THE DOCTRINE OF GERMS,

OR

THE INTEGRATION OF CERTAIN PARTIAL DIFFERENTIAL EQUATIONS WHICH OCCUR IN MATHEMATICAL PHYSICS.

BY

S. EARNSHAW, M.A.,

AUTHOR OF "ETHERSPHERES A VERA CAUSA OF NATURAL PHILOSOPHY."

CAMBRIDGE:
DEIGHTON, BELL, AND CO.
LONDON: GEORGE BELL AND SONS
1881

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE.

THE method of integration by means of Germs adopted in this Treatise is based on the admitted principle that in the work of integrating a proposed differential equation we are free to avail ourselves of the advantages offered by any distinctive peculiarities that are perceived to exist in the equation itself prior to its integration. Any such peculiarity will, as a matter of course, impress a corresponding peculiarity on the integral to be found. Equations will therefore be classified according to their distinctive peculiarities, those peculiarities being indicated by the particular ways in which germs may be connected with the variables of a differential equation without disturbing or in any way affecting its form.

In this Treatise the differential equations that will be brought before the reader are all linear and partial, in consequence of which the doctrine of germs suitable for such equations admits of being presented in a form that is easily reduced to a system of singular efficiency. But there are certain other equations that are not linear, and which therefore do not fall under the system that will be developed in the following pages. The equation $\left(\frac{du}{dx}\right)^2 \frac{d^2u}{dt^2} = \frac{d^2u}{dx^2}$ is one of this kind, for its form will not be affected if ax be written for x; neither will it be affected if bu and bt be written simultaneously for u and t; and these arbitrary constants a, b, will therefore necessarily find a place (either explicitly or implicitly) in its integral. Also as only x takes the constant a, this indicates a peculiarity of x as to the manner in which it can appear in the integral. That u and t are alike related to b is indicative of the existence of a peculiar relation of u to t. And, further, we may write $u+A$ for u, $x+g$ for x, and $t+h$ for t (the constants A, g, h being perfectly arbitrary) without affecting the form of the equation. This shews that if U be an integral of it, so likewise will be the following,

$$\epsilon^{\frac{gd}{dx}+\frac{hd}{dt}}(U+A).$$

We may therefore presume that as there has been found for equations that are linear a "*Doctrine of Germs*," so there may be a possible "*Doctrine of Germ-like Constants*" for equations that are not linear.

In Chap. III. is introduced a theory of "Symbolical Equivalences." The subject is regarded from a point of view which may be considered as in some degree new. The exigencies of this Essay did not seem likely to require the complete development of this Theory; and in consequence of this only so much detail is given as was likely to be wanted in subsequent chapters. The Theory is capable of throwing light on several troublesome known paradoxes which have often been a source of perplexity to the Mathematical Student.

As the author was induced to undertake the development of the Doctrine of Germs by a desire to accomplish the complete integration of Laplace's Equation, and the consequent discovery of the general form of Laplace's Functions, he has deemed it to be of some possible advantage to obtain and to exhibit those results in several forms and aspects; hoping also that by this diversity the reader would be the more disposed to accept results which so confirm one another.

It has been taken as an admitted definition of Laplace's Functions, that any quantity which satisfies the equation (1) of

Art. 127, is a Laplace's Function. Laplace's Equation is usually given in two forms, viz. those in Arts. 124, 127; and to both of these results have been adapted. The first of these forms does not contain r; the second does; and it will be seen that r enters the latter in a manner that demands peculiar management, and when so managed leads to remarkable results, which can hardly fail to throw some light on the usually received theory of Laplace's Functions.

There appears to be an exceptional case of these functions which the reader will find in Art. 131.

It is needful to advise the reader before he enters upon the task of reading some parts of this Treatise, that when arbitrary constants occur in a general integral of a linear equation it will consist of the sum of several subgeneral integrals; and each of these constants being arbitrary by nature will retain their arbitrary character when multiplied by a definite numerical quantity; and if such a constant be separated into several arbitrary parts, each part will be as arbitrary as the original constant. Hence in altering the form of a general integral it will not be necessary to preserve the identity of any such constants, but only the quality of their independent arbitrariness; and thus we need not observe with respect to them the usual requirements of algebraic rules of reduction in passing from step to step. The arbitrary constants referred to are used merely to indicate the absolute independence of the subintegrals or subgeneral integrals of which they are the respective coefficients. (See Art. 42 for an example of what is here alluded to.)

The author has pleasure in acknowledging the valuable assistance rendered him by Mr Greenhill, M.A., Fellow of Emmanuel College, in supervising this Essay in its passage through the press.

SHEFFIELD, *Jan.* 1, 1881.

TABLE OF CONTENTS.

CORRIGENDA.

Page 22, line 6; *for* subsequent *read* subgeneral.

,, 25, line 3; *for* $+c'z)$ *read* $+c'z)\,u$.

,, 29, line 8; *for* algbraic *read* algebraic.

,, 46, line 11; *for* ϵ^{Mt^2} *read* ϵ^{M^2t}.

THE INTEGRATION

OF

LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

CHAPTER I.

INTRODUCTORY REMARKS.

In the present work we have not to deal with Linear Partial Differential Equations in general, but only with such as are known to be of difficult integration, and which have been found to present themselves in connexion with the application of Mathematics to various branches of Natural Philosophy.

This task we have undertaken not as a branch of analytical enterprise, but as a contribution to the resources of those philosophers who think it a matter of importance that Physical Theories should be subject to the severe test of mathematical confirmation. And in the execution of this task we believe that we shall have the privilege of developing a method of integration which may be regarded as new, and that is singularly well adapted to the integration of certain equations which have been found intractable by ordinary methods.

1. In some cases a remarkable degree of uncertainty and intricacy besets the answer to the question,—when may an integral of a linear differential equation be rightly styled *its general integral?* The following are some of the reasons of this uncertainty.

If one of the independent variables (as x) of a linear differential equation occur therein not as a symbol of quantity but only as a symbol of differential operation $\left(\text{as } \frac{d}{dx}\right)$, then (supposing U an integral of the equation) not only will U be an integral but so likewise will each one of the quantities

$$A\frac{dU}{dx}, \quad B\frac{d^2U}{dx^2}, \quad C\frac{d^3U}{dx^3}, \ldots \text{ ad infinitum,}$$

and (speaking generally) we thus can out of a single known integral create an unlimited number of new integrals all different from the original and from each other.

And not only can we thus create new integrals, but out of these new ones we can by addition of all of them, or of a few of them selected arbitrarily, create an unlimited number of fresh integrals different from U and from each of those previously found from U by differentiation.

Now as we have the right of forming known integrals into groups as we please, and from the nature of the integrals of a linear equation each group will be a new integral, we are obliged to come to the conclusion that it is not easy to see on what principle we can say of some one of the infinite mass of integrals a proposed linear differential may have, that it is the general integral.

2. We shall be able to shew, in the case of every linear differential equation of the class supposed in the preceding article, that it admits of integrals of a peculiar kind, forming a distinct class, and they are generally infinite in number, and when added together form a sum which is equivalent to the general integral.

For distinctness of reference we shall denominate these integrals *subintegrals;* and when we speak of those collectively which belong to the same differential equation we shall denominate them a *family* of subintegrals.

If then $P, Q, R, S, \ldots$ be the individual members of a family of subintegrals, and u be the general integral of the same equation as that to which they all individually belong, we shall represent the relation between u and $P, Q, R, \ldots$ by the following equation,

$$u = AP + BQ + CR + DS + \ldots \text{ ad infin.} \ldots\ldots\ldots (1),$$

$A, B, C, D, \ldots$ being arbitrary constants, the use of which in this equation is, to indicate the absolute independence of $P, Q, R, \ldots$ as integrals of the proposed equation.

3. Sometimes the general integral (1) will divide itself by some peculiarity of form into two or more distinct parts; and to these independent parts we intend to refer under the designation of *subgeneral integrals.*

The number of such subgeneral integrals that belong to a proposed differential equation is generally dependent on the order or some other peculiarity of the equation.

4. When we know a family of subintegrals (as $P, Q, R, \ldots$) we can by grouping them into different heaps, and finding the sum of each group, take the various sums thus formed as a family of subintegrals; and as a family it will be symbolically equivalent to the original family ($P, Q, R, \ldots$), though the members of the new family may happen to have no similitude of form to the members of the other.

Their equivalence results from the one fact that each of them explicitly or implicitly contains all the members of the subintegral family of which u (the general integral) is known to be constituted.

Thus a family of subintegrals always admits of being recast by grouping, by summation of series, and other means whereby a change of the forms of its members is effected.

This is important because it brings forward the question,—what will be the most convenient forms in which a family of subintegrals can be obtained?

One answer to this would be;—let the members of the family be cast in such forms, that if any one member of the family can be found all the other members may be obtained from it by simple and repeated differentiations or integrations of that one member with respect to one or more of the independent variables contained in the proposed differential equation.

We shall in due time shew how and under what conditions this may be done.

5. As the existence of subintegrals is but little known, we shall here add the following illustration.

Let $\frac{d^2u}{dx^2}=\frac{du}{dy}$ be a differential equation; then we can at a glance detect the following as independent integrals of it:

$$1;\ \frac{x}{1};\ \frac{x^2}{1.2}+y;\ \frac{x^3}{1.2.3}+\frac{xy}{1.1};\ \frac{x^4}{1.2.3.4}+\frac{x^2y}{1.2.1}+\frac{y^2}{1.2};\ \ldots\ldots$$

each of which can be obtained from the one before it by integration with regard to x, and correction with regard to y. If this be the whole family then

$$u=A+B\frac{x}{1}+C\left(\frac{x^2}{1.2}+\frac{y}{1}\right)+D\left(\frac{x^3}{1.2.3}+\frac{xy}{1.2}\right)+\ldots \text{ ad infin.},$$

$A, B, C, \ldots$ being arbitrary and absolutely independent constants.

6. We shall see in future articles that changes of the independent variables of a proposed differential equation can be sometimes made without producing any effect whatever on the *form* of the equation.

Whenever this can be done, the same changes of the same variables may be made in any known integral of the proposed equation without depriving that integral (however much changed in form thereby) of its property of being still an integral, or of its generality as an integral.

Also if the known integral should happen to be a particular and not a general integral the change of variables just described would introduce such a change of form of the integral itself as might bring it nearer to the form of a general integral by introducing new arbitrary constants which we should be at liberty to treat as germs not existing in the original particular integral.

7. We have just used the word *germ;* let us now explain what we mean by it.

We are aware that integration generally introduces to our notice in the integral certain constant quantities which have no existence in the differential equation itself. Such constants are in fact the offspring of integration; and are generally denominated *arbitrary constants.* The use of such constants in problems is well known.

This designation however is not sufficient for our purpose, and we intend to speak of them, under certain circumstances, as *germs,* or *germ-constants.* For as each of the variables of a proposed linear differential equation is constant with reference to every operating differential symbol contained therein except its own, so an arbitrary constant (*germ*) is constant with reference to all the differential symbols except its own; and it or any function of it contained in u may be operated on by its own symbol of differential operation, though no such symbol is contained in the proposed equation.

We may therefore consider a germ as being a *new* independent variable, i.e., an independent variable that is not contained in the differential equation itself, but only in its integral.

Thus ϵ^{ax+a^2y} is an integral of the equation $\frac{d^2u}{dx^2}=\frac{du}{dy}$; but the constant a is constant only in reference to $\frac{d}{dx}$ and $\frac{d}{dy}$, but not in reference to $\frac{d}{da}$; and therefore in this integral we may consider a either an arbitrary constant, or a new independent variable additional to x and y; and this is the property to which we refer when we call a a *germ.*

8. We shall find it convenient to be able to speak of certain germs under specific names, which will refer to the manner in which a germ in an integral may happen to stand (actually or virtually) connected with its independent variable. Thus if a germ g and an independent variable x stand connected in an integral by addition (as distinguished from multiplication), (as in the form $x \pm g$) we shall refer to g as the *minor germ of x.*

But if the connexion be of the nature of multiplication $\left(\text{as } gx \text{ or } \frac{x}{g}\right)$ we shall speak of g as the *major germ of* x.

9. Germs may also be regarded as being *general*, or *real*.

A *general germ* may receive any value whether real or imaginary.

A *real germ* may receive only such values as are not imaginary.

Nevertheless a general germ may be perfectly represented by means of two real germs. Thus if K be a general germ, the equation

$$K = M + im,$$

in which M and m are independent real germs, will perfectly represent K. Hence a general germ is equivalent symbolically to two real germs.

We shall throughout this Treatise use the two quantities i and j in the ambiguous senses implied by the two independent equations following,

$$i^2 = -1, \text{ and } j^2 = +1;$$

and both i and j will be regarded as independently carrying with them their proper double algebraic signs.

10. There are functions of x which cannot be expanded by MacLaurin's Theorem; and therefore the series $A + Bx + Cx^2 + \ldots$, in which the coefficients $A, B, C, \ldots$ are all arbitrary, does not symbolically represent a perfectly arbitrary function of x; but $Ax^\alpha + Bx^\beta + Cx^\gamma + \ldots$ in which $A, B, C, \ldots$ are arbitrary and $\alpha, \beta, \gamma, \ldots$ not limited by the condition that they are to be positive integers, symbolically represents a perfectly arbitrary function.

We may distinguish these cases, when necessary, by denominating the former a MacLaurin's arbitrary function.

11. To find the potentiality of a germ when it occurs in an integral only as an index or power of a function of the independent variables.

Let $W\omega^m$ be an integral of a proposed equation, W and ω being functions of the independent variables, and m being a germ that occurs only as the index of the quantity denoted by ω.

We may give to m an infinite series of different values $\alpha, \beta, \gamma, \ldots$ at pleasure; and each one of these values of m will furnish us with an independent integral; and all the integrals so obtained we may unite in a single integral in the following manner,

$$W\omega^m = W(A\omega^\alpha + B\omega^\beta + C\omega^\gamma + \ldots \text{ ad infin.}),$$

$A, B, C, \ldots$ being independent arbitrary constants.

But $\alpha, \beta, \gamma, \ldots$ being arbitrary also, the series

$$A\omega^\alpha + B\omega^\beta + C\omega^\gamma + \ldots$$

will represent an arbitrary function of ω of the most perfectly arbitrary kind;

$$\therefore \; W\omega^m = WF(\omega).$$

Hence a germ when it occurs in an integral as an index only is potentially equivalent to an arbitrary function of the most general kind.

The converse is manifestly true, viz., if $F(\omega)$ be a perfectly general arbitrary function of ω, then will $F(\omega) = \omega^m$ symbolically.

But if $F(\omega)$ represent a MacLaurin's series only, then it does not follow that $F(\omega) = \omega^m$ symbolically, for in this case $F(\omega)$ is clogged with the condition that the powers of ω in the expansion of $F(\omega)$ must be positive integers. For such a function we shall therefore when it occurs have to find a potential equivalent clogged with the same condition. ω^m is, as we have said, too general, and may consequently (if incautiously used) lead us into error when we come to the generalizing of results obtained.

12. It will be a convenience to be allowed sometimes to represent the product $1 \,.\, 2 \,.\, 3 \ldots n$ by the symbol $n!$.

CHAPTER II.

SOME GENERAL PROPERTIES OF GERMS.

13. LET $\varpi \,.\, u = 0$ represent a general linear partial differential equation of any number of independent variables, u being the dependent variable, and ϖ denoting the compound operating symbol. Also let U denote any integral of this equation containing a germ. Denote the germ by c, and expand U in a series according to the powers of c;

$$\therefore\; U = Pc^p + Qc^q + Rc^r + \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

in which p, q, r, ... are definite indices, and P, Q, R, ... are functions of the independent variables.

Operate on each member of this equation with ϖ, noting that $\varpi\,(U) = 0$;

$$\therefore\; 0 = (\varpi\,.\,P)\,c^p + (\varpi\,.\,Q)\,c^q + (\varpi\,.\,R)\,c^r + \ldots.$$

Now c being a germ is an arbitrary independent variable, and consequently this must be an identical equation;

$$\therefore\; 0 = \varpi\,.\,P, \quad 0 = \varpi\,.\,Q, \quad 0 = \varpi\,.\,R, \ldots$$

that is, P, Q, R, ... are independent integrals of the proposed equation; they are, in fact, the family of subintegrals, the members of which are rendered independent by the fact that the powers of c in equation (1) are all different. They owe their independence to the presence of a germ in U.

But we can preserve their independence another way, and at the same time unite the subintegrals in a single integral, thus

$$U = AP + BQ + CR + \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

in which A, B, C, ... are arbitrary constants.

Comparing this with (1) we perceive that the different powers of a germ in an expanded integral are symbolically equivalent to independent arbitrary constants.

We have now obtained the power of eliminating a germ by expansion of an integral according to the powers of that germ.

And, conversely, we can eliminate arbitrary constants, which belong to a series each term of which is a subintegral, by means of an extemporized germ.

14. If the integral U in the preceding article should happen to contain a second independent germ (n), then as only m has been eliminated the subintegrals P, Q, R, ... will each contain the germ n constituting each of them a germ-integral of the proposed equation. From each of these n may therefore be independently eliminated, and each of them will be thereby resolved into its own constituent subintegrals. Thus we have before us the fact that a subintegral may be also a germ-integral, and in this character resolvable into subintegrals of a more elementary class: and the ultimate subintegrals are those which do not contain a germ, and into which a germ cannot be introduced.

We may arrive by one step at the ultimate subintegrals of U by expanding U in the first instance in a series according to the powers of both the germs m and n, and then writing independent arbitrary constants for the germs and their powers and different combinations.

The converse is also true, viz. that we may eliminate arbitrary constants by means of the powers and the combinations of the powers of two or more independent germs.

Thus if

$$U = A + (Bx + B'y) + \left(C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \dots$$

we may (if A, B, C, ... be independent arbitrary constants) express a symbolical equivalent to this series by means of two independent germs m, n, thus,

$$U = A\left\{1 + (mx + ny) + \left(m^2\frac{x^2}{1.2} + mn\frac{xy}{1.1} + n^2\frac{y^2}{1.2}\right) + \dots\right\}$$
$$= A\epsilon^{mx+ny},$$

We may reverse this process and at one step assume, if m, n be independent germs, the following symbolical equivalence,

$$\epsilon^{mx+ny} = A + (Bx + B'y) + \left(C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \dots$$

15. As a matter of experience we know that different physical and geometrical problems lead us to the same forms of partial differential equations. The equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$$

is a well-known example of this. Hence each member of a family of subintegrals being a complete integral in itself of its kind, expresses the solution of a particular problem in physics or geometry, which can exist independently; and it also expresses what may be called an independent elementary state of matter or some affection thereof; or some imaginable independent geometrical condition of an elementary nature. The subintegrals being independent, represent properties which can exist independently in nature.

Hence whenever an integral can be resolved into elementary subintegrals we have in such cases this fact before us; that the problem which brought us to the corresponding differential equation is really of a compound nature and capable of being resolved into a number of elementary problems, the superposition of which in their proper proportions is equivalent to the original problem.

Hence, also, one problem may require for its complete representation one set (or group) of members selected from the whole family of subintegrals; and another problem another set of members. And thus comes into use that important property of all linear differential equations, that the sum of any of the members of a subintegral family is an integral of the same differential equation and represents the superimposed action of so many different geometrical or physical properties.

The selection of the group of individual members of a family of subintegrals suitable for the solution of a given problem is sometimes a work of difficulty, and will always make a demand upon the investigator's ingenuity.

16. The connexion between the general integral of an equation and its family of subintegrals is exhibited in Art. 13, in the equation

$$U = AP + BQ + CR + \ldots$$

Now as these coefficients A, B, C, ... are arbitrary and independent it would appear on the face of this series that it is incapable of being expressed in a finite form, there being no law connecting the coefficients with one another. We have seen however that by means of a germ and its powers we can symbolically represent simultaneously in a finite form both the integral itself and the arbitrariness and the independence of the coefficients. There are cases in which this can be accomplished in more forms than one.

It is always an important object to assume a germ and its powers in such forms and with such a law of coefficients as may enable us to sum the series which is constituted of subintegrals in a finite form. This cannot always be done; and when it cannot, then the substitution of a germ and its powers for A, B, C, ... is generally useless; and recourse must be had to the use of other means.

To make our meaning in this matter quite clear, we will produce an example (of a very simple kind) of the process of gathering up a whole family of subintegrals into one finite form which contains them all, and yet at the same time implicitly preserves the individuality and the independence of every member of the family. This is the chiefest of all the properties of a germ, and renders the doctrine of germs of much importance. Nothing can well exceed their utility in the discovery of symbolical equivalences; and in the transformation of integrals by means of these equivalences.

17. Let it be granted that the following quantities constitute as a whole a complete family of subintegrals, belonging to a certain linear differential equation:

$$1;\ \frac{x}{1};\ \frac{x^2}{1.2}+\frac{y}{1};\ \frac{x^3}{3!}+\frac{xy}{1.1};\ \frac{x^4}{4!}+\frac{x^2y}{1.2.1}+\frac{y^2}{1.2};\ \&c.$$

We combine them on the principle of superposition into a single integral U by means of arbitrary constants thus,

$$U=A+B\frac{x}{1}+C\left(\frac{x^2}{1.2}+\frac{y}{1}\right)+D\left(\frac{x^3}{3!}+\frac{xy}{1.1}\right)$$
$$+E\left(\frac{x^4}{4!}+\frac{x^2y}{1.2.1}+\frac{y^2}{1.2}\right)+\dots$$

in which $A, B, C, \dots$ are absolutely arbitrary and independent.

We now assume an extemporised germ m, and use it and its powers to replace (or eliminate) the arbitrary constants $A, B, C, \dots$

$$\therefore\ U=A\left\{1+m\frac{x}{1}+m^2\left(\frac{x^2}{1.2}+\frac{y}{1}\right)+m^3\left(\frac{x^3}{3!}+\frac{xy}{1.1}\right)+\dots\right\}$$
$$=A\left(1+\frac{mx}{1}+\frac{m^2x^2}{1.2}+\dots\right)\left(1+\frac{m^2y}{1}+\frac{m^4y^2}{1.2}+\dots\right)$$
$$=A\epsilon^{mx+m^2y} \qquad\qquad (1).$$

We have here prefixed the common coefficient A, because every integral of a linear equation takes an arbitrary general coefficient. In this simple form of the complete integral are contained (without loss of their individual independence) all the members of the subintegral family because m is a germ.

We may also now obtain the differential equation of which the above are the independent subintegrals. For by differentiating equation (1) with respect to its independent variables x, y, we find

$$\frac{d^2U}{dx^2}=\frac{dU}{dy} \qquad\qquad (2).$$

Now in reference to the form of this differential equation we may remark; that it would not be changed or in any way affected were we to write $x+g$ and $y+h$ for x and y; neither would its form be changed by writing mx and m^2y for x and y. We shall express this by saying that this equation allows its independent variables to take both major and minor germs.

In the above integral (1) the major germ m enters explicitly; but if we write $x+g$, $y+h$ for x, y in it, it takes the following form,

$$A\epsilon^{m(x+g)+m^2(y+h)} = A\epsilon^{mg+m^2h} \cdot \epsilon^{mx+m^2y} = A\epsilon^{mx+m^2y}.$$

Hence the exponential form of the integral (1) can be in no way affected in generality by the introduction of major and minor germs, the former being already present in it explicitly; and the latter implicitly, inasmuch as they may be said to lie hidden in the external arbitrary coefficient A.

18. If one of the independent variables of a proposed linear differential equation can take a minor germ, the family of subintegrals can by means of that germ be cast in such a form that all the subintegrals can be obtained from any one of the family by simple integration and differentiation with respect to that variable. (See Art. 4.)

Let $\varpi . u = 0$ be a linear differential equation which allows one of its independent variables (as x) to take a minor germ g.

Then as the writing of $x+g$ for x in $\varpi . u = 0$ produces no change, we may do the same in any integral of $\varpi . u = 0$ without destroying it as an integral; and as the substitution of $x+g$ for x in an integral would introduce the new germ g the generality of the integral would not be diminished; but on the contrary it would be increased, unless the integral in which the substitution is made be itself perfectly general. Hence the general integral must of necessity be of such a form that the substitution of $x+g$ for x in it cannot affect its perfect generality,

$$\therefore\ u = F(x+g, y, z, \ldots).$$

Let this be expanded by Taylor's Theorem in powers of g;

$$\therefore\ u = \left(1 + \frac{g}{1} \cdot \frac{d}{dx} + \frac{g^2}{2!} \cdot \frac{d^2}{dx^2} + \frac{g^3}{3!} \cdot \frac{d^3}{dx^3} + \ldots\right) F(x, y, z, \ldots)$$

$$= \left(A + B\frac{d}{dx} + C\frac{d^2}{dx^2} + D\frac{d^3}{dx^3} + \ldots\right) F(x, y, z, \ldots).$$

The last step is by Art. 13; and A, B, C, ... are arbitrary constants.

Hence A, B, C, ... are the coefficients of the subintegrals, which therefore are in their order as follows [we denominate $F(x, y, \ldots)$ the *first* subintegral],

$$F(x, y, \ldots); \quad \frac{d}{dx} F(x, y, \ldots); \quad \frac{d^2}{dx^2} F(x, y, \ldots); \text{ \&c.}$$

If any one of this family become known, then the whole family may be found from that one, by integration and by differentiation with regard to x.

From this property of minor germs, it becomes a matter of no little importance, whenever it can be done, to reduce a proposed equation which does not allow any of its independent variables to take a minor germ to a form that will allow a minor germ.

19. The potentiality of a minor germ may always be represented in an equivalent form by means of an arbitrary function; that function being, however, not one of quantitative symbols but of symbols of operation.

For when one of the independent variables (as x) takes a minor germ (as g) we have seen that

$$\begin{aligned} u &= F(x + g, y, \ldots) \\ &= \left(1 + \frac{g}{1} \frac{d}{dx} + \frac{g^2}{1 \, . \, 2} \frac{d^2}{dx^2} + \ldots\right) F(x, y, \ldots) \\ &= \left(A + B \frac{d}{dx} + C \frac{d^2}{dx^2} + \ldots\right) F(x, y, \ldots) \\ &= \phi\left(\frac{d}{dx}\right) . F(x, y, \ldots). \end{aligned}$$

If another variable (as y) take an independent minor germ h, we should find in the same way that

$$u = \phi\left(\frac{d}{dx}, \frac{d}{dy}\right) . F(x, y, \ldots),$$

and so on to any number of variables taking independent minor germs.

20. If an independent variable (as x) takes a minor germ g, the general integral of the equation will always admit of perfect symbolical expression in the form of an infinite series according to positive integer powers of that variable.

For in this case

$$\begin{aligned} u &= F(x+g, y, \dots) \\ &= F(g+x, y, \dots) \\ &= \left(1 + \frac{x}{1}\frac{d}{dg} + \frac{x^2}{1.2}\frac{d^2}{dg^2} + \dots\right) F(g, y, \dots), \end{aligned}$$

which is a series that contains x in positive integer powers only; and this series is symbolically the complete general integral by hypothesis.

If two of the independent variables (x, y) take independent minor germs, then the general integral will always admit of being expressed in the form of an infinite series containing both x and y in positive integer powers only.

For in this case

$$\begin{aligned} u &= F(x+g, y+h, z \dots) \\ &= F(g+x, h+y, z \dots), \end{aligned}$$

from which the proposition follows by expanding F by Taylor's Theorem.

It is evident the proposition may be extended to all the independent variables that take independent minor germs.

21. If $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right) u = 0$ be understood to represent any linear differential equation of two independent variables (x, y) with constant coefficients, then will each of these variables take an independent minor germ; and consequently the complete general integral of the equation may be fully expressed in any one of the three following equivalent forms,

$$\text{(i)} \dots u = A + \left(B\frac{x}{1} + B'\frac{y}{1}\right) + \left(C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \dots$$

or $$\text{(ii)} \ldots u = P + Q\frac{y}{1} + R\frac{y^2}{1.2} + \ldots$$

in which P, Q, R, ... represent series of the general form

$$A + B\frac{x}{1} + C\frac{x^2}{1.2} + \ldots$$

or $$\text{(iii)} \ldots u = P + Q\frac{x}{1} + R\frac{x^2}{1.2} + \ldots$$

in which P, Q, R, ... represent series of the general form

$$A + B\frac{y}{1} + C\frac{y^2}{1.2} + \ldots$$

When hereafter we assume the first of these forms as representative of the complete general integral of a proposed differential equation, we must remember that the sole authority for the truth of this assumption lies in the fact that we know from the form of the differential equation itself that both x and y take independent minor germs. The second form supposes that y takes a minor germ; and the third form supposes x to take a minor germ.

22. Let $\varpi\left(x, \frac{d}{dx}, \frac{d}{dt}\right) u = 0$, or briefly $\varpi \,.\, u = 0$ denote a linear differential equation in which one of the two independent variables (as t) occurs only in the form of a differential symbol of operation.

In this case the general integral is completely represented by the following form,

$$u = F(x, t + g),$$

g being a minor germ of t.

Now if we integrate such a differential equation as the one before us by the method of infinite series, it will sometimes happen that we shall obtain a result which may be represented by the following:

$$u = \psi\left(x, \frac{d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1.2} + \ldots\right),$$

in which A, B, C, ... are independent arbitrary constants; and they are therefore the coefficients of the members of the family of subintegrals of which u is constituted.

Hence we are at liberty to eliminate these arbitrary constants by means of the powers of a germ.

Now $A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots$ is a MacLaurin's infinite series (see Art. 10) and its symbolical equivalent representative will be under a certain restriction of form corresponding to this fact (Art. 11); and the proper form may be found in the following manner.

Equating the two preceding forms of u we have the following symbolical equivalence:

$$F(x, t+g) = \psi\left(x, \frac{d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots\right);$$

and the left-hand member being a function of $t+g$, the right-hand member must be so likewise;

$$\therefore\ A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots = f(t+g) \quad \dots\dots\dots\dots\dots(1).$$

Now $f(t+g) = f(g+t)$, and whichever of these two forms we adopt the result of the expansion of it in a series by Taylor's Theorem must be symbolically equivalent to the left-hand member of equation (1).

Hence

$$A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots = f(g) + f'(g)\cdot\frac{t}{1} + f''(g)\cdot\frac{t^2}{1.2} + \dots \ (2).$$

But in order that (2) may be a symbolical equivalent of the series on the left-hand, $f(g), f'(g), f''(g), \dots$ must be different powers of the germ g; a condition that requires the following supposition,

$$f(g+t) = (g+t)^p,$$

where p must be of such a value as shall render the expansion of $(g+t)^p$ an infinite series. Hence p must not be a positive integer.

$$\therefore\ A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots = (g+t)^p, \text{ or} = (t+g)^p.$$

Hence $$u = \psi\left(x, \frac{d}{dt}\right) . (g+t)^p \text{ ....}(3),$$

or $$= \psi\left(x, \frac{d}{dt}\right) . (t+g)^p \text{ ....}(4).$$

There are therefore two forms in which u may be presented, a circumstance which is notable for the following reason.

If p were a positive integer (which it cannot be) these two forms of u would be identical, because in that case the expansions of $(g+t)^p$ and $(t+g)^p$ would be identical, with the exception only that their respective terms would be in a reverse order. But as p is not a positive integer the expansions of $(g+t)^p$ and $(t+g)^p$ are dissimilar, though symbolically equivalent (see Chap. III.).

Hence that the integrals marked (3) and (4) are dissimilar though symbolically equivalent, is due to the circumstance that p is not a positive integer; and further that $(t+g)^p$ and $(g+t)^p$ are symbolically equivalent.

Consequently we have two dissimilar though symbolically equivalent forms in which we may finally present the general integral of the proposed equation, whenever that integral can be found in the form

$$u = \psi\left(x, \frac{d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1 . 2} + \dots\right).$$

23. If we expand $(g+t)^p$ in order to eliminate the germ g and obtain the family of subintegrals of which u is constituted, we obtain

$$\text{the first subintegral} = \psi\left(x, \frac{d}{dt}\right) . A = \psi(x, 0).$$

The other subintegrals may all be obtained from this by *integrating* it with respect to t successively (Art. 4 contains an example of this).

But if we expand $(t+g)^p$ the first subintegral will be equal to $\psi\left(x, \frac{d}{dt}\right) . t^p$; and the other members of the family will be obtained from this by successive *differentiations* with respect to t.

As it is always possible to differentiate, and not always possible to integrate, a given function of t, there will be an advantage in using the form

$$\text{first subintegral} = \psi\left(x, \frac{d}{dt}\right) . t^p \quad\ldots\ldots\ldots\ldots(1).$$

But here crops up the question,—how are we to assign a proper value to p? for the preceding Article tells us nothing respecting it but that it is not to be a positive integer. It does not even tell us distinctly whether *zero* is to be classed among *positive* or among *negative* integers. There is however no difficulty in seeing that we must assign to p as small a value (apart from its algebraic sign) as possible.

In some degree p is therefore a disposable numerical quantity; and we shall follow the rule of assigning to it the *least* value (apart from algebraic sign) that will enable us *to express the first subintegral* (1) *in finite terms*, for our object is to find the integral of a proposed equation in finite terms.

24. One possible case must here be noticed. There being nothing to fix a definite value of p in the investigation of Art. 22, in the formula

$$\text{first subintegral} = \psi\left(x, \frac{d}{dt}\right) t^p,$$

if it should ever happen that this leads to a general subintegral of the form $W . \omega^p$, without the necessity of our assigning to p any definite value, then p may be considered to be a germ, and the first subintegral will be *inclusive* of the whole family of subintegrals.

In this particular case therefore we have (by Art. 11),

$$\begin{aligned} u &= W . \omega^p \\ &= W . \phi(\omega). \end{aligned}$$

25. We have said that the smallest possible value (apart from algebraic sign) must be assigned to p in order to obtain

the *first* of the family of subintegrals, and that from the subintegral so found all the other members of the family may be obtained by differentiation with $\frac{d}{dt}$.

From this it is obvious that if we can obtain a finite subintegral by assigning to p a value which is not the least possible (apart from sign) the subintegral so obtained will be one of the family of subintegrals; and we may ascend from it to the *first* subintegral by successive integrations with regard to t. We shall know when we have arrived at the first by the circumstance that we have arrived at a subintegral which is not integrable in finite terms.

26. The general principle that we shall adopt in the integration of linear differential equations is that of taking advantage of any peculiarity that may be perceived to exist in their forms, favoring the introduction of germs into their integrals; for as an integral that is perfectly general cannot be made more general, the introduction of a germ, though it may affect the generality of an integral that is not perfectly general, cannot make it less general; but on the contrary every germ introduced brings it one step nearer to perfect generality.

When therefore a differential equation is proposed for integration we begin by changing (if necessary) the dependent and independent variables (see Chap. IV.) with the object of bringing the equation to its simplest form, or to a form which will enable us to detect the possible existence of germs in the integral.

The preceding Articles will have made it evident that it would be a great point gained if the reduction and transformation can be carried on till we have arrived at a form in which one at least of the independent variables shall occur only in the form of a differential symbol of operation, for such a variable will take a minor germ. The following will illustrate the method of proceeding with such an equation, and will also be useful for reference.

27. To integrate $\frac{du}{dt} = \varpi \left(x, y, \ldots \frac{d}{dx}, \frac{d}{dy}, \ldots \right)$, or simply $= \varpi . u.$

This equation allows t to take a minor germ, and therefore (Art. 21) the following will be a perfectly general form of its integral:

$$u = P + Q\frac{t}{1} + R\frac{t^2}{1.2} + S\frac{t^3}{1.2.3} + \ldots$$

in which $P, Q, R, \ldots$ are functions not of t but of the independent variables that occur in the operating function ϖ.

Substitute this form of u in the proposed equation $\frac{du}{dt} = \varpi . u$;

$$\therefore\ u = \left(1 + \frac{t}{1}\varpi + \frac{t^2}{1.2}\varpi^2 + \frac{t^3}{1.2.3}\varpi^3 + \ldots\right) P.$$

28. To integrate $\frac{d^2u}{dt^2} = \varpi \left(x, y, \ldots \frac{d}{dx}, \frac{d}{dy}, \ldots \right) u$, or briefly, $= \varpi . u.$

By the same method as the above, and with the additional consideration that this equation allows us to write in any integral jt instead of t, we obtain the following general form of integral:

$$u = \left(1 + \frac{t^2}{1.2}\varpi + \frac{t^4}{4!}\varpi^2 + \ldots\right) P$$
$$+ j\left(\frac{t}{1} + \frac{t^3}{3!}\varpi + \frac{t^5}{5!}\varpi^2 + \ldots\right) Q,$$

in which P, Q are independent functions of the variables contained in ϖ.

The two serial members of u are independent subgeneral integrals; and their independence is due to the circumstance that $\frac{d}{dt}$ occurs in the differential equation only in the form $\left(\frac{d}{dt}\right)^2$, and their independence is secured symbolically by the existence of the ambiguous symbol j in the latter of them.

It will be noticed that one of the subgeneral integrals contains even powers of t only, and the other odd powers only. But we are at liberty to construct out of them, by addition and subtraction, two other equivalent subsequent integrals, each of which shall contain both the odd and the even powers of t.

The following form of differential equation, though it belongs to the case of two independent variables only, will be found important, for many equations that occur in physical enquiries can be made to depend upon it.

29. To integrate $\phi\left(\frac{d}{dt}\right)u = \varpi\left(x, \frac{d}{dx}\right)u.$

By the usual method of integration by series the integral of this equation can generally be obtained in a form equivalent to the following:

$$u = \psi\left(x, \frac{d}{dt}\right)T,$$

in which T is an arbitrary function of t.

Now since the proposed equation allows t to take a minor germ,

$$\therefore\ T = A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots$$

in which $A, B, C, \dots$ are the arbitrary coefficients of the members of the family of subintegrals which constitute u. Their places may therefore (Art. 13) be supplied by the powers of an extemporized germ m;

$$\therefore\ u = \psi\left(x, \frac{d}{dt}\right)\left(1 + \frac{mt}{1} + \frac{m^2t^2}{1.2} + \dots\right)$$

$$= \psi\left(x, \frac{d}{dt}\right)\epsilon^{mt}$$

$$= \epsilon^{mt}\psi(x, m) = \epsilon^{mt}X.$$

When X is found by the substitution of this value of u in the proposed equation, then u is known from the equation,

$$u = \epsilon^{mt} X.$$

30. Let $\varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right) u = 0$ be a linear partial differential equation of any number of independent variables and having constant coefficients; then will

$$U = A\epsilon^{Lx+My+Nz+\ldots}$$

be an integral of it; $L, M, N, \ldots$ being germs subject only to the following equation of condition,

$$0 = \varpi(L, M, N, \ldots).$$

For

$$\varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right) U = \varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right) \epsilon^{Lx+My+Nz+\ldots}$$
$$= \epsilon^{Lx+My+Nz+\ldots}\, \varpi(L, M, N, \ldots).$$

Now as $L, M, N, \ldots$ are germs, we are at liberty to assume such a relation to exist among them as will render the right-hand member of this equation equal to zero; and the only condition necessary for that purpose is $\varpi(L, M, N, \ldots) = 0$. Hence subject to this condition U represents a quantity which satisfies the proposed equation.

We have not said that U is the general integral of the equation; but as it contains independent germs it needs must be one of a great degree of generality. As a matter of fact it fails to be the general integral in such cases only as are distinguished by the recurrence of one or more of the operative factorials into which $\varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \ldots\right)$ can in some cases be resolved.

We shall be careful to prove the perfect generality of U in every case in which we shall use it; and then only shall we cite it as *the general exponential* integral.

31. The germs $L, M, N, \ldots$ being of the nature of general germs are liable to contain imaginary quantities; it will some-

times be desirable to express the general exponential integral in terms of *real germs* only. Let us therefore assume their forms to be

$$L+il, \quad M+im, \quad N+in, \; \ldots\ldots$$

in which $L, M, N, \ldots l, m, n, \ldots$ are all real quantities.

By this change the exponential integral takes the following form

$$U = A\epsilon^{K+iI} = A\epsilon^{K}\cos I + B\epsilon^{K}\sin I$$
$$= A\epsilon^{K}\cos(I+B);$$

in which $$K = Lx + My + Nz + \ldots$$

and $$I = lx + my + nz + \ldots$$

and the germs $L, M, N, \ldots l, m, n, \ldots$ are subject to the two equations of condition into which the following equation necessarily divides itself,

$$0 = \varpi\,(L+il,\; M+im,\; N+in,\; \ldots).$$

32. If c be a germ contained in an integral U of a linear differential equation $\varpi\,.\,u=0$, containing any number of independent variables and its coefficients not being necessarily constant; then will $\frac{dU}{dc}, \frac{d^2U}{dc^2}, \frac{d^3U}{dc^3}, \ldots$ and generally $\phi\left(\frac{d}{dc}\right)U$ be integrals of $\varpi\,.\,U=0$.

The function ϕ is conditioned by the equation $\phi\left(\frac{d}{dc}\right)0=0$.

Now c being a germ is not contained in ϖ; and therefore c and ϖ are commutative symbols. Also $\varpi\,.\,U=0$.

$$\therefore\; 0 = \phi\left(\frac{d}{dc}\right)0 = \phi\left(\frac{d}{dc}\right).\,\varpi U = \varpi\,.\,\phi\left(\frac{d}{dc}\right)U.$$

Hence $\phi\left(\frac{d}{dc}\right)U$ is an integral of $\varpi\,.\,u=0$.

33. The following indicates the possible existence of quasi-minor germs in some cases.

Suppose we have before us a linear equation of the following form,

$$0 = \varpi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, ax + by + cz,\ a'x + b'y + c'z\right) u,$$

in which a, b, c, a', b', c' are definite constants.

It is evident that we may, without affecting this equation at all, write $x + g$, $y + h$, $z + k$ for x, y, z respectively, provided the values of g, h, k are restricted by the two following conditions,

$$0 = ag + bh + ck, \text{ and } 0 = a'g + b'h + c'k.$$

Now as g, h, k are subject to these two linear conditions only, each of them may be described as a definite multiple of some one indefinite quantity l, which we may designate an independent germ. This independent germ will be divided among the three independent variables in certain definite proportions, and be to each of them a minor germ, or rather a quasi-minor germ; for we have defined a minor germ (Art. 8) as belonging exclusively to an individual independent variable.

Major Germs and Homogeneity.

34. By means of major germs we may extend the usual definition of homogeneity in the following manner.

If a mathematical expression $F(x, y, z, \ldots)$ be of such a form that when $m^\alpha x, m^\beta y, m^\gamma z, \ldots$ are written in it for $x, y, z \ldots$ the germ m becomes a mere factor or coefficient of the whole; i.e. if the following form of expression holds good,

$$F(m^\alpha x, m^\beta y, m^\gamma z, \ldots) = m^p F(x, y, z, \ldots),$$

in which $\alpha, \beta, \gamma \ldots$ have definite values; then we say that $F(x, y, z, \ldots)$ is a homogeneous expression of p dimensions.

We may also say that $x, y, z, \ldots$ are respectively of the dimensions $\alpha, \beta, \gamma, \ldots$ and we shall speak of m as being a major germ in this case.

The following proposition will be found very important in future operations.

35. Every homogeneous linear partial differential equation, whether its coefficients be, or be not, constant, will have all its

subintegrals (that are due to the elimination of a major germ) homogeneous according to the above definition; and they will all be of different dimensions.

Let $u = F(x, y, z, \ldots)$ be the integral of a homogeneous differential equation. Then since a general integral is not affected as to its generality by any change of the independent variables which does not affect the differential equation, we may write $m^\alpha x$, $m^\beta y$, $m^\gamma z$, ... in both the equation and its integral without affecting them;

$$\therefore\ u = F(m^\alpha x, m^\beta y, m^\gamma z, \ldots) \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and by differentiation of this we obtain the following equation,

$$\left(\alpha x \frac{d}{dx} + \beta y \frac{d}{dy} + \gamma z \frac{d}{dz} + \ldots\right) u = m \frac{du}{dm} \ldots\ldots\ldots (2).$$

Now expand the right-hand member of equation (1) in powers of m;

$$\therefore\ u = Pm^p + Qm^q + Rm^r + \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

in which P, Q, R ... are functions of x, y, z ... but not of m; they in fact constitute the family of subintegrals due to the elimination of the germ m.

Hence each of them (i.e. of P, Q, R ...) is an integral of the proposed homogeneous equation; and consequently each term of (3) will satisfy the equation (2).

Taking the first term Pm^p and substituting it in (2) we find

$$\alpha x \frac{dP}{dx} + \beta y \frac{dP}{dy} + \gamma z \frac{dP}{dz} + \ldots = pP \ldots\ldots\ldots\ldots(4),$$

the meaning of which equation is, that the subintegral P is homogeneous and of p dimensions.

In the same way we learn that Q, R, ... are homogeneous subintegrals of q, r, ... dimensions respectively.

The members of the family of subintegrals obtained by the elimination of m have therefore this common property,—*they are all homogeneous;* but being of different dimensions their sum, i.e. the general integral which contains them all, is not homogeneous.

Homogeneity is, therefore, the distinctive feature of a sub-integral.

36. As a major germ generally (though not always) belongs to at least two independent variables, if a proposed differential equation contains more than two such variables it may admit of more than one independent major germ; or if it admits of one only, there may then be some independent variables that do not take a major germ at all.

Hence it may happen that a differential equation may be homogeneous with regard to only a portion of its independent variables: and being homogeneous it may be of different dimensions in reference to its different major germs.

Thus in the equation $\frac{d^2u}{dxdy} = \frac{du}{dt}$, we may write mx, my, m^2t for x, y, t respectively, and the equation is therefore homogeneous.

Or we may write nx, nt for x and t, and consider the equation homogeneous with respect to x and t. So it is homogeneous with respect to y and t. And we may write lx for x, and $l^{-1}y$ for y. But all these results are included when we write lmx for x, $l^{-1}ny$ for y, and mnt for t, there being three germs involved in this case. This is therefore the most general assumption of major germs; and it implies that the equation is independently homogeneous with regard to x, y, t^2; and to x, t; and y, t. It therefore possesses a triple homogeneity; and to obtain general results all three must be taken account of.

It will now be manifest that the existence of major and minor germs can oftentimes be discovered prior to integration from the form of the proposed differential equation by mere inspection. We shall see hereafter, however, that there may be possible major germs which are not so easily discovered.

And it is always to be remembered that we are at liberty to introduce into a known integral any possible germs, and that the result will be still an integral of the proposed equation, which may be thereby rendered one of increased generality.

CHAPTER III.

ON SYMBOLICAL EQUIVALENCE.

37. We consider the elementary quantities and magnitudes with which we have to do as being measurable by numbers; and an essential property of every such quantity or magnitude is, that "the whole is greater than a part of it."

Zero, which is usually denoted by the symbol 0, we consider to be "the negation of quantity or magnitude." The absence or negation of a quantity cannot be divided into parts; and what has no existence cannot be treated as having properties.

But zero, though non-existent as a measurable quantity, admits of symbolical representation by means of real quantities in an infinite variety of ways; as for example,

$$0 = x - x;\ 0 = 1 + \cos 2x - 2\cos^2 x;$$

$$0 = \left(\frac{d^2}{dx^2} + 1\right)\epsilon^{ix};\quad 0 = \left(\frac{d}{dx} - \frac{d}{dy}\right)\phi\,(x + y).$$

These are called equations, but we here speak of their right-hand members as the symbolical equivalents of zero; and hence the mathematical sign (=) is to be understood not as always denoting numerical equality, since zero is not a number, but as (in such cases as these) denoting symbolical equivalence.

Also such a question as this,—find the integral of $\frac{du}{dx} + \frac{du}{dy} = 0$, may be enunciated in the following equivalent form,—find the most general form of u in terms of x, y which will render the following equation a symbolical equivalence $\left(\frac{d}{dx} + \frac{d}{dy}\right)u = 0$.

The reader will kindly keep in mind, whenever he finds the sign (=) connecting two quantities, or two steps in an investigation, which are not equal algebraically, that that sign is in this case to be read as signifying *symbolical* or *integral equivalence.* We might preserve at every step, and in every equation, *both algebraic and symbolic* equivalence, but this would have to be done oftentimes at an inconvenient expenditure of time and new algbraic symbols. After a little practice no inconvenience will be found in the system employed in these investigations which are chiefly about integrals. The sign (=) has three meanings:—algebraic equality;—symbolical equivalence;—and equal in generality as integrals.

38. For a reason analogous to that which leads us to reject zero as a numerical quantity we reject *infinity,* for it cannot be numerically increased by addition nor diminished by subtraction, since it is not measurable by numbers.

Nevertheless there is a case of infinitude which can be dealt with to advantage, viz., the case of series the number of whose terms is infinite.

Infinite series are of two kinds:—

1. A series may have a first term but no last term; or, in other words, it may have a beginning but no end.

2. A series may have neither a first term nor a last term.

$1+2+2^2+2^3+\ldots$ ad infin. is an example of the former kind, and $\ldots+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2}+1+2+2^2+2^3+\ldots$ of the latter kind.

39. The meaning of the word equivalence which it will be necessary to attach to the sign (=) in some of the subsequent articles is so unusual that we shall add a few more illustrations.

The late Professor De Morgan proposed the following equation for solution,

$$x=2x.$$

If x be in this equation a numeral quantity, divide both sides of the equation by x. Then on the ground that if equals be divided by equals the quotients are equal, we find $1=2$, a result which we are obliged to reject though obtained according to the acknowledged principles of numerical reasoning.

Hence the only remaining inference is that the equation before us is not one of numerical magnitudes. The equation when put in a verbal form is this, "Find a numerical magnitude that shall be *numerically* equal to the double of itself." When thus stated the equation is seen at once to involve of necessity a property irreconcilable with the properties of numerical magnitudes.

40. But it remains to be ascertained whether the equation $x=2x$ admits of a solution reconcilable with *symbolical equivalence.* Find x so that it shall be symbolically equivalent to $2x$, is now the problem before us.

There is no particular difficulty in finding the following answer to this question,

$$x=A\,(\ldots+\tfrac{1}{8}+\tfrac{1}{4}+\tfrac{1}{2}+1+2+4+8+\ldots),$$

where A is an arbitrary constant.

Hence the right-hand member of this equation is a symbolical equivalent of zero, which is all that is meant by the equation

$$0=A\,(\ldots+\tfrac{1}{8}+\tfrac{1}{4}+\tfrac{1}{2}+1+2+4+8+\ldots).$$

41. Let it be required to find x such that it shall exceed its double by unity.

The algebraic equation for this case is

$$x=2x+1,$$

of which there are three solutions, viz.

$$x=-1,$$

$$x=1+2+4+8+\ldots$$

and $$x=-(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots).$$

The first and last of these may be numerically equal; but it is evident that the second being a positive quantity cannot be numerically equal to either of the others. Hence the following are nothing but symbolical equivalences,

$$-1 = 1 + 2 + 4 + 8 + \ldots$$

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots = -(1 + 2 + 4 + 8 + \ldots) \ldots\ldots\ldots\ldots (1).$$

Both the terms of this last equation are legitimate expansions of the same symbolical expression, $\frac{1}{2-1}$.

It is only in reference to the problem algebraically expressed by the equation $x = 2x + 1$, that we maintain these equivalences to be real. They all satisfy this equation; yet they are not its *roots* but its *equivalences.*

Another important matter is that the equivalence marked (1) shews that two infinite series may be strictly equivalent though one of them may be convergent, and the other divergent.

42. We come now to speak of another matter which we shall denominate "*integral-equivalence,*" as being distinct from algebraic equality. Brevity of expression is the chief object to be attained by the use of this kind of equivalence. An example will best explain its nature.

In integrating the equation $\frac{d^2u}{dx^2} = u$ by the method of infinite series we find

$$u = A\left(1 + \frac{x^2}{1.2} + \ldots.\right) + B\left(\frac{x}{1} + \frac{x^3}{1.2.3} + \ldots.\right),$$

in which the arbitrary constants A, B indicate that the entire integral u consists of the sum of two independent integrals, which we denominate subgeneral integrals. Each of these is a perfect integral in itself and expressive of properties or relations peculiar to itself. One of them contains only odd powers and the other only even powers of x.

Having found the integral of the proposed equation in the above serial form we proceed to introduce integral equivalences in the following manner, with the object of presenting the integral in the briefest possible form.

$$
\begin{aligned}
u &= A\left(1+\frac{x^2}{1.2}+\ldots.\right)+B\left(\frac{x}{1}+\frac{x^3}{1.2.3}+\ldots.\right)\\
&= A\left(\epsilon^x+\epsilon^{-x}\right)+B\left(\epsilon^x-\epsilon^{-x}\right)\\
&= A\epsilon^x+B\epsilon^{-x}\\
&= A\epsilon^{jx}.
\end{aligned}
$$

This is our final result, and simple as it is, it is perfectly equivalent as an integral to the two infinite series which constitute the entire value of u. In deducing it from those series the arbitrary constants have suffered changes of identity at every step, but we have been careful to preserve their only essential quality, that they are arbitrary constants all through the process of reduction.

Had the equation to be integrated been $\dfrac{d^2u}{dx^2}+u=0$, our result would have been

$$u = A\epsilon^{ix}.$$

It will be seen from the above example that we shall hereafter feel at liberty to use the sign $(=)$, as denoting integral equivalence; and that in so using it we shall consider not the identity of the quantities denoted by A, B, C, ..., but merely take care that each shall preserve its only essential quality, viz., that it denotes a perfectly arbitrary and independent quantity.

43. If $f(x)$ be expanded in an infinite series it is usual to represent the result thus,

$$f(x) = Ax^\alpha + Bx^\beta + Cx^\gamma + \ldots \text{ ad inf.}$$

Professor De Morgan proposed that the left-hand member should be denominated the *Invelopment* of the right-hand member. We shall adopt this designation. It has been usual to speak of $f(x)$ as the *sum* of the series, but unless the series be convergent this designation is incorrect.

When we meet with two symbolically equivalent series, if we can find the invelopment of one of them we shall use that series in preference to the other without reference to its convergence or non-convergence.

44. A series that has no last term may possess properties not possessed by the sum of any number of its terms. Take the following example:

$$U = 1 - x + x^2 - x^3 + \dots \text{ ad inf.} \quad \dots\dots\dots\dots (1).$$

If we multiply this series by x and subtract the product from unity it remains unchanged; and this is not a property of the series continued to n terms only. Hence the following is true of the infinite series only, viz.

$$U = 1 - xU;$$

$$\therefore \; U = \frac{1}{1+x}.$$

This is the involopment of the series; i.e. it represents the whole infinite series, and if it be expanded according to the usual rules it will be found to produce the whole series.

But $\dfrac{1}{1+x} = \dfrac{1}{x+1}$, and the latter expression is the involopment of the following infinite series,

$$\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \dots \text{ ad inf.} \quad \dots\dots\dots\dots (2).$$

Now since the involopments of these respective series are symbolically and algebraically equal, we say that the following is both a symbolical and an integral equivalence,

$$1 - x + x^2 - \dots \text{ ad inf.} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \dots \text{ ad inf.};$$

and therefore in reducing an integral to its simplest or most manageable form we should not hesitate, if necessary to secure ultimate success, to introduce this equivalence, or any other which rests on the same basis.

45. We shall now generalize the above results by shewing that the two following infinite series are symbolically equivalent,

$$A + Bx + Cx^2 + \dots = A + Bx^{-1} + Cx^{-2} + \dots$$

in which A, B, C, ... are constant quantities, definite or indefinite.

Instead of C, D, E, ... in the left-hand member write respectively $C' - 2B$, $D' - 4C' + 3B$, $E' - 6D' + 10C' - 4B$, &c.

$$\begin{aligned}\therefore\ A + Bx + Cx^2 + Dx^3 + \ldots = A &+ B(x - 2x^2 + 3x^3 - 4x^4 + \ldots)\\ &+ C'(x^2 - 4x^3 + 10x^4 - \ldots)\\ &+ D'(x^3 - 6x^4 + \ldots)\\ &+ \&c.\end{aligned}$$

$$= A + \frac{Bx}{(1+x)^2} + \frac{C'x^2}{(1+x)^4} + \frac{D'x^3}{(1+x)^6} + \ldots$$

$$= A + \frac{Bx^{-1}}{(1+x^{-1})^2} + \frac{C'x^{-2}}{(1+x^{-1})^4} + \frac{D'x^{-3}}{(1+x^{-1})^6} + \ldots$$

$$= A + Bx^{-1} + Cx^{-2} + Dx^{-3} + \ldots$$

The validity of this investigation depends entirely on the series being infinite, and it cannot hold good for n terms, with the single exception of $n = 1$.

It is to be noticed also that the quantities C', D', E', ... are used in the proof merely as artificial means of distributing the terms of the left-hand series into groups suitable for our purpose. And it is obvious that a different grouping would have led us to another type of symbolical equivalence; as will be seen in the following Article.

46. To shew that the two following infinite series are symbolically equivalent to each other;

$$A + Bx + Cx^2 + \ldots = x^p(A + Bx^{-1} + Cx^{-2} + \ldots),$$

the index p being subject to the sole condition that it must not be a positive integer.

Instead of B, C, D, ... in the left-hand series substitute the following quantities,

$$B = B' + \frac{p}{1}A,$$

$$C = C' + \frac{p-2}{1}B' + \frac{p(p-1)}{1\,.\,2}A,$$

$$D = D' + \frac{p-4}{1} C' + \frac{(p-2)(p-3)}{1.2} B' + \frac{p(p-1)(p-2)}{1.2.3} A,$$

&c. = &c.,

the law of these substitutions being obvious, and requiring, as the series are infinite, that p shall not be a positive integer.

On making these substitutions and proceeding step by step as in the preceding Article we arrive at the following symbolical equivalence,

$$\begin{aligned} A + Bx + Cx^2 + \ldots &= x^p (A + Bx^{-1} + Cx^{-2} + \ldots) \\ &= Ax^p + Bx^{p-1} + Cx^{p-2} + \ldots \end{aligned}$$

47. This result may be presented in the following form,

$$\begin{aligned} A + Bx + Cx^2 + \ldots &= \left\{A + \frac{B}{p}\frac{d}{dx} + \frac{C}{p(p-1)}\left(\frac{d}{dx}\right)^2 + \ldots\right\} x^p \\ &= F\left(\frac{d}{dx}\right) . x^p \ldots\ldots\ldots\ldots\ldots\ldots(1). \end{aligned}$$

And if $A, B, C, \ldots$ are absolutely arbitrary and independent, then is also the function $F\left(\frac{d}{dx}\right)$ an arbitrary function of $\frac{d}{dx}$, subject only to $F\left(\frac{d}{dx}\right) 0 = 0$.

The value of this result in the discovery of subintegrals will be seen when we come to the actual integration of equations.

The reader may compare these results with Art. 23.

CHAPTER IV.

THE TRANSFORMATION OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

I. *Two independent variables; coefficients constant.*

48. Our object in this chapter is to reduce equations to their most simple forms with the view of discovering those forms which present peculiar integrational difficulties.

We may classify any linear differential equation of two independent variables and having constant coefficients under some one of the four following heads,

$$(\alpha)\ldots 0 = \frac{d^2u}{dx\,dy} + A\frac{du}{dx} + B\frac{du}{dy} + Cu.$$

$$(\beta)\ldots 0 = \frac{d^2u}{dx^2} + (a+b)\frac{d^2u}{dx\,dy} + ab\frac{d^2u}{dy^2} + A\frac{du}{dx} + B\frac{du}{dy} + Cu.$$

$$(\gamma)\ldots 0 = \left(\frac{d}{dx} + \frac{ad}{dy}\right)^2 u + A\frac{du}{dx} + B\frac{du}{dy} + Cu.$$

$$(\delta)\ldots 0 = \left(\frac{d}{dx} + \frac{ad}{dy}\right)^2 u + A\left(\frac{d}{dx} + \frac{ad}{dy}\right) u + Cu.$$

49. To reduce the form (α) assume a new dependent variable v such that

$$u = v\epsilon^{-Ay-Bx}.$$

This gives the following reduced form when substituted for u,

$$0 = \frac{d^2v}{dx\,dy} + (C - AB)\,v,$$

which comprehends the two following elementary forms,

$$(1)\ldots\ldots 0=\frac{d^2\omega}{dx\,dy}, \text{ and } \frac{d^2\omega}{dx\,dy}=\omega\ldots\ldots(2).$$

The former of these presents no integrational difficulty; and the latter we shall integrate in a future Article.

50. To reduce the form (β) we change the dependent variables by assuming two new variables ξ, η such that

$$x=\xi+\eta \text{ and } y=a\xi+b\eta.$$

$$\therefore\ \frac{du}{d\xi}=\left(\frac{d}{dx}+a\frac{d}{dy}\right)u, \text{ and } \frac{du}{d\eta}=\left(\frac{d}{dx}+b\frac{d}{dy}\right)u,$$

and the reduced equation is

$$0=\frac{d^2u}{d\xi\,d\eta}+\frac{B-bA}{a-b}\frac{du}{d\xi}+\frac{B-aA}{b-a}\frac{du}{d\eta}+Cu,$$

which being of the form (α) can be reduced to the forms (1) and (2), and therefore furnishes no new integrational difficulty.

51. To reduce form (γ) we assume $x=\xi+\eta$ and $y=a\xi+\frac{B}{A}\eta$;

$$\therefore\ \frac{du}{d\xi}=\left(\frac{d}{dx}+a\frac{d}{dy}\right)u, \text{ and } \frac{du}{d\eta}=\left(\frac{d}{dx}+\frac{B}{A}\frac{d}{dy}\right)u,$$

and the following is the form of the reduced equation,

$$0=\frac{d^2u}{d\xi^2}+A\frac{du}{d\eta}+Cu;$$

and by changing the dependent variable, if necessary, by writing $v\epsilon^{m\eta}$ for u, m being such as to satisfy the equation $Am+C=0$, we obtain the following form,

$$0=\frac{d^2v}{d\xi^2}+A\frac{dv}{d\eta}.$$

The following are therefore the ultimate forms furnished by form (γ),

$$(3)\ldots\ldots 0=\frac{d^2\omega}{dx^2}+\omega,\quad \frac{d^2\omega}{dx^2}=\frac{d\omega}{dy}\ldots\ldots(4),$$

and

$$\frac{d^2\omega}{dx^2}=0\ldots\ldots(5),$$

of which (4) only presents any new integrational difficulty.

52. To reduce form (δ) we assume $x = \xi + \eta$ and $y = a\xi + b\eta$;

$$\therefore \frac{du}{d\xi} = \left(\frac{d}{dx} + a\frac{d}{dy}\right)u,$$

and the reduced equation is

$$0 = \frac{d^2u}{d\xi^2} + A\frac{du}{d\xi} + Cu,$$

a form which does not contain η $\left(\text{which is equal to } \frac{ax-y}{a-b}\right)$ and presents no integrational difficulty. It is in fact an equation of one independent variable; and consequently, when it is integrated, arbitrary functions of η, or rather of $(ax - y)$ must be used instead of arbitrary constants.

53. Hence gathering together the forms that are of difficult integration we find only the two following,

$$\frac{d^2u}{dx\,dy} = u, \text{ and } \frac{d^2u}{dx^2} = \frac{du}{dy}.$$

These forms it will be our business to integrate in the following chapter.

II. *The case of three independent variables.*

54. We may arrange any equation of this class under some one of the four following heads,

$$(\alpha)\ldots 0 = \frac{d^2u}{dx\,dy} + A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

$$(\beta)\ldots 0 = \frac{d^2u}{dx\,dy} + a\frac{d^2u}{dx\,dz} + A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

$$(\gamma)\ldots 0 = \frac{d^2u}{dx\,dy} + a\frac{d^2u}{dx\,dz} + b\frac{d^2u}{dy\,dz} + A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

$$(\delta)\ldots 0 = \frac{d^2u}{dx^2} + \alpha\frac{d^2u}{dy^2} + \beta\frac{d^2u}{dz^2} + a\frac{d^2u}{dx\,dy} + b\frac{d^2u}{dx\,dz} + c\frac{d^2u}{dy\,dz}$$
$$+ A\frac{du}{dx} + B\frac{du}{dy} + C\frac{du}{dz} + Ku.$$

55. To reduce the form (α) we assume $u = v\epsilon^{-Ay-Bx-mz}$, where m is a constant that satisfies the equation $mC + AB = K$, and the following is the reduced form of the equation,

$$0 = \frac{d^2v}{dx\,dy} + C\frac{dv}{dz},$$

which includes the two following elementary forms,

$$(1)\ldots\ldots \frac{d^2\omega}{dx\,dy} = 0, \text{ and } \frac{d^2\omega}{dx\,dy} = \frac{d\omega}{dz} \ldots\ldots(2).$$

The former of these presents no integrational difficulty.

56. To reduce the form (β) let x, y, ζ be a new set of independent variables, in which $\zeta = z - ay$. The reduced form of the equation is

$$0 = \frac{d^2u}{dx\,dy} + A\frac{du}{dx} + B\frac{du}{dy} + (C - aB)\frac{du}{d\zeta} + Ku,$$

which coinciding with form (α) introduces no additional integrational difficulty.

57. To reduce form (γ), for u write $v\epsilon^{mx+ny+pz}$, the constants m, n, p being such as will satisfy the three following equations:

$$0 = A + n + ap, \quad 0 = B + m + bp, \quad \text{and } 0 = C + am + bn.$$

The reduced equation is the following,

$$0 = \frac{d^2v}{dx\,dy} + a\frac{d^2v}{dx\,dz} + b\frac{d^2v}{dy\,dz} + K'v,$$

in which $K' = Cp - mn + K$.

Let now the independent variables be changed to x, y, ζ where $\zeta = z - ay - bx$. By this means the reduced equation becomes

$$\frac{d^2v}{dx\,dy} = ab\frac{d^2v}{d\zeta^2} + K'v,$$

which includes the two following new elementary forms,

$$(3)\ldots\ldots \frac{d^2\omega}{dx\,dy} = \frac{d^2\omega}{d\zeta^2}, \text{ and } \frac{d^2\omega}{dx\,dy} = \frac{d^2\omega}{d\zeta^2} + \omega \ldots\ldots(4).$$

58. To reduce the form (δ), assume a new set of independent variables ξ, η, ζ such that

$$\xi = x - gy, \quad \eta = y - hz, \quad \text{and} \quad \zeta = z - kx,$$

the constants g, h, k being such as will satisfy the following conditional equations,

$$0 = \alpha g^2 - ag + 1, \quad 0 = \beta h^2 - ch + 1, \quad \text{and} \quad 0 = k^2 - bk + \beta.$$

By these means the form (δ) will be reduced to form (γ), and consequently introduces no new elementary forms.

59. Gathering together the elementary forms which present integrational difficulties, we find that they are the three following:

$$\frac{d^2u}{dx\,dy} = \frac{du}{dz}, \quad \frac{d^2u}{dx\,dy} = \frac{d^2u}{dz^2}, \quad \text{and} \quad \frac{d^2u}{dx\,dy} = \frac{d^2u}{dz^2} + u.$$

In this chapter we are therefore presented with five difficult linear differential equations of the second order with constant coefficients; viz. two when there are two independent variables, and three when there are three independent variables.

CHAPTER V.

INTEGRATION OF EQUATIONS OF TWO INDEPENDENT VARIABLES.

In the preceding chapter we have seen that the two following equations present the only difficulties that are experienced in the integration of linear equations of the second order with constant coefficients. In this chapter we shall bring in the properties of germs to our aid in the task of effecting their complete integration.

60. To integrate $\dfrac{d^2u}{dx^2} = \dfrac{du}{dt}$.

According to Art. 21 the following is a series which may be assumed for the complete integral of this equation,

$$u = P + Q\frac{t}{1} + R\frac{t^2}{1\,.\,2} + \ldots,$$

which being substituted in the proposed equation gives the following complete form of u,

$$u = \left(1 + \frac{t}{1}\,\frac{d^2}{dx^2} + \frac{t^2}{1\,.\,2}\cdot\frac{d^4}{dx^4} + \ldots\right)P,$$

where
$$P = A + B\frac{x}{1} + C\frac{x^2}{1\,.\,2} + \ldots,$$

the constants A, B, C, ... which are absolutely arbitrary, being the coefficients of the subintegral constituents of u. Hence we may write $M, M^2, M^3, \ldots$ for them (Art. 13), M being an extemporized germ; and then we shall have the following equivalence,

$$P = A + B\frac{x}{1} + C\frac{x^2}{1.2} + \ldots = A\epsilon^{Mx};$$

$$\therefore\ u = A\left(1 + \frac{t}{1}\frac{d^2}{dx^2} + \frac{t^2}{1.2}\cdot\frac{d^4}{dx^4} + \ldots\right)\epsilon^{Mx}$$

$$= A\epsilon^{M^2t+Mx}.$$

Thus it is proved, for the proposed equation, that the general exponential integral of Art. 30 is the *complete* integral.

In this form of the general integral the minor germs of x and t are implicitly contained in A, the general coefficient; and M is a general germ, i.e. it is liable to contain both real and imaginary quantities. The major germ is explicitly contained in the integral, on which account the integral takes a form which we may refer to as the major-germ form.

61. In Art. 31 we have shewn the general method of expressing an exponential integral, that contains general germs, in an equivalent integral containing real germs only.

In the integral just found we have merely to write $M + im$ for M; and the following is the form in real germs M, m;

$$\therefore\ u = A\epsilon^{(M^2-m^2)t+Mx}\cos m\,(2Mt + x + B).$$

The minor germs of x and t are in this integral implicitly contained in the arbitrary constants A, B. The form is a major-germ form.

62. To find the integral of $\dfrac{d^2u}{dx^2} = \dfrac{du}{dt}$ in a minor-germ form, i.e. in a form which renders the major germ latent in the arbitrary constants of the integral.

From Art. 34 we learn that the subintegrals obtained by the elimination of a major germ will all be homogeneous and of different dimensions. This therefore suggests the following method of obtaining the subintegrals required.

Let V be a function of x and t which is of zero dimensions. The general representative of such a function in the case of the proposed example will be $V=\phi\left(\frac{x}{\sqrt{t}}\right)$, for if Mx, M^2t be written in this for x, t, the germ M will disappear. Denote $\frac{x}{\sqrt{t}}$ by v, and then the following general form of homogeneity will represent any one of the subintegrals,

$$P=t^pV,$$

the dimensions of this subintegral being p.

This being an integral of the proposed equation must satisfy it, and being substituted therein, the following is the resulting equation for the determination of V,

$$\frac{d^2V}{dv^2}+\frac{v}{2}\frac{dV}{dv}=pV.$$

Now we wish to obtain subintegrals in a finite form, or if that be not possible, then in a form that shall give a finite expression for u.

We make use of p (which is disposable) for this purpose; and enquire what value of p will give a finite expression for V. We can see at once that $p=-\frac{1}{2}$ will answer our purpose. The above equation being integrated on this supposition, we find

$$V=A\epsilon^{-\frac{v^2}{4}}+B\epsilon^{-\frac{v^2}{2}}\int\epsilon^{\frac{v^2}{4}}\,dv;$$

$$\therefore P=t^pV=At^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}}+Bt^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}}\int\epsilon^{\frac{v^2}{4}}\,dv.$$

The last term we reject because it is not in a finite form;

$$\therefore\ u=F\left(\frac{d}{dx},\ \frac{d}{dt}\right).t^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}}.$$

Now the proposed equation shews that $\frac{d}{dt}$ when applied to any integral of the proposed equation is equivalent to $\left(\frac{d}{dx}\right)^2$ applied to the same integral;

$$\therefore\ F\left(\frac{d}{dx},\ \frac{d}{dt}\right) \doteq F\left(\frac{d}{dx}\right);$$

$$\therefore\ u = F\left(\frac{d}{dx}\right) . t^{-\frac{1}{2}} \epsilon^{-\frac{x^2}{4t}},$$

and this is the general integral of the proposed equation, in a form that renders the major germ latent. (Here the sign $\doteq$ denotes integral equivalence; and F stands for the *words* "arbitrary function of.")

Lest the reader should have any doubt of the generality of this result we will obtain it in another manner.

63. The proposed equation $\dfrac{d^2u}{dx^2} = \dfrac{du}{dt}$ has constant coefficients, consequently the following is by Art. 21 the general assumption for its perfect integral,

$$u = P + Q\frac{x}{1} + R\frac{x^2}{1 . 2} + S\frac{x^3}{1 . 2 . 3} + \ldots,$$

in which P, Q, R, ... are serial functions of t of the general form $A + B\dfrac{t}{1} + C\dfrac{t^2}{1 . 2} + \ldots$

The substitution of this value of u in the proposed equation furnishes the following form of the general integral,

$$\left.\begin{aligned} u = &\left(1 + \frac{x^2}{2\,!}\,\frac{d}{dt} + \frac{x^4}{4\,!}\,\frac{d^2}{dt^2} + \ldots\right) P \\ &+ \left(\frac{x}{1} + \frac{x^3}{3\,!}\,\frac{d}{dt} + \frac{x^5}{5\,!}\,\frac{d^2}{dt^2} + \ldots\right) Q \end{aligned}\right\}\ldots\ldots\ldots\ldots(1).$$

These are the two subgeneral integrals; and the former contains only even powers of x, and the latter only its odd powers; and this is due to the fact that $\dfrac{d}{dx}$ occurs in the proposed equation in the form $\left(\dfrac{d}{dx}\right)^2$ only.

The first subgeneral

$$= \left(1 + \frac{x^2}{2!}\frac{d}{dt} + \dots\right) P$$

$$= \psi\left(\frac{x^2 d}{dt}\right)\left(A + B\frac{t}{1} + C\frac{t^2}{1.2} + \dots\right)$$

$$= \psi\left(\frac{x^2 d}{dt}\right)(t+g)^p \text{......see Art. 22,}$$

$$= \left\{1 + \frac{p}{2!}\left(\frac{x^2}{t+g}\right) + \frac{p(p-1)}{4!}\left(\frac{x^2}{t+g}\right)^2 + \dots\right\}(t+g)^p.$$

Our wish is to obtain the subgeneral integrals in a finite form, and therefore we now ask what value of p will enable us to find the involopment of this infinite series. There is no particular difficulty in seeing that $p = -\frac{1}{2}$ will enable us to do this;

$$\therefore \text{ first subgeneral} = \epsilon^{-\frac{x^2}{4(t+g)}}(t+g)^{-\frac{1}{2}}$$

$$= F\left(\frac{d}{dt}\right).\, t^{-\frac{1}{2}}\epsilon^{-\frac{x^2}{4t}} \text{......(Art. 19).}$$

And from this we can deduce the *form* of the second subgeneral integral.

For $\quad F\left(\frac{d}{dt}\right).\, t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}} = \left(1 + \frac{x^2}{2!}\frac{d}{dt} + \frac{x^4}{4!}\frac{d^2}{dt^2} + \dots\right)P.$

Differentiate with $\frac{d}{dx}$.

$$\therefore\ F\left(\frac{d}{dt}\right).\frac{d}{dx}t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}} = \left(\frac{x}{1} + \frac{x^3}{3!}\frac{d}{dt} + \frac{x^5}{5!}\frac{d^2}{dt^2} + \dots\right).\frac{d}{dt}P$$

$$= \left(\frac{x}{1} + \frac{x^3}{3!}\frac{d}{dt} + \dots\right)\left(B + C\frac{t}{1} + D\frac{t^2}{1.2} + \dots\right).$$

Now the right-hand member of this is of precisely the same *form* and *generality* as the second subgeneral integral in (1);

$$\therefore \text{ second subgeneral} = f\left(\frac{d}{dt}\right).\frac{d}{dx}t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}}$$

$$= f\left(\frac{d}{dx}\right)t^{-\frac{1}{2}}\epsilon^{\frac{-x^2}{4t}},$$

with the understanding that this integral shall contain only odd powers of x.

Hence if we gather the two subgeneral integrals together we have two terms, of which one contains only even powers of x, and the other only odd powers;

$$\therefore\ u = \left\{F\left(\frac{d}{dt}\right) + f\left(\frac{d}{dx}\right)\right\} . t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}}$$

$$= F\left(\frac{d}{dx}\right) . t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}},$$

which agrees with the result obtained in the previous Article.

64. Hence we have found the two following forms of the general integral of the equation $\dfrac{d^2u}{dx^2} = \dfrac{du}{dt}$,

$$(1) \ldots\ u = A\epsilon^{Mt^2+Mx},$$

in which the major germ M is explicit, and the minor germs latent; and

$$(2) \ldots\ u = F\left(\frac{d}{dx}\right) . t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}},$$

in which both the major and minor germs are latent in the general operative function $F\left(\dfrac{d}{dx}\right)$;

$$\therefore\ A\epsilon^{M^2t+Mx} = F\left(\frac{d}{dx}\right) . t^{-\frac{1}{2}} \epsilon^{\frac{-x^2}{4t}}.$$

It will not be forgotten by the reader, that when (=) does not denote algebraic equality, it denotes the words "symbolical equivalence."

65. To integrate $\dfrac{d^2u}{dx\,dy} = u$.

Both x and y take minor germs. Hence the general integral can be completely expressed in a series containing only positive integer powers of x and y.

The following may therefore be assumed as a general form of u,

$$u = P + Q\frac{y}{1} + R\frac{y^2}{1.2} + S\frac{y^3}{1.2.3} + \ldots,$$

P, Q, R, ... being serial functions of x of the general form

$$A + B\frac{x}{1} + C\frac{x^2}{1.2} + \ldots$$

Substitute this form of u in the proposed equation; and the following is the result (in which we take the liberty of using $\frac{dx}{d}$ for the symbol of integration),

$$u = \left\{1 + \frac{y}{1}\frac{dx}{d} + \frac{y^2}{1.2}\left(\frac{dx}{d}\right)^2 + \ldots\right\} P$$

$$= \psi\left(y\frac{dx}{d}\right)\left(A + B\frac{x}{1} + C\frac{x^2}{1.2} + \ldots\right)$$

$$= \psi\left(y\frac{dx}{d}\right).\epsilon^{cx} \ldots\ldots \text{Art. (29)}$$

$$= A\left(1 + \frac{y}{1}.\frac{1}{c} + \frac{y^2}{1.2}.\frac{1}{c^2} + \ldots\right)\epsilon^{cx}$$

(1) ...... $= A\epsilon^{cx+c^{-1}y}$, c being a general germ.

Hence the exponential integral

$$u = A\epsilon^{Mx+Ny}, \text{ subject to } MN = 1,$$

is perfectly general in the example before us in this Article.

We may obtain the first subintegral in the following manner by the elimination of the germ c from (1).

The form of the proposed equation shews that the product (xy) is of zero dimensions. Let $v^2 = xy$, and let V be a function of v. We may assume the following as the general representative of subintegrals,

$$P = x^p V.$$

This being substituted in the proposed equation gives the following for the determination of V,

$$\frac{d^2V}{dv^2} + \frac{2p+1}{v}\frac{dV}{dv} - 4V = 0.$$

This will be integrable in a finite form if we assume $2p+1=0$, and therefore $p=-\frac{1}{2}$;

$$\therefore \quad V = A\epsilon^{2v} + B\epsilon^{-2v}$$
$$= A\epsilon^{2jv},$$

and the first subintegral $P = x^{-\frac{1}{2}}V$

$$= x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}};$$

$$\therefore \; u = F\left(\frac{d}{dx}, \frac{d}{dy}\right) . \, x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} \quad \text{.................} \quad (2).$$

Now it appears from the form of the proposed equation that the symbolical product of $\dfrac{d}{dx}$ and $\dfrac{d}{dy}$ is equivalent to unity, when applied to any integral of that equation; the above form of u may therefore be presented in the following equivalent form,

$$u = F\left(\frac{d}{dx}\right) . \, x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} + f\left(\frac{d}{dy}\right) . \, y^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} \; \text{.........} (3).$$

The following equation is obviously true, and it gives rise to the latter of these two subgeneral integrals,

$$\frac{d}{dy} . \, x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}} = y^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}}.$$

It has therefore been proved above that the elimination of the major germ c from the exponential integral

$$u = A\epsilon^{cx+c^{-1}x} \text{.............................} (4)$$

gives the following form of the first subintegral, to which we shall often have occasion to refer,

$$P = x^{-\frac{1}{2}}\epsilon^{2j\sqrt{xy}}.$$

Now the integral (4) is expressed in terms of c as a general germ; but we may express it in real germs by writing

$$c\,(\cos m + i \sin m) \text{ for } c,$$

and $\qquad c^{-1}(\cos m - i \sin m)$ for c^{-1},

in which new forms of the germs, c and m are to be considered real germs.

Let $K=cx+c^{-1}y$, and $I=cx-c^{-1}y$, then the integral (4) will take the following equivalent general form of expression in real germs,

$$u=A\epsilon^{K\cos m}\cos(I\sin m+B)\ \ldots\ldots\ldots\ldots\ldots(5).$$

66. The various integrals of $\dfrac{d^2u}{dx\,dy}+u=0$ may be deduced from the two preceding Articles by writing therein $-y$ for y.

$$\therefore\ u=A\epsilon^{Mx+Ny}=F\left(\frac{d}{dx},\frac{d}{dy}\right).x^{-\frac{1}{2}}\cos(2\sqrt{xy}+B)$$

subject to $MN+1=0$.

The following Article is introduced for the purpose of future reference.

67. To change the independent variables of the expression

$$\frac{d^2u}{dx\,dy}.$$

Let ξ and η the new independent variables be such that

$$\frac{d}{dx}=\frac{d}{d\xi}+a\frac{d}{d\eta},\ \text{ and }\ \frac{d}{dy}=\frac{d}{d\xi}+b\frac{d}{d\eta}.$$

These assumptions require that a, b shall not be equal.

$$\therefore\ \xi=x+y,\ \text{ and }\ \eta=ax+by,$$

and also, $$x=\frac{\eta-b\xi}{a-b},\ \text{ and }\ y=-\frac{\eta-a\xi}{a-b}.$$

$$\therefore\ \frac{d^2u}{dx\,dy}=\left(\frac{d}{d\xi}+a\frac{d}{dy}\right)\left(\frac{d}{d\xi}+b\frac{d}{d\eta}\right)u$$

$$=\frac{d^2u}{d\xi^2}+(a+b)\frac{d^2u}{d\xi\,d\eta}+ab\frac{d^2u}{d\eta^2}\ \ldots\ldots\ldots\ldots(1).$$

Also $$(a-b)^2xy=-(\eta-a\xi)(\eta-b\xi)$$

$$=-(\eta^2-\overline{a+b}\,\eta\xi+ab\xi^2)\ \ldots\ldots\ldots\ldots(2).$$

68. To integrate

$$\frac{d^2u}{d\xi^2}+(a+b)\frac{d^2u}{d\xi\,d\eta}+ab\frac{d^2u}{d\eta^2}=u.$$

We deduce the required integral from Art. 65 by writing therein the above values of x and y in terms of ξ and η.

$$\therefore\ u = F\left(\frac{d}{d\xi} + a\frac{d}{d\eta}\right) . (\eta - b\xi)^{-\frac{1}{2}} \epsilon^{\frac{2i}{a-b}(\eta^2 - \overline{a+b}\eta\xi + ab\xi^2)^{\frac{1}{2}}}$$

$$+ f\left(\frac{d}{d\xi} + b\frac{d}{d\eta}\right) . (\eta - a\xi)^{-\frac{1}{2}} \epsilon^{\frac{2i}{a-b}(\eta^2 - \overline{a+b}\eta\xi + ab\xi^2)^{\frac{1}{2}}}.$$

69. To integrate the equation $\left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)^n u = 0$, in which the compound operative symbol $\left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)$ is repeated n times.

Changing the dependent variable, assume either $u = \epsilon^{jmx} Y$, or $u = \epsilon^{m^2 y} X$.

We begin with the former (m being a general germ).

$$\therefore\ 0 = \left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)^n \epsilon^{jmx} Y$$

$$= \epsilon^{jmx}\left(m^2 - \frac{d}{dy}\right)^n Y,$$

$$\therefore\ 0 = \epsilon^{m^2 y}\left(\frac{d}{dy}\right)^n . Y\epsilon^{-m^2 y},$$

$$\therefore\ 0 = \left(\frac{d}{dy}\right)^n . Y\epsilon^{-m^2 y},$$

$$\therefore\ Y\epsilon^{-m^2 y} = A + B\frac{y}{1} + C\frac{y^2}{1.2} + \ldots + N\frac{y^{n-1}}{(n-1)!}$$

$$= A(h+y)^{n-1},$$

h being a minor germ of y.

$$\therefore\ u = \epsilon^{jmx} Y$$

$$= A\epsilon^{jmx + m^2 y}(h+y)^{n-1} \qquad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Had we taken the form $u = \epsilon^{m^2 y} X$ we should have found

$$0 = \left(\frac{d^2}{dx^2} - \frac{d}{dy}\right)^n \epsilon^{m^2 y} X,$$

$$\therefore\ 0 = \left(\frac{d^2}{dx^2} - m^2\right)^n X$$

$$= \left(\frac{d}{dx} - m\right)^n \left(\frac{d}{dx} + m\right)^n X,$$

which is equivalent to the following integrals,

$$\left(\frac{d}{dx}-m\right)^n X=0, \text{ and } \left(\frac{d}{dx}+m\right)^n X=0,$$

$$\therefore \left(\frac{d}{dx}\right)^n . X\epsilon^{-mx}=0, \text{ and } \left(\frac{d}{dx}\right)^n . X\epsilon^{mx}=0,$$

$$\therefore u=\epsilon^{m^2y}\{A\epsilon^{mx}(x+g)^{n-1}+B\epsilon^{-mx}(x+l)^{n-1}\} \text{ .... (2)},$$

which agrees with (1); since $A\epsilon^{Jmx}$ represents both $A\epsilon^{mx}$ and $B\epsilon^{-mx}$.

A similar method of treatment will succeed with the equation

$$\left(\frac{d^2}{dx\,dy}-1\right)^n u=0.$$

We have hitherto confined ourselves to equations with constant coefficients; but in the following examples the coefficients are functions of one of the independent variables.

70. To integrate

$$\frac{d^2u}{dx\,dy}=\frac{a}{x}\frac{du}{dx}+\frac{b}{x}\frac{du}{dy}.$$

In this equation x and y can take a major germ; and y can take a minor germ also.

Hence changing the dependent variable we assume the following general form for the integral of this equation,

$$u=\epsilon^{my}X,$$

m being the major germ, and X being a function of x.

This value of u being substituted in the proposed equation gives the following for the determination of X:

$$\left(x-\frac{a}{m}\right)\frac{dX}{dx}=bX,$$

$$\therefore X=A\left(x-\frac{a}{m}\right)^b$$

$$=A(mx-a)^b,$$

$$\therefore u=A(mx-a)^b\epsilon^{my} \text{ .... (1)}.$$

In this integral the minor germ is latent, and the major germ is explicitly involved in it. Also m is a general germ.

Again, to find the general integral in a form which renders the major germ latent.

We assume $x^p V$ as the general type of the subintegrals; V being a function of v, and $v = \frac{y}{x}$.

Substituting $x^p V$ for u in the proposed equation we find the following equation for the determination of V in a finite form;

$$\frac{d}{dv}\left(v\frac{dV}{dv}\right) + (b - p - av)\frac{dV}{dv} + apV = 0.$$

That this may be integrable immediately the following condition must be satisfied; $ap = -a$.

Consequently in the general case $p = -1$; but when $a = 0$, p will be subject to no condition. $a = 0$ is therefore an exceptional case.

$$\therefore \frac{d}{dv}\left(v\frac{dV}{dv}\right) + (1 + b - av)\frac{dV}{dv} - aV = 0.$$

This equation being integrated, gives the following as the first subintegral,

$$P = \frac{V}{x} = \frac{Ax^b}{y^{b+1}}\epsilon^{\frac{ay}{x}} + \frac{Bx^b}{y^{b+1}}\epsilon^{\frac{ay}{x}}\int v^b\epsilon^{-av}\,dv \text{............} (2),$$

and
$$u = F\left(\frac{d}{dy}\right)(x^{-1}V).$$

Thus the proposed equation is completely integrated in a form that renders the major germ m latent; but the term multiplied by B will not be in a finite form, and will therefore have to be rejected, unless b be a positive integer.

We will now take the exceptional case, viz. when $a = 0$.

71. To integrate $\dfrac{d^2u}{dx\,dy} = \dfrac{b}{x}\dfrac{du}{dy}$.

$$\therefore u = x^b F(y) + f(y).$$

CHAPTER VI.

EQUATIONS NEARLY RELATED TO LAPLACE'S EQUATION.

Coefficients constant.

72. To integrate $\dfrac{d^2u}{dx^2} = \dfrac{d^2u}{dt^2}$.

Let M be a general germ.

$$\therefore\ u = A\epsilon^{M(x+jt)}$$

$$= \phi(x+jt) \text{ .... } (1).$$

Expressed in terms of real germs only, we have by the method of Art. 31,

$$u = A\epsilon^{M(x+jt)} \cos m(x+jt+B) \text{ .... } (2),$$

M and m being in this form independent real germs.

To obtain the integral from which major germs are eliminated, let us assume $v = \dfrac{x}{t}$, and $V = \phi(v)$. Then all the sub-integrals will be of the form $P = t^p V$, which substituted in the proposed equation gives,

$$\frac{d^2V}{dv^2} = \frac{d}{dv}\left(v^2 \frac{dV}{dv}\right) - 2pv\frac{dV}{dv} + p(p-1)V.$$

This equation will be integrable at once, and therefore in finite terms, if we assume $-2p = p(p-1)$,

$$\therefore\ p(p+1) = 0.$$

The two roots of this are $p=0$, and $p=1$. The former is of a doubtful character as to whether zero is or is not to be considered to be *not a positive integer;* the latter we see is allowable.

We try the former, rejecting that part of the result which is not in finite terms, and find

$$V = P = A \log \frac{v-1}{v+1}$$

$$= A \log \frac{x-t}{x+t} \quad \text{..........................} \quad (3).$$

To find the second subintegral, or rather the first subintegral corresponding to the second subgeneral integral, we assume $p=-1$.

$$\therefore \quad \frac{d^2 V}{dv^2} = \frac{d}{dv}\left(v^2 \frac{dV}{dv}\right) + 2v \frac{dV}{dv} + 2V.$$

$$\therefore \quad V = \frac{Av+B}{v^2-1},$$

and

$$P = t^{-1} V = \frac{Ax+Bt}{x^2-t^2} \quad \text{......................} \quad (4),$$

$$= A(x+t)^{-1} + B(x-t)^{-1},$$

$$\therefore \quad u = F\left(\frac{d}{dx}\right) P$$

$$= F(x+t) + f(x-t),$$

which agrees with equation (1).

But as the value of P in (4) is of the dimension (-1) we may by integrating it with regard to x raise it to the dimension zero; in which case the first subintegral will be

$$P = A \log \left(\frac{x-jt}{x+jt}\right)^{-\frac{1}{2}} + B \log \sqrt{x^2-t^2}.$$

73. Change the independent variables of the equation

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2};$$

and let the new variables ξ, η be such that

$$\xi = \log \sqrt{x^2 - t^2}, \text{ and } \eta = \log \left(\frac{x - jt}{x + jt}\right)^{\frac{1}{2}},$$

then

$$\frac{d^2u}{d\xi^2} = \frac{d^2u}{d\eta^2}.$$

Hence the form of the proposed equation is not changed by this change of variables; from which it follows that we are at liberty to write $\log \sqrt{x^2 - t^2}$ for x, and $\log \left(\frac{x - jt}{x + jt}\right)^{\frac{1}{2}}$ for t in any integral of

$$\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2},$$

and the resulting formula will be an integral of the same equation.

74. We shall now consider the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0.$$

We begin with the following proposition respecting this equation and its integral. If in an integral of the proposed equation we write $ax + jby$ for x, and $ay - jbx$ for y, the resulting formula will be an integral of the same equation; a, b being arbitrary constants.

Let $\xi = ax + jby$, and $\eta = ay - jbx$, and let ξ, η be the new independent variables. We find that the result of this change is the following differential equation

$$\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2} = 0.$$

Hence the form of the equation is not affected by this change of the independent variables; and consequently we may write the above values of ξ, η instead of x, y in any known integral of the proposed equation and the resulting formula will also be an integral of it.

On this we may remark that the substitution of $ax + jby$ and $ay - jbx$ for x and y will introduce two germs a, b into the integral; and if the integral in which this substitution is made had been deficient in the number of germs it contained, the integral that results from these substitutions will contain two additional germs, and may possibly now contain the requisite number to render the integral general.

An example will illustrate this.

$$U = \epsilon^x \cos y$$

is manifestly an integral of the proposed equation, and it contains no germs. Make the above substitutions for x and y; then the following is also an integral of the proposed equation, and it contains two germs a, b,

$$U = \epsilon^{ax+jby} \cos(ay - jbx).$$

If into this we introduce the minor germs of x and y we have the following result which is (as we shall presently prove) the general integral of the proposed equation,

$$U = A\epsilon^{ax+jby} \cos(ay - jbx + B) \ldots\ldots\ldots\ldots (1).$$

75. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} = 0$.

The general exponential integral is

$$u = A\epsilon^{M(x+iy)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$= \phi(x + iy),$$

(in which we are at liberty to write $ax + jby$ for x and $ay - jbx$ for y).

Let $r^2 = x^2 + y^2$, and $\tan\theta = \dfrac{y}{x}$, $\therefore\ x = r\cos\theta,\quad y = r\sin\theta$,

$$\begin{aligned}
\therefore\ u &= \phi(r\,.\,\overline{\cos\theta + i\sin\theta}) \\
&= \phi(r\epsilon^{i\theta}) \\
&= \phi(r\epsilon^{i\theta}) + \psi(r\epsilon^{-i\theta}) \\
&= \phi(r^j\epsilon^{i\theta}) \\
&= Ar^{jn}\epsilon^{in\theta} \\
&= (Ar^n + Br^{-n})(a\epsilon^{n\theta} + b\epsilon^{-n\theta}) \\
&= (Ar^n + Br^{-n})(a\cos n\theta + b\sin n\theta) \ldots\ldots\ldots (2),
\end{aligned}$$

n being a germ; and A, B, a, b being independent arbitrary constants.

76. Let r and θ be made the independent variables instead of x and y; then the equation of the preceding Article takes the following form,

$$\left(\frac{rd}{dr}\right)^2 u + \frac{d^2u}{d\theta^2} = 0,$$

from which we learn, that we may write in any integral of the equation of the preceding Article r^j for r, and $j\theta$ for θ.

Also θ takes a minor germ, and r a major germ.

If we seek the first subintegral after the manner of Art. 72, we find

$$\text{first subintegral} = A \log \sqrt{x^2+y^2} + B \tan^{-1}\frac{y}{x}.$$

$$\therefore\ u = F\left(\frac{d}{dx}\right) \log \sqrt{x^2+y^2} + f\left(\frac{d}{dx}\right) \tan^{-1}\frac{y}{x} \ldots\ldots\ldots(3).$$

The integrals (1) and (3) are symbolically equivalent.

If $\xi = \log \sqrt{x^2+y^2}$,

$$\frac{d^2u}{d\xi^2} + \frac{d^2u}{d\theta^2} = 0.$$

Hence we may write $\log \sqrt{x^2+y^2}$, $\tan^{-1}\frac{y}{x}$ for x, y in any integral of the equation of the preceding Article, and the resulting formula will be an integral of the same.

77. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} = u.$

The general exponential integral may be presented in the following form,

$$u = A\epsilon^{(M+N)x+(M-N)iy}$$

$$= A\epsilon^{M(x+iy)+N(x-iy)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

subject to the condition $4MN = 1$.

This equation of condition will be satisfied if we assume

$$2M = c\,(\cos m + i \sin m)$$
$$2N = c^{-1}\,(\cos m - i \sin m)$$

in which c and m are real germs, and also independent.

Let $$K = \tfrac{1}{2}\,(x \cos m - y \sin m)$$

and $$I = \tfrac{1}{2}\,(y \cos m + x \sin m)$$

$$\therefore\ u = A\epsilon^{(c+c^{-1})K} \cos \{(c - c^{-1})\, I + B\} \ldots\ldots\ldots\ldots (2).$$

78. If we now consider c a general germ, and assume $2M = c$, and $2N = c^{-1}$, we find the exponential integral in this form,

$$u = A\epsilon^{\frac{1}{2}c(x+iy)+\frac{1}{2}c^{-1}(x-iy)}.$$

This form of the exponential integral agrees, as to its germ c, with equation (1) of Art. 65, from which we learn that the following is the form of the first subintegral P,

$$P = (x + iy)^{-\frac{1}{2}}\, \epsilon^{j\sqrt{(x+iy)(x-iy)}}.$$

This comprehends the two forms, (r^2 being equal to $x^2 + y^2$),

$$P = \{A\,(x + iy)^{-\frac{1}{2}} + B\,(x - iy)^{-\frac{1}{2}}\}\, \epsilon^{jr}.$$

Now $$x + iy = r\,(\cos \theta + i \sin \theta) = r\epsilon^{i\theta}.$$

$$\therefore\ u = \left(\frac{A}{\sqrt{r}}\,\epsilon^{\frac{i\theta}{2}} + \frac{B}{\sqrt{r}}\,\epsilon^{-\frac{i\theta}{2}}\right)\epsilon^{jr}$$

$$= r^{-\frac{1}{2}}\,(A\epsilon^{\frac{i\theta}{2}} + B\epsilon^{-\frac{i\theta}{2}})\,(a\epsilon^{r} + b\epsilon^{-r}) \ldots\ldots\ldots (1),$$

in which A, B, a, b are independent arbitrary constants.

This value is symbolically represented by the following brief equivalent,

$$u = Ar^{-\frac{1}{2}}\,\epsilon^{jr}\,\epsilon^{\frac{i\theta}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

79. The integral of $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + u = 0$ may be deduced from the preceding Article by changing the algebraic sign of c^{-1} but not that of c.

$$\therefore\ u = A\epsilon^{(c-c^{-1})K} \cos \{(c + c^{-1})\, I + B\},$$

and
$$P = (x+iy)^{-\frac{1}{2}}\,\epsilon^{j\sqrt{(x+iy)\,(iy-x)}}$$
$$= (x+iy)^{-\frac{1}{2}}\,\epsilon^{i\sqrt{(x+iy)\,(x-iy)}}$$
$$\therefore\ u = Ar^{-\frac{1}{2}}\epsilon^{\frac{1}{2}i\theta}\epsilon^{ir}.$$

80. There are several important equations related to Laplace's equation which are reducible to the following type,

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx}.$$

We shall denominate this, when a is a positive quantity, the *standard* equation of this type, for a reason which will be seen presently.

For a small number of particular values of a this equation has been integrated, but for general values of a it has not been integrated.

The value of a admits of reduction in the following manner.

Let $a = 2n + b$, $2n$ being the greatest even integer in a; and let ω be an auxiliary dependent variable such that

$$\frac{d^2\omega}{dt^2} = \frac{d^2\omega}{dx^2} + \frac{b}{x}\frac{d\omega}{dx} \quad \text{..........................(1)}$$
$$= \frac{d}{dx}\left(x\frac{d\omega}{xdx}\right) + b\left(\frac{d\omega}{xdx}\right).$$

Operate on both sides of this equation with $\dfrac{d}{xdx}$,

$$\therefore\ \frac{d^2}{dt^2}\left(\frac{d\omega}{xdx}\right) = \frac{1}{x}\cdot\frac{d^2}{dx^2}\cdot x\left(\frac{d\omega}{xdx}\right) + \frac{b}{x}\frac{d}{dx}\left(\frac{d\omega}{xdx}\right)$$
$$= \frac{d^2}{dx^2}\left(\frac{d\omega}{xdx}\right) + \frac{b+2}{x}\frac{d}{dx}\left(\frac{d\omega}{xdx}\right).$$

On comparing this equation with (1) we perceive that we have here $\dfrac{d\omega}{xdx}$ instead of ω, and $b+2$ instead of b. These changes are simultaneous, and if repeated n times the following would necessarily be the result,

$$\frac{d^2}{dt^2}\left(\frac{d}{xdx}\right)^n\omega = \frac{d^2}{dx^2}\left(\frac{d}{xdx}\right)^n\omega + \frac{b+2n}{x}\frac{d}{dx}\left(\frac{d}{xdx}\right)^n\omega.$$

But $a = 2n + b$,

P

$$\therefore\ u = \left(\frac{d}{xdx}\right)^n \omega \text{ .... (2).}$$

81. When the integral of the standard equation is known for any positive value of a, the integral for an equal negative value can be deduced from it.

For the equation

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx}$$

is immediately reducible to the following form,

$$\frac{d^2u}{dt^2} = x^{-a}\frac{d}{dx} \cdot x^a \frac{du}{dx}.$$

Now operate on this with $\left(x^a \dfrac{d}{dx}\right)$,

$$\therefore\ \frac{d^2}{dt^2}\left(x^a \frac{du}{dx}\right) = x^a \frac{d}{dx} \cdot x^{-a} \frac{d}{dx}\left(x^a \frac{du}{dx}\right).$$

If now we write ω for $x^a \dfrac{du}{dx}$ we have

$$\frac{d^2\omega}{dt^2} = x^a \frac{d}{dx} \cdot x^{-a} \frac{d\omega}{dx}$$

$$= \frac{d^2\omega}{dx^2} - \frac{a}{x}\frac{d\omega}{dx} \text{ .... (1),}$$

which agrees with the equation in u, with the exception of having $-a$ instead of a. If therefore u the integral of the standard equation be known the integral of (1) will be known from the equation,

$$\omega = x^a \frac{du}{dx} \text{ .... (2).}$$

It will therefore be a sufficient solution of the problem of integrating the class of differential equations of the type

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx},$$

if in subsequent articles we confine ourselves to positive values of a.

82. The forms of the following differential equations are all deducible from the general form

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx}.$$

$$x\frac{d^2u}{dt^2} = \left(x\frac{d}{dx} + a\right)\frac{du}{dx}$$

$$= x^{-a}\left(x\frac{d}{dx}\right)x^a \cdot \frac{du}{dx},$$

$$\therefore \frac{d^2u}{dt^2} = x^{-a}\frac{d}{dx} \cdot x^a\frac{du}{dx} \quad \text{.......................(1).}$$

In this write ξ for $(a-1)t$ and η for x^{1-a}.

$$\therefore \frac{d^2u}{d\xi^2} = \eta^{\frac{2a}{a-1}} \cdot \frac{d^2u}{d\eta^2} \quad \text{........................(2).}$$

This form fails when $a=1$, but in that case the following is the reduced form;

$$x^2\frac{d^2u}{dt^2} = \left(x\frac{d}{dx}\right)^2 u \quad \text{.....................(3).}$$

If in this we write ξ for $\log x$, it takes the following form,

$$\frac{d^2u}{dt^2} = \epsilon^{-2\xi}\frac{d^2u}{d\xi^2} \quad \text{.......................(4).}$$

83. The integral of the equation

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{a}{x}\frac{du}{dx} + \frac{bu}{x^2}$$

can also be deduced from that of the standard equation.

Multiply it by x^2,

$$\therefore x^2\frac{d^2u}{dt^2} = \left(\frac{xd}{dx}\right)^2 u + (a-1)\left(\frac{xd}{dx}\right)u + bu$$

$$= \left(\frac{xd}{dx} + m\right)\left(\frac{xd}{dx} + n\right)u,$$

m, n being the roots of the equation

$$m^2 + (1-a)m + b = 0.$$

$$\therefore\ x^2\frac{d^2u}{dt^2}=x^{-m}\left(\frac{xd}{dx}\right)x^m\,.\,x^{-n}\left(\frac{xd}{dx}\right)x^nu$$

$$\therefore\ \frac{d^2\,(ux^n)}{dt^2}=x^{-(m-n+1)}\,\frac{d}{dx}\,.\,x^{(m-n+1)}\,\frac{d\,(ux^n)}{dx}\ldots\ldots(1),$$

which it will be observed corresponds to the form (1) of the preceding Article, $m-n+1$ taking the place of a, and ux^n of u.

This reduction fails, however, when m and n are equal. In this case $m=\frac{1}{2}(a-1)$,

$$\therefore\ x^2\frac{d^2u}{dt^2}=\left(\frac{xd}{dx}+m\right)^2u$$

$$=x^{-m}\left(\frac{xd}{dx}\right)^2.\,x^mu,$$

$$\therefore\ x^2\frac{d^2\,(x^mu)}{dt^2}=\left(\frac{xd}{dx}\right)^2(x^mu)\ldots\ldots\ldots\ldots(2),$$

which agrees with form (3) of the preceding Article.

84. To integrate

$$\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{a}{x}\frac{du}{dx}$$

when a is a positive even integer.

In this case on referring to Art. 80 we find that $b=0$, and $2n=a$. Hence the auxiliary equation (1) of that article takes the following form

$$\frac{d^2\omega}{dt^2}=\frac{d^2\omega}{dx^2},$$

$$\therefore\ \omega=F(t+x)+f(t-x)=F(t+jx).$$

$$\therefore\ u=\left(\frac{d}{xdx}\right)^n\omega$$

$$=\left(\frac{d}{xdx}\right)^{\frac{a}{2}}F(t+jx).$$

85. To integrate the same equation when $a = 1$, i.e. when the proposed equation is

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx}\cdots$$

$$\therefore\ x^2\frac{d^2u}{dt^2} = \left(x\frac{d}{dx}\right)^2 u.$$

Now only t can take a minor germ; but this equation is homogeneous on the supposition that t and x are of equal dimensions. Hence $v = \frac{t}{x}$ and V, which is a function of v, are of zero dimensions. We may therefore assume $P = x^p V$ to represent any one of the members of the family of subintegrals. This being substituted in the proposed equation gives the following equation for the determination of V in finite terms;

$$\frac{d^2V}{dv^2} = \frac{d}{dv}\left(v^2\frac{dV}{dv}\right) - (2p+1)\,v\frac{dV}{dv} + p^2V.$$

It is evident that this equation will be integrable if

$$-(2p+1) = p^2,$$

$$\therefore\ 0 = (p+1)^2.$$

We have therefore to deal with a case of equal roots. One integration gives the following result,

$$\frac{dV}{dv} = v^2\frac{dV}{dv} + vV + B.$$

We have now to introduce suppositions, since the form of the integral of this equation will turn upon the relation between t and x.

1. If t^2 is less than x^2, v^2 is less than unity, then the equation to be integrated is

$$(1-v^2)\frac{dV}{dv} - vV = B.$$

$$\therefore\ V = \frac{A}{\sqrt{1-v^2}} + \frac{B\sin^{-1}v}{\sqrt{1-v^2}}.$$

Hence the first subintegral, corresponding to the two sub-general integrals, is

$$\therefore\ P = x^{-1} V = \frac{A}{\sqrt{x^2 - t^2}} + \frac{B \sin^{-1} \frac{t}{x}}{\sqrt{x^2 - t^2}} \quad \ldots\ldots\ldots\ldots\ldots\ (1).$$

But this subintegral can be raised to zero dimensions by integrating it with respect to t, for $(x^2 - t^2)^{-\frac{1}{2}} = \frac{d}{dt} . \sin^{-1} \frac{t}{x}$ (see Art. 25). We may therefore take the following as the first subintegral form of zero dimensions,

$$P = A \left(\sin^{-1} \frac{t}{x}\right) + B \left(\sin^{-1} \frac{t}{x}\right)^2 + B (\log mx)^2,$$

in which m is an extemporized major germ.

2. Again, let us now take the case when t^2 is greater than x^2, and therefore v^2 greater than unity.

The equation to be integrated is in this case,

$$(v^2 - 1) \frac{dV}{dv} + vV = B.$$

$$\therefore\ \sqrt{v^2 - 1} . V = A + B \log (\sqrt{v+1} + \sqrt{v-1}),$$

$$\therefore\ P = (t^2 - x^2)^{-\frac{1}{2}} \left\{A + B \log \left(\sqrt{\frac{t}{x} + 1} + \sqrt{\frac{t}{x} - 1}\right)\right\} \ldots (2).$$

This integral can be raised to zero dimensions by integrating it with respect to t, for

$$(t^2 - x^2)^{-\frac{1}{2}} = \frac{d}{dt} . \log \left(\sqrt{\frac{t}{x} + 1} + \sqrt{\frac{t}{x} - 1}\right).$$

As only the variable t takes a minor germ, the family of subintegrals can be obtained from (1) and (2) by differentiating or integrating with respect to t.

The following is therefore the general integral required in this Article,

$$u = F\left(\frac{d}{dt}\right) (x^2 - t^2)^{-\frac{1}{2}} + f\left(\frac{d}{dt}\right) \frac{\sin^{-1} \frac{t}{x}}{\sqrt{x^2 - t^2}}, \text{ if } x^2 > t^2,$$

or $$u = F\left(\frac{d}{dt}\right) (t^2 - x^2)^{-\frac{1}{2}} + f\left(\frac{d}{dt}\right) \log \left(\sqrt{\frac{t}{x} + 1} + \sqrt{\frac{t}{x} - 1}\right),$$
if $t^2 > x^2$.

86. To integrate $\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{a}{x}\frac{du}{dx}$, when a is a positive odd integer.

Referring to Art. 80 we find that $b=1$ in this case, and $n=\frac{1}{2}(a-1)$. Hence the auxiliary equation of that Article takes the following form,

$$\frac{d^2\omega}{dt^2}=\frac{d^2\omega}{dx^2}+\frac{1}{x}\frac{d\omega}{dx},$$

which is the form integrated in the preceding Article;

$$\therefore\ u=\left(\frac{d}{xdx}\right)^{\frac{1}{2}(a-1)}\omega.$$

87. To integrate $\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{a}{x}\frac{du}{dx}$ when a is not an integer.

The differential equation for the determination of V in this case is

$$\frac{d^2V}{dv^2}=\frac{d}{dv}\left(v^2\frac{dV}{dv}\right)-(2p+a)\,v\frac{dV}{dv}+(p^2-p-ap)\,V,$$

and that this may be integrable so as to give V in a finite form we must have $-(2p+a)=p^2-p-ap$;

$$\therefore\ (p+1)(p+a)=0.$$

In this example therefore the two values of p are not equal; and one part of each subintegral will correspond to $p=-1$, and the other to $p=-a$.

1. Let $p=-1$.

Integrating the above on this supposition we find,

$$\frac{dV}{dv}-v^2\frac{dV}{dv}=(2-a)\,vV=B.$$

Multiply this by $(1-v^2)^{-\frac{a}{2}}$ and integrate;

$$\therefore\ (1-v^2)^{1-\frac{a}{2}}V=A+B\int(1-v^2)^{-\frac{a}{2}}dv.$$

We reject the last term, being not integrable in finite terms, and we only require one integral form for this value of p;

$$\therefore\ P = x^{-1} V = Ax^{1-a} (x^2 - t^2)^{\frac{a}{2}-1} \ \ldots\ldots\ldots\ldots\ (1).$$

2. Let $p = -a$.

Integrating the above equation on this supposition we find

$$(1-v^2)^{\frac{a}{2}}\, V = A' \int (1-v^2)\, dv + B.$$

We reject the former term of this because it does not give an integral in finite terms and we require only one integral form for this value of p;

$$\therefore\ P = x^{-a} V = B (x^2 - t^2)^{-\frac{a}{2}} \ \ldots\ldots\ldots\ldots\ (2).$$

Gathering the two parts of P together we have the following complete value of the first subintegral,

$$P = Ax^{1-a} (x^2 - t^2)^{\frac{a}{2}-1} + B (x^2 - t^2)^{-\frac{a}{2}}.$$

The other members of the subintegral family are to be derived from this equation by differentiation with $\dfrac{d}{dt}$.

If t^2 be greater than x^2, we may write $(t^2 - x^2)$ for $(x^2 - t^2)$ in this subintegral.

It will be noticed also that the above subintegrals contain only even powers of t. Subintegrals containing only odd powers of t will be obtained from the above by differentiating once with $\dfrac{d}{dt}$. We may however pass from P to the general integral which (as only t can take a minor germ) will be (Art. 19)

$$\therefore\ u = F\left(\frac{d}{dt}\right) . x^{1-a} (x^2 - t^2)^{\frac{a}{2}-1} + f\left(\frac{d}{dt}\right) (x^2 - t^2)^{-\frac{a}{2}}.$$

88. To find an integral of the equation

$$\frac{d^2u}{dt^2} + \left(\cos x . \frac{d}{dx}\right)^2 u + n(n+1) \cos^2 x . u = 0.$$

Only t takes a minor germ; and therefore, changing the dependent variable, we may assume $u = \epsilon^{mt} X$, in which m is a disposable constant.

$$\therefore \left(\cos x \,.\, \frac{d}{dx}\right)^2 X + \{n(n+1)\cos^2 x + m^2\} X = 0.$$

An equation of one variable only, of which a particular integral may be found by assuming $X = \cos^l x$, l being a disposable constant.

$$\therefore\ l^2 \sin^2 x - l\cos^2 x + n(n+1)\cos^2 x + m^2 = 0\,;$$

$$\therefore\ (l^2 + m^2) + (n^2 + n - l^2 - l)\cos^2 x = 0\,;$$

$$\therefore\ l^2 + m^2 = 0, \text{ and } n^2 + n = l^2 + l\,;$$

$$\therefore\ l = n \text{ or } -(n+1), \text{ and } m = il = in \text{ or } -i(n+1)\,;$$

$$\therefore\ u = A\epsilon^{int}\cos^n x + B\epsilon^{-i(n+1)t}\sec^{n+1} x.$$

We have introduced this example here chiefly for the following reason, and it will be hereafter referred to.

The integral of the equation does not depend directly upon the given value of n, but upon the value of the product $n(n+1)$. Now this product will remain unchanged if we write $-(n+1)$ for n; and consequently the two terms of the above integral belong to it of necessity.

The four following Articles are not specially connected with Laplace's Equation, and therefore do not properly form a part of the present Chapter; but are here introduced as illustrations of the principles laid down in Art. 33 respecting quasi-minor germs.

89. Equations are occasionally met with which are of the following type,

$$\varpi'\left(ax + by,\ \frac{d}{dx},\ \frac{d}{dy}\right) u = 0.$$

We may simplify this form by writing x, y for ax, by. This change of the independent variables will reduce this equation to a form which we may represent by

$$\varpi\left(x + y,\ \frac{d}{dx},\ \frac{d}{dy}\right) u = 0 \dots\dots\dots\dots\dots\dots (1),$$

and this is the equation which we shall now shew how to reduce to a more convenient form for integration; our object being to obtain an equation in which one of the independent variables shall appear only as a differential symbol of operation (see Art. 18).

In the equation as now before us, though neither of the independent variables can take a minor germ, they can take a quasi-minor germ g; for the equation (1) is in no way affected when $x+g$ is written for x and $y-g$ for y simultaneously.

Hence the general integral of (1) must be of the following form,

$$\begin{aligned} u &= F(x+g, y-g) \\ &= F(x, y) + \frac{g}{1}\left(\frac{d}{dx}-\frac{d}{dy}\right)F(x, y) + \frac{g^2}{1.2}\left(\frac{d}{dx}-\frac{d}{dy}\right)^2 F(x, y) + \dots \\ &= AF(x, y) + B\left(\frac{d}{dx}-\frac{d}{dy}\right)F(x, y) + C\left(\frac{d}{dx}-\frac{d}{dy}\right)^2 F(x, y) + \dots \\ &= \phi\left(\frac{d}{dx}-\frac{d}{dy}\right)F(x, y) \quad \dots\dots\dots (2). \end{aligned}$$

And $F(x, y)$ being the first subintegral, all the other subintegrals are deducible from it by successive differentiations with $\left(\frac{d}{dx}-\frac{d}{dy}\right)$.

Now the relation between $\left(\frac{d}{dx}-\frac{d}{dy}\right)$ and $(x+y)$ is such that $\left(\frac{d}{dx}-\frac{d}{dy}\right)(x+y)=0$, and therefore in reference to the compound operation $\left(\frac{d}{dx}-\frac{d}{dy}\right)$ the quantity $(x+y)$ is constant. This suggests the following assumption of new independent variables ξ, η.

Let $\qquad \xi = ax - by, \qquad$ and $\quad \eta = x + y$;

$$\therefore\ (a+b)\frac{d}{d\xi} = \frac{d}{dx} - \frac{d}{dy};$$

and therefore ξ and η are independent variables which satisfy the above conditions.

Also $\frac{d}{dx}=\frac{d}{d\eta}+a\frac{d}{d\xi}$, and $\frac{d}{dy}=\frac{d}{d\eta}-b\frac{d}{d\xi}$,

and the equation becomes

$$\varpi\left(\eta,\ \frac{d}{d\eta}+a\frac{d}{d\xi},\ \frac{d}{d\eta}-b\frac{d}{d\xi}\right)u=0 \quad\ldots\ldots\ldots\ldots\ldots (3),$$

from which we see that equation (1) is now reduced to a form in which one of its independent variables (ξ) occurs only as a differential symbol of operation, and will consequently take a minor germ.

The following example is one of historical interest.

90. To integrate $\frac{d^2u}{dt^2}=\frac{d^2u}{dx^2}+\frac{4a}{t+x}\frac{du}{dx}$.

We assume $\xi=t-x$, and $\eta=t+x$;

$$\therefore\ \frac{d^2u}{d\xi\,d\eta}=\frac{a}{\eta}\left(\frac{du}{d\eta}-\frac{du}{d\xi}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We may write $m\xi$, $m\eta$ for ξ, η in this equation without affecting it; hence ξ, η take a major germ; and ξ takes also a minor germ.

This equation (1) has already been integrated in Art. 70;

$$\therefore\ u=A\,(jm\eta-a)^{-a}\epsilon^{jm\xi} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$=A\,(m\eta-a)^{-a}\epsilon^{m\xi}+B\,(m\eta+a)^{-a}\epsilon^{-m\xi}.$$

Also assuming $\eta^p V$ for the first subintegral, where V is a function of v, and $v=\frac{\xi}{\eta}$, we find $p=-1$, and the first subintegral

$$P=V\eta^{-1}=\frac{\xi^{a-1}}{\eta^a}\epsilon^{\frac{a\xi}{\eta}}\left(A+B\int\epsilon^{-av}v^{-a}dv\right) \quad\ldots\ldots (3),$$

$$u=F\left(\frac{d}{d\xi}\right)(P).$$

This integral will be in a finite form only when a is a negative integer. When a is not negative the last term must be rejected.

91. An inspection of the integral just found shews that the case of $a=1$ is peculiar, for then it takes the following form,

$$P=\eta^{-1}\epsilon^{\frac{\xi}{\eta}}\left(A+B\int\epsilon^{-v}\frac{dv}{v}\right);$$

omitting the last term as not being a finite form, we have

$$P=A\eta^{-1}\epsilon^{\frac{\xi}{\eta}}.$$

Let this be differentiated with $\left(\frac{d}{d\xi}\right)^n$, or (more generally still) with $\phi\left(\frac{d}{d\xi}\right)$;

$$\therefore\ u=\phi\left(\frac{d}{d\xi}\right)P=A\eta^{-1}\epsilon^{\frac{\xi}{\eta}}\phi\left(\frac{1}{\eta}\right)$$

$$=\phi(t+x)\,.\,\epsilon^{\frac{t-x}{t+x}}.$$

92. To integrate the equation

$$(x+y)^2\frac{d^2u}{dx^2}+(x+y)\left(a\frac{du}{dx}+b\frac{du}{dy}\right)+cu=0.$$

We here assume $x=\eta+\xi$, and $y=\eta-\xi$;

$$\therefore\ \frac{d^2u}{d\xi^2}=\frac{d^2u}{d\eta^2}+\frac{a-b}{\eta}\frac{du}{d\xi}+\frac{a+b}{\eta}\frac{du}{d\eta}+\frac{cu}{\eta^2}\ \dots\dots\dots\dots\ (1).$$

If $a=b$ this form of equation has been dealt with in Art. 83; but if a and b are unequal we may proceed in the following manner.

The variable ξ takes a minor germ; and also we may write $m\xi$, $m\eta$ for ξ, η without affecting it.

We may therefore assume $v=\frac{\xi}{\eta}$ and $V=$ a function of v; and the general type of subintegrals is $P=\eta^pV$; and then the following will be found to be the differential equation for the determination of V:

$$\frac{d^2V}{dv^2}+(b-a)\frac{dV}{dv}=\frac{d}{dv}\left(v^2\frac{dV}{dv}\right)-(2p+a+b+1)\left(v\frac{dV}{dv}\right)$$

$$+(p^2+ap+bp-p+c)\,V,$$

which will be integrable in finite terms if the coefficients of the last two terms are equal. This gives the following equation for the determination of the two values of p corresponding to the two subgeneral integrals:

$$p^2 + (1 + a + b)\,p + (1 + a + b + c) = 0.$$

Represent the roots of this equation by $m + jn$; their sum $= 2m$;

$$\therefore\ 2m = -(1 + a + b);$$

also the equation in V being integrated once gives the following:

$$\frac{dV}{dv} + (b - a)\,V = v^2 \frac{dV}{dv} - 2jnvV + B.$$

We omit B as not leading to a finite integral, and also because each value of p, i.e. each sign of jn is required to furnish only a single integral form of V.

Omitting B and integrating, we find

$$V = \left(\frac{1 + v}{1 - v}\right)^{\frac{a-b}{2}} (1 - v^2)^{jn};$$

$$\therefore\ P = \eta^p V = \eta^{m+jn} \left(\frac{\eta + \xi}{\eta - \xi}\right)^{\frac{a-b}{2}} \left(1 - \frac{\xi^2}{\eta^2}\right)^{jn}$$

$$= \eta^m \left(\frac{\eta + \xi}{\eta - \xi}\right)^{\frac{a-b}{2}} \{A\eta^{-n} (\eta^2 - \xi^2)^n + B\eta^n (\eta^2 - \xi^2)^{-n}\} \ldots\ldots (2).$$

Also $\quad u = F\left(\dfrac{d}{d\xi}\right) P.$

P may be expressed in terms of x and y as follows:

$$P = \left(\frac{x}{y}\right)^{\frac{a-b}{2}} (x + y)^m \left\{A \left(\frac{xy}{x + y}\right)^n + B\left(\frac{xy}{x + y}\right)^{-n}\right\} \ldots\ldots\ldots (3),$$

and in this case

$$u = F\left(\frac{d}{dx} - \frac{d}{dy}\right) P.$$

93. The preceding Article fails if the roots be equal, in which case $n = 0$, and $m = -\frac{1}{2}(1 + a + b)$.

Let $Q = \log \dfrac{xy}{x+y}$; we reduce equation (3) as follows; (= means equivalence);

$$A\left(\frac{xy}{x+y}\right)^n + B\left(\frac{xy}{x+y}\right)^{-n} = A\epsilon^{nQ} + B\epsilon^{-nQ}$$

$$= A\left(\epsilon^{nQ} + \epsilon^{-nQ}\right) + \frac{B}{n}\left(\epsilon^{nQ} - \epsilon^{-nQ}\right)$$

$$= A + BQ, \text{ when } n = 0.$$

For equal roots therefore

$$P = \left(\frac{x}{y}\right)^{\frac{a-b}{2}} (x+y)^{-\frac{1+a+b}{2}} \left(A + B \log \frac{xy}{x+y}\right) \text{ .... } (4).$$

There still remains the case of imaginary roots, which will be represented by writing in for jn;

$$\therefore\ P\left(\frac{x}{y}\right)^{\frac{a-b}{2}} (x+y)^{-\frac{1+a+b}{2}} \left\{A\left(\frac{xy}{x+y}\right)^{in} + B\left(\frac{xy}{x+y}\right)^{-in}\right\}$$

$$= \text{........................} \left\{A \cos nQ + B \sin nQ\right\} \text{......} (5).$$

CHAPTER VII.

EQUATIONS OF THREE INDEPENDENT VARIABLES.

Coefficients constant.

94. All the independent variables of equations of this class take minor germs; and therefore a general integral of any such equation will be expressible in an infinite series, every term of which contains only positive integer powers of the variables.

As a general rule the more independent variables are contained in a proposed linear differential equation the more independent major germs may there possibly be; but this is not necessarily the case always. A major germ may perchance belong to only *one*, or to *two* only, or to *all* the independent variables; and thus an individual independent variable may be under the influence of so many as there are different major germs. When the major germs have been introduced into the general exponential integral (Art. 34), we can then eliminate them one by one in any order that shall be found most convenient.

It will be found that the final result of the elimination of all the germs will sometimes depend upon the manner in which major germs were introduced into the original exponential integral; for sometimes they may be introduced in more ways than one. And thus we may obtain in more forms than one a general integral of the proposed equation free from major germs.

95. To integrate the class of equations represented by

$$\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)u = \frac{du}{dt}.$$

In this class of equations x, y, t take independent minor germs, and therefore the general integral u is completely expressible in a series containing only positive integer powers of these variables. We may, therefore, assume

$$u = P + Q\frac{t}{1} + R\frac{t^2}{2!} + S\frac{t^3}{3!} + \ldots$$

in which P, Q, R, ... are series containing only positive integer powers of x and y, the general type of them all being the following,

$$P = A + \left(B\frac{x}{1} + B'\frac{y}{1}\right) + \left(C\frac{x^2}{1.2} + C'\frac{xy}{1.1} + C''\frac{y^2}{1.2}\right) + \ldots$$

Let the above series for u be substituted in the proposed equation;

$$\therefore\ u = \left(1 + \frac{t\varpi}{1} + \frac{t^2\varpi^2}{2!} + \frac{t^3\varpi^3}{3!} + \ldots\right)P,$$

in which ϖ is used, for brevity, to represent $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)$.

Now in the series which P represents the coefficients are, every one of them, absolutely independent and arbitrary. They are therefore the coefficients of the family of subintegrals of which u is constituted. We may therefore replace them with two independent general germs M, N in the usual manner.

$$\therefore\ P = A\left\{1 + \frac{Mx + Ny}{1} + \frac{(Mx + Ny)^2}{1.2} + \ldots\right\}$$

$$= A\epsilon^{Mx+Ny};$$

$$\therefore\ u = \left(1 + \frac{t\varpi}{1} + \frac{t^2\varpi^2}{1.2} + \ldots\right).A\epsilon^{Mx+Ny}$$

$$= A\epsilon^{Mx+Ny}\left\{1 + \frac{t.\varpi(M, N)}{1} + \frac{\overline{t.\varpi(M, N)}^2}{1.2} + \ldots\right\}$$

$$= A\epsilon^{Mx+Ny+\varpi(M, N)t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which is the usual form of the general exponential integral. That form of the general integral is therefore proved to hold good for all equations of three independent variables of the class proposed in this Article.

We have supposed $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)$ not resolvable into equal factorials, for such a case requires a somewhat different treatment. See Art. 69.

96. To integrate the class of equations represented by

$$\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right) u = \frac{d^2u}{dt^2},$$

the operative symbol $\varpi\left(\frac{d}{dx}, \frac{d}{dy}\right)$ being supposed to be not resolvable into any factors that are equal.

Following the method of the preceding Article we find in this case,

$$u = \left(1 + \frac{t^2\varpi}{2!} + \frac{t^4\varpi^2}{4!} + \dots\right) P + \left(\frac{t}{1} + \frac{t^3\varpi}{3!} + \frac{t^5\varpi^2}{5!} + \dots\right) Q,$$

P representing the same series as before, and Q being an independent series of precisely the same form.

The two terms of which u consists are the subgeneral integrals, one of them containing only even, and the other only odd powers of t; and we notice that the *form* of the latter subgeneral integral may be deduced from the former by differentiating with $\frac{d}{dt}$ once.

Now for the same reason as in the preceding Article we may assume, as was there proved, that

$$P = \epsilon^{Mx+Ny}.$$

Let $$\varpi(M, N) = L^2.$$

$\therefore$ first subgeneral integral $= \left(1 + \frac{t^2}{2!}\varpi + \frac{t^4}{4!}\varpi^2 + \dots\right)\epsilon^{Mx+Ny}$

$$= \epsilon^{Mx+Ny}\left(1 + \frac{t^2}{2!}L^2 + \frac{t^4}{4!}L^4 + \dots\right)$$

$$= A\epsilon^{Mx+Ny}\left(\epsilon^{Lt} + \epsilon^{-Lt}\right);$$

$\therefore$ form of second subgeneral integral is

$$B\epsilon^{Mx+Ny}\left(\epsilon^{Lt} - \epsilon^{-Lt}\right).$$

Both of these terms are comprehended in the one form

$$A\epsilon^{Mx+Ny} \, . \, \epsilon^{jLt}.$$

Hence both the subgeneral integrals may be included in the single form

$$u = A\epsilon^{Mx+Ny+jLt}, \text{ subject to } L^2 = \varpi\,(M, N).$$

Hence the exponential integral of Art. (30) is general and complete for all linear differential equations that belong to the class proposed in this Article.

The existence of the two independent subgeneral integrals in this one expression for u is secured and indicated by the symbol j, which always carries with it the double sign $\pm$.

97. To integrate $\dfrac{d^2u}{dt^2} = \dfrac{du}{dx} + \dfrac{du}{dy}$.

Let the independent variables x, y be changed, the new variables ξ, η being such that $x = \xi + \eta$, and $y = \xi + m\eta$.

$$\therefore \frac{du}{dx} + \frac{du}{dy} = \frac{du}{d\xi}, \text{ and } \frac{d^2u}{dt^2} = \frac{du}{d\xi} \quad \text{..............(1)}.$$

By this change of variables the proposed equation has become an equation of only two independent variables t, ξ; and therefore the remaining variable $\eta = \dfrac{x-y}{1-m}$ is to take the places of the arbitrary constants in the integration of (1).

The integral of (1) we find in Art. 60 to be

$$\therefore u = A\epsilon^{M^2\xi + Mt} = \phi\,(x-y) \, . \, \epsilon^{M^2\xi+Mt} \quad \text{...........(2)}.$$

This is the general exponential integral; and M is a general germ.

By the same Article we have the following form of the first subintegral,

$$P = A\xi^{-\frac{1}{2}}\,\epsilon^{-\frac{t^2}{4\xi}} = \frac{\phi\,(x-y)}{\sqrt{mx-y}}\,\epsilon^{\frac{1}{4}\frac{(m-1)\,t^2}{mx-y}} \quad \text{...........(3)}.$$

The value of P contains only even powers of t, and therefore it gives us only one of the subgeneral integrals. But the other subgeneral integral will be obtained from this by differentiating with $\frac{d}{dt}$ (Art. 96). Hence both subgeneral integrals are contained in the following,

$$u = F\left(\frac{d}{dt}\right) \cdot \frac{\phi(x-y)}{\sqrt{mx-y}} \cdot \epsilon^{\frac{1}{4}\frac{(m-1)t^2}{mx-y}}.$$

In this m is arbitrary and may be put equal to (-1).

98. To integrate $\frac{du}{dt} = \left(\frac{d}{dx} + \frac{d}{dy}\right)^2 u.$

Proceeding exactly as in the preceding Article we find

$$\frac{du}{dt} = \frac{d^2u}{d\xi^2}.$$

$$\therefore\ u = \phi(x-y) \cdot \epsilon^{M^2t+M\xi} \quad \text{.........................(1)},$$

and

$$P = At^{-\frac{1}{2}}\epsilon^{-\frac{\xi^2}{4t}} = \frac{\phi(x-y)}{\sqrt{t}} \cdot \epsilon^{-\frac{(mx-y)^2}{4(m-1)^2t}};$$

$$\therefore\ u = F\left(\frac{d}{d\xi}\right) \cdot t^{-\frac{1}{2}}\epsilon^{-\frac{\xi^2}{4t}} \cdot \phi(x-y) \quad \text{.............(2)}.$$

99. To integrate $\frac{d^2u}{dt^2} = \left(\frac{d}{dx} + \frac{d}{dy}\right)^2 u.$

This may be resolved into the two following independent simple equations,

$$\frac{du}{dt} = \pm\left(\frac{du}{dx} + \frac{du}{dy}\right),$$

which can be integrated in the usual manner.

If we proceed with the proposed equation after the method of the two preceding Articles we find $\frac{d^2u}{dt^2} = \frac{d^2u}{d\xi^2}$, which has been integrated in Art. 72, whence we shall obtain the integral in its various forms; but arbitrary functions of $x-y$ will have to be written instead of the arbitrary constants contained in them.

100. To integrate $\dfrac{d^2u}{dt^2}+\left(\dfrac{d}{dx}+\dfrac{d}{dy}\right)^2 u=0.$

By the same method the integral of this equation will be obtained from Art. 75.

101. To integrate $\dfrac{du}{dt}=\dfrac{d^2u}{dx\,dy}.$

The general exponential integral of this equation takes the following form,

$$u=A\epsilon^{Mx+Ny+MNt}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$=A\epsilon^{Mx}\,.\,\epsilon^{N(y+Mt)}$$

$$=\epsilon^{Mx}\phi(y+Mt)\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Similarly $$u=\epsilon^{Ny}\psi(x+Nt)\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the last two being the results of the elimination of one germ only.

To eliminate both the germs from (1) the simplest method will be to change the forms of the germs M, N by assuming $M=m+n$, and $N=m-n$; m, n being two independent germs.

$$\therefore\ u=A\epsilon^{m(x+y)+m^2t}\,.\,\epsilon^{n(x-y)+n^2(-t)}.$$

Both m and n may be eliminated by Art. 64; and the chief subintegral is

$$P=t^{-\frac{1}{2}}\epsilon^{-\frac{(x+y)^2}{4t}}\,.\,t^{-\frac{1}{2}}\epsilon^{\frac{(x-y)^2}{4t}}=t^{-1}\epsilon^{-\frac{xy}{t}}.$$

This form of P is such that we can at once obtain from it the form of the general integral u.

For $$u=F\left(\frac{d}{dx}\right)P+f\left(\frac{d}{dy}\right)P$$

$$=\frac{1}{t}\left\{F\left(\frac{y}{t}\right)+f\left(\frac{x}{t}\right)\right\}\epsilon^{-\frac{xy}{t}}\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

102. The integral of the equation

$$\frac{du}{dt}=\frac{d^2u}{d\xi^2}+(a+b)\frac{d^2u}{d\xi\,d\eta}+ab\frac{d^2u}{d\eta^2}$$

may be deduced from the preceding Article by assuming as in Art. 67,

$$x = \frac{\eta - b\xi}{a - b}, \text{ and } y = -\frac{\eta - a\xi}{a - b};$$

$$\therefore\ u = \frac{1}{t}\left\{F\left(\frac{\eta - a\xi}{t}\right) + f\left(\frac{\eta - b\xi}{t}\right)\right\}\epsilon^{\frac{(\eta - a\xi)(\eta - b\xi)}{(a-b)^2 t}}.$$

103. To integrate $\dfrac{d^2u}{dt^2} = \dfrac{d^2u}{dx\,dy}$.

The general exponential integral is

$$u = A\epsilon^{Mx + Ny + jLt} \text{ ....} (1),$$

subject to the condition $L^2 = MN$.

We may eliminate the germs, and obtain the first subintegral in the following manner,

$$u = A\epsilon^{Mx} \,.\, \epsilon^{Ny + \sqrt{N}(jt\sqrt{M})}.$$

By Art. 64 this gives the following subintegral by the elimination of N,

$$P = A\epsilon^{Mx} \,.\, y^{-\frac{1}{2}}\epsilon^{-\frac{Mt^2}{4y}};$$

$$\therefore\ P = Ay^{-\frac{1}{2}} \,.\, \epsilon^{M\left(x - \frac{t^2}{4y}\right)}$$

$$= y^{-\frac{1}{2}}\phi\left(x - \frac{t^2}{4y}\right).$$

This contains only even powers of t corresponding to one subgeneral integral, and the odd powers which are contained in the other will be contained in $\dfrac{dP}{dt}$. Both subgeneral integrals are contained in the following formula,

$$u = F\left(\frac{d}{dt}\right) \cdot y^{-\frac{1}{2}}\phi\left(x - \frac{t^2}{4y}\right).$$

But x and y are interchangeable in the proposed equation, and therefore also in P and u. Hence the complete values of P and u are the following:

$$\therefore\ P = y^{-\frac{1}{2}}\phi\left(\frac{4xy-t^2}{y}\right) + x^{-\frac{1}{2}}\psi\left(\frac{4xy-t^2}{x}\right)$$

$$u = F\left(\frac{d}{dt}\right).\, y^{-\frac{1}{2}}\phi\left(\frac{4xy-t^2}{y}\right) + f\left(\frac{d}{dt}\right).\, x^{-\frac{1}{2}}\psi\left(\frac{4xy-t^2}{x}\right) \ldots (2).$$

Again, since $MN = L^2$, we may assume in this case,

$$M = Lc\,(\cos m + i \sin m),$$

and

$$N = Lc^{-1}\,(\cos m - i \sin m);$$

$$\therefore\ u = A\epsilon^{L\{(cx+c^{-1}y)\,(\cos m + i\,(cx-c^{-1}y)\sin m + jt\}},$$

in which L, m and c are independent real germs.

Let $K = (cx + cx^{-1}y)\cos m + jt$, and $I = (cx - c^{-1}y)\sin m$.

$$\therefore\ u = A\epsilon^{L(K+iI)}$$

$$= \phi(K + iI) + \psi(K - iI) \ldots\ldots\ldots\ldots\ldots (3).$$

104. In the first part of the preceding Article we took no account of the fact that the proposed equation allows us to write cx for x, and $c^{-1}y$ for y quite independently of t. The variables x, y have therefore a special relation, and we have therefore to consider the following form of the general exponential integral of that equation,

$$u = A\epsilon^{L(cx+c^{-1}y+jt)}$$

$$= \phi(cx + c^{-1}y + t) + \psi(cx + c^{-1}y - t) \ldots\ldots\ldots (4),$$

in which c is a general germ.

We may eliminate c by Art. 64, in the following manner:

$$\therefore\ u = A\epsilon^{jLt}.\,\epsilon^{c(Lx)+c^{-1}(Ly)};$$

$$\therefore\ P = \epsilon^{jLt}.\,x^{-\frac{1}{2}}\,\epsilon^{2L\sqrt{xy}}$$

$$= x^{-\frac{1}{2}}.\,\epsilon^{L(2\sqrt{xy}+jt)}$$

$$= x^{-\frac{1}{2}}\phi(2\sqrt{xy} + jt).$$

As x and y are interchangeable, the general integral may be presented in the following form,

$$u = F\left(\frac{d}{dx}\right).\, x^{-\frac{1}{2}}\phi(2\sqrt{xy} + jt) + f\left(\frac{d}{dy}\right).\, y^{-\frac{1}{2}}\psi(2\sqrt{xy} + jt) \ldots (5).$$

105. To integrate $\frac{d^2u}{dt^2} + \frac{d^2u}{dx\,dy} = 0.$

We have merely to write $-x$ for x, or $-y$ for y in the results of the two preceding Articles. Or we may write it for t.

106. The integral of $\frac{d^2u}{dt^2} = \frac{d^2u}{d\xi^2} + (a+b)\frac{d^2u}{d\xi\,d\eta} + ab\,\frac{d^2u}{d\eta^2}$ may be deduced from Art. 103 by means of the same change of the variable x, y as occurs in Art. 67;

$$\therefore\ x = \frac{\eta - b\xi}{a-b} \text{ and } y = -\frac{\eta - a\xi}{a-b},$$

and

$$u = F\left(\frac{d}{dt}\right).(a\xi-\eta)^{-\frac{1}{2}}\,\phi\left\{\frac{\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2 + \frac{1}{4}(a-b)^2t^2}{a\xi-\eta}\right\}$$

$$+ f\left(\frac{d}{dt}\right).(b\xi-\eta)^{-\frac{1}{2}}\,\psi\left\{\frac{\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2 + \frac{1}{4}(a-b)^2t^2}{b\xi-\eta}\right\} \ldots (1).$$

We may derive the following form of u from Art. 104:

$$u = F\left(\frac{d}{d\xi} + a\frac{d}{d\eta}\right).(b\xi-\eta)^{-\frac{1}{2}}\,\phi\,\{(\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2)^{\frac{1}{2}}$$
$$+ \tfrac{1}{2}(a-b)\,it\}$$

$$+ f\left(\frac{d}{d\xi} + b\frac{d}{d\eta}\right).(a\xi-\eta)^{-\frac{1}{2}}\,\psi\,\{(\eta^2 - \overline{a+b}\,\xi\eta + ab\xi^2)^{\frac{1}{2}}$$
$$+ \tfrac{1}{2}(a-b)\,it\} \ldots (2).$$

107. Let $a = 1$ and $b = -1$ in the preceding Article; then the integrals of the equation

$$\frac{d^2u}{d\xi^2} = \frac{d^2u}{dt^2} + \frac{d^2u}{d\eta^2}$$

will be of the following forms,

$$u = F\left(\frac{d}{dt}\right).(\xi-\eta)^{-\frac{1}{2}}\,\phi\left(\frac{\eta^2 - \xi^2 + t^2}{\xi-\eta}\right)$$
$$+ f\left(\frac{d}{dt}\right).(\xi+\eta)^{-\frac{1}{2}}\,\psi\left(\frac{\eta^2 - \xi^2 + t^2}{\xi+\eta}\right).$$

But t and η are interchangeable, and the two terms of this integral may be represented as one; the following is therefore the form in greater detail,

$$\therefore\; u = F\left(\frac{d}{dt}\right).(\xi + j\eta)^{-\frac{1}{2}}\,\phi\,\frac{t^2+\eta^2-\xi^2}{\xi+j\eta}$$

$$+ f\left(\frac{d}{d\eta}\right).(\xi+jt)^{-\frac{1}{2}}\,\psi\left(\frac{t^2+\eta^2-\xi^2}{\xi+jt}\right) \;\ldots\ldots\; (1).$$

The second form of u will be the following,

$$u = F\left(\frac{d}{d\xi} + j\frac{d}{d\eta}\right).(\xi+j\eta)^{-\frac{1}{2}}\,\phi\,\{(\eta^2-\xi^2)^{\frac{1}{2}} + it\}$$

$$+ f\left(\frac{d}{d\xi} + j\frac{d}{dt}\right).(\xi+jt)^{-\frac{1}{2}}\,\psi\,\{(t^2-\xi^2)^{\frac{1}{2}} + i\eta\} \ldots\ldots (2).$$

108. If in the preceding Article we write $i\xi$ for ξ, the integrals of the equation

$$\frac{d^2u}{dt^2} + \frac{d^2u}{d\xi^2} + \frac{d^2u}{d\eta^2} = 0$$

will be

$$u = F\left(\frac{d}{dt}\right).(\eta+i\xi)^{-\frac{1}{2}}\,\phi\left(\frac{t^2+\xi^2+\eta^2}{\eta+i\xi}\right) \ldots\ldots\ldots\ldots (1),$$

and

$$u = F\left(\frac{d}{d\eta} + i\frac{d}{d\xi}\right).(\eta+i\xi)^{-\frac{1}{2}}\,\phi\,(\sqrt{\xi^2+\eta^2} + it) \ldots\ldots (2).$$

It being understood in these results that t, ξ, η are all interchangeable.

109. To integrate $\dfrac{du}{dt} = \dfrac{d^2u}{dx\,dy} + ju$.

This equation takes the form

$$\frac{d\,.\,u\epsilon^{-jt}}{dt} = \frac{d^2\,.\,u\epsilon^{-jt}}{dx\,dy},$$

a form which is integrated in Art. 101.

110. To integrate $\dfrac{d^2u}{dt^2} = \dfrac{d^2u}{dx\,dy} + u.$

The general exponential integral of this equation is

$$u = A\epsilon^{Mx+Ny+jLt},$$

subject to the conditional equation

$$L^2 = MN + 1 \quad\text{or}\quad L^2 - 1 = MN.$$

We may here assume

$$M = c\,(L+1), \quad\text{and}\quad N = c^{-1}\,(L-1),$$

c being a general germ;

$$\therefore\ Mx + Ny + jLt = L\,(cx + c^{-1}y + jt) + cx - c^{-1}y;$$

$$\therefore\ u = A\epsilon^{cx-c^{-1}y}\,.\,\epsilon^{L(cx+c^{-1}y+jt)}$$

$$= A\epsilon^{cx-c^{-1}y}\,\phi\,(cx + c^{-1}y + jt)\,\ldots\ldots\ldots(1).$$

The germ c still remains uneliminated; we shall therefore now shew how to eliminate both the germs (L and c) contained in the exponential integral,

$$u = A\epsilon^{jLt}\,.\,\epsilon^{c(L+1)x+c^{-1}(L-1)y}.$$

Hence by Art. 65, eliminating c, the following is the corresponding first subintegral,

$$P = \epsilon^{jLt}\,.\,x^{-\frac{1}{2}}\,\epsilon^{\pm 2\sqrt{(L^2-1)xy}};$$

$$\therefore\ P\sqrt{x} = \epsilon^{jLt \pm 2\sqrt{(L^2-1)xy}}.$$

Let now $\xi = 2\sqrt{xy}$, then eliminating L by differentiation of the last equation we find

$$\therefore\ \frac{d^2(P\sqrt{x})}{dt^2} = \frac{d^2(P\sqrt{x})}{d\xi^2} + (P\sqrt{x}),$$

and by the method of Art. 78 the first subintegral of this equation is

$$= (\xi + jt)^{-\frac{1}{2}}\,\epsilon^{\pm\sqrt{t^2-\xi^2}}$$

$$= (2\sqrt{xy} + jt)^{-\frac{1}{2}}\,\epsilon^{\pm\sqrt{t^2-4xy}}.$$

Hence the first subintegral of the proposed equation is

$$P = x^{-\frac{1}{2}} (2\sqrt{xy} + jt)^{-\frac{1}{2}} \epsilon^{\pm\sqrt{t^2-4xy}} \ldots\ldots\ldots\ldots\ldots (2),$$

whence u is known. The algebraic signs j and $\pm$ in this integral are independent; and x, y are interchangeable.

111. In the preceding Article c is a general germ; but if we wish to have the integral which is equivalent to (1) in real germs, we may write

$$c(\cos m + i \sin m) \text{ for } c, \quad \text{and} \quad c^{-1}(\cos m - i \sin m) \text{ for } c^{-1},$$

c and m being now real germs.

112. To integrate $\dfrac{d^2u}{dt^2} = \dfrac{d^2u}{dx\,dy} - u.$

Here the exponential integral is

$$u = A\epsilon^{Mx+Ny+Ljt},$$

subject to the condition

$$L^2 = MN - 1, \quad \text{or} \quad L^2 - i^2 = MN.$$

We now assume

$$M = c(L+i), \quad \text{and} \quad N = c^{-1}(L-i),$$

c being a general germ;

$$\therefore\ u = A\epsilon^{L(cx+c^{-1}y+jt)} \,.\, \epsilon^{i(cx-c^{-1}y)}$$

$$= \phi(cx + c^{-1}y + jt) \,.\, \cos(cx - c^{-1}y + B) \ldots\ldots (1).$$

Following the method of Art. 110 we have

$$u = A\epsilon^{jLt} \,.\, \epsilon^{c(L+i)x+c^{-1}(L-i)y};$$

$$\therefore\ P = \epsilon^{jLt} \,.\, x^{-\frac{1}{2}} \epsilon^{\pm 2\sqrt{(L^2+1)xy}};$$

$$\therefore\ \frac{d^2(P\sqrt{x})}{d\xi^2} = \frac{d^2(P\sqrt{x})}{dt^2} + (P\sqrt{x});$$

$$\therefore\ P = x^{-\frac{1}{2}} (2\sqrt{xy} + jt)^{-\frac{1}{2}} \epsilon^{\pm\sqrt{4xy-t^2}} \ldots\ldots\ldots\ldots (2),$$

whence u is known. As before x, y are interchangeable.

113. In Art. 74 we have shewn that the independent variables of the equation $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}=0$ may be changed without affecting the form of the equation itself. We shall now prove a corresponding property for the more general class of equations included in the following form,

$$\frac{d^2u}{dx^2}\pm\frac{d^2u}{dy^2}=\varpi\left(t,\ \frac{d}{dt}\right)u \;\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Instead of x and y assume two new independent variables ξ, η, such that $\xi = Mx + jmy$, and $\eta = mx \mp jMy$; the disposable constants M, m being subject to the following condition,

$$M^2 \pm m^2 = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Let the integral of (1) be $u = F(x, y, t)$; and let $W = F(\xi, \eta, t)$.

$$\therefore\ \frac{d^2W}{d\xi^2}\pm\frac{d^2W}{d\eta^2}=\varpi\left(t,\ \frac{d}{dt}\right)W.$$

But $$\frac{d^2W}{dx^2}=M^2\frac{d^2W}{d\xi^2}+2Mm\frac{d^2W}{d\xi\,d\eta}+m^2\frac{d^2W}{d\eta^2},$$

and $$\frac{d^2W}{dy^2}=m^2\frac{d^2W}{d\xi^2}\mp 2Mm\frac{d^2W}{d\xi\,d\eta}+M^2\frac{d^2W}{d\eta^2};$$

$$\therefore\ \frac{d^2W}{dx^2}\pm\frac{d^2W}{dy^2}=\frac{d^2W}{d\xi^2}\pm\frac{d^2W}{d\eta^2}=\varpi\left(t,\ \frac{d}{dt}\right)W.$$

On comparing the last result with the equation (1) we see that W is an integral of (1). And as ξ, η contain a germ that is not contained in $F(x, y, t)$, the integral $F(\xi, \eta, t)$ will contain that germ, and be at least as general as $F(x, y, t)$.

Hence without diminishing the generality (and with a chance of increasing it) we may write $Mx + jmy$ and $mx \mp jMy$ for x and y in any integral of equation (1).

In equation (2) we may substitute $\frac{1}{2}(c + c^{-1})$ for M, and $\frac{i}{2}(c - c^{-1})$ for m when the upper sign of m^2 is used in (2); but when the lower sign is used we must substitute $\frac{j}{2}(c - c^{-1})$ for m, c being a general germ.

The double sign in this Article is regulated by that in equation (1).

114. It will be observed as a property connecting the two sets of independent variables used in the preceding Article, that

$$\xi^2 \pm \eta^2 = x^2 \pm y^2.$$

We may represent either of these quantities by r^2. Hence in passing from the equation in terms of x, y to the equivalent equation in ξ, η, and expressing the results in terms of r and another independent variable, the quantity r will occur in the two resulting forms in the same manner, and be of the same value in both.

115. To integrate $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{du}{dt}$.

The general exponential integral may be written in the following form:

$$u = A\epsilon^{Mx+Ny+(M^2+N^2)t}$$

$$= A\epsilon^{M^2t+Mx}\,\epsilon^{N^2t+Ny}.$$

By Art. 64 the germs M, N may be eliminated, and the following is the general form of the subintegrals,

$$P = t^{-\frac{1}{2}}\,\epsilon^{-\frac{x^2}{4t}} \,.\, t^{-\frac{1}{2}}\epsilon^{-\frac{y^2}{4t}}$$

$$= t^{-1}\,\epsilon^{-\frac{x^2+y^2}{4t}} = t^{-1}\,\epsilon^{\frac{(ix+y)(ix-y)}{4t}}.$$

From this we may find u in the following manner,

$$u = F\left(\frac{d}{dx} - i\frac{d}{dy}\right)P + f\left(\frac{d}{dx} + i\frac{d}{dy}\right)P$$

$$= t^{-1}\left\{F\left(\frac{y+ix}{t}\right) + f\left(\frac{y-ix}{t}\right)\right\}\epsilon^{-\frac{x^2+y^2}{4t}} \quad \text{......... (1).}$$

If now we assume $x = r\sin\theta$, and $y = r\cos\theta$, this integral takes the following form, since

$$y + ix = r(\cos\theta + i\sin\theta) = r\epsilon^{i\theta},$$

$$u = \frac{1}{t}\epsilon^{-\frac{r^2}{4t}}\left\{F\left(\frac{r}{t}\epsilon^{i\theta}\right) + f\left(\frac{r}{t}\epsilon^{-i\theta}\right)\right\}$$

$$= \frac{1}{t}\epsilon^{-\frac{r^2}{4t}}F\left(\frac{r}{t}\epsilon^{i\theta}\right) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

116. To integrate

$$\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2} = \frac{du}{dt}.$$

We might deduce this from the preceding Article by merely writing iy for y; but we shall integrate this equation in an independent manner.

Let ξ, η be a new set of independent variables such that $\xi = x + y$ and $\eta = x - y$;

$$\therefore \frac{d^2u}{d\xi\, d\eta} = \frac{du}{4dt}.$$

This agrees in form with the equation integrated in Art. 101;

$$\therefore u = \frac{1}{t}\left\{F\left(\frac{\eta}{t}\right) + f\left(\frac{\xi}{t}\right)\right\}\epsilon^{-\frac{\xi\eta}{4t}}$$

$$= \frac{1}{t}\epsilon^{\frac{y^2-x^2}{4t}}\left\{F\left(\frac{x-y}{t}\right) + f\left(\frac{x+y}{t}\right)\right\}.$$

117. To integrate

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2u}{dt^2}.$$

This has been integrated in Art. 107, but the following method will serve as an illustration of the variety of ways in which germs may be introduced into the general exponential integral:

$$u = A\epsilon^{(M+N)t + (M-N)x + 2\sqrt{MN}.y}$$

$$= A\epsilon^{M(t+x)}\,.\,\epsilon^{N(t-x) + \sqrt{N}(2y\sqrt{M})}.$$

Eliminate N; then the first subintegral is

$$P = \epsilon^{M(t+x)} . (t-x)^{-\frac{1}{2}} \epsilon^{\frac{My^2}{x-t}}$$

$$= (t-x)^{-\frac{1}{2}} . \epsilon^{M\left(t+x+\frac{y^2}{x-t}\right)}$$

$$= (t-x)^{-\frac{1}{2}} F\left(\frac{x^2+y^2-t^2}{x-t}\right).$$

The proposed equation shews that in any integral we may write jt for t. We also notice that x and y are interchangeable. Introducing these properties, we find the following general integral,

$$u = (x+jt)^{-\frac{1}{2}} F\left(\frac{x^2+y^2-t^2}{x+jt}\right) + (y+jt)^{-\frac{1}{2}} f\left(\frac{x^2+y^2-t^2}{y+jt}\right).$$

118. The integral of the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0$$

will be deduced from the preceding Article by writing iz for jt;

$$\therefore\ u = (x+iz)^{-\frac{1}{2}} F\left(\frac{x^2+y^2+z^2}{x+iz}\right) + (y+iz)^{-\frac{1}{2}} f\left(\frac{x^2+y^2+z^2}{y+iz}\right).$$

This is a complete general integral of Laplace's equation.

119. By a different distribution of the major germs from that in Art. 117 we may obtain in another form a general integral of the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{d^2u}{dt^2}.$$

Let
$$u = A\epsilon^{L\{(c+c^{-1})\frac{x}{2} + i(c-c^{-1})\frac{y}{2} + jt\}}$$

$$= A\epsilon^{jLt} . \epsilon^{c(x+iy)\frac{L}{2} + c^{-1}(x-iy)\frac{L}{2}}.$$

Eliminate c by Art. 65; then the following is the first of the subintegrals,

$$P = \epsilon^{jLt} . (x+iy)^{-\frac{1}{2}} \epsilon^{\pm L\sqrt{x^2+y^2}}$$

$$= (x+iy)^{-\frac{1}{2}} \epsilon^{L(jt\pm\sqrt{x^2+y^2})};$$

$$\therefore\ u = (x+iy)^{-\frac{1}{2}} F(jt \pm \sqrt{x^2+y^2}).$$

120. If in preceding Article we write iz for jt, we find the integral of the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0$$

in the following form,

$$u=(x+iy)^{-\frac{1}{2}}F\left(\sqrt{x^2+y^2}\pm iz\right),$$

which is a form of the general integral of Laplace's equation agreeing with (2) in Art. 108.

121. To extend to three independent variables the property proved in Art. 113 for two.

Our equation is now of the following form,

$$\varpi\left(t,\frac{d}{dt}\right)u=\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2} \quad\text{.................}\quad (1).$$

Let $r^2=x^2+y^2+z^2$; and let ξ, η, ζ be new independent variables such that

$$\xi=ax+a'y+a''z,$$
$$\eta=bx+b'y+b''z,$$
$$\zeta=cx+c'y+c''z.$$

The nine constants in these expressions are disposable; and we are to dispose of them in such a way that when these values of ξ, η, ζ are written for x, y, z in any integral of equation (1) the resulting formula will also be an integral of the same equation.

Let $u=F(x, y, z, t)$ be any integral of equation (1), then is $V=F(\xi, \eta, \zeta, t)$ an integral of the equation

$$\varpi\left(t,\frac{d}{dt}\right)V=\frac{d^2u}{d\xi^2}+\frac{d^2u}{d\eta^2}+\frac{d^2u}{d\zeta^2} \quad\text{.................}\quad (2).$$

But $$\frac{dV}{dx}=a\frac{dV}{d\xi}+b\frac{dV}{d\eta}+c\frac{dV}{d\zeta}=\left(\frac{ad}{d\xi}+\frac{bd}{d\eta}+\frac{cd}{d\zeta}\right)V;$$

$$\therefore\ \frac{d^2V}{dx^2}=\left(\frac{ad}{d\xi}+\frac{bd}{d\eta}+\frac{cd}{d\zeta}\right)^2V.$$

Similarly $$\frac{d^2V}{dy^2} = \left(\frac{a'd}{d\xi} + \frac{b'd}{d\eta} + \frac{c'd}{d\zeta}\right)^2 V,$$

and $$\frac{d^2V}{dz^2} = \left(\frac{a''d}{d\xi} + \frac{b''d}{d\eta} + \frac{c''d}{d\zeta}\right)^2 V.$$

Expanding and adding together the right-hand members of these three equations, and assuming the following six relations among the nine disposable constants,

$$1 = a^2 + a'^2 + a''^2 = b^2 + b'^2 + b''^2 = c^2 + c'^2 + c''^2$$

and $$0 = ab + a'b' + a''b'' = ac + a'c' + a''c'' = bc + b'c' + b''c'',$$

we have the following general result,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = \frac{d^2V}{d\xi^2} + \frac{d^2V}{d\eta^2} + \frac{d^2V}{d\zeta^2}$$

$$= \varpi\left(t, \frac{d}{dt}\right) V, \text{ by equat. (2).}$$

Hence V, which is equal to $F(\xi, \eta, \zeta, t)$, is an integral of equation (1) when the above values of ξ, η, ζ are written for x, y, z in $F(x, y, z, t)$, which is equal to u.

122. It will be observed that the nine disposable constants have to satisfy only six conditional equations; and moreover that those six equations are such as prove the following general relation between x, y, z and ξ, η, ζ,

$$x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2.$$

Hence r^2, which is equal to $x^2 + y^2 + z^2$, does not change in value when we pass from x, y, z to the values represented by ξ, η, ζ.

123. In reference to the equation (1) of Art. 121 we are aware that x, y, z which it contains are not necessarily the co-ordinates of a point P in space, nor is there necessarily any system of co-ordinates to which they are referred; but for convenience in what follows we will suppose them referred to an arbitrary rectangular system of co-ordinates Ox, Oy, Oz, and we will set off upon these axes the values of x, y, z; and suppose P

the point in space of which x, y, z are thus constituted the rectangular co-ordinates;

$$\therefore\ OP^2 = r^2 = x^2 + y^2 + z^2.$$

Now on these same co-ordinate axes set off a, a', a'' as the co-ordinates of a fixed point A; and let b, b', b'' be those of B, and c, c', c'' those of C.

Then if we assume $OA = OB = OC = 1$, these assumptions satisfy the first three of the conditions to be satisfied by the nine disposable constants; and if these lines OA, OB, OC be now supposed to be at right angles to each other, the following equations shew that the constants will then satisfy the remaining three conditions also. For then

$$ab + a'b' + a''b'' = \cos AOB = \cos\frac{\pi}{2} = 0,$$

$$ac + a'c' + a''c'' = \cos AOC = \cos\frac{\pi}{2} = 0,$$

$$bc + b'c' + b''c'' = \cos BOC = \cos\frac{\pi}{2} = 0.$$

Hence, if OA, OB, OC be each equal to unity and mutually at right angles, the six conditional equations are all satisfied; and OA, OB, OC may be taken as an arbitrary system of rectangular co-ordinate axes. We say an *arbitrary* system, because of the nine disposable constants on which the positions of OA, OB, OC depend three are still disposable, and these render the position of this system so far arbitrary.

Now as x, y, z are the co-ordinates of P, and $OP = r$,

$$x = r\cos xOP,\quad y = r\cos yOP,\quad z = r\cos zOP;$$

$$\therefore\ \cos AOP = a\frac{x}{r} + a'\frac{y}{r} + a''\frac{z}{r} = \frac{\xi}{r};$$

$$\therefore\ \xi = r\cos AOP.$$

Similarly $$\eta = r\cos BOP,$$

and $$\zeta = r\cos COP.$$

Therefore ξ, η, ζ are the co-ordinates of P referred to the arbitrary system of rectangular co-ordinate axes OA, OB, OC; the position of which in reference to the original fixed system Ox, Oy, Oz depends upon the values arbitrarily assigned to the remaining three still disposable constants; which we may in fact describe as three disposable real germs.

When therefore we have before us an equation of the class comprehended in the equation

$$\varpi\left(t,\ \frac{d}{dt}\right)u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}$$

we are at liberty to substitute, in any integral of it, in the places of x, y, z, the values of ξ, η, ζ in terms of x, y, z given in Art. 121, and the formula produced by such substitution will be an integral of the same equation; and will virtually contain three real germs which were not contained in the original form of the integral. And moreover the value of r^2 will not thereby be affected.

Laplace's Equation, $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0.$

124. The following is the simplest form of the general exponential integral,

$$u = A\epsilon^{L(Mx+Ny+iz)},$$

subject to the condition $M^2 + N^2 = 1$.

This integral is equivalent to the following form,

$$u = A\epsilon^{L(Mx+Ny)} \cos (Lz + B).$$

If we now write in this for x, y, z the values of ξ, η, ζ found in Art. 121, we shall have a general *form* which may be thus represented,

$$u = A\epsilon^{L(ax+a'y+a''z)} \cos L\,(bx + b'y + b''z + B),$$

(a, a', a'', b, b', b'' are not here the same as in Art. 121, but at present they are disposable).

Let the cosine be replaced by its exponential equivalent;

$$\therefore\ u = A\epsilon^{L(\overline{a+ib}\,.\,x+\overline{a'+ib'}\,.\,y+\overline{a''+ib''}\,.\,z)}.$$

That this may be the general integral, the following condition must be satisfied,

$$0 = (a+ib)^2 + (a'+ib')^2 + (a''+ib'')^2,$$

which separates itself into the two following independent conditions,

$$a^2 + a'^2 + a''^2 = b^2 + b'^2 + b''^2,$$

$$ab + a'b' + a''b'' = 0.$$

Now the presence of the arbitrary germ L permits us, without any loss of generality, to assume

$$a^2 + a'^2 + a''^2 = b^2 + b'^2 + b''^2 = 1:$$

and this assumption being made we find, on reference to Art. 121, that these six disposable constants are identical with the corresponding six in that Article;

$$\therefore\ ax + a'y + a''z = \xi,$$

$$\text{and}\ \ bx + b'y + b''z = \eta,$$

and consequently the general exponential integral may be expressed in the following brief form,

$$u = A\epsilon^{L\xi}\cos(L\eta + B) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the above equations of conditions among the six constants a, a', a'', b, b', b'', shew that ξ and η are interchangeable;

$$\therefore\ u = A'\epsilon^{L\eta}\cos(L\xi + B') \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Either of these results may be regarded as a general integral of Laplace's equation; or by addition they may be combined in a single integral.

The six disposable constants are subject to only three conditional equations.

On reference to Art. 123 we perceive that ξ, η are the co-ordinates of P referred to the two lines OA, OB; while x, y, z are the co-ordinates of P referred to the three fixed rectangular axes Ox, Oy, Oz.

Hence ξ, η are the projections of OP (i.e. of r) upon OA, OB respectively; and as the positions of these two rectangular axes are dependent on three arbitrary germs, therefore the values of ξ, η involve those three germs, and consequently the integral

$$u = A\epsilon^{L\eta} \cos (L\xi + B)$$

is a germ integral.

If we assign *particular* values to the three germs we obtain from this a *particular* integral: and as particular values of germs may be infinite in number, we may thus obtain an unlimited number of particular integrals, which we may denominate a new family of subintegrals: and out of them it may be possible by proper management to construct a general integral in finite terms in a form suitable to a physical or geometrical problem which we may have in hand.

125. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + \dfrac{d^2z}{dz^2} = 0.$

Assume the following form of the general exponential integral,

$$u = A\epsilon^{m(iy-x)} . \epsilon^{M(iy+x)+\sqrt{M}(2z\sqrt{m})}.$$

Eliminate M; then the form of the resulting subintegral is,

$$P = \epsilon^{m(iy-x)} . (iy+x)^{-\frac{1}{2}} \epsilon^{\frac{-mz^2}{iy+x}}$$

$$= (iy+x)^{-\frac{1}{2}} \epsilon^{m\left(iy-x-\frac{z^2}{iy+x}\right)};$$

$$\therefore\ u=(iy+x)^{-\frac{1}{2}}F\left(iy-x-\frac{z^2}{iy+x}\right)$$

$$=(x+iy)^{-\frac{1}{2}}F\left(\frac{r^2}{x+iy}\right)$$

$$=\frac{1}{r}\left(\frac{r^2}{x+iy}\right)^{\frac{1}{2}}F\left(\frac{r^2}{x+iy}\right)$$

$$=\frac{1}{r}F\left(\frac{r^2}{x+iy}\right),\ \text{or,}=\frac{1}{r}F\left(\frac{x+iy}{r^2}\right)\text{............}(1)$$

$$=\frac{A}{r}\left(\frac{x+iy}{r^2}\right)^m,\ m\text{ being a germ,}$$

$$=\frac{A\,(x+iy)^m}{r^{2m+1}}\text{....................................}(2).$$

This integral contains only even powers of z; consequently this is only one of the subgeneral integral forms. The other subgeneral form will be found from this by differentiating with $\dfrac{d}{dz}$.

Let us now assume $x=r\cos\theta\cos\phi$, $y=r\cos\theta\sin\phi$ $z=r\sin\theta$;

$$\therefore\ x+iy=r\cos\theta\,(\cos\phi+i\sin\phi)=r\cos\theta\,\epsilon^{i\phi};$$

$$\therefore\ u=\frac{A}{r}\left(\frac{x+iy}{r^2}\right)^m=\frac{A}{r}\left(\frac{\cos\theta}{r}\,.\,\epsilon^{i\phi}\right)^m$$

$$=\frac{1}{r}F\left(\frac{\cos\theta}{r}\,.\,\epsilon^{i\phi}\right)\text{..............................}(3).$$

This subgeneral integral contains two independent arbitrary functions by reason of the double sign of i.

We may obtain another form of the subgeneral integral in the following manner:

126. To integrate $\dfrac{d^2u}{dx^2}+\dfrac{d^2u}{dy^2}+\dfrac{d^2u}{dz^2}=0.$

Assume the following form of the general exponential integral,

$$u=A\epsilon^{N(Mx+my+iz)},$$

the germs M, m being subject to the equation $M^2 + m^2 = 1$;

$$\therefore\ u = A\epsilon^{iNz} . \epsilon^{M(Nx)+m(Ny)}.$$

Eliminate M and m; the following is the corresponding form of the subintegral:

$$P = (x \pm iy)^{-\frac{1}{2}} \epsilon^{N(j\sqrt{x^2+y^2}+iz)};$$

$$\therefore\ u = (x \pm iy)^{-\frac{1}{2}} F(\sqrt{x^2+y^2} + iz) \text{..............} (1)$$

$$= A\,(x \pm iy)^{-\frac{1}{2}} (\sqrt{x^2+y^2} + iz)^m,$$

m being here an extemporized germ. Also y and z are interchangeable.

But $x \pm iy = r\epsilon^{\pm i\phi} \cos\theta$, and $\sqrt{x^2+y^2} + iz = r\epsilon^{i\theta}$;

$$\therefore\ u = A\,(r\epsilon^{\pm i\phi} \cos\theta)^{-\frac{1}{2}} (r\epsilon^{i\theta})^m$$

$$= (r\epsilon^{\pm i\phi} \cos\theta)^{-\frac{1}{2}} F(r\epsilon^{i\theta}) \text{ ....} (2).$$

Also $$u = Ar^{m-\frac{1}{2}} \sqrt{\cos\theta . \epsilon^{\pm i\phi}} . \epsilon^{im\theta} \text{ ....} (3).$$

The form of this integral differs from that found in the preceding Article, and illustrates the power we have over the form of the general integral, since we can introduce the germs into the general exponential integral in more than one set of different relationships to the independent variables.

Laplace's Functions.

127. Let the independent variables in Laplace's equation be changed from x, y, z to r, θ, ϕ; these being defined by the equations before given, viz.,

$$x = r\cos\theta\cos\phi,\ y = r\cos\theta\sin\phi,\ z = r\sin\theta;$$

$$\therefore\ x^2 + y^2 + z^2 = r^2;$$

and the transformed equation is

$$0 = r^2\frac{d^2u}{dr^2} + 2r\frac{du}{dr} + \frac{d^2u}{d\theta^2} - \tan\theta . \frac{du}{d\theta} + \sec^2\theta\frac{d^2u}{d\phi^2},$$

or more conveniently in the following form,

$$0 = \cos^2\theta \, . \, \frac{rd}{dr}\left(\frac{rd}{dr} + 1\right) u + \left(\cos\theta \, . \, \frac{d}{d\theta}\right)^2 u + \frac{d^2 u}{d\phi^2} \ldots\ldots (1).$$

Concerning this form of Laplace's equation we remark that r alone takes a major germ; and ϕ alone takes a minor germ. Also $j\theta, j\phi$ may at any time be independently written for θ, ϕ in any integral of it.

Now as mr may be written for r (m being a major germ) in the integral of (1), that integral (Art. 13) may be expanded in a series in powers of m;

$$\therefore \; u = m^p \, . \, r^p W_p + m^q \, . \, r^q W_q + \ldots$$

and the powers of m in this equation may be replaced with independent arbitrary constants, which are in fact the coefficients of the subintegrals of u. The quantities denoted by W are functions, not of r, but of θ and ϕ.

We may therefore take the following as the general representative of the subintegrals of u,

$$P = r^n W.$$

To determine W corresponding to a given value of n we substitute this subintegral in equation (1), and the result is,

$$0 = n(n+1)\cos^2\theta \, . \, W + \left(\cos\theta \, . \, \frac{d}{d\theta}\right)^2 W + \frac{d^2 W}{d\phi^2} = 0 \ldots (2).$$

This is the Equation of Laplace's Functions, and from this equation we perceive that r has been divided out, leaving W, the representative of Laplace's Functions, dependent on the value of the product $n(n+1)$, the only remanet of r.

128. *Lemma.* The numerical value of the product $(n+a)$ $(n+b)$ will suffer no change if we write in it $-(n+a+b)$ for n.

Hence $n(n+1)$ remains unchanged in value when $-(n+1)$ is written for n.

This shews that instead of assuming the general form of the subintegrals in the preceding Article to be $P = r^n W$, this will not be a general assumption unless we write $Ar^n + ar^{-n-1}$ for r^n;

$$\therefore\ P = (Ar^n + ar^{-n-1})\, W$$

is the general form of the subintegrals; and if it be substituted for u in equation (1) of the preceding Article the result will be found to agree with equation (2).

Consequently W, the n^{th} Laplace's function, is also the $-(n+1)^{\text{th}}$ function.

Hence u admits of expansion in two independent series, one comprising only positive powers of r, and the other only negative powers; and the corresponding terms of these two series will have the same forms of W. We say *forms* because W will contain at least two independent terms, the equation (2) being of the second order.

129. In Art. 125 we have found the following subgeneral integral of the equation

$$0 = \cos^2\theta \,.\, \frac{rd}{dr}\left(\frac{rd}{dr} + 1\right) u + \left(\cos\theta \,.\, \frac{d}{d\theta}\right)^2 u + \frac{d^2u}{d\phi^2},$$

$$u = \frac{A}{r}\left(\frac{\cos\theta}{r} \,.\, \epsilon^{i\phi}\right)^m,$$

in which m is a germ.

As a particular case take $m = n$; then the general term in the expansion of u will be

$$= \frac{A}{r^{n+1}} \,.\, (\cos\theta \,.\, \epsilon^{i\phi})^n.$$

But we have shewn that this term is equal to $Ar^{-n-1}W$; consequently one part of W is $(\cos\theta \,.\, \epsilon^{i\phi})^n$; and the other part will be found from this by writing $-(n+1)$ for n;

$$\therefore\ W = B\,(\cos\theta \,.\, \epsilon^{i\phi})^n + b\,(\cos\theta \,.\, \epsilon^{i\phi})^{-n-1} \dots\dots\dots\dots (1);$$

$$\therefore\ P = (Ar^n + ar^{-n-1})\,\{B\,(\cos\theta \,.\, \epsilon^{i\phi})^n + b\,(\cos\theta \,.\, \epsilon^{i\phi})^{-n-1}\},$$

and if we write m (a germ) for n, we may write u for P.

$$\therefore\ u = (Ar^m + ar^{-m-1})\,\{B\,(\cos\theta\,.\,\epsilon^{i\phi})^m + b\,(\cos\theta\,.\,\epsilon^{i\phi})^{-m-1}\}$$

$$= F\,(r\cos\theta\,.\,\epsilon^{i\phi}) + \frac{1}{r} f\,(r^{-1}\cos\theta\,.\,\epsilon^{i\phi}) \text{ ....(2).}$$

130. We may find W in another general form by means of Art. 126. For according to that Article we have

$$u = A\,(r\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,(r\epsilon^{i\theta})^m.$$

Assume $m = n + \frac{1}{2}$ as a particular case; the corresponding term in the expansion of u will be

$$r^n\,.\,(\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\epsilon^{i\,(n+\frac{1}{2})\theta}.$$

Hence the part of W corresponding to this is

$$W = (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\epsilon^{i\,(n+\frac{1}{2})\theta}.$$

If in this we write $-(n+1)$ for n, we find the remaining part of W to be

$$W = (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\epsilon^{-i(n+\frac{1}{2})\theta}.$$

Hence the complete value of W is

$$W = (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\{B\epsilon^{i(n+\frac{1}{2})\theta} + b\epsilon^{-i(n+\frac{1}{2})\theta}\}$$

$$= (\epsilon^{\pm i\phi}\cos\theta)^{-\frac{1}{2}}\,\{B\cos\overline{n+\tfrac{1}{2}}\theta + b\sin\overline{n+\tfrac{1}{2}}\theta\} \text{ ......(1),}$$

and the value of the general term in the expansion of u corresponding to this will be

$$= (Ar^n + ar^{-n-1})\,W \text{ .... (2).}$$

Thus we have in this and the preceding Article found two distinct forms of Laplace's functions.

We may write m (a germ) for n, and the results will then take the form of arbitrary functions, and (2) will be equal to u.

Exceptional case of Laplace's Functions.

131. This occurs when the value of n is such as renders $n(n+1)=0$, for then the first term of equation (1) of Art. 127 vanishes, and the equation for the determination of W takes the following form,

$$0=\left(\cos\theta\,.\,\frac{d}{d\theta}\right)^2 W+\frac{d^2W}{d\phi^2}.$$

Assume τ such a function of θ as shall satisfy the equation $d\theta=\cos\theta\,.\,d\tau$;

$$\therefore\ \tau=\log\tan\left(\frac{\pi}{4}+\frac{j\theta}{2}\right),$$

$$\text{and } 0=\frac{d^2W}{d\tau^2}+\frac{d^2W}{d\phi^2};$$

$$\therefore\ W=F(\tau+i\phi)$$

$$=F(\epsilon^{\tau+i\phi})$$

$$=F\left(\epsilon^{i\phi}\tan\overline{\frac{\pi}{4}+\frac{j\theta}{2}}\right)\ \ldots\ldots\ldots\ldots\ldots(1).$$

Also $Ar^n+ar^{-(n+1)}$ becomes $A+\dfrac{a}{r}$ when $n=0$;

$$\therefore\ u=\left(A+\frac{a}{r}\right)F\left(\epsilon^{i\phi}\tan\overline{\frac{\pi}{4}+\frac{j\theta}{2}}\right)\ \ldots\ldots\ldots\ldots(2).$$

This gives us that portion of u which corresponds to two terms of its expansion in integer powers of r, viz., A and $\dfrac{a}{r}$.

The algebraic signs of j and i are independent in these results.

132. To integrate the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=u.$$

We may assume the following as the form of the general exponential integral,

$$u = A\epsilon^{NMx+Nmy+nz},$$

subject to the two conditional equations

$$M^2 + m^2 = 1, \text{ and } N^2 + n^2 = 1\,;$$

$$\therefore\ u = A\epsilon^{nz}\,.\,\epsilon^{M(Nx)+m(Ny)}.$$

We may eliminate M and m from this integral, thereby obtaining the following general subintegral form,

$$P = \epsilon^{nz}\,(x+iy)^{-\frac{1}{2}}\,\epsilon^{jN\sqrt{(x+iy)(x-iy)}}$$

$$= (x+iy)^{-\frac{1}{2}}\,\epsilon^{nz+jN\sqrt{x^2+y^2}}.$$

By the same method we now eliminate n and N, and obtain the following as the first subintegral,

$$P = (x+iy)^{-\frac{1}{2}}\,.\,(\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}}\,\epsilon^{j\sqrt{x^2+y^2+z^2}}$$

$$= (x+iy)^{-\frac{1}{2}}\,(\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}}\,\epsilon^{jr}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$= (r\cos\theta\,.\,\epsilon^{i\phi})^{-\frac{1}{2}}\,(r\epsilon^{\pm i\theta})^{-\frac{1}{2}}\,\epsilon^{jr}$$

$$= r^{-1}\,\epsilon^{jr}\,(\cos\theta\,.\,\epsilon^{i\,.\,\overline{\phi\pm\theta}})^{-\frac{1}{2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Written out more fully this is equivalent to the following,

$$u = \frac{1}{r}\,(A\epsilon^{r} + a\epsilon^{-r})\sqrt{\sec\theta}\left(B\cos\frac{\phi\pm\theta}{2} + b\sin\frac{\phi\pm\theta}{2}\right)\ldots\ldots(3).$$

It is not to be forgotten that the form of the proposed equation shews that in integral (1) y and z are interchangeable.

In (3) there are no interchangeable variables.

133. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + \dfrac{d^2u}{dz^2} = u.$

We assume the following as a form which is equivalent to the general exponential integral,

$$u = A\epsilon^{\mu(Lx+My+Nz)}\cos\lambda\,(lx+my+nz+B)\ldots\ldots\ldots(1),$$

subject to the following condition (which the two disposable constants μ, λ enable us to assume)

$$(L+il)^2 + (M+im)^2 + (N+in)^2 = 0.$$

This resolves itself into the two following independent equations,

$$\left.\begin{array}{rl} & L^2+M^2+N^2=l^2+m^2+n^2 \\ \text{and} & Ll+Mm+Nn=0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2).$$

If the above value of u be substituted in the proposed equation we find that the following relation must hold good between μ and λ,

$$\mu^2-\lambda^2=1.$$

Hence if c be an extemporized germ this condition will be satisfied by assuming

$$\mu=\tfrac{1}{2}(c+c^{-1}) \text{ and } \lambda=\tfrac{1}{2}(c-c^{-1}) \ldots\ldots\ldots\ldots(3).$$

Hence the integral form in (1) is fully determined.

Again, let $S=Lx+My+Nz$, and $T=lx+my+nz$, subject to the equations (2);

$$\begin{aligned} \therefore\ u &= A\epsilon^{\mu S}\cos\lambda T \\ &= A\epsilon^{\mu S+i\lambda T} \\ &= A\epsilon^{\frac{1}{2}(c+c^{-1})S+\frac{i}{2}(c-c^{-1})T} \\ &= A\epsilon^{c\,.\,\frac{1}{2}(S+iT)+c^{-1}\,.\,\frac{1}{2}(S-iT)}. \end{aligned}$$

Eliminating c from this integral we find the following form of the general subintegral,

$$P=(S+iT)^{-\frac{1}{2}}\,\epsilon^{j\sqrt{S^2+T^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

134. The integrals of the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}+u=0$$

may be deduced from the above integrals by changing the algebraic sign of c^{-1} but not that of c.

135. To integrate $\dfrac{d^2u}{dx^2}+\dfrac{d^2u}{dy^2}+\dfrac{d^2u}{dz^2}=\dfrac{du}{dt}$.

The general exponential integral may be put in the following form,

$$u=A\epsilon^{Lx+My+Nz+(L^2+M^2+N^2)t},$$

L, M, N being independent germs;

$$\therefore\ u=A\epsilon^{L^2t+Lx}\,.\,\epsilon^{M^2t+My}\,.\,\epsilon^{N^2t+Nz}.$$

The germs being all eliminated by Art. 64, we obtain the following form of the first subintegral,

$$P = t^{-\frac{1}{2}} \epsilon^{-\frac{x^2}{4t}} . t^{-\frac{1}{2}} \epsilon^{-\frac{y^2}{4t}} . t^{-\frac{1}{2}} \epsilon^{-\frac{z^2}{4t}}$$

$$= t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}} .$$

We may pass from this to the general integral in the following manner:

$$u = t^{-\frac{3}{2}} \epsilon^{-\frac{z^2}{4t}} F\left(\frac{d}{dx} + i\frac{d}{dy}\right) \epsilon^{-\frac{x^2+y^2}{4t}} + t^{-\frac{3}{2}} \epsilon^{-\frac{y^2}{4t}} f\left(\frac{d}{dx} + i\frac{d}{dz}\right) \epsilon^{-\frac{x^2+z^2}{4t}}$$

$$= t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}} F\left(\frac{x-iy}{t}\right) + t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}} f\left(\frac{x-iz}{t}\right)$$

$$= t^{-\frac{3}{2}} \epsilon^{-\frac{r^2}{4t}} \left\{ F\left(\frac{x+iy}{t}\right) + f\left(\frac{x+iz}{t}\right) \right\} \text{......................(1).}$$

136. To integrate $\dfrac{d^2u}{dx^2} + \dfrac{d^2u}{dy^2} + \dfrac{d^2u}{dz^2} = \dfrac{d^2u}{dt^2}$.

The general exponential integral is

$$u = A\epsilon^{L(MNx+mNy+nz+jt)},$$

the germs being subject to the following conditions:

$$M^2 + m^2 = 1, \text{ and } N^2 + n^2 = 1;$$

$$\therefore\ u = A\epsilon^{L(nz+jt)} . \epsilon^{M(LNx)+m(LNy)}.$$

Eliminating M and m we find the following subintegral,

$$P = \epsilon^{L(nz+jt)} . (x+iy)^{-\frac{1}{2}} \epsilon^{LN\sqrt{x^2+y^2}}$$

$$= \epsilon^{Ljt} (x+iy)^{-\frac{1}{2}} \epsilon^{n(Lz)+N(L\sqrt{x^2+y^2})};$$

and eliminating N and n in the same manner we find the following subintegral,

$$P = \epsilon^{Ljt} (x+iy)^{-\frac{1}{2}} (\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}} \epsilon^{L\sqrt{x^2+y^2+z^2}}$$

$$= (x+iy)^{-\frac{1}{2}} (\sqrt{x^2+y^2} \pm iz)^{-\frac{1}{2}} \epsilon^{L(jt+r)};$$

$$\therefore\ u = (x+iy)^{-\frac{1}{2}}\left(\sqrt{x^2+y^2} \pm iz\right)^{-\frac{1}{2}} F(r+jt) \ldots\ldots\ldots\ldots\ldots(1)$$

$$= r^{-1}\left(\cos\theta \,.\, \epsilon^{i(\phi\pm\theta)}\right)^{-\frac{1}{2}} F(r+jt)$$

$$(2)\ldots\ldots = \frac{\sqrt{\sec\theta}}{r}\left(A\cos\frac{\phi\pm\theta}{2} + B\sin\frac{\phi\pm\theta}{2}\right)\{F(r+t)+f(r-t)\}.$$

137. Laplace's equation is rendered important beyond most other equations by the circumstance that many problems of great interest in various branches of Natural Philosophy lead to it, and for their perfect solution render a knowledge of its general and complete integral a matter of necessity; for without that knowledge the investigator is obliged to assume some particular integral known to him, and this he fixes upon as being likely to answer the object he has in view. But this is a method which cannot but limit the generality of his results, and so far limit their authority in any case of appeal.

We shall, therefore, conclude this Essay with the following summary of the method and principles by which we have been enabled to accomplish the complete integration of the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0.$$

(1). The form of this equation allows all the independent variables to take independent minor germs.

(2). From this we learn that any general integral of it may be perfectly expressed in the form of an infinite series in which the powers of x, y, z are positive integers.

(3). From this it follows that, subject to the condition

$$0 = L^2 + M^2 + N^2,$$

the quantities L, M, N being otherwise arbitrary, and not functions of any of the independent variables, the following is a complete and perfect integral of the above equation of Laplace,

$$u = A\epsilon^{Lx+My+Nz}.$$

When we call this the general integral of Laplace's equation, we must remember that the word *general* as thus used is dependent for its propriety on the fact that L, M, N are indeterminate quantities and not mere arbitrary constants.

The equation $y^2 = x$ may on the same principle be called a "parabola"; but when it is so called we assume that x and y are indefinite quantities that simultaneously belong to every individual point of the parabolic curve: for as x, y represent simultaneously all the values they can possibly have that are consistent with the equation $y^2 = x$, they simultaneously represent the co-ordinates of every point of the parabola.

On precisely the same principle we say that L, M, N simultaneously represent all the values that can be given to them which are consistent with the equation $L^2 + M^2 + N^2 = 0$; and accepting their significance in this general sense we denominate the integral above written the *general exponential integral* of Laplace's equation, just as we call $y^2 = x$ a *parabola.*

(4). There are many systems or sets of values of L, M, N in terms of two independent germs which will satisfy the equation $L^2 + M^2 + N^2 = 0$. Each set of such values will give a general integral in a form answering to that particular set by means of which it is obtained. Examples of this occur in the preceding Articles.

The following set of values of L, M, N containing two germs only, leads to a simple result;

$$\begin{aligned}&2Lx + 2My + 2Nz \\ &\qquad = (L - iM)(x + iy) + (L + iM)(x - iy) + 2Nz.\end{aligned}$$

Let $H^2 = L - iM$, and $K^2 = L + iM$;

$$\therefore\ H^2K^2 = L^2 + M^2 = -N^2;$$

$$\therefore\ N = iHK.$$

And $2Lx + 2My + 2Nz = H^2(x + iy) + K^2(x - iy) + 2iHKz$;

$$\therefore\ u = A\epsilon^{(H^2+K^2)x} \cos\{(H^2 - K^2)y + 2HKz + B\} \ldots (\alpha).$$

$$\text{Also } u = A\epsilon^{H^2(x+iy)+2HKiz+K^2(x-iy)}.$$

From this form of u we may first eliminate H and then K, which will give one part of u's first subgeneral integral. The other part of that same subgeneral will be obtained by first eliminating K and then H. The result of elimination is the complete first subgeneral

$$= (x+iy)^{-\frac{1}{2}} F\left(\frac{x^2+y^2+z^2}{x+iy}\right).$$

The second subgeneral may be obtained from this by differentiating with $\frac{d}{dz}$. (See Art. 125.)

(5). We are tempted to pronounce the integral just found to be perfectly general, and so it is in one sense, because it is the type of the missing terms. But in another sense it is not the complete general integral, for the form of the differential equation of which it is the integral shews that in the complete general integral x, y, z must be interchangeable; they must therefore be made to enter the general integral in a symmetrical form.

This we may accomplish by means of Art. 121, and the result will be that we shall have the following instead of the integral in (4), (see Art. 125, equat. 1).

$$ru = F\left(\frac{Ax+By+Cz}{r^2}\right),$$

A, B, C being germs subject to the equation

$$A^2+B^2+C^2=0.$$

We may therefore write L, M, N instead of A, B, C.

But in (3) we have the following form of integral in terms of the same quantities,

$$u = A\epsilon^{Lx+My+Nz}$$
$$= F(Lx+My+Nz).$$

Hence the complete form of the general integral of Laplace's equation which involves x, y, z in a manner which allows

x, y, z to be interchanged without affecting its form is the following:

$$u = F(Lx + My + Nz) + \frac{1}{r} f\left(\frac{Lx + My + Nz}{r^2}\right) \text{ ...... } (\beta).$$

We may therefore announce this as the complete general integral of Laplace's equation in terms of x, y, z.

(6). We now make a change of the independent variables from x, y, z to r, θ, ϕ.

By this change the differential equation itself takes the following form:

$$\cos^2\theta \cdot \frac{rd}{dr}\left(\frac{rd}{dr} + 1\right) u + \left(\cos\theta \cdot \frac{d}{d\theta}\right)^2 u + \frac{d^2 u}{d\phi^2} = 0.$$

Also, since $2\cos\phi = \epsilon^{i\phi} + \epsilon^{-i\phi}$, and $2\sin\phi = i(\epsilon^{-i\phi} - \epsilon^{i\phi})$;

$$\begin{aligned}
\therefore\ & 2Lx + 2My + 2Nz \\
&= 2r(L\cos\theta\cos\phi + M\cos\theta\sin\phi + N\sin\theta) \\
&= r\{(L + iM)\epsilon^{-i\phi} + (L - iM)\epsilon^{i\phi}\}\cos\theta + 2rN\sin\theta \\
&= r(K^2\epsilon^{-i\phi} + H^2\epsilon^{i\phi})\cos\theta + 2rN\sin\theta \\
&= \text{........................} + 2riHK\sin\theta.
\end{aligned}$$

If we now eliminate $\cos\theta$ and $\sin\theta$ by means of their exponential equivalents, we find

$$\begin{aligned}
& 4(L\cos\theta\cos\phi + M\cos\theta\sin\phi + N\sin\theta) \\
&\quad = \frac{4(Lx + My + Nz)}{r} \\
&\quad = (H\epsilon^{\frac{i\phi}{2}} + K\epsilon^{-\frac{i\phi}{2}})^2\epsilon^{\pm i\theta} + (H\epsilon^{\frac{i\phi}{2}} - K\epsilon^{-\frac{i\phi}{2}})^2\epsilon^{\mp i\theta}.
\end{aligned}$$

As H and K are independent germs, we may omit the factor 4 on the left hand.

Hence assuming Q to represent the right-hand member of this equation the integral in (β) may be thus expressed in terms of r, θ, ϕ, *and* two germs H, K,

$$u = F(rQ) + \frac{1}{r} f\left(\frac{Q}{r}\right) \text{ .... } (\gamma).$$

By comparing (4) and (6) with each other it will be seen that the set of germs arbitrarily adopted in (4) is forced on us in (6) by the assumption, not altogether peculiar to this Essay, of the following change of variables,

$$x = r\cos\theta\cos\phi,\quad y = r\cos\theta\sin\phi,\quad \text{and}\quad z = r\sin\theta.$$

(7). If by Art. 24 we express the arbitrary functions in (γ) by means of an extemporized germ m we find the following equivalent integral,

$$u = Ar^m Q^m + ar^{-m-1}Q^m.$$

It is shewn in Art. 129 that the two subgeneral integrals of the equation

$$0 = m(m+1)\cos^2\theta \, . \, Q^m + \left(\cos\theta \, . \, \frac{d}{d\theta}\right)^2 Q^m + \frac{d^2Q^m}{d\phi^2}$$

(which is the equation of Laplace's functions) are Q^m and Q^{-m-1};

$$\begin{aligned} \therefore\ u &= (Ar^m + ar^{-m-1})(BQ^m + bQ^{-m-1}) \\ &= \{A(rQ)^m + B(rQ)^{-m-1}\} + \left\{\frac{C}{r}\left(\frac{r}{Q}\right)^{m+1} + \frac{D}{r}\left(\frac{Q}{r}\right)^m\right\} \\ &= F(rQ) + \frac{1}{r}f\left(\frac{Q}{r}\right), \text{ by Art. 24.} \end{aligned}$$

Hence $(Ar^m + ar^{-m-1})(BQ^m + bQ^{-m-1})$ is exactly equivalent to the integral in (6); and consequently the general expression for Laplace's m^{th} function is the following,

$$m^{\text{th}} \text{ function} = BQ^m + bQ^{-m-1} \ldots\ldots\ldots\ldots\ldots (\delta),$$

B and b being independent arbitrary constants.

The independence of B and b indicates that there are two distinct Laplace's m^{th} functions, viz. Q^m and Q^{-m-1}; the product of which is constant, i.e. independent of m, being equal to Q^{-1}; where

$$\begin{aligned} Q &= (H^2\epsilon^{i\phi} + K^2\epsilon^{-i\phi})\cos\theta + 2iHK\sin\theta \\ &= (H\epsilon^{\frac{i\phi}{2}} + K\epsilon^{-\frac{i\phi}{2}})^2\,\epsilon^{\pm i\phi} + (H\epsilon^{\frac{i\phi}{2}} - K\epsilon^{-\frac{i\phi}{2}})^2\,\epsilon^{\mp i\theta}. \end{aligned}$$

(8). There is an exception to (δ) corresponding to $m = 0$, or -1. This is pointed out in equat. 1 of Art. 131.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A., AT THE UNIVERSITY PRESS.

∂