Faculty Working Papers

DYNAMIC THEORY OF RENEWABLE RESOURCE ECONOMICS WITH ECONOMY POPULATION OF SIZE; OPTIMAL CONTROL THEORETIC APPROACH

T. Takayama, Professor of Economics, and M. Simaan, Associate Professor of Electrical Engineering at the University of Pittsburgh

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Summary:

The problem the economists in the field of renewable resources face is that of determining and recommending the harvest intensity that will maximize the economic value to the consuming society and also maximize the producers' surplus at a level of production in perpetuity.

The optimal control theoretic approach was employed in this paper to theoretically answer this problem within the framework of a one country-one renewable resource with economy of size of resource population.

In order to derive quantitative as well as qualitative results, a quadratic objective function-linear population-harvest dynamics model and a quadratic objective function-quadratic population-harvest dynamic model are solved for the optimal harvest paths over time.

A general conclusion is that if the initial resource population is smaller than the target population to be determined in the text, the present harvest intensity must be curtailed to an extent that the resource population can grow to a level that can sustain the optimal or target harvest. This establishes a principle of conservation of renewable resources. In this case, a government regulation over the total harvest is supported. In case the initial population is already greater than the target population, the competitive harvest is proved to be the optimal harvest. Detailed analyses of the economy of size effects and the future discount rate effects are presented.

Dynamic Theory of Renewable Resource Economics
with Economy Population of Size;
Optimal Control Theoretic Approach

T. Takayama** and M. Simaan***

Introduction

Renewable resources such as fish, whale, deer, forest, etc., constitute an increasingly important class of economic resources for the sustenance and improvement of human welfare on this planet, Earth.

The common characteristics of these resources are (1) that they are for direct human consumption, and (2) that they can reproduce themselves with a specific speed of renewal given a specific environment.

In this paper, we develop a dynamic theory of renewable resource economics that takes these common characteristics into consideration to establish principle of conservation of this class of resources. Other important theoretical as well as practical results in this field are obtained.

Since this class of resources embraces a large number of animals, fish, trees, etc., we deal with them as one species, namely, "fish" in this paper, without loss of substance.

The problem the renewable resource economists face could be summarizedas that of determining and recommending the intensity of withdrawal (harvesting) that will maximize the economic value to the consuming societies

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**Professor of Economics, University of Illinois, at Urbana-Champaign.

***Associate Professor of the Department of Electrical Engineering at the University of Pittsburgh, Pittsburgh, PA.

and also maximize the producers' profit or surplus at a level of production in perpetuity, if it exists at all.[1] Stated differently and narrowly, some economists consider the following questions are of (theoretical) importance; "(1) What is the optimal rate at which to withdraw fish? (2) Why might the maximum sustainable yield not be optimal? (3) How do optimal and competitive behavior differ? and (4) Under what conditions will extinction occur?" [Peterson and Fisher (1976) [1]]. There are many other questions that are of practical and theoretical importance. Such questions are: (5) Is catch regulation necessary (this is actually related to (3) above, and this will be duscussed fully later)? (6) Is mesh control regulation necessary? (7) What is the optimal catch when economy of size of fish population exists? and (8) What are the effects of the two hundred miles territorial waters limit on these questions raised above?

Even though the last question is of overwhelming importance at this stage of development of international regulations over the intensity of the catch of fish, it lies outside of the range of the tool that we employ in this paper. However, in the near future we plan to grapple with this question by using the differential game theoretic approach [10].

As will be revealed later, question (7) embraces questions (1), (3), and (4), and our theoretical investigation in this paper will be completed if the questions (2), (6), and (7) are answered.

In the next section we formulate our renewable resource economics problem as that of maximization of the social pay-off subject to fish population-catch dynamics and briefly discuss some general properties of the optimal catch, population, and other related variable, the Lagrangian.

In Section 2 we employ a quadratic social pay-off functional which

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[1]We avoid our own review of the existing literature in this field. The reader is referred to a comprehensive review by Peterson and Fisher [1].

contains the economy of size effect (for empirical implications of this effect, see [11]) in its industry supply functional, and a linear first-order fish population-catch dynamics, and within this framework we answer the three questions analytically and quantitatively.

In Section 3, the same social pay-off functional as in the previous section is used, but the population-catch dynamics is assumed as quadratic in population. Properties of the optimal catch paths such as multiple equilibria and the related convergence patterns (co-existence of a monotone stable or unstable convergence and a cyclical asymptotic convergence) are pointed out in relation to Samuelsonian "universal cycles" [4]. Policy implications of the conclusions derived from this model are discussed.

In conclusion, we summarize the results and point to future research topics and directions.

1. Dynamic Formulation of Renewable Resource Economics Problem with Economy of Population Size

We conceive that the renewable resource population follows a typical dynamics of the form

(1.1) $\dot{p} = f(p, x, t),\ p(0) = p_0$ given

where

- p denotes the population (more clearly, the unit must be expressed in pounds or tons of the resource at or older than the recruitable age),
- $\dot{p}$ denotes the time derivative of the population,
- x denotes the intensity of catch or harvest of the resource, and
- t denotes the real time over which the population and catch are moving and measured.

In this paper we assume that the function $f(p(t), x(t), t)$ can be separated into the following two parts:

(1.2) $f(p(t), x(t), t) = g(p(t), t) - x(t)$

where $g(t(t), t)$ is usually expressed as a function of $p(t)$ only, $g(p(t))$. and $\dot{p} = g(p(t))$ itself is called the biological growth law [1]. However, in due recognition of the fact that the population of a renewable economic resource cannot be observed and measured without interference of human endeavor for harvesting the resource, we write (1.1) via (1.2) as

(1.3) $\dot{p} = g(p(t)) - x(t),$[2] $p(0) = p_0$ given.

The society (or societies treated as an integrated single body in this paper) is considered to maximize its objective. We define the objective of our model as the present value of social pay-off, $SP(p,x)$, (or the sum of the consumers' and producers' surpluses) over the time horizon $[0, T)$, $T > 0$, following the market-oriented formulation of Samuelson [3] and Takayama [9, 10]. That is, the society is assumed to maximize

(1.4) $$TSP(p, x) \equiv \int_0^T e^{-rt} \int_0^x \{P_d(\xi, t) - P_s(p, \xi, t)\} d\xi dt$$

$$\int_0^T e^{-rt} SP(p,x)dt$$

where

$P_d(\xi, t)$ is the market demand function of the catch ($\equiv$ consumption) ξ up to x and time at t, and expresses the dollar value of the fish consumed, and $P_s(p, \xi, t)$ denotes the industry supply function of the catch time, and the fish population level. The introduction of p in the industry supply function is a reflection of "economy of population size" that this industry can enjoy relative to the increasing size of the resource.

We now define our dynamic renewable resource economics problem with economy of population size as

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[2]Hereafter, unless otherwise stated or special emphases are needed to do so, the time, t, in p(t) and x(t) (and other variables that may be introduced later) will be omitted.

Problem: Find $x^*(t)$ that maximizes (1.4) subject to (1.3), $p \geq 0$, and $x \geq 0$.

The Hamiltonian for this problem (ignoring $p \geq 0$ and $x \geq 0$ at this stage) can be written as

$$(1.5) \quad H(p, x, t) \equiv SP(p, x) + \lambda(g(p) - x)$$

where

$\lambda(\equiv \lambda(t))$ denotes the costate variable or Lagrangian. Based on (1.5) we can derive the necessary conditions for the optimality of $x(t)$ as follows:

$$(1.6)\begin{cases} \text{(i)} & \dot{p} = g(p) - x,\ p(0) = p_0 \text{ given} \\ \text{(ii)} & \dot{\lambda} = (r - \frac{\partial g}{\partial p})\lambda - \frac{\partial SP}{\partial p} \\ \text{(iii)} & \frac{\partial SP}{\partial x} - \lambda = 0 \\ \text{(iv)} & \lambda(T) = 0 \text{ or } \lim_{t\to\infty} e^{-rt}\lambda(t) = 0. \end{cases}$$

From (1.6, iii) we can identify the costate variable λ as the discrepancy between the market value (price) of the unit resource and the cost of producing or harvesting the same by an individual producer in the industry, that is,

$$(1.7) \quad \lambda(t) = P_d(x(t)) - P_s(p(t), x(t), t).$$

Thus, if $\lambda(t)$ is found to be positive for an optimal harvest path or trajectory $x^*(t)$ for any finite t, then one can state that until the target population p^* or target catch x^* when t tends to infinity is reached, a total catch or harvest control regulation must be implemented to protect the resource from being privately exploited. We discuss this point in more detail in the next section.

The external economy effect is reflected in the $\partial SP/\partial p$ term in (1.6, ii) and also $\partial SP/\partial x$ term. In the next section we will fully discuss this effect when we explicitly solve our quadratic-linear optimal control problem with economy of size of the resource population.

Instead of using a phase-diagram in $\dot{p} - \lambda$ space of the Quirk and Smith type [2], we have shown that the population-catch ($\equiv$ harvest) relationships can be best illustrated on a phase-diagram in p - x space [7, 9]. For this purpose, one can derive $\dot{x} = h(p, x)$ function from (1.6), and trace out optimal trajectories diagramatically if no analytical solution is available

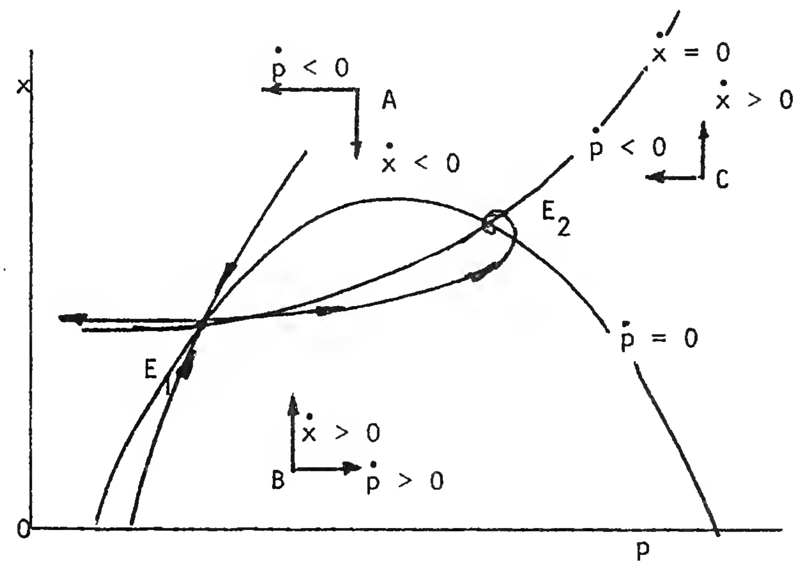

Figure 1. Phase-Diagram in P - x Space with Two Equilibria.

In Figure 1 a nonlinear population-catch dynamics and a nonlinear ($\dot{x}$ =)h(p, x) function are used to heuristically derive dynamic trajectories on the p - x space. More detailed analyses of a problem falling in this category will be made in Section 3 of this paper.

One characteristic that makes optimal control theoretical approach attractive is its capability of answering practical real-life problems quantitatively. In the next section, we will turn to this type of problem formulation that renders analytical solutions and conclusions on which policy decision makers can base decisions.

2. A Quadratic Social Pay-off-Linear Population-Harvest Dynamics Formulation with Economy of Size

In order to see clearly and quantitatively, the conditions under

which the optimal catch does exist, their quantitative results and the qualitative implications, we now turn to a quadratic, market-oriented formulation of our social pay-off function and linear population-catch dynamics case.

As a basis of constructing the quadratic social pay-off function, let us assume that the market-demand function and the industry supply function take the following form: demand function

(2.1) $$P_d(x) = \alpha - \beta x$$

where $\alpha > 0$ and $\beta > 0$, and supply function

(2.2) $$P_s(p, x) = \mu - wp + \Theta x$$

where $\mu > 0$, $w > 0$, and $\Theta > 0$.

Here, in (2.2), we assume that the (instantaneous) supply function shifts to the right as the fish population increases.

Now the social pay-off function can be written as

(2.3) $$TSP(p, x) = \int_0^T e^{-rt}\{(\alpha - \mu)x - \frac{1}{2}(\beta + \Theta)x^2 - wpx\}dt.$$

The linear population-catch dynamics is expressed as

(2.4) $$\dot{p} = a + bp - x, \quad p(0) = p_0 \text{ given,}$$

where $a \leq 0$, and $b > 0$ are assumed.

We can now formulate our problem as:

Problem $Q - L_w$: Find $x^*(t)$ that maximizes (2.3) subject to (2.4), $x \geq 0$, and $p \geq 0$.

The necessary conditions accompanying the optimality of $x^*(t)$ are:

$$
(2.5)\quad \begin{cases} \text{(i)} & \dot{p} = a + bp - x,\ p(0) = p_0 \text{ given} \\ \text{(ii)} & \dot{\lambda} = -(b - r)\lambda - wx \\ \text{(iii)} & (\alpha - \mu) - (\beta + \Theta)x\ wp - \lambda = 0 \\ \text{(iv)} & \lambda(T) = 0 \text{ or } \lim_{t\to\infty} e^{-rt}\lambda(t) = 0. \end{cases}
$$

It is possible to derive solutions of (2.5) for a finite time horizon case, and we are planning to develop a computer program to do just this in the near future. In this paper, however, we solve (2.5) for $T = \infty$. The detailed derivation of the solutions is documented in the Appendix at the end of this paper, and the interested reader is referred to it.

For this problem, Problem Q - L_w, there exist two solutions: (i) an unstable and divergent catch solution, and (ii) a stable, monotonically convergent solution.

The unstable, divergent solution can be expressed in a closed-loop form as

$$
(2.6)\qquad x_w = \frac{(2b - r)(1 - \delta)p}{2} + \frac{2(b - r)}{(2b-r)\delta - r}\left\{\frac{\alpha - \mu}{\beta + \Theta} + \frac{aw}{(\beta + \Theta)(b-r)}\right\} - \frac{a(2b - r)(1 - \delta)}{(2b-r)\delta - r}
$$

where

$$
(2.7)\qquad \delta = \sqrt{1 - \frac{4w}{(\beta + \Theta)(2b-r)}}\ (> 0)
$$

and is assumed to be positive. That is, we assume that

$$
(2.8)\qquad \{(\beta + \Theta)(2b-r) > 4w \text{ and } b - r > 0
$$

One can identify this solution as the path of instantaneous social pay-off maximization.

The accompanying population dynamics can be written as

$$(2.9)\quad \dot{p} = -\frac{2(b-r)}{(2b-r)\delta - r}\left\{\frac{\alpha+\mu}{\beta+\Theta} - a + \frac{aw}{(\beta+\Theta)(b-r)}\right\} + \frac{(2b-r)\delta + r}{2}p$$

which has a positive coefficient for p. (2.9) can be integrated out to give us the following divergent population growth:

$$(2.10)\quad p(t) = (p(0) - p_w^*)e^{\frac{(2b-r)\delta + r}{2}t} + p_w^*$$

where

$$(2.11)\quad p_w^* = \left\{\frac{1}{b - \frac{w(2b-r)}{(\beta+\Theta)(b-r)}}\right\}\left\{\frac{\alpha+\mu}{\beta+\Theta} - a + \frac{aw}{(\beta+\Theta)(b-r)}\right\}.$$

This solution, (2.6), is therefore not stable, and unless the initial condition is already at p_0^* (the optimal catch at this population is the same as (2.15) and if $p_0 < p_w^*$, which is most likely in the case of many fish species, and especially almost all whale species, the catch strategy (2.6) will drive the population toward certain extinction.

A stable, monotonically convergent catch solution can be expressed in a closed-loop form as

$$(2.12)\quad x_w^*(p(t)) = \frac{(2b-r)(1+\delta)}{2}p - \frac{2(b-r)}{(2b-r)\delta + r}\left\{\frac{\alpha+\mu}{\beta+\Theta} + \frac{aw}{(\beta+\Theta)(b-r)}\right\} + \frac{a(2b-r)(1+\delta)}{(2b-r)\delta + r}$$

with the following population dynamics

$$(2.13)\quad p = \frac{2(b-r)}{(2b-r)\delta + r}\left\{\frac{\alpha+\mu}{\beta+\Theta} - a + \frac{aw}{(\beta+\Theta)(b-r)}\right\} - \frac{(2b-r)\delta - r}{2}p$$

$$\equiv \frac{2(b-r)}{(2b-r)\delta + r}\Omega - \frac{(2b-r)\delta - r}{2}p$$

where

$$(2.14)\quad \begin{cases} (2b-r)\delta - r > 0 \\ b - r > 0 \end{cases}$$

are assumed to guarantee the stability of the population growth and positivity of the population when t tends to infinity, as the integral of (2.13) can be written explicitly as

$$(2.15)\quad p(t) = (p(0) - p_w^*)e^{\frac{-(2b-r)\delta + r}{2}t} + p_w^*$$

with p_w^* already defined in (2.11).

The terminal target population is exactly (2.11), and the corresponding catch is

$$(2.16)\quad x_w^* = \frac{1}{1 - \frac{w(2b-r)}{b(\beta + \Theta)(b-r)}}\left[\frac{\alpha - \mu}{\beta + \Theta} + \frac{aw}{(\beta + \Theta)(b-r)}\left\{1 - \frac{2b-r}{b(\beta + \Theta)(b-r)}\right\}\right].$$

In order to get a clearer picture of this case with economy of population size, let us compare these results we obtained in this paper with those without such economy which were already obtained in [9]. The results without economy of size can also be obtained by driving w to zero throughout the results obtained in this paper. Thus, since $\delta = 1$ when $w = 0$, we have for (2.6) a singular solution (2.6') $x^* = \frac{\alpha - \mu}{\beta + \Theta}$ which is unstable and drive the population to zero if $p_0 < p^*$ where

$$(2.11')\quad p^* = \frac{1}{b}\left\{\frac{\alpha - \mu}{\beta + \Theta} - a\right\},$$

and the population in this case moves along the following path

$$(2.10')\quad p(t) = (p(0) - p^*)e^{(b-r)\ t} + p^*$$

which is unstable due to (2.8).

There is a stable optimal catch solution in this case and is

$$(2.12')\quad x^*(p(t)) = (2b-r)p + \frac{\alpha - \mu}{\beta + \Theta} - \frac{(2b-r)}{b}\left(\frac{\alpha - \mu}{\beta + \Theta} - a\right),$$

with the corresponding population movement

$$(2.15') \quad p(t) = (p(0) - p^*)e^{-(b-r)t} + p^* ,$$

which is obviously stable due to (2.8).

The target population and catch are (2.11') and (2.6'), respectively. In order to facilitate the comparisons between these two situations, one with economy of size and the other one without it, let us assume that a = 0. Due to assumptions (2.14) on demand and supply parameters and on the positive future discount rate, one can conclude

$$(2.17) \begin{cases} p_w^* > p^* \\ x_w^* > x^*. \end{cases}$$

In other words, if economy of size can be expolited by a new fishing technology, then, in the long run, the fish population and the catch in perpetuity, and eventually the social pay-off will be larger than when the economy of size is not exploited.

The effect of the size economy expressed by w on the target population is positive as long as w satisfies (2.8). To confirm this statement, take the partial derivative of p_w^* with respect to w to obtain

$$(2.18) \quad \frac{\partial p_w^*}{\partial w} = \frac{\dfrac{2b-r}{(\beta + \Theta)(b-r)}}{\left[b - \left\{\dfrac{w(2b-r)}{(\beta + \Theta)(b-r)}\right\}\right]^2} \Omega + \frac{a}{(\beta + \Theta)(b-r)} \left\{\frac{1}{b - \left\{\dfrac{w(2b-r)}{(\beta + \Theta)(b-r)}\right\}}\right\}.$$

The sign $\partial p_w^*/\partial w$ depends on the economy of size parameter w; that is,

$$(2.19) \quad \frac{\partial p_w^*}{\partial w} \lesseqgtr \text{ accordingly as}$$

$$w \gtreqless \frac{(\beta + \Theta)(b-r)(3b-r)}{2(2b-r)} - \frac{(\alpha - \mu)(b-r)}{2a} .$$

This expression reduces, due to (2.8), to a more positive statement:

(2.20) $\frac{\partial p_w^*}{\partial w} > 0$ for any positive w satisfying (2.8), confirming our conclusion.

Figure 2 below is based on (2.5)(i) and

$$(2.22)\quad \dot{x} = \frac{1}{\beta + \theta}\{aw + (b-r)(\alpha - \mu)\} - (b-r)x + \frac{(2b-r)w}{\beta + \theta}p,$$

which is derived from (2.5), and summarizes the analytical results of our case with economy of size in p - x space in comparison with the case without the size economy.

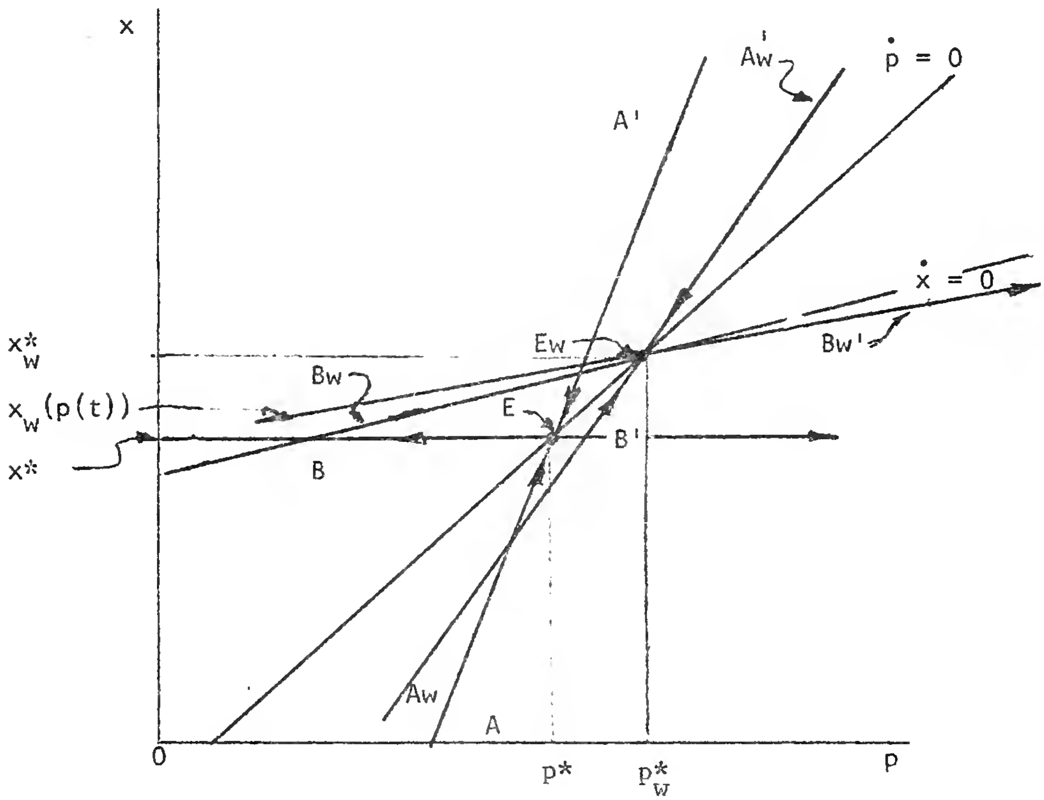

Figure 2. Stable and Unstable Solution Paths of the Cases with and without Economy of Size: Ew and E are Two Target Population-Target Catch Combinations

Based on the analytical results obtained so far, we can now derive the following conclusions: (1) Under certain conditions the optimal catch strategy exists and is smaller than the instantaneous social pay-off maximization catch (competitive behavior [1]) $X_w(p(t))$ on $\overrightarrow{EBw}$, (2.6), which leads the fish population toward extinction. This shows, within the restriction of the model assumptions, that to enjoy a larger social payoff in the

long run in perpetuity, the society must limit the catch or conserve the scarce resource.

Another way to reach this conclusion is to utilize (1.7) in relation to (2.5). Since for any initial population $p(0)$ $(< p_w^*)\lambda^*(p(t)) > 0$, and as t tends to infinity, $\lambda^*(p(t))$ converges to zero, the marginal social value of the unit resource is always larger than the marginal individual cost of producing (catching) the same. This, under competitive assumptions, especially the free-entry assumption, will drive the catch to $x_w(p(t))$, (2.6), leading the resource population to zero.

This prompts us to conclude that in the field of renewable resource economy, the catch control regulation must be implemented. This conclusion answers questions (1), (3), (4), (5), and (7).

(2) The mesh size control or regulation argument can be advanced along the line of [9]. The basic reason for our support for this regulation is that the optimal mesh-size information is external to individual fishermen or producers, and for the best benefit of the whole consuming and producing society, this information should be disseminated in the form of regulation similar arguments about the methods of harvesting other renewable resources can be advanced. This answers question (6).

(3) The effects of the future discount rate on the optimal catch and target population are not as straight forward as those of the case without economy of population size. However, the evaluation of the magnitude of $\partial p_w^*/\partial r$ indicates that the effect of a larger r is negative; as the future discount rate increases. In the latter case referred to above, the future discount rate has no effect on the target catch or population, but affects the rate of convergence of the optimal catch and the fish population related to the catch. More specifically, we concluded [9] that the larger the rate, the greater the present catch, and thus the slower the fish population growth.

It is impossible to answer question (2), "Why might the maximum sustainable yield not be optimal?", raised in the Introduction of this paper by using quadratic-linear model of this chapter. This leads us to a nonlinear formulation of our population--catch dynamics, and we now turn to a quadratic social pay-off-quadratic population-catch dynamic model to answer this question and explore other implications of the model results.

3. A Quadratic Social-Payoff-Quadratic Population-Catch Dynamic Model with Economy of Size

Economists have been accustomed to the stability arguments of the general equilibrium theory, and tend to define the dynamics in the first place and then argue about the stability of this defined system. This is typically seen in [8], and [4, 5]. Samuelson, later, made one step further away from this tradition and developed a two species-(predator-prey)-model into a minimum problem [6]. In this section we develop a quadratic-quadratic model with economy size, and point out, in this more general case, that the maximum sustainable yield cannot be optimal, and then show a way of generalizing the Samuelsonian universal cycle theory.

The social pay-off function can be defined exactly the same as (2.3), that is

$$(3.1) \qquad SP(p, x) = \int_0^T e^{-rt} \left\{ (\alpha - \mu)x - \frac{1}{2}(\beta + \Theta)x^2 - wpx \right\} dt.$$

The fish population-catch dynamics that this society ought to observe conscientiously can be written as

$$(3.2) \quad \dot{p} = f(p, x) = a + bp - cp^2 - x, \quad p(0) = p_0 \text{ given},$$

where $a \not> 0$ and b, and c are assumed to be positive constants.

The society is assumed to maximize the social pay-off, (3.1), constrained by the dynamics equation (3.2). Within this framework we define our dynamic optimization problem as:

Problem $Q - Q_w$: Find $x^*(t)$ that maximizes (3.1) subject to (3.2), $p \geq 0$ and $x \geq 0$.

The necessary conditions accompanying the optimality of x(t) are:

$$(3.3)\quad \begin{cases} \text{(i)} & \dot{p} = a + bp - cp^2 - x,\ p(0) = p_0 \text{ given} \\ \text{(ii)} & \dot{\lambda} = -(b - r - 2cp)\lambda - wx \\ \text{(iii)} & (\alpha - \mu + wp) - (\beta + \Theta)x - \lambda = 0 \\ \text{(iv)} & \lambda(T) = 0 \text{ or } \lim_{t\to\infty} e^{-rt}\lambda(t) = 0 \end{cases}$$

By differentiating (iii) with respect to time t and eliminating $\dot{\lambda}$, λ, and $\dot{p}$ (using (i) and (ii)), one can get (see [7] for a formula).

$$(3.4)\quad \dot{x} = \frac{1}{\beta + \Theta}[aw + (b-r)(\alpha - \mu) + \{(2b-r) - 2c(\alpha - \mu)\}p - 3cwp^2 + 2c(\beta + \Theta)px - \{(b-r)(\beta + \Theta) + 2w\}x] \equiv h(p, x).$$

The quadratic equations represented by

$$(3.5)\quad f(p, x) = a + bp - cp^2 - x = 0$$

and

$$(3.6)\quad h(p, x) = 0,$$

once solved simultaneously for x^* and p^* will give us the steady-state solutions at which the system may be stable or unstable. There are, in general, three solutions for the pair of equations above. All three solutions may be imaginary, or real, or any combination of these extremes.

Equation (3.5) is a parabola such as shown in Figure 1. While (3.6) can be expressed by the following rectangular hyperbola

$$(3.7)\quad x = \frac{aw + (b-r)(\alpha - \mu) + \Delta\{(b-r)(\beta + \Theta) + 2w\}/2c(\beta + \Theta)}{(b-r)(\beta + \Theta) + 2w - 2c(\beta + \Theta)p}$$

with Δ defined in (3.9) below and with the following two axes;

A vertical axis represented by

$$(3.8) \qquad p = \frac{(b-r)(\beta + \Theta) + 2w}{2c(\beta + \Theta)}$$

and another axis represented by

$$(3.9) \quad x = -\frac{(2b-r) - 2c(\alpha - \mu) - \frac{3w}{2(\beta + \Theta)}\left\{(b-r)(\beta + \Theta) + 2w\right\}}{2c(\beta + \Theta)}$$

$$+ \frac{3w}{2(\beta + \Theta)}p \equiv -\frac{\Delta}{2c(\beta + \Theta)} + \frac{3w}{2(\beta + \Theta)}p \; .$$

The most likely cases are

(a) The numerator in (3.7) is negative, or

(b) The numerator in (3.7) is zero, or

(c) The numerator in (3.7) is positive and p in (3.8) is positive, Δ in (3.9) is negative, and the coefficient of p in (3.9) is positive.

It is almost certain that the maximum sustainable yield (M's in Figure 3) is not the optimal population or the optimal catch. This answers question (2) raised in the Introduction.

In Figure 3(A), E_1 is a saddle-point-like equilibrium point similar to those we dealt with in Section 2 of this paper. E_2 and E_3 are asymptotically stable equilibrium points similar to E_2 in Figure 1. In Figure 3 (A'), E is an asymptotically unstable equilibrium point.

In Figure 3(B), E_1 is like E_1 in Figure 3(A), and E_2 is like E_3 in Figure 3(A).

In Figure 3(C), E_1 and E_2 are saddle-point-like equilibria. and E_3 is a stable equilibrium. Figure 3(C) shows only one saddle-point-like equilibrium.

A feature clearly different from the results we obtained in the previous model is the asymptotically (cyclically) converging or divergent catch paths or even a limit cycle generated by this model. For instance, in Figures 3(A), (A'), or (B), we have one such equilibrium. In Figures 3(A) or (B), if the initial fish

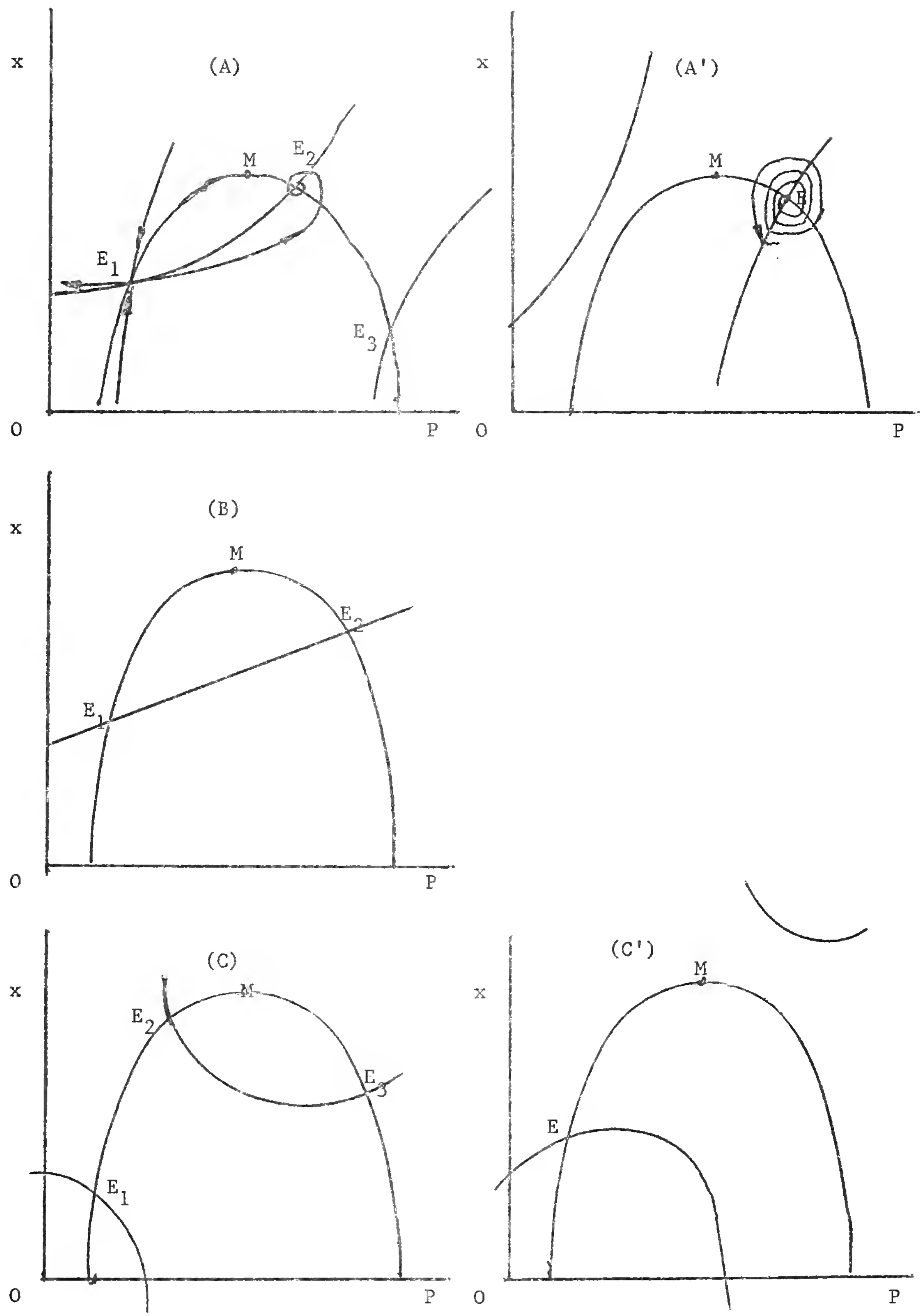

Figure 3. Phase-Diagrams and Possible Optimal Paths of Problem $Q\text{-}Q_w$.

population is smaller than p_1, the stable monotonically converging path will lead the catch-population to E_1. However, if the initial population is greater than p_1, say p_2, then a completely different path will lead the catch-population toward E_2 with ever oscillating movements around E_2. A local limit cycle is also a possibility [5].

As a limiting case of Problem Q - Qw we can construct a model with a quadratic social pay-off function without economy of size and a quadratic population-fish dynamics (Problem Q - Q). As one can easily prove that the phase-diagram can be constructed by the following $\dot{p}$ - $\dot{x}$ dynamics:

$$(3.10) \begin{cases} \text{(i)} \ \dot{p} = a + bp - cp^2 - x, \ p(0) = p_0 \text{ given} \\ \text{(ii)} \ \dot{x} = (b-r)\left(\frac{\alpha - \mu}{\beta + \Theta} - x\right). \end{cases}$$

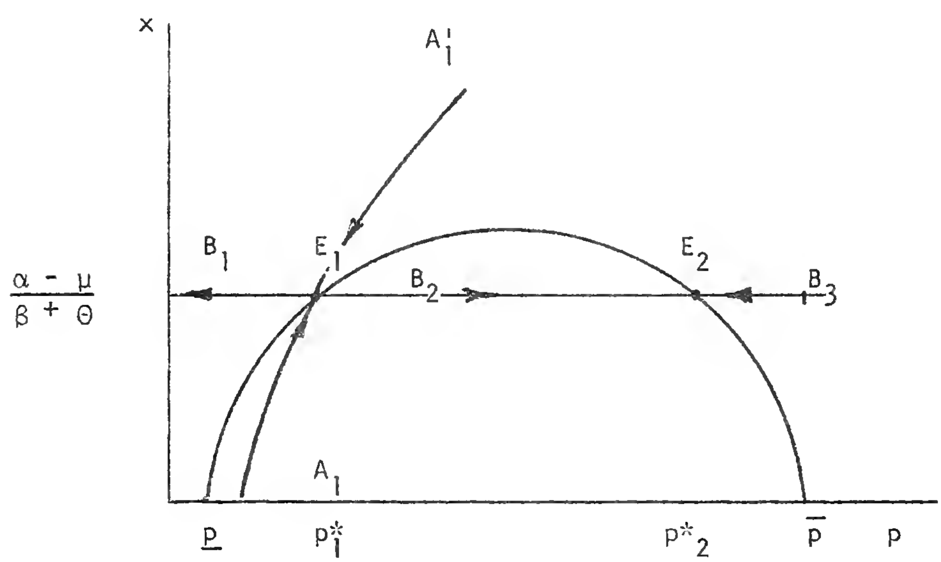

Figure 4. Equilibria of Problem Q - Q and Optimal Catch-Population Characteristics.

By setting $\dot{x}$ = 0, we have a familiar expression already derived elsewhere [9]. In this case, E_1 equilibrium possesses a similar stability and instability properties as a saddle-point-like equilibrium point. However, E_2 exhibits a completely stable equilibrium. For instance, if the population is greater than p_1^*, the optimal catch is exactly the same as the "competitive" catch. As the industry maintains this competitive catch, the

population monotonically converges to p_2^* along $\overrightarrow{B_2E_2}$. If the population is beyond p_2^* (but smaller than $\overline{P}$), the optimal catch will be still the competitive catch, which will bring the population to p_2^* along $\overrightarrow{B_3E_2}$ path.

Thus $\overrightarrow{B_2E_2}$, and $\overrightarrow{B_3E_2}$ paths are for the resource population abundance case. Due to the lack of economy of size, the target catches are exactly the same irrespective of the population as long as it lies in between $\underline{P}$ and $\overline{P}$ In contrast to Problem $Q - Q_w$, no oscillating, asymptotic convergence is observed. The comparison above reveals that the asymptotically stable path converging to E_2 is also a competitive path. Thus, in the case in which the population is already large enough and the population-catch dynamics exhibits an absolute decreasing return to scale, the optimal catch coincides with the competitive catch.

It may be almost impossible to solve (3.3) for clean analytical solutions. However, for practical problems falling in this category, graphical methods or some other methods may be available to derive the optimal trajectories.

This model clearly points out a way of generating a Voltera-Lolka type differential equations system extensively treated by Samuelson [4, 5, 6]. A prey is a prey, and follows its own population dynamics (although this population dynamics itself may depend on how the predator catches it--mesh-size regulations, hunting regulations, etc. (for further discussion see [9]). The predator catches the prey with certain technology and related cost. And, finally, the consuming society pays for the prey to enjoy the social benefit. Naturally, a large number of cases may emerge if we carefully investigate individual preys--human consumption relationships. Eventually, we may be able to identify where the fish population is located on the p axis, what are the market demand and supply relationships, etc. to determine and recommend at what intensity should harvest our scarce, renewable resource, and how to adjust the catch intensity so that we will be able to

enjoy the maximum social benefit in the long run.

Concluding Remarks

In this paper we have developed a dynamic model of renewable resource economics with economy of size of resource population, and solved one model with a quadratic social pay-off function and a linear population-catch dynamics. We have also attempted to analyze a model with a quadratic social pay-off function and a quadratic population-catch dynamics.

Several conclusions were drawn within the framework of our models, and they will be summarized below.

1. As the socially most desirable catch or harvest intensity indicates, the total catch or harvest must be regulated and controlled along the optimal catch trajectory, due to the clear-cut discrepancy between the marginal social value product (market price) of a unit of the resource and the marginal individual cost of producing the same.
2. The larger the effect of economy of size (Problem $Q - L_W$), the larger the target resource population and the target catch (if the society follows the optimal catch trajectory).
3. The so-called "competitive" catch is always larger than the socially optimal catch, and eventually leads the resource population to sure extinction.
4. The effects of the future discount rate on the target population and the target catch are nil when there is no economy of size, but is most likely negative with economy of size. However, the larger future discount rate increases the present catch and delays the convergence of the population to the target population, in the no economy of size case. In the economy of size case (Problem $Q - L_W$), the effects are not definite.
5. Other conclusions such as the need for mesh-size regulations or those related to catch or harvest methods must be implemented due mainly to the

fact that the information of such a technical nature is external to the individual producers.

6. The maximum sustainable yield is not likely for the (steady-state) optimal target catch (Problem $Q - L_w$ or Problem $Q - Q_w$).

7. Our nonlinear model (Problem $Q - Q_w$) suggests that there are (steady-state) equilibrium points other than the saddle-point-like equilibrium point (Problem $Q - L_w$). Oscillating, asymptotically stable or unstable equilibrium or even a limit cycle can coexist with the other.

The dynamic optimal control formulations of renewable resource economics, developed in this paper, open up various avenues to future research in this field.

A natural extension of our single resource-single country formulations is a multiple resources-single country formulation. Another extension is a single resource (or multiple resources)-multiple nations formulation. A one fish species-two countries formulation has been attempted and some results have been obtained in [10], when there is no size economy. This can be extended to the case in which economy of size exists.

It goes without saying that it is more to the satisfaction of many theorists and practitioners of economics _if_ the quadratic-linear form can be generalized and still obtain rich quantitative results. We can strive for it, and here we have an ever increasing need for interdisciplinary work ahead of us.

Fisheries theorists and technicians, those in forestry science, other disciplines related to renewable resources, economists specializing or interested in these fields, optimal control theorists, differential game theorists and practitioners in these fields, and policy decision makers in these fields, can work together to make regulations and controls in these areas technically, economically, and politically viable and sound.

Appendix

The Hamiltonian of Problem $Q - L_w$ is defined as

$$H_w = (\alpha - \mu)x - \frac{1}{2}(\beta + \Theta)x^2 + wxp + \lambda(a + bp - x).$$

The necessary conditions for the optimality of the catch x are given by (2.5) or

$$\text{(A.1)} \begin{cases} \text{(i)} & \dot{p} = a + bp - x,\ p(0) = p_0 \text{ given} \\ \text{(ii)} & \dot{\lambda} = -(b-r)\lambda - wx \\ \text{(iii)} & (\alpha - \mu) - (\beta + \Theta)x - \lambda = 0 \\ \text{(iv)} & \lambda(T) = 0 \text{ or } \lim_{t\to\infty} e^{-rt}\lambda(t) = 0 \end{cases}$$

Assuming the closed-loop (feedback) control of the following form

$$\text{(A.2)} \quad x = Kp + E$$

hold, we can derive the following identity:

$$\left\{\dot{K} + bK - K^2 + (b-r)K - \frac{(2b-r)w}{\beta + \Theta}\right\}p + \dot{E} + (b-r)E - KE + aK$$

$$- \frac{1}{\beta + \Theta}\{aw + (b-r)(\alpha - \mu)\} = 0 .$$

(for detailed derivation procedures, see [7]).

Assuming K(t) and E(t) converges to some constant as t tends to infinity, that is $\dot{K} = 0$, $\dot{E} = 0$, we get the following two equations:

$$\text{(A.4)} \begin{cases} \text{(i)} & -K^2 + (2b-r)K - \frac{(2b-r)w}{\beta + \Theta} = 0 \\ \text{(ii)} & (b-r)E - KE + aK - \frac{1}{\beta + \Theta}\left\{(b-r)(\alpha - \mu) - aw\right\} = 0. \end{cases}$$

Solving (A.4)(i), we get

$$\text{(A.5)} \begin{cases} \text{(i)} & K_1 = \frac{2b-r}{2}\left\{1 + \sqrt{1 - \frac{4w}{(\beta + \Theta)(2b-r)}}\right\} \\ & \text{and} \\ \text{(ii)} & K_2 = \frac{2b-r}{2}\left\{1 - \sqrt{1 - \frac{4w}{(\beta + \Theta)(2b-r)}}\right\} \end{cases}$$

E_1 and E_2, below, are obtained by solving (A.4)(ii) by using K_1 and K_2, respectively.

$$
\text{(A.6)}\begin{cases}
E_1 = \dfrac{a(2b-r)(1+\delta)}{(2b-r)\delta + r} - \dfrac{2(b-r)}{(2b-r)\delta + r}\left\{\left(\dfrac{\alpha-\mu}{\beta+\Theta}\right) + \dfrac{aw}{(\beta+\Theta)(b-r)}\right\} \\
\text{and} \\
E_2 = -\dfrac{a(2b-r)(1-\delta)}{(2b-r)\delta - r} + \dfrac{2(b-r)}{(2b-r)\delta - r}\left\{\left(\dfrac{\alpha-\mu}{\beta+\Theta}\right) - \dfrac{aw}{(\beta+\Theta)(b-r)}\right\}.
\end{cases}
$$

By using K_1 and E_1, we get $x_w^*(p(t))$, (2.12), and K_2 and E_2 result in x_w in (2.6). p and p* expressions corresponding $x_w^*(p(t))$ and x_w are easy to obtain.

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