

† Art. 211. $\frac{bb}{a}$ †, or as $\frac{xv\dot{u}}{bb\dot{x}} + \frac{2uv}{bb}$ to Unity, so is the Action of the Rod upon the Ring, to the (given) Centrifugal Force at A (or the Force that would retain a Body in the Circle ACK, with the Velocity *b*.) *Q. E. I.*

Corollary I.

482. If the angular Motion be uniform, the Equations found above, will become $v\dot{v} = \frac{b^2x\dot{x}}{a^2} - w\dot{x}$, and $v = \frac{b\dot{x}}{\dot{z}}$. From the latter of which, by taking the Fluxion, we have $\dot{v} = \frac{b\ddot{x}}{\dot{z}}$; whence (by Substitution) $\frac{b^2\dot{x}\ddot{x}}{\dot{z}\dot{z}} = \frac{b^2x\dot{x}}{aa} - w\dot{x}$, and consequently $\ddot{x} - \frac{x\dot{z}^2}{a^2} = -\frac{w\dot{z}^2}{b^2}$; from the Solution of which, the Relation of x and z will be given. And then, the Value of v $\left(\frac{b\dot{x}}{\dot{z}}\right)$ being also known, the Action upon the Rod, which in this Case is barely $= \frac{2bv}{a}\left(= \frac{2b^2\dot{x}}{a\dot{z}}\right)$ will be given likewise, being

being to $\left(\frac{bb}{a}\right)$ the centrifugal Force in the Circle ACK, as $\frac{2\dot{x}}{\dot{z}}$ to Unity.

Corollary II.

483. But if the Angular Celerity be proportional to any Power (x^m) of the Diſtance, and the Centripetal Force w be, alſo, ſuppoſed to vary according to ſome Power (x^n) of the ſame Diſtance: Then, putting p to denote the Centripetal, and q the Centrifugal, Force, at the given Point A, the Value of w will, here, be expounded by $\frac{x^n}{a^n} \times p$, and That of u by $\frac{x^m}{a^m} \times b$: And therefore, the paracentric Velocity of the Ring at D being $= \frac{x^m}{a^m} \times b \times \frac{x}{a} \left(= \frac{bx^{m+1}}{a^{m+1}}\right)$ it will be as $\frac{bb}{a} : \frac{b^2 x^{2m+2}}{xa^{2m+2}} :: q : \frac{x^{2m+1}}{a^{2m+1}} \times q$, the Centrifugal Force at D *. Hence $v\dot{v} = \frac{qx^{2m+1}\dot{x}}{a^{2m+1}} - \frac{px^n\dot{x}}{a^n}$; whereof the (corrected) Fluent is $\frac{1}{2}vv - \frac{1}{2}cc = \frac{qx^{2m+2}}{2m+2 \times a^{2m+1}} - \frac{px^{n+1}}{n+1 \times a^n} - \frac{qa}{2m+2} + \frac{pa}{n+1}$: From whence v is found $=$

$$\sqrt{cc - \frac{qa}{m+1} + \frac{2pa}{n+1} + \frac{qx^{2m+2}}{m+1 \,.\, a^{2m+1}} - \frac{2px^{n+1}}{n+1 \,.\, a^n}}$$

and $\dot{z}\left(= \frac{u\dot{x}}{v} = \frac{bx^m\dot{x}}{a^m v}\right) =$

$$\frac{bx^m\dot{x}}{a^m\sqrt{cc - \frac{qa}{m+1} + \frac{2pa}{n+1} + \frac{qx^{2m+2}}{m+1 \,.\, a^{2m+1}} - \frac{2px^{n+1}}{n+1 \,.\, a^n}}}$$

* Art. 211.

Moreover, by substituting for u, and its Fluxion, we get $\frac{xv\dot{u}}{a\dot{x}} + \frac{2uv}{a} = \overline{m+2} \times \frac{bx^m v}{a^{m+1}}$, expressing the Action of the Rod upon the Ring: Which, therefore, when m is expounded by -2, will intirely vanish: And, in that Case, $\dot{z}$ will become =

$$\frac{a^2 b\dot{x}}{x\sqrt{cc + qa + \frac{2pa}{n+1} \times x^2 - qa^3 - \frac{2px^{n+3}}{\overline{n+1} \cdot a^n}}};$$

expressing the Nature of the Trajectory described by means of a Centripetal Force, varying according to any Power (x^n) of the Distance. But this Equation will be rendered somewhat more commodious, by substituting the Values of b and c: For, if OQ (perpendicular to the Tangent at A) be denoted by h, it will be, h : $\sqrt{a^2 - h^2}$ (AQ) :: b (the Celerity in the Direction AC) to $c = \frac{b\sqrt{a^2 - h^2}}{h}$ = the Celerity in the Direction AH *. Therefore, b being $= \sqrt{aq}$ †, we have $c^2 = \frac{a^3 q}{hh} - aq$;

* Art. 35.

† Art. 211.

and $\dot{z} = \dfrac{a^2\dot{x}}{x\sqrt{\frac{aa}{hh} + \frac{2p}{\overline{n+1} \cdot q} \times x^2 - a^2 - \frac{2px^{n+3}}{\overline{n+1} \cdot qa^{n+1}}}}$:

Which Equation is the same, in effect, with that given in *Art.* 242. by a different Method.

Corollary III.

484. If the Angular Celerity be supposed uniform, and the Ring to have no other Motion along the Rod than what it acquires from its Centrifugal Force; then c, m and p being all of them equal to Nothing, $\dot{z}$ will here become, barely $= \frac{b\dot{x}}{\sqrt{-qa + \frac{qx^2}{a}}} = \frac{a\dot{x}}{\sqrt{x^2 - a^2}}$: And there-

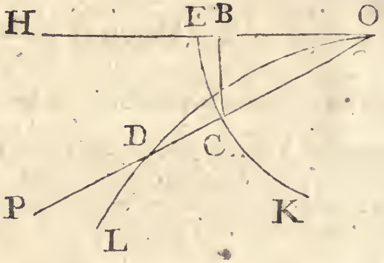
therefore $z = a \times$ Hyp. Log. $\frac{x + \sqrt{xx - aa}}{a}$. Hence if the Number whose Hyp. Log. is $\frac{z}{a}$ be denoted by N, we shall have $\frac{x + \sqrt{xx - aa}}{a} = N$: From which x is found $= a \times \overline{\frac{N}{2} + \frac{1}{2N}}$; whence $\dot{x}$ is, also, had $= \frac{a\dot{N}}{2} - \frac{a\dot{N}}{2N^2} = \frac{N\dot{z}}{2} - \frac{\dot{z}}{2N}$ (because $\frac{\dot{N}}{N} = \frac{\dot{z}}{a}$). Therefore, it will be (*by Corol.* 1.) as Unity is to $\frac{N}{2} - \frac{1}{2N}$, so is the Angular Velocity (b) in the Arch ACK to the Velocity with which the Body recedes from the Center of Motion: And so, likewise, is the Centrifugal Force in that Arch to half the Pressure upon the Rod—By taking $z =$ the whole Periphery, or $\frac{z}{a} = 2 \times \overline{3.145}$ &c. N will come out $= 535.5$, and $x = 267.7 \times a$: From whence it appears that the Distance of the Ring from the Center at the End of one intire Revolution will be almost 268 times as great as at first.

Corollary IV.

485. If a Body be suppos'd to descend from the Point O, (*see the next Fig.*) by the Force of its own Gravity, along an inclin'd Plane OCP; whilst the Plane itself moves uniformly about that Point, from an horizontal Position OEH; then the Place, and the Pressure of the Body upon the Plane, in any given Position OCP, may, also, be derived from the Equations in *Corollary* 1. For let CB (perpendicular to OH) be put $= y$; and let the Ratio of the Centrifugal Force in the Circle ECK, to the Force of Gravity (*given by Art.* 217.) be as r to *Unity*: Then, as the Measure of the former Force is expressed by $\frac{bb}{a}$,

 That

That of the latter must be represented by $\frac{bb}{ra}$; and, consequently, its Efficacy in the Direction PO, by $\frac{bby}{raa}\left(=\frac{bb}{ra}\times\frac{CB}{OC}\right)$: Which Value being substituted for $-w$, in the aforesaid Corollary, we have $\ddot{x} - \frac{x\dot{z}^2}{aa} = \frac{y\dot{z}^2}{raa}$. But now, in order to the Solution of this



Equation, put the Radius OC (a) $= 1$ (that the Operation may be as simple as possible) also, instead of y, let its Equal $z - \frac{z^3}{2 \cdot 3} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5}$ * &c. be substituted, and let x be assumed $= Az^3 + Bz^5 + Cz^7 + Dz^9$ &c.

* Art. 425.

Then, by proceeding as is taught in *Art.* 267. the Value of x will come out $= \frac{1}{r}$ into $\frac{z^3}{2 \cdot 3} + \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \frac{z^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}$ + &c. Whence the Velocity $\left(\frac{b\dot{x}}{\dot{z}}\right)$ in the Plane, is, also, found $= \frac{b}{r}$ into $\frac{z^2}{2} + \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$ &c. Which, therefore, is

to

to (*b*) the angular Velocity of the Plane, in the Arch ECK, as $\frac{z^2}{z} + \frac{z^6}{2.3.4.5.6} + \&c.$ to *r*. Moreover, the Centrifugal Force in the said Arch being denoted by *r* (the Force of Gravity being Unity) it will likewise be (by the above-mentioned *Corol.*) as $1 : \frac{2\dot{x}}{\dot{z}} :: r : \left(\frac{2r\dot{x}}{\dot{z}} = \right)$ $z^2 + \frac{z^6}{3.4.5.6} + \frac{z^{10}}{3.4.5.6.7.8.9.10} + \&c. =$ the Force sufficient to keep the Body upon the Plane. But the Force of Gravity in a Direction perpendicular to the Plane (the Weight of the Body being represented by *Unity*) is $\frac{OB}{OC} = 1 - \frac{z^2}{2} + \frac{z^4}{2.3.4}$ * &c. From which deducting the Quantity last found, there rests $1 - \frac{3z^2}{2} + \frac{z^4}{2.3.4} - \frac{3z^6}{2.3.4.5.6}$ &c. for the true Pressure of the Body upon the Plane. By putting of Which equal to Nothing, z^2 will be found $= 0.67715$; answering to an Angle (EOC) of $47° : 9'$: Which Angle is therefore the Inclination, when the Force of Gravity is no longer sufficient to keep the Body upon the Plane. * Art. 425.

Though the Value of *x*, given above, is found by an Infinite Series, yet the Sum of that Series is easily exhibited by the Measures of Angles and Ratios. For, putting *N* to denote the Number whose hyperbolical Logarithm is *z*,

we have †
$$1 + z + \frac{z^2}{2} + \frac{z^3}{2.3} + \frac{z^4}{2.3.4} \&c. = N$$
$$1 - z + \frac{z^2}{2} - \frac{z^3}{2.3} + \frac{z^4}{2.3.4} \&c. = \frac{1}{N}:$$
† Art. 424.

Half the Difference of which two Equations is $z + \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} + \frac{z^7}{2.3.4.5.6.7} \&c. = \frac{N}{2} - \frac{1}{2N}$:

 From

From which taking $z - \frac{z^3}{2\cdot3} + \frac{z^5}{2\cdot3\cdot4\cdot5} - \frac{z^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}$ *&c.* $= y$; and dividing the remainder by $2r$, there results $\left(\overline{\frac{1}{r}\times\frac{z^3}{2\cdot3}+\frac{z^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7}\text{ \&c.}}\right)\frac{1}{2r}\times\overline{\frac{N}{2}-\frac{1}{2N}-y}$, for the true Value of x. Which, if required, may be expressed independent of r; by putting d for the Distance through which a Body, freely, descends in the first Second of Time, and taking b to denote the Velocity of the Plane, *(per Second)* in the Arch *ECK*: For then, the Ratio of the Centrifugal Force, in the said Arch, to the Force of Gravity (or That of r to 1) being as $\frac{bb}{1}\left(=\frac{bb}{OC}\right)$ to $2d$ *, we shall have $r=\frac{bb}{2d}$, and consequently $x=\frac{d}{bb}\times\overline{\frac{N}{2}-\frac{1}{2N}-y}$.

* Art. 211.

By Computations, not very unlike Those above, the Motion of the Moon's *Apogee*, and the principal Equations of the Lunar Orbit may be exhibited; by means of proper Approximations, derived from the general Equations in *Art.* 481. But this is a Consideration that would require a Volume of itself, to treat it, from first Principles, with all the Attention and Perspicuity suitable to the Importance of the Subject. I shall conclude this Work with the following short Table of *Hyperbolical Logarithms*, drawn up and communicated by my ingenious Friend Mr. *John Turner*: Whereof the Use, in finding Fluents, will sufficiently appear from the foregoing Pages. In the said Table we have given the Hyperbolical Logarithms of every whole Number and hundredth Part of an Unit, from 1 to 10 (which Form is best adapted to the Purposes above-mentioned) by Help whereof, and the following Observations the Hyperbolical Logarithm of any Number, not exceeding ſeven Places of Figures, may be found with very little Trouble.

1°. *If the Number given be between* 1 *and* 10 *(ſo as to fall within the Limits of the Table.)*

Then take from it the next inferior Number in the Table, and divide the Remainder by the ſaid inferior Number increaſed by half the Remainder; and let the Quotient be added to the Logarithm of the ſaid inferior Number, the Sum will be the Logarithm ſought.

Thus, let the Hyperbolical Logarithm of 3.45678 be required; then the Operation will ſtand thus: 3.45339).005678(.0016442: Which added to 1.2383742, the Log. of 3.45, gives 1.2400184 for the Logarithm ſought.

2°. *When the Number propoſed exceeds* 10.

Find the Logarithm thereof, ſuppoſing all the Figures after the Firſt to be Decimals; then to the Logarithm, ſo found, let 2.3025851, 4.6051702, or 6.9077553 &c. be added, according as the whole Number conſiſts of 2, 3, or 4 &c. Places: The Sum will be the Logarithm ſought.

Thus, the Hyperbolical Logarithm of 345.678 will be found to be 5.8451886: For That of 3.45678 being 1.2400184; the ſame, added to 4.6051702, gives the very Quantity above exhibited. The Reaſon of which, as well as of the Operation in the preceding Caſe, is evident from the Nature and Conſtruction of Logarithms.

N	Logarithm	N	Logarithm	N	Logarithm
1.01	.0099503	1.34	.2926696	1.67	.5128236
1.02	.0198026	1.35	.3001045	1.68	.5187937
1.03	.0295588	1.36	.3074846	1.69	.5247285
1.04	.0392207	1.37	.3148107	1.70	.5306282
1.05	.0487902	1.38	.3220834	1.71	.5364933
1.06	.0582689	1.39	.3293037	1.72	.5423242
1.07	.0676586	1.40	.3364722	1.73	.5481214
1.08	.0769610	1.41	.3435897	1.74	.5538851
1.09	.0861777	1.42	.3506568	1.75	.5596157
1.10	.0953102	1.43	.3576744	1.76	.5653138
1.11	.1043600	1.44	.3646431	1.77	.5709795
1.12	.1133287	1.45	.3715635	1.78	.5766133
1.13	.1222176	1.46	.3784364	1.79	.5822156
1.14	.1310283	1.47	.3852624	1.80	.5877866
1.15	.1397619	1.48	.3920420	1.81	.5933268
1.16	.1484200	1.49	.3987761	1.82	.5988365
1.17	.1570037	1.50	.4054651	1.83	.6043159
1.18	.1655144	1.51	.4121096	1.84	.6097655
1.19	.1739533	1.52	.4187103	1.85	.6151856
1.20	.1823215	1.53	.4252677	1.86	.6205764
1.21	.1906203	1.54	.4317824	1.87	.6259384
1.22	.1988508	1.55	.4382549	1.88	.6312717
1.23	.2070141	1.56	.4446858	1.89	.6365768
1.24	.2151113	1.57	.4510756	1.90	.6418538
1.25	.2231435	1.58	.4574248	1.91	.6471032
1.26	.2311117	1.59	.4637340	1.92	.6523251
1.27	.2390169	1.60	.4700036	1.93	.6575200
1.28	.2468600	1.61	.4762341	1.94	.6626879
1.29	.2546422	1.62	.4824261	1.95	.6678293
1.30	.2623642	1.63	.4885800	1.96	.6729444
1.31	.2700271	1.64	.4946962	1.97	.6780335
1.32	.2776317	1.65	.5007752	1.98	.6830968
1.33	.2851789	1.66	.5068175	1.99	.6881346
1.34	.2926696	1.67	.5128236	2.00	.6931472

N	Logarithm	N	Logarithm	N	Logarithm
2.01	.6981347	2.34	.8501509	2.67	.9820784
2.02	.7030974	2.35	.8544153	2.68	.9858167
2.03	.7080357	2.36	.8586616	2.69	.9895411
2.04	.7129497	2.37	.8628899	2.70	.9932517
2.05	.7178397	2.38	.8671004	2.71	.9969486
2.06	.7227059	2.39	.8712933	2.72	1.0006318
2.07	.7275485	2.40	.8754687	2.73	1.0043015
2.08	.7323678	2.41	.8796267	2.74	1.0079579
2.09	.7371640	2.42	.8837675	2.75	1.0116008
2.10	.7419373	2.43	.8878912	2.76	1.0152306
2.11	.7466879	2.44	.8919980	2.77	1.0188473
2.12	.7514160	2.45	.8960880	2.78	1.0224509
2.13	.7561219	2.46	.9001613	2.79	1.0260415
2.14	.7608058	2.47	.9042181	2.80	1.0296194
2.15	.7654678	2.48	.9082585	2.81	1.0331844
2.16	.7701082	2.49	.9122826	2.82	1.0367368
2.17	.7747271	2.50	.9162907	2.83	1.0402766
2.18	.7793248	2.51	.9202827	2.84	1.0438040
2.19	.7839015	2.52	.9242589	2.85	1.0473189
2.20	.7884573	2.53	.9282193	2.86	1.0508216
2.21	.7929925	2.54	.9321640	2.87	1.0543120
2.22	.7275071	2.55	.9360933	2.88	1.0577902
2.23	.8020015	2.56	.9400072	2.89	1.0612564
2.24	.8064758	2.57	.9439058	2.90	1.0647107
2.25	.8109302	2.58	.9477893	2.91	1.0681530
2.26	.8153648	2.59	.9516578	2.92	1.0715836
2.27	.8197798	2.60	.9555114	2.93	1.0750024
2.28	.8241754	2.61	.9593502	2.94	1.0784095
2.29	.8285518	2.62	.9631743	2.95	1.0818051
2.30	.8329091	2.63	.9669838	2.96	1.0851892
2.31	.8372475	2.64	.9707789	2.97	1.0885619
2.32	.8415671	2.65	.9745596	2.98	1.0919233
2.33	.8458682	2.66	.9783261	2.99	1.0952733
2.34	.8501509	2.67	.9820784	3.00	1.0986123

N	Logarithm	N	Logarithm	N	Logarithm
3.01	1.1019400	3.34	1.2059707	3.67	1.3001916
3.02	1.1052568	3.35	1.2089603	3.68	1.3029127
3.03	1.1085626	3.36	1.2119409	3.69	1.3056264
3.04	1.1118575	3.37	1.2149127	3.70	1.3083328
3.05	1.1151415	3.38	1.2178757	3.71	1.3110318
3.06	1.1184149	3.39	1.2208299	3.72	1.3137236
3.07	1.1216775	3.40	1.2237754	3.73	1.3164082
3.08	1.1249295	3.41	1.2267122	3.74	1.3190856
3.09	1.1281710	3.42	1.2296405	3.75	1.3217558
3.10	1.1314021	3.43	1.2325605	3.76	1.3244189
3.11	1.1346227	3.44	1.2354714	3.77	1.3270749
3.12	1.1378330	3.45	1.2383742	3.78	1.3297240
3.13	1.1410330	3.46	1.2412685	3.79	1.3323660
3.14	1.1442227	3.47	1.2441545	3.80	1.3350010
3.15	1.1474024	3.48	1.2470322	3.81	1.3376291
3.16	1.1505720	3.49	1.2499017	3.82	1.3402504
3.17	1.1537315	3.50	1.2527629	3.83	1.3428648
3.18	1.1568811	3.51	1.2556160	3.84	1.3454723
3.19	1.1600209	3.52	1.2584609	3.85	1.3480731
3.20	1.1631508	3.53	1.2612978	3.86	1.3506671
3.21	1.1662709	3.54	1.2641266	3.87	1.3532544
3.22	1.1693813	3.55	1.2669475	3.88	1.3558351
3.23	1.1724821	3.56	1.2697605	3.89	1.3584091
3.24	1.1755733	3.57	1.2725655	3.90	1.3609765
3.25	1.1786549	3.58	1.2753627	3.91	1.3635373
3.26	1.1817271	3.59	1.2781521	3.92	1.3660916
3.27	1.1847899	3.60	1.2809338	3.93	1.3686394
3.28	1.1878434	3.61	1.2837077	3.94	1.3711807
3.29	1.1908875	3.62	1.2864740	3.95	1.3737156
3.30	1.1939224	3.63	1.2892326	3.96	1.3762440
3.31	1.1969481	3.64	1.2919836	3.97	1.3787661
3.32	1.1999647	3.65	1.2947271	3.98	1.3812818
3.33	1.2029722	3.66	1.2974631	3.99	1.3837912
3.34	1.2059707	3.67	1.3001916	4.00	1.3862943

N	Logarithm	N	Logarithm	N	Logarithm
4.01	1.3887912	4.34	1.4678743	4.67	1.5411590
4.02	1.3912818	4.35	1.4701758	4.68	1.5432981
4.03	1.3937663	4.36	1.4724720	4.69	1.5454325
4.04	1.3962446	4.37	1.4747630	4.70	1.5475625
4.05	1.3987168	4.38	1.4770487	4.71	1.5496879
4.06	1.4011829	4.39	2.4793292	4.72	1.5518087
4.07	1.4036429	4.40	1.4816045	4.73	1.5539252
4.08	1.4060969	4.41	1.4838746	4.74	1.5560371
4.09	1.4085449	4.42	1.4861396	4.75	1.5581446
4.10	1.4109869	4.43	1.4883995	4.76	1.5602476
4.11	1.4134230	4.44	1.4906543	4.77	1.5623462
4.12	1.4158531	4.45	1.4929040	4.78	1.5644405
4.13	1.4182774	4.46	1.4951487	4.79	1.5665304
4.14	1.4206957	4.47	1.4973883	4.80	1.5686159
4.15	1.4231083	4.48	1.4996230	4.81	1.5706971
4.16	1.4255150	4.49	1.5018527	4.82	1.5727739
4.17	1.4279160	4.50	1.5040774	4.83	1.5748464
4.18	1.4303112	4.51	1.5062971	4.84	1.5769147
4.19	1.4327007	4.52	1.5085119	4.85	1.5789787
4.20	1.4350845	4.53	1.5107219	4.86	1.5810384
4.21	1.4374626	4.54	1.5129269	4.87	1.5830939
4.22	1.4398351	4.55	1.5151272	4.88	1.5851452
4.23	1.4422020	4.56	1.5173226	4.89	1.5871923
4.24	1.4445632	4.57	1.5195132	4.90	1.5892352
4.25	1.4469189	4.58	1.5216990	4.91	1.5912739
4.26	1.4492691	4.59	1.5238800	4.92	1.5933085
4.27	1.4516138	4.60	1.5260563	4.93	1.5953389
4.28	1.4539530	4.61	1.5282278	4.94	1.5973653
4.29	1.4562867	4.62	1.5303947	4.95	1.5993875
4.30	1.4586149	4.63	1.5325568	4.96	1.6014057
4.31	1.4609379	4.64	1.5347143	4.97	1.6034198
4.32	1.4632553	4.65	1.5368672	4.98	1.6054298
4.33	1.4655675	4.66	1.5390154	4.99	1.6074358
4.34	1.4678743	4.67	1.5411590	5.00	1.6094379

N	Logarithm	N	Logarithm	N	Logarithm
5.01	1.6114359	5.34	1.6752256	5.67	1.7351891
5.02	1.6134300	5.35	1.6770965	5.68	1.7369512
5.03	1.6154200	5.36	1.6789639	5.69	1.7387102
5.04	1.6174060	5.37	1.6808278	5.70	1.7404661
5.05	1.6193882	5.38	1.6826882	5.71	1.7422189
5.06	1.6213664	5.39	1.6845453	5.72	1.7439687
5.07	1.6233408	5.40	1.6863989	5.73	1.7457155
5.08	1.6253112	5.41	1.6882491	5.74	1.7474591
5.09	1.6272778	5.42	1.6900958	5.75	1.7491998
5.10	1.6292405	5.43	1.6919391	5.76	1.7509374
5.11	1.6311994	5.44	1.6937790	5.77	1.7526720
5.12	1.6331544	5.45	1.6956155	5.78	1.7544036
5.13	1.6351056	5.46	1.6974487	5.79	1.7561323
5.14	1.6370530	5.47	1.6992786	5.80	1.7578579
5.15	1.6389967	5.48	1.7011051	5.81	1.7595805
5.16	1.6409365	5.49	1.7029282	5.82	1.7613002
5.17	1.6428726	5.50	1.7047481	5.83	1.7630170
5.18	1.6448050	5.51	1.7065646	5.84	1.7647308
5.19	1.6467336	5.52	1.7083778	5.85	1.7664416
5.20	1.6486586	5.53	1.7101878	5.86	1.7681496
5.21	1.6505798	5.54	1.7119944	5.87	1.7698546
5.22	1.6524974	5.55	1.7137979	5.88	1.7715567
5.23	1.6544112	5.56	1.7155981	5.89	1.7732559
5.24	1.6563214	5.57	1.7173950	5.90	1.7749523
5.25	1.6582280	5.58	1.7191887	5.91	1.7766458
5.26	1.6601310	5.59	1.7209792	5.92	1.7783364
5.27	1.6620303	5.60	1.7227666	5.93	1.7800242
5.28	1.6639260	5.61	1.7245507	5.94	1.7817091
5.29	1.6658182	5.62	1.7263316	5.95	1.7833912
5.30	1.6677068	5.63	1.7281094	5.96	1.7850704
5.31	1.6695918	5.64	1.7298840	5.97	1.7867469
5.32	1.6714733	5.65	1.7316555	5.98	1.7884205
5.33	1.6733512	5.66	1.7334238	5.99	1.7900914
5.34	1.6752256	5.67	1.7351891	6.00	1.7917594

N	Logarithm	N	Logarithm	N	Logarithm
6.01	1.7934247	6.34	1.8468787	6.67	1.8976198
6.02	1.7950872	6.35	1.8484547	6.68	1.8991179
6.03	1.7967470	6.36	1.8500283	6.69	1.9006138
6.04	1.7984040	6.37	1.8515994	6.70	1.9021078
6.05	1.8000582	6.38	1.8531680	6.71	1.9035985
6.06	1.8017098	6.39	1.8547342	6.72	1.9050881
6.07	1.8033586	6.40	1.8562979	6.73	1.9065751
6.08	1.8050047	6.41	1.8578592	6.74	1.9080600
6.09	1.8066481	6.42	1.8594181	6.75	1.9095425
6.10	1.8082887	6.43	1.8609745	6.76	1.9110228
6.11	1.8099267	6.44	1.8625285	6.77	1.9125011
6.12	1.8115621	6.45	1.8640801	6.78	1.9139771
6.13	1.8131947	6.46	1.8656293	6.79	1.9154509
6.14	1.8148247	6.47	1.8671761	6.80	1.9169226
6.15	1.8164520	6.48	1.8687205	6.81	1.9183921
6.16	1.8180767	6.49	1.8702625	6.82	1.9198594
6.17	1.8196988	6.50	1.8718021	6.83	1.9213247
6.18	1.8213182	6.51	1.8733394	6.84	1.9227877
6.19	1.8229351	6.52	1.8748743	6.85	1.9242486
6.20	1.8245493	6.53	1.8764069	6.86	1.9257074
6.21	1.8261608	6.54	1.8779371	6.87	1.9271641
6.22	1.8277699	6.55	1.8794650	6.88	1.9286186
6.23	1.8293763	6.56	1.8809906	6.89	1.9300710
6.24	1.8309801	6.57	1.8825138	6.90	1.9315214
6.25	1.8325814	6.58	1.8840347	6.91	1.9329696
6.26	1.8341801	6.59	1.8855533	6.92	1.9344157
6.27	1.8357763	6.60	1.8870696	6.93	1.9358598
6.28	1.8373699	6.61	1.8885837	6.94	1.9373017
6.29	1.8389610	6.62	1.8900954	6.95	1.9387416
6.30	1.8405496	6.63	1.8916048	6.96	1.9401794
6.31	1.8421356	6.64	1.8931119	6.97	1.9416152
6.32	1.8437191	6.65	1.8946168	6.98	1.9430489
6.33	1.8453002	6.66	1.8961194	6.99	1.9444805
6.34	1.8468787	6.67	1.8976198	7.00	1.9459101

N	Logarithm	N	Logarithm	N	Logarithm
7.01	1.9473376	7.34	1.9933387	7.67	2.0373166
7.02	1.9487632	7.35	1.9947002	7.68	2.0386195
7.03	1.9501866	7.36	1.9960599	7.69	2.0399207
7.04	1.9516080	7.37	1.9974177	7.70	2.0412203
7.05	1.9530275	7.38	1.9987736	7.71	2.0425181
7.06	1.9544449	7.39	2.0001278	7.72	2.0438143
7.07	1.9558604	7.40	2.0014800	7.73	2.0451088
7.08	1.9572739	7.41	2.0028305	7.74	2.0464016
7.09	1.9586853	7.42	2.0041790	7.75	2.0476928
7.10	1.9600947	7.43	2.0055258	7.76	2.0489823
7.11	1.9615022	7.44	2.0068708	7.77	2.0502701
7.12	1.9629077	7.45	2.0082140	7.78	2.0515563
7.13	1.9643112	7.46	2.0095553	7.79	2.0528408
7.14	1.9657127	7.47	2.0108949	7.80	2.0541237
7.15	1.9671123	7.48	2.0122327	7.81	2.0554049
7.16	1.9685099	7.49	2.0135687	7.82	2.0566845
7.17	1.9699056	7.50	2.0149030	7.83	2.0579624
7.18	1.9712993	7.51	2.0162354	7.84	2.0592388
7.19	1.9726911	7.52	2.0175661	7.85	2.0605135
7.20	1.9740810	7.53	2,0188950	7.86	2.0617866
7.21	1.9754689	7.54	2.0202221	7.87	2.0630580
7.22	1.9768549	7.55	2.0215475	7.88	2.0643278
7.23	1.9782390	7.56	2.0228711	7.89	2.0655961
7.24	1.9796212	7.57	2.0241929	7.90	2.0668627
7.25	1.9810014	7.58	2.0255131	7.91	2.0681277
7.26	1.9823798	7.59	2.0268315	7.92	2.0693911
7.27	1.9837562	7.60	2.0281482	7.93	2.0706530
7.28	1.9851308	7.61	2.0294631	7.94	2.0719132
7.29	1.9865035	7.62	2.0307763	7.95	2.0731719
7.30	1.9878743	7.63	2.0320878	7.96	2.0744290
7.31	1.9892432	7.64	2.0333976	7.97	2.0756845
7.32	1.9906103	7.65	2.0347056	7.98	2.0769384
7.33	1.9919754	7.66	2.0360119	7.99	2.0781907
7.34	1.9933387	7.67	2.0373166	8.00	2.0794415

N	Logarithm	N	Logarithm	N	Logarithm
8.01	2.0806907	8.34	2.1210632	8.67	2.1598687
8.02	2.0819384	8.35	2.1222615	8.68	2.1610215
8.03	2.0831845	8.36	2.1234584	8.69	2.1621729
8.04	2.0844290	8.37	2.1246539	8.70	2.1633230
8.05	2.0856720	8.38	2.1258479	8.71	2.1644718
8.06	2.0869135	8.39	2.1270405	8.72	2.1656192
8.07	2.0881534	8.40	2.1282317	8.73	2.1667653
8.08	2.0893918	8.41	2.1294214	8.74	2.1679101
8.09	2.0906287	8.42	2.1306098	8.75	2.1690536
8.10	2.0918640	8.43	2.1317967	8.76	2.1701959
8.11	2.0930984	8.44	2.1329822	8.77	2.1713367
8.12	2.0943306	8.45	2.1341664	8.78	2.1724763
8.13	2.0955613	8.46	2.1353491	8.79	2.1736146
8.14	2.0967905	8.47	2.1365304	8.80	2.1747517
8.15	2.0980182	8.48	2.1377104	8.81	2.1758874
8.16	2.0992444	8.49	2.1388889	8.82	2.1770218
8.17	2.1004691	8.50	2.1400661	8.83	2.1781550
8.18	2.1016923	8.51	2.1412419	8.84	2.1792868
8.19	2.1029140	8.52	2.1424163	8.85	2.1804174
8.20	2.1041341	8.53	2.1435893	8.86	2.1815467
8.21	2.1053529	8.54	2.1447609	8.87	2.1826747
8.22	2.1065702	8.55	2.1459312	8.88	2.1838015
8.23	2.1077861	8.56	2.1471001	8.89	2.1849270
8.24	2.1089998	8.57	2.1482676	8.90	2.1860512
8.25	2.1102128	8.58	2.1494339	8.91	2.1871742
8.26	2.1114243	8.59	2.1505987	8.92	2.1882959
8.27	2.1126343	8.60	2.1517622	8.93	2.1894163
8.28	2.1138428	8.61	2.1529243	8.94	2.1905355
8.29	2.1150499	8.62	2.1540851	8.95	2.1916535
8.30	2.1162555	8.63	2.1552445	8.96	2.1927702
8.31	2.1174596	8.64	2.1564026	8.97	2.1938856
8.32	2.1186622	8.65	2.1575593	8.98	2.1949998
8.33	2.1198634	8.66	2.1587147	8.99	2.1961128
8.34	2.1210632	8.67	2.1598687	9.00	2.1972245

N	Logarithm	N	Logarithm	N	Logarithm
9.01	2.1983350	9.34	2.2343062	9.67	2.2690282
9.02	2.1994443	9.35	2.2353763	9.68	2.2700618
9.03	2.2005523	9.36	2.2364452	9.69	2.2710944
9.04	2.2016591	9.37	2.2375130	9.70	2.2721258
9.05	2.2027647	9.38	2.2385797	9.71	2.2731562
9.06	2.2038691	9.39	2.2396452	9.72	2.2741856
9.07	2.2049722	9.40	2.2407096	9.73	2.2752138
9.08	2.2060741	9.41	2.2417729	9.74	2.2762411
9.09	2.2071748	9.42	2.2428350	9.75	2.2772673
9.10	2.2082744	9.43	2.2438960	9.76	2.2782924
9.11	2.2093727	9.44	2.2449559	9.77	2.2793165
9.12	2.2104697	9.45	2.2460147	9.78	2.2803395
9.13	2.2115656	9.46	2.2470723	9.79	2.2813614
9.14	2.2126603	9.47	2.2481288	9.80	2.2823823
9.15	2.2137538	9.48	2.2491843	9.81	2.2834022
9.16	2.2148461	9.49	2.2502386	9.82	2.2844211
9.17	2.2159372	9.50	2.2512917	9.83	2.2854389
9.18	2.2170272	9.51	2.2523438	9.84	2.2864556
9.19	2.2181160	9.52	2.2533948	9.85	2.2874714
9.20	2.2192034	9.53	2.2544446	9.86	2.2884861
9.21	2.2202898	9.54	2.2554934	9.87	2.2894998
9.22	2.2213750	9.55	2.2565411	9.88	2.2905124
9.23	2.2224590	9.56	2.2575877	9.89	2.2915241
9.24	2.2235418	9.57	2.2586332	9.90	2.2925347
9.25	2.2246235	9.58	2.2596776	9.91	2.2635443
9.26	2.2257040	9.59	2.2607209	9.92	2.2945529
9.27	2.2267833	9.60	2.2617631	9.93	2.2955604
9.28	2.2278615	9.61	2.2628042	9.94	2.2965670
9.29	2.2289385	9.62	2.2638442	9.95	2.2975725
9.30	2.2300144	9.63	2.2648832	9.96	2.2985770
9.31	2.2310890	9.64	2.2659211	9.97	2.2995806
9.32	2.2321626	9.65	2.2669579	9.98	2.3005831
9.33	2.2332350	9.66	2.2679936	9.99	2.3015846
9.34	2.2343062	9.67	2.2690282	10.00	2.3025851

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