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The Effects of Transaction Costs and Different Borrowing and Lending Rates on the Option pricing Model

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Abstract

This paper solves a stochastic differential equation to demonstrate that the market imperfections of transaction costs and different borrowing and lending rates partially offset each other to yield a range of equilibrium prices for an option. The Black-Scholes model price is shown to be in the lower portion of, or entirely below, the equilibrium range. These observations are used to explain several of the mythical anomalies found in the option pricing literature.

The paper also points out that under some conditions there may be _no_ equilibrium option price. Instead there may be a bounded disequilibrium within which a single option will offer a risk free return above the Treasury bill rate, while _simultaneously_ permitting borrowing below the borrowing rate.

I. Introduction

The Black-Scholes model requires an investor to create a risk free hedge by taking a position in an option and the opposite position in the underlying stock. The stocks and options are held in proportions such that any price movement in the stock is perfectly offset by an opposite movement in the option. These proportions are readjusted continuously throughout the life of the hedge. The hedge is therefore risk free and yields the risk free rate.

If the hedge consists of a long position in common stock and a short position in options the hedge will require a net investment on which the investor will earn the risk free rate (an "investment hedge"). If the hedge consists of a long position in an option and a short position in the stock, the hedge supplies funds to the investor for which he pays interest (a "borrowing hedge").[1]

The original Black-Scholes (1972) option pricing model assumes zero transaction costs and implicitly assumes borrowing and lending at the risk free rate. Under these assumptions the option price appropriate to an investment hedge is equal to the option price appropriate to a borrowing hedge and this determines a unique equilibrium option price. This will be shown to be a special case of a more general model.

If transaction costs are ignored, the effects of different borrowing and lending rates are relatively obvious. The option price appropriate to an investment hedge is the traditional Black and Scholes price (and therefore earns the risk free rate) and the option price appropriate to a borrowing hedge can be calculated from the Black-Scholes model equation with the investor's borrowing rate substituted for the Treasury bill rate (the hedge therefore costs the borrowing rate). Obviously, the option price appropriate to a borrowing hedge will be greater than the option price appropriate to an investment hedge.

When transaction costs are considered, option prices must be adjusted so as to earn the investor the risk free rate on an investment hedge or cost him the borrowing rate on a borrowing hedge net of transaction costs.

In essence, the profit from an investment hedge comes from the declining time premium of an option that has been short sold (or "written"). The additional revenue needed to pay transaction costs can be generated by raising the initial option price to provide for a greater decline in time premium and a greater profit for the hedge's short position in options.

In essence, the cost of a borrowing hedge results from the deteriorating time premium of the hedge's long position in options. To provide for borrowing at the market rate after transaction costs the initial option price must be reduced so as to provide for less deterioration in time premium and more funds available to pay transaction costs.

The reader will note that if transaction costs and the borrowing and lending rate spread are of precisely the right size, the transaction cost adjusted option price for an investment hedge can equal the transaction cost adjusted option price for a borrowing hedge. In this case the market imperfection of different borrowing and lending rates and the market imperfection of positive transaction costs cancel to produce a unique equilibrium option price.

This precise cancellation is, of course, rare. Usually one of the two imperfections dominates yielding a bounded range of option prices. Section IV and the conclusion of the paper point out the potentially bizarre nature of some of these situations.

Section I of the paper presents an introduction and a general description of the problem. Section II modifies the Black-Scholes option pricing model to include rebalancing transaction costs. Section III suggests ways in which some investors may reduce the cost of acquiring and terminating the hedge position. Section IV calculates option prices with transaction costs and different borrowing and lending rates. Section V presents a conclusion and summary.

II. Transaction Costs of Hedge Rebalancing

The Black-Scholes model assumes that the price of an option, $w(x,t)$, is a function of stock price x, and time, t. In this case, the equity in an investing hedge of one stock share long and $n = 1/w_1$ options short is $x - wn$ (where the subscript refers to the partial derivative of $w(x,t)$ with respect to its first argument). The equity change in a short interval Δt can be expressed as:

$$\Delta(x - w/w_1) = \Delta x - \Delta(wn) \tag{1}$$

$$= \Delta x - [w(x+\Delta x,\ t+\Delta t)n(x+\Delta x,\ t+\Delta t) - w(x,t)n(x,t)]$$

which can be expanded to:

$$= \Delta x - \{[w+w_1\Delta x+\tfrac{1}{2}w_{11}(\Delta x)^2 + w_2\Delta t][n+n_1\Delta x+\tfrac{1}{2}n_{11}(\Delta x)^2+w_2\Delta t] - wn\}$$

Substituting $v^2 = (\Delta x/x)^2/\Delta t$, and keeping only the terms of Δx and Δt, equation (1) becomes:

$$\begin{aligned}\Delta(x-w/w_1) &= (-\tfrac{1}{2}v^2x^2w_{11}-w_2)n\Delta t + \\ &(-\Delta xwn_1/\Delta t-wn_2-\tfrac{1}{2}v^2x^2wn_{11}-v^2x^2w_1n_1)\Delta t\end{aligned} \tag{2}$$

The first term on the right side of equation (2) is the part of the equity change which yields the risk-free rate as derived in the Black and Schole model:

$$\Delta x - \Delta w/w_1 = \Delta x - [w(x+\Delta x, t+\Delta t) - w(x,t)]/w_1$$

$$= \Delta x - [w+w_1\Delta x+\tfrac{1}{2}w_{11}(\Delta x)^2 + w_2\Delta t-w]/w_1$$

$$= (-\tfrac{1}{2}w_{11}x^2v^2\Delta t-w_2\Delta t)/w_1$$

$$= (-\tfrac{1}{2}v^2x^2w_{11}-w_2)n\Delta t \qquad (3)$$

The second term of equation (2) is the extra capital required to maintain the hedge position.[2] The extra capital is composed of the change in the number of options Δn at the changed price $-w(x+\Delta x,\ t+\Delta t)$ i.e., $-w(x+\Delta x,\ t+\Delta t)\Delta n$

$$= -[w+w_1\Delta x+\tfrac{1}{2}w_{11}(\Delta x)^2+w_2\Delta t]\ [n+n_1\Delta x+\tfrac{1}{2}n_{11}(\Delta x)^2+n_2\Delta t-n]$$

$$= (-\Delta x w n_1/\Delta t-wn_2-\tfrac{1}{2}v^2x^2wn_{11}-v^2x^2w_1n_1)\Delta t \qquad (4)$$

Accompanying the extra capital, the transaction cost by which the equity change should be reduced is $\alpha\ |-w(x+\Delta x,t+\Delta t)\Delta n\ |$, where α is the transaction cost rate for options. Therefore, the equity change $\Delta x - \Delta w/w_1$ yields the risk-free investing rate r^i on the equity $x - w/w_1$ after the cost, $\alpha\ |-w(x+\Delta x,t+\Delta t)\Delta n\ |$, has been deducted. Therefore:

$$\Delta x - \Delta w/w_1 - \alpha|-w(x+\Delta x,t+\Delta t)\Delta n| = (x-w/w_1)r^i\Delta t \qquad (5)$$

Substituting equations (3) and (4) into equation (5) and replacing n, n_1, n_{11} and n_2 by $1/w_1$, $-w_{11}/w_1^2$, $(2w_{11}^2-w_1w_{111})/w_1^3$ and $-w_{12}/w_1^2$ respectively, yields:

$$r^i w - r^i x w_1 - \frac{1}{2} v^2 x^2 w_{11} - w_2 = \alpha g \tag{6}$$

where

$$g = E(w_1 \mid - \delta x w n_1/\Delta t - w n_2 - \frac{1}{2} v^2 x^2 w n_{11} - v^2 x^2 w_1 n_1 \mid)$$

$$= E \mid \Delta x w w_{11}/(w_1 \Delta t) + w w_{12}/w_1 + v^2 x^2 (-w w_{11}{}^2/w_1{}^2 + \tfrac{1}{2} w w_{111}/w_1 + w_{11}) \mid \tag{7}$$

Equation (6) is the differential equation for the option price from an investing hedge. Similarly, one could be derived from a borrowing hedge, i.e.,

$$r^b w - r^b x w_1 - \frac{1}{2} v^2 x^2 w_{11} - w_2 = -\alpha g, \tag{8}$$

where r^b is the appropriate borrowing rate.

Equation (7) makes it clear that as $\Delta t \to 0$, g approaches infinity, yielding the horrifying (though not entirely unexpected) result that with continuous rebalancing, transaction costs will be infinite for any positive transaction cost rate.

Fortunately, this result is more of a mathematical artifact than a practical problem. As the tables presented in Section IV demonstrate, short finite interval rebalancing (e.g., daily) results in very reasonable transaction costs.[3,4]

To solve for equation (6), it is reasonable to assume the solution w^i differs only slightly from the Black and Scholes model solution because the transaction cost rate, α, is quite small. Let w^0 be the Black and Scholes solution and $\alpha w'$ the correction, then

$$w^i = w^0 + \alpha w' \quad (w^0 >> \alpha w') \tag{9}$$

Replacing w by w^i in equation (6):

$$r^i w' - r^i x w_1' - \frac{1}{2}v^2 x^2 w_{11}' - w_2' = g \tag{10}$$

Since $w^i \approx w^0$, g can be approximated by

$$g \approx E \mid \Delta x w^0 w_{11}^0/(w_1^0 \Delta t) + w^0 w_{12}^0/w_1^0 + v^2 x^2(-w^0 w_{11}^{0\ 2}/w_1^{02} + \frac{1}{2} w^0 w_{111}^0/w_1^0 + w_{11}^0) \mid \tag{11}$$

Because the hedge will not change when $t = t^*$, $x \to \infty$ or $x = 0$, there will be no transactions costs. Therefore, $w^i = w^0$ and

$$w'(x,t^*) = w'(\infty,t) = w'(0,t) = 0 \tag{12}$$

The same substitution for w' used in the Black and Scholes model yields:

$$w'(x,t) = e^{r^i(t^*-t)} Z(u,s) \tag{13}$$

where:

$$u = \frac{2}{v^2}(r^i - \frac{v^2}{2}) \; [\ell n \frac{x}{c} - (r^i - \frac{v^2}{2}) \; (t-t^*)], \tag{14}$$

$$s = - \frac{2}{v^2}(r^i - \frac{v^2}{2})^2 \; (t-t^*) \tag{15}$$

and c is the exercise price of the option.

Equation (15) implies:

$$t = t^* - sv^2/(2(r^i - \frac{v^2}{2})^2)$$

Equation (14) implies:

$$x = c \exp \{(u-s)/[r^i - \frac{v^2}{2})]\}$$

Substituting x and w' into equation (10) and (11):

$$Z_2 - Z_{11} = h \tag{16}$$

with h = h(u,s) being the function g after multiplying by $e^{r^i(t^*-t)}v^2/(2(r^i - \frac{v^2}{2})^2)$ and substituting in x and t. The boundary conditions, equation (12), become:

$$Z\ (u,0) = Z\ (\infty,0) = Z(-\infty,0) = 0$$

The solution for equation (16) is given by Butkov (1968, pp. 525-526):

$$Z(u,s) = \int_0^s \int_{-\infty}^{\infty} \frac{e^{-(u-u')^2/4(s-s')}}{(4\pi(s-s'))^{1/2}}\ h\ (u',s')\ du'ds' \tag{17}$$

Substituting (17) into (13) and then (9), yields the solution w^i. Similarly, w^b can be solved for by using interest rate r^b and replacing equations (9) and (10) with:

$$w^b = w^0 + \alpha w' \text{ and} \tag{18}$$

$$r^b w' - r^b\ x\ w_1' - \tfrac{1}{2}v^2x^2w_{11}' - w_2' = -g \tag{19}$$

III. Initiation and Termination Costs

In option hedging, the lowest cost market participant will usually be the investor for whom acquiring (or short selling) the common stock portion of the hedge is a by-product of other activities.

An investment hedge consists of a long position in common stock and a short position in options. An investor who owns but wishes to sell the common stock for which the hedge is to be written can form the hedge without incurring a marginal cost for buying or selling the stock. In this case, instead of selling the stock immediately the stock is retained and the usual hedged position of w/w_1 worth of options are written for each share of stock held. This hedge is held until either:

1) The option price drops below 1/16 point at which time C.B.O.E. trading in the option is halted. A hedge can no longer be formed and the common stock is (finally) sold.

2) The option is in the money at expiration and the stock is called away. In this case transaction costs are calculated as if the stock were sold at the exercize price, C rather than the actual stock price X*.

As soon as the hedge is formed, stock price movements are neutralized and the stock used in the hedge is <u>in effect</u> sold. The initial cash flows (including the savings from not actually selling the stock) are:

$$\alpha w/w_1 - \alpha_x x$$

where α_x is the transaction cost rate for common stock transactions. When the stock is finally (actually) sold and the hedge position closed out the flows are:

$$\alpha_x \operatorname{Min}(x^y;c) = \alpha_x (x^y - w^y) \tag{20}$$

where the superscript y indicates values at the time the hedge is closed out (not necessarily at expiration; see contingency 1 above).[5]

In equilibrium (ignoring dividends) the discounted present value of the expected value of x^y is x. Therefore, when the hedge is constructed the risk adjusted present value of the total cash outflows are:

$$\alpha w/w_1 - \alpha_x e^{-K\Delta t} E(w^y) \tag{21}$$

where K is the discount rate appropriate to the option and Δt is the time until the option hedge is closed out. Equation (21) shows that the investor has, in effect, paid $\alpha(w/w_1)$ plus continuous rebalancing costs to save the transaction costs on the amount by which x* might exceed c at expiration (i.e., w*). Under reasonable assumptions this will involve a net outflow, but the costs will be small relative to the size of the hedge. Moreover, these costs relate to more than one option position. When the hedge begins it consists of $1/w_1$ options and if there are transaction cost savings at the dissolution of the hedge it is because the option is in the money at expiration and therefore has a hedge ratio of one (i.e., equation (21) then relates to only one option).

Similarly, a borrowing hedge can be formed without the marginal cost of stock sales and purchases. In this case an investor who wishes to purchase a stock does not purchase it immediately, instead he buys the usual hedged position of w/w_1 options and continuously rebalances as if he actually held the stock. Eventually one of two things happens:

1) The option price drops below 1/16 point at which time trading is halted. A hedge can no longer be formed and the stock is finally, actually, purchased.

2) The option is in the money at expiration at which time the option is exercised and the stock is (finally) acquired.

Since transaction costs are the same for buying and selling, equation (21) will also describe the investor's costs for a borrowing hedge. The investor has, in effect, paid $\alpha(w/w_1)$ per share plus continuous rebalancing

TABLE 1

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

ANNUAL STD. DEV. OF C.S. RETURNS = .15 — EXERCISE PRICE = $40

Stock Price	Time To Exp (Months)	Investing at 12%: B&S Price (1)	Rebal Adj (2)	Init -End Adj (3)	Net Adj Price (4)	Net Price Spread (5)	Borrowing at 15%: Net Adj Price (6)	Init -End Adj (7)	Rebal Adj (8)	B&S Price (9)
28	1	.000	.000	.000	.000	.000	.000	.000	.000	.000
	5	.001	.000	.000	.001	-.000	.001	-.000	-.000	.000
	9	.031	.014	.001	.046	-.020	.026	-.002	-.020	.048
34	1	.000	.000	.000	.000	-.000	.000	-.000	-.000	.000
	5	.211	.048	.007	.266	-.062	.204	-.009	-.058	.270
	9	.825	.173	.020	1.019	-.174	.845	-.025	-.196	1.066
40	1	.905	.030	.014	.949	-.031	.918	-.015	-.030	.963
	5	2.679	.150	.035	2.863	-.042	2.821	-.040	-.150	3.011
	9	4.160	.260	.049	4.470	.016	4.485	-.058	-.237	4.780
46	1	6.398	.000	.048	6.446	.000	6.447	-.050	-.000	6.497
	5	7.989	.045	.063	8.097	.246	8.343	-.071	-.036	8.450
	9	9.525	.106	.078	9.709	.429	10.138	-.091	-.080	10.308
52	1	12.398	.000	.093	12.491	-.089	12.402	-.095	-.000	12.497
	5	13.952	.003	.105	14.059	.249	14.308	-.114	-.002	14.424
	9	15.449	.018	.118	15.584	.532	16.116	-.132	-.011	16.259

TABLE 2

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

ANNUAL STD. DEV. OF C.S. RETURNS = .30 — EXERCISE PRICE = $40

		Lending at 12%					Borrowing at 15%			
Stock Price	Time To Exp (Months)	B&S Price (1)	Rebal Adj (2)	Init -End Adj (3)	Net Adj Price (4)	Net Price Spread (5)	Net Adj Price (6)	Init -End Adj (7)	Rebal Adj (8)	B&S Price (9)
28	1	.000	.000	.000	.000	-.000	.000	-.000	-.000	.000
	5	.152	.040	.005	.197	-.072	.125	-.006	-.044	.176
	9	.655	.165	.020	.840	-.285	.554	-.022	-.186	.763
34	1	.050	.006	.002	.058	-.013	.045	-.002	-.007	.054
	5	1.208	.171	.030	1.409	-.289	1.120	-.032	-.179	1.331
	9	2.508	.369	.051	2.928	-.572	2.355	-.056	-.388	2.799
40	1	1.583	.058	.026	1.667	-.117	1.550	-.027	-.058	1.636
	5	4.085	.276	.060	4.421	-.401	4.019	-.064	-.274	4.358
	9	5.911	.473	.081	6.465	-.620	5.845	-.088	-.468	6.401
46	1	6.461	.024	.053	6.538	-.060	6.478	-.054	-.023	6.555
	5	8.650	.240	.086	8.976	-.259	8.718	-.092	-.228	9.038
	9	10.538	.431	.108	11.078	-.427	10.650	-.118	-.411	11.179
52	1	12.399	.001	.093	12.493	-.092	12.401	-.095	-.001	12.498
	5	14.144	.137	.117	14.398	-.058	14.340	-.124	-.125	14.588
	9	15.897	.313	.138	16.349	-.160	16.189	-.150	-.289	16.627

TABLE 3

The Effects of Transaction Costs and Different Borrowing and Lending Rates on Option Prices

ANNUAL STD. DEV. OF C.S. RETURNS = .45 EXERCISE PRICE = $40

		Investing at 12%					Borrowing at 15%			
Stock Price	Time To Exp (Months)	B&S Price (1)	Rebal Adj (2)	Init -End Adj (3)	Net Adj Price (4)	Net Price Spread (5)	Net Adj Price (6)	Init -End Adj (7)	Rebal Adj (8)	B&S Price (9)
28	1	.005	.001	.000	.006	-.002	.004	-.000	-.001	.005
	5	.705	.134	.021	.860	-.262	.598	-.023	-.141	.761
	9	1.810	.352	.044	2.206	-.650	1.556	-.048	-.367	1.971
34	1	.282	.027	.009	.318	-.062	.255	-.009	-.028	.292
	5	2.441	.289	.052	2.782	-.547	2.235	-.055	-.296	2.585
	9	4.265	.554	.078	4.896	-.981	3.916	-.083	-.561	4.559
40	1	2.266	.085	.038	2.390	-.198	2.192	-.039	-.086	2.317
	5	5.548	.396	.083	6.027	-.715	5.312	-.087	-.395	5.793
	9	7.790	.673	.110	8.572	-1.136	7.437	-.116	-.666	8.218
46	1	6.740	.067	.064	6.870	-.176	6.694	-.065	-.066	6.825
	5	9.818	.403	.112	10.333	-.694	9.639	-.116	-.394	10.149
	9	12.157	.690	.139	12.986	-1.108	11.878	-.147	-.672	12.697
52	1	12.436	.021	.096	12.554	-.139	12.415	-.098	-.020	12.533
	5	14.875	.339	.140	15.354	-.559	14.796	-.146	-.325	15.267
	9	17.117	.635	.169	17.921	-.962	16.959	-.178	-.605	17.743

costs to postpone the cost of acquiring the stock and save $\alpha_x w^*$ (per share) when the stock is finally acquired.

Since the continuously rebalanced option hedge position mimics every price movement of the underlying stock, the investor has, in effect bought the stock immediately without paying for the stock until the option expires or becomes worthless. This procedure is therefore a substitute for margin borrowing but without a margin requirement (or collateral in the usual sense of the word).[6]

IV. The Combined Effects of Transaction Costs and Different Borrowing and Lending Rates

The effects of transaction costs and different borrowing and lending rates are presented in Tables 1, 2, and 3. Column (1) is the Black-Scholes option price calculated under the assumptions specified in the table. Column (9) is the value the Black-Scholes model gives if the specified borrowing rate is substituted for the risk free rate. The common stock standard deviations listed in the tables roughly correspond to the 10th, 50th and 90th deciles of the standard deviations observed by Whaley (1982).

Columns (2) and (8) ("Rebal Adj") are the difference between the traditional B&S option price and the daily (i.e, $\Delta t = 1/260$) rebalancing transaction cost price derived in Section II. In addition to the assumptions specified in the table, one way transaction costs for options are assumed to be 2% and the underlying stock's expected return is 17% per year.

<u>INSERT TABLES 1-3 ABOUT HERE</u>

Columns (3) and (7) ("Init-End Adj") are estimates of the adjustment to the option price required to cover the cost of acquiring the hedge and finally liquidating it. These costs are based on the trading techniques presented in the previous section and embodied in equation (21). Common stock transaction costs (α_x) are assumed to be 1.25% and option transaction costs (α) are assumed to be 2%.[7] The expected value of the future value of the option is calculated from Sprenkle's equation (see Smith, 1976, page 17) and this value is discounted back to the present using a discount rate, K, derived from the CAPM under the assumptions that the market return is 17% per year, the risk free rate is 12%, the beta of the underlying stock is one and the beta of the option (as pointed out by Black and Scholes, 1972) is:[8]

$$\beta_w = xw_1\beta_x/w$$

Columns (4) and (6) ("Net Adj Price") are Black & Scholes option prices (columns (1) and (7)) with both types of transaction costs added (for the investment hedge, columns (2) and (3)) or subtracted (for the borrowing hedge, columns (7) and (8)). Columns (4) and (6) therefore show the prices the options must sell for to net the specified borrowing and lending rates after transaction costs.[9] Needless to say, these transaction costs would be different for different sets of assumptions. The costs presented are illustrative and can be helpful in understanding the nature of the phenomena. The reader is encouraged to analyse the effects of his own assumptions.[10]

Column (5) ("Net Price Spread") is the result of subtracting column (4) from column (6). It should be interpreted as follows:

1) Negative values indicate that the option price is the bounded range between the prices specified in columns (4) and (6). For option prices within this range neither investment hedges nor borrowing hedges are particularly attractive. The reader will note that the highest option price which produces an attractive borrowing hedge (Column (6), Net Adj Price - Borrowing) is frequently higher than the traditional Black-Scholes price (Column (1)). When this occurs, the Black-Scholes price isn't an equilibrium value. If an option sells for the Black-Scholes price, excess profits can be made by forming a borrowing hedge and (in effect) borrowing at below the market rate.

2) A zero value indicates a unique equilibrium option price (i.e., the value shown in both column (6) and column (4)). This occurs when transaction cost effects and borrowing and lending effects precisely cancel. This unique option price is <u>never</u> the Black-Scholes price except in the trivial case where all prices are zero.

3) A positive value indicates that the option hedge can be viewed as a financial intermediary with a lower spread than traditional intermediaries. If the option sells at a price between the prices specified in columns (4) and (6) the option hedge is <u>simultaneously</u> a higher return risk free investment than treasury bills and a lower cost source of funds than traditional borrowing. In this case, there is <u>no</u> price to which the option can adjust which will eliminate excess profits. For example, if the option price were to drop low enough for the investment hedge to no longer be attractive, this would only make a borrowing hedge even <u>better</u>. It may be that the only thing that prevents all short term borrowing and lending from being sucked into these financial "black holes" is the limited number of investors in the special transaction cost situations described in the previous section.

   This permanent disequilibrium is bounded between the column (6) and column (4) prices. If the option price is above the column (6) price the option is <u>not</u> attractive as a part of a borrowing hedge but a short position in the option will be <u>very</u> desirable as part of an investment hedge. This unbalanced selling pressure should drive the option price below the column (6) price at which time the option is desirable as <u>both</u> an investment hedge <u>and</u> a borrowing hedge. This presumably results in a better balance between supply and demand for the option.

   Similarly, if the option were to sell below the column (4) price it would be very attractive as a part of a borrowing hedge but there would be no interest in forming investment

> hedges. The resulting net buying pressure should push the option price back above the column (4) value.
>
> Therefore, when the column (5) value is positive, it indicates a bizarre form of bounded disequilibrium.[11]

Transaction costs and different borrowing and lending rates may help to explain some empirical anomalies. Specifically:

Some empirical findings contradict each other. For example, Black (1976) vs. Macbeth and Merville (1979) on the direction of the bias for in the money, relative to out of the money options. This paper shows that options usually have a range of equilibrium (or bounded disequilibrium) values. It is therefore not surprising that studies which assume unique equilibrium values sometimes contradict each other.

Merton (1976) says that practitioners believe that the B-S model underprices both in the money and out of the money options. Latané and Rendleman (1976) conclude that "... the preponderance of evidence would be toward options being over priced." (i.e., the B-S price is too low). Tables 1, 2 and 3 show that when the transaction cost/interest rate effect is considered, the B-S price (column (1)) is in the lower portion or entirely below the equilibrium range of option values. Empirical tests *should* show that the B-S model underprices options. It does.

Although empirical results conflict, the more modern, sophisticated and exhaustive studies seem to show that the B-S model underprices in the money options relative to out of the money options (see Macbeth and Merville, 1979) and underprices options on low risk stocks relative to high risk stocks (see Whaley, 1982).

This is the *average* effect a combination of transaction costs and divergent interest rates should produce. If the effect of different borrowing and lending rates is large relative to the size of transaction costs (Tables 1, 2 and 3 and others calculated by the authors suggest this is the case for options which are in the money and/or written on low variance stocks) the B-S price will be near the bottom of, or below the range of equilibrium prices. Therefore, the B-S model will be found to underprice these options. On the other hand, options for which the interest rate effect is small relative to the transaction cost effect (out of the money options and options on high variance stocks), the B-S price will be near the center of the equilibrium range and such options may appear overpriced for one sample and underpriced for another; thus explaining the conflicting results in the literature. However, for these options the Black-Scholes price is near the middle of the equilibrium range so, on average (and presumably for large sample sizes), the B-S price may be a relatively unbiased description. Therefore, the transaction cost/interest rate model may help to explain Macbeth and Mervilles' (1979) and Whaley's (1982) observed biases; but the transaction cost/interest rate model suggests that the range of biases is more likely to extend from the (roughly) correctly priced to the underpriced rather than from the overpriced to the underpriced; a distinction their weighted implied standard deviation techniques would have difficulty detecting.[12]

V. Conclusion

When transaction costs and different borrowing and lending rates are taken into consideration, options (in general) take on a range of equilibrium values. The traditional Black-Scholes price is in the bottom portion of, or entirely below, this range. These observations are used to explain empirical anomalies found in the option pricing literature.

The paper also makes the startling observation that for in the money options on low variance stocks, there may be no equilibrium option price. Any price these options might assume will offer a risk free investment at above the risk free rate or borrowing at below the market rate or both simultaneously. The option exchange becomes society's lowest spread financial intermediary. All short term borrowing and lending might be sucked into these financial "black holes"[13] were it not for the special nature of the situations needed to produce low enough transaction costs.

Appendix A

The Implications of Hedge Rebalancing Using Adjustments to the Option Position

Throughout this paper the authors assume that rebalancing is done by adjusting the option portion of the hedge. This is generally the cheapest way to rebalance because option rebalancing involves smaller dollar amounts than rebalancing with common stock. This cost advantage is partially offset by the fact that the average bid-ask spread in the options market is greater than in the stock market (see Phillips and Smith, 1980).

The hedge acquisition and dissolution techniques described in the text assume that the hedge contains the same number of shares of stock at the beginning and end of the life of the hedge. Therefore, equation (24) will only (usually) be an accurate description of costs if all rebalancing is done with options (thus leaving the number of shares in the hedge unchanged throughout the life of the hedge).

Needless to say, an investor should not rebalance by buying an overpriced option or selling an underpriced option. Therefore, the assumption that all rebalancing is done with options is unrealistic.

However, one suspects that the advantage of being able to rebalance with options when they are favorably priced and avoid them (with stock rebalancing) when they are unfavorably priced more than offsets the additional rebalancing cost of common stock rebalancing and the additional hedge dissolution cost which may result from acquiring an unwanted common stock position to liquidate at the end of the life of the hedge. Moreover, common stock rebalancing can also result in the acquisition of part of the desired stock position prior to dissolution thus reducing costs below those assumed in equation (24).

Moreover, footnote 10 shows how some investors can reduce transaction costs to a level generally below those presented in this paper. Finally, the reader may feel that the option rebalancing assumption is unrealistic because it is not possible to trade options in odd lots. The authors suggest that this is not a real problem because, if the investor's hedge is so small that rebalancing involves trades of less than several thousand dollars each, transaction costs will destroy the investor no matter how he rebalances.

Appendix B

A Demonstration of the Validity of the Proposed Solution to Equation (9).

The authors present this example in hopes of convincing the reader that the proposed solution to equation (6) is correct. In order to provide a simple example, the authors have chosen parameters which are realistic but relatively easy to calculate:

$$x = \$50$$

$$c = \$50$$

$$v^2 = .25$$

$$(t^* - t) = .5 \text{ (years)}$$

$$r = .10$$

$$\delta = .15$$

$$\alpha = .01$$

The g function (equation (11)) on the right side of equation (6) is the dollar amount of rebalancing required. It can be calculated from Black & Sholes pricing theory based on the parameters listed above. This yields:

$$w = 8.1316$$

$$d_1 = (\ln \frac{x}{c} + (r + .5v^2)(t^* - t))/(v^2(t^* - t))^{1/2} = .31820$$

$$w_1 = N(d_1) = .62483$$

For simplicity define

$$m = d_1/(2)^{1/2} = .225$$

w_{11} can then be expressed as:

$$w_{11} = e^{-m^2}/(xv(2\pi(t^* - t))^{1/2})$$

$$= .02145$$

Also:

$$w_{12} = ((\ln \frac{x}{c})/(t^* - t) - r - \frac{v^2}{2})(e^{-m^2}))/(2v(2\pi(t^* - t))^{1/2})$$

$$= -.12068$$

$$w_{111} = -e^{-m^2}(v/(2(t^* - t))^{1/2} + m/(t^* - t))/(v^2x^2\pi^{1/2})$$

$$= -.00081523$$

Substituting these values into equation (11) of the paper and assuming daily rebalancing (i.e., $\Delta t = 1/260$) yields:

$$g = 89.82$$

(the approximation listed in footnote 4 yields 89.76).

The right side of equation (6) is therefore:

$$\alpha g = .8982$$

The left side of equation (6) includes partial derivatives of the daily rebalancing transaction cost option price derived in this paper. These derivatives must be approximated by taking small interval values about the $50 stock price and the .5 year time to expiration:

Parameter	Estimation Interval	Estimated Value
w	----	8.4503
w_1	\$50 $\pm$ \$.50	.63184
w_{11}	\$50 $\pm$ \$2.00	.02033
w_2	.5 year $\pm$ .005 year	-9.5952

The width of the interval (column 2 above) used to approximate each partial derivative is a function of the 5 to 6 significant digit accuracy of the option price, w, as calculated from the numerical methods solution to equation (17).

When the estimated values from column 3 above are substituted into the left side of equation (6) they yield .9278. Considering the inherent inaccuracy of small interval approximations, this seems to be a good approximation of the value previously calculated for the right side of equation (6) (i.e., they differ by less than 4%).

Footnotes

[1]Thorpe (1973) demonstrated that option hedges can be sources of funds despite restrictions on short selling.

[2]These results are derived under the assumption that rebalancing is done by buying or selling options (rather than stock). See Appendix A for a discussion of the implications of this assumption.

[3]Discrete rebalancing interval applications of the continuous time option pricing model seem to pose no great problem. Boyle and Emanuel (1980) have demonstrated that the risk created by short interval rebalancing is uncorrelated with the market; and Rubenstein (1976) and Brennan (1979) have demonstrated that discrete interval applications of the Black-Scholes model will be valid under assumptions of constant proportional risk aversion and a bivariate lognormal distribution between market and underlying asset returns.

[4]The RHS of (7) is evaluated by separating the term inside the absolute value into a positive lognormal distribution truncated at zero and a negative lognormal distribution truncated at zero. Sprenkle's formula (see Smith (1976), pp. 17) is used to evaluate the expectation of each truncated distribution and the absolute values of these expectations are then added.

For rebalancing intervals of one day or less and common stock standard deviations of .3 (annual) or more, the procedure can be greatly simplified. In this case, the first term inside the absolute value (which represents short term stock fluctuations) will be much greater than the other terms (which represent longer term shifts in hedge ratio).

g can then be approximated by applying the formula for a full wave linear detector:

$$g \approx E \mid \Delta x/x \mid xw w_{11}/(w_1 dt) \doteq (2\sigma^2 dt/\pi)^{1/2} \, xww_{11}/w_1 dt$$

[5]When an option is exercized the commission is based on the exercise price not the stock price. Therefore the investor pays α_x Min$(x^y;c)$ when the hedge is terminated.

[6]One problem with the Black Scholes model is its dependence on the assumption that short positions in common stock are an immediate source of funds. Thorpe (1973) has argued that if an investor currently owns the stock for which a hedge is to be created, selling the stock is equivalent to short selling and is a source of funds. The procedure for forming a minimum transaction cost borrowing hedge described in this paper is another way in which a borrowing hedge can be a source of funds.

[7]The reader is encouraged to make his own transaction cost assumptions. The rebalancing adjustment is approximately proportional to the transaction cost rate, α throughout the relevant range of values and the initiation and ending transaction costs can be approximated from equations (21) and (22).

The transaction costs used to create Tables 1, 2 and 3 were chosen to illustrate the lower end of the reasonable range of costs. For example, the 2% transaction cost rate for options can be interpreted as the rate for an investor who has 1.1% (one way) transaction costs, who is investing in an option series with an average bid-ask spread of

3% (about the 35th percentile of the option spreads observed by Phillips and Smith, 1980) for which 40% of the trades require no market maker involvement and a zero spread (Phillips and Roberts, 1979). Thus yielding a one way spread cost of 3% x .6 x .5 = .9% and an average total trading cost of 2%. Similarly, the 1.25% common stock transaction cost rate can be interpreted as 1% transaction costs plus a one way average spread of .25% (i.e., the total spread, including the probability of specialist involvement is .5%).

[8]Equation (22) describes an option's beta at a specific instant in time and is therefore only an approximation of the beta actually needed. Fortunately, beta is only used to calculate the discount rate to be applied to a small amount of money to be realized a short time in the future. Therefore, more precision is probably unnecessary.

[9]In effect Tables 1, 2 and 3 assume that the costs of initiating and terminating the option hedge are prepaid by adding or subtracting them from the initial option price. This change in initial option price will slightly alter rebalancing transaction costs and is <u>not</u> taken into consideration in the derivation of the continuous rebalancing model (equations (1) through (19)).

Given the generally small size of initiation and termination costs relative to the option price, and the approximate nature of other aspects of the option pricing model, this should be acceptable for most purposes.

[10]Some investors will be able to reduce transaction costs to levels below those assumed in this paper by rebalancing less frequently. For

a borrowing hedge using the acquisition and dissolution technique suggested in the paper, a failure to rebalance means that the equivalent number of shares being mimicked by the option position changes as the hedge ratio changes but is not immediately rebalanced (i.e., the investor may have the option equivalent of 96 shares rather than the 100 shares he originally intended, etc.). Some investors may not care about the exact number of common stock equivalents his option position equals. If not, he can save money by rebalancing only after the hedge ratio has changed substantially. He can then avoid the transaction costs resulting from all of the reversals of the hedge ratio occurring between rebalancing. Moreover, this strategy allows the investor to save money by rebalancing in larger amounts.

The same less frequent rebalancing strategy can be applied to the investment hedge strategy described in the paper except that in this case a failure to rebalance means that the investor will have a small long or short position in the stock he originally wished to sell (i.e., instead of having the equivalent of no stock, his equivalent position might fluctuate between long and short positions amounting to a few percent of his original holding.)

[11]The existence of these bounded disequilibrium situations is not particularly sensitive to the assumptions used to create Tables 1, 2 and 3. For example, the phenomenon does not disappear even if the transaction cost rate is doubled. Moreover, if the bid-ask spread is positively correlated to the risk of the option, the options exhibiting bounded disequilibrium should have unusually low transaction costs because of their low variances (i.e., they are deep in the money options on low variance stocks).

[12]The weighted implied standard deviation (WISD) technique explicitly (as in Macbeth and Merville, 1979) or implicitly (Whaley, 1982) assumes the average option is correctly priced. This technique is therefore excellent at detecting relative biases but ill suited to detecting biases effecting all options.

[13]No pun intended.

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