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The Effects of the Sample Size, the Investment Horizon, and Market Conditions on the Validity of Composite Performance Measures: A Generalization

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ABSTRACT

In our previous study, the empirical relationship between Sharpe's performance measure and its risk proxy has been shown to be dependent on the sample size, the investment horizon and the market conditions. This paper generalizes this important result to include Treynor's and Jensen's performance measures. In addition, it is shown that the relationship between the estimated Sharpe's measure and its risk proxy is a special case of the relationship between the estimated Treynor's measure and its risk measure. Moreover, the conventional sample estimate of ex-ante Treynor's measure is biased. The ranking of mutual fund performance using the biased estimate is not an unbiased ranking as implied by the ex-ante Treynor's measure. It is shown that a significant relationship between the estimated Jensen's measure and its estimated risk measure may produce a potential bias associated with the cumulative average residual technique with which to be used for efficient market hypothesis testing. Finally, the impact of the dependence between estimated risk and average return in Friend and Blume's findings is also investigated.

THE EFFECTS OF THE SAMPLE SIZE, THE INVESTMENT HORIZON, AND MARKET CONDITIONS ON THE VALIDITY OF COMPOSITE PERFORMANCE MEASURES: A GENERALIZATION

I. Introduction

The capital asset pricing theory developed by Sharpe (1964), Lintner (1965) and Mossin (1966) (SLM) has been extensively used in pricing risky assets. Using the capital asset pricing theory, Sharpe (1966), Treynor (1965) and Jensen (1968, 1969) provided three one-parameter measures of mutual fund performance. These three risk-adjusted performance measures were later provided with the theoretical rationale by Friend and Blume (1970). However, in their empirical study, Friend and Blume (1970) documented evidence exhibiting a strong relationship between the estimated performance measures and their corresponding risk surrogates. Klemkosky (1973) and Kim (1978) subsequently found the existence of this significant relationship. The possible biases associated with the estimated performance measures and their possible implications were not carefully investigated until Chen and Lee (1981) recently provided the possible sources of the bias associated with the empirical relationship between the estimated Sharpe's performance measure and its estimated risk proxy.[1] They showed that the sample size and the investment horizon are two important factors in determining the degree of the empirical relationship between the estimated Sharpe's measure and the estimated risk proxy. In addition, the afore-mentioned empirical relationship is generally not independent of the market conditions associated with the sample period selected for empirical studies.

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[1]The term "bias" used in this study refers to the deviation of the empirical relationship from the theoretical relationship. Theoretically, one parameter performance measures are not expected to depend upon their risk proxies. However, it is empirically found thatthe estimated composite performance measures are generally highly correlated with their estimated risk proxies. To test the bias associated with the capital asset pricing theory, Black, Jensen and Scholes (1972), Blume and Friend (1973), Fama and MacBeth (1973) and others have done numerous empirical studies. Most recently, Roll (1977) has carefully re-examined these empirical tests.

This paper attempts to show that the conclusions associated with the relationship between the estimated Sharpe's measure and its risk proxy can be generalized to include Treynor's and Jensen's performance measures. The relationship between the estimated Sharpe's measure and its risk proxy is shown to be a special case of the realtionship between the estimated Treynor's measure and its risk measure. Moreover, the conventional sample estimate of ex-ante Treynor's measure is biased. The ranking of mutual fund performance using the biased estimate of Treynor's measure is not an unbiased ranking as implied by the ex-ante Treynor's measure. In addition, it is shown that a significant relationship between the estimated Jensen's measure and its estimated risk measure may produce a potential bias associated with the cumulative average residual technique with which to be used for efficient market hypothesis testing. Finally, the implications of the ex-post relationships between the estimated composite performance measures and their risk proxies on portfolio managements are also explored.

In section II, the empirical relationship between the estimated Treynor's measure and its risk proxy is studied in detail. Section III examines the possible problems associated with the estimated Jensen's measure. The possible implications of Roll's (1977, 1978) critique on asset pricing theory tests are also discussed. Section IV examines Friend and Blume's (1970) empirical findings in terms of the dependence assumptions. The results of this study are summarized in Section V.

II. The Empirical Relationship Between the Estimated Treynor's Measure and Its Risk Proxy

Following Friend and Blume (1970), the theoretical relationship of the capital asset pricing model [CAPM] developed by SLM can be defined as:

$$E(R_i) - R_f = \beta_i[E(R_M) - R_f] \tag{1}$$

where R_f is the risk-free rate for borrowing or lending, R_i is the rate of return on portfolio or asset i, and R_M is the market rate of return. Equation

(1) can be rewritten in an ex post model [see Jensen (1968)]:

$$R_{it} - R_f = \alpha_i + \beta_i[R_{Mt} - R_f] + \epsilon_{it} \tag{2}$$

where R_{it} is the rate of return on portfolio or asset i in period t, R_{Mt} is the rate of return in period t and ϵ_{it} is a random disturbance with mean zero and variance σ_ϵ^2 and is independent of R_{Mt}. If n observations are used to estimate the parameters by ordinary least squares (OLS), equation (2) can be summed over n and averaged to obtain:

$$\bar{R}_i - R_f = \hat{\alpha}_i + \hat{\beta}_i[\bar{R}_M - R_f] \tag{3}$$

where the bar indicates an average and $\hat{\alpha}_i$ and $\hat{\beta}_i$ are least-squares estimates of α_i and β_i respectively. The estimated intercept $\hat{\alpha}_i$ is the Jensen's measure. The estimated Treynor's measure is defined as $(\bar{R}_i - R_f)/\hat{\beta}_i$. In the following paragraphs, it is demonstrated that the conclusions associated with the relationship between estimated Sharpe's measure and its risk proxy obtained by Chen and Lee(1981) can be generalized to include Treynor's performance measure. The covariance between the estimated Treynor's measure and its estimated risk measure is first explored.

We assume that the holding period rate of return on security i (R^*_{it}) is lognormally distributed. Then the rate of return ($R_{it} = \ln R^*_{it}$) is normally distributed.[2] Under the normality of security return, theory of least squares has indicated that the estimated beta coefficient, $\hat{\beta}$, and the sample average excess rate of return, $\bar{R} - R_f$, are independently, normally distributed. This implies that the estimated Treynor's measure, $(\bar{R} - R_f)/\hat{\beta}$, is a ratio of two independent normal variables. Fieller (1932) and others have shown that the ratio of two normal variables does not have finite moments. To find the covari-

[2]In case of finite and long holding period horizon, it might be better to use the lognormal instead of the normal assumption. In addition, under the lognormality assumption the multiperiod return $[\prod_{t=1}^{n} R^*_{it}]$ is also lognormally distributed. We are grateful to two refees' suggestion to use the lognormality instead of the normality assumption for holding period returns.

ance between $(R - R_f)/\hat{\beta}$ and $\hat{\beta}$, it is assumed that $\hat{\beta}$ takes values in some positive range.[3] This assumption eliminates the existence of infinite moments. Then, the truncated distribution of the estimated risk measure, $\hat{\beta}$, can be written as

$$f(\hat{\beta}) = \begin{cases} \frac{1}{k} \cdot \frac{1}{\sqrt{2\pi}\, \sigma_{\hat{\beta}}} \cdot \exp[-\frac{1}{2}(\frac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}})^2], \\ \text{for } a \leq \hat{\beta} \leq b,\ a > 0 \\ 0 \text{ elsewhere} \end{cases} \tag{4}$$

where,

$$\sigma^2_{\hat{\beta}} = Var(\hat{\beta}) = \frac{\sigma^2_\varepsilon}{\sum_{t=1}^{n} (R_{Mt} - \bar{R}_M)^2},$$

$\beta = E(\hat{\beta})$, the expected value of non-truncated $\hat{\beta}$,

and

$$k = \int_a^b \frac{1}{\sqrt{2\pi}\, \sigma_{\hat{\beta}}} \cdot \exp[-\frac{1}{2}(\frac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}})^2] d\beta \quad = \text{a constant} > 0.$$

Since $\hat{\beta}$ and $(\bar{R} - R_f)$ are independently distributed, the truncated distribution of $\hat{\beta}$ will also be independent of the distribution of $(\bar{R} - R_f)$. With this independence property, the covariance between the estimated Treynor's measure and its estimated risk measure can be easily obtained as follows:[4]

$$Cov(\frac{\bar{R} - R_f}{\hat{\beta}}, \hat{\beta}) = E(\bar{R} - R_f) - E(\hat{\beta})E(\frac{\bar{R} - R_f}{\hat{\beta}})$$

$$= E(\bar{R} - R_f) - \beta^+ \cdot E(\bar{R} - R_f)E(\frac{1}{\hat{\beta}}),$$

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[3] For example, β may be assumed to take values between 0.0001 and 10. This will guarantee the existence of finite moments. This assumption is realistic since a large percentage of securities have positive betas.

[4] The distribution function of the estimated Treynor's measure can be easily obtained from Fieller's (1932) results. See Appendix (A) for the derivation.

by using independence property

$$= (\beta^{+}C - 1)[E(R_f - \bar{R})]$$

$$= (\beta^{+}C - 1)(R_f - \mu),^{5} \tag{5}$$

where, $\mu = E(\bar{R})$, the population (market) mean rate of return,

$C = E(\frac{1}{\hat{\beta}})$ is a constant defined in equation (B-5) of Appendix (B),

β^{+} = the expected value of the truncated $\hat{\beta} = \beta + \ell$,[6]

$$\ell = \frac{1}{k}[Z(\frac{a - \beta}{\sigma_{\hat{\beta}}}) - Z(\frac{b - \beta}{\sigma_{\hat{\beta}}})] > 0,$$

and

$$Z(\frac{a - \beta}{\sigma_{\hat{\beta}}}) = \frac{1}{\sqrt{2\pi}} \exp\ [\frac{-(a - \beta)^2}{2\sigma_{\hat{\beta}}^2}].$$

Equation (5) is similar to Chen and Lee's equation (8) except the scalar term. Similarly, the correlation coefficient between the estimated Treynor's measure and its estimated risk measure can be expressed as

$$\rho(\frac{\bar{R} - R_f}{\hat{\beta}}, \hat{\beta}) = \frac{(\beta^{+}C - 1) \cdot (R_f - \mu)}{\sigma_T\ \sigma_\beta} \tag{6}$$

where σ_T denotes the standard deviation of the estimated Treynor's measure with the truncated value of $\hat{\beta}$.[7] Equation (6) is also similar to Chen and Lee's equation (13) except the scalars, $(\beta^{+}C - 1)$, σ_T and σ_β. Furthermore, following

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[5]Following Friend and Blume (1970), $\bar{R}$ is the average rate of return of a random sample portfolio drawn from a population containing all securities with mean μ and variance σ^2. In other words, μ represents the population (market) mean return and σ^2 denotes the population variance associated with market rates of return. Thus, $E(\bar{R}) = \mu$.

[6]See Johnson and Kotz (1970) for the derivation of the expected value of a truncated normal distribution.

[7]The explicit form of σ_T is not necessary for the analysis.

Chen and Lee's analysis of the impact of the investment horizon on the degree of the association between the estimated Sharpe's measure and its estimated risk proxy,[8] the m-period covariance and correlation coefficient between the estimated Treynor's measure and the estimated risk measure can be easily written, respectively, as

$$\mathrm{Cov}\left(\frac{\bar{R}^* - R_f^*}{\hat{\beta}^*}, \hat{\beta}^*\right) = (\beta^+ C - 1)[(1 + R_f)^m - (1 + \mu)^m] \tag{7}$$

and

$$\rho\left(\frac{\bar{R}^* - R_f^*}{\hat{\beta}^*}, \hat{\beta}^*\right) = \frac{(\beta^+ C - 1)[(1 + R_f)^m - (1 + \mu)^m]}{\sigma_T^* \; \sigma_{\hat{\beta}}^*} \tag{8}$$

where the asterisks denote the m-period sample estimates (or the m-period parameters, σ_T^* and $\sigma_{\hat{\beta}}^*$). Thus, equations (7) and (8) are similar to Chen and Lee's equations (20) and (21) except scalar constants. Applying the similar analysis as employed by Chen and Lee (1981) to equations (5) through (8), we conclude that the sample size, the investment horizon and market conditions determine the degree of association between the estimated Treynor's measure and its estimated risk measure ($\hat{\beta}$). Specifically, the conclusions are as follows:

(1) The estimated Treynor's measure is uncorrelated with the estimated risk measure only either when the risk-free rate is equal to the mean rate (μ)

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[8] Under the assumption of stationary returns over time and all investors having the same investment horizon, one has $\mu_j = \mu$ and $\sigma_j^2 = \sigma^2$ for all j = 1, 2, ..., m where μ_j and σ_j^2 are the expected rate of return and the return variance associated with j-period observations (the μ and σ^2 have the same definitions as indicated in footnote 6). In additon, following Tobin (1965), distribution stationarity over time and return independence are assumed. It can be shown that $\tilde{E} = (1 + \mu)^m - 1$ and $\tilde{\sigma}^2 = [\sigma^2 + (1 + \mu)^2]^m - (1 + \mu)^{2m}$, $m > 0$ where, for the m-period case, $\tilde{E}$ and $\tilde{\sigma}^2$ are the expected rate of return and the return variance of the market portfolio, respectively.

of the return on the market portfolio or when the sample size n is infinite. The risk-free rate of interest is generally not equal to the mean rate of return on the market portfolio and the sample size associated with empirical work is generally finite. Therefore, the estimated Treynor's measure is in general correlated with the estimated risk measure (the beta coefficient).

(2) If the risk-free rate is less than the mean rate of return on the market portfolio, the estimated Treynor's measure is negatively correlated with the estimated risk measure $[(B^{+}C-1) > 0]$. This result explains an unusual relationship found by Friend and Blume (1970). Their empirical study showed that the estimated Treynor measure was negatively related to its estimated risk proxy $(\hat{\beta}_j)$ during the period from January 1960 to June 1968 and the subperiod from January 1960 to March 1964 over which the riskless rate of interest was less than the market rate of return.[9] The explanation of a positive relationship found by Friend and Blume (1970) in the second subperiod [April 1964 - June 1968 over which $R_f = .0034 < \mu = .0088$] will be examined in Section IV.

(3) An improper observation horizon will have a significant impact on the covariance between the estimated Treynor's measure and the estimated risk measure.[10] An observation horizon shorter than the true investment horizon will reduce the dependence of the estimated Treynor's performance measure on its estimated risk measure. An observation horizon longer than the true investment horizon will

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[9] During the whole period (1960-1968), $R_f = .0028 < \mu = .0104$ (the average monthly return). During the first subperiod (January 1960 - March 1964), $R_f = .002 < \mu = .0119$. The R_f and μ are, respectively, the average monthly returns on Treasury bills and the NYSE stock prices index.

[10] The observation horizon refers to either one day, one week, one month, one quarter or one year. The concept or the "true" investment horizon implies that investors will all share the same horizon. The justification and the implication of this assumption can be found in either Lee (1976) or Levy (1972). Merton (1978) has argued that there exist three kinds of true horizons, i.e., trading horizon, decision horizon and planning horizon. The true horizon discussed in this paper is the decision horizon.

An "improper" observation horizon refers to that the time horizon used in empirical research does not coincide with the true investment horizon.

magnify the dependence.[11] This ex-post relationship does not coincide with the ex-ante relationship derived by Levy (1972) and Levihari and Levy (1977). The ex-post relationship examined in the present study is based on the sampling distributions of the estimators of performance measures and their risk proxies while Levy (1972) and Levihari and Levy (1977) investigate the relationship in terms of ex-ante parameters without specifically considering the sample variation in the estimators. Moreover, their results determines the relationship between m-period Treynor (or Sharpe) measure and its one-period risk measure as the length of the investment horizon varies. In contrast, we examine the m-period correlation between the estimated m-period performance measure and its m-period estimated risk proxy. This m-period correlation can be explained by means of one-period market conditions (R_f and μ) and varying investment horizon. Thus, their studies and our analysis examine two different relationships.[12] Nevertheless, the result of our analysis has strengthened the understanding of this important relationship in ex-post terms.

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[11]This conclusion is shown as follows:

$$\frac{d(Cov)}{dm} = (B^+C-1)\ [(1+R_f)^m \ln(1+R_f) - (1+\mu)^m \ln(1+\mu)].$$

Then $d(Cov)/dm > 0$ if $R_f > \mu$. And $d(Cov)/dm < 0$ if $R_f < \mu$. In addition $Cov(\cdot) > 0$ for all m if $R_f > \mu$, and $Cov(\cdot) < 0$ for all m if $R_f < \mu$. Thus, when $R_f > \mu$, $Cov(\cdot)$ decreases (increases) with decreasing (increasing) m. For a shorter horizon, $Cov(\cdot)$ gets smaller when $R_f > \mu$. Similarly, for a longer horizon, $Cov(\cdot)$ increases when $R_f < \mu$. See Chen and Lee (1981, p.616, Figure 2) for a similar, detailed graphic analysis of this result.

[12]Levy (1972) and Levihari and Levy's (1977) results are determined by $dTm/d\beta$ and $dSm/d\sigma$ where Tm and Sm are, respectively, m-period Treynor and Sharpe measures, and β and σ are one-period risk measures. In addition, they examined the direction of the ex-ante relationship between m-period Sharpe (or Treynor) measure and its one-period risk measure, while our analysis investigated the degree of m-period association between the estimated m-period performance measure and the estimated m-period risk proxy. Thus, our analysis should be regarded as a complementary instead of substitutional result to their analysis in understanding the impacts of the investment horizon on the relationship between performance measures and their risk proxies.

In sum, a shorter observation horizon and a large sample size should be used in empirical research to test the relationship between estimated Treynor measure and estimated systematic risk.

It is important to note that the estimated Treynor's measure is a biased estimator of ex-ante Treynor's measure, as indicated by equation(9):[13]

$$E\left(\frac{\bar{R} - R_f}{\hat{\beta}}\right) = \left(\frac{\mu - R_f}{\beta}\right) \cdot e_\beta, \tag{9}$$

where

$$e_\beta = \frac{\beta}{k\sqrt{2\pi}(m^* \sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[\left(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta}\right)^i \cdot \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{(i-j)} 2^{(j+1)/2}\, \Gamma\left(\frac{j+1}{2}\right) m_j \right], \tag{10}$$

$$m^* = (a' + b'),$$

$$a' = \frac{a - \beta}{\sigma_{\hat{\beta}}}, \quad b' = \frac{b - \beta}{\sigma_{\hat{\beta}}},$$

and

$$m_j = \int_{a'^2}^{b'^2} \frac{x^{\frac{(j+1)}{2} - 1} e^{-x/2}}{\Gamma\left(\frac{j+1}{2}\right) 2^{(j+1)/2}}\, dx$$

= a chi-square probability (integral).

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[13] See Appendix (B) for the derivation.

The bias factor, e_{β}, associated with the expected value of the estimated Treynor's measure depends on the value of a mutual fund's (or portfolio's) systematic risk. The bias factor will vary from one mutual fund (or portfolio) to another because of different systematic risks associated with different mutual funds (or portfolios). Therefore, the ranking of mutual fund performance using the biased estimate, $(\bar{R} - R_f)/\hat{\beta}$, of Treynor's measure will not be an unbiased ranking of ex-ante Treynor's measure, $(\mu - R_f)/\beta$. In addition, the absolute difference between two estimated Treynor's measures will also be affected by the bias factor, e_{β}. Note that $[(\bar{R} - R_f)/\hat{\beta}]/e_{\beta}$ cannot be used as an unbiased estimate of ex-ante Treynor's measure since the e_{β} involves unknown parameters.

The above analysis is applicable to any individual securities and portfolios with positive beta coefficients. Therefore, it is also applicable to efficient portfolios. In other words, for efficient portfolios, the relationship between the estimated Treynor's measure and the $\hat{\beta}$ provides a direct generalization for the relationship between the estimated Sharpe's measure and its estimated risk measure. This result is next explored. For an efficient portfolio, the total risk (S^2) is equal to the square of beta coefficient times the variance of the market return (S_M^2). This is because non-systematic risk is diversified away. That is,

$$S^2 = \hat{\beta}^2 S_M^2, \qquad \hat{\beta} > 0. \tag{11}$$

Based on the relation described by (11) for efficient portfolios, it is easy to show that equations (5) and (6) become Chen and Lee's (CL) equations (8) and (13), respectively.[14] That is, following (5) and (6), we have

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[14] This analysis is conditional on the market return.

(a) $\beta^{+}C - 1 = E(\hat{\beta})E(\frac{1}{\hat{\beta}}) - 1$

$$= E(S/S_m) \cdot E(\frac{1}{S/S_M}) - 1 \text{ using (11)}$$

$$= E(S)E(\frac{1}{S}) - 1 = d \text{ which is defined in footnote 7.}$$

(b) $$\sigma_T^2 = Var(\frac{\bar{R} - R_f}{\hat{\beta}}) = Var(\frac{\bar{R} - R_f}{S/S_M}) = S_M^2\ \sigma_{sp}^2$$

(c) $$\sigma_{\hat{\beta}}^2 = Var(\hat{\beta}) = Var\ (S/S_M) = \sigma_S^2/S_M^2$$

Substituting (a), (b), and (c) into (5) and (6) gives

$$Cov(\frac{\bar{R} - R_f}{\hat{\beta}}, \hat{\beta}) = Cov\ (\frac{\bar{R} - R_f}{S}, S), \text{ which is CL's equation (8)} \qquad (12)$$

and

$$\rho(\frac{\bar{R} - R_f}{\hat{\beta}}, \hat{\beta}) = \rho(\frac{\bar{R} - R_f}{S}, S), \text{ which is CL's equation (13).} \qquad (13)$$

Similarly, it can be shown that for efficient portfolios equations (7) and (8) are equal to CL's equations (20) and (21), respectively. Therefore, the relationship between the estimated Sharpe's measure and its estimated risk proxy is a special case of the relationship between the estimated Treynor's measure and the estimated systematic risk, $\hat{\beta}$.

The empirical relationship between the estimated Jensen's measure and its estimated risk proxy is next examined.

III. The Empirical Relationship Between The Estimated Jensen's Measure And Its Estimated Risk Proxy

Follwoing Heinen (1969) and the definitions defined in this paper, the covariance between the estimated Jensen's measure and its estimated risk proxy can be defined as

$$Cov(\hat{\alpha}_i, \hat{\beta}_i) = \frac{-\bar{X}_M\ \sigma_\varepsilon^2}{\sum_{t=1}^{n} (X_{Mt} - \bar{X}_M)^2} \qquad (14)$$

where,

$$X_{Mt} = R_{Mt} - R_f,\ \bar{X}_M = \bar{R}_M - R_f \qquad (15)$$

σ_ε^2 is the residual variance associated with equation (2). Equation (14) indicates that the estimated Jensen's performance measure is correlated with its estimated risk proxy unless $\bar{R}_M = R_f$. This relationship has been found by both Friend and Blume (1970) and Klemkosky (1973). It is clear from (14) and (15) that the relationship can also be affected by the sample size, the market conditions and the investment horizon. The results from observing (14) and (15) are given as follows:

(i) The larger the sample size (n), the greater the sum of the squared deviations, $\sum_{t=1}^{n} (x_{Mt} - \bar{x}_M)^2$, and hence the smaller the Cov $(\hat{\alpha}_i, \hat{\beta}_i)$. Moreover, when the sample size (n) approaches infinity, the Cov$(\hat{\alpha}_i, \hat{\beta}_i)$ goes to zero.

(ii) The estimated Jensen's measure $(\hat{\alpha}_i)$ is uncorrelated with its estimated risk proxy $(\hat{\beta}_i)$ only when $\bar{R}_M = R_f$ (i.e. $\bar{x}_M = 0$). The $\hat{\alpha}_i$ is negatively correlated with the $\hat{\beta}_i$ when R_f is less than $\bar{R}_M$, as found by Friend and Blume (1970) in the whole period and the first subperiod. The explanation for their positive relationship in the second subperiod is discussed in Section IV.

(iii) The Cov$(\hat{\alpha}_i, \hat{\beta}_i)$ is influenced by the length (m) of the observation horizon. This can be observed by deriving the expected value of the m-period covariance, $E[\text{cov}^*(\alpha_i, \hat{\beta}_i)]$, between the estimated Jensen's measure and its estimated risk proxy:[15]

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[15] For the one-period case,

$$E[\text{Cov}(\hat{\alpha}_i, \hat{\beta}_i)] = \sigma_\varepsilon^2 \, E[\frac{-\bar{x}_M}{nS_M^2}], \text{ where } S_M^2 = \sum_{t=1}^{n} (x_{Mt} - \bar{x}_M)^2/n.$$

$$= \frac{\sigma_\varepsilon^2}{n} E(-\bar{x}_M) \cdot E(\frac{1}{s_M^2}),$$ using the normality assumption and the independence property of $\bar{x}_M$ and S_M^2 see Hogg and Craig (1970).

$$= c\ (R_f - \mu), \text{ where } c = \frac{\sigma_\varepsilon^2}{n} E(\frac{1}{S_M^2}).$$

Similarly for the m-period case,

$$E[\text{Cov}^*(\hat{\alpha}_i, \hat{\beta}_i)] = c^*(R_f^* - \mu^*),\ c^* = \frac{\sigma_\varepsilon^{*2}}{n^*} E(\frac{1}{S_M^{*2}})$$

$$= c^*[(1 + R_f)^m - (1 + \mu)^m],$$ using the result in footnote 11.

$$E[\mathrm{Cov}^*(\hat{\alpha}_i,\ \hat{\beta}_i)] = c^*[(1 + R_f)^m - (1 + \mu)^m], \tag{16}$$

where "*" indicates m-period parameters or estimates. The m-period $\mathrm{Cov}^*(\hat{\alpha}_i, \hat{\beta}_i)$ in (16) has an expression similar to $\mathrm{Cov}[(\bar{R}^* - R_f^*)/\hat{\beta}^*, \hat{\beta}^*]$ in equation (7). Therefore, the analysis and conclusions for the relationships associated with the estimated Treynor's and Sharpe's measures can be generalized to include the association between the estimated Jensen's measure and its estimated risk proxy.

If the estimated Jensen's measure, $\hat{\alpha}_i$, and the estimated systemaric risk, $\hat{\beta}_j$, are significantly correlated, the estimate of the residual return $(\hat{\varepsilon}_{it})$ used in testing the efficient market hypothesis will be biased. The residual return is estimated by

$$\hat{\varepsilon}_{jt} = (R_{it} - R_f) - \hat{\alpha}_i - \hat{\beta}_i(R_{Mt} - R_f) \tag{17}$$

Following the above conslusions, the $\hat{\alpha}_i$ is negatively correlated with the $\hat{\beta}_i$ during the time period over which R_f is smaller than the expected market return (μ). Then, a negative correlation between $\hat{\alpha}_i$ and $\hat{\beta}_i$ means that securities having low beta coefficients are favorably evaluated to have high non-market returns (or intercept terms), $\hat{\alpha}_i$. As a result, the use of equation (17) leads to under-estimate the residual returns. This result can be easily shown as follows: let $\hat{\alpha}_i'$ be the estimate of non-market return when $\hat{\beta}_i$ and $\hat{\alpha}_i$ are negatively correlated. Then α_i being favorably evaluated implies that $\hat{\alpha}_i'$ is greater than the true estimate of α_i, say $\hat{\alpha}_i^*$. One may then write

$$\hat{\alpha}_i' = \hat{\alpha}_i^* + \delta_i,\ \delta_i > 0. \tag{18}$$

Using equations (17) and (18) leads to the equation

$$\hat{\varepsilon}_{it}' = \hat{\varepsilon}_{it}^* - \delta_i,\ \delta_i > 0. \tag{19}$$

where $\hat{\varepsilon}_{it}' = (R_{it} - R_f) - \hat{\alpha}_i' - \hat{\beta}_i(R_{Mt} - R_f)$ and $\varepsilon_{it}^* = (R_{it} - R_f) - \hat{\alpha}_i^* - \hat{\beta}_i(R_{Mt} - R_f)$. Equation (19) indicates that ε'_{it} is less than ε^*_{it}. In other words, the residual return is under-estimated if the non-market return is favorably assessed. This implies that the estimated average residual return and the estimated cumulative average residual return will be downward biased. On the other hand, high beta securities are unfavorably evaluated to have low non-market returns. The unfavorable assessment of non-market returns, in turn, results in over-estimating the estimated average residual return and the estimated cumulative average residual return.[16] However, during the time period the $\hat{\alpha}_i$ being positively correlated with the $\hat{\beta}_i$ [see Section IV] implies that high (low) beta securities are assessed to have high (low) non-market returns. As a result, the estimated average residual return and the estimated cumulative average residual return are downward biased for high beta securities whose non-market returns are assessed to be favorably high, and upward biased for low beta securities whose non-market returns are evaluated to be unfavorably low. Thus the result from the efficient market hypothesis testing may produce a potential bias associated with the cumulative average residual technique if the estimated Jensen's measure and its risk proxy are significantly correlated.

The present study has shown that the estimated composite performance measures can be highly correlated with their estimated risk proxies. In general, the sample size, the investment horizon and the market conditions are major factors in determining the above-mentioned relationships. Johnson and Burgess (1975) and Burgess (1975) and Johnson (1976) examined the effects of sample sizes and sampling fluctuation on the accuracy of both portfolio and security analyses. They concluded that the number of historical observation is important to produce efficient portfolio

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[16]This analysis holds true for the market model if the estimated intercept term is significantly correlated with the estimated beta coefficient.

performance characteristics. Their conclusion is similar to the results associated with the impact of the sample size on the relationship between the estimated composite measures and the risk proxies derived in this study.

The problems found in the present study associated with the empirical relationships of composite performance measures remain existent even if we assume away the problems as indicated by Roll (1977, 1978). In his well-known studies, Professor Roll has found two major problems in performing the asset pricing theory tests: (i) the index measurement and (or) the specification problem, and (ii) the theoretical tautology problem. The impacts of the index measurement problem on the estimated Jensen's performance measure have been investigated by Chen and Lee (1984) and Roll (1978). Their results have indicated that the bias of the estimated Jensen's measure generally increases the covariance as indicated in equation (14). The possible effects of the theoretical tautology problem of asset pricing tests on the issues investigated in this paper still remains an open question.

VI. Explanation of Friend and Blume's Positive Empirical Relationships

In this section, we attempt to explain what causes the positive ex-post relationship in the second subperiod found in Friend and Blume's study. In our previous derivation, the sampling distribution of $(\bar{R} - R_f)$ is independent of the distributions of S and $\hat{\beta}$, where S is the sample standard deviation of return. However, in the ex-post sample study, some circumstance may arise that $(\bar{R} - R_f)$ is not independent of S and $\hat{\beta}$. Then, Chen and Lee's (1981) equation (5) can be easily shown to be

$$\mathrm{Cov}\left(\frac{\bar{R} - R_f}{S}\right) = d(R_f - \mu) - E(S) \cdot \mathrm{Cov}\left(\bar{R} - R_f, \frac{1}{S}\right) \tag{20}$$

where $d = E(S) \cdot E(\frac{1}{S}) - 1$.

Similarly, equations (5) and (14) in this paper can be rewritten, respectively, as follows:

$$\mathrm{Cov}\left(\frac{\bar{R} - R_f}{\hat{\beta}}, \hat{\beta}\right) = (B^{+}C-1)\ (R_f - \mu) - \beta^{+} \cdot \mathrm{Cov}(\bar{R} - R_f, \frac{1}{\hat{\beta}}) \qquad (5')$$

and

$$\mathrm{Cov}\ (\hat{\alpha}, \hat{\beta}) = \mathrm{Cov}\ (\bar{R} - R_f, \hat{\beta}) - \frac{\bar{X}_M\ \sigma_\varepsilon^2}{\sum_{t=1}^{n} (X_{Mt} - \bar{X}_M)^2} . \qquad (14')$$

By observing equations (20), (5'), and (14'), it is obvious that the signs of Friend and Blume's empirical relationship in the whole period and the first subperiod are negative if the distribution of $(\bar{R} - R_f)$ is independent of S and $\hat{\beta}$ [as indicated in Section II and III]. However, in the ex-post sample study, the covariance terms on the right-hand side of (20), (5'), and (14'), [Cov $(\bar{R} - R_f, \frac{1}{S})$, Cov $(\bar{R} - R_f, \frac{1}{\hat{\beta}})$, and Cov $(\bar{R} - R_f, \hat{\beta})$], might not be trivial in the second subperiod. Thus, the second-subperiod relationship can be positive as a result of these non-zero convariance terms.

V. Conclusions

The empirical relationships between the one-parameter composite performance measures and their risk proxies have been derived and analyzed in detail in accordance with statistical distribution theory. These empirical relationships are generally affected by the sample size, the investment horizon and the market conditions associated with the sample period selected for empirical studies. Moreover, the conventional sample estimate of Treynor's measure is not an unbiased estimate of the ex-ante measure. The ranking based on the estimated Treynor's measure does not represent an unbiased ranking as implies by the ex-ante Treynor's measure. However, a large sample of historical observations and an appropriate investment horizon can generally be used to improve the usefulness of composite performance measures for both portfolio and mutual fund managements.

Finally, using the cumulative average residual technique may lead to a potential bias in testing the efficient market hypothesis if the estimated Jensen's measure is significantly correlated with its risk measure, the estimated beta coefficient.

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APPENDIX

(A) Fieller (1932) has obtained a marginal density function of a ratio of two normal random variables. The distribution function $\psi(\nu)$ of the estimated Treynor's measure can be obtained from Fieller's equation (24) by letting the correlation coefficient be zero:

$$\psi(\nu) = \frac{1}{\pi} \cdot \frac{\sigma_1 \sigma_2}{\sigma_2^2 + \nu^2\sigma_1^2} \cdot e^{-\frac{1}{2}[\frac{\beta^2}{\sigma_1^2} + \frac{(\mu - R_f)^2}{\sigma_2^2}]}$$

$$+ e^{-\frac{1}{2}\frac{[(\mu - R_f) - a\beta]^2}{\sigma_2^2 + \nu^2\sigma_1^2}} \times \frac{-\beta\sigma_2^2 - \nu\sigma_1^2(\mu - R_f)}{\pi(\sigma_2^2 + \nu^2\sigma_1^2)^{3/2}}$$

$$\times \int_0^h e^{-\frac{1}{2}\mu^2} d\mu , \qquad -\infty < \nu < \infty,$$

where $\nu = \dfrac{\bar{R} - R_f}{\hat{\beta}}$,

$$\sigma_1^2 = \text{Var}(\hat{\beta}) = \frac{\sigma_\varepsilon^2}{\sum_{t=1}^{n} (R_{Mt} - \bar{R}_M)^2} ,$$

and $\sigma_2^2 = \text{Var}(\bar{R} - R_f) = \frac{\sigma^2}{n}$ $(\sigma = \text{Var}(R_{it}))$.

(B) The derivation of $E(\frac{\bar{R} - R_f}{\hat{\beta}})$:

By independent property of $(\bar{R} - R_f)$ and truncated $\hat{\beta}$,

$$E(\frac{\bar{R} - R_f}{\hat{\beta}}) = E(\bar{R} - R_f)E(\frac{1}{\hat{\beta}}) = (\mu - R_f)\cdot E(\frac{1}{\hat{\beta}}). \qquad \text{(B-1)}$$

The last equality is obtained using the following lemma: if x_1 and x_2 are independent, then $E[g_1(x_1)\cdot g_2(x_2)] = E[g_1(x_1)].E[g_2(x_2)]$ where $g_1(x_1)$ and $g_2(x_2)$ are real-valued (measurable) functions of x_1 and x_2. In our case, we identify $\bar{R}$ and $1/\hat{\beta}$ as $g_1(x_1)$ and $g_2(x_2)$, respectively. See Roussas [1973, p.123, Lemma 3].

Thus, $E(1/\hat{\beta})$ is determined as follows.

$$E(\frac{1}{\hat{\beta}}) = \int_a^b \frac{1}{\hat{\beta}} \cdot \frac{1}{k\sqrt{2\pi}\sigma_{\hat{\beta}}} \cdot \exp[-\frac{1}{2}(\frac{\hat{\beta}-\beta}{\sigma_{\hat{\beta}}})^2]d\hat{\beta} \qquad \text{(B-2)}$$

Let $y = (\hat{\beta} - \beta)/\sigma_{\hat{\beta}}$. The Jacobian of the transformation is $|d\hat{\beta}/dy| = \sigma_{\hat{\beta}}$. Thus, equation (B-2) becomes

$$E(\frac{1}{\hat{\beta}}) = \frac{1}{k\sqrt{2\pi}} \int_{a'}^{b'} (\frac{1}{y\sigma_{\hat{\beta}} + \beta})e^{-y^2/2}\, dy, \qquad \text{(B-3)}$$

where $a' = (a - \beta)/\sigma_{\hat{\beta}}$ and $b' = (b - \beta)/\sigma_{\hat{\beta}}$. (Here, $a < \beta$.) To integrate the integrand in (B-3) it is necessary to express the function, $g(y) = 1/(y\sigma_{\hat{\beta}} + \beta)$, in terms of an infinite series. Since $1/(y\sigma_{\hat{\beta}} + \beta)$ is defined for all values of y in the interval (a', b'). Note that the point, $y = -\beta/\sigma_{\hat{\beta}}$, at which the function $g(y)$ is undefined is not in (a', b'). The Taylor's series expansion of $g(y)$ at a positive point, say $m^* = (a'+b')$, is

$$g(y) = \frac{1}{y\sigma_{\hat{\beta}} + \beta} = \frac{1}{(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} [-\frac{\sigma_{\hat{\beta}}(y-m^*)}{(m^*\sigma_{\hat{\beta}} + \beta)}]^i, \qquad \text{(B-4)}$$

where $a' \leq y \leq b'$.

It can be shown that $|\frac{-\sigma_{\hat{\beta}}(y-m^*)}{(m^*\sigma_{\hat{\beta}} + \beta)}| < 1$ as follows:

$$\frac{-\sigma_{\hat{\beta}}(y-m^*)}{m^*\sigma_{\hat{\beta}} + \beta} = \frac{-\sigma_{\hat{\beta}}y + m^*\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta} = \frac{(m^*\sigma_{\hat{\beta}} + \beta) - \hat{\beta}}{m^*\sigma_{\hat{\beta}} + \beta}, \quad \text{since } \sigma_{\hat{\beta}}y = \hat{\beta} - \beta < 1,$$

$$< 1$$

since $\hat{\beta} > 0$ and $-\hat{\beta} < 0$. (It is the truncated $\hat{\beta}$.)

Also,

$$\frac{-\sigma_{\hat{\beta}}(y-m^*)}{m^*\sigma_{\hat{\beta}} + \beta} > \frac{-\sigma_{\hat{\beta}}y}{m^*\sigma_{\hat{\beta}} + \beta} = \frac{-\hat{\beta}+\beta}{(a+b) - \beta}, \quad \text{since } m^*\sigma_{\hat{\beta}} + \beta = (a+b) - \beta > 0$$

$$> \frac{-\hat{\beta}}{(a+b) - \beta} > \frac{-\hat{\beta}}{(a+b)} > -1,$$

**b is chosen such that $m^* > 0$, i.e., $(a + b) > 2\beta$.

since β is assumed to be positive in the truncated case and $\hat{\beta}<(a+b)$.

This implies that

$$\left|\frac{-\sigma_{\hat{\beta}}(y - m^*)}{(m^*\sigma_{\hat{\beta}} + \beta)}\right| < 1 \quad \text{for} \quad a' \leq y \leq b'.$$

Since every term of the infinite series in (B-4) has an absolute magnitude less than one, the series converges uniformly and absolutely by advanced calculus. Then, equation (B-3) can be rewritten as

$$E(\frac{1}{\hat{\beta}}) = \frac{1}{k\sqrt{2\pi}} \int_{a'}^{b'} \frac{1}{(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} [-\frac{\sigma_{\hat{\beta}}(y - m^*)}{m^*\sigma_{\hat{\beta}} + \beta}]^i \cdot e^{-y^2/2}\, dy$$

$$= \frac{1}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} [(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta})^i \int_{a'}^{b'} \sum_{j=0}^{i} \binom{i}{j} y^j (-m^*)^{i-j} e^{-y^2/2} dy],$$

since the series converges uniformly.

Let $x = y^2$. Then $y = \pm\sqrt{x}$ and $dy/dx = \pm \frac{1}{2} x^{-1/2}$. Thus,

$$E(\frac{1}{\hat{\beta}}) = \frac{1}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} [(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta})^i \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{i-j} \cdot$$

$$\int_{a'^2}^{b'^2} x^{\frac{(j+1)}{2} - 1} e^{-x/2}\, dx$$

$$= \frac{1}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} [(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta})^i \cdot \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{i-j} 2^{(j+1)/2} \Gamma(\frac{j+1}{2}) m_j] \qquad (B-5)$$

where $m_j = \int_{a'^2}^{b'^2} \frac{x^{\frac{(j+1)}{2} - 1} e^{-x/2}}{\Gamma(\frac{j+1}{2}) 2^{(j+1)/2}}\, dx.$

Hence, substituting equation (B-5) into equation (B-1) leads to

$$E\left(\frac{\bar{R} - R_f}{\hat{\beta}}\right) = \left(\frac{\mu - R_f}{\beta}\right) \cdot e_\beta \tag{B-6}$$

where
$$e_\beta = \frac{\beta}{k\sqrt{2\pi}(m^*\sigma_{\hat{\beta}} + \beta)} \cdot \sum_{i=0}^{\infty} \left[\left(\frac{-\sigma_{\hat{\beta}}}{m^*\sigma_{\hat{\beta}} + \beta}\right)^i \sum_{j=0}^{i} \binom{i}{j} (-m^*)^{i-j} \cdot 2^{(j+1)/2}\, \Gamma\left(\frac{j+1}{2}\right) m_j \right]. \tag{B-7}$$