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A DYNAMIC MODEL OF HOUSING PRODUCTION

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College of Commerce and Business Administration
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Kotler, Philip, Marketing Management (Third edition, 1976), Chapter 19.

Myers, John G., Stephen Greyser and William Massy, "Th ness of Marketing's R & D for Marketing Management: ment," Journal of Marketing, Vol. 43, January, 1979,

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A DYNAMIC MODEL OF HOUSING PRODUCTION

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Summary:

This paper derives the optimal development strategy for a housing producer with perfect foresight in a steady-state environment where dwellings deteriorate as they age. Under the assumption of zero demolition costs, the solution is an infinite sequence of identical buildings. Building abandonment is shown to be possible with positive demolition costs. A solution highlighting the model's spatial properties is computed using Cobb-Douglas functions.

A Dynamic Model of Housing Production

by

Jan K. Brueckner

1. Introduction

The last few years have seen the development of the first formal housing models which explicitly recognize the long-lived, durable nature of structures. Many of these studies emerged out of the recognition that important features of urban housing markets cannot be explained by models which assume that housing capital is perfectly malleable. While the models reflect a diversity of approaches, the most important differences lie in the determination of building lives and in the effects of the aging process. While Anas [1], Arnott [2], and Fujita [7] avoid modelling demolition and redevelopment by assuming that structures have infinite lives, a building's retirement age is endogenous in the models of Brueckner [3] and [4], Evans [5], Fisch [6], Muth [8], and Sweeney [10]. Brueckner and Muth explicitly analyse the producer's demolition decision, while Evans and Fisch deduce retirement ages indirectly. The assumption that quality deterioration is a result of aging also differentiates the models of Brueckner, Fisch, Muth, and Sweeney from the other analyses, in which dwellings of different ages are qualitatively the same.

The present paper, in common with my earlier work and the work of Muth, reflects the belief that a realistic model of durable housing must incorporate quality deterioration, which appears to be the most important effect of aging, and must make building lives endogenous by explicitly treating the producer's decision to demolish a structure. The paper differs from my earlier work in the behavioral assumption for housing producers which underlies the analysis. While producer myopia about future housing prices was assumed in the earlier papers, producers in the present analysis are able to predict future prices exactly. While this perfect foresight assumption makes the model somewhat similar to Muth's, important differences remain. First, the analysis is conducted under the open city assumption, so that utility is exogenous. This simplifies the characterization of market equilibrium, eliminating a source of confusion in Muth's paper. Second, the producer has an infinite time horizon, and the further assumption of a steady-state environment yields a simple objective function. It appears impossible to similarly justify the form of the producer's objective function in Muth's analysis. Finally, the housing production technology is specified in more detail than in Muth's paper, and a spatial interpretation of the model is presented.

A principal result of the analysis is that when demolition costs are zero, the producer constructs an infinite sequence of identical buildings which are occupied throughout their lives. Building abandonment may occur, however, when demolition costs are positive. In addition, solving the model using Cobb-Douglas functions generates a city whose spatial structure closely resembles that of the familiar static city.

The paper is organized as follows. Sections 2 and 3 develop the general model, and Section 4 discusses extensions. Section 5 contains the Cobb-Douglas solution, while Section 6 sketches the structure of the myopia model and compares the Cobb-Douglas solutions under myopia and perfect foresight. Section 7 presents conclusions.

. The π function

It is assumed in the analysis that all consumers have the differentiable strictly quasi-concave utility function $u(Q,x)$, where Q is consumption of housing services and x is consumption of a numeraire non-housing good. A fundamental assumption is that the utility level of urban residents is constant over time and equal to $\bar{u}$. This is the familiar open city assumption: $\bar{u}$ is the constant level of utility in the rural hinterland and costless migration assures that urban and rural residents are equally well off. The income y of an urban resident is constant over time, and $w(k) \equiv y-c(k)$ gives the net income of a commuter living k miles from the CBD, where c is the time-invariant commuting cost function, with $c' > 0$. Although strong, the steady-state assumptions of constant utility and net income are crucial in the following analysis.

The condition $\bar{u} = u(Q,x)$ implies $x = x(Q)$, with $-x'(Q)$ equal to the MRS at the given point on the $\bar{u}$ indifference curve. To avoid needless complexity in the analysis, it will be assumed that the $\bar{u}$ indifference curve approaches the x and Q axes asymptotically. This assumption may be relaxed without affecting the basic arguments which follow. Now if the price per unit of housing services in a dwelling with service level Q located k miles from the CBD equals

$$\frac{w(k) - x(Q)}{Q}, \tag{1}$$

then the occupant's rental payment is $w(k) - x(Q)$ and his consumption of the non-housing good is $x(Q)$. Eq. (1) thus gives the price per unit of services which permits the occupant of the dwelling to reach utility level $\bar{u}$. Note that (1) is negative for Q sufficiently small; consumers will require a negative price to inhabit a dwelling with a low service level.

Although it is assumed that structural modification of buildings an the individual dwelling units they contain is possible only through demo lition and redevelopment, quality deterioration means that the housing services provided by a dwelling at age τ equal a fraction $f(\tau)$ of the original service level. The function f satisfies $f(0) = 1$, $f'(\tau) < 0$, a $f(\tau) > 0$ for $\tau \geq 0$. In reality, the rate of deterioration of dwellings is effected by maintenance, but consideration of this possibility is pos poned until Section 4. It is clear from (1) that the shrinkage of a dwelling's service level over time means that its price per unit of services varies with time. The goal of the housing producer is to choose an optimal development strategy taking this variation into account. The producer in particular will optimize by choosing the operating life and structural characteristics of each of an infinite sequence of buildings.

Before characterizing the solution to this problem, it is necessary to derive the function which relates a building's present value of profi (PVP) per acre (gross of land cost) to the length of its operating life and structural characteristics. First, it is clear that since the pro- iucer prefers a zero to a negative rent, a dwelling for which (1) is negative will be uninhabited. Therefore, the price per unit of housing services at age τ for a dwelling with initial service level q is given b

$$\max \left\{\frac{w - x(qf(\tau))}{qf(\tau)}, 0\right\}. \tag{2}$$

Now the initial output of housing services in a building is given by $H(N,\ell)$, where N and ℓ are the non-land capital and land used in the structure and H is concave and exhibits constant returns to scale. A building's initial output of housing services per acre of land is $h(S) = H(S,1)$, where $S \equiv N/\ell$ is structural density, and services per acre for a building of age τ is $h(S)f(\tau)$. Finally, the product of (2) and $h(S)f(\tau)$ gives revenue per acre for a building of age τ, a quantity which depends on both the initial dwelling size in the building and its structural density. Assuming that demolition costs are zero and letting n denote the constant price per unit of non-land capital, a building's PVP per acre (gross of land cost)[1] as a function of q, S, and T, the length of its operating life, is

$$\pi(T,q,S) \equiv \int_0^T \max \{\frac{w - x(qf(\tau))}{qf(\tau)}, 0\} \, h(S)f(\tau)e^{-rT} \, d\tau - nS, \qquad (3)$$

where r is the discount rate. The first term in (3) is the present value of revenue (PVR) per acre for the structure, while the second term is the initial non-land capital cost per acre. Let $\overline{Q}$ be the service level of the dwelling which calls forth a zero rent, satisfying $w - x(\overline{Q}) = 0$.[2] Not that $\overline{Q}$ implicitly depends on k. Since $-x' = u_2/u_1 > 0$, $w - x(Q) \gtreqless 0$ as $Q \gtreqless \overline{Q}$. Thus $w - x(qf(\tau)) \gtreqless 0$ as $qf(\tau) \gtreqless \overline{Q}$ or as $\tau \lesseqgtr f^{-1}(\overline{Q}/q) \equiv m(q)$, where $m' > 0$. When $q > \overline{Q}$, $m(q)$ is positive and gives the age when the ren for a dwelling of initial size q reaches zero. The function m implicitly depends on k. When $q > \overline{Q}$, (3) becomes

$$\pi(T,q,S) = \frac{h(S)}{q} \int_0^{\min\{T,m(q)\}} (w - x(qf(\tau)))e^{-r\tau} d\tau - nS. \qquad (4)$$

The equivalence of (3) and (4) follows because when $m(q) < T$, the upper limit T in (3) may be replaced by m(q) since the integrand is zero for

$m(q) \leq \tau \leq T$. Note that the condition $q > \overline{Q}$ merely requires that a new dwelling call forth a positive rent. If this were not the case, the PVR for the dwelling would be zero since a zero initial rent implies a zero rent thereafter.

The term $h(S)/q$ in (4) is the initial housing service output per acre for the building divided by initial services per dwelling, which equals the number of dwellings per acre in the building. Since the integral is the present value of rent per dwelling, the whole expression is the PVR per acre for the building. Note finally that as a result of the steady-state assumptions, the PVP per acre for a building does not depend on its construction date.

In computing the derivatives of π with respect to q and T, special attention must be paid to the min function in the upper limit of integration in (4). For $T < m(q)$,

$$\pi_2(T,q,S) = h(S) \int_0^T \frac{\partial}{\partial q} \left[\frac{w - x(qf(\tau))}{q}\right] e^{-r\tau} d\tau \quad . \tag{5}$$

For $T > m(q)$,

$$\pi_2(T,q,S) = h(S) \int_0^{m(q)} \frac{\partial}{\partial q} \left[\frac{w - x(qf(\tau))}{q}\right] e^{-r\tau} d\tau$$

$$+ \frac{w - x(qf(m(q)))}{q} \; e^{-rm(q)} \; m'(q)$$

$$= h(S) \int_0^{m(q)} \frac{\partial}{\partial q} \left[\frac{w - x(qf(\tau))}{q}\right] e^{-r\tau} d\tau \; , \tag{6}$$

where the last equality follows because $w - x(qf(m(q))) = 0$ by the definition of $m(q)$. As $q \rightarrow m^{-1}(T)$ from either direction, the derivatives

tives in (5) and (6) both approach the expression in (5) evaluated at $q = m^{-1}(T)$. Thus, π_2 exists when $q = m^{-1}(T)$ and is given by (5). In view of these results,

$$\pi_2(T,q,S) = h(S) \int_0^{\min\{T,m(q)\}} \frac{\partial}{\partial q} \left[\frac{w - x(qf(\tau))}{q}\right] e^{-r\tau} d\tau \ . \qquad (7)$$

Similarly, for $T < m(q)$,

$$\pi_1(T,q,S) = \frac{w - x(qf(T))}{q} h(S)e^{-rT} > 0 \ , \qquad (8)$$

while for $T > m(q)$, $\pi_1(T,q,S) = 0$. As $T \to m(q)$ from below, $\pi_1(T,q,S) \to 0$, and hence π_1 exists and equals zero when $T = m(q)$.

An important concern in the following analysis is whether a building which is part of an optimal development strategy can have $T > m(q)$, implying that there exists an interval at the end of a building's life where rent is zero and the building is vacant. It is clear from (8) that if the optimal development strategy calls for $\pi_1 > 0$ for a given building, then the building is torn down before its dwelling rent falls to zero: abandonment will not occur.

3. The producer's optimization problem

Suppose a housing producer acquires a plot of land in an urban area and considers development strategies consisting of infinite sequences of buildings. Letting T_j, q_j, S_j be the operating life and structural characteristics of the j^{th} building, the PVP per acre at time zero from the development strategy characterized by the infinite sequence $\{T_i, q_i, S_i\}_{i=1,2,3,...}$ is given by

$$\pi(T_1,q_1,S_1) + \pi(T_2,q_2,S_2)e^{-rT_1} + \pi(T_3,q_3,S_3)e^{-r(T_1+T_2)} + \ldots \qquad (9)$$

Note in (9) that $T_1+\ldots+T_{i-1}$ gives the construction date of the i^{th} building and hence that $\pi(T_i,q_i,S_i)\exp(-r(T_1+\ldots+T_{i-1}))$ is the PVP per acre from the i^{th} building discounted back to time zero. Note also that (9) implicitly requires that buildings be constructed back-to-back, disallowing intervals where the land sits vacant. It is easy to see, however, that any development strategy with vacant intervals is dominated by one without vacant intervals.

The producer's problem is to maximize (9) by choice of the infinite sequence $\{T_i,q_i,S_i\}_{i=1,2,3,\ldots}$ subject to the conditions $T_i,S_i > 0$, $q_i > \overline{Q}$, $i=1,2,3,\ldots$ This is a dynamic programming problem with an infinite horizon, and we have

<u>Theorem</u>: The sequence $\{T_i^*,q_i^*,S_i^*\}_{i=1,2,3,\ldots}$ which maximizes (9) is given by

$$(T_i^*,q_i^*,S_i^*) = (T^*,q^*,S^*) \qquad i=1,2,3,\ldots$$

where (T^*,q^*,S^*) maximizes

$$\pi(T,q,S)(1+e^{-rT}+e^{-r2T}+e^{-r3T}+\ldots) = \frac{\pi(T,q,S)}{1-e^{-rT}}. \qquad (10)$$

<u>Proof</u>: By the Principle of Optimality, the sequence $\{T_i^{**},q_i^{**},S_i^{**}\}_{i=2,3,\ldots}$ which maximizes

$$\pi(T_2,q_2,S_2)e^{-rT_1^*} + \pi(T_3,q_3,S_3)e^{-r(T_1^*+T_2)} + \pi(T_4,q_4,S_4)e^{-r(T_1^*+T_2+T_3)} + \ldots$$

$$= e^{-rT_1^*}\left(\pi(T_2,q_2,S_2) + \pi(T_3,q_3,S_3)e^{-rT_2} + \pi(T_4,q_4,S_4)e^{-r(T_2+T_3)} + \ldots\right) . \qquad (11)$$

is equal to $\{T_i^*,S_i^*,q_i^*\}_{i=2,3,\ldots}$ But since the infinite series on the RHS of (11) is identical to (9) except for the index of summation, it follows that $(T_i^{**},q_i^{**},S_i^{**}) = (T_{i-1}^*,q_{i-1}^*,S_{i-1}^*)$, $i=2,3,\ldots$, which gives $(T_i^*,q_i^*,S_i^*) = (T_{i-1}^*,q_{i-1}^*,S_{i-1}^*)$, $i=2,3,\ldots$ This means $T_2^* = T_1^*$, $T_3^* = T_2^*$, and so on, implying $T_1^* = T_2^* = T_3^* = \ldots$, and similarly for q_i^* and S_i^*, $i=1,2,3,\ldots$ Thus, since the optimal sequence is constant, the objective function may be written as (10). Q.E.D.

The constancy of the optimal sequence means that the housing producer constructs an infinite sequence of identical buildings. This result is due, of course, to the steady-state assumptions; the above proof requires that the π function is independent of time. In a changing environment, this independence would disappear, and objective function would fail to collapse into a simple expression. In this case, advanced techniques from dynamic programming might be used to find the limit of the optimal sequence as $i \to \infty$, which could be used to approximate the characteristics of a given building in the sequence. However, the complexity of the π function would make this a difficult undertaking.

It is interesting to note that the objective function (10) is of the same form as the one used by Samuelson [9] in his well-known attempt to settle the controversy over how long a forest should grow before it is cut. He argued that any formulation of the problem which is not based on an infinite time horizon ignores the correct opportunity costs.

Maximizing (10) with respect to T, q, and S yields the first-order conditions[3]

$$\pi_1(T,q,S)(1-e^{-rT}) = re^{-rT}\pi(T,q,S) \tag{12}$$

$$\pi_2(T,q,S) = 0 \tag{13}$$

$$\pi_3(T,q,S) = 0 \ . \tag{14}$$

It is interesting to note that Eq. (12) follows from two opposing effects of an increase in T. Dividing through by $(1-e^{-rT})^2$, the LHS becomes $\pi_1/(1-e^{-rT})$, which gives the increase in PVP per acre from an increase in T holding all the discount weights fixed. The RHS becomes $re^{-rT}\pi/(1-e^{-rT})^2$, which gives the decrease in PVP resulting from a decrease in all the discount weights, holding π fixed. Only when the increase in PVP from increasing π is balanced by the decrease in PVP from decreasing the discount weights is T at an optimal level for given values of q and S. Eq. (12) also directly gives an important property of the solution:

> PROPERTY 1: NO BUILDING ABANDONMENT. Each building in the optimal sequence is occupied throughout its life.

This follows because (12) implies $\pi_1 > 0$ at the optimum, which, referring to (8), gives $T^* < m(q^*)$. This mean that dwelling rental is positive throughout a building's life and hence that the building is always occupied. The intuition behind this result is that since there is no benefit from allowing a building to stand beyond age m(q) and the opportunity cost of doing so (the foregone revenue from a new building) is positive, abandonment can never be optimal.

Building abandonment is a serious problem in a number of American central cities, but Property 1 establishes that the phenomenon is not part of an optimal development strategy under the assumptions of the model. It will be seen in Section 4 that the introduction of demolition costs leads to a different conclusion.

Using the result $T^* < m(q^*)$ and (4), (7), and (8), the conditions (12)-(14) become

$$h(S)\ \frac{w - x(qf(T))}{q}\ \frac{1-e^{-rT}}{r} = \frac{h(S)}{q}\int_0^T (w - x(qf(\tau)))e^{-r\tau}d\tau - nS \qquad (15)$$

$$\int_0^T [-\frac{w - x(qf(\tau))}{qf(\tau)} - x'(qf(\tau))]f(\tau)e^{-r\tau}d\tau = 0 \qquad (16)$$

$$\frac{h'(S)}{q}\int_0^T (w - x(qf(\tau)))e^{-r\tau}d\tau = n \qquad (17)$$

Eq. (17) says that holding T and the initial dwelling size fixed, the present value of the marginal cost per acre of increasing S, given by n, should equal the present value of the marginal revenue per acre from doing so. A diagram is useful for deriving the implications of (16). Figure 1 shows the $\bar{u}$ indifference curve and the point (0,w) on the x axis. It is easy to see that the absolute value of the slope of the line connecting (0,w) to any point (Q,x(Q)) on the indifference curve is $(w - x(Q))/Q$, the price per unit of housing services for the dwelling with service level Q. For example, the line connecting (0,w) to $(\bar{Q},x(\bar{Q}))$ is horizontal since $w - x(\bar{Q}) = 0$. Further inspection of Figure 1 shows that the slope of the line tangent to the indifference curve and the slope of the line connecting the curve to $(0,w)$ are equal only at the point $(\hat{Q},x(\hat{Q}))$. Formally, $-(w - x(\hat{Q}))/\hat{Q} = x'(\hat{Q})$. For $Q > \hat{Q}$, the line connecting $(0,w)$ to the indifference curve is steeper than the tangent line $(-(w - x(Q))/Q < x'(Q))$, while for $Q < \hat{Q}$, the reverse is true $(-(w - x(Q))/Q > x'(Q))$. Note finally that $(w - x(Q))/Q$ is maximal for $Q = \hat{Q}$; the slope of the line connecting $(0,w)$ to the indifference curve decreases monotonically as Q approaches $\hat{Q}$ from above, reaches a minimum at $\hat{Q}$, and increases monotonically as Q decreases below $\hat{Q}$. This means that a dwelling with service level $\hat{Q}$ calls forth a higher price per unit of services than a dwelling of any other size.[4] Note that $\hat{Q}$ implicitly depends on k. These facts yield

> PROPERTY 2: INITIAL AND TERMINAL DWELLING SERVICE LEVELS BRACKET $\hat{Q}$. The initial and terminal dwelling service levels q^* and $q^*f(T^*)$ satisfy $q^*f(T^*) < \hat{Q} < q^*$.

To establish Property 2, suppose, $q^* \leq \hat{Q}$, which gives $q^*f(\tau) < \hat{Q}$ for $\tau > 0$. This inequality implies, using the above results from Figure 1, that $-(w - x(q^*f(\tau)))/q^*f(\tau) > x'(q^*f(\tau))$ for $\tau > 0$, which means the integrand in (16) with $q = q^*$ is positive for $\tau > 0$ and hence that the integral itself is positive. Thus, for fixed T, the solution to (16) must satisfy $q^* > \hat{Q}$. This argument can be repeated, however, unless $q^*f(T^*) < \hat{Q}$. If this inequality fails to hold, then $q^*f(\tau) > \hat{Q}$ for $0 \leq \tau < T^*$, with the results that the integrand in (15) with $q = q^*$ is negative for $0 \leq \tau < T^*$ and the integral with $T = T^*$ is negative. Therefore, the solution to (15)-(17) must satisfy $q^*f(T^*) < \hat{Q} < q^*$. An implication of Property 2 is that since the dwelling service level starts above and ends below $\hat{Q}$, the price per unit of services in the dwelling first increases and then decreases, reaching a maximum when the service level equals $\hat{Q}$.

Since the optimal building characteristics q^*, S^*, and T^* are all implicit functions of k, the spatial properties of an urban area described by the model may be derived in principle through comparative static calculations. Unfortunately, these calculations yield ambiguous results; systematic variation of q^*, S^*, and T^* over space cannot be established. In spite of this general indeterminary, solution of the model using specific functional forms can produce a city with straightforward spatial properties, as will be seen below in Section 5. In addition, if it is assumed that the price per acre of agricultural land is constant and equal to R_A, the size of the urban area described by the general model may be deduced in a standard fashion. Since the urban land price equates the PVP per acre net of land cost to zero, the land price R is given by $\pi(T^*,q^*,S^*)/(1-e^{-rT^*})$, which is implicitly a function of k. Noting $w'(k) < 0$ and using the envelope theorem, $\partial R/\partial k < 0$ follows directly from (4). As usual, the distance $\bar{k}$ to the urban periphery is the value of k at which the urban land price falls to R_A.

The urban history implied by the model is easily sketched given the previous discussion. The city occupies the land out to $\bar{k}$ indefinitely, with each ring being rebuilt at constant intervals exactly as it was originally constructed. Structural density and initial dwelling size need not be simple functions of distance, and the possibility of variation of T^* with k means that the age of buildings at a given instant in the city's history may vary erratically over distance. Finally, urban population is constant at its original level throughout the city's history.

4. Extensions of the model

This section discusses two extensions of the model: positive demolition costs and endogenous building maintenance. If demolition costs per acre are given by $D(S)$, an increasing function of structural density, then the PVP per acre for a building is $\pi(T,q,S) - D(S)e^{-rT}$, and the optimality conditions are (13) and

$$\pi_1(T,q,S)(1-e^{-rT}) = re^{-rT}(\pi(T,q,S) - D(S)) \qquad (18)$$

$$\pi_3(T,q,S) - D'(S)e^{-rT} = 0 \; . \qquad (19)$$

It may be shown that any solution to (13), (18), and (19) with $\pi_1 = \pi - D = 0$ fails to satisfy the second-order condition.[5] Therefore, an interior maximum must be characterized by $\pi_1 > 0$, implying $T^* < m(q^*)$. A second possibility is that the optimal T is infinite. If (13) and (19) are solved for q and S as functions of T, then $T^* = \infty$ will result if, substituting for q and S, the LHS of (18) exceeds the RHS for all $T > 0$. This will be the case, for example, if $\pi(T,q,S) - D(S) < 0$ for all T, q, and S. Generally, it appears that if D is sufficiently large compared to π, it will be optimal to avoid ever incurring demolition costs by setting $T^* = \infty$. The producer will construct one building which is abandoned at age $m(q^*)$ and sits vacant thereafter. Summarizing these results gives[6]

> PROPERTY 1': POSSIBLE BUILDING ABANDONMENT. When demolition costs are positive, the optimal development strategy consists of either an infinite sequence of identical buildings which are always occupied or one building which is eventually abandoned.

The striking feature of Property 1' is that it offers an explanation of urban blight which differs from the common market failure hypothesis. Abandoned buildings may be part of an optimal development strategy when demolition costs are positive.

A second modification of the model is the assumption that the rate of shrinkage of a dwelling's service level is a function of the level of maintenance z, which is constant over the dwelling's life. In particular, it is assumed that z is an argument of f, with $\partial f/\partial z > 0$. Letting the per acre flow of costs associated with a maintenance level z in a structure be $g(S,z)$, with g_1, $g_2 > 0$, the π function is modified accordingly, and the first-order condition for choice of z is

$$\int_0^T [-h(S)x'(qf(\tau;z))f_2(\tau;z) - g_2(S,z)]e^{-r\tau}d\tau = 0 \ . \tag{20}$$

Eq. (20) says that the present value per acre of marginal maintenance costs equals the present value per acre of marginal revenue due to maintenance. Although (17) must be modified slightly, introduction of building maintenance does not alter any earlier results.

5. A model solution using Cobb-Douglas functions

Assuming $H(N,\ell) \equiv N^{\beta}\ell^{1-\beta}$, $0<\beta<1$, which gives $h(S) = S^{\beta}$, $u(Q,x) = Q^{1-\theta}x^{\theta}$, $0<\theta<1$, and $\bar{u} = 1$, which give $x(Q) = Q^{(\theta-1)/\theta}$, and $f(\tau) = e^{-\alpha\tau}$, an explicit solution to (15)-(17) can be calculated.[7] Equation (16) may be solved for q as a function of T, yielding

$$q = (\theta w)^{\frac{\theta}{\theta-1}} \left[\frac{a(1-e^{-rT})}{r(1-e^{-aT})}\right]^{\frac{\theta}{\theta-1}}, \tag{21}$$

where $0 \neq a = r - \alpha(1-\theta)/\theta < r$.[8] Setting $(w - Q^{(\theta-1)/\theta})/Q = ((\theta-1)/\theta)Q^{-1/\theta}$ yields $\hat{Q} = (\theta w)^{\theta/(\theta-1)}$. Since the discussion in Section 3 showed that the q which solves (16) for arbitrary $T > 0$ exceeds $\hat{Q}$, the second term on the RHS of (21) must exceed one, which, noting $\theta/(\theta-1) < 0$, requires

$$\frac{a(1-e^{-rT})}{r(1-e^{-aT})} < 1 , \tag{22}$$

for $T > 0$. Using $a < r$, (22) may be verified directly.[9]

Substituting (21) into (17) yields

$$S = \left(\frac{\beta(1-\theta)}{n}\right)^{\frac{1}{1-\beta}} (\theta^{\theta} w)^{\frac{1}{(1-\theta)(1-\beta)}} \left[\frac{a(1-e^{-rT})}{r(1-e^{-aT})}\right]^{\frac{1}{(1-\theta)(1-\beta)}} \left[\frac{1-e^{-aT}}{a}\right]^{\frac{1}{1-\beta}}, \tag{23}$$

and substituting (21) and (23) into (15) yields, after considerable manipulation, the equation which gives T^*:

$$1 + \frac{\beta(1-\theta)}{\theta} = \frac{a(1-e^{rT})}{r(1-e^{aT})} . \tag{24}$$

It is easily seen that the RHS of (24) is increasing in T as a result of (22), and l'Hopital's rule establishes that the RHS approaches unity as $T \to 0$. Since the LHS of (24) exceeds one, these results establish that there exists a unique positive T^* which satisfies (24). Substituting T^* into (23) gives S^*, and substitution of (24) in (21) gives $q^* = [(\theta+\beta(1-\theta))w]^{\theta/(1-\theta)}$.

The spatial properties of the Cobb-Douglas city are easily inferred. First, since (24) does not involve k, T^* is independent of k. This fact, in addition to $w'(k) < 0$, gives $\partial q^*/\partial k > 0$ and $\partial S^*/\partial k < 0$ using (21) and (23). The city thus bears a striking resemblance to the familiar static urban area: structural density falls off as distance to the CBD increases and initial dwelling service levels are larger farther from the CBD. Furthermore, the constancy of T* means that at any point in the city's history, all its structures will have the same age. These results suggest the interesting and natural conclusion that the only important qualitative difference between a static city and a dynamic Cobb-Douglas city in a steady-state environment is the uniform cyclical aging of structures in the latter.

6. A model with producer myopia

While perfect foresight is in many ways a more attractive behavioral assumption than myopia, a model with perfect foresight is not useful for exploring dynamic processes such as the spatial growth of a city and residential succession in a multi-class city, which are investigated in Brueckner [3] and [4]. The reason is that these phenomena cannot be generated in a steady-state environment like the one assumed in this paper. In order to contrast the perfect foresight model with its more versatile counterpart, this section develops the myopia model under the steady-state assumptions. The earlier applications of the model assumed, of course, that income and utility change over time.

A myopic producer assumes that a dwelling's price per unit of services will remain stationary forever at its current level. In addition, at the construction date, the producer expects a building to last forever. Under these assumptions, the expected PVP per acre for a new building when $f(\tau) = e^{-\alpha\tau}$ is

$$\int_0^{\infty} \frac{w - x(q)}{q} h(S) e^{-(\alpha+r)\tau} d\tau - nS = \frac{w - x(q)}{q} \frac{h(S)}{\alpha+r} - nS. \quad (25)$$

While $(w - x(q))/q$, the initial price per unit of housing services, is expected to persist forever, (25) reflects the producer's awareness that a building's service level will decline with age. To maximize (25), the producer sets q equal to $\hat{Q}$, the dwelling size which calls forth the highest price per unit of housing services. The condition which gives structural density is then

$$\frac{w - x(\hat{Q})}{\hat{Q}} h'(S) = (\alpha+r)n . \qquad (26)$$

As the building ages, its price per unit of services falls, contradicting the producer's expectations. When the expected PVR per acre, based on myopic extrapolation of the current price per unit of services, equals the price per acre for the land used in the structure, the producer is indifferent between continuing to operate the building and demolishing it and selling the land (demolition costs are zero). The land price R is equal to the maximized value of (25), the expected PVP per acre for a new developer. Letting $\tilde{S}$ denote the solution to (26), a building's demolition age is consequently given by the T which satisfies

$$\frac{w - x(\hat{Q}e^{-\alpha T})}{\hat{Q}} \frac{h(\tilde{S})}{\alpha+r} e^{-\alpha T} = \frac{w - x(\hat{Q})}{\hat{Q}} \frac{h(\tilde{S})}{\alpha+r} - n\tilde{S} \equiv R . \qquad (27)$$

The LHS of (27), which is expected PVR at T, comes from integrating $\frac{(w - x(\hat{Q}e^{-\alpha T}))}{\hat{Q}e^{-\alpha T}} h(S)e^{-\alpha\tau}$, expected revenue per acre at τ, weighted by the discount factor $e^{-r(\tau-T)}$, from $\tau = T$ to $\tau = \infty$. Note that the price per unit of housing services at T, $(w - x(\hat{Q}e^{-\alpha T}))/\hat{Q}e^{-\alpha T}$, is expected to persist indefinitely.

The comparison of initial dwelling sizes under myopia and perfect foresight is immediate since $\hat{Q} < q^*$. While the myopic housing producer chooses the initial dwelling size to maximize the initial price per unit of services, a producer with perfect foresight avoids the resulting monotonic decrease in price by choosing a larger initial dwelling size.

Structural densities may be compared using the Cobb-Douglas assumptions of Section 5, which give

$$\tilde{S} = \left(\frac{\beta(1-\theta)}{n}\right)^{\frac{1}{1-\beta}} (\theta^{\theta} w)^{\frac{1}{(1-\beta)(1-\theta)}} (\alpha+r)^{-\frac{1}{1-\beta}} . \qquad (28)$$

It is easily shown that $\tilde{S} > S^*$ as long as $\alpha+r < 1$. Furthermore, under the Cobb-Douglas assumptions, the demolition-age condition (27) reduces t

$$1 + \frac{\beta(1-\theta)}{\theta} = e^{(r-a)T} . \qquad (29)$$

Since the condition (24) which solves for T in the perfect foresight case may be written

$$1 + \frac{\beta(1-\theta)}{\theta} = \frac{a(1-e^{-rT})}{r(1-e^{-aT})} e^{(r-a)T} < e^{(r-a)T} , \qquad (30)$$

where the inequality follows from (22), it follows that $\exp((r-a)T^*)$ exceeds the LHS of (29). Since $r > a$, this means that $\tilde{T}$, the value of T which satisfies (29), lies below T^*. Thus, the operating life of buildings is longer under perfect foresight than under myopia. Since the myopic producer does not correctly take account of opportunity cost, we would expect $T^* \neq \tilde{T}$. Intuition, however, appears incapable of predicting the direction of the inequality relating T^* and $\tilde{T}$.

For a detailed development of the myopia model without the steady-state assumptions, see Brueckner [3].

7. Conclusion

In this paper, the assumptions of constant income, commuting cost, and utility allowed the rigorous formulation of the optimization problem for a housing producer with perfect foresight and an infinite time horizon. Under the assumption of zero demolition costs, the solution called for an infinite sequence of identical buildings, each of which is occupied throughout its life. Although dwelling rent declines over the life of a building, the solution required that the price per unit of housing services in each dwelling first increase and then decrease as the dwellin ages. Building abandonment was shown to be possible with positive demolition costs. Solution of the model for Cobb-Douglas utility and production functions showed that the spatial properties of an urban area described by the model can be similar to the properties of a static city.

Although the difficulties created by relaxing the steady-state assumptions were noted above, the characterization of an optimal development strategy in a dynamic environment is obviously an important goal for future research. If this problem can be solved, our understanding of urban dynamics will be more nearly complete.

Footnotes

[1]For simplicity in the sequel, PVP per acre is taken to be gross of land cost unless otherwise specified.

[2]The existence of $\overline{Q}$ is guaranteed by the assumption that the indifference curve approaches the axes asymptotically.

[3]We assume the second-order condition, which requires that the Hessian matrix of $\pi/(1-e^{-rT})$ is negative definite at the solution to (12)-(14), is satisfied.

[4]Note that the existence of $\hat{Q}$ is guaranteed by the assumption that the indifference curve is convex and approaches the axes asymptotically.

[5]Letting $L(T,q,S) = (\pi(T,q,S) - D(S)e^{-rT})/(1-e^{-rT})$, it is easily shown that $L_{11} = L_{21} = L_{31} = 0$ when both sides of (18) equal zero. Thus the determinant of the Hessian matrix of L is zero and the second-order condition is not fulfilled at a solution where $\pi_1 = \pi - D = 0$.

[6]I am indebted to Randolph Lyon for an intuitive argument which suggested Property 1'.

[7]Computations with a CES utility function proved intractable.

[8]When a = 0, (20) becomes $q = (\theta w)^{\theta/(\theta-1)}[(1-e^{-rT})/rT]^{\theta/(\theta-1)}$, and similar modification of (22), (23), and (24) is also necessary.

[9]The following argument establishes (22). Consider the function $f(v) \equiv (1-e^{-vT})/v$. For $v \neq 0$, $f'(v)$ has the same sign as $1+\gamma-e^{\gamma}$, where $\gamma \equiv vT$. It is easily seen that since the exponential function is convex and tangent to the line $1+\gamma$ at $\gamma = 0$, the inequality $1+\gamma<e^{\gamma}$ holds for $\gamma \neq 0$, implying that $f'(v)$ is negative for $v \neq 0$. Although f(0) is undefined, l'Hopital's rule establishes that $\lim_{v\to 0} f(v)$ exists and equals T. Therefore , f is decreasing monotonically and $a < r$ implies $f(a) > f(r)$, establishing (22). A similar argument verifies $(1-e^{-rT})/rT < 1$ for the case where a = 0 (see footnote 8).

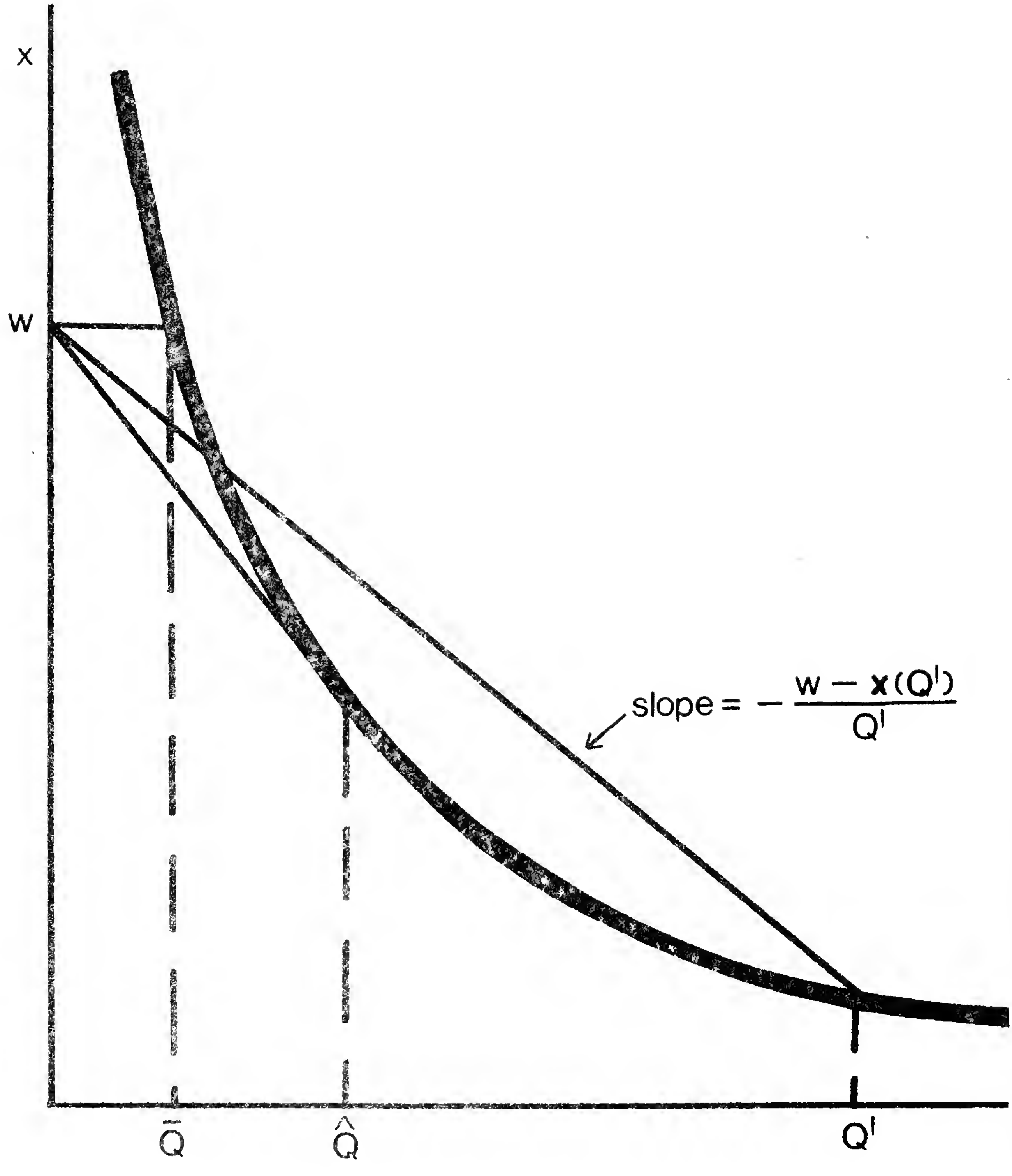

FIGURE 1.

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