



UNIVERSITY OF  
ILLINOIS LIBRARY  
AT URBANA-CHAMPAIGN  
BOOKSTACKS

**CENTRAL CIRCULATION BOOKSTACKS**

The person charging this material is responsible for its renewal or its return to the library from which it was borrowed on or before the **Latest Date** stamped below. **You may be charged a minimum fee of \$75.00 for each lost book.**

*Theft, mutilation, and underlining of books are reasons for disciplinary action and may result in dismissal from the University.*

**TO RENEW CALL TELEPHONE CENTER, 333-8400**


**UNIVERSITY OF ILLINOIS LIBRARY AT URBANA-CHAMPAIGN**

1 1997

When renewing by phone, write new due date below previous due date.

L162





FACULTY WORKING  
PAPER NO. 1121

Does the CAPM Under Inflation Differ  
from the APT Under Inflation?

*Kuo C. John Wei*  
*Cheng F. Lee*  
*Chen-Chin Chu*

College of Commerce and Business Administration  
Bureau of Economic and Business Research  
University of Illinois, Urbana-Champaign

THE LIBRARY OF THE  
APR 17 1985  
UNIVERSITY OF ILLINOIS  
URBANA-CHAMPAIGN



# BEBR

FACULTY WORKING PAPER NO. 1121

College of Commerce and Business Administration

University of Illinois at Urbana-Champaign

March, 1985

Does the CAPM Under Inflation  
Differ from the APT Under Inflation?

Kuo C. John Wei  
The University of Mississippi

Cheng F. Lee, Professor  
Department of Finance

Chen-Chin Chu  
Memphis State University

Digitized by the Internet Archive  
in 2011 with funding from  
University of Illinois Urbana-Champaign

<http://www.archive.org/details/doescapmunderinf1121weik>



## Abstract

This paper generalizes the CAPM under inflation of Chen and Boness [3] and Friend, Lanskroner and Losq [8]. Then we compare this generalized CAPM under inflation with the APT under inflation of Elton, Gruber and Rentzler [6]. Interestingly, both two models come out to be the same. We also show that inflation is always priced in different theoretical models, and that either the CAPM or the APT under inflation can be expressed by several different ways.



## Does the CAPM Under Inflation Differ From APT Under Inflation?

In the 1981 Survey of Investment Performance, Peat, Marwick, Mitchell and Company [16] found that investment decisions over the past decade have increasingly been made with an acute awareness of the effects of inflation on asset values, reported earnings, and stock prices. Thus, the classical Capital Asset Pricing Model (CAPM) of Sharpe [25], Lintner [11] and Mossin [15] without an explicit consideration of inflation will be misleading in the guidance of investment decisions, especially during high inflation periods such as 1970s and early 1980s. Fortunately, several articles concerning the impact of inflation on the linear asset pricing models have been derived in recent years. Chen and Boness (CB) [3], Long [12] and Roll [18], and Hagerman and Kim [9] have derived the models under inflation by utility maximization approach in the discrete-time world.<sup>1</sup> Later on, Friend, Landskroner and Losq (FLL) [8] derive the model in the continuous-time world.<sup>2</sup> All of these authors derive the CAPM under inflation in a mean-variance world.<sup>3</sup> Recently, Elton, Gruber and Rentzler (EGR) [6] and Wei [27] have utilized the Arbitrage Pricing Theory (APT) of Ross [20, 21, 22] to derive the complete asset pricing models under inflation. The EGR and the Wei studies differ in that the former assumes that the return generating process is based upon real terms, while the latter assumes that the return generating process is based upon nominal terms.<sup>4</sup> EGR also argue that inflation should be priced in the asset pricing models.<sup>5</sup> Indeed, inflation is priced in the APT framework in both EGR's and Wei's models. Nevertheless, inflation is not priced in

the mean-variance models of CB, FLL and Roll. We believe the mean-variance approach to be creditable in that it has been utilized to derive voluminous models in economics and finance. There is, however, a logical inconsistency in the mean-variance derivations of CB, FLL and Roll.

The purpose of this paper is to clarify the inclusion of inflation into the mean-variance framework. Since inflation risk is a systematic risk, the investor cannot diversify this risk away by simply forming a portfolio. As a consequence, individual's utility should be a function of price level. In other words, the covariance of inflation and individual's wealth should be one of the arguments in the expected utility function assuming that returns on assets and inflation are a multivariate normal distribution or that the utility function is quadratic. Unfortunately, CB, FLL and Roll did not incorporate this feature into their models. This is essentially the reason why inflation is not priced in their models. In this paper, both discrete-time and continuous-time approaches are used to derive asset pricing models under uncertain inflation. In addition, the model is also derived in the APT framework assuming that the return generating process is based on nominal rather than real returns. Since all of the other models assume that investors use real returns to choose between assets, it will be shown that even though the forms of the models are different, all of them are equivalent.

In Section I, five different approaches are utilized to derive the CAPM under inflation. Section II shows that all different models

derived in Section II are equivalent. A brief summary is contained in Section III.

### I. Derivation of the Linear Asset Pricing Models under Uncertain Inflation

For the purpose of clarity and simplicity, throughout the paper  $R_i$  represents nominal return on asset  $i$ , while  $r_j$  represents real return on asset  $i$ . Meanwhile,  $M$  represents the value-weighted market portfolio of all assets (including risk-free asset). This portfolio consists of a fraction  $\alpha$  of assets  $m$  which is risky in nominal returns and a fraction  $(1-\alpha)$  of an asset  $F$  risk-free in nominal returns.

#### A. The Continuous-Time Framework--Merton's Approach

Fisher [7] utilized Merton's [14] continuous-time approach to derive demand for index bonds. Breeden [1] also used this approach to derive the consumption-base CAPM. This section applies the approach used in Merton, Fisher and Breeden to derive the CAPM under inflation. It is assumed that there exist  $N$  risky assets, one risk-free nominal bond, and one state variable, price level ( $\pi$ ). Assuming that the rate of inflation and returns on risky assets are stochastic, and price level ( $\pi$ ) and the price of risky asset ( $P_i$ ) follow a continuous-time Markov process of the Ito type as

$$\frac{d\tilde{\pi}}{\tilde{\pi}} = \mu_{\pi}(\pi, t)dt + \sigma_{\pi}(\pi, t)d\tilde{Z}_{\pi}, \quad (1)$$

$$\frac{d\tilde{P}_i}{\tilde{P}_i} = R_i(\pi, t)dt + \sigma_i(\pi, t)d\tilde{Z}_i, \quad i = 1, \dots, N \quad (2)$$

$$\frac{dB}{B} = R_f(\pi, t)dt, \quad (3)$$

where  $B$  represents the value of the discount bond, which is assumed non-stochastic in nominal returns.  $R_i(\pi, t)$  and  $\sigma_i(\pi, t)$  are drift and diffusion coefficients, respectively. The  $u_\pi(\pi, t)$  represents the expected rate of inflation at time  $t$  while the price level at that time is  $\pi$ .  $R_i(\pi, t)$  is the expected nominal return on asset  $i$ . Similarly,  $\sigma_\pi(\pi, t)$  is the standard deviation of inflation rate, which depends on time and the price level and  $\sigma_i(\pi, t)$  is the standard deviation of nominal return on asset  $i$ .  $\tilde{Z}_i$  and  $\tilde{Z}_\pi$  are Weiner processes.

For simplicity, we assume that the preferences of investors can be summarized by a representative investor with a utility function for lifetime consumption.<sup>6</sup> Let  $X_1, \dots, X_n$ , and  $X_F$  be proportions of the portfolio held in risky asset 1,  $\dots$ ,  $N$ , and the risk-free asset, respectively. The flow budget constraint, given the dynamic change in nominal wealth ( $W$ ), is

$$\begin{aligned} d\tilde{W} = & \{[\sum X_i (R_i - R_F) + R_F]W - \pi C\} dt \\ & + W \sum_{i=1}^n X_i \sigma_i d\tilde{Z}_i, \end{aligned} \quad (4)$$

where  $C$  is the rate of consumption.

The representative investor is assumed to maximize the expected value at each instant in time of a time-additive and state-independent von Neumann-Morgenstein utility function for lifetime consumption. That is,

$$\max_{\{C, X_i\}} E_0 \int_0^{\infty} U(C(t), t) dt, \quad (5)$$

subject to equation (4) and  $W(0) = W_0$ , and with  $U(C(t), t)$  strictly quasiconcave in  $C$ . Let  $J(W, \pi, t)$  be the maximum expected utility of

lifetime consumption in equation (5) that is attainable with nominal wealth  $W$  for price level  $\pi$ . That is,

$$J(W, \pi, t) = \max_{\{C, X_i\}} E_t \int_t^{\infty} U(C(\tau), \tau) d\tau. \quad (6)$$

The first-order conditions for the optimal control of equation (5) subject to the dynamic wealth constraints of equation (4) through the utilization of equation (6) are

$$0 = U_C(C, t) - \pi J_W, \quad (7)$$

$$0 = J_W(R_i - R_F) + J_{WW} W \sigma_{iM} + \pi J_{W\pi} \sigma_{i\pi}, \quad i=1, \dots, N \quad (8)$$

where  $\sigma_{iM}$  is covariance between nominal returns on risky asset  $i$  and on the market portfolio  $M$ , and  $\sigma_{i\pi}$  is the covariance of nominal return on risky asset  $i$  with rate of inflation. Equation (8) can be rewritten as

$$R_i - R_F = T(W, \pi) \sigma_{iM} + H(W, \pi) \sigma_{i\pi}, \quad i=1, \dots, N, \quad (9)$$

where  $T(W, \pi) = -WJ_{WW}/J_W$ , is relative risk tolerance (the reciprocal of relative risk aversion), and  $H(W, \pi) = -\pi J_{W\pi}/J_W$ . Since equation (9) is a pricing relation for any asset including the market portfolio, substituting the market portfolio into equation (9), we have

$$R_M - R_F = T(W, \pi) \sigma_M^2 + H(W, \pi) \sigma_{M\pi} \quad (10)$$

Likewise, consider the index bond if it exists, or an inflation hedge portfolio.<sup>7</sup> As noted in Fisher [7], the nominal return on the index bond ( $R_I$ ) is a real return of  $r_I$  (which is a constant) plus the

realized rate of inflation ( $d\pi/\pi$ ). Substituting the index bond into equation (9), we arrive at

$$R_I - R_F = T(W, \pi)\sigma_{M\pi} + H(W, \pi)\sigma_{\pi}^2 \quad (11)$$

Solving  $T(W, \pi)$  and  $H(W, \pi)$  from equations (10) and (11) and substituting the solutions of  $T(W, \pi)$  and  $H(W, \pi)$  into equation (9), the asset pricing behavior may be written as

$$R_i - R_F = [\sigma_{iM}, \sigma_{i\pi}] \begin{bmatrix} \sigma_M^2 & \sigma_{M\pi} \\ \sigma_{M\pi} & \sigma_{\pi}^2 \end{bmatrix}^{-1} \begin{bmatrix} (R_M - R_F) \\ (R_I - R_F) \end{bmatrix} \quad (12a)$$

$$= b_{iM}(R_M - R_F) + b_{i\pi}(R_I - R_F) \quad (12)$$

where

$$b_{iM} = \frac{\sigma_{iM}\sigma_{\pi}^2 - \sigma_{i\pi}\sigma_{M\pi}}{\sigma_M^2\sigma_{\pi}^2 - \sigma_{M\pi}^2}$$

$$b_{i\pi} = \frac{\sigma_{i\pi}\sigma_M^2 - \sigma_{iM}\sigma_{M\pi}}{\sigma_M^2\sigma_{\pi}^2 - \sigma_{M\pi}^2}$$

If  $\sigma_{M\pi} = 0$ , then  $b_{iM}$  reduces to the traditional beta coefficient without considering inflation effect. However,  $b_{i\pi}(R_I - R_F)$  is an additional term needed to determine the value of  $R_i - R_F$  unless  $\sigma_{i\pi}$  is zero.

Now, if we consider the real wealth change rather than the nominal wealth change, the budget constraint of equation (4) should be written as<sup>8</sup>



$$dW = \{ [\sum X_i (R_i - R_F - \sigma_{i\pi}) + (R_F - u_\pi + \sigma_\pi^2)] W - C \} dt + W [\sum X_i \sigma_i dZ_i - \sigma_\pi dZ_\pi]. \quad (13)$$

Here we assume the price level is one at time 0. The first-order conditions for the optimal control of equation (5), given the real wealth dynamics of equation (13), are

$$0 = U_C(C, t) - J_W \quad (14)$$

$$0 = J_W (R_i - R_F - \sigma_{i\pi}) + J_{WW} W (\sigma_{iM} - \sigma_{i\pi}) + \pi J_{W\pi} \sigma_{i\pi}, \quad i = 1, \dots, N \quad (15)$$

Recall that  $W$  wealth here is a real term, not a nominal term. Utilizing the same techniques as the above, the following four different forms of pricing equations can be derived. Depending on how one manipulates the  $\sigma_{i\pi}$  term, the results are

$$R_i - R_F = [\sigma_{iM}, \sigma_{i\pi}] \begin{bmatrix} \sigma_M^2 & \sigma_{M\pi} \\ \sigma_{M\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} (R_M - R_F) \\ (R_I - R_F) \end{bmatrix} \quad (12a)$$

$$R_i - R_F = [(\sigma_{iM} - \sigma_{i\pi}), \sigma_{i\pi}] \begin{bmatrix} (\sigma_M^2 - \sigma_{M\pi}), \sigma_{M\pi} \\ (\sigma_{M\pi} - \sigma_\pi^2), \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} (R_M - R_F) \\ (R_I - R_F) \end{bmatrix} \quad (12b)$$

$$R_i - R_F - \sigma_{i\pi} = [\sigma_{iM}, \sigma_{i\pi}] \begin{bmatrix} \sigma_M^2 & \sigma_{M\pi} \\ \sigma_{M\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} (R_M - R_F - \sigma_{M\pi}) \\ (R_I - R_F - \sigma_\pi^2) \end{bmatrix} \quad (12c)$$

$$R_i - R_F - \sigma_{i\pi} = [(\sigma_{iM} - \sigma_{i\pi}), \sigma_{i\pi}] \begin{bmatrix} (\sigma_M^2 - \sigma_{M\pi}), \sigma_{M\pi} \\ (\sigma_{M\pi} - \sigma_\pi^2), \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} (R_M - R_F - \sigma_{M\pi}) \\ (R_I - R_F - \sigma_\pi^2) \end{bmatrix} \quad (12d)$$

Later, we shall show that equations (12a) through (12d) are all equivalent (i.e., they can be transformed into each other).

### B. The Continuous-Time Framework--FLL's Approach

Following a similar approach to FLL's, the first-order conditions can be expressed as follows

$$E[U'(W_{t+dt}, \pi_{t+dt})W_t \{(R_i - R_F - \sigma_{i\pi})dt + \sigma_i dZ_i\}] = 0, \quad i=1, \dots, N, \quad (16)$$

where  $W_t$  is the real wealth at time  $t$ , and  $E$  is the expectation operator. Since the investment opportunities depend on the price level  $\pi_t$ , utility and, in turn, marginal utility should be influenced by the price level.<sup>9</sup> Unfortunately, FLL did not incorporate the price level into the marginal utility. Expanding the marginal utility function in equation (16) in a Taylor series about  $W_t$  and  $\pi_t$ , we obtain

$$\begin{aligned} U'(W_{t+dt}, \pi_{t+dt}) &= U'(W_t, \pi_t) + U''(W_t, \pi_t) \\ &\quad \cdot \{([\sum_i (R_i - R_F + \sigma_{i\pi}) + (R_F - \mu_\pi + \sigma_\pi^2)]W_t - C)dt \\ &\quad + W_t [\sum_i \sigma_i dZ_i - \sigma_\pi dZ_\pi]\} \\ &\quad + \frac{\partial U'(W_t, \pi_t)}{\partial \pi_t} (\pi_t \sigma_\pi dt + \pi_t \sigma_\pi dZ_\pi) + \varepsilon, \end{aligned} \quad (17)$$

where  $\varepsilon$  stands for terms of order higher than  $dt$ . Equation (17) is the same as equation (7) of FLL except for the last term which does not

appear in FLL. However, that term should be included in that marginal utility must be a function of price level; otherwise, the result will be biased. Substituting equation (17) into equation (16), taking expected values, eliminating terms of order higher than  $dt$  and rearranging the result yields

$$R_i - R_F - \sigma_{i\pi} = T(W_t, \pi_t)(\sigma_{iM} - \sigma_{i\pi}) + H(W_t, \pi_t)\sigma_{i\pi} \quad (18)$$

where  $T(W_t, \pi_t) = -\frac{W_t U''(W_t, \pi_t)}{U'(W_t, \pi_t)}$  is the relative tolerance of risk, and

$$H(W_t, \pi_t) = -\pi_t \frac{\partial U'(W_t, \pi_t)}{\partial \pi_t} / U'(W_t, \pi_t). \quad \text{Equation (18) is similar to}$$

equation (15). Therefore, the pricing relations of equations (12a) through (12d) can be easily derived from equation (18).

### C. The Discrete-Time Approach

We make the same assumptions as those in the derivation of the classical CAPM except for the following<sup>10</sup>

- (1) Investors are risk-averse, single-period expected utility of real terminal wealth maximizers.
- (2) Either returns on assets and inflation rate are normally distributed or the utility function is quadratic. However, the former assumption is made to derive the model in this paper.
- (3) At the end of the period, investors are still concerned about the investment opportunities, or the individual's utility is a function of price level.<sup>11</sup>

CB make all the above assumptions except the last one. They assume that the utility function is quadratic rather than that returns are normally distributed. Breeden [1] has argued that in the classical single-period model, all wealth is assumed to be consumed at the end of period, so investment opportunities are irrelevant. From this argument, we can infer that utility is not a function

of price level in addition to the real wealth in the single-period, single-good model. However, in the real world, investors do not consume all of their wealth at the end of period. They are still concerned with the reinvestment opportunities. Furthermore, there are more than one good in the real world. From a different point of view, inflation risk is a systematic one and it is never diversified away. Consequently, the utility function should be influenced by the price level. Moreover, the last assumption is very reasonable even in the single-period model multi-good model.

Since returns on risky assets and inflation rate are assumed to be a multivariate normal distribution, to maximize the representative investor's expected utility is equivalent to<sup>12</sup>

$$\text{Max}_{\{C, X_i\}} U(E(\tilde{W}), \text{Var}(\tilde{W}), \text{Cov}(\tilde{W}, \tilde{\pi})) \quad (19)$$

where  $\tilde{W}$  is the real wealth at the end of period, and  $\tilde{\pi}$  is the price level at the end of period. As defined in CB, the real return on asset  $i$  ( $\tilde{r}_i$ ) is the nominal return ( $\tilde{R}_i$ ) minus the inflation rate ( $\tilde{\pi}$ ) in the discrete-time model.<sup>13</sup> The real wealth at the end of period is

$$\tilde{W} = W[1 + \sum X_i (\tilde{R}_i - R_F) + R_F - \tilde{\pi}], \quad (20)$$

where  $W$  is the initial wealth.

The first-order conditions for equation (19), given the random real wealth of equation (20), are

$$\frac{\partial U}{\partial E(\tilde{W})} [R_i - R_F] + 2W \frac{\partial U}{\partial \text{Var}(\tilde{W})} [\sigma_{iM} - \sigma_{i\pi}] + \pi \frac{\partial U}{\partial \text{Cov}(\tilde{W}, \tilde{\pi})} \sigma_{i\pi} = 0 \quad i=1, \dots, N \quad (21)$$

where  $R_i$  is the expected value of  $\tilde{R}_i$ . Equation (21) is similar to equation (15). In consequence, from equation (21), we can easily derive equations (12a) through (12d).

#### D. The APT Approach under a Nominal Return Generating Process

This section employs the APT approach to derive the CAPM under inflation. First, it is assumed that the nominal return on an asset can be generated from the nominal return on the market portfolio M and an uncertain inflation rate. This two-factor model of Ross [20, 21] can be described as

$$\tilde{R}_i = R_i + b_{iM}(\tilde{R}_M - R_M) + b_{i\pi}(\tilde{\mu}_\pi - \mu_\pi) + \tilde{e}_i \quad (22)$$

where  $\tilde{e}_i$  is a random error term. The e's are assumed to be independent of each other as well as uncorrelated with the market return  $\tilde{R}_M$  and inflation  $\tilde{\mu}_\pi$ . The  $R_i$ ,  $R_M$ , and  $\mu_\pi$  are expected values of  $\tilde{R}_i$ ,  $\tilde{R}_M$ , and  $\tilde{\mu}_\pi$  respectively. We further assume that the price level follows a continuous-time Markov process of Ito type as was described in equation (1), the real return of equation (22) is thus<sup>14</sup>

$$\begin{aligned} \tilde{r}_i &= \frac{d(P_i/\pi)}{(P_i/\pi)} = R_i + b_{iM}[(R_M - R_M) - \sigma_{M\pi}] \\ &\quad + b_{i\pi}[(\bar{\mu}_\pi - \mu_\pi) - \sigma_\pi^2] - \bar{\mu}_\pi + \sigma_\pi^2 - \text{Cov}(\bar{\mu}_\pi, \tilde{e}_i) \\ &\quad + \tilde{e}_i, \end{aligned} \tag{23}$$

where the  $\text{Cov}(\bar{\mu}_\pi, \tilde{e}_i)$  is zero by assumption. To determine the pricing behavior, a well-diversified arbitrage portfolio is formed with weight  $X_i$  invested in asset  $i$  so that:

$$\sum X_i = 0$$

$$\sum X_i b_{iM} = \sum X_i b_{i\pi} = 0$$

$$\sum X_i \tilde{e}_i \cong 0$$

This arbitrage portfolio is constructed as well as diversified in the sense that the residual risk is negligible.<sup>15</sup> As a result, the real return on the arbitrage portfolio is<sup>16</sup>

$$\begin{aligned} \sum X_i \tilde{r}_i &= \sum X_i R_i + (\sum X_i b_{iM})[(R_M - R_M) - \sigma_{M\pi}] \\ &\quad + (\sum X_i b_{i\pi})[(\bar{\mu}_\pi - \mu_\pi) - \sigma_\pi^2] + (\sum X_i)(\sigma_\pi^2 - \bar{\mu}_\pi) \\ &\quad + \sum X_i \tilde{e}_i \cong \sum X_i R_i \end{aligned} \tag{25}$$

Since the arbitrage portfolio is almost risk-free and has a zero investment, its real return should be zero. Namely,  $\sum X_i \tilde{r}_i = 0$ , and from equation (25)  $\sum X_i R_i = 0$ . As pointed by Ross [20, 21] and Roll and Ross [19], the expected return  $R_i$  should be a linear combination of 1,  $b_{iM}$  and  $b_{i\pi}$  as follows

$$R_i = \lambda_0 + \lambda_1 b_{iM} + \lambda_2 b_{i\pi}, \quad i=1, \dots, N \quad (26)$$

Substituting the nominal risk-free asset, the market portfolio and the index bond into equation (26), we can have equation (12).

#### E. The APT Approach under Real Return Generating Process

Elton, Gruber and Rentzler [6] have derived the CAPM under inflation by the APT approach assuming that the real rate of return is generated by the real return on market and uncertain inflation. (See Appendix A for the proof that  $r_I$  is the same as  $R_Z$  in EGR),

$$r_i - r_I = \beta_{im}(r_m - r_I) + \beta_{i\pi}(r_I - r_F) \quad (27)$$

where

$$\beta_{iM} = \frac{\text{Cov}(\tilde{r}_i, \tilde{r}_M)\sigma_\pi^2 - \text{Cov}(\tilde{r}_i, \tilde{u}_\pi)\text{Cov}(\tilde{r}_M, \tilde{u}_\pi)}{\text{Var}(\tilde{r}_M)\sigma_\pi^2 - (\text{Cov}(\tilde{r}_M, \tilde{u}_\pi))^2}$$

$$\beta_{i\pi} = \frac{\text{Cov}(\tilde{r}_i, \tilde{u}_\pi)\text{Var}(\tilde{r}_M) - \text{Cov}(\tilde{r}_i, \tilde{r}_M)\text{Cov}(\tilde{r}_M, \tilde{u}_\pi)}{\text{Var}(\tilde{r}_M)\sigma_\pi^2 - (\text{Cov}(\tilde{r}_M, \tilde{u}_\pi))^2}$$

Equation (27) is expressed by real expected returns and beta coefficients are also expressed in real terms. Equation (27) is different from equation (12) at the first glance. But, as it will be shown in the next section, they are actually equivalent.

## II. A Comparison of the Models

This section compares the models derived in the preceding section. Moreover, the models expressed by the whole market portfolio M are also compared with those expressed only by the risky market portfolio m.

A. Proof that Equations (12a) through (12d) are Equivalent

In the preceding section, we derived the CAPM under inflation in nominal terms as was shown in equations (12a) through (12d). Here we want to examine that all four of these equations are equivalent.

First of all, it is easily shown that

$$[(\sigma_{iM} - \sigma_{i\pi}), \sigma_{i\pi}] \begin{bmatrix} (\sigma_M^2 - \sigma_{M\pi}), \sigma_{M\pi} \\ \sigma_{M\pi} - \sigma_\pi^2 & \sigma_\pi^2 \end{bmatrix}^{-1} = [\sigma_{iM}, \sigma_{i\pi}] \begin{bmatrix} \sigma_M^2 & \sigma_{M\pi} \\ \sigma_{M\pi} & \sigma_\pi^2 \end{bmatrix}^{-1}$$

Since both row vectors are equal to  $[b_{iM}, b_{i\pi}]$  (see equation (12)). From this equality, one can easily see that equation (12a) is the same as equation (12b); likewise, equation (12c) is equivalent to equation (12d).

Next, we want to exemplify that equation (12a) equals to equation (12c). Equation (12c) can be rewritten as

$$R_i - R_F - \sigma_{i\pi} = (\sigma_{iM}, \sigma_{i\pi}) \begin{bmatrix} \sigma_M^2 & \sigma_{M\pi} \\ \sigma_{M\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} R_M - R_F \\ R_I - R_F \end{bmatrix}$$

$$- (\sigma_{iM}, \sigma_{i\pi}) \begin{bmatrix} \sigma_M^2 & \sigma_{M\pi} \\ \sigma_{M\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{M\pi} \\ \sigma_\pi^2 \end{bmatrix}$$

and



$$(\sigma_{iM}, \sigma_{i\pi}) \begin{bmatrix} \sigma_M^2 & \sigma_{M\pi} \\ \sigma_{M\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{M\pi} \\ \sigma_\pi^2 \end{bmatrix} \\
 = \frac{(\sigma_{iM}\sigma_\pi^2 - \sigma_{i\pi}\sigma_{M\pi})\sigma_{M\pi} + (\sigma_{i\pi}\sigma_M^2 - \sigma_{iM}\sigma_{M\pi})\sigma_\pi^2}{\sigma_M^2\sigma_\pi^2 - \sigma_{M\pi}^2} = \sigma_{i\pi}$$

Thus, equation (12c) is reduced to equation (12a). This completes the proof that equations (12a) through (12d) are equivalent.

Q.E.D.

B. Proof that Models Expressed by the Whole Market Portfolio M has the same form as by the Risky Market Portfolio m

The whole market portfolio M can be expressed by the risky market portfolios m as follows:

$$\tilde{R}_M = \alpha \tilde{R}_m + (1-\alpha)R_F,$$

and

$$\sigma_M^2 = \alpha^2 \sigma_m^2; \sigma_{M\pi} = \alpha \sigma_{m\pi}; \sigma_{iM} = \alpha \sigma_{im}.$$

Substituting these relations into equation (12a), we have

$$\begin{aligned}
 R_i - R_F &= (\alpha\sigma_{im}, \sigma_{i\pi}) \begin{bmatrix} \alpha^2\sigma_m^2 & \alpha\sigma_{m\pi} \\ \alpha\sigma_{m\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha(R_m - R_F) \\ (R_I - R_F) \end{bmatrix} \\
 &= (\sigma_{im}, \sigma_{i\pi}) \begin{bmatrix} \sigma_m^2 & \sigma_{m\pi} \\ \sigma_{m\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} (R_m - R_F) \\ (R_I - R_F) \end{bmatrix} \\
 &= b_{im}(R_m - R_F) + b_{i\pi}(R_I - R_F) \tag{28}
 \end{aligned}$$

When the whole market portfolio M is replaced by the risky market portfolio m (see equation (12)), the results derived in the preceding section still hold. However, the beta coefficients are expressed in terms of the risky market portfolio m, and the market portfolio return is expressed by  $R_m$  rather than  $R_M$ . O.E.D.

C. Proof that the Models in Real Terms are Equivalent to the Models in Nominal Terms

When the CB approximation between real and nominal return is used, EGR derived equation (28) (the model in nominal terms which is equation (14) in EGR) from equation (27) (the model in real terms which is equation (16) in EGR). We have also derived equation (28) (the model in nominal terms with the risky market portfolio m) from equation (12a) (the model in nominal terms with the whole market portfolio M). In addition, we have shown that equations (12a) through (12d) are all

equivalent. Hence, equations (12a) through (12d), (27) and (28) are all equivalent (i.e., one can be derived from the other).

Furthermore, EGR have shown that if the FLL or Fisher [7] approximation between real and nominal returns is used, their equation (15) [p. 534] can be derived from our equation (27) (equation (6) in EGR). It is obvious that  $\text{cov}(\tilde{R}_I, \tilde{u}_\pi) = \sigma_\pi^2$  from the definition of index bonds as described in Fisher [1975]. Following Section II.A, it is easily shown that equation (15) in EGR can also be derived from our equation (28) or equation (14) of EGR<sup>17</sup> (see Appendix A). Consequently, when either CB or FLL approximation between real and nominal returns is used, the same results can be reached. Q.E.D.

In sum, even though several forms of linear asset pricing models under inflation were derived by different academicians, all of them are equivalent (i.e., one can be derived from the other).

#### D. A Special Case

As argued by Fisher [7], if consumption is only a function of real wealth,<sup>18</sup> note that:

$$C = f\left(\frac{W}{\pi}\right), \quad \partial C / \partial \pi = -W f'(W/\pi) / \pi^2$$

and

$$\partial C / \partial W = f'(W/\pi) / \pi$$

Hence,

$$\frac{\partial C}{\partial \pi} = -\frac{W}{\pi} \frac{\partial C}{\partial W} \tag{29}$$

where  $W$  is nominal wealth. To differentiate the first-order condition (7) with respect to  $\pi$  and with respect to  $W$ , and then applying the relation of equation (29), Fisher has shown

$$\pi J_{W\pi} = -J_W - WJ_{WW} \quad (30)$$

Combining equations (8) and (30), we have

$$R_i - R_F - \sigma_{i\pi} = T(W, \pi)(\sigma_{iM} - \sigma_{i\pi}) \quad (31)$$

Considering the market portfolio, equation (31) can be expressed as

$$R_i - R_F - \sigma_{i\pi} = \frac{\sigma_{iM} - \sigma_{i\pi}}{\sigma_M - \sigma_{M\pi}} (R_M - R_F - \sigma_{i\pi}) \quad (32)$$

which is the same as FLL's result.

Likewise, the indirect utility function  $J$  is only a function of real wealth and time so that  $J_{W\pi}$  in equation (15) and  $\partial U'(W_t, \pi_t) / \partial \pi_t$  in equation (17) are zero. From this special case of equation (15) and (17), equation (32) is easily derived. In the discrete-time case, if consumption is only a function of real wealth, there is no  $\text{Cov}(\tilde{W}, \tilde{\pi})$  term in equation (19), and equation (21) reduces to

$$\frac{\partial U}{\partial E(\tilde{W})} [R_i - R_F] + 2W \frac{\partial U}{\partial \text{Var}(\tilde{W})} (\sigma_{iM} - \sigma_{i\pi})$$

Hence, we can derive

$$R_i - R_F = \frac{\sigma_{iM} - \sigma_{i\pi}}{\sigma_M - \sigma_{M\pi}} (R_M - R_F) \quad (33)$$

which is the same as Chen and Boness's [3].

### III. Conclusions

The impact of inflation on investments has attracted increasing attention with regard to asset pricing models under uncertain inflation. The CAPM under inflation is the simplest among them. Unfortunately, inflation in the models of CB, FLL and Roll is not priced. This fact leads EGR to argue that the asset pricing models under inflation in the mean-variance world is inappropriate and the only correct model is derived from the APT framework. This paper has shown that inflation in the mean-variance world is still priced. There is nothing wrong with the mean-variance approach. There was wrong was that CB and FLL did not realize that the investor's utility is a function of both real wealth and inflation, not just real wealth. When utility is defined to be a function of both real wealth and inflation, the expected rate of return on an asset should be a linear function of a market risk premium and an inflation risk premium. This model is a generalization of CB's, FLL's and Roll's models; however, if utility was defined to be only a function of real wealth (and not a function of inflation), our generalized model can be reduced to the models of CB, FLL and Roll. Furthermore, we showed that all models under inflation, even in different forms, are eventually equivalent. This paper deals with the theoretical arguments of these models; thus empirical tests should be explored in a future research.

Footnotes

<sup>1</sup>Long derives the model under a multiperiod framework. In addition, Manaster [13] and Serco [24] compare the real and nominal efficient sets.

<sup>2</sup>Friend, Landskroner and Losq's continuous-time approach is somewhat different from the method adopted by Merton [14].

<sup>3</sup>Elton, Gruber and Rentzler [6] view the models derived by CB, Long, Roll and FLL as mean-variance models.

<sup>4</sup>More specifically, the model derived by Wei is the APT under inflation, not the CAPM under inflation, and the market portfolio may or may not be one of the common factors.

<sup>5</sup>The word "priced" in this paper refers to "a systematic risk." See Roll and Ross [19] and Elton, Gruber and Rentzler [6].

<sup>6</sup>This assumption has been discussed in Rubinstein [1974] and used in Fisher [7], Brennan [2], Stapleton and Subrahmanyam [26], Chen and Ingersoll [4], and others.

<sup>7</sup>Here, we assume that the inflation hedge portfolio earns the same returns as the index bond.

<sup>8</sup>The real returns on risky assets and risk-free asset in continuous-time framework has been derived in Fisher [7]. Here, we use his results directly. Friend, Landskroner and Losq [8] have also derived the same results.

<sup>9</sup>Breeden [1] has discussed this concept in detail. Also see Merton [14].

<sup>10</sup>See Jensen's [10] review article for a detail. Elton and Gruber [5] derived the CAPM under inflation in a different way.

<sup>11</sup>This assumption is similar to the implicit assumption made in the multiperiod world.

<sup>12</sup>Variance of inflation is treated as exogenous in determining the utility of investors. However, Pinky [17] has recently showed that variance of inflation can affect market cost of equity. The justification of omitting variance can be found in Long [12].

<sup>13</sup>The real return on an asset in the discrete-time model is different from that in the continuous-time model. Interested readers can compare the real returns defined in Chen and Boness [3] with Fisher's [7].

<sup>14</sup>The real return can also be derived in the discrete-time framework by

$$\begin{aligned}
 \tilde{r}_i &= \frac{1 + \tilde{R}_i}{1 + \tilde{\mu}_\pi} - 1 \approx (1 + \tilde{R}_i)(1 - \tilde{\mu}_\pi) - 1 \\
 &= R_i - \tilde{\mu}_i + b_{iM}[(\tilde{R}_M - R_M) - \sigma_{M\pi}] \\
 &\quad + b_{i\pi}[(\tilde{\mu}_\pi - \mu_\pi) - \sigma_\pi^2] - \text{Cov}(\tilde{\epsilon}_i, \tilde{\mu}_\pi) \\
 &\quad + \tilde{\epsilon}_i.
 \end{aligned}$$

If the time interval is very short, the approximation is very good.

<sup>15</sup> Stapleton and Subrahmanyam [26] have shown that given a linear market model assumption, the same relationship can be derived without the approximation implicit in the APT argument. Wei [27] has generalized this argument to a k-factor model with one of the factors to be market portfolio. Equation (26) is only a special case of Wei. Therefore, the pricing relationship derived in following is an exact relation rather than an approximation.

<sup>16</sup> From Footnote 13, we can derive the same result as equation [25] in a discrete-time framework.

<sup>17</sup> There is a type-error in equation (15) of EGR. The sign of  $\text{Cov}(r_i, \pi)$  in EGR's equation (15) should be positive rather than negative.

<sup>18</sup> It is not always true. Breeden [1] has argued that consumption is also dependent on the investment opportunities. This implies that consumption is a function of both real wealth and price level in our models.

References

1. D. Breeden. "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities." Journal of Financial Economics (September 1979), 265-96.
2. M. J. Brennan. "The Pricing of Contingent Claims in Discrete Time Model." Journal of Finance (March 1979), 53-68.
3. A. H. Chen and A. J. Boness. "Effects of Uncertain Inflation on the Investment and Financing Decisions of a Firm." Journal of Finance (May 1975), 469-83.
4. N. F. Chen and J. E. Ingersoll. "Exact Pricing in Linear Factor Models with Finitely Many Assets: A Note." Journal of Finance (June 1983), 985-88.
5. E. Elton and M. Gruber. "Non-Standard C.A.P.M.'s and the Market Portfolio." Journal of Finance (July 1984), 911-24.
6. E. Elton, M. Gruber and J. Rentzler. "The Arbitrage Pricing Model and Returns on Assets Under Uncertain Inflation." Journal of Finance (May 1983), 525-37.
7. S. Fischer. "The Demand for Index Bonds." Journal of Political Economy (June 1975), 509-34.
8. I. Friend, Y. Landskroner and E. Losq. "The Demand for Risky Assets under Uncertain Inflation." Journal of Finance (December 1976), 1278-97.
9. R. L. Hagerman and E. H. Kim. "Capital Asset Pricing with Price Level Changes," Journal of Financial and Quantitative Analysis (September 1976), 381-391.
10. M. Jensen. "Capital Markets: Theory and Evidence." Bell Journal of Economics and Management Science (Autumn 1972), 357-98.
11. J. Lintner. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolio and Capital Budgets." Review of Economics and Statistics (February 1965), 87-99.
12. J. B. Long, Jr. "Stock Prices, Inflation, and the Term Structure of Interest Rate." Journal of Financial Economics (July 1974), 131-70.
13. S. Manaster. "Real and Nominal Efficient Sets", Journal of Finance (March 1979), 93-102.
14. R. Merton. "An Intertemporal Capital Asset Pricing Model." Econometrica (September 1973), 867-88.



15. J. Mossin. "Equilibrium in a Capital Asset Market." Econometrica (October 1966), 768-83.
16. Peat, Marwick, Mitchell, and Company, Survey of Investment Performance. 1981.
17. Pinkyck, R. S. "Risk, Inflation and Stock Market," The American Economic Review (June 1984), 335-362.
18. R. Roll. "Assets, Money and Commodity Price Inflation Under Uncertainty." Journal of Money, Credit and Banking (November 1973), 903-23.
19. R. Roll and S. Ross. "An Empirical Investigation of the Arbitrage Pricing Theory." Journal of Finance (December 1980), 1073-103.
20. S. Ross. "Return, Risk, and Arbitrage." In I. Friend and J. Bicksler (eds.) Risk and Return in Finance. Cambridge: Ballinger, 1977, 189-218.
21. S. Ross. "The Arbitrage Theory of Capital Asset Pricing." Journal of Economic Theory (December 1976), 341-60.
22. S. Ross. "A Simple Approach to the Valuation of Risky Streams." Journal of Business (July 1979), 254-286.
23. M. Rubinstein. "An Aggregation Theorem for Securities Markets." Journal of Financial Economics (September 1974), 225-44.
24. P. Serco. "A Note on Real and Heal Efficient Sets," The Journal of Finance, 36 (June 1981), 721-737.
25. W. F. Sharpe. "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk." Journal of Finance (September 1974), 425-42.
26. R. Stapleton and M. Subrahmanyam. "The Market Model and Capital Asset Pricing Theory: A Note." Journal of Finance (December 1983), 1637-42.
27. K. Wei. "The Factor Models and Arbitrage Pricing Theory: Linearity, Normality, and Approximation." Memo, University of Mississippi (1984).

Appendix A

$R_i$  in our paper represents nominal return, while in EGR represents real return. Likewise,  $r_i$  in our paper represents real return, while in EGR stands for nominal return.

1. Proof of that  $r_I$  in our paper (real return for inflation hedge portfolio or index bond) =  $R_Z$  in EGR (real return on zero-beta portfolio).

Since real return for inflation hedge portfolio is assumed to be constant, the covariance between real market portfolio return and  $r_I$  is zero, so is the covariance between inflation rate and  $r_I$ . This implies that  $r_I$  is a real return on zero-beta portfolio. This proves that  $r_I$  in our paper is equal to  $R_Z$  in EGR.

2. Proof of that equation (15) in EGR = (14) in EGR (or (28) in ours). Equation (14) in EGR can be written as (using EGR notations)

$$(A) \quad r_i - r_F = [\sigma_{im}, \sigma_{i\pi}] \begin{bmatrix} \sigma_m^2 & \sigma_{m\pi} \\ \sigma_{m\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} r_m - r_F \\ r_Z - r_F \end{bmatrix}$$

Equation (15) in EGR can be written as

$$(B) \quad r_i - r_F - \sigma_{i\pi} = (\sigma_{iM}, \sigma_{i\pi}) \begin{bmatrix} \sigma_m^2 & \sigma_{m\pi} \\ \sigma_{m\pi} & \sigma_\pi^2 \end{bmatrix}^{-1} \begin{bmatrix} r_m - r_F - \sigma_{m\pi} \\ r_Z - r_F - \sigma_{Z\pi} \end{bmatrix}$$

Notice that the sign in front of  $\sigma_{i\pi}$  should be negative (there was a typing error in EGR (15)). Furthermore,

$$\sigma_{Z\pi} = \text{Cov}(\tilde{r}_Z, \tilde{\pi}) = \text{Cov}(R_Z + \tilde{\pi}, \tilde{\pi}) = \sigma_{\pi}^2$$

therefore, (B) can be written as

$$(C) \quad r_i - r_F - \sigma_{i\pi} = (\sigma_{im}, \sigma_{i\pi}) \begin{bmatrix} \sigma_m^2 & \sigma_{m\pi} \\ \sigma_{m\pi} & \sigma_{\pi}^2 \end{bmatrix}^{-1} \begin{bmatrix} r_m - r_F - \sigma_{m\pi} \\ r_Z - r_F - \sigma_{\pi}^2 \end{bmatrix}$$

Equation (A) is similar to our (12a), while Equation (C) is similar to our (12c). And we have proved that equation (12a) = (12c), thus (A) = (C).













**HECKMAN**  
BINDERY INC.



**JUN 95**

Bound-To-Place® N. MANCHESTER,  
INDIANA 46962

UNIVERSITY OF ILLINOIS URBANA



3 0112 005704041