WORKS OF PROF. WM. H. BURR

PUBLISHED BY

JOHN WILEY & SONS, Inc.

Ancient and Modern Engineering and the Isthmian Canal.

xv+476 pages, 6 by 9, profusely illustrated, including many half-tones. Cloth, $3.50 net.

Elasticity and Resistance of Materials of Engineering.

For the use of Engineers and Students. Containing the latest engineering experience and tests. Seventh Edition, revised. 946 pages, 6 by 9. Cloth, $5.50 net.

Suspension Bridges—Arch Ribs and Cantilevers.

xi+417 pages, 6 by 9, 68 figures, and 6 full-page and folding plates. Cloth, $4.50 net.

BY PROF. BURR and DR. FALK

The Graphic Method by Influence Lines for Bridge and Roof Computations.

xi+253 pages, 6 by 9, 4 folding plates. Cloth, $3.00.

The Design and Construction of Metallic Bridges.

xiii+532 pages, 6 by 9, many figures in the text and 4 folding plates. Cloth, $5.00.

BY DR. MYRON S. FALK

PUBLISHED BY

MYRON C. CLARK PUBLISHING CO.,
355 DEARBORN ST., CHICAGO, ILL.

Cements, Mortars and Concretes—Their Physical Properties.

Containing the results of late investigations upon these materials. 176 pages. 6 by 9. Cloth, $2.50.

A view of the 400-ton testing-machine at the U. S. Arsenal, Watertown, Mass., immediately after compressing a timber column to failure.

(*Frontispiece.*)

THE

ELASTICITY AND RESISTANCE

OF THE

MATERIALS OF ENGINEERING.

BY

WM. H. BURR, C.E.,

PROFESSOR OF CIVIL ENGINEERING IN COLUMBIA UNIVERSITY IN THE CITY OF NEW YORK;
CONSULTING ENGINEER; MEMBER OF THE AMERICAN SOCIETY OF CIVIL ENGINEERS;
MEMBER OF THE INSTITUTION OF CIVIL ENGINEERS OF
GREAT BRITAIN.

SEVENTH EDITION, THOROUGHLY REVISED
TOTAL ISSUE, SEVEN THOUSAND

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED
1915

PREFACE TO SEVENTH EDITION.

The rapid development which has characterized all branches of engineering construction during the past decade carries with it corresponding advances in experimental and analytic work in that field of engineering science known as the Elasticity and Resistance of Materials. In the present edition of this book, prepared to meet the advancing requirements of the profession, it will be observed that much of the older matter has been canceled and displaced by many new topics now become of practical importance, so that new material constitutes probably not less than three-quarters of the volume. These new parts will readily be discovered by a glance at the contents. It may be well, however, to state that the treatment of reinforced concrete, the general analysis of which as a development of the common theory of flexure was first given in a prior edition of this book, has been extended to cover substantially all the principal features of that special field. The analysis given is general, but simple and free from the superfluous and labor-increasing accretions which, for some not obvious reasons, have found place in some of the commonly used formulæ.

Results of the most recent experimental investigations have been used for the requisite empirical data, so as to make the book a real work on the Elasticity and Resistance of the Materials of Engineering rather than a mere matter of applied mechanics.

W. H. B.

COLUMBIA UNIVERSITY,
Oct. 1, 1915.

CONTENTS.

PART I.

ANALYTICAL.

CHAPTER I.

ELEMENTARY THEORY OF ELASTICITY IN AMORPHOUS SOLID BODIES.

CHAPTER II.

FLEXURE.

CHAPTER III.

TORSION.

CHAPTER IV.

HOLLOW CYLINDERS AND SPHERES.

CHAPTER V.

RESILIENCE.

CHAPTER VI.

COMBINED STRESS CONDITIONS.

PART II.

TECHNICAL.

CHAPTER VII.

TENSION.

CHAPTER VIII.

COMPRESSION.

CHAPTER IX.

RIVETED JOINTS AND PIN CONNECTION.

CHAPTER X.

LONG COLUMNS.

CHAPTER XI.

SHEARING AND TORSION.

CHAPTER XII.

BENDING OR FLEXURE.

CHAPTER XIII.

CONCRETE-STEEL MEMBERS.

CHAPTER XIV.

ROLLED AND CAST-FLANGED BEAMS.

CHAPTER XV.

PLATE GIRDERS.

CHAPTER XVI.

MISCELLANEOUS SUBJECTS.

CHAPTER XVII.

THE FATIGUE OF METALS.

CHAPTER XVIII.

THE FLOW OF SOLIDS.

APPENDIX I.

ELEMENTS OF THEORY OF ELASTICITY IN AMORPHOUS SOLID BODIES.

CHAPTER I.

GENERAL EQUATIONS.

CHAPTER II.

THICK, HOLLOW CYLINDERS AND SPHERES, AND TORSION.

CHAPTER III.

THEORY OF FLEXURE.

APPENDIX II.

APPENDIX III.

These stresses and strains vary in character according to the method of application of the external forces. Each stress, however, is accompanied by its own characteristic strain and no other. Thus there are shearing stresses and shearing strains, tensile stresses and tensile strains, compressive stresses and compressive strains. Usually a number of different stresses with their corresponding strains are coexistent at any point in a body subjected to the action of external forces.

It is a matter of experience that strains always vary continuously and in the same direction with the corresponding stresses. Consequently the stresses are continuously increasing functions of the strains, and any stress may be represented by a series composed of the ascending powers (commencing with the first) of the strains multiplied by proper coefficients. When, as is usually the case, the displacements are very small, the terms of the series whose indices are greater than unity are exceedingly small compared with the first term, whose index is unity. Those terms may consequently be omitted without essentially changing the value of the expression. Hence follows what is ordinarily termed Hooke's law:

The ratio between stresses and corresponding strains, for a given material, is constant.

This law is susceptible of very simple algebraic representation. If a piece of material, whose normal cross-section is A, is subjected to either tensile or compressive stress, its length L will be changed by the amount ΔL. If P be the external force or loading which produces that deformation or change of length, the amount of force or stress, supposed to be uniformly distributed, acting on 1 square inch of normal cross-section of the piece, will be found by dividing the total force P by the area of cross-section A. This amount of uniformly distributed stress is called the "intensity of stress," and it is a most important quantity. In dealing with the effects of forces or stresses in all engineering work, the amount of such force or stress on a square unit of area, usually a square inch in American practice, and called the intensity, is often the main object sought, for it determines the question whether material is carrying too much or too little load, as well as many other related questions.

Again, the important consideration as to strain is the fractional change in length of the entire piece, and not the total change in length expressed in the unit adopted, ordinarily an inch. This fractional change of length is the same as the amount of actual change of each linear unit of the piece, as found by dividing ΔL by L. Inasmuch as that fraction expresses the amount of change in length for each unit, it is frequently called the rate of change of length or rate of deformation. Hooke's law is to the effect that the intensity of stress is proportional to the rate of strain, and its analytic expression may readily be written.

Let p represent the intensity of any stress and l the strain per unit of length, or, in other words, the rate of strain. If E is a constant coefficient, Hooke's law will be given by the following equation:

$$p=\frac{P}{A}=\frac{\Delta L}{L}E=El. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the intensity of stress varies from point to point of a body, Hooke's law may be expressed by the following differential equation:

$$\frac{dp}{dl}=E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If p and l are rectangular coordinates, eqs. (1) and (2) are evidently equations of a straight line passing through the origin of coordinates. It will hereafter be seen that the line under consideration is essentially straight for comparatively small strains in any case, and for some materials it has no straight portions.

Art. 2.—Coefficient or Modulus of Elasticity.

In general the coefficient E in eq. (1) of the preceding article is called the *coefficient of elasticity*, or, more usually, *modulus of elasticity*. The coefficient of elasticity varies both with the kind of material and kind of stress. It simply expresses *the ratio between the rates of stress and strain.*

The characteristic strain of a tensile stress is evidently an *increase* of the linear dimensions of the body in the direction of action of the external forces.

Let this increase per unit of length be represented by l, while p and E represent, respectively, the corresponding intensity and coefficient. Eq. (1) of the preceding article then becomes

$$p = El, \quad \text{or} \quad E = \frac{p}{l}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

E is then the coefficient of elasticity for tension.

The characteristic strain for a compressive stress is evidently a *decrease* in the linear dimensions of the body in the direction of action of the external forces. Let l_1 represent this decrease per unit of length, p_1 the intensity of compressive stress, and E_1 the corresponding coefficient. Hence

$$p_1 = E_1 l_1, \quad \text{or} \quad E_1 = \frac{p_1}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

E_1 consequently is the coefficient of elasticity for compression.

The characteristic strain for a shearing stress may be determined by considering the effect which it produces on the layers of the body parallel to its plane of action.

In Fig. 1 let $ABCD$ represent one face of a cube, another of whose faces is fixed along AD. If a shear acts in the face BC, whose plane is normal to the plane of the paper, all layers of the cube parallel to the plane of the shearing stress, i.e., BC, will slide over each other, so that the faces AB and DC will take the positions AE and DF. The amount of distortion or strain per unit of length will be represented by the angle $EAB=\phi$. If the strain is small, there may be written ϕ, $\sin\phi$, or $\tan\phi$ indifferently.

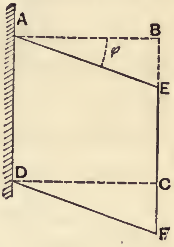


Fig. 1.

Representing, therefore, the intensity of shear, coefficient, and strain by S, G, and ϕ, respectively, eq. (1) of Art. 1 becomes

$$S=G\phi, \quad \text{or} \quad G=\frac{S}{\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It will be seen hereafter that there are certain limits of stress within which eqs. (1), (2), and (3) are essentially true, but beyond which they do not hold; this limit is called the *limit of elasticity*, and is not in general a well-defined point.

The line *Okghn* exhibited in Fig. 2 represents the actual strains in a piece of structural steel 1 inch in length with 1 square inch of cross-section. O is the origin of coordinates, and the loads per square inch, i.e., intensities of stresses, are shown by the vertical ordinates drawn parallel to OC from OD to the strain curve, while the strains per unit of length, that is, per inch, are laid off as horizontal ordinates of the curve parallel to OD. If Op' is the intensity of stress, p' corresponding to the point k of the strain curve, while Ol' is the resulting strain per unit of length, then $p'=El'$. Again, if g is at the upper limit of the straight portion of the curve for which the intensity of stress and rate of strain are p and l respectively, the relation between those two quantities is shown by eq. (1). Since E, also as shown by eq. (1), is equal to the quotient of p divided by l, Fig. 2 shows that it is equal to the tangent of the angle between OD and the straight portion Og of the strain curve, it being supposed that the rates of strain are laid down at their actual or natural sizes. If the strain line is curved, the first term of eq. (2) of Art. 1, the differential ratio, will represent the tangent of the angle between the curve and the horizontal axis OD in Fig. 2. The point g, being at the upper limit of constant proportionality between intensity of stress and rate of strain, is called the elastic limit, above which it is seen that the strains increase far more rapidly than the stresses until the point n is reached, where actual rupture takes place. The nearly horizontal portion of the curve between g and h and a little

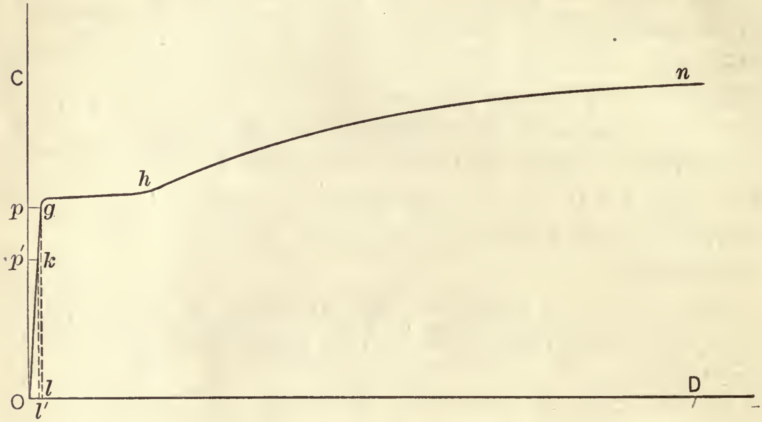


FIG. 2.

above g indicates the "yield point," an intensity of stress where the material is said first to "break down" or stretch rapidly under tensile stress without much increase of the latter.

Art. 3.—Direct Stresses of Tension and Compression.

The direct stresses of tension and compression always produce shearing stresses and strains on all planes in the interior of a body except those perpendicular and parallel to those direct stresses. If, in Fig. 1, a straight piece of material CD is subjected to the tensile stress induced by the forces P equal and opposite to each other, there will be pure tension only on all planes or sections of the piece at right angles to the direction of the forces P, such as HK. On all planes passing through the longitudinal axis of the piece there will be no stress whatever, if, as is supposed, the forces P are uniformly distributed over the sections of application DF and BC.

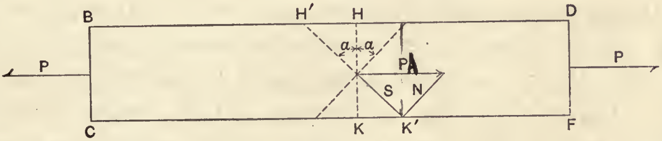


FIG. 1.

On every oblique plane or section in all parts of the piece as $H'K'$, supposed to be perpendicular to the plane of the diagram, there will be shear as well as direct stress of tension normal to it, the intensities of both the shear and the normal stress being dependent upon the angle α between HK and $H'K'$. The force P may be resolved by the triangle of forces into two components, one at right angles to $H'K'$, represented by N, and the other along or tangential to $H'K'$, represented by S. If

A represents the area of the normal section HK, the area of the oblique section $H'K'$ will be $A \sec \alpha$. The value of the normal stress N will be $N = P \cos \alpha$, but $S = P \sin \alpha$. The intensity of the normal tensile stress on $H'K'$ will be, therefore,

$$n = \frac{N}{A \sec \alpha} = \frac{P \cos \alpha}{A \sec \alpha} = p \cos^2 \alpha. \quad . \quad . \quad (1)$$

The intensity of shear on the same plane $H'K'$ will be

$$s = \frac{S}{A \sec \alpha} = \frac{P \sin \alpha}{A \sec \alpha} = p \sin \alpha \cos \alpha. \quad . \quad . \quad (2)$$

When the angle α is zero, s in eq. (2) becomes zero, while n in eq. (1) becomes equal to p, i.e., the intensity of direct tensile stress on the normal section. On the other hand, when the angle α has the value of 90°, both n and s become zero, i.e., there is no stress whatever on a longitudinal, axial plane.

Inasmuch as the angle α may have any value whatever from zero to 90° on either side of HK, it is clear that both shearing and normal tensile stresses will be found concurrently on every oblique plane in the piece. As has been observed in the preceding article, these shearing stresses induce the lateral strains under which the normal cross-sections of a piece subjected to pure tension decrease in area while they increase under the action of pure compression.

Eqs. (1) and (2) have been written on the assumption that the external forces P produce tension in the material, but precisely the same equations apply to the condition of pure compression, the only difference being that in the latter case the external forces P would be directed toward each other from the ends of the piece, instead of away from each other.

Art. 4.—Lateral Strains.

If a body, as indicated in Fig. 1, be subjected to tension, it has been shown in Art. 3 that all of its oblique cross-sections, such as *FE* and *GH*, will sustain shearing stresses in consequence of the component of the tension tangential to those oblique sections. These tangential stresses will cause the oblique sections, in both directions, to slide over each other. Consequently *the normal cross-sections of the body will be decreased;* and if the normal cross-sections of the body are made less, its capacity to resist the external forces acting on *AB* and *CD* will be correspondingly diminished.

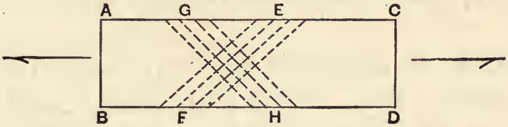


Fig. 1.

If the body is subjected to compression, oblique sections of the body will be subjected to shears, but in directions *opposite* to those existing in the previous case. The effect of such shears will be an *increase* of the lateral dimensions of the body and a corresponding increase in its capacity of resistance.

These changes in the lateral dimensions of the body are termed "lateral strains"; they always accompany direct strains of tension and compression.

It is to be observed that lateral strains *decrease* a body's resistance to tension, but *increase* its resistance to compression. Also, that if they are prevented, both kinds of resistance are *increased.*

Consider a cube, each of whose edges is a, in a body subjected to tension. Let r represent the ratio between the lateral and direct strains,* and let it be supposed to be the same in all directions. If l, as in Art. 2, represents the direct unit strain, the edges of the cube will become, by the tension, $a(1+l)$, $a(1-lr)$, and $a(1-rl)$. Consequently the volume of the resulting parallelopiped will be

$$a^3(1+l)(1-rl)^2 = a^3[1+l(1-2r)] \quad . \quad . \quad . \quad (1)$$

if powers of l higher than the first be omitted. *With r between* 0 *and* ½, *there will be an increase of volume*, but not otherwise.

If the body is subjected to compression, the edges of the cube become $a(1-l_1)$, $a(1+r_1l_1)$, and $a(1+r_1l_1)$; while the volume of the parallelopiped takes the value

$$a^3(1-l_1)(1+r_1l_1)^2 = a^3[1+l_1(2r_1-1)]. \quad . \quad . \quad (2)$$

As before, the higher powers of l_1 are omitted. If the volume of the cube is decreased, r_1 must be found between 0 and ½.

If a be unity in eq. (1), it is then clear that the expression $l(1-2r)$ is the change of volume of a unit cube, i.e., it is the rate of change of volume when the intensity of stress is $p=El$. Hence if this rate of change of volume be multiplied by a definite volume V the result will be the total change of that definite volume produced by the uniform intensity of stress p.

If the intensity of stress varies from point to point the total change of volume will become:

$$\int \frac{p}{E}(1-2r)dV = \left(\frac{1-2r}{E}\right)\int pdV. \quad . \quad . \quad . \quad (3)$$

Evidently the volume V must be expressed in the same independent variable, or variables, as p. The integral must then be made to cover the desired limits.

---

* Frequently called Poisson's ratio.

Art. 5.—Relation between the Coefficients of Elasticity for Shearing and Direct Stress in a Homogeneous Body.

A body is said to be homogeneous when its elasticity, of a given kind, is the same in all directions.

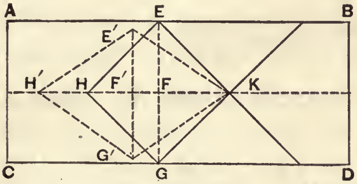


FIG. 1.

Let Fig. 1 represent a body subjected to tension parallel to *CD*. That oblique section on which the shear has the greatest intensity will make an angle of 45° with either of those faces whose traces are *CD* or *BD*; for if α is the angle which *any* oblique section makes with *BD*, *P* the total tension on *BD*, and *A′* the area of the latter surface, the total shear on any section whose area is $A' \sec \alpha$ will be $P \sin \alpha$. Hence the intensity of shear is

$$\frac{P \sin \alpha}{A' \sec \alpha} = \frac{P}{A'} \sin \alpha \cos \alpha. \quad . \quad . \quad . \quad . \quad (1)$$

The second member of eq. (1) evidently has its greatest value for $\alpha = 45°$. Hence if the tensile intensity on *BD* is represented by $\frac{P}{A'}, = p$, the greatest intensity of shear will be

$$S = \frac{p}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then by eq. (3) of Art. 2,

$$\phi = \frac{p}{2G}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In Fig. 1 *EK* and *KG* are perpendicular to each other, while they make angles of 45° with either *AB* or *CD*. After stress, the cube *EKGH* is distorted to the oblique parallelopiped *E′KG′H′*. Consequently *EKGH* and *E′KG′H′* correspond to *ABCD* and *AEFD*, respectively, of Fig. 1, Art. 2. The angular difference $EKG - E'KG'$ is then equal to ϕ; and $EKE' = GKG' = \frac{\phi}{2}$. Also, $E'KF' = 45^\circ - \frac{\phi}{2}$.

Using, then, the notation of the preceding articles, there will result, nearly,

$$\tan\left(45^\circ - \frac{\phi}{2}\right) = \frac{1 - rl}{1 + l} = 1 - l(1 + r); \quad . \quad . \quad (4)$$

remembering that

$$F'K = FK(1 + l), \quad \text{and} \quad E'F' = FK(1 - rl).$$

From a trigonometrical formula there is obtained, very nearly,

$$\tan\left(45^\circ - \frac{\phi}{2}\right) = \frac{\tan 45^\circ - \tan\frac{\phi}{2}}{\tan 45^\circ + \tan\frac{\phi}{2}} = \frac{1 - \frac{\phi}{2}}{1 + \frac{\phi}{2}} = 1 - \phi. \quad . \quad (5)$$

From eqs. (4) and (5),

$$\phi = l(1 + r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting from eq. (3), as well as from eq. (1) of Art. 2,

$$G = \frac{E}{2(1 + r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It has already been seen in the preceding article that r must be found between 0 and $\frac{1}{2}$, consequently *the coefficient of elasticity for shearing lies between the values of* $\frac{1}{3}$ *and* $\frac{1}{2}$ *of that of the coefficient of elasticity for tension.*

This result is approximately verified by experiment.

Since precisely the same form of result is obtained by treating compressive stress, instead of tensile, there will be found, by equating the two values of G,

$$\frac{E}{1 + r} = \frac{E_1}{1 + r_1}, \quad \text{or} \quad \frac{E_1}{E} = \frac{1 + r_1}{1 + r}. \quad . \quad . \quad . \quad . \quad (8)$$

It is clear, from the conditions assumed and operations involved, that the relations shown by eqs. (7) and (8) can only be approximate.

Art. 6.—Shearing Stresses and Strains.

In the preceding Arts. the more simple and ordinary relations between stress and strain are shown, but in this and following Arts. it is desirable to give a more extended treatment.

Materials are rarely used in structures and machines under conditions in which the stress is wholly shear. The usual conditions are such as to produce shear concurrently with stresses of tension and compression. Even in the use of rivets, where shearing stress acts prominently, tension and compression in the form of flexure and direct compression are concurrent. Again in the case of flexure or the bending of beams, the shearing stress is sufficiently high in intensity in some cases to produce failure, but concurrently with relatively high intensities of tension and compression.

Figs. 1 and 2 show a rectangular parallelopiped of material of depth b at right angles to the plane $ABCD$ firmly held on the face AD, while the intensities of shear s and s' act on the faces AB, BC, CD, and AD. It is supposed that no other stresses act upon the exterior faces of the prism of material. Let the prism be imagined to be divided into indefinitely thin vertical slices at right angles to the face $ABCD$ when in its original position shown by $AB'C'D$. Similarly let the prism be imagined to be divided into indefinitely thin horizontal slices at right angles to the same face.

Before considering the distortion of the prism due to the action of the shearing stresses an important but simple principle must be established. As there are no stresses acting upon the prism except the opposite pairs of shearing stresses whose intensities are s and s' as shown, it is clear that the prism must be held in equilibrium by the two couples acting in opposite directions whose lever arms are AB' and AD. Let l represent the length AB' of the prism, while $AD=d$, as shown in Fig. 2. Then since the prism is in equilibrium there will result the equation,

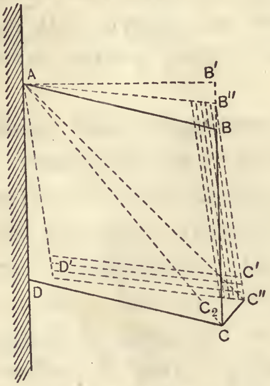


FIG. 1.

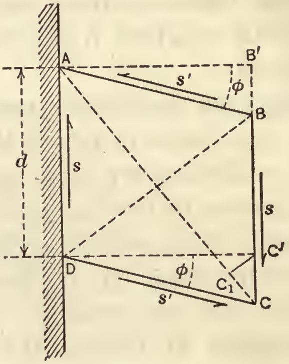


FIG. 2.

$$s'bl.d = sbd.l$$

$$\therefore \quad s = s'. \qquad \qquad \qquad (1)$$

This equation shows that the intensities of two shears acting on planes at right angles to each other and parallel to a third plane at right angles to the other two must be equal. Furthermore, it is clear from Fig. 2 that the shears on the faces of the prism must act in pairs toward two of the corners of the prism diagonally opposite each other and away from the other diagonally opposite pair of corners.

The rectangular prism of Figs. 1 and 2 may be considered indefinitely small under ordinary conditions of stress in structural material in order to have the stress uniformly distributed on the four faces. Whatever may be the condition of stress at any point in the interior of a piece of material, the stresses acting upon the four faces of the rectangular prism, when all stress is parallel to one plane, may be resolved into normal and tangential components. The normal components will act opposite to each other producing no moments about any point, but the tangential components will produce precisely the moments shown in Figs. 1 and 2. The equilibrium of the indefinitely small prism invariably requires therefore the action of two pairs of shears of equal intensity, as established above.

The complete distortion of the rectangular prism *ABCD* may be considered as produced first by the sliding over each other of the indefinitely thin vertical sections parallel to *BC*, so as to produce the oblique prism $AB''C_2D$, Fig. 1, then by the subsequent sliding over each other of the indefinitely thin horizontal sections parallel to *DC*, so as to produce the oblique prism $AB''C''D'$. This last movement of the horizontal slices will bring the line *AD* into the position of AD', then swinging the latter line about *A* to the original position *AD*, the completely distorted prism will take the form *ABCD*.

$B'B''$, Fig. 1, is the characteristic shearing strain produced by the vertical shearing stress whose intensity is *s* acting in the planes parallel to *BC*. DD' is the characteristic shearing strain produced by the action of the horizontal shearing intensity s' in sliding the thin horizontal slices over each other. These detrusive movements are so small that $B'B''$ may be considered at right angles to *AB* and DD' at right angles to *AD*. The total detrusive strain $B'B$ is the sum of $B'B''$, due to the vertical shear, and $B''B$ due to the horizontal shear, and $B'B''=B''B$, if $AB' = AD$. The total shearing strain per unit of length of AB will therefore be,

$$\frac{B'B}{AB} = \frac{B'B'' + B''B}{AB}. \quad . \quad . \quad . \quad . \quad (2)$$

This is the expression for the characteristic resultant shearing strain and it is seen to be measured at right angles to the original face AB', i.e., it is a small arc measurement in radians. It is important to remember that this total detrusive strain due to shear is the sum of two equal effects, one of horizontal shear and the other of vertical shear, i.e., of the two shears on planes at right angles to each other.

If $b = 1$ and if $AB'C'D$, Fig. 2, now be considered square so that $AB = BC$, then will the tension T acting perpendicular to the plane BD be equal to the sum of the components of the shear $s = s'$, on the planes BC and DC, normal to the diagonal plane BD. Since the angle BCA is 45° and its cosine $\frac{1}{\sqrt{2}}$, the following equation at once results:

$$T = 2s \cos 45° = s\sqrt{2}. \quad . \quad . \quad . \quad . \quad (3)$$

Similarly the compression on the diagonal plane AC is:

$$C = -s\sqrt{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As the area of each diagonal plane section AC and BD is $\sqrt{2}$, the intensity of the tension T and compression C on the planes AC and BD respectively will be:

$$\frac{T}{\sqrt{2}} = -\frac{C}{\sqrt{2}} = s. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence it is seen that when the stress at any point is wholly shear on two planes at right angles to each other and perpendicular to the plane to which the shearing stress is parallel, the stress on two planes at right angles to each other and making angles of 45° with the two planes on which the shears act, will be wholly tension on one and compression on the other, and both will have the same intensity as the two shears.

Inasmuch as the prism whose section is shown in Fig. 2 is subjected to a normal stress of tension in the direction of AC and an equal normal stress of compression in the direction BD, it is obvious that there will be no change in volume due to those stresses, since the change in intensity caused by one stress will be exactly neutralized by the other. Again the sliding over each other of the thin slices of the material will not change its density or volume, although a change of shape is produced. Hence it is to be carefully observed that shearing stresses produce no change of volume, but change of shape only.

If ϕ is the angle $B'AB=C'DC$, then in general, the resultant shearing strain $B'B=C'C=AB'\phi=AB' \sin \phi =AB' \tan \phi$, since the angle ϕ is exceedingly small. If $AB=BC=1$, $B'B=\phi=\sin \phi=\tan \phi$.

In Fig. 2 if the total detrusive strain CC' be projected on the diagonal AC the change CC_1 in length of that diagonal will result. As the angle $C'CC_1$ is 45°, the change of length CC_1 will be $\frac{\phi}{\sqrt{2}}$, and the strain per unit of length of the diagonal will be,

$$\frac{\phi}{\sqrt{2}}\frac{1}{\sqrt{2}}=\frac{\phi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is clear that the diagonal BD will be shortened by the same amount. Indeed Eq. 6 shows the tensile strain in the diagonal AC, while the same value with a minus sign would show the compressive strain for the diagonal BD. If the diagonal AC were subjected to the tensile intensity s only the strain per unit of length would be $\frac{s}{E}$.

If G is the modulus of elasticity for shearing, the intensity of shearing stress may be written,

$$s = s' = G\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Inasmuch as the total detrusive strain ϕ per linear unit is the sum of the equal effects of the shears on the two faces of the prism, it would be more rational to call $\frac{\phi}{2}$ the detrusive strain per linear unit for the shear on one face of the prism. This would make the modulus G of elasticity for shearing double the value usually employed, but it would represent accurately the rigidity of the material, since one half of the total shearing strain ϕ, Fig. 2, is produced by a rotation of the prism as a whole. In other words the total strain is the sum of two separate but equal strains. This doubling of the value of G would obviously change no results of computation for practical purposes since the strain ϕ would be halved. It is interesting to observe in connection with this feature of the matter that the shearing rigidity of the material in this case, would become the same as the apparent rigidity in tension or compression.

Art. 7.—Relation between Moduli of Elasticity and Rate of Change of Volume.

The preceding analyses yield some simple relations between the moduli of elasticity for tension, compression and shearing and the rate of change of volume v,* i.e., the change of unit volume for unit intensity of stress.

In Fig. 2 of the preceding Art. CC' shows the total shearing strain ϕ, and the elongation or strain $CC_1\left(=\frac{\phi}{\sqrt{2}}\right)$ of the diagonal AC. It has also been shown that the intensity of tension on BD or compression on AC is the same as the shear $s=s'$. Remembering that the compression s on AC will produce a unit positive lateral strain $r\frac{s}{E}$ parallel to AC, the two equal values of the unit strain of the diagonal AC may be written,

$$\frac{\phi}{2}=\frac{s}{2G}=\left(\frac{s}{E}+r\frac{s}{E}\right).$$

Hence,

$$G=\frac{E}{2(1+r)}=\frac{E_1}{2(1+r_1)}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the modulus of elasticity for compression, E_1, should be different from that for tension it is evident that the third member of Eq. 1 would be required.

If the value of r is $\frac{1}{3}$ or $\frac{1}{4}$ then will,

$$G=\tfrac{3}{8}E \quad \text{or} \quad E. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The relation between v and E can readily be written by considering a cube (indefinitely small if necessary) subjected to uniformly distributed tensile stress of intensity p normal to each of its six faces. Each edge of the cube, assumed to be of unit length, will be lengthened by the normal stress parallel to it to the extent $\frac{p}{E}$, and it

---

* The reciprocal of what is sometimes called the volume or bulk modulus.

will be decreased in length $r\frac{p}{E}$ by each of the two normal stresses p acting at right angles to it.

The resultant change in length of each edge will then be,

$$\frac{p}{E}(1-2r).$$

Hence the change of unit volume in terms of the unit rate v will be,

$$pv=3\frac{p}{E}(1-2r).$$

$$\therefore \quad E=\frac{3(1-2r)}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If V be any volume, the total change of volume will be pvV.

The equation preceding Eq. (3) shows that the unit rate of change of volume v is simply the sum of the three linear rates of change of the edges of the cube, since $\frac{1-2r}{E}$ is the change of length of each edge of the cube for each unit of p, i.e., $p\left(\frac{1-2r}{E}\right)$ is the change in length of each such edge under the action of the intensity of stress p. If the intensity of stress parallel to each edge of the cube should be different from the others the preceding analysis shows that the rate of variation of volume would still be the sum of the three coordinate linear rates of variation.

By the aid of Eq. (1),

$$E=2G(1+r)=\frac{3(1-2r)}{v}. \quad . \quad . \quad . \quad . \quad (4)$$

Therefore:

$$r=\frac{3-2Gv}{6+2Gv}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Finally, placing r from Eq. (5) in Eq. (3),

$$E = \frac{9G}{3 + Gv}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These simple relations will enable the various moduli to be determined with the least possible amount of experimental work.

Art. 8.—All Stresses Parallel to One Plane—Resultant Stress on any Plane Normal to the Plane of Action of the Stresses.

In Fig. 1 let XOY be the plane parallel to which all stresses act. Then OX and OY being any rectangular coordinate directions, consider the two planes OA and OB normal to each other and at right angles to the plane XOY and let the width of each of those planes at right angles to XOY be unity.

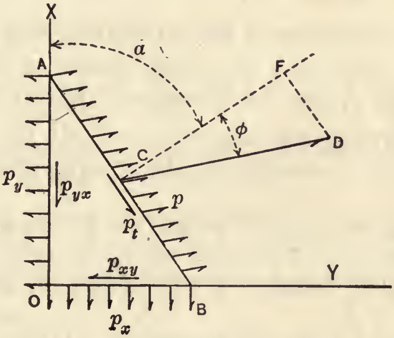


FIG. 1.

Again let it be supposed that the normal stress on the plane AO has the intensity p_y and that the intensity of the tangential or shearing stress on the same plane is p_{yx}. Similarly let it be supposed that the intensity of the normal stress on the plane OB is p_x, the intensity of the tangential or shearing stress being p_{xy}. It is known from the principles already established in the preceding articles that the two intensities of shear p_{yx} and p_{xy} are equal to each other. The problem is to determine the intensity and direction of the resultant stress on any plane AB, taken at right angles to XOY. In general the resultant stress CD will make the angle ϕ with the normal CF to the plane AB, i.e., the resultant stress will have the obliquity ϕ.

The direction of the plane AB will be fixed by the angle which its normal CF makes with the axis OX. In order that the stresses on the three planes in question may be taken as uniformly distributed let it be assumed that $OA = dx$ and $OB = dy$. Then will

$$AB = dy \sec \alpha = dx \operatorname{cosec} \alpha. \quad . \quad . \quad . \quad . \quad (1)$$

If p is the intensity of the uniformly distributed resultant stress on AB, then the equilibrium of the indefinitely small triangular prism OAB requires that the two following equations, representing the sums of all the forces acting upon it in the two coordinate directions, shall be true.

$$p_x dy + p_{xy} dx = p \cos (\alpha + \phi) . \, dy \sec \alpha \quad . \quad (2)$$

$$p_y dx + p_{xy} dy = p \sin (\alpha + \phi) . \, dy \sec \alpha \quad . \quad (3)$$

Fig. 1 shows that $dy \tan \alpha = dx$. Hence Eqs. (2) and (3) become Eqs. (4) and (5), respectively:

$$p_x \cot \alpha + p_{xy} = p \cos (\alpha + \phi) \operatorname{cosec} \alpha \quad . \quad . \quad (4)$$

$$p_y + p_{xy} \cot \alpha = p \sin (\alpha + \phi) \operatorname{cosec} \alpha. \quad . \quad . \quad (5)$$

It is sometimes convenient to express the normal and tangential components of the resultant intensity p in terms of the known intensities p_x, p_y and p_{xy}. If in Fig. 1 the stresses on the faces OA and OB be resolved into components normal and parallel to the plane AB the sum of the normal components will be equal to the normal stress on AB while the sum of the parallel components will be equal to the tangential or shearing stress on AB. This procedure will give,

$$p_y dx \sin \alpha + p_{yx} dx \cos \alpha + p_x dy \cos \alpha + p_{xy} dy \sin \alpha = p dy \sec \alpha \cos \phi.$$

$$p_y dx \cos \alpha - p_{yx} dx \sin \alpha - p_x dy \sin \alpha + p_{xy} dy \cos \alpha = p dx \operatorname{cosec} \alpha. \quad \sin \phi.$$

Using the values already given for dy and AB the following expressions for the normal and tangential components of p ($p \cos \phi$ and $p \sin \phi$) will result:

$$p_y \sin^2 \alpha + p_x \cos^2 \alpha + 2 p_{xy} \sin \alpha \cos \alpha = p \cos \phi \quad . \quad (4a)$$

$$(p_y - p_x) \sin \alpha \cos \alpha + p_{xy}(\cos^2 \alpha - \sin^2 \alpha) = p \sin \phi. \quad (5a)$$

These two equations will be used in establishing the ellipse of stress in the next Art.

If the stress p is a principal stress its obliquity ϕ, i.e., the angle between its direction and the normal to the plane on which it acts, will be zero. If $\phi = o$ Eqs. (4) and (5) become,

$$p - p_x = p_{xy} \tan \alpha, \quad . \quad . \quad . \quad . \quad (6)$$

$$p - p_y = p_{xy} \cot \alpha. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Subtracting Eq. 6 from Eq. 7,

$$\cot \alpha - \tan \alpha = \frac{2}{\tan 2\alpha} = \frac{p_x - p_y}{p_{xy}}.$$

$$\therefore \quad \tan 2\alpha = \frac{2 p_{xy}}{p_x - p_y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the angle α_1 satisfies this equation, then will $\alpha_1+90°$ also satisfy it. Hence, there will always be two principal planes at right angles to each other on each of which a normal stress only acts, i.e., there is no shearing stress on either principal plane.

Eq. 8 will at once locate, by the two values of α, the two principal planes, while the same two values of α introduced into either Eq. 6 or Eq. 7 will give the two intensities of principal stresses to be called p_1 and p_2, it being supposed that the normal and shearing stresses on the planes OA and OB are completely known.

The two principal stresses can however readily be found without computing the angle α. Multiplying Eq. 7 by Eq. 6,

$$p^2-p(p_x+p_y)=p_{xy}{}^2-p_xp_y.$$

The solution of this quadratic equation gives,

$$p=\frac{p_x+p_y}{2}\pm\sqrt{p_{xy}{}^2+\tfrac{1}{4}(p_x-p_y)^2} \quad . \quad . \quad . \quad . \quad (9)$$

The two roots of this equation will give the two principal intensities at any point in terms of the known intensities p_x, p_y and p_{xy}.

The two stress intensities p_x and p_y have been taken of the same kind, tension or compression, and considered positive. If one, as p_y, be considered compression or negative, its sign would be changed in the preceding equations, but there would be no other change.

Sum of Normal Components.

If any other plane be taken at right angles to XOY, Fig. 1, and at right angles to the plane whose trace is AB, the preceding equations are made applicable to it by writing

$90^\circ + \alpha$ for α in Eqs. (4*a*) and (5*a*), since the new plane is at right angles to that whose trace is *AB*.

Then in Eqs. (4*a*) and (5*a*) there must be written,

For $\sin \alpha$, $\sin (90 + \alpha) \ = \cos \alpha$.

For $\cos \alpha$, $\cos (90 + \alpha) \ = -\sin \alpha$.

Hence by Eq. (4*a*), writing p' and ϕ' for p and ϕ;

$$p_y \cos^2 \alpha + p_x \sin^2 \alpha - 2 p_{xy} \sin \alpha \cos \alpha = p' \cos \phi'.$$

Then by adding this equation to Eq. (4*a*);

$$p_x + p_y = p \cos \phi + p' \cos \phi'. \quad . \quad . \quad . \quad (10)$$

This equation shows that on any two planes at right angles to each other the sum of the normal intensities will be constant and equal to $p_x + p_y$. Furthermore, inasmuch as there is no shear on the principal planes, i.e., the stress is wholly normal, it is thus seen that the sum of the normal intensities on any two planes at right angles to each other is always equal to the sum of the two principal intensities.

If the above values of $\sin \alpha$ and $\cos \alpha$ are written in Eq. (5*a*), the following equation will result:

$$(p_y - p_x) \sin \alpha \cos \alpha + p_{xy}(\cos^2 \alpha - \sin^2 \alpha) = -p' \sin \phi'.$$

This equation is identical with Eq. (5*a*), except that the sign of the second member is changed. This result simply shows what is already known that the intensities of the shears on planes at right angles to each other are equal. The change of sign indicates the direction only of the shear.

In all the usual cases of stress arising in the subject of Resistance of Materials the internal stresses produced by external loading may be considered parallel to one plane. The preceding investigation shows that without considering the elastic properties of the material there are two equations of condition (Eqs. 4 and 5) from which the two rectangular components of the resultant stress p (or intensity p and obliquity ϕ) may be found. If the general case of internal stress be taken in which stresses may act in three rectangular coordinate directions, there obviously will be three equations of condition from which the three rectangular components of the resultant stress on any plane may be found.

The triangle *OAB* may be considered the side of a wedge whose edge is at *O*. The two faces *OB* and *OA* are acted upon by the stresses indicated and their resultant holds in equilibrium the stress p on the head *AB* of the wedge. The surface *AB* may be considered a part of the exterior surface of a body acted upon by the stress whose intensity is p, while the faces *OA* and *OB* are interior surfaces of the body acted upon by the internal stresses shown.

Art. 9.—The Ellipse of Stress—Greatest Intensity of Shearing Stress—Equivalence of Pure Shear to Two Principal Stresses of Opposite Kinds but Equal Intensities—Greatest Obliquity of Resultant Stress on any Plane.

The analysis of the preceding article makes it comparatively easy to express the relation between the stress on any plane whatever at right angles to the plane parallel to which the principal stresses act and those principal stresses, all stresses still acting parallel to one plane. In Fig. 1 let *OX* and *OY* be taken in the direction of the principal stresses, *OA* representing the intensity p_1 of the principal stress at *O* on the plane *OD*, while *OB* represents the intensity of the principal stress p_2, acting at O on the plane OC. OCD represents an indefinitely small triangular prism whose face CD normal to XOY makes the angle β with the principal plane OD. The intensity of the resultant stress on any plane CD is represented by p, whose obliquity is ϕ, the normal N to the plane CD making the angle β with the axis OX. The resultant intensity p may at once be written by the aid of Eqs. 4 and 5 or 4a and 5a of the preceding article if the principal intensities p_1 and p_2 be written in the place of p_x and p_y, respectively, in those equations while p_{xy} is made equal to zero. This procedure with Eqs. (4a) and (5a) will give the following Eqs. (1) and (2).

$$p_2 \sin^2 \beta + p_1 \cos^2 \beta = p \cos \phi, \quad . \quad . \quad . \quad (1)$$

$$(p_2 - p_1) \sin \beta \cos \beta = \frac{p_2 - p_1}{2} \sin 2\beta = p \sin \phi. \quad . \quad (2)$$

Squaring each of those equations and adding the results:

$$p_2{}^2 \sin^2 \beta + p_1{}^2 \cos^2 \beta = p^2. \quad . \quad . \quad . \quad (3)^*$$

This is the equation of an ellipse with the origin of coordinates at the centre, the rectangular coordinates being $p_2 \sin \beta$ and $p_1 \cos \beta$. Fig. 1 shows the ellipse of stress, the intensities of the principal stresses being represented by the semi axes of the ellipse.

$$OB = p_2 \quad \text{and} \quad OA = p_1.$$

In this Fig. p_2 represents the intensity of the principal stress on the plane OC, while p_1 is the intensity on the principal plane OD. The intensity p on any plane as CD

---

[*] Precisely the same result will be obtained by making $p_{xy} = 0$ in eqs. 4 and 5 of the preceding Art. and then squaring and adding them.

perpendicular to XOY and whose normal ON makes the angle β with OX is represented in Fig. 2 by OH, the curve AHB being an ellipse. Let the partial circles shown be described by the radii OB and OA. Then if OCD be considered indefinitely small the normal ON, and the line OH representing the intensity of the resultant stress on the plane CD, will both pass through the origin O. Then OG will represent p_2 and $OK = p_2 \sin \beta$. The same construction shows that $HK = p_1 \cos \beta$ since $OJ = p_1$. The square of $OH = p$ will then obviously be equal to the square of HK added to the square of OK, an expression identical with Eq. (3).

Any radius vector of the ellipse therefore represents the intensity of a resultant stress on a plane whose normal makes the angle β with the axis of X. The obliquity of the resultant stress in question is represented by the angle ϕ.

The two principal stresses have been taken of the same kind in finding the ellipse of stress, but the results are essentially the same if the principal stresses are of opposite kind. If for example, p_2 were negative while p_1 remains positive $p_2 = OB$ would be laid off in Fig. 1 to the left of O instead of laying it off to the right of the same point. Similarly if the sign of p_1 should be considered negative that intensity would be laid off downward from O to A' instead of upward to A.

If the two intensities of principal stresses p_1 and p_2 are equal to each other and of the same kind Eq. 3 becomes $p_1 = p_2 = p$.

Under the same conditions Eq. (2) shows that the shearing intensity is zero, whatever value the angle β may have, since in such a case $p_1 - p_2 = 0$. Hence all stresses are principal stresses and of equal intensity. This condition of stress is the same as that which holds in a perfect fluid.

An examination of the ellipse of stress as given in Fig. 1 shows that the intensity of one principal stress is greater than that of any other stress at the point for which the ellipse is drawn, while the intensity of the other principal stress is the least of all the intensities at the same point, since the semi-major and semi-minor axes of the ellipse are the greatest and least, respectively, of all the semi-diameters. If therefore in the design or construction of any machine or structure the principal stress at any point is provided for by the use of a proper working stress, no further provision for the direct stresses of tension and compression will be needed. If there may be either a reversal of stress or rapid repetition of stresses the intensity of working stress must be prescribed under a proper recognition of those conditions. Similarly provision must be made for the greatest shearing stress at the point under consideration.

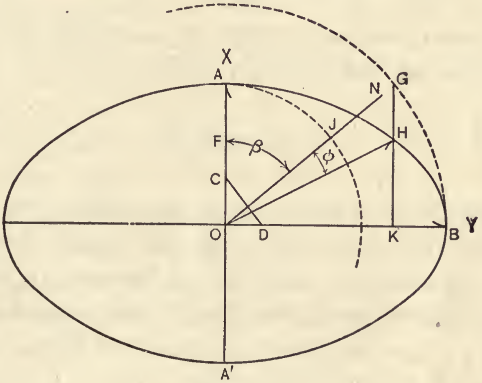


Fig. 1.

Greatest Intensity of Shearing Stress.

The intensity of shear on any plane *CD* at the point *O*, Fig. 1, is $p \sin \phi$ as given by Eq. 2. Its greatest value and the plane on which it acts are readily determined by differentiating that equation:

$$\frac{d(p \sin \phi)}{d\beta} = (p_2 - p_1)(\cos^2 \beta - \sin^2 \beta) = 0 = (p_2 - p_1) \cos 2\beta.$$

Hence,

$$\cos \beta = \sin \beta; \text{ or, } \beta = 45^\circ. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$, the greatest intensity of shear at any point, as O, Fig. 1, is found by substituting $\beta = 45^\circ$ in the second member of Eq. (2):

$$p_t = \frac{p_2 - p_1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The planes of greatest shear, therefore, are at the angle of 45° *from each of the two principal planes, and the greatest intensity of shear is half the difference of the principal intensities, both of the latter being of the same kind.*

As $\beta = 45^\circ$ the resultant intensity of stress on the plane of greatest shear will be, by Eq. (3),

$$p^2 = \frac{p_2^2 + p_1^2}{2} \quad \therefore \quad p = \pm\sqrt{\frac{p_2^2 + p_1^2}{2}}. \quad . \quad . \quad . \quad (5a)$$

If $p_2 = \pm p_1$; $p = \pm p_2 = \pm p_1$.

Equivalence of Pure Shear to Two Principal Stresses of Opposite Kinds but Equal Intensities.

If the principal stresses are of opposite kinds, i.e., if p_1 is negative while p_2 is positive, then by Eq. (5) the greatest shear becomes:

$$p_t = \frac{p_2 + p_1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The greatest intensity of shear is half the sum of the principal intensities.

Obviously the planes of greatest shear remain as established by Eq. (4).

If the principal stresses of opposite kinds have the same intensities Eq. (6) shows that:

$$p_t = p_2 = p_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence the intensity of the greatest shear is the same as that of the principal stresses of opposite kinds. It is therefore sometimes stated that a pair of normal stresses of opposite kinds and equal intensities on two planes at right angles to each other are equivalent to two pure shears of the same intensity as the normal stresses on planes at right angles to each other, but at 45° with the planes on which the normal stresses act, all planes under consideration being perpendicular to one plane. This simple condition of stress exists in both flexure or bending and torsion, and some important results are based on it.

Greatest Obliquity of Resultant Stress on any Plane.

If Eq. (2) be divided by Eq. 1:

$$\tan\phi = (p_2 - p_1)\frac{\sin\beta\cos\beta}{p_2\sin^2\beta + p_1\cos^2\beta}. \quad . \quad . \quad (8)$$

It is desired to find that value of β which will make ϕ (or $\tan\phi$) a maximum. By differentiating Eq. (8) and placing $\frac{d(\tan\phi)}{d\beta} = 0$, there will result,

$$(\cos^2\beta - \sin^2\beta)(p_2\sin^2\beta + p_1\cos^2\beta)$$
$$= 2\sin^2\beta\cos^2\beta(p_2 - p_1). \quad . \quad . \quad . \quad (9)$$

Remembering that $\cos^2\beta - \sin^2\beta = \cos 2\beta$ and that $2 \sin\beta \cos\beta = \sin 2\beta$, Eq. (9) will become Eq. (10);

$$\cos 2\beta(p_2 \sin^2\beta + p_1 \cos^2\beta) = \sin 2\beta \sin\beta \cos\beta(p_2 - p_1). \quad (10)$$

Calling the normal component of the intensity p, i.e., $p \cos\phi = p_n$ and the tangential or shearing component $p \sin\phi = p_t$, those values taken from Eqs. (1) and (2) placed in Eq. (10) will give,

$$\cos 2\beta p_n = \sin 2\beta p_t.$$

Hence,
$$\tan 2\beta = \frac{p_n}{p_t} = \cot\phi = \tan(90° - \phi). \quad . \quad . \quad (11)$$

And,
$$\beta = 45° - \frac{\phi}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Eq. (12) gives the relation between β and ϕ when the obliquity ϕ is the greatest possible.

By the aid of Eq. (12),

$$\sin\beta \cos\beta = \tfrac{1}{2} \sin 2\beta = \tfrac{1}{2} \cos\phi.$$

Then as,
$$\sin^2\left(45° - \frac{\phi}{2}\right) = \frac{1}{2}(1 - \sin\phi),$$

and
$$\cos^2\left(45° - \frac{\phi}{2}\right) = \frac{1}{2}(1 + \sin\phi),$$

Eq. (8) gives,

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = (p_2 - p_1)\frac{\cos\phi}{p_2(1 - \sin\phi) + p_1(1 + \sin\phi)}.$$

Hence,

$$\frac{p_1}{p_2} = \frac{1 - \sin\phi}{1 + \sin\phi}, \quad \text{and,} \quad \sin\phi = \frac{p_2 - p_1}{p_2 + p_1}. \quad . \quad . \quad (13)$$

The relation shown in the first of Eqs. (13) is used in the theory of earth pressure. The second of Eqs. (13) gives the value of the greatest obliquity ϕ in terms of the known principal intensities p_1 and p_2.

The angle β locating the plane on which the obliquity is greatest may also be expressed in terms of p_1 and p_2.

Using Eqs. (12) and (13),

$$\frac{p_1}{p_2}=\frac{1-\sin\phi}{1+\sin\phi}=\frac{1-\cos 2\beta}{1+\cos 2\beta}=\frac{\sin^2\beta}{\cos^2\beta}.$$

$$\therefore \quad \tan\beta=\pm\sqrt{\frac{p_1}{p_2}}. \quad . \quad . \quad : \quad . \quad . \quad (14)$$

The intensity, p', of this stress of greatest obliquity is, by Eq. (3), since by Eq. (14) $\sin^2\beta=\frac{p_1}{p_1+p_2}$ and $\cos^2\beta=\frac{p_2}{p_1+p_2}$,

$$p^1=\sqrt{p_1p_2.} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Art. 10.—Ellipse of Stress and Resulting Formulæ for the Special Case of Zero Intensity of One of the Known Direct Stresses.

If in the second preceding article it be supposed that the intensity of one of the direct stresses as p_x is zero while the other intensity p_y and the two shearing intensities $p_{xy}=p_{yx}$ have known values, the formulæ will be correspondingly simplified. This is the condition of stress in a bent beam as will be seen later on. The intensity of direct stress p_y is what is ordinarily called the fibre stress at any point in the beam and this intensity varies directly as the distance from an intermediate plane (before flexure) in the beam called the neutral surface. The plane *OY* of Fig. 1, representing part of a beam, is supposed to be a horizontal plane coincident with or parallel to the neutral surface of the bent beam at any point, while the plane whose trace is OX is the plane of vertical (normal) transverse section of the beam at any point. Both the direct intensity p_y and the intensity of shear p_{xy} are readily determined from the known conditions of loading and flexure. The analysis of this condition of stress therefore is of much practical importance in connection with the design or other treatment of beams subjected to transverse bending.

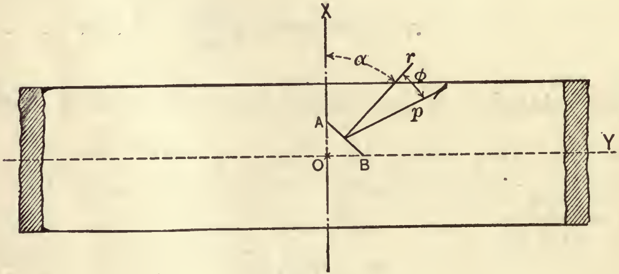


Fig. 1.

If the stress p in Eq. (4a) of Art. 8 is a principal stress and if the intensity $p_x = 0$, the principal intensity p will become,

$$p = p_y \sin^2 \alpha + 2 p_{xy} \sin \alpha \cos \alpha. \quad . \quad . \quad . \quad (1)$$

Or, Eq. (9) of the same Art. will give for the two principal intensities,

$$p = \tfrac{1}{2} p_y \pm \sqrt{p_{xy}^2 + \tfrac{1}{4} p_y^2}. \quad . \quad . \quad . \quad . \quad (2)$$

Also Eq. (8) of the same Art. will give,

$$\tan 2\alpha = -\frac{2 p_{xy}}{p_y}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the point O, Fig. 1, is in the neutral surface of the bent beam $p_y = 0$; and, hence,

$$\tan 2\alpha = -\infty, \quad \text{or}, \quad 2\alpha = \pm 90°. \quad . \quad . \quad . \quad (4)$$

Therefore, $\alpha = \pm 45°$.

If the stress p_y is negative or compression, $\alpha = \begin{cases} +\ 45° \\ +135° \end{cases}$. The direct fiber stresses in a bent beam are tensile on one side of the neutral surface and compressive on the other. As in this special case $\alpha = -45°$, $\sin \alpha = -\cos \alpha = -\frac{1}{\sqrt{2}}$ and the intensity p of the principal stress becomes by the aid of Eq. (1), since $p_y = 0$.

$$p = -p_{xy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It has already been seen that α and $90° + \alpha$ will satisfy Eq. (3); but $90° + \alpha = 90° - 45° = 45°$. Hence placing $\alpha = +45°$ in Eq. (1),

$$p = +p_{xy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

Therefore at O, Fig. 1, where there is no direct stress (but shear only) on the two planes OX and OY the principal stresses are of equal intensities, but of opposite kinds and they act on planes making angles of 45° with the planes OX and OY. This is the same condition shown by Eq. (7) of the preceding Art.

Again at the exterior surface of the beam p has its greatest value and the shearing intensity $p_{yx} = 0$. Eq. (2) then gives,

$$2\alpha = 0 \quad \text{or} \quad 180°.$$

Hence, $\alpha = 0$ or $90°$. . . . . . . (6)

Eq. 1 gives $p=0$ for the principal stress corresponding to $\alpha=0$; and, for $\alpha=90°$,

$$p=p_y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

There is therefore only one principal stress p_y, the fiber stress acting on the normal section of the beam for which $\alpha=90°$.

For intermediate points of the beam between the neutral surface and the exterior surface the principal stresses will have varying values between p_{xy} and p_y as shown by Eqs. (5) and (7) with planes of action located by values of α between ± 45° and +90°.

A graphical representation of this condition of stress for a bent beam may be found in Art. 34.

Art. 11.—General Condition of Stress—Ellipsoid of Stress.

The conditions of stress in structural material as found within the experience of engineers seldom include more than the action of stresses parallel to one plane. There may, however, be occasional cases in which an elementary consideration of stresses acting in any direction whatever becomes necessary or at least helpful. In this Article therefore only the most elementary results of the action of such stresses will be treated, including the ellipsoid of stress.

In the preceding articles both the determination and the application to a number of useful cases of the ellipse of stress have been made. That ellipse is simply a special case of the more general ellipsoid of stress. In other words, if the action of stresses in space, i.e., on three rectangular coordinate planes be considered it will be found that there will be three such planes at any point on which there will be no shear and which therefore are called principal planes, the resultant normal stresses being called the principal stresses at that point. The semi-diameter of the ellipsoid of stress drawn with its center at the point under consideration will be the intensity of stress in that direction, acting upon a plane whose position may be determined. For this elementary treatment let the three rectangular coordinate planes in Fig. 1 be drawn.

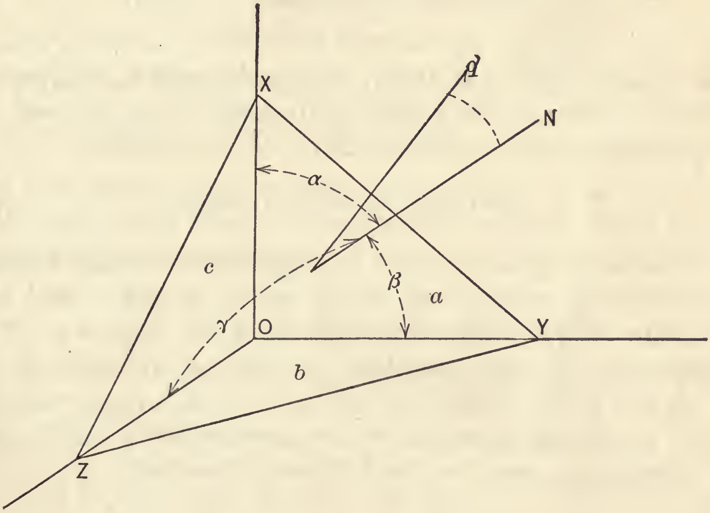


Fig. 1.

In that Fig. the normal stresses on the planes *XOY*, *YOZ*, and *ZOX* have the intensities p_z, p_x and p_y, respectively. The intensities of the shearing stresses on the planes *XOY* and *XOZ*, parallel to *YOZ*, are $p_{zy} = p_{yz}$; and those on the planes *XOY* and *YOZ*, parallel to *ZOX*, are $p_{zx} = p_{xz}$; and finally those on the planes *YOZ*, and *XOZ*, parallel to *XOY*, are $p_{yx} = p_{xy}$. If these normal and shearing or tangential stresses on the three faces, *AOB*, *BOC*, and *AOC* of the elementary tetrahedron *ABCO* are given, it is required to find the resultant intensity of stress on the plane surface ABC, the base of the tetrahedron, and its obliquity. It may be considered that $AO = dx$, $BO = dy$, and $CO = dz$. It will simplify the resulting equations if there be written for the areas of the faces of the elementary tetrahedron;

$$a = \frac{dxdy}{2}; \quad b = \frac{dydz}{2}; \quad \text{and} \quad c = \frac{dxdz}{2}.$$

Also if area $ABC = \varDelta$, and if the angles which the normal N to the face ABC makes with axes of X, Y, and Z, respectively, are α, β, and γ, there may be written:

$$\varDelta = a \sec \gamma = b \sec \alpha = c \sec \beta. \quad . \quad . \quad . \quad (1)$$

The tetrahedron is held in equilibrium by the normal and shearing stresses on the faces a, b, and c and the resultant stress whose intensity is q on $ABC = \varDelta$. The components of that resultant parallel to the axes of X, Y, and Z whose intensities are q_x, q_y, and q_z are respectively equal and opposite to the corresponding axial sums of stresses as shown by the following equations:

$$p_x b + p_{zx} a + p_{yx} c = q_x \varDelta, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$p_y c + p_{zy} a + p_{xy} b = q_y \varDelta, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p_z a + p_{xz} b + p_{yz} c = q_z \varDelta, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As these are rectangular components, if their squares are added the sum will be equal to $q^2 \varDelta^2$. If both sides of the resulting equation be divided by $\varDelta^2$, remembering that

$$\frac{a^2}{\varDelta^2} = \cos^2 \gamma; \qquad \frac{b^2}{\varDelta^2} = \cos^2 \alpha; \qquad \frac{c^2}{\varDelta^2} = \cos^2 \beta;$$

$$\frac{ab}{\Delta^2}=\cos\alpha\cos\gamma;\quad \frac{bc}{\Delta^2}=\cos\alpha\cos\beta;\quad \text{and}\quad \frac{ac}{\Delta^2}=\cos\beta\cos\gamma;$$

and that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1; \quad . \quad . \quad . \quad . \quad (5)$$

there will be found:

$$p_x^2\cos^2\alpha+p_y^2\cos^2\beta+p_z^2\cos^2\gamma+2\cos\alpha\cos\gamma(p_xp_{zx}+ \\ p_{xy}p_{zy}+p_zp_{xz})+2\cos\alpha\cos\beta(p_xp_{yx}+p_{xz}p_{yz}+p_yp_{xy})+ \\ 2\cos\beta\cos\gamma(p_zp_{yz}+p_{zx}p_{xy}+p_yp_{zy})+p^2{}_{xz}(1-\cos^2\beta)+ \\ p^2{}_{yx}(1-\cos^2\gamma)+p^2{}_{zy}(1-\cos^2\alpha)=q^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The square root of the first member of eq. (6) will give the desired value of the intensity q on any given plane.

If both members of eqs. (2), (3), and (4) be divided by Δ:

$$p_x\cos\alpha+p_{zx}\cos\gamma+p_{yx}\cos\beta=q_x, \quad , \quad , \quad . \quad (7)$$

$$p_y\cos\beta+p_{zy}\cos\gamma+p_{xy}\cos\alpha=q_y, \quad . \quad . \quad . \quad (8)$$

$$p_z\cos\gamma+p_{xz}\cos\alpha+p_{yz}\cos\beta=q_z. \quad . \quad . \quad . \quad (9)$$

If ρ be the angle between the axis of X and the direction of q, then will

$$\cos\rho_1=\frac{q_x}{q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eqs. (8) and (9) give similar values of the cosines of the angles between the direction of q and the axes of Y and Z, thus fixing the direction of q.

Using the values of q_x, q_y, and q_z as given in eqs. (7), (8), and (9), the component of q normal to its plane of action ($ABC=\Delta$) will be:

$$q_n=q_x\cos\alpha+q_y\cos\beta+q_z\cos\gamma. \quad . \quad . \quad . \quad (11)$$

Hence the obliquity ϕ of q can at once be determined by the equation

$$\cos \phi = \frac{q_n}{q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

The triangular face XYZ of the tetrahedron Fig. 1 may be considered a part of the exterior surface of a body on which acts the stress whose intensity is q. The three rectangular coordinates faces XOY, YOZ, and ZOX are then to be taken as interior surfaces of the body on which act the internal stresses indicated. The stress on the external face XYZ must be in equilibrium with the stresses acting on the three interior rectangular coordinate faces of the tetrahedron.

Principal Stresses and Ellipsoid of Stress.

The preceding equations are general and relate to stresses on any planes whatever. If, however, the stress q is normal to its plane of action it is a principal stress. In that case the obliquity is zero and there is no shear. Hence,

$$q_x = q \cos \alpha; \quad q_y = q \cos \beta; \quad q_z = q \cos \gamma. \quad . \quad . \quad . \qquad (13)$$

Substituting these values in the second members of eqs. (7), (8), and (9), and then eliminating $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ from the three resulting equations, the following equation of the third degree will be found:

$$q^3 - (p_x + p_y + p_z)q^2 + (p_x p_y + p_x p_z + p_y p_z - p^2_{xy} - p^2_{zx} - p^2_{zy})q \\ + p_x p^2_{zy} + p_y p^2_{zx} + p_z p^2_{xy} - p_x p_y p_z - 2 p_{xy} p_{zx} p_{yz} = 0. \quad . \quad . \quad . \qquad (14)$$

Or, indicating the coefficients of q and the part of this equation independent of that quantity by A, B, and C, respectively:

$$q^3 - Aq^2 + Bq - C = 0. \quad . \quad . \quad . \quad . \quad . \quad (15)^*$$

The three roots of this cubic equation are the intensities of the three principal stresses, and the equation shows that at every point three such principal stresses exist, each normal to its plane of action on which there is no shear.

If in eq. (6) the coordinate axes of X, Y, and Z be taken as the principal axes so that the intensities p_x, p_y, and p_z become the principal intensities q_1, q_2, and q_3, then will $p_{xy} = p_{yz} = p_{xz} = 0$, and q will be the intensity of stress in any direction on a plane whose normal makes the angles α, β, and γ with the coordinate axes, i.e., with q_1, q_2, and q_3. Hence

$$q = \sqrt{q_1^2 \cos^2 \alpha + q_2^2 \cos^2 \beta + q_3^2 \cos^2 \gamma} \quad . \quad . \quad (16)$$

Again, if q_x, q_y, and q_z are the rectangular components of q, Eqs. (7), (8), and (9), will, under the same conditions, give:

$$q_1 \cos \alpha = q_x; \quad q_2 \cos \beta = q_y; \quad \text{and} \quad q_3 \cos \gamma = q_z.$$

Then, squaring and adding:

$$\frac{q_x^2}{q_1^2} + \frac{q_y^2}{q_2^2} + \frac{q_z^2}{q_3^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

---

[*] Rankine observed in his Applied Mechanics that if q_1, q_2, and q_3 are the roots of a cubic equation, then:

$$(q - q_1)(q - q_2)(q - q_3) = q^3 - q^2(q_1 + q_2 + q_3) + q(q_1q_2 + q_2q_3 + q_1q_3) - q_1q_2q_3 = 0.$$

This equation shows that the quantities A, B, and C remain the same whatever may be the directions of the three rectangular axes at a given point. Hence, by using A it is seen that $q_1 + q_2 + q_3 = p_x + p_y + p_z$, i.e., the sum of the normal components of the intensities of stress on any three rectangular coordinate planes is constant and equal to the sum of the intensities of the three principal stresses.

By dividing this equation through by q^2 it may be written in terms of the angles between q and the coordinate axes (eq. 18) and the reciprocal of q^2.

Eq. (17) is the usual form of the equation of an ellipsoid with the origin of coordinates at its center in which q_1, q_2, and q_3 are the semi axes and q_x, q_y and q_z are the coordinates of any point in the surface.

The intensity of stress q in any direction, represented by the semi-diameter of the ellipsoid in that direction, is given by eq. (16), and the angles between its direction and the coordinate axes X, Y, and Z may, by the aid of eq. (10), be written:

$$\left.\begin{array}{c} \cos\rho_1 = \frac{q_x}{q} = \frac{q_1 \cos\alpha}{q}; \quad \cos\rho_2 = \frac{q_y}{q} = \frac{q_2 \cos\beta}{q}; \\ \cos\rho_3 = \frac{q_z}{q} = \frac{q_3 \cos\gamma}{q}. \end{array}\right\} \quad . \quad (18)$$

The component of q normal to its plane of action is given by eq. (11):

$$q_n = q_1 \cos^2\alpha + q_2 \cos^2\beta + q_3 \cos^2\gamma. \quad . \quad . \quad (19)$$

The cosine of the obliquity of q is therefore:

$$\cos\phi = \frac{q_n}{q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

These elementary considerations are sufficient for the purpose of outlining to some extent at least the general subject of stress in any or all directions in solid bodies. The results may easily be developed, so as to be applicable to the solution of any required problem. The equations (2), (3), and (4) are frequently applied to the discussion of the action of external forces q_x, q_y, and q_z, in connection with the internal stresses p_x, p_y and p_z, etc., as will be indicated later.

It is obvious that if all the internal stresses act parallel to one plane, eq. (14) and those which follow it will relate to the ellipse of stress, showing that the latter is a special case of the ellipsoid of stress.

Art. 12.—Ellipse and Ellipsoid of Strain.

It has been shown that the intensity of stress at any point in a solid homogeneous body may be represented by the semi-diameter of an ellipsoid in the general case or the semi-diameter of an ellipse in the special case of all stress being parallel to a plane. Inasmuch as strains are proportional to the corresponding stresses below the elastic limit, the strain of a very short but constant length of a solid element at any point would be represented by the semi-diameter of an ellipsoid or ellipse having the same direction as the corresponding intensity, which also might be represented by the same semi-diameter at a proper scale. It follows from these simple considerations that strains in all directions may be represented by ellipsoids and ellipses as well as stresses. While such ellipsoids and ellipses possess analytic interest in connection with the theory of elasticity in solid bodies, they are not of sufficient importance in the structural operations of engineering to require even elementary analytic treatment.

Art. 13.—Orthogonal Stresses.

When stresses of tension or compression at right angles to each other concur either in one plane or on three coordinate planes making right angles with each other, as in the cases of the ellipse and ellipsoid of stress, they are said to be orthogonal stresses. Such stresses produce partially independent strains in the directions in which they act, but the resultant stress on any one plane is a single stress obviously accompanied by its characteristic strain. This is true whether the stress is wholly parallel to one plane or if it acts in all directions. The fact that lateral and direct strains in the same directions may concur has induced some engineers and writers to attempt to provide rather arbitrarily for the supposed effects of orthogonal stresses and strains.

If in the case of stress wholly parallel to one plane p_x and p_y represent the intensities of the principal stresses, as in Art. 7, the unit strain parallel to the axis of x will be,

$$l_x = \frac{p_x}{E} \pm r\frac{p_y}{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly the unit strain in the direction of y will be,

$$l_y = \frac{p_y}{E} \pm r\frac{p_x}{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the preceding eqs. (1) and (2) the plus sign is to be used if the intensities p_x and p_y are of opposite kinds, but the minus sign is to be written if the two stresses are of the same kind, i.e., both tension or both compression.

Two intensities of stress p'_x and p'_y are then assumed, each of which if acting separately would produce the strains in the two coordinate directions, respectively, shown by eqs. (1) and (2). These two intensities must have the following values:

$$p_x \pm r p_y \quad \text{and} \quad p_y \pm r p_x. \quad . \quad . \quad . \quad . \quad (3)$$

These are called "equivalent" intensities of stress, and it is postulated that the working intensity of stress prescribed for any member of a structure must not exceed the greatest of the two values given by eq. (3).

In the special case of two principal stresses being of opposite kinds but of equal intensity, the greatest shear will be of the same intensity as the principal stresses, or by the aid of eq. (3)

$$p_x = p_t = -p_y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or, combining eqs. (3) and (4),

$$p'_x = p_t + rp_t = (1+r)p_t, \quad . \quad . \quad . \quad . \quad (5)$$

hence,

$$p_t = \frac{p'_x}{1+r}.$$

In the latter case it is said that the greatest shear must not exceed p_t in eq. (6), p'_x representing the prescribed working intensity in tension or compression as the case may be.

This arbitrary substitution of an intensity of stress corresponding to the sum of two coordinate strains, in the place of an actual greatest intensity of stress acting on its proper plane, is not supported by any substantial analytical or experimental basis. The maximum intensity of stress at any point in a piece of material subjected to loading may readily be determined and the position of the plane on which it acts may be ascertained by the methods given in the preceding articles, and it is difficult to imagine any sufficient reason for not making that actual maximum intensity of stress equal to the prescribed working stress of the same kind. The maximum intensity of stress at any point will of course be accompanied by the maximum unit strain and a proper limitation of that strain will be coincident with a proper limitation of the stress producing it. These observations are equally true whether the kind of stress involved be tension, compression, or shearing.

The substitution of an artificial " equivalent " stress, therefore, in the place of the actual maximum stress at any point remains to be justified and will not be employed in this work. All the design work involving the employment of a prescribed working stress will be based upon the greatest actual intensity of stress in the structural member under consideration.

Again, the significance of lateral strains has been expressed by stating that if a straight bar of structural steel with square cross-section, for illustration, be subjected to a tensile stress of intensity p, the lateral strains will be negative, as they decrease the lateral dimensions of the bar, and hence that if the ratio of lateral to direct strains be taken as one-fourth, then those lateral strains are each precisely the same as would be produced by an intensity of compression equal to $\frac{1}{4}p$, acting at right angles to the bar and on either pair of opposite sides. Hence, it has been said that such a bar is not only subjected to the axial tension, but also to a " true internal stress which acts as a compression at right angles to the axis of the bar." It is further stated that such a bar " suffers a true internal compressive unit stress . . . in all directions at right angles to its length . . ."

It is still further stated that " The injury done to a body does not depend upon the actual stress or pressure, but upon the actual deformations produced, and the true stresses are those corresponding to these deformations."

It is difficult to imagine how the " actual " stress existing at any point in a body fails to be the "true" stress. If the "true" stresses are different from the actual they must be imaginary or at least not actual or real.

It cannot be admitted that the lateral strains accompanying the direct strains of a bar subjected to axial tension are produced by "a true internal" compression, for no such corresponding external compressive forces or pressure at right angles to the axis of the bar exist. If the lateral strains were due to such compressive stress, the corresponding external compressive forces would perform work and would make the total resilience of the bar two-ninths greater than the resilience due to direct tensile stress only, if the ratio r be taken as one-third.

This species of confusion seems to arise at least partly from a failure to distinguish between molecular conditions below the elastic limit and those above that limit.

If a bar is subjected to axial tension producing corresponding axial and lateral strains, in consequence of which the lateral dimensions of the bar decrease, it by no means follows that actual compression has produced that decrease. In fact, since the molecules have been separated to a slight degree axially, the transverse movement of the molecules may easily be conceived to take place without any compression whatever, and the fact that the density of the material is decreased by tensile stress makes that view reasonable, and perhaps conclusively confirms it. It should be remembered that all these analytic investigations relate only to stresses and strains existing below the elastic limit.

While it is true that experimental investigation is still lacking to give complete information regarding the effects of orthogonal stresses and strains below the elastic limit (as well as above it) there is lacking material evidence showing the existence of any such stress conditions consequent upon the existence of lateral strains as those to which allusions are made above, and they will not be recognized in the analytic work which is to follow.

In discussing the stresses in the walls of thick cylinders in Appendix II, the bearing of these considerations on the formula of Clavarino will be fully set forth.

PROBLEMS FOR CHAPTER I.

Problem 1.—A wrought iron bar $4'' \times \frac{1}{2}''$ in section is subjected to a tensile force of 28,000 pounds. The stretch for a gaged length of 20 feet was 0.12 inch. Find the intensity of tensile stress in the material, the modulus of elasticity E, and the rate of strain, i.e., the strain per linear inch.

Ans. Intensity of stress = 14,000 lbs. per square inch.
$E = 28,000,000$ pounds per square inch.
Rate of strain = 0.0005 inch per inch.

Problem 2.—A steel eye-bar $8'' \times 2''$ in section carries a total load of 128,000 pounds, under which there is a stretch of 0.016″ in a gaged length of 5 ft. Find the intensity of stress, rate of strain, and modulus of elasticity E.

Problem 3.—Steel has a modulus of elasticity of 30,000,000 pounds per square inch, and a coefficient of expansion of 0.0000065 per degree F. If a steel bar $2'' \times 4''$ in cross-section has a length of 30′ 0″ at a temperature of 40° F., find the length of the bar at 10° F. and at 110° F. Suppose the ends of this bar had been fastened rigidly at the temperature of 40° F. Find the intensity of tensile stress at 10° F. and intensity of compressive stress at

110° F., supposing the bar to be firmly held against lateral deflection.

Partial Ans. Length of bar at 10° F. = $29'.99415$.

Intensity of tensile stress in bar at a temperature of 10° F. = 5850 pounds per square inch.

Problem 4.—A concrete pillar 24″×24″ in section and 8 ft. high carries a total (compressive) load of 115,200 pounds. If the modulus of elasticity for the concrete is 2,500,000 pounds per square inch, what will be the rate of compressive strain and the shortening, first, for the total height 8 ft. of pillar, and, second, for 12″, under the preceding load?

Problem 5.—In Problems 1 and 2, if Poisson's ratio r (i.e., the ratio of lateral to direct strain) is 0.3, find the new cross-dimension of the bars and also the change in volume for a portion of each bar 1 foot long.

Ans. for Problem 1.

$d = 3''.99916$; $b = 0''.499895$;

change in volume = 0.00908 cubic inch decrease.

Problem 6.—In Problem 3, the cross-dimensions of the bar, under the compressive stress, become $2''.000114$ and $4''.000228$. Find the ratio r between direct and lateral unit strains, and also the increase of volume of 3 ft. length of the bar.

Problem 7.—In Problems 5 and 6 find the modulus of elasticity, G, for shearing in terms of the direct modulus of elasticity E.

Problem 8.—In Problem 2 find the total normal and tangential stresses and their intensities on plane sections making angles of 18°, 35°, and 53° with the axis of the piece.

Problem 9.—In Problem 3 find the total normal and tangential stresses and their intensities on plane sections making angles of 31°, 45°, and 72° with the axis of the piece.

Problem 10.—A round steel bar 3 inches in diameter is subjected to a tensile stress of 212,100 pounds. If the diameter of the bar decreases 0.00105 inch, find the ratio r between the direct and lateral strains, and also the increase of volume in a 4-ft. length of bar. Assume modulus of elasticity E as 30,000,000 pounds per square inch.

Problem 11.—Given three planes, AO, OB, and BA, Art. 8, so placed that $AOB=90°$ and $ABO=\alpha=60°$.

The tensile stress on OB is $p_x=3500$ pounds per square inch and the tensile stress on OA, $p_y=5600$ pounds per square inch. The shearing stresses on OA and OB are equal, i.e., $p_{xy}=p_{yx}=1750$ pounds per square inch.

Find the normal and tangential components of the resultant intensity p, when p makes an angle $\phi=10°$, below the normal to the plane AB. Also find the intensity of the principal stress on the plane AB.

Problem 12.—In Fig. 1, Art. 8, let the intensity of the normal tensile stress on the plane OB be 8000 pounds per square inch, while the intensity of normal compressive stress on the plane OA is 12,000 pounds per square inch, and let the intensities of shearing stresses on the same planes OB and OA be 3500 and 6500 pounds per square inch respectively. Find the principal stresses and the principal planes on which they act. Then, by means of the formulæ of Art. 9, find the greatest intensity of shearing stress on any plane at O, and the position of that plane. Finally, determine the intensity of the stress of greatest obliquity at the point O, and the plane on which it acts, together with the intensity of shearing stress on that plane.

CHAPTER II.

FLEXURE.

Art. 14.—The Common Theory of Flexure.

A STRAIGHT piece or bar of material is subjected to flexure or bending when it is acted upon by loads or forces at right angles to its axis, the loads and supporting forces taken as a whole constituting a system in equilibrium.

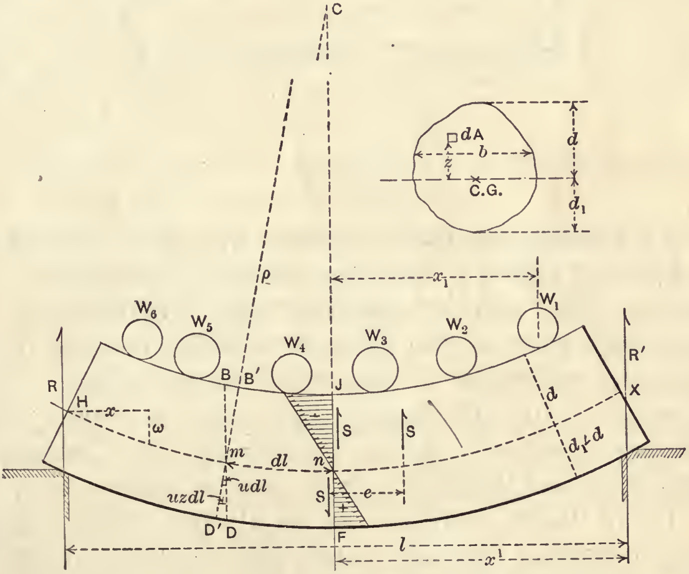


FIG. 1.

The beam shown in Fig. 1 may be taken to illustrate the general condition of flexure or bending.

Each end of the beam is supported as shown at R and R', the reactions at those points constituting the supporting forces, while the weights W_1 and W_2, etc., constitute the loading. The reactions are in reality just as much loads on the beam as the weights carried by it, but it is convenient always to make the distinction between loads and reactions or supporting forces.

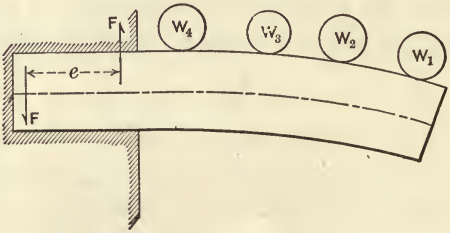


FIG. 2.

An overhanging beam is shown in Fig. 2 carrying the weights W_1 and W_2, etc., one end being firmly fixed in a wall or other similar supporting mass. In this case the supporting effect of the material in which one end of the beam is embedded is equivalent to the couple whose moment is Fe. Obviously there may be many other different cases of bending, according to the manner of supporting and loading the bent piece or beam.

In all these analyses and in all that follow, except when otherwise specially noted, the beams are supposed to be horizontal with the loads and reactions vertical, all external forces thus acting at right angles to the axis of the beam, and they are further supposed to lie all in a vertical plane passing through the same axis. When the loading acts as shown in Fig. 1, it is evident that the beam will be bent so as to become convex downward and concave upward, thus causing the upper portion of the beam to be in compression while the lower portion is in tension. Hence if any normal section of the beam as *BD* be considered, in passing from *B* where there is compression to *D* where there is an opposite stress of tension it is clear that at some point, as *m*, there will be a zero stress, or, in other words, no stress at all. The horizontal line passing through that point *m* of no stress, and normal to the vertical plane through the axis of the beam, is called the neutral axis of the section and its locus *HX* throughout the entire beam is called the neutral surface. On one side of the neutral axis in any normal section there will be direct stresses of compression and on the other direct stresses of tension. There are two fundamental assumptions in the common theory of flexure:

First, that all plane normal sections of the beam remain plane after flexure or bending.

Second, that the intensity (amount uniformly distributed on a square unit) of either the tensile or compressive stress in any normal section acting parallel to the axis of the beam varies directly as the distance from the neutral axis of the section.

In Fig. 1 the shaded triangles above and below *m*, having the common vertex at that point, represent the stresses of tension and compression induced in the normal section *BD* by the bending.

The loads and supporting forces act normally to the axis of the beam upon either portion of it, as *HBD*, while the internal stresses of tension and compression in the section *BD* act parallel to that axis. If the equilibrium of the same portion *HBD* be considered, it will be seen that the only horizontal forces acting upon it are the internal stresses of tension and compression shown by the two shaded triangles. Hence in order that there may be equilibrium the sum of those stresses of tension and compression must be equal to zero. This latter condition will determine in a simple manner the position of the neutral axis. If a is the intensity of either the tensile or compressive stress at the distance unity from the neutral axis, then by the second of the preceding fundamental assumptions the intensity N, at any other distance z from the same axis or line of no stress, will be $N = az$. Again, if A is the area of the normal section of the beam, dA will be the area of an indefinitely small portion of that section, so that the amount of internal stress acting on it will be $az.dA$. If this differential amount of stress be integrated for the entire section, the preceding condition of equilibrium for either portion of the beam requires that the sum represented by that total integration shall be equal to zero; or if d_1 and d represent the distances of the most remote fibres on either side of the neutral axis, the following equations may be written:

$$\int_{-d}^{d_1} azdA = a\int_{-d}^{d_1} zdA = 0,$$

or

$$\int_{-d}^{d_1} zdA = 0. \quad \dots\dots\dots \quad (1)$$

Eq. (1) shows that the static moment of the entire section about the neutral axis is equal to zero, and therefore that the neutral axis passes through the centre of gravity or the centroid of the normal section.

It is next necessary to determine the expression for the bending moment of the internal stresses of any section, such as JF of Fig. 1, which is induced by and must be equal to the moment of the external forces acting upon either one of the two portions into which the beam is divided by that section.

In Fig. 1, let *mn* represent a differential length, *dl* of the neutral surface, and let ρ represent the radius of curvature of *dl* after flexure, also as shown in Fig. 1, *C* being the centre of curvature. If *u* is the direct or longitudinal strain of a unit length of fibre at the distance unity from the neutral axis, when stressed with the intensity *a*, the strain in *dl* under that intensity will be *udl*. *BD* is drawn parallel to *JF*, and represents the position of *BD* before flexure. The triangle *D′mD* is, therefore, similar to *Cmn*. Consequently there may be written

$$\frac{udl}{1}=\frac{dl}{\rho}; \quad \therefore u=\frac{1}{\rho}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Or *the rate of strain*, i.e., the strain of a unit length of fibre at distance unity from the neutral axis, *is equal to the reciprocal of the radius of curvature.*

By the fundamental law or assumption of the common theory of flexure already given

$$\textit{Rate of strain at distance } z=\frac{z}{\rho}.$$

Then, by the fundamental law between stress and strain, the intensity N of the direct stress at any distance z is

$$N=E\frac{z}{\rho}=Euz. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If b is the variable breadth of section, the differential of the total stress is

$$Nbdz=\frac{E}{\rho}.(bdz).z. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The differential moment of the internal stresses about the neutral axis will be

$$dM = N . bdz . z = \frac{E}{\rho} . (bdz) . z^2; \quad . \quad . \quad . \quad . \quad (5)$$

$$\therefore M = \frac{E}{\rho} \int_{-d}^{d_1} (bdz) . z^2 = \frac{EI}{\rho}; \quad . \quad . \quad . \quad . \quad (6)$$

in which I is the moment of inertia of the section of the beam about the neutral axis.

If x is the horizontal coordinate of the neutral surface, and w the deflection or sag of the beam at any point, as indicated in Fig. 1, when the curvature is small

$$\frac{1}{\rho} = \frac{d^2w}{dx^2}$$

and

$$M = EI \frac{d^2w}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eq. (7) is the fundamental equation by which the deflection of a bent beam is found, whatever may be the character or amount of the loading. As indicated, it is strictly true only when the deflections are small; in other words, when they are produced by strains within the elastic limit of such beams as are ordinarily used in engineering practice. That equation is easily integrated in all ordinary cases, if the value of the external bending moment M is expressed in terms of x, as will be abundantly illustrated in succeeding articles.

Another equation of great practical value remains to be established. Let it first be observed that the intensity of stress a, at the distance of unity from the neutral surface of a bent beam is $a = Eu$, by Hooke's law, and further by eq. (2)

$$a = Eu = \frac{E}{\rho}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

If the value of $\frac{E}{\rho}$ from eq. (8) be substituted in eq. (6) there will result

$$M = aI. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

If the greatest intensity of stress in a normal section of a bent beam at the distance d_1 from the neutral axis be represented by k, then $a = \frac{k}{d_1}$, and eq. (9) will take the form

$$M = \frac{kI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Eq. (10) is one of the most important equations in the whole subject of the resistance of materials in consequence of its frequent use in the practical operation of designing beams or girders. Its employment is rendered exceedingly simple and convenient by tables in which may be found computed the moments of inertia I for all the rolled sections, as well as values of the quantity $\frac{I}{d_1}$, called the "section modulus." These tables are found in the various "Hand-books" published by steel-producing companies, and they obviate essentially all numerical computations for the determination of either moment of inertia or section modulus. Other tables may also be found which give the moments of inertia of a great variety of built sections, i.e., composite sections formed of various commercial rolled shapes such as plates, angles, channels, and I beams.

In all the preceding expressions where the quantity

M appears it is to be taken to represent the bending moment of the external forces, including the reactions, applied to a beam, the moment being taken about the neutral axis of the section under consideration. This external moment must necessarily be equal to the moment of the internal stresses represented by the last members of the preceding moment equations involving the greatest intensity of stress *k* of the section and the moment of inertia *I* of the latter.

There are one or two approximate features involved in the preceding analysis, the character of which is not discoverable when the fundamental laws of the theory of flexure are assumed rather than demonstrated, but which appear plainly evident in the true demonstration of the theory of flexure in App. I. It is obvious that the compression produced at the exterior surface of a bent beam at the points of loading is neglected or ignored in the preceding demonstrations; but this does not sensibly affect the accuracy of the formulæ which have been reached. There is, however, one result of the assumptions made which materially affects the accuracy of the formulæ of the common theory of flexure for comparatively short beams. If the accurate analysis be followed it will be found that the formulæ of that theory involve in reality the further assumption that the depth of the beam, i.e., in the direction of the loading, is small in comparison with the length of span. The limit of ratio of length of span to depth above which the formulæ may be applied with strict accuracy cannot be definitely assigned, but there are many beams, especially of timber, employed in engineering practice which are much too short in comparison with their depths to permit an accurate application of the formulæ of the common theory of flexure. This observation bears with special emphasis on computations for pins in pin-connected bridges which are treated as short beams. As a matter of fact, the common theory of flexure cannot be applied to such short thick beams with any degree of accuracy whatever. It is, however, entirely permissible to use these formulæ as general expressions, even under such loosely approximate conditions, into which empirical quantities established under the actual conditions of use are introduced, but they are not to be used in any other way. By such a procedure the formulæ of the common theory of flexure have become of inestimable value to the civil engineer, but it is imperative to realize under what conditions they may be employed with strict accuracy and under what conditions the introduction of quantities established by practical tests is required.

Art. 15.—The Distribution of Shearing Stress in the Normal Section of a Bent Beam.

The longitudinal fibres of a beam under loading take their stresses of tension and compression from the shearing stresses which are induced on vertical and horizontal planes in the interior of the beam. In order to realize what takes place in the interior of a beam let it be supposed to be divided into an indefinitely large number of small rectangular portions like those shown in the upper part of Fig. 1, and on a somewhat larger scale in the lower part. The vertical loading and reactions induce transverse shears, i.e., shearing stresses on vertical transverse planes, which, as known from the general theory of stresses in solid bodies, induce shears of equal intensity on horizontal planes. The result is that which is shown in the lower portion of Fig. 1. On the faces of the indefinitely small rectangular portions of the beam there are induced shears in pairs having the same intensity and acting either toward or from a given edge. Each horizontal layer of the beam is, therefore, made to slide a little over the adjoining layers above and below it, as shown at *A* and *A′* in the lower part of Fig. 1.

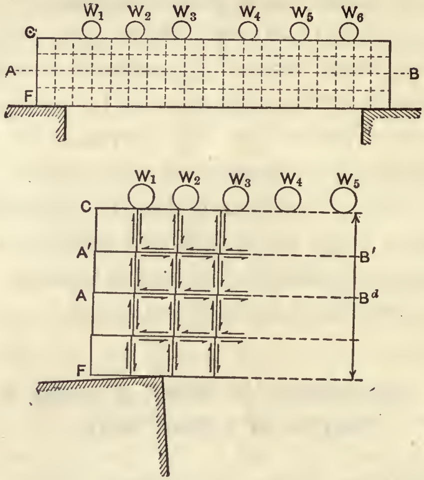


FIG. 1.

Carefully remembering these general conditions, let the bending moment in the section *ad* of the beam in Fig. 2 be represented by *M* and let the total transverse shear at

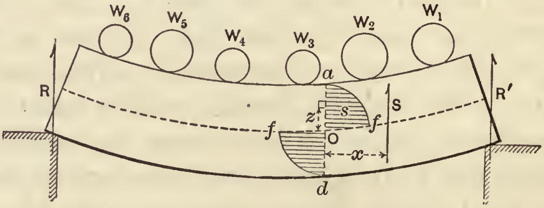


FIG. 2.

the same section be represented by *S*. Then if *x* measured horizontally from the section *ad* be so taken that $x = \frac{M}{S}$, and if the intensity of the direct stress of tension or compression at the distance z from the neutral axis be represented by k, there may at once be written

$$M = Sx = \frac{kI}{z}; \quad \therefore \ k = \frac{Sz}{I}x. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

k is thus seen to be a function of both z and x. If z be unchanged while x varies, the small variation of k for an indefinitely small variation of x will be

$$\frac{dk}{dx}dx = \frac{Sz}{I}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If s is the intensity of the transverse shear at the distance z from the neutral axis, the variation of that intensity for the indefinitely short distance dz (x remaining unchanged) will be $\frac{ds}{dz}dz$, and if the breadth or width of the beam is b, the variation of longitudinal shear on the small horizontal area bdx for the small distance dz will be

$$\frac{ds}{dz}dz(bdx). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The small shear given by expression (3) is equal to the variation of k found by multiplying the members of eq. (2) by bdz, hence

$$\frac{ds}{dz}dz \, . \, bdx = \frac{Sz}{I}dx \, . \, bdz; \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\therefore \ \frac{ds}{dz} = \frac{Sz}{I}, \quad \text{or} \quad ds = \frac{S}{I}zdz. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is obvious that the intensity of the shear at the exterior surface of the beam is zero; in other words, $s = 0$, when $z = d$ the distance of the extreme fibre of the section from the neutral axis. Hence eq. (5) must be integrated between the limits of z and d, and that integration will give

$$\therefore \quad s = \frac{S}{I}\int_{z}^{d} zdz = \frac{S}{2I}(d^2 - z^2). \quad . \quad . \quad . \quad (6)^*$$

---

* The intensity of shear s is sometimes found with a partial regard only to the laws of the Common Theory of Flexure. In Fig. 3 the piece *abcd* of a beam subjected to flexure whose neutral surface is *NN* is held in equilibrium by the direct stresses on the faces *bc* and *ad* in combination with the longitudinal shear on the face *dc*. If *ab* is equal to *dx* and if y be the normal distance of any fibre from *NN*, obviously the difference between the direct stresses on the two sides *bc* and *ad* will be $\int_{y}^{y_1} dk.bdy$ in which b is the variable width of the section. By the common theory of flexure, however, $dk = \frac{dM}{I}y$. Hence the above expression becomes $\frac{dM}{I}\int_{y}^{y_1} ybdy$. If s is the intensity of shear on the face *dc* the following equation at once results:

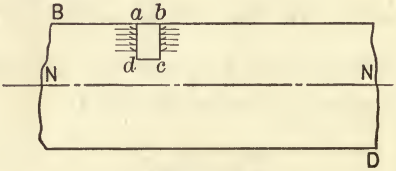


FIG. 3.

$$sbdx = \frac{dM}{I}\int_{y}^{y_1} ybdy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$\therefore$

$$s = \frac{dM}{dx}\frac{1}{bI}\int_{y}^{y_1} ybdy = \frac{S^y}{bI}\int_{y}^{1} ybdy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

This equation differs from eq. (6) in that b, considered as a variable, appears in the second member. If the section is rectangular, b is constant and eq. (6) at once results. In fact if y_1 and y be taken as consecutive in eq. (a), which is the differential method of establishing s, that equation will become

$$dsbdx = \frac{dM}{I}ybdy.$$

The quantity b now disappears from the equation whether the width of the section be considered constant or variable. Then dividing both sides of the

The intensity s has its maximum value where $z=0$, i.e., at the neutral axis; hence

$$(\text{max.})\ s=\frac{Sd^2}{2I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the section is rectangular $I=\frac{8bd^3}{12}=\frac{2bd^3}{3}$

and

$$(\text{max.})\ s=\frac{3}{2}\cdot\frac{S}{2bd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In other words, the maximum intensity of shear found at the neutral axis is $\frac{3}{2}$, the average shear of the entire section.

It is to be remembered that this intensity of shear s, at all points in the entire beam, acts on both the vertical and horizontal planes, i.e., this shear acts on longitudinal or horizontal planes parallel to the neutral surface as well as upon the vertical section of the beam.

Eq. (6) is the equation of a parabola with its vertex in the neutral surface. Hence if *Of* be laid off, as shown in Fig. 2, at any convenient scale to represent the maximum value of s, as given in eq. (7), and if from f as vertices the two branches of parabolic curves *fa* and *fd* be described as shown, any horizontal abscissa of the curves drawn from the line *ad* will represent the intensity of shear at that point. The origin of coordinates for eq. (6) is at O in Fig. 2.

---

equation by dx and integrating, eq. (6) of the text will be established. This means that all fibres equidistant from the neutral axis being stressed uniformly and hence without longitudinal shear along their vertical sides, the beam may be considered, so far as this analysis is concerned, as composed of vertical rectangular strips of width b, which may be of finite value or indefinitely small.

Distribution of Shear in Circular and Other Sections.

A number of special approximate investigations have been made to determine the distribution of shear in the circular cross-section of a bent beam, involving more or less complicated consideration of stresses. While these investigations recognize the straight line variation of the intensities of normal stresses in the section under consideration, they are based on other conditions which are not closely consistent with the fundamental assumptions of the Common Theory of Flexure.

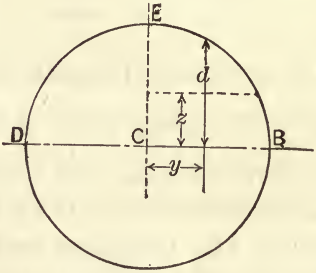


FIG. 4.

If the intensity of normal stress is the same at a uniform distance from the neutral axis of the section, adjacent fibres equidistant from that axis will stretch the same amount, eliminating all shearing stresses between such fibres. If therefore a circular section whose area is A be divided into vertical strips each with the width dy as shown in Fig. 4, and if the notation shown in that figure be observed, eq. (6) may be adapted to the circular section by placing in the second member of that equation, $S\frac{2ddy}{A}$ for S and $\frac{(2d)^3}{12}dy$ for I, resulting as follows:

$$s = \frac{3S}{2A}\left(1 - \frac{z^2}{d^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This equation gives the value of the intensity of shear in all parts of the circular section. If $z=d$, i.e., at all points of the surface, the intensity s is zero. The maximum intensity is found by making $z=0$, giving $s=\frac{3S}{2A}$, i.e., the maximum intensity of shear is $\frac{3}{2}$ the mean, as was to be expected. The same result will necessarily follow the same mode of treatment of any form of section whatever, as each such section is assumed to be made up of vertical rectangular strips between which no shear exists. The difference between this simple approximate method based upon results for a rectangular section and one of the special analyses for a circular section is shown by the maximum intensity of shearing stress at the neutral surface being found equal to $\frac{4}{3}$ (instead of $\frac{3}{2}$) of the mean by one of those special methods. If, however, the ordinary assumptions of the Common Theory of Flexure are to be made at all the advantage or increased accuracy of such special or more complicated analyses is not obvious.

With such material as timber, in the case of beams, the longitudinal shear represented by s in either eq. (7) or eq. (8) may be the governing quantity in design. The capacity of timber to resist shear along its fibres is comparatively so small that where the spans are relatively short failure will take place by shearing along the neutral surface before the extreme fibres yield either in tension or compression. In the design of timber beams, therefore, and in other similar cases, it is necessary to test by computation the maximum value of s as well as to determine the greatest intensity of tensile or compressive stress in the extreme fibres, as will be completely shown in a later article.

Art. 16.—External Bending Moments and Shears in General.

Beams subjected to pure bending only will be treated here.

A beam is said to be *non-continuous* if its extremities simply *rest* at each end of the span and *suffer no constraint whatever.*

A beam is said to be *continuous* if its length is equal to two or more spans, or if its ends, in case of one span (or more) suffer constraint.

A *cantilever* is a beam which overhangs its span, one end of which is in no manner supported. Each of the overhanging portions of an open swing bridge is a cantilever truss.

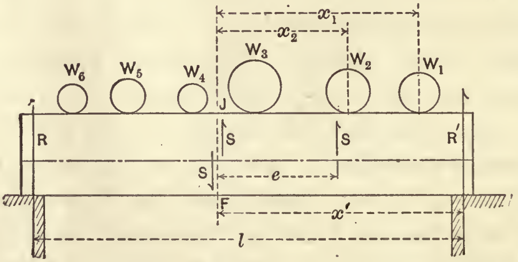


Fig. 1.

Fig. 1 represents a beam simply supported at each end, carrying the loads W_1, W_2, W_3, etc. Let bending moments be taken for any section, as JF, at the distance x' from the right-hand abutment, at which location the reaction R' acts. The load W_1 is at the distance x_1 from the section, W_2 at the distance x_2, and W_3 at the distance x_3 from the same section, the last distance not being shown in the figure. The bending moment desired will be the following:

$$M = R'x' - W_1x_1 - W_2x_2 - W_3x_3. \quad . \quad . \quad . \quad (1)$$

This equation is typical of all external bending moments for a beam simply supported at each end, whatever may be the system of loading or its amount, or whatever may be the location of the section. This equation is frequently written in the following form:

$$M = \Sigma Wx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The summation sign indicates that the sum is to be taken of the products formed by multiplying each external force, whether loading or reaction, by its lever-arm or normal distance from the section in question. It is a common and convenient mode of expressing the general value of the bending moment in any case whatever.

In eq. (1) the differentials of x', x_1, x_2, and x_3 are all equal, so that if that equation be differentiated, the first derivative of M will have the following form:

$$\frac{dM}{dx} = R' - W_1 - W_2 - W_3 = \Sigma W = S. \quad . \quad . \quad (3)$$

It will be at once evident that S in eq. (3) is the total transverse shear in the section for which the bending moment M is written, since the algebraic sum of R' and the loads between the end of the beam and the section constitutes that shear. Indeed, the usual manner of determining the total transverse shear is the simple operation of summing up all the external forces acting on one of the portions of the beam formed by the section in question; the external forces, such as the reaction, acting in one direction being given one sign, and those, like the loading, acting in the other direction being given the opposite sign. The shear, therefore, becomes the numerical difference of the two sets of forces having opposite directions.

Eq. (3) thus establishes the following important principle: *The total transverse shear at any section is equal* *to the first differential coefficient of the bending moment considered a function of x.*

In Fig. 1 the force S is supposed to be the resultant of the three loads W_1, W_2, and W_3, and the reaction R', i.e., the force S is supposed to represent that resultant both in line of action and magnitude. The bending moment M is, therefore, equal to Se, e being the normal distance of the line of action of S from the section, so that the actual bending moment upon any section of a bent beam may always be represented by the transverse shear, located as the resultant of all the external forces producing the bending moment, multiplied by its lever-arm. This is a simple but important matter of observation.

In the section JF let the two equal and opposite forces S and $-S$, numerically equal, act in opposite directions; they will not, therefore, affect the equilibrium of the beam or any portion of it in any way whatever. As far as the equilibrium of the portion of the beam JF is concerned, the loads and the reactions may be supposed to be displaced by the couple S, $-S$, with the lever-arm e, and the shear S acting upward in the section JF. The importance of this particular feature of the analysis consists in showing that in every bent beam carrying loads the action of the external forces (including the reaction) producing the bending is equivalent to a couple whose moment is Se acting about the neutral axis of the section and the total transverse shear S acting in the section. The shear S evidently tends to move or slide one portion of the beam past the other, and an essential part of the operation of designing beams and trusses is its determination at various sections with correspondingly various positions of loading.

As is well known, the analytical condition for a maximum or minimum bending moment in a beam is

$$\frac{dM}{dx}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From eqs. (3) and (4) is to be deduced the following principle: *The greatest or least bending moment in any beam is to be found in that section for which the shear is zero.*

The greatest bending moment obviously is the only one of importance in the design of beams and trusses, and eq. (4) shows that the section in which it will be found can be located by simple inspection of the loading. It is only necessary to sum up the reaction at one end and the loads adjacent to it, until the point is reached where the summation is zero. This point will usually be found where a load is supported. In that case the single load may arbitrarily be divided into two parts, supposed to act indefinitely near to each other, so that one of the parts may be just sufficient to make the zero summation desired. A single practical operation will make this feature perfectly clear and simple.

If the loading is uniformly continuous and of the intensity p, in each of the equations (1), (2), and (3) pdx is to be used for each of the separate loads W_1, W_2, W_3, etc. The bending moment thus becomes

$$M=R'x'-\Sigma Wx=R'x'-\int_0^x x\,.\,pdx=R'x'-\tfrac{1}{2}px^2. \quad (5)$$

The expression for the shear then becomes

$$\frac{dM}{dx}=S=R'-px. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A second differentiation gives

$$\frac{d^2M}{dx^2}=-p.. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Or, the second differential coefficient of the moment considered a function of x is equal to the intensity of the continuous load.

This method of passing from formulæ for concentrated loads to those for continuous loads is perfectly simple and frequently employed.

Art. 17.—Intermediate and End Shears.

The beam shown in Fig. 1 is supposed to carry any loading whatever, and the figure is consequently intended to exhibit a uniform load in addition to a load of concentrations. Inasmuch as all beams and other similar pieces have considerable weight, and sometimes great weight, ordinarily considered uniformly distributed over the span, this condition of loading is that which exists in all actual cases. The amount of uniform loading per linear unit, usually a foot, is represented by p, while the concentrations, as heretofore, are represented by W_1, W_2, etc.

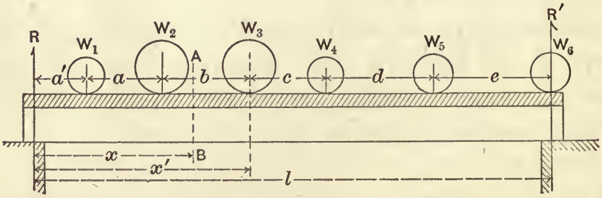


FIG. 1.

The determination of the transverse shear at any section of a beam or truss is one of the most frequent as well as one of the most important computations required in the design of structures. As has already been indicated, it is an extremely simple computation. It is first necessary, after knowing the position of the loading, to find the reactions at both ends of the span. In Fig. 1 the various weights or loads are separated by the distances shown, a' being the distance from W_1 to the reaction R or end of the span. W_6 is supposed to rest at the right end of the span for a purpose that will presently appear. The reaction R'' at the left end of the span (not shown) resulting from the concentrated loads only will have the following value:

$$R'' = W_1\left(\frac{a+b+\ .\ .\ .\ +e}{l}\right) + W_2\left(\frac{b+c+\ .\ .\ .\ +e}{l}\right)$$
$$+ W_3\left(\frac{c+d+e}{l}\right) + W_4\frac{d+e}{l} + W_5\frac{e}{l}. \quad (1)$$

The reaction R''' at the other end of the span (not shown) can be expressed by a similar equation, but it is simpler and more direct to write it as follows:

$$R''' = W_1 + W_2 + W_3 + W_4 + W_5 - R''. \quad . \quad . \quad (2)$$

Obviously the sum of the two reactions R'' and R''' must be equal to the total concentrated loading.

That part of the reaction due to the uniform load extending over the span l will clearly be one half of that load or

$$R_1 = \tfrac{1}{2}pl = R_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The reaction R_1 is supposed to be found at the left end of the span and R_2 at the right end. The total reactions then will be as follows. At left end of the span:

$$R = R'' + \tfrac{1}{2}pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At right end of the span:

$$R' = R''' + \tfrac{1}{2}pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The transverse shear at any intermediate section of the beam whatever may now readily be written. Let the section AB at the distance x from the left end of the span first be considered. The total loading between that section and the end of the span is W_1+W_2+px, and it acts downward. As the reaction R acts upward the expression for the shear will be

$$S=R-W_1-W_2-px. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In this case the section considered has been taken between two weights; let the section at the weight W_2 be considered, that weight being at the distance x' from the end of the span. The amount of uniform load over the length x' is simply px', but inasmuch as the weight W_3 is located at the section under consideration, the portion of that weight which may be taken as resting on the left of the section considered is indeterminate. In such cases it is proper and customary to take any portion or all of the weight as resting on either side of the section, but indefinitely near to it. If it is a case where the maximum shear is desired, the single weight should be taken in such a position as to make the transverse shear as great as possible. If the case is one where it is desired to find the section at which the total load from that section to the end of the span is equal to the reaction, any portion may be taken which is found necessary to make the equality. If, for instance, $px'+W_1+W_2$ is less than R while $px'+W_1+W_2+W_3$ is greater than R, then that portion of W_3 which would be considered on the left of the section but indefinitely near to it would be $R-px'-W_1-W_2$. The remaining portion of W_3 would be considered as resting at the right of the section but indefinitely near to it. In such a case the transverse shear is zero at the weight W_3.

Again, let it be desired to find the greatest upward shear at W_3, it being supposed that R is greater than the total load between W_3 and the left end of the span. In this case no portion of W_3 would be considered as acting to the left of the section, but the expression for the shear would be

$$S = R - px' - W_1 - W_2. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It can be seen from the preceding statements that the maximum transverse shear in the beam will occur at the ends of the span where the value of the shear is the end reaction. Inasmuch as the end reaction R or R' is thus the greatest shear in the entire span, it is a most important quantity to determine in the design of beams and trusses; it is the most important single factor in the determination of the amount of material required at the end sections of both beams and trusses. The value of this end shear is given by the values for R and R' in eqs. (4) and (5).

Since the total transverse shear in any section of a beam is simply the summation of all the external loads, including the reactions from one end of the span up to the section considered, it is evident, first, that that summation may be made from either end of the span, and second, that the amounts so found will be equal numerically but affected by opposite signs. In determining the shear, therefore, in any given case, it is usual to make the summation from that end of the span which can be used with the greatest convenience in computation.

Fig. 2 exhibits a graphical representation of the preceding treatment of intermediate and end shears, MN being the length of span shown in Fig. 1. MF is the reaction R laid off at a convenient scale. The weights or loads W_1, W_2, W_3, etc., are laid off vertically downward in their proper locations at the same scale, as shown. The vertical distance of G below F is the amount of uniform load pa' between R and W_1 in Fig. 1, also laid down by the same scale. GG_1 is, therefore, the shear in the beam of Fig. 1 immediately to the left of W_1, and H_1G_1 is the shear immediately to the right of the same load. Similarly, H_1H being drawn horizontally, HK is the amount of uniform loading pa between W_1 and W_2. The remainder of the diagram is drawn in the same manner.

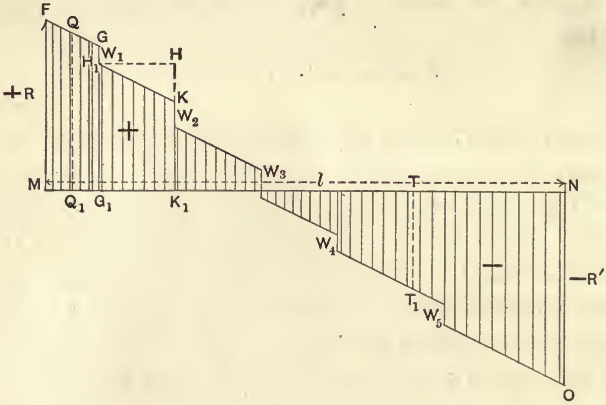


FIG. 2.

Any vertical ordinate drawn from MN either up or down to the broken line $FGH_1K \dots O$ represents the shear at the corresponding point in the span at the same scale used in laying off the reactions and loads. QQ_1 is the shear at the point or section of beam at Q_1, while TT_1 is the shear at the section T. The shear is zero at W_3 where it changes its sign. At that point also will be found the greatest bending moment in the beam.

As the diagram is drawn the shears on the left of W_3 and above MN are positive, those on the right of W_3 and below MN being negative; but the diagram might have been drawn with equal propriety so as to have made R' and the shears between it and W_3 positive and those between that load and R negative.

A glance at the diagram shows that the end shears, equal to the reactions, are the greatest in the span.

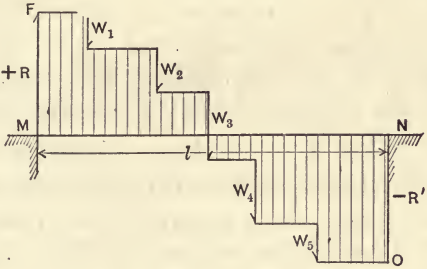


FIG. 3.

If a beam carries a load of concentrations only its shear diagram will be illustrated by Fig. 3, in which there are five loads, the diagram being composed of rectangles only. If, again, the load is wholly uniform Fig. 4 will represent the shear diagram composed of two triangles with their apices at C, the centre of the span and point of no shear. Any vertical ordinate drawn from MN in either figure

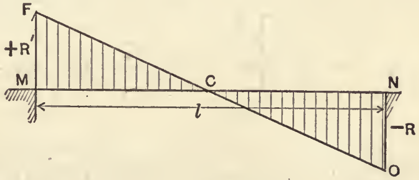


FIG. 4.

to the stepped line in the one case and to the straight line in the other will represent the shear at the section of beam from which the ordinate is drawn. Those diagrams represent completely the graphical treatment of shears in all cases.

Art. 18.—Maximum Reactions for Bridge Floor Beams.

Three transverse floor beams of a railroad bridge are represented in Fig. 1 separated by the two spans l_1 and l which, in a bridge, represent the panel lengths. The members *AB* and *BC* supporting the weights W_1, W_2, etc., indicate the stringers which carry the railroad track and the train. The two beams or stringers *AB* and *BC* are considered simple non-continuous beams resting on the floor beams, but not necessarily nor usually on their tops. The problem is to determine the position of the locomotive or other train loads on the adjacent two short spans l_1 and l, so that the reaction *R* on the floor beam between shall have its greatest value.

In Fig. 1 let a section of the beam be shown at *R*, and let x and x_1 be measured from the right ends of the two spans as shown in Fig. 1, while W_1, W_2, . . . W_4 represent a train of weights or wheel concentrations passing over the two spans from right to left. If R' and R are the reactions at *A* and *B*, respectively:

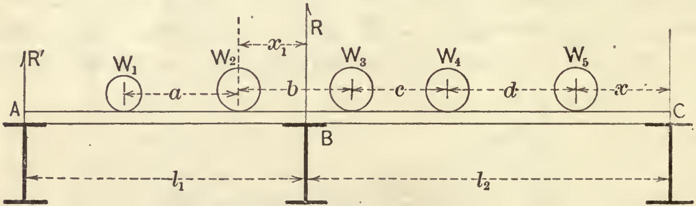


FIG. 1.

$$R'l_1 = W_1a + (W_1 + W_2)x_1 \;\therefore\; R' = \frac{W_1 + (W_1 + W_2)x_1}{l_1} \quad . \quad (1)$$

Then if the moments of weights and reactions be taken about C at the right-hand end of span l_2:

$$\begin{aligned} &R'(l_1+l_2)-(W_1a+(W_1+W_2)x)-(W_1+W_2)(b-x_1) \\ &\quad -(W_1+W_2+W_3)c-(W_1+ . \ . \ . +W_4)d \\ &\quad -\overset{l_1+l_2}{\Sigma} Wx+Rl_2=0. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2) \end{aligned}$$

Hence, since $R'l_1$ is equal to the quantity within the second parenthesis of the first member of eq. (2):

$$\begin{aligned} &(W_1a+(W_1+W_2)x)\frac{l_2}{l_1}-(W_1+W_2)(b-x_1)-(W_1+W_2+W_3)c \\ &\quad -(W_1+ . \ . \ . +W_4)d-\overset{l_1+l_2}{\Sigma} Wx+Rl_2=0. \quad . \ . \ . \ . \ . \ . \quad (3) \end{aligned}$$

In order that the reaction R may have its greatest value it must remain unchanged when a small movement of the train is made. If therefore $x+\Delta x$ and $x_1+\Delta x$ be written for x and x_1, respectively, in eq. (3) and if eq. (3) be subtracted from the result so obtained, the following equations will be found:

$$(W_1+W_2)\frac{l_2}{l_1}+(W_1+W_2)=\overset{l_1+l_2}{\Sigma} W,$$

$$\therefore \ \frac{l_1}{l_1+l_2}=\frac{W_1+W_2+\text{etc.}}{\overset{l.+l_2}{\Sigma} W} \quad . \ . \ . \ . \ . \quad (4)$$

Eq. (4) shows the position of loading for the greatest value of the reaction R. It means simply that the ratio between the amount of loading on span l_1 and the total load on both spans shall be the same as the ratio between the span l_1 and the sum of the two spans (l_1+l_2). Inasmuch as the load may move in either direction l_2 may be written for l_1 in the numerator of the first member of eq. (4).

Clearly the two weights W_1 and W_2 in the preceding equations represent all the loads resting on span l_1 whether there be two such weights or any number whatever. Similarly the weights indicated by the summation sign in the second member of eq. (4) represent the total load on both spans. If $l_1 = l_2$, as is usually the case, the first member of eq. (4) has the value of one-half.

As in all such cases there may be more than one position of the loading which will satisfy the criterion eq. (4); in that case it is necessary to determine which of those conditions will give the maximum of the " greatest values " of R.

Inasmuch as the sum of the weights on the span l_1 does not change for any value of x_1 equal to or less than b, it follows that a weight may be taken at the point of support B in satisfying eq. (4). This will simplify the use of eq. (3) in writing the expression for R. If $x_1 = b$ there may at once be written from eq. (3):

$$R = \frac{-(W_1 a + (W_1 + W_2) b)\frac{l_2}{l_1} + (W_1 + W_2 + W_3) c + (W_1 + \ldots + W_4) d + \sum^{l_1 + l_2} Wx}{l_2} \quad (5)$$

This equation gives the value of R desired, and it is so written that numerical values may readily be computed by the use of tables. If $l_1 = l_2$, as is usual, the ratio of those two quantities becomes unity.

Art. 19.—Greatest Bending Moment Produced by Two Equal Weights.

Fig. 1 represents a non-continuous beam with the span l supporting two equal weights P, P. These two weights or loads are to be kept at a constant distance apart denoted by a.

It is required to find that position of the two loads which will cause the greatest bending moment to exist in the beam, and the value of that moment. The reaction R is to be found by the simple principle of the lever. Its value will therefore be

$$R = \frac{l - \left(x + \frac{a}{2}\right)}{l} . 2P. \quad . \; . \; . \; . \; . \; . \quad (1)$$

Since the reaction R can never be equal to $2P$, ΣP, or the shear, must be equal to zero at the point of application of one of the loads P. In searching for the greatest moment, then, it will only be necessary to find the moment about the point of application of one of the forces P. It will be most convenient to take that one nearest R.

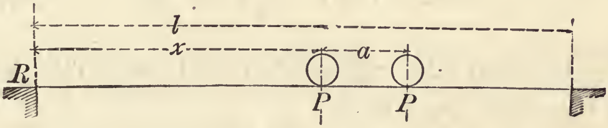


Fig. 1.

The moment desired will be

$$M = Rx = 2P\left(x - \frac{x^2}{l} - \frac{ax}{2l}\right). \quad . \; . \; . \; . \; . \quad (2)$$

$$\therefore \; \frac{dM}{dx} = 0 = 2P\left(1 - \frac{2x}{l} - \frac{a}{2l}\right);$$

$$\therefore \; x = \frac{l}{2} - \frac{a}{4}.$$

This value in eq. (2) gives

$$M_1 = \frac{P}{2l}\left(l - \frac{a}{2}\right)^2. \quad . \; . \; . \; . \; . \; . \; . \quad (\;\;)$$

Since

$$\frac{d^2M}{dx^2}=-\frac{4P}{l},$$

it appears that M_1 is a maximum.

The shear S in the section RP of the span will be the reaction R as given by eq. (1):

$$S=2P-\frac{2P}{l}\left(x+\frac{a}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Throughout the section a the shear S' will be

$$S'=S-P=P-\frac{2P}{l}\left(x+\frac{a}{2}\right). \quad . \quad . \quad . \quad . \quad (5)$$

Finally, between the right abutment and the nearest weight the shear S_1 will be

$$S_1=S-2P=-\frac{2P}{l}\left(x+\frac{a}{2}\right). \quad . \quad . \quad . \quad . \quad (6)$$

If the separating distance, a, between the two weights be increased a value may be reached so great as to make the bending moment of the pair of weights less than that produced by placing one of them at the centre of the span. This limiting value of a may easily be found. The moment at the centre of span produced by placing a single weight P there is

$$M'=\frac{P}{2}\cdot\frac{l}{2}=\frac{Pl}{4}.$$

By using eq. (3)

$$M'=M_1; \quad \therefore \frac{Pl}{4}=\frac{P}{2l}\left(l-\frac{a}{2}\right)^2. \quad . \quad . \quad . \quad (7)$$

By solving this equation

$$a = l(2 - \sqrt{2}) = .586l. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Whenever, therefore, the separating distance a is equal to or greater than .586 span length, the moment should be found by placing a single weight P at the centre of the span.

Art. 20.—Position of Uniform Load for Greatest Shear and Greatest Bending Moment at any Section of a Non-Continuous Beam — Bending Moments of Concentrated Loads.

A continuous load of uniform density is frequently employed in structural operations both for beams and trusses, and it is essential to place such a load so as to produce the greatest effect both for shears and moments. The position of loading for the greatest shear will first be found.

A continuous train of any given uniform density advances along a simple beam of span l. It is required to determine what position of loading will give the greatest shear at any specified section.

In Fig. 1, CD is the span l, and A is any section for

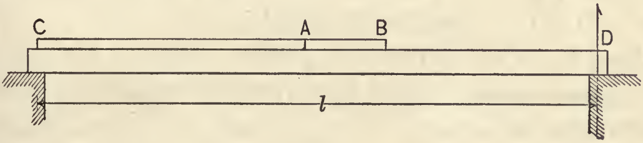


Fig. 1.

which it is required to find the position of the load for the greatest transverse shear. The load is supposed to advance continuously from C to any point B. Let R be the reaction at D, and ΣP the load between A and B. The shear S' at A will be

$$R - \Sigma P = S'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let R' be that part of R which is due to ΣP, and R'' that part due to the load on CA; evidently R' is less than ΣP. Then

$$R' + R'' - \Sigma P = S'.$$

If AB carries no load, R' and ΣP disappear in the value of S. Hence

$$R'' = S$$

is the shear for the head of the train at A. S is greater than S' because ΣP is greater than R'. But no load can be taken from AC without decreasing R''. Hence *the greatest shear at any section will exist when the load extends from the end of the span to that section, whatever be the density of the load.*

In general, the section will divide the span into two unequal segments. The load also may approach from either direction. The greater or smaller segment, then, may be covered, and, according to the principle just established, either one of these conditions will give a maximum shear. A consideration of these conditions of loading in connection with Fig. 1, however, will show that *these greatest shears will act in opposite directions.*

When the load covers the greater segment the shear is called a *main* shear; when it covers the smaller, it is called a *counter* shear.

The determination of the greatest bending moment at any section A of a beam or truss, exemplified by Fig. 1, traversed by a continuous train of uniform density is a very simple matter. It is clear that every part of the uniform load resting on the beam will produce bending at any section considered; and it is further obvious that every part of that uniform loading will create a bending moment at *A* of the same sign. It follows, therefore, that the entire span should be covered by the uniform train in order to produce a maximum bending moment at any section of the beam or truss, and that this single position of the train will give the maximum bending moment throughout the entire span.

The preceding position of moving load is taken only for a train of uniform density or for a series of uniform concentrations, each pair of which is separated by the same distance as every other pair, i.e., for a uniformly distributed system of uniform concentrations.

The general case of a simple beam loaded with any system of weights may be represented by Fig. 2, in which the beam carries three loads W_1, W_2, and W_3, spaced as shown. The reactions or supporting forces R and R' are determined in the usual manner by the law of the lever. Hence

$$R = W_3\frac{d}{l} + W_2\frac{d+c}{l} + W_1\frac{d+c+b}{l}. \quad . \quad . \quad . \quad (2)$$

A similar value may be written for R', but it is simpler after having found one reaction to write

$$R' = W_1 + W_2 + W_3 - R. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The beam itself being supposed to have no weight, the bending moments at the points of application of the loads will be

$$\left.\begin{aligned} M_1 &= Ra, \\ M_2 &= R(a+b) - W_1 b, \\ M_3 &= R(a+b+c) - W_1(b+c) - W_2 c. \end{aligned}\right\} \quad . \quad . \quad (4)$$

After substituting the value of R from eq. (2) in eqs. (4) the moments in the latter equations will be completely known.

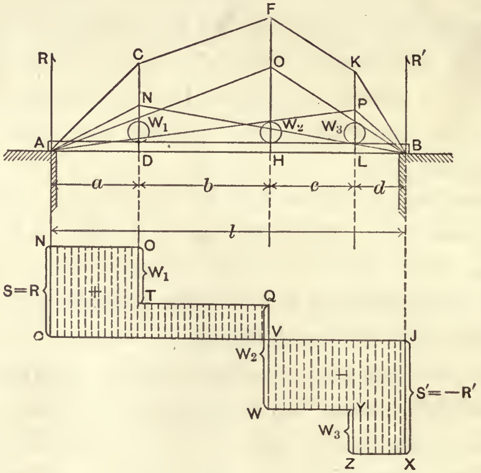


FIG. 2.

The bending moment produced by each weight will be represented by the ordinates of the triangles shown in Fig. 2, the resultant moments at the points of application of the weights being given by eqs. (4). The ordinate CD represents M_1 in eqs. (4) by any convenient scale. Similarly FH represents M_2 in eqs. (4), and KL, M_3. The lines AC, CF, FK, and KB are then drawn. Any vertical intercept between AB and the polygon $ACFKB$, found in the manner explained, will represent the bending moment at the point where the intercept is drawn, and to the scale at which M_1, M_2, and M_3 are laid down. This intercept is simply the sum of the intercepts of the triangles, each representing the partial bending moment due to a single weight.

Obviously the bending moments of any number of loads of any magnitude or of a uniform load, even, may be treated or represented in the same manner.

The lower portion of Fig. 2 is the shear diagram drawn precisely as explained for Fig. 3 of Art. 17.

Art. 21.—Greatest Bending Moment in a Non-Continuous Beam Produced by Concentrated Loads.

The position of the moving load for the greatest bending moment at any section of a non-continuous beam may be very simply determined. In Fig. 1, let FG represent any such beam of the span l, and let any moving load whatever, as $W_1 \ldots W_{n'} \ldots W_n$ advance from F toward G. Let C be the section at which it is desired to determine the maximum bending moment, and let n' loads rest to the left of C, while n is the total number of loads on the span. Finally, let x' represent the distance of $W_{n'}$ from C and to the left of that point, while x is the distance of W_n to the left of F. If a is the distance between W_1 and W_2, b the distance between W_2 and W_3, c the distance between W_3 and W_4, etc., the reaction R at G will be

$$R = \left\{ \begin{array}{l} W_1 \dfrac{a+b+c+\ldots+x}{l} \\ + W_2 \dfrac{b+c+\ldots+x}{l} \\ \ldots\ldots\ldots\ldots\ldots \\ + W_n \dfrac{x}{l} \end{array} \right. \quad \ldots\ldots \quad (1)$$

The bending moment M about C will then take the value

$$M = Rl' - \begin{cases} \quad W_1(a+b+c+\ \ldots\ +x') \\ +W_2(\quad b+c+\ \ldots\ +x') \\ \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot \\ +W_{n'}x'. \end{cases}$$

Or, after inserting the value of R from above,

$$\left.\begin{aligned} M = \frac{l'}{l}[&W_1a+(W_1+W_2)b+(W_1+W_2+W_3)c \\ &+\ \ldots\ +(W_1+W_2+W_3+\ \ldots\ +W_n)x] \\ &-W_1a-(W_1+W_2)b-(W_1+W_2+W_3)c \\ &-\ \ldots\ -(W_1+W_2+W_3+\ \ldots\ +W_{n'})x'. \end{aligned}\right\} \quad . \quad (2)$$

If the moving load advances by the amount $\varDelta x$, the moment becomes, since $\varDelta x = \varDelta x'$,

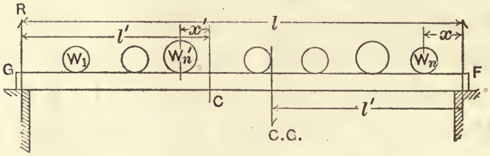


FIG. 1.

$$M' = M + \frac{l'}{l}(W_1+W_2+W_3+\ \ldots\ +W_n)\varDelta x$$
$$-(W_1+W_2+\ \ldots\ +W_{n'})\varDelta x. \quad (3)$$

Hence, for a maximum, the following value must never be negative:

$$M' - M = \varDelta x\left\{\frac{l'}{l}(W_1+W_2+W_3+\ \ldots\ +W_n)\right.$$
$$\left. -(W_1+W_2+\ \ldots\ +W_{n'})\right\} = 0. \quad (4)$$

Or the desired condition for a maximum takes the form

$$\frac{l'}{l} = \frac{W_1+W_2+\ \ldots\ +W_{n'}}{W_1+W_2+W_3+\ \ldots\ +W_n}. \quad . \quad . \quad . \quad (5)$$

It will seldom or never occur that this ratio will exactly exist if $W_{n'}$ is supposed to be a *whole* weight; hence $W_{n'}$ will usually be that part of a whole weight at C which is necessary to be taken in order that the equality (5) may hold.

It is to be observed that if the moving load is very irregular, so that there is a great and arbitrary diversity among the weights W, there may be a number of positions of the moving load which will fulfil eq. (5), some one of which will give a value greater than any other; this is the absolute maximum desired.

From what has preceded, it follows that $W_{n'}$ may always be taken at the point C in question; hence x' in eq. (2) may always be taken equal to zero when that equation expresses the greatest value of the moment. The latter may then take either of the two following forms:

$$\left.\begin{aligned} M = \frac{l'}{l}[W_1 a + (W_1 + W_2)b + \ldots + (W_1 + W_2 \\ + \ldots + W_n)x] - W_1 a - (W_1 + W_2)b \\ - \ldots - (W_1 + W_2 + \ldots + W_{n'-1})(?) \end{aligned}\right\}. \quad (6)$$

$$\left.\begin{aligned} M = \frac{l'}{l}[W_1(a + b + \ldots + x) + W_2(b + c + \ldots + x) \\ + W_3(c + d + \ldots + x) + \ldots + W_n x] \\ - W_1(a + b + \ldots + ?) - W_2(b + \ldots + ?) \\ - \ldots - W_{n'-1}(?) \end{aligned}\right\}. \quad (6a)$$

In these equations x corresponds to the position of maximum bending, while the sign (?) represents the distance between the concentrations $W_{n'-1}$ and $W_{n'}$.

The preceding equations give the greatest bending moments at any arbitrarily assigned points in the span. There remains to be determined the point at which the greatest moment in the entire span exists, and the magnitude of that greatest moment.

It has already been shown that for any given condition of loading the greatest bending moment in the beam will occur at that section for which the shear is zero. But if the shear is zero, the reaction R must be equal to the sum of the weights $(W_1+W_2+. . .+W_{n'})$ between G and C, the latter now being the section at which the greatest moment in the span exists.

Hence for that section eq. (5) will take the form

$$\frac{l'}{l}=\frac{R}{W_1+W_2+W_3+. . .+W_n}. \quad . \quad . \quad . \quad (7)$$

Hence

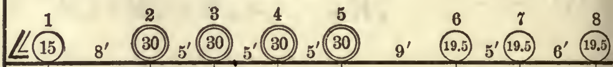
$$R=\frac{l'}{l}(W_1+W_2+. . .+W_n). \quad . \quad . \quad . \quad (8)$$

The relations existing in eqs. (7) and (8) can obtain only if the centre of gravity CG in Fig. 1 is at the distance l' from F, showing that the centre of gravity of the load is at the same distance from one end of the beam as the section or point of greatest bending is from the other. In other words, *the distance between the point of greatest bending for any given system of loading and the centre of gravity of the latter is bisected by the centre of span.*

If the load is uniform, therefore, it must cover the whole span.

It is to be observed that eq. (6) is composed of the sums W_1, W_1+W_2, etc., multiplied by the distances a, b, c, etc. Again, as in the equation immediately preceding eq. (2), the expression for the moment, M, may be taken as composed of the positive products of each of the single weights W_1, W_2, etc., multiplied by its distance from any point distant x to the right of W_n and of the negative products similarly taken in reference to the section located by x', as shown by eq. (6a).

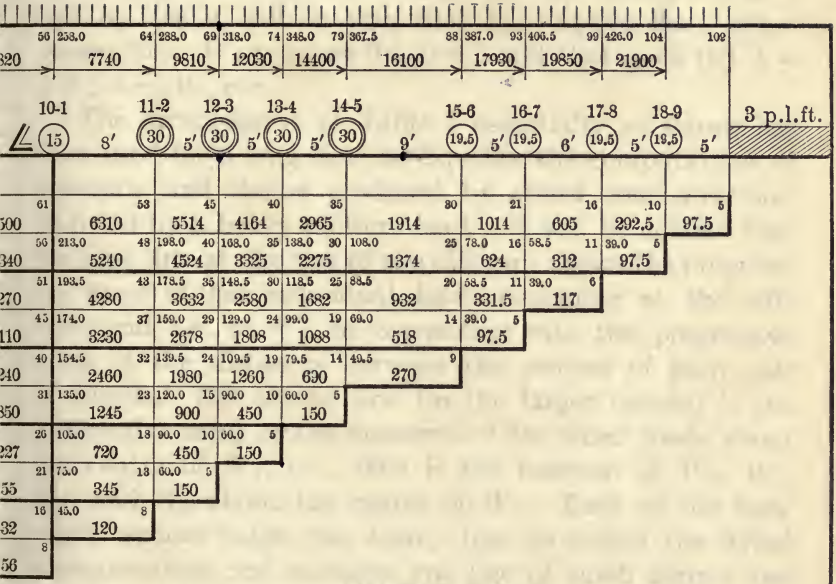
Art. 21.]



	1	2	3	4	5	6	7
	45.0 8 240	75.0 .13 630	105.0 18 1170	135.0 23 1860	154.5 32 2485	174.0 37 3205	193.5 43 4040
	109 24550	101 22910	96 19880	91 17000	86 14270	77 11690	72 10190
18	426.0 104 22420	411.0 96 20860	381.0 91 17980	351.0 86 15250	321.0 81 12670	291.0 72 10240	271.5 67 8830
17	406.5 99 20380	391.5 91 18900	361.5 86 16170	331.5 81 13590	301.5 76 11160	271.5 67 8880	252.0 62 7570
16	387.0 93 18060	372.0 85 16670	342.0 80 14120	312.0 75 11720	282.0 70 9470	252.0 61 7370	232.5 56 6180
15	367.5 88 16220	352.5 80 14910	322.5 75 12500	292.5 70 10250	262.5 65 8150	232.5 56 6200	213.0 51 5110
14	348.0 79 13090	333.0 71 11900	303.0 66 9780	273.0 61 7800	243.0 56 5970	213.0 47 4290	193.5 42 3370
13	318.0 74 11500	303.0 66 10400	273.0 61 8410	243.0 56 6580	213.0 51 4900	183.0 42 3370	163.5 37 2555
12	288.0 69 10060	273.0 61 9030	243.0 56 7200	213.0 51 5520	183.0 46 3990	153.0 37 2605	133.5 32 1885
11	258.0 64 8770	243.0 56 7810	213.0 51 6130	183.0 46 4600	153.0 41 3220	123.0 32 1992	103.5 27 1368
10	228.0 56 6950	213.0 48 6110	183.0 43 4670	153.0 38 3380	123.0 33 2240	93.0 24 1248	73.5 19 780

Loads and moments are for one rail
Loads given in thousands of pounds
Moments " " " " foot pounds
Moments are expressed to a limit of error of 0.1 per cent

LE I.



MOMENT TABLE
COOPER'S E-60 LOADING
Two 213-ton Engines+6000 lbs. p.l.ft.
Scale: 1″=15′

(*To face page* 87.)

The practical application of the preceding formulæ can therefore best be effected by means of a tabulation of moments like that shown in Table I, taken from the standard specifications of the N. Y. C. R. R. Co. for 1915. The wheel weights and train loads shown in the table are for one rail only, i.e., they are half those for one track. By comparing the weights and spacings with those in Fig. 1 and eq. (6) it will be seen that $W_1 = 15{,}000$ lbs.; $W_2 = 30{,}000$ lbs.; $W_3 = 30{,}000$ lbs., etc., and that $a = 8$ ft.; $b = 5$ ft.; $c = 5$ ft., etc.

The arrangement of Table I essentially as shown has been used for a long time to expedite the computations of moments and shears produced by wheel concentrations, followed by a heavy uniform load. It will be noticed that the first line at the top of the diagram shows the progressive sums of the individual loads beginning at the left-hand end, i.e., at W_1, in connection with the progressive sums of the distances between the centres of each pair of wheels. The second line (in the larger figures) is the progressive sums of the moments of the wheel loads about the centre of W_1; i.e., 1860 is the moment of W_2, W_3, W_4, and W_5 about the centre of W_1. Each of the horizontal spaces below the heavy line on which the wheel concentrations rest contains one line of small figures and one line of large figures. The small figures are the progressive sums of the distances from the head of the uniform moving load or from each successive wheel to each of the wheel weights in the series. The larger figures give the progressive sums of the moments of the wheel weights beginning with W_{18} about the head of the uniform load, i.e., $19.5 \times 5 = 97.5$, and $19.5 \times 10 + 97.5 = 292.5$. Each horizontal space is seen to begin at the vertical heavy line under each weight taken in succession and to contain the progressive sums of the moments, weights, and distances about or from each such weight, as is clear on examining the diagram. At the left of each horizontal line there is found the number of the wheel load under which the right-hand end of the line begins.

The diagrammatic exhibit of these various numerical quantities will enable the reactions, shears, and greatest moments at any point in the span to be readily determined.

When a uniform train load is a part of the system of loading it is only necessary to consider any section of it as acting through its centre of gravity, i.e., through its mid-point. Taking that centre as its point of application the separating space is the distance from that point to the nearest concentration. If in Table II 20 ft. of train load be used, that train weight will be 60,000 lbs. applied at the distance $10+5=15$ ft. from load 18. This simple operation is all that is needed for any uniform load or for a series of sections of uniform load.

Table II is a table of maximum moments, end shears, and floor-beam reactions for girders having spans up to 125 ft., and it is taken from the New York Central Railroad Specifications for 1915. The shears and floor-beam reactions, like the results shown in Table I, are given in thousands of pounds and are for one rail only. The moments are given in thousands of foot-pounds, like the moments shown in Table I. The loading is the same as that shown by the diagram in Table I, except that the results for spans up to a maximum of 11 ft. are found by using a special loading of two 72,000-lb. axle loads 7 ft. apart, or 36,000 lbs. for each rail. The maximum moments are found for the conditions of loading given by the criterion, eq. (5), of this article. The maximum floor-beam reactions are found by eq. (5) of Art. 18, in accordance with the criterion, eq. (4), of the same article.

Table II.

TABLE OF MAXIMUM MOMENTS, END SHEARS AND FLOOR-BEAM REACTIONS FOR GIRDERS.

Moments in Thousands of Foot-pounds.

Shears and Floor-beam Reactions in Thousands of Pounds.

Loading Two E 60 Engines and Train Load of 6000 lbs. per Foot or Special Loading Two 72,000-lb. Axle Loads 7 Ft. C to C.

Results for One Rail. Results from Special Loading Marked *.

Span. Ft.	Maximum Moments.	End Shear.	Floor-beam Reaction.	Span.	Maximum Moments.	End Shear.	Floor-beam Reaction.
5	* 45.0	*36.0	*36.0	35	784.5	103.8	146.4
6	* 54.0	*36.0	40.0	36	823.0	105.9	149.3
7	* 63.0	38.6	47.1	37	861.6	107.8	152.2
8	* 72.0	41.3	52.5	38	900.0	109.7	155.6
9	* 81.0	*44.0	56.7	39	940.0	111.4	158.8
10	* 90.0	*46.8	60.0	40	983.4	113.1	162.0
11	* 99.0	49.1	65.5	41	1027.0	115.2	....
12	120.0	52.5	70.0	42	1070.4	117.2	....
13	142.5	55.4	73.9	43	1113.9	119.0	....
14	165.0	57.8	78.2	44	1157.4	120.8	....
15	187.5	60.0	82.0	45	1201.1	122.5	....
16	210.0	63.8	85.3	46	1244.4	124.2	....
17	232.5	67.1	88.2	47	1287.9	125.9	....
18	255.0	70.0	91.0	48	1331.4	127.5	....
19	280.0	72.6	94.3	49	1378.3	129.2	....
20	309.5	75.0	98.3	50	1426.3	130.8	....
21	339.0	77.1	101.9	51	1474.7	132.5	
22	368.5	79.1	105.2	52	1522.8	134.1	
23	398.2	80.9	108.2	53	1571.0	135.7	....
24	427.8	83.1	110.9	54	1622.2	137.4	....
25	457.5	85.2	113.5	55	1675.2	139.0	....
26	487.2	87.1	116.6	56	1728.6	140.6	....
27	516.9	88.9	120.1	57	1781.9	142.2	....
28	548.3	90.6	123.4	58	1835.1	143.8	....
29	582.0	92.3	126.5	59	1891.4	145.4	....
30	615.8	94.6	129.4	60	1949.4	147.0	....
31	649.3	96.6	132.7	61	2007.5	148.6	....
32	683.2	98.6	136.5	62	2065.4	150.2	....
33	716.9	100.4	140.0	63	2123.4	152.0	....
34	750.6	102.1	143.2	64	2183.3	153.8	....

Table II.—(*Con.*)

Span. Ft.	Maximum Moments.	End Shear.	Floor beam Reaction.	Span.	Maximum Moments.	End Shear.	Floor-beam Reaction.
65	2246.3	155.7	....	95	4408.4	215.4	....
66	2309.3	157.5	....	96	4490.7	217.2	....
67	2372.3	159.6	....	97	4573.5	219.2	....
68	2435.4	161.7	....	98	4659.8	221.2	....
69	2498.4	163.8	....	99	4743.8	223.1	....
70	2560.4	165.8	....	100	4830.0	225.0	....
71	2624.5	167.7	....	101	4916.9	226.8	....
72	2688.3	170.0	....	102	5004.0	228.6	....
73	2750.9	172.2	....	103	5115.5	230.4	....
74	2819.4	174.4	....	104	5212.8	232.3	....
75	2888.6	176.5	....	105	5306.5	234.1	....
76	2958.0	178.6	....	106	5401.3	235.9	....
77	3028.6	180.6	....	107	5499.2	237.7	....
78	3096.6	182.5	....	108	5617.0	239.4	....
79	3168.2	184.4	....	109	5727.6	241.2	....
80	3240.7	186.3	....	110	5829.6	243.0	....
81	3311.4	188.4	....	111	5937.4	244.8	....
82	3385.1	190.4	....	112	6040.0	246.6	....
83	3459.6	192.3	....	113	6148.2	248.3	....
84	3534.6	194.2	....	114	6258.0	250.0	....
85	3610.4	196.1	....	115	6366.8	251.8	....
86	3689.4	198.1	....	116	6478.0	253.6	....
87	3766.5	200.1	....	117	6586.1	255.3	....
88	3846.0	202.1	....	118	6696.6	257.0	....
89	3924.3	204.0	....	119	6808.3	258.8	....
90	4005.8	205.8	....	120	6921.6	260.5	....
91	4084.4	207.7	....	121	7030.5	262.2	....
92	4164.0	209.7	....	122	7143.8	264.0	....
93	4246.6	211.6	....	123	7260.1	265.7	....
94	4328.0	213.5	....	124	7376.4	267.4	....
				125	7495.2	269.1	....

Problem.

Let a single-track railroad plate girder with an effective span of 88 ft. be traversed from right to left by the moving load shown in Table I. It is required to find the greatest bending moments and shears at the centre and quarter-points of the span, the dead load or own weight of the girder, floor system and track being taken at 1800 lbs. per linear foot.

Dead Load.

By eq. (6) of Art. 22 the bending moments at the quarter-point and centre are, since the reaction R is $44 \times 900 = 39,600$ lbs.;

	Quarter-point.	Centre.
	$x = \frac{1}{4}l = 22$ ft.	$x = \frac{1}{2}l = 44$ ft.
$M = \frac{p}{2}(lx - x^2)$	654,000 ft.-lbs.	871,000 ft.-lbs.

By eq. (7) of Art. 22, the shears at end, quarter-point, and centre are:

	End.	Quarter-point.	Centre.
	$x = 0$	$x = 22$ ft.	$x = 44$ ft.
Shear =	39,600 lbs.	19,800 lbs.	zero

Moving Load.

If weight W_4 be placed at the quarter-point of the span, 14 wheel weights will rest on the girder with W_{14} 5 ft. from the right-hand end of the span. As $\frac{l'}{l} = \frac{1}{4}$, the criterion, eq. (5), gives either $\frac{l'}{l} = \frac{75,000}{367,500}$ or, $\frac{105,000}{367,500}$, the first being too small and the second too large. Hence W_4 at the quarter-point is the proper position for the maximum bending moment. W_1 will be 84 ft. from the right-hand end of the span. Taking moments of all the wheels about that point, by the aid of Table I, the reaction R at the left end of the span is:

$$R = \frac{14,830,000}{88} = 168,500 \text{ lbs.}$$

Eq. (6) will then give the bending moment at W_4, but having the reaction R and using Table I the bending moment becomes:

$$M = 168{,}500 \times 22 - 720{,}000 = 2{,}987{,}000 \text{ ft.-lbs.}$$

The end shear with the load placed so as to produce the greatest bending moment at the quarter-point is obviously the reaction $R = 168{,}500$ lbs. The shear immediately at the left of the quarter-point will be $168{,}500 - 75{,}000 = 93{,}500$ lbs.

The greatest bending moment at the centre of span is similarly found. If W_{13} be placed at the centre of the span the wheel weights $W_6 \ldots W_{18}$ and 9 ft. in length of the uniform train load will rest on the span. The ratio representing the criterion, eq. (5), is $\frac{l'}{l} = \frac{183}{318}$ or $\frac{153}{318}$. The first of these values is too large and the latter is too small, showing that W_{13} at the centre of the span is the correct position for the greatest bending moment at that point. The reaction R for this position of the load is at once written by the aid of Table I as follows:

$$R = \frac{11{,}695 + 2619 + 121.5}{88} \times 1000 = 164{,}000 \text{ lbs.}$$

The bending moment M for the centre of the span is as follows, using the preceding value of R and Table I:

$$M = (\tfrac{1}{2} \times 14{,}440.5 - 3370) \times 1000 = 3{,}848{,}000 \text{ ft.-lbs.}$$

The end shear for this position of the loading is the reaction R, i.e., 164,000 lbs. The shear indefinitely near to but at the left of the centre is $164{,}000 - 153{,}000 = 11{,}000$ lbs. This small shear shows that the moment at the centre of the span is the greatest in the entire span for this position of loading.

Assembling the preceding results, the total dead and moving load moments and shears will be as follows:

Moments.

	Quarter-point.	Centre.
Dead Load........	654,000 ft.-lbs.	871,000 ft.-lbs.
Moving Load....	2,987,000 ft.-lbs.	3,848,000 ft.-lbs.
	3,641,000 ft.-lbs.	4,719,000 ft.-lbs.

Shears.

	End.	Quarter-point.	Centre.
Dead Load......	39,600 lbs.	19,800 lbs.	zero
Moving Load....	168,500 lbs.	93,500 lbs.	11,000 lbs.
Total...........	208,100 lbs.	113,300 lbs.	11,000 lbs.

The expression "equivalent uniform load," for moments or shears, as the case may be, is sometimes used. It simply means that the uniform load is such as to produce the moments or shears equivalent to those found under given conditions. A uniform load p per linear foot acting on the entire span l will produce a centre-moment of $\frac{pl^2}{8}$. Hence if there be written $\frac{pl^2}{8}=3,848,000$, then, if $l=88$:

$$p=\frac{8}{7744}\times 3,848,000=3980 \text{ lbs. per linear foot.}$$

The equivalent uniform load therefore for the greatest bending moment at the centre of the span is 3980 lbs. per linear foot. Similarly as the bending moment at any distance x from one end of the span is $\frac{p}{2}(lx-x^2)$, if x be made 22 in the present case, l being 88 feet, there will be found by placing this expression equal to 2,987,000 ft.-lbs:

$$p=\frac{2,987,000}{726}=4114 \text{ lbs. per linear foot.}$$

The end shear for a uniform load over the whole span is equal to the load on half the span. Hence by placing $p\times 44=164,000$ lbs., there will result:

$$p=\frac{164,000}{44}=3727 \text{ lbs. per linear foot.}$$

This is the equivalent uniform load for the end shear with the load so placed as to give the greatest bending moment at the centre of the span.

In the same way the equivalent uniform load for the end shear 168,600 lbs., with the load placed so as to give the greatest bending moment at the quarter-point, will be found to be 3830 lbs. per linear foot.

These simple instances show that the equivalent uniform load varies from one case to another according to the amount, distribution and position of the loading.

Art. 22.—Moments and Shears in Special Cases.

Certain special cases of beams are of such common occurrence, and consequently of such importance, that a somewhat more detailed treatment than that already given may be deemed desirable. The following cases are of this character:

Case I.

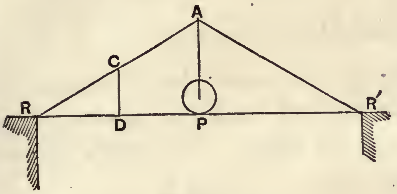


FIG. 1.

Let a non-continuous beam supporting a single weight P at any point be considered, and let such a beam be represented in Fig. 1. If the span RR' is represented by

$$l = a + b = RP + R'P,$$

the reactions R and R' will be

$$R = \frac{b}{l}P, \quad \text{and} \quad R' = \frac{a}{l}P. \quad . \quad . \quad . \quad . \quad (1)$$

Consequently, if x represents the distance of any section in RP from R, while x' represents the distance of any section of $R'P$ from R', the general values of the bending moments for the two segments a and b of the beam will be

$$M = Rx, \quad \text{and} \quad M' = R'x'. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These two moments become equal to each other and represent *the greatest bending moment in the beam* when

$$x = a \quad \text{and} \quad x' = b,$$

or *when the section is taken at the point of application of the load P.*

Eq. (2) shows that the moments vary directly as the distances from the ends of the beam. Hence if AP (normal to RR') is taken by any convenient scale to represent the greatest moment, $\frac{ab}{l}P$, and if RAR' is drawn, any intercept parallel to AP and lying between RAR' and RR' will represent the bending moment for the section at its foot by the same scale. In this manner CD is the bending moment at D.

The shear is uniform for each single segment; it is evidently equal to R for RP and R' for $R'P$. It becomes zero at P, where is found the greatest bending moment.

Case II.

Again, let Fig. 2 represent the same beam shown in Fig. 1, but let the load be one of uniform intensity, p, extending from end to end of the beam. Let C be placed at the centre of the span, and let R and R', as before, represent the two reactions. Since the load is symmetrical in reference to C,

$$R=R'.$$

For the same reason the moments and shears in one half of the beam will be exactly like those in the other; consequently reference will be made to one half of the beam only. Let x and x_1 then be measured from R toward C. The forces acting upon the beam are R and p, the latter being uniformly continuous. Applying the formulæ for the bending moment at any section x, remembering that x_1 has all values less than x,

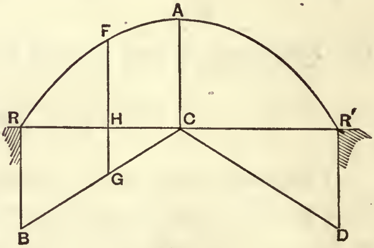


Fig. 2.

$$M=Rx-p\int_0^x(x-x_1)dx_1;$$

$$\therefore\ M=Rx-\frac{px^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If l is the span, at C, M becomes

$$M_1=\frac{Rl}{2}-\frac{pl^2}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But because the load is uniform

$$R=\frac{pl}{2}.$$

Hence

$$M_1 = \frac{pl^2}{8} = \frac{Wl}{8} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

if W is put for the total load. Placing

$$R = \frac{pl}{2},$$

in eq. (3),

$$M = \frac{p}{2}(lx - x^2). \quad . \; . \; . \; . \; . \; . \quad (6)$$

The moments M, therefore, are proportional to the abscissæ of a parabola whose vertex is over C, and which passes through the origin of coordinates R. Let AC, then, normal to RR', be taken equal to M_1, and let the parabola RAR' be drawn. Intercepts, as FH, parallel to AC, will represent bending moments in the sections, as H, at their feet.

The shear at any section is

$$S = \frac{dM}{dx} = R - px = p\left(\frac{l}{2} - x\right), \quad . \; . \; . \; . \; . \quad (7)$$

or *it is equal to the load covering that portion of the beam between the section in question and the centre.*

Eq. (7) shows that the shear at the centre is zero; it also shows that $S = R$ at the ends of the beam. It further demonstrates that *the shear varies directly as the distance from the centre.* Hence, take RB to represent R and draw BC. The shear at any section, as H, will then be represented by the vertical intercept, as HG, included between BC and RC.

The shear being zero at the centre, the greatest bending moment will also be found at that point. This is also evident from inspection of the loading.

Eq. (2) of *Case I* shows that if a beam of span l carries a weight $\frac{W}{2}$ at its centre, the moment M at the same point will be

$$M_1 = \frac{W}{4} \cdot \frac{l}{2} = \frac{Wl}{8}. \quad \ldots \ldots \quad (8)$$

The third member of eq. (8) is identical with the third member of eq. (5). It is shown, therefore, *that a load concentrated at the centre of a non-continuous beam will cause the same moment, at that centre, as double the same load uniformly distributed over the span.*

Eqs. (5) and (8) are much used in connection with the bending of ordinary non-continuous beams, whether solid or flanged; and such beams are frequently found.

Case III.

The third case to be taken is a cantilever uniformly loaded; it is shown in Fig. 3. Let x be measured from the free end A, and let the uniform intensity of the load be represented by p. The load px acts with its centre at the distance $\frac{1}{2}x$ from the section x. Hence the desired moment will be

$$M = -px \cdot \frac{x}{2} = -\frac{px^2}{2}. \quad . \quad (9)$$

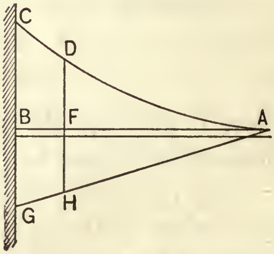


FIG. 3.

If $AB = l$, the moment at B is

$$M_1 = -\frac{pl^2}{2}. \quad \ldots \ldots \quad (10)$$

The negative sign is used to indicate that the *lower* side of the beam is subjected to compression. In the two preceding cases, evidently the *upper* side is in compression.

The shear at any section is

$$S = \frac{dM}{dx} = -px. \quad \ldots \ldots \quad (11)$$

Hence the shear at any section is the load between the free end and that section.

Eq. (9) shows that the moments vary as the *square* of the distance from the free end; consequently the moment curve is a parabola with the vertex at A, and with a vertical axis. Let BC, then, represent M_1 by any convenient scale and draw the parabola CDA. Any vertical intercept, as DF, will represent the moment at the section, as F, at its foot.

Again, let BG represent the shear pl at B, then draw the straight line AG. Any vertical intercept, as HF, will then represent the shear at the corresponding section F.

Art. 23.—Recapitulation of the General Formulæ of the Common Theory of Flexure.

It is convenient for many purposes to arrange the formulæ of the Common Theory of Flexure in the most general and concise form. In this article the preceding general formulæ for shear, strains, resisting moments, and deflections will be recapitulated and so arranged. In order to complete the generalization, the summation sign Σ will be used instead of the sign of integration.

In Fig. 1, let ABC represent the centre line of any bent beam; AF, a vertical line through A; CF, a horizontal line through C, while A is the section of the beam at which the deflection (vertical or horizontal) in reference to C, the bending moment, the shearing stress, etc., are to be determined. As shown in figure, let x be the horizontal coordinate measured from A, and y the vertical one measured from the same point; then let x_1 be the horizontal distance from the same point to the point of application of any external vertical force P. To complete the notation, let D

be the deflection desired; M_1, the moment of the external forces about A; S, the shear at A; u, the strain (extension or compression) per unit of length of a fibre parallel to the neutral surface and situated at a normal distance of unity from it; I, the general expression of the moment of inertia of a normal cross-section of the beam, taken in reference to the neutral axis of that section; E, the coefficient of elasticity for the material of the beam; and M the moment of the external forces for any section, as B.

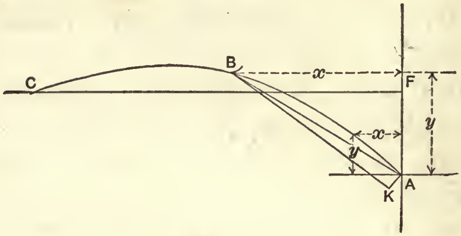


FIG. 1.

Again, let Δ be an indefinitely small portion of any normal cross-section of the beam, and let z be an ordinate normal to the neutral axis of the same section. By the "common theory" of flexure, the intensity of stress at the distance z from the neutral surface is $\underset{zuE}{(zP'E)}$. Consequently the stress developed in the portion Δ of the section is $EP'z\Delta$, and the resisting moment of that stress is $EP'z^2\Delta$.

The resisting moment of the whole section will therefore be found by taking the sum of all such moments for its whole area.

Hence

$$M = Eu\Sigma z^2\Delta = EuI.$$

Hence, also,

$$u = \frac{M}{EI}.$$

If n represents an indefinitely short portion of the neutral surface, the strain for such a length of fibre at unit's distance from that surface will be nu.

If the beam were originally straight and horizontal, n would be equal to dx.

u being supposed small, the effect of the strain nu at any section, B, is to cause the end A of the chord BA to move vertically through the distance nux.

If BK and BA (taken equal) are the positions of the chords before and after flexure, nux will be the vertical distance between K and A.

By precisely the same kinematical principle the expression nuy will be the horizontal movement of A in reference to B.

Let Σnux and Σnuy represent summations extending from A to C, then will those expressions be the vertical and horizontal deflections respectively of A in reference to C. It is evident that these operations are perfectly general, and that x and y may be taken in any direction whatever.

The following general but strictly approximate equations relating to the subject of flexure may now be written:

$$S = \Sigma P. \qquad (1)$$

$$M_1 = \Sigma P x_1. \qquad (2)$$

$$u = \frac{M}{EI}. \qquad (3)$$

$$\Sigma nu = \Sigma n \frac{M}{EI}. \qquad (4)$$

$$D_0 = \Sigma nux = \Sigma \frac{nMx}{EI}. \qquad (5)$$

[illegible] $= \int$ [illegible]

$$D_h = \Sigma nuy = \Sigma \frac{nMy}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

D_h represents horizontal deflection.

The summation ΣPz must extend from A to a point of no bending, or from A to a point at which the bending moment is M_1'. In the latter case

$$M_1 = \Sigma Pz + M_1'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

M_1' may be positive or negative.

Art. 24.—The Theorem of Three Moments.

The object of this theorem is the determination of the relation existing between the bending moments which are found in any continuous beam at any three adjacent points of support. In the most general case to which the theorem applies, the section of the beam is supposed to be variable, the points of support are not supposed to be in the same level, and at any point, or all points, of support there may be constraint applied to the beam external to the load which it is to carry; or, what is equivalent to the last condition, the beam may not be straight at any point of support before flexure takes place.

Before establishing the theorem itself, some preliminary matters must receive attention.

If a beam is simply supported at each end, the reactions are found by dividing the applied loads according to the simple principle of the lever. If, however, either or both ends are not simply supported, the reaction in general is greater at one end and less at the other than would be found by the law of the lever; a portion of the reaction at one end is, as it were, transferred to the other. The transference can only be accomplished by the application of a couple to the beam, the forces of the couple being applied at the two adjacent points of support; the span, consequently, will be the lever-arm of the couple. The existence of equilibrium requires the application to the beam of an equal and opposite couple. It is only necessary, however, to consider, in connection with the span AB, the one shown in Fig. 1. Further, from what has immediately preceded, it appears that the force of this couple is equal to the difference between the actual reaction at either point of support and that found by the law of the lever. The bending caused by this couple may evidently be of an opposite kind to that existing in a beam simply supported at each end.

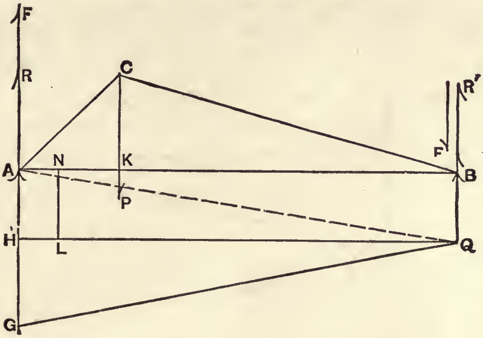


FIG. 1.

These results are represented graphically in Fig. 1. A and B are points of support, and AB is the beam; AR and BR' are the reactions according to the law of the lever; $RF = R'F$ is the force of the applied couple; consequently

$$AF = AR + RF \quad \text{and} \quad BF = BR' - (R'F = RF)$$

are the reactions after the couple is applied. As is well known, lines parallel to CK, drawn in the triangle ACB, represent the bending moments at the various sections of the beam, when the reactions are AR and BR'. Finally, vertical lines parallel to AG, in the triangle QHG, will represent the bending moments caused by the force $R'F$.

In the general case there may also be applied to the beam two equal and opposite couples having axes passing through A and B respectively. The effect of such couples will be nothing so far as the reactions are concerned, but they will cause uniform bending between A and B. This uniform or constant moment may be represented by vertical lines drawn parallel to AH or LN (equal to each other) between the lines AB and HQ. The resultant moments to which the various sections of the beam are subjected will then be represented by the *algebraic* sum of the three vertical ordinates included between the lines ACB and GQ. Let that resultant be called M. This composition of the resultant moment M will be made clearer by reference to Figs. 2 and 3. Fig. 2 shows the component moment due to the single force F acting with

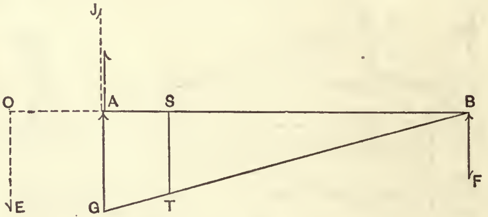


FIG. 2.

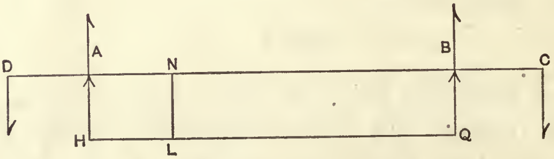


FIG. 3.

the lever-arm l so that its moment increases directly as the distance from B. Fig. 3, on the other hand, shows the component moment due to the two equal and opposite couples acting at the ends of the span. The resultant moment M is the algebraic sum of the three component moments, shown combined in Fig. 1.

Let the moment GA be called M_a, and the moment

$$BQ = LN = HA = M_b.$$

Also designate the moment caused by the load P, shown by lines parallel to CK in ACB, by M_1. Then let x be any horizontal distance measured from A toward B; l the horizontal distance AB; and z the distance of the point of application, K, of the force P from A. With this notation there can be at once written

$$M = M_a\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 . \quad . \quad . \quad . \quad . \quad (1)^*$$

Eq. (1) is simply the general form of eq. (2), Art. 23.

It is to be noticed that Fig. 1 does not show all the moments M_a, M_b, and M_1 to be the same sign, but for convenience they are so written in eq. (1).

The formula which represents the theorem of three moments can now be written without difficulty. The method to be followed involves the improvements added by Prof. H. T. Eddy, and is the same as that given by him in the "American Journal of Mathematics," Vol. I., No. 1.

Fig. 4 shows a portion of a continuous beam, including two spans and three points of supports. The deflections will be supposed measured from the horizontal line NQ. The spans are represented by l_a and l_c; the vertical dis-

---

* This equation is used in the next Art. for a short demonstration of the common form of the Theorem of Three Moments.

tances of *NQ* from the points of support by c_a, c_b, and c_c; the moments at the same points by M_a, M_b, and M_c, while the letters *S* and *R* represent shears and reactions respectively.

In order to make the case general, it will be supposed that the beam is curved in a vertical plane, and has an elbow at *b*, before flexure, and that, at that point of support, the tangent of its inclination to a horizontal line, toward the span l_a, is *t*, while *t′* represents the tangent on the other side of the same point of support; also let *d* and *d′* be the vertical distances, before bending takes place, of the points *a* and *c*, respectively, below the tangents at the point *b*.

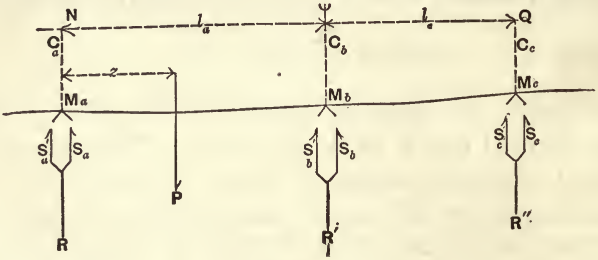


FIG. 4.

A portion of the difference between c_a and c_b is due to the original inclination, whose tangent is *t*, and the original lack of straightness, and is not caused by the bending; that portion which is due to the bending, however, is, remembering eq. (5), Art. 23,

$$D=c_a-c_b-l_at-d=\sum_b^a\frac{Mxn}{EI}.$$

Fig. 5 will make clear the component parts of the value of *D* in the preceding equation.

By the aid of eq. (1) this equation may be written:

$E(c_a-c_b-l_at-d)$

$$= \Sigma_b^a \left[\left\{ M_a\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 \right\} \frac{xn}{I} \right]. \quad (2)$$

In this equation, it is to be remembered, both x and z (involved in M_1) are measured from support a toward support b. Now let a similar equation be written for the span l_c, in which the variables x and z will be measured from c toward b. There will then result

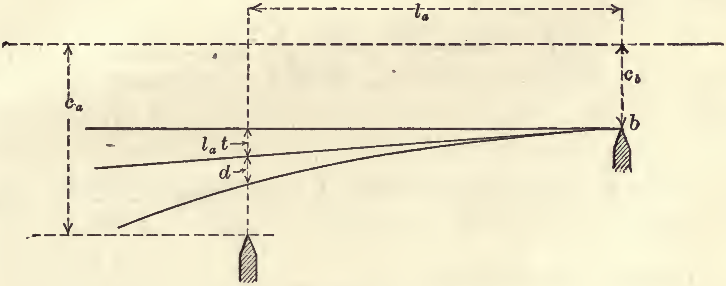


FIG. 5.

$$E(c_c - c_b - l_c t' - d')$$

$$= \Sigma_b^c \left[\left\{ M_c\left(\frac{l-x}{l}\right) + M_b\left(\frac{x}{l}\right) + M_1 \right\} \frac{xn}{I} \right]. \quad (3)$$

When the general sign of summation is displaced by the integral sign, n becomes the differential of the axis of the beam, or ds. But ds may be represented by udx, u being such a function of x as becomes unity if the axis of the beam is originally straight and parallel to the axis of x. The eqs. (2) and (3) may then be reduced to simpler forms by the following methods:*

---

* These analytic transformations are of the nature of convenient but arbitrary notation and are not to any degree whatever analytic demonstrations.

In eq. (2) put

$$\Sigma_b^a\left(\frac{l-x}{l}\right)\frac{xn}{I}=\frac{1}{l_a}\int_b^a\frac{u(l_a-x)xdx}{I}=\frac{x_a}{l_a}\int_b^a\frac{u(l_a-x)dx}{I}. \qquad (4)$$

Also

$$\frac{x_a}{l_a}\int_b^a\frac{u(l_a-x)dx}{I}=\frac{i_ax_a}{l_a}\int_b^a u(l_a-x)dx. \quad . \quad . \qquad (5)$$

Also

$$\frac{i_ax_a}{l_a}\int_b^a u(l_a-x)\,dx=\frac{i_ax_au_a}{l_a}\int_b^a(l_a-x)\,dx=\frac{i_ax_au_al_a}{2}. \qquad (6)$$

In the same manner

$$\Sigma_b^a\frac{x^2n}{l_aI}=\frac{1}{l_a}\int_b^a\frac{ux^2dx}{I}=\frac{x_a'}{l_a}\int_b^a\frac{uxdx}{I}. \quad . \quad . \qquad (7)$$

Also

$$\frac{x_a'}{l_a}\int_b^a\frac{uxdx}{I}=\frac{i_a'x_a'}{l_a}\int_b^a uxdx. \quad . \quad . \quad . \quad . \qquad (8)$$

And

$$\frac{i_a'x_a'}{l_a}\int_b^a uxdx=\frac{i_a'x_a'u_a'}{l_a}\int_b^a xdx=\frac{i_a'x_a'u_a'l_a}{2}. \quad . \qquad (9)$$

Again, in the same manner,

$$\Sigma_b^a\frac{M_1xn}{I}=i_{1a}u_{1a}\Sigma M_1x\Delta x. \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Using eqs. (4) to (10), eq. (2) may be written:

$$E(c_a-c_b-l_at-d)=\frac{l_a}{2}(M_au_ai_ax_a+M_bu_a'i_a'x_a')$$
$$+u_{1a}i_{1a}\Sigma_b^a M_1x\Delta x. \qquad (11)$$

Proceeding in precisely the same manner with the span l_c, eq. (3) becomes

$$E(c_c - c_b - l_c t' - d') = \frac{l_c}{2}(M_c u_c i_c x_c + M_b u_c' i_c' x_c') + u_{1c} i_{1c} \overset{c}{\underset{b}{\Sigma}} M_1 x \Delta x. \qquad (12)$$

The quantities x_a and x_c are to be determined by applying eq. (4) to the span indicated by the subscript; while u_a, i_a, u_c, and i_c are to be determined by using eqs. (5) and (6) in the same way. Similar observations apply to u_a', i_a', x_a', u_c', i_c', and x_c' taken in connection with eqs. (7), (8), and (9).

If I is not a continuous function of x, the various integrations of eqs. (4), (5), (7), and (8) must give place to summations (Σ) taken between the proper limits.

Dividing eqs. (11) and (12) by l_a and l_c respectively, and adding the results,

$$\left.\begin{aligned} &E\left(\frac{c_a - c_b}{l_a} + \frac{c_c - c_b}{l_c} - T - \frac{d}{l_a} - \frac{d'}{l_c}\right) \\ &\quad = \frac{u_{1a} i_{1a}}{l_a} \Sigma_b^a M_1 x \Delta x + \frac{u_{1c} i_{1c}}{l_c} \Sigma_b^c M_1 x \Delta x \\ &\quad + \tfrac{1}{2}(M_a u_a i_a x_a + M_b u_a' i_a' x_a' + M_c u_c i_c x_c + M_b u_c' i_c' x_c') \end{aligned}\right\} \qquad (13)$$

in which $T = t + t'$.

Eq. (13) is the most general form of the theorem of three moments if E, the coefficient of elasticity, is a constant quantity. Indeed, that equation expresses, as it stands, the "theorem" for a variable coefficient of elasticity if (ie) be written instead of i; e representing a quantity determined in a manner exactly similar to that used in connection with the quantity i.

In the ordinary case of an engineer's experience $T = 0$, $d = d' = 0$, *I = constant*, *$u = u_a = u_c =$ etc., $= c' =$ secant of the inclination for which $t = -t'$ is the tangent;* consequently

$$i_a = i_a' = i_c = i_c' = i_{,a} = i_{,c} = \frac{1}{I}.$$

From eq. (4)

$$x_a = \frac{2l_a}{6}, \qquad x_c = \frac{2l_c}{6}.$$

From eq. (7)

$$x_a' = \frac{4l_a}{6}, \qquad x_c' = \frac{4l_c}{6}.$$

The summation $\Sigma M_1 x \Delta x$ can be readily made by referring to Fig. 1.

The moment represented by CK in that figure is

$$P\left(\frac{l-z}{l}\right).z;$$

consequently the moment at any point between A and K, due to P, is

$$M_1 = P\left(\frac{l-z}{l}\right).z.\frac{x}{z} = P\left(\frac{l-z}{l}\right)x.$$

Between K and B

$$M_1' = \left(\frac{l-x}{l-z}\right).CK = P\frac{z}{l}(l-x).$$

Using these quantities for the span l_a,

$$\overset{a}{\underset{b}{\Sigma}} M_1 x \Delta x = \int_0^z M_1 x dx + \int_z^{la} M_1' x dx = \tfrac{1}{6}P(l_a^2 - z^2)z.$$

For the span l_c the subscript a is to be changed to c.

Introducing all these quantities eq. (13) becomes, after providing for any number of weights, P:

$$\frac{6EI}{c'}\left(\frac{c_a-c_b}{l_a}+\frac{c_c-c_b}{l_c}\right)=M_al_a+2M_b(l_a+l_c)+M_cl_c$$

$$+\frac{1}{l_a}\overset{a}{\Sigma}P(l_a^2-z^2)z+\frac{1}{l_c}\overset{c}{\Sigma}P(l_c^2-z^2)z. \qquad (14)$$

Eq. (14), with c' equal to unity, is the form in which the theorem of three moments is usually given; with c' equal to unity or not, *it applies only to a beam which is straight before flexure*, since

$$T=t+t'=0=u=d'.$$

If such a beam rests on the supports a, b, and c, before bending takes place,

$$\frac{c_a-c_b}{l_a}=-\frac{c_c-c_b}{l_c},$$

and the first member of eq. (14) becomes zero.

If, in the general case to which eq. (13) applies, the deflections c_a, c_b, and c_c belong to the beam in a position of no bending, the first member of that equation disappears, since it is the sum of the deflections *due to bending only* for the spans l_a and l_c, divided by those spans, and each of those quantities is zero by the equation immediately preceding, eq. (2). Also, if the beam or truss belonging to each span is straight between the points of support (*such points being supposed in the same level or not*), $u_a=u_a'=u_{1a}=$ *constant*, and $u_c=u_c'=u_{1c}=$ *another constant*. If, finally, I be again taken as constant, x_a and x_c, as well as $\Sigma M_1x\Delta x$, will have the values found above.

From these considerations it at once follows that the second member of eq. (14), put equal to zero, expresses the theorem of three moments for a beam or truss straight between points of support, when those points are not in the same level, but when they belong to a configuration of no bending in the beam. Such an equation, however, does not belong to a beam not straight between points of support.

The shear at either end of any span, as l_a, is next to be found, and it can be at once written by referring to the observations made in connection with Fig. 1. It was there seen that the reaction found by the simple law of the lever is to be increased or decreased for the continuous beam, by an amount found by dividing the difference of the moments at the extremities of any span by the span itself. Referring, therefore, to Fig. 4, for the shears S, there may at once be written:

$$S_a = \overset{a}{\Sigma} P \frac{l_a - z}{l_a} - \frac{M_a - M_b}{l_a} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$S_b' = \overset{a}{\Sigma} P \frac{z}{l_a} + \frac{M_a - M_b}{l_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$S_b = \overset{c}{\Sigma} P \frac{z}{l_c} + \frac{M_c - M_b}{l_c} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$S_c' = \overset{c}{\Sigma} P \frac{l_c - z}{l_c} - \frac{M_c - M_b}{l_c} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The negative sign is put before the fraction

$$\frac{M_a - M_b}{l_a}$$

in eq. (15) because in Fig. 1 the moments M_a and M_b are represented opposite in sign to that caused by P, while in eq. (1) the three moments are given the same sign, as has already been noticed.

Eqs. (15) to (18) are so written as to make an upward reaction positive, and they may, perhaps, be more simply found by taking moments about either end of a span. For example, taking moments about the right end of l_a,

$$S_a l_a - \overset{a}{\Sigma} P(l_a - z) + M_a = M_b.$$

From this, eq. (15) at once results. Again, moments about the left end of the same span give

$$S_b' l_a - \overset{a}{\Sigma} Pz + M_b = M_a.$$

This equation gives eq. (16), and the same process will give the others.

If the loading over the different spans is of uniform intensity, then, in general, $P = wdz$, w being the intensity. Consequently

$$\Sigma P(l^2 - z^2)z = \int_0^l w(l^2 - z^2)z dz = w\frac{l^4}{4}.$$

In all equations, therefore, for

$$\frac{1}{l_a} \overset{a}{\Sigma} P(l_a{}^2 - z^2)z$$

there is to be placed the term $w_a \frac{l_a{}^3}{4}$; and for

$$\frac{1}{l_c} \overset{c}{\Sigma} P(l_c{}^2 - z^2)z$$

the term $w_c \frac{l_c{}^3}{4}$. The letters a and c mean, of course, that reference is made to the spans l_a and l_c.

From Fig. 4, there may at once be written:

$$R \;=S_a'+S_a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$R' =S_b'+S_b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$R''=S_c'+S_c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

etc. = etc. + etc.

Art. 25.—Short Demonstration of the Common Form of the Theorem of Three Moments.

The general demonstration of the Theorem of Three Moments given in the preceding article has the great advantage of showing the influence of all the elements which enter the complete problem, including variability of moment of inertia, lack of straightness of beam, and points of support not at the same elevation. An adequate conception of the influences of the assumptions made in establishing the common or approximate form of the theorem can be obtained only by the employment of the general analysis, but it is convenient to establish the usual or approximate form of the theorem by a short direct method like the following.

Eq. (1) of the preceding article gives the general value of the bending moment in any span whatever of a continuous beam such as that shown in Fig. 1. The notation given in that figure explains itself and is essentially the same as that already used. It should be remembered that each reaction R, R', and R'' is composed of two shears as indicated, one acting at an indefinitely short distance to the left of a point of support and the other at an indefinitely short distance to the right of the same support. It is supposed that one load acts in each span at the distance

z from the left-hand end of the left-hand span, or from the right-hand end of the right-hand span.

Using eq. (1) of the preceding article and representing the deflection at any point in the span l_1 by w, eq. (1) may be at once written:

$$EI_1\frac{d^2w}{dx^2}=M_a\left(\frac{l_1-x}{l_1}\right)+M_b\frac{x}{l_1}+M_1. \quad . \quad . \quad . \quad (1)$$

The quantity I_1 is the moment of inertia of the cross-section of the beam about its neutral axis and E is the modulus of elasticity. It is assumed that the beam is straight and horizontal and that the moment of inertia does not vary in either span. If t_1 is the tangent of the inclination of the neutral surface of the beam at the right-hand end of the span l_1, then integrating eq. (1) between the limits of x and l_1 eq. (2) will at once result:

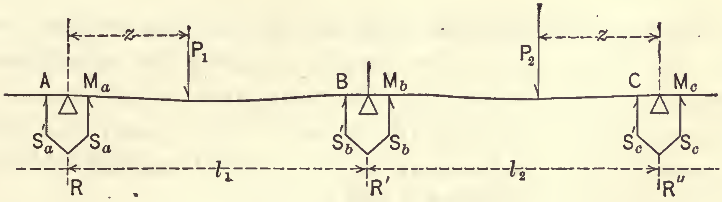


FIG. 1.

$$\frac{dw}{dx}=\frac{1}{EI_1}\left\{\frac{M_a}{l_1}\left(l_1x-\frac{x^2}{2}-\frac{l^2_1}{2}\right)+\frac{M_b}{2l_1}(x^2-l^2_1)+\int_{l_1}^{x}M_1dx\right\}+t_1.(2)$$

The integration of M_1dx is indicated only in eq. (2) for the reason that in general M_1 is a discontinuous function. The double integral $\int_0^{l_1}\int_{l_1}^{x}M_1dx^2$ cannot therefore generally be completed by the usual procedures, but it must be taken as $\frac{1}{EI_1}\Sigma nM_1x$, as given by eq. (5) of Art. 23. The value of this expression for a single load P_1 is shown in detail on the lower half of page 110 of the preceding Art. as $\frac{1}{6}P_1(l_1^2-z^2)z$, which appears in eq. (3). By integrating eq. (2) between the limits of l_1 and o, remembering that the points of support are supposed to be at the same elevation and hence that $w=o$ for $x=l_1$:

$$w=-\frac{1}{EI_1}\left(M_al_1+2M_bl_1+\frac{P_1}{l_1}(l_1^2-z^2)z\right)+6t_1=0. \qquad (3)$$

An equation identical with eq. (3) may be written for the right-hand span l_2 by simply changing the subscripts, remembering, however, that the origin from which z and x are measured is the point of support C, Fig. 1, and that the tangent of the inclination of the neutral surface at the left-hand end of the span l_2 will be $-t_1$.

Hence:

$$w=-\frac{1}{EI_2}\left(M_cl_2+2M_bl_2+\frac{P_2}{l_2}(l_2^2-z^2)z\right)=-6t_1=0. \qquad (4)$$

If eqs. (3) and (4) be added the usual and approximate form of the Theorem of Three Moments will at once result, except that the moments of inertia I_1 and I_2 are different. Assuming $I_1=I_2$ and writing the summation sign before P_1 and P_2 to indicate that any number of loads may act on every span, the Theorem of Three Moments as usually employed will at once result:

$$M_al_1+2M_b(l_1+l_2)+M_cl_2=-\frac{1}{l_1}\Sigma P_1(l_1^2-z^2)z$$

$$-\frac{1}{l_2}\Sigma P_2(l^2{}_2-z^2)z \quad . \quad . \quad . \quad . \qquad (5)$$

It will be observed that eq. (5) is identical with the second member of eq. (14) of the preceding article, and it is the equation sought. The expressions for the shears composing each of the reactions may now easily be written.

Taking moments about the right-hand end of the span l_1:

$$S_a l_1 - \Sigma P_1(l_1 - z) + M_a = M_b. \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Hence:

$$S_a = \Sigma P_1 \frac{l_1 - z}{l_1} - \frac{M_a - M_b}{l_1}. \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Again taking moments about the left-hand end of the same span:

$$S'_b l_1 - \Sigma P_1 z + M_b = M_a. \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Hence:

$$S'_b = \Sigma P_1 \frac{z}{l_1} + \frac{M_a - M_b}{l_1}. \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Eqs. (7) and (9) give the shears at the two ends of the span l_1 and they also give the shears at the two ends of the span l_2 by simply changing the notation so as to apply to the span l_2 as shown in eqs. (10) and (11):

$$S_b = \Sigma P_2 \frac{z}{l_2} + \frac{M_c - M_b}{l_2}. \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

$$S'_c = \Sigma P_2 \frac{l_2 - z}{l_2} - \frac{M_c - M_b}{l_2} \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Each reaction will be the sum of the appropriate pair of shears as shown by eqs. (19), (20), and (21) of the preceding article.

These equations are given in their most general forms;

that is, for any disposition of loads of any magnitude. They may be adapted to uniform loading either partial or entire, as indicated on the lower half of page 113.

Art. 26.—Reaction under Continuous Beam of any Number of Spans.

The general value of the reactions at the points of support under any continuous beam have been given in eqs. (19), (20), (21), etc., of article 24. Before those equations, however, can be applied to any particular case, the values of the bending moments, which appear in the expressions S_a, S_b', S_b, etc., for the shears, must be determined. In the application of the theorem of three moments, it is usually assumed that the continuous beam before flexure is straight between the points of support, and that the latter belong to a configuration of no bending. The moment of inertia I is also assumed to be constant. This is frequently not strictly true, yet it will be assumed in what follows, since the method to be used in finding the moments is independent of the assumption, and remains precisely the same whatever form for the theorem of three moments may be chosen.

Agreeably to the assumption made, eq. (5)* of the preceding article takes the following form:

$$M_al_a+2M_b(l_a+l_c)+M_cl_c=-\frac{1}{l_a}\overset{a}{\Sigma}P(l_a^2-z^2)z$$

$$-\frac{1}{l_c}\overset{c}{\Sigma}P(l_c^2-z^2)z. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

---

* Or eq. (14) of Art. 24.

Let Fig. 1 represent a continuous beam of n spans equal or unequal in length. At the points of support,

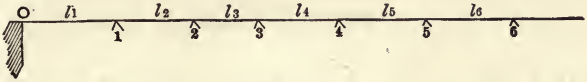


FIG. 1.

0, 1, 2, 3, 4, 5, etc., let the bending moments be represented by M_0, M_1, M_2, M_3, etc. The moment M_0 is always known; it is ordinarily zero, and that will be considered its value.

An examination of Fig. 1 shows that, by repeated applications of eq. (1), the number of resulting equations of condition will be one less than the number of spans. If the two end moments are known (here assumed to be zero), the number of unknown moments will also be one less than the number of spans. Hence the number of equations will always be sufficient for the determination of the unknown moments.

For the sake of brevity let the following notation be adopted:

$$u_1 = -\frac{1}{l_1}\overset{1}{\Sigma}P(l_1^2 - z^2)z - \frac{1}{l_2}\overset{2}{\Sigma}P(l_2^2 - z^2)z.$$

$$u_2 = -\frac{1}{l_2}\overset{2}{\Sigma}P(l_2^2 - z^2)z - \frac{1}{l_3}\overset{3}{\Sigma}P(l_3^2 - z^2)z.$$

$$u_3 = -\frac{1}{l_3}\overset{3}{\Sigma}P(l_3^2 - z^2)z - \frac{1}{l_4}\overset{4}{\Sigma}P(l_4^2 - z^2)z.$$

$$\text{etc.} = \qquad \text{etc.} - \qquad \text{etc.}$$

$$\begin{array}{lll} & a_1 = 2(l_1 + l_2); & b_1 = l_2. \\ a_2 = l_2; & b_2 = 2(l_2 + l_3); & c_2 = l_3. \\ b_3 = l_3; & c_3 = 2(l_3 + l_4); & d_3 = l_4. \\ c_4 = l_4; & d_4 = 2(l_4 + l_5); & f_4 = l_5. \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ p_i = l_i; & q_i = 2(l_i + l_{i+1}); & s_i = l_{i+1}; \end{array}$$

i denoting any number of the series 1, 2, 3, 4, . . . n. It is thus seen that, in general,

$$q_i = 2(p_i + s_i);$$

also that $a_2 = b_1$, $c_2 = b_3$, $d_3 = c_4$, etc. These relations can be used to simplify the final result.

By repeated applications of eq. (1) the following n equations of condition, involving the notation given above, will result:

$$\left.\begin{array}{lr}
a_1 M_1 + b_1 M_2 & = u_1 \\
a_2 M_1 + b_2 M_2 + c_2 M_3 & = u_2 \\
\quad + b_3 M_2 + c_3 M_3 + d_3 M_4 & = u_3 \\
\qquad + c_4 M_3 + d_4 M_4 + f_4 M_5 & = u_4 \\
\qquad\quad + d_5 M_4 + f_5 M_5 + g_5 M_6 & = u_5 \\
\qquad\qquad . \; . \; . \; . \; . \; . & = . \, . \\
\qquad\qquad\quad . \; . \; . & = u_n
\end{array}\right\} \quad . \quad (2)$$

These simultaneous equations may be treated in various ways in order to determine the values of the moments M_1, M_2, M_3, etc. The preceding notation is adapted to the method by determinants, which is probably as simple as any. As these procedures are purely algebraic they will not be further developed here.

In American engineering practice, as exemplified in the theory of revolving-swing bridges, it is necessary to consider at most, two simultaneous equations of condition whose solution requires the simplest process of elimination only.

This last case may be simply illustrated by referring to Fig. 1, in which $M_0=0$. If there are three spans $M_3=0$ as one of the end spans. The first two of eq. (2) will be needed:

$$a_1M_1+b_1M_2=u_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$a_2M_1+b_2M_2=u_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Simple elimination will then give:

$$M_1=\frac{b_2u_1-b_1u_2}{a_1b_2-a_2b_1}; \quad \text{and} \quad M_2=\frac{a_1u_2-a_2u_1}{a_1b_2-a_2b_1}. \quad . \quad (5)$$

Reactions.

After the moments are found, either by the general or special method, for any condition of loading, the reactions will at once result from the substitution of the values thus found in the eqs. (15) to (21) of Art. 24, which it is not necessary to reproduce here.

Art. 27.—Deflection by the Common Theory of Flexure.

The deflection or sag of a beam subjected to loading at right angles to its axis is the displacement of the neutral surface in the direction of the loading. Ordinarily the beam is horizontal and the loading vertical, so that the deflection is also vertical. The entire deflection is due both to the lengthening and the shortening of the fibres on the two sides of the netural surface and to the action of the transverse shear throughout the beam. The equation leading directly to the former portion is eq. (7) of Art 14, but the equations of Art. 24 must be used to determine the deflection due to shear.

Let x_0 be the coordinate of some point at which the tangent of the inclination of the neutral surface to the axis of x is known; then from eq. (7) of Art. 14

$$\frac{dw}{dx} = \int_{x_0}^{x} \frac{M}{EI} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$\frac{dw}{dx}$ will be at once recognized as the general value of the tangent of the inclination just mentioned, or, in the case of curved beams, as approximately the difference between the tangent, before and after flexure.

Again, let x_1 represent the coordinate of a point at which the deflection w is known, then from eq. (1):

$$w = \int_{x_1}^{x} \int_{x_0}^{x} \frac{M}{EI} dx^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The points of greatest or least deflection and greatest or least inclination of neutral surface are easily found by the aid of eqs. (1) and (2).

The point of greatest or least deflection is evidently found by putting

$$\frac{dw}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and solving for x. Since $\frac{dw}{dx}$ is the value of the tangent of the inclination of the neutral surface, it follows that *a point of greatest or least deflection is found where the beam is horizontal.*

Again, the point at which the inclination will be greatest or least is found by the equation

$$\frac{d\left(\frac{dw}{dx}\right)}{dx} = \frac{d^2w}{dx^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

But, approximately, $\frac{d^2w}{dx^2}$ is the reciprocal of the radius of curvature; hence *the greatest inclination will be found at that point at which the radius of curvature becomes infinitely great*, or, *at that point at which the curvature changes from positive to negative or vice versa.* These points are called points of "contra-flexure." Since:

$$M = EI\frac{d^2w}{dx^2},$$

there is no bending at a point of contra-flexure.

The moment of the external forces, M, will always be expressed in terms of x. After the insertion of such values, eqs. (1) and (2) may at once be integrated and (3) and (4) solved.

The coefficient of elasticity, E, is always considered a constant quantity; hence it may always be taken outside the integral signs. In all ordinary cases, also, I is constant throughout the entire beam. In such cases, then, there will only need to be integrated the expressions:

$$\int_{x_0}^{x} Mdx \quad \text{and} \quad \int_{x_1}^{x}\int_{x_0}^{x} Mdx^2.$$

It is sometimes convenient to express the tangent of inclination of the neutral surface and the deflection in terms of some known intensity k_0 of fibre stress at the distance d from the neutral surface and at a section of the beam where the known external bending moment is M_0. The desired expressions may readily be written by simply transforming eqs. (1) and (2) to the proper shape. It has been shown by eq. (10) of Art. 14 that $k_0 = \frac{M_0 d}{I}$, and hence that $I=\frac{M_0 d}{k_0}$. By substitution of this value of I first in eq. (1) and then in eq. (2), there will result:

$$\frac{dw}{dx}=\frac{k_0}{EM_0 d}\int_{x_0}^{x} Mdx \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$w=\frac{k_0}{EM_0 d}\int_{x_1}^{x}\int_{x_0}^{x} Mdx^2. \quad . \quad . \quad . \quad . \quad (6)$$

Eqs. (5) and (6) give the desired expressions in which I and d are considered constant in accordance with all ordinary practice. In the use of these last two equations it is supposed that the conditions of any given problems will enable k_0 and M_0 to be computed as known quantities.

The general form of the integral in the second member of eq. (6) is easily determined. The quantities M_0 and M are exactly similar expressions with the same number of terms and of the same degree. The effect of the integration of M twice between the limits indicated is to raise the degree of each term of which it is composed by two, so that the double integration of Mdx^2 divided by M_0 will be a simple product al^2, a being a numerical quantity depending upon the manner of loading, the condition of the ends of the beam, or other attendant circumstances of the same general character. Inserting these results in eq. (6), the expression for the deflection will become

$$w=k_0\frac{al^2}{Ed}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

Eq. (6*a*) is not often used, but there are some practical applications of formulæ in which it must be employed.

Deflection Due to Shearing.

That portion of the deflection due to transverse shearing may be determined as readily as that due to the lengthening and shortening of the fibres of the bent beam. In determining the requisite equations it is necessary to consider only the intensity of shear in the neutral surface, as it is the deflection of that surface which is sought.

Let w' be the deflection due to shearing and let ϕ represent the transverse shearing strain for a unit of length of the beam. The transverse strain for an indefinitely short portion dx of the neutral surface will then be $dw' = \phi dx$. If G represents the coefficient of elasticity for shear, while s represents the intensity of shear, eq. (3) of Art. 2 shows that $\phi = \frac{s}{G}$. There may then be written:

$$dw' = \phi dx = \frac{s}{G} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By using the value of s given in eq. (7) of Art. 15,

$$dw' = \frac{Sd^2}{2IG} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The general expressions for the shearing deflection will, therefore, take the form:

$$w' = \frac{d^2}{2IG} \int Sdx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The integration required in eq. (9) can be made with ease in any given case, as it is necessary only to express the value of the total transverse shear S in terms of x. The application of that equation to special cases will be made in a later article. Obviously the total deflection in any bent beam will be the sum:

$$w+w'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Art. 28.—The Neutral Curve for Special Cases.

The curved intersection of the neutral surface with a vertical plane passing through the axis of a loaded, and originally straight, beam may be called the "neutral curve." The neutral curve is the locus of the extremities of the ordinates w of Art. 27; it therefore gives the deflection at any point of the beam due to the direct stresses of tension and compression in it, but not due to the effect of transverse shear, which will be treated in a subsequent article.

The method of finding the neutral curve for any particular case of beam or loading can be well illustrated by the operations in the following three cases:

Case I.

This case is shown in the accompanying figure, which represents a cantilever carrying a uniform load with a single weight W at its free end. As usual, the intensity of the uniform loading will be represented by p.

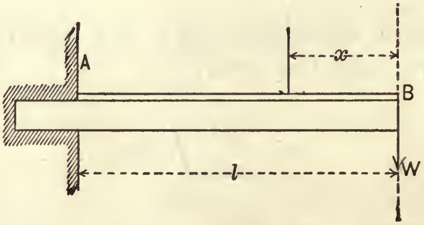


Fig. 1.

Measuring x and w from B, as shown, the general value of the bending moment is

$$M = EI\frac{d^2w}{dx^2} = Wx + \frac{px^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating between x and l, remembering that:

$$\frac{dw}{dx} = 0$$

for $x = l$:

$$EI\frac{dw}{dx} = \frac{W}{2}(x^2 - l^2) + \frac{p}{6}(x^3 - l^3). \quad . \quad . \quad . \quad (2)$$

Hence

$$\overset{\Delta}{w} = \frac{1}{EI}\left\{\frac{W}{2}\left(\frac{x^3}{3} - xl^2\right) + \frac{p}{6}\left(\frac{x^4}{4} - l^3x\right)\right\}. \quad . \quad . \quad (3)$$

The greatest deflection, w_1, occurs for $x = l$. Hence

$$\overset{\Delta}{w_1} = -\frac{1}{EI}\left(\frac{Wl^3}{3} + \frac{pl^4}{8}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This value of w_1 is the deflection of B below A. The general value of w in eq. (3) is the vertical distance (deflection) of B below the point located by x; as an ordinate it is measured upward from B as the origin of coordinates.

The greatest moment, M_1, exists at A, and its value is:

$$M_1 = Wl + \frac{pl^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

These equations are made applicable to a cantilever with a uniform load by simply making $W = 0$. They then become

$$M = EI\frac{d^2w}{dx^2} = \frac{px^2}{2}, \quad \ldots \ldots \quad (6)$$

$$EI\frac{dw}{dx} = \frac{p}{6}(x^3 - l^3), \quad \ldots \ldots \quad (7)$$

$$w = \frac{p}{6EI}\left(\frac{x^4}{4} - l^3x\right), \quad \ldots \ldots \quad (8)$$

$$w_1 = -\frac{pl^4}{8EI}, \quad \ldots \ldots \quad (9)$$

$$M_1 = \frac{pl^2}{2}. \quad \ldots \ldots \quad (10)$$

Again, for a cantilever with a single weight only at its free end, p is to be made equal to zero in the first set of equations. Those equations then become:

$$M = EI\frac{d^2w}{dx^2} = Wx, \quad \ldots \ldots \quad (11)$$

$$EI\frac{dw}{dx} = \frac{W}{2}(x^2 - l^2), \quad \ldots \ldots \quad (12)$$

$$w = \frac{W}{2EI}\left(\frac{x^3}{3} - xl^2\right), \quad \ldots \ldots \quad (13)$$

$$w_1 = -\frac{Wl^3}{3EI}, \quad \ldots \ldots \quad (14)$$

$$M_1 = Wl. \quad \ldots \ldots \quad (15)$$

The general expressions for the shear and the intensity of loading are:

$$S = EI\frac{d^3w}{dx^3} = W + px, \quad . \quad . \quad . \quad . \quad (16)$$

$$EI\frac{d^4w}{dx^4} = p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Case II.

This case, shown in the figure, is that of a non-continuous beam, supported at each end, and carrying both a uniform load (whose intensity is p) and a single weight W at its middle point. The reaction R, at either end, will then be

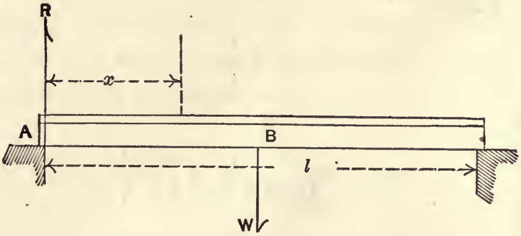


FIG. 2

$$R = \frac{pl + W}{2}.$$

The general value of the moment will then be

$$M = EI\frac{d^2w}{dx^2} = Rx - \frac{px^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The origin of x and w is taken at A.

Remembering that

$$\frac{dw}{dx} = 0 \quad \text{for} \quad x = \frac{l}{2},$$

and integrating between the limits x and $\frac{l}{2}$,

$$EI\frac{dw}{dx}=\frac{R}{2}\left(x^2-\frac{l^2}{4}\right)-\frac{p}{6}\left(x^3-\frac{l^3}{8}\right). \quad . \quad . \quad . \quad . \qquad (19)$$

Again integrating

$$w=\frac{1}{EI}\left\{\frac{R}{2}\left(\frac{x^3}{3}-\frac{xl^2}{4}\right)-\frac{p}{6}\left(\frac{x^4}{4}-\frac{xl^3}{8}\right)\right\}. \quad . \quad . \qquad (20)$$

The greatest deflection w_1 occurs at the centre of the span, for which

$$x=\frac{l}{2}.$$

Hence

$$w_1=-\frac{l^3}{48EI}\left\{W+\frac{5}{8}pl\right\}. \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

The greatest moment, also, is found by putting

$$x=\frac{l}{2}.$$

It has the value

$$M_1=\frac{l}{4}\left(W+\frac{pl}{2}\right). \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

These formulæ are made applicable to a non-continuous beam carrying a uniform load only, by putting $W=0$. They then become

$$R=\frac{pl}{2},$$

$$M=EI\frac{d^2w}{dx^2}=\frac{px}{2}(l-x), \quad . \quad . \quad . \quad . \qquad (23)$$

$$EI\frac{dw}{dx}=\frac{p}{2}\left(\frac{x^2l}{2}-\frac{x^3}{3}-\frac{l^3}{12}\right), \quad . \quad . \quad . \quad . \quad (24)$$

$$w=\frac{p}{24EI}(2x^3l-x^4-l^3x), \quad . \quad . \quad . \quad . \quad (25)$$

$$w_1=-\frac{5pl^4}{384EI}=-\frac{5}{8}\cdot\frac{pl^4}{48EI}, \quad . \quad . \quad . \quad . \quad (26)$$

$$M_1=\frac{pl^2}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The formulæ for a beam of the same kind carrying a single weight at the centre are obtained by putting $p=0$ in the first set of equations. Those for the greatest deflection and greatest moment, only, however, will be given. They are

$$w_1=-\frac{Wl^3}{48EI}, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$M_1=\frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The general values of the shear and intensity of loading are

$$S=\frac{dM}{dx}=R-px, \quad . \quad . \quad . \quad . \quad . \quad (30)$$

$$\frac{d^2M}{dx^2}=-p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Case III.

The general treatment of continuous beams requires the use of the theorem of three moments. The particular case to be treated is shown in Fig. 3. The beam covers the three spans, DA, AB, and BC, and is continuous over the two points of support, A and B.

$$\left.\begin{array}{ll}\text{Let } DA = l_1 \\ \text{`` } AB = l_2 \\ \text{`` } BC = l_3\end{array}\right\} \text{Let } l_2 = nl_1 = n'l_3.$$

Let the intensity of the uniform load on AB be represented by p and let the two single forces P and P' *only*, act in the spans DA and BC respectively. Also let the two distances

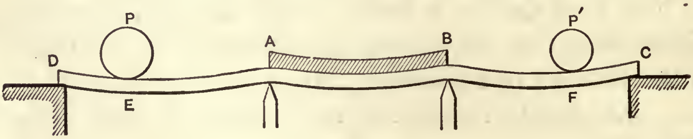


FIG. 3.

$$DE = z_1 = al_1 \quad \text{and} \quad CF = a'l_3$$

be given. *It is required to find the magnitudes of the forces P and P′, if the beam is horizontal at A and B.*

Since the beam is horizontal at A and B, the bending moments over those two points of support will be equal to each other, for the load on AB is both uniform and symmetrical. Let this bending moment, common to A and B, be represented by M_2. As the ends of the beam simply *rest* at D and C, the moments at those two points reduce to zero.

Because the four points D, A, B, and C are in the same level, the first member of eq. (14) of Art. 24 becomes equal to zero.

If that equation be applied to the three points D, A, and B, the conditions of the present problem produce the following results:

$$M_a = 0, \quad M_b = M_c = M_2,$$

and

$$\frac{1}{l_c} \overset{c}{\Sigma} P(l_c^2 - z^2)z = p\frac{l_2^3}{4}.$$

Hence the equation itself will become

$$M_2(2l_1 + 3l_2) + \frac{P}{l_1}(l_1^2 - z_1^2)z_1 + p\frac{l_2^3}{4} = 0. \quad . \quad . \quad (32)$$

$$\therefore \quad M_2 = -\frac{4P(l_1^2 - z_1^2)z_1 + pl_1l_2^3}{4l_1(2l_1 + 3l_2)};$$

$$\therefore \quad M_2 = -l_1\frac{4P(1 - a^2)a + pn^3l_1}{4(2 + 3n)}. \quad . \quad . \quad . \quad . \quad (33)$$

$$\therefore \quad \text{Reaction at } D = R_1 = P\frac{l_1 - z_1}{l_1} + \frac{M_2}{l_1}. \quad . \quad . \quad (34)$$

As the origin of z_1 is at D, x will be measured from the same point.

Separate expressions for moments must be obtained for the two portions, DE and EA of l_1, because the law of loading in that span is not continuous.

Taking moments about any point of EA

$$EI\frac{d^2w}{dx^2} = R_1x - P(x - z_1). \quad . \quad . \quad . \quad . \quad (35)$$

Remembering that

$$\frac{dw}{dx} = 0$$

for $x=l_1$, and integrating between the limits x and l_1

$$EI\frac{dw}{dx}=\frac{R_1}{2}(x^2-l_1^2)-\frac{P}{2}(x^2-l_1^2)+Pz_1(x-l_1). \quad . \quad (36)$$

Again, remembering that $w=0$ for $x=l_1$, and integrating between the limits x and l_1,

$$EIw=\frac{R_1}{2}\left(\frac{x^3}{3}-l_1^2x+\frac{2l_1^3}{3}\right)-\frac{P}{2}\left(\frac{x^3}{3}-l_1^2x+\frac{2l_1^3}{3}\right)$$
$$+Pz_1\left(\frac{x^2}{2}-l_1x+\frac{l_1^2}{2}\right). \quad (37)$$

Taking moments about any point in DE

$$EI\frac{d^2w}{dx^2}=R_1x; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$\therefore \quad EI\frac{dw}{dx}=R_1\frac{x^2}{2}+C. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Making $x=z_1$ in eqs. (36) and (39), then subtracting

$$C=-\frac{R_1}{2}l_1^2-\frac{P}{2}(z_1^2-l_1^2)+Pz_1(z_1-l_1);$$

$$\therefore \quad EI\frac{dw}{dx}=\frac{R_1}{2}(x^2-l_1^2)-\frac{P}{2}(z_1^2-l_1^2)+Pz_1(z_1-l_1). \quad (40)$$

Remembering that $w=0$ for $x=0$, and integrating between the limits x and 0,

$$EIw=\frac{R_1}{2}\left(\frac{x^3}{3}-l_1^2x\right)-\frac{P}{2}(z_1^2-l_1^2)x+Pz_1(z_1-l_1)x. \quad (41)$$

Making $x=z_1$ in eqs. (37) and (41), then subtracting

$$\frac{R_1l_1^3}{3}-\frac{P}{3}(l_1^3-z_1^3)+\frac{Pz_1}{2}(l_1^2-z_1^2)=0. \quad . \quad . \quad . \quad (42)$$

Putting the value of M_2 from eq. (33) in eq. (34), then inserting the value of R_1, thus obtained, in eq. (42), after making $z_1 = al_1$,

$$P\left[(1-a) - \frac{a(1-a^2)}{2+3n} - (1-a^3) + \frac{3}{2}a(1-a^2)\right] = \frac{pn^3l_1}{4(2+3n)};$$

$$\therefore\ P = \frac{pn^2l_1}{6a(1-a^2)} = \frac{pnl_2}{6a(1-a^2)}. \quad . \quad . \quad . \quad (43)$$

This is the desired value of P, which will cause the beam to be horizontal over the two points of support A and B when the span AB carries a uniform load of the intensity p.

By the aid of eq. (43), eq. (33) now gives

$$M_2 = -pl_1^2\frac{(2n^2+3n^3)}{12(2+3n)} = -\frac{pn^2l_1^2}{12} = -\frac{pl_2^2}{12}. \quad . \quad (44)$$

It is to be noticed that M_2 is entirely independent of l_1 or l_3. Eq. (43) also gives

$$p = P\frac{6a(1-a^2)}{n^2l_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Hence

$$M_2 = -\frac{Pl_1}{2}(1-a^2)a. \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Thus any of the preceding equations may be expressed in terms of p or P.

R_1 also becomes

$$R_1 = \frac{pnl_2}{6a(1+a)} - \frac{pnl_2}{12}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

or

$$R_1 = P(1-a)[1 - \tfrac{1}{2}a(1+a)]. \quad . \quad . \quad . \quad . \quad . \quad (48)$$

It is clear that there cannot be a point of no bending in *DE*. Hence the point of contra-flexure must lie between *E* and *A*, Fig. 3. In order to locate this point, according to the principles already established, the second member of eq. (35) must be put equal to zero. Doing so and solving for x,

$$x = \frac{P}{P - R_1} z_1 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Since P is always greater than R_1, there will always be a point of contra-flexure.

All these equations will be made applicable to the span *BC* by simply writing a' for a, l_3 for l_1, and n' for n.

As an example, let

$$a = \tfrac{1}{2} \quad \text{and} \quad n = 1.$$

Eqs. (43), (44), and (47) then give

$$P = \tfrac{4}{9} pl,$$

$$M_2 = -\frac{pl^2}{12} = -\frac{3Pl}{16},$$

$$R_1 = pl_1 \left(\tfrac{2}{9} - \tfrac{1}{12}\right) = \tfrac{5}{36} pl = \tfrac{5}{16} P;$$

after writing,

$$l_1 = l_2 = l_3 = l.$$

In general, the span l_1 is called "a beam fixed at one end, simply supported at the other and loaded at any point with the single weight, P."

Let it, again, be required *to find an intensity, "p'," of a uniform load, resting on the span l_1, which will cause the beam to be horizontal at the points A and B.*

Since the load is continuous, only one set of equations will be required for the span. The equation of moments will be

$$EI\frac{d^2w}{dx^2} = R_1 x - \frac{p'x^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Integrating between the limits x and l_1,

$$EI\frac{dw}{dx} = \frac{R_1}{2}(x^2 - l_1{}^2) - \frac{p'}{6}(x^3 - l_1{}^3). \quad . \quad . \quad . \quad . \quad (51)$$

Integrating between the limits x and 0,

$$EIw = \frac{R_1}{2}\left(\frac{x^3}{3} - l_1{}^2 x\right) - \frac{p'}{6}\left(\frac{x^4}{4} - l_1{}^3 x\right). \quad . \quad . \quad . \quad (52)$$

But, also, $w = 0$, when $x = l_1$. Hence

$$R_1\frac{l_1{}^3}{3} = \frac{p'l_1{}^4}{8}; \quad \therefore \; R_1 = \tfrac{3}{8}p'l_1. \quad . \quad . \quad . \quad . \quad . \quad (53)$$

This equation gives the value R_1 when p' is known. Making $x = l_1$ in eq. (50), and using the value of R_1 from eq. (53),

$$M_2 = p'l_1{}^2(\tfrac{3}{8} - \tfrac{1}{2}) = -\frac{p'l_1{}^2}{8}. \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Adapting eq. (32) to the present case,

$$M_2(2l_1 + 3l_2) + \tfrac{1}{4}(p'l_1{}^3 + pl_2{}^3) = 0;$$

$$\therefore \; M_2 = -\frac{(p' + pn^3)l_1{}^2}{4(2 + 3n)}. \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Equating these two values of M_2,

$$p' = \tfrac{2}{3}pn^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Thus is found the desired value of p'. In this case the span l_1 is called "a beam fixed at one end, simply supported at the other and uniformly loaded."

The points of contra-flexure are found by putting the second member of eq. (50) equal to zero and solving for x, after introducing the value of R_1 from eq. (53). Hence

$$\tfrac{3}{4}l_1x - x^2 = 0,$$

or

$$x = 0 \quad \text{and} \quad x = \tfrac{3}{4}l_1.$$

Between the simply supported end and point of contra-flexure the beam is evidently convex *downward*, and convex upward in the other portion of the spans l_1 and l_3, whether the load is single or continuous. Moments of different signs will then be found in these two portions, and there will be a maximum for each sign. The location of the sections in which these greatest moments act may be made in the ordinary manner by the use of the differential calculus; but the *negative* maximum is evidently M_2, given by eqs. (44) and (55). On the other hand, the *positive* maximum is clearly found at the point of application of P in the case of a single load, and at the point

$$x = \tfrac{3}{8}l_1$$

in the case of a continuous load. These conclusions will at once be evident if it be remembered that the portion of the beam between the supported end and point of contra-flexure is, in reality, *a beam simply supported at each end.* These moments will have the values

$$M_1 = Pl_1(1-a)a - l_1\frac{4P(1-a^2)a^2 + pan^3l_1}{4(2+3n)}, \quad . \quad (57)$$

$$M_1' = \tfrac{9}{128}p'l_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

In case of a single load if P is given, and not p, eq. (45) shows

$$M_1 = Pl_1(1-a)a[1-\tfrac{1}{2}a(1+a)].$$

The points of greatest deflection are found by putting the second members of eqs. (36), (40), and (51) each equal to zero, and then solving for x. They are not points of great importance, and the solutions will not be made.

The following are the general values of the shears for a single load on l_1:

$$\text{In } AE, \quad S = EI\frac{d^3w}{dx^3} = R_1 - P; \quad \text{[from eq. (35)]}.$$

$$\text{In } ED \quad S_1 = EI\frac{d^3w}{dx^3} = R_1; \quad \text{[from eq. (38)]}.$$

The shear in ι_1 for the uniform load p' is

$$S' = EI\frac{d^3w}{dx^3} = R_1 - p'x; \quad \text{[from eq. (50)]}.$$

Also

$$\textit{Intensity of load} = EI\frac{d^4w}{dx^4} = -p'.$$

As has already been observed, all the equations relating to the span l_1 may be made applicable to the span l_3 by changing a to a' and n to n'.

The span l_2 remains to be considered.

Since the bending moments at A and B are equal to each other, and since the loading is uniformly continuous, half of it (the load pl_2) will be supported at A and the other half at B. In other words, the vertical shear at an indefinitely short distance to the right of A, also to the left of B, will be equal to $\frac{pl_2}{2}$. Let x be measured to the right and from A. The bending moment at any section x will be

$$EI\frac{d^2w}{dx^2}=M_2+\frac{pl_2}{2}x-\frac{px^2}{2},$$

or

$$EI\frac{d^2w}{dx^2}=M_2+\frac{p}{2}(l_2x-x^2). \quad . \quad . \quad . \quad (59)$$

Integrating between the limits x and 0,

$$EI\frac{dw}{dx}=M_2x+\frac{p}{2}\left(\frac{l_2x^2}{2}-\frac{x^3}{3}\right). \quad . \quad . \quad . \quad (60)$$

Again, integrating between the same limits,

$$EIw=\frac{M_2x^2}{2}+\frac{p}{12}\left(l_2x^3-\frac{x^4}{2}\right). \quad . \quad . \quad . \quad (61)$$

Since

$$\frac{dw}{dx}=0$$

for $x=l_2$, eq. (60) will give M_2 independently of preceding equations. Following this method, therefore,

$$M_2=-\frac{pl_2^2}{12}.$$

This is the same value which has already been obtained. Introducing the value of M_2,

$$EI\frac{d^2w}{dx^2}=\frac{p}{2}\left(l_2x-x^2-\frac{l_2^2}{6}\right), \quad . \quad . \quad . \quad (62)$$

$$EI\frac{dw}{dx}=\frac{p}{2}\left(\frac{l_2x^2}{2}-\frac{x^3}{3}-\frac{l_2^2}{6}x\right), \quad . \quad . \quad . \quad (63)$$

$$EIw=\frac{px^2}{12}\left(l_2x-\frac{x^2}{2}-\frac{l_2^2}{2}\right). \quad . \quad . \quad . \quad (64)$$

The points of contra-flexure are found by putting the second member of eq. (62) equal to zero. Hence

$$x^2 - l_2 x = -\frac{l_2{}^2}{6};$$

$$\therefore \quad x = l_2\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{6}}\right) = \begin{cases} 0.789 l_2. \\ 0.211 l_2. \end{cases}$$

The moment at the centre of the span is found by putting

$$x = \frac{l_2}{2}$$

in eq. (62):

$$M_1 = \frac{p l_2{}^2}{24}.$$

This is the greatest *positive* moment

The general value of the shear is

$$S = EI\,\frac{d^3w}{dx^3} = p\left(\frac{l_2}{2} - x\right);$$

and the intensity of load

$$EI\,\frac{d^4w}{dx^4} = -p.$$

The span l_2 is generally called "a beam fixed at both ends and uniformly loaded."

It is sometimes convenient to consider a single load at the centre of the span l_2, while the beam remains horizontal at A and B; in other words, to consider "a beam fixed at each end and supporting a weight at the centre."

Let W represent this weight; then a half of it will be the shear at an indefinitely short distance to the right of

A and left of B. As before, let x be measured from A, and positive to the right. The moment at any point will be

$$EI\frac{d^2w}{dx^2}=M_2-\frac{Wx}{2}. \quad . \quad . \quad . \quad . \quad . \quad .(65)^*$$

Integrating between x and o,

$$EI\frac{dw}{dx}=M_2x-\frac{Wx^2}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

If $x=\frac{l_2}{2}$, then will

$$\frac{dw}{dx}=0;$$

hence $$M_2=\frac{Wl_2}{8}.$$

The general value of the moment then becomes

$$M=EI\frac{d^2w}{dx^2}=\frac{Wl_2}{8}-\frac{Wx}{2}. \quad . \quad . \quad . \quad . \quad . \quad (67)$$

If $x=\frac{l_2}{2}$ in this equation, the bending moment at the centre (where W is applied) has the value

$$\textit{Centre moment}=-\frac{Wl_2}{8}.$$

Hence *the bending moments at the centre and ends are each equal to the product of the load by one eighth the span, but have opposite signs.*

A second integration between x and o gives

$$w=\frac{1}{EI}\left(\frac{Wl_2x^2}{16}-\frac{Wx^3}{12}\right). \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Hence the deflection at the centre has the value

$$\textit{Centre deflection}=\frac{Wl_2^3}{192EI}.$$

---

* The use of the signs in this and the following equations is changed from the preceding to show that either procedure may be employed.

By placing $M = 0$, the points of contra-flexure are found at the distance from each end,

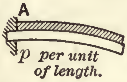
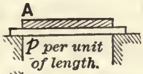
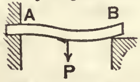
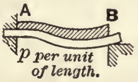
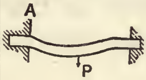
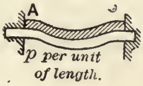
$$x_1 = \frac{l_2}{4}.$$

Addendum to Art. 28.

The formulæ of this article furnish the solutions of many practical questions of maxima deflections and moments. The latter for several ordinary cases are given in the following tabulation:

P is the weight in pounds at *end of beam* or *centre of span.*

p is the load in pounds *per lin. ft.* of beam.

	Beam.	Maximum Moment.	Maximum Deflection.	Point of Contra-flexure.
I		Pl at A	$576\frac{Pl^3}{EI}$ at A	
II		$\frac{1}{2}pl^2$ at A	$216\frac{pl^4}{EI}$ at A	
III		$\frac{1}{4}Pl$ at centre	$36\frac{Pl^3}{EI}$ at centre	
IV		$\frac{1}{8}pl^2$ at centre	$22.5\frac{pl^4}{EI}$ at centre	
V		$-\frac{3}{16}Pl$ at A $\frac{5}{32}Pl$ at centre	$16.16\frac{Pl^3}{EI}$ at $0.447l$ from B	$\frac{8}{11}l$ from B Reaction at B $= \frac{5}{16}P$
VI		$-\frac{1}{8}pl^2$ at A $+\frac{9}{128}pl^2$ at $\frac{3}{8}l$ from B	$9.35\frac{pl^4}{EI}$ at $0.4215l$ from B	$\frac{3}{4}l$ from B Reaction at B $= \frac{3}{8}pl$
VII		$-\frac{1}{8}Pl$ at A $\frac{1}{8}Pl$ at centre	$9\frac{Pl^3}{EI}$ at centre	$\frac{1}{4}l$ from each end
VIII		$-\frac{1}{12}pl^2$ at A $\frac{1}{24}pl^2$ at centre	$4.5\frac{pl^4}{EI}$ at centre	$0.211l$ from each end

l is the length of beam or of span in *feet*.

E is the coefficient of elasticity in pounds per sq. inch.

I is the moment of inertia of the normal section of the beam with all dimensions of section in *inches*.

The "Max. Moments" will be in *foot pounds*, and the "Max. Deflections" will be in *inches*.

In the use of flexure formulæ, in many practical applications, it is best to have the moment M in inch-pounds, which will result from simply multiplying the "Max. Moments" of the preceding table by 12.

Case I results from eqs. (14) and (15); Case II from eqs. (9) and (10); Case III from eqs. (28) and (29); Case IV from eqs. (26) and (27). In Case V the reaction is found by putting $a = \frac{1}{2}$ in eq. (48); the point of "Max. Deflection" is found by placing $z_1 = \frac{1}{2}l$ in eq. (40), and the resulting value of $\frac{dw}{dx}$ equal to zero and solving for x, which latter value in eq. (41) will give "Max. Deflection." Case VI results from treating eqs. (53), (51), and (52) in precisely the same manner. Case VII results directly from the formulæ on page 142. Case VIII results directly from the equations on pages 140 and 141.

The preceding cases are those which commonly occur with constant values of E and I. Other cases, such as a single load at any point, or partial uniform load over any part of span, are to be treated by the same general principles.

Art. 29.—Direct Demonstration for Beam Fixed at One End and Simply Supported at the Other Under Uniform and Single Loads.

A beam is said to be fixed at one end when it is under such constraint that the neutral surface does not change its direction at that end whatever may be the loading.

This fixedness, as has been fully shown in Art. 28, is equivalent to the application of a suitable constraining moment. Beams with one or both ends under such constraint have been fully treated in Art. 28, but it is desirable to establish the formulæ for such cases directly, i.e., without the employment of the theorem of three moments.

In Fig. 1 a beam is shown fixed at one end B and simply supported at the other end A, while it carries a uniform load p, per linear unit and the single load P at the distance al from A. The length of span is l and the coordinate x is measured horizontally to the right from A. The two reactions are R and R'. E is the modulus of elasticity, I the moment of inertia of the normal section of the beam, and w is the deflection at any point. The bending moment for any point in the segment al of the beam is:

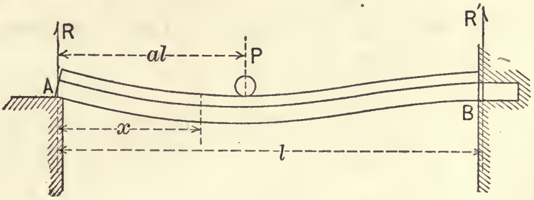


Fig. 1.

$$EI\frac{d^2w}{dx^2}=Rx-p\frac{x^2}{2}. \quad \dots\dots \quad (1)$$

The bending moment for the section $l-al$ of the beam is:

$$EI\frac{d^2w}{dx^2}=Rx-p\frac{x^2}{2}-P(x-al). \quad \dots \quad (2)$$

Integrating eq. (1) and representing by C the constant of integration:

$$EI\frac{dw}{dx}=R\frac{x^2}{2}-p\frac{x^3}{6}+C. \quad \dots\dots \quad (3)$$

Integrating eq. (3) between x and 0, remembering that $w=0$, when $x=0$;

$$EIw=R\frac{x^3}{6}-\frac{px^4}{24}+Cx. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Integrating eq. (2) between x and l, remembering that $\frac{dw}{dx}=0$ when $x=l$,

$$EI\frac{dw}{dx}=R\frac{x^2-l^2}{2}-\frac{p}{6}(x^3-l^3)-\frac{p}{2}(x^2-l^2-2al(x-l)) \quad . \qquad (5)$$

If $x=al$ in eqs. (3) and (5), the first members of those equations will be equal, hence:

$$\left(EI\frac{dw}{dx}\right)=R\frac{a^2l^2}{2}-p\frac{a^3l^3}{6}+C \quad . \quad . \quad . \quad . \qquad (6)$$

$$\left(EI\frac{dw}{dx}\right)=R\frac{l^2}{2}(a^2-1)-\frac{pl^3}{6}(a^3-1)+\frac{pl^2}{2}(a-1)^2. \qquad (7)$$

Taking the difference between (6) and (7) and solving for C:

$$C=-\frac{Rl^2}{2}+\frac{pl^3}{6}+\frac{Pl^2}{2}(a-1)^2. \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Placing this value of C in eq. (4):

$$EIw=\frac{R}{6}(x^3-3l^2x)-\frac{p}{24}(x^4-4l^3x)+\frac{Pl^2}{2}(a-1)^2x. \quad . \qquad (9)$$

Integrating eq. (5) between the limits of x and l:

$$EIw=\frac{R}{6}(x^3-3l^2x+2l^3)-\frac{p}{24}(x^4-4l^3x+3l^4)$$
$$-\frac{P}{6}(x^3-3l^2x+2l^3-3al(x-l)^2) \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Making $x=al$ in eqs. (9) and (10) and subtracting the former from the latter, there will result:

$$R=\tfrac{3}{8}pl+\frac{p}{2}(a^3-3a+2). \quad . \quad . \quad . \quad . \quad (11)$$

This equation gives the reaction required to enable any of the preceding formulæ to be applied to actual computations. The loads P and p, as well as the quantity a are obviously known for any particular case or problem. With the value of the reaction now established by eq. (11) the deflection or the tangent of inclination of the neutral surface $\frac{dw}{dx}$ may be at once computed for any point in either part of the beam. The fixing or constraining moment required to keep the beam horizontal at B can be at once determined by making $x=l$ in eq. (2) and it has the value;

$$M=Rl-\frac{pl^2}{2}-Pl(1-a). \quad . \quad . \quad . \quad . \quad (12)$$

If the load is wholly uniform or $P=0$, eqs. (11) and (1) give:

$$R=\tfrac{3}{8}pl \text{ and } M=-\frac{pl^2}{8}. \quad . \quad . \quad . \quad . \quad (13)$$

This value of M is the constraining moment required at B when the load is wholly uniform and is identical with eq. 54 of Art. 28. Indeed the preceding equations are the same as those established for the continuous span, conditioned similarly to the beam treated in this article.

In all the preceding equations if the load is wholly uniform it is only necessary to make $P=0$. On the other hand, if there is a single load with no uniform loading $p=0$.

Inasmuch as the beam is convex downward over its left-hand part and convex upward in the vicinity of B, there must be a point of contraflexure either to the right or to the left of P, according to its location. If that point is between P and B, the second member of eq. (2) must be placed equal to 0, giving;

$$x^2+\frac{2(P-R)}{p}x=2\frac{Pal}{p}. \quad . \quad . \quad . \quad . \quad (14)$$

Solving this quadratic equation;

$$x=\frac{R-P}{p}\pm\sqrt{\frac{2Pal}{p}+\left(\frac{P-R}{p}\right)^2}. \quad . \quad . \quad . \quad (15)$$

Eq. (15) gives the location of the point of contraflexure by the value of x measured from A. There are two roots of the equation, but evidently the positive value of the radical only is required.

If the point of contraflexure is between P and A, which would be the case if the single load were near the right-hand end of the span, the second member of eq. (1) must be placed equal to 0, giving;

$$X=\frac{2R}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In case the point of contraflexure is at the point of application of P, $x=al$, hence,

$$x=\frac{2R}{p}=al \text{ and } a=\frac{2R}{pl}. \quad . \quad . \quad . \quad . \quad (17)$$

If it is desired to find the point at which the deflection is geratet, it is osnly necessary to place $\frac{dw}{dx}=0$ in either eqs. (3) or (5), as the case may be, and solve the resulting eq. for x.

The reaction R, i.e., the end shear at B, is;

$$R' = pl + P - R \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The sum of the two reactions must be equal to the total load on the beam.

Special Case, $a = \frac{1}{2}$.

In this case eq. (11) will give the reaction R at A as follows:

$$R = \tfrac{3}{8}pl + \tfrac{5}{16}P. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Hence, the reaction R' at B will be:

$$R' = pl + P - R = \tfrac{5}{8}pl + \tfrac{11}{16}P. \quad . \quad . \quad . \quad . \quad (20)$$

The fixing or constraining moment M_1 at B is by eq. (12):

$$M_1 = -\frac{pl^2}{8} - \frac{3}{16}Pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Eq. (15) shows that the position of the point of contraflexure will depend upon the magnitude of P. If $P = 0$ that equation shows that the point of contraflexure will be $\frac{3}{4}l$ from A:

$$x = \tfrac{3}{4}l. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The part $\frac{3}{4}l$ of the span will be in the condition of a beam simply supported at each end and uniformly loaded. Hence the greatest positive bending moment at the distance $\frac{3}{8}l$ from A is:

$$M' = \tfrac{9}{128}pl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The point of greatest deflection will be found by placing the second member of eq. (5) = 0 and solving for x.

Art. 30.—Direct Demonstration for Beams Fixed at Both Ends under Uniform and Single Loads.

Fig. 1 shows a horizontal beam with both ends fixed, so that whatever may be the magnitudes of the uniform loading and the single load, or the position of the latter, the neutral surface at each end of the beam remains horizontal. The coordinate x is measured from the left-hand end A of the beam as is also the distance al of the single load P from the left end of the span. The length of the span is l and the reactions or shears at the ends of the span are indicated by R and R'. The fixing or constraining moment at A is indicated by M_0 and the uniform load per linear unit by p. If as before w represents the deflection at any point, the equation of moments for the part al of the beam may at once be written:

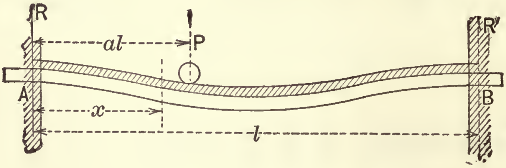


FIG. 1.

$$EI\frac{d^2w}{dx^2}=M_0+Rx-p\frac{x^2}{2}. \quad . \quad . \quad . \quad . \quad (1)$$

Integrating eq. (1) between the limits x and 0, and remembering that $\frac{dw}{dx}=0$ for $x=0$;

$$EI\frac{dw}{dx}=M_0x+R\frac{x^2}{2}-\frac{px^3}{6}. \quad . \quad . \quad . \quad . \quad (2)$$

Integrating eq. (2) between the limits x and 0, eq. (3) may be at once written as $w=0$ for $x=0$:

$$EIw=M_0\frac{x^2}{2}+R\frac{x^3}{6}-p\frac{x^4}{24}. \quad . \quad . \quad . \quad . \qquad (3)$$

Proceeding in the same manner for that part of the beam between B and the load P the equation of moments is:

$$EI\frac{d^2w}{dx^2}=M_0+Rx-p\frac{x^2}{2}-P(x-al). \qquad . \quad . \qquad (4)$$

Integrating eq. (4) between the limits of x and l, since $\frac{dw}{dx}=0$ for $x=l$;

$$EI\frac{dw}{dx}=M_0(x^2-l^2)+\frac{R}{2}(x^2-l^2)-\frac{p}{6}(x^3-l^3)$$
$$-\frac{P}{2}(x^2-l^2-2al(x-l)). \quad . \quad . \quad . \quad . \qquad (5)$$

Again integrating between the limits x and l:

$$EIw=\frac{M_0}{2}(x-l)^2+\frac{R}{6}(x^3-3l^2x+2l^3)-\frac{p}{24}(x^4-4l^3x+3l^4)$$
$$-\frac{P}{2}\left(\frac{x^3}{3}-l^2x-alx^2+2al^2x+\tfrac{2}{3}l^3-al^3\right). \quad . \quad . \qquad (6)$$

The two unknown quantities M_0 and R are to be found. By placing $x=al$ in eqs. (2) and (5), then subtracting the former from the latter:

$$0=-M_0-\frac{R}{2}l+\frac{pl^2}{6}+\frac{Pl}{2}(a-1)^2. \quad . \quad . \quad . \qquad (7)$$

Again making $x=al$ in eqs. (3) and (4), then subtracting the former from the latter;

$$0=-\frac{M_0}{2}(2a-1)-\frac{Rl}{6}(3a-2)+\frac{pl^2}{24}(4a-3)+\frac{Pl}{3}(a-1)^3. \qquad (8)$$

If eq. (7) be multiplied by $\frac{1}{2}(2a-1)$ and then subtracted from eq. (8), the following value of the reaction or end shear will at once result:

$$R=\frac{pl}{2}+P(a-1)^2(2a+1). \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

By placing this value of R in eq. (7), the value of M_0 at once follows:

$$M_0=-\frac{pl^2}{12}-Pla(a-1)^2. \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

In order to determine the moment M_1 at the end B of the span, it is only necessary to substitute the preceding values of M_0 and R in eq. (4):

$$M_1=-\frac{pl^2}{12}-Pla^2(1-a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

These equations give all the quantities required for the complete solution of the case. The reaction or end shear at B is simply:

$$pl+P-R=\frac{pl}{2}+P(1-(a-1)^2(2a+1)) \quad . \quad . \qquad (12)$$

The greatest negative bending moment will obviously be found at either one end or the other of the span, depending upon the value of a and the amount of the load P. The greatest positive bending moment will be found where the shear is zero.

There will be two points of contraflexure, one in each segment of the span. That point located in the part al will be determined in the usual manner by placing the second member of eq. (1) equal to zero and solving the quadratic equation. This simple operation will give eq. (13):

$$x=\frac{R}{p}\pm\sqrt{\frac{2M_0}{p}+\frac{R^2}{p^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Proceeding in the same manner with eq. (4) there will result:

$$x = \frac{R-P}{p} \pm \sqrt{\frac{2(M+Pal)}{p} + \frac{(R-P)^2}{p^2}}. \quad . \quad . \quad (14)$$

This last value of x will indicate the point of contraflexure for the right-hand part of the beam.

Special Case, $a = \frac{1}{2}$.

If P be placed at the center of the span, $a = \frac{1}{2}$ and eqs. (9), (10), and (11) will give eq. (15):

$$R = \frac{pl}{2} + \frac{P}{2} \text{ and } M_0 = M_1 = -\frac{pl^2}{12} - \frac{Pl}{8}. \quad . \quad . \quad (15)$$

The moment M^1 at the centre of the span will be given by the aid of eq. (1):

$$M' = \frac{pl^2}{24} + \frac{Pl}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The greatest deflection w_1 is at the centre of the span and it is given by placing $x = \frac{l}{2}$ in eq. (3).

$$w_1 = \frac{l^2}{8EI}\left(M_0 + \frac{Rl}{6} - \frac{pl^2}{48}\right). \quad . \quad . \quad . \quad . \quad (17)$$

The values of M_0 and R are given by eq. (15).

Art. 31.—Deflection Due to Shearing in Special Cases.

The deflection due to transverse shearing only in all the ordinary cases of loaded beams can readily be computed by aid of the general eq. (9) of Art. 27. If d is the distance from the most remote fibre from the neutral axis of any normal section whose moment of inertia about the same axis is I, and if G and S are the coefficient of elasticity and total transverse shear respectively, the deflection, w', sought is

$$w' = \frac{d^2}{2IG}\int Sdx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The limits of the integration must be indicated for each particular case.

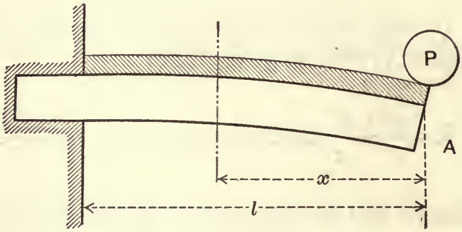


Fig. 1.

In Fig. 1 let the cantilever, whose length is l, carry the single load P at its end, and the uniform load p per linear unit. The shear at any section distant x from A is $S = P + px$. The substitution of this value of S in eq. (1) will give

$$w' = \frac{d^2}{2IG}\int_0^l (P + px)dx = \frac{d^2}{2IG}\left(Pl + \frac{pl^2}{2}\right). \quad . \quad (2)$$

If the uniform load only acts, $P = 0$; and if P only acts, $p = 0$.

Fig. 2 shows the case of a simple beam supported at each end, carrying a uniform load p per linear unit and the single load P at the centre of the span. The reaction $R = \frac{1}{2}(P + pl)$, and the shear $S = R - px$. Hence eq. (1) gives the general value of the deflection

$$w' = \frac{d^2}{2IG}\int_0^x (R - px)dx = \frac{d^2}{2IG}\left\{\frac{x}{2}(P + pl) - \frac{px^2}{2}\right\}. \quad (3)$$

And for the centre of the span:

$$w' = \frac{d^2}{2IG}\int_0^{l/2}(R-px)dx = \frac{d^2}{8IG}\left(Pl+\frac{pl^2}{2}\right). \quad . \quad (4)$$

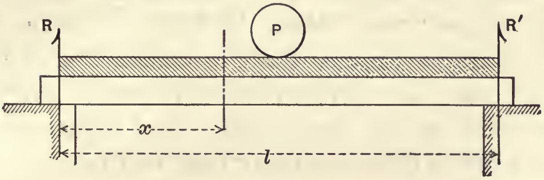
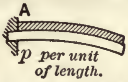
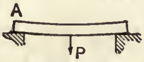
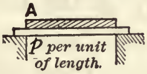
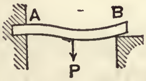
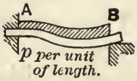
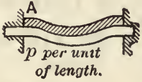


FIG. 2.

The values of the deflection w' may be similarly written for other cases. The following table gives the results for the cases indicated, which are those commonly required.

	Beam.	End Shear.	Shear S.	Section for Deflection w'.	Deflection w'.
I		P	P	l from end	$\frac{d^2Pl}{2IG}$
II		0	px	l " "	$\frac{d^2pl^2}{4IG}$
III		$\frac{1}{2}P$	$\frac{1}{2}P$	$\frac{1}{2}l$ " "	$\frac{d^2Pl}{8IG}$
IV		$\frac{1}{2}pl$	$p(\frac{1}{2}l-x)$	$\frac{1}{2}l$ " "	$\frac{d^2pl^2}{16IG}$
V		$\frac{5}{16}P$	$\frac{5}{16}P$	$.447l$ " "	$\frac{d^2Pl}{14.32IG}$
		$\frac{5}{16}P$	$\frac{5}{16}P$	$.5l$ " "	$\frac{d^2Pl}{12.8IG}$
VI		$\frac{3}{8}pl$	$p(\frac{3}{8}l-x)$	$.4215l$ " "	$\frac{.0347d^2pl^2}{IG}$
VII		$\frac{1}{2}P$	$\frac{1}{2}P$	$.5l$ " "	$\frac{d^2Pl}{8IG}$
VIII		$\frac{1}{2}pl$	$p(\frac{1}{2}l-x)$	$.5l$ " "	$\frac{d^2pl^2}{16IG}$

The end shears in this table are the reactions taken from the table of the preceding article, the "Beams" in the two tables being the same.

The total deflection for any particular beam is to be found by adding the "Max. Deflection" from the table of the preceding article to the w' found in the above table.

In the notation of the preceding article, if w_1 is the deflection due to the lengthening and shortening of the fibres the total deflection in any case will be

$$w = w_1 + w'. \qquad \ldots \ldots \quad (5)$$

These formulæ for shearing deflection, like all the formulæ relating to the distribution of transverse shearing in a bent beam, are more accurately applicable to rectangular or circular sections than to others.

Art. 32.—The Common Theory of Flexure for a Beam Composed of Two Materials.

The common theory of flexure as set forth in the preceding articles is applicable to a beam composed of two or more materials with minor changes only in the formulæ established, but two different materials only will be considered here, as that number are frequently used in engineering works.

Two such materials, concrete and steel, are widely used in reinforced concrete beams. Let E be the modulus of elasticity for steel and E_1 for concrete, and let e represent the ratio between the two moduli, i.e., $e = \frac{E}{E_1}$. This ratio for concrete and steel is generally taken as 15, although 12 is sometimes used. Let A be the area of that part of the section with the modulus E and A_1, the area of section having the modulus E_1. If $u = \frac{1}{\rho}$ (the reciprocal of the radius of curvature) be the strain of a unit length of fibre at unit distance from the neutral axis, then will the intensities of the direct stresses of tension or compression at the distance z from the neutral surface be:

$$N_1 = E_1uz \text{ and } N = Euz = eE_1uz.$$

Inasmuch as the two materials are supposed to act together as a unit, the rate of strain will be the same for both at a given distance from the neutral axis.

The amounts of direct stress on the two differential areas dA_1 and dA will be as follows:

$$E_1uzdA_1 + EuzdA = a_1zdA_1 + ea_1zdA. \quad . \quad . \quad (1)$$

a_1 and no_1 are intensities of stsess at unit distance from the nutral axis.

The sum of the direct stresses of tension and compression in any normal section of the beam, if the beam is horizontal and all loading vertical, will be zero. Hence:

$$\int zdA_1 + \int ezdA = 0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The limits of the integrations indicated will depend upon the form of cross-section and the distribution of the two materials. Frequently the section of one material, such as the steel in reinforced concrete work, is but a small percentage of the total cross-section, and it is sufficiently accurate to consider it concentrated at the distance d_2 from the neutral axis on one side of the latter and at the distance d_3 from the same axis on the opposite side. If d_2 is considered positive, d_3 must be taken as negative.

Finally, if A_2 and A_3 be taken as the small areas of section of the material, eq. (2) will take the form of eq. (3):

$$\int z dA_1 + (A_2 d_2 - A_3 d_3) = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Invariably the small sections A_2 and A_3 belong to a material with a far higher modulus than the other. In reinforced concrete the sum of A_2 and A_3 is usually about 1 per cent or less of the entire cross-sectional area of the beam with $E = 30,000,000$ and $E_1 = 2,000,000$.

When the form of cross-section of the beam, i.e., the cross-section of both materials of which the beam is composed, is known, the position of the neutral axis of the section can at once be found by either eq. (2) or eq. (3). It is obvious from these equations that the neutral axis will not pass through the centre of gravity of the section. Whether it will be at one side or the other of that point will depend upon the amount and distribution of the materials and the greater modulus of elasticity.

Frequently the steel is omitted on the compression side of reinforced concrete beams and in such case either A_2 or A_3 will be zero.

The bending or resisting moment of the internal stresses in any normal section of a beam can be written at once by the aid of the second member of eq. (1). If that second member be multiplied by z, the differential resisting moment will at once result. Hence:

$$M = a_1 \int z^2 dA_1 + e a_1 \int z^2 dA. \quad . \quad . \quad . \quad . \quad (4)$$

As indicated in eq. (1), a_1 is the intensity of the direct stress of either tension or compression in a fibre at unit distance from the neutral axis for the material with the modulus E_1. The integrals in eq. (4) will be recognized at once as the moments of inertia of the cross-section of the two different materials about the neutral axis established by eq. (2) or eq. (3). If the same assumptions made in connection with eq. (3) are known in connection with eq. 4 this latter equation will take the following form:

$$M = a_1 \int z^2 dA_1 + ea_1 (A_2 d^2{}_2 + A_3 d^2{}_3). \quad . \quad . \quad (5)$$

Again since $a_1 = \frac{k}{d_1} = \frac{k_2}{d_2} = \frac{k_3}{d_3}$ eq. (5) may take the following form:

$$M = \frac{k}{d_1} I + n \frac{k_2}{d_2} I_2 + e \frac{k_3}{d_3} I_3. \quad . \quad . \quad . \quad . \quad (6)$$

It is to be observed that k, k_2 and k_3 are intensities of stress at the distances from the neutral axis indicated by d_1, d_2, and d_3 in the material whose modulus of elasticity is E_1.

These equations indicate completely the only modifications to be made in the common theory of flexure as applied to one material for a beam composed of two different materials, and they indicate also the corresponding changes necessary to adapt the common theory of flexure to a beam composed of more than two different materials.

In eqs. (4), (5), and (6) the moment M is simply the ordinary expression for the external bending moment to which a beam is subjected in terms of the horizontal co-ordinate x and given loads.

The formulæ to be used to compute the deflection of a beam composed of two materials are readily written by means of the preceding equations. As

$$a_1 = \frac{k}{d_1} = \frac{k_2}{d_2} = \text{etc.} = E_1 u = \frac{E_1}{\rho} = E_1 \frac{d^2 w}{dx^2},$$

eq. (6) gives:

$$E_1(I+eI_2+eI_3)\frac{d^2w}{dx^2}=M. \quad . \quad . \quad . \quad . \quad (7)$$

As already explained, M, the external bending moment, is expressed in terms of the loads and the coordinate x. Eq. (7) therefore can be integrated precisely as in the case of a beam of a single material. Indeed there is no difference between the two cases except that istead of the moment of inertia I for a single material, the term $I+eI_2+eI_3$ must take its place, the latter expression being the sum of the three components of the resultant moment of inertia of the combined normal section.

The first integration of eq. (7) will obviously give the tangent of the inclination of the neutral surface at any point, while the second will give the deflection.

Art. 33.—Graphical Determination of the Resistance of a Beam.

The graphical method is well adapted to the treatment of beams whose normal sections are limited either wholly or in part by irregular curves. In Fig. 1 is represented the normal section of such a beam, the centre of gravity of the section being situated at C. The lines HL, AB, and DF are parallel. As is known by the common theory of flexure, the neutral axis will pass through C.

Let aa be any line on either side of AB, then draw the lines aa' normal to AB, having made MN and HL equidistant from AB. From the points a' thus determined draw straight lines to C. These last lines will include intercepts, bb, on the original lines aa. Let every linear element parallel to AB, on each side of C, be similarly treated. All the intercepts found in this manner will compose the shaded figure.

This operation in reality, and only, determines an amount of stress with a uniform intensity identical with that developed in the layer of fibres farthest from the neutral axis, and equal to the total bending stress existing in the section; this latter stress, of course, having a variable intensity. *HL* represents the layer of fibres farthest from the neutral surface, consequently *MN* was taken at the same distance from *AB*. Any other distance might have been taken, but the intensity of the uniform stress would then have had a value equal to that which exists at that distance from the neutral axis. Again, a different intensity might have been chosen for the stress on each side of *AB*. It is most convenient, however, to use the greatest intensity in the section for the stress on both sides of the neutral axis; this intensity, which is the modulus of rupture by bending, will be represented, as heretofore, by *K*.

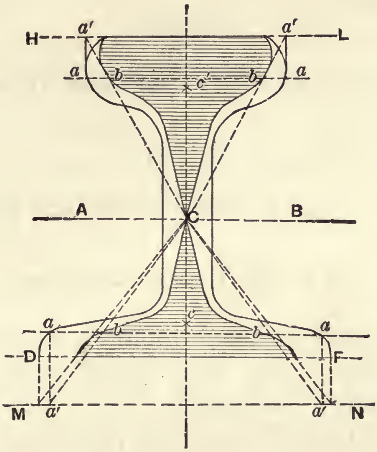


FIG. 1

Let *c* and *c′* be the centres of gravity of the two shaded figures. These centres can readily and accurately be found by cutting the figures out of stiff manilla paper and then balancing on a knife-edge. Let *s* represent the area of the shaded surface below AB, and s' the area of that above AB.

Because this is a case of pure bending, the stresses of tension must be equal to those of compression. Hence

$$Ks = Ks', \quad \text{or} \quad s = s'. \quad . \quad . \quad . \quad . \quad (1)$$

The moment of the compression stresses about AB will be

$$Ks \times c'C.$$

The moment of the tensile stresses about the same line will be

$$Ks \times cC.$$

Consequently the resisting moment of the whole section will be

$$M = Ks(c'C + cC) = Ks \times cc'. \quad . \quad . \quad . \quad . \quad (2)$$

Thus the total resisting moment is completely determined. In some cases of irregular section the method becomes absolutely necessary.

It is to be observed that the centre of gravity, c or c', is at the same normal distance from AB as the centre of the actual stress on the same side of AB with c or c'.

Art. 34.—Greatest Stresses at any Point in a Beam.

Any beam under transverse loading is subjected to internal stresses determined by the Common Theory of Flexure, the intensities of fibre stresses varying directly as the distance from the neutral axis while the transverse and longitudinal shears are distributed as indicated in Art. 60. 14, The maximum intensities of the direct stresses and shears at any point, however, must be determined by the aid of the procedures given in Arts. 8 and 9.

The intensity of the direct tensile and compressive stresses in any normal section may readily be determined when the conditions of loading are known. The only stresses acting on any two transverse planes at right angles to each other, one horizontal and the other vertical, are the direct fibre stress p_v and the longitudinal and transverse shear p_{xy}. It is shown in Art. 8 that the two intensities of principal stresses are given by the following equation for all points:

$$p = \tfrac{1}{2}p_v \pm \sqrt{p_{xy}^2 + \tfrac{1}{4}p_v^2}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Again, if α is the angle which the axis of X (vertical) makes with the direction of one of the principal stresses it is shown in the same article that

$$\tan 2\alpha = -\frac{2p_{xy}}{p_v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By the use of these equations it is shown in Art. 10 that at the neutral surface of the bent beam where the intensity of the transverse and longitudinal shear has its maximum value, i.e., $\frac{3}{2}$ the mean intensity on the entire section, there will be two principal stresses of equal intensity, and of the same intensity as the shear, but of opposite kinds, one being tension and one compression, each making an angle of 45° with the neutral surface. This determines completely the state of stress at the neutral surface. In the same article it is shown that there is but one principal stress at the exterior surface and that is the ordinary fibre stress of flexure whose intensity is determined by the bending moment at the normal section considered. This intensity may be called *k*. The greatest intensity of shearing stress at the surface of the beam where the intensity *k* exists is given by eq. (5) of Art. 9. One of the principal stresses, i.e., that one normal to the exterior surface of the beam will be zero. Hence the maximum shear will be found on two planes at right angles to each other and each at 45° to the surface of the beam, the intensity of the shear being one-half of the principal stress k. These considerations determine completely the greatest stresses at the neutral surface and at the exterior surface, upper or lower, of the beam. There remain to be found the intensity of principal stress at all other points by means of eqs. (1) and (2).

To illustrate the necessary procedures, let a steel beam of rectangular normal section be taken with an effective span of 20 feet, and with a depth of 16 inches. For the purpose of these computations the beam may be considered to have a lateral thickness or width of 1 inch, making the area of cross-section 16 square inches. The load per linear foot may be taken at 1140 pounds, producing an extreme fibre stress of $k=16,000$ pounds per square inch. If x be measured from one end of the span and if $\pm z$ be measured upward and downward, respectively, from the neutral surface, the greatest value in either direction being 8 inches, and if I be the moment of inertia of the normal section of the beam about its neutral axis, there may be written the following values for the bending moment and intensity of fibre stress at any distance z from the neutral axis, g being the load per unit of span:

$$k=\frac{Mz}{I},$$

$$M=\frac{g}{2}(l-x)x \qquad \therefore\ k=p_v=\frac{g}{2I}zx(l-x). \quad . \quad . \quad (3)$$

The transverse shear at any section x from the end of the span is

$$S = g\left(\frac{l}{2} - x\right).$$

It is found by eq. (6) of Art. 15 that the intensity of transverse and longitudinal shear at any point in a section of the beam is

$$s = p_{xy} = \frac{g\left(\frac{l}{2} - x\right)}{2I}\left(\frac{h^2}{4} - z^2\right). \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The value of tan 2α giving the direction in which the principal stresses act now becomes

$$\tan 2\alpha = -\frac{2p_{xy}}{p_v} = -\frac{(l - 2x)\left(\frac{h^2}{4} - z^2\right)}{zx(l - x)}. \quad . \quad . \quad . \qquad (5)$$

Fig. 1 shows a part of one-half of the beam under consideration, the effective span being 20 feet $= 240$ inches. One end support is at *B* while *CD* is at the centre of the span. *NN* is a trace of the neutral surface.

Normal sections of the beam were taken 2 feet apart at *F*, *G*, *H* and *C* and the directions and intensity of the principal stresses p were computed by means of eqs. (1) and (2) at four points 2 inches apart vertically, including the neutral surface and exterior surfaces at each of those sections. The curved lines drawn in Fig. 1 are each laid down in the direction of the principal stresses acting at each point, the curves having the plus sign representing the directions of principal tensile stresses, while those indicated by the minus sign show the directions of the principal compressive stresses at each point. Along the neutral surface *NN* all lines are inclined at an angle of 45° to that surface, while at each exterior surface one set of lines is parallel to that surface and the other at right angles to it. Wherever the curved lines cross they are at right angles to each other. The plus stresses at the upper surface of the beam and the minus stresses at the lower surface have zero intensities at those surfaces. At the centre of span

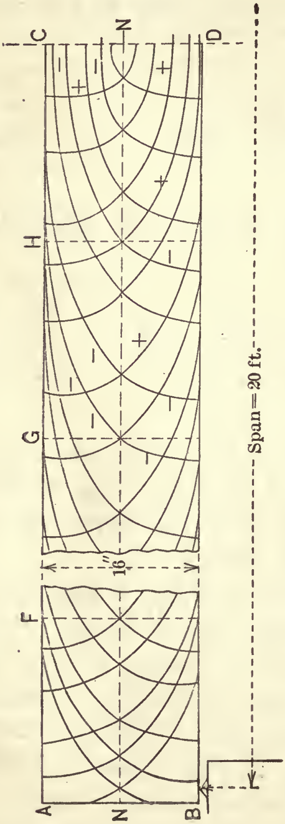


FIG. 1.

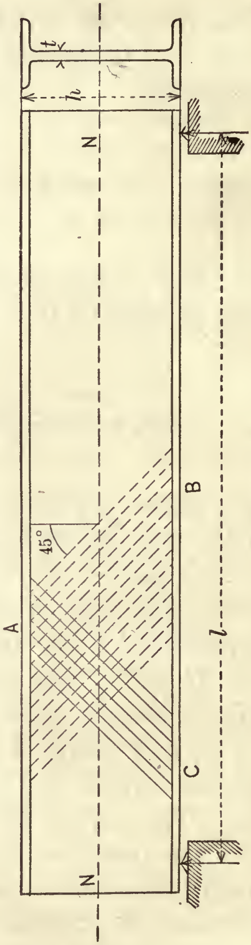


FIG. 2.

CD all lines are horizontal, as the shear at that point is zero. They are horizontal whatever may be the character of loading at the point where the bending moment is greatest, i.e., where the shear is zero. These curved lines representing the direction of the principal stresses at all points are sometimes called stress trajectories.

Some important practical matters are based upon the existence of the principal stresses of tension and compression at the neutral surface of a bent beam, those principal stresses making angles of 45° with that surface. In Fig. 2 is shown a rolled I-beam, although this discussion is equally applicable to the web of a plate girder.

Inasmuch as the inclined principal stresses of tension and compression act at the neutral surface, let that distribution of principal stresses be supposed to exist throughout the entire web of the rolled beam. This condition may be represented by the sets of lines drawn in Fig. 2, each at an angle of 45° with the vertical line (or with a horizontal line). Let it be supposed that the entire web of the beam is composed of the strips shown, those indicated by the broken lines *AB* being subjected to tension in the left half of the beam and those represented by full lines, to compression. Inasmuch as the strips *AC* will be subjected to compression they may approximately be considered columns with the length h sec $45° = h\sqrt{2}$. The thickness of the web of flanged beams such as plate girders is sometimes determined by an empirical formula based upon this long-column condition of stress. Any part *AC* of the web is in fact not in a true long-column condition because the parts parallel to *AB* are in tension and tend to hold the parts *AC* in position.

Again, it is sometimes supposed that the web of a flanged beam may be considered approximately to be composed of a system of tension and compression web members like a truss represented by such sets of strips of metal as AB and AC.

The condition of compression in which the web exists in the direction AC tends to buckle a thin web into corrugations with their axes parallel to AB, and such girders exhibit that result when tested to destruction if the web is insufficiently stiffened. For this reason it has sometimes been proposed to place the stiffeners on the webs of plate girders in the direction AC, Fig. 2, so as to prevent any buckling of the kind described. Such a method, however, is not satisfactory for a number of reasons.

If the total transverse shear in any normal sections of the beam such as a vertical section through A or C be called S then the average intensity of shear assumed uniformly distributed over the section of the web would be $s=\frac{S}{th}$. Since such a vertical section would cut the same number of inclined strips in tension and compression, the shear $\frac{S}{2}\sqrt{2}$ (sec. 45°) would be carried by each of the sets of inclined strips whose normal section would be

$$th \cos 45^\circ = \frac{th}{\sqrt{2}}.$$

Hence, the intensity of stress in each of the two sets of strips would be

$$\frac{S}{2}\sqrt{2} \div \frac{th}{\sqrt{2}} = \frac{S}{th}.$$

This is the same intensity as the mean transverse shear on the section of the web. According to this mode of treatment, therefore, it is seen that the intensities of stress throughout the assumed 45° strips is the same as the intensity of the average transverse shear. This again is simply the condition which exists at the neutral surface of the solid beam as already found, except in that case the intensity of transverse shear at the neutral surface is one and one-half times the average intensity.

Art. 35.—The Flexure of Long Columns.

A "long column" is a piece of material whose length is a number of times its breadth or width, and which is subjected to a compressive force exerted in the direction of its length. Such a piece of material will not be strained or compressed directly back into itself, but will yield laterally as a whole, thus causing flexure. If the length of a long column is many times the width or breadth, the failure in consequence of flexure will take place while the pure compression is very small and neglected.

As with beams, so with columns, the ends may be "fixed," so that the end surfaces do not change their position however great the compression or flexure. Such a column is frequently, perhaps usually, said to have fixed ends. If the ends of the column are free to turn in any direction, being simply supported, as flexure takes place, the column is said to have "round" ends. It is clear that if the column has freedom in one or several directions only, it will be a "round" end column in that one direction, or those several directions, only. It is also evident that a column may have one end round and one end flat or fixed.

In Fig. 1 let there be represented a column with flat ends, vertical and originally straight. After external pressure is

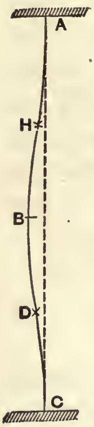


FIG. 1.

imposed at A, the column will take a shape similar to that represented. Consequently the load P, at A, will act with a lever-arm at any section equal to the deflection of that section from its original position. Let y be the general value of that deflection, and at B let $y=y_1$. Let x be measured from A, as an origin, along the original axis of the column. In accordance with principles already established, the condition of fixedness at each of the ends A and C is secured by the application of a *negative* moment, $-M$. It is known from the general condition of the column that the curve of its axis will be convex toward the axis of x at and near A, while it will be concave at and near B (the middle point of the column). Hence, since y is positive toward the left, and since the ordinate and its second derivative must have the same sign when the curve is convex toward the axis of the abscissæ, the general equation of moments must be written as follows:

$$EI\frac{d^2y}{dx^2}=M-Py. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Multiplying by $2dy$,

$$EI\frac{2dyd^2y}{dx^2}=2Mdy-P2ydy;$$

$$\therefore\ EI\left(\frac{dy}{dx}\right)^2=2My-Py^2+(c=0); \quad . \quad . \quad . \quad (2)$$

$c=0$, because the column has flat ends, and

$$\frac{dy}{dx}=0$$

when $y=0$. Also

$$\frac{dy}{dx}=0$$

when $y=y_1$;

$$\therefore\ M=\frac{Py_1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eq. (2) now becomes

$$\sqrt{\frac{EI}{P}}\ \frac{dy}{\sqrt{y_1y-y^2}}=dx;$$

$$\therefore\ x=\sqrt{\frac{IE}{P}}\ \text{versin}^{-1}\frac{2y}{y_1}. \quad . \quad . \quad . \quad . \quad (4)$$

If $y=y_1$,

$$x=\frac{l}{2}=\pi\sqrt{\frac{EI}{P}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In this equation l is the length of the column. From eq. (5) there may be deduced

$$P=\frac{4\pi^2EI}{l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It is to be observed that P is *wholly independent of the deflection*, i.e., it remains the same, whatever may be the amount of deflection, after the column begins to bend. Consequently, if the elasticity of the material were perfect, the weight P would hold the column in any position in which it might be placed after bending begins. This result is for pure flexure, direct compression being neglected.

Eq. (6) forms the basis of some old long column formulæ now out of use. It was first established by Euler.

Some very important results follow from the consideration of Fig. 1 in connection with the preceding equations.

The bending moment at the centre, B, of the column is obtained by placing $y=y_1$ in eq. (1); its value is, consequently,

$$M'=-M+Py_1=M. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence *the bending at the centre of the column is exactly the same (but of opposite sign) as that at either end.* Between A and B, then, there must be a point of contra-flexure.

Putting the second member of eq. (1) equal to zero, and introducing the value of M from eq. (3),

$$y=\frac{y_1}{2}.$$

Introducing this value of y in eq. (4), and bearing in mind eq. (5),

$$x=\frac{\pi}{2}\sqrt{\frac{EI}{P}}=\frac{l}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The points of contra-flexure, then, are at H and D, $\frac{1}{4}l$ and $\frac{3}{4}l$ from A.

Hence *the middle half of the column (HD) is actually a column with round ends*, and it is equal in resistance to a fixed-end column of double its length.

Hence writing l' for $\frac{l}{2}$ and putting $2l'$ for l in eq. (6),

$$P=\frac{\pi^2EI}{l'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (9) gives the value of P for a round-end column.

Again, either the upper three quarters (AD) or the lower three quarters (CH) of the column is very nearly equivalent to a column with one end flat and one end round, and its resistance is equal to that of a fixed-end column whose length is $\frac{4}{3}$ its own. Putting, therefore,

$$l_1 = \tfrac{3}{4}l,$$

and introducing

$$l = \frac{4}{3}l_1$$

in eq. (6),

$$P = 2.25\frac{\pi^2 EI}{l_1^{\,2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The last case is not quite accurate, because the ends of the columns *HC* and *AD* are not exactly in a vertical line.

In reality, the column under compression may be composed of any number of such parts as *HD*, with the portions *HA* and *CD* at the ends, thus taking a serpentine shape, so far as pure equilibrium is concerned. In such a condition the column would be subjected to considerably less bending than in that shown in the figure. In ordinary experience, however, the serpentine shape is impossible, because the slightest jar or tremor would cause the column to take the shape shown in Fig. 1. Hence the latter case only has been considered.

If r is the radius of gyration and S the area of normal section of the column, eqs. (6) and (9) will take the forms

$$\frac{P}{S} = \frac{4\pi^2 Er^2}{l^2} \quad \text{and} \quad \frac{P}{S} = \frac{\pi^2 Er^2}{l'^2}.$$

Eq. (10) will, of course, take a corresponding form.

These equations evidently become inapplicable when $\frac{P}{S}$

approaches C, the ultimate compressive resistance of the material in short blocks. The corresponding values of $\left(\frac{l}{r}\right)$ at the limit are

$$\frac{l}{r}=2\pi\sqrt{\frac{E}{C}} \quad \text{and} \quad \frac{l}{r}=\pi\sqrt{\frac{E}{C}} \quad . \quad . \quad . \quad (11)$$

for fixed and round ends respectively; other conditions of ends will be included between those two.

If for structural steel

$$E=30,000,000 \quad \text{and} \quad C=60,000,$$

the above values become 140 and 70, nearly.

Euler's formula, therefore, is strictly applicable only to structural steel columns, with ends fixed or rounded, for which $l \div r$ greatly exceeds 140 and 70, respectively.

If for cast iron

$$E=14,000,000 \quad \text{and} \quad C=100,000,$$

eqs. (11) give

$$\frac{l}{r}=74 \quad \text{and} \quad \frac{l}{r}=37, \text{ nearly.}$$

Euler's formula evidently becomes inapplicable considerably above the limits indicated, since columns in which $\frac{l}{r}$ has those values will not nearly sustain the intensity C.

The analytical basis of "Gordon's Formula" for the resistance of long columns is so closely associated with the empirical that both will be treated together hereafter.

Art. 36.—Special Cases of Flexure of Long Columns.

There are a few cases of flexure of columns which, while not frequently found in engineering experience, may be of some practical importance. The two or three which follow involve the integration of linear differential equations treated in advanced works on the integral calculus; consequently the operations of integration will not be given here, but the general integrals will be assumed.

Flexure by Oblique Forces.

In Fig. 1 let OA represent a column acted upon by the oblique force P, which makes the angle α with the axis of X. The column is supposed to be fixed in the direction of OX at O, but the coordinates x and y are measured from the point of application A of the load P as shown in the figure. If right-hand moments are positive, and left-hand negative, the component $P \sin \alpha$ will have the negative moment $-P \sin \alpha x$ about any point O'. The lever arm of $P \cos \alpha$, if the deflection y is positive, is $+y$, and its moment $-P \cos \alpha y$ is also negative. Hence the resultant moment of any force, P, in reference to the point O' is

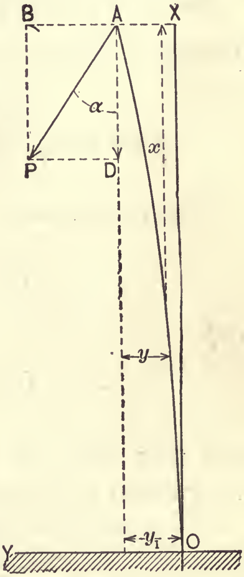


FIG. 1.

$$\left. \begin{aligned} M = EI\frac{d^2y}{dx^2} = -P \sin \alpha \\ x - P \cos \alpha y \end{aligned} \right\} \quad . \quad . \quad . \quad (1)$$

For any number of forces or loads P there will obviously be a corresponding number of pairs of terms in the third member of eq. (1). It will therefore be sufficient to treat one force P only.

Eq. (1) may be put in the form,

$$\frac{d^2y}{dx^2}+n^2y=-\frac{P\sin\alpha}{EI}x=mx \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In this equation $n^2=\frac{P\cos\alpha}{EI}$ and $m=\frac{P\sin\alpha}{EI}$. Eq. (2) may readily be integrated so as to give the following equation, C_1 and C_2 being constants of integration:

$$y=\frac{m}{n^3}\{nx+C_1\sin nx-C_2\cos nx\} \quad . \quad . \quad . \quad (3)$$

Using the values of m and n given above y may take the following form, observing that $\frac{m}{n^2}=-\tan\alpha$:

$$y=C\sin\sqrt{\frac{P\cos\alpha}{EI}}x-C'\cos\sqrt{\frac{P\cos\alpha}{EI}}x-x\tan\alpha \quad . \quad (4)$$

The coefficients C and C' have the values

$$C=-C_1\tan\alpha\sqrt{\frac{EI}{P\cos\alpha}}$$

and

$$C'=-C_2\tan\alpha\sqrt{\frac{EI}{P\cos\alpha}},$$

and they may be treated as arbitrary constants to be determined by the conditions of each problem

As x and the deflection y are measured from the point of application of the load P, if $x=0$ then must $y=0$. Hence by eq. (4), $C'=0$. Consequently

$$y=C\sin\sqrt{\frac{P\cos\alpha}{EI}}x-x\tan\alpha \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If α is greater than 90°, $\cos \alpha$ will be negative and the exponential value of the sine may be used as follows: Placing $b = \sqrt{\frac{P \cos \alpha}{EI}}$, and e being the base of the Naperian logarithms:

$$y = \frac{C}{2\sqrt{-1}}\left(e^{bx\sqrt{-1}} - e^{-bx\sqrt{-1}}\right) - x \tan \alpha. \quad . \quad . \quad (6)$$

When $\cos \alpha$ is negative $b\sqrt{-1}$ is the square root of a positive quantity, and $\frac{C}{\sqrt{-1}}$ will be rational.

Column Free at Upper End and Fixed Vertically at Lower End with either Inclined or Vertical Loading at Upper End.

In this case the axis of x, Fig. 1, is to be considered vertical with the column fixed at its base O. In accordance with the latter condition $\frac{dy}{dx} = 0$ at O, i.e., when $x = l =$ length of column.

From eq. (5),

$$\frac{dy}{dx} = C\sqrt{\frac{P \cos \alpha}{EI}} \cos \sqrt{\frac{P \cos \alpha}{EI}}x - \tan \alpha. \quad . \quad . \quad (7)$$

It is to be observed that P is not yet determined and that $\cos\sqrt{\frac{P \cos \alpha}{EI}}\ x$ may vary largely (and periodically) while $\frac{dy}{dx}$ remains unchanged.

If the column carries a vertical load at its upper end $\alpha = 0 = \tan \alpha$, and when $x = l$, $\frac{dy}{dx} = 0$. Eq. (7) then gives:

$$\text{Cos}\sqrt{\frac{P}{EI}}\,l = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If f is any whole odd number from 1 to infinity, then there may be placed by the aid of eq. (8):

$$\sqrt{\frac{P}{EI}}=\frac{f\pi}{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If this value be substituted in eq. (7) after making $\alpha=\tan\alpha=0$:

$$\frac{dy}{dx}=C\sqrt{\frac{P}{EI}}\cos\frac{f\pi}{2l}x. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eq. (10) shows that when $x=\frac{l}{f}$ (f being any whole odd number) $\frac{dy}{dx}=0$, for $\cos\frac{\pi}{2}=\cos 90^\circ=0$.

Obviously P must have the smallest value which will satisfy eq. (9); but f cannot be smaller than 1. Therefore

$$P=EI\frac{\pi^2}{4l^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The carrying capacity of the column is thus seen to be independent of the deflection as was the case in Art. 35, but it must be observed that the effect of direct compression is neglected, i.e., it is a case of pure bending of excessively long columns. The end of the column considered here which carries the vertical load is free to deflect laterally, whereas in Art. 35 both ends are supposed to be held against lateral movement. In the latter case the resistance is seen to be nine times as great as in the present.

Eq. (11) can be found in a direct and simple manner by making $M=0$ in eq. (1) of Art. 34 and integrating the resulting equation.

Since by eq. (8), $\cos\sqrt{\frac{P}{EI}}l=0$, $\sin\sqrt{\frac{P}{EI}}l=1$.

If therefore $\alpha = 0$ and $x = l$ in eq. (5), and if y is the deflection of the free end of the column in reference to the base, Fig. 1, that equation will give:

$$C = y_1. \qquad (12)$$

Then

$$y = y_1 \sin \sqrt{\frac{P}{EI}}x. \qquad (13)$$

For a given value of x, therefore, y varies directly as y_1 and the relative deflections at the base and any point may be computed by the equation:

$$\frac{y}{y_1} = \sin\left(\frac{f\pi}{2l}x\right). \qquad (14)$$

Or in the ordinary case:

$$\frac{y}{y_1} = \sin \frac{\pi x}{2l}. \qquad (15)$$

It should be remembered that deflection is initiated by the load P determined by eq. (11) and that the deflection may take any subsequent value without increase of load.

PROBLEMS FOR CHAPTER II.

Problem 1.—A beam simply supported at each end carries a load of 850 pounds per linear foot over a span of 26 feet. Find the bending moment and transverse shears at the end and centre of span and at 2 points 3 feet and 11 feet 6 inches respectively from the end.

Ans. Moment at end is 0; at 3 feet, 29,325 ft.-lbs.; at 11.5 feet, 70,868.75 ft.-lbs.; at centre, 71,825 ft.-lbs. Shear at end is 11,050 lbs.; at 3 feet, 8500 lbs.; at 11.5 feet, 1275 lbs.

Problem 2.—A beam or girder having a span length of 41 feet carries a uniform load of 1200 pounds per linear foot and a single weight of 1800 pounds at the centre. Find the bending moments and the shears due to the uniform load and the single load separately at the ends and at the centre and at points 6 and 14 feet from the end.

Problem 3.—In Problem 2 find the single weight which placed at the centre of the span will produce the same centre bending moment as the uniform load.

Ans. 24,600 pounds.

Again, find two weights placed 6 feet apart, i.e., one 3 feet either way from the centre, which will produce the same centre bending moment as the uniform load.

Ans. Each of the two weights is 14,406 pounds.

Problem 4.—A beam or girder with a span length of 31 feet carries a uniform load of 300 pounds per linear foot in addition to five loads, the first weighing 7000 pounds at a distance of 3 feet from the end; the second weighing 10,000 pounds 7 feet from the end; the third weighing 11,000 pounds 14 feet from the end; the fourth weighing 17,000 pounds 21 feet from the end, and the fifth weighing 6400 pounds 27 feet from the end.

Construct the shear and moment diagrams for this case, Fig. 2 of Art. 15 and Fig. 2 of Art. 12.

Problem 5.—Find a uniform load for the same beam considered in Problem 4 which will have a centre bending moment equal to the greatest bending moment of that problem; also another uniform load whose end shear shall equal the greatest of the two end shears of Problem 4. Such uniform loads are called "equivalent uniform loads."

Problem 6.—In Problem 2 the moment of inertia I

is 3570 (the unit being the inch), while $E = 30,000,000$, the beam being of steel. Find the tangent of inclination of the neutral surface at the end and at 10 feet from the end. Also find the deflection at the centre of span and at 10 feet from the end. Use eqs. (19), (20), and (21) of Art. 22.

Partial Ans. Tangent 10 feet from end is .00344. The deflection at the same point is .53 inch.

Problem 7.—In Problem 6 let it be required to ascertain how much additional deflection is produced by the transverse shear at the centre of the span and at 10 feet from the end. Let the coefficient of elasticity for shear (G) be taken at 12,000,000 pounds, while $I = 3570$ and $d = 14$ inches.

Ans. Deflection at 10 feet is .0054 inch, and at the centre of span .0075 inch.

CHAPTER III.

TORSION.

Art. 37.—Torsion in Equilibrium.

THE state of stress called torsion is produced when a straight bar of material, like a piece of round shafting, is twisted. Such a bar is represented in Fig. 1, the axis of the piece being AB, and its normal cross-section having any shape whatever. In engineering practice the outline of that normal section is usually circular, although it is occasionally square.

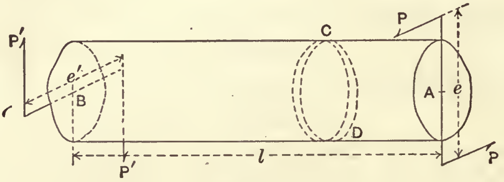
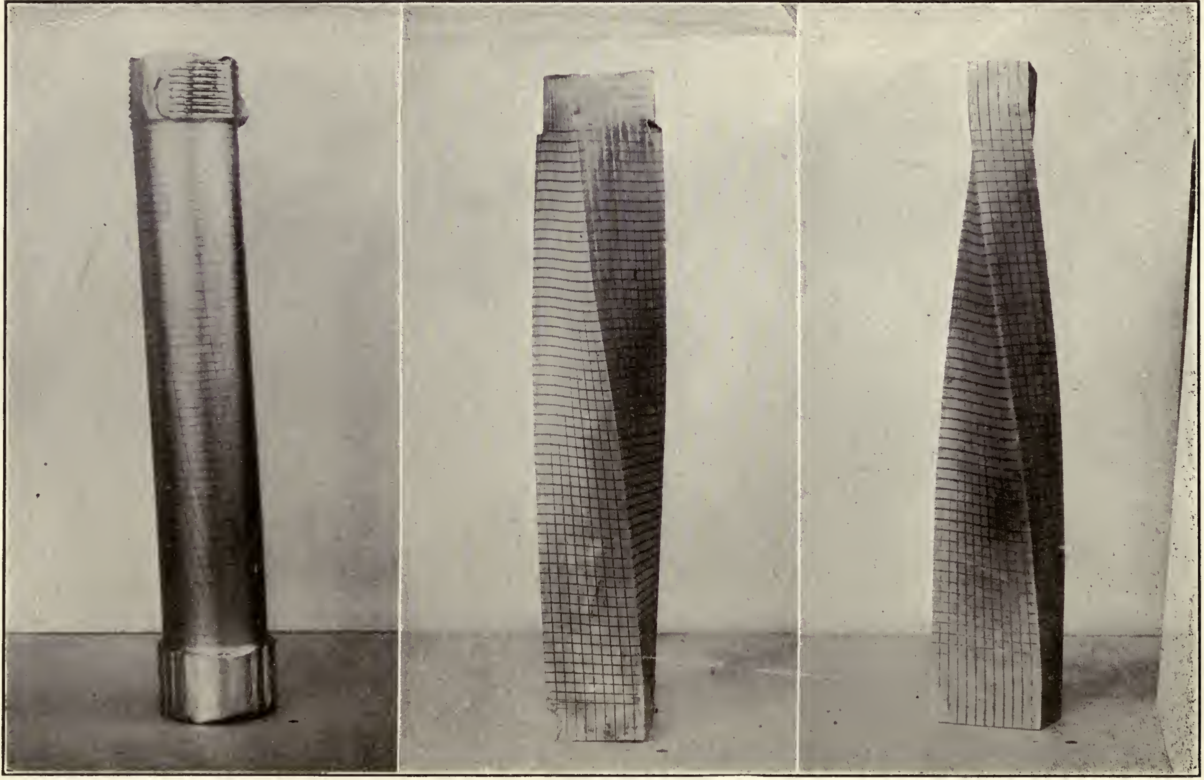


FIG. 1.

The twisting of the bar is done by the action of two equal and opposite couples acting in two planes, each normal to the axis, but at any desired distance apart. The two couples are represented in Fig. 1 at each end of the piece in the two normal sections A and B. The forces and lever-arm of one couple are respectively P and e, and P' and e' of the other. The moment of the first couple



View of three cylinders subjected to torsion beyond the elastic limit. The lines shown were drawn at right angles to each other on the original test pieces before being subjected to torsion. Those lines show the characteristic distortion of sections normal to the axes of the pieces.

(*To face page* 183.)

will be Pe and that of the second couple $P'e'$, and if pure torsion is to be produced these two moments must be equal, but opposite to each other. Inasmuch as the moment of a couple is the product of the force by the lever-arm, the forces and lever-arms of the two twisting couples may vary to any extent as long as the moments remain unchanged.

Although the system of forces to which a bar in torsion is subjected is such as to be in equilibrium, any portion of the piece will tend to have its normal sections like those at CD rotated over each other, the result being a small sliding motion around the axis of the piece. Hence a torsive stress is wholly a shearing stress on normal sections of the piece subjected to torsion. It is further important to observe that inasmuch as a couple produces the same effect wherever it may act in its own plane, the actual twisting moment need not be applied with its forces symmetrically disposed in reference to the axis of the piece; indeed, both of those forces may be anywhere on one side of the piece without varying the conditions of torsion or torsive stress to any extent whatever.

It is known from the general theory of stress in a solid body that although there can be no stresses of tension and compression parallel to the axis of a bar under torsion, or at right angles to it, there will be such stresses of varying intensities on oblique planes. Inasmuch as the result of torsion is to slide normal sections each past its neighbor, the elastic torsive shear like any other shear will not change the volume of the body. The principal shearing strains will produce deformation without changing the dimensions whose product gives the volume.

The exact and complete mathematical theory of torsion deduced from the general equations of equilibrium of stresses in an elastic solid, without extraneous assumptions, will be found in App. I. Those formulæ show accurately the state of torsive stress in bars of any elastic material and of various shapes of cross-section. For the general purposes of engineering practice that general demonstration is rather complicated. Hence it is often avoided by making certain approximate assumptions based to some extent on experimental observations which lead to an approximate and simpler theory, yielding formulæ accurate only for the circular normal section, but which are not materially in error for the square section. These formulæ are, however, far from accurate for certain other sections. In this article only the formulæ of the simpler theory, called the common theory of torsion, will be given.

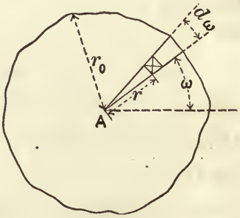


FIG. 2.

Fig. 2 is supposed to represent the normal section of a bar of material of any shape, subjected to torsion by the application of couples as shown in Fig. 1. The fundamental assumptions of the common theory of torsion are that the intensity of shearing stress varies directly as the distance from a central point at which that intensity is zero, and that that central point is located at the centre of gravity or the centroid of the section. It is also implicitly assumed that the normal sections which are plane before torsion remain plane during torsion. In Fig. 2, *A* is supposed to be the centre of gravity of the section at which the intensity of shear, i.e., the shear per square unit of section, is zero. The distance from the centre *A* to any point of the section is represented by r, and to the most remote point in the perimeter of the section by r_0. In accordance with the assumed law, the greatest intensity of shear T_m in the section will be found at the distance r_0 from its centre. While this is accurately true for the circular section, it is quite erroneous for a number of other sections.

Hence the intensity at the distance unity from the centre A will be $\frac{T_m}{r_0}$, and at the distance r from the centre it will have the value

$$s = \frac{r}{r_0} T_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The element of the section at the distance r from A will be

$$r d\omega . dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the shear on that element is

$$dS = \frac{r}{r_0} T_m . r d\omega . dr = \frac{T_m}{r_0} r^2 dr . d\omega. \quad . \quad . \quad . \quad (3)$$

The direction of action of this torsive shear is around the circumference of a circle whose radius is r; hence if moments of all these small shears, dS, be taken about the centre or point of no shear, A, the lever-arm of each small force, dS, will be r, and the differential moment will be

$$dM = r dS = \frac{T_m}{r_0} r^3 dr . d\omega. \quad . \quad . \quad . \quad . \quad (4)$$

The total moment of torsion therefore will be

$$M = \int_0^{2\pi} \int_0^{r} \frac{T_m}{r_0} r^3 dr . d\omega = \frac{T_m}{r_0} \int_0^{2\pi} \int_0^{r_0} r^3 dr \, d\omega = \frac{T_m}{r_0} I_p. \quad (5)$$

The quantity I_p is the polar moment of inertia of the section.

For a circular section

$$I_p = \frac{\pi r_0^4}{2} = \frac{\pi d^4}{32} \; (d = \text{diameter}) = \frac{Ad^2}{8}. \quad . \quad . \quad (6)$$

For a square section (b = side of square)

$$I_p = \frac{b^4}{6} = \frac{Ab^2}{6}. \quad \dots \dots \quad (7)$$

For a rectangular section (b = one side and c = the other side)

$$I_p = \frac{bc^3 + b^3c}{12} = \frac{A(b^2 + c^2)}{12}. \quad \dots \dots \quad (8)$$

For an elliptical section (a_1 and b_1 being semi-axes)

$$I_p = \frac{\pi(a_1^3b_1 + a_1b_1^3)}{4} = \frac{\pi a_1b_1(a_1^2 + b_1^2)}{4}. \quad \dots \quad (9)$$

Using the notation of Fig. 1, the following equation of moments may be written, Pe being the moment of the external twisting couple and M the moment of the internal torsive shearing stresses in any normal section:

$$Pe = M = \frac{T_m}{r_0} I_p. \quad \dots \dots \quad (10)$$

It is clear from Art. 2, if ϕ_0 is the shearing strain at the distance r_0 from the centre, that $T_m = G\phi_0$, G being the coefficient of elasticity for shearing. Also, since the intensity of shearing varies directly as the distance from the centre A, it is equally clear that the shearing strain ϕ varies directly as the distance from the centre, so that if α represents the shearing strain at unit's distance from A

$$\phi = r\alpha \quad \text{and} \quad \phi_0 = r_0\alpha. \quad \dots \dots \quad (11)$$

Hence in general

$$T = Gr\alpha, \quad \dots \dots \quad (12)$$

and as a maximum

$$T_m = Gr_0\alpha. \quad \dots \dots \quad (13)$$

α is evidently *the angle through which one end of a fibre of unit's length and at unit's distance from the centre or axis is turned.* It is called the *angle* of torsion.*

If l is the length of the piece twisted, the total angle through which the end of the fibre at unit's distance from the axis will be turned is

$$\text{Total angle of torsion} = \alpha l. \quad . \quad . \quad . \quad (14)$$

If the fibre is at the distance r_0 from the axis one end will be twisted around beyond the other by an amount equal to

$$\text{Total strain of torsion} = r_0 \alpha l. \quad . \quad . \quad . \quad (15)$$

By the aid of eq. (13) eq. (5) may be written

$$Pe = M = G\alpha I_p = \frac{G\phi_0}{r_0} I_p. \quad . \quad . \quad . \quad . \quad (16)$$

If ϕ_0 is observed experimentally

$$G = \frac{Pe \,.\, r_0}{\phi_0 I_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The angle through which a shaft will be twisted by the moment Pe, if the length is l, is

$$\alpha l = \frac{Pel}{GI_p} = \frac{lT_m}{Gr_0}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If G is in pounds per square inch, as is usual, the preceding formulæ require all dimensions to be in inches, while α will be arc distance at radius of one inch.

If $12l$ is written for l the unit for the latter quantity must be the foot.

By inserting the value of I_p from eq. (6) in eq. (5),

---

* This small angle is measured in radians. Strictly speaking it is an indefinitely short arc with unit radians,

$$Pe = M = \frac{2T_m}{d} \cdot \frac{\pi d^4}{32} = \frac{\pi T_m d^3}{16};$$

$$\therefore \ d = 1.72 \sqrt[3]{\frac{Pe}{T_m}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Eq. (19) will give the diameter of a shaft capable of resisting the twisting moment represented by Pe with the maximum torsive shear in the extreme fibres of T_m.

The main cross dimensions of other sections may be found similarly by the use of eqs. (7), (8), and (9).

It is frequently convenient to compute the greatest intensity T_m from the twisting moment M. For this purpose the equation preceding eq. (19) gives

$$T_m = 5.1 \frac{M}{d^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

These equations complete all that are required for the practical use of the common theory of torsion. In some cases it may be necessary to use accurate formulæ for other shapes of section than the circular. In those cases the exact formulæ of App. I should be employed. The practical applications of the preceding formulæ to such

Twisting Moment in Terms of Horse-power H.

It is sometimes convenient to express the twisting moment M in terms of horse-power transmitted by the snafting. If H is the number of horse-powers transmitted by a shaft making n revolutions per minute, the inch-pounds of work will be $12 \times 33{,}000 \times H$, since each horse-power represents 33,000 foot-pounds of work performed per minute. Again if e is the lever arm of the twisting couple, the path of the force P per minute will be $2\pi en$ and the work performed by the couple must therefore be $P \times 2\pi en = M 2\pi n$. Equating these two expressions for the work or energy transmitted;

$$\frac{12 \times 33{,}000 \times H}{2\pi n} = 63{,}025\frac{H}{n} = M. \quad . \quad . \quad . \quad (21)$$

If this value of M be placed in eqs. (19) and (20), the values of d, the diameter of the shaft and T_m, the greatest intensity of shear will take the following forms in terms of the horse-power and the number of revolutions per minute:

$$d = 68.5\sqrt[3]{\frac{H}{T_m n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$T_m = 321{,}443\frac{H}{nd^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Hollow Circular Cylinders.

If the exterior diameter of a hollow cylinder is d and the interior diameter $d_1 = jd$, j being simply the ratio between the two diameters, the equation preceding eq. (19) may be written:

$$M = \frac{\pi T_m}{16}(d^3 - d_1^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Hence

$$M = Pe = \frac{\pi T_m(1 - j^3)}{16}d^3 \quad . \quad . \quad . \quad . \quad (25)$$

Eq. (25) shows that any of the preceding equations may be made applicable to a hollow cylinder by writing $T_m(1 - j^3)$ in the place of T_m.

Eqs. (19) and (22) therefore take the following forms for a hollow cylinder:

$$d = 1.72\sqrt[3]{\frac{Pe}{T_m(1 - j^3)}} = 68.5\sqrt[3]{\frac{H}{T_m(1 - j^3)n}}. \quad . \quad . \quad (26)$$

The resistance of the hollow cylinder is obviously the difference between the resistances of two solid cylinders, one having the exterior diameter and the other the interior diameter of the hollow cylinder.

Art. 38.—Practical Applications of Formulæ for Torsion.

There has been comparatively little experimental investigation in the resistance of structural materials to torsion and practically none of that has been done in connection with pieces of considerable size. Such results as have been obtained appear to justify the following data.

Steel.

Some of the older tests, as those of Kirkaldy, indicate that the ultimate intensity of torsional shear, T_m, may be taken as high as 75,000 pounds to 90,000 pounds per square inch for special grades of steel like those used for tires, rails, and crucible steel, but lower values must be employed for mild structural steel and for the ordinary grades of shafting.

Torsion tests on circular pieces of spring and cold-drawn steel about $\frac{5}{8}$ inch and $1\frac{1}{4}$ inches in diameter made in the testing laboratory of the Dept. of Civil Engineering at Columbia University by Mr. J. S. Macgregor gave the following results, which are shown rather fully in order to exhibit clearly their main features. There were either four or six tests in each group from which the "max.," "mean" and "min." were taken. All these test specimens except those of mild steel were heat treated. Part of these were heated to 1350° F. and then plunged in oil at 70° until cold. They were then temper drawn in hot oil at 575° F. and part were again heated to 1350° F. and immersed in oil at 575° F. They were then allowed to cool in air at normal temperature.

		Diam. Inches.	*Pe.* In.-Lbs.	T_m. Elastic.	T_m. Ult.	Modulus. *G.*
Spring steel......	max.	.617	5,640	41,600	122,310	13,010,000
	mean	.....	5,427	40,710	118,110	12,455,000
	min.	.614	5,290	31,050	114,620	11,292,000
Spring steel......	max.	1.252	45,260	46,100	117,320	13,954,500
	mean	.....	44,040	43,130	115,070	?12,659,000
	min.	1.246	42,720	41,500	111,900	11,830,000
Cold-drawn steel.	max.	1.252	43,500	46,200	113,200	12,445,000
	mean	.....	37,990	39,300	99,000	11,602,000
	min.	1.25	34,270	33,000	89,500	10,534,000
Mild steel.......	max.	1.257	24,500	22,000	62,800	12,600,000
	mean	.....	23,200	20,800	60,950	12,110,000
	min.	1.233	21,520	19,700	57,300	11,700,000

It has been shown in Art. 5 that $r = \frac{E}{2G} - 1$. Hence if $E = 30,000,000$, which is essentially correct for steel, and if $G = 12,000,000$ as the mean, approximately, of the values in the preceding table, then will

$$r = .25$$

Direct torsion tests of six small nickel steel specimens by Prof. E. L. Hancock and described by him in Vol. VI (1906) of Proceedings of the American Society for Testing Materials gave elastic limits:

	Max.	Mean.	Min.
Nickel Steel . .	36,000	32,900	30,500 pounds per sq.in.

He also found for mild carbon steel the two following elastic limits:

Mild Carbon Steel. . . . 29,000. . . . 25,500 pounds per sq.in.

As the ultimate resistance of mild carbon steel to torsive or ordinary shear may be taken at about three-quarters the ultimate tensile resistance, and approximately the same ratio between the elastic limits, it is reasonable to take the elastic limit in torsion at 25,000 pounds to 28,000 pounds per square inch for that grade of material having an ultimate tensile resistance of 60,000 pounds to 68,000 pounds per square inch.

Nickel steel has a higher ratio of the elastic limit divided by the ultimate, and a mean value of 33,000 pounds per square inch for the elastic limit is reasonable.

If the greatest intensity of torsive shear T_m allowed in the design of a shaft of diameter d is fT_e in which T_e is the elastic limit and f a suitable fraction, perhaps .5 in some cases, then eq. (19) of the preceding article will take the form:

$$d = 1.72\sqrt[3]{\frac{Pe}{fT_e}} = \left(1.365\sqrt[3]{\frac{Pe}{T_e}}\right)_{f=\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Similarly eq. (22) of the same Art. will become:

$$d = 68.5\sqrt[3]{\frac{H}{fT_e n}} = \left(54.37\sqrt[3]{\frac{H}{T_e n}}\right)_{f=\frac{1}{2}} \quad . \quad . \quad . \quad . \qquad (2)$$

Wrought Iron.

Wrought iron is now seldom used for shafting or similar purposes, but such tests as have been made show that the torsive elastic limit of wrought iron may be taken from 20,000 pounds to 25,000 pounds per square inch and used as indicated in eqs. (1) and (2). From 10 per cent to 20 per cent higher values may be taken for cold-rolled shafting.

Cast Iron.

Cast iron is ill adapted to resist torsion and is not commonly used for that purpose, yet it has been tested in torsion, although generally in special grades such as were formerly employed in making cannon or car wheels. Such grades of cast iron gave ultimate values of T_m from 24,000 pounds to 45,000 pounds per square inch or even more, but they are far too high for ordinary castings used in engineering practice. Probably half the preceding values would be large enough for the best quality of ordinary castings, although the highly variable and erratic qualities of cast iron make it exceedingly difficult to assign exact data for purposes of design. The modulus of elasticity, G, may be taken at 7,000,000 for ordinary grades of cast iron, or at 6,000,000 for the lower grades.

Alloys of Copper, Tin, Zinc and Aluminum.

The torsional resistance of this class of alloys varies greatly with the relative proportions of their constituent elements in a manner quite similar to that exhibited by the corresponding resistance to tension.

Professor R. H. Thurston was probably the earliest thorough investigator of the torsional resistances of many of these alloys. He found the ultimate intensity of torsive stress T_m to vary from a few hundred pounds per square inch to nearly 48,000 pounds per square inch for alloys of copper and tin running by gradual variation from pure copper to 10% of that metal alloy to 90% of tin. The alloy 80–90% Cu with 20–10% Sn gave T_m varying from about 47,700 pounds to 43,900 pounds per square inch with a maximum twist of 114.5 degrees. Similarly he found the ultimate T_m for pure copper to range from 28,400 to 35,900 pounds per square inch with a total twist of over 150 degrees. On the other hand, pure tin gave the ultimate $T_m = 3200$ pounds (nearly), the total angle of twist running as high as 691 degrees. The elastic limit of the more ductile of these alloys was found to vary from about 35% to 60% of the ultimate T_m. The alloys running from 70% Cu with 30% Sn to 29% Cu with 71% Sn were brittle, giving low values of T_m from about 700 pounds per square inch to less than 6000 pounds per square inch; those alloys failed at the elastic limit with a total angle of twist of only 1 to 2 degrees.

Similar results with like erratic variations were found by Professor Thurston for alloys of copper and zinc. The greatest values of T_m ran from about 35,000 to 52,000 pounds per sq. in. for 90.58% Cu with 9.42% Zn to 49.66% Cu with 50.14% Zn.

It should be observed that the test specimens used by Prof. Thurston were .625 inch in diameter with a torsion length of 1 inch only and they were tested in his torsion machine.

TABLE I.

ALUMINUM ALLOYS—TORSIONAL RESISTANCE.

Composition Per Cent.			Angle of Torsion Deg.		Torsive Shear per Sq.In.		General Character.
Al.	Sn.	Cu.	Elastic Limit.	Maximum.	Elastic Limit.	Maximum.	
—	—	100	2	130	4,300	25,000	
2.5	2.5	95	4	200	10,710	33,075	Very soft; ductile.
2.75	3.75	92.5	6	198	11,827	35,802	Soft; ductile.
5	5	90	7	175	15,525	45,155	Slightly tough; ductile.
6.25	6.25	87.5	4	37	30,282	63,440	Tough; medium ductility.
7.5	7.5	85	3.5	22	25,447	37,062	Very tough; rather hard.
8.75	8.75	82.5	7	10	18,413	18,413	Hard; somewhat brittle.
10	10	80	6	8	15,230	15,230	Very hard; brittle.
*11	11	78	5.8	5.8	13,717	13,717	Very hard; exceed'gly brittle
*20	20	60	1	1	2,321	2,321	Very hard; exceed'gly brittle
Scattering.							
2	10	88	3	147.5	14,000	43,987	Somewhat soft; ductile.
10	1	89	5	52	21,740	50,000	Tough; medium ductility.
12	2	86	9	9	32,984	32,984	Very tough; hard.
13	2	85	8	12	32,723	37,003	Very tough; hard.
85	7.5	7.5	3	37	8,703	17,630	Very soft; somewhat ductile.
27.1	119	69.6	2.5	20	2,800	2,800	Very soft; spongy.
100	—	—	2	160	4,005	12,911	

*Could not be machined.

Table I contains experimental values of the elastic limit and ultimate torsion shearing resistance of the alloys of aluminum, tin, and copper shown in the table. They were determined by Messrs. Gebhardt and Ward in the mechanical laboratory of Sibley College at Cornell University and reported to the Am. Soc. Mech. Engrs. in 1898.

The results of the table show that the alloys yielding other resistances of considerable value will also exhibit proportionate torsion resistances, as might be anticipated.

The Eighth Report to the Alloys Research Committee of the Institution of Mechanical Engineers of Great Britain by Prof. H. C. H. Carpenter, M.A., Ph.D., and Mr. C. A. Edwards in 1907 contains some interesting torsion tests on specimens of copper-aluminum, the pieces being .624 inch in diameter and 3 inches in length with the exception of No. 3, which was 2.8 inches in length. Table II gives the results of these tests. It will be observed that alloys with a comparatively small percentage of aluminum give much higher torsional ductility than pure copper. This is probably due to the fact that rolled copper generally contains

TABLE II.

Cu. Per cent.	Al. Per cent.	Greatest Twisting Moment and Stress.		Twist on Whole Length Degrees.	Ratio, $\frac{\text{Torsion } (T_m)}{\text{Tension}}$
		Moment. In.-Lbs.	Stress, Lbs. per Sq.In.		
99·96	0	1,792	37,500	2,736	1.51
99·9	.1	2,293	47,960	5,184	1.15
98.94	1.06	2,359	49,350	4,345	1.41
97·9	2.1	2,464	51,450	3,600	1·34
95·95	4·05	2,813	58,870	2,316	1.2
93·23	6·73	3,306	69,170	1,623	1.18
92.61	7·35	3,373	70,580	1,374	1.15
90.06	9·9	3,351	70,110	234	0.89
88.2	11.72	3,584	74,970	51	1.04

some dissolved oxygen which diminishes its ductility. The addition of a small amount of aluminum removes the oxygen and enhances the ductility. The authors of the report express the conclusion that "Alloys containing aluminum up to $7\frac{1}{2}$ per cent behave extremely well under the torsion test but beyond this percentage there is a rapid deterioration of properties." The ratio between the ultimate resistance T_m to torsional shear and the ultimate tensile resistance is shown in the last column of the table.

Other Sections than Circular.

The common theory of torsion is correct only for circular sections. The general demonstration for other sections than circular shows that for square, rectangular, triangular and elliptical sections, the maximum intensity of torsive stress T_m will be found at the middle point of a side of a square section or of the longest side of a rectangular section, or at the middle point of the side of an equilateral triangular section and at the extremities of the minor axis of an elliptical section. If, however, for approximate purposes the formulæ of the common theory of torsion should be used for the sections indicated above the polar moments of inertia I_p would be taken from eqs. (6), (7), (8) and (9) of Art. 37. The maximum torsive shear T_m, in this procedure, should be taken as existing at the extreme points of the section. The results by this approximate method will be sufficiently near for most ordinary purposes, at least with the square section, but the exact theory should be used for oblong sections or where the highest degree of accuracy is desired for non-circular sections.

CHAPTER IV.

HOLLOW CYLINDERS AND SPHERES

Art. 39.—Thin Hollow Cylinders and Spheres in Tension.

If a straight closed hollow cylinder be subjected to an interior pressure having the intensity q' sufficiently greater than that of the exterior pressure q_1, there will be a tendency to split the cylinder longitudinally.

Fig. 1 represents such a cylinder with sides so thin that the stress to which they are subjected may be considered uniformly distributed throughout any diametral section. If a cylindrical shell has much thickness relatively to its interior radius the tensile annular stress due to inner pressure will not be uniformly distributed throughout the shell. The excess of inner pressure over the outer, if the latter exists, will cause the inside part of the annular section of metal to be stressed to a higher intensity than the outside and that difference will be greater as the thickness of the shell increases relatively to the radius. It becomes necessary therefore to distinguish between these two classes of cylindrical shells in their analytic treatment.

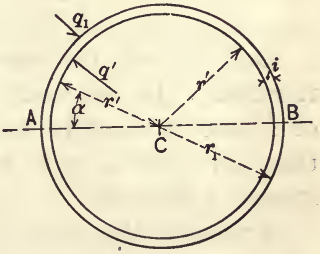


FIG. 1.

AB represents the diametral plane through the axis of the cylinder, the thickness i of the shell being supposed in this case to be so small that the cylindrical shell may be considered " thin."

As the notation shows r' is the interior radius and r_1 the exterior radius. If C, the centre of the cylindrical section, be taken as the origin of the circular coordinates r' and α, and if a unit length of cylinder be considered, the indefinitely small amount of pressure on a differential of the interior surface $r'\alpha$ will be $q'r'd\alpha$ and it will have a component at right angles to the diametral plane AB expressed by $q'r'd\alpha \sin \alpha$. The integral of this expression between 180° and 0 will be the total normal pressure acting on the two longitudinal sections of metal at A and B, as shown by the following equation:

$$\int_0^\pi q'r' \sin \alpha d\alpha = 2q'r'.$$

One-half of the second member of this equation, $q'r'$, represents the tendency to split the cylinder at either A or B and it must be resisted by the sections of metal at those two points, or at any other two points at the extremities of a diameter.

Precisely the same integration made for the exterior pressure will obviously give the quantity q_1r_1 representing the tendency to give the metal compression at the extremities of any diameter.

The resultant tendency to split the cylinder per unit of length will then be $q'r' - q_1r_1$, it being supposed that the interior pressure is so much greater than the exterior that tension only will be induced in the material. Obviously if the exterior pressure were much larger than the interior, compression would exist instead of tension. The intensity of tensile stress t in the sides of the cylinder will therefore be

$$t = \frac{q'r' - q_1r_1}{i}. \quad \dots \dots \dots \quad (1)$$

This value of t expresses the tendency of the cylinder to split along a diametral plane under the action of the interior pressure q'.

If the ends of the cylinder are closed, the internal pressure against them will tend to force them off or to pull the cylinder apart around a section normal to the axis. The force F tending to produce this result will be

$$F = \pi(q'r'^2 - q_1 r_1^2). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The area of normal section of the cylinder will be $\pi(r_1^2 - r'^2)$. Hence the intensity of stress developed by this force will be

$$f = \frac{F}{\pi(r_1^2 - r'^2)} = \frac{q'r'^2 - q_1 r_1^2}{r_1^2 - r'^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the exterior pressure is so small that it may be considered zero, eqs. (1) and (3) give

$$t = \frac{q'r'}{i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$f = \frac{q'r'^2}{r_1^2 - r'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When the thickness of the shell is small r' may be placed equal to $\frac{r' + r_1}{2}$, and this value introduced in eq. (5) will give

$$f = \frac{q'r'}{2(r_1 - r')} = \frac{q'r'}{2i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

f in eq. (6) is seen to be but half as much as t in eq. (4). In this case, therefore, if the material has the same ultimate resistance in both directions, the cylinder will fail longitudinally when the interior intensity is only half great enough to produce transverse rupture.

In designing thin cylinders it will usually be necessary to determine the thickness i, so that the tensile stress t in the metal shall not exceed the prescribed value h. After writing h for t in eq. (1), also $r_1 - r'$ for i, then dividing both sides of the equation by r', there will result

$$h\frac{r_1}{r'} = h + q' - q_1\frac{r_1}{r'}.$$

This equation readily gives

$$r_1 - r' = i = r'\left(\frac{h+q'}{h+q_1}\right) - r'. \quad . \quad . \quad . \quad . \quad (7)$$

If the exterior pressure q_1 is so small that it may be considered zero, the thickness given by eq. (7) takes the following form:

$$i = \frac{q'r'}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This is the same value that will be found by solving eq. (4) for i.

The expression for the thickness of the material of the cylinder to resist the longitudinal tension having the intensity f can be found with equal ease. If f_1 be written for f in eq. (3), as the greatest permissible longitudinal tension, then if both numerator and denominator of the second member of that equation be divided by r'^2, there will result

$$\frac{r_1^2}{r'^2} = \frac{f_1 + q'}{f_1 + q_1}.$$

The solution of this equation at once gives the desired thickness:

$$r_1 - r' = i = r'\left(\frac{f_1 + q'}{f_1 + q_1}\right)^{\frac{1}{2}} - r'. \quad . \quad . \quad . \quad . \quad (9)$$

If q_1 is so small that it may be neglected, it is simply to be made zero in eq. (9).

If the exterior pressure q_1 were considerably larger than q', the resulting stresses in the sides of the cylinder would be compression, but the formulæ for the resulting intensities would be precisely the same as the preceding, as long as the cylinder retained its circular shape.

The case of stresses in a thin hollow sphere or thin spherical shell may be treated in the same general manner. The hemispherical ends of a metallic cylindrical tank or reservoir may be illustrated by the skeleton section in Fig. 2.

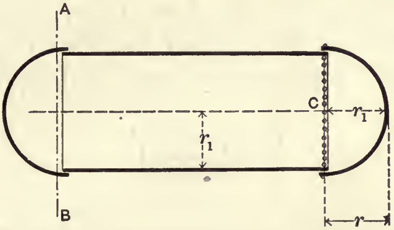


FIG. 2.

As indicated in the figure the internal radius of each end is r', while r_1 is the external radius. The internal and external intensities of pressure are as shown in Fig. 1. The force tending to tear off the hemispherical ends of the tank along the line AB, Fig. 2, is $\pi(qr'^2 - qr_1^2)$. The section of metal resisting this force with the intensity t is $\pi(r_1^2 - r'^2)$. The intensity of stress developed in the metal will therefore be

$$t = \frac{q'r'^2 - q_1r_1^2}{r_1^2 - r'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the external pressure is so small that there may be taken $q_1 = 0$, eq. (10) will take the form

$$t = \frac{q'r'^2}{r_1^2 - r'^2} = \frac{q'r'}{2i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In this last equation $i = r_1 - r'$, and the interior radius is placed equal to one-half the sum of the interior and exterior radii, as may be done without sensible error. The interior radius being given, the thickness of metal required to withstand a given internal pressure q' without stressing the metal above a given working value t may be written as follows from eq. (11):

$$i = \frac{q'r'}{2t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If the value of the thickness i should be desired in terms of both the interior and exterior pressures, it can easily be written by the aid of eq. (10); if both numerator and denominator of the second member of that equation be divided by r'^2, there may at once be found

$$\frac{r_1}{r'} = \left(\frac{t + q'}{t + q_1}\right)^{\frac{1}{2}}.$$

After multiplying this equation through by r', then subtracting that quantity from each side of the resulting equation, the desired value of the thickness will be

$$i = r_1 - r' = r'\left(\frac{t + q'}{t + q_1}\right)^{\frac{1}{2}} - r'. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

By giving a proper working value to the tensile intensity t and inserting the values of the pressures, the thickness i will at once result.

In all these equations no allowance is made for the metal taken out by the rivet holes in riveted work. This does not, however, affect in any way the equations found. It is only necessary to remember that the cross-section of metal required by the preceding equations is to be regarded as the net section, i.e., the section remaining after the rivet holes have been made. This is equivalent to making the thickness i great enough to give the required section as net section.

Art. 40. Thick Hollow Cylinders.

If the thickness of sides or walls of hollow cylinders and spheres subjected to high internal pressures is great in comparison with the internal radius, the tensile stress in the metal may not be assumed to be uniformly distributed, and it is necessary to determine entirely different formulæ from those established in the preceding article.

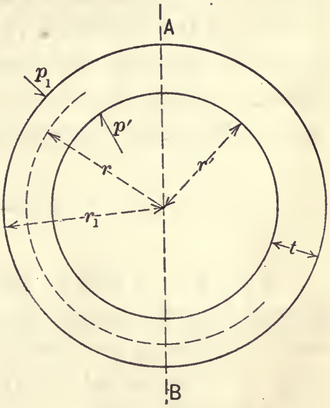


FIG. 1.

The normal section of a thick hollow cylinder is shown in Fig. 1, r' being the internal radius and r_1 the external, with the intensities of internal and external pressures p' and p_1 respectively. It is supposed that the internal pressure so greatly exceeds the external that the metal sustains tensile stress only. If the cylinder be supposed to be divided into a great number of thin concentric portions, the elastic stretching of the metal will cause a much higher tension to exist in the interior portions than in the exterior. If any diametral section, such as AB, Fig. 1, be assumed, it is clear that the sum of all the tensile stresses developed in that section must be equal to the excess of the internal pressure over the external. A unit length of cylinder will be considered in the following formulæ.

The tensile stress in the sides of the cylinder, whose intensity will be represented by h, and which is developed in any diametral section, as AB, has a circumferential direction, and for that reason it is sometimes called "hoop tension."

The variation of this tensile intensity h carries with it a corresponding variation in intensity of the radial pressure whose intensity is p, having the values p' in the interior of the cylinder and p_1 at the external surface.

The amount of tension on a radial section of thickness dr will be hdr, and if that differential expression be integrated so as to extend over the entire thickness of one wall or side of the cylinder, it must be equal to the effort of the internal pressure in excess of the external to split the cylinder along one of its sides. The following equation is the analytical expression of this condition:

$$p'r' - p_1r_1 = \int_{r'}^{r_1} hdr. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If p', p_1, r', and r_1 be considered variable so as to refer to any interior points in the wall of the cylinder, and if r' and r_1 become so nearly equal to each other that $r_1 - r'$ may be considered as dr, then will $p'r' - p_1r_1 = d(pr)$ and eq. (1) will become:

$$d(pr) = pdr + rdp = hdr. \quad . \quad . \quad . \quad (2)$$

Eqs. (1) and (2) will be in no way changed if the ends of the cylinder are closed, it being assumed in that case that the longitudinal stress is uniformly distributed over a normal section like that shown in Fig. 1.

Eq. (2) is a differential equation expressing a relation between the two intensities p and h. Another equation of condition is required in order to determine the two unknown quantities. This second equation can be written by expressing the relation existing between the direct and lateral strains due to the stresses p and h, so as to leave the radial longitudinal sections of the walls of the cylinder plane under the conditions of stress due to the assumed internal and external pressures. The establishment of such an equation, however, will lead to, or express, precisely the same conditions involved in the analysis of Art. 5 of Appendix I, which therefore need not be repeated here. Those conditions may be expressed by stating that the sum of the two intensities p and h, i.e., $(p+h)$, is a constant for given intensities of pressure. If therefore, a be such a constant there will be assumed the equation:

$$p+h=a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\therefore \quad dp=-dh, \text{ and } p=a-h.$$

By the aid of these expressions eq. (2) will take the form:

$$2hdr+rdh=adr.$$

By multiplying both sides of this equation by r there will result:

$$d(r^2h)=\frac{a}{2}dr^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If b is a constant the integration of eq. (4) will give

$$h=\frac{a}{2}+\frac{b}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Also

$$p=a-h=\frac{a}{2}-\frac{b}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The interior and exterior pressures p' and p_1 are known, and eq. (6) will give the two equations:

$$p'=\frac{a}{2}-\frac{b}{r'^2} \text{ and } p_1=\frac{a}{2}-\frac{b}{r_1{}^2}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By subtracting p_1 from p'

$$b=\frac{p'-p_1}{r'^2-r_1^2}r'^2r_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then by the second of eqs. (7):

$$\frac{a}{2}=p_1+\frac{b}{r_1^2}=\frac{p'r'^2-p_1r_1^2}{r'^2-r_1^2}. \quad . \quad . \quad . \quad . \quad (9)$$

The substitution of these values of b and $\frac{a}{2}$ in eqs. (5) and (6) will give the following values of the intensities p and h. Inasmuch as the preceding equations involving h and p have been written without giving distinctive signs to either tension or compression and as the constants b and $\frac{a}{2}$ may be regarded either as positive or negative, the sign of each one will be changed by writing $r_1^2-r'^2$ for $r'^2-r_1^2$, which will make the tensile stress h positive and the compressive stress p negative after substituting the values of the constants b and $\frac{a}{2}$ in eqs. (5) and (6).

$$\therefore \quad p=\frac{p'r'^2-p_1r_1^2}{r_1^2-r'^2}-\frac{p'-p_1}{r_1^2-r'^2}\frac{r'^2r_1^2}{r^2}. \quad . \quad . \quad . \quad (10)$$

$$h=\frac{p'r'^2-p_1r_1^2}{r_1^2-r'^2}+\frac{p'-p_1}{r_1^2-r'^2}\frac{r'^2r_1^2}{r^2}. \quad . \quad . \quad . \quad (11)$$

Eqs. (10) and (11) can be put in more convenient form for use in numerical computations by dividing both numerator and denominator of all the terms in the second members of those equations by r_1^2. This simple operation will give eqs. (12) and (13):

$$p = \frac{p'\frac{r'^2}{r^2_1} - p_1}{1 - \frac{r'^2}{r^2_1}} - \frac{p' - p_1}{1 - \frac{r'^2}{r^2_1}}\frac{r'^2}{r^2}. \quad . \quad . \quad . \quad (12)$$

$$h = \frac{p'\frac{r'^2}{r^2_1} - p_1}{1 - \frac{r'^2}{r^2_1}} + \frac{p' - p_1}{1 - \frac{r'^2}{r_1^2}}\frac{r'^2}{r^2}. \quad . \quad . \quad . \quad . \quad (13)$$

Eqs. (12) and (13) are the general values of the intensities of the internal stresses in the walls of the cylinder, p acting in a radial direction and h in a circumferential direction. The greatest tensile intensity h' will exist at the interior surface of the cylinder and it will be found by making $r=r'$ in eq. (13) as shown by eq. (14):

$$h' = \frac{p'\left(1 + \frac{r'^2}{r_1^2}\right) - 2p_1}{1 - \frac{r'^2}{r_1^2}} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Similarly the intensity of tensile stress at the outer surface of the cylinder (the least intensity of tensile stress) will be given by making $r=r_1$.

$$h_1 = \frac{2p'\frac{r'^2}{r_1^2} - p_1\left(1 + \frac{r'^2}{r_1^2}\right)}{1 - \frac{r'^2}{r_1^2}}. \quad . \quad . \quad . \quad . \quad (15)$$

The thickness t of the wall of the cylinder which must be provided if the greatest intensity of tensile stress h' is not to be exceeded by a given intensity p' of interior pressure, can readily be found by solving eq. (14) for the quantity $\frac{r'^2}{r_1^2}$, which will give eq. (16).

$$\frac{r_1}{r'} = \left(\frac{h'+p'}{2p_1-p'+h'}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Then by adding (-1) to each side of eq. (16) and multiplying both members by r' eq. (17) will at once result:

$$r_1 - r' = t = r'\left(\frac{h'+p'}{2p_1-p'+h'}\right)^{\frac{1}{2}} - r' \quad . \quad . \quad . \quad . \quad (17)$$

As the internal radius r' will always be known, eq. (17) gives the thickness t desired in terms of the known pressures and the intensity of working stress h'.

Eq. (17) shows that if $2p_1 + h' = p'$, t will be infinity. This shows that when the intensity of the internal pressure is equal to or greater than twice the intensity of the external pressure added to the greatest allowed tensile stress in the metal, it is impossible to make the wall of the cylinder thick enough to resist that internal pressure.

If the external pressure is so small that it may be neglected, it is necessary only to place $p_1 = 0$ in the preceding equations.

If p_1 exceeds p' it is obvious that the internal stress h will be compression, i.e., there will be hoop compression as the circumferential stress in the cylinder wall instead of hoop tension.

The complete solution of the problem of the thick cylinder including expressions for the distortions or strains of the material at all points will be found in Art. 5 of Appendix I.

The application of the preceding formulæ can be expedited by the use of the following tabular values which explain themselves. A curve more useful than the table can readily be constructed from the numerical values in the latter, so that any value whatever for the ratio of the radii indicated can be read at sight.

$\frac{r'}{r}$	$\frac{r'^2}{r_1^2}$	$\frac{r'}{r}$	$\frac{r'^2}{r_1^2}$
1.	1.	.5	.25
.95	.9025	.45	.2025
.9	.81	.4	.16
.85	.7225	.35	.1125
.8	.64	.3	.09
.75	.5625	.25	.0625
.7	.49	.2	.04
.65	.4225	.15	.0225
.6	.36	.1	.01
.55	.3025	.05	.0025

As an illustration of the laws of variation of the intensities h and p the following data may be used:

$r' = 10$ inches;
$h' = 20,000$ pounds per square inch;
$p' = 10,000$ pounds per square inch;
$p_1 = 1000$ pounds per square inch.

Eq. (17) will give, after a substitution in it of the above numerical data, $t = 5.81$ inches.

$$\therefore \quad r_1 = 15.81 \text{ inches.}$$

The quantity $\frac{r'^2}{r^2}$ will have values ranging from unity for the interior of the cylinder to $\frac{r'^2}{r_1} = .4$. Inserting these values in eqs. (12) and (13) there will result the two equations:

$$p = 5000 - 15,000 \frac{r'^2}{r^2};$$

$$h = 5000 + 15{,}000\,\frac{r'^2}{r^2}.$$

Taking the varying values of $\frac{r'^2}{r^2}$ given in the above table the following values of p and h will result:

$\frac{r'}{r}$	Pounds per Sq.in.	
	p	h
1.	−10,000	20,000
.95	− 8,540	18,540
.9	− 7,150	17,150
.85	− 5,840	15,840
.8	− 4,600	14,600
.75	− 3,440	13,440
.7	− 2,350	12,350
.65	− 1,340	11,340
.63	− 1,000	11,000

Fig. 2 represents these results graphically. The straight line GAF is laid off tangent at any point A to the circle representing the interior of the cylinder subjected to the pressure of 10,000 pounds per square inch. Similarly HBD

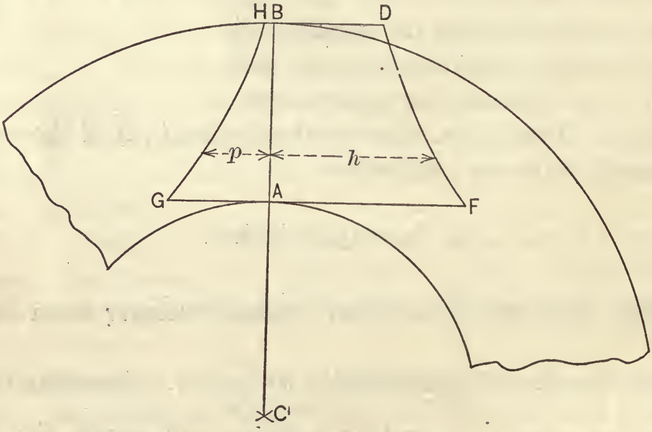


FIG. 2.

is a straight line laid off tangent at the point B on any radius CB of the exterior surface of the cylinder, the distance AB being equal to 5.81 inches. AF is then laid off by scale equal to $h' = 20,000$ pounds, while AG is similarly laid off to represent $p' = 10,000$ pounds per square inch, but it must be remembered that it acts in a radial direction, i.e., along AB. BD and BH are the corresponding quantities for the exterior surface of the cylinder, equal respectively to 11,000 pounds and 1000 pounds. Curves DF and HG are then constructed by laying off the ordinates p and h at right angles to AB as shown.

Case of Exterior Pressure Greater than Interior Pressure.

If the exterior pressure p_1 is greater than the interior pressure p', it is evident that the preceding equations will need no change whatever, but the difference $p' - p_1$ will now be negative. As $p'\frac{r'^2}{r_1^2}$ is less than p', p will still be negative and represent compression. On the other hand, h will now be negative and represent circumferential or hoop compression as shown by eq. (13). Eqs. (12) and (13) are used in connection with this case in designing modern heavy guns where thick cylinders are raised to a high temperature and slipped over a close-fitting interior thick cylinder at ordinary temperature, so that when the outside hot cylinder cools it contracts and puts the interior cylinder under a high compression. In fact, the lining of the gun may be enclosed by two or more such cylinders successively shrunk into place. One interior cylinder with slightly conical interior surface may be forced by a high pressure at ordinary temperature into the interior of a corresponding exterior cylinder with similar results. These matters will be treated more extendedly in the next article.

Art. 41.—Radial Strain or Displacement in Thick Hollow Cylinders.—Stresses Due to Shrinkage of One Hollow Cylinder on Another.

Radial Strain or Displacement.

Inasmuch as all diametral sections of thick hollow cylinders remain plane for all conditions of stress due to internal or external pressure, the only strain or displacement in such a cylinder is that in a radial direction due to either increase or decrease of the diameter of any elementary thin cylinder or shell with radius r. This radial displacement will be indicated by ρ and the expression for it can only be established by the analysis shown in Art. 5 of Appendix I, or by some equivalent analysis. By referring to eq. (10) and the two equations preceding eq. (15) of that article it will be seen that the desired displacement is given by the following equation:

$$\rho = \frac{(1-2\bar{r})\left(p_1 - p'\dfrac{r'^2}{r_1^2}\right)r + (p_1 - p')\dfrac{r'^2}{r}}{2G\left(\dfrac{r'^2}{r_1^2} - 1\right)}. \quad . \quad . \quad (1)$$

In this equation G is the modulus of elasticity for shearing, while p_1 and r_1 represent the intensity of exterior pressure and the exterior radius, respectively, and p' and r' similar quantities for the interior of the cylinder. The quantity $\bar{r}$ represents the ratio of the lateral strain divided by the direct strain, i.e., Poisson's ratio. Obviously if $r = r'$, the increase or decrease of the interior radius will be given by ρ and a similar observation applies to the increase or decrease of the exterior radius when $r = r_1$.

It is clear that if r be made equal to either r' or r_1 in eq. (1) either p' or p_1 may be written from that equation in terms of the corresponding radial displacement ρ' or ρ_1. It is sometimes desirable to express the intensities of interior or exterior pressures in this manner, after having determined the radial displacement corresponding to a known change of temperature or in some other manner. In the operation of shrinking one cylinder on another the difference in diameters required for the operation may be prescribed by some empirical rule.

Stresses Due to Shrinkage.

It has been shown in the preceding article that when a thick hollow cylinder is subjected to a high internal pressure the intensity of circumferential or hoop tension is much greater at and near the interior surface of the cylinder than at the exterior surface and that if the thickness is great the intensity of the interior tension may be high, while that of the exterior surface will be extremely low, showing the use of the metal to be uneconomical. In heavy gun making this undesirable condition is overcome by dividing the body of the gun into a number of concentric thick cylinders, each being shrunk over those inside of it, after making the interior diameter at ordinary temperature less than the exterior diameter of that over which it is shrunk into place. Each tube is heated so as to enlarge its diameter until it can be slipped over the tube, or tubes, inside of it, so that when it cools it will itself be subjected to high internal pressure with correspondingly high circumferential or hoop tension, while the tube, or tubes, inside of it will be correspondingly compressed at ordinary temperature. The body of the gun thus composed of a series of concentric tubes shrunk in place in series will form a combination in the interior of which there will be relatively high circumferential or hoop compression, decreasing outwardly though not regularly or continuously, with circumferential or hoop tension in the outer part or parts. When the intensely high pressures of modern explosives are produced in firing the gun the metal will be more nearly uniformly stressed in circumferential tension and thus act more effectively throughout the entire thickness of the wall of the gun. It will not be attempted here to give the details required to secure the best effects by shrinking into place a series of thick hollow cylinders in the manufacture of ordnance, but the general analytic procedure in deducing the proper results for the shrinkage of one cylinder on another either in gun making or in the making of other compound cylinders for high internal pressures will be illustrated by a single computation only.

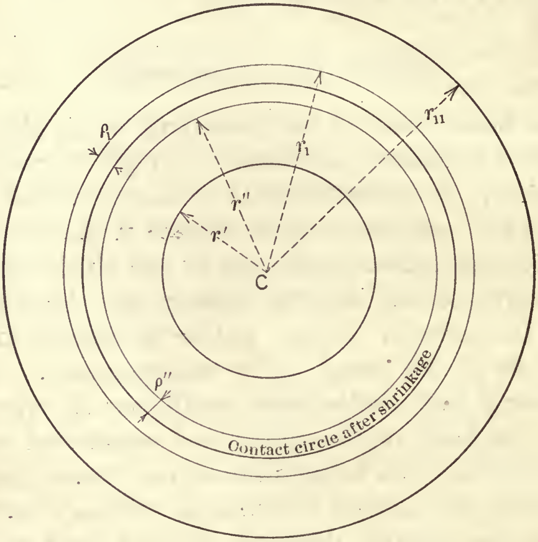


FIG. I.

Fig. 1 represents a thick hollow steel cylinder with internal diameter of 12 inches and total thickness of wall of 12 inches composed of an outer cylinder 6 inches thick shrunk on an interior hollow steel cylinder with wall also 6 inches thick. It will be supposed that the coefficient of expansion of steel per degree Fahr. is $\delta = .0000065$. The increase in diameter due to a change of 225° Fahr. of the interior 24-inch cylinder will be $225 \times 24 \times \delta = .0351$ inch. The change in radius will be one-half of this amount. The interior diameter of the exterior thick cylinder at ordinary temperature must be $24 - .0351 = 23.9649$ inches.

If r'' be the interior radius of the exterior cylinder before being heated and $r_{,,}$ the exterior radius also before being heated, while r' and r_1 represent the interior and exterior radii of the interior cylinder at ordinary temperature and before shrinkage, as shown in Fig. 1, the data required will be as follows:

$$r' = 6''; \quad r_1 = 12''; \quad r'' = 11.98245; \quad r_{,,} = 17.98245.$$

The interior pressure of the inner cylinder will be simply that due to atmosphere. Similarly the exterior pressure on the exterior cylinder will also be that due to the atmosphere. Hence, both these pressures will be considered zero. There will then be acting the shrinkage pressure on the exterior surface of the inner cylinder and the same pressure on the interior surface of the outer cylinder. The intensity of this common shrinkage pressure will be indicated by p_1.

As indicated in Fig. 1, after the properly heated outer cylinder has been slipped over the inner cylinder at ordinary temperature and the two allowed to cool, the radius r_1 will be decreased by the radial displacement ρ_1, while the radius r'' will be increased by the amount ρ''. Inasmuch as ρ_1 will be intrinsically negative, eqs. (2) will at once result.

$$r_1+\rho_1=r''+\rho'' \quad \therefore \quad \rho''-\rho_1=r_1-r''. \quad . \quad . \quad . \quad (2)$$

By making $p'=0$ in eq. (1) and $r=r_1$ there will result eq. (3):

$$\rho_1=\frac{\left((1-2\bar{\ })r_1+\frac{r'^2}{r_1}\right)p_1}{2G\left(\frac{r'^2}{r_1^2}-1\right)} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly by making $p_1=0$ in eq. (1), $r'=r''$, $r_1=r_{,,}$, $r=r''$ and remembering that $p'=p''=p_1$, eq. (4) will represent the increase of the radius of the exterior cylinder after the operation of shrinkage is complete:

$$\rho''=\frac{\left((1-2\bar{r})\frac{r''^3}{r_{,,}^2}+r_{,,}\right)p_1}{2G\left(\frac{r''^2}{r_{,,}^2}-1\right)}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By substituting the second members of eqs. (3) and (4) for the first member of eq. (2) an equation will result giving the value of p_1 as shown by the following equation and eq. (5):

$$-\frac{p_1}{2G}\left\{(1-2\bar{r})\left(\frac{r_1}{\frac{r'^2}{r_1^2}-1}+\frac{\frac{r''^3}{r_{,,}^2}}{\frac{r''^2}{r_{,,}^2}-1}\right)+\frac{\frac{r'^2}{r_1^2}}{\frac{r'^2}{r_1^2}-1}+\frac{r''}{\frac{r''^2}{r_{,,}^2}-1}\right\}=r_1-r''$$

Or

$$-p_1\frac{Z}{2G}=r_1-r''.$$

Hence

$$p_1=-\frac{2G(r_1-r'')}{Z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The quantity represented by Z is clear.

It should be observed that the relation between the changes of the radii at the cylindrical surface of shrinkage contact and the original radii (eq. (2)) is general and holds for all conditions of shrinkage stresses whatever may be the thicknesses of the two cylindrical walls.

In computing the value of p_1 by eq. (5) there will be taken:

$$G = 12{,}000{,}000 \text{ and } \bar{r} = .25.$$

If the values already given for the four radii of the cylinders be inserted in the equation immediately following eq. (4), there will result:

$$Z = -38.4.$$

Hence

$$p_1 = \frac{.0351 \times .5 \times 24{,}000{,}000}{38.4} = 10{,}970 \text{ lbs. per square inch.}$$

In computing the value of Z it is essentially accurate to use the inner and outer radii of the outer cylinder as they exist before it is heated for the shrinkage process. This will save much labor and simplify the application of the formulæ, but the difference $r_1 - r''$ must of course be expressed as accurately as possible.

The stresses in the walls of the two cylinders due to shrinkage may now be readily computed, since the outer cylinder is subjected to an inner pressure of 10,970 lbs. per square inch and the inner cylinder to an exterior pressure of the same intensity. The resulting values of p and h for the two cylinders are as follows:

Inner Cylinder in Compression.

$p_1 = 10{,}970$ lbs. per square inch; $p' = 0$; $r' = 6$ inches; $r_1 = 12$ inches.

Eqs. (12) and (13) of the preceding article will give at once eqs. (6) and (7) for this case:

$$p = p_1 \frac{1 - \frac{r'^2}{r}}{\frac{r'^2}{r_1^2} - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$h = p_1 \frac{1 + \frac{r'^2}{r^2}}{\frac{r'^2}{r_1^2} - 1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the intensities p and h are computed at six equidistant points at the two surfaces and at intermediate points one-fifth of the thickness of the wall of the cylinder apart, the results given in the following tabulation will be found and they are shown graphically in Fig. 2.

$\frac{r}{r'}$	$\frac{r'^2}{r^2}$	Pounds per Sq. In.	
		p	h
1	1.	0	−29,250
1.2	.6944	− 4,470	−24,780
1.4	.5102	− 7,164	−22,090
1.6	.3906	− 8,913	−20,340
1.8	.3086	−10,113	−19,089
2	.25	−10,970	−18,283

Outer Cylinder in Tension.

$p' = 10{,}970$ lbs. per square inch; $p_1 = 0$; $r' = 12$ inches; $r_1 = 18$ inches.

By making $p_1 = 0$ in eqs. (12) and (13) of the preceding article there will result the following two formulæ for the intensities p and h:

$$p = -p'\frac{\frac{r'^2}{r_1^2} - \frac{r'^2}{r^2}}{\frac{r'^2}{r_1^2} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$h = -p'\frac{\frac{r'^2}{r_1^2} + \frac{r'^2}{r^2}}{\frac{r'^2}{r_1^2} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The two intensities p and h will be computed for six equidistant points, including those on the two cylindrical surfaces, by taking corresponding values of r. The results of these computations are given in the following tabulation and they are shown graphically in Fig. 2, as will be explained further on.

$\frac{r}{r'}$	$\frac{r'^2}{r^2}$	Pounds per Sq.In.	
		p	h
1	1.	−10,970	+28,054
1.1	.8264	− 7,543	+25,093
1.2	.6944	− 4,937	+22,486
1.3	.5917	− 2,908	+20,459
1.4	.5102	− 1,299	+18,848
1.5	.4444	0	+17,552

Combined Cylinder under High Internal Pressure.

The stresses induced by shrinkage in making the combined cylinder of two concentric shells have been explained and computed in the preceding sections; those stresses are permanent and they must be combined with stresses which may be produced usually temporarily by subjecting the combined cylinder to a high internal pressure such as that caused by the discharge of a gun. The internal pressure produced by a modern high explosive may reach 50,000 or 60,000 lbs. per square inch, but as an illustration in this case the internal pressure will be taken as 40,000 lbs. per square inch. Hence the following data are required:

$p' = 40{,}000$ lbs. per square inch; $p_1 = 0$; $r' = 6$ inches; $r_1 = 18$ inches.

As this case is similar to that expressed by eqs. (8) and (9), those equations will yield the results shown in the following tabulation when the above data are substituted in them.

$\frac{r}{r'}$	$\frac{r'^2}{r^2}$	Pounds per Sq.In.	
		p	h
1	1.	−40,000	+50,000
$1\frac{1}{3}$	.5625	−20,313	+30,312
$1\frac{2}{3}$	.36	−11,196	+21,200
2	.25	− 6,252	+16,248
$2\frac{1}{3}$	.1837	− 3,268	+13,268
$2\frac{2}{3}$	.1406	− 1,328	+11,328
3	.1111	0	+10,000

It will be observed that the intensities p and h have been computed at points 3 inches apart throughout the 12-inch thickness of the combined cylinder wall.

The results of computations shown in the three preceding tabulations may now be shown graphically in Fig. 2. That figure shows part of a normal section of the two cylinders, C being the center and CD being the internal radius of 6 inches. The separate walls each 6 inches thick are shown by the parts of concentric circles with radii 6 inches, 12 inches, and 18 inches. The line ABD represents the trace of a longitudinal diametral plane at right angles to which the intensities of the circumferential or hoop stresses shown in the preceding tabulations are laid off.

Tensile stresses indicated by the plus sign are laid off to the left of *AD* and compressive stresses to the right of *BD* as indicated by the minus sign.

Referring to the tabulated results for the inner cylinder in compression, *DQ* represents 29,250 and *BP* 18,283, both pounds per square inch. Intermediate ordinates of the curved line *PQ* are laid off by the same scale to represent the other intensities *h* given in the table.

The ordinate *AL* represents by the same scale the intensity 17,552 lbs. per square inch and *MB* the intensity 28,054 lbs. per square inch, both shown in the tabulation for the outer cylinder in tension. The other intensities laid off as ordinates give the curved line *LM*.

The tensile intensities *h* for the combined cylinder under the internal pressure of 40,000 lbs. per square inch are shown by the ordinates to the curved line *EF*, *FD* representing 50,000 lbs. per square inch and *EA* 10,000 lbs. per square inch.

The resultant intensities at various points in the wall of the combined cylinder are found by taking the algebraic sum at each point of the three results shown. The resultant hoop stress at *D* is found by laying off $KF = DQ$, the resultant intensity being $DK = 50,000 - 29,250 = 20,750$ lbs. per square inch. Similarly $BH = MB - BP = 16,248 - 18,283 = -2035$ lbs. per square inch, showing that the tensile stress developed by the high internal pressure was not quite enough to overcome the shrinkage compression. The intensities of hoop stress in the wall of the inner cylinder are therefore the intercepts of ordinates at right angles to *BD* between *FM* and *KH*.

All stress in the outer cylinder is tension equal in intensity at any point to the sum of the ordinates between *AB* and *ME* added to those between *AB* and *LS* represented by the ordinates drawn from *AB* to *ON*. Thus it is seen that the shaded parts of the diagram represent at each point the intensity of stress existing at that point.

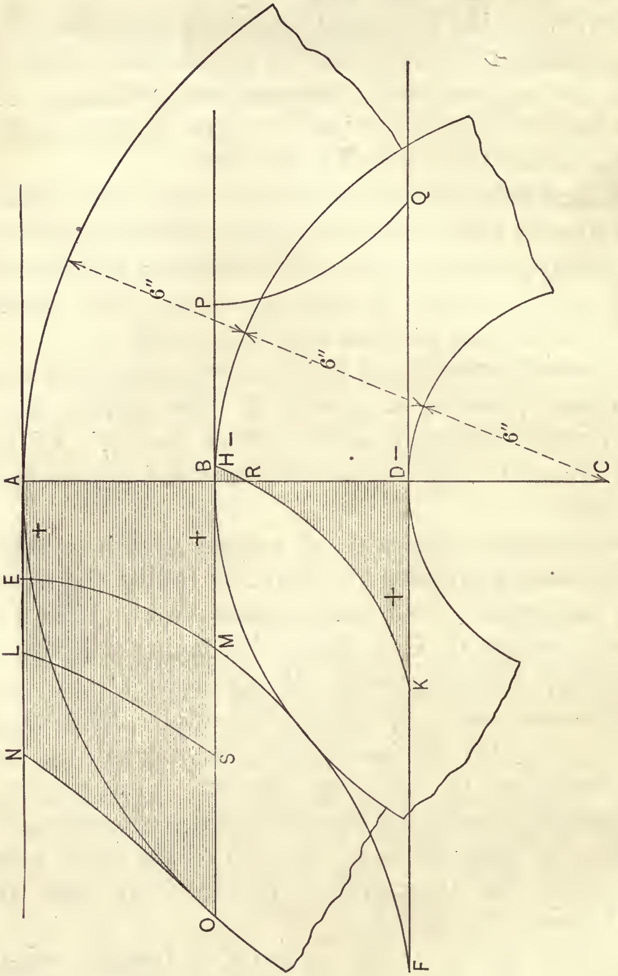


Fig. 2.

The highest tension exists in the outer cylinder at *B* and is equal to 28,054 + 16,248 = 44,302 lbs. per square inch. At the outer point *A* the tensile intensity of hoop stress is seen to be 27,552 lbs. The intensity of hoop stress at the interior surface of the cylinder has been found to be 20,750 lbs. per square inch, materially less than at the outer surface, which is desirable, as the radial normal pressure at the inner point is 40,000 lbs. per square inch.

The high tensile intensity 44,302 lbs. per square inch, found at the inner surface of the outer cylinder and the compression of about 2000 lbs. per square inch at the adjacent point on the inner cylinder show the desirability of a redesign for the assumed internal pressure with adjustment of the amount of shrinkage and with the wall composed perhaps of three cylinders instead of two. In this manner the undesired extremes of stress in the vicinity of the middle of the wall can be avoided. The results, however, exhibit completely the procedures to be followed where it is desired to make a combined cylinder with a number of concentric shells with shrinkage so employed as to produce a more nearly uniform, though not continuous, stress condition than can be attained in a single wall without shrinkage. In a single wall of 12-inch thickness in this case the hoop tension would have varied from 50,000 lbs. per square inch at the interior surface to only 10,000 lbs. per square inch at the exterior surface.

The radial compressive intensities p have not been plotted in Fig. 2, as the resultant intensity in every case is found by adding the intensities due to each condition as given in the tabulations. At the interior surface the maximum intensity of pressure is 40,000 lbs. per square inch. At 3 inches from the interior surface the maximum intensity will be about 20,000 lbs. per square inch and at the common surface of the two shells that intensity will be about 17,000 lbs. per square inch, thus decreasing outward until the value 0 is found at the outer surface.

Art. 42.—Thick Hollow Spheres.

When the thickness of wall of a hollow sphere is so great that the stresses may not be considered uniformly distributed over a diametral section of the shell the approximate formulæ of Art. 39 cannot be used; it becomes necessary to make an investigation similar to that required for thick hollow cylinders.

It will be supposed that the interior of the spherical shell is subjected to an intensity of pressure p' greater than the exterior normal pressure p_1 as shown in Fig. 1. As the intensity of the interior pressure, produced possibly by a fluid, is greater than that of the exterior pressure the material of the shell will be subjected to an internal stress of tension as well as the radial compression, but the formulæ as demonstrated will be equally applicable to the case of the exterior pressure being greater than the interior without any modification whatever. In the latter case, however, the internal stress acting around a great circle will be compression instead of tension. The formulæ will be so written that a tensile stress is positive and a compressive stress negative.

If a diametral section of the spherical shell be taken as in Fig. 1, it is clear that for a given radius r there will be a uniform intensity of tension normal to that section and no other stress, i.e., this tension at every point will be in the direction of the circumference of a great circle. Furthermore, since that observation is true of all possible diametral sections of the shell it is equally obvious that at any point in the shell there will be two circumferential or hoop stresses at right angles to each other and a third radial stress of compression with no other stress on its surface of action, the three stresses being principal stresses at the assumed point. The three principal planes on which these principal stresses act are two of them diametral and at right angles to each other, while the third is tangent to the spherical surface with radius r, and it is at right angles to the other two planes. The state of stress in the interior of the shell is also obvious from the fact that the interior and exterior fluid or normal pressures are each the same in intensity at all points making the resulting condition of stress completely symmetrical. As every diametral plane section of the shell is a principal plane of stress there will be no shear on any such plane and for the same reason there will be no shear on any of the concentric spherical surfaces within the limits of the shell.

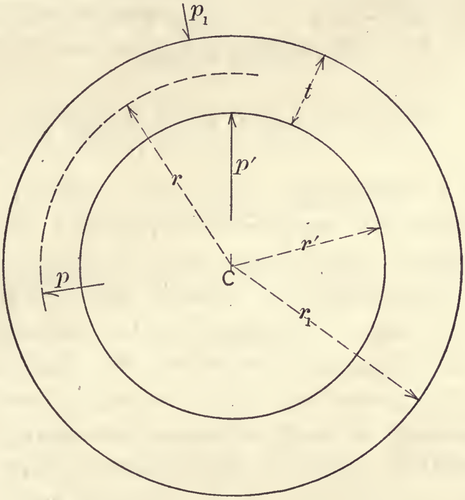


FIG. 1.

Remembering that the interior radius of the shell is r' and the exterior radius r_1 and that the tendency to tear the shell apart in any diametral annular section is due to the excess of the interior pressure over the exterior the following equation may be at once written, if h represents the intensity of the internal tensile stress developed at any point in the annular section:

$$\pi(p'r'^2 - p_1r_1^2) = \int_{r_1}^{r'} h \, . \, 2\pi r \, . \, dr. \quad . \quad . \quad . \quad (1)$$

If in eq. (1), r' and r_1 be considered variable and of so nearly the same value that they differ from each other only by dr, the quantity $p'r'^2 - p_1r_1^2$ becomes equal to $d(pr^2)$. Hence, eq. (1) for that supposition may take the following form:

$$d(pr^2) = 2hrdr = 2prdr + r^2dp. \quad . \quad . \quad . \quad . \quad (2)$$

This is a differential equation between h and p. Another equation of condition is required to determine those two variable quantities. Such an equation may be written by so expressing the relation between the internal distortions or strains accompanying the stresses h and p as to make the diametral sections of the shell plane whatever may be the intensities of the internal stresses h and p. The consideration of such relations between the strains produced would be precisely the same as given in Art. 8 of Appendix I, and hence it is repeated here. If it be remembered that the intensities of the two circumferential stresses at any interior point of the shell are equal to each other and indicated by h, as in eqs. (1) and (2), while p represents the intensity of the internal radial stress at the same point, the relation between the internal strains or distortions necessary to make all diametral sections of the shell plane for all intensities of stress is equivalent to the condition that the sum of the three principal stresses must be constant at all points as expressed by eq. (3), a being constant:

$$p + 2h = a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From eq. (3):

$$p = a - 2h \text{ and } dp = 2dh. \quad . \quad . \quad . \quad . \quad (4)$$

Substituting from eq. (4) in the second and third members of eq. (2):

$$2hrdr = 2ardr - 4hrdr - 2r^2dh.$$

By arranging terms the preceding equation takes integrable form as given by eq. (5):

$$3hrdr + r^2dh = ardr. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If b is a constant of integration, eq. (5) may be integrated so as to take the form of eq. (6):

$$r^3h = \frac{1}{3}ar^3 + b; \quad h = \frac{a}{3} + \frac{b}{r^3}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

By using the first of eqs. (4) and eq. (6), eq. (7) at once follows:

$$p = \frac{a}{3} - \frac{2b}{r^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

At the inner and outer surface of the spherical shell $p = p'$ and $p = p_1$, respectively. Eq. (7) will then give:

$$p' = \frac{a}{3} - \frac{2b}{r'^3} \text{ and } p_1 = \frac{a}{3} - \frac{2b}{r_1^3}. \quad . \quad . \quad . \quad . \quad (8)$$

Hence:

$$p' - p_1 = 2b\left(\frac{1}{r_1^3} - \frac{1}{r'^3}\right) = 2b\frac{r'^3 - r_1^3}{r'^3r_1^3}.$$

The preceding equation will at once give the following value of b, which in turn substituted in the second of eqs. 8 will give the value of $\frac{a}{3}$, following that of b:

$$b = \frac{(p' - p_1)r'^3r_1^3}{2(r'^3 - r_1^3)}; \quad \text{and } \frac{a}{3} = \frac{p'r'^3 - p_1r_1^3}{r'^3 - r_1^3}. \quad . \quad . \quad . \quad (9)$$

These values of b and $\frac{a}{3}$ substituted in eqs. (6) and (7) will give the following values of radial intensity p and circumferential intensity h at any point in the shell distant r from the centre. In writing these final expressions it is to be remembered that the constants a and b may be either positive or negative and their signs are changed so as to make all positive stress tension and all negative stress compression, as was done in the case of thick cylinders.

$$p = \frac{p_1 r_1^3 - p' r'^3}{r'^3 - r_1^3} - \frac{(p_1 - p') r'^3 r_1^3}{(r'^3 - r_1^3)} \frac{1}{r^3}. \quad . \quad . \quad (10)$$

$$h = \frac{p_1 r_1^3 - p' r'^3}{r'^3 - r_1^3} + \frac{(p_1 - p') r'^3 r_1^3}{2(r'^3 - r_1^3)} \frac{1}{r^3}. \quad . \quad . \quad (11)$$

These equations can be put in more convenient shape for computation by dividing all terms in the second members by r_1^3, which will give eqs. (10a) and (11a):

$$p = \frac{p_1 - p' \frac{r'^3}{r_1^3}}{\frac{r'^3}{r_1^3} - 1} - \frac{(p_1 - p') r'^3}{\frac{r'^3}{r_1^3} - 1} \frac{1}{r^3}. \quad . \quad . \quad . \quad (10a)$$

$$h = \frac{p_1 - p' \frac{r'^3}{r_1^3}}{\frac{r'^3}{r_1^3} - 1} + \frac{(p_1 - p') r'^3}{\frac{r'^3}{r_1^3} - 1} \frac{1}{r^3}. \quad . \quad . \quad . \quad (11a)$$

It is necessary to determine a thickness t of shell which will resist a given intensity of internal pressure. Eq. (11) shows that the circumferential tension h will be greatest when $r = r'$ in eq. (11). Making this substitution

$$2h(r'^3 - r_1^3) = 3p_1 r_1^3 - p'(2r'^3 + r_1^3).$$

Dividing by r'^3 and solving:

$$\frac{r_1^3}{r'^3}=\frac{2(h+p')}{2h-p'+3p_1}.$$

Hence there may be at once written:

$$r_1-r'=t=r'\sqrt[3]{\frac{2(h+p')}{2h-p'+3p_1}}-r'. \quad . \quad . \quad . \quad (12)$$

This value of t will give the thickness of material required so that the maximum intensity of circumferential tensile stress shall not exceed a prescribed value h at the interior surface of the sphere when the interior pressure is p' and the exterior pressure p_1, the latter being smaller than the former.

A similar treatment may be given to eq. (11) after making $r=r_1$ in order to determine a thickness t such that the circumferential compressive stress shall not exceed a given prescribed value when the exterior pressure p_1 exceeds the interior pressure p'.

In eq. (12) if $p'=2h+3p_1$, the value of t becomes infinitely great, showing that if the interior pressure reaches or exceeds the value indicated no thickness of shell whatever will prevent the circumferential or hoop tension exceeding the prescribed limit h.

If either internal or external pressure become zero, while the other has any assigned value, it is only necessary to make either p' or p_1 equal zero in all the preceding equations. Furthermore, it is a matter of indifference whether p' or p_1 is numerically greater in the application of any of the preceding equations except eq. (12).

Radial Displacement at any Point in the Spherical Shell

The general analysis of Art. 8 of Appendix I, gives an expression for the radial strain or displacement of the material at any point within the spherical shell. It has already been seen that no other displacement occurs, as all diametral sections of the shell remain plane for any degree of stress whatever. If this radial displacement or strain at any point be indicated by ρ, the analysis indicated shows that the value of this displacement will be given by eq. (13):

$$\rho = \frac{r}{4G}\left\{ \frac{2(1-2\bar{r})}{3(1+\bar{r})} \frac{p_1 - p'\dfrac{r'^3}{r_1{}^3}}{\dfrac{r'^3}{r_1{}^3} - 1} + \frac{(p_1 - p')r'^3}{\dfrac{r'^3}{r_1{}^3} - 1} \frac{r'^3}{r^3} \right\}. \quad . \quad (13)$$

Knowing the internal and external pressures to which the shell is subjected eq. (13) will give the value of the radial displacement of any indefinitely small piece of material at the distance r from the center. If $r = r_1$ the corresponding value of ρ given by eq. (13) will indicate the increase or decrease, as the case may be, of the external radius r_1; and if $r = r'$ the increase or decrease in length of the interior radius r' will result. In eq. (13) G is obviously the modulus of elasticity of the material for shear, while $\bar{r}$ is the ratio of lateral over direct strains.

CHAPTER V.

RESILIENCE.

Art. 43.—General Considerations.

The term resilience is applied to the quantity of work required to be expended in order to produce a given state of strain in a body. If a piece of material is subjected to tension, that state of strain will be simply the stretching of the piece or the amount of compression, if the piece is subjected to compressive stress. In precisely the same manner the resilience of a bent beam is the amount of work performed upon it by its load in producing deflection. There may also be the resilience of shearing or of torsion.

In the ordinary use of the expression, resilience refers to the amount of work expended within the elastic limit, whether of torsion, compression, or tension, but it may properly be extended in its meaning to include the total amount of work required to rupture the material under any one of the preceding conditions of stress. Elastic resilience may easily be computed by means of exact formulæ, but if the total work required to cause rupture in any case is desired, a graphical record of the total strains produced between the elastic limit and failure must be obtained by actual tests. In these articles the formulæ for elastic resilience only will be given; in other subsequent articles the method of computing the total resilience of failure will be illustrated by computations from actual strain records.

Art. 44.—The Elastic Resilience of Tension and Compression and of Flexure.

Let it be supposed that a piece of material whose length is L and the area of whose cross-section is A is either stretched or compressed by the weight or load W applied so as to increase gradually from zero to its full value. If E is the coefficient of elasticity, the elastic change of length will be $\frac{WL}{AE}$. The average force acting will be $\frac{1}{2}W$, hence the work performed in producing the strain will be

$$Resilience = \frac{W^2L}{2AE}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If $\frac{W}{A}$, the intensity of stress in the metal, be represented by t, eq. (1) may be written

$$Resilience = \tfrac{1}{2}At^2\frac{L}{E}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, eq. (2) may take the following form:

$$Resilience = \tfrac{1}{2}AE\frac{t^2}{E^2}L = \tfrac{1}{2}AEl^2L. \quad . \quad . \quad . \quad (3)$$

In eq. (3) the quantity $l^2 = \frac{t^2}{E^2}$ is obviously the square of the strain (stretch or compression) per unit of length.

If a bar of material 1 inch in length and 1 square inch in cross-section be considered, $A = 1$ and $L = 1$ must be inserted in the preceding equations, and there will result

$$\textit{Unit resilience} = \tfrac{1}{2}\frac{t^2}{E} = \tfrac{1}{2}El^2. \quad . \quad . \quad . \quad . \quad (4)$$

The quantity $\frac{t^2}{E} = El^2$ is called the "Modulus of Resilience." The expression is ordinarily employed when t is the greatest intensity of stress allowed in the bar.

The preceding equations are applicable whether the bar or piece of material is in tension or compression, the coefficient of elasticity E being used for either stress, while t represents the intensity of either tension or compression, as the case may be.

Inasmuch as the values of t and E are usually taken in reference to the square inch as the unit of area, it is generally convenient to take L in inches, although any other unit of length may be taken when multiplied by the proper numerical coefficient.

The Resilience of Bending or Flexure.

It has already been shown, in considering the common theory of flexure, as applied to the flexure or bending of beams, that the intensities of the stresses of tension and compression vary from point to point throughout the entire beam. In determining the elastic resilience of flexure, therefore, it is necessary to find the work performed in producing the varying strains corresponding to the stresses in the interior of the beam. The resilience due to the direct stresses of tension and compression will first be considered and then that due to the shearing stresses.

In order to obtain the expression for the work performed by the direct stresses of tension and compression in a beam bent by loads acting at right angles to its axis, a differential of the length, dL, is to be considered at any normal section in which the bending moment is M, the total length of span or beam being L. Let I be the moment of inertia of the normal section, A, about its neutral axis, and let k be the intensity of stress (usually the stress per square inch) at any point distant d from the axis about which I is taken. The elastic change produced in the indefinitely short length dL when the intensity k exists is $\frac{k}{E}dL$. If dA is an indefinitely small portion of the normal section, the average force or stress, either of tension or compression, acting through the small elastic change of length just given, can be written by the aid of a familiar equation of flexure as

$$\frac{1}{2}k.dA=\frac{Md}{2I}.dA. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the work performed in any normal section of the member, for which M remains unchanged, will be, since $\int k.dA.d=M$,

$$\int\frac{M}{2IE}kd.dA.dL=\frac{M^2}{2EI}dL. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The work performed throughout the entire piece will then be

$$\int\frac{M^2}{2EI}dL. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The integration indicated in eq. (7) is readily made in all ordinary cases by substituting the value of the bending moment M in terms of the variable horizontal ordinate or abscissa x and the load, it being remembered that dL is precisely the same as dx. If, for example, the beam is non-continuous, simply supported at each end and carries uniformly distributed load p per unit of length throughout the whole span, $M = \frac{p}{2}(Lx - x^2)$. By the insertion of this value of M in eq. (7), there will result

$$Resilience = \frac{1}{2EI}\int_0^L \frac{p^2x^2}{4}(L-x)^2dx = \frac{p^2L^5}{240EI} = \frac{W^2L^3}{240EI}, \quad (8)$$

W representing pL, the entire load on the beam.

This equation gives the value of the total work performed by the direct stresses of tension and compression in the interior of a simple beam uniformly loaded and supported at each end, under the assumption that the moment of inertia I of the cross-section is constant throughout the entire span.

If a single load W rests at the centre of the span, the reaction at each end being $\frac{W}{2}$, the value of the bending moment at any point will be $\frac{W}{2}x$. By inserting this value of M in eq. (7), there will result

$$Resilience = \frac{1}{2EI} \cdot 2\int_0^{\frac{L}{2}} \frac{W^2}{4}x^2dx = \frac{W^2L^3}{96EI}. \quad . \quad . \quad (9)$$

Similarly equations of the elastic resilience of the direct stress of tension and compression in beams loaded in any manner whatever may easily be written. In some cases like the last the deflection at the point of application of a single load may easily be determined. Let that deflection be represented by w; when a single load W rests at the centre of the span the work performed by this load in producing the deflection is $\frac{1}{2}Ww$. Hence that amount of work must be equal to the resilience given by eq. (9), and

$$w = \frac{WL^3}{48EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The Resilience Due to the Vertical or Transverse Shearing Stresses in a Bent Beam.

The work performed by the vertical shearing stresses in a bent beam of any shape of cross-section may readily be found. Let S be the total transverse shear in a normal section, I being the moment of inertia of the latter about its neutral axis, b the width or breadth (constant or variable) of the section, ϕ the unit shearing strain defined in Art. 2, d and d_1 the distances of the extreme fibres from the neutral axis, and G the coefficient of elasticity for shearing. By eq. (6) of Art. 15 the intensity s of the shear at any point in the section at the distance z from the neutral axis will be

$$s = \frac{S}{2I}(d^2 - z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Again, by eq. (3) of Art. 2,

$$\phi = \frac{s}{G} = \frac{S}{2GI}(d^2 - z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The amount of shearing stress on the indefinitely small portion of the section $b.dz$ will be $sb.dz$, and its path in performing the work will be ϕdx, x being the horizontal ordinate of the section of the beam from any convenient origin, as the end or the centre of the span, i.e., in this case the end of the span. The differential work performed in the section will be, by the aid of eqs. (11) and (12),

$$\tfrac{1}{2}(sbdz) . (\phi dx) = \frac{S^2 b}{8I^2 G}(d^2 - z^2)^2 dzdx. \quad . \quad . \quad (13)$$

In this equation it is easy to express the breadth b of the section in terms of z, whatever may be its shape, by the aid of the equation of the perimeter of the section. In all the ordinary and important cases of engineering practice involving this resilience of shearing the shape of the section is rectangular for which b is constant, and it will be so regarded in the following equations. Remembering that x and z are independent variables, and that the first integration will be made in reference to z, that integration will give

$$\frac{b}{8I^2 G}\int_0^l S^2 dx \checkmark \int_{-d_1}^{d} (d^2 - z^2)^2 dz \checkmark = \frac{b(d^5 + d_1^5)}{15 I^2 G}\int S^2 dx. \quad (14)$$

As the section is taken to be rectangular in outline, with the breadth b and depth h, $d = d_1 = \frac{h}{2}$ and $I = \frac{bh^3}{12}$. Eq. (14) will then become

$$Resilience = \frac{3}{5bhG}\int S^2 dx. \quad . \quad . \quad . \quad . \quad (15)$$

The total transverse shear S will have varying values depending upon the amount of loading on the beam and its distribution, i.e., in general it will vary with x, and when not constant it must be expressed in terms of that variable before the remaining integration can be made.

If a single weight W rests on the beam at the distance of l' from one end where the reaction is R', and at the distance l_1 from the other end where the reaction is R_1, the shear S will be constant for each of the segments into which the point of loading divides the span; in one of those segments $S=R'$, and in the other $S=R_1$. The complete integration of eq. (15) will be, therefore,

$$Resilience=\frac{3}{5bhG}(R'^2l'+R_1{}^2l_1).$$

If there be substituted in the parentheses of the second member of the preceding equation the values $R'=W\frac{l_1}{l}$ and $R_1=W\frac{l'}{l}$, there will result

$$Resilience=\frac{3}{5bhG}\,\frac{l_1l'}{l}W^2. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If the weight W rest at the centre of the span $l_1=l'=\frac{l}{2}$ and

$$Resilience=\frac{3}{20Gbh}W^2l. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Eq. (17) affords a simple method of finding the deflection w_1 of the point of loading due to the transverse shear. As the weight W is supposed to be gradually applied the expended work $\frac{1}{2}Ww_1$ must be equal to the shearing resilience given in eq. (17). Hence

$$w_1=\frac{3}{10G}\,\frac{Wl}{bh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When a non-continuous beam simply supported at each end carries a uniform load over the entire span, it has been shown in Art. 22, eq. (7), that the transverse shear at any section is equal to the load between the centre of span and that section. If, therefore, the origin of x be taken at the centre of span and if p represents the load per unit of length of the beam, $S = px$. By substituting this value of S in eq. (15), and remembering that twice the integral must be taken for the whole beam,

$$Resilience = \frac{3}{5Gbh} \cdot 2\int_0^{\frac{l}{2}} p^2x^2dx = \frac{p^2l^3}{20Gbh} = \frac{W^2l}{20Gbh}. \quad (19)$$

The shearing resilience, therefore, in a non-continuous beam carrying a uniform load is only one third as much as that due to the same load concentrated at the centre of the span.

If, as is usual, G is expressed in pounds per square inch the unit for l, b, and h will be the linear inch.

Other modes of loading than those taken can be treated in precisely the same general manner.

As the intensity of the longitudinal shear at any point of a beam is the same as that of the transverse shear, the total work of the longitudinal shear throughout the beam is the same as the work of the transverse shear. The total work of the shearing stresses in a beam is therefore composed of those two equal parts.

The Total Resilience Due to Both Direct and Shearing Stresses.

The general expression for the total resilience of a bent beam due to both shearing and direct stresses will be the sum of the second members of eqs. (7) and (13), expressed by the following equation:

$$Total\ resilience = \int \frac{M^2}{2EI}dL + \int\int \frac{S^2b}{8I^2G}(d^2 - z^2)^2dzdx.$$

Or, by eqs. (7) and (15), since $dL=dx$,

$$Total\ resilience = \int \frac{M^2}{2EI}dx + \frac{3}{5bhG}\int S^2dx. \quad . \quad (20)$$

By the aid of eqs. (8) and (19) the total resilience for a simple non-continuous beam may be as follows:

If the uniform load $pl=W$,

$$Total\ resilience = W^2\left(\frac{l^3}{240EI}+\frac{l}{20Gbh}\right). \quad . \quad (21)$$

For the same beam carrying a single load W at the centre, by eqs. (9) and (17)

$$Total\ resilience = W^2\left(\frac{l^3}{96EI}+\frac{3l}{20Gbh}\right). \quad . \quad . \quad (22)$$

As has been explained, the last two equations are applicable to beams with rectangular sections only.

In a similar manner the total deflection of a beam supported at each end and loaded with a single weight W at the centre of the span, due to bending and flexure, will be found by the sum of the two expressions given in eqs. (10) and (18):

$$w+w' = W\left(\frac{l^3}{48EI}+\frac{3l}{10Gbh}\right). \quad . \quad . \quad . \quad (23)$$

Art. 45.—Resilience of Torsion.

The work expended in producing elastic strains of torsion constitutes the resilience of torsion and is a special case of shearing resilience. The twisting moment which produces the angle of torsion α is given by eq. (16) of

Art. 37 and is $M=G\alpha I_p$. When the piece twisted has the length l the total angle of torsion is αl and the differential amount of work performed by the moment M in producing the indefinitely small twist $d(\alpha l)=l.d\alpha$ is $Ml.d\alpha$. Hence

$$Resilience=\int Mld\alpha=GlI_p\int_0^{\alpha_1}\alpha.d\alpha=GlI_p\frac{\alpha_1^{\ 2}}{2}. \quad . \quad (1)$$

If P and e are the force and lever-arm of the twisting couple, eq. (18) of Art. 37 shows that

$$\alpha_1=\frac{Pe}{GI_p}.$$

Substituting this value of α_1 in eq. (1),

$$Resilience=\frac{P^2e^2l}{2GI_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the normal section of the piece is circular $I_p=\frac{\pi r^4}{2}$. Hence, for a shaft with circular section,

$$Resilience=\frac{P^2e^2l}{\pi Gr^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the section of the shaft is a square, $I_p=\frac{b^4}{6}$, b being the side of the square. Hence, for a square section,

$$Resilience=\frac{3P^2e^2l}{Gb^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In some cases shafts are subjected to combined torsion and bending. In such cases, if it is desired to compute the total elastic resilience it is only necessary to take the sum of the two resiliences, each found as if existing independently of the other.

The resilience of torsion beyond the elastic limit or between the elastic limit and the ultimate resistance must be determined, as in all cases of distortion beyond the elastic limit, from an actual strain record, as given by the testing machine when the piece is strained up to any given degree of permanent stretch or to rupture.

Art. 46.—Suddenly Applied Loads.

A load is considered suddenly applied when its full amount acts instantly upon any piece of material loaded by it. In the preceding articles relating to resilience the loads are treated as being gradually increased from zero to their full values. In such cases the amount of external loading at any instant is supposed to be equal only to the internal stress or stresses opposing it, so that the work performed is equivalent to one half the total load multiplied by the total resulting strain. When the loads are suddenly applied, on the other hand, the internal stresses produced are exactly equal to the external forces only when the strains corresponding to the latter are reached, and the work performed up to that point is just double the work expended when the loads are gradually applied. It follows from this last consideration that the strains produced by the suddenly applied loads will be double those found under gradual application. Inasmuch as the elastic strains are proportional to the corresponding stresses, it further follows that the stresses produced by suddenly applied loads will be double in intensity those which are produced by the same loads gradually applied.

The work expended by a suddenly applied load up to the point of strain corresponding to its amount being double the work performed by the internal stresses, the total stress induced in the material at the limit of the final strain produced by such a load will be double the amount of the latter. The internal stresses in the piece will, therefore, cause it to recover from its strained condition and vibrations will result, the treatment of which constitutes an important branch of the theory of elasticity in solid bodies. Some general features of that treatment will be given in Art. 12, App. I, but as they are seldom used in engineering practice they will not be considered here. It is only important at this point to note carefully the distinction between the effects of a given load gradually applied and suddenly applied, the strains and stresses in the latter condition being double those in the former.

Again, it is also important to distinguish between loads suddenly applied, and shocks, as they are called in engineering practice. A shock is produced when the load falls freely before acting upon a piece of material sustaining it. The cause of shock, therefore, is a suddenly applied load with the effect of a free fall of the latter superimposed. These matters must be carefully taken into account and allowed for in such structures as bridges carrying rapidly moving trains, and those allowances are incorporated in the provisions of specifications covering bridge construction.

PROBLEMS FOR CHAPTER V.

Problem 1.—A 6-inch by 1.75-inch steel eye-bar 48 feet long is subjected to a stress of 117,500 pounds. If that load is gradually applied what is the work performed in the total length of the bar, if $E=30,000,000$ pounds? Also what is the unit resilience?

$t = \frac{117,500}{10.5} = 11,190$. $L = 48 \times 12 = 576$ inches. Eq. (2) of Art. 44 then gives

$$\text{Resilience} = \text{work performed} = \frac{10.5 \times (11,190)^2 \times 576}{2 \times 30,000,000}$$

$$= 12,621 \text{ in.-lbs.}$$

Eq. (4) of Art. 44 gives

$$\text{Unit resilience} = \frac{(11,190)^2}{2 \times 30,000,000} = 2.09 \text{ in.-lbs.}$$

Problem 2.—A cast-iron column 18 feet long having an area of cross-section of 40.8 sq. in. carries a load of 245,000 pounds. If the coefficient of elasticity E is 14,000,000 pounds, how much work is performed in compressing the column if the load is gradually applied.

Problem 3.—A 30-pound 10-inch rolled steel I beam carries a uniform load of 1000 pounds per linear foot in addition to its own weight with a span of 16 feet. What will be the resilience or work performed in the material of the beam under the gradual application of that total load of 1030 pounds per linear foot, the moment of inertia I of the beam being 134.2 and $E = 30,000,000$ pounds? Eq. (8) of Art. 44 is to be used, in which L is 192 inches. Incidentally, what will be the greatest intensity of stress, k, in the extreme fibres?

Ans. Resilience = 1987 in.-lbs; $k = 15,000$ lbs. per square inch.

Problem 4.—In Problem 3 if the thickness of the web of the 10-inch rolled beam is .5 inch, find the resilience of the vertical or transverse shearing stresses in the beam, the coefficient of shearing elasticity, G, being taken at 12,000,000 pounds. The remaining data are $l = 192$ inches; $h = 10$ inches; $b = 0.5$ inch, and $W = 16,480$ pounds, and they are to be used in eq. (19) of Art. 44.

Problem 5.—A round bar of steel $2\frac{7}{8}$ inches in diameter is twisted by a force of 2100 pounds acting with a lever-arm of 17 inches. Two sections 25 ft. apart are turned 0.185 inch in reference to each other, i.e., the total strain of torsion for a length of bar of 25 feet has that value. Find the *total angle of torsion*, the *angle of torsion* and the coefficient of elasticity, G, for shearing (i.e., for torsion).

Ans. $\alpha = 0.00043$; $\alpha l = 0.129$; and $G = 13{,}000{,}000$ lbs.

Problem 6.—The greatest permitted working intensity of torsive shearing is 8000 pounds per square inch. Design a steel shaft to carry a twisting moment produced by a force of 1900 pounds, acting with a lever-arm of 84 inches. If the coefficient of elasticity for shearing is 12,000,000 pounds, what will be the angle of torsion? Also what will be the total angle of torsion and total strain of torsion for a length of shaft of 13 feet?

Problem 7.—In Problems 5 and 6 find the work performed in twisting the two steel shafts, i.e., the resilience for 25 feet length in the one case and 13 feet in the other. Use equations of Art. 45.

Problem 8.—In Problem 5 suppose the load suddenly applied, what will be the resulting resilience and greatest intensity of extreme fibre stress?

CHAPTER VI.

COMBINED STRESS CONDITIONS.

Art. 47.—Combined Bending and Torsion.

PROBABLY the most important case of combined bending or flexure and torsion is that of the ordinary crank-shaft represented in Fig. 1.

The centre of the thrust of a connecting-rod is at A, on the crank-pin journal against which the connecting-rod bears. The centre of the shaft-bearing is at B. If the thrust at A is represented by P, then the actual resultant moment about the centre of the bearing B will be $P \times AB$. The problem is to determine the maximum stresses developed by this resultant moment in the section of the shaft at B. Two methods may be employed in both of which the resultant moment of P multiplied by the lever arm AB is resolved into its two components, one of which is the ordinary bending moment represented by $M = P \times CB$, and the other is the twisting moment $M' = P \times AC$. The latter produces torsion in the journal at B and the former produces pure flexure or bending at the same section.

Let CB be represented by l while e represents AC. The moment of pure bending at B will be

$$M = Pl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The twisting moment producing pure torsion will be

$$M' = Pe. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If d represents the distance of the most remote fibre in the section B from the neutral axis of the latter, and if k is the greatest intensity of bending stress at the distance d from the neutral axis, while I is the moment of inertia of the normal section of the shaft at B about the same neutral axis, the following will be the value of k:

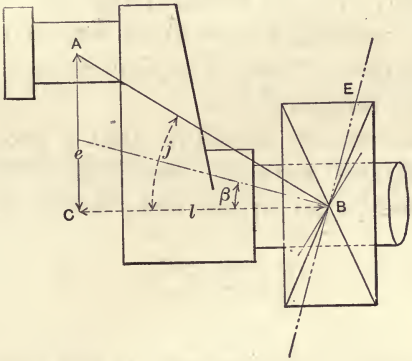


FIG. 1.

$$k = \frac{Md}{I} = \frac{Pld}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, if T is the greatest intensity of torsional shear in the normal section of the shaft at B, at the greatest distance r, in the perimeter, from the centre of gravity or the centroid of the same section, the value of the maximum intensity T will be

$$T = \frac{M'r}{I_p} = \frac{Per}{I_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In eq. (4) I_p is the polar moment of inertia of the normal section at B.

First Method.

In this method it is only necessary to consider the intensities k given by eq. (3) and T given by eq. (4), the greatest allowed working stresses of direct tension and of shearing respectively. k would have the value of the greatest tensile working stress of the material of the shaft for the reason that if tested to failure the shaft would yield first on the tension side.

It being understood, therefore, that k and T represent the greatest allowed working intensities of stress, usually expressed in pounds per square inch, eq. (3) will give

$$\frac{I}{d}=\frac{M}{k}=\frac{Pl}{k}. \quad \ldots \ldots \quad (5)$$

Under the same conditions eq. (4) will give

$$\frac{I_p}{r}=\frac{M'}{T}=\frac{Pe}{T}. \quad \ldots \ldots \quad (6)$$

For the circular section

$$I=\frac{\pi r^4}{4} \quad \text{and} \quad I_p=\frac{\pi r^4}{2}. \quad \ldots \ldots \quad (7)$$

For a square section

$$I=\frac{b^4}{12} \quad \text{and} \quad I_p=\frac{b^4}{6}, \quad \ldots \ldots \quad (8)$$

b being the side of the square. In eq. (5) for a circular section $d=r$ and for a square section $d=\frac{b}{\sqrt{2}}$. In eq. (6), $r=r$ for the circular section, but for the square section

$r = \frac{b}{\sqrt{2}}$. Making those substitutions in eqs. (5) and (6) for the circular section, there will result, D being the diameter of the shaft,

$$\left.\begin{aligned} &\text{For bending} \ldots r = \frac{D}{2} = \sqrt[3]{\frac{4Pl}{\pi k}} = 1.08\sqrt[3]{\frac{Pl}{k}} \\ &\text{For torsion} \ldots . r = \frac{D}{2} = \sqrt[3]{\frac{2Pe}{\pi T}} = .86\sqrt[3]{\frac{Pe}{T}} \end{aligned}\right\} \quad . \quad (9)$$

In the practical use of eqs. (9) that one of the two values of r should be taken which is the greatest. This will insure that both the direct stress of tension and the shearing stress shall not exceed the prescribed values of k and T.

The substitution of the values of I and I_p for the square section in eqs. (5) and (6) will give, remembering that d and r are each one half the diagonal of the square,

$$\left.\begin{aligned} &\text{For bending} \ldots b = \sqrt[3]{\frac{6\sqrt{2}Pl}{k}} = 2.04\sqrt[3]{\frac{Pl}{k}} \\ &\text{For torsion} \ldots . b = 1.62\sqrt[3]{\frac{Pe}{T}} \end{aligned}\right\} \quad . \quad (10)$$

In eq. (10), also, the greatest value of b given by the application of the two formulæ is to be taken, so that, as in the case of the circular section, neither of the two intensities k and T shall exceed the values prescribed for them.

This method involves only the consideration of the simple formulæ of the common theories of flexure and torsion.

Second Method.

The second method of treatment of this case of the crank-shaft consists in determining the greatest intensity of the direct stress of tension in the section B of the shaft at the journal-bearing. This resultant maximum intensity is produced by the combination of the same component moments, $M = Pl$ and $M' = Pe$, as in the preceding method. With the sections of shafting always employed the maximum intensity of bending stress k and the maximum intensity of torsional shear T exist at the same point and on the same plane, i.e., the plane of normal section of the shaft. The existence of the shear T on the normal section at the distance r from its centre of gravity carries with it the same intensity of shear at the same point on a longitudinal plane passing through the axis of the shafting. At the point considered, therefore, on two indefinitely small planes at right angles to each other, one normal to the axis of the shaft and the other parallel to it, there exist the direct intensity of tension k on the first, and the intensity of shear T on the second. The problem is to determine at the same point the greatest intensity of the direct stress of tension on any plane whatever, and the angle β between the direction of that stress and the axis of the shaft. Reference may best be made to the general formulæ of internal stresses in a solid body for its solution, and those are eqs. (8) and (9) of Art. 8. Those equations are adapted to this case by making $p_x = k$, $p_{xy} = T$, $\tan \alpha = \tan \beta$, and $p = t$, the latter quantity being the greatest intensity of tension desired. These substitutions give the following two equations:

$$t = \sqrt{\frac{k^2}{4} + T^2} + \frac{k}{2}. \qquad \ldots \ldots \quad (11)$$

$$\tan 2\beta = \frac{2T}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Eq. (11) gives the greatest intensity of direct tension in the shaft in terms of known stresses.

By eq. (12) the position of the plane or section of the shaft on which the maximum intensity t exists may at once be found. Inasmuch as β is the angle between the direction of the stress t and the axis of the shaft, the angle between the plane on which t acts and the axis of the shaft will be $90° + \beta$.

Under this method of treatment it would be necessary to design the shaft so that t should not exceed the greatest prescribed tensile working stress for the material employed.

The greatest intensity of compressive stress in the shaft would be found by giving the negative sign to the radical in the second member of eq. (11).

The preceding formulæ have been established in a manner to make them applicable to any form of shaft section or any values of k and T. It is only necessary to insert in those formulæ any intensities of those stresses that may exist. If, for example, it were considered desirable to add the shear $\frac{P}{\pi r^2}$ due to the thrust P to the torsional shear it would only be necessary to take $T + \frac{P}{\pi r^2}$ for T wherever the latter quantity occurs.

If a shaft is circular in section, as is almost universally the case, so that $I_p = 2I$, and if the shearing effect of P in the section at B, Fig. 1, be omitted, useful and extremely simple relations may be deduced. In that case $D = 2r$, being the diameter of the shaft, and j the angle ABC

of Fig. 1, M as before being the resultant moment, or $M=P\times AB$:

$$k=\frac{rM\cos j}{I}\quad\text{and}\quad T=\frac{rM\sin j}{2I}.\quad . \quad . \quad (13)$$

By the substitution of these values in eq. (11),

$$t=\frac{rM}{2I}(1+\cos j)=\frac{5.1M}{D^3}(1+\cos j).\quad . \quad . \quad (14)$$

Hence

$$D=1.72\sqrt[3]{\frac{M}{t}(1+\cos j)}.\quad . \quad . \quad . \quad . \quad (15)$$

Eq. (14) gives, by the aid of the first of eqs. (13),

$$t=\frac{rM}{2I}(1+\cos j)=\frac{k}{2}(\sec j+1).\quad . \quad . \quad . \quad (16)$$

The second of eqs. (13) gives, after substituting the value of $\frac{r}{2I}=\frac{16}{\pi D^3}$,

$$T=\frac{5.1M\sin j}{D^3}.\quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The substitution of the values of T and k from eqs. (13) in eq. (12) gives

$$\tan 2\beta = \frac{2T}{k} = \tan j; \quad \therefore \beta = \tfrac{1}{2}j. \quad . \quad . \quad . \quad (18)$$

This last set of results relating to circular shafts will, in all ordinary cases, supply everything required for the operations of design or of investigations regarding conditions of stress in existing shafts.

Eqs. (13), first of (14), (16), and (18) apply as they stand to square shafts.

The first method involves simpler considerations than the second, not only analytically, but also in respect to empirical quantities required to be used. The test pieces from which the ultimate resistance of the material is determined are always taken parallel to the axis of the shaft, but the greatest intensity of stress t found in the second method has a direction inclined to that axis by the angle β. In general, therefore, it will probably be found more practicable to use the first method rather than the second.

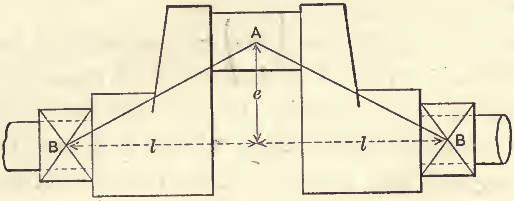


FIG. 2.

In the case of the double crank-shaft shown in Fig. 2, it is only necessary to treat each half precisely as if it were the single crank-arm in Fig. 1.

Art. 48.—Combined Bending and Direct Stress.

There are a considerable number of practical problems of combined flexure and direct stress of sufficient importance to merit careful examination, and among them is the flexure of long columns treated in Art. 24. In this place the particular cases to be considered are those in which the bending is produced by a uniform load at right angles to the axis of the member, or by eccentricity of longitudinal loading, the direct stress (or external force) being applied in a direction parallel to the same axis. Lower chord eye-bars and other horizontal or inclined chord members of pin bridges belong to this class.

Let M_1 represent the bending moment in the member at that section where the deflection is greatest, produced by loading at right angles to the member's axis or by eccentricity in the application of the longitudinal loading; let w' represent the greatest deflection resulting from the total bending moment and direct stress; also, let P be the total direct stress acting upon the member whose length is l, while k represents the greatest intensity of stress due to bending alone and at the distance d of the most remote fibre from the neutral axis of the section at which the deflection w' is found. Finally, let A be the area of cross-section of the member which, together with the moment of inertia I, is supposed to be constant throughout the entire length; and let $q=\frac{P}{A}$, the intensity of uniform stress in the member due to the direct stress or force P.

The resultant maximum bending moment in the member will then be

$$M=M_1 \pm Pw'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If P is tension it will tend to pull the member straight, thus producing a moment opposite to M_1. In the second member of eq. (1), therefore, the negative sign is to be used for a member in tension and the positive sign for a member in compression.

The greatest resultant intensity of stress, t, in the member will then take the value

$$t=\frac{P}{A}+\frac{Md}{I}=\frac{1}{A}\left(P+\frac{Md}{r^2}\right). \quad . \quad . \quad . \quad . \quad (2)$$

The quantity r is the radius of gyration, so that

$$I=Ar^2.$$

When the intensity t is prescribed, the required area of section A is

$$A=\frac{1}{t}\left(P+\frac{Md}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These equations are perfectly general and may be applied to all cases of combined bending and direct stress.

Art. 49.—The Eye-bar Subjected to Bending by Its Own Weight or Other Vertical Loading.

Let Fig. 1 represent a lower chord eye-bar of a pin-connected bridge with the length l and carrying the total tension P. The depth of the bar is h and the thickness b, so that the area of the normal section is bh. The bar acts as a beam carrying its own weight as a uniform load over the span l. That load deflects the bar as a beam while the direct stress of tension (P) decreases that deflection by tending to pull the bar straight. The problem is to determine the greatest stress in the bar and incidentally its centre deflection.

There are several methods of procedure. The first and simplest method is approximate in its results, although sufficiently close for some purposes. It consists in treating the bending and direct stresses as existing independently, so that results are obtained by simply adding the bending to the direct intensities. This method will be treated first.

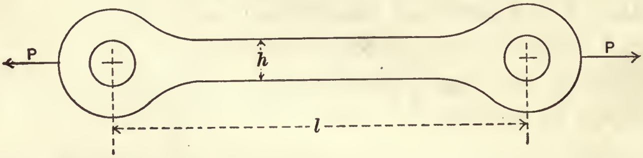


FIG. 1.

The more exact method consists in recognizing the bending moment as the resultant of those due to the transverse load acting on the bar as a simply supported beam, and to the direct stress P acting with the greatest deflection as its lever-arm.

Approximate Method.

Although reference will be made to Fig. 1, the formulæ as written will be equally applicable to compression members in which P would be the total force of compression.

If the total weight of the bar or compression member is W, and if I is the moment of inertia of its cross-section about the neutral axis, while k is the greatest intensity of bending stress at the distance d from the same axis, the theory of flexure gives

$$M_1=\frac{Wl}{8}=\frac{kI}{d}; \therefore \ k=\frac{Wld}{8I}. \quad . \quad . \quad . \quad . \quad (1)$$

If the area of cross-section is represented by A, while the radius of gyration is r, $I=Ar^2$. Again, the quantity $I \div d$ is called the "section modulus," and tabulated values of it for rolled sections may be found in hand-books. Let m be that modulus, then eq. (1) may take the form

$$k=\frac{Wld}{8Ar^2}=\frac{Wl}{8m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The intensity of direct tension is

$$q=\frac{P}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Obviously k will be tension on the lower side of the bar or other member and compression on the upper side. The greatest intensity of stress in the piece will be the sum of q and k. Eqs. (2) and (3) will, therefore, give the value of that greatest intensity, t, of stress as follows:

$$t=q+k=\frac{1}{A}\left(P+\frac{Wld}{8r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the greatest value of t is prescribed, the required area of section, A, can be at once written from eq. (4)

$$A=\frac{1}{t}\left(P+\frac{Wld}{8r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In the case of an eye-bar with the cross-section bh, $d=\frac{h}{2}$ and $\frac{d}{r^2}=\frac{6}{h}$. Hence

$$t=\frac{1}{bh}\left(P+\frac{3Wl}{4h}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$bh=\frac{1}{t}\left(P+\frac{3Wl}{4h}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the bar carries any other uniform load than its own, it is only necessary to make W represent the total uniform load, including the weight of the bar itself.

Finally the direct force P may act with the eccentricity e. In this case the moment Pe produces uniform bending throughout the length of the bar, and it is only needful to write $\left(\frac{Wl}{8} \pm Pe\right)$ for $\frac{Wl}{8}$ in the preceding formulæ, the double sign showing that Pe may act either with or against the moment of the uniform load.

The formulæ of this article are not sufficiently exact for the usual cases of engineering practice.

Art. 50.—The Approximate Method Ordinarily Employed.

The method commonly employed in practical work for the treatment of compound bending and direct stress is a much closer approximation than the preceding method, although not exact. Its chief feature is the introduction of the bending moment produced by the direct or longitudinal force multiplied by the actual maximum deflection. In the same manner the moment due to the eccentricity of the line of action of that force is introduced wherever necessary.

Eq. (6a) of Art. 27 gives the following expression for the deflection w' due to pure bending and in terms of the greatest intensity of bending stress k, a being a constant depending, among other things, upon the distribution of loading:

$$w' = a\frac{kl^2}{Ed}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the deflection as given in eq. (1) be placed equal to each of the two parts of the deflection given in eq. (21)

of Art. 28, it will be found for a beam simply supported at each end and loaded uniformly, that $a = \frac{5}{48}$, and for the same beam loaded by a single weight only at the centre of the span, $a = \frac{4}{48}$. The cases which occur in practice conform nearly to that of a load uniformly distributed over the length l. Hence for such a beam there is ordinarily taken

$$w' = \frac{kl^2}{10Ed}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The moment produced by the direct force or stress P acting with the lever arm w' will have the opposite sign to that of M_1 (the moment due to transverse loading or to eccentricity), if the member is in tension, but if the member is in compression those two moments will have the same sign. The resultant equation of moments may, therefore, be written

$$M = \frac{kI}{d} = M_1 \pm Pw'. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As stated, the plus sign is to be used for a compression member and the negative sign for a tension member.

If the value of w', given by eq. (2), be substituted in eq. (3), the following value of k will result:

$$k = \frac{M_1 d}{I \mp \frac{Pl^2}{10E}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In eq. (4) the plus sign is to be used for tension members and the minus sign for compression members. This equation is general and adapted to all forms of cross-section under the conditions virtually assumed. Although not explicitly stated, it is essentially assumed that the ends of the member remain absolutely fixed in distance apart. This is frequently not the case, especially in the lower chord eye-bar of a pin-connected bridge subjected to direct tension and to bending due to its own weight, the bar usually being horizontal.

If the ends of the beam or member, uniformly loaded, are fixed, $a = \frac{1}{16}$, when k is the greatest intensity of bending stress at the mid-point of the member, or $\frac{1}{32}$ if k is the intensity of the bending stress at the fixed ends. One of those values (usually $\frac{1}{32}$) is to be substituted therefore for $\frac{1}{10}$ in the formulæ which follow when the fixed-end condition exists.

The resultant maximum intensity of stress t in the member will obviously be

$$t = k + q, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in which equation q is the uniform intensity $P \div A$.

Eq. (4) will be immediately applicable to any particular case by substituting in it the values of I and M_1 for that special case.

If the case of the lower chord eye-bar mentioned in a preceding paragraph be considered, the total weight of the bar being W, while b and h represent its thickness and depth respectively, $I = \frac{bh^3}{12}$ and $M_1 = \frac{Wl}{8}$. These values substituted in eq. (5) will give the desired value of the resultant intensity, as follows:

$$t = \frac{P}{bh} + \frac{W\frac{l}{h}}{\frac{4}{3}bh + \frac{8Pl^2}{5Eh^2}}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Eq. (6) gives the value of the maximum intensity of tension in the extreme lower fibres of the eye-bar when subjected to the total direct tension P and to the bending due to its own weight.

The greatest intensity of bending stress in the bar is evidently the second term of the second member of eq. (6), and it has the following value if the weight of the bar per unit of length is $\frac{W}{l}=g$, or if the weight of a cubic unit of the metal is i:

$$k=\frac{\frac{5}{8}\frac{gE}{b}}{\frac{5}{6}E\frac{h^2}{l^2}+\frac{P}{bh}}=\frac{\frac{5}{8}ihE}{\frac{5}{6}E\frac{h^2}{l^2}+q}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It is frequently important to observe what depth of bar with a constant area of cross-section, subjected to a prescribed working stress, will give the maximum bending stress due to its own weight when the length is fixed. That depth can readily be determined by taking the first derivative of k, as given by eq. (7), with h as the variable. By performing that operation and placing $\frac{dk}{dh}=0$, there will at once result

$$h=\frac{l}{\sqrt{\frac{5}{6}E}}\sqrt{q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The value of h resulting from an application of eq. (8) gives the depth of bar which, with a given value of l, will under the conditions of the case yield the greatest intensity of bending stress k; it indicates, therefore, a limit of depth to be avoided as far as practicable.

Steel is the usual structural material for eye-bars for which E may be taken at 29,000,000. For this value of E, h will become, by eq. (8),

$$h=\frac{l}{4900}\sqrt{q}.$$

In this equation $q=\frac{P}{A}$ is the intensity of uniform stress in the bar, or the "working stress."

By placing the value of h, as given by eq. (8), in the value of k, eq. (7), there will result the maximum possible bending stress in a bar of given length l and given area of cross-section A:

$$k=\tfrac{1}{3}i\sqrt{E}\frac{l}{\sqrt{q}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If $E=29{,}000{,}000$ and $i=.286$ lb. per cubic inch for steel, eq. (9) will take the value, for the corresponding values of h in the equation preceding eq. (9),

$$k=\frac{510l}{\sqrt{q}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The following table shows at a glance the greatest possible fibre stresses in eye-bars of different lengths and depths when the working tensile stresses in pounds per square inch are those given in the extreme left-hand column of the table:

Working Tensile Stresses in Pounds per Square Inch.	Length of Eye-bars in Feet.											
	15		20		25		30		35		40	
	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.	Depth.	Fibre Stress.
	Ins.	Lbs. p. Sq. In.	Ins.	Lbs. p Sq. In.	Ins.	Lbs. p. Sq. In	Ins.	Lbs. p. Sq. In.	Ins.	Lbs. p. Sq. In.	Ins.	Lbs. p. Sq. In.
8,000	3.3	1030	4.4	1370	5.5	1710	6.6	2050	7.7	2400	8.8	2740
10,000	3.7	920	4.9	1220	6.1	1530	7.3	1840	8.6	2140	9.8	2450
12.000	4.0	840	5.4	1120	6.7	1400	8.0	1680	9.4	1960	10.7	2240
14,000	4.3	780	5.8	1030	7.2	1290	8.7	1550	10.1	1810	11.6	2070
16,000	4.6	730	6.2	970	7.7	1210	9.3	1450	10.8	1690	12.4	1940

In using the preceding formulæ it is to be remembered that the ordinary unit of length, as well as the unit of cross-section, is the linear inch, and that the weight i of a cubic unit will then be the weight of a cubic inch. This investigation will yield results sufficiently accurate for all the usual cases of engineering practice, although it does not provide for the straightening effect of the pull P, except as producing a bending moment opposite to that of the uniformly distributed load W.

Allowance for any other distributed loading than the weight of the bar itself, and for any eccentricity of the line of action of P that may exist, are made precisely as explained in the two paragraphs following eq. (7) of Art. 49.

Art. 51.—Exact Method of Treating Combined Bending and Direct Stress.

In this method of finding the results of direct stress combined with bending it is necessary to determine an expression for the centre deflection of the bar, or compression member, considered as simply supported at each end. As the line of action of the direct stress P is supposed to coincide with the original centre line or axis of the bar, if g is the weight per linear unit of the latter, the bending moment M_1 in the second member of eq. (1), Art. 48, becomes

$$M_1 = \frac{g}{2}x(l-x).$$

As this case is one in which P is tension the general eq. (1) of Art. 48 will take the following form by the aid of eq. (7) of Art. 14:

$$\frac{d^2w}{dx^2} = \frac{1}{EI}\left(\frac{g}{2}x(l-x) - Pw'\right). \quad . \quad . \quad . \quad . \quad (1)$$

In this equation $g = \frac{W}{l}$ is the weight per linear inch, or other unit, of the bar or member producing a bending moment opposite to that induced by the direct stress P acting with the lever-arm w'. The integration indicated in eq. (1) may be completed, but as it is not a simple integration it will not be made here. As the greatest bending stress is found at the centre of span the centre deflection only is needed and a different procedure may be followed.

Let w_1 represent the centre deflection of the member considered, a beam simply supported at each end and carrying its own weight only, or any other total weight W uniformly distributed. It is necessary to use the expression for the work performed, or resilience of the beam in being deflected at the centre by the amount w_1. Eq. (8) of Art. 44 gives that resilience as

$$Resilience = \frac{W^2 l^3}{240EI}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In producing the centre deflection w_1 the centre of gravity of the weight W will descend through the distance w_0 found by placing Ww_0 equal to the resilience given in eq. (2). Hence

$$w_0 = \frac{Wl^3}{240EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also, since by eq. (26) of Art. 28 $w_1 = \frac{5Wl^3}{384EI}$,

$$\frac{w_0}{w_1} = \frac{8}{25}; \quad \therefore \ w_0 = \tfrac{8}{25} w_1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the resilience becomes

$$Resilience = W \tfrac{8}{25} w_1. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If the value of W in terms of w_1 be taken from eq. (26) of Art. 28 and substituted in eq. (5),

$$Resilience = \frac{3072EI}{125l^3}w_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence *the resilience of a bent beam varies as the square of the centre deflection.*

If the actual centre deflection of the bar or member considered be w', the resilience of the beam when deflected to that extent will be

$$Resilience = \left(\frac{w'}{w_1}\right)^2 \frac{W^2l^3}{240EI}. \quad . \quad . \quad . \quad . \quad (7)$$

The curvature of the bar or member being slight, the lengths (equal to each other) of the neutral surface with the deflections w' and w_1 will be, if l' and l_1 are the corresponding lengths of span or horizontal projections of the neutral surface,

$$l'\left(1 + \frac{8}{3}\frac{w'^2}{l'^2}\right) = l_1\left(1 + \frac{8}{3}\frac{w_1^2}{l_1^2}\right). \quad . \quad . \quad . \quad (8)$$

Hence

$$l' - l_1 = \frac{8}{3}\left(\frac{w_1^2}{l_1} - \frac{w'^2}{l'}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The difference $l' - l_1$ represents the movement toward or from each other of the two ends of the bar or member under the action of the direct stress or force P.

In the case of the eye-bar, the pull of the force P removes a part of the deflection w_1, and in so doing performs work in aiding to lift the weight W of the bar, the remainder of the work of lifting W being performed by the elastic efforts of the bar to straighten itself from the deflection

w_1 to w', the latter portion of the work being represented by the quantity $\frac{W^2l^3}{240EI}\left(1-\frac{w'^2}{w_1^2}\right)$. Hence the following equation of work may be written,

$$\frac{P}{2}\frac{8}{3}\left(\frac{w_1^2-w'^2}{l}\right)+\frac{W^2l^3}{240EI}\left(1-\frac{w'^2}{w_1^2}\right)=W\frac{8}{25}(w_1-w'). \quad (10)$$

The conditions under which the work represented by eq. (10) is performed are such that either (w_1-w') or (w_1+w') may be written in the second member. The resulting numerical value of w' will be the same in both cases but affected by different signs. As the equation is written the numerical value of w' will be negative.

In eq. (10) there is taken $l'=l_1=l$, the length of panel, which may be done with essential accuracy.

Dividing both sides of eq. (10) by (w_1-w') and solving for w',

$$w'=-\frac{w_1}{1+\frac{6}{25}\frac{Wl}{P}\frac{1}{w_1}}. \quad . \quad . \quad . \quad . \quad (11)$$

The deflection $w_1=\frac{5Wl^3}{384EI}$ appearing in eq. (11) is a known quantity.

After w' is determined, the resultant bending moment at the centre of the bar will be

$$M'=\frac{Wl}{8}-Pw'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If the area of cross-section of the bar is A, the maximum intensity of stress t in it will be, by eq. (2) of Art. 48,

$$t=\frac{1}{A}\left(P+\frac{M'd}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Or if the maximum value of t is specified

$$A = \frac{1}{t}\left(P + \frac{M'd}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If the section is rectangular, so that $A = bh$ and $d = \frac{h}{2}$,

$$t = \frac{1}{bh}\left(P + \frac{6M'}{h}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$bh = \frac{1}{t}\left(P + \frac{6M'}{h}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

When the depth of the bar is small in comparison with the length l, it may happen that the resultant or final deflection w' will be such as to make the bending moment M' equal to zero. Or

$$M' = \frac{Wl}{8} - Pw' = 0; \quad \therefore \; w' = \frac{Wl}{8P}. \quad . \quad . \quad (17)$$

When w' found by eq. (17) is less than w' given by eq. (11), eq. (17) is to be employed. This result shows that the bar will be subject to no bending, but that it will hang like a flexible cable. The conditions thus developed are those which indicate when a horizontal or inclined bar stressed in tension ceases to act partially as a beam and becomes purely or wholly a tie.

These formulæ are perfectly general for all cases of bars or members in tension, even for such small sections as wire. Their application to individual cases will show that excessive intensities will not exist where simple tension members are held under stress in a nearly horizontal position.

Art. 52.—Combined Bending and Direct Stress in Compression Members.

If the ordinary approximate method of Art. 50 be employed, eq. (4) of that article is immediately applicable, using the minus sign in the denominator, P being the total direct stress of compression and M_1 the bending moment due to the uniform transverse load and to eccentricity of the line of action of P, if there be any. The greatest intensity of bending stress as represented by that formula would then be

$$k = \frac{M_1 d}{I - \frac{Pl^2}{10E}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation, d is the distance from the neutral axis of the section to the extreme fibre in which the intensity k exists.

If e be the eccentricity of the line of action of P, and if W be the weight of the compression member whose length is l,

$$M_1 = \frac{Wl}{8} \pm Pe. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

When the moment of P produces bending of the same sign with the transverse load W, the plus sign is to be used in eq. (2), and the minus sign when those moments are opposite. If the line of action of P coincides with the axis of the member, the moment Pe disappears from eq. (2). Again, if the member is vertical, so that there is no transverse bending due to the load W, when the line of action of P has the eccentricity e,

$$M_1 = Pe. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This latter case exists very frequently in the columns of buildings.

Eq. (1) is thus seen to represent the greatest intensity of bending stress with M_1 taken from either eq. (2) or eq. (3) for the cases of transverse loading, no transverse loading, eccentric longitudinal loading, or any combination of those cases.

The resultant intensity of stress, i.e., the greatest intensity of compressive stress in the entire compression member, will be

$$t=\frac{P}{A}+\frac{M_1 d}{I-\frac{Pl^2}{10E}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As A is the area of cross-section, $I=Ar^2$, r being the radius of gyration of the cross-section of the compression member. If $q=\frac{P}{A}$, eq. (4) will take the form

$$t=\frac{P}{A}+\frac{M_1 d}{Ar^2-\frac{Pl^2}{10E}}=q+\frac{M_1 d}{Ar^2-\frac{Pl^2}{10E}} \quad . \quad . \quad . \quad . \quad (5)$$

In the use of this equation, the intensity q must obviously never exceed the working value given by the column formula employed. Indeed, if there is suitable eccentricity q may be much less than that working long column value.

In practical operation the principal use of eq. (5) may be the determination of the area of cross-section A with some prescribed value of t. It is usually feasible to assign general outside dimensions of the proposed column section and that will enable a close approximate value of r to be assigned. If, at the same time, an approximate value of q may also be taken, the resolution of the first and third members of eq. (5) will at once give

$$A = \frac{P}{10E}\frac{l^2}{r^2} + \frac{M_1}{t-q}\frac{d}{r^2}. \quad . \quad . \quad . \quad . \quad (6)$$

If, on the other hand, such an assignment of q may not be made, it will be necessary to solve the first and second members of eq. (5), as a quadratic equation, for A. Bringing both terms of the second member of eq. (5) over a common denominator and solving the resulting equation of the second degree in the usual manner, the following general value of A will be found:

$$A = \frac{1}{2}\left(\frac{P}{10E}\frac{l^2}{r^2} + \frac{P}{t} + \frac{M_1}{t}\frac{d}{r^2}\right)$$
$$\pm\sqrt{\frac{1}{4}\left(\frac{P}{10E}\frac{l^2}{r^2} + \frac{P}{t} + \frac{M_1 d}{tr^2}\right)^2 - \frac{P}{10Et}\frac{l^2}{r^2}}. \quad (7)$$

Frequently there may be written $d = \frac{h}{2}$ and $r = .4h$. Hence

$$\frac{d}{r^2} = \frac{3}{h} \text{ (nearly).}$$

If, again, $d = \frac{h}{2}$ and $r = .35h$,

$$\frac{d}{r^2} = \frac{4}{h} \text{ (nearly).}$$

The preceding values of the radius of gyration r represented in terms of the depth h of the compression member are closely approximate for practical design work.

Eqs. (6) and (7) will give the desired area of section of the compression member carrying both direct stress and bending produced by transverse loading under the assumptions of the method ordinarily employed. Those formulæ are sufficiently accurate for their purposes, but it may be desirable to use the more exact formulæ given in the next section.

Exact Method for Combined Compression and Bending.

The exact procedure for combined compression and bending is identical with that used in Art. 51, the formulæ determined there simply being adapted to a compressive longitudinal force instead of a force of tension. It is to be observed, as in the case of the tension member, that the compression member may be horizontal or inclined, so as to be subjected to bending either from its own weight or from some other form of loading in addition to that weight. The member may also be subjected to uniform bending throughout its length by the eccentric application of the longitudinal force P concurrently with the preceding cross bending, or, as in the case of a vertical column carrying eccentric loading, by that force P alone.

It is essential to recognize in this connection that while the columns may occasionally be in the pin-end condition, usually their ends are essentially in a condition of at least partial fixedness, although the degree of fixedness is indeterminate. It will conduce to simplicity of treatment if the transverse bending, either from distributed loading or by the eccentricity of application of the column load, be treated as if the ends of columns are hinged. It has been shown in Art. 28 that the centre deflection of a beam of given length and cross-section with ends simply supported and with the loading uniformly distributed is five times as great as when the ends of the same beam are fixed. In the following analysis, therefore, the bending from both the sources named may be considered as produced in a column with hinged ends by a total uniformly distributed load W, sufficient in amount to cause one fifth of the actual bending moment acting on the column with ends fixed. In this manner the fixed or constrained end condition of the actual column is provided for, while the simplicity of the hinged end computations is retained. The bending moment produced by P, acting with the lever-arm of the greatest deflection, will concur with the bending moment produced by the own weight of the member or other vertical uniform loading, instead of being opposed to it, as was the case with the tension member of Art. 51. The work performed, therefore, by P and the uniform loading W will be equal to the resilience or elastic work performed in the member in changing the deflection from w_1 to w', it being remembered, in this case, that w' may be less than w_1. Under these conditions, then, eq. (10) of Art. 51, expressing the work done on the beam in changing the deflection from the w_1 to w' will become the following, the second member representing the resilience or the work done by the elastic stresses throughout its volume:

$$\frac{8}{3}\frac{P}{2}\left(\frac{w'^2 - w_1^2}{l}\right) + \frac{8}{25}W(w' - w_1) = \frac{W^2 l^3}{240EI}\left(\frac{w'^2}{w_1^2} - 1\right). \quad (8)$$

Dividing both members of this equation by $(w' - w_1)$, then solving for w', the following value of the latter will immediately result:

$$w' = \frac{w_1}{\frac{6}{25}\frac{Wl}{P}\frac{1}{w_1} - 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which

$$w_1 = \frac{5Wl^3}{384EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

Having found the deflection w', the general equation for the resultant maximum bending moment, eq. (1) of Art. 48, will take the following form, in which the coefficient c is introduced to provide for fixedness of ends in the manner shown in Prob. 4, at the end of this chapter. If the ends are hinged, corresponding to the end condition of a beam simply supported, $c=1$, but if the ends are fixed, c may be taken as .5:

$$M=c\left(\frac{Wl}{8}\pm P(e\pm w')\right). \quad . \quad . \quad . \quad . \quad (10)$$

In this equation care must be exercised in using the double signs, observing that both plus signs are to be taken together as are both minus signs; also, that the eccentricity e in a vertical column is taken in a direction opposite to the deflection w', in which case e is to be considered positive and the lever-arm of P is $(e+w')$. In the upper chord of bridges e may be given such value that

$$M=\frac{Wl}{8}-P(e-w')=0 \text{ (nearly)}. \quad . \quad . \quad . \quad (11)$$

In the case of vertical columns, like those in buildings, ordinarily the term $\frac{Wl}{8}$ disappears, leaving the bending moment in the column:

$$M=P(e+w'). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the great majority of cases w' is so small in comparison with e as to make it negligible, so that

$$M=Pe. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

These various values of the bending moment M cover all that usually occur in practical operations.

If, in accordance with the preceding notation, t is the maximum resultant intensity of stress in the member, there will result

$$t=\frac{P}{A}+\frac{Md}{I}=\frac{1}{A}\left(P+\frac{Md}{r^2}\right). \quad . \quad . \quad . \quad . \quad (14)$$

Evidently the uniform intensity of compressive stress $\frac{P}{A}$ must not exceed the intensity of working stress given by a suitable long column formula. When the greatest working intensity t is prescribed, the desired area of cross-section of the compression member will be

$$A=\frac{1}{t}\left(P+\frac{Md}{r^2}\right). \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The closely approximate values of $\frac{d}{r^2}$ given immediately following eq. (7) may be used in a precisely similar manner in eq. (15), so as to simplify the practical use of that equation.

Problems for Chapter VI.

Problem 1.—A steel eye-bar 8 ins. by $1\frac{1}{2}$ ins. in section and 32 feet long sustains, in a horizontal position, a tensile stress of 144,000 pounds, i.e., 12,000 pounds per square inch. Find the greatest bending tensile intensity of stress and the resultant intensity of tensile stress at its centre section by the ordinary approximate method of Art. 50, and by the exact method of Art. 51. In this problem, $E=30,000,000$; $l=32\times 12=384$ ins.; $P=144,000$ and $W=40.8\times 32=1306$ pounds. Also $I=\frac{1.5\times 8\times 8\times 8}{12}=64.$

By eq. (6) of Art. 50 the resultant intensity of tensile stress required is

$$t = 12{,}000 + \frac{1306 \times 48}{16 + 17\ 7} = 12{,}000 + 1860 = 13{,}860 \text{ lbs. per sq. in.}$$

The centre deflection w_1, due to own weight only, used in the exact method, is $w_1 = .5$ inch. Hence, by eq. (11) of Art. 51, the centre deflection under tensile stress is

$$w' = \frac{.5}{1 + 1.67} = .19 \text{ inch.}$$

The resultant intensity of tensile stress at the centre section of the eye-bar, is therefore,

$$t = 12{,}000 + \frac{6 \times 35{,}328}{1.5 \times 8 \times 8} = 12{,}000 + 2208 = 14{,}208 \text{ lbs. per sq. in.}$$

The approximate method, therefore, gives an intensity 2208 − 1860 = 348 pounds per sq. in. too small.

Problem 2.—A horizontal square 2 in. by 2 in. steel bar 30 ft. long is subjected to a tensile stress of 48,000 pounds, i.e., 12,000 pounds per square inch. Find the same quantities as in Prob. 1. $E = 30{,}000{,}000$; $l = 360$ inches; own weight, $W = 408$ pounds, and $P = 48{,}000$ pounds.

By eq. (6) of Art. 35

$$t = 12{,}000 + 835 = 12{,}835 \text{ lbs. per sq. in.}$$

In the exact method the centre deflection due to own weight is

$$w_1 = 6.2 \text{ inches.}$$

Eq. (11) of Art. 51 gives

$$w' = 5.55 \text{ inches.}$$

On the other hand, the criterion, eq. (17) of Art. 51, gives

$$w' = \frac{408 \times 360}{8 \times 48{,}000} = .3825 \text{ inch.}$$

The bar, therefore, will be subject to no bending and its stress will be simply that of tension, the centre deflection being .3825 inch. If the deflection were sufficient to give the bar sensible inclination, it would be necessary to multiply the horizontal force $P = 48{,}000$ by the secant of that inclination to obtain the actual tensile stress in the bar.

The results given by the ordinary approximate method are thus seen to be quite erroneous.

Problem 3.—A 1.5-inch round steel bar 48 ft. long, carrying a tensile stress of 10,000 pounds per square inch, is inclined at an angle of 51° to the horizontal. Will it be subjected to any bending, and what will be its centre deflection at right angles to its axis if $\alpha = 51°$? The component of the bar's weight producing the deflection named is $W \cos \alpha$, in which $W = 288$ pounds is the bar's weight; $W \cos \alpha = 224$ pounds; $l = 48$ ft. $= 576$ ins. By the usual formula, $w_1 = 74$ ins. Eq. (11) of Art. 51 then gives $w' = 72$ ins.; but eq. (17) of Art. 51 gives

$$w' = \frac{224 \times 576}{8 \times 17{,}700} = .91 \text{ inch.}$$

Hence this latter deflection is the true value and the bar is subjected to no bending in its stressed condition.

Problem 4.—A steel column 18 ft. long sustains a load of 240,000 pounds and carries a transverse load (i.e., perpendicular to its axis) of 300 pounds per linear foot, the latter total being 5400 pounds. The column has a section like that shown as "top chord latticed," Page 476, Art. 81, and it is composed of two 15-in. by $\frac{1}{2}$-in. web plates, two 3-in. by 3-in. 7-lb. angles, two 3-in. by 4-in. 14-lb. angles, and one 18-in. by $\frac{7}{16}$-in. top plate. The sectional area is 35 sq. ins. The moment of inertia I is 1255, and the radius of gyration r is 6. The loading is applied to the latticed side of the column, so that the eccentricity of application is 8.5 inches. It is required to find the deflection at the centre of the column length, the bending moment and greatest intensity of stress at the same section. Also, if the area of section were not given, find that area if the greatest allowed intensity of compression is 12,000 pounds per square inch. The details of the column at top and bottom are first to be assumed such as to make those ends essentially fixed and then hinged.

If the ends of the column were hinged, the centre bending moment would be

$$M = \frac{5400 \times 216}{8} + 240{,}000 \times 8.5 = 2{,}185{,}800 \text{ in.-lbs.}$$

As the ends of the column are first to be taken as fixed, and as the deflection in that condition will be but one fifth of that existing with ends hinged, it will be necessary to take one fifth of the preceding bending moment and place it equal to the expression for the centre bending moment produced by a uniformly distributed load acting on a column supposed to be with hinged ends. If W represents that uniformly distributed load,

$$\frac{Wl}{8} = 437{,}160 \text{ in.-lbs.}$$

Hence

$$W = 16{,}200 \text{ pounds.}$$

By eq. (9*a*) of Art. 52, E being 30,000,000 and $l = 216$ ins.,

$$w_1 = .0565 \text{ in.}$$

Remembering that $P=240,000$, eq. (9) then gives

$$w' = .00093 \text{ in.}$$

These deflections are so small in comparison with $e=8.5$ inches, that they will have no sensible effect upon the result and they may be neglected.

In consequence of the elastic motions of the members of a steel structure it is difficult to estimate accurately the effect of such degree of fixedness of the ends of a column as may be attained in an actual structure, but it is probable that the resulting bending moment at the centre of the column due to eccentricity and lateral loading is not less than one half that existing with hinged ends, and that ratio will be employed. In eq. (10) of Art. 52, therefore, $c=.5$, and the bending moment will be

$$M=\frac{1}{2}\left(\frac{5400\times 216}{8}+240,000\times 8.5\right)=1,092,900 \text{ in.-lbs.}$$

Hence by eq. (14) of the same Article the greatest intensity of compression will be

$$t=\frac{240,000}{35}+\frac{1,092,900\times(d=8.5)}{1255}$$
$$=6857+7402=14,259 \text{ lbs. per sq. in.}$$

This computation shows the serious effect of eccentric application of loading.

If the greatest allowed intensity of compression is 12,000 pounds per square inch, eq. (15) of Art. 52 shows that the area of cross section required is

$$A=\frac{1}{12,000}\left(240,000+\frac{1,092,900\times 8.5}{36}\right)=41.3 \text{ sq. ins.}$$

The moment of inertia I will now become 1481 instead of 1255.

These results may be compared with those of the ordinary approximate method by finding the greatest intensity of compression, t, by eq. (4) of Art. 52, as follows, after displacing $\frac{1}{10}$ by $\frac{1}{32}$ on account of the fixed end condition:

$$t = \frac{240,000}{41 \cdot 3} + \frac{1,092,900 \times 8.5}{1481 - 12} = 5811 + 7570$$
$$= 12,135 \text{ lbs. per sq. in.}$$

There is, therefore, no material discrepancy.

Results corresponding to the preceding, but under the supposition that the ends of the column are hinged, may readily be found as follows:

$$\frac{Wl}{8} = 2,185,800; \quad \therefore \ W = 81,000 \text{ pounds.}$$

Hence

$$w_1 = .2825 \text{ in.}$$

and

$$w' = .0047 \text{ in.}$$

While these deflections are five times as large as before, w' is still too small to affect sensibly the results and it will be neglected. The bending moment at the centre of the column will then be

$$M = \frac{5400 \times 216}{8} + 240,000 \times 8.5 = 2,185,800 \text{ in.-lbs.}$$

and the greatest intensity of compression

$$t = \frac{240,000}{35} + \frac{2,185,800 \times 8.5}{1255}$$
$$= 6857 + 14,800 = 21,657 \text{ lbs. per sq. in.}$$

If the greatest allowed intensity of compression is 12,000 pounds per square inch, the area of cross-section becomes

$$A=\frac{1}{12,000}\left(240,000+\frac{2,185,800\times 8.5}{36}\right)=63 \text{ sq. ins.}$$

The moment of inertia I will now become 2259 instead of 1255.

Comparing these results with those of the ordinary approximate method by finding the greatest intensity of compression, t, by eq. (4) of Art. 52,

$$t=\frac{240,000}{63}+\frac{2,185,800\times 8.5}{2259-37}=12,170 \text{ lbs. per sq. in.}$$

This result is a close agreement with the other.

Problem 5.—The pin of a crank-shaft like that shown in Fig. 1 of Art. 47 sustains a maximum thrust, P, of 32,000 pounds, the length of crank, e, being 20 inches, and the axial distance, l, between the centre of the thrust and shaft bearings being 18 inches. Find the diameter of the steel shaft at the bearing B if the greatest allowed bending tension, k, is 10,000 lbs. per sq. in. and the greatest allowed torsional shear, T, is 7000 lbs. per sq. in.

In using the formulæ of Art. 47 the data will be as follows:

$e=20$ ins.; $l=18$ ins.; $AB=26.9$ ins.; $\tan j=\frac{20}{18}=1.111$; $j=48°$; $\cos j=.669$; $\sin j=.743$; $P=32,000$ lbs.; $k=10,000$ lbs. per sq. in.; $T=7000$ lbs. per sq. in.; bending moment $M=576,000$ in.-lbs.; twisting moment $M'=640,000$ in.-lbs.

The first method of Art. 47 gives for bending, by using the first of eqs. (9),

$$D=2.16\sqrt[3]{\frac{32,000\times 18}{10,000}}=8.34 \text{ ins.}$$

PART II.—TECHNICAL.

CHAPTER VII.

TENSION.

Art. 53.—General Observations.—Limit of Elasticity.—Yield Point.

Hitherto certain conditions affecting the nature of elastic bodies and the mode of applying external forces to them, have been assumed as the basis of mathematical operations, and from these last have been deduced the formulæ to be *adapted* to the use of the engineer. These conditions are never realized in nature, but they are approached so closely that, by the introduction of empirical quantities, the formulæ give results of sufficient accuracy for all engineering purposes; at any rate, they are the only ones available in the study of the resistance of materials.

In determining the quantity called the "coefficient or modulus of elasticity," it is supposed that the body is perfectly elastic, i.e., that it will return to its original form and volume when relieved of the action of the external forces, also that this "modulus" is constant. There is reason to believe that no body known to the engineer is either perfectly elastic or possesses a perfectly constant modulus of elasticity. Yet within certain limits, the deviations from these assumptions are not sufficiently great to vitiate their great practical usefulness.

These limits for any given material are in the vicinity of the "limit of elasticity" or "elastic limit." The limit of elasticity or elastic limit of a material may be defined as that point of stress below which the intensity of stress divided by the rate of strain, i.e., strain per unit of length, is essentially constant. This point or limit is fairly well defined for most grades of structural steel and for some other ductile metals, but in other materials like stone or timber it is difficult to assign any degree of stress as the limit of elasticity. In such material the intensity of stress divided by the rate of strain sometimes fails to be constant at all. If the intensities of stress and rates of strain for such materials be plotted so as to exhibit the relation between those quantities the resulting line will be found to be a curve without any point which can properly be considered the limit of elasticity. Frequently when such materials are relieved of loads, the dimensions of the piece subjected to stress will not return to their original values.

Between the extreme limits of these materials exhibiting such a range of elastic or physical qualities, all degrees of imperfect elastic characteristics may be found. Fortunately, however, the structural materials commonly employed in engineering operations may be treated as if possessing at least approximately elastic characteristics sufficient to make applicable useful formulæ based upon Hooke's Law.

It should be stated that some authorities have given arbitrary definitions of the elastic limit, and that these definitions have been much used. Wertheim and others have considered the elastic limit to be that force which produces a permanent elongation of 0.00005 of the length of bar. Again, Styffe defines, as the limit of elasticity, a much more complicated quantity. He considers the external load to be gradually increased by increments, which may be constant, and that each load, thus attained, is allowed to act during a number of minutes given by taking 100 times the quotient of the increment divided by the load. Then the "limit of elasticity" is "that load by which, when it has been operating by successive small increments as above described, there is produced an increase in the permanent elongation which bears a ratio to the length of the bar equal to 0.01 (or approximates most nearly to 0.01) of the ratio which the increment of weight bears to the total load." (Iron and Steel, p. 30.)

These rather artificial expressions for limit of elasticity, however, have now been abandoned in favor of what seems to be the most natural value, i.e., the point where the ratio between intensity of stress and rate of strain ceases to be essentially constant.

The preceding observations relate to the limit of elasticity as determined by tests of materials under direct tension or compression. Obviously, however, the coefficient or modulus of elasticity and elastic limit as well as other physical qualities may be determined by subjecting beams to flexure. Observed deflections under known loads, which do not bend the tested beam beyond the elastic limit, will enable the coefficient of elasticity to be computed by using formulæ of the common theory of flexure. Similarly the observed increments of transverse loading will yield data from which the limit of elasticity may be determined.

By precisely similar procedures the coefficient of elasticity and elastic limit of material subjected to torsion may be found. All such results will be well defined in proportion to the elastic properties of the materials. If those elastic properties are nearly perfect the results will be well defined. On the other hand, they will be obscure and ill defined if the material possesses only a low degree of elastic properties.

Yield Point.

In the ordinary testing of materials for engineering purposes the true elastic limit is not determined. The true elastic limit of any test piece is found by carefully computing the ratio between intensity of stress and rate of strain for a loading continually increasing by comparatively small increments. Such a procedure is too slow for what may be termed the commercial purposes of engineering. A much more rapid and convenient procedure consists in carefully observing the scale beam of the testing machine. As the load is gradually increased the scale beam may easily be kept in a horizontal position by moving the scale weights until a point of stress in the specimen is reached at which the beam drops in consequence of the relatively sudden stretching of the material. This stretching continues with such a material as structural steel with a slight addition of loading, or none at all, to a remarkable extent. Finally, after much stretching of the test piece, the strained material appears to take on renewed resistance, requiring additional loading to produce much elongation. The intensity of stress in the specimen when this sudden stretching begins is called the "yield point" or sometimes the "stretch limit." It is but little above the elastic limit. In soft or mild steels, or in high structural steel the yield point may not be more than two or three thousand pounds above the elastic limit. The elastic limit itself is from one-half to six-tenths the ultimate resistance for small specimens or about one-half the ultimate resistance for large members like eye-bars, or a little less than that after annealing.

The ease with which the yield point may be determined has led to its wide use under the name of elastic limit in much engineering literature, but the distinction should always be observed.

In the case of some structural materials with erratic or defective elastic properties, like some grades of cast iron, it is practically impossible to find any well-defined elastic limit or even yield point.

Art. 54.—Ultimate Resistance.

After a piece of material, subjected to stress, has passed its elastic limit, the strains increase until failure takes place. If the piece is subjected to tensile stress, there will be some degree of strain, either at the instant of rupture or somewhat before, accompanied by an intensity of stress greater than that existing in the piece in any other condition. This greatest intensity of internal resistance is called the " Ultimate Resistance."

In ductile materials this point of greatest resistance is found considerably before rupture; the strains beyond it increasing rapidly while the resistance decreases until separation takes place.

These phenomena are highly marked in ductile materials like wrought iron and structural steel, particularly in the latter. In such cases if the application of stress to the test piece is carefully controlled a considerable stretching of the piece may be produced beyond the point of ultimate resistance without actually separating the metal, the load per square inch of original section of the piece decreasing rapidly. It is not difficult to obtain such results with soft or mild steel.

The ultimate resistances of different materials used in engineering constructions can only be determined by actual tests, and they have been the objects of many experiments.

It has been observed in these experiments that many influences affect the ultimate resistance of any given material, such as mode of manufacture, condition (annealed or unannealed, etc.), size of normal cross-section, form of normal cross-section, relative dimensions of test piece, shape of test piece, etc. In making new experiments or drawing deductions from those already made, these and similar circumstances should all be carefully considered.

Art. 55.—Ductility.—Permanent Set.

One of the most important and valuable characteristics of any material is its "ductility," or that property by which it is enabled to change its form, beyond the limit of elasticity, before failure takes place. It is measured by the permanent "set," or stretch, in the case of a tensile stress, which the test piece possesses after fracture; also, by the decrease of cross-section which the piece suffers at the place of fracture.

In general terms, i.e., for any degree of strain at which it occurs, "permanent set" is the strain which remains in the piece when the external forces cease their action. It will be seen hereafter that in many cases, and perhaps all, permanent set decreases during a period of time immediately subsequent to the removal of stress. Indeed, in some cases of small strains it is observed to disappear entirely.

Art. 56.—Cast Iron.

Modulus of Elasticity and Elastic Limit.

Cast iron is a metal produced by fusion without subsequent working such as forging or rolling. Except when made for special purposes under conditions of careful control of the elements entering it, the quality of the product is irregular and variable. Bubbles of gases not escaping from the molten mass will leave voids or " blow-holes " in the final product and carbon exists both in the graphitic and combined condition, but in varying proportions. The mode of production and the practically unavoidable irregularities in cooling induce both variable conditions of crystallization and internal stresses which are sometimes high enough to fracture the completed casting.

There are some grades of cast iron like those formerly used for car wheels and ordnance which give high ultimate resistance and comparatively high moduli of elasticity and which exhibit an approximation at least to an elastic limit, although the latter point is never well defined as in wrought iron and steel. The ordinary soft castings used in engineering practice for water pipes, machine frames and other similar purposes disclose under test such erratic properties that they cannot be said to have either a well-defined modulus of elasticity or any real elastic limit. The irregular behavior of cast iron under stress is well shown for different grades of the material by the stress-strain diagrams shown in Fig. 1, in which the vertical ordinates are intensities of stress, while the horizontal ordinates or abscissæ are the strains or elongations per linear inch. These curves are typical of what may be considered good grades of cast iron for their purpose. The line *oq* represents a fair grade of ordinary soft cast iron, while *on* and *oe* belong to a higher grade and *od* a still stronger metal for special purposes. The amounts written at the extreme upper ends of the curves indicate the loads or stresses per square inch at which the test specimens failed. The two curves *Of* and *Oe* were constructed from data given on pages 597 and 605 of the " U. S. Report of Tests of Metals and Other Materials " for 1899. These two cast-iron test specimens were of metal of superior or special grades, proposed to be used for ordnance purposes, as is indicated by the high ultimate resistances, 22,300 and 35,280 pounds per square inch.

There is seen to be the greatest diversity in the inclination and general character of the four strain curves

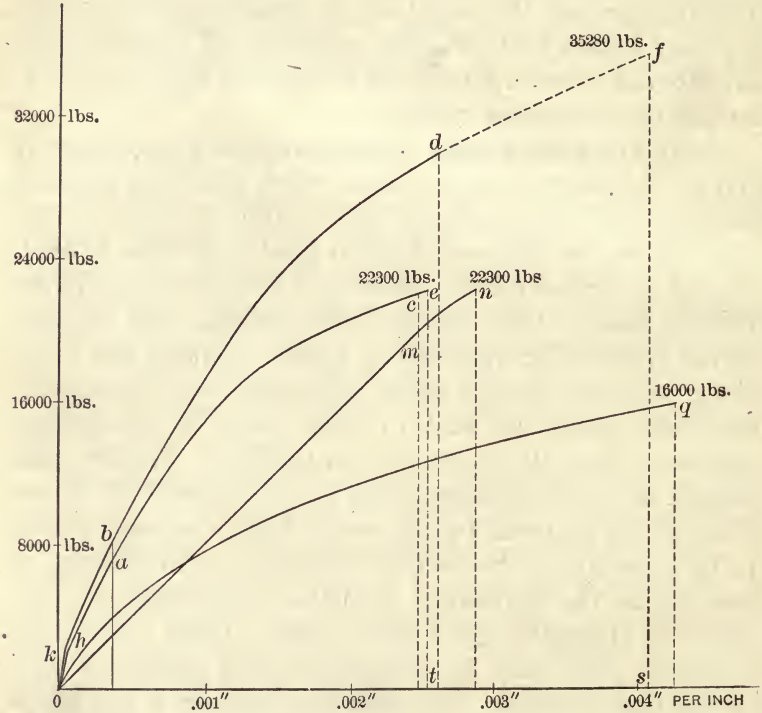


FIG. 1.

The curve *Oe* has a fairly straight portion *ha*, the point *a* representing an intensity of stress of 7000 pounds, while the point *h* represents an intensity of 2000 pounds per square inch. The cross-sectional area of this test specimen was 1 square inch. The difference in strains at the two points *a* and *h*, or for a range in intensity of 5000 pounds, was .0003 inch. Hence the coefficient of elasticity for these data would be

$$E = \frac{5000}{.0003} = 16,667,000 \text{ pounds.}$$

In the same manner the increase of strain per linear inch of test specimen resulting from increasing the stress of 2000 pounds per square inch at k to 8000 pounds per square inch at b was .00032. Hence with these data the coefficient of elasticity would be

$$E = \frac{6000}{.00032} = 18,750,000 \text{ pounds.}$$

The strain curve On is an extraordinary one for cast iron, as it is straight for nearly its entire length. For the intensity of stress of 16,200 pounds the strain or stretch is seen to be .002 inch; hence the coefficient of elasticity would be

$$E = \frac{16,200}{.002} = 8,100,000 \text{ pounds.}$$

The metal represented by the strain curve Oq cannot be said to have any coefficient of elasticity at all, as no part of the curve is straight. These instances selected from a large number of tests are representative of what may be expected in elastic behavior of cast iron. As a rule, the grades possessing the higher ultimate resistances exhibit a more nearly normal elastic character and possess what may be termed not very well-defined coefficients of elasticity running from about 14,000,000 to perhaps 18,000,000 pounds per square inch, while the usual grades or quantities employed in engineering castings may have no coefficient of elasticity at all or as low as 8,000,000 or 10,000,000 pounds per square inch. In view of all experimental data available at the present time it is probably about as near correct as practicable to take the tensile coefficient of cast iron for ordinary engineering purposes, as

$$E = 12{,}000{,}000 \quad \text{to} \quad 14{,}000{,}000 \text{ pounds},$$

or one half that of wrought iron. For the special grades of stronger cast iron, such as are used for ordnance and car-wheel purposes, a coefficient or modulus of 16,000,000 pounds to 18,000,000 pounds per square inch may be used.

As is usually the case in cast iron, the elastic limits of the curves in Fig. 1 are so ill-defined that they cannot be placed with certainty even on the curves *Of* and *Oe*, or scarcely on *On*, and not at all on curve *Oq*. If the points are approximately located on the first three of these curves they may perhaps be placed at *b* (8000 pounds per square inch), at *a* (7000 pounds per square inch), and at *m* (19,000 pounds per square inch). In none of these cases, however, can the metal be said to have either a well-defined limit of elasticity or a true yield point, and that observation is in general true of all cast iron.

Resilience, or Work Performed in Straining Cast Iron.

As the scale of the original of Fig. 1 was 8000 pounds to each inch of vertical ordinate and .001 inch to each inch or horizontal ordinate or abscissa, and as the strains shown in Fig. 1 belong to a test piece 1 inch square in section and 1 inch long, each square inch of area on the original diagram between any one of the strain curves and the axis of abscissæ drawn through *O* will represent $8000 \times .001 = 8$ inch-pounds of work performed in stretching that test piece. The strain at the point *b* on the curve *Of* is .00036 inch, as shown in the figure, while the mean intensity of stress in producing that strain is 4400 pounds. Hence if *b* represents the elastic limit the resilience or work performed in stretching the metal up to the elastic limit of 8000 pounds per square inch is

$$4400 \times .00036 = 1.58 \textit{ inch-pounds per cubic inch.}$$

Similarly, if *a* is the elastic limit in the strain curve *Oc*, the total strain for each inch in length of the test specimen is .00038 inch and the mean intensity of stress is 3750 pounds, all as shown in Fig. 1. Hence the resilience or work performed was

$$3750 \times .00038 = 1.43 \textit{ inch-pounds per cubic inch.}$$

A similar computation may be made for the straight portion of the strain curve *On*, but the preceding operations sufficiently illustrate the procedure.

The total work performed in breaking each specimen may readily be found in precisely the same manner. In the case of the curve *Of* the strains or elongations of the specimen were actually observed only up to the point *d*, although failure actually took place at *f* or at the intensity, 35,280 pounds per square inch. The part *df* of the curve is drawn approximately as a continuation of the observed curve and therefore is shown as a broken line. The area included between the curve *Of* and the horizontal ordinate *Os*, i.e. the area of the figure *Ofs*, is 11.97 square inches. Hence the work performed in rupturing the test piece was

$$11.97 \times 8000 \times .001 = 95.76 \textit{ inch-pounds per cubic inch.}$$

Again, in the case of the strain curve *Oe* the area of the figure *Oet* is 4.69 square inches. The total work expended, therefore, in rupturing the specimen was

$$4.69 \times 8000 \times .001 = 37.5 \textit{ inch-pounds per cubic inch.}$$

In the latter case the short portion *ce* of the strain curve is drawn approximately, as the strain observations ceased at *c*. It is to be remembered, as is indicated in each of these cases, that when the data apply to each linear inch of test piece and each square inch of sectional area, the work computed will be for 1 cubic inch of material. It is only necessary to multiply by the number of cubic inches in the test piece in order to obtain the work performed in the entire piece.

Ultimate Resistance.

The ultimate tensile resistance of cast iron is an exceedingly variable quantity; it may range from not more than 8000 or 10,000 pounds in castings of indifferent quality to values of nearly 50,000 pounds per square inch in such special grades of metal as those which have been used for car wheels and ordnance. Cast iron has passed completely out of use for the manufacture of heavy guns, but there are other ordnance purposes for which it is still used. The castings usually employed by civil engineers are generally of soft-grade iron; they are such as water pipes, frames, beds of machines, and other similar purposes which do not require special grades produced by special mixtures of raw material or special processes of manufacture. The ultimate resistances will, therefore, be considerably less than those belonging to ordnance and car-wheel irons, or for specially strong grades of metal. As with all material, the character of cast iron affects to a great extent its resistance, i.e., whether it is fine or coarse grained, as does also the character of the ore from which it is produced.

Three specimens turned down to a diameter of about .625 inch taken from iron used in the Boston water pipes and broken at the Warren Foundry, Phillipsburg, New

Jersey, gave the following ultimate resistances in pounds per square inch:

18,300, 15,470, 13,070.

These results represent fairly the ultimate resistance of ordinary cast-iron pipe and other castings commonly used in civil engineering practice. It has sometimes been stated that the outer surface or "skin" of iron castings has a greater capacity of resistance to stress than the interior parts. Investigations carefully conducted, however, by the late Professor J. B. Johnson and others do not show that to be the case. Indeed it is practically certain that there is no essential difference between the resistances of the exterior and interior parts of a casting unless it has been subjected to some special treatment. It is not unlikely that this erroneous impression may have arisen from the results of irregular cooling of castings producing internal stresses sometimes sufficient to produce fracture.

The "Report of the Tests of Metals and Other Materials" at the United States Arsenal, Watertown, Mass., for 1900, contains a mass of tensile tests of pig irons and ordnance castings of a great variety of grades and qualities, from which the following tabular statement of greatest and least values have been taken. There are also given the results of two tests of gear teeth taken from the same source.

TENSILE TESTS OF CAST IRON.

Iron.	Ultimate Resistance. Lbs. per Sq. In.	
	Greatest.	Least.
Pig	31,890 Fine granular, gray.	11,820 Coarse granular, dark gray.
Ordnance Castings .	33,500 " " "	14,900 " " " "
Gear teeth.	12,200 Fine or medium granular, gray.	12,080 Fine or medium granular, gray.

As a recapitulation there may be written:

For ordinary castings:

$$\textit{Modulus of elasticity}\left\{\begin{array}{c}12{,}000{,}000 \text{ lbs. per sq. in.}\\ \text{to}\\ 14{,}000{,}000 \text{ lbs. per sq. in.}\end{array}\right.$$

Ultimate tensile resistance, 15,000 to 18,000 lbs. per sq. in.

For specially excellent grades:

$$\textit{Modulus of elasticity}\left\{\begin{array}{c}16{,}000{,}000 \text{ lbs. per sq. in.}\\ \text{to}\\ 18{,}000{,}000 \text{ lbs. per sq. in.}\end{array}\right.$$

Ultimate tensile resistance, 20,000 to 35,000 lbs. per. sq. in.

Tensile working resistances in pounds per square inch may be taken as follows:

For water pipes and other similar purposes:

3000 to 3500 lbs. per sq. in.

With higher grades of cast iron for special purposes:

4000 to 7000 lbs. per sq. in.

Effects of Remelting, Continued Fusion, Repetition of Stress, and High Temperatures.

The physical qualities of cast iron may be much improved by remelting and continued fusion. The product of the blast furnace is commercial pig iron. These pigs remelted, as in a cupola furnace, form the ordinary castings of engineering work. If this remelting should be continued so as to secure third or fourth fusion metal the resisting properties of the iron would be enhanced, but the cost would at the same time be materially increased, and hence second fusion metal only is ordinarily used.

Again, experience has shown that if molten metal be held in fusion, even for a period of three hours or more, its physical quality continues to improve, but the cost of such a procedure renders it prohibitive for ordinary purposes.

Many investigations have been made to determine the resisting power of structural materials to frequent and continued repetition of stresses, not only below, but above the elastic limit, the relief from stress between two applications sometimes being partial and sometimes complete. It has been found that such repeated stresses, when as high as one-half to three-quarters of the ultimate resistance, produce material fatigue in cast iron and final failure much below the ordinary ultimate resistance as determined by a gradual application of load. Such tests have shown that cast iron is somewhat more sensitive to fatigue than the ductile structural materials of higher ultimate resistance.

The effect of high temperatures upon the resisting capacity of cast iron is not in general different from that found for steel and wrought iron. Little, if any, softening is observed until a temperature of 500° F. is approached, but beyond that limit it is liable to begin to lose capacity of resistance to a material extent if not rapidly.

Art. 57.—Wrought Iron.—Modulus of Elasticity.—Limit of Elasticity and Yield Point.—Resilience.—Ultimate Resistance and Ductility.

Wrought iron as a structural material has been completely displaced by the various grades of structural steel, although it is still used in relatively small quantities for special purposes. Again, many bridge and other structures built of wrought iron are still standing, and it is essential to retain a record of its physical qualities.

Wrought iron differs fundamentally from steel in its manner of production, as it is a product of the puddling furnace. A white-hot spongy mass was brought out of a bath of molten slag and passed between rolls, resulting in what were known as puddle bars. These were cut in suitable lengths, and placed in rectangular packages or piles of proper size to produce the finished bar or beam by subsequent heating and rolling.

This process of production gave to wrought iron a fibrous internal structure of much greater ultimate resistance in the direction of the fibre than at right angles to the fibre or direction of rolling, and this was true whatever shape was produced, such as plates, beams, bars, etc.

Modulus of Elasticity.

The coefficient or modulus of elasticity of wrought iron was determined by many tests of both small and full-size bars when it was the principal structural material in bridges and other similar structures. The adjoining table gives the results of tests of four bars only. The two 1-inch square bars were of fine quality of wrought iron and were tested many years ago by Eaton Hodgkinson. The results of tests of the 5-inch and 3-inch bars are taken from the "Report of Tests of Metals for 1881" made on the large testing machine at the U. S. Arsenal, Watertown, Mass. The table below gives full information as to the total strain, gage length and stress per square inch for the various bars. If p is the stress per square inch and l the strain per linear inch of gaged length, the coefficient of elasticity E will have the value,

$$E = \frac{p}{l}.$$

Size of Bar, Inches.	Gaged Length, Inches.	Total Strain, Inches.	Stress per Sq. Inch, Pounds.	E.
1×1	120	.04556	10,670	28,101,000
1×1	120	.043	10,095	28,198,000
5.04×1.27	80	.029	20,000	27,586,000
3.05×1	80	.0279	20,000	28,674,000

It will be observed that the four values shown are more nearly the same than will be found in a long series of determinations in the early tests of engineering materials when wrought iron was in general use. As a result of such determinations, the value of

$$E = 26,000,000$$

may be taken as a fair average value for wrought iron members of structures. For small specimens, or for some special grades of wrought iron, 27,000,000 or possibly 28,000,000 may be used.

Obviously all values of E must be computed for intensities of stress less than the elastic limit.

Limit of Elasticity and Yield Point—Resilience.

The limit of elasticity for wrought iron is not nearly so well defined as for structural steel. The diagram Fig. 1 has been constructed from the test of the one-inch square wrought iron bar with a gaged length of 10 feet and with a load increasing by small increments. The horizontal ordinates represent the total strains in inches, while the vertical ordinates represent intensities of stress per square inch.

That part of the curve from the origin o to a is straight and its equation is,

$$p = El.$$

Above a the line begins to curve and at e the curvature becomes about as sharp as at any point. The point a, elastic limit, may be taken at 26,000 pounds per square inch, while e, the yield point, may be considered as 29,000 pounds per square inch, although this latter point is not well defined. Above e the curve becomes much less inclined to a horizontal line, showing that for small increments of load the stretch of the specimen is relatively great.

While these results belong to one specimen only of wrought iron they are characteristic of the metal. Approximately the elastic limit may be considered half the ultimate resistance and the yield point possibly 2000 to 4000 pounds more.

FIG. 1.

For all ordinary cases of wrought-iron structures the elastic limit may safely be considered 22,000 to 24,000 pounds per square inch and the yield point from 25,000 to 28,000 pounds per square inch, as it is to be remembered that the elastic limit and yield point will be higher for small test specimens than for full-size structural members.

Ductility and Resilience.

In Fig. 1 the horizontal coordinates of the stress-strain curve are the strains for 120 inches in length of a wrought-iron test bar, corresponding at each point to the intensities of stress per square inch shown on the vertical line through *o*. This curve exhibits fully the physical characteristics of the material under test. The straight part *oa* of the curve belongs to that part of the loading below the elastic limit *a*, i.e., below 26,000 pounds per square inch. The point *e* indicates the stretch limit at about 29,000 pounds per square inch. There is no constant proportionality between stress and strain above *a* nor is there any great increase in the strain for a given small increment of loading until the point *e* is passed, but above that point the stretch for each constant increment of loading becomes relatively large. Beyond the point *b*, the inclination of the stress-strain curve to horizontal is relatively small. At or near *c* the curve becomes horizontal, showing the maximum intensity of resistance, i.e., the ultimate resistance, and the broken line *cf* indicates a rapidly decreasing load if the testing machine is properly manipulated prior to the actual parting of the material at *l*. Usually the actual failure of the material will take place at the highest point of the curve unless special pains be taken to operate the decrease of loading and even under such conditions the material must be highly ductile to produce the part of the curve shown by the broken line.

The resilience of work expended below the elastic limit *a* can readily be computed by the aid of Fig. 1, as it is represented by the triangular area between the straight part of the stress-strain curve and a vertical line through its upper limit. The strain at the elastic limit of 26,000 pounds per square inch is .11744 inch. The average force acting upon the specimen up to the elastic limit would be half the value of the latter. Hence the elastic resilience or the work performed on the specimen up to the elastic limit is

$$.11744 \times \frac{26,000}{2} = 1527.6 \text{ inch-pounds.}$$

Inasmuch as the test specimen was 120 inches long, the elastic resilience of the bar would be 12.73 inch-pounds per cubic inch of its volume. Similarly, the area of the irregular figure *oebch* is 4.97 square inches, and as the scale of force is 20,000 pounds per linear inch, that figure represents $4.97 \times 20,000$ pounds $= 99,360$ inch-pounds of work; or $\frac{99,360}{120} = 828$ inch-pounds of work per cubic inch of volume of the test specimen. If this test-bar, therefore, were to be broken by a falling weight of 100 pounds, that weight would be required to fall through a height of

$$\frac{99,360}{100} = 993.6 \text{ inches.}$$

It is clear from the figure that if the metal possessed little ductility so that its strain curve extended no further than the point *b*, the work required to be expended in breaking it would be very small compared with that needed for rupturing the actual wrought-iron piece. The effect of a falling weight may represent a shock or blow, or be taken as the equivalent of what is usually called a suddenly applied load. These considerations show why a ductile material requiring so much more work to be performed to break it is much better adapted to sustain shock than a non-ductile or brittle material. The latter class of materials can be strained so little before failure that little work is required to be expended to break them.

Ultimate Resistance.

The ultimate resistance of wrought iron depends to some extent, like structural steel, on the size of the test specimen or bar, its treatment during manufacture, and whether the piece is tested in the direction in which it was rolled or at right angles to that direction. Wrought iron being a fibrous material, its ultimate resistance is materially greater in the direction of the fiber than at right angles to that direction or in inclined directions. Structural specifications usually prescribed that when used in tension, wrought iron should take its load parallel to the direction of rolling, particularly for wrought-iron plates.

Round and rectangular bars of wrought iron of ordinary structural sizes showed under tests ultimate resistances, generally varying from about 45,000 to 50,000 pounds per square inch, the smaller values applying to large bars and the large values to bars of small section.

A series of tests of round bars found in the "Report of the Committee of the U. S. Board Appointed to Test Iron, Steel, and other Metals, etc.," showed that the ultimate resistances ran from about 60,000 per square inch for $\frac{1}{4}$-inch rounds down to about 46,000 to 47,000 pounds per square inch for bars 4 inches in diameter.

The ultimate resistance of such wrought-iron shapes as angles, eye bars, channels, tees, and others were shown by many tests to be about the same as bars and flats of the same quality and size, i.e., many test specimens showed ultimate resistances running from about 45,000 to 50,000 pounds per square inch. If the shapes or plates were small, so that the temperature was relatively low during final passes between the rolls, the hardening effect of such treatment would raise the ultimate resistance to some extent, resulting in higher values than for similar shapes of large section which suffered less reduction of temperatures during the process of rolling. Thin plates showed markedly higher ultimate resistances than thick plates for this reason.

Ductility.

From what has been stated it is evident that wrought iron would show the greatest final contraction of fractured area and final stretch when tested in the direction of rolling than in any other direction. Again it is equally clear that the percentage of final stretch would be materially greater for short specimens than for long ones, because the necking-down at the section of fracture would add a much greater percentage to the length of a short specimen than to a long one. While both final contraction and final stretch varied greatly in different test pieces, it may be stated that for gage lengths ranging from about 5 feet to 20 feet, full-size wrought-iron bars gave a final contraction of 20 per cent to 30 per cent and a final stretch of about half these values.

Test specimens of plates, angles and other shapes, the final stretch being measured over a gage length of 8 inches, would generally yield about 20 per cent to 30 per cent of final contraction and about 10 per cent to 20 per cent of final stretch.

The preceding may be considered fairly representative values of ductility of the best quality of wrought iron used in bridge and other structures. They show that the metal was highly ductile and well adapted to structural purposes, although possessing these desirable qualities to a less degree than structural steel.

Fracture of Wrought Iron.

The characteristic fracture of wrought iron broken in tension either directly or transversely is rather coarsely fibrous, not infrequently exhibiting a few bright granular spots, which, in rare cases, may possibly be crystalline. This characteristic fibrous fracture is produced by the steady application of load, but a piece of wrought iron will exhibit a granular fracture if broken suddenly. Many statements have been made that wrought iron may become crystalline and lose both ultimate resistance and ductility under certain conditions of use, but bright granular fracture has probably been mistaken in such cases for crystalline.

Art. 58.—Steel.

Modulus of Elasticity.

The great number of varieties and grades of steel brings into existence a correspondingly great number of physical quantities and coefficients or moduli used in its consideration in connection with the " Resistance of Materials."

Notwithstanding the number of varieties of steel used at the present time for engineering purposes, it is fortunate in the interests of simplified computations to find their moduli of elasticity varying so little that they may be taken as practically the same. Again, it is further fortunate that the moduli for tension and compression also appear to be the same, and they are so taken.

That class of steel generally to be considered here is included under the term ''Structural Steel,'' which may be divided into low, medium, and high steel. These three grades of structural steel are mainly based upon the amounts of carbon which they contain. While each class shades insensibly into another without well-defined limits, it may be approximately stated at least that low or soft steel will have carbon ranging from about .1 to .2 per cent., and that the carbon in medium steel will run from about .2 to .3 per cent., while high steel will show about .3 to .45

per cent. of carbon. The ultimate resistance of low steel may run from 52,000 to 60,000 pounds per square inch, medium steel from 60,000 to 68,000 pounds per square inch, and high steel from 68,000 to about 76,000 pounds per square inch, or possibly higher. Experimental investigations have shown that the coefficient of elasticity is essentially the same for all grades of steel used in construction. This observation holds true also for nickel steel, which has within the past few years come into use for special structural purposes. A considerable number of tests of nickel-steel specimens, in some cases containing 3.375 per cent. of nickel with .3 per cent. of carbon and .73 per cent. of manganese, given in the U. S. Report of Tests of Metals for 1898 and 1899, show that the coefficient of elasticity for this metal may be taken at values ranging from 28,700,000 pounds to 30,385,000 pounds per square inch. In other words, the coefficient of elasticity of this nickel steel may be taken between the usual limits for ordinary structural steel of 28,000,000 and 30,000,000 pounds per square inch.

Table I gives a condensed statement of the results of an extended investigation made to determine the "constants" of structural steel by Prof. (now President) P. C. Ricketts, at the mechanical laboratory of the Rens. Pol. Inst. in 1886. Although these tests were made before as many varieties and grades of steel had been developed as at present, the values given in the table are accurately characteristic of the same grades of structural steel produced at the present time, 1915. As no corresponding determinations have been made of such wide range nor with such a wide scope of purpose since that early date, the table has unique value and is worthy of careful study. Although this table contains other values than those immediately desired, the opportunity of directly comparing

Tensile-test specimens of steel and wrought iron broken in the testing laboratory of the Department of Civil Engineering of Columbia University. The four steel specimens on the left show the characteristic "cupping" fracture. The four specimens on the right are of wrought iron and show the characteristic fibrous fracture of that material.

(*To face page* 304)

TABLE I.

	Mark.	Per Cent. Carbon.	TENSION					
			Specimen.			Pounds per Square Inch.		
			Diam. Inches.	Per Cent. Reduc. of Area	Per Cent. Elong in 8 Ins.	Elastic Limit	Ultimate Resistance.	Coefficient of Elas.
Rivet steel * . . .	1_1	.09	0.756	61.7	30.5	39.600	63,600	30,039,000
" . . .	1_2	"	0.758	61.7	30.5	38,800	63,300	30,010,000
" . . .	1_3	"	0.757	60.8	28.9	37,800	63,000	31,160,000
" . . .	4_1	"	0.757	65.3	29.6	37,800	62,000	31,063,000
" . . .	4_2	"	0.758	65.1	29.4	38,600	63,200	30,471,000
" . . .	4_3	"	0.758	62.3	29.9	39,400	62,800	29,965,000
" . . .	6_1	"	0.760	61.6	30.1	37,400	60,600	30.456,000
" . . .	6_2	"	0.760	60.6	29.6	36,900	61,300	30,885.000
" . . .	6_3	"	0.760	61.8	32.2	39,100	61,900	27,335,000
" . . .	8_1	"	0.760	57.9	29.2	38,100	62,500	30,618,000
" . . .	8_2	"	0.759	62.4	28.4	37,100	62,300	30,172,000
" . . .	8_3	"	0.758	61.0	28.2	36.600	61,400	30,424,000
" . . .	10_1	"	0.756	65.7	28.6	35 600	61,700	29 696,000
" . . .	10_2	"	0.755	64.7	29.0	36,800	61,600	30,075,000
" . . .	10_3	"	0.754	64.3	29.1	36,900	62,100	30,371.000
" . . .	3_1	"	0.757	63.4	27.9	36,700	61,200	30,918,000
" . . .	3_2	"	0.758	64.0	30.4	37,700	61,900	30,801,000
" . . .	3_3	"	0.758	64.3	29.2	37,100	61,800	31,091,000
" . . .	5_1	"	0.757	51.7	30.1	37,800	62,900	30,032,000
" . . .	5_2	"	0.755	49.4	29.2	38,500	63,600	31,646,000
" . . .	5_3	"	0.757	51.2	28.1	37,800	61,300	30,031,000
" . . .	7_1	"	0.750	62.1	30.9	36,200	61,200	30,166,000
" . . .	7_2	"	0.749	60.5	29.6	36,800	62,400	30,415,000
" . . .	7_3	"	0.751	61.3	31.7	37,800	62,000	30,232,000
" . . .	11_1	"	0.752	64.3	29.4	36,400	62,400	30,030,000
" . . .	11_2	"	0.754	63.0	29.4	36,400	61,700	30,556,000
" . . .	11_3	"	0.749	62.3	29.2	36,700	62,200	30,011,000
" . . .	2_1	"	0.752	55.1	29.9	37,200	61,600	30,210,000
" . . .	2_2	"	0.757	53.7	31.0	36,700	60,100	32,965,000
" . . .	2_3	"	0.753	53.2	32.0	39,300	61,000	30,097,000
Bessemer † . . .	N_1	.11	0.748	60.3	28.4	41,500	66,600	28,950,000
" . . .	N_2	"	0.754	58.3	28.2	41,400	65,200	29,391,000
" . . .	N_3	"	0.750	57.0	28.2	43,400	67,000	29,899,000
" . . .	O_1	.12	0.751	59.7	27.4	41,500	65,300	29,186,000
" . . .	O_2	"	0.750	59.2	28.5	41,100	65,100	29,252,000
" . . .	O_3	"	0.750	57.4	27.0	41,400	65,700	29,464,000
" . . .	T_1	"	0.747	57.3	30.6	42,000	66,100	29,907,000
" . . .	T_2	"	0.750		30.1	41,900	65.400	29.899,000
" . . .	T_3		0.751	57.1	28.7	41,300	65,400	20.270,000
" . . .	S_1	.13	0.763	58.1	26.8	48,100	69,400	20,706,000
" . . .	S_2	"	0.760	59.5	27.0	47.400	69,300	20,500;000
" . . .	S_3	"	0.760	56.4	27.1	47,100	70,100	20,238,000
" . . .	U_1	"	0.763	59.1	28.2	42,200	65,300	29,430,000
" . . .	U_2	"	0.760	56.6	27.6	42,300	65,600	29,678,000
" . . .	U_3	"	0.756	58.3	27.0	42,300	66,400	29,390,000
" . . .	R_1	.16	0.747	54.8	28.9	42,000	68,300	30,083,000
" . . .	R_2	"	0.745	55.7	27.6	41,700	68,500	30,266,000
" . . .	R_3	"	0.745	55.0	27.4	41,000	68,600	20,442,000
" . . .	P_1	.17	0.746	56.3	27.1	42.100	70,400	29,375,000
" . . .	P_2	"	0.744	57.2	27.4	42,700	70,500	30.158,000
" . . .	P_3	"	0.749	55.8	27.1	41,500	79,600	30,784,000
" . . .	V_1	.36	0.761	40.7	20.5	60,900	97,500	29,045 000
" . . .	V_2	"	0.756	38.5	19.1	60,400	99.600	30,236,000
" . . .	V_3	"	0.759	39.5	19.4	69,700	99,100	29,989,000
" . . .	W_1	.39	0.763	39.0	20.0	69.500	95,800	30,025 000
" . . .	W_2	"	0.762	36.8	19.2	69,600	96,200	30,944,000
. . .	W_3	"	0.765	36.7	19.0	69,100	95,200	29,291,000

* Open hearth from Steelton, Pa.

† From Troy, N. Y.

Table I.—*Continued.*

COMPRESSION.		SHEAR.				Double Shear Ultimate Over Single Shear Ultimate.
Pounds per Square Inch.		Pounds per Square Inch.				
		Single Shear.		Double Shear.		
Elastic Limit.	Coefficient of Elasticity.	Elastic Limit.	Ultimate Resist.	Elastic Limit.	Ultimate Resist.	
39,000	29,897,000					
39,500	27,113,000	39,600	45,440	43,600	46,460	1.022
39,000	28,444,000					
41,100	29,110,000					
41,100	29,025,000	34,600	45,260	38,200	47,450	1.048
41,000	29,045,000					
40,200	30,045,000					
40,200	28,853,000	31,500	46,020	33,800	47,590	1.034
40,400	29,411,000					
41,600	30,192,000					
41,600	29,302,000	31,700	46,910	33,500	48,390	1.032
41,600	29,216,000					
38,600	29,013,000					
38,600	29,963,000	31,100	44,780	34,000	46,590	1.040
38,600	29,478,000					
38,300	29,090,000					
38,300	29,807,000	35,900	44,600	38,500	47,350	1.062
38,300	28,961,000					
41,700	29,630,000					
41,700	28,941,000	33,800	46,440	39,400	48,890	1.053
41,700	29,696,000					
39,900	29,437,000					
40,000	30,009,000	33,700	45,190	35,700	47,210	1.045
40,000	28,730,000					
39,500	29,005,000					
39,700	29,740,000					
39,900	29,963,000					
40,000	31,433,000					
40,000	29,782,000	35,800	46,100	40,700	47,210	1.024
39,700	29,391,000					
41,800	28,567,000					
41,700	29,144,000	30,500	49,210	38,600	51,000	1.036
41,700	28,747,000					
41,100	28,503,000					
41,400	29,531,000	34,400	51,470	39,500	51,470	1.000
41,200	28,730,000					
42,600	29,162,000					
42,400	29,210,000	37,000	49,740	40,300	50,940	1.024
41,900	28,635,000					
44,400	28,070,000					
44,800	28,729,000					
45,000	29,025,000					
44,100	29,281,000					
44,300	29,830,000	36,600	51,000	40,800	51,510	1.010
44,200	29,324,000					
41,100	28,812,000					
41,400	29,342,000	36,700	51,280	43,800	52,550	1.025
41,000	28,666,000					
41,400	28,860,000					
41,600	29,241,000	41,500	53,260	46,000	53,390	1.002
41,800	29,802,000					
55,200	29,162,000					
54,400	29,454,000	52,500	70,190			
54,400	29,281,000					
59,500	28,602,000					
59,200	28,981,000	51,900	67,760			
59,500	29,281,000					

different physical constants from the same quality of steel is a sufficient reason for inserting the entire table at this place. All the test pieces were uniformly about three-quarters of an inch in diameter, and the stretch was in all cases measured on 8 inches. The elongations given are per cents of the original length of 8 inches.

The reductions of area are the per cents of original sections of the test pieces which indicate the differences between the original and fractured areas.

As indicated, the first half of the table belongs to specimens of open-hearth rivet steel from Steelton, Pa., while the second half contains results drawn from tests on a comparatively wide range of metal from the Bessemer process of the Troy Steel and Iron Co., of Troy, N. Y. The open-hearth rivet steel is all seen to contain only .09 per cent. of carbon, while the Bessemer metal had carbon varying from 0.11 per cent. to 0.39 per cent., with a wide gap between 0.17 and 0.36 per cent.

The specimens 1_1, 1_2, and 1_3 were cut from the two ends and centre of bar 1, and those subjected to tension were located adjacent to specimens of the same name subjected to compression. Similar observations apply to other sets of specimens affected by the same figure or same letter. Hence there is shown in this table the relation of different physical quantities belonging to as nearly identically the same material as the possibilities of the case admit.

The coefficients of tensile elasticity exhibit unusual uniformity. Those for the open-hearth steel show no variation with the small variation in carbon. Although the tensile coefficients for the Bessemer steel are slightly lower for the lowest per cents of carbon than for the highest, yet some of the lowest coefficients are found for the highest carbons, and it is difficult to determine any essential variation with varying proportions of that element.

While the average of the tensile coefficients is a very little more for the open hearth than for the Bessemer steel, there is really no sensible difference between them. The average tensile coefficient may be taken at 30,000,000 pounds per square inch.

Too much importance should not be attached to the percentage of carbon alone in these specimens, as the presence of other elements not given, such as manganese, phosphorus, etc., exert marked influences on the physical characteristics of steel.

The modulus of elasticity of the steel wire used in the cables of long span, stiffened suspension bridges also has the value of about 30,000,000 pounds, the ultimate tensile resistance of such wire varying from about 200,000 to 220,000 pounds per square inch. The resisting capacity of this material is largely affected by the process of cold drawing in its manufacture, but the modulus of elasticity seems to experience little or no effect of the cold working.

Variation of Ultimate Resistance with Area of Cross-section.

The ultimate resistance of a ductile material like steel depends to some extent upon the area of cross-section for a number of reasons.

Generally the work put upon a bar of small cross-section in reducing between the rolls from the ingot or bloom to the finished bar will be greater for a bar of small section than for a similar bar of large section. Other things being equal, the greater amount of such work put upon the material the higher will be its physical qualities, including the ultimate resistance. Again, the temperature of a small bar or thin plate during its last passes between the rolls will generally be lower than for a bar of larger cross-section or for a thicker plate. In other words, the slight tendency toward cold rolling tends to enhanced ultimate resistance and elastic limit.

Finally at the section of ultimate failure there is a "necking down" to the final reduction of area of fracture within a short length of bar. This means a rather violent movement or flow of molecules of the material toward the axis of the bar, distinctly greater in distance for a larger bar than one of smaller section for the same percentage of final reduction. This corresponds to a greater longitudinal separation of the molecules near the axis of the specimen for a large bar than for a small one, which induces a little earlier rupture in the former bar than in the latter.

For all these reasons the somewhat smaller ultimate resistance per square inch of cross-section is to be anticipated for bars of large section, or plates of greater thickness than for bars of smaller sectional area, or for thin plates. This difference, however, is much less for steel bars and plates at the present time than in the case of wrought iron when that material was widely or even exclusively used for structural purposes.

Influence of Shortness of Specimen.

If the dimensions of a test specimen are such as to make exceedingly short that part within which failure will occur if a test is carried to rupture, there is less opportunity for the molecules of the material to move in toward the axis of the piece as failure is approached, thus preventing an unrestrained final reduction of fractured area. The result is an abnormal enhancement of the ultimate resistance. If the specimen is exceedingly short, as in the case of its being made by a groove, as shown in Fig. 1, it is readily seen that the planes of shear indicated lie mostly in the enlarged part of the test piece. This condition prevents the free movement of the molecules along the oblique planes, required to produce the necking down or final reduction of area of section. In other words, the material at and in the vicinity of the section of failure is substantially supported by that in the enlarged part of the piece, thus enabling the ultimate section of fracture to retain an abnormally large area, which correspondingly raises the ultimate resistance. Many tests have been made with wrought-iron specimens to determine the limits of this influence of shortness. These tests show that the length of the reduced part of a test piece in which the section of fracture will be found should not be less than about four times the diameter in any case and that with ductile material five or six times would be preferable. As a matter of actual engineering practice, the length of the reduced part of a test piece is never less than about eight to ten times the diameter. In the case of a test piece of rectangular section, the length should not be less than five or six times the greatest dimension of the cross-section, or preferably six to eight times that dimension.

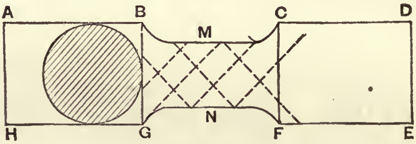


FIG. 1.

This matter of influence of shortness in test specimens is of the utmost importance in determining the true ultimate resistance of materials. If the test piece be too short the ultimate resistance will be unusually high.

Elastic Limit, Resilience, and Ultimate Resistance.

In scrutinizing the results of tests of specimens and full-size members of this section, it is to be observed that the elastic limit is almost invariably the "stretch-limit," or, as it is commonly called, "the yield-point," and not the true "elastic limit," below which the ratio between intensity of stress and rate of strain is essentially constant. It has already been shown and stated that the true elastic limit is from 2000 to 3000 or 4000 pounds per square inch below the stretch-limit or yield-point. The stretch-limit is so readily observed without delaying the ordinary routine of testing that it has come to be called, although erroneously, the elastic limit, in spite of the fact that it is a little above the intensity of stress to which that term should be applied.

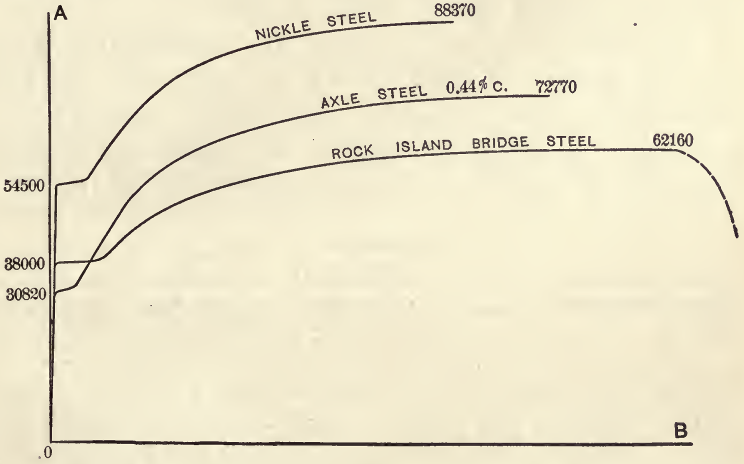


FIG. 2.

The elastic properties of three grades of steel are exhibited graphically in Fig. 2. The curved lines represent the tensile strains of the steel specimens at the intensities of stresses shown. The vertical ordinates are intensities of stress and the horizontal ordinates the rates of stretch, i.e., the stretches per unit of length, the latter being drawn 20 times their actual amounts. The Rock Island Steel belongs to a specimen of steel used for the combined railroad and highway structure across the Mississippi River at Rock Island, Ill., the data being taken from the U. S. Report of Tests of Metals for 1896. The lines for axle steel and nickel steel are the graphical representations of data taken from the "U. S. Report of Tests of Metals" for 1899. As in the previous case, the horizontal ordinates are the stretches per lineal inch shown at 20 times their actual values. The figures at the right-hand extremities of the curves are the ultimate resistances per square inch. The elastic limits and stretch-limits or yield-points are shown with clear definition. The remarkably high elastic limit of the nickel steel is well indicated.

By taking areas first between the horizontal axis *OB* and the inclined straight portion of each line, and then between the same horizontal axis and the entire line in each case, the following values of the elastic and ultimate resilience per cubic inch of each specimen will be found:

	Rock Island Steel.	Axle-steel.	Nickel-steel.
Elastic Resilience	24 in. lbs.	15.7 in. lbs.	49.6 in. lbs.
Ultimate "	10,500 in. lbs.	10,860 in. lbs.	11,040 in. lbs.

The three stress-strain lines or curves of Fig. 1 illustrate completely the physical characteristics of the various grades of steel indicated under all degrees of stress up to actual failure, except that the lines are carried only to the maximum intensities of stress sustained. If those lines were prolonged to the actual parting of the metal, they would show rapidly descending portions like the broken portion of the Rock Island Bridge steel line. That portion of the curve, however, has little practical value, although considerable scientific interest.

Table I contains a synopsis of the valuable series of tests of specimens by Prof. P. C. Ricketts. This table has already been explained on page 305. The tension tests show remarkably uniform results in elastic limit and ultimate resistance, and characterize a most excellent material. With the exception of the two Bessemer specimens containing 0.36 and 0.39 per cent carbon, all specimens were of mild steel.

Table II exhibits results of tests of a number of unusually large eye-bars 12 inches wide with other 8-inch and 7-inch bars used in the Pennsylvania Railroad bridge across the Delaware River a short distance above Philadelphia and completed in 1896. There will also be found in the table tests of specimens taken from the same bars, together with the chemical composition. This table is interesting as disclosing the ultimate tensile resistance of large bars of mild steel having the chemical composition shown. The decrease in ultimate tensile resistance and elastic limit between the original bar and the finished eye-bar, due to the process of manufacture of the latter, is also evident at a glance. Although the steel in the original bars shows ultimate resistances revealed by the tests of specimens running from 58,300 pounds to 69,500 pounds per square inch, no ultimate resistance of the completed bars exceeds 59,500 pounds per square inch, while as small a value as 52,300 pounds per square inch is found. This table is taken from the description of the Delaware River bridge by Mr. F. C. Kunz, Assistant to Vice-President of the American Bridge Company, Engineering Department, published at Vienna, 1901.

Table III gives the results of testing a remarkable series of large steel eye-bars. The table exhibits not only the physical results of the tests but the chemical composition of the metal and the relative results for annealed and unannealed bars. The table was supplied

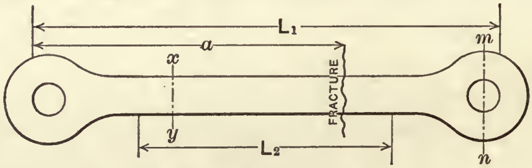
TABLE II.*

RESULTS OF TESTS OF EYE-BARS AND OF TEST SPECIMENS

Full-size Eye-bars.								Location of Fracture "a" in Inches.	Pounds per Square Inch.	
No.	Cross-sections $\overline{xy}$ in Inches.	Pin-hole in Inches.	Percentage of Excess of Section $\overline{mn}$ over Section $\overline{xy}$.	Length L_1 in Inches.		Length L_2 in Inches.			Ultimate Strength.	Elastic Limit.
				Original.	Final.	Original.	Final.			
1	$12 \times 2\frac{1}{2}$	9	37	410	472	360	414	348	55,000	29,200
2	$12 \times 2\frac{1}{4}$	9	41	407	478	360	423	434	53,500	25,900
3	$12 \times 2\frac{3}{16}$	9	41	408	471	360	416	433	53,750	27,800
4	$12 \times 2\frac{1}{16}$	9	42	422	487	360	418	280	58,300	30,600
5	$12 \times 2\frac{1}{16}$	9	42	420	485	360	417	284	57,600	30,000
6	$12 \times 2\frac{1}{16}$	9	42	407	483	360	430	296	56,300	30,000
7	$12 \times 1\frac{7}{8}$	9	41	403	485	348	423	356	53,000	29,200
8	$12 \times 1\frac{3}{4}$	9	41	400	470	348	411	332	58,000	31,400
9	$12 \times 1\frac{11}{16}$	9	38	407	471	360	419	337	54,800	29,800
10	$12 \times 1\frac{5}{8}$	9	44	407	482	360	430	256	54,900	29,300
11	$12 \times 1\frac{9}{16}$	9	38	400	447	348	387	414	52,600	29,200
12	$12 \times 1\frac{9}{16}$	9	41	415	490	360	427	266	52,300	29,900
13	$12 \times 1\frac{9}{16}$	9	42	411	489	360	430	293	58,500	30,600
14	$12 \times 1\frac{9}{16}$	—	—	—	—	—	—	—	—	—
15	$8 \times 1\frac{7}{8}$	$8\frac{1}{4}$	47	428	483	384	434	446	53,500	29,900
16	$8 \times 1\frac{13}{16}$	$7\frac{1}{2}$	34	328	394	276	334	351	53,300	29,300
17	$8 \times 1\frac{13}{16}$	$6\frac{1}{2}$	46	328	390	276	332	199	53,000	29,300
18	$8 \times 1\frac{9}{16}$	$7\frac{1}{2}$	40	328	388	276	328	324	58,500	34,000
19	8×1	$6\frac{1}{2}$	38	470	545	432	501	306	53,300	32,900
20	$7 \times 1\frac{1}{2}$	—	—	—	—	—	—	—	—	—
21	$7 \times 1\frac{1}{2}$	$8\frac{1}{4}$	44	501	575	456	525	495	59,500	32,700

---

* From page 5, "The Delaware River Bridge, Built by Pencoyd Bridge Co.," by F. C. Kunz, "Allgem. Bauzeitung," Heft 1, 1901.



by Mr. Henry W. Hodge, C.E., of the firm of consulting engineers, Boller & Hodge, of New York City. The variation in chemical composition is accounted for by the fact that the bars tested were not specimens of any actual lot, but were forged and broken for the purpose of an investigation to determine specifications under which 12 and 14 inch eye-bars for the Monongahela River cantilever bridge should be manufactured. The machine in which the eye-bar heads were formed was not of sufficient

TABLE II.—*Continued.*

TAKEN FROM THE SAME EYE-BARS, DELAWARE RIVER BRIDGE.

Percentage of			Test Specimen 8″ Long, Approx. 1″ Square.								Process of Manufacture.	No.
			Chemical Composition in Per Cent.				Pounds per Square Inch.		Per Cent.			
Stretch in 12 Inches, including Fractured Section.	Stretch for Length L_2.	Reduction of Area at Fractured Section.	C.	S.	P.	Mn.	Ultimate St'gth.	Elastic Limit.	Str'ch.	Reduction.		
48.0	15.0	49.1	0.18	0.06	0.04	0.49	65,600	33,200	26.0	45.4	basic	1
36.0	17.7	44.7	—	—	—	—	—	—	—	—	"	2
41.0	15.7	38.8	—	—	—	—	—	—	—	—	"	3
43.0	16.1	43.2	—	—	—	—	—	—	—	—	"	4
40.0	15.9	45.3	—	—	—	—	—	—	—	—	"	5
36.0	19.5	38.4	0.24	0.05	0.03	0.62	68,000	40,300	26.75	51.7	"	6
44.0	21.5	41.4	—	—	—	—	—	—	—	—	"	7
49.0	18.3	49.3	—	—	—	—	—	—	—	—	"	8
44.0	16.5	41.1	0.24	0.08	0.06	0.51	69,500	36,200	25.25	42.9	acid	9
42.0	19.4	41.2	0.23	0.04	0.07	0.55	66,700	36,300	27.25	55.2	"	10
30.0	11.2	37.4	0.21	0.07	0.06	0.51	63,800	35,800	27.00	39.5	"	11
45.0	18.8	54.3	—	—	—	—	—	—	—	—	basic	12
47.0	19.5	51.2	0.24	0.05	0.05	0.65	66,500	39,000	28.50	47.9	"	13
—	—	—	0.17	0.07	0.06	0.58	65,000	36,800	26.00	57.3	acid	14
39.0	13.1	44.9	0.21	0.04	0.02	0.45	59,500	33,700	33.50	45.6	basic	15
40.0	21.2	41.9	0.20	0.05	0.03	0.56	60,300	32,000	28.00	58.9		16
33.0	20.5	29.7	0.20	0.05	0.03	0.56	58,300	30,800	29.25	52.7		17
42.0	19.7	48.6	—	—	—	—	—	—	—	—		18
42.0	16.1	49.3	—	—	—	—	—	—	—	—		19
—	—	—	0.22	0.08	0.06	0.59	62,400	35,600	30.25	58.0	acid	20
42.0	15.4	45.8	—	—	—	—	—	—	—	—	basic	21

capacity to give satisfactory results, and hence it will be observed that most of the bars broke in the head or neck. The actual bars for the structure were to be forged in a new machine of greater capacity.

The results are highly interesting as indicating what excellent results may be obtained even for the largest bars under satisfactory conditions of manufacture.

Shape Steel and Plates.

A number of specimen tests were made, during 1899–1901, of the open-hearth acid and basic steel shapes and plates for the construction of the City Island bridge and the 145th Street bridge across the Harlem River, both in the city of New York, under the direction and super-

TABLE III.

RESULTS OF FULL-SIZE EYE-BAR TESTS ON TRIAL STEEL. MONONGAHELA RIVER CANTILEVER. BOLLER & HODGE, CONSULTING ENGINEERS.

The steel was basic open-hearth metal manufactured and rolled by the Carnegie Steel Company, 1902. All bars were about 30 feet long.

"A" means annealed and "N" not annealed.

Bar.	Chemical Composition.				Specimen.			
	Car.	Phos.	Mang.	Sulph.	Elas.	Ult.	Elongation, Per cent in 8″.	Reduct.
					Pounds per Sq. In.			
12″ × 1¾″	.30	.021	.62	.026	A 37380	67600	27.0	46.5
					N 44330	71080	21.5	40.0
12″ × 1¾″	.28	.020	.54	.022	A 32050	60230	26.5	54.1
					N 37230	68000	25.7	39.8
12″ × 1⅝″	.26	.03	.52	.02	A 38610	69700	28.7	43.9
					N 39700	72400	27.2	46.2
12″ × 1⅝″	.36	.019	.57	.035	A 31250	70720	28.5	45.6
					N 37740	76980	18.7	26.6
12″ × 1⅝″	.32	.03	.51	.03	A 29140	67120	28.5	45.5
					N 35600	72120	25.0	38.5
12″ × 1⅜″	.28	.03	.46	.04	A 37050	69820	30.5	45.0
					N 40450	73340	27.5	44.7

Bar.	Full-size Bar.				Remarks.
	Elas.	Ult.	Elongation, Per cent. in 20 Feet.	Reduct., Per cent.	
	Pounds per Sq. In.				
12″ × 1¾″	38190	64140	15.05	55.54	Broke in body.
12″ × 1¾″	37480	58670	5.7	....	" " neck.
12″ × 1⅝″	33330	61090	11.55	....	" " head.
12″ × 1⅝″	34320	63030	7.05	....	" " "
12″ × 1⅝″	34210	59170	6.0	5.96	" " body.
12″ × 1⅜″	35930	65520	9.56	....	" " head.

vision of the author. Table IV contains the results of a portion of such tests for the quality of material used. It will be seen that the specimens were taken from a wide range of shapes and plates, and that a large portion of the material was produced by the basic open-hearth process. The table is of special value in consequence

A 15 × 2-in. steel eye-bar forged at the shops of the Phœnix Bridge Co., Phœnixville, Pa. The bar developed an ultimate resistance of 50,160 lbs. per sq. in. and 28,000 lbs. per sq. in. at elastic limit. The elongation in 8 ins. of the bar, including the section of failure, was 25.6 per cent. and the elongation of the pin-hole was 5.26 inches. The reduction of area at the section of fracture was 52.9 per cent.

TABLE IV.

SPECIMEN TENSILE TESTS OF OPEN-HEARTH STEEL SHAPES AND PLATES. 1899 to 1901.

Specimen from	Loads in Pounds per Square Inch.		Elongation, Per Cent. in 8 Inches.	Final Contraction, Per Cent.	Chemical Analysis.				Character of Fracture.	Remarks.
	Elastic Limit.	Ultimate.			C.	Mn.	P.	S.		
8″ × 6″ × ½″ angle	37,880	63,260	30.5	58.7	.15	.48	.04	.04	Silky ½ cup	Basic open hearth
8 × 6 × 5/8 “	38,650	62,840	29.8	59.1	.18	.46	.03	.04	“ “ “	“ “ “
7 × 3½ × ½ “	41,720	63,120	28.7	56	.16	.41	.025	.027	“ cup	“ “ “
6 × 6 × 7/8 “	42,030	65,330	26.5	46.9	.25	.48	.023	.027	“ ½ “	“ “ “
6 × 6 × 9/16 “	40,560	62,290	26.8	55.5	.28	.42	.022	.029	“ cup	“ “ “
6 × 4 × 5/8 “	38,350	62,680	31.4	53.9	.15	.42	.02	.05	“ “	“ “ “
6 × 4 × 3/8 “	40,000	62,570	27.5	53.6	.15	.42	.04	.05	“ “	“ “ “
5 × 5 × 7/16 “	40,850	62,150	26.2	51.5	.24	.38	.02	.029	“ ½ cup	“ “ “
5 × 5 × 3/8 “	40,260	64,710	26.2	55.3	.21	.43	.038	.036	“ ang.	“ “ “
4 × 4 × ¾ “	40,970	62,500	27.5	54.6	.16	.43	.014	.030	“ cup	“ “ “
4 × 4 × ½ “	38,070	62,610	29	55.7	.21	.46	.020	.05	“ ang.	“ “ “
3½ × 3½ × 5/8 “	40,040	61,930	28	57.6	.16	.50	.016	.03	“ cup	“ “ “
3 × 2½ × 5/16 “	40,240	61,490	28.8	55.7	.12	.53	.03	.05	“ ang.	“ “ “
20″ 80 lb. I	38,300	56,400	30.4	52.6	.16	.47	.03	.05	“ “	“ “ “
20 65 “ “	36,810	55,900	30.5	50.9	.17	.50	.03	.05	“ “	“ “ “
15 55 “ “	38,030	62,690	29.4	55.5	.18	.44	.03	.05	“ cup	“ “ “
15 42 “ “	40,440	63,230	29.6	58.5	.15	.42	.04	.05	“ “	“ “ “
10 35 “ “	40,390	70,250	15.9	17.1	.24	.44	.018	.03	Granular	“ “ “
10 30 “ “	40,850	64,890	28.1	54.3	.23	.45	.037	.04	Silky cup	“ “ “
12 30 “ “	40,812	61,810	28.7	55	.26	.38	.024	.035	“ ang.	“ “ “
10 30 “ “	37,450	63,000	29.8	57.2	.16	.42	.03	.05	“ cup	“ “ “
9 20 “ “	41,070	63,920	30.4	57.4	.15	.46	.03	.04	“ ½ cup	“ “ “
7 20 “ “	45,240	67,380	30.4	57.4	.18	.39	.02	.05	“ ang.	“ “ “
6 8 “ “	39,640	67,820	28.8	59	.17	.43	.02	.05	“ cup	“ “ “
3″ × 3/8″ Z	38,300	65,040	27	47.6	.17	.60	.035	.039	“ ang.	“ “ “
97¾″ × 5/8″ plate	37,390	63,400	30	59.3	.24	.50	.017	.025	“ “	“ “ “
90¼ × ½ “	38,530	68,910	25	52.3	.23	.47	.064	.05	“ ½ cup	Acid “ “
78½ × 5/16 “	40,340	68,035	24.8	53	.19	.36	.072	.05	“ “	“ “ “
60¼ × 3/8 “	36,270	60,080	28.7	56.9	.21	.54	.028	.024	“ “	Basic “ “
48 × 3/8 “	38,870	62,860	27.5	57.9	.22	.53	.015	.03	“ “	“ “ “
36 × 5/16 “	42,220	63,730	26	53.1	.17	.39	.077	.025	“ ang.	Acid “ “
26¼ × 5/16 “	37,420	63,380	27.5	58.2	.23	.60	.028	.029	“ ½ cup	Basic “ “
18 × ¾ “	39,150	63,420	27.5	48.9	.24	.52	.02	.024	“ ang.	“ “ “
18 × ½ “	39,750	64,100	27.5	55	.23	.45	.013	.022	“ “	“ “ “
13 × 5/16 “	39,120	60,115	28.2	59.6	.15	.37	.016	.04	“ “	Acid “ “
11½ × ¾ “	40,640	67,580	26.2	52.4	.24	.50	.018	.022	“ cup	Basic “ “
7 × 5/8 “	40,500	63,070	27.5	57.2	.19	.40	.024	.027	“ “	“ “ “

of the wide range of sections covered by it, as well as for the chemical data which it contains, showing the percentages of carbon, manganese, phosphorus, and sulphur contained by the steel. Both the chemical analyses and the physical results indicate that many of the shapes are of mild steel, while the remaining portion is of low steel.

The quality of metal either in steel shapes or plates depends largely upon the amount of reduction reached in the passage of the blooms through the rolls before the final area of section is attained. In the early days of rolling steel sufficient work between the rolls was not always done, and the quality of the metal suffered correspondingly. This defect is seldom or never found at the present time and the corresponding variations in certain physical qualities are avoided. In the case of wide and thin plates, in which the temperature of the metal may be lower than in thicker plates at the last pass through the rolls, increased hardness may sometimes be found, but as a rule there will be little, if any, difference, as the preceding tables show, in the physical results for the thick and thin sections ordinarily used in engineering construction.

Tests of specimens from a large variety of shapes, plates, and bars used in the towers and stiffening trusses of the Manhattan Suspension Bridge across the East River at New York City, as given in the Report of Mr. Ralph Modjeski, consulting engineer, 1909, show the following results:

Carbon Steel for Towers.

> Metal from plates, bars, channels, bulb angles, and rivet rounds gave average elastic limits for different sets of tests varying from a maximum of 43,040 pounds per square inch down to 31,137

pounds per square inch. The average ultimate resistances of the same sets of tests gave a maximum of 65,880 pounds per square inch with intermediate values running down to 51,380 pounds per square inch. The smaller of each of these sets of results belongs to the low steel used for rivets; the higher values belong to shapes, plates and bars.

Carbon Steel for Suspended Structures.

Tests of specimens cut from shapes, plates, bars and rivet rounds gave average elastic limits running from a maximum of 44,505 pounds per square inch down to 33,907 pounds per square inch. The corresponding ultimate resistances varied from 68,652 pounds per square inch down to 52,411 pounds per square inch. Again, the smaller values are found for the low-carbon rivet steel.

Nickel Steel for Stiffening Trusses.

Tests of specimens cut from nickel-steel shapes, bars and rivet rounds used in the suspended structure gave average elastic limits for different sets of tests varying from a maximum of 61,355 pounds per square inch down to a minimum of 55,400 pounds. The corresponding ultimate resistances varied from a maximum of 90,760 pounds per square inch down to 77,268 pounds per square inch.

The preceding results for the Manhattan Suspension Bridge show values which may reasonably be expected for such carbon and nickel steels as are now in use for the best types of large bridge structures. The carbon steel for the plates, shapes and bars belongs to the grade of medium the 145th Street bridge across the Harlem River in New York City. The tests were made in 1901. The left-hand column of the table shows the particular (cast) members of the turntable from which the specimens were taken. They also show that, in steel castings, a sensibly higher grade (in the sense of containing more carbon and manganese) of steel is used than in rolled shapes. As indicated in the heading of the table, the material was acid open-hearth steel. The ultimate tensile resistance runs from about 67,000 pounds to nearly 76,000 pounds per square inch. The elastic limit is also observed to be high, in consequence of the rather large percentage of manganese. The quality of metal exhibited by the physical results of the table is fairly representative of that ordinarily used in steel castings. Obviously the ductility exhibited is less than that found in connection with rolled shapes.

TABLE V.

TENSILE TESTS OF ACID OPEN-HEARTH STEEL CASTINGS, 1901.

Specimen from	Loads in Pounds per Sq. In.		Elongation, % in 8 Ins.	Final Contraction, %.	Chemical Analysis.					Character of Fracture.
	Elastic Limit.	Ultimate.			C.	Mn.	P.	S.	Si.	
Turntable wheel.	47,510	72,300	31.2	39.8	.23	.70	.052	.004	.27	Silky cup.
Track segments..	49, 500	67,200	30.4	29.	.27	.65	.045	.018	.26	" "
" " ..	47,500	67,900	34.3	46.5	.27	.60	.047	.007	.28	" ang.
" " ..	47,500	71,100	32.0	40.8	.30	.65	.040	.004	.28	Irregular.
Rack segments..	49,775	72,115	31.2	39.8	.23	.70	.052	.004	.27	Silky ang.
Track segments .	47,500	68,100	32.6	44.5	.27	.60	.047	.008	.28	" "
" " ..	46,270	71,920	29.6	38.5	.23	.60	.052	.004	.27	" cup.
" " ..	48,900	74,700	30.4	42.5	.30	.65	.041	.009	.26	Irregular.
Turntable wheel.	46,130	71,860	28.1	37.6	.29	.70	.046	.006	.25	Silky cup.
" "	48,640	71,600	31.4	39.4	.29	.70	.046	.006	.25	" "
Rack segments...	49,775	71,335	21.9	37.1	.29	.70	.046	.006	.25	" "
Track segments..	47,600	76,020	31.6	31.7	.30	.65	.037	.004	.27	" ang.
Shoes...........	45,200	68,000	29.6	37.7	.25	.50	.043	.02	.20	Irregular.
"	47,500	68,700	31.2	40.	.29	.65	.05	.024	.27	"
"	49,800	75,700	29.6	45.5	.31	.60	.036	.003	.26	"

Rail Steel.

The grade of steel adapted to railroad rails is much higher in the hardeners carbon and manganese, and correspondingly higher in physical quantities than structural steel, at the same time it is a quite different metal from that adapted to the finer purposes of tools; it is manufactured by the Bessemer process. The great increase in the immediate past in the weight and speed of railroad locomotives and trains has subjected rails to intensely severe duties which can be performed without deterioration of metal only by steel of the highest powers of endurance, which means a steel of high ultimate resistance, elastic limit, and corresponding ductility. The grades of steel used for rail purposes at the present time are well illustrated by the following tabular statement, which shows the chemical composition of the rails of various weights and sizes used by the N. Y. C. & H. R. R. R. Co., the pounds at the head of the columns indicating the weight

NEW YORK CENTRAL & HUDSON RIVER R. R. SPECIFICATIONS.

	65-Lb.	70-Lb.	75-Lb.	80-Lb.	100-Lb.
Carbon	0.45 to 0.55	0.47 to 0.57	0.50 to 0.60	0.55 to 0.60	0.65 to 0.70
Silicon	0.15 to 0.20	0.15 to 0.20	0.15 to 0.20	0.15 to 0.20	0.15 to 0.20
Manganese	1.05 to 1.25	1.05 to 1.25	1.10 to 1.30	1.10 to 1.30	1.20 to 1.40
Sulphur not to exceed	0.069	0.069	0.069	0.069	0.069
Phosphorus not to exceed	0.06	0.06	0.06	0.06	0.06
Rails having carbon below will be rejected	0.43	0.45	0.48	0.53	0.60
Rails having carbon above will be rejected	0.57	0.59	0.62	0.65	0.70

The numbers represent the per cents of the various elements.

of rail per yard. The metal of the lightest or 65-pound rail corresponds to an ultimate resistance of 85,000 to 90,000 pounds per square inch, with an elastic limit of .5 to .7 of that value. The highest or 100-pound rail corresponds to metal having an ultimate tensile resistance of probably 110,000 to 120,000 pounds per square inch, with an elastic limit of .6 to .7 of those amounts. In these chemical compositions it is pertinent to observe the high carbon and manganese, and the low phosphorus and sulphur.

After several years' experience in the effort to secure a most enduring steel for a railroad rail weighing 135 pounds per yard, Mr. James O. Osgood, Chief Engineer of the Central Railroad of New Jersey, states in a paper published in the Official proceedings of the New York Railroad Club for May 21, 1915, that the following chemical composition has yielded the most satisfactory results within the experience of that road, on which, where these heavy rails are laid, the traffic is of excessive intensity.

Carbon .85 to 1.00 per cent or carbon .8 to .95 per cent.

The rails having the latter carbon content also contain chromium 0.2 to 0.4 per cent and nickel 0.2 to 0.4 per cent. It will be observed that this rail section, i.e., 135 pounds per yard, is the heaviest yet rolled and used in the United States up to the date of Mr. Osgood's paper.

Rivet Steel.

The grade of steel ordinarily used for rivets is the softest, or lowest in hardeners, employed in engineering construction; it should thus be correspondingly low in phosphorus and carbon. In Table I of this article there will be found the measures of ductility and other physical properties of a number of specimens of rivet steel, which are fairly representative of that metal, except that the ultimate resistance is frequently much lower than is shown there. In much of the rivet metal used at the present time the ultimate tensile resistance may run from 52,000 to 60,000 pounds per square inch. In such steel the carbon may run down to .06 or .08 per cent. with sulphur between .02 and .03 per cent., and phosphorus even lower. The treatment to which rivet metal must be subjected in the heading of rivets makes it imperative that the metal possess qualities of ductility and toughness to an unusual degree and that the variations of temperature in the rivet shall not reduce its resisting capacity. In other words, rivet steel must possess physical properties enabling it to resist torturing treatment to the highest practicable degree.

Nickel Steel.

The alloy, nickel steel, to which the allusion has already been made in connection with the subject of the modulus of elasticity of steel, possesses marked characteristics of high ultimate resistance and elastic limit, the latter usually running from $\frac{6}{10}$ to $\frac{3}{4}$ of the former. The amount of nickel in the alloy is usually about 3.25 per cent, while the carbon content may frequently be .25 to .30 per cent, although higher values of the nickel content will be found in the table following, which shows the results of tests of both full-size eye-bars and specimens cut from those bars. That table* shows the high ultimate resistance and elastic limit yielded by this material, with but little if any decrease in ductility. The effects of annealing may be observed to be practically the same as for carbon steel.

---

* The results in this table were courteously given to the author by Mr. HenryW. Hodge, C. E., of the firm of consulting engineers, who designed and built the St. Louis Municipal Bridge, at St. Louis, Mo.

TABLE VI.

FULL SIZE, EYE-BAR TESTS ST. LOUIS MUNICIPAL BRIDGE, ST. LOUIS, MO.

BARS MADE BY AMERICAN BRIDGE CO, AMBRIDGE PLANT, 1910-1911.

NICKEL STEEL MADE BY CARNEGIE STEEL CO. TESTED AT AMBRIDGE WORKS.

No.	Specimen Tests.								Chemical Analysis.					Dimensions of Full Size Bars.	
	Unannealed.				Annealed.										
	Elastic Limit per sq.in.	Breaking Stress per sq.in.	Elongation, % in 8 ins.	Reduction of Area, %	Elastic Limit per sq.in.	Breaking Strain per sq.in.	Elongation % in 8 ins.	Reduction of Area. %	C.	P.	S.	Mn.	Nl.	Section. ins.	Length, c. to c.
1	60,140	100,700	18.2	37.1	52,000	91,950	21.5	43.1	.40	.014	.023	.64	3.43	$16\times 2\frac{1}{16}$	30′ 0″
2	59,120	96,620	17.0	37.7	54,880	90,500	23.2	39.4	.40	.010	.026	.55	3.70	$16\times 2\frac{1}{16}$	38 0
3	61,060	96,880	20.0	34.2	60,840	90,800	25.2	49.6	.37	.010	.030	.59	3.81	$16\times 2\frac{1}{16}$	38 0
4	63,810	108,900	15.0	30.6	57,660	92,680	21.2	39.5	.40	.010	.027	.57	3.41	$14\times 1\frac{15}{16}$	43 4
5	60,480	96,960	19.5	33.6	55,880	92,940	24.7	44.6	.41	.010	.030	.55	3.60	$14\times 1\frac{5}{8}$	38 0
6	57,560	94,900	20.7	41.8	54,060	91,160	21.7	41.2	.35	.012	.030	.50	3.57	$14\times 1\frac{5}{8}$	38 0
7	63,810	108,900	15.0	30.6	57,660	92,680	21.2	39.5	.40	.010	.027	.57	3.41	$14\times 1\frac{15}{16}$	42 4
8	59,940	98,450	18.5	27.2	55,560	92,680	22.5	35.2	.36	.019	.034	.59	3.29	$16\times 2\frac{1}{16}$	30 0
9	61,160	96,820	16.0	31.9	61,220	91,200	21.2	33.9	.37	.019	.030	.57	3.28	$14\times 1\frac{7}{8}$	10 0
10	59,580	101,600	17.5	28.0	58,720	91,000	20.2	39.1	.37	.010	.027	.54	3.33	$16\times 2\frac{1}{16}$	30 0
11	53,580	91,600	24.0	40.5	53,560	84,120	26.2	47.6	.35	.013	.030	.54	3.44	$16\times 1\frac{15}{16}$	30 0
12	61,080	98,320	22.0	30.5	54,800	90,600	23.0	39.7	.32	.010	.035	.65	3.26	$16\times 1\frac{15}{16}$	30 0
13	60,080	102,900	20.0	32.7	57,140	95,040	22.0	42.9	.39	.010	.030	.62	3.47	$16\times 1\frac{3}{4}$	30 0
14	61,200	112,100	17.5	36.9	61,220	104,400	22.0	38.4	.42	.010	.034	.65	3.63	$16\times 1\frac{15}{16}$	30 0
15	60,560	103,400	16.7	26.5	60,220	95,620	22.2	37.2	.40	.012	.030	.55	3.51	$16\times 1\frac{15}{16}$	30 0
16	59,040	109,150	10.0	22.0	60,980	98,430	21.2	41.4	.38	.010	.029	.60	3.65	$12\times 1\frac{5}{8}$	31 6
17	58,900	93,630	21.2	37.2	53,520	90,460	23.2	50.7	.34	.012	.033	.58	3.44	$14\times 1\frac{5}{8}$	32 10
18	62,060	100,600	21.7	38.7	53,900	91,510	22.5	40.6	.36	.020	.032	.63	3.31	$16\times 1\frac{15}{16}$	32 0
19	63,540	107,900	17.5	32.5	60,400	101,300	22.5	41.7	.39	.012	.025	.63	3.48	$12\times 1\frac{5}{8}$	33 6
20	58,880	95,180	19.2	32.9	54,450	83,920	23.5	41.8	.34	.016	.032	.50	3.34	$16\times 2\frac{1}{16}$	30 0
21	59,560	102,800	17.0	27.9	54,040	97,460	21.2	37.2	.39	.010	.034	.65	3.47	$16\times 2\frac{1}{16}$	30 0
22	62,490	99,700	21.7	39.6	54,360	93,120	23.7	43.3	.36	.025	.030	.62	3.43	$16\times 2\frac{1}{16}$	38 0
23	57,200	96,120	18.0	30.4	57,260	95,660	20.7	37.6	.40	.011	.028	.55	3.42	$16\times 1\frac{15}{16}$	30 0
24	59,360	98,900	17.2	31.5	54,440	92,250	19.5	35.9	.37	.018	.022	.55	3.25	$14\times 1\frac{15}{16}$	42 4
25	55,000	95,000 to 110,000	16.0	25.0	52,000	90,000 to 105,000	20.0	35.0	.45	.040	.040	.70	3.25	Required by Specification	

TABLE VI.—*Continued.*

No.	Dimension of Full Size Bars.			Full Size Bar Tests.								Remarks.
	Size of Pin. Ins.	Size of Head. Ins.	Excess in Heads %	Elastic Limit per sq.in.	Breaking Stress per sq.in.	Reduction of Area, Per Cent.	Elongation.					
							Per cent in 12 ins.	Measured		Pin Holes		
								Feet.	%	*A*, %	*B*, %	
1	15	35½	35.70	51,400	98,800	32.7	31.0	18	12.4	20.1	14.5	Square granular.
2	13	35½	45.30	56,060	93,180	40.0	40.0	18	15.8	24.4	22.4	100% silky.
3	13	35½	42.10	54,920	85,000	46.9	41.0	18	14.1	21.3	19.5	90% silky, ½ cup.
4	13	33	46.30	56,540	97,040	23.7	24.6	18	16.2	12.0	16.4	Square granular.
5	12	31	34.07	53,930	85,210	37.2	28.0	18	16.2	25.6	17.5	75% silky irregular.
6	12	31	37.80	48,920	80,470	40.7	39.6	18	16.3	22.1	19.3	90% silky.
7	13	33	42.30	56,420	89,030	....	6.7	18	6.11	....	9.8	Broke in head.
8	13	35½	39.00	55,150	90,900	30.4	......	18	14.1	20.0	28.5	30% silky, 70% fine granular.
9	10	31	52.40	57,850	92,190	33.4	25.0	5	18.4	24.2	20.1	75% silky, ½ cup.
10	13	35½	43.90	58,250	94,370	....	..10.1	18	8.4	....	20.5	Broke in head.
11	15	36	43.50	54,828	87,360	28.7	32.5	18	17.5	18.7	19.2	Special treatment after rejecting specimen
12	13	35½	45.40	57,660	88,270	44.1	35.8	18	13.1	16.0	14.0	100% silky.
13	15	35½	38.90	57,520	92,680	35.2	31.6	18	16.3	13.4	11.4	Silky.
14	13	35½	40.70	63,140	107,920	32.1	25.8	18	10.5	16.0	18.9	Special treatment after rejecting specimen
15	13	35½	41.80	62,990	94,480	22.7	22.4	18	13.6	18.3	19.8	Special treatment after rejecting specimen
16	10	26	40.50	59,430	99,640	12.7	15.0	18	11.0	17.8	....	Broke in head.
17	12	31	37.50	56,870	88,180	44.1	32.3	18	16.3	24.0	16.5	Special treatment after rejecting specimen
18	13	35½	46.90	53,840	89,030	43.7	31.8	18	15.3	17.8	15.8	80% silky.
19	10	26	40.60	58,670	96,500	36.5	28.6	18	10.2	17.2	17.7	75% silky.
20	13	35½	48.20	49,640	80,710	31.5	33.6	18	17.1	14.7	19.8	75% silky.
21	13	35½	48.90	60,680	94,350	43.9	30.0	18	13.9	15.7	12.6	70% silky.
22	13	35½	45.90	59,350	95,320	44.4	35.0	18	12.0	13.5	20.9	95% silky, ½″ cup, pin hole $\frac{5}{16}$″ off center.
23	15	35½	37.70	54,030	87,410	42.3	35.8	18	12.8	19.4	16.5	75% silky.
24	13	33	46.90	58,050	90,710	36.9	29.8	18	11.0	9.8	12.9	80% silky.
25	Required by Specification			48,000	85,000 to 100,000	35.0	......	18	10.0			

The following tabular statements give the physical qualities of nickel steel adapted to the various purposes indicated. They are taken from results published in the *Railroad Gazette* for August 8th, 1902.

NICKEL-STEEL FORGINGS.

	Tensile Strength, Lbs.	Elastic Limit, Lbs.	Exten., Per Cent.	Cont., Per Cent
Driving-wheel axles............	99,310	64,170	25.00	53.76
Piston-rods....................	90,140	60,090	25.50	54.08
Main crank-pins................	93,570	65,450	24.00	49.37
Front crank-pins...............	92,180	64,170	24.50	51.00
Connecting-rods and guides....	92,040	59,820	26.00	53.01

NICKEL-STEEL CASTINGS.

Crosshead......................	84,540	53,980	18.50	31.10
Furnace-bearer, bearer-guide...	85,050	54,490	18.00	26.04
Annealed:				
Carbon steel................	109,500	51,440	19.50	36.31
Nickel steel................	100,330	66,720	25.00	54.56
Oil-tempered:				
Carbon steel................	129,360	67,230	17.50	38.53
Nickel steel................	103,890	76,390	25.00	61.56

SMALL RIFLE BARRELS—NICKEL STEEL.

Tensile Strength, Lbs.	Elastic Limit, Lbs.	Ext. in 2 Inches, Per Cent.	Cont. or Area, Per Cent.
115,100	99,820	23	64.00
114,080	97,780	23	64.95
114,590	99,820	23	65.45
116,620	96,770	22.50	62.05
116,120	97,780	23	64.00
114,590	98,800	24	62.53

Vanadium Steel.

The alloy called vanadium steel contains when used for many purposes some chromium, which frequently gives it the name Chrome Vanadium Steel. This grade of steel contains carbon and manganese about in the proportion of ordinary structural steel. Indeed it may be considered ordinary structural steel alloyed with chromium and vanadium. The addition of these latter materials gives to the resulting product great toughness with high ultimate resistance and an elastic limit remarkably high in proportion to the ultimate resistance. It is used largely for such special purposes as locomotive parts, both as castings and in the forged condition. In either case, however, it requires heat treatment. It is largely used for locomotive frames, axles, piston rods, crank pins, tires, as well as for many parts of automobiles.

Many physical tests of small specimens have been made giving elastic limits of about 40,000 pounds per square inch (for castings) up to about 100,000 pounds per square inch, the corresponding ultimate tensile resistance being about 70,000 pounds per square inch up to about 150,000 pounds per square inch. These variations in physical qualities depend upon chemical contents of the alloy and upon the condition of the material as cast or rolled, and finally upon the heat treatment of the material.

In a paper on "Vanadium Steel in Locomotive Construction" by George L. Norris, Engineer of Tests of the American Vanadium Co., published in the Official Proceedings of the New York Railroad Club, 1915, he gives the following chemical contents as meeting the requirements for the locomotive parts indicated.

Chemical Contents of Chrome Vanadium Steel.

	Castings	Axles, Piston Rods and Crank Pins	Tires
Carbon........	.20 to .30%	.30 to .40%*	.50 to .65%
Manganese.....	.50 to .70	.40 to .60	.60 to .80
Chromium.....	0	.75 to 1.25	.80 to 1.10
Silicon.........	.20 to .30	Not over .20	.20 to .35
Vanadium......	Over .16	Over .16	Over .16
Phosphorus....	Not over .05	Not over .04	Not over .05
Sulphur........	Not over .05	Not over .04	Not over .05

* Preferred .35%

The elastic limit, ultimate resistance, final stretch and final reduction of area corresponding to the grades of material indicated by the chemical contents are shown in the next table.

PHYSICAL REQUIREMENTS (After Heat Treatment).

	Elastic Limit Lbs. per sq. in.	Ult. Resist. Lbs. per sq. in.	Stretch in 2 ins.	Reduction of Area.
Castings......	40,000– 50,000	70,000– 85,000	25%	45%
Axles, Piston Rods and Crank Pins...	80,000–100,000	95,000–125,000	25	55
Tires 56″ diam. and under....	110,000–125,000	140,000–160,000	Min. 12	Min. 30
Tires over 56″ diam........	95,000–115,000	120,000–140,000	Min. 15	Min. 35

These physical requirements correspond closely to the usual results of tests. They show the high elastic limit of the material and its high degree of ductility.

Castings must be carefully annealed by heating slowly to about 1550° F. and then slowly cooling.

The heat treatment for chrome vanadium driving axles consists of:

" (1) annealing the rough forging by heating carefully and cooling slowly, (2) reheating, forging, and quenching in water or oil, preferably the latter, (3) then promptly reheating slowly and uniformly to a temperature sufficiently high to give the desired properties. The forging must be held at this final or draw-back temperature for at least two hours. The axle should then be allowed to cool slowly.

" The recommended temperature for annealing is 1475–1525° F., and for quenching from 1600° F. to 1650° F. The final heating for obtaining the physical properties should be approximately 1100° F. to 1200° F."

The heat treatment to which vanadium side rods, piston rods, and crank pins are submitted is the same as that given above for driving axles.

In the manufacture of locomotive tires, the heat treatment is somewhat different from that set forth above, as it consists of:

"(1) In reheating the tires after rolling, and then quenching in oil, (2) then reheating slowly and uniformly to a temperature sufficiently high to obtain the desired physical properties. The tire must be held at this final temperature at least two hours, which is considered the minimum time required for the changes to be effected throughout the tire section. The tire should then be withdrawn from the furnace and allowed to cool in still air.

"The recommended temperature for quenching is about 1600° F. The final heating for obtaining the physical properties specified should be approximately 1100 to 1200° F."

It is obvious that material with such physical properties possesses unusual toughness and resilience. For that reason it is specially adapted to locomotive springs and other similar uses. For such a purpose the carbon contained is relatively high. Mr. Norris in the paper already indicated gives the following as a suitable chemical composition:

Chemical Composition.

	Per cent.
Carbon	0.52 to 0.60
Manganese	0.70 to 0.90
Chromium	0.80 to 11.0
Vanadium	Over 0.16
Phosphorus	Not over 0.04
Sulphur	Not over 0.04

This material requires heat treatment consisting of:

"(1) Heating and quenching in oil, (2) then reheating or drawing back, preferably in a lead bath, and cooling slowly. The time in the lead bath should be 10 to 15 minutes.

"The recommended temperature for quenching is from 1575 to 1650° F. The drawback or annealing temperature should be approximately from 900 to 1100° F."

When such material is tempered for railway springs it has the following physical properties:

"Elastic limit, lbs. per sq.in.......	160,000–180,000
Tensile strength, lbs. per sq.in....	190,000–230,000
Elongation in 2 inches..........	10–15%
Reduction of area..............	30–45%"

This material possesses the highest physical properties of the steels yet used for commercial purposes.

Some recent tests, June, 1915, reported by the American Vanadium Company, show excellent results for carbon-vanadium steel both in the natural condition of the specimens and after simple annealing as well as after heat treatment, the latter yielding highest results generally, but not for ultimate resistance, the ductility, however, being distinctly lower in the natural condition. The following table gives the results of the tests as well as the chemical analysis and treatment. The first six sets of values belong to test specimens taken from 7-inch and 11-inch axles, while the last three belong to specimens from connecting rods.

TESTS OF CARBON–VANADIUM STEEL

	Carbon	Manganese	Phosphorus	Sulphur	Vanadium
Chem. Analysis..	0.47%	0.90%	0.012%	0.020%	0.15%

PHYSICAL PROPERTIES

Treatment.	Yield Point lbs. per sq. in.	Elastic Limit lbs. per sq. in.	Ultimate Resist. lbs. per sq. in.	Stretch in 2 in. Per-cent.	Reduction of Area Per-cent.
Natural	71,200	68,000	123,000	16.0	30.0
Annealed 1450° F.	56,000	52,000	90,000	24.0	50.0
O. Q. 1600; T. 1160° F.	85,000	82,000	112,500	22.0	55.0
Natural	75,000	70,000	117,000	16.0	28.5
Annealed 1450° F.	58,000	54,000	94,000	22.0	47.0
O. Q. 1600; T. 1160° F.	87,000	80,000	115,000	20.5	52.0
Natural	92,000	85,000	131,000	17.0	44.0
Annealed 1450° F.	71,500	67,000	105,000	23.5	52.0
O. Q. 1600; T. 1160° F.	92,500	86,000	123,000	20.5	50.0

Effect of Low and High Temperatures on Steel.

There has been much difference of opinion expressed upon the effect of low temperature upon steel, especially upon steel rails. The high number of breakages in steel rails during the winter, particularly in the early days of the use of steel for such a purpose, has given the impression that low temperatures in the vicinity of zero degrees F., or lower, make steel brittle and hence subject to sudden fracture without warning in the cold weather of winter. This impression has been shown to be without material foundation in rails of the best quality, but phosphorus makes iron and steel "cold short." If, therefore, there should be a sufficient amount of phosphorus present steel or iron would undoubtedly become more liable to fracture at low temperatures. In the early days of rail making, when the constituent elements were not so carefully controlled as at present, it is highly probable if not practically certain that the presence of phosphorus accounted for many breakages at low temperatures. For many years, however, the effects of the prejudicial hardeners phosphorus and sulphur have been well recognized and they have been kept so low as to have no material effect upon the finished products.

Again, frozen ground in the winter adds somewhat to the rigidity of a roadbed, enhancing to some extent the effects of shocks or blows to which rails are subjected under rapidly moving heavy train loads. Some of the increased breakages in the winter are probably due to this cause and it is possible that a great majority of them may be accounted for in this way.

On the whole the latest experiences do not seem to indicate that with the excellent quality of steel now produced for engineering purposes the effects of low temperatures are at all serious, but that they may be ignored when suitable precautions are taken in the processes of manufacture.

The effect of high temperature, on the other hand, is a matter of some concern in connection with building construction, since the ultimate carrying capacity of iron or steel may be seriously affected or even destroyed by the high temperatures of conflagrations unless the supporting members are protected against the effects of intense heat.

Figs. 3 and 4 represent the results of investigations by Prof. R. C. Carpenter, formerly of Cornell University, who made tensile tests on wrought iron and steel circular specimens .5 inch in diameter. Fig. 3 is self-explanatory. It shows the graphical relation between the temperatures of the specimens and the ultimate tensile resistance per square inch.

The ductility represented by the final elongations or stretches in 8 inches at the corresponding temperatures of rupture are exhibited in Fig. 4.

Prof. Carpenter observes "that all the curves have a point of contraflexure at about 70° F., and another at a temperature not far from 500°. The maximum strength is found at temperatures of 400° to 550°. At temperatures higher than this all the materials show a rapidly decreasing strength."

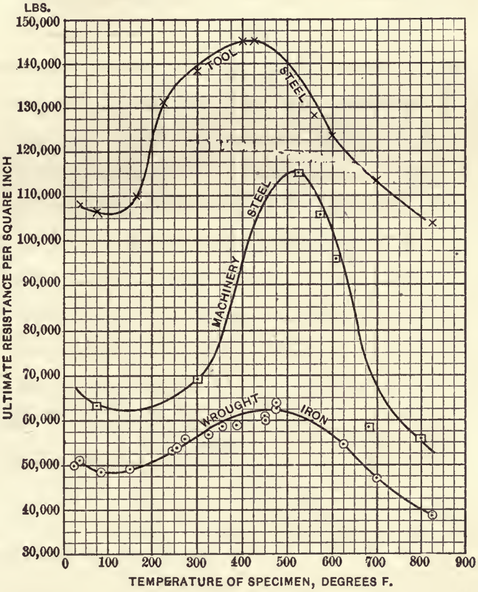


FIG. 3.

As a general result or consensus of all results, including the older and the later, it may be stated that iron and steel lose no sensible portion of their resisting capacity under about 500° Fahr., but that softening is liable to begin when the temperature rises much above that limit. At a temperature of about 800° Fahr. these metals may lose as much as 20 per cent. of ultimate resistance.

Hardening and Tempering.

The processes of hardening and tempering are not usually applied to structural steel, but to those higher grades of metal used for such special purposes as tools or wire. The hardening process consists in heating the steel to such temperature as may be desired to accomplish a given purpose and then quenching in water, brine, oil, molten lead, or other proper bath. The temperature from which the quenching is done may be that indicated

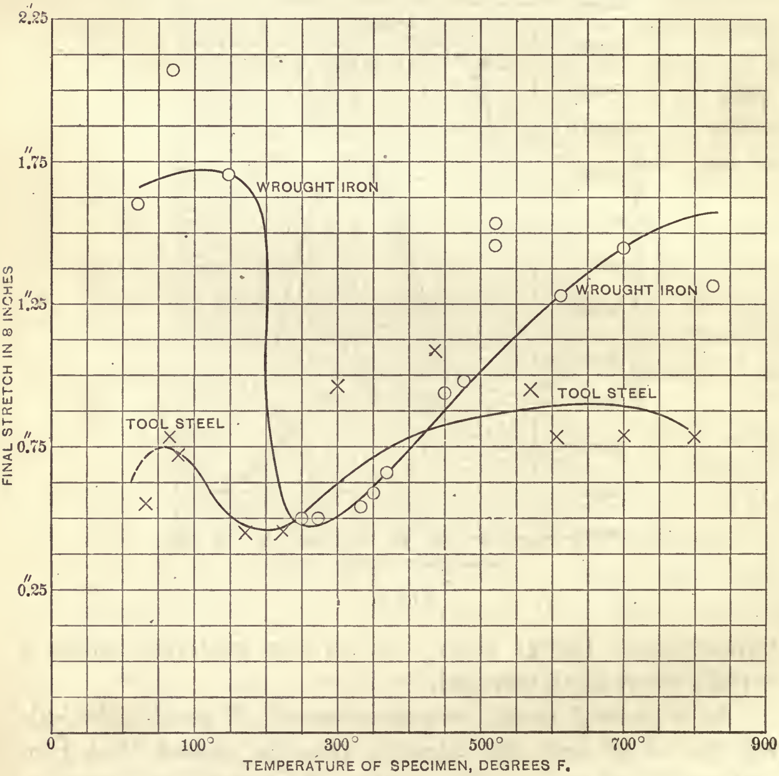


FIG. 4

by an orange color; it depends upon the size or character of grain of metal desired. In general terms, the higher the content of carbon, the more marked will be the results of the hardening processes. Quenching has a comparatively small effect upon low or medium structural steel.

The process of tempering is, in reality, supplementary to the process of hardening in the manner just described. After a piece of steel has been hardened by quenching so that its temperature is that of the air, if it be again heated it will exhibit different colors as the temperature is increased. The first noticeable color will be a light delicate straw, then deep straw, light brown, dark brown, brownish blue, called "pigeon wing," light bluish, light brilliant blue, dark blue, and black, after which the temper is completely removed. The preceding colors are due to thin films of oxide that form on the exterior surfaces of the pieces as the temperature increases. When this heating is stopped at any color and the steel allowed to cool, the metal is said to be drawn to the temper shown by the corresponding color.

The tempers at different colors for different processes are sometimes stated as follows:

Light straw	For lathe-tools, files, etc.
Straw	“ “ “ “ “
Light brown	“ taps, reamers, drills, etc.
Darker brown	“ “ “ “ “
Pigeon wing	“ axes, hatchets, and some tools.
Light blue	“ springs.
Dark blue	“ some springs, occasionally.

Tempering or hardening increases both the elastic limit and ultimate resistance, but decreases the ductility.

Annealing.

The processes of annealing, like those of hardening and tempering, produce more marked results in the higher steels than in the lower. Steel has a sensibly varying density at different temperatures; in other words, a given weight of metal will occupy sensibly different volumes at different temperatures. Hence if a piece of steel be subjected to any operation, such as forging, which gives to different portions concurrently widely varying temperatures, those portions will necessarily be subjected to considerable intensities of internal stresses, and if those stresses are not removed they may reduce greatly both the ultimate resistance and ductility. In the higher grades of steel and in special steels it is, therefore, imperative to anneal members which have been subjected to such operations. These observations are specially pertinent to such high steels as those adapted to the manufacture of tools or other similar purposes. In general it is necessary in structural engineering practice to resort to annealing only in the case of eye-bars, or other members which have been subjected to the operations of forging. The process consists simply in heating the member to be annealed to about a cherry-red temperature until the piece is heated through, and then allowing it to cool gradually to a normal temperature. At the cherry-red heat the metal is sufficiently softened to allow the molecules to readjust their relative positions so as to remove the internal stresses. After the operation of cooling is completed the metal will be at least approximately, if not entirely, in a condition of no internal stresses, i.e., if the annealing has been properly done. The more gradually and uniformly the cooling is accomplished the more excellent will be the results. Sometimes resort is made to such special means to accomplish these ends as covering the members, after bringing them to a proper temperature, with sand, ashes, or other similar material, to insure a slow and uniform cooling.

The preceding tables show what is always found in a comparison of results for the natural and the annealed metal. The process of annealing will diminish the ultimate resistance of structural steel in general from about 4,000 to 6,000 or 8,000 pounds per square inch, and the elastic limit will be reduced correspondingly. These effects will be found more marked as the metal is finished between the rolls at lower temperatures. In general, steel which is hardened by the conditions of manufacture, like that of comparatively low temperature in rolling, will exhibit greater decreases of ultimate resistance and elastic limit under annealing.

The process of annealing increases the ductility of the steel, since it softens the metal. In spite of the reduction in ultimate resistance and elastic limit, therefore, the operation gives a valuable quality to the steel.

Effect of Manipulations Common to Constructive Processes; Also Punched, Drilled and Reamed Holes.

The shop treatment of steel must in some respects be peculiar to that metal and different from that which characterizes the manufacture of wrought-iron bridge members. While the processes of punching and shearing may not be specially injurious to comparatively thin plates and shapes of low steel and of the lower carbon grades of mild steel (perhaps up to a limit of 65,000 pounds per square inch) they are sufficiently injurious to heavier sections and to the higher grades of steel to necessitate the avoidance of their effects. If punches and dies are kept in good sharp condition, as they should be, the prejudicial effects are lessened. The effect of a punch, however, under the best conditions of operation is not to make a smooth-sheared surface, but one of somewhat ragged or serrated character in which incipient cracks are started and which may be continued indefinitely into the interior of the metal unless some curative procedure is employed.

It has been found by actual test that the region affected by the punch or by the jaw of the shear extends but a short distance from the cutting-edge of the tool. Within that region, however, the metal is much hardened and the loss of ductility and elevation of elastic limit is due to that hardening. The decreased ultimate resistance is probably due to the violent disturbance of the molecules and the resulting minute fissures in the metal within the same region. In riveted work, the prejudicial effect is therefore removed by reaming the punched hole to a diameter about $\frac{1}{8}$ inch larger than made by the punch. This removes a thin ring of injured metal about $\frac{1}{16}$ of an inch thick, and it is found sufficient for the purpose.

In large and heavy work it has come to be the practice by the best shops to make drilled holes in which cases no question of the injury of metal can arise. The use of the drill leaves a sharp edge at each surface of the plate which tends to produce a shearing effect upon the corresponding rivet sections. Some specifications require this to be overcome by a quick application of a proper tool to remove the sharp edge.

The general effects of the cutting edge of the shear are precisely the same as those of the punch, as the operation in each case is a shear. Hence, if sheared edges are planed off to a depth of one-sixteenth to one-eighth of an inch, the injured metal will be entirely removed. The hardening effects of both shearing and punching may also be removed by the process of annealing, although less effectually than by reaming and planing. As naturally would be inferred by experience in punching, higher steel and thicker plates are more injuriously affected by shearing than low steels and thinner plates.

In consequence of the irregular edge of a large sheared plate, bridge specifications frequently require that at least one-quarter of an inch of metal shall be removed from the edge of such plates by planing.

Steel seems to be very sensitive to the effects of hammering or working at what is termed a " blue heat." Consequently it is necessary to heat the rivet to such a temperature as will enable the operation of heading to be completed before the rivet cools to the blue stage. A bright red or yellow heat is requisite for good work, and the rivet should be held under a pressure of fifty or sixty tons per square inch of the shaft section until the metal has time to flow throughout the rivet length and thus completely fill the hole, otherwise the upsetting will be complete at and in the vicinity of the rivet-heads only. An additional advantage in holding the rivet under the greatest pressure of the riveter for a short time is the fact that the rivet becomes cool enough to prevent the separation of the plates.

The forging of steel requires unusual skill and experience. When a piece has been heated to a proper temperature it should be kept under work until it has fallen in temperature to a proper point to secure all the advantages of working, but of course not below red heat. The forging should be done with a hammer whose weight is suitably proportionate to the mass to be forged. If the hammer is too light, the result will be a surface effect only, with the interior but little changed. Pressure forging, with appropriate facilities for attaining great pressures, is probably capable of producing the best results.

The operation of annealing, particularly as applied to full-size bars, is one of great importance in the manufacture of structural steelwork. The metal is heated as uniformly as possible, so that undue stresses will not be developed, to a bright cherry-red, corresponding probably to about 1100 or 1200 degrees Fahr., and then allowed to cool gradually. By this means any internal stresses that may have been produced by the process of forging, or any other shop manipulation, are eliminated. The metal is sufficiently softened at the highest temperature to allow the molecules to adjust themselves to a condition of essentially no stress, and if the cooling is gradual the internal stresses will not be re-developed.

Change of Ultimate Resistance, Elastic Limit and Modulus of Elasticity by Retesting.

It has been observed from the earliest experiences in testing steel and wrought iron that if a piece of material be subjected to an intensity of tensile stress higher than the elastic limit, thus producing permanent stretch, the ultimate resistance will be materially increased, although the ductility is generally decreased. Sufficient investigation has not even yet been undertaken to gage the full significance of such phenomena, but enough has been done to show some important results.

It is yet uncertain whether an indefinitely long rest may not diminish to some extent at least the enhanced ultimate resistance of a piece of metal stressed beyond the elastic limit. Professor Bauschinger made some investigations in this special field many years ago which indicate that the elastic limit is considerably decreased by immediate retesting, but that such a decrease does not take place if a period of at least twenty-four hours or possibly more elapses before retesting. Some tests indicate that the elastic limit may be much increased even by suitable periods of rest between applications of loading.

The yield point appears to be raised materially by retesting and the same observation as already indicated is equally applicable to the ultimate resistance.

Fracture of Steel.

The character of steel fractures has been incidentally noticed, in some cases, in the different tables.

If the steel is low, or if some of the higher grades are thoroughly annealed, the fracture is fine and silky, provided the breakage is produced gradually. In other cases the fracture is partly granular and partly silky, or wholly granular.

In any case a sudden breakage may produce a granular fracture.

The Effects of Chemical Elements on the Physical Qualities of Steel.

Anything more than a meagre statement of the influence of the chemical composition of steel on its physical properties is obviously out of place here, but a knowledge, however slight it may be, of the influence of certain elements on those properties is so essential to the engineer in his structural work that attention should at least be called to it.

Although other elements exert highly important influences upon the resisting qualities of steel, carbon is undoubtedly the most prominent hardener. The effect of a given percentage of carbon, at least within certain rather wide limits, is to give greater toughness and resisting qualities to steel with less concurrent brittleness than any other contained element. It is made, therefore, the basis of classification of structural steel, the low steels being low in carbon and the high steels high in carbon.

The metal manganese also gives to steel some advantageous qualities. At the present time it seldom enters steel to an amount less than .5 per cent., nor more than about 1 per cent. Its presence seems to confer the capacity of resisting the effects of high temperatures in shop processes. Metal low in phosphorus and sulphur appears to require less manganese than that which is higher in those impurities. It has been found that the influence of manganese upon steel depends in a rather extraordinary manner upon its amount. If the content reaches 1.5 or 2 per cent. steel becomes practically worthless on account of its brittleness, but when a content of 6 or 7 per cent. of manganese is reached, the metal becomes extremely hard and acquires to a high degree the property of toughness by quenching in water without becoming much harder.

When steel is alloyed with more than about 7 per cent. of manganese, manganese-steel is the product, which, in its natural state, may have an ultimate tensile resistance running from 74,000 to over 116,000 pounds per square inch. When quenched in water the ultimate tensile resistance of the same metal may run from about 90,000 pounds per square inch up to nearly 137,000 pounds per square inch. Before quenching the final stretch ranged from 1 to 4 or 5 per cent., and after quenching from 4 to 44 per cent. The preceding figures belong to a range in manganese from about 7 per cent. to over 19 per cent. concurrently with carbon from about .61 per cent. up to 1.83 per cent. This metal is an interesting alloy, but is never used in structural engineering work.

Opinions vary much as to the influence of silicon on steel, but it seems now to be well established that that influence within the limits ordinarily found is of minor consequence, or at least not prejudicial to either resistance or ductility. In structural steel it usually ranges from less than .03 to .05 per cent., while in rail steel it may run as high as .3 per cent. In some excellent tool-steel it may run even from .2 to .75 per cent.

Sulphur is an impurity carrying with it highly prejudicial effects. It essentially injures metal for rolling, as it makes the steel liable to crack and tear at the usual temperatures found between the rolls. It also diminishes capacity to weld. Its effects may, to some extent, be overcome by the presence of manganese and by proper care in heating. It is, however, highly prejudicial as an element and is usually kept below about .04 per cent.

Of all the objectionable elements found in steel, phosphorus has the position of primacy. Although it is a hardener which may increase the ultimate resistance to some extent, it produces brittleness and diminishes most materially the capacity to resist shock, and it is one of the chief purposes of the best methods of steel production to reduce phosphorus to the lowest practicable limit. Its effects are sometimes erratic, being occasionally found in excess in apparently good material. In structural steel it is seldom permitted to run over .08 per cent., and in the basic processes of manufacture it frequently falls to .03 or .04 per cent.

The presence of .1 to .25 per cent. copper appears to have no deleterious effect upon steel and may even be beneficial. As high as 1 per cent. of copper has been found in steel without serious effects where sulphur was low.

Aluminum steel is an alloy containing at times as high as 5 to 6 per cent. of aluminum. The effect of aluminum on ultimate resistance does not seem to be prejudicial, nor, again, is it of any special advantage; nor does it act seriously upon the ductility until its amount approaches about 2 per cent. or more. On the whole it does not seem to be a valuable element for steel.

There are other special alloys such as tungsten and chromium steel. They are used for the special purposes of tools on account of their hardness, which is so extreme that neither quenching nor tempering is required. They do not, however, enter into structural use.

Art. 59.—Copper, Tin, Aluminum, and Zinc, and their Alloys—Alloys of Aluminum—Phosphor-bronze—Magnesium.

Anything like a complete knowledge of the physical properties of the alloys of copper, tin, aluminum, zinc, etc., is still lacking, although many investigations have been made in the past by the late Prof. R. H. Thurston and others, while other investigations are still in progress. The character of many of these alloys changes so radically for different proportions of the constituent elements and under different conditions of heat and other treatment that the results of tests are as varied as the relative amounts of the constituents and the physical conditions which attend the tests. Some of the results which follow belong to the earlier work of Prof. Thurston, but as they exhibit the same physical qualities as the corresponding alloys now used and as the later investigations do not cover the same field, they possess real value and are retained.

Table I gives the tensile coefficients of elasticity (E) of copper and the alloys indicated as determined by Prof. Thurston.

TABLE I.

Metal.	Authority.	E.	Remarks.
Gun-bronze....	Thurston	11,468,000	Copper, 0.90; tin, 0.10 (nearly).
Alloy.........	"	13,514,000	Copper, 0.80; zinc, 0.20.
Alloy.........	"	14,286,000	Copper, 0.625; zinc, 0.375.
Tobin's alloy...	"	4,545,000	Composition, below table.
Copper.......	"	9,091,000	Cast metal

Tobin's alloy is a composition of copper, tin, and zinc, in the proportions (very nearly) of 58.2, 2.3, and 39.5, respectively. The value of E for this metal, and those for the two preceding and one following it, are calculated for small stresses and strains given by Prof. Thurston in the "Trans. Am. Soc. Civ. Engrs.," for Sept., 1881.

There will also be found in Tables VIII, IX, X and XI coefficients of elasticity for aluminum-zinc, aluminum magnesium, and other alloys, and for magnesium, aluminum, and zinc.

TABLE II.

CAST TIN.

p.	*E.*	*p.*	*E.*
1,950	1,147,000	3,200	96,400
2,360	472,000	4,000	41,540
2,580	172,000	Broke at 4,200 lbs.	

TABLE III.

CAST COPPER.

p.	*E.*	*p.*	*E.*
800	10,000,000	12,000	18,750,000
2,000	9,091,000	13,600	8,193,000
4,000	9,091,000	16,000	2,235,000
8,000	14,815,000	22,000	137,000

Broke at 29,200 lbs.

The values of E (stress over strain) for different intensities of stress (pounds per square inch) for cast tin, cast copper, and Tobin's alloy, are given in Tables II, III, and IV.

"p" is the intensity of stress in pounds per square inch, at which the ratio E exists.

Each of these metals is seen to give a very irregular elastic behavior.

Tables II, III, and IV are computed from data given by Prof. Thurston in the United States Report (page 425) and "Trans. Am. Soc. Civ. Engrs.," already cited.

TABLE IV.

TOBIN'S ALLOY.

p.	E.	p.	E.
2,000	4,545,000	18,000	5,455,000
4,000	4,545,000	24,000	5,941,000
6,000	4,688,000	30,000	6,250,000
8,000	4,938,000	40,000	6,390,000
10,000	5,263,000	50,000	4,744,000
14,000	5,110,000	60,000	3,436,000

Broke at 67,600 lbs.

Ultimate Resistance and Elastic Limit.

Table V is abstracted from the results of the experiments of Prof. Thurston as given in the "Report of the U. S. Board Appointed to Test Iron, Steel, and other Metals," and "Trans. Am. Soc. of Civ. Engrs.," Sept., 1881. The composition of the various alloys was as given in the table, which also contains results for pure copper, tin, and zinc. All the specimens were of cast metal.

The mechanical properties of the copper-tin-zinc alloys have been very thoroughly investigated by Prof. Thurston ("Trans. Am. Soc. of Civ. Engrs.," Jan. and Sept., 1881). As results of his work he has found that the ultimate

TABLE V.

Percentage of			Pounds Stress per Square Inch at		Per Cent., Fina	
Copper.	Tin.	Zinc.	Elastic Limit.	Ultimate Resistance.	Stretch.	Contraction.
100	00	00	11,620	19,872	0.05	10.0
100	00	00	11,000	12,760	0.005	8.0
100	00	00	14,400	27,800	0.065	15.0
90	10	00	15,740	26,860	0.037	13.5
80	20	00	———	32,980	0.004	00.0
70	30	00	5,585	5,585	———	00.0
62	38	00	688	688	———	00.0
52	48	00	2,555	2,555	———	00.0
39	61	00	2,820	2,820	———	00.0
29	71	00	———	1,648	———	00.0
21	79	00	———	4,337	———	00.0
10	90	00	3,500	6,450	0.07	15.0
00	100	00	1,670	3,500	0.36	75.0
	Queensl'd					
00	100	00	———	2,760	———	47.0
	Banca					
00	100	00	2,000	3,500	0.36	86.0
Gun	Bronze					
90	10	00	10,000	31,000	4.6	———
80	00	20	———	33,140	32.4	40.0
62.5	00	37.5	———	48,760	31.0	29.5
58.2	2.3	39.5	———	67,600	4.0	8.0
100	0.0	0.0	———	29,200	7.5	16.0
90.56	0.0	9.42	———	———	———	———
81.91	0.0	17.99	10,000	32,670	31.4	43.0
71.20	0.0	28.54	9,000	30,510	29.2	38.0
60.94	0.0	38.65	16,470	41,065	20.7	28.0
58.49	0.0	41.10	27,240	50,450	10.1	17.0
49.66	0.0	50.14	16,890	30,990	5.0	11.5
41.30	0.0	58.12	3,727	3,727	———	———
32.94	0.0	66.23	1,774	1,774	———	———
20.81	0.0	77.63	9,000	9,000	0.16	0.0
10.30	0.0	88.88	14,450	14,450	0.39	0.0
0.0	0.0	100.00	4,050	5,400	0.69	0.0
70.0	8.75	20.25	18,000 (?)	31,600	0.36	0.0
57.50	21.25	21.25	1,300	1,300	———	———
45.0	23.75	31.25	2,196	2,196	———	———
66.25	23.75	10.00	3,294	3,294	———	———
58.22	2.30	39.48	30,000 (?)	66,500	3.13	7.0
10.00	50.00	40.00	5,000 (?)	9,300	0.7	0.0
60.00	10.00	30.00	21,780 (?)	21,780	0.15	0.0
65.00	20.00	15.00	———	3,765	———	———

TABLE V.—*Continued.*

Percentage of			Pounds Stress per Square Inch at		Per Cent., Final	
Copper.	Tin.	Zinc.	Elastic Limit.	Ultimate Resistance.	Stretch.	Contraction.
70.00	10.00	20.00	24,000(?)	33,140	0.31	—
75.00	5.00	20.00	12,000(?)	34,960	3.2	5.4
80.00	10.00	10.00	12,000(?)	32,830	1.6	4.0
55.00	0.50	44.50	22,000	68,900	9.4	25.0
60.00	2.50	37.50	22,000	57,400	4.9	6.6
72.50	7.50	2.00	11,000	32,700	3.7	11.0
77.50	12.5	10.00	20,000	36,000	0.7	0.0
85.00	12.5	2.50	12,000(?)	34,500	1.3	3.0

The values of the elastic limit in the lower part of the table were not at all well defined.

tensile resistance, in pounds per square inch, of "ordinary bronze, composed of copper and tin, ... as cast in the ordinary course of a brassfounder's business," may be well represented by

$$T_c = 30,000 + 1,000t;$$

"where t is the percentage of tin and not above 15 per cent."

"For brass (copper and zinc) the tenacity may be taken as:

$$T_z = 30,000 + 500z,$$

where z is the percentage of zinc and not above 50 per cent."

He found that a large portion of the copper-tin-zinc alloys is worthless to the engineer, while the other, or valuable portion, may be considered to possess a tenacity, in pounds per square inch, well represented by combining the above formulæ as follows:

$$T_{zt} = 30,000 + 1,000t + 500z.$$

These formulæ are not intended to be exact, but to give safe results for ordinary use within the limits of the circumstances on which they are based.

Prof. Thurston found the "strongest of the bronzes" to be composed of:

Copper	55.0
Tin	0.5
Zinc	44.5
	100.00

This alloy possessed an ultimate tensile resistance of 68,900 pounds per square inch of original section, an elongation of 47 to 51 per cent. and a final contraction of fractured section of 47 to 52 per cent.

The first and sixth alloys of copper, tin, and zinc, in Table V, are called by Professor Thurston "Tobin's alloy." "This alloy, like the maximum metal, was capable of being forged or rolled at a low red heat or worked cold. Rolled hot, its tenacity rose to 79,000 pounds, and when moderately and carefully rolled, to 104,000 pounds. It could be bent double either hot or cold, and was found to make excellent bolts and nuts."

As just indicated for the particular case of the Tobin alloy, the manner of treating and working these alloys exerts great influence on the tenacity and ductility.

Professor Thurston states: "brass, containing copper 62 to 70, zinc 38 to 30, attains a strength in the wire mill of 90,000 pounds per square inch, and sometimes of 100,000 pounds."

All of Professor Thurston's specimens were what may be called "long" ones, i.e., they were turned down to a diameter of 0.798 inch for a length of five inches, giving an area of cross-section of 0.5 square inch.

Alloys of Aluminum.

Prof. R. C. Carpenter, of Cornell University, in the transactions of the Am. Soc. Mech. Engrs., vol. xix, has reported a number of interesting and valuable tests of alloys of aluminum, as well as tests of pure magnesium.

Table VI.

ALLOYS OF GREATEST RESISTANCE.

Percentage of			Ultimate Resistance, Lbs. per Square Inch.	Specific Gravity.	Per Cent. of Final Stretch.
Aluminum.	Copper.	Tin.			
85.	7.5	7.5	30,000	3.02	4.
6.25	87.5	6.25	63,000	7.35	3.8
5.	5.	90.	11,000	6.82	10.1

The greater part of the results for the aluminum-tin-copper alloys are given in Table VII, but the composition of those giving the greatest ultimate resistances are exhibited in Table VI. It will be observed that the highest ultimate resistance belongs to the alloy of greatest density but the alloy of least resistance has nearly as great density. The ductility of none of these alloys of greatest ultimate resistance is specially marked; indeed, the ductility is very low except in the case of the least ultimate resistance.

The composition and corresponding elastic limits and ultimate resistances of aluminum-tin-copper alloys will be found in Table VII. Like all the aluminum alloys the specific gravity varies between wide limits, being low where there is much aluminum and high where there is little. The ductility is low in all cases except in that of pure tin or the alloy in which it appears to the extent of 90 per cent. There is in this table the usual wide range of physical qualities belonging to such a series of mixtures.

TABLE VII.

ALUMINUM ALLOYS.

Composition, Per Cent. by Weight.			Ultimate Resistance, Lbs. per Square Inch.		Elastic Limit, Lbs. per Square Inch.	Specific Gravity.	Final Stretch, Per Cent in 6 Inches.
Al.	Tn.	Cu.	*A.*	*B.*			
......	....	100	27,000	28,330	12,000	6.5	6.5
5	5	90	40,815	42,038	13,832	7.6	4.0
10	10	80	32,209	34,200	24,829	6.5	0.8
20	20	60	1,966	2,225	*	5.7	
30	30	40	849	1,077	*	5.05	
40	40	20	4,800	5,672	*	4.91	
100	......	....	15,000	14,316	6,432	2.67	5.6
90	5	5	15,476	17,070	8,227	2.82	3.
80	10	10	18,580	21,140	13.329	3.09	1.2
60	20	20	4,416	5,950	*	3.53	.3
40	30	30	915	1,123	*	4.4	
20	40	40	2,221	2,622	*	5.21	
......	100	....	3,505	3,933	....	7.3	35.51
5	90	5	11,582	10,418	4,823	6.77	10.15
10	80	10	5,999	5,922	2,988	6.24	1.1
20	60	20	1,198	1,200	*	5.55	
30	40	30	993	961	*	4.96	
40	20	40	3,798	3,997	*	4.48	

A. Results of first melting. *B.* Results of second melting.
Test pieces 6 in. between shoulders, diam. ½ inch.
* Could not be turned in the lathe.

The results in this table were obtained by Messrs. Gebhardt and Ward, at the testing laboratory of Sibley College of Mechanical Engineering, Cornell University, in 1896.

The physical properties of aluminum-zinc alloys, including those metals unalloyed, are equally fully given in Table VIII, as well as the values of the coefficients of elasticity. There is not as wide variation of results in this table as in Table VII, although there is a considerable range of ultimate resistance, especially if the results for unalloyed zinc be included. It will be observed that this table also includes the intensity of stress found in

TABLE VIII.

ALUMINUM-ZINC ALLOYS.

Percentage.		Specific Gravity.	Ultimate Resistance, Lbs. per Sq. In.	Transverse Tests. Maximum Fibre Stress Lbs. per Sq. In.	Coefficient of Elasticity.	Remarks.
Aluminum.	Zinc.					
100	0	2.67	14,460	14,500	6,535,000	Shrinkage uneven.
100	0	2.67	16,750	14,150	....	" "
90	10	2.77	17,940	18,950	7,710,000	
90	10	2.74				
85	15	2.918	....	28,091	9,260,000	
85	15	2.918	18,100	....	....	Shrinkage uneven
80	20	2.998	21,850	....	9,110,000	" "
80	20	2.975	....	34,600	....	" "
75	25	3.15	22,940	....	8,210,000	" "
75	25	3.14	....	45,080	....	" "
70	30	3.191	24,400	43,200	8,178,000	Shrinkage even.
70	30	3.24	23,950	41,200		
66⅔	33					
65	35	3.326	....	....	....	Poor specimen.
60	40	3.471				
60	40	3.57	19,770	40,350	8,540,000	
50	50	....	19,300	38,100		
50	50	....	19,060	39,850	8,500,000	
25	75	....	13,175	25,500		
25	75	....	14,150	....	8,670,000	
0	100	7.19	2,522	7,556	6,680,000	Elongation of all the specimens less than 1 per cent.

NOTE.—The experimental results given in Table IX are those of Messrs. Hunt and Andrews, obtained at Sibley College of Mechanical Engineering, Cornell University, in 1894.

TABLE IX.

TENSILE TESTS OF MAGNESIUM—CAST METAL.

Number of Test Piece.	Diameter.	Ultimate Resistance, Lbs. per Square Inch.	Elastic Limit, Lbs. per Square Inch.	Final Extension, Per Cent.	Coefficient of Elasticity.
1.........	.433	23,800	8,800	4.2	2,040,000
2.........	.433	22,050	....	....	1,860,000
3.........	.442	20,900	10,780	1.8	2,060,000
4.........	.435	19,500	8,400	2.5	1,830,000
5	.424	24,800	7,090	3.1	1,930,000
6.........	.432	22,500	....	2.3	

TABLE X.

ALLOYS OF ALUMINUM AND MAGNESIUM

Number of Test Piece.	Percentage of Magnesium.	Specific Gravity.	Ultimate Resistance, Lbs. per Square Inch.	Elastic Limit, Lbs. per Square Inch.	Coefficient of Elasticity.
1.........	0	2.67	13,685	4,900	1,690,000
2.........	2	2.62	15,440	8,700	2,650,000
3.........	5	2.59	17,850	13,090	2,917,000
4.........	10	2.55	19,680	14,600	2,650,000
5.........	30	2.29	5,000		

the extreme fibres of beams subjected to transverse loading. Although these values are not required at this point, it is more convenient to insert them here and refer to them in the article devoted to the flexure of such beams. The sizes of the specimens subjected to transverse loading are not given, but they were small.

TABLE XI.

Character of Alloys.						Resistance, Pounds per Square Inch.			
	Test Piece.	Composition and Remarks.				Tension.		Transverse.	
		Al.	Cu.	Tin.		Elastic.	Ultimate.	Elastic.	Ultimate.
		%	%	%					
Cast.	1	100	....	....	....	4,000	12,055	2,345	......
	2	93	7	....	....	6,250	18,555	9,000	25,250
	3	75.7	3	....	20% zinc, 1.3 man.	....	35,075	....	23,420
Rolled.	4	100	....	....	1 inch bars.......	12,500	17,185	17,154	......
	5	100	....	....	¾ " "	....	....	18,870	......
	6	98	2	....	1 " "	9,000	18,647	13,720	......
	7	98	2	....	¾ " "	....	....	18,870	......
	8	96	4	....	1 " "	16,000	23,045	22,300	......
	9	96	4	....	¾ " "	....	....	30,880	......
	10	96.5	2	....	1½% chromium.	19,000	26,310	26,313	......
Cast Tin Alloys.	11	98	....	2	....	2,150	8,622	....	......
	12	96	....	4	....	2,400	9,565	....	......
	13	94	....	6	....	2,250	9,315	....	......
	14	92	....	8	....	2,000	7,270	....	......
	15	90	....	10	....	1,750	7,352	....	......

The experimental results given in Tables IX and X were also established at the testing laboratory of Sibley College of Mechanical Engineering of Cornell University. The tests were made by Messrs. Marks and Barraclough, graduate students in 1893. Table IX gives results for pure magnesium, including the coefficients of elasticity and the final stretch, while Table X exhibits the results for alloys of aluminum and magnesium, the per cent. of magnesium being shown in one of the columns, the remaining per cent. being aluminum. The ultimate resistances given in Table IX show that magnesium is a metal of considerable tensile resistance, especially in comparison with its density, its specific gravity being but 1.74, that of aluminum being 2.67.

Table XI exhibits the elastic limits and ultimate

TABLE XI.—*Continued.*

Final Stretch Per Cent. (Tension Pieces).	Final Contraction, Per Cent. (Tension Pieces).	Hardness (Relative).	Specific Gravity of Specimen.		Coefficient of Elasticity, Lbs. per Square Inch.	
			Tension.	Transverse.	Tension.	Transverse.
5.62	10.93	3.61	2.670	2.654	8,385,000	8,440,000
1.00	3.08	12.87	2.830	2.810	11,115,000	8,065,000
.15	1.77	35.56	3.117	3.055	9,685,000	8,060,000
8.49	38.30	7.12	2.710		9,780,000	10,110,000
....	....	6.94	2.715		....	10,000,000
19.49	39.02	6.79	2.725		9,505,000	10,330,000
....	....	12.30	2.756		....	9,600,000
3.62	10.10	12:42	2.774		10,440,000	10,595,000
....	....	13.35	2.773		....	10,070,000
1.31	9.78	14.09	2.759		9,850,000	9,813,000
4.00	8.64	3.71	2.689		5,435,000	
5.38	6.86	3.74	2.739		6,210,000	
5.19	7.97	3.49	2.771		5,035,000	
3.06	5.41	3.33	2.804		5,175,000	
3.87	8.89	3.09	2.856		6,675,000	

resistances of all the different alloys shown in the table, and in the conditions also exhibited by the table, i.e., whether cast or rolled. There are also given coefficients of elasticity for both tension and transverse tests, as well as elastic limits and ultimate stresses (intensities) in the extreme fibres of small beams, to which reference will be made in the article devoted to transverse resistance.

It will be observed that both the elastic limits and the ultimate resistances of Table XI are found within the range exhibited by the results already shown in the preceding tables.

If desired, diagrams can readily be constructed from the results of each table which will show the variations of physical quantities corresponding to the variations of composition of the alloys.

In 1895 the Fairbanks Company tested at their New York office four specimens of Tobin bronze manufactured by the Ansonia Brass and Copper Co., with the following results.

ROLLED TOBIN BRONZE PLATES—SPECIMENS 8 INCHES LONG.

Specimen, Inches.	Resistance in Pounds per Sq. Inch.		Per Cent., Final	
	Elastic.	Ultimate.	Stretch.	Contraction.
1×.185	51,350	78,920	20.5	45.4
1×.185	51,350	78,810	17.5	44.33
1×.25	56,000	79,200	17.5	43.2
1×.25	56,450	79,640	16.25	40.72

Alloys of Aluminum and Copper.

In 1907, Prof. H. C. H. Carpenter, M.A., Ph.D., and Mr. C. A. Edwards, made their Eighth Report on alloys of aluminum and copper to the Alloys Research Committee of the Institution of Mechanical Engineers of Great Britain.

This alloy is known as "aluminum bronze" or "gold." These investigators made over a thousand tests in tension and torsion and in other ways, including heat treatment for both cast and rolled material. The investigation is one of the most important ever made with this class of alloys. Out of the great number of tests contained in the report, Table XII has been selected as sufficiently typical for the purpose of conveying a correct impression of the character of the work done.

Table XII.

The percentage of aluminum only is given in the Table, as the alloy is of aluminum and copper, the remaining percentage being copper.

No.	Al. per cent.	Yield Point lbs. per sq. in.	Ult. Resist. lbs. per sq. in.	Elastic ratio.	Elongation in 2 inches per cent.
1	0.1	8,512	25,760	.33	46
2	1.06	6,720	30,020	.22	52
3	2.1	7,616	30,240	.25	53.5
4	2.99	8,512	32,480	.26	60
5	4.05	7,840	37,410	.21	83
6	5.07	9,632	40,540	.24	75
7	5.76	10,752	39,870	.27	67
8	6.73	10,752	41,780	.26	....
9	7.35	14,784	47,710	.31	71
10	8.12	17,248	55,800	.31	58
11	8.67	21,952	62,944	.35	48
12	9.38	21,728	68,050	.32	36.2
13	9.9	25,312	71,010	.36	21.7
14	10.78	31,584	59,750	.48	9.0
15	11.73	31,360	56,960	.55	5
16	13.02	44,240	44,240	....	1

It will be observed that the specimens were of cast metal. While the rolled specimens give somewhat higher ductility, in the main there is much less difference than would probably have been anticipated. Although the elastic ratio, i.e., the ratio of the elastic limit over the ultimate, is somewhat higher for the rolled specimens, the difference on the whole is not great, except in a comparatively few instances. In fact, the differences in results found by the investigators between the cast and rolled metal are much smaller than might have been expected.

The authors of the report state, among other observations:

"(*a*) The limit of industrially serviceable alloys must be placed at 11 per cent. of aluminum. For most purposes the limit might be put at 10 per cent., beyond which there is a rapid fall of ductility with no rise of ultimate resistance. . . .

"(*b*) Between these limits the alloys fall into two classes: 1. those containing from 0 to 7.35 per cent. of aluminum: 2. Those containing from 8 to 11 per cent. of aluminum. Class 1 represents material of apparently low yield point and moderate ultimate stress, but of very good ductility. The introduction and further addition of aluminum causes a gradual increase of strength but hardly affects the ductility. It is true that as regards the steadiness of the ductility this has only been established for the rolled bars. But the sand and chill castings have shown the same kind of variations as the rolled bars in all the properties examined. . . .

"Into Class 2 come alloys of relatively low yield point but good ultimate stress. From 8 to 10 per cent. of aluminum the ductility is also good. . . ."

To gain an adequate idea of the physical properties of the various grades of this alloy of aluminum and copper requires a full scrutiny of the entire report.

Bronzes and Brass Used by the Board of Water Supply of New York City.

In the construction of the Additional Catskill Water Supply for the city of New York by the Board of Water Supply a large amount of bronze castings and rolled bronze, as well as brass, was used for a great variety of large and small articles varying from a number of tons in weight each to a few pounds, such as small bolts. The specifications prescribed that " Whenever the term ' bronze ' is used in these Specifications in a general way or on the drawings, without qualification, it shall mean manganese or vanadium bronze or monel metal. . . .

" The minimum physical properties of bronze shall, except as otherwise specified, be as follows:

Castings:

Ultimate tensile strength	65,000 lbs. per sq.in.
Yield point	32,000 lbs. per sq.in.
Elongation	25 per cent.

Rolled Material:

Ultimate strength	72,000 lbs. per sq.in.
Yield point	36,000 lbs. per sq.in.
Elongation	28 per cent.

Rolled material, thickness above one inch:

Ultimate strength	70,000 lbs. per sq.in.
Yield point	35,000 lbs. per sq.in.
Elongation	28 per cent."

The modulus of elasticity E for tension and compression was about 14,000,000.

The requirements of these specifications were even exceeded both in resistances and in ductility. Much trouble, however, was experienced by the rolled metal exhibiting cracks and failures in articles large and small, in many cases even before put in place in the work and subjected to duty. Such difficulties, however, were not experienced in castings. Investigations intended to discover the origin of these difficulties have not yet been completed, but they are probably due to some feature of manipulation of material during processes of manufacture, including the treatment of the molten metal.

Phosphor-Bronze.

Phosphor-bronze possesses merit not only as a structural material on account of its high elastic limit and ultimate resistance, but also because it is a good anti-friction metal. Its elastic limit may be taken from 45,000 to 55,000 pounds per square inch and its ultimate resistance from 50,000 to 75,000 pounds per square inch, both values being given for unannealed material. The same material as unannealed wire with a diameter of one-tenth to one-sixteenth of an inch may give ultimate resistances varying from 100,000 to 150,000 pounds per square inch, or if annealed not more perhaps than 50,000 to 60,000 per square inch. In the latter case, however, the final stretch may run from 30 to 40 per cent.

Bauschinger's Tests of Copper and Brass as to Effects of Repeated Application of Stress.

The late Professor Bauschinger made some investigations regarding the effect on elastic limit and yield point of repeated application of loading similar to those made on steel and wrought iron. The grade of brass used in his tests was called "red brass."

With the exception of one case of brass the elastic limit and the yield point were both materially elevated by repeated application of loading, whether the repetition was made without a period of rest between two consecutive applications or not. Some repetitions were made immediately and some after periods of 17½ to 53 hours of rest.

The effect on the modulus of elasticity was small and irregular, i.e., in some cases there was a small increase and in others a small decrease and in some cases no material change.

Art. 60.—Cement, Cement Mortars, etc.—Brick.

The ultimate tensile resistance of cements and cement mortars depends upon many conditions. The two great divisions of cements, i.e., natural and Portland, possess very different ultimate resistances whether neat or mixed with sand, the latter being much the stronger. With given proportions of sand or neat, the ultimate resistances of cement mortar or cement will vary with the amount of water. used in tempering and with the pressure under which the moulds are filled. Again, the character of the sand used will obviously influence largely the tensile resistance of the mortar produced, and not only the degree of cleanliness, but the size of grain and the variety of sizes are elements which must be considered. It has also been maintained by some that silica-sand will give better results, other things being equal, than other sand. Finally, the shape of briquette used will affect the results to some extent. Fig. 1, on page 370, shows the form of briquette recommended by the Committee of the American Society of Civil Engineers, and it is the form generally used in American practice. It is foreign to the purpose of this work to enter into the consideration of all these influences; they are only mentioned to enable the few typical experimental results which follow to be interpreted properly.

As the fineness of grinding is an important quality of a cement, it is usually noted by stating the percentage of weight of the cement which either passes through or is retained upon a sieve having a stated number of meshes per linear inch, which number squared gives the number of meshes per square inch. The sizes of the grains of sand used are graded in the same way. The "No." of a sieve to which reference may be made in what follows indicates, therefore, the number of meshes per linear inch.

Modulus of Elasticity.

In consequence of the fact that cement, mortars, and concrete begin to exhibit permanent stretch at comparatively low tensile stresses there is a little uncertainty as to the value of the modulus of elasticity unless distinct statement is made of the intensities of stress at which those values are obtained, and whether the total stretch is used or that total less the permanent set. It is not possible to make such statement in connection with all the values which follow, except that they have been reached at low intensities of stress unless otherwise stated, and with elongations which may be considered wholly elastic. Although cement mortars and concrete do not exhibit a perfectly elastic behavior their stress-strain lines for intensities of stress even exceeding those used in practice are essentially straight and, on the whole, exhibit elastic properties at least equal to those of cast iron.

Comparatively few tests have been made to determine either the tensile or compressive modulus of elasticity of cement, mortar and concrete, although that quantity is a most important element in the theory and design of much concrete work and reinforced concrete members. Mr. W. H. Henby of St. Louis, made a number of determinations of the tensile modulus of elasticity of Portland cement concrete of 1–2–4, 1–2–5, 1–3–6, and 1–4–8 mixtures and gave the results in a paper read before the Engineers Club of St. Louis in 1900. He obtained values varying from less than 2,000,000 to 8,360,000. Other tests, however, indicate that values above perhaps 3,000,000 should not be used. While higher values of the modulus of elasticity for rich mixtures of concrete may exist, the more important considerations of design usually bear upon work in which concrete must take serious loading when less than thirty days of age.

For all these reasons it will seldom be advisable to take the modulus of elasticity of even as rich a mixture as 1 cement, 2 sand, and 4 broken stone higher than about 2,500,000, and it will be seen later that in concrete steel work where portions of a structure are liable to be loaded to a material extent within a comparatively short time after removal of the forms, it is the usual practice to consider the modulus as having a value of 2,000,000 only. These considerations are confirmed by the results of tests given below.

Professor W. Kendrick Hatt, of Purdue University, in a paper read before the American Section of the International Association for Testing Materials, at its convention, 1902, gave the following values for the tensile coefficient of elasticity and ultimate tensile resistances of Portland cement concrete composed of 1 cement, 2 sand, and 4 broken stone at the ages of 25, 26, 28, and 33 days:

	Coefficient of Elasticity, Lbs. per Sq. in.	Ultimate Tensile Resistance, Lbs. per Sq. In.
Maximum	2,700,000	360
Average	2,100,000	311
Minimum	1,400,000	280

It will be found in discussing the compressive modulus of elasticity that both moduli probably acquire nearly their full value in about three months' time. It would appear that moduli do not increase in value with the lapse of time to the same extent as the ultimate resistance to compression, although conclusive data as to this point are not complete.

Such tests as have been made show that the modulus of elasticity in tension or compression for cinder concrete should not be taken higher than about 1,250,000 for 1–2–5 mixtures. Some tests show somewhat lower values and others values running over 2,000,000, but the latter results are too high for cinder concrete as ordinarily made and put in place.

Ultimate Resistance.

The ultimate resistances of neat Portland cement and mortar made with the same cement have been somewhat increased within the past half dozen years; but, upon the whole, those resistances as exhibited in the following tables are fairly representative of the best grades of cement used at the present time (1915). The conditions of manufacture are now so well controlled that a high 7-day or 28-day test cement may readily be produced; but that is not always desirable; the main purpose in masonry construction being rather the attainment of an ultimate resistance possibly less high under a short-time test but which continues to increase indefinitely. A cement showing a high ultimate resistance on a short-time test may not continue to increase its ultimate resistance satisfactorily, or that resistance may even recede for a time.

The following tabular statement is of interest and value as indicating the character of the cement used in the construction of the first subway for the Rapid Transit Railroad in the City of New York. It will be observed that the ultimate resistances of both the neat cement and the mixture of 1 cement, 2 sand, are practically as found a dozen years later. The number of briquettes broken during the years 1900 and 1901 was over 18,000. The average ultimate tensile resistances in pounds per square inch found by that series of tests of both Portland and natural cements, as given in the report of the Chief Engineer, are the following:

	Year.	Neat Cement.			Sand 2, Cement 1*	
		1 Day.	7 Days.	28 Days.	7 Days.	28 Days.
Portland:						
Average result......	1900	229	552	714	276	434
Average result......	1901	300	645	763	380	525
Spec. requirements..	....	150	400	500	200	300
Natural:						
Average result......	1900	....	172	249	118	215
Average result......	1901	....	215	322	218	350
Spec. requirements..	....	....	125	200	100	150

* For natural cement a 1 cement 1 sand mortar was used.

The results for the natural cement are of interest, as that material has at present (1915) practically disappeared from use in consequence of the low prices for which Portland can be produced.

Table I exhibits the results of tests of briquettes of different brands of domestic Portland cement as made in the testing laboratory of the Bureau of Surveys of the Department of Public Works of Philadelphia, Pa., for the year 1912. This table gives the fineness of the cements in terms of the percentages by weight which were retained on sieves with 2500, 10,000 and 40,000 meshes per square inch; it also shows the amount of water used for the different mixtures, as well as the specific gravities of the material. It will be observed that the briquettes were made of neat cement and of mortar with a mixture of 1 cement to 3 sand. The results, therefore, show the effect of the presence of sand on the ultimate tensile resistance of the matrix. The periods at which the briquettes were tested are the standard 24 hours, 7 days and 28 days.

TABLE I.

Average Results of Tests of Portland Cement Made during 1912—Phila., Pa.

Brand	Number of Briquettes.	Fineness in per cent.			Specific Gravity.	Percent of Water Neat	Tensile strength in pounds per square inch.				
		No. 50.	No. 100.	No. 200.			Neat			1:3	
							24 hrs.	7 dys.	28 dys.	7 dys.	28 dys.
Allentown...	816	0.0	3.6	19.7	3.174	20.0	377	721	797	379	499
Alpha........	500	0.0	4.8	23.5	3.161	19.9	399	701	770	367	450
Atlas........	168	0.0	4.1	23.2	3.151	19.8	480	656	741	348	430
Bath........	388	0.0	3.4	20.5	3.130	20.3	434	710	741	384	468
Dexter......	582	0.0	2.3	17.8	3.128	20.6	434	767	820	376	450
Dragon.....	532	0.1	4.0	20.9	3.106	21.6	398	704	718	370	436
Edison......	630	0.2	2.7	18.9	3.114	23.1	261	598	670	334	412
Giant........	28	0.0	4.3	22.8	3.202	20.0	563	672	751	366	398
Lehigh......	2,026	0.0	3.1	19.4	3.172	20.0	363	752	812	410	498
Nazareth....	1,956	0.0	1.9	16.8	3.151	20.6	453	776	830	403	467
Northampton	28	0.0	3.4	22.1	3.138	20.0	497	727	812	393	445
Paragon.....	42	0.3	5.3	21.8	3.082	23.3	355	644	674	328	429
Penn Allen...	514	0.0	4.8	23.2	3.156	20.0	446	686	735	373	475
Phoenix.....	572	0.0	3.0	20.1	3.146	20.7	372	670	723	364	456
Saylor's.....	1,984	0.0	3.1	21.1	3.127	19.9	277	706	801	327	436
Vulcanite....	150	0.0	3.5	21.7	3.165	20.0	267	717	746	376	467
Whitehall ...	1,162	0.0	3.5	21.2	3.155	20.0	429	713	759	389	477

Table II shows the maximum, mean and minimum results of the tests of briquettes of various brands used by the Board of Water Supply of the City of New York during 1914. During the past few years American Portland cement has been improved in uniformity of quality and fineness of grinding. These tests, therefore, show the latest results of the best practice in cement production and use. The tabulated values show the variations occurring in systematic testing of large quantities of cements at 7-day and 28-day periods. The results are all in pounds per square inch and so arranged, as is evident, that in each vertical group of three in each column the highest value is the maximum and the lowest, the minimum, the mean occupying the middle position.

Table II.

Brand.	Approx. Bbls.	No. Briquettes.	Neat Lbs. per Sq. In.		1 c. 3 Ottawa Sand Lbs. per Sq. In.	
			7 Day.	28 Day.	7 Day.	28 Day.
Alpha..........	170,000	852	866	857	320	458
			700	716	285	380
			572	601	246	336
Alsen...........	324,000	1620	744	864	263	360
			615	747	192	300
			453	650	147	219
Atlas............	520,000	2790	755	785	324	414
			643	662	279	356
			549	575	221	279
Saylor's.........	45,000	243	815	883	265	392
			714	773	247	354
			669	694	208	322

The large quantities of cement used with the corresponding large number of briquettes tested give the Table special value and interest. The Ottawa sand is the standard silica sand of that name so extensively used in cement mortar testing.

The preceding tabular results give ultimate tensile resistances for periods no longer than 28 days, but both neat cement and cement mortars go on acquiring additional resistance for long periods, although at slow rates after a period of 28 days; indeed, it may be stated without exaggeration generally after a period of only 7 days. Table III therefore is used to show the increase of ultimate resistance up to a period of six months. The results of this table are taken from the Annual Report of the Bureau of Surveys of Philadelphia, Pa., for the year 1901. It will be observed that the values are not greatly different from those given in Table I at a date 10 years later. In fact, the earlier values are a little higher than the later, showing the tendency to secure a higher degree of permanency in the setting of the cement rather than higher ultimate tensile resistances.

TABLE III.

AVERAGE RESULTS OF PORTLAND CEMENT TESTS MADE DURING 1901.

Brand.	No. of Tests.	Ultimate Tensile Resistance in Pounds per Square Inch.						
		Neat.						
	Broken.	24 Hrs.	7 Days.	28 Dys.	2 Mos.	3 Mos.	4 Mos.	6 Mos.
Alpha.	136	357	770	834	885	813	785	827
Atlas.	820	542	728	790	802	761	815	825
* Castle.	16	235	336	387	443	——	——	——
Dexter.	28	363	826	932	778	——	——	——
Giant.	816	424	669	719	713	745	776	786
Krause's.	72	418	830	864	775	——	——	——
Lehigh.	112	377	699	747	684	735	774	760
Phœnix.	36	345	721	723	——	——	——	——
Reading..	16	460	800	955	775	——	——	——
Saylor's.	764	295	697	766	756	766	733	745
Star.	4,012	437	721	746	731	715	727	——
Vulcanite.	204	290	748	767	707	807	710	740
Whitehall.	356	524	713	765	788	796	775	——

Brand.	No. of Tests.	Ultimate Tensile Resistance in Pounds per Square Inch.						
		1 to 3 Standard Quartz Sand.						
	Broken.	24 Hrs.	7 Days.	28 Dys.	2 Mos.	3 Mos.	4 Mos.	6 Mos.
Alpha.	136	81	252	314	344	312	302	262
Atlas.	820	104	204	289	324	321	337	308
* Castle.	16	65	121	176	215	——	——	——
Dexter.	28	68	298	336	312	——	——	——
Giant.	816	87	227	309	328	317	328	329
Krause's.	72	74	229	285	270	——	——	——
Lehigh.	112	76	233	329	296	310	303	325
Phœnix.	36	94	264	343	——	——	——	——
Reading.	16	150	263	301	338	——	——	——
Saylor's.	764	64	217	296	319	301	311	286
Star.	4,012	77	219	298	321	301	286	——
Vulcanite.	204	45	226	287	269	298	280	330
Whitehall.	356	87	232	313	295	295	343	——

During the construction of a number of dams in the Croton basin supplying the water works of the City of New York, briquettes of neat cement and of mortar 1 to 2 and 1 to 3 were tested after periods beginning with one week and extending up to five years. There was a continuous increase of ultimate resistance throughout the entire period, although at a very slow rate after about six months. At the end of five years the neat Portland cement attained an ultimate resistance of 840 pounds per square inch and the 1 to 2 mortar, 700 pounds per square inch, while the 1 to 3 mortar reached 590 pounds per square inch.

Other tests of briquettes up to two years of age and more confirm the preceding results.

The recent cement product, called silica-Portland cement, is manufactured by grinding together certain portions of clean silicious sand and Portland cement. The results given below are taken from the tests of such silica-Portland cement, manufactured by the Silica-Portland Cement Co., of Long Island City, N. Y. One part, by weight, of Aalborg Portland cement was ground to-

TABLE IV.

SILICA-PORTLAND CEMENT.

Ultimate Tensile Resistance in Pounds per Square Inch.

Mixture.	Per Cent. of Water.	Age.							
		Seven Days.		Fifteen Days.		Twenty-one Days.		Two Hundred and Nineteen Days.	
Neat..........	18–21%	8	148	6	172	8	166		——
			130		165		149		
			121		147		121		
(1–6) s. c.–2 q...	11%	23	81		——	8	114	5	220
			69				98		204
			58				88		194

All specimens one day in air and remainder in water.

gether with six parts, by weight, of clean silicious sand to such a degree of fineness that essentially all of the product passed through a 32,000-mesh sieve. This finely ground mixture of 1 cement to 6 sand, by weight, is called "neat" in what follows, while "(1–6) s. c.–2 q." is 1 part, by weight, of the "neat" silica-Portland cement to 2 parts, by weight, of crushed quartz, or "standard" sand, all of which passes a No. 20 sieve and is retained on a No. 30 sieve. The results were obtained in the cement-testing laboratory of the department of civil engineering of Columbia University. The figures on the left of the brackets show the number of tests of which the ultimate resistances are the greatest, mean, and least in each case.

Five seven-day tests of the Aalborg Portland cement used in the manufacture of the silica-Portland cement gave the following greatest, mean, and least ultimate tensile resistances, the specimens having been one day in air and six days in water:

Greatest.	Mean.	Least.
594 lbs. per sq. in.	536 lbs. per sq. in.	441 lbs. per sq. in.

Four specimens of the neat silica-Portland cement (1–6), one day in air and the remainder of the time in water, gave the following results:

		Age.
Neat (1–6)	308 lbs. per sq. in.	199 days.
	264 " "	190 "
	294 " "	189 "
	260 " "	185 "

All the preceding tensile tests of cement and cement mortars, unless otherwise stated, were made with the shape of briquette shown in Fig. 1, which was recommended for use in the report of the "Committee on a Uniform System for Tests of Cement" of the American Society of Civil Engineers. That report was made in 1912, and the briquette recommended has become the standard in American practice for the testing of cements and mortars.

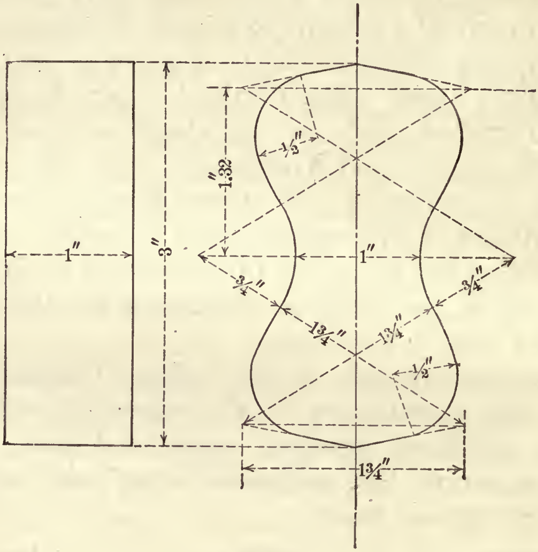


Fig. 1.

Weight of Concrete.

As concrete is frequently used in masses where weight is an important element, it is always desirable to use an aggregate of high specific gravity. Concrete when made of cement, sand and silicious gravel or broken limestone, trap-rock or granitic rock in such mixtures as are commonly employed, will weigh from 140 to 155 pounds per cubic foot with the greater part running from 145 to 150 pounds per cubic foot.

The weight of cinder concrete will necessarily vary much with the character of the cinders. It may usually be taken as weighing about two-thirds as much as ordinary concrete made with gravel or broken stone, i.e., from 100 to 110 pounds per cubic foot.

Adhesion between Bricks and Cement Mortar.

General Q. A. Gillmore many years ago investigated the adhesion of bricks to the cement mortar joint between them and also the adhesion of fine-cut granite to a similar joint. As might be expected in connection with such tests his results varied greatly, the highest belonging to a rich cement mortar and the lowest to the lean mortar of 1 cement to 6 sand. He found the adhesion to vary from about 31 pounds per square inch for neat cement to brick to nearly 3.3 pounds per square inch for a lean mortar of 1 cement to 6 sand. With fine-cut granite the adhesion for neat cement was 27.5 pounds per square inch and for cement mortar of 1 cement to 4 sand about 8 pounds per square inch. It is highly probable that the actual adhesion of bricks and cut stone to the usual joints made of 1 cement to 2 sand or 1 cement to 3 sand would be materially less in a mass of masonry than as arranged for a laboratory test. Nevertheless these early investigations would indicate that such joints might be worth from 8 to 12 pounds per square inch for bricks and but little different for granite.

Mr. Emil Kuichling prepared a paper in 1888 from all available sources for the purpose of disclosing what all experimental investigation had determined up to that time. These results indicated that neat cement might give adhesion to bricks or cut stone varying from about 20 pounds up to over 200 pounds per square inch, with values from 29 pounds up to 146 pounds per square inch for mortar of 1 cement to 1 sand; and 38 pounds to 73 pounds per square inch for a mortar of 1 cement to 2 sand. Further, according to his table a mortar of 1 cement to 3 sand would yield adhesion from 13 pounds up to 48 pounds per square inch and but little less for a mortar of 1 cement to 4 sand. Nearly all these results, however, are undoubtedly too high for the usual masses of masonry in engineering construction.

Other experimental determinations of the adhesive resistance of natural and Portland cement mortars to brick and stone may be found in the report of the Chief of Engineers, U. S. A., for 1895. At the age of 28 days the adhesive resistance of neat Portland cement to the surface of sawn limestone was about 270 pounds per square inch; about 240 pounds per square inch with a mortar of 1 cement to ½ sand; about 225 pounds per square inch with a mortar of 1 cement to 1 sand, and about 170 pounds per square inch with a mortar of 1 cement to 2 sand.

Table V exhibits the average results of three and six months' tests of the adhesion of Portland and natural cement mortars to bricks which were cemented to each other at right angles and then pulled apart normally at the ends of the periods named. These average results are taken from the same report of the Chief of Engineers, U. S. A., for 1895.

TABLE V.

AVERAGE ADHESIVE RESISTANCE OF BRICKS CEMENTED TOGETHER AT RIGHT ANGLES TO EACH OTHER.

Cement.	Mortar.	Adhesion, Pounds per Square Inch.
Portland	Neat	60
"	1 c., ½ s.	60
"	1 c., 1 s.	40
"	1 c., 2 s.	20
"	1 c., 3 s.	20
Natural	Neat	55
"	1 c., ½ s.	50
"	1 c., 1 s.	45
"	1 c., 2 s.	30
"	1 c., 3 s.	15

There will also be found in that report average values of the shearing adhesion of plain 1-inch round bolts to neat Portland cement and to Portland cement mortars of 1 month's age, the bolts having been embedded at various depths from 2 to 10 inches in the mortars. The shearing adhesion for the neat cement varied from a maximum of 345 pounds per square inch for a depth of insertion of 4 inches down to 230 pounds per square inch for a depth of insertion of about 8¾ inches. In the case of the Portland cement mortar of 1 cement to 2 sand the shearing adhesion varied from a maximum of 280 pounds per square inch for a depth of insertion of the bolt of 2½ inches down to 250 pounds per square inch for a depth of insertion of about 7¾ inches. When the bolt was embedded in the Portland cement mortar of 1 cement to 4 sand the shearing adhesion ranged from a maximum of about 145 pounds per square inch for a depth of insertion of 10 inches to a minimum of about 70 pounds per square inch for a depth of insertion of 2 inches. These values of shearing adhesion are important results in the theory and design of concrete-steel members.

The Effect of Freezing Cements and Cement Mortars.

There have been many attempts made to determine the effect of freezing neat cements and cement mortars after having been mixed for use at various ages and under various conditions. Some valuable data have been accumulated, but the conditions attending such investigations are so complicated and so difficult to be analyzed quantitatively that many most discordant conclusions have been reached. Different results will follow if the freezing is done immediately after the mixing of the cement or mortar, or after the initial set has taken place, or after the considerable hardening which takes place at the age of

12 to 24 hours. Probably the best data in this connection arise from an engineer's practical experience in laying masonry when the temperature of the air is below the freezing-point. Under such circumstances it is rarely the case that anything more than surface freezing takes place before the hardening of Portland cement. With the slower action of the natural cements similar conditions do not exist. It is undoubtedly prejudicial even with Portland cements to have alternate freezing and thawing take place at comparatively short intervals of time. On the other hand, the great majority of laboratory investigations indicate that Portland cement or cement mortars may be severely frozen and remain so for long periods of time without essential injury. It is probable that setting usually proceeds during a frozen condition, but at an exceedingly slow rate, and that the operation of setting is actively renewed after thawing.

While it has been stated in some quarters that natural cements may be frozen similarly and thawed without essential injury, there is considerable laboratory evidence as well as that of practice which indicates that conclusion to be erroneous, especially if it be given any considerable application. There may be cases in which natural cements can be or have been frozen without essential injury, but the author's experience in extended practical operations in masonry construction induces him to believe that any natural cement severely frozen before being thoroughly hardened is so seriously injured as to be practically destroyed. On the other hand, his extended observations not only on his own work, but on those of others, lead him to believe that, as a rule, Portland cement will not be sensibly injured under the conditions of actual masonry construction by being frozen. It is customary in most large works to permit no masonry to be laid at a temperature much below about 26° Fahr. above zero, but with precautions easily attained it is certain that concrete and other masonry laid in Portland cement mortar may properly and safely be put in place several degrees below that temperature.

It has also been stated in some quarters that natural cements and some Portlands have been actually improved by being frozen. Such conclusions should be received with exceeding caution. The author believes that there is no conclusive evidence that any cement or cement mortar can be improved by freezing.

In cold weather it is customary on some works to use salt water for mixing mortars and concretes, and that practice when suitably conducted may be resorted to with safety and propriety. Such solutions generally run from 2 to 8 or 10 per cent. by weight of salt. Occasionally, also, soda is dissolved in water at the rate of 2 pounds per gallon. Before using this solution an equal volume of water is added so that the final solution contains about 1 pound of soda to a gallon of water. This solution expedites the setting of the cement with a view to accomplishing a safe degree of hardening before the mortar is frozen. It is doubtful whether this practice should be encouraged.

The Linear Thermal Expansion and Contraction of Concrete and Stone.

Satisfactory investigations regarding the expansion and contraction of concrete and stone are exceedingly few in number, and the data by which variations in the dimensions of large masses of masonry due to temperature changes can be computed are correspondingly meagre. Professor William D. Pence, of Purdue University, has made such investigations and presented the results in a valuable paper read before the Western Society of Engineers, November 20, 1901. In his experimental work he compared the thermal linear changes of concrete bars and bars of steel and copper, basing the coefficients of expansion of the concrete and mortar on the relative changes of the two materials for the same range of temperature. These experiments were conducted with great care, but the resulting values might perhaps have been at least better defined had two materials been employed with a greater difference in their rates of thermal expansion and contraction. Professor Pence employed two kinds of concrete and one bar of Kankakee limestone, seven experiments having been performed on a concrete of 1 Portland cement, 2 sand, and 4 broken stone; one on a concrete of 1 Portland cement, 2 sand, and 4 gravel; and three on a concrete composed of 1 cement and 5 of sand and gravel, making the mixture essentially equivalent to the preceding concrete of 1 cement, 2 sand, and 4 gravel. The maximum, mean, and minimum coefficients of linear expansion per degree Fahr. found in these tests were as follows:

Kind of Concrete.	Maximum.	Mean.	Minimum.
Broken stone, 1 : 2 : 4.	.0000057	.0000055	.0000052
Gravel, 1 : 2 : 4.	——	.0000054	——
Gravel, 1 : 5.	.0000055	.0000053	.0000052
Kankakee limestone.	——	.0000056	——

Between January and June, 1902, Messrs. J. G. Rae and R. E. Dougherty, graduating students in Civil Engineering at Columbia University, with the aid of Professor Hallock of the Department of Physics of the same university, determined with great care by the most accurate direct measurements the coefficients of linear thermal expansion of one bar of concrete of 1 Portland cement, 3 sand and 5 gravel, and one bar of mortar of 1 Portland cement and 2 sand, each bar being 4 inches by 4 inches in cross-section and about 3 feet long, both bars being tested at the age of about 5½ years. The coefficients of linear thermal expansion for each degree Fahr. found in these investigations were as follows:

For 1:3:5 concrete	.00000655
" 1:2 mortar	.00000561

It is believed that these last two determinations were made with the utmost accuracy attainable at the present time in an unusually well equipped physical laboratory and under most favorable conditions.

When it is remembered that the coefficient of linear thermal expansion of such iron and steel as are used in engineering structures is about .0000066,* it is apparent that structures of combined concrete or other masonry and steel may be expected to act under thermal changes essentially as a unit, a conclusion which is justified at the present time by extended experience.

Art. 61.—Timber in Tension.

The ultimate resistance of timber in general is much affected by the moisture which it contains, except that the amount of moisture does not appear to affect sensibly the ultimate tensile resistance. At this point, therefore, no further attention will be given to the effect of moisture or sap on the tensile resistance, but the influence of moisture on the

---

* A large number of determinations of the thermal expansion of iron and steel per degree Fahr. may be found in the U. S. Report of Tests of Metals and Other Materials for 1887. The maximum, mean, and minimum for steel bars are as follows:

.000006756	.000006466	.00000617

Other coefficients of thermal expansion are also given as follows:

Wrought iron	.00000673
Cast iron	.000005926
Copper	.000009129

compressive and bending resistances will be fully set forth in the articles devoted to timber in compression and bending.

There are few results of investigations which give satisfactory moduli of elasticity for timber in tension. Values are given in the annual " U. S. Report of Tests of Metals and Other Materials," but these results are generally for small selected sticks which are quite different from commercial sizes of lumber as generally used. Some of these moduli run up to nearly 3,000,000, which is much too high for any ordinary commercial timber as used in structural work.

In " Tests of Structural Timbers," by McGarvey Cline, Director of Forest Products Laboratory, and A. L. Heim, Engineer of Forest Products, issued as Bulletin 108 of the U. S. Department of Agriculture, 1912, a large number of determinations are made of ultimate resistance, elastic limit and modulus of elasticity for commercial sizes of lumber of nine different kinds of generally used timber. The moduli of elasticity, however, are determined from bending tests, which makes them a kind of composite of both tension and compression values. The results found, however, are among the best available.

The following tabular statement gives the moduli for green and air-seasoned structural sizes:

TABLE I.

	Green	Air-seasoned	Wt. per cu. ft. oven-dry
Long-leaf Pine	1,463,000	1,705,000	35
Douglas Fir	1,517,000	1,549,000	28
Short-leaf Pine	1,473,000	1,726,000	30
Western Larch	1,301,000	1,487,000	28
Loblolly Pine	1,387,000	1,487,000	31
Tamarack	1,220,000	1,341,000	30
Western Hemlock	1,445,000	1,737,000	27
Red Wood	1,042,000	890,000	22
Norway Pine	1,133,000	1,418,000	25

It will be noticed that redwood gives the lowest modulus of elasticity and Norway pine next above it except the value for air-seasoned tamarack. Long-leaf pine, short-leaf pine, and Douglas fir give nearly the same results.

In determining the tensile resistance, and, indeed, other resistances of timber, the size of the specimen plays a more important part, probably, than in any other class of materials used by the engineer. Small specimens, such as are usually employed in tensile tests, are inevitably so selected as to eliminate such defects as decay and decayed or other knots, wind shakes, season cracks, and other deteriorating features, so that the results exhibit physical properties belonging to the best parts of full-size sticks. In engineering practice, on the other hand, large pieces of timber must be used as furnished in the timber market. However close the inspection may be such pieces invariably include within their volumes many spots of weakness due to those features which in the small specimen are carefully excluded. It is of the utmost consequence, therefore, in dealing with physical data belonging to timber to realize that results determined by the testing of small specimens are almost without exception materially misleading in consequence of reaching higher values than those which can possibly belong to the average stick used in structural work. These observations must be carefully remembered in considering the experimental data which follow.

While there exists a large amount of data on the tensile tests of timber it relates largely to small selected sticks or is otherwise scarcely available for engineering construction. The best recent data are given by Messrs. Cline and Heim from which Table I was taken. On page 57 of that Bulletin tabulated data of a large number of bending tests of green and dry structural timbers are found, the failures being by tension in the fibres subjected to that kind of stress. Those data are shown in Table II. The modulus of rupture is simply the intensity of stress in the most remote fibre of the timber.

TABLE II.

Species.	Average modulus of rupture lbs. per sq. in.	Modulus of rupture in per cent. of average green modulus of rupture. First failure by tension (per cent.)					
		All tension failures.	Failure due to				
			Large knots.	Small knots.	Irregular grs.	Pitch pockets	Nothing apparent
Long-leaf pine:							
Green	6,140	112	......	......	......	......	112
Dry	5,749	121	......	......	......	......	121
Douglas fir:							
Green	5,983	83	82	80	77	......	104
Dry	6,372	82	69	78	82	......	136
Short-leaf pine:							
Green	5,548	94	76	90	100	......	109
Dry	6,573	117	96	......	115	......	132
Western larch:							
Green	4,948	73	166	77	71	......	100
Dry	5,856	110	102	112	103	......	124
Loblolly pine:							
Green	5,084	86	73	86	90	90	98
Dry	6,118	120	39	......	114	......	138
Tamarack:							
Green	4,556	90	71	90	96	......	98
Dry	5,498	106	......	......	112	......	100
Western hemlock:							
Green	5,296	74	71	......	......	85	81
Dry	6,420	108	......	92	106	......	119
Redwood:							
Green	4,472	81	77	......	55	......	90
Dry	3,891	80	......	58	48	......	87
Norway pine:							
Green	3,864	94	94	......	73	......	105
Dry	6,054	134	136	......	......	......	129

The moduli of rupture are the averages of all failures whether by tension, compression or shear, but the figures given in the table after the second column represent the percentages of the average "green" moduli of rupture at which the extreme fibres failed in tension uuder influence of "large knots," "small knots," "irregular grain" or "nothing apparent" as indicated at the head of each column. Although these values are not found by direct tests of tension, they may be accepted as fair and suitable ultimate resistances of the different kinds of timber in tension.

TABLE III.

Kind of Timber.	Ultimate Resistance, Pounds per Square Inch.		Working Stresses, Pounds per Square Inch.	
	With Grain.	Across Grain.	With Grain.	Across Grain.
White oak	10,000	2,000	1,000	200
White pine	7,000	500	700	50
Southern long-leaf or Georgia yellow pine	12,000	600	1,200	60
Douglas, Oregon, and yellow fir	12,000	—	1,200	—
Washington fir or pine (red fir)	10,000	—	1,000	—
Northern or short-leaf yellow pine	9,000	500	900	50
Red pine	9,000	500	900	50
Norway pine	8,000	—	800	—
Canadian (Ottowa) white pine	10,000	—	1,000	—
Canadian (Ontario) red pine	10,000	—	1,000	—
Spruce and Eastern fir	8,000	500	800	50
Hemlock	6,000	—	600	—
Cypress	6,000	—	600	—
Cedar	8,000	—	800	—
Chestnut	9,000	—	900	—
California redwood	7,000	—	700	—
California spruce	—	—	—	—

Reviewing all the experimental work which has been done up to the present time (1902) in determining the ultimate tensile resistance of timber, and keeping in view experience with the resistance of full-size timber sticks in completed structures, the best representative series of values of the ultimate and working tensile intensities of timbers is that recommended by the Committee on

" Strength of Bridge and Trestle Timbers " of the Association of Railway Superintendents of Bridges and Buildings at the Fifth Annual Convention in New Orleans, 1895. That series is given in Table III.

The ultimate resistances of the table are much too high for full size pieces, but the working stresses may be accepted as they stand.

It will be noticed that the ultimate tensile resistance of the various timbers across the grain, so far as they are given, are but small fractions of the ultimate resistances along the grain. A corresponding large decrease in resistance across the grain will also be found in connection with the compressive resistance of the same timbers. The working resistances given in this table are those employed in the great bulk of engineering timber structures.

CHAPTER VIII.

COMPRESSION.

Art. 62.—Preliminary.

With the exception of material in the shape of long columns, but few experiments, comparatively speaking, have been made upon the compressive resistance of constructive materials.

Pieces of material subjected to compression are divided into two general classes—"short blocks" and "long columns"; the first of these, only, afford phenomena of *pure compression.*

A "short block" is such a piece of material that if it be subjected to compressive load it will fail by pure compression.

On the other hand, a long column (as has been indicated in Art. 35) fails by combined compression and bending.

Short blocks only will be considered in the articles immediately succeeding, while long columns will be separately considered further on.

The length of a short block is usually about three times its least lateral dimension or less.

It has already been shown in Art. 5 that the greatest shear in a short block subjected to compression will be found in planes making an angle of 45° with the surfaces of the block on which the compressive force acts, i.e., with its ends. If the material is not ductile this shear will frequently cause wedge-shaped portions to separate from the block. But the friction at these end surfaces, and in the surfaces of failure will prevent those wedge portions shearing off at that angle. In fact the friction will cause the angle of separation to be considerably larger than 45°; let it be called α. Then, in order that there may be perfect freedom in failure, the length of the block must not be less than its least width or breadth multiplied by $2 \tan \alpha$. In some cases, α has been found to be about 55°, for which value.

$$2 \tan \alpha = 2 \times 1.43 = 2.86.$$

If the bearing faces of the short block under compression are of much area, for such a purpose, it will be difficult in many cases, especially with large loads, to secure a uniform application of those loads. The resulting ultimate resistance for the entire block will give an average intensity of pressure which may be quite different from the greatest intensity. These simple considerations are particularly pertinent to such materials as blocks of concrete or of natural stone, which may be 12 inches square or more in section.

Again, in such material as natural or artificial stone the friction between the head of the testing machine and the bearing surface of the specimen, or along the planes of greatest ultimate shear will tend to support laterally to some extent the material as it approaches failure, thus raising the apparent ultimate resistance of the material. The shorter the block the greater will be this frictional supporting tendency. This effect has been marked where the tests specimens have been cubes varying from 2 inches on their edges to 12 inches, the large cubes showing materially greater resisting capacity.

Failure of short cylinders of cast iron showing the shearing of the metal on the plane of maximum shear.

View exhibiting the failure of short cylinders of Connecticut brown sandstone.

(*To face page* 386.)

Art. 63.—Wrought Iron.

It is difficult to fix the point of failure of a short block of wrought iron or other ductile material. As the load increases above the elastic limit, the cross-sections of the test piece increase in lateral dimensions or "bulge out," so that increase of compressive force simply produces an increased area of resistance, while the material never truly fails by crumbling or shearing off in wedges.

It is comparatively easy to determine the elastic limit, but at what degree of loading the material may be said to fail after permanent distortion begins is not clear unless some arbitrary limit should be fixed by convention.

In an actual structure obviously failure may be said to take place when the degree of distortion is such that the structure fails to discharge safely its function as a load carrier, but that degree of distortion would vary much in different structures or in different parts, possibly, of the same structure.

For the present purpose it may perhaps be assumed tentatively that a ductile material fails when its distortion under compressive loading becomes apparent to the unaided eye.

Modulus of Elasticity.

As wrought iron is no longer a structural material, there are practically no recent tests to determine the compressive modulus of elasticity, but earlier investigators made sufficient tests when the material was in general use to establish the modulus with reasonable accuracy. Those investigations show that there is no essential difference between moduli for compression and tension. Hence the modulus of elasticity for wrought iron in compression may be taken at 26,000,000. Small specimens would in some cases yield results perhaps as high as 28,000,000, but for general use the former or smaller value is preferable.

Limit of Elasticity and Ultimate Resistance.

Investigations for determining the elastic limit of wrought iron in compression are almost entirely lacking, but its value may safely be taken the same as for tension, i.e., depending upon the area of cross-section and the amount of work put upon the material in its manufacture, from 22,000 to perhaps 26,000 pounds per square inch, the former for large sections and the latter for small sections. The difficulties met in the effort to determine a well-defined ultimate compressive resistance for wrought iron have already been noticed, but such compression tests as were made during the general use of wrought iron for structural purposes indicate that what may be termed the ultimate compressive resistance may reasonably be taken at about the ultimate tensile resistance. The amount of permanent distortion taking place at that degree of loading has not been satisfactorily determined, but it would certainly be apparent to the unaided eye and it might run from 1 per cent. to 5 per cent. or possibly more. It may be assumed, therefore, that the ultimate compressive resistance of wrought iron will range generally from 45,000 to 50,000 pounds per square inch.

Art. 64.—Cast Iron.

The behavior of cast iron under compression as found in ordinary casting is not less erratic than in tension. When this material was used for such purposes as heavy ordnance and car wheels it was so produced as to possess excellent physical qualities for a cast metal, especially after remelting and being held in fusion. Even then, however, the modulus of elasticity was not much higher than for the best qualities of ordinary castings. It may be said generally that the modulus of elasticity for cast iron in either tension or compression may be taken from 12,000,000 to 14,000,000. These values are about half of the corresponding values for wrought iron and little less than half the corresponding values for structural steel.

Inasmuch as cast iron is a brittle material failing suddenly at the limit of its resisting capacity, either in tension or compression, it can scarcely be said to have an elastic limit except for special grades of unusual excellence, and even with such material it is not well defined.

The ultimate resistance of cast iron to compression is fairly well defined, but it varies greatly in value according to its quality. Special grades for ordnance and car wheels may have compressive resistances running from 100,000 per square inch up to 150,000 pounds per square inch. For many years when cast-iron columns were used in engineering practice it was customary to consider the ultimate compressive resistance for such members as 100,000 pounds per square inch, but that value is far too high. Although the quality of ordinary castings is variable, it is reasonable to take the ultimate compressive resistance at 80,000 pounds per square inch for such material as may be used under good and effective specifications for columns, machine frames and similar purposes, although there are modern cast-iron column tests which appear to indicate that even that value is too high.

Art. 65.—Steel.

Table I of Art. 58 contains the results found by Prof. Ricketts in testing cylindrical specimens of mild steel in compression. These specimens were six inches long between carefully faced ends, and, as the table shows, their diameter was about 0.75 inch. The coefficients of elasticity for compression were found by measurements very carefully made with a micrometer on a length of four inches. The elastic limits, however, were determined by operating with a cylinder two inches long, and were taken at those points where the material of the specimens ceased to hold up the scale beam, and may have been somewhat above that point where the ratio between stress and strain ceases to be essentially constant.

The coefficients of elasticity are found to be quite uniform, irrespective of the per cents of carbon, within the limits of the table, and they are seen to be a very little less than the coefficients for tension. The difference is so small that no essential error will arise if, for all engineering purposes, they are assumed the same.

A comparison of the elastic limits for tension and compression presents some irregularities; yet with the exception of the high percentages of carbon in the last two grades of Bessemer metal, the two sets of elastic limits as wholes are not very different from each other. In the Bessemer steel with the two high per cents of carbon, the tensile elastic limits are materially higher than those for compression. The following very important conclusion results from this comparison of the elastic limits for the mild structural steels: since these elastic limits are essentially equal it is not only permissible but wholly rational to increase the working resistances of mild steel bridge columns over those for iron in at least the same proportion that the tensile working stress of the same steel is increased over that of iron in tension. Experiments on a sufficient number of full-size steel columns are yet lacking to verify this conclusion.

It appears from such data on the compressive resistance of steel as exist that not only the coefficient of elasticity but, also, the limit of elasticity in compression may be taken the same as that for tension for the same grade of steel. This was practically true in the older investigations of Kirkaldy, and it is essentially confirmed in the few later investigations available.

The ultimate compressive resistance of steel, like the ultimate tensile resistance, varies with the content of carbon, being comparatively low with a small percentage of carbon, and correspondingly large with a high percentage of that element. It is also much affected by the operations of tempering and annealing.

Special grades of steel adapted to heat treatment have after such treatment given ultimate compressive resistances of various values up to nearly or quite 400,000 pounds per square inch and values ranging from 150,000 pounds up to 300,000 pounds per square inch are not uncommon in the records of the older testing. Such high results, however, are only obtained with hardened and tempered metal.

There is the same uncertainty as to the point at which compressive failure takes place in steel which attaches to the ultimate compressive resistance of all ductile metals and which was commented upon in Art. 63. It is probably safe, however, if not entirely correct, to take the ultimate compressive resistances of different grades of steel equal to their ultimate tensile resistances in the absence of explicit determinations; and a similar observation may be applied to the working resistances in pure compression of same grades of steel.

Art. 66.—Copper, Tin, Zinc, Lead, and Alloys.*

Table I shows some coefficients of elasticity (i.e., ratios between stress and strain), computed from data deter-

---

* As this field of investigation has not been worked since Prof. Thurston left it his results are allowed to stand (1915).

mined by Prof. Thurston, and given by him in the "Trans. Amer. Soc. of Civ. Engrs.," Sept., 1881. The gun bronze contained copper, 89.97; tin, 10.00; flux, 0.03. The cast copper was cast very hot.

TABLE I.

Stress in Pounds per Square Inch.	Coefficients of Elasticity in Pounds per Square Inch.	
	Gun Bronze.	Cast Copper.
1,620	——	1,254,000
3,260	3,622,000	1,415,000
6,520	4,075,000	1,651,000
9,780	6,113,000	1,795,000
13,040	6,520,000	1,824,000
16,300	5,433,000	1,842,000
19,560	5,148,000	1,845,000
22,820	3,935,000	1,735,000
26,080	2,308,000	1,503,000
29,340	——	1,144,000
32,600	1,073,000	815,000
48,900	463,600	332,500

The ratios of stress over strain are far from being constant. Strictly speaking, therefore, there is no elastic limit in either case. In that of the gun bronze, however, it may be approximately taken at 20,000 pounds per square inch (Prof. Thurston takes it 22,820), and in that of the copper at 25,000 pounds. The test specimens were two inches long and turned to 0.625 inch in diameter.

At 38,000 pounds per square inch the gun bronze specimen was shortened about 41 per cent. of its original length, while its diameter had become 0.77 inch.

The copper specimen failed at 71,700 pounds per square inch, having been shortened about one third of its length.

The results of a series of tests by Prof. Thurston, in connection with the United States testing commission, are given in Table II; they were abstracted from "Mechanical

TABLE II.

Composition.		Pounds per Square Inch Causing a Shortening of			Greatest Load in Pounds per Square Inch.	Per Cent. of Shortening Caused by Greatest Load.	Ultimate Crushing Resistance in Lbs. per Square Inch.	Manner of Failure.
Copper.	Tin.	5 Per Cent.	10 Per Cent.	20 Per Cent.				
97.83	1.92	29,340	34,000	46,000	46,260	0.37	34,000	Flattened
95.96	3.80	39,200	42,050	52,150	52,150	0.30	42,050	"
92.07	7.76	31,500	42,000	65,000	84,100	0.45	42,000	"
90.43	9.50	32,000	38,000	60,000	61,930	0.34	38,000	"
87.15	12.77	39,000	53,000	80,000	89,640	0.39	53,000	"
80.99	18.92	65,000	78,000	103,490	103,490	0.20	78,000	"
76.60	23.23	101,040	——	——	114,080	0.09	114,080	Crushed
69.90	29.85	——	——	——	146,680	0.04	146,680	"
65.31	34.47	——	——	——	84,750	0.03	84,750	"
61.83	37.74	——	——	——	39,110	0.02	39,110	"
47.72	51.99	——	——	——	84,750	0.02	84,750	"
44.62	55.15	——	——	——	35,850	0.01	35,850	"
38.83	60.79	——	——	——	39,110	0.02	39,110	"
38.37	61.32	——	——	——	29,340	0.01	29,340	"
34.22	65.80	19,560	——	——	19,560	0.06	19,560	"
25.12	74.51	17,930	17,930	17,930	17,930	0.28	17,930	"
20.21	79.62	16,300	16,300	16,300	16,300	0.29	16,300	"
15.12	84.58	6,520	6,520	6,520	9,450	0.51	6,520	Flattened
11.48	88.50	10,100	10,100	10,100	14,020	0.50	10,100	"
8.57	91.39	6,500	——	——	9,780	0.06	9,780	"
3.72	96,31	6,520	6,520	6,520	9,780	0.34	9,780	"
0.74	99.02	6,520	6,520	6,520	9,780	0.36	9,780	"
0.32	99.46	6,520	6,520	6,520	9,780	0.38	9,780	"
Cast copper		26,000	39,000	51,000	74,970	0.45	39,000	"
"	"	33,000	45,500	58,670	78,230	0.43	45,500	"
"	"	34,000	42,000	58,000	71,710	0.32	42,000	"
"	"	30,000	36,000	50,000	104,300	0.52	36,000	"
"	"	30,000	37,000	50,000	91,270	0.48	37,000	"
"	"	35,000	48,000	65,000	97,790	0.41	48,000	"
Cast tin		6,030	6,400	6,530	7,500	0.44	6,400	"

and Physical Properties of the Copper-tin Alloys," United States Report, edited by Prof. R. H. Thurston, 1879. All the specimens were 0.625 inch in diameter and 2 inches long. Scarcely one of them can be said to possess an elastic limit.

The series of alloys presents some interesting results. About the middle third of the series are seen to be brittle compounds giving (as a rule) high ultimate compressive resistances, while the other two thirds are ductile, and give at the copper end high results, and low ones at the tin end.

It will be observed that Prof. Thurston took the load per square inch which gave a shortening of 10 per cent. of the original length as the ultimate resistance to crushing of the ductile alloys and metals, since such materials cannot be said to completely fail under any pressure, but spread laterally and offer increased resistance.

TABLE III.

Per Cent. of		Pounds per Square Inch for		Per Cent. of Shortening.	Manner of Failure.
Copper.	Zinc.	E_1.	Ultimate Resistance.		
96.07	3.79	305,500	29,000	10.0	Flattened
90.56	9.42	342,100	30,000	10.0	"
89.80	10.06	——	29,500	10.0	"
76.65	23.08	656,500	42,000	10.0	"
60.94	38.65	1,772,500	75,000	10.0	"
55.15	44.44	——	78,000	10.0	"
49.66	50.14	1,345,500	117,400	10.0	"
47.56	52.28	1,500,000	121,000	10.0	"
25.77	73.45	4,232,800	110,822	5.85	Crushed
20.81	77.63	2,485,000	52,152	2.75	"
14.19	85.10	897,000	48,892	10.8	"
10.30	88.88	——	49,000	10.0	Flattened
4.35	94.59	——	48,000	10.0	"
0.00	100.00	318,500	22,000	10.0	"

Table III contains the results of Prof. Thurston's tests of the copper-zinc alloys made while he was a member of the United States Board. The data are taken from "Ex. Doc. 23, House of Representatives, 46th Congress, 2d Session." The specimens were two inches long and 0.625 inch in diameter of circular cross-section.

The values of E_1 (ratios of stress over strain) are computed for about one quarter the ultimate resistance. This ratio is so very variable for different intensities of stress that these alloys can scarcely be said to have a proper "elastic limit."

Two specimens of tobin bronze, each .75 inch in diameter and 1 inch long, tested by the Fairbanks Company of New York City in 1891, were compressed about .8 per cent. at 45,000 pounds per square inch, and a little over 10 per cent.

at 90,000 pounds per square inch. Tobin bronze contains 58.2 per cent. copper, 2.3 per cent. tin, and 39.5 per cent. zinc.

Art. 67.—Cement—Cement Mortar—Concrete.

The ultimate compressive resistances of mortars and concrete determine the carrying power of many engineering works, and it is of much importance to ascertain those resistances and the conditions under which they may be made the greatest possible. Obviously, the carrying power in compression of both mortars and concretes will depend upon a considerable number of elements such as the character of the cement, the proportions of mixture of the sand and cement for mortar or of the cement, sand, and gravel or broken stone for concrete, the thoroughness of the admixture, the amount of water used, the conditions under which the mortar and concrete are maintained while the operation of setting is taking place, the temperature, and other various influences.

The modulus of elasticity of concrete must necessarily depend chiefly upon the proportions of the mixture and the age of the concrete when tested. It will also depend to a material extent upon the intensity of compressive stress at which the strain is observed. At this point a clear understanding of the elastic behavior of the mortars and concrete is necessary to a correspondingly clear view of what takes place in a concrete-steel beam under loading. In many cases of concrete under compression of varying intensities a careful measurement of the resulting strains shows that a permanent deformation or compression remains at least for the time being after the removal of the load, even when the latter is sometimes not more than 100 or 200 pounds per square inch. This permanent set is dependent upon the age of the material and usually, perhaps always, decreases as age increases. In many other cases a permanent set is observable only under intensities of stress as high as 1000 or 1200 pounds per square inch, or even considerably more. When these sets occur they are frequently found far below what may probably be termed the elastic limit of the material, and in some quarters they have given the impression that mortar and concrete have little or no true elastic behavior. This, however, is an erroneous view, as in the testing of concrete and mortar cubes equal increments of stress intensities quite uniformly give equal increments of strain or deformation over a considerable range. Although the upper limit of this essentially constant ratio between stress and strain is usually not very clearly defined, it is so defined in a considerable percentage of cases, and in almost all tests of well-made concrete and mortar that limit may readily be assigned near enough for all practical purposes.

A large amount of data bearing upon these points will be found in the "Report of Tests of Metals and Other Materials" at the Watertown Arsenal for 1899. Twelve-inch cubes with a great variety of proportions of constituent elements ranging from a few days up to six months in age were employed in those investigations. Figs. 1 and 2 exhibit graphically the results of twelve of those tests so taken as to be fairly representative of all. The vertical ordinates of the curves represent compressive stress intensities up to failure, while the horizontal ordinates represent the total compressive strains or deformation under the corresponding stresses also up to the point of failure. These strains are shown in the figures one hundred times their actual amounts. In Fig. 1 the concrete nine days old shows only little resisting power and a low coefficient of elasticity, as would be expected. In nearly all the other cases, on the other hand, the ratio between stress and strain is reasonably constant up to nearly 1000 pounds per square inch. The two exceptions are found in Fig. 2, belonging to 1 to 3 Portland-cement mortar and to 1, 2, and 4 steel-cement concrete, the former four months old and the latter three months old.

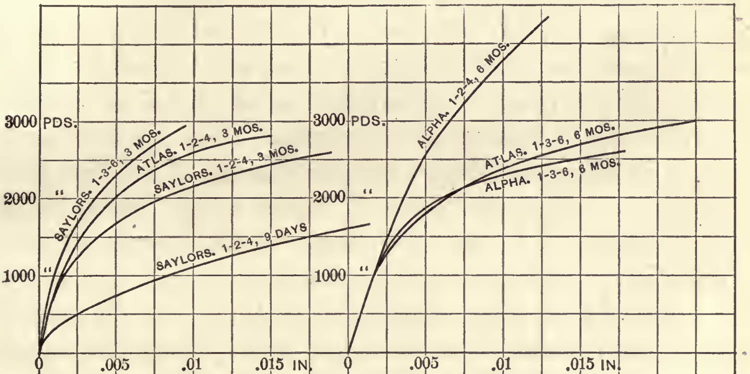


FIG. 1

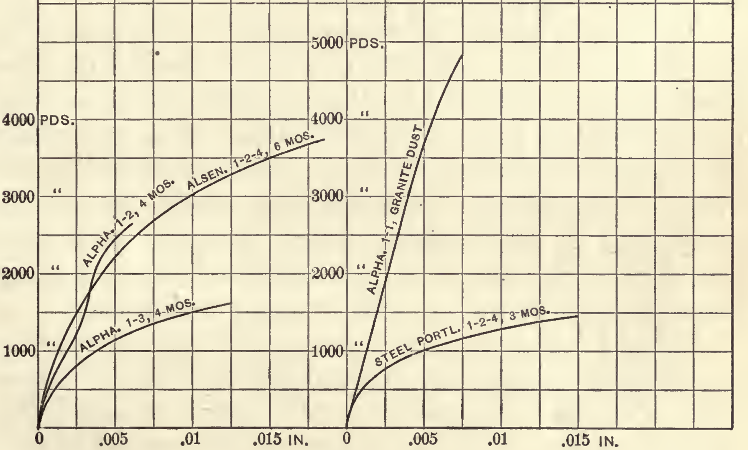


FIG. 2

On the other hand, the 1, 2, and 4 concrete six months old in the right-hand group of Fig. 1 discloses constant proportionality between stress and strain up to 2000 pounds per square inch, and the same observation may apply to a similar concrete represented by one of the curves in the left-hand group of Fig. 2. Again the 1 to 1 granite-dust mortar four months old represented by one of the curves in the right-hand group of Fig. 2 shows a constant ratio up to nearly 4000 pounds per square inch. Indeed, the whole group of curves probably shows a more satisfactory approach to a constant ratio between stress and strain than do similar curves for cast iron. It should be stated, as will be observed by referring to the report cited, that some of the curves shown in Fig. 1 and Fig. 2 belong to groups for which small permanent sets were observed below elastic limits, while others belong to those which show no such permanent set. This observation does not appear from the test records to be applicable to any particular character of curves, but sometimes to those which are more nearly straight and sometimes to those which are less so.

The results deduced from the tests of cubes covered by the 1899 and other "Reports of Tests of Metals and Other Materials" are confirmed by the investigations of such foreign authorities as M. Considére, Melan, Brik, and others. They show conclusively that it is reasonable and safe to apply to concrete and concrete-steel beams the formulæ established by the common theory of flexure after introducing into them empirical quantities established by experiment precisely as is done with iron and steel beams.

Table I is a condensed statement of average values of the modulus of elasticity for concrete of different proportions of mixture prepared by Mr. Edwin Thacher from original sources, including the annual Reports of Tests of Metals and Other Materials carried on by U. S. officers at the Watertown Arsenal for a lecture given by him at the College of Civil Engineering of Cornell University, 1902.

This table exhibits as reasonable values for the coefficient of elasticity in compression as can be determined at

TABLE I.

COEFFICIENTS OF ELASTICITY OF PORTLAND-CEMENT CONCRETE IN COMPRESSION FROM TESTS MADE AT THE UNITED STATES ARSENAL, WATERTOWN, MASS., FOR GEORGE A. KIMBALL, CHIEF ENGINEER, 1899. THE CONCRETE WAS HAND-MIXED. EACH RESULT IS A MEAN OF FIVE OR MORE TESTS.

Cement.	Age.	Concrete, 1, 2, 4, between Loads of			Concrete, 1, 3, 6, between Loads of			Concrete, 1, 6, 12, between Loads of		
		100 and 600.	100 and 1000.	1000 and 2000.	100 and 600.	100 and 1000.	1000 and 2000.	100 and 600.	100 and 1000.	1000 and 2000.
Atlas.	7 days	2,778,000	1,875,000	———	1,667,000	———	———	———	———	———
Germania.	“	2,500,000	2,143,000	1,351,000	2,273,000	1,607,000	———	———	———	———
Alsen.	“	2,500,000	2,143,000	———	1,667,000	1,452,000	———	———	———	———
Mean.	“	2,592,000	2,033,000	1,351,000	1,869,000	1,529,000	———	———	———	———
Atlas.	1 mo.	3,125,000	2,812,000	1,724,000	3,125,000	2,647,000	———	1,316,000	———	———
Alpha.	“	2,083,000	1,875,000	1,190,000	2,083,000	2,143,000	———	1,667,000	1,607,000	———
Germania.	“	———	———	———	2,273,000	1,875,000	1,219,000	961,000	———	———
Alsen.	“	2,778,000	2,647,000	1,471,000	2,273,000	1,875,000	———	1,562,000	———	———
Mean.	“	2,662,000	2,444,000	1,462,000	2,438,000	2,135,000	1,219,000	1,376,000	*1,140,000	———
Atlas.	3 mos.	4,167,000	3,214,000	2,083,000	2,778,000	2,500,000	1,471,000	1,136,000	833,000	———
Alpha.	“	4,167,000	3,750,000	2,778,000	3,571,000	2,812,000	2,000,000	1,786,000	1,800,000	———
Germania.	“	3,571,000	3,214,000	2,381,000	2,778,000	2,812,000	2,083,000	2,083,000	1,667,000	———
Alsen.	“	2,778,000	2,500,000	1,389,000	2,778,000	2,500,000	1,667,000	1,562,000	1,154,000	———
Mean.	“	3,670,000	3,170,000	2,157,000	2,976,000	2,656,000	1,805,000	1,642,000	1,363,000	———
Atlas.	6 mos.	3,125,000	3,214,000	2,273,000	3,571,000	3,461,000	2,174,000	1,786,000	1,607,000	———
Alpha.	“	3,125,000	3,214,000	3,571,000	4,167,000	4,091,000	1,724,000	1,923,000	1,667,000	———
Germania.	“	4,167,000	3,750,000	1,852,000	3,125,000	3,000,000	1,852,000	1,786,000	1,364,000	———
Alsen.	“	4,167,000	4,091,000	2,631,000	3,571,000	3,461,000	1,724,000	1,786,000	1,452,000	———
Mean.	“	3,646,000	3,567,000	2,581,000	3,608,000	3,503,000	1,868,000	1,820,000	1,522,000	———

* Probable.

the present time. The value to be selected for any particular case will depend upon the proportions of mixture and upon the degree of balancing of the sand and gravel or broken stone, although the influence of the latter cannot be definitely stated. It is not improbable that a considerable portion at least of the variations in the results of the table are due to the varying degrees of natural balancing in the different test blocks. The value will also depend upon the age of the concrete. For all ordinary engineering constructions it is reasonable to take the coefficient of compressive elasticity at 2,500,000 to 3,000,000 pounds per square inch for a concrete mixture of 1 cement, 2 sand, and 4 gravel or broken stone. This table shows that practically the same value may be taken for a concrete of 1 cement, 3 sand, and 6 gravel or broken stone, especially if the materials are well selected and balanced. If the concrete is mixed in the proportions of 1 cement, 6 sand, and 12 gravel or broken stone, the coefficient of elasticity is seen to decrease materially and should not be taken higher than 1,500,000 pounds per square inch. Suitable quantities for mixtures other than those named in the table can be reasonably and safely selected from those afforded in it.

These values show that the ratio of the coefficient of elasticity for steel over that for concrete may range from 10 to 20 for the varying conditions described.

The more common practice is to make this ratio 15, i.e., on the basis of 30,000,000, for the modulus of elasticity for steel and 2,000,000 for concrete. The ratio of 12, however, is sometimes found by taking the same value as before for the modulus of steel, but 2,500,000 for the modulus of elasticity for concrete. The ratio of the two moduli is constantly used in the treatment of reinforced concrete work.

A further consideration must be kept in view in connection with the value of the modulus of elasticity for concrete, and that is the fact alluded to in previous pages that nearly all concrete and reinforced concrete work must usually carry considerable loading, in the exigencies of construction, when it has attained no greater age than perhaps 10 to 30 days, i.e., before the modulus of elasticity (or ultimate resistance) has attained its full value. Again, a large mass of concrete, as actually built, cannot reasonably be expected to have as high a modulus as 12-inch cubes or other comparatively small pieces made and tested in a laboratory. For all these reasons it is prudent to take a rather low value of the modulus of elasticity for the analytic work of design.

The following tabulated statement shows ultimate resistances per square inch of 12-inch cubes of concrete obtained in the Testing Laboratory of the Department of Civil Engineering of Columbia University in 1912 by Mr. James S. Macgregor, in charge of the laboratory.

GRAVEL CONCRETE; 1 *Cement,* 2½ *Sand,* 5 *Gravel.*

	Ult. Resistance Pounds per Sq. In.			
	Max.	Mean.	Min.	
Alsen	1,917	1,773	1,557	Age of all cubes 42 days
Atlas	1,905	1,796	1,706	
Atlas	2,223	2,191	2,152	
Iron Clad	1,789*	1,553*	1,431*	
Iron Clad	1,848	1,778	1,657	
Lehigh	2,717	2,584	2,431	
Lehigh	2,278	2,139	2,007	
Vulcanite	1,162*	1,097*	1,047*	
Vulcanite	1,735	1,593	1,518	
Alsen	2,322	2,088	2,006	
Alsen*	1,202	1,023	944	

* Gravel unwashed.

The coarse aggregate for all cubes was river gravel with stones up to 1-inch size. Some of the gravel contained an excessive amount of dirt or other fine material, which

TABLE II.

MEAN ULTIMATE COMPRESSIVE RESISTANCES OF 12-INCH PORTLAND-CEMENT CONCRETE CUBES.

Portland Cements. Brand; Composition.		Mean Ultimate Resistance, Pounds per Square Inch at Age.				Coefficient of Elasticity in Pounds per Square Inch at Age.		
		7 Days.	1 Mo.	3 Mos.	6 Mos.	1 Mo.	3 Mos.	6 Mos.
Saylors. .	1 c., 2 s., 4 b. st.	1,724	2,238	2,702	3,506	2,500,000	3,571,000	5,000,000
	1 c., 3 s., 6 b. st.	1,625	2,568	2,882	3,567	2,778,000	4,167,000	2,500,000
	1 c., 6 s., 12 b. st.	675	800	1,128	1,542	833.000	2,273,000	2,083,000
Atlas . . .	1 c., 2 s., 4 b. st.	1,387	2,428	2,966	3,953	3,125,000	4,167,000	3,125,000
	1 c., 3 s., 6 b. st.	1,050	1,816	2,538	3,170	3,125,000	2,778,000	3,571,000
	1 c., 6 s., 12 b. st.	594	1,090	1,201	1,583	1,316,000	1,136,000	1,786,000
Alpha. . .	1 c., 0 s., 2 b. st.	3,294	5,053	5,047	——	3,125,000	5,000,000	——
	1 c., 2 s., 4 b. st.	902	2,420	3,123	4,411	2,083,000	4,167,000	3,125,000
	1 c., 3 s., 6 b. st.	892	5,150	2,355	2,750	2,083,000	3,571,000	4,167,000
	1 c., 6 s., 12 b. st.	564	1,218	1,257	1,532	1,667,000	1,786,000	1,923,000
Germania	1 c., 0 s., 2 b. st.	2,734	3,246	3.858	5,129	3,571.000	2,778,000	3,571,000
	1 c., 2 s., 4 b. st.	2,219	2,642	3,082	3,643	——	3,571,000	4,167,000
	1 c., 3 s., 6 b. st.	1,550	2,174	2,486	2,930	2,273,000	2,778,000	3,125.000
	1 c., 6 s., 12 b. st.	759	987	963	815	961,000	2,083,000	1,786,000
Alsen . . .	1 c., 0 s., 2 b. st.	3,118	3,240	3,710	5,332	2,273,000	2,273,000	3,571,000
	1 c., 2 s., 4 b. st.	1,592	2,269	2,608	3,612	2,788,000	2,778,000	4,167,000
	1 c., 3 s., 6 b. st.	1,438	2,114	2,349	3,026	2,273,000	2,778,000	3,571,000
	1 c., 6 s., 12 b. st.	417	873	844	1,323	1,562,000	1,562,000	1,786,000

10-INCH CUBES.

Alpha. . .	1 c., 0 s., 2 b. st.	——	——	5,463	6,556	——	——	5,000,000

In this table each ultimate resistance is a mean of four to six tests.

TABLE III.

MEAN ULTIMATE COMPRESSIVE RESISTANCES OF 12-INCH PORTLAND-CEMENT CONCRETE CUBES WITH LOAD TAKEN ON 8″ BY 8″.25 PLATE ON ONE FACE.

Portland Cements. Brand; Composition.		Mean Ultimate Resistance, Pounds per Square Inch at Age.			
		1 Month.	3 Months.	6 Months.	
Alpha. . . .	1 c., 0 s., 2 b. st. . .	5,089	——	——	Each ultimate resistance is a mean of three tests.
	1 " 2 " 4 " . . .	3,287	4,531	5,669	
Germania	1 c., 0 s., 2 b. st. . .	4,327	——	6,671	
	1 " 2 " 4 " . . .	3,587	3,522	4,582	
Alsen	1 c., 0 s., 2 b. st. . .	4,087	——	6,382	
	1 " 2 " 4 " . . .	3,233	3,426	4,983	

A view exhibiting the failure under compression of a 12-in. concrete cube. The composition is 1 Portland cement, 1 sand, and 4.5 broken stone. The age of the concrete was 1 year, 8 months, 23 days, and the ultimate compressive resistance attained was 4481 lbs. per sq. in.

(*To face page* 402.)

accounts for the low values of the starred ultimate resistances per square inch, as indicated by the footnote. The age of all the cubes was 42 days, also as indicated in the table. These results are unusually valuable in one respect, in that the cubes were not mixed in the laboratory, but in the field, where actual work was being done, and hence received no special care in the operation.

Tables II and III contain the results taken from the "U. S. Report of Tests of Metals and Other Materials" for 1899. They exhibit the ultimate compressive resistances of cubes of Portland-cement concrete, the cements being among the well-known brands. The ages of these cubes vary from seven days to six months. The data show clearly the increase of ultimate resistance with the ages of the cubes, and the same observation applies to the three columns showing the coefficients of elasticity at one month, three months, and six months. The compositions of the different concretes of Table II are those quite generally employed in engineering practice.

Table III exhibits the ultimate resistances of the same concretes, but with the pressure applied to the 12-inch cubes on areas 8 inches by 8¼ inches, this end being attained by the use of steel plates. As would be expected, the ultimate resistances are seen to be considerably greater than are found with the total load distributed over the entire surface of a cube.

The broken stone used in the cubes, the results of whose tests are given in Tables II and III, was a conglomerate from Roxbury, Mass., and the sand was coarse, clean, and sharp. The voids of the broken stone measured 49.5 per cent. of their total volume.

Table IV, taken from the same volume of the "U. S. Report of Tests of Metal and Other Materials" as Tables II and III, exhibits the ultimate compressive resistances of

TABLE IV.

MEAN ULTIMATE COMPRESSIVE RESISTANCES OF MORTAR AND CONCRETE 12-INCH CUBES.

Brand; Composition.		Mean Ultimate Resistance, Pounds per Square Inch at Age of Four Months.	Weight per Cubic Foot, Pounds.	Coefficient of Elasticity, Pounds per Square Inch.
Alpha Portland	1 c., 1 s., 0 b. st.	4,371	136.5	3,571,000
	1 " 2 " 0 "	2,506	134.2	3,125,000
	1 " 3 " 0 "	1,812	133.8	1,786,000
	1 " 4 " 0 "	829	120.9	——
	1 " 5 " 0 "	484	119.5	——
	1 " 6 " 0 "	185	116.9	——
	1 " 7 " 0 "	118	111.5	——
Atlas Portland	1 " 1 " 0 "	5,570	141.5	6,250,000
Star Portland	1 " 1 " 0 "	5,045	134.5	4,167,000
Saylors Portland	1 " 1 " 0 "	3,979	134.7	3,125,000
Germania Portland	1 " 1 " 0 "	4,353	134.7	2,500,000
Alpha Portland	1 " 1* 0 "	5,306	137.3	3,571,000
Steel slag	1 " 1 s., 0 "	1,743	126.6	1,190,000
" "	1 " 2 " 4 "	1,939 †	152.1 ‡	2,500,000
Hoffman Rosendale	1 " 1 " 0 "	741	127.7	——
Norton Rosendale	1 " 1 " 0 "	643	125.2	——
	1 " 2 " 0 "	277 †	120.7	——
	1 " 2 " 4 "	332 †	146.2 ‡	——

* Granite dust. † Age, 3 months. ‡ Trap rock, broken stone.

TABLE V.

CHEMICAL ANALYSES OF PORTLAND AND STEEL-SLAG CEMENTS.

Cement.	Silica.	Oxide of Iron.	Alumina.	Lime.	Magnesia.	Sulphur Trioxide.	Carbon Dioxide.
Alpha. . .	20	2.8	10.87	58.66	3.35	1.34	2.56
Star. . . .	21.73	2.5	9.47	56.34	3.61	1.91	3.94
Standard	22.5	2.6	11.98	51.44	3.61	1.57	5.96
Alsen. . .	20.67	2.1	14.6	42.16	2.32	2.32	4.45
Steel. . . .	31.02	Trace	10.9	57.31	4.05	3.36	4.81

the mortar and concrete 12-inch cubes described therein. These results need no explanation, as they are similar to those which have already been given, but it is well to note that the last four lines of the table give results belonging to two brands of natural cement. There are also shown one test of a steel-slag cement mortar cube and one of concrete.

Table V exhibits the chemical analyses of the Portland and steel-slag cements named in Table IV. These analyses exhibit about the usual composition of the various grades of cement to which they belong.

TABLE VI.

COMPRESSION TESTS OF 12-INCH CUBES OF PORTLAND-CEMENT CINDER CONCRETE.

Brand.	Composition.	Age when Tested, Days.	No. Cubes Tested.	Weight per Cu. Ft., Lbs.	Ultimate Resistance in Lbs. per Sq. In.			Coefficient of Elasticity, Pounds.
					Max.	Mean.	Least.	
Germania .	1 c., 1 s., 3 cinder	99 and 102	3	110.4	2,023	2,001	1,975	———
" .	1 c., 2 s., 3 "	102	3	112.8	1,701	1,634	1,589	———
" .	1 c., 2 s., 4 "	98	3	107.9	1,344	1,325	1,295	———
" .	1 c., 2 s., 5 "	98 and 101	3	106.3	1,114	1,084	1,052	———
" .	1 c., 3 s., 6 "	91	3	103.5	854	788	749	———
Alpha.	1 c., 1 s., 3 "	90	3	114.1	2,988	2,834	2,780	2,500,000
"	1 c., 2 s., 5 "	90	3	110	1,715	1,600	1,402	1,279,000
Atlas.	1 c., 1 s., 3 "	90	3	116.3	2,580	2,414	2,295	3,125,000
"	1 c., 2 s.,.5 "	90	3	109.9	1,263	1,223	1,200	857,000

The results exhibited in Table VI are interesting as belonging to Portland-cement cinder concrete and they are of practical importance because such concrete is used in many buildings especially for floors, in consequence of its weighing much less than ordinary broken-stone concrete. The ages of these cinder concrete cubes is seen to run from 90 to 102 days, which is sufficient to give nearly the full ultimate resistance of such material. It is seen, however, that cinder concrete is materially less strong or capable of ultimate compressive resistance than either broken-stone or gravel concrete having the same proportions of mixture in its composition. The column giving the weight in pounds per cubic foot shows that cinder concrete weighs but about three fourths as much as that made with gravel and broken stone. The data contained in this table were taken from the "U. S. Report of Tests of Metal and Other Materials" for 1898.

Messrs. Harold Perrine, C.E. and George E. Strehan, C.E. presented a paper to the Am. Soc. C. E. in 1915 describing their extended investigation* in "Cinder Concrete for Floor Construction between Steel Beams." The Table VII is taken from that paper and each value is a mean of ten results, except those in the second column

TABLE VII.

Proportions........ Method........	C. S. Cin. 1 : 2 : 5 Continuous mixer. Coltrin.	C. S. Cin. 1 : 1 : 5 By hand turned twice.	C. S. Cin. 1 : 2 : 5 Batch mixer.	C. S. Cin. 1 : 2 : 5 Ransome. Mixer,
Cement........	Alsen.	Dragon.	Vulcanite.	Atlas.
Sand........	Long Island Bank Sand, North Shore.			
Cinders.	Anthracite.			
	Ice plant.	Local hotel steam plant.	Local.	Local office building steam plant.
Weight, lbs. per cu. ft.......	107	100	107	109
One month test:				
Ult. Resist., lbs. per sq. in.	407	507	818	980
E, lbs. per sq. in.........	924,600	857,400	1,230,000	1,492,000
Two months test:				
Ult. Resist., lbs. per sq. in.	701	662	1,254	1,035
E, lbs. per sq. in.........	1,134,000	1,030,000	1,740,000	1,428,250
Six months test:				
Ult. Resist., lbs. per sq. in.	933	754	1,744	1,478
E, lbs. per sq. in.........	971,000	1,050,000	1,348,000	1,276,000
One year test:				
Ult. Resist., lbs. per sq. in.	913	813	1,465	1,475
E, lbs. per sq. in.........	993,000	956,000	1,200,000	1,320,000

---

* Made in the testing laboratory of the Dept. of Civil Engineering, Columbia University by the aid of the Wm. R. Peters, Jr. memorial research fund.

from the right side of the Table, which are means of nearly that number. The compressive test specimens were cinder-concrete cylinders 8 inches in diameter and 16 inches long. The values given in the Table are representative of good structural cinder concrete.

A large number of tests, the results of which need not be given here, have shown that gravel may advantageously be used, in the interests of economy, in the place of broken stone for concrete. On the whole, the broken-stone concrete is probably stronger than that made with gravel, but the difference is not material for all ordinary cases. The gravel should not be water-worn, but have sharp, gritty surfaces to which the setting cement may strongly bond itself. All sizes from the largest permissible down to coarse sand should be taken, and when so balanced the voids may be reduced as low as 20 per cent. of the total volume of the gravel or even lower. This balancing of the broken stone or gravel enhances both economy and resisting qualities.

A careful examination of all the Tables, I to V, shows that reasonably well-made broken-stone concrete may carry a load of 300 to 500 pounds per square inch without exceeding $\frac{1}{7}$ to $\frac{1}{8}$, or possibly $\frac{1}{9}$, of its ultimate resistance, the composition of the mixture being 1 cement, 2 sand, and 4 broken stone, or perhaps 1 cement, 3 sand, and 5 broken stone. It is possible that this may be an under statement of the capacity of the concrete if the mixture is as well balanced as it should be. It is a mistake, as has been shown repeatedly by actual test, to screen out the finer portions of the broken stone or to attempt to secure an approximately even sand grain. It is conducive to an increased resistance as it is to increased economy to balance the sand, gravel, or broken stone by using all the varying sizes between the least and the greatest. Indeed, in many

TABLE VIII.

COMPRESSIVE RESISTANCES OF 12″×12″ CONCRETE COLUMNS.

Height, Feet.	Age, Days.	Composition.	W'ght in Lbs. per Cu. Ft.	Ult. Resist. in Lbs. per Sq. In.	
2	47	1 cement, 3 sand, 4–1¼″ broken stone, 2–½″ broken stone	145	1,072	Hand mixed
2	47		145	917	
4	47	do.	144	1,067	
4	47	do.	144	1,132	
6	46	do.	——	844	
6	46	do.	143	1,048	
8	42	do.	145	935	
8	42	do.	145	900	
10	40	do.	142	909	
10	41	do.	143	807	
12	39	do.	144	947	
12	39	do.	144	980	
14	34	do.	145	936	
14	35	do.	145	907	
2	47	1 cement, 3 gravel, 4–1¼″ broken stone, 2–⅜″ broken stone	145	1,185	Machine mixed
2	47		147	1,183	
4	48	do.	143	980	
4	48	do.	144	936	
6	48	do.	146	1,131	
6	48	do.	146	1,200	
8	42	do.	146	1,108	
8	42	do.	146	1,086	
10	41	do.	146	1,015	
10	42	do.	146	1,000	
12	37	do.	149	1,400	
12	39	do.	148	1,500	
14	35	do.	148	858	
14	35	do.	148	807	
6	42	1 cement, 6 gravel, 8–1¼″ broken stone, 4–⅜″ broken stone	143	500	
6	42		144	467	
6	42	1 cement, 7 gravel, 8⅔–1¼″ br'k'n stone, 4⅓–⅜″ br'k'n stone	141	427	
6	42		142	436	
6	45	1 cement, 5 gravel, 6⅔–1¼″ br'k'n stone, 3½–⅜″ br'k'n stone	146	708	
6	45		146	747	
6	46	1 cement, 4 gravel, 5⅓–1¼″ br'k'n stone, 2⅔–⅜″ br'k'n stone	146	900	
6	46		145	797	
12	36	1 cement, 3 gravel, 6–⅜″ broken stone	150	1,250	Reinforced with 4–¾″ cold-twisted steel rods embedded in the concrete
12	39		149	1,700	
9	580	1 Silica Portland cement, 2 coarse clean sand, 3 quartz gravel (½″–2″)	148	2,548	

cases it may be advisable to use the entire product of the crusher.

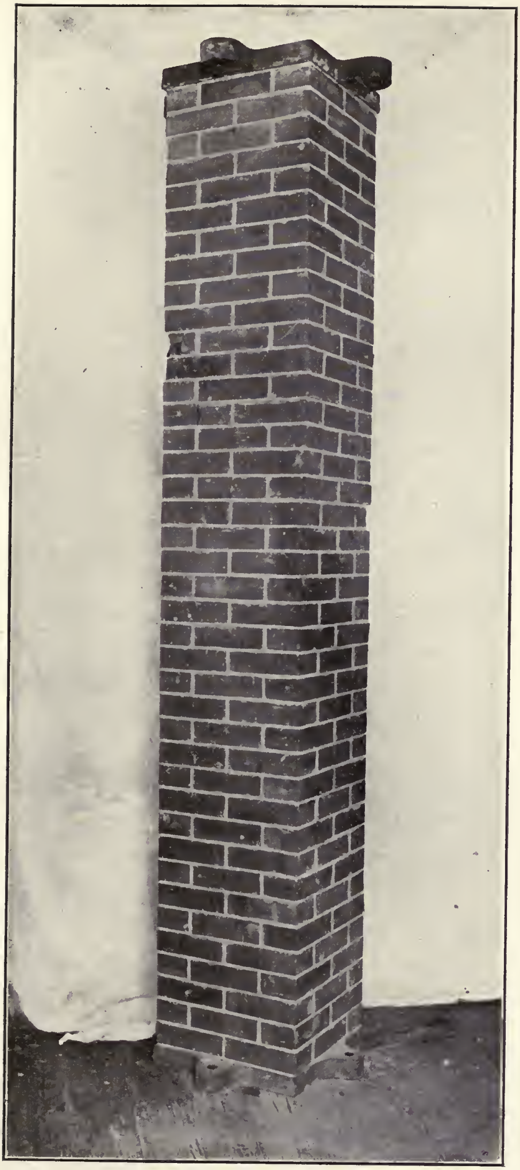
The relation between the ultimate compressive resistance of concrete made with balanced material and the length of column is illustrated by the results given in Table VIII, which has been collated and arranged from the "U. S. Report of Tests of Metal and Other Materials" for 1897. The heights of column range from 2 to 14 feet. While there are some exceptions, the rule is general that, other things being equal, the ultimate resistance decreases as the length or height of column increases. On the whole, the machine-mixed material appears to be a little stronger than the hand-mixed, but the difference is not substantial except for the 8, 10, and 12 feet lengths.

Art. 68.—Bricks and Brick Piers.

The ultimate compressive resistance of bricks depends largely upon the manner in which they are tested and the care with which the surfaces pressed are filled out with a proper cushion and made truly parallel to the bearing surfaces of the testing machine. The best of bricks as produced for the market do not have opposite faces truly parallel, and hence when they are placed in a testing machine for testing to failure the pressure will be concentrated at different points and the bricks will be broken partly by bending before the full ultimate compressive resistance is developed unless the pressed surfaces are made true by some kind of a cushion. This cushioning is frequently and perhaps usually done with plaster of paris, as in the case of the tests of bricks at the U. S. Arsenal, Watertown, Mass., the results of which are given in Table II. Again, a brick tested on edge will give a less ultimate resistance per square inch than when tested flat and the resistance on end per square inch of section will be less than that on edge. When the brick is tested flatwise, even when truly surfaced with a cushion such as plaster of paris, it is a very short block and the friction of the pressed surfaces on the bearing faces of the testing machine is sufficient to give the compressed material substantial lateral support, not permitting it to separate and crush away readily. It will be found, therefore, that when blocks are tested flatwise the ultimate resistances per square inch, as a whole, will be much higher than when tested on edge. This condition of things holds to some extent when the bricks are tested on edge, so that an endwise test will give the ultimate compressive resistance per square inch somewhat less than that found when the brick is tested on edge. An endwise test of the brick more truly represents the ultimate compressive resistance of the material than a test either flatwise or on edge.

A series of tests of a variety of bricks and terra-cotta made in 1896 at the U. S. Arsenal at Watertown, Mass., gave moduli of elasticity about as follows: Pressed brick, 1,000,000 to 3,000,000 pounds per square inch, the hardest varieties giving the higher values and the softer material, the lower values; hard buff brick and terra-cotta, 4,000,000 to 4,800,000 pounds per square inch. Some soft-face brick gave moduli of elasticity varying from about 400,000 to 890,000 pounds per square inch. These determinations of the modulus were made with intensities of pressure from about 1000 to 4000 or 5000 pounds per square inch. Such experimental results ordinarily show some erratic or abnormal features and these tests were no exception to that rule.

The coefficients of thermal expansion and contraction per degree Fahr., were at the same time found to range from .00000205 to .00000754, the larger of these values



A solid 16-inch square-face brick pier laid in lime mortar It was tested at the U. S. Arsenal, Watertown, Mass., and gave an ultimate compressive resistance of 1337 lbs. per sq. in. The pier is shown as it existed after failure.

(*To face page* 410.)

being about 25 per cent. higher than the coefficient for concrete.

In the Proceedings of the Am. Soc. C. E. for March, 1903, Mr. S. M. Turrill, Assoc. Am. Soc. C. E., gives the results of a large number of tests of common building brick, 2 in. by 4 in. by 8 in. in size, manufactured at Horseheads, N. Y. The following table is fairly representative of the results of Mr. Turrill's tests, made with great care at the civil-engineering laboratories of Cornell University:

TEST OF COMMON BUILDING BRICK.

Brick Tested.	No. of Tests.	Ultimate Compressive Resistance, Pounds per Square Inch.		
		Greatest.	Mean.	Least.
On end.	12	3,763	2,628	1,234
On edge.	12	3,913	2,832	1,897
Flat.	12	5,463	3,995	2,665

These bricks were tested in their natural condition as delivered from the kiln ready for use.

Other tests were made of the same brick saturated with water and after being reheated in a suitable oven. This latter test was designed to disclose the quality of brick after having passed through a conflagration. The saturated bricks tested on end and on edge showed material loss of resistance below that of their natural condition, but those tested flat showed large gains. The reheated bricks exhibited large gains in all three modes of testing. These bricks were obviously not of hard-burned, high-resisting character.

The coefficient of elasticity of twelve of these bricks ran from 540,000 to 1,815,000 pounds per square inch, with a mean value of 1,305,000 pounds.

A large number of determinations of the ultimate compressive resistances of bricks were made among the earlier experimental investigations at the U. S. Arsenal at Watertown, Mass. These results showed values for hard-burned bricks varying from about 8,000 to about 12,000 pounds per square inch with an average of about 9,000 pounds per square inch when tested on edge. What may be termed medium bricks, i.e., intermediate between hard-burned strongest bricks and common building bricks, gave results varying from about 4,000 to about 8,000 pounds per square inch, with an average value of about 5,500 pounds per square inch when tested on edge.

The following results of tests of three different kind of brick and hollow tile were obtained by Mr. J. S. Macgregor in the testing laboratory of the Department of Civil Engineering at Columbia University. The ultimate resistances given are the means of seven sets of tests, eight in each set. Half bricks were tested flatwise. This mode of testing obviously yields much higher values than if the bricks were tested on edge.

	Lbs. per sq. in.		
	Max.	Mean.	Min.
Common Hudson River, moulded......	4,357	3,203	2,006
Stiff Clay, side cut..................	2,537	2,305	2,072
Harvard, over-burned...............	....	6,642	....

The hollow tiles were of two types, six-core and two-core. The cross-sections were 10 inches by 12 inches, 8 inches by 12 inches, 8 inches by 16 inches, and 12 inches by 12 inches. The length or height of each set of tiles was 12 inches with one exception of 8 inches. The tiles were all tested with the webs (or cores) vertical and the net sectional areas varied from about 41 square inches to 60 square inches. The ultimate resistances per square inch on both the net sections and the gross sections are as given below. There were five sets of ten tests each and the results given are the greatest, mean and least results of the five sets.

	Lbs. per sq. in.		
	Max.	Mean.	Min.
Net section	5,718	4,598	3,826
Gross section	2,680	2,090	1,710

Brick Piers.

Inasmuch as tests of brick piers have shown that their ultimate compressive resistances run only from about 1000 to 4500 pounds per square inch, depending upon the character of the mortar, it is seen that in such masonry a small portion only of the compressive resistance of the bricks is developed in piers and other similar brick-masonry masses.

These latter results doubtless depend largely upon the cementing material. There is no question that the ultimate resisting capacity of brick masonry is affected greatly by the resisting capacity of the mortar, and the same general observation can be applied to other classes of masonry. There is more than this, however, affecting the carrying capacity of brick and other grades of masonry as compared with the ultimate compressive resistance of the bricks used in the one case of masonry or of the individual stones employed in the other. The texture or character of the mass of burned clay composing the brick is exceedingly variable, both in consequence of the varying mixture of the material in the bricks before being burned and in consequence of the varying degree of burning in each individual brick. Again, whatever may be the care in placing the bricks in a testing-machine, including the cushioning of the ends, it is practicably impossible to secure anything like a uniform bearing upon either the ends, sides, or beds. Their irregular dimensions and exterior surfaces and the varying quality of the materials, even in the best of brick, introduce into their resisting capacity elements of variation which are frequently so great as to lead to abnormal results. While the mortar used in forming a mass of brick masonry undoubtedly fills up many irregularities of surface, voids of considerable magnitude frequently remain unfilled. The consequence of these uncontrollable elements in a mass of brick masonry is always a material reduction of ultimate carrying capacity and frequently a large reduction. However excellent in quality, therefore, the mortar or binding material in a brick-masonry pier may be, it is inevitable that there will be not only a wide range in ultimate compressive resistance, but in all cases a material reduction below that exhibited by the individual bricks when tested by themselves.

Profs. Arthur N. Talbot and Duff A. Abrams reported, in Bulletin No. 27 (1908) of the University of Illinois, the results of a series of sixteen tests of brick piers and the same number of hollow terra-cotta block piers. Two grades of brick were used, a hard-burned shale brick and a soft under-burned clay brick. Eighteen of the former tested on beds gave:

	Lbs. per sq. in.		
	Max.	Mean.	Min.
Ult. Comp. Resist....................	14,150	10,690	7,030

Sixteen of the soft bricks similarly tested gave:

	Lbs. per sq. in.		
	Max.	Mean.	Min.
Ult. Comp. Resist.	5,670	3,920	2,190

The hollow terra-cotta blocks were about 4 inches by 8 inches, 4 inches by 8½ inches and 4 inches by 8¼ inches in cross-section, the height or length being generally 8 inches, but 4 inches in some cases. These blocks had three cores, two 1½ inches square each and one 1½ inches by ½ inch.

TABLE I

AVERAGE VALUES FOR BRICK COLUMNS

Columns.	Average Ultimate. Load, lb. per sq. in.	Ratio of Ultimate of Column to Ultimate of Brick.	Ratio of Ultimate of Column to Ultimate of "A"	E Initial Modulus of Elasticity.	Number of Tests.
Shale Building Brick.					
A–Well laid, 1–3 portland cement mortar, 67 days	3365	.31	1.00	4,780,000	3
Well laid, 1–3 portland cement mortar, 6 months.	3950	.37	1.18	5,025,000	2
Well laid, 1–3 portland cement mortar, eccentrically loaded, 68 days.	2800	.26	.83	4,400,000	2
Poorly laid, 1–3 portland cement mortar, 67 days	2920	.27	.87	3,525,000	2
Well laid, 1–5 portland cement mortar, 65 days. .	2225	.21	.66	3,250,000	2
Well laid, 1–3 natural cement mortar, 67 days. .	1750	.16	.52	800,000	1
Well laid, 1–2 lime mortar, 66 days.	1450	.14	.43	104,000	2
Under-burned Clay Brick.					
Well laid, 1–3 portland cement mortar, 63 days. .	1060	.27	.31	433,000	2

The brick columns were about 12½ inches by 12½ inches in section and 10 feet long. The mortar joints were about ⅜ inches thick. Failure of these columns took place chiefly by vertical cracks through joints and bricks. Table I gives the mean results of these tests.

The characteristics and dimensions of the terra-cotta columns or piers and the average results of tests per square inch of gross area are given in Table I*a*.

TABLE IA.

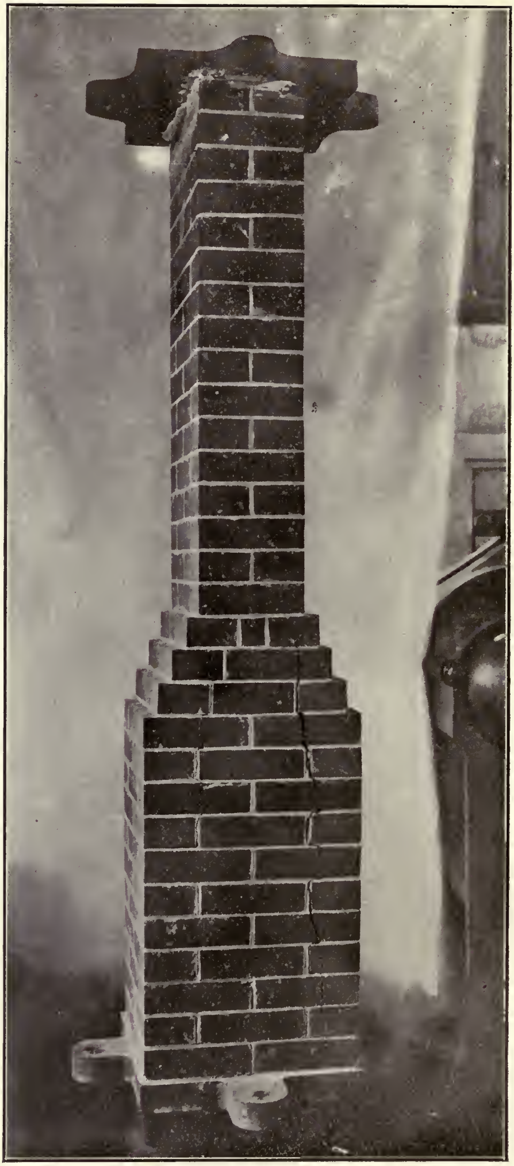
AVERAGE VALUES FOR TERRA COTTA COLUMNS

Characteristics of columns.	Number of Columns in Average	Average Ultimate Unit Load lb. per sq. in.	Ratio Ultimate of Column over Ultimate of Block (Gross area).	E Initial Modulus of Elasticity.
1–2 portland cement mortar. All well laid and centrally loaded.				
8½×8½ in.	2	2885	.83	2,194,000
8½×13 in.	2	3070	.89	2,194,000
13×13 in.	2	2955	.85	2,194,000
12½ × 12½ in., 1–3 portland cement mortar, well laid unless noted.				
Central load	2	3790	.74	2,765,000
	....	4300*	.83*	....
Eccentric load	1	3470	.65	2,330,000
Poorly laid, central load	1	3305	.64	3,200,000
Poorly laid, eccentric load	1	3110	.60	2,500,000
Inferior blocks, central load	1	3050	.59	2,300,000
1–5 mortar, central load	2	3350	.65	2,690,000

* Estimated.

The average age of columns when tested was 67 days.

The joints of the columns were about ⅜ inch thick and the blocks were laid on end. Failures were sudden and accompanied or caused by longitudinal cracks. In fact,



An 8 × 16-in.-face brick pier with 16-in. square base laid in lime mortar. It was tested at the U. S. Arsenal, Watertown, Mass., and gave an ultimate compressive resistance of 1233 lbs. per sq. in. on the upper section and 601 lbs. per sq. in. on the lower section. The cracks due to failure are clearly seen.

(*To face page* 417.)

the chief manner of fracture of both brick and terra-cotta columns or piers is by longitudinal cracking.

Table II exhibits the results of testing piers of brick masonry in the Gov't testing machine at Watertown, Mass. It is taken from "Ex. Doc. No. 35, 49th Congress, 1st Session." The dimensions of piers are shown in the table; also the kinds of mortar used and the grades of brick. The "common" and "face" brick, both hard burnt, were from North Cambridge, Mass. The other bricks

Table II.

CRUSHING STRENGTH OF BRICK PIERS.

No.	Height of Pier,		Section of Pier, Ins.	Composition of Mortar.	Weight per Cu. Ft., Lbs.	Ultimate Resistance, Lbs. per Sq. In.	
	Ft.	Ins.					
1	1	4	8×8	1 lime, 3 sand.	137.4	2,520	Face bricks.
2	6	8	8×8	1 “ 3 “	133.5	1,877	
3	1	4	8×8	1 Portland cement, 3 sand.	136.3	3,776	
4	6	8	8×8	1 “ “ 3 “	133.5	2,249	
5	2	0	12×12	1 lime, 3 sand.		1,940	
6	2	0	12×12	1 “ 3 “		1,900	
7	10	0	12×12	1 “ 3 “	131.7	1,511	
8	10	0	12×12	1 “ 3 “	125.0	1,807	
9	2	0	12×12	1 Portland cement, 2 sand.		3,670	
10	10	0	12×12	1 “ “ 2 “	132.2	2,253	
11	1	4	8×8	1 lime, 3 sand.	135.6	2,440	Common bricks.
12	6	8	8×8	1 “ 3 “	133.6	1,540	
13	2	0	12×12	1 “ 3 “		2,150	
14	2	0	12×12	1 “ 3 “		2,050	
15	9	9	12×12	1 “ 3 “	131.5	1,118	
16	10	0	12×12	1 “ 3 “	136.0	1,587	
17	10	0	12×12	1 Portland cement, 2 sand.	131.0	2,003	
18	2	8	16×16	1 “ “ 2 “		2,720	
19	10	0	16×16	1 “ “ 2 “		1,887	
20	2	0	12×12	1 lime, 3 sand.		1,370	Bay State bricks.
21	6	0	12×12	1 “ 3 “		1,133	
22	6	0	12×12	1 “ 3 “	119.7	1,210	
23*	6	0	12×12	1 lime, 3 sand.	118.2	1,331	Face bricks.
24†	6	0	12×12	1 “ 3 “	118.1	1,211	
25	7	10	12×12	1 “ 3 “	120.3	1,174	
26	10	0	12×12	1 “ 3 “	118.0	924	
27	10	0	8×12	1 “ 3 “	107.0	940	
28	10	0	12×16	1 “ 3 “	118.7	773	
29	6	0	12×12	1 “ 3 “ , 1 Rosendale cement.	120.6	1,646	
30	6	0	12×12	1 Rosendale cement, 2 sand.	123.0	1,972	
31	6	0	12×12	1 lime, 3 sand, 2 Portland cement.	120.3	1,411	
32	6	0	12×12	1 Portland cement, 2 sand.	119.7	1,792	
33	6	0	12×12	Clear Portland cement.	126.6	2,375	

* Joints broken every 6 courses.

† Bricks laid on edge.

were from the Bay State Brick Co., of Boston and Cambridge, Mass., and were medium burnt.

The brick piers were built of bricks "laid on beds and joints broken every course, with the exception of two 12 by 12 piers, one of which had joints broken every sixth course, and one had bricks laid on edge.

"They were built in the month of May, 1882," and "their ages when tested ranged from 14 to 24 months." They were all tested between cast-iron plates.

"Loads were gradually applied in regular increments, . . . returning at regular intervals to the initial load. . . . Cracks made their appearance at the surfaces of the piers and were gradually enlarged before the maximum loads were reached. Final failure occurred by the partial crushing of some of the bricks, and by the enlargement of these cracks, which took a longitudinal direction and occurred in the bricks of one course opposite the end joints of the bricks in the adjacent courses. This manner of failure was common to all piers.

It is important to notice that the resistance of the piers varies with the strength of the mortar used in the joints.

Brick piers, 8 inches by 8 inches in cross-section and 6 feet high, built of Hudson River common brick, and others of Sykesville face brick were tested to destruction in the testing laboratory of the Department of Civil Engineering of Columbia University in 1915 by Mr. J. S. Macgregor, in charge of the laboratory, with the following results, two of the piers being built of Hudson River common brick and three of the Sykesville face brick.

	Lbs. per sq. in.		
	Max.	Mean.	Min.
Hudson River Common..............	902	812	722
Sykesville Face......................	3,436	3,363	3,289

These piers also gave the two following values for the modulus of elasticity in compression:

Hudson River Common........$E =$ 748,000 lbs. per sq. in.
Sykesville Face................$E = 2{,}860{,}000$ lbs. per sq. in.

The age of the columns was 60 days. The ends were finished with plaster of paris to secure square and uniform bearings. The two moduli were determined at intensities of stress less than 250 pounds per square inch.

Mr. Macgregor also obtained the ultimate resistances of three piers, 74 inches high built up of single, approximately 8-inch by 12-inch hollow tile giving a gross horizontal cross-section of 94 square inches and a net section of actual tile material of 50 square inches.

These tile piers had $\frac{3}{8}$-inch joints filled with Portland cement mortar, 1 cement, 3 sand, the age of the piers being 60 days.

The ultimate compressive resistances per square inch for the three piers were as follows:

Gross Section.......1,236; 1,239; and 1,117 lbs. per sq. in.
Net Section.........2,324; 2,329; and 2,100 lbs. per sq. in.

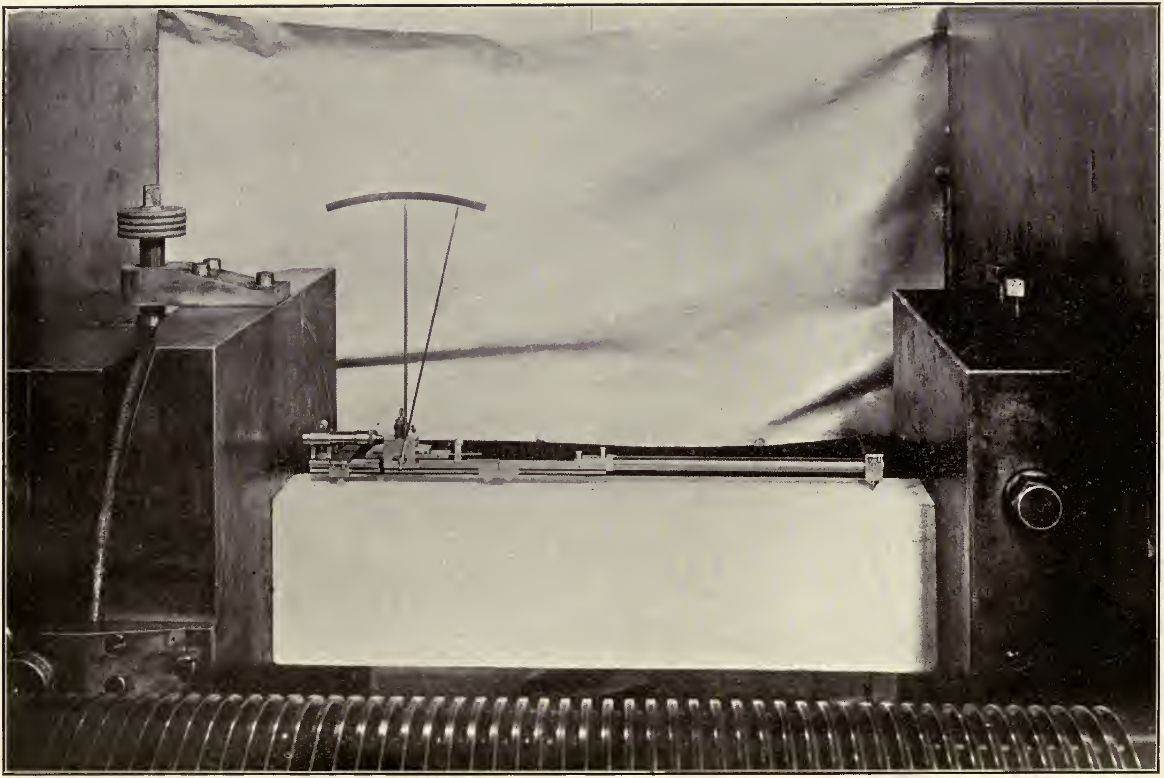
These tile piers failed in the blocks in most cases, but in other cases in the joints. The failures of the blocks showed vertical cracks as well as horizontal and some spalling.

The results of all the experimental investigations available in connection with brick masonry and experiences in the best class of engineering work indicate that masonry laid up of good hard-burnt common brick may safely carry a working load of 15 to 20 tons per square foot or 210 to 280 pounds per square inch. In the construction of this class of masonry where the duties are to be severe it is of the utmost importance that the best class of Portland cement mortar be employed, as the carrying capacity of brick masonry depends largely if not chiefly upon the character of the mortar.

Art. 69.—Natural Building Stones.

The ultimate compressive resistance of natural building stones is affected greatly by the condition of the rock from which the cube or other test-piece is taken. That portion of a ledge exposed to the weather may be much weakened and, in fact, even disintegrated, but the material at a short distance from the exterior surface may have the greatest resistance of which the particular kind of stone is capable of yielding. Again, the compressive resistance of stones on their natural beds is much greater than when tested on edge. In the tests which follow the test-pieces were fairly representative of such quality of stones as would pass inspection in first-class engineering work, and it is to be assumed that they were compressed on their beds unless otherwise stated.

Table I taken from the " U. S. Report of Tests of Metals and Other Materials " for 1894, exhibits the coefficients of elasticity, ultimate compressive resistances, weights per cubic foot and coefficients of thermal expansion per degree Fahr., as well as the ratio, r, between lateral and direct strains for the granites, marbles, limestones, sandstones, and other stones shown in the left-hand column. The coefficients of elasticity and of thermal expansion were determined by employing blocks of stone about 24 ins. long and 6 ins. by 4 ins. in cross-section, the gauged length being 20 inches, but the ultimate compressive resistances were found by testing 4-inch. cubes. The number of tests for each coefficient of elasticity and ultimate resistance varied from one to nine but were generally two or three. The general run of values of ultimate resistance will be found to conform as well as could be expected with results for the same kind of stones in the tables which follow.



The micrometer used with the 400-ton Emery testing-machine at the U. S. Arsenal, Watertown, Mass. The view shows the micrometer in position for determining the compressibility of a piece of stone.

(*To face page* 421.)

It will be observed that the marbles are the heaviest stones, although the granites are not much lighter. There is a large difference, however, between the sandstones and the marbles or granites.

TABLE I.

NATURAL STONES IN COMPRESSION ON BEDS.

Stone.	Coefficient of Elasticity, Lbs. per Sq. In.	Ultimate Compressive Resistance, Lbs. per Sq. In.			Weight per Cu. Ft., Lbs.	Coefficient of Expansion per Degree Fahr.	*r.*
		Max.	Mean.	Min.			
Branford granite, Conn. . . .	8,712,100	15,854	15,707	15,560	162	.00000398	$\frac{1}{4}$
Milford granite, Mass.	7,676,750	25,738	23,773	19,258	162.5	.00000418	$\frac{1}{5.8}$
Troy granite, N. H.	6,118,850	28,768	26,174	23,580	164.7	.00000337	$\frac{1}{5.1}$
Milford pink granite, Mass. .	6,200,350	22,162	18,988	15,756	161.9	——	
Pigeon Hill granite, Mass. . .	8,095,250	20,716	19,670	17,772	161.5	——	
Creole marble, Ga.	7,993,700	15,512	13,466	11,420	170	——	$\frac{1}{2.9}$
Cherokee marble, Ga.	10 427,800	13,415	12,619	11,822	167.8	.00000441	$\frac{1}{3.7}$
Etowah marble, Ga.	8,792,600	14,217	14,053	13,888	169.8	——	$\frac{1}{3.6}$
Kennesaw marble, Ga.	8,217,950	10,771	9,563	8,354	168.1	——	$\frac{1}{3.9}$
Lee marble, Mass.	——	——	——	——	——	.00000454	
Marble Hill marble, Ga. . . .	9,950,850	11,532	11,505	11,478	168.6	.00000202	$\frac{1}{3.4}$
Tuckhoe marble, N. Y.	15 173,200	19,223	16,203	11,640	178	.00000441	$\frac{1}{4.5}$
Mount Vernon limestone, Ky	3,278,400	11,566	7,647	5,247	139.1	.00000464	$\frac{1}{4}$
Oolitic limestone, Ind.	——	——	——	——	——	.00000437	
North River bluestone, N. Y.	5,475,300	——	22,947	——	——	.00000519	
Manson slate, Maine.	——	——	——	——	——	——	
Cooper sandstone, Oregon . . .	——	——	14,920	——	——	——	
Cooper sandstone, Oregon . .	3,021,350	16,366	15,284	14,203	159.8	.00000177	$\frac{1}{11}$
Maynard sandstone, Mass. . .	2,034,650	10,538	9,880	9,223	133.5	.00000567	$\frac{1}{3}$
Kibbe sandstone, Mass.	2,066,800	10,663	10,363	10,063	133.4	.00000577	$\frac{1}{3.3}$
Worcester sandstone, Mass. .	2,668,750	9,869	9,763	9,656	136.6	.00000517	$\frac{1}{4.4}$
Potomac sandstone, Md. . .	——	——	——	——	——	.000005	
Olympia sandstone, Oregon.	——	13,441	12,665	12,061	——	.0000032	
Chuckanut sandstone, Wash.	——	12,790	11,389	10,276	——	——	
Dyckerhoff's cement.	——	——	——	——	——	.00000578	
* Yammerthal flint limestone, Buffalo.	——	28,951	23,724	18,496	——	——	
* Red granite.	——	——	28,647	——	——	——	

* From Report of 1899.

The coefficients of elasticity generally range considerably higher than those for concrete in Art. 67, but the sandstones form an exception to this observation. The coefficients of thermal expansion vary between rather wide limits but they are mostly a little lower only than those determined for concrete. The coefficient for the Dyckerhoff cement is very close to those exhibited for cement mortar and concrete in Art. 60. The column headed *r*, giving the ratios between lateral and direct strains, contains interesting data. From what has been shown in Art. 4 it is apparent that the total volume of the test-pieces was considerably reduced by the compression to which the cubes were subjected.

The coefficients of elasticity were determined at intensities of pressure running from 1000 or 2000 pounds per square inch up to 8000 or 10,000 pounds per square inch.

A coefficient would first be determined at comparatively low pressures, as from 1000 to 3000 pounds per square inch, and then at higher pressures, as from 7000 to 9000 or 10,000 pounds per square inch. As a rule, the coefficients determined at the higher pressures were materially higher in value than the others, the stiffness of the stone increasing with the loads within the limits of the test. The values in the table are the means of those at the low and high pressures.

With the ordinary working values of pressures in masonry, probably not more than two thirds of the values of the coefficients of elasticity given in the table should be employed.

In the "U. S. Report of Tests of Metals and Other Materials" for 1900 there may be found the results of compressing 4-inch cubes of Tennessee marble and of granite from the Mount Waldo Quarries at Frankfort, Maine. The ultimate compressive resistances of the 4-inch Tennessee marble cubes expressed in pounds per square inch, were as follows:

Maximum.	Mean.	Minimum.
25,478	20,329	16,309

The preceding three results cover twenty tests.

The ultimate resistances in pounds per square inch of the "Black Granite" from the Waldo Quarries, as determined from four tests of 2-inch cubes, were as follows:

Maximum.	Mean.	Minimum.
32,635	30,949	29,183

Again, in the same report, the ultimate resistances in pounds per square inch of four 4-inch cubes of limestone from Carthage, Mo., are as follows·

Maximum.	Mean.	Minimum.
17,130	14,947	13,660

The preceding tests and the results of others given in Table II have been determined by compressing cubes 4 inches and 5 inches on the edge and it has been generally customary to use a cube for a test piece for either natural or artificial stones. It has already been indicated, however, in Art. 62 that such a short test piece in compression must necessarily give higher results than should be credited to the material.

The use of compressive test specimens with lengths two to two and one-half times the diameter is just beginning, but that use has not become sufficiently general, nor has it been long enough the practice, to make available results from such desirable tests.

Furthermore, some tests have shown that ultimate compressive resistances may be materially higher for large cubes than for small ones. This is probably due to the lateral supporting effect given to parts of the test piece by the friction between the bearing head of the machine and the face of the material under test with which it is in contact. Preferably no cube tested for engineering purposes should be less than 12 by 12 inches in section, nor should any test piece be shorter than twice its diameter.

The results found in Table II are taken from the "U. S. Report of Tests of Metals and Other Materials," for 1894. They relate to the various kinds of rock indicated and were found by testing 4-inch to 5-inch cubes on their beds.

TABLE II.

State.	Stone.	Ultimate Compressive Resistance, Pounds per Square Inch.
Minnesota	Ortonville granite	20,415
"	Kasota pink limestone	10,833
"	Faribault marble	17,780
"	Duluth brownstone	4,353
"	Mankato sandstone	9,606
"	Mantorville sandstone	8,775
"	Frontinac sandstone	10,114
"	Luverne quartzite	21,556
"	" "	19,875
Iowa	Rubble rock	9,465
"	Firestone	4,834
"	Gypsum, Fort Dodge	2,899

The ultimate resistances of the sandstones are relatively low, while the higher values are found for granites, limestones, and quartzites, as is usual.

In 1906 the Carnegie Institution of Washington published An Investigation into the Elastic Constants of Rocks, More Especially with Reference to Cubic Compressibility, by Mr. Frank D. Adams and Dr. Ernest G. Coker. The experimental part of this investigation was made atMcGill University under the auspices of the Carnegie Institution.

Although this investigation was made as a contribution more to physics than to engineering, the results obtained are of both interest and value to engineers and it is well to make use even for engineering purposes of results determined with so much care and such extreme accuracy in spite of the fact that the specimens used were only 1 inch square in section or 1 inch in diameter and 3 inches long. If E is the ordinary modulus of elasticity in compression G the modulus of elasticity for shearing, V the so-called bulk modulus, i.e., the reciprocal of the rate of change of unit volume for unit intensity of stress, and r the ratio of the rate of lateral strain of the specimen divided by the rate of direct strain under compression, Table III gives the results of these experimental determinations for those materials which American engineers more commonly use.

TABLE III.

Specimen.	E.	r.	G.	$V=\frac{E.}{3(1-2r)}$
Black Belgian marble.	11,070,000	0.2780	4,330,000	8,303,000
Carrara marble......	8,046,000	0.2744	3,154,000	5,946,000
Vermont marble......	7,592,000	0.2630	3,000,000	5,341,000
Tennessee marble.....	9,006,000	0.2513	3,607,000	5,967,000
Montreal limestone...	9,205,000	0.2522	3,636,000	6,167,500
Baveno granite......	6,833,000	0.2528	2,724,800	4,604,000
Peterhead granite....	8,295,000	0.2112	3,399,000	4,792,000
Lily Lake granite.....	8,165,000	0.1982	3,380,000	4,517,500
Westerly granite.....	7,394,500	0.2195	3,019,700	4,397,500
Quincy granite (1)....	6,747,000	0.2152	2,781,600	3,984,000
Quincy granite (2)....	8,247,500	0.1977	3,445 000	4,555,000
Stanstead granite.....	5,685,000	0.2585	2,258,700	3,940,000
Ohio sandstone......	2,290,000	0.2900	888,000	1.816,000
Plate glass..........	10,500,000	0.2273	4,290,000	6,448,000

Art. 70.—Timber.

The ultimate compressive resistance, coefficient of elasticity, and other physical properties of timber in compression are affected greatly by the amount of moisture in the timber and by the size of stick. The investigations of Professor J. B. Johnson, acting for the Forestry Division of the U. S. Department of Agriculture, have shown that when the amount of moisture exceeds about 30% by weight of the timber the physical properties are not much affected by any increased saturation. The walls of the wood cells at that point seem to experience their maximum softening. Green timber may be considered as carrying about one third of its weight in moisture, and it seems to matter little whether that moisture is water or sap, timber once dried and resaturated appearing to suffer the same diminished resistance as in its original green condition. Professor Johnson's tests showed that the Southern pines increased their ultimate compressive resistance in some cases as much as 75% by the process of drying or seasoning from 33% of moisture down to 10%, the general rule being a greatly increased compressive resistance with a decrease of moisture. It follows from these results, therefore, that green timber will be much weaker in compression than seasoned timber. Ordinary air seasoning even under cover seldom reduces moisture below about 15% in weight of the timber itself, although under favorable circumstances of seasoning the moisture may sometimes drop to 12% of that weight. As a matter of precision, therefore, or accuracy, the ultimate compressive resistance of timber should always be stated in connection with the percentage of moisture carried by the timber. This will be found to be the case in all of Professor Johnson's experimental work, to which reference has already been made and the results of which are chiefly found in bulletins Nos. 8 and 15 of the Division of Forestry of the U. S. Department of Agriculture, the former being dated 1893.

The earlier tests of Professor Johnson were made on a basis of 15% moisture, but in his later work a basis of 12% moisture was adopted, and he states in Circular No. 15 that in reducing the moisture from 15% to 12% the corresponding increases in the ultimate compressive resistance in pounds per square inch of Southern pines are approximately as follows:

	Endwise.	Across Grain.
Long-leaf pine	1,100	180
Cuban pine	800	220
Loblolly pine	900	150
Short-leaf pine	600	60

While it is important as a matter of physics to recognize clearly the effect of moisture upon the compressive resistance of timber, it is of equal importance, and possibly of greater importance, to recognize the fact that in engineering practice, except in specially protected cases, the timber used in structures is more or less exposed and can seldom or never be depended upon to contain even as little as 15% of moisture, and with some conditions of weather and at some seasons of the year it may contain considerably more. It follows, also, that the condition of timber as to moisture in most structures will change materially from time to time. It would be unwise, therefore, and perhaps dangerous to use working compressive resistances based upon the results of tests of small pieces with moisture reduced to 15% or 12%.

Again, it has been frequently stated as a result of the timber investigations by the Forestry Division of the U. S. Department of Agriculture, that the ultimate compressive resistance of large sticks may be taken as practically identical with that belonging to small selected test pieces, the quality of the material being the same in both cases. It is possible, if the quality of material throughout all portions of every large stick were identical with the quality of small selected specimens, that the ultimate compressive resistance per square inch might be the same; but that is radically different from the facts as they are. There is probably no stick of timber whose condition is permanent at any given time. If it is seasoning, its quality is improving, but after reaching a maximum of excellence it begins to depreciate by decay or from other causes. Any large stick of timber as used by the engineer is seldom free from some point of incipient decay and it is never free from knots, large or small, wind shakes, cracks from one cause or another, or from some other defective condition, at some point. Small specimens for testing are invariably so selected as to eliminate such spots as militating against a comparatively high resistance. The inevitable result for full-size sticks is a decreased resistance materially below that of the small specimen. For all these reasons, therefore, in engineering practice it would be a radical error to accept the ultimate compressive resistance per square inch of small test specimens as practically identical with that of large sticks. Values for the latter class of timber should be determined upon pieces as large as those used in structures and under the same conditions in which they are used, which means an indefinite amount of moisture ordinarily sensibly larger than 12% or 15%.

In the "U. S. Report of Tests of Metals and Other Materials" for 1896 and 1897 there may be found results of compressive tests for coefficients of elasticity for sticks of timber as shown in Table I. Those sticks were many of them large enough to form full-size posts. They appear to

The fracture of a piece of Douglass fir or Oregon pine loaded tangentially to the rings of growth. The ultimate compressive resistance was found to be 600 lbs. per sq. in.

(*To face page* 429.)

TABLE I.

TIMBER IN COMPRESSION.

Kind of Wood.	Coefficient of Elasticity, Pounds per Square Inch.			No. of Tests.	Remarks.
	Maximum.	Mean.	Minimum.		
Douglas fir:					
Endwise.	3,461,000	2,358,000	1,915,000	4	Not well seasoned.
Tangentially.	112,000	74,600	40,000	9	“ “ “
Radially.	207,000	158,000	134,300	6	“ “ “
White oak:					
Endwise.	1,789,000	1,554,000	1,338,000	6	“ “ “
Long-leaf pine:					
Endwise.	1,890,000	1,657,000	1,488,000	4	From tops of trees
“	2,252,000	2,175,000	2,049,000	4	From butts of trees
Short-leaf pine: *					
Endwise.	1,655,000	1,469,500	1,202,000	6	Not well seasoned
Spruce: *					
Endwise.	1,623,000	1,531,000	1,437,000	10	“ “ “
Old yellow-pine posts: *					
Endwise.	2,300,000	2,251,000	2,207,000	12	Very dry.

* These results are means of determinations at intensities varying from 500 to 5,000 pounds per square inch.

have been of merchantable timber of about such quality as is used in first-class engineering works. They had the usual supply of knots and other features which, while not material defects, prevented the pieces from being of selected quality.

As also shown in the table, there were a considerable number of tests in each case. "Endwise" compression means compression parallel to the fibres of the timber, while "Tangentially" means a direction tangent to the rings of growth. That compression indicated by "Radially" was in a radial direction, i.e., passing through the centre of the tree trunk. The determinations were made at intensities of pressure varying from one third to one half the ultimate resistance. It will be noticed that in the values for long-leaf pine the highest results belong to sticks from the butts of trees, while those from the tops give materially less values. It will also be observed that the values for the very dry yellow-pine posts in the last line of the table are high, showing the increased stiffness due to the absence of moisture. The coefficients of elasticity in the last five lines of the table were computed from the resilience of the compressed columns by means of a formula similar to eq. (2) of Art. 44.

The values of the elastic limit, ultimate resistance and modulus of elasticity in compression along the fibres as well as the elastic limit in compression across the fibres of nine of the prominent structural timbers of the United States, both for large or structural sizes and small specimens, as shown in Table II, are taken from Tests of Structural Timbers, Forest Service-Bulletin 108, U. S. Department of Agriculture, by Messrs. McGarvey Cline and A. L. Heim, 1912, and exhibit some of the latest experimental investigations in the elasticity and resistance of timber. The large or structural sizes had cross-sections up to 10 inches by 16 inches and the small sizes down to 2 inches by 2 inches. The resistances parallel to the fibres, i.e. on end, were determined for pieces whose lengths were three to four times the cross dimensions.

The authors of the paper properly observe that the " Results of tests made only on small thoroughly seasoned specimens free from defects "—" may be from one and one-half to two times as high as stresses developed in large timbers and joists." This is an important conclusion and a number of results in Table II confirm the observations of the authors.

It is essential to observe the small resisting capacity of the various timbers when compressed across the grain, the resistance in the latter condition being but a small fraction of that along the grain.

Table III contains the results of tests by Colonel Laidley,

TABLE II.

	Wt.* per Cu. Ft Lbs.	R'gs per In.	Green Timber. Parallel to Grain. Pounds per Sq. In.			Elas. Limit Perp. to grain	R'gs per In.	Air Seasoned Timber. Parallel to Grain. Pounds per Sq. In.			Elas. limit Perp. grain.
			Elas. Limit.	Ult. Resist.	E.			Elas. Limit.	Ult. Resist.	E.	
Longleaf pine:											
Large sizes	35	13.8	3,480	4,800	....	568	15.4	3,480	4,800	....	572
Small sizes	....	....	....	4,400	....	....	....	....	....	....	....
Douglas fir:											
Large sizes	28	11	2,770	3,495	1,414,000	570	13.1	3,271	4,258	1,038,000	639
Small sizes	....	....	3,500	4,030	1,925,000	....	....	3,842	5,002	1,084,000	....
Shortleaf pine:											
Large sizes	30	12.1	2,460	3,435	1,548,000	351	12.4	4,070	6,030	1,951,000	796
Small sizes	....	....	....	3,570	....	400	....	....	6,380	....	926
Western larch:											
Large sizes	28	24.3	2,675	3,510	1,575,000	456	22.7	....	5,746	....	597
Small sizes	....	....	3,026	3,696	1,545,000	....	....	....	5,934	....	....
Loblolly pine:											
Large sizes	31	5.9	2,050	2,940	548,000	500	6.3	3,011	4,292	1,206,000	655
Small sizes	....	....	....	3,240	....	....	....	....	5,547	....	....
Tamarack:											
Large sizes	30	14	2,400	3,230	1,373,000	....	12.7	3,349	4,320	1,351,000	....
Small sizes	....	....	....	3,190	....	....	....	....	4,790	....	697
Western hemlock:											
Large sizes	27	15.6	2,905	3,355	1,617,000	434	17.7	4,840	5,814	2,140,000	473
Small sizes	....	....	2,938	3,392	1,737,000	....	....	4,560	5,403	1,923,000	....
Redwood:											
Large sizes	22	18.8	3,194	3,882	1,240,000	434	20.1	....	4,276	....	525
Small sizes	....	....	3,490	3,980	1,222,000	569	....	....	5,119	....	564
Norway pine:											
Large sizes	25	13.7	2,065	2,555	1,002,000	....	10	3,047	4,228	1,367,000	....
Small sizes	....	....	....	2,504	....	....	....	....	7,550	....	924

* Oven dry.

TABLE III.

No.	Kind of Wood.	Length, Inches.	Compressed Section in Inches.	Ultimate Resistance, Pounds per Square Inch.	Perpendicular to or with Grain.	Remarks.
1	Oregon pine	16.5	2.46 × 2.0	8,496	With	
2	Oregon pine	19.9	1.21 × 1.21	9,041	"	
3	Oregon pine	19.9	1.21 × 1.21	8253	"	
4	Oregon maple	8.0	3.63 × 3.63	6,661	"	
5	Oregon spruce	24.02	3.92 × 5.75	5,772	"	Unseasoned
6	California laurel	8.0	3.58 × 3.58	6,734	"	Worm-eaten
7	Ava Mexicana	8.0	3.69 × 3.69	6,382	"	
8	Oregon ash	8.0	3.64 × 3.64	5,121	"	
9	Mexican white mahogany	8.0	3.77 × 3.77	6,155	"	
10	Mexican cedar	8.0	3.75 × 3.75	4,814	"	
11	Mexican mahogany	8.0	3.75 × 3.75	10,043	"	
12	White maple	12.0	4.00 × 4.00	7,140	"	
13	White maple	12.0	4.00 × 4.00	7,210	"	
14	Red birch	13.0	4.26 × 4.26	8,030	"	
15	Red birch	13.0	4.26 × 4.26	7,820	"	
16	Whitewood	12.0	4.00 × 4.00	4,440	"	
17	Whitewood	12.0	4.00 × 4.00	4,330	"	
18	White pine	12.0	4.00 × 4.00	5,475	"	
19	White pine	12.0	4.00 × 4.00	5,760	"	
20	White oak	12.0	4.00 × 4.00	7,375	"	
21	White oak	12.0	4.00 × 4.00	7,010	"	
22	Ash	12.0	4.00 × 4.00	7,940	"	
23	Ash	12.0	4.00 × 4.00	7,640	"	
24	Oregon pine	1.95	3.45 × 3.00	1,150	Perp.	
25	Oregon maple	3.63	3.63 × 3.00	1,875	"	
26	Oregon spruce	3.92	5.75 × 4.75	710	"	Unseasoned
27	Oregon spruce	3.92	4.75 × 4.00	680	"	Unseasoned
28	California laurel	3.58	3.58 × 3.00	2,000	"	
29	Ava Mexicana	3.69	3.69 × 3.00	2,100	"	
30	Oregon ash	3.64	3.64 × 3.00	2,200	"	
31	Mexican white mahogany	3.77	3.77 × 3.00	2,150	"	
32	Mexican cedar	3.75	3.75 × 3.00	1,950	"	
33	Mexican mahogany	3.75	3.75 × 3.00	4,500	"	
34	White pine	3.06	6.20 × 4.75	875	"	
35	White pine	2.90	4.75 × 4.00	1,012	"	Mean of two
36	Whitewood	3.15	4.75 × 6.20	900	"	Mean of two
37	Whitewood	3.15	4.75 × 4.00	1,000	"	Mean of four
38	Black walnut	0.875	4.75 × 4.00	2,450	"	Mean of two
39	Black walnut	0.875	4.00 × 3.94	2,200	"	Mean of two
40	Black walnut	0.875	4.00 × 2.50	2,525	"	Mean of two
41	White oak	2.40	4.75 × 4.00	3,550	"	
42	Spruce	3.70	4.75 × 4.00	970	"	Mean of four
43	Yellow pine	3.90	4.00 × 4.00	1,900	"	
44	Black walnut	0.75	4.05 × 4.00	2,800	"	
45	Black walnut	1.00	4.05 × 4.00	2,560	"	
46	Black walnut	1.25	4.05 × 4.00	2,400	"	
47	Black walnut	1.50	4.05 × 4.00	2,500	"	
48	Black walnut	1.75	4.05 × 4.00	2,400	"	
49	Black walnut	2.06	4.05 × 4.00	2,360	"	
50	White pine	0.75	4.05 × 4.00	1,120	"	
51	White pine	1.00	4.05 × 4.00	1,100	"	
52	White pine	1.25	4.05 × 4.00	1,160	"	
53	White pine	1.50	4.05 × 4.00	1,070	"	
54	White pine	1.75	4.05 × 4.00	1,060	"	
55	White pine	2.00	4.05 × 4.00	1,000	"	
56	Yellow birch	4.25	4.25 × 3.00	2,000	"	
57	Yellow birch	4.25	5.98 × 3.00	1,650	"	
58	White maple	4.00	3.95 × 3.00	1,700	"	
59	White maple	4.00	5.98 × 3.00	1,900	"	
60	White oak	3.95	3.96 × 3.00	2,500	"	Mean of two

U.S.A., "Ex. Doc. No. 12, 47th Congress, 2d Session." A few other tests of short blocks from the same source will be found in the article on "Timber Columns." Unless otherwise stated, all the specimens were thoroughly seasoned.

In this table the "length" of all those pieces which were compressed in a direction perpendicular to the grain might, with greater propriety, be called the thickness, since it is measured across the grain.

In the tests (24–60) the compressing force was distributed over only a portion of the face of the block on which it was applied; thus the compressed area was supported, on the face of application, by material about it carrying no pressure. In some cases this rectangular compressed area extended across the block in one direction, but not in the other. In all such instances the ultimate resistance was a little less than in those in which the area of compression was supported on all its sides.

The "ultimate resistance" was taken to be that pressure which caused an indentation of 0.05 inch.

Nos. (44–55) show the effect of varying thickness of blocks. Within the limits of the experiments, the ultimate resistance is seen to decrease somewhat as the thickness increases.

The best series of values of the ultimate compressive resistance of timbers as actually used in large pieces and for engineering structures that can be written at the present time is that given in Table IV.

That table shows values for railway bridges and trestles adopted by the American Railway Engineering Association.

As in the case of tension, the compressive resistances across the grain are but small fractions of those with the grain. Values are given for columns under 15 diameters in length for the reason that such columns fail essentially by compression and without the bending which characterizes long columns. The table is one of great practical value.

TABLE IV.

TIMBER IN COMPRESSION.

Kind of Timber.	Unit Stresses in Lbs. per Sq. In.					
	Perpendicular to the Grain.		Parallel to the Grain.		Working Stresses for Columns.	
	Elastic Limit.	Working Stress.	Mean Ult.	Working Stress.	Length Under $15 \times d$.	Length Over $15 \times d$.
Douglas fir....	630	310	3,600	1,200	900	$1,200(1-l/60d)$
Longleaf pine..	520	260	3,800	1,300	975	$1,300(1-l/60d)$
Shortleaf pine..	340	170	3,400	1,100	825	$1,100(1-l/60d)$
White pine....	290	150	3,000	1,000	750	$1,000(1-l/60d)$
Spruce........	370	180	3,200	1,100	825	$1,100(1-l/60d)$
Norway pine...	.....	150	2,600*	800	600	$800(1-l/60d)$
Tamarack.....	.....	220	3,200*	1,000	750	$1,000(1-l/60d)$
Western Hemlock........	440	220	3,500	1,200	900	$1,200(1-l/60d)$
Redwood......	400	150	3,300	900	675	$900(1-l/60d)$
Bald cypress...	340	170	3,900	1,100	825	$1,100(1-l/60d)$
Red cedar.....	470	230	2,800	900	675	$900(1-l/60d)$
White oak.....	920	450	3,500	1,300	975	$1,300(1-l/60d)$

Unit stresses are for green timber and are to be used without increasing the live load stresses for impact. Values noted * are partially air-dry timbers.

In the formulas given for columns, l = length of column, in inches, and d = least side or diameter, in inches.

CHAPTER IX.

RIVETED JOINTS AND PIN CONNECTION.

Art. 71.—Riveted Joints.

ALTHOUGH riveted joints possess certain characteristics under all circumstances, yet those adapted to boiler and similar work differ to some extent from those found in the best riveted trusses. The former must be steam- and water-tight, while such considerations do not influence the design of the latter, consequently far greater pitch may be found in riveted-truss work than in boilers. Again, the peculiar requirements of bridge and roof work frequently demand a greater overlap at joints and different distribution of rivets than would be permissible in boilers.

Kinds of Joints.

Some of the principal kinds of joints are shown in Figs. 1 to 6. Fig. 1 is a "lap-joint" single-riveted; Fig. 2 is a "lap-joint" double-riveted; Fig. 3 is a "butt-joint" with a single butt-strap and single-riveted; while Figs. 4, 5, and 9 are "butt-joints" with double butt-straps, Fig. 4 being single-riveted, while the others are double-riveted. Fig. 5 shows zigzag riveting, and Fig. 6 chain riveting. All these joints are designed to resist tension and to convey stress from one single thickness of plate to another. Two or

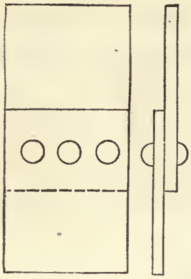


FIG. 1.

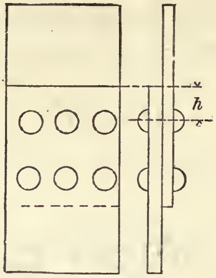


FIG. 2.

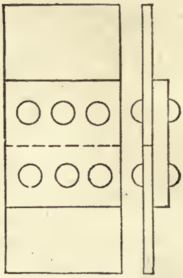


FIG. 3.

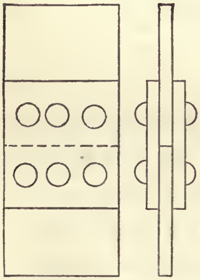


FIG. 4.

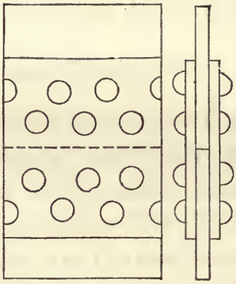


FIG. 5.

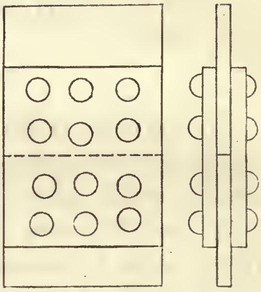


FIG. 6.

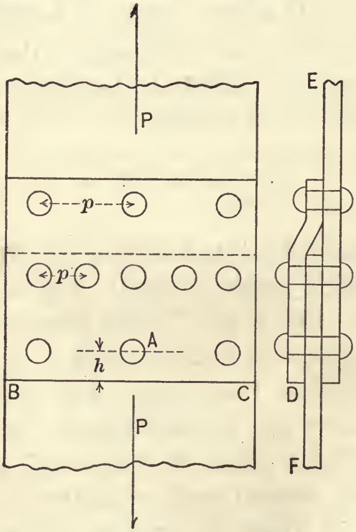


FIG. 7.

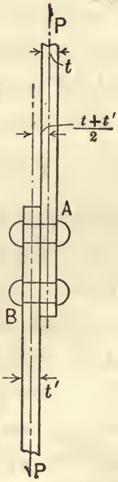


FIG. 8.

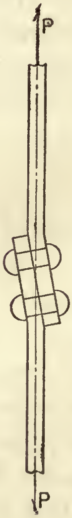


FIG. 9.

three other joints peculiar to bridge and roof work will hereafter be shown.

In the cases of bridges and roofs these "butt-straps" are usually called "cover-plates."

Art. 72.—Distribution of Stress in Riveted Joints.

Bending of the Plates.

In order that rivets, butt-straps or cover-plates and different parts of the main plates may take their proper stresses, an accurate adjustment of these different parts to the external forces or loads must be attained; but all shop work is necessarily more or less imperfect and the varying stresses at different parts of the joint produce at least elastic deformations so that the requisite conditions for a proper distribution of stresses as computed cannot be maintained. The precise amount of stress, therefore, carried by each rivet, cover-plate or other part of the joint including the main plates cannot be computed. By means of reasonable assumptions, however, and by the introduction of factors or coefficients determined by the actual testing of riveted joints, simple and sufficiently accurate formulæ for all engineering purposes may be established.

The shafts of the rivets of any joint compress or bear against the walls of the rivet holes in the transference of loading from one main plate to the other. This condition will necessarily subject the metal on either side of the hole and adjacent to it to a higher degree of tension than the metal midway between two neighboring holes. This makes the average intensity of stress over the minimum section of either the main plate or the cover-plate materially less than the maximum intensity at or near the wall of the rivet hole. On the other hand, the removal of the metal for the rivet holes makes that part of the plates between two consecutive holes at right-angles to the direction of loading a " short " specimen with a higher ultimate resistance than a long specimen.

Again let Fig. 8, like Fig. 2 of the preceding article, represent a longitudinal section of a double riveted lap-joint, the thicknesses of the two plates being t and t'. The two opposite loads P would produce a bending moment about an axis at right-angles to the plane of section of $P\frac{t+t'}{2}$. Usually the two thicknesses of plate are the same making t the lever arm of the couple. This moment causes bending in the plates in the vicinity of A and B of equal amount and the bending intensities of stresses may be computed in the usual manner if the joints were not distorted so as to change the lever arm of the couple. As the load is increased, however, the joint tends to take the shape shown in Fig. 9, the two plates tending to pull into the same straight line, making it impossible to compute accurately the bending moment. It is sufficient, however, to recognize this condition of flexure in the joint.

This eccentric action of the load P produces also the same bending moment in the rivets of the joint, in the aggregate, as that impressed upon the plates. The assumed bending moment carried by each rivet will be the moment $P\frac{t+t'}{2}$ or Pt divided by the number of rivets in the joint. This bending moment is seldom or never computed for rivets but it is always computed in the design of pins of a pin-connected truss bridge.

For all these reasons and others shortly to be considered it is obvious that if a riveted joint of any type be tested to destruction, it is essentially impossible to compute accurately what the intensity of stress will be in any part of it at any stage of loading. Such tests, however, yield most valuable empirical quantities to be used in formulæ to be established and without which it would be essentially impossible to design a riveted joint in a rational manner.

Although these considerations are based upon the characteristics of a double-riveted lap-joint, they apply to all riveted joints of any type whatever. If the butt-joint with double cover-plates shown in Fig. 5 of the preceding article be considered, it will be clear at once that if a line be drawn centrally through the section of the two main plates, each half of the actual joint will be divided into two equal double-riveted lap-joints in each of which the plates will be subjected at least approximately to the same condition of stress as that found in connection with Fig. 9 and the bending of the rivets will be precisely the same. There will be, however, no bending of the main plates.

The special form of joint shown in Fig. 7, which has come to be much used, will also have its parts subjected to the same general condition of stresses including the bending of rivets and main plates.

It is clear that the bending of the plates illustrated in Fig. 9 will increase with their thickness.

Net Section of Plates

The net section of any main plate or cover-plate in a riveted joint is the gross section along any transverse line of rivets less the metal taken out by the rivet holes. In Fig. 2 of the preceding article, the net section of either main plate will be its gross section less three rivet holes. The pitch p of the rivets in any transverse line of rivet holes in a riveted joint is the distance between the centres of two consecutive rivets as shown in Fig. 7. In the centre line of rivets in that figure, the pitch is one-half that in the outer line. The net section of any plate, therefore, per rivet will be $(p-d)t$, d being the diameter of the rivet hole and t the thickness of the plate. If n is the number of rivets in one main plate and if q is the number of rows of rivets in it, then the number of rivets in each row will be $\frac{n}{q}$ and the total net section along any transverse row of rivets will be $\frac{n}{q}(p-d)t$.

Bending of the Rivets.

It has already been seen that the rivets of any riveted joint are subjected to bending. It is assumed that the total bending moment, $M=P\frac{t+t'}{2}$ or $M=Pt$ is divided uniformly among all the rivets of the joint. Hence the bending moment to which a single rivet is subjected is

$$\frac{M}{n}=\frac{kAd}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which A is the area of cross section of one rivet and k the greatest intensity of tension or compression in the extreme fibre due to bending. By introducing in eq. (1) the values of M already used, eqs. (2) and (3) at once result.

$$k=4Tt(p-d)\frac{(t+t')}{nAd}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $t=t'$,

$$k=8Tt^2\frac{(p-d)}{nAd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This equation is approximate because it is virtually assumed that the pressure on the rivet is uniformly distributed along its axis.* This is a considerable deviation from the truth, particularly as failure is approached. The true bending moment is much less than that given by eq. (1) after the rivet has deflected a little.

When the joint takes the position shown in Fig. 9, it is clear that the rivet is also subject to some direct tension.

The Bearing Capacity of Rivets.

There is a very high intensity of pressure between the shaft of the rivet and the wall of the hole. This intensity is not uniform over the surface of contact, but has its greatest value at, or in the vicinity of, the extremities of that diameter lying in the direction of the stress exerted in the plate. At and near failure this intensity may be equal to the crushing resistance of the material over a considerable portion of the surface of contact.

The intricate character of the conditions involved renders it quite impossible to determine the law of the distribution of this pressure. The bending of the rivets under stress tends to a concentration of the pressure near the surface of contact of the joined plates, while the unavoidably varying "fit" of the rivet in its hole, even in the best of work, throws the pressure towards the front portion of the surface of the rivet shaft. The intensity thus varies both along the axis and around the circumference of the rivet.

If any arbitrary law is assumed, the greatest intensity of pressure is easily determined. Such laws, however, are mere hypotheses and possess no real value. All that can be done is to determine, by experiment, the mean safe

---

* In accordance with this assumption, strictly speaking, $\frac{1}{2}t$ (thickness of main plate) should be taken instead of t in the sum $(t+t')$ in the above formulæ for bending, when applied to the double butt-joint, Figs. 5 and 6.

working intensity on the diametral plane of the rivet which is equivalent to a fluid pressure of the same intensity against its shaft.

Thus, if f is this mean (empirically determined) intensity, d the diameter of the rivet, and t the thickness of the plate, the total pressure carried by one rivet pressing against one plate is

$$R=fdt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Bending of Plate Metal in Front of Rivets.

In addition to the bending of the plates of a riveted joint about an axis parallel to the plates and at right angles to the direction of loading, there is further bending of the metal immediately in front of a rivet about an axis parallel to the axis of the rivet. If a rivet, such as A, Fig. 7, be considered, the metal on that side of the hole nearest to the line BC will be in the condition approximately of a beam fixed at each end of the diameter of the hole parallel to BC, the bearing load fdt being the load resting upon it and assumed to be uniformly distributed over the span d. Manifestly the depth of this beam is not uniform, but it is assumed to have a depth $h-\frac{d}{2}$, Fig. 7, throughout the span d. If t is the thickness of the plate, p the pitch of the rivets and T the mean intensity of tension between the rivet holes, the load on this beam will be $(p-d)Tt$ and the moment of inertia of the cross-section will be

$$I=\frac{t\left(h-\frac{d}{2}\right)^3}{12}.$$

It will be shown in the chapter on bending that k may here be taken at $\frac{3}{2}T$.

In Art. 30 the moments at the centre and end of a span fixed at each end and uniformly loaded were shown to be $\frac{1}{12}$ of the load into the span for the end moments and $\frac{1}{24}$ of the load into the span for the centre moment.

Hence, by the usual formulæ,

$$M=\frac{d}{12}(p-d)Tt=\frac{2kI}{\left(h-\frac{d}{2}\right)}=\frac{3}{2}T\frac{t\left(h-\frac{d}{2}\right)^2}{6};$$

$$\therefore \quad h=0.58\sqrt{(p-d)d}+0.5d \quad . \quad . \quad . \quad . \quad (5)$$

Shearing of Rivets.

The shearing of the rivets in a riveted joint takes place in the plane of the surface of contact between any two plates tending to move in opposite directions. In Fig. 8 the plane of shear would be the surface of contact between the main plates A and B, and in Fig. 7 on both sides of the main plate, F, i.e., between the main plates E and F and at the surface of contact between the main plate F and the bent cover-plate D. It is assumed that the total shear is divided uniformly between all the shear sections of the rivets so that if n were the total number of rivets carrying the load P and if d be the diameter of the rivet while S is the intensity of shearing stress in the normal sections of the rivets, there would result for single shear the expression $P=n.7854d^2S$. The rivets shown in Fig. 8 and Figs. 1, 2, and 3 of the preceding article are in single shear. If each rivet must be sheared at two normal sections in order that the joint may fail (by shear), as in Figs. 4, 5, and 6 of the preceding article, the rivets are said to be in double shear. In the latter case in the preceding expression $2n$ must be written for n for all rivets in double shear. In

Fig. 7 the two lower rows of rivets are in double shear and the upper row in single shear.

In Fig. 8 and in Figs. 5 and 6 of the preceding article, each row of rivets is assumed to take half the total load carried by the joint. That condition, if the cover-plates of Figs. 5 and 6 are of half the thickness of the main plates, makes the intensity of stress the same in the main plate and in the two covers between the two rows of rivets on either side of the joint. If, however, the thickness of the cover-plate is greater than one-half the thickness of the main plate, as is always the case in such joints, then if each row of rivets carries half the load, the intensity of stress in the two covers between each two rows of rivets will be less than in the main plate causing the rate of stretch in the latter to be greater than in the former. This condition would throw more than half the load, as shear, on the outer row of rivets. In other words, the tendency will be to make the stretch of the plates within the joint added to the distortion due to bending and shearing of the rivets equal to each other between each pair of rows of rivets parallel to the joint line between the main plates. If again there are three or more rows of rivets on either side of an abutting joint, there will be a corresponding tendency to overload the outer rows of rivets and relieve those nearest the centre or abutting line of the joint. There are further conditions in addition to those already discussed, militating against perfect uniformity in the stress conditions of the complete joint. It is impossible, however, to make allowance for these complicated and more or less obscure stress conditions in the operations of design or development of formulæ. Hence, as already indicated, the usual assumptions of uniformity in the three principal methods of failure of riveted joints are made leaving the working stresses to be determined by the results of tests of actual joints.

Art. 73.—Diameter and Pitch of Rivets and Overlap of Plate. Distance between Rows of Riveting.

Diameter of Rivets.

The diameter of rivet may at least approximately be expressed in terms of the thickness of the plate which it pierces. There are various arbitrary or conventional rules based upon this method of determining the rivet diameter. If the unit is the inch, the diameter d may be expressed as ranging between the two following values for ordinary thicknesses of plate:

$$\left.\begin{array}{l} d=.75t+.375, \\ d=.875t+.375, \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which t is the thickness of the plate. Unwin gives the following expression for the diameter of somewhat different form from that which precedes:

$$d=1.2\sqrt{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Neither of the preceding expressions can be applied for all thicknesses of plates. If the thickness is great, those expressions make the diameter of the rivet too large, the diameter rarely exceeding 1 inch even for the heaviest plates. The application of eq. (1) to different thicknesses of plates will give the following diameters of rivets expressed by the nearest $\frac{1}{16}$ in.:

t	d
$\frac{1}{4}$ in.	$\frac{9}{16}$ in.
$\frac{3}{8}$	$\frac{3}{4}$
$\frac{1}{2}$	$\frac{7}{8}$
$\frac{5}{8}$	$\frac{15}{16}$
$\frac{3}{4}$	$1\frac{1}{16}$
$\frac{7}{8}$	$1\frac{1}{8}$
1	$1\frac{3}{16}$

In structural work for ordinary thicknesses of metal the prevailing diameters of rivets are $\frac{3}{4}$ in. and $\frac{7}{8}$ in. For light work, such as sidewalk railings or light highway construction, rivets as small as $\frac{1}{2}$ in. or $\frac{5}{8}$ in. in diameter are used. On the other hand, 1 to $1\frac{1}{8}$-inch rivets are employed for specially heavy sections.

Pitch of Rivets.

It is possible to determine the pitch of rivets approximately by an equation expressing equality between the tensile resistance of the net section between two adjacent rivets and the shearing or bearing capacity of a single rivet, but it is scarcely practicable to proceed in that manner as a rule. Again, the pitch will vary to some extent with the number of lines of riveting on either side of the joint. In single-riveting the pitch must be less than in double- or other multiple-riveting. In boiler or other similar riveting, also, the pitch must be usually less than in structural work, as questions of steam- and water-tightness or other similar considerations are involved in the former class of joints. Finally, the pitch will also obviously depend largely upon the thickness of plates. In single-riveting for comparatively thin plates the following relation may be taken, p being the pitch in inches:

$$p = 1.75 \text{ in. to } 2.25 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For comparatively thick plates in single-riveting the following relation may hold:

$$p = 2.375 \text{ in. to } 3 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In double-riveting, p and t still being the pitch and thickness respectively, the following relation may be taken for comparatively thin plates:

$$p = 2.6875 \text{ in. to } 3.25 \text{ in.} \quad . \quad . \quad . \quad . \quad (5)$$

Again, for comparatively thick plates in double-riveting,

$$p = 3.375 \text{ in. to } 3.75 \text{ in.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The values given by eqs. (3) to (6) are for boiler or other similar work.

In structural work the pitch of rivets is seldom less than about three times the diameter of the rivet, and it is frequently specified not to exceed sixteen times the thickness of the thinnest plate pierced by the rivet.

Overlap of Plate.

The overlap of a plate, h in Fig. 2, Art. 71, in a riveted joint is the distance from the edge of the plate to the centre line of the nearest row of rivets. This distance, like other elements of riveted joints, will depend somewhat upon the thickness of the plate as well as the diameter of rivet and other similar considerations. It is a common practice to make the overlap not less than about $1.5d$, d being the diameter of the rivet. Occasionally in riveted joints it is made a little less, but 1.5 times the diameter of the rivet is about as small as the overlap should be made. Sometimes $\frac{1}{8}$ in. is added to the preceding value of the overlap.

The width of overlap (h) may also be determined in terms of d by the aid of eq. (11) of Art. 72. Since the load on the rivet is represented by $(p-d)Tt$, p must be taken in terms of d for a single-riveted joint, in which $p = 2\frac{1}{3}d$ to $2\frac{3}{4}d$. As a margin of safety, and as it will at the same time simplify the resulting expression, let $p = 3d$.

Eq. (5) of Art. 72 then gives, in confirmation of the preceding rule,

$$h = 1.31d^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

---

* In consequence of the direct tension in the metal on either side of the rivet this value of h should be increased, i.e., to perhaps $1.5d$.

Experience has shown that this rule gives ample strength, and is about right for calking in boiler joints.

It is to be remembered that the preceding conventional rules for the diameter of rivet, pitch, and overlap of plate are necessarily to a large extent conventional or approximate, and in special cases they cannot be applied with mathematical exactness. As practical rules, however, they are sufficiently close to give good general ideas of those features of riveted joints.

Distance between Rows of Riveting.

The distance between the rows of riveting is not susceptible of accurate expression by formulæ, although the considerations involved in the establishment of eq. (11) of Art. 72 would lead to an approximate value. It is evident, however, that this distance should never be as small as h. Apparently, in more than double-riveted joints, this distance should increase as the centre line of the joint is receded from, in consequence of the bending action of the rivet. There are other reasons, however, besides that of inconvenience, why such a practice is not advisable.

In chain riveting the distance between the centre lines of the rows of rivets may be taken equal to the pitch in a single-riveted joint, or, as a mean, at 2.5 *the diameter of a rivet.*

In zigzag riveting (Fig. 5) *this distance may be taken at three quarters its value for chain riveting.*

Art. 74.—Lap-joints, and Butt-joints with Single Butt-strap for Steel Plates.

A butt-joint with single butt-strap, similar to that shown in Fig. 3, Art. 71, is really composed of two lap-joints in contact, since each half of the butt-strap or cover-plate with its underlying main plate forms a lap-joint. It is unnecessary, therefore, to give it separate treatment.

From these considerations it is clear that the thickness of the butt-strap or cover-plate should be at least equal to that of the main plate; it is usually a little greater.

Let t = thickness of plates;
d = diameter of rivets;
p = pitch of rivets (i.e., distance between centres in the same row);
T = mean intensity of tension in net section of plates between rivets;
T' = mean intensity of tension in main plates;
f = mean intensity of pressure on diametral plane of rivet;
S = mean intensity of shear in rivets;
n = number of rivets in one main plate;
q = number of rows in one main plate;
h = lap as shown in Fig. 2, Art. 71.

If all the dimensions are in inches, then T, T', f, and S are in pounds per square inch.

The starting-point in the design of a joint is the thickness t of the plate. The rivet diameter may then be expressed in terms of t, and the pitch in terms of the diameter. Such rules, like those given in Art. 72, may be useful within a certain range of application, but they cannot be depended upon in all cases.

The thickness t of boiler-plate depends upon the internal pressure, and is to be determined in accordance with the principles laid down in Art. 39, after having made allowance for the metal punched out at the holes to find the net section.

In truss work the thickness depends upon the amount of stress to be carried, and the same allowance is to be made for rivet-holes in finding the net section.

The relation existing between T and T' is shown by the following equations:

$$t(p-d)T = tpT'; \quad \therefore \quad \frac{T}{T'} = \frac{p}{p-d},$$

or

$$\frac{T'}{T} = \frac{p-d}{p} = 1 - \frac{d}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In order that the joint may be equally strong in reference to all methods of failure, the following series of equalities must hold:

$$\frac{n}{q}tpT' = \frac{n}{q}t(p-d)T = nfdt = 0.7854nd^2S.$$

$$\therefore \; tpT' = t(p-d)T = qfdt \doteq 0.7854qd^2S. \quad . \quad . \quad (2)$$

It is probably impossible to cause these equalities to exist in any actual joint, but none of the intensities T', T, f, or S should exceed a safe working value.

The method of failure by tearing through the gross section of the main plate is practically impossible under ordinary circumstances, and it is neglected in designing riveted joints. This neglect is expressed by dropping the first member of eq. (2) and thus reaching eq. (3):

$$t(p-d)T = qfdt = 0.7854qd^2S. \quad . \quad . \quad . \quad . \quad (3)$$

This equation shows that the usual design of a riveted joint must provide against failure in three principal ways:

1. *Tearing through the net section of the plate.*
2. *Compression of the metal where the rivets bear against the plate.*
3. *Shearing of the rivets.*

Although these are the three principal methods of failure of riveted joints, whatever may be their type or form, the proper design of such joints should be so performed as to afford provision also against the secondary stresses caused by rivet bending, bending of the plates, and other indirect influences discusssd in preceding articles. This latter end is attained by determining the empirical intensities T, f, and S of eq. (3) by testing to failure actual riveted joints in which those secondary stresses exist. In that manner the design against the three principal methods of failure, described above, will also afford provision against the secondary or indirect stresses of rivet and plate bending or other similar conditions. The determination of the intensities T, f, and S by tests of actual riveted joints will be fully shown in the following articles.

It may be stated here, however, that an approximate relation between the ultimate intensities of resistance to shear and tension for steel has been used in engineering practice in accordance with which

$$S = .75T. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It will be found hereafter that f may be taken at least 1.25 T. If these values be substituted in the third and fourth members of eq. (3) in which $q = 2$, there will result

$$d = 2t \text{(nearly)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This value of d is too large for thick plates.

The rivet diameter, therefore, for steel plates may be said to vary from $2t$ for thin plates to $1.6t$ for thick ones, with a maximum diameter of $1\frac{1}{8}$ to $1\frac{1}{4}$ inches. The distance between the centre lines of the rows of rivets may be taken at $2.5d$ to $3d$ for chain riveting and three fourths of that distance for zigzag riveting.

The best designed single-riveted lap-joints give from 55 to 64 per cent. the strength of the solid plates.

Well-designed double-riveted lap-joints should give from 65 to 75 per cent. the resistance of the solid plate.

Equally well-constructed treble- and quadruple-riveted joints should have an efficiency of 70 to 80 per cent. of the solid plate.

It is therefore seen that there is little economy in more than double-riveting ordinary joints.

Art. 75.—Steel Butt-joints with Double Cover-plates.

Butt-joints with double butt-straps or covers differ in two respects, and advantageously, from lap-joints and butt-joints with a single cover; i.e., in the former the rivets are in double shear and the main plates are subjected to no bending. The cover-plates, however, are subjected to greater flexure than the plates of a lap-joint, for there is no opportunity to decrease the leverage by stretching. As the covers form only a small portion of the total material, these, with economy, may be made sufficiently thick to resist this tendency to failure.

Let t' = thickness of each cover-plate; and let the remaining notation be the same as in Art. 74. The intensity of compression between the walls of the holes in the cover-plates and the rivets, and the tension in the former, will be ignored on account of the excess in thickness of the two cover-plates combined over that of the main plate. This excess in thickness is required on account of the bending in the covers noticed above.

The thickness of each cover should be from $\frac{5}{8}$ to $\frac{7}{8}$ the thickness of the main plates, or $t' = .625$ to $.875t$.

The combined thickness of the covers will thus be from 1.25 to 1.75 that of the main plates.

The four principal methods of rupture in the main plate will then lead to the following equations, corresponding to eq. (2), Art. 74:

$$\frac{n}{q}tpT' = \frac{n}{q}t(p-d)T = nfdt = 1.5708nd^2S.$$

$$\therefore\ tpT' = t(p-d)T = qfdt = 1.5708qd^2S. \quad . \quad . \quad . \quad (1)$$

As in Art. 74, and for the reasons there given, the first member of eq. (1) may be omitted, thus giving

$$t(p-d)T = qfdt = 1.5708qd^2S. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Tests of steel butt-joints with double cover-plates as well as other tests in bearing and tension in net section of plates make it reasonable to take $f=1.25T$, with T having values from 55,000 to 60,000 pounds per square inch for thick plates to perhaps 65,000 to 70,000 pounds per square inch for thin plates.

With this value of f, and $q=2$, the first and second members of eq. (2) give for double-riveted butt-joints with two covers;

$$p=3.5d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the same value of f be preserved, there will result for *single-riveted butt-joints with two covers*

$$p=2.5d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If, as in the preceding article, there be taken $S=.75T$ and $f=1.25T$, the second and third members of eq. (2) give

$$d=1.06t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This value of the rivet diameter is too small for thin plates, but about right for thick plates.

Double-riveted butt-joints designed in accordance with the foregoing deductions should give a resistance ranging from 65 to 75 per cent. of that of the solid plate.

Single-riveted joints will give an efficiency somewhat less; perhaps from 60 to 65 per cent.

It is to be supposed, in applying the rules just established, that all steel plates are drilled or punched and reamed.

As in the preceding cases, the distance between the centre lines of the rows of rivets may be taken at 2.5 to 3*d* for chain riveting, and three quarters that distance for zigzag.

Art. 76.—Tests of Full-size Riveted Joints.

There have not been many tests of full-size riveted joints of either iron or steel, and those which have been made seldom include such heavy steel plates as are now frequently employed both in boiler work and for structural purposes. The most valuable tests available and with the greatest range in size of rivet and thickness of plate are those which have been made at the U. S: Arsenal, Watertown, Mass. The results shown in Table I were taken from "Senate Ex. Doc. No. 1, 47th Congress, 2d Session," while those in Table II are taken from "Senate Ex. Doc. No. 5, 48th Congress, 1st Session." The results shown in Table III are from the same source and are given in the "U. S. Report of Tests of Metals and Other Materials" for 1896. The character of plates, rivets, and holes is shown in the tables, and the intensities of tension in the net sections of plates, compression or bearing on diametral surface, and shearing on rivets are those which existed at the instant of failure. The bold-face figures show the kind of failure, and when such figures are found, for the same test, in two or three columns, they show that the same two or three kinds of failure took place simultaneously.

TABLE I.

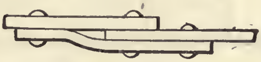
RIVETED JOINTS—IRON AND STEEL.

No.	Size of Rivet and Kind.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	Remarks.
			Tension on Net Area of Plate (T)	Compression on Diametral Surface (f).	Shearing on Rivets (S).		
Single-riveted lap-joints; 1/4-inch iron plates.							
35	5/8″ iron	2 ins.	43,230	76,140	34,900	57.7	11/16″ punched holes.
36	5/8″ “	2 “	45,520	82,910	38,640	61.4	“ “ “
37	5/8″ “	2 “	38,580	73,260	34,870	52.8	“ drilled “
38	5/8″ “	2 “	41,790	79,360	38,660	57.1	“ “ “
39	5/8″ “	1 5/8 “	52,160	65,420	33,420	60.6	“ punched “
40	5/8″ “	1 5/8 “	54,930	68,890	35,200	64.0	“ “ “
41	5/8″ steel	2 “	49,420	87,670	39,640	65.9	“ “ “
42	5/8″ “	2 “	47,260	83,940	40,610	63.1	“ “ “
43	1/2″ “	1 5/8 “	45,890	78,220	45,300	60.3	9/16″ “ “
44	1/2″ “	1 5/8 “	49,720	84,660	48,420	65.5	“ “ “
45	7/16″ iron	1 5/16 “	41,095	66,778	44,204	53.1	1/2″ drilled “
46	7/16″ “	1 5/16 “	37,500	60,886	42,038	48.3	“ “ “
Single-riveted lap-joints; 1/4-inch steel plates.							
426	5/8″ iron	2 ins.	46,340	82,480	37,890	53.2	11/16″ punched holes.
427	5/8″ “	2 “	46,010	81,780	37,860	52.8	“ “ “
436	5/8″ steel	2 “	60,250	107,260	49,270	69.2	“ “ “
437	5/8″ “	2 “	59,240	105,290	48,750	68.0	“ “ “
428	5/8″ iron	2 “	40,950	77,870	36,350	48.2	“ drilled “
429	5/8″ “	2 “	42,370	80,200	36,710	49.6	“ “ “
438	5/8″ steel	2 “	63,190	120,160	56,100	74.3	“ “ “
439	5/8″ “	2 “	61,310	116,090	52,460	71.8	“ “ “
430	5/8″ “	1 5/8 “	66,860	90,000	41,790	68.8	“ “ “
31	5/8″ “	1 5/8 “	70,000	94,230	43,750	72.0	“ “ “
47	7/16″ “	1 5/16 “	62,496	101,180	65,220	69.0	1/2″ “ “
48	7/16″ “	1 5/16 “	58,338	94,800	60,382	64.8	“ “ “
49	5/8″ “	2 “	60,184	114,603	52,742	70.6	11/16″ “ “
50	5/8″ “	2 “	57,439	109,650	50,645	67.6	“ “ “
Double-riveted lap-joints; 1/4-inch plates.							
85	7/16″ iron	2 ins.	38,535	64,120	43,110	60.3	1/2″ drilled holes.
86	7/16″ “	2 “	41,750	69,710	41,750	65.3	“ “ “
617	1/2″ “	1 5/8 “	50,592	42,118	28,691	65.8	9/16″ punched “
618	1/2″ “	1 5/8 “	49,950	41,660	28,660	65.3	“ “ “
Double-riveted lap-joints; 1/4-inch steel plates.							
432	5/8″ iron	2 ins.	61,510	54,640	25,400	70.4	11/16″ punched holes.
433	5/8″ “	2 “	60,300	53,715	25,530	69.4	“ “ “
434	5/8″ “	2 “	65,400	64,600	30,430	74.9	“ “ “
435	5/8″ “	2 “	64,600	63,430	30,430	74.3	“ “ “
87	7/16″ steel	2 “	56,944	94,910	57,910	76.3	1/2″ “ “
88	7/16″ “	2 “	59,130	98,360	61,130	79.5	“ “ “
Double-welt butt-joints; 1/4-inch iron plates.							
615	5/8″ iron	1 5/8 ins.	53,475	67,321	16,944	62.2	11/16″ punched holes.
616	5/8″ “	1 5/8 “	50,959	64,138	16,719	59.3	“ “ “

TABLE I.—*Continued.*

No.	Size of Rivet and Kind.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	Remarks.
			Tension on Net Area of Plate (*T*)	Compression on Diametral Surface (*f*).	Shearing on Rivets (*S*).		
Single-riveted lap-joints; ⅜-inch iron plates.							
62	11/16″ iron	2 ins.	37,460	**60,340**	**38,280**	49.0	¾″ punched holes.
63	" "	2 "	**36,130**	**58,150**	35,520	47.2	" " "
64	" "	2 "	38,190	60,730	**37,530**	49.7	" drilled "
65	" "	2 "	36,210	57,530	**36,050**	47.1	" " "
66	" "	1¾ "	**41,750**	54,130	34,230	50.0	" punched holes.
67	" "	1¾ "	**41,290**	53,400	34,150	49.3	" " "
720	1″ "	2 1/16 "	**61,700**	52,970	26,180	60.4	1 1/16″ " "
721	1″ "	2 1/16 "	**58,510**	50,220	24,830	57.1	" " "
Single-riveted lap-joints; ⅜-inch steel plates.							
51	11/16″ iron	2 ins.	39,220	63,210	**39,740**	45.4	¾″ punched holes.
52	" "	2 "	37,700	60,760	**38,190**	43.6	" " "
53	" steel	2 "	55,215	**89,580**	**56,430**	64.1	" " "
54	" "	2 "	54,740	**88,660**	**55,460**	63.5	" " "
55	" "	1¾ "	**63,650**	80,930	**50,650**	66.7	" drilled "
56	" "	1¾ "	**63,976**	81,600	**50,900**	67.2	" " "
238	¾″ "	2 "	65,460	**89,490**	**53,560**	70.9	13/16″ punched "
239	¾″ "	2 "	65,210	**88,990**	**53,600**	70.6	" " "
718	1″ iron	2 5/16 "	**73,394**	**79,510**	**36,614**	71.4	1 1/16″ " "
719	1″ "	2 5/16 "	**73,970**	**80,200**	36,590	72.0	" " "
Double-riveted lap-joints; ⅜-inch iron plates.							
68	11/16″ iron	2 ins.	**48,450**	39,160	24,760	63.5	¾″ punched holes.
69	" "	2 "	**50,730**	41,070	26,150	66.4	" " "
58	" "	2 "	**50,220**	40,640	25,330	65.7	" " "
70	" "	2 "	**46,255**	41,480	27,550	60.5	" " "
71	" "	2 "	**46,110**	41,270	27,010	60.4	" " "
81	" "	3¼ "	30,920	58,700	**39,130**	50.4	" drilled "
82	" "	3¼ "	30,130	57,340	**38,410**	49.1	" " "
Double-riveted lap-joints; ⅜-inch steel plates.							
57	11/16″ iron	2 ins.	**62,800**	50,760	32,310	73.2	¾″ punched holes.
59	" "	2 "	**64,720**	52,450	32,930	75.2	" " "
60	" "	2 "	**63,210**	56,860	34,710	73.2	" " "
61	" "	2 "	**54,930**	49,530	30,830	63.8	" " "
83	" steel	3¼ "	44,660	84,460	**52,750**	64.4	" drilled "
84	" "	3¼ "	43,650	83,000	**51,845**	63.0	" " "
Reinforced riveted lap-joints; ⅜-inch iron plates. (See figure next page.)							
244	⅝″ iron	2 ins. joint 4 " welt	38,870	59,080	**40,360**	67.6	11/16″ drilled hole, ⅜″ welt.
245	¾″ "	" " " " " "	**43,770**	56,640	34,460	74.0	13/16″ " " " "
296	¾″ "	" " " " " "	**44,840**	57,910	33,890	75.7	" " " ¼″ "
297	¾″ "	" " " " " "	**42,680**	55,350	31,810	71.9	" " " " "

TABLE I.—*Continued.*



No.	Size of Rivet and Kind.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	**Remarks.**
			Tension on Net Area of Plate (*T*)	Compression on Diametral Surface (*f*).	Shearing on Rivets (*S*).		
			Reinforced riveted joints; ⅜-inch steel plates. (See above figure.)				
246	⅞″ steel	2 ins. joint 4 " welt	62,050	67,320	32,960	89.0	15/16″ drilled holes.
247	⅞″ "	" " " " " "	62,880	68,135	33,900	90.1	" " "
298	⅞″ iron	" " " " " "	61,020	67,300	34,250	87.8	" " "
299	⅞″ "	" " " " " "	61,710	68,040	34,750	88.9	" " "
			Single-riveted lap-joints; ½-inch iron plates.				
240	¾″ iron	2 ins.	31,100	41,500	34,280	39.8	13/16″ punched holes.
241	" "	2 "	31,395	41,955	34,960	39.7	" " "
292	" "	2 "	32,376	47,850	38,020	42.9	" drilled "
293	" "	2 "	33,180	48,890	39,220	44.3	" " "
327	" steel	2 "	39,900	58,880	47,020	52.2	" " "
328	" "	2 "	40,500	59,900	47,830	54.2	" " "
			Single-riveted lap-joints; ½-inch steel plates.				
242	¾″ iron	2 ins.	38,204	50,940	41,100	38.2	13/16″ punched holes.
243	¾″ "	2 "	35,915	47,890	38,636	35.9	" " "
294	15/16″ "	2 "	60,210	56,980	36,770	51.2	1″ " "
295	15/16″ "	2 "	49,590	47,060	30,540	42.2	" " "
			Double-riveted lap-joints; ½-inch iron plates.				
329	¾″ iron	2 ins.	44,320	59,640	25,280	57.0	13/16″ punched holes.
635	¾″ "	2 "	42,920	57,950	24,560	55.2	" " "
			Double-riveted lap-joints; ½-inch steel plates.				
619	15/16″ iron	2 ins.	64,602	29,354	19,670	53.8	1″ punched holes.
620	15/16″ "	2 "	64,519	29,371	19,644	53.8	" " "
			Single-riveted lap-joints; ⅝-inch iron plates.				
730	1″ iron	2⅝ ins.	34,680	47,510	35,460	44.9	1 1/16″ punched holes.
731	1″ "	2⅝ "	34,230	46,790	34,930	42.0	" " "
			Double-riveted lap-joints; ⅝-inch iron plates.				
732	1″ iron	2⅝ ins.	43,580	29,740	22,960	56.3	1 1/16″ punched holes.
733	1″ "	2⅝ "	45,850	31,310	23,670	59.3	" " "
			Single-riveted lap-joints; ⅝-inch steel plates.				
734	1″ steel	2⅜ ins.	49,650	56,760	43,490	50.5	1 1/16″ punched holes.
735	1″ "	2⅜ "	52,770	60,150	46,080	53.6	" " "
			Double-riveted lap-joints; ⅝-inch steel plates.				
736	1″ steel	2⅜ ins.	69,680	39,780	30,470	70.9	1 1/16″ punched holes.
737	1″ "	2⅜ "	67,100	38,300	29,340	68.3	" " "

TABLE II.

RIVETED JOINTS—IRON AND STEEL.

No.	Thickness of Plate and Kind.	Diameter and Kind of Rivet.	Pitch of Rivet.	Maximum Stresses, Pounds per Square Inch.			Efficiency of Joint, Per Cent.	Remarks.
				Tension on Net Area of Plate (T)	Compression on Diametral Surface (f).	Shearing on Rivets (S).		
				Single-riveted iron lap-joints.				
1	⅜″ iron	11/16″ iron	1¾ ins.	39,300	50,850	33,710	47.0	¾″ punched holes.
2	" "	" "	" "	41,000	53,050	35,170	49.0	1 1/16″ " "
3	½″ "	¾″ "	2 "	35,650	47,350	37,300	45.6	13/16″ " "
4	⅜″ "	" "	" "	35,150	46,690	36,780	44.9	" " "
				Single-riveted iron butt-joints.				
5	⅜″ iron	11/16″ iron	2 ins.	46,360	72,390	25,380	59.9	¾″ punched holes.
6	" "	" "	" "	46,875	73,050	25,450	60.5	" " "
7	½″ "	¾″ "	" "	46,400	61,940	24,630	59.4	13/16″ " "
8	" "	" "	" "	46,140	61,740	24,310	59.2	" " "
9	⅝″ "	1″ "	2⅝ "	44,260	60,330	23,010	57.2	1 1/16″ " "
10	" "	" "	" "	42,350	58,080	22,310	54.9	" " "
11	¾″ "	1⅛″ "	2.9 "	42,310	57,000	21,870	52.1	1 3/16″ " "
12	" "	" "	" "	41,920	56,540	22,140	51.7	" " "
				Single-riveted steel lap-joints.				
13	⅜″ steel	¾″ iron	1¾ ins.	61,270	65,760	40,390	59.5	13/16″ punched holes.
14	" "	" "	" "	60,830	65,320	39,900	59.1	" " "
15	½″ "	15/16″ iron	2 "	47,530	44,590	29,390	40.2	1″ " "
16	⅜″ "	" "	" "	49,840	46,960	31,070	42.3	" " "
				Single-riveted steel butt-joints.				
17	⅜″ steel	11/16″ iron	2 ins.	62,770	97,940	31,240	71.7	¾″ punched holes.
18	" "	15/16″ "	" "	61,210	95,210	31,020	69.8	1″ " "
19	½″ "	" steel	" "	68,920	62,220	20,370	57.1	" " "
20	2″ "	" "	" "	66,710	59,580	19,890	55.0	" " "
21	⅝″ "	1″ "	2⅜ "	62,180	71,450	27,750	63.4	1 1/16″ " "
22	" "	" "	" "	62,590	71,930	27,940	63.8	" " "
23	¾″ "	1⅛″ "	2½ "	54,650	55,610	23,190	54.0	1 3/16″ " "
24	" "	" "	" "	54,200	55,840	22,810	53.4	" " "

It is important to notice that in general the highest ultimate resistances of tension and compression or bearing are found with the thin plates, and that those quantities diminish appreciably as the thickness of plate increases, both for iron and steel. This law is not so well defined in reference to the diameter of rivet, if indeed these tests show it at all, except for steel.

The length of these test joints varied from 9.75 to 13 inches for Tables I and II, and from 10 to 27 inches for Table III.

Although the results of these tables are somewhat irregular, they confirm the general accuracy of the relations established between the values of T, f, and S in the preceding articles, as well as other general rules and conclusions for boiler work.

Some efficiencies are lower than those given for similar joints in Art. 94, but such instances can, by the aid of the tables, be traced either to indifferent design or a phenomenally low value of some one of the three resistances. In general the results compare well with those given in that article.

The pitches of rivets are seen to be adapted to boiler work, being much less than are ordinarily used in bridge work; yet the corresponding resistances show what may legitimately be done and expected when unusual conditions demand a departure from ordinary rules.

Before deducing working intensities for bridge construction from the preceding results it is to be first explained that those results are as given in the government reports, and that the net section used is the gross section of the plate, less the actual metal removed by the punch or drill, with no allowance for deterioration by the former in the immediate vicinity of the hole. Again, in Tables I and II the diametral bearing surface and the shearing area of the rivet are taken to be those of the drill, or a mean between the punch and die in case of punched holes. In bridge work, in determining the net section, metal is deducted for a diameter equal to that of the cold rivet before driving plus one eighth of an inch; and the shearing and bearing are computed for the section and diameter of the cold rivet before driving.

TABLE III.

TESTS OF STEEL-RIVETED JOINTS; ½-INCH PLATES.

Joint.	Rivet.	Maximum Stresses: Pounds per Square Inch.			Efficiency of Joint, Per Cent.	Remarks.
		Tension on Net Area of Plate (*T*).	Compression on Diametral Surface (*t*).	Shearing on Rivets (*S*).		
A	⅞″ steel	38,940	57,960	41,760	47.1	$\frac{15}{16}$″ drilled holes.
B	" "	39,450	81,530	35,560	57	" " "
C	" "	62,200	59,950	22,480	83.5	" " "
"	" "	56,410	77,900	29,640	80.3	" " "
"	" "	63,000	88,510	20,930	85.5	" " "
"	" "	59,330	78,900	29,410	85.3	" " "
" *	" "	55,050	71,890	29,850	79.4	" " "
H	" "	51,340	76,550	36,030	78	" " "
"	" "	52,150	50,170	20,790	78.6	" " "
"	" "	62,390	54,660	21,530	90.1	" " "
"	" "	58,550	51,350	20,620	84.7	" " "
"	" "	55,030	67,490	27,030	82.5	" " "

* Joint not fractured.

A. Double-riveted lap-joint; ½-inch plate.
B. Double-riveted butt-joint, two splice-plates; ½-in. plate.
C. Treble-riveted " " " " "
H. Quadruple-riveted butt-joint, two splice-plates; ½-in. plate.

The pitch of the outside rows of rivets in joints B, C, and H was double that of the adjoining rows. In the same joints one splice-plate was narrower than the other, so that it took one less row of rivets on either side of the joint than the other.

With these explanations in view, the preceding tests justify the following working stresses for the plate-girder floor-beams and stringers of railway bridges with machine-driven rivets.

Rivet shearing............ { 7,500 lbs. per sq. in. for iron.
10,000 " " " " " steel.

Rivet bearing............ { 14,000 lbs. per sq. in. for iron.
18,000 " " " " " steel.

Tension in net section of plate { 8,000 lbs. per sq. in. for iron.
10,000 " " " " " steel.

The bearing resistances are taken rather low, especially for steel, for the reason that thick plates are frequently used in bridge construction, and the ultimate bearing resistance for them is appreciably less than for the thin plates used in most of the preceding tests.

The preceding working stresses are based on steel for rivets giving from 56,000 to 64,000 pounds per square inch tensile resistance, while the steel for plates, in test specimens, should offer from 58,000 to 66,000 pounds per square inch ultimate tensile resistance.

In the government report from which Table I is abstracted, can be found a large number of tests made for the purpose of determining the proper minimum distance from the centres of rivet-holes to the edge of plates. As a result of those tests and other experience on the same subject, it may be stated that the least distance from the centre of a rivet-hole to the edge of a plate may be taken at one and one half the diameter of the *hole* for steel and one and five eighths the diameter of the *hole* for iron, in cases where it is important to secure the maximum resistance of the joint.

Efficiencies.

The values of the quantity which has been termed the "efficiency" of the joint, i.e., the ratio of the resistance of a given width of joint over that of an equal width of solid plate, in the preceding investigations, are those actually determined by experiments with the joints themselves. They may, therefore, be relied upon. Some values which have for many years been considered as standard, but which in reality are of a somewhat arbitrary nature, and at best belonging to a limited class of joints, have been disregarded.

The tests of full-size wrought-iron and steel-riveted joints exhibited in Art. 76 show, as a rule, that thin plates give materially higher efficiencies than thick plates. Although there are irregularities, single-riveted lap-joints may yield efficiencies running from 50 to 74 per cent. for ¼-inch plates, but dropping to 50 to 54 per cent. for ⅝-inch plates and materially lower for ½-inch plates. On the whole, the double-riveted lap-joints show somewhat higher efficiencies than the single-riveted, but not quite the same relative differences between ¼-inch and ⅝-inch plates, the values being found more generally between about 60 and 80 per cent.

The single-riveted butt-joints of Table II, Art. 76, give efficiencies ranging from about 52 to 72 per cent.

Some unusually high efficiencies are found in Table III of the same article for butt-joints, i.e., about 78 to 90 per cent. Those high values are due to the special design of the joints, and they cannot ordinarily be attained in practice, but they show that well-considered designs will yield greatly increased efficiencies.

In general, efficiencies running from 65 to 70 per cent. may be considered excellent for the usual conditions of practice.

Art. 77.—Tests of Joints for the American Railway Engineering and Maintenance of Way Association and for the Board of Consulting Engineers of the Quebec Bridge.

In "Proceedings of the American Railway Engineering and Maintenance of Way Association," Vol. 6, 1905, there are given the results of a series of tests of carbon-steel riveted joints and a duplication of that series of tests in both nickel and chrome-nickel steel made for the Board of Consulting Engineers of the Quebec Bridge by Profs. Arthur N. Talbot and Herbert F. Moore of the University of Illinois, also fully described in Bulletin No. 49 (1911) of that institution. There were 144 joints tested in the latter two series. Furthermore, there were tested in alternate tension and compression 16 other nickel-steel joints and the same number of chrome-nickel steel joints.

All the main plates of these joints were 6.5 inches or 7.5 inches wide with thicknesses from $\frac{3}{8}$ inch to $\frac{3}{4}$ inch except the 32 joints subjected to compression, for which the plates were 2 inches thick. There were 24 lap joints, the same number of butt-joints with double covers or butt-straps and an equal number each of the same type of joint with one filler and two fillers on each side of both main plates. The remaining joints for tension loads only ($7\frac{1}{2} \times \frac{3}{4}$-inch main plates), with the exception of two sets of eight each, were also made with one or two fillers, but the latter extended beyond the end of the cover far enough to take one rivet.

All rivets were $\frac{7}{8}$-inch in diameter, and those driven by a hydro-pneumatic riveter were called "shop" rivets while those driven by a hand-pneumatic riveter were designated

TABLE I.

CHEMICAL COMPOSITION OF RIVET AND PLATE MATERIAL

Element.	Nickel-steel Riveted Joints.		Chrome-nickel-steel Riveted Joints.	
	Rivet Material Per Cent.	Plate Material Per Cent.	Rivet Material Per Cent.	Plate Material Per Cent.
Carbon	0.141	0.258	0.136	0.191
Sulphur	0.0023	0.008	0.038	0.035
Phosphorus	0.037	0.044	0.032	0.042
Manganese	0.442	0.700	0.696	0.485
Nickel	3.33	3.330	0.986	0.733
Chromium	....	....	0.240	0.170

TABLE II.

PHYSICAL PROPERTIES OF RIVET AND PLATE MATERIAL

All stresses in pounds per square inch.

Item.	Nickel-Steel.		Chrome-Nickel-Steel.	
	Rivet Material.	Plate Material.	Rivet Material.	Plate Material.
Number of specimens tested	2	9	2	8
Elastic limit	....	40,200	....	27,200
Stress at yield point	45,000	51,700	38,400	36,300
Stress at ultimate	68,500	89,700	59,000	63,900
Elongation in 2 inches, per cent	33.5	25.0	35.2	31.7
Reduction of area, per cent	63.4	55.8	63.3	59.9
Modulus of elasticity	....	29,950,000	....	30,750,000

as "field" rivets. The difference in resistance of the shop and field rivets was not material.

Tables I and II show the chemical composition and the physical properties of the nickel and chrome-nickel steels used.

The following statement shows in a condensed form the results of the tests.

TABLE III.

Nickel-Steel Joints

	Lbs. per Sq. In.
Av. Ult. shear shop and field rivets	52,440 to 60,140
Max. tension in plates	16,850 to 50,800

Chrome-Nickel-Steel Joints

	Lbs. per Sq. In.
Av. Ult. shear in rivets	48,190 to 56,650
Max tension in plates	16,170 to 49,500

Carbon Steel (Main. of Way Assoc. Tests)

	Lbs. per Sq. In.
Av. shear stress	44,940 to 52,060
Max. tension in plates	15,190 to 48,400

The shearing of the rivets caused the failures of all the nickel-steel and chrome-nickel steel joints.

The "carbon steel" used in the American Railway Engineering and Maintenance of Way Association tests was low basic open hearth material conforming to the specifications of that Association. Some of these joints failed by the yielding of the plates but the greater part of them failed by the shearing of the rivets and the results are all given in terms of the maximum shearing stress in the rivets at the instant of failure.

The lower values in the ultimate and final shear stresses in these series of tests belong to the longer rivets, i.e. to the joints in which fillers were used. This was to be expected in consequence of the increased bending in those rivets. Indeed, these tests indicate that with ordinary thicknesses of plates the carrying capacity of the rivets begins to be seriously affected when the "grip" of the rivets, i.e. the aggregate thickness of plates pierced by them, exceeds about four diameters. It should be stated, however, that this depends much upon the design of the joint.

Friction of Riveted Joints.

Careful observations were made by Profs. Talbot and Moore as well as in the tests of joints for the American Railway Engineering and Maintenance of Way Association to determine the friction of riveted joints which experienced engineers have long known to exist. These observations indicate that a material slipping of the plates took place in some of these joints when the shearing stress in the rivets was not greater than about 6,000 pounds per square inch. In other cases this slipping took place when the rivet shear was as high as 15,000 pounds per square inch. It was observed, as might be anticipated, that the quality of the material of the joints had little effect upon the degree of stress at which slipping began. The results were about the same for the low carbon steel joints as for the chrome-nickel steel joints. As might be expected in a well-proportioned joint, the friction between the plates depends upon the force with which they are held in contact by the rivets. The motion of the plates is obviously due to the fact that the shaft of the rivet in cooling contracts more than the comparatively cool plate around it leaving a small annular space between the rivet and the wall of the hole. As the load on the joint increases a degree of direct stress of tension (or of compression in joints under compression) is reached at which the plates slip on each other bringing the rivet shafts successively, or more or less simultaneously, in contact with the bearing side of the hole.

After the load increases still more, a higher stage of stress is reached at which the yield point of the joint is found when relatively rapid distortion takes place. As an average the yield point of the nickel steel joints was found at an intensity of shearing stress in the rivets of about 35,000 pounds per square inch and not much different from that for the chrome-nickel steel joints. Material bending of the rivets appears to be an influential element in the increased deformation at the yield point of a joint and it is reasonable to suppose that, other things being the same, the longer the rivets the lower will be the degree of stress at which the yield point is found, although it is doubtful whether the rivets are long enough in the well-designed riveted joints of good engineering practice to show much effect upon the yield point. Profs. Talbot and Moore state that "The ratio of the yield point of riveted joints to ultimate shearing strength in these tests was about the same as the ratio of the yield point of the plate material in tension to the ultimate tensile strength of the plate material."

The results obtained from the joints tested in alternate tension and compression were not markedly different from those obtained in tension. The yield point seems to be slightly lowered after a few alternations of tensile and compressive loads. If these alternations took place rapidly, doubtless the joints would show much diminution of resisting capacity but the actual alternations were few in number and not rapidly made.

These tests show that the friction between plates of a riveted joint cannot properly be considered as enhancing the resisting capacity. Furthermore, this slipping has a direct bearing upon the computations of secondary stresses in trusses with riveted connections. The corresponding deformation may militate materially against the accuracy or reliability of such computations.

Art. 78.—Riveted-truss Joints.

The circumstances in which riveted joints are used in truss work render permissible many special forms which can find no place in boiler-riveting. If joints are found under the same circumstances, as far as the transference of stress is concerned, precisely the same forms would be used, except that calking is, of course, only required in boiler work.

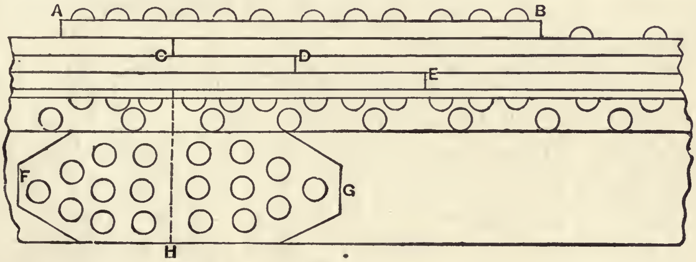
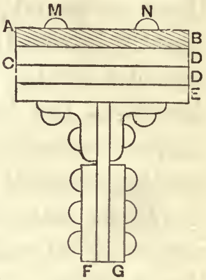


FIG. 1.

Fig. 1 shows a common form of chord construction in riveted-truss work, with the relative proportions exaggerated.



The lower portion of the figure shows a section of the chord in which the cover- or splice-plate is shaded. The joint is supposed to be in tension.

In this form of joint the splice-plate material is reduced to a minimum. These are, in reality, two lap-joints *CD* and *DE* with the two plates *C* and *E* to be spliced. In each lap-joint there should be sufficient rivets determined by the methods of Art. 74. The splice-plate *AB* should be long enough to give the requisite plate *AC* to the left of *C*, with the same length from *B* to a point vertically over *E*.

In most cases one or two plates only should be spliced at the same point.

The joint in the vertical plate should be formed as at *FG*; i.e., it should be a double-cover butt-joint. The principles already established in a preceding section, in regard to the thickness of covers and diameter of rivets, should be observed here.

The two or more full rows of rivets on either side of the joint may as well be chain-riveted with a pitch of $3\frac{1}{2}$ to 4 diameters. Other rivets should then be staggered in until the group of rivet centres on each side is brought to a point, as shown in the upper part of Fig. 1. In this manner the available section of a width of plate equal to that of the cover becomes approximately equal to the total, less the material from one rivet-hole. Hence the efficiency of the joint becomes correspondingly increased.

If the joint is in compression the preceding observations hold without change, except that all covers should have the same thickness as the plates covered.

Even if the joints *C*, *D*, *E*, and *H* are of planed edges, little or no reliance should be placed upon their bearing on each other, since the operation of riveting will draw them apart more or less, however well the work may be done.

Unless great caution is observed and excellence of design secured, there will frequently be excessive bending in the riveted joints of truss work, on account of the great variety of connections required.

Diagonal Joints.

Diagonal riveted joints have from time to time been proposed, the line of the joint making an angle of perhaps 45° to the line of action of the loading. Such joints when properly designed have high efficiencies for the reason that a normal section of the joint taken anywhere within its extreme limits will lie wholly within the main plates except at the point where the oblique joint line cuts it. In designing such a joint, however, the rows of rivets should be placed parallel to the joint line and extend across the entire main plates, or some other arrangement may be employed which will make the centre of gravity of the group of rivets on the two sides of the joint lie in the centre line of the main plates or other connected members. If this condition is not attained, there will be eccentricity of the aggregate resistance of the rivets on either side of the joint line resulting in serious bending about an axis perpendicular to the main plates. The added cost of this type of joint and the inconvenience of its use in many cases prohibits its general employment as a detail in riveted structural work.

Riveted Joints in Angles.

It has been found by tests of full-size angles that if a riveted joint be formed by riveting one leg only, the ultimate tensile resistance per square inch of the net angle section may be but 75 per cent. of the ultimate tensile resistance of test specimens cut from the same angle. On the other hand, if both legs are riveted the ultimate tensile resistance per square inch of the net section may easily be 90 per cent. of the ultimate resistance of test specimens cut from the same angle. These results show that both legs of angles should always be riveted at joints.

Hand and Machine Riveting.

The development of the pneumatic and other power riveters for both shop and field purposes has practically eliminated hand riveting from all structural work except in rare cases. When hand riveting was done its inferiority to power riveting was recognized by specifying that at least one-third more rivets should be used when they were driven by hand.

Art. 79.—Welded Joints.

Welded joints, as a rule, have never been permitted in first class structural work. Fairly good joints of that type, however, were made where necessary in wrought iron, but it is difficult, if not essentially impossible, to make a satisfactory weld in structural steel by ordinary procedure. In cases where welding of steel is done, some method is necessary in which the metal at the weld is brought into a state of fusion for a material depth. The thermit and other processes accomplish satisfactory welded joints in both steel castings and rolled bars for many purposes although they are not used in structural work.

Art. 80.—Pin Connections.

A pin connection consists of two sets of eye-bars or links, through the heads at one end of each of which a single pin passes. Fig. 1 shows a pin connection; *A*, *A*, *B*, *B*, are eye-bars or links, and *P* is the pin.

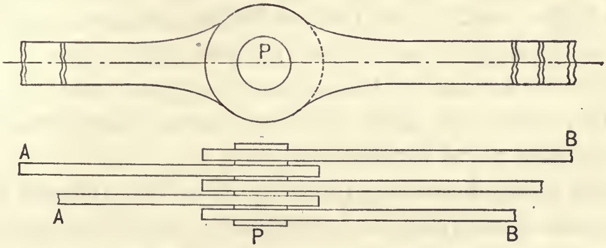


Fig. 1.

The head of the eye-bar (one is shown in elevation in Fig. 2) requires the greatest care in its formation. It is imperfect unless it be so proportioned that when the eye-bar is tested to failure, fracture will be as likely to take place in the body of the bar as in the head; in other words, unless its efficiency is unity.

In Fig. 2 the head of the eye-bar, or link, is supposed

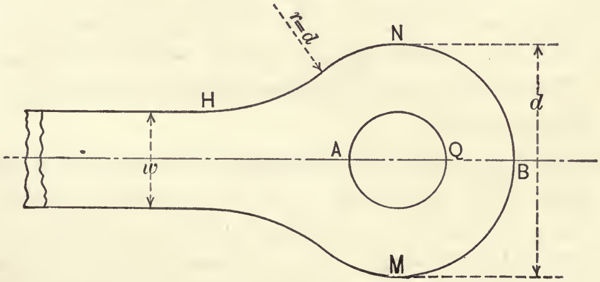


Fig. 2.

to be of the same thickness as that of the body of the bar whose width is *w*.

If *t* is the thickness of the bar so that *wt* is the area of its normal section, then *t* is generally included between the limits of $\frac{1}{3}w$ and $\frac{1}{8}w$ for ordinary sizes of eye-bars. These limits, however, are exceeded both for the smaller sizes used and the larger sizes. A bar for which $w=3$ inches, may have a thickness of $1\frac{1}{2}$ inches, while the maximum thickness of a bar 16 inches wide may be no more than 2 inches. Similarly the minimum thickness of a 3-inch wide bar may be $\frac{5}{8}$ inch while the least thickness of a 16-inch wide bar may be taken at $1\frac{3}{4}$ inches or $t=\frac{1}{9}w$.

In the early days of eye-bar manufacture earnest efforts were made to analyze the complicated condition of stresses in the eye-bar head so as to give it a rational outline, and an approximate treatment of the problem may be found in the "Trans. Am. Soc. of Civ. Engrs." Vol. VI, 1877, the results of which agree essentially with those of experiment.

After much experimenting, including the thickening of the head, it has for many years been the practice to make the heads of eye-bars circular in outline as shown in Figs. 1 and 2. In Fig. 2 the front part of the head *NBM* is a semicircle and it is extended on both sides to the left of *NM* so as to be tangent to the circular curves of the neck drawn with the radius equal to the width *d* of the entire head. The latter curves are also tangent to the body of the bar as shown at H.

The head is formed by heating the end of the bar to a white heat, then upsetting it in a properly-formed die as closely as possible to the finished shape. A little finishing work is then usually done under a power hammer or between rolls. The head is seldom thicker than the body of the bar.

The normal section of the head taken through the centre of the pinhole is usually from 35 per cent. to 40 per cent. in excess of the section of the bar. All steel eye-bars are thoroughly annealed after the completion of manufacture so as to remove all internal stresses in the head and any undue hardness that may have been acquired during that process.

The diameter of the pin should never be less than about 80 per cent. of the width *w* of the bar, and it may be from $1\frac{1}{8}$ to $2\frac{1}{2}$ times that width, the greater of those factors belonging to bars of small width and the smaller to bars of the greatest width used.

In pin connections the pin is subjected to heavy bending for which it is carefully designed as well as for the shear in its normal section and for the bearing or compression between it and the pin hole. The pin and the pin hole are accurately machine finished, the diameter of the latter being from perhaps $\frac{1}{100}$ inch (for the smallest pins) to $\frac{1}{32}$ inch (for the largest pins) greater than the former.

If M is the bending moment to which the pin is subjected, k the greatest intensity of bending stress developed, and A the area of the normal section of the pin, eq. (4) of Art. 90 gives

$$M = k\frac{Ad}{8} = 0.1kd^3 \text{ (nearly)}, \quad . \quad . \quad . \quad . \quad (1)$$

or

$$d = 2.16\sqrt[3]{\frac{M}{k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Values of k, for circular sections, may be found in Art. 90.

CHAPTER X.

LONG COLUMNS.

Art. 81.—Preliminary Matter.

There is a class of members in structures subjected to compressive stress which do not fail entirely by compression. The axes of these pieces coincide, as nearly as possible, with the line of action of the resultant of the external forces, yet their lengths are so great compared with their lateral dimensions that they deflect laterally, and failure finally takes place by combined compression and bending. Such pieces are called "long columns," and the application to them, of the common theory of flexure, has been made in Art. 35.

Two different formulæ were first established for use in estimating the resistance of long columns; they are known as "Gordon's Formula" and "Hodgkinson's Formula." Neither Gordon nor Hodgkinson, however, gave the original demonstration of either formula.

The form known as Gordon's formula was originally demonstrated and established by Thomas Tredgold ("Strength of Cast Iron and other Metals," etc.), for rectangular and round columns, while that known as Hodgkinson's formula (demonstrated in Art. 35) was first given by Euler.

In 1840, however, Eaton Hodgkinson, F.R.S., published the results of some most valuable experiments made by himself on cast and wrought-iron columns (Experimental Researches on the Strength of Pillars of Cast Iron, and other Materials; Phil. Trans. of the Royal Society, Part II, 1840), and from these experiments he determined empirical coefficients applicable to Euler's formula, on which account it has since been called Hodgkinson's formula.

Prof. Lewis Gordon deduced from the same experiments some empirical coefficients for Tredgold's formula, since which time Gordon's formula has been known.

The latter has been quite generally used, but it has lately been largely displaced by the straight-line formula to be given later. Hodgkinson's coefficients and formula have now been abandoned.

Before taking up the subject of long columns it is desirable to establish some important properties of the moments of inertia of surfaces used in the analytic treatment of long columns and in some problems of flexure.

It will also be both convenient and important to determine the conditions which exist with an isotropic character of section in respect to the moment of inertia.

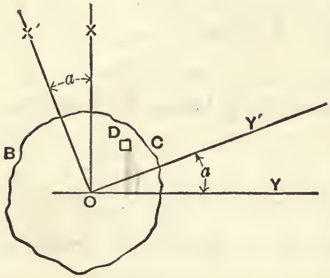


Fig. 1.

In Fig. 1 let *BC* be any figure whose area is *A*, and whose centre of gravity is at *O*. In the plane of that figure let any arbitrary system of rectangular coordinates *X'*, *Y'* be chosen and let *XY* be any other system having the same origin; also, let x', y' and x, y be the coordinates of the element *D* of the surface *A* in the two systems. There will then result

$$x = x' \cos \alpha + y' \sin \alpha,$$
$$y = y' \cos \alpha - x' \sin \alpha.$$

The moments of inertia of the surface about the axes y and x will then be

$$\int x^2dA = \cos^2 \alpha \int x'^2dA + 2 \sin \alpha \cos \alpha \int x'y'dA + \sin^2\alpha \int y'^2dA, \quad . \quad . \quad . \quad . \quad (1)$$

$$\int y^2dA = \cos^2 \alpha \int y'^2dA - 2 \sin \alpha \cos \alpha \int x'y'dA + \sin^2 \alpha \int x'^2dA . \quad . \quad . \quad . \quad . \quad (2)$$

If x and y are to be so chosen that they are *principal* axes, then must $\int xydA = 0$, or

$$0 = \int xydA = \sin \alpha \cos \alpha \int y'^2dA + (\cos^2 \alpha - \sin^2 \alpha) \int x'y'dA - \sin \alpha \cos \alpha \int x'^2dA; \quad (3)$$

$$\therefore \tan 2 \alpha = \frac{2\int x'y'dA}{\int x'^2dA - \int y'^2dA}.$$

Hence, since $\tan 2\alpha = \tan (180 + 2\alpha)$, there will *always* be two principal axes 90° apart.

Now, if $\int x'y'dA = 0$, while no other condition is imposed. $\tan 2\alpha = 0$. This makes $\alpha = 0$ or $90°$; i.e., $X'Y'$ are the principal axes.

If, however, $\int x'y'dA = 0$, while α is neither 0 nor 90°, eq. (3) becomes

$$\int y'^2dA - \int x'^2dA = 0;$$

or

$$\tan 2\alpha = \frac{0}{0}, \text{ i.e., indeterminate.}$$

This shows that any axis is a *principal* axis; also that

$$\int x^2 dA = \int y^2 dA = \int x'^2 dA = \int y'^2 dA.$$

Hence the surface is completely isotropic in reference to its moment of inertia, or *its moment of inertia is the same about every axis lying in it and passing through its centre of gravity.*

It has been seen that this condition exists where there are two different rectangular systems, for which

$$\int xy dA = \int x'y' dA = 0;$$

but the first of these holds true if either x or y is an axis of symmetry, and the latter if either x' or y' is an axis of symmetry.

Hence, *if the surface has two axes of symmetry not at right angles to each other, its moment of inertia is the same about all axes passing through its centre of gravity and lying in it.*

Eqs. (3) and the two preceding it also show that the same condition obtains *if the moments of inertia about four axes at right angles to each other, in pairs, are equal.*

In the case of such a surface, therefore, it will only be necessary to compute the moment of inertia about such an axis as will make the simplest operation.

Principal Moments of Inertia.

If the moments of inertia I' about the axis of Y' and I'' about the axis of X' be expressed in terms of the principal moments I_1 about the axis of Y and I_2 about the axis of X, eqs. (1) and (2) will give by simply changing the primes from the second to the first members of the equations;

$$\int x'^2 dA = I' = I_1 \cos {}^2\alpha + I_2 \sin^2 \alpha. \quad . \quad . \quad . \qquad (4)$$

$$\int y'^2 dA = I'' = I_2 \cos^2 \alpha + I_1 \sin^2 \alpha. \quad . \quad . \quad . \qquad (5)$$

If the principal moments of inertia I_1 and I_2 are known eqs. (4) and (5) show that the moments I' and I'' about any axes making the angle α with the principal axes may at once be computed.

Adding eqs. (4) and (5);

$$I' + I'' = I_1 + I_2 = I \text{ (Polar moment)}.. \quad . \quad . \qquad (6)$$

Hence the sum of the two moments of inertia about any two axes at right angles to each other is constant and equal to the polar moment of inertia.

If the second members of eqs. (4) and (5) be divided by the area A of the cross section, and if the radii of gyration be represented by r', r'', r_1 and r_2;

$$r'^2 = r_1^2 \cos^2 \alpha + \mathrm{r}_2^2 \sin^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

$$r''^2 = r_2^2 \cos^2 \alpha + r_1^2 \sin^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Each of eqs. (7) and (8) is the equation of an ellipse in which r_1 is the semi-axis in the direction of the coordinate axis X and r_2 is the semi-axis of the ellipse in the direction of the coordinate axis Y, while r' and r'' are two semi-diameters OD' and OD, all as shown in Fig. 2.

If eqs. (7) and (8) be added, eq. (9) will result;

$$r'^2 + r''^2 = r_1^2 + r_2^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

This equation is the expression of one characteristic of the ellipse, viz., the sum of the squares of any two conjugate semi-diameters is equal to the sum of the squares of the two semi-axes. The two radii of gyration therefore about any two inertia axes at right angles to each other, except

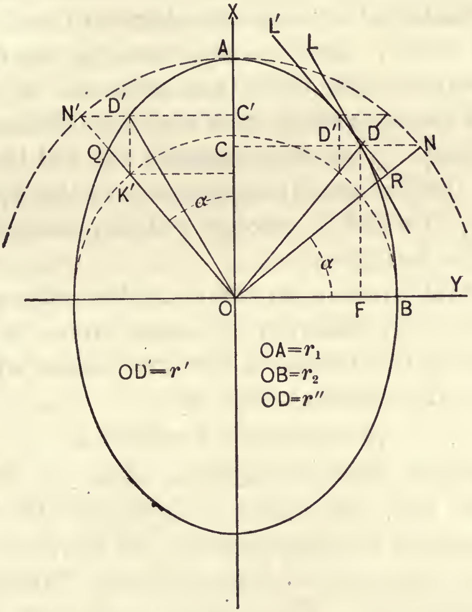


FIG. 2.

the principal axes, are semi-conjugate diameters of the ellipse.

Eqs. (7) and (8) are precisely the same in character as eq. (3) of Art. 9 and the ellipse of Fig. 2 is constructed precisely as was the ellipse of stress. The two principal radii of gyration r_1 and r_2 are represented by the semi-axes *OA* and *OB*, while the semi-conjugate diameters *OD′* and *OD* represent the radii of gyration r' and r'' taken about any two axes at right angles to each other, represented by *ON* and *ON′*. The construction lines of Fig. 2 show how the ellipse is constructed from eqs. (7) and (8), precisely as was the ellipse of stress in Art. 9.

If it is desired to find the radius of gyration about any axis, as the semi-diameter OQ, the construction of the ellipse shows that it is only necessary to describe the two circles with radii r_1 and r_2, as shown in the figure, then erect ON perpendicular to OQ and draw the horizontal and vertical lines respectively from N and K to their intersection D on the ellipse. The semi-diameter OD will be the radius of gyration desired and its direction on the figure of the cross-section to which it belongs will obviously be ON, i.e., at right angles to OQ.

It is a well-known property of the ellipse that the square of the perpendicular p drawn from the center to the tangent to the curve, if the inclination of that perpendicular to the semi-axis is α, is;

$$p^2 = r_1^2 \cos^2 \alpha + r_2^2 \sin^2 \alpha. \quad . \quad . \quad . \quad (10)$$

This value of p^2 is precisely the same as r'^2 in eq. (7) and it shows that the radius of gyration $OR = OD$ about any semi-diameter OQ considered as an inertia axis is equal to the normal distance between that semi-diameter and the parallel tangent RL'. This simple result finds an important application in the problem of the flexure of a beam of unsymmetrical cross-section.

This same normal distance between a semi-diameter of the ellipse and the parallel tangent RL' is also equal to $\frac{r_1 r_2}{r^1}$, the semi-major axis of the ellipse being represented by r_1 and the semi-minor axis by r_2, while r' represents the semi-diameter.

The preceding equations indicate the principal properties of every form of cross-section which may affect the value of the moment of inertia about any axis whatever passing through its centre of gravity.

Art. 82.—Gordon's Formula for Long Columns.

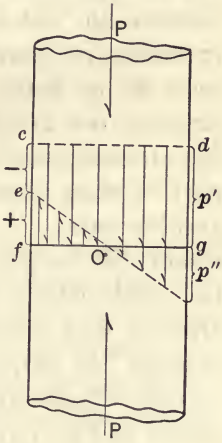


FIG. 1.

Since flexure takes place in a long column subjected to a thrust in the direction of its length, the greatest intensity of stress in a normal section of the column may be considered as composed of two parts, one a uniform compression over the whole section the total of which is equal to the load on the column, and the other the usual uniformly varying stress due to flexure the total of which is zero and the intensity of which is also zero along the neutral axis of the section. Fig. 1, which is supposed to represent a longitudinal axial section of a column, shows completely this composite stress. The line *fg* is the trace of the normal section and $gd = cf = p'$ is the uniform intensity of compression due to the compressive load *P*. The bending moment is represented by the stresses of flexure varying uniformly in intensity from p'' on the right-hand side of the section to *ef* on the left side, *O* being at the neutral axis. The compressive stresses are indicated by − and the tensile stresses by +. The resultant of these two composite stresses is a uniformly varying stress with the greatest compressive intensity $p' + p''$ on one side of the section and the small compressive intensity *ec* on the left side. The bending tension neutralizes exactly the same amount of uniform compression, making the resultant intensity uniformly varying. There is no resultant tensile stress in the section, but it is obvious that there would be if the bending moment were sufficiently large. In that case *fe* would be larger than *fc*. This condition, however, seldom occurs in actual structural columns and never unless they are slender and too heavily loaded.

The condition of stress as described above is that ordinarily assumed for columns, but the actual condition of stress is frequently, if not almost invariably, much more complicated. The details and the different main parts of columns do not act with perfect concurrence nor are the processes of manufacture even in the most careful shops such as to leave the finished members without internal stresses, nor are they perfectly straight. In fact the best of columns may be a little convex in one direction at one part of their length and concave in the same direction at another part. It is imperative, however, to have some reasonably simple rational analysis on which formulæ may be based leaving the erratic stress conditions which are too obscure and uncertain to be reached by analysis to be covered by empirical coefficients determined by tests of actual full-size columns and the stress assumptions illustrated in Fig. 1 fulfill this requisite at least reasonably.

In order to determine the two parts of the resultant stresses shown in Fig. 1, let S represent the area of the normal section; I, its moment of inertia about a neutral axis normal to the plane in which flexure takes place; r, its radius of gyration in reference to the same axis; P, the magnitude of the imposed thrust; f, the greatest intensity of stress allowable in the column, and Δ, the deflection corresponding to f. Let p' be that part of f caused by the direct effect of P, and p'' that part due to flexure alone. Then, if h is the greatest normal distance of any element of the column from the axis about which the moment of inertia is taken, by the "common theory of flexure,"

$$c'P\Delta = \frac{p''I}{h}; \quad \therefore \quad p'' = \frac{c'P\Delta h}{I}. \quad . \quad . \quad . \quad . \quad (1)$$

If the column ends are round, $c' = 1$; but if the ends are fixed, the value of c' will depend upon the degree of fixedness.

Also

$$p' = \frac{P}{S}; \quad \therefore \ p' + p'' = f = \frac{P}{S}\left(1 + \frac{c'S\Delta h}{I}\right). \quad . \quad (2)$$

Hence

$$P = \frac{fS}{1 + \frac{c'S\Delta h}{I}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eq. (3) may be considered one form of Gordon's formula.

In order to make eq. (3) workable in actual computations, it is necessary to express the deflection Δ in terms of known dimensions of the column. By referring back to eq. ($6a$) in Art. 27 the desired expression for the deflection may be found and by its aid, introducing the notation of this article, eq. (4) may be at once written;

$$\Delta = \frac{ak_0 l^2}{Ed} = \frac{a'p''}{Eh} l^2 = a_1 \frac{l^2}{h}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It is seen, therefore, that the quantity a_1 depends upon both p'' and E, but it is ordinarily considered constant.

Since $I = Sr^2$, eqs. (1) and (7) give

$$p'' = a_1 \frac{c'Pl^2}{I} = a \frac{P}{S} \frac{l^2}{r^2}; \quad \therefore \ f = p' + p'' = \frac{P}{S}\left(1 + a\frac{l^2}{r^2}\right). \quad (5)$$

Eq. (8) shows that $a_1 c' = a$.

Hence

$$P = \frac{fS}{1 + a\frac{l^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The integration by which eq. (4) is obtained, being taken between limits, causes everything to disappear which depends upon the condition of the ends of the column. Consequently eq. (6) applies to all columns, whether the ends are rounded or fixed. Let the latter condition be assumed, and let it be represented in the adjoining figure. Since the column must be bent symmetrically, there must be *at least* two points of contraflexure. Two such points only may be supposed, since such a supposition makes the distance between any two adjacent points the greatest possible and induces the most unfavorable condition of bending for the column.

FIG. 2.

If B and C are the points of contraflexure supposed, then BC will be equal to a half of AD, for each half of BC must be in the same condition, so far as flexure is concerned, as either AB or CD. Also the bending moment at the section midway between B and C must be equal to that at A or D. Consequently the hinge- or round-end column BC must possess the same resistance as the fixed- or flat-end column AD. In eq. (6), therefore, let $l = 2BC = 2l_1$,

$$P = \frac{fS}{1 + 4a\frac{l_1^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eq. (7) is, consequently, the formula for free- or round-end columns with length l_1.

The flat- or fixed-end column AD is also of the same resistance as the column AC, with one end flat and one end round. Hence in eq. (6) let there be put $l = \frac{4}{3}AC = \frac{4}{3}l'$, and there will result, nearly,

$$P = \frac{fS}{1 + 1.8a\frac{l'^2}{r^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eq. (8) is, then, the formula for a column with one end flat and the other round. A slight element of approximation will ordinarily enter eq. (8) on account of the fact that C is not found in the tangent at A just as eqs. (6) and (7) are based on the supposition that A and D lie exactly in the line of action of the imposed load.

Eqs. (7) and (8) have been and are now generally accepted as representing the resistances of columns with the end conditions to which they are intended to apply. As a matter of fact, however, tests of full-size members have demonstrated that those conditions are not realized in the actual use of columns. They have further shown that essentially but one condition of column ends need be recognized, and that is the actual pin-end condition, as realized in pin-connected structures. In that condition the end of the column is not free to turn. The compression between the pin and the metal bearing against it caused by the load carried by the column creates a considerable surface of contact over a substantial portion of which the intensity of pressure is high. This produces a condition of great frictional resistance to any motion between the pin and the end of the column, but not sufficient probably to induce a fixed-end condition. It has been found by test that flat-end columns, as a rule, give less ultimate resisting capacity than pin-end columns of the same length and same radius of gyration of cross-section. This is doubtless due to the practical impossibility to secure a central application of loading when flat ends are employed, the resulting eccentricity reducing the ultimate carrying capacity of the members. While, therefore, the classes of columns represented by eqs. (7) and (8) are still recognized, it would be more rational and more in accordance with experience to use only the general form of eq. (6) with a determined from actual pin-end tests.

Although the quantities f and a, in eqs. (6), (7), and (8) are usually considered constant, they are strictly variable. Eq. (4) shows that a is a function of $p'' \div E$. It is by no means certain that p'' is the same for different forms of cross-section, or even for different sections of the same form. While the modulus of elasticity E varies slightly it may properly be taken as constant.

Again, the greatest intensity of stress, f, which can exist in the column varies not only with different grades of material, but there is some reason to believe that it must also be considered as varying with the length of the column. The law governing this last kind of variation, for many sections, still needs empirical determination. It is clear, therefore, that both f and a must be considered empirical variables.

Since f and a are to be considered variable quantities, let y take the place of f and x that of a; also, let $p = \frac{P}{S}$ represent the mean intensity of stress. Eq. (6) then takes the form

$$p = \frac{y}{1 + cx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

in which $c = l^2 \div r^2$.

In eq. (9) there are two unknown quantities, y and x, consequently two equations are required for their determination. If two columns of different ultimate resistances per unit of section, and with different values of c, are broken in a testing machine, and the two sets of data thus established separately inserted in eq. (9), two equations will result which will be sufficient to give y and x. Those two equations may be written as follows:

$$y = p'(1 + c'x), \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$y = p''(1 + c''x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The simple elimination of y gives

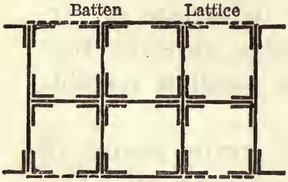
$$x = \frac{p'' - p'}{c'p' - c''p''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Either eq. (10) or (11) will then give y.

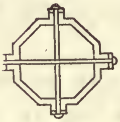
In selecting experimental results for insertion in eq. (12), care should be taken to make the differences $p'' - p'$ and $c' - c''$ as large numerically as possible, in order that the errors of experiment may form the smallest possible proportion of the first.

In consequence of the more or less erratic results of tests of full-sized columns, if two or more pairs of values of f and a be found as indicated above, they will not agree with each other and some of them may differ largely. Consequently the procedures illustrated by eqs. (10), (11) and (12) are not sufficient for a satisfactory determination of the quantities desired. The method of probabilities has been employed, but it also is unsatisfactory because of the small number of tests available if for no other reason. The usual process is to plot the results of tests using $\frac{l}{r}$ for a horizontal coordinate and the mean load per square inch of cross-section of column for the vertical ordinate. The results of a series of tests will in this manner be represented graphically by a more or less extended group of points depending upon the range of $\frac{l}{r}$. A curved or straight line, as the case may be, is then drawn through such a plotted group of points so as to give it a mean position among them. The quantities *f* and *a* are then so determined by trial as to produce a curve lying as close as possible to the experimental curve and the resulting equation will then be Gordon's formula for that particular set of tests or type of columns. This operation is fully illustrated and will be further considered in the next article in connection with a series of tests of Phoenix columns and columns of other shapes.

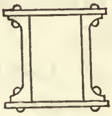
The accompanying diagrams represent some cross-sections of columns which have been much used.



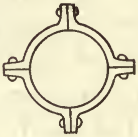
Quebec Bridge



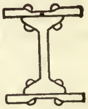
OPEN



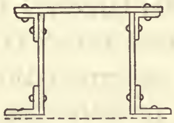
SQUARE



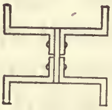
PHOENIX



AM.BR.CO.



TOP CHORD LATTICED.



Z-BAR.

SQUARE-LATTICED.

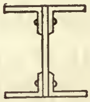


PLATE AND ANGLES

There are a large number of other sections which have also been employed either for wrought iron or steel columns. For large columns it is occasionally necessary to build up cross-sections consisting of a number of webs and angles, all so secured to each other as to act as a unit. The Quebec Bridge section is such a one.

Occasionally a so-called "swelled" column, i.e. with a considerably enlarged cross-section at and in the vicinity of the centre of the column length, the outline of section gradually but not uniformly decreasing from the centre towards the ends, is required. A formula for such a column similar to Gordon's formula may be written for a varying moment of inertia, but it is too complicated to be of practical use. In the case of such columns the judgment of the engineer must be used in applying a column formula, but it will generally be sufficient to take the radius of gyration at the middle section of such a member in computing the ratio $\frac{l}{r}$.

The preceding formulæ and the considerations on which they are based imply without qualification that all parts of a column must be so rigidly bound together that each such member will act as a perfect unit under loading and they include the condition that the cross-section of the column is maintained in its proper shape and proportions without material distortion up to actual failure of the tested columns. It is imperative, therefore, in the design of these members that the details, including rivets, lattice bars, batten plates and other spacing details, shall be sufficient in number and dimensions to maintain the column as a unit up to its full carrying capacity. A failure to meet these conditions may greatly and perhaps fatally reduce the carrying capacity of the column and result in disaster, as in the case of the first Quebec Bridge, caused by the weak latticing of a compression member. If a column more or less weak in its spacing or other details is tested to its ultimate resistance, it will yield in some of its weak details instead of failing as a whole, i.e., as a unit.

The general principles which govern the *resistance* of built columns may, then, be summed up as follows.

The material should be disposed as far as possible from the neutral axis of the cross-section, thereby increasing r;

There should be no initial internal stress;

The individual parts of the column should be mutually supporting;

The individual parts of the column should be so firmly secured to each other that no relative motion can take place, in order that the column may fail as a whole, thus maintaining the original value of r.

These considerations, it is to be borne in mind, affect the *resistance* of the column only; it may be advisable to sacrifice some elements of resistance, in order to attain accessibility to the interior of the compression member, for the purpose of painting. This point may be a very important one, and should never be neglected in designing compression members.

Art. 83.—Tests of Wrought Iron Phoenix Columns, Steel Angles and Other Steel Columns.

During the period of use of wrought iron as a structural material many full-size wrought-iron columns were tested to failure giving data on which to base long column formulæ, but as yet few steel columns of full size have been tested to failure and the data on which to base proper long column formulæ, either for ordinary structural carbon steel or for nickel steel, are correspondingly meagre. At this time (1915) full-size steel columns are in process of testing at the National Bureau of Standards, Washington, D. C., and when they are completed, the desired data will be much increased.

In view of this condition of experimental work on steel columns it seems best to give the results of tests of an extended series of wrought iron Phoenix columns made with much care at the U. S. Arsenal at Watertown, Mass. in order to illustrate fully the method of graphical treatment of such results in the process of seeking proper column formulæ. The complete account of this series of tests is given in the Transactions of the American Society of Civil Engineers for 1882 and the numerical data relating both to the dimensions of the columns and to the results of the tests are given in Table I. It will be noticed that the ratio of length over radius of gyration ranges from less than 3 up to 112 which more than includes the values of that ratio in practically all steel structural work. The ends of the columns were flat, a condition which usually introduces some erratic results, but apparently the care with which the columns were tested eliminated this defect. The Phoenix column is a particularly advantageous section for testing as its different parts are effectively self-supporting and furthermore it has a section whose radius of gyration is the same in all directions as the latter has two or more axes of symmetry not at right angles to each other.

TABLE I.

No.	Length.	Area.	r^2.	$l \div r$.	$l^2 \div r^2$.	*E. L.*	*Exp.*	p_1.	p'.	p''.
	Feet.	Sq. In.	Ins.			Lbs.	Lbs.	Lbs.	Lbs.	Lbs.
1	28	12.062	8.94	112	12,544	——	35,150	32,550	34,488	——
2	28	12.181	8.94	112	12,544	——	34,150	32,550	34,488	——
3	25	12.233	8.94	100	10,000	27,960	35,270	34,000	35,040	——
4	25	12.100	8.94	100	10,000	——	35,040	34,000	35,040	——
5	22	12.371	8.94	88	7,744	——	35,570	35,420	35,592	——
6	22	12.311	8.94	88	7,744	——	34,360	35,420	35,592	——
7	19	12.023	8.94	76	5,776	——	35,365	36,800	36,144	——
8	19	12.087	8.94	76	5,776	29,290	36,900	36,800	36,144	——
9	16	12.000	8.94	64	4,096	——	36,580	38,130	36,696	——
10	16	12.000	8.94	64	4,096	——	36,580	38,130	36,696	——
11	13	12.185	8.94	52	2,704	28,890	36,857	39,400	37,248	——
12	13	12.069	8.94	52	2,704	——	37,200	39,400	37,248	——
13	10	12.248	8.94	40	1,600	26,940	36,480	40,700	37,800	——
14	10	12.339	8.94	40	1,600	28,360	36,397	40,700	37,800	——
15	7	12.265	8.94	28	784	29,350	38,157	42,200	38,352	40,360
16	7	11.962	8.94	28	784	29,590	43,300	42,200	38,352	40,360
17	4	12.081	8.94	16	256	——	49,500	44,770	——	46,300
18	4	12.119	8.94	16	256	28,050	51,240	44,770	——	46,300
19	8 ins.	11.903	8.94	2.7	7.29	——	57,130	69,600	——	57,140
20	8 ins.	11.903	8.94	2.7	7.29	——	57,300	69,600	——	57,140
21	25′ 2.65″	18.300	19.37	68.8	4,733	——	36,010	37,600	36,666	——
22	8′ 9″	18.300	19.37	24	576	29,510	42,180	42,840	——	42,160

The numerical quantities in Table I are self-explanatory, particularly in connection with eq. (6) of the preceding article.

The five columns in the right-hand half of the Table are pounds per square inch for the different purposes shown by the headings of the columns, i.e. *E. L.* represents the compressive stress in the column at the elastic limit, while the column headed *Exp.* indicates the compressive load per square inch of section at which it failed in the testing machine. The headings, p_1, p' and p'' are computed values from eqs. (1), (2), and (3) to be explained immediately.

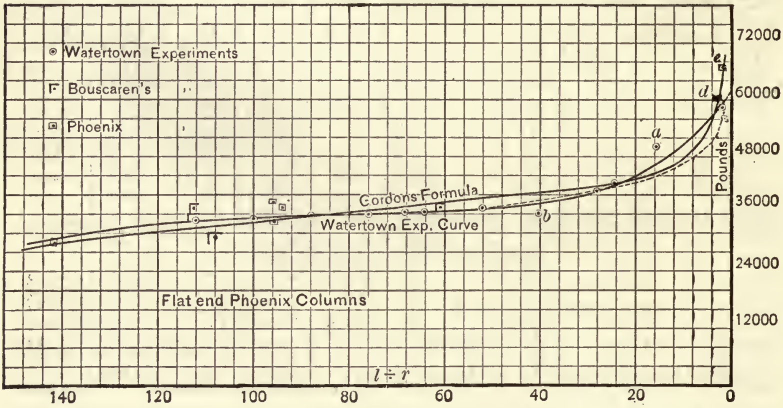


FIG. 1.

The numerical values in the column headed *Exp.* are accurately plotted in the diagram, Fig. 1, by laying off the ratios $\frac{l}{r}$ from O to the left as horizontal ordinates and erecting at their extremities the corresponding ultimate resistances given in that column as vertical ordinates with the scale as shown in Fig. 1. It should be observed that in the majority of cases in Table I, there are two experimental results for each value of $\frac{l}{r}$ and each vertical ordinate in Fig. 1 represents the mean of these two results.

The full-curved line marked " Watertown Exp. Curve " is then drawn so as to represent as accurately as possible the actual experimental results which, as shown in the figure, include a few tests other than those made at Watertown. This experimental curve rises rapidly for small values of $\frac{l}{r}$, i.e., for what are actually short blocks. At the left end of the curve where $\frac{l}{r}$ equals 140, the slope of the curve is but little more than for intermediate values of that ratio.

After a number of trials it was found that the value of p_1, as given in eq. (1), agrees quite closely with the experimental curve for all values between $\frac{l}{r}=28$ and $\frac{l}{r}=112$, and the results computed from it are shown in the column headed p_1 of the Table

$$p_1 = \frac{40000\left(1+\frac{2r}{l}\right)}{1+\frac{1}{50000}\cdot\frac{l^2}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Eq. (1) is Gordon's formula for this particular set of Phoenix columns except that the value of f (the numerator of the second member) is seen to vary slightly with the ratio $\frac{r}{l}$. In actual engineering practice, however, the numerator shown in eq. (1) was displaced by the numerical value 42,000, as a constant numerator of the second member makes a simpler application of the formula and it was sufficiently accurate for all practical purposes.

Inasmuch as all long columns used in structural work are found within the limits of $\frac{l}{r}=30$ and $\frac{l}{r}=120$ (usually for bridge truss members, 100) Gordon's formula is never used outside of practically these limits.

It may be observed that the experimental curve is nearly a straight line from a point just above b to the extreme left of the diagram. For that portion of the curve, therefore, the following formula applies very closely:

$$p' = 39{,}640 - 46\frac{l}{r}.^{*} \quad \ldots \ldots \quad (2)$$

The results of this formula are given in the column headed "p'." The table, in connection with the diagram, shows that this formula may be used with accuracy for values of $l \div r$ lying between 30 and 140, and further experiments may possibly show that it is applicable above the latter limit.

For values of $l \div r$ less than 30, the following formula will be found to give results approximating very closely to the experimental curve:

$$p'' = 64{,}700 - 4{,}600\sqrt{\frac{l}{r}}. \quad \ldots \ldots \quad (3)$$

The results of the application of this formula are given in the column headed "p''."

It will be observed in Table I that the ultimate resistance per square inch of the Phoenix columns tested for

---

* This equation known as the straight-line formula for long columns was first proposed in a paper by the author before the Annual Convention of the American Society of Civil Engineers in 1881. It was established at that time concurrently, but independently, by the author and Prof. Mansfield Merriman. The formula is sometimes called the Johnson Straight Line Formula, but Mr. Johnson's paper, in which he discussed the straight-line formula, was not given to the American Society of Civil Engineers until 1885, four years after the papers by the author and Professor Merriman had been published.

ratios of $\frac{l}{r}$ between about 40 and 112 ranges from about 34,000 to about 38,000 pounds, which is somewhat above the yield point of the material but far below the ultimate compressive resistance per square inch as found for short blocks.

In built-up sections of columns in which the component parts are less well supported than in the Phoenix section, the ultimate column resistance per square inch will be but little if any above the yield point of the material and with high values of $\frac{l}{r}$ the ultimate resistance may not rise above the elastic limit. This is a most important feature of long column resistance and it shows the effect of bending or flexure which increases as $\frac{l}{r}$ becomes greater.

Many tests of full-size pin-end wrought-iron columns have shown that, when well designed with lattice bars and other spacing details of sufficient capacity, the ultimate resistance of such columns may be represented by eqs. (4) and (4*a*);

$$p = \frac{39,000}{1 + \frac{1}{30,000}\frac{l^2}{r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Or;

$$p = 42,500 - 140\frac{l}{r} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

Although either equation is for columns with pin ends, it may be used generally for such end conditions as are usually found in structures like bridges or buildings. The flat end condition has already been indicated as giving in general somewhat erratic results, but with no advantage over pin ends for ordinary circumstances or for such ratios of $\frac{l}{r}$ as are commonly employed.

For working stresses in wrought-iron columns eqs. (5) or (5*a*) may be used. They are derived from eqs. (4) or (4*a*) by dividing the second members of those equations by a so-called "safety factor" of about 3.5;

$$p = \frac{11,000}{1 + \frac{1}{30,000}\frac{l^2}{r^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Or;

$$p = 12,000 - 40\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

Steel Columns.

The paucity of tests of suitably-designed full-size steel columns, either with pin ends or other end conditions, has already been observed. Some scattered tests of such members have fortunately been made while others have been made upon members so designed as to bring out in exaggerated form certain features of actions of stresses in various parts of the columns without, however, reaching data available for the best designs for general engineering practice.

Among the most valuable of these data are some results of old tests by the late Mr. James Christie and described in the Transactions of the American Society of Civil Engineers for 1884. Mr. Christie tested mild and high steel angle struts with ratios of $\frac{l}{r}$ running from 20 up to 300. The mild steel contained from .11 to .15 per cent. carbon, while the high steel contained .36 per cent. The ultimate tensile resistance of the mild steel ran from 60,000 to 66,000 pounds per square inch with 24 to 26 per cent. stretch in 8 inches. The high steel had an ultimate tensile resistance of about 100,000 pounds per square inch and a stretch of about 16 per cent. in 8 inches.

Table II gives the results of these steel angle tests and

TABLE II.

FLAT-END STEEL ANGLE STRUTS.

$\frac{l}{r}$.	Ultimate Resistance, Pounds per Square Inch.		$\frac{l}{r}$.	Ultimate Resistance, Pounds per Square Inch.	
	Mild Steel.	High Steel.		Mild Steel.	High Steel.
20	72,000	100,000	170	21,000	26,000
30	51,000	74,000	180	19,500	23,800
40	46,000	65,000	190	18,000	21,800
50	43,000	61,000	200	16,500	20,000
60	41,000	58,000	210	15,200	18,400
70	39,000	56,000	220	14,000	16,900
80	38,000	54,000	230	13,000	15,400
90	36,500	51,000	240	12,000	14,000
100	35,000	47,000	250	11,100	12,800
110	33,500	43,500	260	10,300	11,800
120	31,500	40,000	270	9,600	11,000
130	29,000	36,500	280	9,000	10,200
140	27,000	33,500	290	8,400	9,500
150	25,000	30,800	300	7,900	9,000
160	23,000	28,300			

Fig. 2 shows the curves formed by plotting in the usual manner the ultimate resistances found in Table II. The ratios $\frac{l}{r}$ are laid off as horizontal ordinates and the corresponding ultimate resistances as vertical ordinates. These curves are highly interesting as exhibiting the various stages of resistance offered by columns in compression as the lengths increase from small values of $\frac{l}{r}$ up to large values of that quantity. The ultimate resistances decrease rapidly

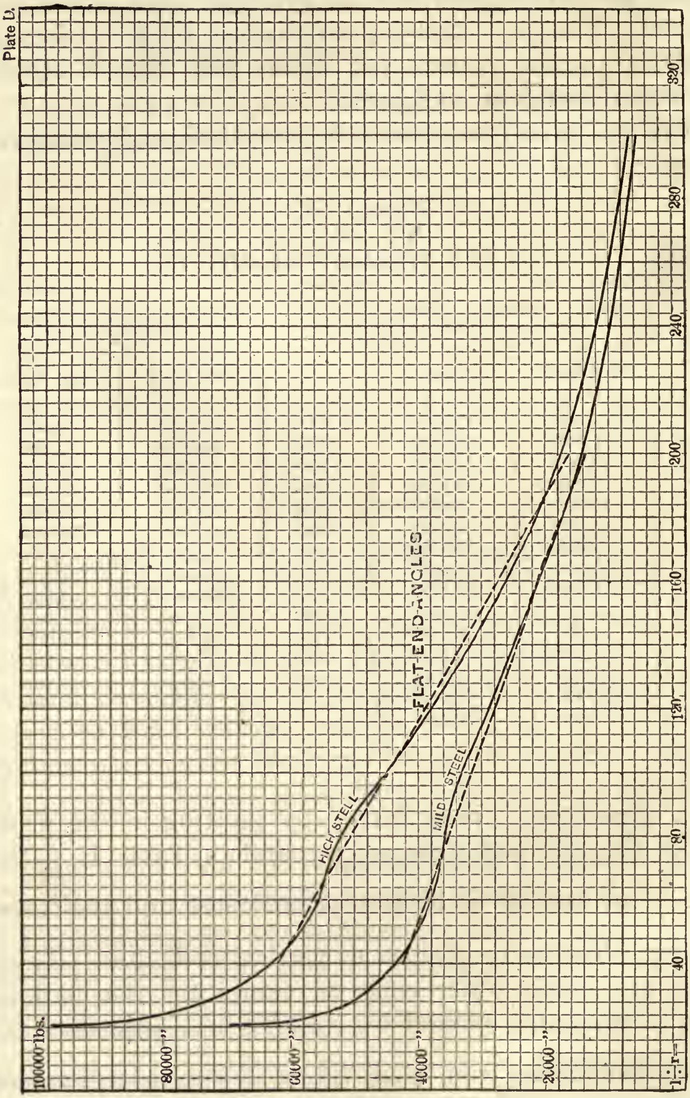


Plate D.
100000 lbs.
80000 "
60000 "
40000 "
20000 "
l : r =
40
80
120
160
200
240
280
320
HICH STELL
MILD STEEL
FLAT-END-ANGLES

FIG. 2.

when the column ratio $\frac{l}{r}$ increases from 20 to 40, then up to a value of at least 140 the curves differ but little from straight lines. Above the latter, the curvature becomes decided but not sharp and the two lines converge so that when $\frac{l}{r}$ becomes equal to 300 the difference between the two resistances is but little over 1000 pounds per square inch.

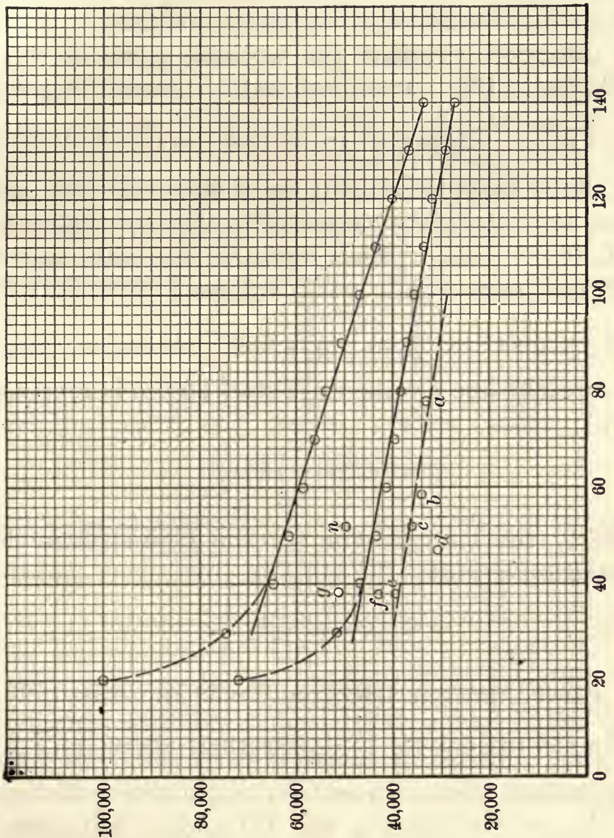


FIG. 3.

This convergence is one element of confirmation of Euler's Formula as the carrying capacity for such high values of $\frac{l}{r}$ depends chiefly upon the modulus of elasticity. With still higher ratios the two curves would probably coincide as both grades of steel have the same modulus.

The difference between the working parts of the two curves shown in Fig. 2 is reproduced on a much larger scale for $\frac{l}{r}$ in Fig. 3. Between $\frac{l}{r}=30$ and $\frac{l}{r}=140$, the two full straight lines may be drawn as shown. As the points represent accurately the numerical values of Table II, it is seen that the straight lines represent the ultimate resistances of the angle struts with sufficient closeness for all practical purposes between $\frac{l}{r}=35$ and $\frac{l}{r}=140$.

The straight line for the mild-steel angles is represented by eq. (6);

$$p=53,000-186\frac{l}{r}. \qquad (6)$$

Similarly the straight line for the high-steel angles is represented by eq. (7);

$$p=79,000-325\frac{l}{r}. \qquad (7)$$

The curved broken lines represent approximately the unit ultimate resistances for $\frac{l}{r}$ less than about 40. If the second members of eqs. (6) and (7) be divided by a so-called "safety factor" of about 3, eqs. (8) and (9) will represent working stresses;

For high steel $$p=25,000-100\frac{l}{r}. \qquad (8)$$

For mild steel $$p=17,000-53\frac{l}{r}. \qquad (9)$$

A number of "model" carbon steel columns of large dimensions have been tested within two or three years in the large testing machine of the Phoenix Bridge Company at Phoenixville, Pa., together with two such nickel steel columns, under the supervision of Mr. James E. Howard, all but three of those tests having been made for the purpose of affording data for the design of the new Quebec Bridge across the St. Lawrence River. The results of these tests, as given in the Transactions of the American Society of Civil Engineers for 1911 and in the Engineering Record for 1914 are shown on Fig. 3. The average of three tests of built up carbon steel columns, 30 inches by 20 inches in outline, as indicated by Fig. 4, are shown at *d*, the value of $\frac{l}{r}$ being 47 and the average ultimate resistance of the three tests (varying but little from each other) being 30,000 pounds per square inch.

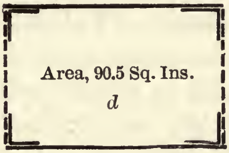


FIG 4.

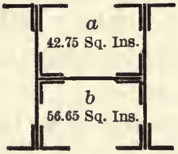


FIG. 5.

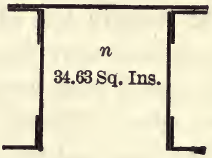


FIG. 6.

The results shown at *e*, *f* and *g* are also for carbon steel columns with built-up sections shown in the diagram on page 488, the cross-sectional area being 70.65 square inches. The length of these columns was 18 feet 9 inches and the ratio $\frac{l}{r}$ was 38.

Again *a* and *b* represent results for carbon-steel columns having $\frac{l}{r}$ equal to 78 and 58 and with cross-sectional areas 42.75 square inches distributed as shown in Fig. 5.

Finally, the point *n* represents the result for two nickel steel columns having an area of cross-section of 34.63 square inches and $\frac{l}{r}=52$, the section being shown in Fig. 6.

The number of tests of the carbon-steel columns is not sufficient to form a proper basis for a straight line long column formula, but the broken line drawn through *a* and *c* and below *e* may, as a tentative matter, be represented by eq. (10);

$$p=44,000-150\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

All these built-up carbon-steel columns were of mild steel, but their ultimate resistances are distinctly lower than the results for Mr. Christie's mild-steel angles. Full-size tests, however, have shown that the built-up column, unless designed with great care so as to act solidly as a unit, will not offer ultimate resistances as high as might be expected from the quality of the steel of which they are composed.

On the same basis used for eqs. (7) and (9), the tentative working stress for built-up mild carbon-steel columns would be;

$$p=14,000-50\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The average for the two nickel-steel columns, shown at *n*, Fig. 3 is about 50,000 pounds per square inch and more than one-third greater than the corresponding result shown for the mild carbon steel at *c*.

In all these column tests the elastic limit or the yield point of the member as a whole appears to be the controlling feature, i.e., the ultimate resistance is not above the yield point of the column and if the ratio $\frac{l}{r}$ is comparatively large it will not be above the limit of elasticity of the column as a whole. It must be remembered also that both the elastic limit and the yield point of built-up columns will be materially lower than the corresponding points of a single piece of the same metal.

These tests appear to indicate that the ultimate resistances of nickel-steel columns exceed those of mild carbon-steel columns in about the same proportion that the elastic limit of nickel steel exceeds the elastic limit of the carbon-steel.

Observations in these tests of full-size columns made at Phoenixville by Mr. Howard indicate that steel columns may be considered to have a true modulus of elasticity of about 29,000,000 or perhaps 29,500,000 for intensities of loading not greater than ordinarily allowed working stresses, i.e., from 8,000 to 12,000 pounds per square inch. While there are not sufficient data to determine precisely such physical elements of steel column resistance, there seems to be a relative motion of the component parts of a built-up member under test, which does not permit the existence of a true modulus of elasticity when loadings exceed about 12,000 to 15,000 pounds per square inch. Obviously the more nearly a column acts as a perfect unit, the better defined will be its elastic properties.

Much more data derived from experimental work with full-size steel columns are imperatively necessary in order to reach definite conclusions regarding actions of stresses in the various parts of such members as well as for the development of such important details as latticing, battens, and other riveted details.

Typical Formulæ Now in Use.

As a result of the present conditions of experimental knowledge of built columns, as well as of those that are not built up, there is a great variety of column formulæ used by engineers, both of the Gordon and straight-line type. The straight-line formula, however, is largely displacing the Gordon formula. The General Specifications for Steel Railway Bridges recommended by the American Railway Engineering Association as applied to the design of cross-sections of steel columns is;

$$p=16,000-70\,\frac{l}{r}. \qquad (12)$$

The New York Central Lines are using the same formula in the design of their bridge work, as are engineering organizations of other railway companies. Under the use of this formula a greater compressive load than 14,000 pounds per square inch is not permitted.

The American Bridge Company Specifications for Steel Structures 1913, uses the following formula in its design work;

$$p=19,000-100\,\frac{l}{r}. \qquad (13)$$

A provision for impact is made and 13,000 pounds per sq. in. is the maximum allowed under the use of eq. (13).

A form of Gordon's formula still appearing in engineering practice is

$$p=\frac{12,500}{1+\frac{1}{36,000}\frac{l^2}{r^2}}. \qquad (14)$$

This formula is really an old wrought-iron column formula and should not be used without reducing the 36,000 in the denominator to 30,000.

The New York Building Law gives for a steel column;

$$p=15,200-58\,\frac{l}{r}. \qquad (15)$$

The formula used by the City of Philadelphia for its buildings is of the Gordon type as follows:

$$p=\frac{16,250}{1+\frac{1}{11,000}\frac{l^2}{r^2}}. \qquad (16)$$

Other formulæ could be cited but enough is shown to indicate the pronounced lack of uniformity in this practice.

None of the preceding formulæ should be used for $\frac{l}{r}$ less than 30 nor more than about 120.

In every case where a column formula is used, it would be much more convenient to employ a diagram with the curves accurately drawn to represent the desired formulæ. The actual results, without computations, could be read directly from such long column curves.

Details of Columns.

In addition to the data already given in another portion of this article, the tests cited in this chapter show that the unsupported width of no plate in a compression member should exceed 30 to 35 times its thickness. These tests have usually been made with plates or metal $\frac{1}{4}$ to $\frac{1}{2}$ inch in thickness, and it is altogether probable that the above ratio of width over thickness would be increased with greater thicknesses.

In built columns, however, *the transverse distance between centre lines of rivets securing plates to angles or channels, etc., should not exceed* 35 *times the plate thickness.* If this width is exceeded, longitudinal buckling of the plate takes place, and the column ceases to fail as a whole, but yields in detail.

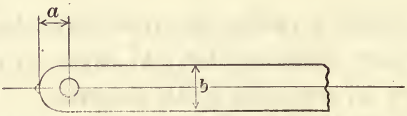
The same tests show that *the thickness of the leg of an angle to which latticing is riveted should not be less than* $\frac{1}{9}$ *of the length of that leg or side*, if the column is purely and wholly a compression member. The above limit may be passed somewhat in stiff ties and compression members designed to carry transverse loads.

The panel points of latticing should not be separated by a greater distance than 60 *times the thickness of the angle leg to which the latticing is riveted*, if the column is wholly a compression member.

The rivet pitch should never exceed 16 *times the thickness of the outside thinnest metal pierced by the rivet,* and if the plates are very thick it should never nearly equal that value.

Art. 84.—Complete Design of Pin-end Steel Columns.

In actual design it is necessary not only to make application of the preceding formulæ for ultimate resistance of columns, but also to proportion a considerable number of details as matters largely of judgment and experience. If the column, like the section shown as the latticed channel or latticed upper chord in the preceding article, has two open sides as in the former or one open side as in the latter latticed, i.e., has small bars of iron running diagonally across those open sides in order to hold the parts of the column in their proper relative positions, those lattice bars vary in size with the size of column. While the dimensions vary somewhat among engineers, the following table, which has been largely used, illustrates effectively sizes that may properly be employed.



For	6	inch rolled or built channels	$1\frac{3}{4}'' \times \frac{5}{16}''$
“	7	“ “ “ “ “	$1\frac{3}{4} \times \frac{5}{16}$
“	8	“ “ “ “ “	$1\frac{3}{4} \times \frac{5}{16}$
“	9	“ “ “ “ “	$1\frac{3}{4}$ and $2 \times \frac{3}{8}$
“	10	“ “ “ “ “	$1\frac{3}{4}$ “ $2 \times \frac{3}{8}$
“	11	“ “ “ “ “	$2 \times \frac{3}{8}$
“	12	“ “ “ “ “	$2 \times \frac{3}{8}$
“	13	“ “ “ “ “	$2\frac{1}{4} \times \frac{3}{8}$
“	14	“ “ “ “ “	$2\frac{1}{4} \times \frac{3}{8}$
“	15	“ “ “ “ “	$2\frac{1}{4} \times \frac{3}{8}$
“	16	“ “ “ “ “	$2\frac{1}{2} \times \frac{3}{8}$
“	18	“ “ “ “ “	$2\frac{1}{2} \times \frac{3}{8}$
“	19–23	“ “ “ “ “	$2\frac{1}{2} \times \frac{7}{16}$
“	24–29	“ “ “ “ “	$2\frac{1}{2} \times \frac{1}{2}$
“	30	“ “ “ “ “	$3 \times \frac{1}{2}$

$a = \frac{b}{2} + \frac{1}{4}''$

	b	*a*
for	$1\frac{1}{2}''$	$1''$
“	$1\frac{3}{4}$	$1\frac{1}{8}$
“	2	$1\frac{1}{4}$
“	$2\frac{1}{2}$	$1\frac{1}{2}$

These bars or lattices may be used in single system, in which case each one should make an angle of about 60° with the centre line of the side of the column on which they are placed. If they are used in double system each pair of bars will intersect at their mid-points, and in this case the bars may make angles of 45° with the centre line of the side of the column on which they are employed. In the case of double latticing the intersecting pairs of bars are riveted at their intersections. Lattice bars are held at their ends by one rivet or by two rivets according to the size of the column, as shown in the next table.

Figs. 1, 2, and 3 illustrate different modes of riveting the ends of lattice bars. The size and number of rivets will obviously depend upon the size of the lattice bars employed and to some extent upon the manner in which their ends are held.

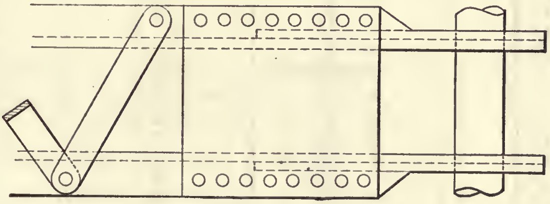


FIG. 1.

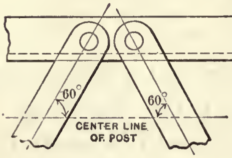


FIG. 2.

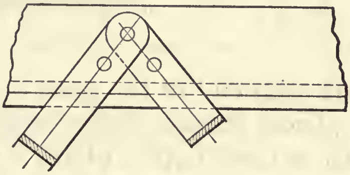


FIG. 3.

The following table has been used in actual structural practice and exhibits good practice in the design of single latticing. It is based on the supposition that the lattice bars are flats. In very large columns or in some exposed situations it is necessary to use steel angles for latticing, the ends of which must be secured by rivets proportionate in number and diameter to the size of angle.

Size of Lattice.	Rivets: Number and Size.	Number of Rivets at Lattice Point.	Limiting Length of Lattice Centre to Centre of Inner Rivets.
$1\frac{3}{4}\times\frac{5}{16}$ and $\frac{3}{8}$	1 . . . $\frac{5}{8}''$	1	13 inches
$2\times\frac{5}{16}$	1 . . . $\frac{5}{8}$	1	16 “
$2\times\frac{5}{16}$	1 . . . $\frac{3}{4}$	1	10 “
$2\times\frac{3}{8}$	1 . . . $\frac{3}{4}$	1	23 “
$2\times\frac{3}{8}$	1 . . . $\frac{3}{4}$	1	16 “
$2\frac{1}{2}\times\frac{3}{8}$	1 . . . $\frac{3}{4}$	1 or 2	20 “
$2\frac{1}{2}\times\frac{3}{8}$	1 . . . $\frac{7}{8}$	1 “ 2	15 “
$2\frac{1}{2}\times\frac{7}{16}$	1 . . . $\frac{7}{8}$	1 “ 2	20 “
$2\frac{1}{2}\times\frac{7}{16}$	1 . . . $\frac{15}{16}$	1 “ 2	17 “
$2\frac{1}{2}\times\frac{1}{2}$	1 . . . $\frac{7}{8}$	1 “ 2	26 “
$2\frac{1}{2}\times\frac{1}{2}$	1 . . . $\frac{15}{16}$	1 “ 2	24 “
$2\frac{1}{2}\times\frac{1}{2}$	2 . . . $\frac{3}{4}$	4	15 “
$3\times\frac{3}{8}$	1 . . . $\frac{7}{8}$	1 or 2	18 “
$3\times\frac{3}{8}$	1 . . . $\frac{15}{16}$	1 “ 2	16 “
$3\times\frac{3}{8}$	2 . . . $\frac{3}{4}$	4	9 “
$3\times\frac{7}{16}$	1 . . . $\frac{7}{8}$	1 or 2	25 “
$3\times\frac{7}{16}$	1 . . . $\frac{15}{16}$	1 “ 2	22 “
$3\times\frac{7}{16}$	2 . . . $\frac{3}{4}$	4	15 “
$3\times\frac{1}{2}$	1 . . . $\frac{7}{8}$	1 or 2	32 “
$3\times\frac{1}{2}$	1 . . . $\frac{15}{16}$	1 “ 2	29 “
$3\times\frac{1}{2}$	2 . . . $\frac{3}{4}$	4	21 “
$3\times\frac{1}{2}$	2 . . . $\frac{7}{8}$	4	11 “
$4\times\frac{7}{16}$	1 . . . $\frac{15}{16}$	1 or 2	28 “
$4\times\frac{7}{16}$	2 . . . $\frac{3}{4}$	4	22 “
$4\times\frac{7}{16}$	2 . . . $\frac{7}{8}$	4	15 “

At each end of the open or latticed sides of the column are placed batten plates which limit the latticing. The width of these batten plates is determined evidently by the width of the column, but the lengths vary somewhat under different specifications. A good and convenient rule is to make the length of a batten plate at least equal to its width. The thickness of a batten plate will depend upon the size of column; it is seldom made less than $\frac{3}{8}$ in. and usually not more than $\frac{5}{8}$ in. for large columns. The size of rivet will also depend upon the size of columns. Rivets less than $\frac{3}{4}$ in. in diameter are seldom used in railroad work and rarely more than 1 in., the prevailing diameter being $\frac{7}{8}$ in.

One of the most important details of a column is the jaw or extension of one side at the end. The two jaws contain the pin holes through which are transferred to the pin the total load carried by the column. These jaws or extensions are formed so as to fit in between the parts of intersecting members, usually the upper or lower chords and eye-bars. It is, therefore, imperative to make them as thin as the bearing upon the pins and the carrying capacity of the jaws themselves acting as short columns will permit. Figs. 4, 5, 6, and 7 exhibit some types of these post jaws as they commonly occur. As the figures show, they are formed by cutting away the flanges of the angles or channels forming parts of the posts and riveting on the pin or thickening plates required to strengthen the detail. The jaws form short columns whose lengths should be taken from the centre of the pin hole to the last centre line of rivets in the body of the column back of the

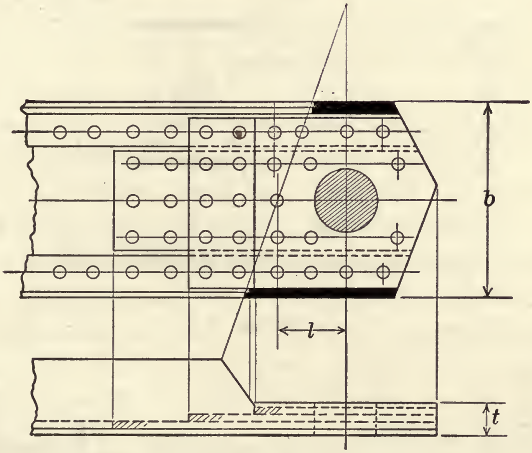


Fig. 4.

cut in the angle or in the flange of the channel. This length indicated by l is shown in each of the figures.

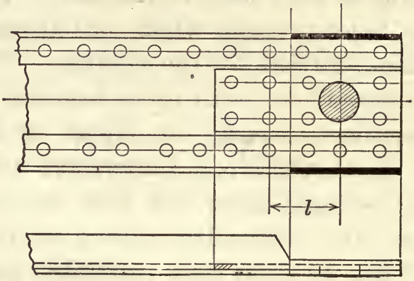


FIG. 5.

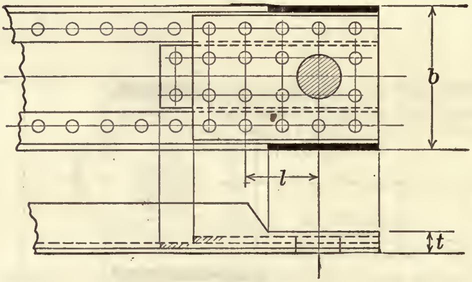


FIG. 6.

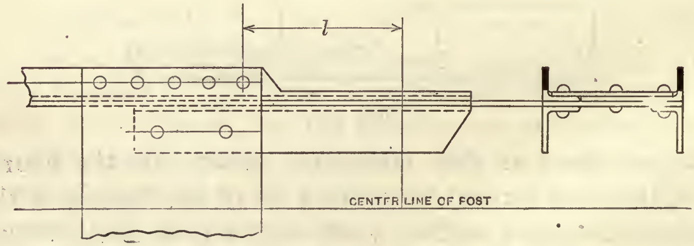


FIG. 7.

There have been but few tests made to determine the resisting capacities of this particular detail, but those which have been made form the basis of the following formula for medium steel columns. Obviously there will usually be at least two jaws at the end of each column. The width of the side of the column will be represented by b, as shown in Figs. 4 and 6, and t will represent the total thickness of metal whose width is b, also as indicated in the same figures. If P represents the total load on one jaw of the post, usually one half the total load carried by the post or column, the average working intensity of pressure on the section of metal bt may be written

$$\frac{P}{bt}=9000-340\frac{l}{t}. \qquad (1)$$

The thickness t of metal is usually the quantity desired, and eq. (1) gives

$$t=\frac{P}{9000b}+\frac{l}{26}. \qquad (2)$$

In these equations P should be taken in pounds, with b, t, and l in inches.

Eq: (2) has been used to a considerable extent in the design of steel railroad bridges, and it is probably as reasonable and safe a value of the thickness t as can be written with the experimental data and experience now available. It is applicable to steel with ultimate tensile resistance running from 60,000 to 68,000 pounds per square inch. For higher steel or for highway bridges, or for other structures where less margin of safety may be justifiable, the value of t may be made correspondingly less than that given in eq. (2).

Prob. 1. It is required to design a mild-steel pin-end column 45 feet long between centres of pins to carry a load of 353,000 pounds. The column formula to be used is essentially that given as eq. (11) of Art. 83:

$$p=16,000-70\frac{l}{r}. \qquad (3)$$

This equation gives the greatest mean intensity allowed on the column, so that p multiplied by the area of cross-section to be determined must be equal or nearly equal to 232,000. ?

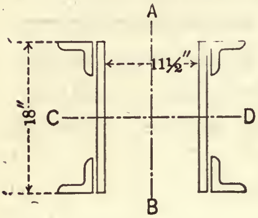


FIG. 8.

The least diameter or width of a built column should not exceed about one thirty-fifth of its length, except where posts or columns are used as lateral members, when the length may reach as much as 40 times the least diameter or width of cross-section. In this case the column is to be built of two plates and four angles, as shown in Fig. 8, and the width of plate *FG* must, therefore, not be less than about 16 inches. A width of 18 inches will make a well-proportioned column and that dimension will be assumed. The separation of the plates is preferably made such that the moment of inertia of the section about the axis *AB* will be a little larger than the moment about the axis *CD*. The pin will pierce the two plates so that its axis will be parallel to *CD*. Under these conditions, if the column is designed so as to be strong enough with the moment of inertia of section taken about *CD*, it will be still stronger in reference to the axis *AB*, and no further attention need be given to possible failure about the latter axis.

If columns of this type are proportioned in the general manner indicated, the radius of gyration of the section about the axis *CD* will be approximately .35 of the width. In this case that trial radius will, therefore, equal 6.3 inches. Hence, inserting the values of $l=540$ inches and $r=6.3$ inches in eq. (3), there will result $p=10,000$ pounds per square inch. The total area of section required, therefore, will be closely $353,000 \div 10,000 = 35.3$ sq. ins. The distribution of this metal between the plates and angles is largely a matter of judgment. Let there be assumed

Two 18″×⅝″ plates.	=22.5 sq. ins.
Four 3½″×3½″×11-pound angles.	=13 " "
Total.	=35.5 sq. ins.

This is a tentative composition of section which must be tested by eq. (3) to determine whether it is as nearly accurate as it should be. In order to do this, the moments of inertia of the section, as indicated, must be taken about the two axes *AB* and *CD*.

MOMENT OF INERTIA ABOUT *CD*:

Two 18″×⅝″ plates. . . . . $=2\times\frac{5}{8}\times\frac{\overline{18}^3}{12}=$	607.50
Four 3½″×3½″×11-lb angles about own axis. . . . . =	14.20
Four 3½″×3½″×11-lb.angles about *CD* $=4\times3.25\times(7.99)^2=$	829.92
Moment of inertia. . . . . =	1451.62

MOMENT OF INERTIA ABOUT *AB*:

Two 18″×⅝″ plates about own axis. . . . . $2\times\frac{18(\frac{5}{8})^3}{12}=$	.74
Two 18″×⅝″ plates about *AB*. . . . . $2\times11.25\times(6.06)^2=$	758.70
Four 3½″×3½″×11-lb. angles about own axis. . . . . =	14.20
Four 3½″×3½″×11-lb.angles about *AB* $=4\times3.25\times(7.38)^2=$	708.38
Moment of inertia. . . . . =	1482.02

These computations show, first, that the moment of inertia about *AB* is a little larger than that about *CD*, which is as it should be. They also show that the radius of gyration r is 6.39 inches. The approximate rule gives $r=$ 6.3 inches. These two values are sufficiently near to accept the former. The trial composition of section may, therefore, be considered satisfactory and final. The thickness of the side plates, .625 inch, is sufficient to insure no buckling in the unsupported width between rivets. Similarly the length of leg of the 3½-inch angles is also far within safe or proper limits. All features of the cross-section are, therefore, so arranged as to meet all the requirements of suitable resistance in detail.

The details of the ends of the columns where they are formed into jaws, as shown by Figs. 9 and 10, still remain to be designed. The diameter of pin will be taken at 7 inches, as shown in Fig. 9. The permissible intensities of shearing and of the bearing on the walls of rivet and pin holes will be taken as follows:

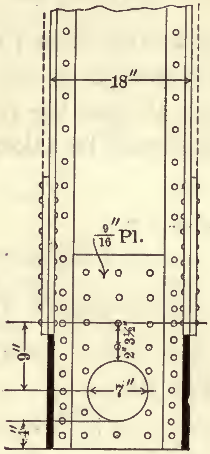


FIG. 9.

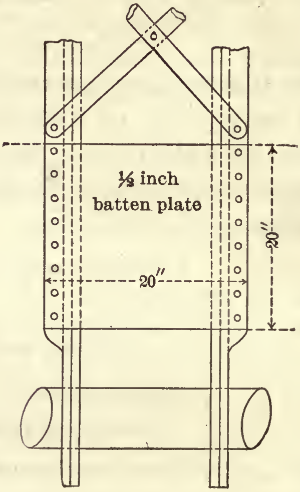


FIG. 10.

Shearing on rivets = 9000 pounds per sq. in.
Bearing on rivets and pins = 16,000 pounds per sq. in.

The total thickness of metal in the two post jaws will, therefore, be

$$\text{Thickness of metal} = \frac{232000}{7 \times 16000} = 2.1 \text{ inches.}$$

The thickness of metal in each jaw must therefore be at least $1\frac{1}{16}$ inches. Inasmuch as the thickness of side plates of the column is $\frac{5}{8}$ inch, the pin plates to be riveted to the side plates must be at least $\frac{7}{16}$ inch thick to supply the proper bearing surface for the pin; but that thickness must be decided by the formula for the jaws, eq. (2). In that equation, $P=116,000$ pounds, while $b=18$ inches and l, from Fig. 9, is 9 inches. Making these substitutions in eq. (2),

$$t=1.13 \text{ inches.}$$

In order to meet the requirements of the post-jaw formula, therefore, the pin plate must be at least $\frac{1}{2}$ inch thick. It is essential however to make these details specially stiff and strong and the thickness will, therefore, be taken at $\frac{9}{16}$ inch, as shown in Fig. 9.

The number of rivets required above the pin hole would ordinarily be computed for the thickness of plate required for bearing on the pin, i.e., with the thickness of pin plate of $\frac{7}{16}$ inch. Assuming that thickness for this purpose, the rivets being taken $\frac{7}{8}$ inch in diameter, the bearing value of a single rivet will be

$$\frac{7}{8}\times\frac{7}{16}\times16,000=6125 \text{ lbs.}$$

The single shear of one $\frac{7}{8}$-inch rivet at 9000 pounds per square inch has a value of 5412 pounds which is less than the bearing value; the shear will, therefore, decide the number of rivets required. The bearing value of the $\frac{5}{8}$-inch side plate on the pin is $7\times\frac{5}{8}\times16,000=70,000$ pounds. Hence the number of rivets required in the pin plate on each side of the column will be

$$\frac{116000-70000}{5412}= \text{nine rivets (nearly).}$$

These nine rivets must be found above the pin. That number, however, is far too small for the pin plate acting as a part of the jaw, and it will be judicious to make the total number of rivets above the pin 12, as shown in Fig. 9.

The jaw plates will extend 5 inches beyond the pin, as shown. The two batten plates above which the latticing begins will each be taken ½ inch thick, and they will be placed as shown in both Figs. 9 and 10.

It is assumed that the ends of the column are to fit into or between other members of the truss, so as to require cutting away the legs of the steel angles, as shown, as this is a common requirement.

The length of a batten plate should not be less than its width. In the present instance the width of batten will be 19.75 inches; the length will, therefore, be taken as 20 inches.

As indicated in the tabular statement at the beginning of this article, the lattice bars, fully shown in Fig. 10, will be 2½×⅜ inches, and the latticing will be taken as double, although this is not always done for the size of column in this particular instance. The lattice bars will be riveted at their intersections also as shown in Fig. 10. The length of lattice bar between rivets will be about 11 inches, as the angle made by each lattice bar with the side of the column will be about 45 degrees. A single ⅞-inch rivet, therefore, at the end of each bar will be sufficient, as shown by the second table of this article. At each panel point of latticing a single ⅞-inch rivet will hold the ends of both lattice bars.

The complete bill of material for the entire column will be as follows:

Four 3½″×3½″×11-lb. angles, 46.42 ft. long. .	185.7×11 =	2,043	lbs.
Two 18″×⅝″ plates, 46.42 ft. long.	93×38.25 =	3,557	"
Four 27″×11″×9/16″ plates	9×21 =	189	"
Four 20″×20″×½″ battens	6⅔×34 =	227	"
240 lin. ft. of 2½″×⅜″ latticing.	240×3.19 =	766	"
1060 ⅞″ rivets.	10.6×54 =	572	"
Total weight of one column.	=	7,354	lbs.

Prob. 2. Let it be required to design a mild-steel column with pin ends, 36 feet long between centres of pins, to carry a load of 225,500 pounds. It is supposed that the column is a member of a railroad bridge, so that the load given includes a full allowance for impact. Gordon's formula as formerly employed in the American Bridge Company's specification will be used:

$$p = \frac{17000}{1 + \frac{l^2}{11000r^2}}.$$

In this formula p is the greatest mean intensity of working pressure allowed on the section of the column, l the length between centres of pins in inches, and r the radius of gyration of the column section in inches. As the length of the column is but 36 ft. $=432$ inches two rolled 15-inch channels latticed may be taken as the principal parts, as shown in Fig. 11. By turning to the tables in any steel handbook, it will be found that the radius of gyration of a 15-inch channel about the axis AB varies from about 5.6 inches to nearly 5.2 inches. The larger of the two values will be tentatively employed. Substituting $l=432$ and $r=5.6$ in the above formula for p,

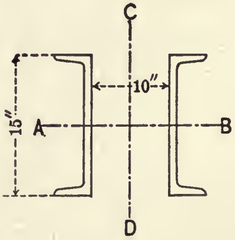


FIG. 11.

$$p = 11,000 \text{ pounds per sq. in.}$$

Hence the total area required is

$$\frac{225500}{11000} = 20.5 \text{ sq. ins.}$$

The table of steel channels in any handbook shows that the combined area of two 15-inch 35-pound channels is 20.58 sq. in., and they will be accepted as correct. The same table gives the radius of gyration r about the axis AB, Fig. 11, as 5.57 inches, which is essentially equal to the trial value 5.6 inches.

As shown in Prob. 1, it is desirable to have the moment of inertia of the section about AB, Fig. 11, a little less than that about CD, the former (AB) being parallel to the axis of the pin. Let the separation of the channels be made 10 inches in the clear. By using the values of the table, the moments of inertia about the two axes may be written:

ABOUT AXIS *AB*:

$$\text{Moment of inertia} = 320 \times 2 = 640.$$

Hence $$r^2 = \frac{640}{20.58} = 31.02; \quad \therefore r = 5.57 \text{ ins.}$$

ABOUT AXIS *CD*:

Moment of two channel sections each about axis parallel to *CD* and through centre of gravity	$2 \times 8.48 = 16.96$
$2 \times 10.29 \times \overline{5.79}^2$	$= 689.84$
Moment of inertia	$= 706.80$

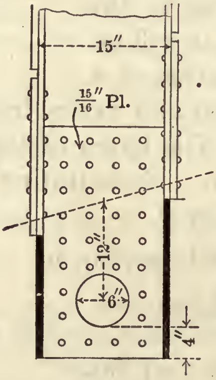


FIG. 12.

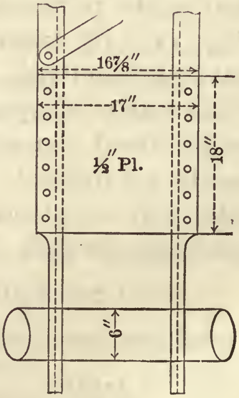
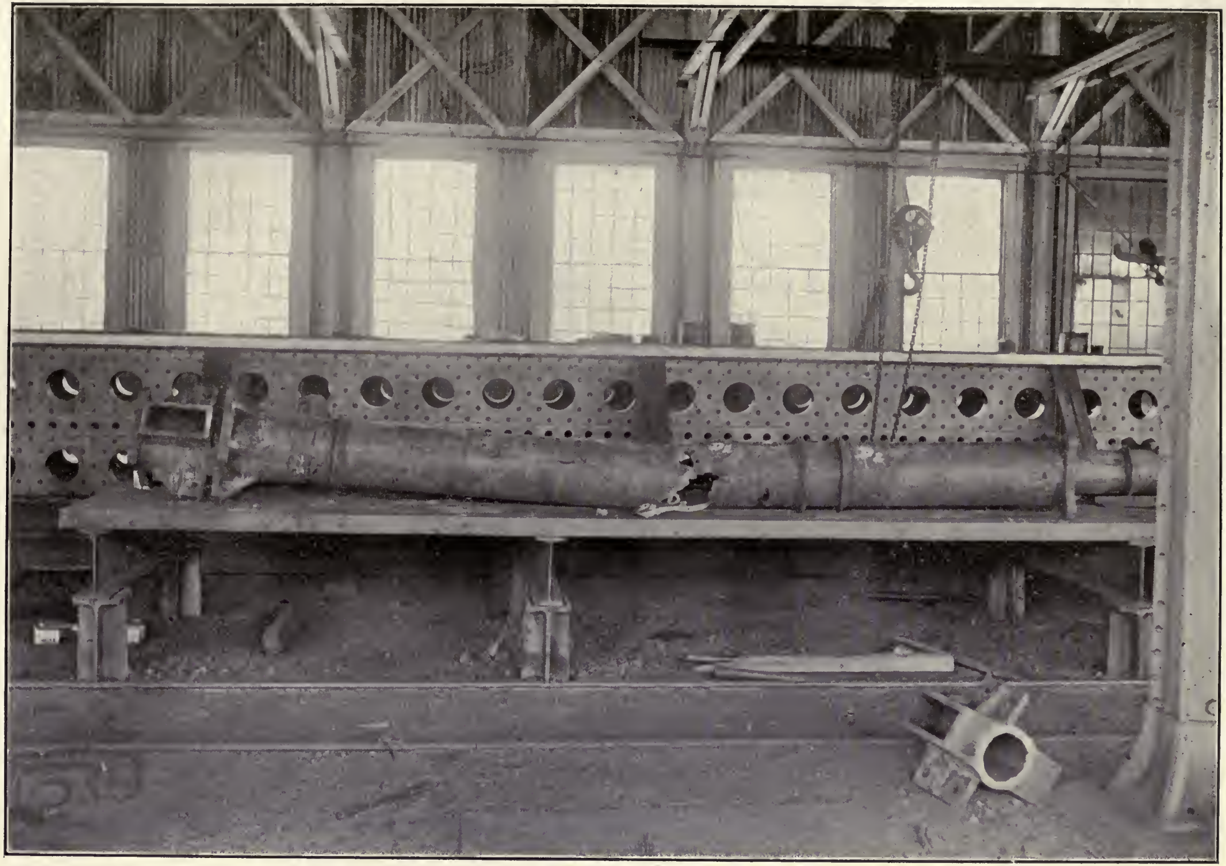


FIG. 13.

These results are all satisfactory and show that no revision of the section as given in Fig. 11 is needed.

The end details and latticing shown in Figs. 12 and 13



View of failure of a cast-iron column tested in the 1000-ton testing-machine of the Phœnix Iron Co. at Phœnixville, Pa.

remain to be considered. The following data will be required:

Thickness of channel web. . . .. $=.43$ inch.
Allowed shearing on rivets and pins. . . . . $=10,000$ lbs. per sq. in.
Allowed bearing on rivets and pins. . . . . $=20,000$ lbs. per sq. in.
Diameter of rivets. . . . . $=\frac{7}{8}$ inch.
Diameter of pin. . . . . $=6$ inches.
Value of one $\frac{7}{8}$-inch rivet in single shear. . . . . $=6,013$ lbs.
Bearing of pin on channel web. . . . . $=6\times .43\times 20,000$
$=51,600$ lbs.

Bearing to be carried by pin plate $=\frac{225500}{2}-51,600=61,150$ lbs.

Thickness of pin plate. . . . . $=\frac{61150}{6\times 20000}=.51$ inch.

Bearing value of one $\frac{7}{8}$-inch rivet on $\frac{1}{2}$-inch plate $=$
$\frac{7}{8}\times\frac{1}{2}\times 20,000=$ 8,750 lbs.

Hence one pin plate needs $\frac{61150}{6013}=$ ten $\frac{7}{8}$-inch rivets.

It is assumed that the ends of the column must be formed into the jaws shown in Figs. 12 and 13. As indicated in Fig. 12 the mean or effective length of the jaw is 12 inches. The load carried by one jaw is 112,750 pounds; hence the thickness of that jaw is by eq. (2)

$$t=\frac{112750}{8000\times 15}+\frac{12}{27}=1\tfrac{7}{16} \text{ inch (nearly).}$$

The thickness of the jaw or pin plate to be riveted to the jaw must therefore be $1\frac{7}{16}-.43=\frac{15}{16}$ inch. In order that these plates may be firmly made a solid extension of the post or column they should be riveted to the webs of the channels with the rivets shown in Fig. 12. The proper design of the jaw, therefore, requires a much longer and thicker plate and more rivets than the simple consideration of the pin and rivet bearing and shearing.

The width of channel flange is 3.43 inches, hence the total width of column over these flanges, as shown in Fig.

13, is 16⅞ inches. Each batten plate is therefore taken as 17 inches by 18 inches.

The length of each lattice bar of the single, 30-degree latticing will be about 16 inches between centres of rivets at their ends. Lattice bars 2½ inches by ⅜ inch in section will, therefore, be used.

The complete bill of material for one column will then be

Two	15″ 35-lb. channels 37⅙ ft. long	$2 \times 35 \times 37\frac{1}{6}$	= 2,602 lbs.
Four	$13'' \times 30'' \times \frac{15}{16}''$ plates	10×41.44	= 415 "
Four	$17'' \times 18'' \times \frac{1}{2}''$ plates	6×28.9	= 173 "
Forty-six	$2\frac{1}{2}'' \times \frac{3}{8}'' \times 19''$ bars	72×3.19	= 230 "
Two hundred and twenty-five	⅞″ rivets	$2\frac{1}{4} \times 54$	= 122 "
	Total weight of one column		= 3,542 lbs.

Art. 85.—Cast-iron Columns.

Cast iron was the earliest form in which the metal iron was used for columns, and it is natural, therefore, that the first long-column formulæ for cast iron should have been among the earliest for that class of members. The first experimenter was Eton Hodgkinson, who published the results of his tests on small cast-iron columns, the greatest length of which was but 60.5 inches, in the "Philosophical Transactions of the Royal Society of London for 1840." He not only recognized the round- and fixed-end conditions, but he also made the distinction between long columns and short blocks, the length of the latter being from 4 to 5 times the diameter or least cross-section dimension. If d be the diameter of the column in inches and l the length in feet, and in the case of hollow round columns if D be the exterior diameter in inches and d the interior diameter in the same unit, while P is the total or ultimate load in pounds on the column, Hodgkinson established the following formulæ for long cast-iron columns:

$$P = 33,379 \frac{d^{3.76}}{l^{1.7}}; \text{ (for rounded ends).} \quad . \quad . \quad . \quad (1)$$

$$P = 98{,}922 \frac{d^{3.55}}{l^{1.7}}; \text{ (for fixed ends).} \quad . \quad . \quad . \quad . \quad (2)$$

For hollow cylindrical columns of cast iron

$$P = 29{,}120 \frac{D^{3.76} - d^{3.76}}{l^{1.7}}; \text{ (for rounded ends).} \quad . \quad (3)$$

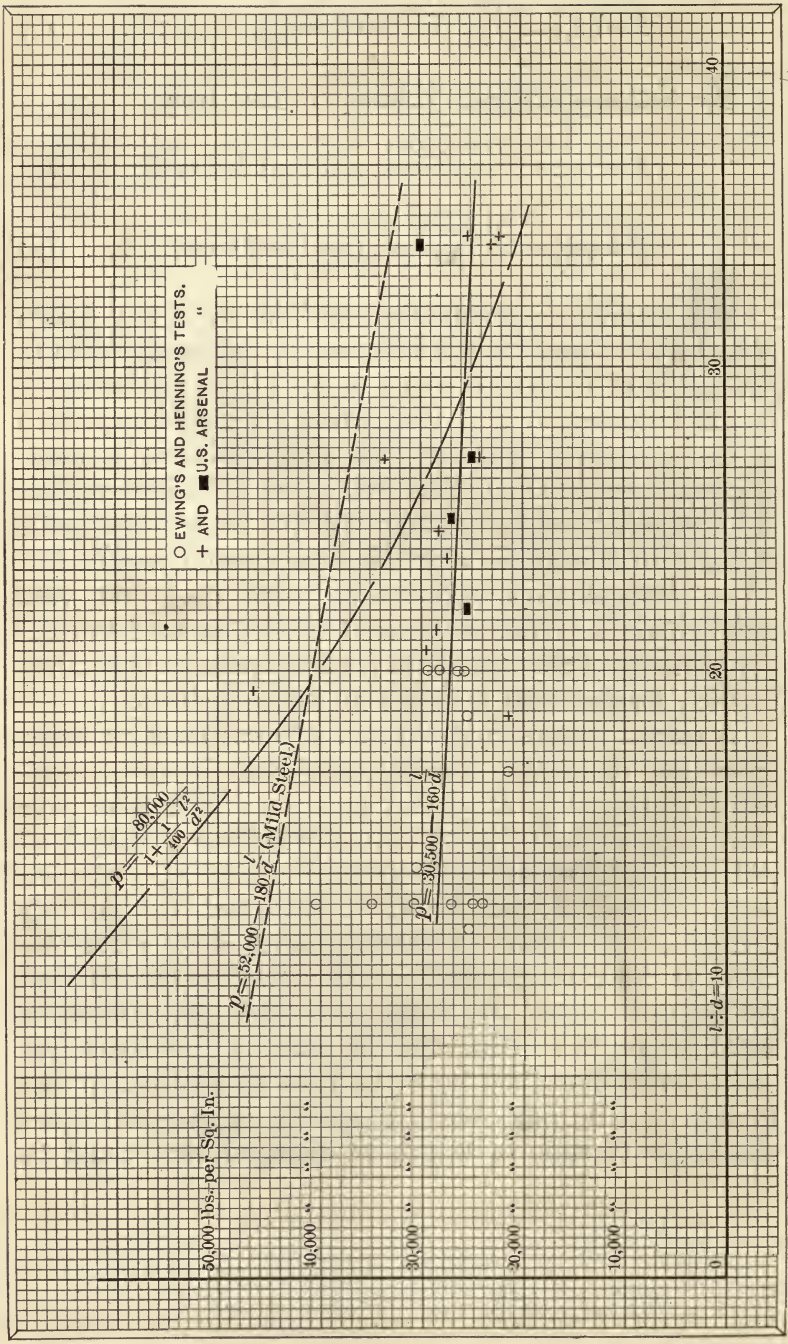
$$P = 99{,}320 \frac{D^{3.55} - d^{3.55}}{l^{1.7}}; \text{ (for fixed ends).} \quad . \quad . \quad (4)$$

The working or maximum load allowed in any design of cast-iron columns would be found by taking one fifth to one eighth of the values given in eqs. (1) to (4) inclusive. It will be observed that Hodgkinson's formulæ expressed in the preceding equations are simply Euler's formulæ as given in eqs. (6) and (9) of Art. 35. with the introduction of an empirical coefficient and with the indices of d and l changed to harmonize with the experimental results.

As Hodgkinson's experiments were made on very small columns of different metal from that used in cast-iron columns of the present day, his formulæ cannot safely be used for practical purposes at the present time.

A correct formula for cast-iron columns must be based upon tests of full-size columns cast with the metal ordinarily employed in structural practice. Such tests have been made at the U. S. Arsenal at Watertown, Mass., and will be found reported in H. R. Ex. Doc. No. 45, 50th Congress, 2d Session, and in H. R. Ex. Doc. No. 16, 50th Congress, 1st Session. A valuable series of tests was also made at Phœnixville, Pa., at the works of the Phœnix Bridge Co., under the auspices of the Department of Buildings of New York City in 1896–97. Although the entire series, including both the tests at Watertown and Phœnixville, do not cover the variety of sectional forms and range of ratio of length to diameter that could be

PLATE I.


○ EWING'S AND HENNING'S TESTS.
+ AND ■ U.S. ARSENAL "
$p = \frac{80,000}{1 + \frac{1}{400}\frac{l^2}{d^2}}$
$p = 52,000 - 180\frac{l}{d}$ (Mild Steel)
$p = 30,500 - 160\frac{l}{d}$
50,000 lbs. per Sq. In.
40,000 "
30,000 "
20,000 "
10,000 "
0
$l : d = 10$
20
30
40

View of failure of a cast-iron column tested in the 1000-ton testing-machine of the Phœnix Iron Co. at Phœnixville, Pa.

(*To face page* 522.)

desired, the results are sufficiently extended to show closely what may be considered the proper ultimate values for hollow round cast-iron columns of full size.

TABLE I.

No.	Length in Inches.	Diameter in Inches.				Area of Section in Square Inches.	Length over Exterior Diameter	Ultimate Resistance in Pounds per Square Inch.
		Large End.		Small End.				
		Ext.	Int.	Ext.	Int.			
1	190.25	15	13	——	——	43.98	12.7	30,830
2	"	15	12.75	——	——	49.03	12.7	27,126
3	"	15	12.75	——	——	49.03	12.7	24,434
4	"	15⅛	12.75	——	——	49.48	12.7	25,182
5	"	15	12.66	——	——	50.91	12.7	35,435
6	"	15	12.63	——	——	51.52	12.7	40,411*
7	160	8	6	——	——	21.99	20	29,604
8	160	8	5.91	——	——	22.87	20	28,229
9	120	6.06	3.78	——	——	17.64	20	25,805
10	120	6.09	3.96	——	——	17.37	20	26,205
11	147.75	8	6.5	——	——	17.08	18.5	25,973
12	150	9	7	——	——	25.14	16.7	21,183
13	162	12	10	——	——	34.55	13.5	30,813
14	159.75	14	12	——	——	40.84	11.4	25,400
15	169	5	4.54	——	——	3.5	34	29,854
16	157	7.17	4.83	——	——	21.8	22	25,470
17	157	6.35	3.9	——	——	17.28	25	27,210
18	156	5.8	4.03	——	——	13.22	27	25,100
19	142.6	7.68	5.52	5.94	4.3	17.49	26.7	29,310
20	146.8	8.01	5.58	5.9	4.35	18.65	21.3	28,520
21	150	6.17	4.85	5.09	3.48	12.08	27	33,500
22	145.5	6	4.35	4.74	2.73	12.81	37.1	24,620
23	133.6	6.02	4.36	4.84	2.88	12.87	24.6	28,060
24	129.3	6.03	4.35	4.87	2.95	12.87	23.7	27,350
25	127.6	7.47	5.97	5.72	4.62	12.13	19.3	46,660
26	118.5	3.98	1.96	2.97	1.49	7.16	34.1	23,090
27	119	3.98	1.96	2.98	1.47	7.17	34.3	22,040
28	118	3.97	1.95	2.99	1.39	7.20	34.2	25,060
29	84.6	4.88	3.03	4.27	2.08	11.25	18.5	31,190

* Not broken.

Table I shows the results of all these tests, while the Plate exhibits the same results graphically. The tests Nos. 1 to 10 inclusive were made at Phœnixville in December, 1897, and Nos. 11 to 14 inclusive in 1896; the former group under the immediate direction of Mr. W. W. Ewing, and the latter under the immediate direction of Mr. Gus C. Henning. The results shown for tests 15 to 18 inclusive were taken from H. R. Ex. Doc. No. 45, 50th Congress, 2d Session, but those for Nos. 19 to 29 inclusive are either taken or digested from H. R. Ex. Doc. No. 16, 50th Congress, 1st Session, being portions of reports of tests of metals and other materials at the United States Arsenal, Watertown, Mass.

As Table I shows, the columns Nos. 19 to 29 inclusive were slightly conical, although probably not enough so to affect appreciably their resistances. The areas of section in square inches for these columns were taken at mid-distance between their ends. As the area of section varied considerably in some columns that operation may be a source of a little error in determining the ultimate resistance per square inch from the result of the tests, but if the error exists at all it must be very small. The mid-external diameter was also taken for these columns in determining the ratio of the length over the diameter shown in the Table and in the Plate.

As will be observed both in the Table and in the Plate, the ultimate resistances per square inch determined by the tests are quite variable, even for the same ratio of length over diameter. Indeed, in a number of cases they are quite erratic. In Nos. 1 to 6, for which the ratio of length over diameter was 12.7, the ultimate resistances vary from a little over 24,000 lbs. per square inch to over 40,000 lbs. per square inch with no failure at the latter value. Again, the ultimate resistance per square inch for No. 25, which shows a ratio of length over diameter of less than 20, is nearly 47,000 lbs. per square inch, which is excessively high as compared with other ultimate resistances with the same or less ratio of length over diameter.

These erratic results are not surprising in view of the ordinary character of the metal. It should be remembered that the failures of these columns are frequently recorded with such "remarks" as the following: "Foundry dirt or honey-comb between inner and outer surfaces," "bad spots," "cinder pockets and blow holes near middle of column," "flaws and foundry dirt at point of break." In other words, it was no uncommon feature to observe that defects, flaws, or blow holes or thin metal had determined the place of failure. There is considerable uncertainty in platting the results of tests affected by these abnormal conditions, but a more or less satisfactory law for the generality of cases may be determined from a graphical representation of the results, as shown on Plate I. On that Plate the ultimate resistances in pounds per square inch, as shown in Table I, have been platted as vertical ordinates, while the ratios of length over diameter given in the same Table are represented by the horizontal abscissas, all as clearly shown. The full straight line drawn in about a mean position among the results of the tests probably represents as near as any that can be found a reasonable law of variation of ultimate resistance with the ratio of length over diameter. It is evident that within the range of these experiments a straight line will represent the ultimate resistances fully as well as any curve, if not better, although the results for the lengths of thirty-four times the diameter begin to indicate a little curvature. The formula which represents this straight line, i.e., which gives the ultimate resistance per square inch, is as follows:

$$p = 30,500 - 160\frac{l}{d}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is to be borne in mind that these columns were round and hollow, and that they were tested with flat ends in all cases. The ordinary formula, based upon Hodgkinson's tests, and frequently used in cast-iron column construction, is as follows:

$$p = \frac{80000}{1 + \frac{1}{400}\frac{l^2}{d^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The curve corresponding to this particular form of Tredgold's formula is also shown on the Plate. It will be seen that at the ratio of length over diameter of 10 to 12 (not an uncommon ratio) the ultimate, as given by this formula, is just about double that shown by actual test. In other words, if a safety factor of 5 were required, as is the case in some building laws, the actual safety factor would be but 2½. The curve represented by eq. (6) is seen to cross the true curve at a ratio of length over diameter of about 29. A glance at the Plate will show how erroneous and dangerous is the use of the usual formula for hollow round cast-iron columns; indeed, that formula is grossly wrong, both as to the law of variation and the values of ultimate resistance.

In view of the working resistances, which have been used in the design of cast-iron columns, it is no less interesting than important to compare the ultimate resistances per square inch of mild-steel columns, as determined by actual tests, with the ultimate resistances of cast-iron columns, as shown by the tests under consideration. The broken line of short dashes represents the formula

$$p = 52{,}000 - 180\frac{l}{d} \text{ - - - - - -} . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

determined by actual tests of mild-steel angles made by Mr. James Christie at the Pencoyd Bridge Works, and given in Art. 60. This line or formula shows that the ultimate resistances per square inch of mild-steel columns are from 40 to 50% greater than the corresponding quantities for cast-iron, the same ratio of length over diameter being taken in each comparison.

When the erratic and unreliable character of cast-iron columns is considered, it is no material exaggeration to state that these tests show that the working resistance per square inch may be taken twice as great for mild-steel columns as for cast-iron; indeed, this may be put as a reasonably accurate statement.

The series of tests of cast-iron columns represented in the Plate constitute a revelation of a not very assuring character in reference to cast-iron columns now standing, and which may be loaded approximately up to specification amounts. They further show that if cast-iron columns are designed with anything like a reasonable and real margin of safety the amount of metal required dissipates any supposed economy over columns of mild steel.

If the average working stress per square inch is one fourth of the ultimate resistance, eq. (5) gives

$$p = 7600 - 40\frac{l}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the working stress is to be taken at one fifth the ultimate, eq. (5) gives

$$p = 6100 - 32\frac{l}{d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In these equations p is the average working intensity of pressure in pounds per square inch. The length l and the exterior diameter d must be taken both in the same unit, ordinarily the inch.

These formulæ may be used between the limits of $\frac{l}{d} = 10$ and $\frac{l}{d} = 35$ or even 40. They may also be applied to hollow rectangular columns with reasonably close approximation, *d* being taken as the smaller exterior side of the rectangular cross-section.

Art. 86.—Timber Columns.

The greater part of available tests of full-size timber columns have been made prior to 1900, and their results have not been obtained either by the aid of improved appliances in testing now employed, or in all respects under the care given in later testing work to secure accuracy or to avoid misinterpretation of the more or less obscure conditions which attend the testing of full-size timber members.

The ratio of the length divided by the radius of gyration is much less in timber columns than those of iron or steel. Furthermore, as sections taken at right angles to the axes of timber columns are almost always rectangular, it is permissible to use the ratio of the length over the least side rather than the length over the least radius of gyration, gaining thereby a little simplicity in the use of column formulæ.

Timber columns are subject to the same vicissitudes of knots, wind-shakes, season cracks and decay as other timber members. Indeed most failures of full-size timber members are induced by some local defect such as a knot, either decayed or sound. Unless in a thoroughly protected place, timber columns are in a condition of almost constant change and in the long run for the worse.

The degree of seasoning is an element of material effect in the resistance of timber columns. The greater the amount of moisture in timber, the less will be its capacity for compressive resistance, other conditions remaining unchanged. As in all other full-size timber tests, the condition of moisture should be known and stated in connection with the results of timber column tests. It makes little or no difference whether the moisture is the original sap or the result of a damp atmosphere or immersion in water.

Among the earliest tests were those of Professor Lanza, who investigated timber mill columns, mostly of circular section and some of them after standing in use in completed buildings for various periods from one year to twenty-five years. These columns varied in length from about 2 to 14 feet, the great majority of them being from 11 to 14 feet. The diameters varied generally from about 5 inches to about 11 inches. A few were square. Neither the shape nor the dimensions of cross-sections appeared to affect materially the results of tests. The principal results of these tests are given in the tabulated statement below:

	Max. Lbs. per Sq. In.	Mean. Lbs. per Sq. In.	Min. Lbs. per Sq. In.
Yellow pine, partially seasoned	5,450	4,370	3,510
Yellow pine, air seasoned.....	4,892	4,690	4,488
Yellow pine, dock seasoned...	5,950	4,563	3,477
White wood, partially seasoned	3,333	3,010	2,687
White oak, partially seasoned.	3,786	3,070	1,964
White oak, in mill 6½ years...	6,029	4,170	2,945
White oak, in mill 25 years...	6,147	4,420	3,266
White oak, thoroughly seasoned	4,450	3,175	1,865

The ends of these columns were usually flat, sometimes with a so-called " pintle " or, in a few cases, one end round. These results show the usual erratic features of full-size timber tests, some of which doubtless are due to undiscovered weaknesses at some point. Prof. Lanza stated that " The immediate location of the fracture was generally determined by knots." Some of the columns were tapered and the reduction of the section at the ends of such columns usually located the failure at those reduced ends.

The greatest ratio of length to radius of gyration in these columns was about 86, but the actual results did not show that there was any discoverable relation between the ratio of the length over the radius of gyration and the ultimate column resistance. The latter was influenced little or none by the length of the columns.

Tables I and II show the results of the early tests of Col. Laidley, Engineer Corps, U. S. A., made many years ago and reported in "Ex. Doc. 12, 47th Congress, 1st Session." They show the large increase in ultimate resistance per square inch with short lengths. Indeed some of the pieces were short blocks. These results indicate the care that should be taken in discriminating between the ultimate compressive resistances of short timber blocks and long columns. The results in Table I for those pieces seasoned twenty years are too high, while those for pieces Nos. 16, 17, and 18 are low, in consequence of the material

TABLE I.

YELLOW PINE.

No.	Length, Inches.	Form of Section.	Section Dimensions, Inches.	Ultimate Resistance per Sq. In.	
				Lbs.	
1	20.4	Circular.	10.2 diam.	6,676	Straight-grained and seasoned 20 years.
2	119.95	Square.	11 × 11	6,230	
3	119.90	“	11 × 11	6,552	
4	20.0	“	10.4 × 10.4	7,936	
5	16.0	“	8 × 8	8,165	
6	8.0	“	4 × 4	7,394	
7	3.0	“	1.5 × 1.5	5,533	
8	6.0	“	3 × 3	8,644	
9	6.0	“	3 × 3	8,133	
10	3.0	“	1.5 × 1.5	8,389	
11	3.0	“	1.5 × 1.5	8,302	
12	3.0	“	1.5 × 1.5	6,355	
13	14.0	“	4.6 × 4.6	9,947	
14	17.2	“	4.6 × 4.6	10,250	
15	19.1	“	5.3 × 5.3	7,820	
16	180.0	Rectangular.	16 × 13.65	3,070	
17	180.0	“	16.2 × 7.0	2,795	
18	180.0	“	17 × 8.75	3,180	

Nos. 13, 14, and 15 were pine of very slow growth.
Nos. 16, 17, and 18 were very green and wet.

TABLE II.

SPRUCE THOROUGHLY SEASONED.

No.	Length, Inches.	Form of Section.	Section Dimensions, Inches.	Ultimate Resistance per Sq. In.
				Lbs.
1	24	Rectangular.	5.4×5.4	4,946
2	24	"	5.4×5.4	4,811
3	36	"	5.4×5.4	4,874
4	36	"	5.4×5.4	4,500
5	60	"	5.4×6.4	4,451
6	60	"	5.4×6.4	4,943
7	120	"	5.4×5.4	3,967
8	120	"	5.4×5.4	4,908
9	60	"	5.4×5.4	5,275
10	30	"	5.4×5.4	5,372
11	15	"	5.4×5.4	5,754
12	121.2	Circular.	12.4 diam.	4,681

being green and wet. The tests pieces in Tables I and II were generally fine straight-grained timber of better quality than ordinarily used in engineering practice.

This condition accounts largely for Col. Laidley's results, being materially higher than Prof. Lanza's for the same kind of timber.

Formula of C. Shaler Smith.

Although these formulæ were deduced from tests made many years ago, they have been so extensively used over such a long period that they may properly be considered among the classics of engineering literature of this kind. Hence they are given here, although not now used so widely as formerly.

The tests of full-size sticks on which the formulæ are based were grouped by Mr. Smith as indicated and the corresponding formulæ are as given below.

"1st. Green, half-seasoned sticks answering to the specification 'good, merchantable lumber.'

"2d. Selected sticks reasonably straight and air-seasoned under cover for two years and over.

"3d. Average sticks cut from lumber which had been in open-air service for four years and over."

If l = length of column in inches,
d = least side of column section in inches,
and p = Ult. Comp. resistance in lbs. per sq. in.;

then the formulæ found for these three groups were:

$$\text{For No. 1: } p = \frac{5400}{1 + \frac{1}{250}\frac{l^2}{d^2}}.$$

$$\text{For No. 2: } p = \frac{8200}{1 + \frac{1}{300}\frac{l^2}{d^2}}.$$

$$\text{For No. 3: } p = \frac{5000}{1 + \frac{1}{250}\frac{l^2}{d^2}}.$$

But in order to provide against ordinary deterioration while in use, as well as the devices of unscrupulous builders, Mr. Smith recommends the formula for group No. 3 as the proper one for general application. He also recommended that the factor of safety be $\sqrt{\frac{l}{d}}$ until 25 diameters are reached, and *five* thenceforward up to 60 diameters. This last limit he regards as the extreme for good practice.

Tests of White Pine and Yellow Pine Full-size Sticks with Flat Ends.

In consequence of the usual manner of simply abutting the end of timber columns against their supports, all such members are practically always assumed to have flat ends, but this expression does not mean accurately squared "flat ends." Tables III and IV have been formed by digesting the results of tests of nearly or quite full-size white and yellow pine timber columns made at the U. S. Arsenal at Watertown, Mass., and reported in "Ex. Doc. No. 1, 47th Congress, 2d Session," constituting one of the best series of timber column tests yet made in this country.

Each result in both Tables is usually a mean of from two to four tests, although a few belong to one test only. All timber, both of yellow and white pine, was ordinary merchantable material, with about the usual defects, knots, etc., and failure frequently took place at the latter; it was all well seasoned, and all columns were tested with flat ends.

Table III.

YELLOW-PINE COLUMNS WITH FLAT ENDS.

Length.		Size of Stick, Inches.	$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.	Length.		Size of Stick, Inches.	$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.
Ft.	Ins.				Ft.	Ins.			
15	0	8.25 × 16.25	21.7	3,445	15	0	5.0 × 12	35.6	3,764
10	0	5.5 × 5.5	22	4,738	23	4	7.7 × 9.7	36.4	3,304
16	8	7.7 × 9.7	26.7	4,384	17	6	5.5 × 5.5	38.2	3,242
15	0	6.6 × 15.6	27.0	3,593	15	0	4.5 × 11.6	41	2,462
12	6	5.5 × 5.5	27.3	5,077	26	8	7.4 × 9.4	43	2,893
15	0	5.9 × 12.0	30.8	3,546	15	0	4.0 × 11.4	44	3,065
20	0	7.6 × 9.6	31.2	3,496	20	0	5.4 × 5.4	44.3	2,867
15	0	5.7 × 11.7	31.9	3,106	22	6	5.5 × 5.5	50	2,065
15	0	5.6 × 15.6	32.1	3,656	25	0	5.5 × 5.5	55	1,856
15	0	5.5 × 5.5	32.8	3,962	27	6	5.3 × 5.3	62.3	1,709

Flat-end yellow-pine columns were observed to begin to fail with deflection at a length of about $22d$, d being the width or least dimension of the normal cross-section. All columns were of rectangular section, and l in the following table is the length. Table III, therefore, includes no short column, i.e., one which failed by compression alone with no deflection.

About sixteen of the latter were tested with the following results:

Short yellow-pine columns; $l \div d$ below 22.........	maximum	= 5,677 lbs. per sq. in.
	mean	= 4,442 " " "
	minimum	= 3,430 " " "

Each of the preceding tests was made on a single rectangular stick. A number of tests, however, were made on compound columns formed by bolting together from two to three rectangular sticks, with bolts and packing or separating blocks at the two ends and at the centre. The bolts were parallel to the smaller sectional dimensions of the component sticks. As was to be expected, those compound columns possessed essentially the same ultimate resistance per square inch as each component stick considered as a column by itself, as the following results show. l is the length of the column and d the smallest dimension or width of one member of the composite column. All had flat ends.

$l \div d$.	Number of Tests.		
32.1	18	maximum	= 4,559 lbs. per sq. in.
		mean	= 3,841 " " "
		minimum	= 2,756 " " "
36	18	maximum	= 3,357 " " "
		mean	= 3,122 " " "
		minimum	= 2,942 " " "

Table IV gives the results for white-pine columns, and corresponds with Table III, in that it shows only the failures with deflection, which was observed to begin with those columns at a length of $32d$. l and d possess the same

TABLE IV.

WHITE-PINE COLUMNS WITH FLAT ENDS.

Length.	Size of Stick, Inches.	$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.	Length.	Size of Stick, Inches.	$\frac{l}{d}$:	Ultimate Compressive Resistance, Lbs. per Sq. In.
Ft. Ins.				Ft. Ins.			
15 0	5.6×15.6	32	1,874	17 6	5.4×5.4	40	1,841
20 3	7.4× 9.3	32.4	2,448	26 8	7.5×9.3	42.7	2,113
15 0	5.6×11.5	32.7	2,432	20 0	5.3×5.3	45	1,455
15 3	5.4× 5.4	33	2,744	22 6	5.2×5.2	52	1,501
23 4	7.7× 9.6	36.4	2,072	25 0	5.3×5.3	57	952
15 0	4.5×11.6	40	1,672	27 6	5.4×5.4	62	1,081

signification as in Table III, the column $l \div d$ showing the ratios between the lengths and least widths.

Thirty columns with lengths less than $32d$ were tested to destruction. These sticks failed generally at knots by direct compression and without deflection. The results of these thirty tests were as follows:

Short white-pine columns; $l \div d$ below 32........ { maximum = 3,700 lbs. per sq. in.
mean = 2,414 " " "
minimum = 1,687 " " "

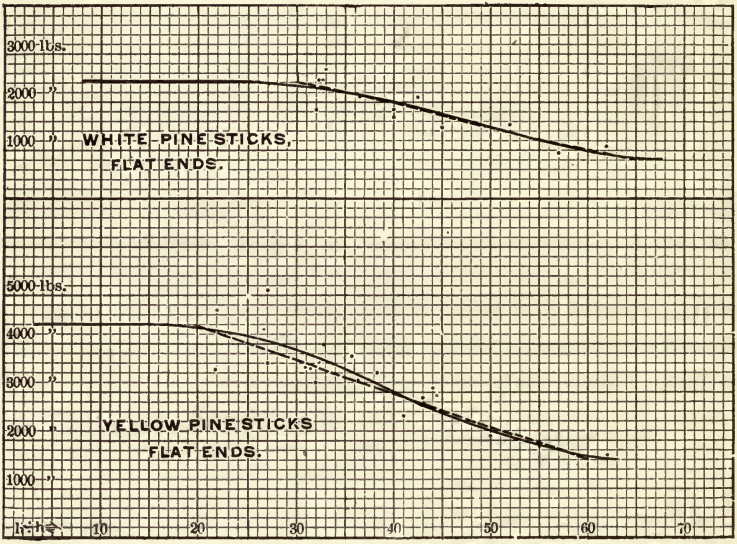
All the preceding white-pine columns were single sticks, but a large number of built posts composed of two to four white-pine sticks bolted together, with spacing blocks at the two ends and at the centre, were also tested with the results shown below. $l \div d$ is the ratio of length over least width of a single stick of the set forming the composite column.

$l \div d$.	Number of Tests.	
32.1........	15.......	maximum = 2,273 lbs. per sq. in. mean = 1,980 " " " minimum = 1,661 " " "
36...........	9.......	maximum = 2,255 " " " mean = 1,999 " " " minimum = 1,804 " " "
40...........	6.......	maximum = 2,021 " " " mean = 1,830 " " " minimum = 1,419 " " "

A comparison of these results with those given in Table IV shows that these composite or built columns were the same in strength per square inch with the single sticks of which they were composed, the latter being considered single columns.

All the white-pine composite columns were tested with flat ends and were built up with the greatest widths of individual sticks adjacent to each other.

Plate F.



The results in Tables III and IV are shown graphically in Plate F. One ordinate gives the values of $l \div d$, and the other the ultimate resistance in pounds per sq. in.

The full curved lines running into horizontal tangents at the left represent about mean lines through the points indicating the actual column tests.

The broken lines represent the following empirical formulæ; in which p is either the ultimate resistance or working stress in pounds per sq. in.

For yellow pine . . . $p = 5800 - 70(l \div d)$
" white " . . . $p = 3800 - 47(l \div d)$

For wooden railway structures there may be used:

For yellow pine . . . $p = 750 - 9(l \div d)$
" white " . . . $p = 500 - 6(l \div d)$

For temporary structures, such as bridge false works *carrying no traffic:*

For yellow pine . . . $p = 1500 - 18(l \div d)$
" white " . . . $p = 1000 - 12(l \div d)$

The preceding formulæ are to be used only between the limits of $\frac{l}{d} = 20$ *and* $\frac{l}{d} = 60$ *for yellow pine and* $\frac{l}{d} = 30$ *and* $\frac{l}{d} = 60$ *for white pine.*

For short columns below $\frac{l}{d} = 20$ and $\frac{l}{d} = 30$ there are to be used for yellow and white pine respectively:

	Ultimate.	Railway Bridges.	Temporary Structures.
Yellow pine....	$p = 4400$........	550........	1100 lbs. per sq. in.
White "	$p = 2400$........	300........	600 " " "

All the preceding values are applicable to good average lumber for the engineering purposes indicated.

Table V exhibits a number of results of the tests of short timber columns taken from the "U. S. Reports of Tests of Metals and Other Materials" for 1894, 1896, 1897, and 1900. It will be observed that the ratios of length over thickness, i.e., minimum dimension of cross-section, are less than 22, and with two exceptions much less. These columns do not, therefore, come within the range of application of such formulæ as those given on the preceding page for yellow pine and white pine.

TABLE V.

SHORT TIMBER COLUMNS.

Timber.	Dimensions, Inches.			$\frac{l}{d}$.	Ultimate Compressive Resistance, Lbs. per Sq. In.			No. of Tests.	
	Leng'h	Bre'dth	Thickness.		Max.	Mean.	Min.		
Long-leaf pine....	120	9.8	9.8	12	4,976	4,574	4,200	5	Butt sticks.
" "	120	9.8	9.8	12	3,800	3,558	3,369	5	Top "
" "	120	9.8	9.8	12	4,200	3,957	3,714	2	Middle "
Short-leaf pine...	120	9.8	9.8	12	3,925	3,481	3,037	2	Butt "
" " ...	120	9.8	9.8	12	3,400	3,000	2,600	2	Top "
" " ...	120	9.8	9.8	12	4,000	3,568	3,135	2	Middle "
Spruce..........	120	9.6	9.6	12	3,174	2,589	1,900	10	
Long-leaf pine*...	131	9.5	9.5	14	7,354	6,093	4,960	12	Old posts.
Cypress.........	120	8	8	15	3,457	3,308	3,113	3	
White pine......	58	9.5	7.9	8	——	3,652	——	1	Probably 170 years old.
" "	71	9.5	7.9	9	——	2,917	——	1	
Red oak.........	48	4 to 6	3.5	14	6,247	5,160	2,917	5	
Douglas fir.......	60	14	2.8	22	——	6,211	——	1	
" ".......	60	8 & 14	3	20	7,882	6,725	5,568	2	
" ".......	60	12	4.1	15	——	6,220	——	1	
White oak.......	121	10	8	15	——	3,697	——	1	
" "	106	10 & 12	10	11	4,230	4,214	4,197	2	
" "	74	7.5	7.9	10	——	4,372	——	1	
" "	69	10	8	9	——	4,042	——	1	

* Well seasoned and dry; 12 years old. Had been in a fire and corners were partially charred.

All posts represented in this table contained probably 15 to 18 per cent. of moisture, or perhaps more.

The long-leaf and short-leaf pine tests show that columns taken from the butts of trees are stronger than those taken either from the middle or the tops, the top sticks, as a rule, having the least ultimate resistance per square inch of all. The white-pine and red-oak sticks yield interesting results on account of their age, as they were taken from some wooden trusses of the Old South Church, Boston, Mass., a building constructed in 1729. The timber was so housed as to be completely protected and kept very dry. The results show no loss of resistance as compared with tests of the same kind of timber at the present time.

The effect of immersion in water on the resistance of timber is illustrated by tests made at the Watertown Arsenal. A post similar to one of the old long-leaf pine columns, 12 of which were tested in a seasoned condition giving the average shown in the Table of 6093 pounds per square inch, was submerged in water for a period of 130 days and then tested with the result of failing at 3800 pounds per square inch.

The values given in Table V correspond closely to the results shown for yellow pine and white pine on pages 534 and 535, so far as they may properly be compared.

CHAPTER XI.

SHEARING AND TORSION.

Art. 87.—Modulus of Elasticity.

It has already been shown in some of the Articles of the first part of this book that the stresses of shearing and torsion are identical, both being shears; hence the modulus of elasticity is the same for both.

As it is much more convenient to make accurate determinations of the modulus of elasticity in torsion than in direct shearing, the former method has been employed in practically all cases. A number of such moduli for four varieties of steel are given in Art. 38. Those values show that the modulus changes but little for the different varieties of steel indicated.

The aggregate of torsion tests so far as they have been made indicate that the two moduli of elasticity, G for shear and E for direct stresses of tension and compression, have the approximate relation:

$$G = (.4 \text{ to } .45)E.$$

Prof. Bauschinger published in "Der Civilingenieur," Heft 2, 1881, the results of some of his tests of cast-iron cylinders or prisms which are still valuable on account of the accuracy with which he made his determinations. The prisms were about 40 inches long, and were subjected to torsion, while the twisting of two sections about 20 inches apart, in reference to each other, was carefully observed. The results for four different cross-sections will be given—i.e., circular, square, elliptical (the greater axis was twice the less), and rectangular (the greater side was twice the less). In each case the area of cross-section was about 7.75 square inches. The angle α is the angle of torsion—i.e., the angle twisted or turned through by a longitudinal fibre whose length is unity and which is at unit's distance from the axis of the bar.

Section.	α.	G.
Circular	0.007 degree	7,466,000 lbs. per sq. in.
	0.07 "	6,157,000 " " "
Elliptical	0.009 "	7,437,000 " " "
	0.076 "	6,228,000 " " "
Square	0.008 "	7,039,000 " " "
	0.073 "	5,987,000 " " "
Rectangular	0.01 "	6,996,000 " " "
	0.08 "	5,716,000 " " "

The formula by which G is computed, when the torsional moment and angle α are given, is the following:

$$G=\frac{M}{\alpha}\cdot c\frac{I_p}{A^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

in which M is the twisting moment, A the area of the cross-section, I_p the polar moment of inertia of that cross-section, and c a coefficient which has the following values

$4\pi^2 = 39.48$ for circle and ellipse,
42.70 " square,
42.00 " rectangle,

as shown in Appendix I.

Bauschinger's experiments show that the coefficient of shearing elasticity for cast iron may be taken from 6,000,000 to 7,000,000 pounds per square inch; also that it varies for different ratios between stress and strain.

It has been shown in Art. 6, that if E is the coefficient of elasticity for direct stress, and r the ratio between direct and lateral strains, for tension and compression, that G may have the following value:

$$G = \frac{E}{2(1+r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Prof. Bauschinger, in the experiments just mentioned, measured the direct strain for a length of about 4 inches, and the accompanying lateral strain along the greater axis of the elliptical and rectangular cross-sections, and thus determined the ratio r between the direct and lateral strains per unit in each direction. The following were the results:

COMPRESSION.

Section.	r.	G.
Circular	0.22	6,541,000 lbs. per sq. in
Elliptical	0.23	6,541,000 " " "
Square	0.24	6,442,000 " " "
Rectangular	0.24	6,499,000 " " "

TENSION.

Circular	0.23	6,570,000 lbs. per sq. in.
Elliptical	0.21	6,811,000 " " "
Square	0.26	6,399,000 " " "
Rectangular	0.22	6,527,000 " " "

The values of E are not reproduced, but they can be calculated indirectly from eq. (2) if desired.

It is seen that the values of G, as determined by the different methods, agree in a very satisfactory manner, and thus furnish experimental confirmation of the fundamental equations of the mathematical theory of elasticity in solid bodies.

The fact that G is essentially the same for all sections is also strongly confirmatory of the theory of torsion in particular.

These experiments show that, for cast iron, the lateral strains are a little less than one quarter of the direct strains. If r were one quarter, then $G = \frac{2}{5}E$, or $E = \frac{5}{2}G$.

Art. 88.—Ultimate Resistance.

It has seemed more convenient to give some values of ultimate and working resistances for the materials iron and steel which are much more commonly used than any others to resist torsion in Arts. 37 and 38, where the complete analyses of the formulæ for the common theory of torsion are given. Those articles should, therefore, be consulted for such formulæ and analytic operations as are involved in the design of shafting to resist torsion. The experimental values set forth in the following articles may be employed in the formulæ of the common theory of torsion for any desired practical operation in the design of torsion members.

Before considering the ultimate shearing resistance of special materials it will be well to notice the two different methods in which a piece may be ruptured by shearing.

If the dimensions of the piece in which the shearing force or stress acts are very small, i.e., if the piece is very thin, the case is said to be that of "simultaneous" shearing. If the piece is thick, so that those portions near the jaws of the shear begin to be separated before those at some distance from it, the case is said to be that of "shearing in detail." In the latter case failure extends gradually, and in the former takes place simultaneously over the surface of separation. Other things being the same, the latter case (shearing in detail), will give the least ultimate shearing resistance per unit of the whole surface.

In reality no plate used by the engineer is so thin that the shearing is absolutely simultaneous, though in many cases it may be essentially so.

Wrought Iron.

There may be found in the Articles on Riveted Joints some experimental determinations of the ultimate shearing resistance of wrought iron which, under the conditions of such joints, may range from about 34,000 to about 43,000 pounds per square inch. It has been observed in the consideration of riveted joints that the ultimate resistance to shear of rivets will generally be less with thick plates than with thin, because the bending stresses of tension and compression will generally be greater for thick plates than for those that are thinner. If the riveted joint is so designed that the bending stresses are not greater for thick plates than for thin ones, the effects of bending will necessarily disappear.

Such tests as have been made on direct shearing resistance show that generally it may safely be taken at 35,000 to 40,000 pounds per square inch, or if S is the ultimate shear per square inch and T the ultimate tensile resistance of wrought iron per square inch, there may be taken approximately

$$S = .8T.$$

Cast Iron.

There are few tests available for the determination of the ultimate shearing resistance of cast iron. For the ordinary grades, such as cast-iron water pipes and similar soft gray-iron castings, the ultimate shearing resistance has sometimes been taken equal to the ultimate tensile resistance, i.e., 15,000 to 18,000 pounds per square inch, but this is probably too high except for the special stronger grades of material.

For general purposes it is probably safe to take the ultimate shearing resistance of cast iron about three-quarters of its ultimate tensile resistance. It should only be used for shearing, however, at a low working stress, depending obviously on the purpose for which its use is contemplated.

Steel.

The results of Prof. Ricketts' shearing tests on both open-hearth and Bessemer steel rounds with different grades of carbon are given in Table I of Art. 43. The elastic limit is the point at which the metal first fails to sustain the scale beam. The double-shear resistance in one case exceeds the single by over six per cent. According to these tests, the ultimate shearing resistance of mild steel may be taken at three quarters of its ultimate tensile resistance. Each shear result is a mean of three tests.

The rivet steel was low, containing but .09 per cent. of carbon. While the specimens of Bessemer steel were a little higher in carbon, ranging from .11 to .17 per cent., except the last six, they were also of low or medium steel. It should be carefully noted that the results in that table show that the ultimate shearing resistances for the low or medium steels running from 44,600 pounds per square inch up to 53,260 pounds per square inch are closely three fourths the corresponding ultimate tensile resistances. On the other hand, the six specimens of high steel give ultimate shearing resistances but little over two thirds of the corresponding ultimate tensile resistances. This is a feature of the relation between the ultimate shearing and ultimate tensile resistances of different grades of steel which is commonly exhibited in tests. The high steel appears to yield an ultimate shearing resistance of sensibly less percentage of the tensile ultimate than low steel.

In the Arts. 74 and 76 on riveted joints there will be found a number of values of ultimate resistance for steel rivets in shear. They constitute important determinations of the ultimate shearing resistance of steel rivets under conditions in which they are frequently used.

Copper, Tin, Zinc, and Their Alloys.

The following values of the ultimate resistance to torsive shear T_m, were determined by Prof. R. H. Thurston in his early experimental work on the bronzes. Although these determinations were made on test specimens only .625 inch in diameter and with a torsion length of 1 inch, they constitute practically the only fairly complete shear and torsion data on the copper-tin and copper-zinc alloys.

TABLE I.

Composition.		Ultimate Torsive Shear, T_m.	Elastic Limit, Per Cent of T_m.	Ultimate Torsion Angle.
Cu.	Sn.			
		Pounds.		Degrees.
100	00	35,910	35	153.0
100	00	28,430	40	52 to 154
00	100	3,196	45	557.0
00	100	3,297	33	691.0
90	10	43,943	41	114.5
80	20	47,671	62	16.3
70	30	4,407	100	1.5
62	38	1,770	100	1.0
52	48	686	100	1.0
39	61	5,881	100	1.7
29	71	5,257	100	2.34
10	90	5,761	63	131.8
90	10	25,027	49	57.2
90	10	31,851	57	72.6

T_m is in pounds per square inch.

Table I relates to alloys of copper and tin, and Table II to alloys of copper and zinc.

None but specimens with circular sections were tested.

An examination of the results given in Tables I and II show that the resisting capacities of each series of alloys vary greatly with the varying elements constituting the alloy. Indeed, the shearing resistances of these alloys in torsion are seen to vary as widely as their tensile resistances.

TABLE II.

Percentage of		Ultimate Torsive Shear, T_m	Elastic Limit, Per Cent of T_m.	Ultimate Torsion Angle.
Copper.	Zinc.			
		Pounds.		Degrees.
90.56	9.42	35,100	17.2	458.0
81.90	17.99	41,575	27.5	345.0
71.20	28.54	41,000	24.0	269.0
60.94	38.65	48,520	29.4	202.0
55.15	44.44	52,320	32.7	109.0
49.66	50.14	43,154	36.0	38.0
41.30	58.12	4,588	100.0	1.8
32.94	66.23	7,241	100.0	1.2
20.81	77.63	16,374	100.0	0.8
10.30	88.88	22,500	85.6	7.1
0.00	100.00	9,186	38.1	141.5

Although the values of T_m are the ultimate intensities of torsive shear, they may be accepted as ultimate resistances for direct shear for the same alloys.

Timber.

The shearing resistance of timber is least along planes parallel to the fibres and greatest when the shearing force acts in planes at right angles to the fibres. Again, the shearing resistance parallel to the fibres is somewhat different in short blocks from that found in full-size beams subjected to flexure. In the latter case it has been shown in Art. 15 that the greatest intensity of shearing stress parallel to the fibres will take place in the neutral surface. It has been found that for relatively short spans timber beams in flexure will fail by shear along the neutral surface. Hence the ultimate resistance to shear along that surface has much practical value and it has been determined in tests of many full-size beams. Among the latter those made by Prof. Arthur N. Talbot at the University of Illinois and described in the University of Illinois Bulletin No. 15, December, 1909, are of unusual value. The full-size beams were 13.5 feet to about 14.5 feet span and with cross-sections of 7 inches by 12 inches, 7 inches by 14 inches and 7 inches by 16 inches. Other smaller beams were, however, used. The beams were of sound merchantable lumber and of about the quality used in good engineering work. The following table gives the results of these tests, showing the number of pieces tested to failure with the highest, average and lowest ultimate shear per square inch along the fibres in or near the neutral surface.

Table III.

ULTIMATE RESISTANCES ARE GIVEN IN POUNDS PER SQUARE INCH

Timber.	No. of Pieces.	Ultimate Shearing Stress.		
		Highest.	Average.	Lowest.
Untreated longleaf pine	25	497	370	188
Untreated shortleaf pine	4	505	364	293
Creosoted shortleaf pine	6	410	302	224
Creosoted loblolly pine	8	391	273	224
Untreated loblolly pine	10	388	314	253
Old Douglas fir	10	383	298	221
New Douglas fir	11	401	323	275

The values given in Table III are somewhat smaller than those which Prof. Talbot found for short blocks. The ultimate shearing stress along the fibres of the neutral axis of the full-size beams ranged from 75 per cent. up to 101 per cent. of the corresponding results for short blocks of the same kind of timber. The new Douglas fir gave the highest of these percentages and untreated longleaf pine together with creosoted loblolly pine gave the smallest.

The American Railway Engineering and Maintenance of Way Association has recommended ultimate and working values for shear along the grain and in the neutral surface of beams as given in Table IV of Art. 90.

Natural Stones.

The ultimate shearing resistance of stones has not as great practical value as the ultimate compressive or the ultimate bending resistance, yet there are occasional structural conditions under which it is necessary to ascertain what shearing capacity may be relied upon. Valuable data for this purpose are shown in Table IV taken from the "U. S. Report of Tests of Metals and Other Materials" for 1894 and 1899. The sheared surfaces were about 6 inches by 4 inches in area. Generally one such surface was sheared, but occasionally two.

TABLE IV.

SHEARING RESISTANCE OF NATURAL STONES.

Stone.	Ultimate Shearing Resistance, Lbs. per Square Inch.		
	Maximum.	Mean.	Minimum.
Brandford granite, Conn.	1,925	1,834	1,742
Milford Granite, Mass.	2,872	2,554	2,236
Troy granite, N. H.	2,231	2,219	2,197
Milford pink granite, Mass.	——	1,825	——
Pigeon Hill granite, Mass.	2,047	1,549	1,052
Creole marble, Ga.	——	1,369	——
Cherokee marble, Ga.	——	1,237	——
Etowah marble, Ga.	——	1,411	——
Kennesaw marble, Ga.	——	1,242	——
Marble Hill marble, Ga.	1,501	1,332	1,163
Tuckahoe marble, N. Y.	1,554	1,490	1,426
Mount Vernon limestone, Ky.	2,016	1,705	1,389
Cooper sandstone, Oregon	——	1,831	——
Maynard sandstone, Mass.	1,287	1,204	1,120
Kibbe sandstone, Mass.	1,308	1,150	992
Worcester sandstone, Mass.	1,383	1,243	1,102
Chuckanut sandstone, Wash.	——	1,352	——
Yammerthal limestone, Buffalo.	2,518	2,127	1,735

All the results except the last are taken from the Report for 1894. Where but one value appears in the table one test only was made. In the other cases two tests were made and the mean values are means of the two shown in the columns containing the greatest and least. It will be observed that the ultimate shearing resistance is scarcely more than ten per cent. of the ultimate compressive resistance of the various stones tested.

The greatest permissible working stresses for natural stones in shear, in design work, will necessarily depend on the duty to be performed. In view of the variable character of even the best of natural stones as delivered ready for use, one eighth to one tenth of the ultimate is as much as should be taken in ordinary cases, and materially less than that under some conditions.

Bricks.

The shearing resistance of bricks, like that of natural stones, is seldom employed, but it is sometimes needed. The ultimate resistances of bricks in shearing shown in Table V are taken from the "U. S. Report of Tests of Metals and Other Materials" for 1894.

TABLE V.

BRICKS IN SHEARING.

Kind of Brick.	Ultimate Shearing Resistance, Lbs. per Square Inch.	Number of Sheared Surfaces.
Hydraulic Press Brick Co., St. Louis, No. 6.	1,011	1
" " " " " " 511.	642	2
" " " " " brown.	1,047	1
" " " " Chicago, red.	784	1
Northern Hydraulic Press Brick Co., Minneapolis, dark red.	714	1
Eastern " " " " Philadelphia, 210	1,167	1
" " " " " " 220.	1,097	1
" " " " " " 390.	988	2
Philadelphia and Boston Face Brick Co., Boston, gray.	433	2
" " " " " " " "	639	1

In these shearing tests the sheared surfaces were each about 2.25 by 4 inches in dimensions.

The ultimate shearing resistances in Table V range scarcely 10 to 20 per cent. of the ultimate compressive resistances of the same materials shown in Art. 68.

Working shearing stresses for design operations should not be taken more than one eighth to one tenth of the ultimate values found in Table V.

CHAPTER XII.

BENDING OR FLEXURE.

Art. 89.—Modulus of Elasticity.

THE modulus of elasticity as determined by experiments in flexure can scarcely be considered other than a conventional quantity. If the span of a beam were very long compared with the depth of the beam and if the moduli of elasticity for tension and compression were equal to each other, and if all the hypotheses involved in the common theory of flexure were true, then the modulus of elasticity for flexure would be a real quantity and essentially the same, at least, as that for either tension or compression.

These conditions, however, do not exist in bent beams and the quantity ordinarily called the modulus of elasticity in flexure possesses value chiefly as an empirical factor which enables deflection, independently of shear, to be estimated with sufficient accuracy for all usual purposes.

The formulæ to be employed in the determination of the modulus of elasticity for flexure have already been established in connection with the common theory of flexure and their use will be shown in succeeding articles.

Art. 90.—Formulæ for Rupture.

The formulæ of the common theory of flexure, available for practical use, are true only within the limits of elasticity. In the testing of beams to failure they are employed precisely as if the elastic properties of the material were maintained up to the degree of loading which causes failure.

While this, strictly speaking, is irrational, it is the only satisfactory procedure available. By placing the analytic expression for the moment of the internal stresses in the normal section of a bent beam equal to the moment of the external loading causing failure, the resulting equation may be solved so as to give the apparent ultimate intensity of stress k in the extreme fibres of the beam. The so-called intensity of fibre stress found in this manner is an empirical quantity which may be introduced into the formulæ of the common theory of flexure and so make them applicable to the operations of engineering practice in connection with loaded beams of any shape of cross-section.

If k and k^1 are the greatest intensities of stress in the section of rupture and at the instant of rupture; y the variable normal distance of any fibre from the neutral surface; y_1 and y' the greatest values of y; b the variable width of the section (normal to y); and M the resisting moment at the instant of rupture; then the general formula for rupture by bending, as given by eq. (1) of Art. 26, is

$$M=\frac{k}{y_1}\int_0^{y_1} y^2bdy+\frac{k'}{y'}\int_{-y'}^{0} y^2bdy. \quad . \quad . \quad . \quad (1)$$

This equation is in reality based on the supposition that the moduli of elasticity for tension and compression are not equal. It is rare, however, that such a supposition is made. It is practically the invariable rule to assume the moduli of elasticity for tension and compression to have equal values and such an assumption is fortunately sufficiently accurate for all ordinary purposes.

If the tensile and compressive moduli of elasticity are the same $\frac{k}{y_1}=\frac{k'}{y'}$ and eq. (1) becomes

$$M=\frac{kI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the usual equation of flexure employed so frequently in connection with the design of bent beams or the investigation of their carrying capacity, I being the moment of inertia of the normal section of the beam d_1 the distance of the most remote fibre from the neutral axis of the section and M the moment of the external forces or loading about the neutral axis of the section in question. In the practical use of this formula it is only necessary to introduce the proper values of I and d_1 for the shape of a section involved.

Art. 91.—Beams with Rectangular and Circular Sections.

These are the simplest forms of sections for bent beams employed in engineering work. Timber beams are with few exceptions of rectangular section and so are many reinforced concrete beams, although in such a case the section is composite, i.e., composed of two materials, and it will receive separate treatment in a later article. The solid circular section belongs to pins in pin-connected truss bridges whose design always involves their consideration as a loaded beam of very short span.

The following are the values of I and d_1 for rectangular and circular sections, h being the side of the rectangle normal and b that parallel to the neutral axis, while r is the radius of the circular section and A the area in each case:

$$\text{Rectangular:} \begin{cases} I = \dfrac{bh^3}{12} = \dfrac{Ah^2}{12}, \quad . \quad . \quad . \quad . & (1) \\ d_1 = \dfrac{h}{2}. \end{cases}$$

$$\text{Circular:} \begin{cases} I = \dfrac{\pi r^4}{4} = \dfrac{Ar^2}{4}, \quad . \quad . \quad . \quad . & (1a) \\ d_1 = r. \end{cases}$$

If the beams are supported at each end and loaded by a weight W at the centre of the span (or distance between supports), which may be represented by l, then the moment at the centre of the beam becomes

$$\Sigma Px = M = \frac{Wl}{4}. \qquad (2)$$

There will then result from eq. (2), Art. 89:

For rectangular sections:

$$M = \frac{Wl}{4} = \frac{kbh^2}{6} = \frac{kAh}{6}. \qquad (3)$$

For circular sections:

$$M = \frac{Wl}{4} = \frac{\pi kr^3}{4} = \frac{kAr}{4}. \qquad (4)$$

The quantity k is called the *modulus of rupture for bending*, and if experiments have been made, so that W is known, eq. (3) gives

$$k = \frac{3}{2}\frac{Wl}{Ah} = \frac{3}{2}\frac{Wl}{bh^2}, \qquad (5)$$

and eq. (4)

$$k = \frac{Wl}{Ar} = \frac{Wl}{\pi r^3}. \qquad (6)$$

If the rectangular section is square, $bh^2 = b^3 = h^3$.

Steel.

If the beam is simply supported at each end and carries a load W at the centre, while E is the coefficient of elasticity and w the deflection at the centre, eq. (28) of Art. 28 gives

$$w = \frac{Wl^3}{48EI}. \qquad (7)$$

If, in any given experiment, w is measured, E may then be found by the following form of eq. (7):

$$E=\frac{Wl^3}{48wI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If the section is rectangular

$$E=\frac{Wl^3}{4wbh^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

These equations enable the coefficient of elasticity E to be computed readily from experimental observations. It is only necessary to measure accurately the deflection w produced by the load or weight W and then introduce all the known quantities in eq. (8) or eq. (9).

A bar of wrought iron 3 inches deep and 1 inch wide was placed on supports 48 inches apart and loaded with a weight of 400 pounds at mid-span. The measured deflection was .0138 inch. Hence

$$E=\frac{400\times48\times48\times48}{4\times1\times3\times3\times3\times.0138}=29,730,000.$$

Other applications may be made in precisely the same way.

High Extreme Fibre Stress in Short Solid Beams.

During the period when wrought iron was used for structural purposes, especially for wrought-iron pins with diameters up to 9 or 10 inches, it was observed that if the ultimate extreme fibre intensity k was computed by eq. (5) or (6) with data obtained by actual test, the result would be excessively high, i.e., far beyond the ultimate resistance to tension. These pins, however, on which are packed the lower chord eye-bars of an ordinary truss bridge, have very short spans, indeed the span is usually much less than the diameter of the pin and sometimes less than one quarter of the diameter of the pin. It should be remembered in this connection that the common theory of flexure is implicitly if not explicitly based upon the condition that the length of span of the bent beam must be long compared with the depth of the beam. In fact the span should be many times that depth, and the longer it is the more nearly correct becomes the common theory of flexure. These observations are equally true whether the cross-section of the beam is circular or rectangular or has any other shape.

The following Table shows the results of tests of a series of short wrought-iron beams of circular section made by the author when wrought-iron pins were used in bridge construction, but which illustrate markedly the intensities of extreme fibre stress found with short spans. It will be observed that the spans were 8 inches and 12 inches only

CIRCULAR BEAMS OF "BURDEN'S BEST" WROUGHT IRON.

Kind.	Diameter.	Span.	*W.*		*K.*	
			Elastic.	Ultimate.	Elastic.	Ultimate.
	Ins.	Ins.	Lbs.	Lbs.	Lbs.	Lbs.
Turned.	1.25	12	3,000	6,000	46,950	93,900
Turned.	1.25	8	4,400	10,500	45,900	109,500
Turned.	1.25	12	——	——	54,760	93,870
Turned.	1.25	8	——	——	52,150	114,700
Rough.	1.00	12	——	——	55,000	91,700
Rough.	1.00	8	——	——	57,000	101,900
Turned.	1.00	12	——	——	55,000	91,600
Turned.	1.00	8	——	——	——	107,000
Rough.	1.00	12	1,700	3,000	51,950	91,680
Rough.	1.00	8	2,800	4,800	57,000	97,800
Turned.	0.75	12	700	1,100	47,100	74,050
Turned.	0.75	8	1,200	1,900	53,880	85,310
Turned.	0.75	12	700	1,100	47,100	74,050
Turned.	0.75	8	1,300	1,900	58,370	85,310

while the diameters of the circular beam sections varied from 1.25 inches down to .75 inch.

W is the centre load and the extreme fibre intensity *k* is computed by eq. (6). The ultimate intensity *k* was assumed to be reached when the deflection at the centre of span amounted to about the diameter of the circular section of the beam. This particular feature of the tests is a matter of judgment, but *k* would differ little whether it be taken at a centre deflection equal to the diameter of the circular section or one half that diameter or even less.

It will be noticed that the ultimate values of *k* are all much larger for the 8-inch span than for the 12-inch, and that all the ultimate values increase materially with the depth of the beam, rising to 107,000 to 114,700 pounds per square inch for diameters (i.e., depths of beams) of 1 inch and $1\frac{1}{4}$ inch. It will also be observed that the elastic limits are greatly increased. The ultimate tensile resistance of the iron used in these tests was about 55,000 pounds per square inch and the elastic limit a little more than half that value.

Steel.

Investigation by actual test has shown that short steel beams with circular or rectangular section will exhibit the same elevation of ultimate intensity of fibre stress *k* as found for wrought iron in the preceding section. This is well illustrated by the following tabular statement of results of tests of Bessemer steel beams with circular cross-section, also made by the author in the early days of the use of steel for bridge building.

The Table is self-explanatory in view of the explanations made for short wrought-iron beams of circular section. The ultimate tensile resistance of the mild Bessemer steel used in these tests was about 65,000 to 70,000 pounds per square

CIRCULAR BESSEMER STEEL BEAMS, EQ. (6).

Kind.	Diameter.	Span	*W.*		*k*,	
			Elastic.	Ultimate.	Elastic.	Ultimate.
	In.	Ins.	Lbs.	Lbs.	Lbs.	Lbs.
Turned...........	1.00	12	——	——	86,000	146,750
"	1.00	8	——	——	85,300	152,800
"	1.00	12	2,500	4,500	76,400	137,520
"	1.00	8	3,750	7,500	76,400	152,800
"	0.75	12	1,150	1,800	77,400	122,200
"	0.75	8	1,800	3,300	80,800	148,200
"	0.75	12	1,150	1,700	77,400	114,400
"	0.75	8	1,800	3,300	80,800	148,200

inch and the elastic limit about 35,000 to 38,000 pounds per square inch. The ultimate intensity of stress in the extreme fibres of these beams ranged, however, from 114,400 up to 152,800 pounds per square inch, the larger values belonging to the greater depth of beam and the smaller values to the smaller depth. The elastic limit is seen to be correspondingly high.

These and the preceding tests show that the apparent ultimate resistance of wrought iron and structural steel in the extreme fibres of very short beams with circular or rectangular cross-section may be even more than twice the ultimate tensile resistance as derived from the testing of ordinary tensile specimens.

This feature becomes even more marked when the spans of the cylindrical beams are still shorter, perhaps as short as the diameter of the circular section.

In the design of pins in pin-connected bridges, this high-resisting capacity of wrought iron or steel in pins is recognized by making the working resistance in the extreme fibres of pins considered as beams as much as 50 per cent. higher than in members subjected to simple or direct tension.

The explanation of this phenomenally high resistance to the tension of flexure (and also the compression) is found, as already indicated, in the fact that the common theory of flexure is not correctly applicable to such excessively short beams. No such high intensity of tensile (or compressive) stress actually exists in the metal as computed by eqs. (5) and (6). When the span becomes very short, not more than perhaps three or four times the depth of the beam, lines of stress run from the point of application of the load at the centre of the span direct to both supports, transverse shear being the vertical components of the stresses acting along these lines. All such or similar stress action reduces the actual flexure and makes the bending stresses of tension and compression correspondingly less; but as the flexure formulæ, eqs. (5) or (6), contain no recognition of this condition, the apparent fibre stresses computed by their use are far above the actual.

Numerous other similar short solid beam tests have confirmed the results given in the preceding two Tables.

Cast Iron.

Although cast iron is rarely ever used to resist flexure except in window and door lintels or other similar members whose duties are light, tests of short cast-iron beams have shown the same phenomena of greatly elevated ultimate resistance as found for the more ductile metals. The apparent ultimate intensity k in the extreme fibres of short cast-iron beams of circular or square section may be taken 50 per cent. above the ultimate tensile resistance of the same metal under ordinary tensile tests.

Alloys of Aluminum.

Table VIII of Art. 59, in the fifth column from the left side, exhibits values of the ultimate stress in the extreme fibres of small beams of varying proportions of aluminum-zinc alloys. As might be anticipated, beams of either of those metals showed comparatively low resistance, but with aluminum varying from 80 down to 50 per cent. and zinc from 20 up to 50 per cent. the resistance was excellent, the maximum being found with *Al* 75 and zinc 25.

Table XI of Art. 59 exhibits the ultimate fibre stresses in small beams of the alloys of aluminum with copper, zinc, manganese and chromium. The rolled bars of *Al* 96 and *Cu* 4 give excellent results; as does the cast bar of *Al* 75.7, *Cu* 3, zinc 20 and *Man* 1.3. The remaining values of the transverse resistances in the table are self-explanatory.

Copper, Tin, Zinc, and their Alloys.

In the following table are given the data and the results of the experiments of Prof. R. H. Thurston, as contained in his various papers, to which reference has already been made. The distance between the points of support was twenty-two inches, while the bars were about one inch square in section, and of cast metal.

The modulus of rupture, k, is found by eq. (5), in which, however, in many of these cases, W is the weight applied at the centre, added to half the weight of the bar. When k is large and the specimens small, this correction for the weight of the bar is unnecessary; otherwise it is advisable to introduce it.

The coefficient of elasticity, E, is found by eq. (9), in which W is the centre load added to five eighths of the weight of the bar.

The manner in which both these corrections arise is completely shown in *Case* 2 of Art. 28.

E, for any particular bar, has a varying value for different degrees of stress and strain. Those given in the table

SQUARE BARS.

Percentage of			*k*, Lbs. per Sq. In.	Elastic over Ultimate.	Final Deflection.	*E*, Lbs. per Sq. In.
Cu.	Sn.	Zn.			Ins.	
100	0.00	0.00	29,850	——	8.00	9,000,000
100	0.00	0.00	25,920	0.14 to 0.41	1.38 to 8.00	10,830,600
100	0.00	0.00	21,251	0.346	2.31	13,986,600
100	0.00	0.00	29,848	0.140	Bent.	10,203,200
90	10.00	0.00	49,400	0.400	Bent.	14,012,135
90	10.00	0.00	56,375	0.41	3.36	——
80	20.00	0.00	56,715	0.657	0.492	13,304,200
70	30.00	0.00	12,076	1.00	0.062	15,321,740
61.7	38.3	0.00	2,761	1.00	0.032	9,663,990
48.0	52.0	0.00	3,600	1.00	0.019	17,039,130
39.2	60.8	0.00	8,400	1.00	0.060	12,302,350
28.7	71.3	0.00	8,067	0.583	0.121	9,982,832
9.7	90.3	0.00	5,305	0.25	Bent.	7,665,988
0.00	100	0.00	3,740	0.273	Bent.	6,734,840
0.00	100	0.00	4,559	0.267	Bent.	5,635,590
80.00	0.00	20.00	21,193	——	3.27	11,000,000
62.50	0.00	37.50	43,216	——	3.13	14,000,000
58.22	2.30	39.48	95,620	——	1.99	11,000,000
55.00	0.50	44.50	72,308	——	——	——
92.32	0.00	7.68	21,784	0.30	Bent.	13,842,720
82.93	0.00	16.98	23,197	0.41	Bent.	14,425,150
71.20	0.00	28.54	24,468	0.51	Bent.	14,035,330
63.44	0.00	36.36	43,216	0.53	Bent.	14,101,300
58.49	0.00	41.10	63,304	0.48	Bent.	11,850,000
54.86	0.00	44.78	47,955	0.39	Bent.	10,816,050
43.36	0.00	56.22	17,691	1.00	0.0982	12,918,210
36.62	0.00	62.78	4,893	1.00	0.0245	14,121,780
29.20	0.00	70.17	16,579	1.00	0.0449	14,748,170
20.81	0.00	77.63	22,972	1.00	0.1254	14,469,650
10.30	0.00	88.88	41,347	0.73	0.5456	12,809,470
0.00	0.00	100.00	7,539	0.57	0.1244	6,984,644
70.22	8.90	20.68	50,541	——	0.4019	14,400,000
56.88	21.35	21.39	2,752	——	0.0146	14,800,000
45.00	23.75	31.25	6,512	——	0.0150	7,000,000*
66.25	23.75	10.00	8,344	——	0.0162	12,000,000*
10.00	50.00	40.00	21,525	——	Bent.	9,000,000
58.22	2.30	39.48	95,623	——	2.000	10,600,000
60.00	10.00	30.00	24,700	——	0.1267	14,500,000
65.00	20.00	15.00	11,932	——	0.0514	17,000,000
70.00	10.00	20.00	36,520	——	0.1837	15,000,000
75.00	5.00	20.00	55,355	——	Bent.	13,000,000
80.00	10.00	10.00	67,117	——	Bent.	13,500,000
55.00	5.00	44.50	72,308	——	Bent.	11,000,000
60.00	2.50	37.50	69,508	——	1.500	13,000,000
72.52	7.50	20.00	51,839	——	Bent.	12,000,000
77.50	12.50	10.00	61,705	——	0.705	13,500,000
85.00	12.50	2.5	62,405	——	Bent.	12,500,000

* These bars were about half the length of the others.

may be considered average values within the elastic limit.

As usual, "*elastic over ultimate*" is the ratio of k at the elastic limit over its ultimate value.

An examination of the ultimate tensile and compressive resistances of these same alloys, as given in preceding pages, shows that the ratio of k over either of those resistances is very variable. It is usually found between them, but occasionally it exceeds both.

Timber Beams.

As timber beams are always rectangular in section, eq. (3) only will be needed. Retaining the notation of that equation, if the beam carries a single weight W at the centre of the span l,

$$W = \frac{2}{3}\frac{kAh}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the total load W' is uniformly distributed over the span,

$$W' = \frac{4}{3}\frac{kAh}{l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

As k is supposed to be expressed in pounds per square inch, all dimensions in eqs. (10) and (11) must be expressed in inches.

In the use of timber beams it is usually convenient to take the span l in feet, and the breadth (b) and depth (h) in inches. Placing $12l$ for l, therefore, in eqs. (10) and (11),

$$W = \frac{kAh}{18l}; \quad \text{and} \quad W' = 2\frac{kAh}{18l} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which formulæ l must be taken in feet and A and h in inches.

If B be put for $\frac{k}{18}$, eq. 12 becomes

$$W=B\frac{Ah}{l}; \quad \text{and} \quad W'=2B\frac{Ah}{l}. \quad . \quad . \quad . \quad . \qquad (13)$$

Hence when W and W' have been determined by experiment,

For single load W at centre

$$B=\frac{Wl}{Ah} \therefore h=\frac{Wl}{AB}=\frac{18Wl}{Ak}=\sqrt{\frac{Wl}{Bb}}=4.24\sqrt{\frac{Wl}{kb}}. \qquad (14)$$

For total load W′ uniformly distributed

$$B=\frac{W'l}{2Ah} \therefore h=\frac{W'l}{2AB}=\frac{9W'l}{Ak}=\sqrt{\frac{W'l}{2Bb}}=3\sqrt{\frac{W'l}{kb}}. \qquad (15)$$

If the beam has a section one inch square and is one foot long, $B=W=\frac{W'}{2}$. B, therefore, may be considered *the unit of transverse rupture*; it is sometimes called *the coefficient for centre-breaking loads.*

If the depth h of the beam is given and the breadth is desired, eq. (14) gives

$$b=\frac{Wl}{Bh^2}=\frac{18Wl}{kh^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Eq. 15 also gives

$$b=\frac{W'l}{2Bh^2}=\frac{9W'l}{kh^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

In general, whatever may be the distribution of the loading, if the bending movement is M (in inch-pounds), eq. (3) gives

$$h = \sqrt{\frac{M}{3Bb}} = 2.45\sqrt{\frac{M}{kb}}; \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

or

$$b = \frac{M}{3Bh^2} = \frac{6M}{kh^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

The general observations which have already been made in connection with the ultimate resistances of timber in tension and compression are equally applicable to the flexure or bending of timber beams. The ultimate resistance of the timber as exhibited by the intensity of stress in the extreme fibre can safely be taken only when determined from tests of full-size beams as actually used in engineering structures. Such resistances or moduli when determined from small pieces selected for the purpose of test are liable to be largely in error for the reasons given in detail in Art. 61. In fact Messrs. Cline and Heim state in Bulletin 108, "Tests of Structural Timbers," U. S. Department of Agriculture, that values obtained from testing small thoroughly seasoned selected specimens "may be from one and one half to two times as high as stresses developed in large timbers and joists," and that statement is rather under than over, as many tests have shown. Furthermore, it is essential to know at least approximately the degree of seasoning to which the timber has been subjected. Ordinary air seasoning will seldom reduce the moisture in full-size timber beams to less than 15 per cent. to 20 per cent. Inasmuch as timber in open engineering structures, like bridges, will at all times be exposed to rainfalls often heavy, working stresses used in the design of such structures should be prescribed for wet or green condition. If the structure is to be protected from atmospheric moisture, values belonging to seasoned timber may properly be employed.

Table II of Art. 61 gives the modulus of rupture for full-size beams tested to failure on a span of 15 feet by concentrated loading at two points one third of the span from each end (Messrs. Cline and Heim, U. S. Dept. Agriculture). These results include failures by tension and compression of fibres as well as failures due to shear along the neutral surface of the beams. Both green and air-seasoned timbers were tested with the sections given in the Article cited.

Table I gives the results of the same series of tests under a proposed grading by which all beams tested were divided into Grade I and Grade II, the higher resistances being found in the former.

TABLE I.

AVERAGE RESISTANCE VALUES OF DIFFERENT SPECIES BY PROPOSED GRADES

Species.	Number of Tests.			Average Modulus of Rupture per Square Inch.		Average Fibre Stress at Elastic Limit per Square Inch.		Average Modulus of Elasticity per Square Inch.	
	Total.	Grade I.	Grade II.	Grade I.	Grade II.	Grade I.	Grade II.	Grade I.	Grade II.
				Lbs.	Lbs.	Lbs.	Lbs.	Lbs.	Lbs.
Longleaf pine..	17	17	....	6,140	....	3,734	....	1,463,000	
Douglas fir....	161	81	80	6,919	5,564	4,402	3,831	1,643,000	1,468,000
Shortleaf pine..	48	35	13	5,849	4,739	3,318	3,005	1,525,000	1,324,000
Western larch..	62	45	17	5,479	3,543	3,662	2,432	1,365,000	1,130,000
Loblolly pine . .	94	45	49	5,898	4,702	3,513	2,793	1,535,000	1,309,000
Tamarack.....	25	9	16	5,469	4,525	3,151	2,847	1,276,000	1,261,000
West. hemlock.	39	26	13	5,615	4,658	3,689	3,172	1,481,000	1,360,000
Redwood......	28	21	7	4,932	3,091	4,031	2,947	1,097,000	877,000
Norway pine...	34	17	17	4,821	3,764	3,082	2,364	1,373,000	1,204,000

The intensities of stresses in extreme fibres are averages for each kind of timber at rupture and at elastic limit. It is to be understood, however, that the elastic limit is approximate only as it is not a well-defined point in timber. The moduli of elasticity are fully as high as should be taken, if, indeed, they are not a little too high for ordinary purposes.

The table does not include results for white pine and spruce, but the resisting and elastic qualities of those two timbers are so near to the corresponding qualities of Norway pine that they may be assumed to be the same under ordinary conditions.

Table II gives a summary of the results of tests of full-size beams made by Prof. Arthur N. Talbot and described by him in Bulletin No. 41 (1909) of the University of Illinois. The cross-sections of these beams varied from 7 inches by 12 inches to 8 inches by 16 inches and the spans were 13.5 feet and 14.5 feet. The loads were applied equally at two points, each one third of the span from each end.

The series into which the program of results is divided were used as a matter of convenience only and have no significance as to quality of material or as to physical features of the results.

It will be observed that small beams and shear blocks were also tested and that the results for these smaller pieces are on the whole materially larger than for the full-size beams and nearly or quite twice as large in some cases.

The extreme fibre stress was computed by means of eq. (5), in which W is the total load at the two points of application at failure and l is two-thirds of the actual length of span in the tests, which makes the bending moment $M=\frac{1}{6}Wl$. If this external bending moment is placed equal to the $\frac{2kI}{h}$, the intensity of stress k will take the value, as indicated by eq. (5):

$$k=\frac{Wl}{bh^2}.$$

In this equation h is the depth of the beam and b its breadth, as already explained in connection with eqs. (1) and (1*a*). W is obviously the load given by the reading

Table II.

SUMMARY OF RESULTS OF TESTS.

All Stresses are Given in Lbs. per Sq. In.

	Series *A*. Longleaf Pine.	Series *B*. Longleaf Pine.	Series *C*.				Series *D*. Loblolly Pine		Series *E*. Old Douglas Fir.	Series *F*. New Douglas Fir.
			Shortleaf Pine		Loblolly Pine					
			Un-treated.	Creo-soted.	Un-treated.	Creo-soted.	Un-treated.	Creo-soted.		
Number of stringers tested	20	14	4	8	4	8	14	12	12	16
Percentage of horizontal shear failures	60	93	100	75	75	62	50	25	83	69
Percentage of tension and compression failures	40	7	0	25	25	38	50	75	17	31
Average horizontal shearing stress at failure	344	386	364	302	368	257	289	330	291	315
Average fibre stress at elastic limit	4,366	4,392	4,343	3,386	4,164	2,330	2,960	3,587	3,823	3,955
Average fibre stress at failure	5,308	5,470	4,886	4,078	4,936	3,446	4,090	4,748	4,284	4,544
Failing by horizontal shear:										
Average horizontal shearing stress at failure	370	390	364	302	379	273	314	363	298	323
Highest horizontal shearing stress at failure	497	474	505	410	388	368	381	391	383	401
Lowest horizontal shearing stress at failure	188	237	293	224	368	224	253	320	221	275
Highest fibre stress at failure	7,260	6,540	6,810	5,550	5,195	4,915	5,740	6,020	5,730	6,040
Average fibre stress at failure	5,527	5,483	4,886	4,082	5,082	3,687	4,387	5,193	4,403	4,751
Lowest fibre stress at failure	2,640	3,620	4,005	3,025	4,890	3,020	3,760	4,460	3,320	3,990
Failing by tension or compression:										
Average horizontal shearing stress at failure	306	344	.....	302	334	229	263	319	252	293
Highest horizontal shearing stress at failure	471	344	.....	357	334	243	326	420	282	355
Average fibre stress at failure	4,978	5,300	.....	4,067	4,500	3,043	3,794	4,600	3,690	4,088
Lowest fibre stress at failure	2,860	5,300	.....	3,405	4,500	2,870	2,530	2,660	3,260	3,250
Highest fibre stress at failure	8,410	5,300	.....	4,730	4,500	3,175	5,130	6,080	4,120	5,180
Small beams. Average horizontal shearing stress.	550	513	.....	.....	.....	.....	375	383	570	460
Small beams. Average fibre stress at elastic limit	6,835	6,110	.....	.....	.....	.....	4,485	4,945	7,145	5,970
Small beams. Average fibre stress at failure	8,810	8,280	.....	.....	.....	.....	6,035	6,210	9,200	7,445
Shear blocks. Average shearing stress	466	393	.....	.....	.....	.....	403	441	486	315

of the scale beam of the testing machine. If W_1 is one of the two equal loads applied to the beam at each one third point of the span, $2W_1$ must be written for W.

The ultimate intensity of shear shown in Table II, which is both the intensity of shear in the neutral surface and on a normal section of the beam at the same point, is found by simply taking one and one half the end reaction divided by the cross-section bh of the beam. As the total transverse shear is greatest at the end of the span, the greatest intensity of shear on the neutral surface will be found at that point at or near which failure by shear will begin unless induced elsewhere by a season crack, wind-shake, decay or some other weakness of the material. Obviously there is neither transverse nor longitudinal shear between the two points, equally loaded, as they are symmetrically located with reference to the centre of the span.

Table III shows the moduli of elasticity computed by Professor Talbot from the data secured by his beam tests. The modulus is found by observing the centre deflection of the beam when loaded within its elastic limit and then inserting the observed value of the deflection and the corresponding observed load in a formula similar to eq. (7). Eq. (7) itself is not applicable for the reason that these

TABLE III.

Timber.	Modulus of Elasticity (E).		
	Max.	Mean.	Min.
Longleaf pine	2,105,000	1,620,000	1,025,000
Shortleaf pine, untreated	1,595,000	1,591,000	1,585,000
Shortleaf pine, creosoted	1,478,000	1,229,000	887,000
Loblolly pine, untreated	1,915,000	1,386,000	944,000
Loblolly pine, creosoted	1,857,000	1,251,000	611,000
Old Douglas fir	2,087,000	1,780,000	1,310,000
New Douglas fir	1,900,000	1,499,000	1,138,000

beams were not loaded at the centre of span. The formula for the centre deflection, however, is readily derived by an analysis similar to that used in Art. 28. That operation will give

$$w = \frac{23Wl^3}{1296EI}; \quad \text{or} \quad E = \frac{23Wl^3}{1296Iw}.$$

The preceding experimental values for timber are among the latest determinations and are representative of the best engineering practice, especially as they are based on tests of full-size timbers of as good quality as can probably be secured in the open market.

The American Railway Engineering Association, after careful scrutiny of all tests of timber made up to 1911, recommended the values given in Table IV for use in the

TABLE IV.

UNIT STRESSES IN POUNDS PER SQUARE INCH

Timber.	Bending.			Shearing.			
	Extreme Fiber Stress.		Modulus of Elasticity.	Parallel to the Grain.		Longitudinal Shear in Beams.	
	Mean Ult.	Working Stress.	Mean.	Mean Ult.	Working Stress.	Mean Ult.	Working Stress.
Douglas fir	6,100	1,200	1,510,000	690	170	270	110
Longleaf pine	6,500	1,300	1,610,000	720	180	300	120
Shortleaf pine	5,600	1,100	1,480,000	710	170	330	130
White pine	4,400	900	1,130,000	400	100	180	70
Spruce	4,800	1,000	1,310,000	600	150	170	70
Norway pine	4,200	800	1,190,000	590*	130	250	100
Tamarack	4,600	900	1,220,000	670	170	260	100
Western hemlock	5,800	1,100	1,480,000	630	160	270*	100
Redwood	5,000	900	800,000	300	80		
Bald cypress	4,800	900	1,150,000	500	120		
Red cedar	4,200	800	800,000				
White oak	5,700	1,100	1,150,000	840	210	270	110

Unit stresses are for green timber and are to be used without increasing the live load stresses for impact. Values noted * are for partially air-dry timbers.

design and construction of timber railway structures for the modulus of elasticity in flexure, the ultimate resistance and working stress in extreme fibres of bent beams, and similar quantities for ordinary shearing parallel to the grain and for longitudinal shearing along the fibres in the neutral surface of beams.

The intensities of working stresses given in this Table are for railway structures. It may be justifiable to use somewhat higher values in other structures where the moving loads are more steady or where perhaps it may be proper to consider all loading as practically quiescent or dead load. It is always to be remembered, however, that timber structures are usually highly combustible and hence that it will frequently be advisable to provide some surplus of sectional area to prolong the carrying capacity of timber members after the beginning of a fire.

Failure of Timber Beams by Shearing Along the Neutral Surface.

In the preceding treatment of timber beams, it has been assumed that when broken under test the extreme fibres will fail, either in tension or compression. As a matter of fact, failure of such beams usually takes place at some weak spot, as a knot, point of incipient or active decay, or at some other point where abnormal weakness is developed. This latter observation holds true whether the failure of the beam takes place by tension or compression in the extreme fibres or by shearing in the neutral surface.

In Art. 15 it was shown that the greatest intensity of either transverse or longitudinal shear in any normal section of a beam takes place at the neutral surface, and hence that the tendency of the fibres there is to separate by longitudinal movement over each other. This is precisely the kind of failure which actually takes place in some short timber beams. If the total transverse shear at any normal section of the beam is S, eq. (8) of Art. 15 shows that the maximum intensity, s, of shear in the neutral surface is

$$s=\frac{3}{2}\frac{S}{bd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In this equation, b is the breadth or width of the beam and d the depth, usually taken in inches.

If W is a single weight or load at the centre of span of a beam simply supported at each end, the shear s, as far as that single load is concerned, is constant throughout the entire length of the beam with the value

$$s=\frac{3W}{4bd}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

If, again, the beam is uniformly loaded with the total load W', the intensity of shear s in the neutral surface has a value which varies from zero at the centre of span to the value given by eq. (21) after making $W=W'$. Whenever the value of the intensity s exceeds the ultimate intensity of shear along the fibres lying in the neutral surface, the beam will fail by the separation of its two halves or parts at the neutral surface.

The mean values for the ultimate resistance to shear along the fibres in the neutral surface of his loaded beams were found by Prof. Talbot and are given in Table II for the best varieties of pine timber and for Douglas fir, including results for creosoted beams of shortleaf pine and loblolly pine. The values for shear and other quantities recommended by the American Railway Engineering Association are found in Table IV..

The average values of the ultimate shear in the neutral surface determined by Messrs. Cline and Heim in their "Tests of Structural Timbers," already cited, are given in Table V for nine varieties of structural timbers, both green and air-seasoned. These results belong to the same full-size beams as the values given in Table I of this Article.

TABLE V.

COMPUTED SHEARING STRESSES DEVELOPED IN STRUCTURAL BEAMS

Species.	Total Number of Tests.		First Failure by Shear. Per cent. of Total and Average per Sq. In.				Shear Following Other Failure. Per cent. of Total and Average per Sq. In.			
			Green.		Dry.		Green.		Dry.	
	Green.	Dry.	%	Lbs.	%	Lbs.	%	Lbs.	%	Lbs.
Longleaf pine......	17	9	54	353	56	272	23	374	0	
Douglas fir.........	191	91	2	166	6	221	22	295	49	294
Shortleaf pine......	48	13	17	332	46	364	6	327	8	418
Western larch......	62	52	8	288	27	340	16	314	21	370
Loblolly pine......	111	25	7	335	28	434	2	356	16	546
Tamarack..........	30	9	10	261	33	299	3	263	0	
Western hemlock...	39	44	5	288	23	307	28	281	68	438
Redwood...........	28	12	7	302	0	...	11	218	17	250
Norway pine.......	49	10	6	232	10	278	6	266	0	

It will be observed in all of these tests that there is much variation in the intensities of the different stresses found and especially in these ultimate intensities of shear in the neutral surfaces of full-size beams. As has already been indicated this is due to the presence of a variety of weakening defects to which timber is subject. This signifies that low working stresses should be used.

It has been found in many cases, and possibly in nearly all, that wind-shakes, season cracks, and other influences which induce at least partial separation of the fibres at the neutral surface, are the sources of incipient failure by shearing in the neutral surface.

In designing timber beams this liability to shear along the neutral surface should always be carefully tested by computations. Relatively short beams are particularly liable to fail in this manner, and the greater part of the timber beams used in engineering work are of this class.

It is a very simple analytical matter to establish such a relation between the methods of failure by longitudinal shearing and rupture of the fibres as to indicate more or less approximately the limit beyond which one mode of failure is more liable to occur than the other, but empirical values for both these ultimate resistances have been seen to be so variable as to make it more advisable to compute the carrying capacity of the beam by both methods, especially as each is a simple procedure.

Influence of Time on the Strains of Timber Beams.

It has been found by actual observation that if a timber beam is loaded to no greater extent than one fourth of its ultimate load, the resulting deflection will continue to increase under continued loading for a long period of time. Sufficient investigations have not yet been made to express these results quantitatively with much accuracy. Enough has been ascertained, however, to show that the influence of time is most important in determining the deflection of timber beams under loads applied for a considerable period of time, and that when the loading becomes a large portion of the ultimate, i.e., perhaps 75 per cent., the beam may fail if the application be sufficiently continued. Indeed, some experiments indicate that failure may possibly take place at .6 or .7 of the ultimate of a single application, if that amount be imposed a sufficient length of time.

It should be understood, therefore, that in using the coefficients of elasticity given in this article for the purpose of computing deflections, such computations may be applicable only when the loads are applied for short periods of time.

Concrete Beams.

When a concrete or a natural stone beam is subjected to transverse loading it fails by tearing apart on the tension side. The failure of the beams, therefore, indicates to some extent the ultimate tensile resistance of the material. Obviously, in the case of concrete beams the ultimate carrying capacity will depend upon a number of elements, such as the kind and quality of cement, sand and broken stone used, and the proportions of the mixture. Table VI contains results of tests of a considerable number of concrete beams 6 ins. by 6 ins. in cross-section and six months of age. For three months these beams were frequently wetted though kept in air. During the remaining three months they were kept in air without wetting. The length of span for some of these beams was 42 ins. and 18 ins. for the remainder. Within the limits of the tests this difference in span appeared to make no essential difference in the ultimate intensities of stress in the extreme fibres. With the cross-sections of the beams, i.e., 6 ins. wide and 6 ins. deep, the ratio of span length divided by the depth was either 7 or 3, making the beams very short. The different columns of the table show the character of the ingredients of the concrete as well as the greatest, mean, and least values of the intensities of extreme fibre stress K. As would be anticipated, the values

TABLE VI.

CONCRETE BEAMS SIX MONTHS OLD.

	Concrete.	Size of Stone in Inches.	No. of Tests.	Ultimate Stress in Extreme Fibres, Lbs. per Sq. In.		
				Max.	Mean.	Min.
	c. s. br.					
B'klyn Bridge Rosendale......	1–2–4	0–2½	6	140	103	76
" " "	1–3–5	"	5	128	80	33
" " "	1–2–4	1–2½	4	153	128	109
" " "	1–3–5	"	3	140	136	134
" " "	1–2–4	0–1	6	134	125	120
" " "	1–3–5	"	3	128	126	122
Atlas Portland................	1–2–4	0–2½	6	647	526	460
" "	1–3–5	"	6	516	449	360
" "	1–2–4	1–2½	6	510	452	335
" "	1–3–5	"	6	458	402	360
" "	1–2–4	0–1	6	560	503	458
" "	1–3–5	"	6	516	420	355
Silica Portland................	1–2–4	0–2½	5	385	349	282
" "	1–3–5	"	6	329	283	238
" "	1–2–4	1–2½	6	554	424	326
" "	1–3–5	"	6	297	268	224
" "	1–2–4	0–1	6	423	377	297
" "	1–3–5	"	5	329	272	238
Alsen Portland................	1–2–4	0–2½	6	560	472	404
" "	1–3–5	"	6	491	404	341
" "	1–2–4	1–2½	6	574	493	453
" "	1–3–5	"	6	460	419	391
" "	1–2–4	0–1	6	654	566	466
" "	1–3–5	"	6	541	484	414
B'klyn Bridge Rosendale	1–2–4	Gravel.	6	192	171	147
" " "	1–3–5	"	1	—	157	—
Atlas Portland................	1–2–4	"	6	554	481	414
" "	1–3–5	"	6	417	352	285
Silica Portland................	1–2–4	"	6	379	344	312
" "	1–3–5	"	6	279	245	195
Alsen Portland	1–2–4	"	5	460	382	326
" "	1–3–5	"	6	373	314	266

"*c.*" indicates cement; "*s.*" indicates sand; "*br.*" indicates broken stone or gravel. An excellent limestone was used for broken stone.

for the Portland cement beams are much higher than those for the Rosendale cement. The table exhibits the usual variations in the results for such material, but on the whole those for gravel are seen to be somewhat less than those for broken stone, the proportions of mixture being the same for the two materials. Even with Portland cement, and with as rich a mixture as 1–2–4, the results show that working values of the greatest intensity in extreme fibres should not exceed 40 to 60 pounds per square inch.

The investigations from which the results in Table VI have been taken were conducted by Messrs. George C. Saunders and Herbert D. Brown, graduating students in the class of Civil Engineering of Columbia University in 1898.

The results of tests of twelve Giant Portland cement concrete beams with 30- and 68-inch spans are given in the " U. S. Report of Tests of Metals and Other Materials " for 1900, and they are shown in Table VII.

TABLE VII.

TRANSVERSE TESTS OF GIANT PORTLAND-CEMENT CONCRETE BEAMS.

Composition: 1 c., 3 s., 5 br. st.

Span, Inches.	Breadth, Inches.	Depth, Inches.	No. of Tests.	Ultimate Stress, *k*, in Extreme Fibres, Pounds per Square Inch.		
				Max.	Mean.	Min.
68	6	6	1	—	472	—
30	6 and 4	6	7	564	493	348
30	6 and 4	6	4	454	415	367

The age of these beams was made up of 2 days in air, 2 months in water, and then 1 one month in air, making a total of 3 months and 2 days. The broken stone included all sizes passing a 2½-inch ring, and retained on a sieve with ½-inch meshes. In these tests the ratio of length over depth was 5, except in the first where it was 11. There seems to be little difference in the values of *k* for the two ratios, but the number is much too small to yield any law of variation.

The values of the ultimate extreme fibre stresses *k*, shown in Table VIII, are the results of testing to failure short Portland cement concrete beams by Mr. H. Von Schon, Chief Engineer of the Michigan Lake Superior Power Company, at Sault Ste. Marie, Mich., and they are taken from his paper in the "Transactions of the American Society of Civil Engineers" for December 1899. The beams were 6 inches by 6 inches in cross-section, with a span of 18 inches. The ratio of length over depth, therefore, was 3.

TABLE VIII.

PORTLAND-CEMENT CONCRETE BEAMS, 6 INS. BY 6 INS. SECTION, 18 INS. SPAN.

Cement.	Broken Stone.	Mixture.	No. of Tests.	Ultimate Fibre Stress, *K*, Pounds per Square Inch.		
				Max.	Mean.	Min.
E	Sandstone	*A*	2	178	176	174
"	"	*B*	2	225	217	209
"	"	*C*	2	288	280	272
"	"	*D*	2	329	325	321
"	"	*E*	2	108	102	97
"	Boulder stone	*A*	2	354	326	298
"	"	*B*	2	358	328	299
"	"	*C*	2	390	373	356
"	"	*D*	2	420	410	400
"	"	*E*	2	350	330	310
R	Sandstone	*A*	2	181	169	158
"	"	*B*	2	183	175	167
"	"	*C*	2	266	262	258
"	"	*D*	2	328	308	288
"	"	*E*	2	195	182	169
"	Boulder stone	*A*	2	390	347	204
"	"	*B*	2	423	406	390
"	"	*C*	2	410	392	374
"	"	*D*	2	411	393	375
"	"	*E*	2	332	322	312

Mixture *A*.... 1 cement, 2.4 sand, 5.3 broken stone.
" *B*.... 1 " 2.4 " 4.8 " "
" *C*.... 1 " 2.4 " 4.4 " "
" *D*.... 1 " 2.4 " 4 " "
" *E*.... 1 " 0.3 lime, 3.1 sand, 5.3 broken stone.

The beams were left from two to eight days in their forms or moulds after being made, and then tested at the age of 60 days in air.

The chief elements in the composition of the Portland cements indicated by *E* and *R* in the Table were as follows:

	Cement *E*.	Cement *R*.
Lime	62.38	63.55
Silica	23.08	21.70
Alumina	5.69	8.76
Magnesia	1.21	2.96
Iron oxide	5.35	1.27
Potash and soda	1.66	1.12

The sand used in Mr. Von Schon's tests was from St. Mary's River, the broken sandstone was the native Potsdam variety, while the broken boulder stone was granitic in character. All broken stone would pass through a 1½-inch ring and be retained on a 1-inch ring; the material was, therefore, little balanced.

In the constructions executed under the supervision of the Boston Transit Commission, large amounts of concrete were needed, and in the report of the Commission for the year ending June 30, 1902, there are exhibited a large number of tests of Portland-cement concrete beams 6 inches by 6 inches in cross-section with 30-inch spans. The ratio of length of span over depth of beam in this case is, therefore, 5. Table IX gives the greatest, average, and least results of these tests with the number of beams broken.

TABLE IX.

PORTLAND-CEMENT CONCRETE BEAMS, 6 INS. BY 6 INS. SECTION, 30 INS. SPAN.

Composition by Volume.			Hours in Compressed Air.	Air Pressure, Lbs. per Sq. In.	No. of Tests.	Ultimate Fibre Stress, *k*, Lbs. per Sq. In.		
Cement.	Stone Dust.*	Broken Stone.*				Max.	Mean.	Min.
1	1.7	2.75	24	7–12	12	999	851	677
1	1.9	2.6	24	12–18	50	924	850	590
1	2	2.4	48	18–25	30	904	731	622
1	2	2.4	28–30 days	20–25	100	900	728	523

* Approximate volumes.

The concrete was machine mixed and Vulcanite-Portland cement was used. The stone dust, to which reference is made in the table, was finely crushed stone varying from impalpable powder up to ⅛ inch diameter, the broken stone, on the other hand, being of ordinary size. It will be noticed that these beams were kept a part of the time in compressed air at pressures varying from 7 to 25 pounds, presumably for the reason that some of the material was to be used under such conditions.

Table X contains results of a number of tests of concrete beams 6 inches by 6 inches in cross-section and with 30-inch spans, made for the purpose of comparing the resistances of concretes made with stone dust and sand. This table is also taken from the Report of the Boston Transit Commission for the year ending June 30, 1902.

TABLE X.

PORTLAND-CEMENT CONCRETE BEAMS, 6 INS. BY 6 INS. SECTION, 30 INS. SPAN.

Composition by Volume (Approximate).				No. of Tests.	Ultimate Fibre Stress, k, Pounds per Square Inch.		
Cement.	Sand.	Stone Dust.	Broken Stone.		Max.	Mean.	Min.
1	—	2	2.4	4	947	848	760
1	.9	.9	2.7	4	846	784	704
1	1.6	—	3	4	773	711	656
1	.9	.9	2.7	4	862	806	759

This concrete was also made with Vulcanite-Portland cement and the mixing was done by hand. The beams were kept in air for the first 24 hours and then 29 days in damp earth.

The results both as to coefficient of elasticity and extreme fibre stress, given in Table XI, were determined at the mechanical laboratory of the Department of Civil

Engineering of Columbia University in 1902 by Mr. Myron S. Falk.* They have special value from the age of the beams, which was about seven years. These beams were originally made under the supervision of Mr. A. Black, Instructor in Civil Engineering, Columbia University, for the purpose of determining thermal linear expansion. They were kept well moistened for several months after being made, but subsequently until tested they were kept under cover without moistening. The gravel used was rounded, varying in size from ½ to 2½ inches.

Table XI.

PORTLAND-CEMENT MORTAR AND CONCRETE BEAMS BROKEN BY CENTRE WEIGHT.

Bar.	Age, Years.	Span in Inches.	Section of Bar in Inches.		Coefficient of Elasticity, Pounds per Sq. In.	Extreme Fibre Stress, Pounds per Square Inch
			Depth.	Width.		
A	7.4	36	4.12	4.06	——	——
A_1	7.4	16	‘‘	‘‘	1,591,000	278
A_2	7.4	16	‘‘	‘‘	1,102,000	315
B	7	36	‘‘	4.00	2,122,000	606
B_1	7	16	‘‘	‘‘	2,440,000	636
B_2	7	16	‘‘	‘‘	1,220,000	530
C	7	36	‘‘	4.05	1,315,000	247
C_1	7	16	‘‘	‘‘	387,000	229
C_2	7	16	‘‘	‘‘	1,023,000	208
D	7.3	36	4.10	4.15	1,165,000	294
D_1	7.3	16	‘‘	‘‘	597,000	415
D_2	7.3	16	‘‘	‘‘	597,000	346

Bars *A*, 1 Aalborg cement, 2 sand, 4 gravel.
‘‘ *B*, 1 Atlas ‘‘ 3 ‘‘
‘‘ *C*, 1 Alsen ‘‘ 3 ‘‘ 5 gravel.
‘‘ *D*, 1 ‘‘ ‘‘ 2 ‘‘

Some of the coefficients of elasticity are abnormally low, those belonging to the beams *B*, B_1, and B_2 are fairly

---

* See Proc. Am. Soc. C. E., February, 1903.

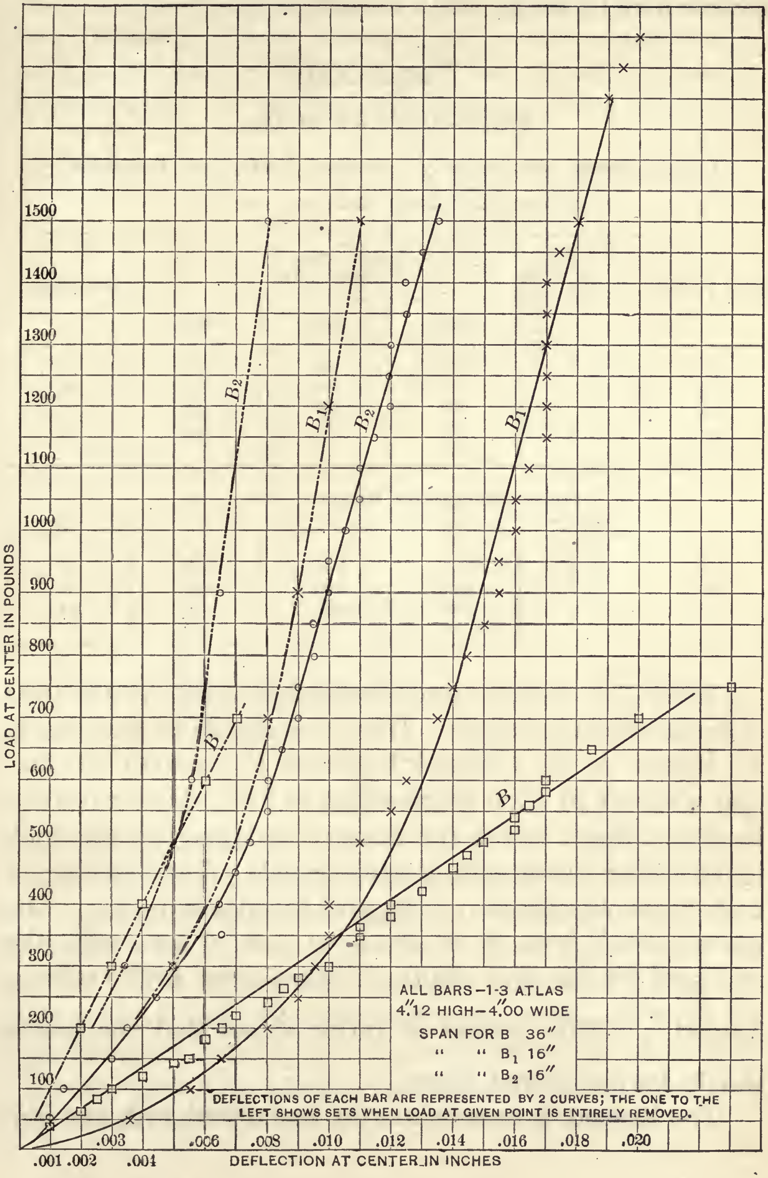
representative of what may be expected with such material in flexure.

Plate A represents graphically the results of the tests of the preceding three bars *B*. As usual, the strain of deflection consisted of two parts in all cases, one permanent, at least for the time being, and one elastic, which disappeared on the removal of the load. This feature is shown by two lines, in each case indicated by the same letter and subscript. The difference between the total and permanent strain or deflection varied very nearly as the centre load, and that difference being the elastic deflection was used in computing the coefficients of elasticity given in Table XI. No coefficient of elasticity was computed for a centre loading less than about 200 pounds. For the purpose of computing deflections under ordinary working stresses from a condition of little or no loading, it would be best to take the coefficient of elasticity at not more than one half of the values given in the Table, in order to allow for that part of the deflection which does not disappear immediately upon the removal of the loading.

Reviewing all the preceding values of the ultimate stress in the extreme fibres of concrete and mortar beams, the working intensities of stress in extreme fibres can probably not be properly taken higher than 50 to 75 pounds per square inch when Portland cement is used for well-balanced mixtures not less rich than 1 cement, 2 sand, and 4 broken stone, or possibly, where exceptionally well made, 1 cement, 3 sand, and 5 broken stone. If gravel is employed, some reduction should be made, depending upon its character, and a similar observation must be applied to mixtures less rich in cement than the preceding.

For natural cements, values of working stress greater than one fourth of the preceding probably should not be used. Indeed, it may be a serious question whether

PLATE A.


1500
1400
1300
1200
1100
1000
900
800
700
600
500
400
300
200
100
LOAD AT CENTER IN POUNDS
.001 .002
.003
.004
.006
.008
.010
.012
.014
.016
.018
.020
DEFLECTION AT CENTER IN INCHES
B
B
B_1
B_1
B_2
B_2
ALL BARS—1-3 ATLAS
4.″12 HIGH—4.″00 WIDE
SPAN FOR B 36″
" " B_1 16″
" " B_2 16″
DEFLECTIONS OF EACH BAR ARE REPRESENTED BY 2 CURVES; THE ONE TO THE LEFT SHOWS SETS WHEN LOAD AT GIVEN POINT IS ENTIRELY REMOVED.

natural cement should be used at all where concrete or mortar may be subjected to flexure.

TABLE XII.

BRICK-MASONRY BEAMS.

(Age of beams about equally 5 months, 8 days, and 6 months.)

ROSENDALE-CEMENT MORTAR: 1 C., 2 S.

Span, Inches.	$\frac{l}{d}$.	Stress in Extreme Fibre, Pounds per Square Inch.			No. of Tests.
		Max.	Mean.	Min.	
96	7.4	67	54	38	5
78	6	—	18	—	—
66	5.1	81	56	23	4
42	3.2	91	73	54	8

PORTLAND-CEMENT MORTAR: 1 C., 3 S.

96	7.4	173	144	124	4
66	5.1	145	120	96	4
42	3.2	229	166	94	10

Table XII exhibits some interesting results of the tests of brick-masonry beams. These investigations were made by Messrs. A. W. Gill and Frederick Coykendall, graduating students in Civil Engineering in Columbia University in 1897. Fig. 1 shows the manner of laying up the brick to form the beams which were tested. The breadth of each beam was about 12 ins. and the depth 13 ins. The spans varied from 8 ft. down to 3 ft. 6 ins., with the ratios of length over depth of beam given in the column headed $\frac{l}{d}$. This column of ratios shows that the beams should be considered short.

The Rosendale-cement mortar was mixed with one volume of cement to two volumes of sand, while the Portland-cement mortar was mixed with one volume of cement to three volumes of sand. During the first three months the beams were kept well wetted, but less so during the last three months. At no time were they dry. The Table gives all the results of tests and shows that the beams had very little resisting capacity, although possibly 15 to 20 pounds per square inch might be justified as working values in the extreme fibres of the beams built with Portland-cement mortar. The bricks were laid by ordinary masons with such care as could be impressed upon them, although the experimenters stated that the brickwork was of very indifferent quality and hence that the results are lower than they should be.

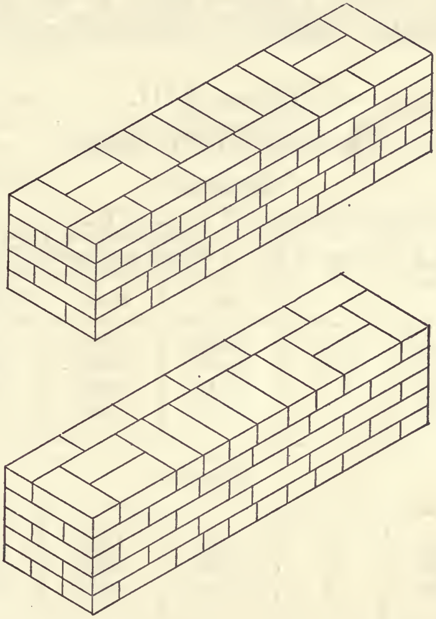


Fig. 1.

Natural-stone Beams.

Table XIII exhibits results found by the same experimenters as in the case of Table XII with a number of natural-stone beams, the spans for which varied from 36 ins. down to 12 ins. The first figure in the second column of the table headed " Section " gives the depth of each beam,

TABLE XIII.

NATURAL-STONE BEAMS.

BLUESTONE.

Span, Inches.	Section, Inches.	$\frac{l}{d}$.	Stress in Extreme Fibre, Pounds per Square Inch.			No. of Tests.
			Max.	Mean.	Min.	
24	4×6	6.15	3,958	3,512	3,054	5
36	6×8	6.2	3,288	2,797	2,906	3
12	4×6	3	4,112	3,237	2,282	11
24	8×6	3	3,929	3,547	2,715	6

GRANITE.

24	4×6	6	2,321	2,250	2,178	3
36	6×8	6	1,861	1,798	1,766	3
12	4×6	3	2,714	2,487	2,086	9

SANDSTONE.

24	4×6	6	1,575	1,354	1,237	3
36	6×4	6	1,204	945	637	3
12	4×6	3	1,907	1,539	1,267	9

MARBLE.

24	4×6	6	2,036	1,880	1,617	3
36	6×8	6	1,683	1,548	1,354	3
12	4×6	3	2,455	2,026	1,696	9

while the second figure gives the width. It will be observed from the ratios of $\frac{l}{d}$ given in the third column that the beams were very short. The extreme fibre stresses are seen to run comparatively high for the bluestone, granite, and marble. Indeed, working values of intensities may reasonably be taken as follows:

For blue tone.............	250 to 400	pounds	per	square	inch.
" granite...............	200 to 300	"	"	"	
" marbl	17 to 225	"	"	"	'
" sandstone.............	100 to 150	"	"	"	"

In the use of sandstone it should be understood that the preceding values apply only to the best qualities of that particular stone.

CHAPTER XIII.

CONCRETE-STEEL MEMBERS.

Art. 92.—Composite Beams or Other Members of Concrete and Steel.

Concrete, like other masonry, is admirably adapted to resist compression. Its capacity of resistance to tension is much less than its ultimate compressive resistance, although if the concrete is well made the tensile resistance may have considerable value. The purpose of the concrete-steel combination is the production of a beam or other member almost entirely of concrete, but which shall have a high capacity to resist tension in those portions which may be subjected to tensile stresses. This result is accomplished by embedding steel bars of desired shape and of suitable cross-sectional area in the proper parts of the concrete. While no general rule can be given for the area of the steel section in comparison with the concrete, it may be stated approximately that the steel section is usually between $\frac{3}{4}$ and $1\frac{1}{2}$ per cent. of the area of a normal section of the concrete. Inasmuch as the presence of the steel is for the purpose of giving tensile resistance to the member it is evident that the re-enforcing steel bars will always be found in those portions of the concrete mass which may be subjected to tension. In such concrete-steel construction as arches the steel re-enforcement is frequently used both on the tension and compression sides of the concrete.

In the case of concrete-steel beams or other similar members, as the steel is entirely embedded in the concrete, the loads and reactions must obviously be applied directly to the latter. When the concrete takes its stress, therefore, at least a portion of that stress must be conveyed to the steel, and that requires that the adhesive joint or bond between the steel and concrete shall be as strong as possible. Hence in laying the steel bars in the concrete it is necessary that the contact between the two materials shall be intimate and essentially continuous. Various means are employed to accomplish these ends. Square bars are frequently twisted, while round bars may be nicked and flat ones either twisted continuously in one direction or have alternate portions twisted in opposite directions, or, finally, rolled with alternately enlarged and contracted sections. Again, where built-up members are embedded in concrete, rivet-heads and other details of construction serve the same general purposes. The efficiency of the concrete-steel construction depends wholly upon the resistance of this bond, and the design must always be such that the adhesive shear, so to speak, or the stress of sliding along the steel surface, shall never exceed per square unit the ultimate resistance of the bond.

In the analysis and computations which follow it is assumed, as it must be, that the bond between the steel and concrete is such as to make the entire mass act as a unit, so that the combination of the two heterogeneous elements shall act as a single whole.

Art. 93.—Physical Features of the Concrete-steel Combination in Beams.

It will be shown later on that so far as can be determined from physical data now available the coefficient of elasticity for concrete in compression for the operations ordinarily employed in designing engineering structures and for mixtures not less rich in cement than 1 cement, 3 sand, and 6 gravel or broken stone, at ages of one to six months, may range from about 2,000,000 pounds per square inch to more than 4,000,000 pounds per square inch, while for concrete beams the coefficient or modulus may range from about 1,500,000 pounds per square inch for comparatively shallow beams to more than 3,000,000 pounds per square inch for beams of comparatively great depths. Values for the coefficient of elasticity for concrete in tension can be found in Art. 60. Further tests for the determination of this quantity are much to be desired, but enough has been done to establish at least closely approximate values. Some authorities assume the tensile coefficient to be much less than the coefficient of elasticity for concrete or mortar in compression. As a matter of fact, the tests of a Monier arch of 75 feet span by a committee of the Austrian Society of Engineers and Architects, which made its report in 1895, showed in that particular case the coefficient of elasticity of concrete in tension to be nearly one fifth greater than the coefficient for compression, although it should be stated that the age of the tensile specimens was materially greater than that of the compression material. The values in Art. 60 indicate that the tensile coefficient is at least equal to the compressive. It is possible that subsequent investigations may show that the tensile coefficient of elasticity is less than that for compression, but at the present time there appears to be practically no basis for that assumption. It seems to be reasonable and safe, as it is more simple to take the two coefficients equal to each other until further investigations have conclusively established a different ratio.

It is important to state in this connection that the results of tests with concrete-steel beams, so far as they have been made, indicate that the elastic or semi-elastic behavior of concrete under stress will in the main characterize the behavior of the same material when under loading in the composite beam of concrete and steel, so that the coefficients of elasticity determined for concrete alone may be used in the composite member.

There is one important respect in which the action of concrete alone is quite different from that which takes place when it is combined with steel. In the latter case the concrete will stretch under a stress nearly or quite equal to its ultimate resistance a comparatively large amount. It is sometimes stated that under such conditions the coefficient of tensile elasticity of the concrete is practically zero, but there is just as much ground, or more, for making the same observation in connection with such ductile materials as structural steel. What is actually meant is simply that the concrete will stretch before parting much more when its deformation is controlled by the corresponding deformation of the steel reinforcement than when it acts by itself or without such reinforcement. This feature of the action under stress of concrete in the composite beam has a most important bearing upon some rather peculiar phenomena connected with the testing of such beams to failure. M. Considére has stated ("Comptes Rendus Académie des Sciences," Paris, Dec. 12, 1898) that mortar will stretch twenty times as much when combined with steel as when unaided by that combination. He further states that the concrete stretches uniformly with uniform increments of bending moment up to about four tenths of the ultimate moment.

As the coefficient of elasticity for concrete is a small fraction only of that of steel the tendency of the concrete in composite beams is to stretch or compress more than the steel embedded in it. Hence the concrete immediately adjacent to the steel tends to slide along the latter, but that tendency is resisted by the adhesive shear at the joint, in consequence of which the steel acquires its stress whether of tension or compression. The normal section of the unloaded beam, therefore, will not remain normal after flexure, but there will be either a cup-shaped depression around the steel or a similar shaped elevation. This is illustrated in Fig. 1.

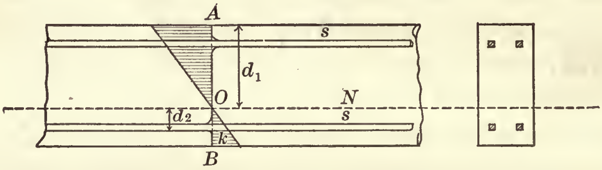


Fig. 1.

In that figure the intensity of stress on either side of the neutral axis is assumed to vary directly as the distance from the axis, but in a subsequent analysis a different law of variation will be assumed in order that the treatment may be complete, although the author is not of opinion that the assumption of any law of variation different from that of the common theory of flexure is at the present time justified. It will further be assumed in the analysis which follows that normal sections of the unloaded beam will remain normal under loading. This is a common procedure, and it is not believed that the amount of variation from a plane section under stress, described above, is sufficient to make the assumption sensibly in error.

Art. 94.—Rate at Which Steel Reinforcement Acquires Stress.

The determination of the rate at which the concrete gives stress to the steel is not of great importance in ordinary design work or in most other practical relations; yet it is desirable in some cases, and it is an element of the action of internal stresses in a composite beam which should be understood as clearly as practicable. The following analysis offers a means of determining that rate as nearly as it can be done at the present time. The notation used is shown also in Fig. 3 on the opposite page.

The intensity of stress in the concrete at the distance d_2, the distance of the steel reinforcement, from the neutral axis is k. Then if I represent the moment of inertia of the entire composite section about its neutral axis (located by d_1, determined hereafter), there may be written

$$M=\frac{kI}{d_2}; \quad \therefore\ dM=\frac{dk.I}{d_2}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If S is the total transverse shear in the normal section in question at the distance x from one end of the beam,

$$dM=Sdx=\frac{dk.I}{d_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let p be the total perimeter of section of the steel reinforcement at the section located by x.

Let A_2 be the area of steel section with perimeter p.

Let s' be the intensity of adhesive shear at the surface or joint between the steel and concrete.

Let k_2 be the intensity of stress in the steel.

The variation of k_2 for the indefinitely small distance dx is dk_2. From what has preceded there may be written

$$p.dx.s'=A_2dk_2; \quad \therefore\ dx=\frac{A_2dk_2}{ps'}. \quad . \quad . \quad . \quad . \quad (3)$$

Inserting the value of dx from eq. (3) in eq. (2),

$$SA_2\frac{dk_2}{ps'}=\frac{dk}{d_2}I.$$

By solving this equation for s' and remembering that

$$\frac{dk_2}{dk}=\frac{E_2}{E_1},$$

$$s'=SA_2\frac{d_2}{Ip}\frac{dk_2}{dk}=S\frac{E_2}{E_1}\frac{d_2}{I}\frac{A_2}{p}. \quad . \quad . \quad . \quad . \qquad (4)$$

This value of s' must never exceed the ultimate adhesive resistance between the steel and concrete.

Tests for the determination of the adhesive shear between concrete and imbedded round rods have been made by Professors Talbot, Withey, Hatt, Duff A. Abrams and others. In view of the inevitable uncertainties of condition of such rods in respect to the bond between them and the concrete, greatly varying values must be anticipated, as they will depend upon the age proportions of the concrete, the smoothness (or roughness) of the surface of the rods, the amount of water used in mixing the concrete and the continuity of contact between the concrete and the rods. The value of adhesive shear has sometimes been taken as 16 to 20 per cent. of the ultimate compressive resistance of the concrete, but this is probably too high, even for the best qualities of concrete.

Again, the ultimate value of adhesive shear as determined by the pulling of rods directly from a block of concrete may be materially different from that developed in a bent beam and, hence, the latter procedure should be the basis of determinations for reinforcing rods for beams. A clear distinction should be drawn between the adhesive shear existing prior to movement of the rod in its mastic and the resistance to that motion after it once begins.

Professor M. O. Withey published in a Bulletin of the University of Wisconsin, No. 321, 1909, the data of a large number of tests in which the results were obtained from loaded beams, the stretch of the rods being accurately measured by an extensometer for a given length of imbedded rod. The diameter of rod was $\frac{5}{8}$ inch and the age of the concrete varied from seven days up to six months. A large number of tests gave the adhesive shear as varying from a minimum of 129 pounds per square inch to a maximum of 362 pounds, a few only of the results falling below 200 pounds per square inch. It would probably be fair to take 250 pounds per square inch as a representative average of these results.

In a series of tests with diameters of bars running from $\frac{3}{8}$ inch to 1 inch, the average results were 278 and 286 pounds per square inch for the two smaller sizes of bars and 163 pounds and 195 pounds per square inch for the 1-inch bars. The age of the 1–2–4 concrete in this case was two months.

There may be found in Bulletin No. 71, University of Illinois, a full account of a large number of " Tests of Bond between Concrete and Steel," by Duff A. Abrams. These tests were made under a great variety of conditions as to age, sizes of rods, surface of rods, i.e., whether plain or deformed, shapes of cross-sections, rods pulled out of blocks and rods stressed in reinforced concrete beams, accompanied by extended observations as to effects of loading including careful measurements of the stretch of steel both in pulling rods from blocks and as they were stressed in beams. In these tests a clear distinction was recognized between the adhesion to the surface of the rods and the resistance of movement after initial slip, the greatest intensity of bond resistance usually being developed after the beginning of slip.

A roughened surface of rod will obviously yield a greater bond resistance than a perfectly smooth surface, the resistance of the latter being almost wholly adhesion.

The following are a few of Mr. Abrams' conclusions:

"(41) The mean computed values for bond stresses in the 6-foot beams in the series of 1911 and 1912 were as given below. All beams were of 1–2–4 concrete, tested at 2 to 8 months by loads applied at the one third points of the span. Stresses are given in pounds per square inch.

	Number of Tests.	First End Slip of Bar.	End Slip of 0.001 In.	Maximum Bond Stress
1 and 1¼-in. plain round.........	28	245	340	375
¾-in. plain round...............	3	186	242	274
⅝-in. plain round...............	3	172	235	255
1-in. plain square...............	6	190	248	278
1-in. twisted square.............	3	222	289	337
1⅛-in. corrugated round.........	9	251	360	488

"(42) In the beams reinforced with plain bars end slip begins at 67 per cent. of the maximum bond resistance; for the corrugated rounds this ratio is 51 per cent., and for the twisted squares, 66 per cent.

"(43) The bond unit resistance in beams reinforced with plain square bars, computed on the superficial area of the bar, was about 75 per cent. of that for similar beams reinforced with plain round bars of similar size.

"(44) Beams reinforced with twisted square bars gave values at small slips about 85 per cent. of those found for plain rounds. At the maximum load, the bond-unit stress with the twisted bars was 90 per cent. of that with plain round bars of similar size.

"(45) In the beams reinforced with 1⅛-inch corrugated rounds, slip of the end of the bar was observed at about the same bond stress as in the plain bars of comparable size. At an end slip of 0.001 inch, the corrugated bars gave a bond resistance about 6 per cent. higher and at the maximum load, about 30 per cent. higher than the plain rounds.

" (46) The beams in which the longitudinal reinforcement consisted of three or four bars smaller than those used in most of the tests gave bond stresses which, according to the usual method of computation, were about 70 per cent. of the stresses obtained in the beams reinforced with a single bar of large size."

As the greatest bond stress was developed after the beginning of slip, the preceding results show that such a maximum value exists beyond a net slip of 0.001 inch.

Again referring to the resistance of deformed bars, he states, " The mean bond resistance for the deformed bars, tested was not materially different from that for plain bars until a slip of about .01 inch was developed; with a continuation of slip, the projections came into action and with much larger slip high bond stresses were developed."

Again referring to a working bond stress, he states:

" (59) A working bond stress equal to 4 per cent. of the compressive strength of the concrete tested in the form of 8- by 16-inch cylinders at the age of 28 days (equivalent to 80 pounds per square inch in concrete having a compressive strength of 2000 pounds per square inch) is as high a stress as should be used. This stress is equivalent to about one third that causing first slip of bar and one fifth of the maximum bond resistance of plain round bars as determined from pull-out tests. The use of deformed bars of proper design may be expected to guard against local deficiencies in bond resistance due to poor workmanship and their presence may properly be considered as an additional safeguard against ultimate failure by bond. However, it does not seem wise to place the working bond stress for deformed bars higher than that used for plain bars."

The preceding results were obtained from statically loaded beams. Professor Withey found no injurious effects on the resistance of adhesive shear under repeated loads until the latter became 50 to 60 per cent. of the ultimate static loads. This last percentage may be raised to 60 to 70 per cent. with corrugated bars. Investigations made by the same authority indicate that the results of static tests on smooth round rods imbedded in beams will give values for the bond or adhesive shear between the concrete and the rods from one half to two thirds only of corresponding results obtained by pulling imbedded steel rods from the concrete cylinders, but Mr. Abrams appears to believe that the results of properly made "pull-out" tests will be about the same as found for beams.

While materially larger values for ultimate resistance of adhesive shear have been reported by some experimenters with small rods, it appears prudent not to take the ultimate resistance greater than perhaps 200 to 350 pounds per square inch for round or square rods from 1¼ inch to ⅝ inch in diameter.

The working value for this bond for adhesive shear should not be taken more than one fourth to one fifth of its ultimate value.

Art. 95.—Ultimate and Working Values of Empirical Quantities for Concrete-steel Beams.

It is necessary for the practical use of the preceding and following analyses that a number of empirical quantities be determined, chiefly for the concrete. The coefficient of elasticity for wrought iron for this purpose may be taken at 28,000,000 pounds per square inch, and 30,000,000 pounds per square inch for structural steel, which is now generally used in the reinforcement of concrete-steel beams.

The modulus of elasticity for concrete at different ages and for different proportions of matrix and aggregate has been fully considered in Art. 67, and Table I of that Article exhibits a full set of values. A mixture of 1 cement, 2 sand and 4 broken stone or gravel is generally used in reinforced concrete work; and for such concrete the Table cited above shows that the modulus of elasticity at the age of one month may be taken from about 1,500,000 to nearly 3,000,000. In view, however, of the uncertain conditions attending the making of concrete on actual work a higher value than 2,000,000 is seldom used. The ratio of the modulus for steel divided by that for concrete is generally taken at 15, although 12 is sometimes employed, the latter value implying a modulus for concrete of 2,500,000.

The ultimate resistances of mortar and concrete in tension and compression will be found in Arts. 60 and 67. These values will also depend upon the proportions and character of mixture or upon the age. The records of tests and experience which have thus far accumulated in connection with concrete-steel construction show that the compressive working stress of concrete in beams, where the mixture is in the proportions of 1 cement, 2 sand, and 4 gravel or broken stone, may probably be taken as high as 500 pounds per square inch. It should be remembered that this intensity will exist in the extreme fibres of the beam only. Mixtures of less strength would require a corresponding reduction in the maximum working intensity of compression. A mixture, for example, of 1 cement, 2½ sand, and 5 broken stone, unless the materials were well balanced, might justify a reduction of the greatest working stress to 400 pounds per square inch.

Some foreign authorities have prescribed two degrees of safety, in the first of which the maximum working stress of compression of 427 pounds per square inch is allowed, and 711 pounds per square inch for safety of the second degree. Structures in which the duty of the concrete is severe might be designed with the smallest of those values, but where the duty is materially less severe, with the larger.

It is not unusual at the present time in the design of concrete-steel arches to allow a maximum intensity of compression of 500 pounds per square inch and 50 to 75 pounds per square inch for the maximum intensity of tension, if tension is allowed.

Tensile tests of concrete show that where proportions of 1 cement, 2 sand, and 4 gravel or broken stone are used a maximum intensity of tension of 50 to 70 pounds per square inch is about $\frac{1}{6}$ to $\frac{1}{8}$ the ultimate tensile resistance at the age of three to six months. These values are reasonable and may be employed in concrete work where it is permitted to avail of the tensile resistance of concrete. In much of the best engineering practice at the present time, however, the tensile resistance of the concrete is neglected in the interests of additional safety in concrete-steel beam construction. Inasmuch as fine cracks may appear in concrete from other agencies than tensile stress, it is undoubtedly advisable in most cases certainly to omit the bending resistance of the concrete in tension, especially as that omission does not sensibly increase the weight or cost of the beam when properly designed.

Art. 96.—General Formulæ and Notation for the Theory of Concrete-steel Beams According to the Common Theory of Flexure.

The application of the common theory of flexure to the bending of concrete-steel beams is in reality the development of the theory of flexure for composite beams of any two materials. The notation to be used and the general formulæ will first be written, therefore, and then the special formulæ for concrete-steel beams will be established in the succeeding articles. These general formulæ, it should be observed, apply to beams of any shapes of cross-section of either material or for any relative areas of cross-section of those materials, although in concrete-steel beams the area of cross-section of the steel is frequently or perhaps usually but one to one and a half per cent. of the area of the concrete.

Again, the formulæ will be so written as to make practicable the use of different coefficients of elasticity for concrete in tension and compression if that should be desired.

The notation to be used in the succeeding articles is chiefly the following:

E_2 = coefficient of elasticity of the steel.
E_1 = " " " " " concrete in compression.
nE_1 = " " " " " concrete in tension.
A_1 and A_2 are the areas of normal section of the concrete and steel respectively.
I_1 and I_2 are the moments of inertia of A_1 and A_2 respectively about the neutral axis of the normal section.
k_1 = greatest intensity of bending compression in the concrete.
k' = greatest intensity of bending tension in the concrete.
c = greatest intensity of bending compression in the steel.
t = greatest intensity of bending tension in the steel.
b = breadth of the concrete.
h and h_2' are total depths of the concrete and steel respectively.
h_2 = vertical distance between the centres of the steel reinforcing members.
d_1 = distance of extreme compression "fibre" of the concrete from the neutral axis.
d_2 = distance of the centre of the compression steel reinforcing member from the neutral axis.

d_3 = distance from the neutral axis to the centre of the tension steel reinforcement.

d_2' = distance from extreme compression fibre of the steel to the neutral axis.

a = distance of the centre of the compression steel reinforcing member from exterior compression surface of concrete.

a_1 = distance of the centre of the tension steel reinforcing member from exterior tension surface of concrete.

rA_2 = area of normal section of reinforcing steel in tension.

$(1-r)A_2$ = area of normal section of reinforcing steel in compression.

k = intensity of compressive stress in the concrete at distance z from the neutral axis.

k'' = intensity of tensile stress in the concrete at distance z from the neutral axis.

k_2 = intensity of stress in the steel at distance z from the neutral axis.

u = tensile or compressive strain in unit length of "fibre" at unit distance from the neutral axis.

In all the theory* of bending of concrete-steel beams it is assumed, as in the common theory of flexure, that any plane, normal section of the beam, before bending takes place, will remain plane (and normal) while the beam is subjected to bending. Hence

$$k = E_1uz, \quad k'' = nE_1uz, \quad \text{and} \quad k_2 = E_2uz. \qquad . \quad (1)$$

Inasmuch as all the loading carried by concrete-steel beams is supposed to act in a direction normal to the axes

---

* Given in Art. 32. Eqs. (1) to (4) are simple adaptations of the equations of that Art. to this case.

of the beams, as is usual in the common theory of flexure, the total stresses of tension and compression in any normal section of a beam induced by the bending must be equal to zero. The expression of this sum, written by the aid of eqs. (1) and by which the neutral axis of the composite section is determined, is the following:

$$E_1 u \left\{ \int_0^{d_1} zdA_1 + n \int_{h_1-d_1}^{0} zdA_1 \right\} + E_2 u \int_{d_2'-h_2'}^{d_2'} zdA_2 = 0. \quad (2)$$

Or

$$\int_0^{d_1} zdA_1 + n \int_{h_1-d_1}^{0} zdA_1 + \frac{E_2}{E_1} \int_{d_2'-h_2'}^{d_2'} zdA_2 = 0. \quad . \quad (3)$$

Eq. (3) is perfectly general, and the position of the neutral axis can always be located by it whatever may be the shape of cross-section of either the concrete or steel.

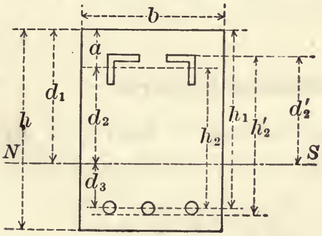


FIG. 1.

Fig. 1 may be taken as an arbitrary typical composite section showing the preceding system of notation applied to it. The outline of the concrete is rectangular, as in the ordinary concrete-steel beam. The steel in compression is represented as two steel angles, while three round rods constitute the steel in tension. In the next article the application of the general eq. (3) to the special case of the ordinary concrete-steel beam will be made.

The general value of the bending moment of the stresses induced in any normal section of a composite beam can be at once written by the aid of eqs. (1). The typical expression of the differential moment is

$$kdA_1 z = E_1 u z^2 dA_1.$$

Hence the value of the moment is

$$M = E_1 u \int_0^{d_1} z^2 dA_1 + n E_1 u \int_{h_1 - d_1}^{0} z^2 dA_1 + E_2 u \int_{h_2' - d_2'}^{d_2'} z^2 dA_2. \quad (4)$$

This equation is also completely general whatever may be the shape of section of either material. It will be developed for the ordinary form of concrete-steel beams in Art. 97.

Eqs. (3) and (4) cover completely the theory of bending or flexure of composite beams of two materials, one of them having different values for the coefficients of elasticity in tension and compression. It will be observed that the position of the neutral axis of any section of the beam, as located by eq. (3), is affected by the values of E_1, E_2, and n, and that it does not in general pass through the centre of gravity of the section.

Art. 97.—T-Beams of Reinforced Concrete.

The general formulæ of Art. 96 belong to beams of any shape of cross-section whatever; it is only necessary, therefore, in this case, to apply them to the T-shaped section. Two conditions may arise, in one of which the neutral

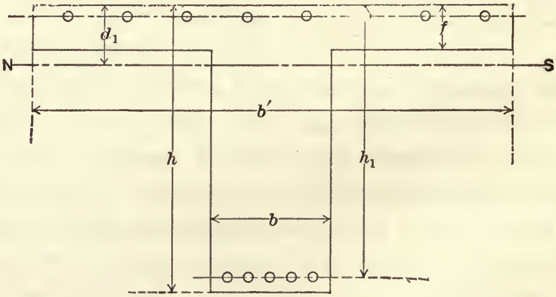


FIG. 1.

axis lies in the flange of the beam whose cross-section is shown in Fig. 1, or, as shown in that figure, it may lie below the flange. As is usually the case in actual work, the tensile resistance of the concrete will finally be neglected. This latter condition makes it necessary to consider only the case shown by Fig. 1.

Position of Neutral Axis.

Using the notation of Art. 96 under the conditions outlined above, but first recognizing the tensile resistance of the concrete,

$$\int_0^{d_1} zdA_1 = \int_{d_1-f}^{d_1} z \cdot b^1 dz + \int_0^{d_1-f} z \cdot bdz$$

$$= b^1 f\left(d_1 - \frac{f}{2}\right) + b\frac{(d_1-f)^2}{2}. \quad . \quad . \quad . \quad . \quad (1)$$

Again,

$$n\int_{h_1-d_1}^{0} zdA_1 = n\int_{h_1-d_1}^{0} zbdz = -\frac{nb}{2}(h_1^2 - 2h_1d_1 + d_1^2). \quad (1a)$$

As the steel section is small it will be essentially correct to consider each part of it concentrated at its centre of gravity. Hence there may be written,

$$\int_{h_2'-d_2'}^{d_2'} zdA_2 = (1-r)A_2d_2 - rA_2(n_2-d_2) = A_2(d_2 - rh_2). \quad (1b)$$

Introducing the values given by eqs. (1), (1*a*) and (2) in eq. (3) of Art 96,

$$b^1fd_1 - b^1\frac{f^2}{2} + \frac{bd_1^{\ 2}}{2} - bd_1f + \frac{bf^2}{2} - \frac{d_1^{\ 2}bn}{2} + nbh_1d_1 - n\,\frac{bh_1^{\ 2}}{2}$$

$$+\frac{E_2}{E_1}A_2\,(d_2 - rh_2) = 0.$$

$$\therefore\quad d_1^{\ 2} + d_1\frac{2\left(f\left(\frac{b^1}{b} - 1\right) + nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)}{1 - n} = \frac{\left(\frac{b^1}{b} - 1\right)f^2 + nh_1^{\ 2}}{1 - n}$$

$$+\ 2\,\frac{E_2}{E_1}\frac{A_2}{b}\frac{(a + rh_2)}{1 - n}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (2)$$

The solution of this quadratic equation will give

$$d_1 = -\frac{\left(f\left(\frac{b^1}{b} - 1\right) + nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)}{1 - n}$$

$$\pm\sqrt{\frac{\left(\frac{b^1}{b} - 1\right)f^2 + nh_1^{\ 2}}{1 - n} + 2\frac{E_2}{E_1}\frac{A_2}{b}\frac{(a + rh_2)}{1 - n} + \frac{\left(f\left(\frac{b^1}{b} - 1\right) + nh_1 + \frac{E_2}{E_1}\frac{A_2}{b}\right)^2}{(1 - n)^2}}.\qquad (3)$$

3

If the two coefficients of elasticity for concrete in tension and compression are the same, as is always assumed in actual work, $n = 1$. This value gives indetermination in Eq. 3, but it is only necessary to multiply both members of Eq. 2 by $(1 - n)$ and then make $n = 1$. These operations give

$$d_1 = \frac{\frac{1}{2}\left(\frac{b^1}{b} - 1\right)f^2 + \frac{h_1^{\ 2}}{2} + \frac{E_2}{E_1}\frac{A_2}{b}(a + rh_2)}{f\left(\frac{b^1}{b} - 1\right) + h_1 + \frac{E_2}{E_1}\frac{A_2}{b}}\quad .\ .\ .\ .\ .\ \ 4$$

If the entire steel reinforcement is on the tension side of the beam, $r = 1$, and in Eqs. 3 and 4, $a + rh_2 = a + h_2 = h_1$

The tensile capacity of the concrete is practically always neglected; hence $n = 0$ in Eq. 3, and

$$d_1 = -f\left(\frac{b^1}{b} - 1\right) - \frac{E_2}{E_1}\frac{A_2}{b} \pm\sqrt{\left(\frac{b^1}{b} - 1\right)f^2 + 2\frac{E_2}{E_1}\frac{A_2}{b}\left(a + rh_2\right) + \left(f\left(\frac{b^1}{b} - 1\right) + \frac{E_2}{E_1}\frac{A_2}{b}\right)^2}. \quad 5$$

These formulæ locate the neutral axis by giving the distance d_1 for all cases.

An important special case arises where the neutral axis NS, Fig. 1, lies in the lower side of the flange, i.e., when $d_1 = f$. Making that substitution in the equation preceding eq. (2),

$$d_1^2\frac{b' - nb}{2} + d_1\left(nbh_1 + \frac{E_2}{E_1}A_2\right) = \frac{nbh_1^2}{2} + \frac{E_2}{E_1}A_2(a + rh_2). \quad (6)$$

Solving this quadratic equation,

$$d_1 = -\frac{nbh_1 + \frac{E_2}{E_1}A_2}{b' - nb} \pm\sqrt{\left(\frac{nbh_1 + \frac{E_2}{E_1}A_2}{b' - nb}\right)^2 + \frac{nbh_1^2 + 2\frac{E_2}{E_1}A_2(a + rh_2)}{b' - nb}}. \quad . \quad (7)$$

If concrete in tension be neglected, $n = 0$ and,

$$d_1 = -\frac{E_2}{E_1}\frac{A_2}{b'} \pm\sqrt{\left(\frac{E_2}{E_1}\frac{A_2}{b'}\right)^2 + 2\frac{E_2}{E_1}\frac{A_2(a + rh_2)}{b'}}. \quad . \quad . \quad . \quad . \quad (8)$$

Eq. (8) shows that the case of a T-beam with neutral axis at the lower surface of the flange and with tensile resistance of concrete neglected is equivalent to a solid rectangular beam of the same width as the flange under the same assumption of the neglect of the concrete in tension. No material error will be committed in assuming any T-beam similarly equivalent to a solid rectangular beam if the neutral axis is near the under side of the flange.

If the neutral axis NS lies in the flange the area $(b'-b)(f-d_1)$ of concrete flange section will be in tension. In that case the term $-n(b'-b)\frac{(f-d_1)^2}{2}$ must be added to the third member of eq. (1a), and hence to the first member of the equation preceding eq. (2). This will add obvious corresponding terms to eqs. (3), (4) and (5), but the special case is so rare that it needs no further attention. Unless $(f-d_1)$ has material value eqs. (7) and (8) may be used.

Balanced or Economic Steel Reinforcement.

In order that there may be economy of material it is necessary that the relation between the cross-sectional areas of the steel and concrete may be such as to make the greatest intensities of stress in each equal to the prescribed working stresses. This condition is said to make a "balanced" section or a balanced percentage of steel reinforcement.

In the general case of tensile and compressive steel reinforcement with the tensile resistance of concrete recognized, the equality of the total tensile and compressive stresses in a normal section of a T-beam gives eq. (9), if the neutral axis lies in the under surface of the flange, as is assumed in establishing eqs. (6), (7) and (8);

$$\tfrac{1}{2}k_1d_1b'+c(1-r)A_2=\tfrac{1}{2}\frac{k_1}{d_1}d_3bd_3+rA_2t. \quad . \quad . \quad (9)$$

Adding $\frac{1}{2}k_1b'd_3$ to each side of eq. (9) and then dividing the resulting equation by $b'(d_1+d_3)=b'h_1$, eq. (10) will result:

$$\tfrac{1}{2}k_1+c(1-r)\frac{A_2}{b'h_1}=\tfrac{1}{2}k_1\left(\frac{d_3}{d_1}\frac{b}{b'}+1\right)\frac{d_3}{h_1}+rt\frac{A_2}{b'h_1}. \quad . \quad (10)$$

or

$$A_2=\frac{k_1}{2}\left\{\frac{\left(\frac{d_3}{d_1}\frac{b}{b'}+1\right)\frac{d_3}{h_1}-1}{c(1-r)-rt}\right\}b'h_1. \quad . \quad . \quad . \quad (10a)$$

It will now be convenient to simplify the forms of the preceding equations by using the following notation:

$e=\frac{E_2}{E_1}$, the ratio of the modulus of elasticity for steel over that for concrete. Usually $e=15$, but occasionally $e=12$.

$p=\frac{A_2}{b'h_1}$, the steel ratio, usually expressed as per cent. of total rectangular section, i.e., in case of the T-beam per cent. of total rectangular outline $b'h_1$.

$\frac{d_1}{h_1}=q$.

The steel ratio or per cent. p, is written in terms of the circumscribing rectangle $b'h_1$ in the interests of simplicity and as being at least as rational as any other method.

The effective depth of the beam is taken as h_1 because the exterior thickness of concrete $(h-h_1)$ is usually a protecting shell against fire, possibly to be partially or wholly destroyed in a conflagration, and, hence, not to be counted as effective beam material. The formulæ may easily be changed so as to be expressed in terms of the full depth h by simply writing $h-o$ for h_1, o being the difference $(h-h_1)$, i.e., the thickness of the concrete from the centre of the tension steel reinforcement to the lower surface of the web or stem of the beam, usually 2 to 3 inches, or more for very large beams. The preceding notation will enable the following formulæ for practical use to be written.

Formulæ to Locate Neutral Axis in T-Beams.

Dividing eq. (3) by h_1 and writing $\frac{E_2}{E_1}\frac{b'}{b}\frac{A_2}{b'}$ for $\frac{E_2}{E_1}\frac{A_2}{b}$;

$$\frac{d_1}{h_1}=q=-\frac{\frac{f}{h_1}\left(\frac{b'}{b}-1\right)+n+\frac{b'}{b}ep}{1-n}$$

$$\pm\sqrt{\frac{\left(\frac{b'}{b}-1\right)\frac{f^2}{h_1^2}+n}{1-n}+2\frac{b'}{b}ep\frac{a+rh_2}{(1-n)h_1}+\frac{\left(\frac{f}{h_1}\left(\frac{b'}{b}-1\right)+n+\frac{b'}{b}ep\right)^2}{(1-n)^2}}. \qquad (11)$$

Doing precisely the same with eq. (4) there will result for the usual condition of the two moduli for tension and compression being the same, but with tensile resistance of the concrete recognized:

$$\frac{d_1}{h_1}=q=\frac{\frac{1}{2}\left(\frac{f^2}{h_1^2}\left(\frac{b'}{b}-1\right)+1+2\frac{b'}{b}\frac{a+rh_2}{h_1}ep\right)}{\frac{f}{h_1}\left(\frac{b'}{b}-1\right)+1+\frac{b'}{b}ep}. \qquad (12)$$

For the special case of neglect of the tensile resistance of concrete, eq. (5) gives, after dividing both sides by h_1:

$$\frac{d_1}{h_1}=q=-\frac{f}{h_1}\left(\frac{b'}{b}-1\right)-\frac{b'}{b}ep$$

$$\pm\sqrt{\left(\frac{b'}{b}-1\right)\frac{f^2}{h_1^2}+2\frac{b'}{b}\frac{a+rh_2}{h_1}ep+\left(\frac{f}{h_1}\left(\frac{b'}{b}-1\right)+\frac{b'}{b}ep\right)^2}. \qquad (13)$$

If the neutral axis lies in the under side of the flange, both sides of eq. (7) are to be divided by h_1, and that equation may then take the form:

$$\frac{d_1}{h_1}=q=-\frac{n+\frac{b'}{b}ep}{\frac{b'}{b}-n}\pm\sqrt{\left(\frac{n+\frac{b'}{b}ep}{\frac{b'}{b}-n}\right)^2+\frac{n+2\frac{a+rh_2}{h_1}\frac{b'}{b}ep}{\frac{b'}{b}-n}}. \quad (14)$$

Or, if concrete in tension be neglected, $n=0$ and eq. (8) then gives

$$\frac{d_1}{h_1}=q=-ep\pm\sqrt{e^2p^2+2\frac{a+rh_2}{h_1}ep}. \quad . \quad . \quad (15)$$

If the reinforcing steel is wholly on the tension side of the beam section $r=1$ and $a+rh_2=a+h_2=h_1$. Hence in eqs. (12), (13), (14) and (15), $\frac{a+h_2}{h_1}=1$, but no other change is needed.

The value of the steel ratio or per cent. for the perfectly general case is given by eq. (10), by placing $p=\frac{A_2}{b'h_1}$ in that equation and then solving for p, which will give:

$$p=\frac{k_1}{2}\,\frac{\left(\frac{d_3}{d_1}\frac{b}{b'}+1\right)\frac{d_3}{h_1}-1}{c(1-r)-rt}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If concrete in tension be neglected, eq. (9) shows that $\frac{k_1}{d_1}d_3=k_1=0$ in eq. (16), and that equation will then take the form

$$p=\frac{k_1}{2}\,\frac{\frac{d_3}{h_1}-1}{c(1-r)-rt}=-\frac{k_1}{2}\,\frac{d_1}{(c(1-r)-rt)h_1}. \quad . \quad (17)$$

If the reinforcing steel is wholly on the tension side of the beam section, $r=1$ and $c(1-r)-rt=-rt$. Eq. (17) will then take the form:

$$p=\frac{k_1}{2}\frac{1-\dfrac{d_3}{h_1}}{t}=\frac{k_1}{2t_1}\frac{d_1}{h_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The laws of the common theory of flexure give the following relations:

$$\frac{k_1}{d_1}=\frac{t}{e}\frac{1}{d_3}; \quad \text{or} \quad \frac{d_1}{d_3}=\frac{ek_1}{t}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

hence:

$$\frac{d_1+d_3}{d_3}=\frac{ek_1+t}{t}=\frac{h_1}{d_3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Also:

$$\frac{c}{t}=\frac{d_2}{d_3}, \quad \text{and} \quad t=\frac{d_3}{d_2}c. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Placing the value of $\dfrac{d_3}{h_1}$ from eq. (20) in eq. (18):

$$p=\frac{1}{2}\frac{1}{\dfrac{t}{k_1}\left(\dfrac{t}{ek_1}+1\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The area of the steel section A_2 can at once be found from p in all cases by simply writing:

$$p=\frac{A_2}{b'h_1}, \text{ and hence, } \quad A_2=pb'h_1. \quad . \quad . \quad . \quad (23)$$

In all these equations for locating the neutral axis of a section NS, Fig. 1, the ratios $\dfrac{b'}{b}$, $\dfrac{f}{h_1}$ and other similar quantities depending on the dimensions of the cross-section will be known, at least tentatively. Indeed in making practical applications of these equations it will in general be necessary to assign trial dimensions of the cross-sections of the beam if the application is made for the design of the latter. Such trial dimensions must be assigned by the aid of prior experience or other beams already designed for more or less similar conditions. After trial dimensions have been tested by actual computations for the assigned loads, such modifications or revision of these dimensions as may be necessary must then be made.

If the neutral axis lies within the section of the flange, the changes in the preceding formulæ already indicated for that case may be easily introduced, but the case is so rare that complete expressions for its treatment need not be written. If the tensile resistance of the concrete is neglected, the formulæ for the special case, only, of the neutral axis lying in the under surface of the flange are needed, simply considering the depth of flange f as the distance from the upper flange surface to the neutral axis. In fact that special case will cover the great majority of T-beams with sufficient accuracy for practical purposes.

The general value of the steel ratio or per cent. for a balanced section may be considered as given by eq. (16) even though the neutral axis does not lie in the under surface of the flange, at least as a reasonably close approximation even when the position of the neutral axis is materially different from that supposition. In determining that ratio or per cent. k_1, c and t must be considered as prescribed working values of those respective intensities of stress, the ratio between c and t being fixed by the distances of the steel reinforcements from the neutral surface. When the steel reinforcement is wholly on the tension side, as in the usual cases, k_1 and t are prescribed working stresses for the concrete in compression and the steel in tension, respectively.

Art. 98.—Bending Moments in Concrete-steel T-Beams by Common Theory of Flexure.

The complete expressions for the bending moments of concrete-steel T-beams may now be written and their values for any particular case estimated, by introducing the notation already employed into eq. (4) of Art. 95. The moment of inertia or integral in the last term of the second member of that equation takes the form:

$$\int_{h'_2-d'_2}^{d'_2} z^2 dA_2 = (1-r)A_2 d_2^2 + rA_2 d_3^2. \quad . \quad . \quad (1)$$

Referring to Fig. 1 of Art. 96, the other two moments of inertia in the first and second terms of the second member of eq. (4) of Art. 95 become:

$$\int_0^{d_1} z^2 dA_1 = \frac{b'd_1^3}{3} - (b'-b)\frac{(d_1-f)^3}{3}. \quad . \quad . \quad (2)$$

$$\int_{h_1-d_1}^{0} z^2 dA_1 = b\frac{(h_1-d_1)^3}{3}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Also,

$$E_1 u = \frac{k_1}{d_1}; \text{ and } E_2 u = \frac{E_2}{E_1}E_1 u = \frac{E_2}{E_1}\frac{k_1}{d_1}. \quad . \quad . \quad (4)$$

Introducing these values in eq. (4) of Art. 95, remembering that $h_1 - d_1 = h_2 - d_2 = d_3$ the bending moment M for a T-beam become:

$$M = \frac{k_1}{d_1}\left[\frac{b'd_1^3}{3} - (b'-b)\frac{(d_1-f)^3}{3} + n\frac{bd_3^3}{3} + \frac{E_2}{E_1}A_2(d_2^2(1-r) + d_3^2 r)\right] (5)$$

This equation is written in terms of one intensity of stress k_1 for convenience in computation, but it will be advisable sometimes to use other intensities, such as the greatest stress t in the tensile steel reinforcement. This can readily be done by the aid of the following relations based upon common theory of flexure, in addition to the relations shown in eq. (4) and remembering that $\frac{E_2}{E_1}=e$.

$$eE_1u=\frac{ek_1}{d_1}=\frac{t}{d_3}=\frac{c}{d_2}. \quad \text{Also } \frac{k'}{d_3}=\frac{k_1}{d_1}. \quad . \quad . \quad . \quad (6)$$

These simple values will enable any intensity to be expressed in terms of any other. The greatest compression in the concrete, k_1, and greatest tension in the steel, t, are those mostly required.

If, as is usual, *the two moduli of elasticity of concrete in tension and compression are equal to each other*, $n=1$ in eq. (5), but no other change is needed.

Neglect of Concrete in Tension.

If the resistance to concrete in tension be neglected, $n=o$ in eq. (5), and:

$$M=\frac{k_1b}{d_1}\left[\frac{b'}{b}\frac{d_1^3}{3}-\left(\frac{b'}{b}-1\right)\frac{(d_1-f)^3}{3}+e\frac{A_2}{b}(d_2^2(1-r)+d_3^2r)\right]. \quad (7)$$

In ordinary T-beams all steel reinforcement is in tension; hence $r=1$ and eq. (7) becomes:

$$M=\frac{k_1b}{3d_1}\left[\frac{b'}{b}d_1^3-\left(\frac{b'}{b}-1\right)(d_1-f)^3+3e\frac{A_2}{b}d_3^2\right] \quad . \quad . \quad (8)$$

Special Case of Neutral Axis in under Surface of Flange.

In this case $(d_1-f)=0$ and eq. (7) will take the form:

$$M=\frac{k_1b}{3d_1}\left[\frac{b'}{b}d_1^3+3e\frac{A_2}{b}(d_2^2(1-r)+d_3^2r)\right]. \quad . \quad . \quad (9)$$

If the steel reinforcement is wholly on the tension side $r=1$, as in eq. (8).

This special case may, without material error, be considered to include all T-beams for which (d_1-f) or $(f-d_1)$ is relatively small.

Art. 99.—Concrete Steel Beams of Rectangular Section.

All formulæ for reinforced concrete beams with rectangular section may be written at once from those for T-beams by simply making $b'=b$ in the latter. A typical rectangular cross-section for the general case is shown in Fig. 1, Art. 96, although in the usual case the steel reinforcement is wholly on the tension side.

Formulæ to Locate Neutral Axis in Beams of Rectangular Section.

The general case requires eq. (11) of Art. 97. Making $b'=b$ that equation becomes:

$$\frac{d_1}{h_1}=q=-\frac{n+ep}{1-n}\pm\sqrt{\frac{n}{1-n}+2ep\frac{a+rh_2}{(1-n)h_1}+\frac{(n+ep)^2}{(1-n)^2}}. \quad (1)$$

If the moduli for tension and compression are the same, as is invariably assumed in engineering practice, $b=b'$ in eq. (12), Art. 97:

$$\frac{d_1}{h_1}=q=\frac{\frac{1}{2}+ep\frac{a+rh_2}{h_1}}{1+ep}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the tensile resistance of the concrete be neglected, the same substitution of $b=b'$ is made in eq. (13) of Art. 97:

$$\frac{d_1}{h_1}=q=-ep\pm\sqrt{2ep\frac{a+rh_2}{h_1}+e^2p^2}. \quad . \quad . \quad (3)$$

When the reinforcing steel is wholly on the tension side $r=1$ and $a+rh_2=a+h_2=h_1$, hence:

$$\frac{d_1}{h_1}=q=-ep\pm\sqrt{2ep+e^2p^2}. \quad . \quad . \quad . \quad . \quad (4)$$

This is the ordinary case.

It will be observed that eq. (3) is identical with eq. (15) of Art. 97.

The steel ratio or per cent., $p=\frac{A_2}{bh_1}$, for the general case of balanced sections is given by eq. (16) of Art. 97 by making $b=b'$:

$$p=\frac{k_1}{2}\frac{\left(\frac{d_3}{d_1}+1\right)\frac{d_3}{h_1}-1}{c(1-r)-rt}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When the tensile resistance of the concrete is neglected, the value of p given by eq. (17) of Art. 97 remains unchanged:

$$p=\frac{k_1}{2}\frac{\frac{d_3}{h_1}-1}{c(1-r)-rt}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Eq. (18) of the same article gives p as it stands if the tensile resistance of the concrete is neglected, i.e., of $r=1$:

$$p=\frac{k_1}{2}\frac{1-\frac{d_3}{h_1}}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eqs. (19), (20) and (21) of Art. 97 hold true for rectangular sections and, hence, eq. (7) may take the form of eq. (22) of that article:

$$p=\frac{1}{2}\frac{1}{\frac{t}{k_1}\left(\frac{t}{ek_1}+1\right)}. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

These values of the steel ratio p will form the basis of economical beam design. The working stresses k_1 for concrete in compression and t for steel in tension will be prescribed in the specifications for the work to be done.

Bending Moments for Rectangular Sections.

The general value of the bending moment, M, i.e., for unequal moduli in tension and compression and with tensile resistance of concrete recognized, is given by eq. (5) of Art. 98 after making $b'=b$.

$$M=\frac{k_1}{d_1}\left[\frac{bd_1^3}{3}+n\frac{bd_3^3}{3}+eA_2(d_2^2(1-r)+d_3^2r)\right]. \quad . \quad . \quad (9)$$

If it be desired to express this equation in terms of other intensities than k_1, the following relations given in eq. (6) of Art. 98 will enable that to be done:

$$\frac{ek_1}{d_1}=\frac{t}{d_3}=\frac{c}{d_2} \quad \text{and} \quad \frac{k'}{d_3}=\frac{k_1}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The moduli for concrete in tension and compression are invariably considered equal, and in that case $n=1$ in eq. (9), but no other change is required.

Neglect of Concrete in Tension.

The neglect of the concrete in tension is affected by making $n=o$ in eq. (9) giving:

$$M=\frac{k_1}{d_1}\left[\frac{bd_1^3}{3}+eA_2(d_2^2(1-r)+d_3^2r)\right]. \quad . \quad . \quad (11)$$

The steel reinforcement is usually wholly on the tension side, i.e., $r=1$. Making this substitution in eq. (11):

$$M=\frac{k_1}{d_1}\left(\frac{bd_1^3}{3}+eA_2d_2^2\right). \quad . \quad . \quad . \quad . \quad (12)$$

All the preceding values of the bending moment M may, if desired, be expressed in terms of the steel ratio p by substituting pbh_1 for A_2.

In the preceding equations the distance d_1 of the neutral axis from the exterior compression surface of the beam is to be found from the appropriate formula for q of this article, since $d_1=qh_1$.

The preceding equations complete all that are necessary in the treatment of practical questions of design or of ultimate carrying capacity.

In all the preceding analyses of Arts. (97), (98) and (99), the total depth h of either the T-beam or the solid rectangular section may be used if desired by making $p=\frac{A}{b'h}$ or $=\frac{A}{bh}$, but in that case in the equations for d_1 the fraction $\frac{a+h_2}{h}$ (when $r=1$) will occur, that fraction having values varying from about .67 for floor slabs to .95, for beams of much depth instead of $\frac{a+h_2}{h_1}=1$. It is rare, however, that such a form of equation will need to be used.

Art. 100.—Shearing Stresses and Web Reinforcements in Reinforced Concrete Beams.

In the case of reinforced concrete T-beams it will be assumed that the stem or web extending from the upper surface of the flange down to the centre of the tension steel, i.e., having the depth h_1 and the width b, will carry the whole transverse shear. In the solid rectangular section, the total sectional area less that part of it below or outside of the centre of the tension steel reinforcement will be assumed to resist transverse shear, i.e., the resisting area will be bh_1 as in the case of the T-beam.

Fig. 1 represents a simple T-beam supported at each end Q and R, having steel reinforcement both in the flange and in the lower or tension part of the beam. In order to illustrate fully the action of the shearing stresses in such a beam, the tensile resistance of the concrete may be recognized. If there were no steel reinforcement, the analysis of Art. 15 shows that in the case of a rectangular section the greatest intensity of either longitudinal or transverse shear exists at the neutral axis of the section and has the value of $\frac{3}{2}$ the average shear on the whole section. If s be that maximum intensity of shear and if S is the total external transverse shear at the given section, then will

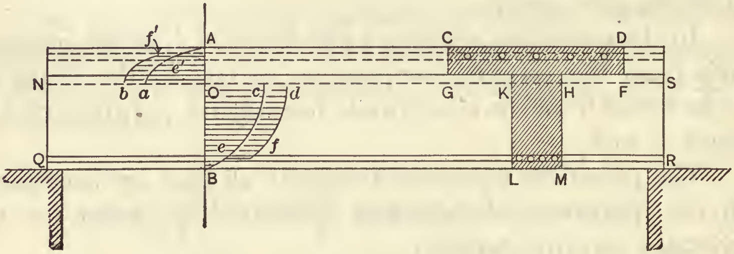


FIG. 1.

$s=\frac{2}{3}\frac{S}{bh_1}$. In Fig. 1, $Oc=Oa=s$ and both of the curves $Ae'a$ and Bec are parabolas with the vertices at a and c, so that horizontal ordinates from AO to the curve in the one case and from BO to the curve in the other case represent intensities of the longitudinal and transverse shear at the points from which those horizontal ordinates are drawn. This is the condition of the shearing stresses in beams of a single material subjected to flexure, and reinforced concrete beams represent similar members, but of two materials. The stresses given to the longitudinal steel reinforcements may be assumed provisionally to be conveyed to them from the neutral surface at a constant intensity s_1 and in Fig. 1 that constant intensity is represented by cd and ab. The curves df and bf' are drawn so as to make a constant horizontal ordinate between them and the parabolas already indicated. The total maximum intensity of longitudinal or transverse shear at the neutral axis will, therefore, be the sum of s and s_1; this may be taken with sufficiently close approximation, at least for practical purposes, as $\frac{3}{2}$ the total average transverse shear at a given section.* Even if the horizontal ordinate between the two curves is not uniform, this value of the maximum intensity may properly be used.

In the case of the tensile resistance of the concrete

---

* It has come to be the practice, for some reason not easily appreciated, to treat the transverse shear in the normal section of a reinforced concrete beam as if it were uniformly distributed over that normal section, which is an error on the side of danger. In the interests of both safety and correct analysis, the established variation of intensity of shear in the normal section of a bent beam should be recognized, for it holds just as much for a resisting concrete section as for a section of any other material. When the bending resistance of the concrete on the tension side is ignored, the law of variation of intensity will change, but the maximum intensity at the neutral axis will be unchanged.

being neglected, Fig. 2, representing a part of a continuous reinforced concrete T-beam, shows the variation of the intensity of the longitudinal and transverse shears. The parabolic curve *Aa* shows the variation of the intensity of the shear in passing from the neutral axis *O* of the section to the exterior surface *A*, *aO* being the maximum intensity and equal to $\frac{3}{2}$ the average intensity for the entire section. Inasmuch as the tensile resistance of the concrete is neglected, the maximum intensity of longitudinal shear $Ob = Oa$ may be considered as varying by some unknown law such, however, as to make the total internal transverse shear equal to the total external, *cd* representing the intensity at the centre of the tension steel reinforcement.

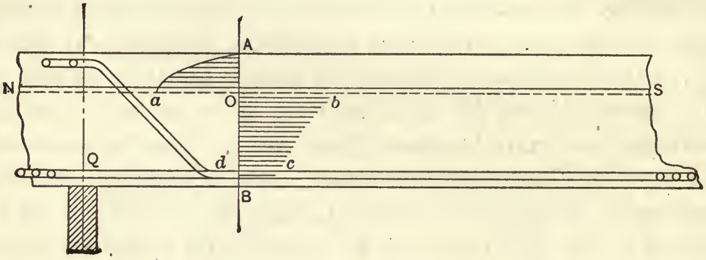


FIG. 2.

It is impossible to analyze with complete accuracy the variation of the intensity of shear in the concrete by which the reinforcing steel, either in tension or compression, acquires its stress, but it cannot have a uniform value equal to the maximium intensity at the neutral surface. It is to be understood that the constant horizontal shear ordinates in both Figs. 1 and 2 are to be interpreted in this manner.

The shearing resistance of concrete in any plain or reinforced concrete structure is of uncertain value, much as is the tensile resistance, although the practice of crediting it with some material amount may be justified. At the same time the incipient surface cracks which are found to form with lapse of time at any point may extend deep enough ultimately to prejudice seriously resistance to shear. It is probably hazardous, therefore, to depend upon concrete alone to resist transverse shear in beams, either of the T form or solid rectangular, or, of any other form. In fact it is prudent to state unqualifiedly that reinforced concrete beams carrying moving loads tending to produce vibrations or shock should be so designed as to provide for the entire transverse shear independently of the shearing resistance of the concrete. This provision for resistance to transverse shear is made chiefly by bending upward, in that part of the spans near the end supports, the steel tension reinforcement as shown in Figs. (2) and (3). The inclination of the bent parts of the rods will depend upon the judgment of the engineer in view of the length of the span, depth of beam or other features of each case. Usually all of the rods constituting the tension reinforcement are not bent upward, as that much provision for shear is not needed. If the span is short, there may be but one set of bent rods, as shown in Fig. 2, but in other cases there may be two or more sets bent upward at different distances from each end of the span, as shown in Fig. 3, the number of such sets of rods being determined, like the angle of inclination, in accordance with the best judgment of the designing engineer. The vertical components of the stresses in these

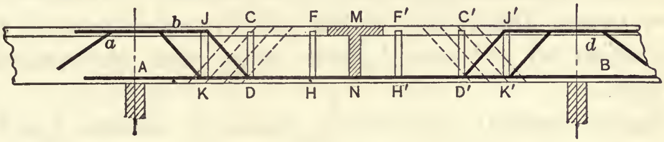


FIG. 3.

bent rods may obviously equal the transverse shear at the section considered. For example, in Fig. 3, the total transverse shear at the section *CD* multiplied by the tangent of the inclination of the rods to the vertical must for good design be at least equal to the required horizontal reinforcement to be afforded by those rods, i.e., the section of the rods must be sufficient to take their stresses without exceeding the prescribed working stress, which is frequently 16,000 pounds per square inch. A similar computation is to be made at other points where rods are to be bent upward. It is not necessary (although usual) that the different sets of inclined parts of rods should be parallel, i.e., those nearer the centre of the span may have a greater inclination to a vertical than those near the points of support.

Again, vertical reinforcing pieces or stirrups, such as those at *FH*, *F'H'*, *CD*, *C'D'*, in Fig. 3, are introduced under the assumption that they will take the vertical transverse shear in tension. These stirrups are of a variety of forms and may be in sets of two or more vertical prongs or parts, but they should be securely fastened to the horizontal steel reinforcement. If the total transverse shear at any section as *JK* is supposed to be carried by the stirrup as tension in that section, the cross-sectional area of the steel stirrups should be sufficient for that purpose at the prescribed working stress. Furthermore, the adhesive shear or bond on the exterior surfaces of these stirrups should be sufficient to give such tension without exceeding the prescribed working stress for that shear or bond. These vertical stirrups are thus supposed to act the part essentially of vertical truss members in tension and so produce diagonal stresses of compression in the concrete as shown by broken lines in the vicinity of *KC* and *K'C'*. It is known that the greatest diagonal stresses of tension and compression exist at angles of 45 degrees with the neutral surface of every bent beam. The function of these stirrups is intended to be such as to relieve the concrete of that tension and induce diagonal stresses of compression. Indeed their function is somewhat similar to that of vertical stiffening members on the web plates of plate girders when those stiffeners are assumed to take tension and produce compression in the web in a 45-degree direction, as was fully shown in Art. 34. The distance apart of these vertical stirrups should certainly not be greater than the depth of the beam from the upper surface down to the tension steel reinforcement; probably a horizontal distance apart of about three-quarters of that depth is advisable.

If any beam carry a set of loads, W_1, W_2, W_3, etc., and if R' is the end shear at A, Fig. 3, and if ΣW be the sum of the loads between the end A and any section at which it is desired to obtain the transverse shear S', then will that transverse shear at any stirrup, as CD, Fig. 3, be $S'=R'-\Sigma W$, and it is assumed that the stirrup will carry that shear as tension. If t' is the allowed tensile stress in the stirrup, the sectional area A_s of the latter will be $A_s=\frac{S'}{t'}$. If the intensity of permitted bond shear is s' and if the circumference of a stirrup section is o, and if l' is the imbedded length of one complete stirrup, including all prongs, then must $s'ol'$ be at least equal to S'. Evidently a form of cross-section like an oblong rectangle will give much more area for bond shear, for a given sectional area, than such a section as a circle or a square and it will have a corresponding advantage for this purpose.

The ends of all stirrup bars as well as all reinforcing rods should be turned or bent at right angles so as to prevent slipping at and near the ends. Furthermore, they should preferably be looped at top and bottom, around the reinforcing rods where they exist, so as to bear directly on the concrete supplementary to the bond shear.

Obviously if both the inclined bent rods shown in Figs. 2 and 3 and the vertical stirrups shown in Fig. 3 effectively perform their functions, both would not be needed at the same part of a beam, but as the effectiveness of each detail by itself is somewhat uncertain, both are frequently used concurrently. The stirrups may judiciously be used in the central part of the span extending well toward the ends where the bent rods are employed.

Another form of steel reinforcement of beams is shown in Fig. 4, which is supposed to be part of a reinforced concrete beam on both sides of *CD*, the centre of span. The beam may be either a T-beam or a beam of rectangular section. The steel reinforcement *AB* is supposed to be on the tension side of the beam only, although a precisely similar reinforcement might be placed on the compression side also. The small inclined bars *ab*, *cd*, *a′b′*, *c′d′*, etc., are usually parts of the main tension reinforcement bent upward in a diagonal direction, as shown, which may be at the angle of 45 degrees of theory or at some other angle. They should extend above the neutral surface *NS* and be carried nearly to the top of the beam.

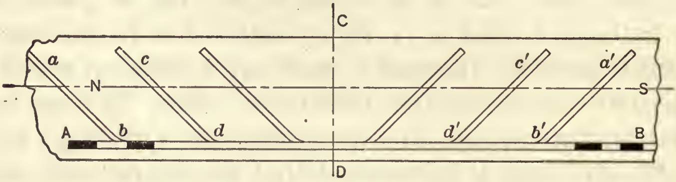


FIG. 4.

As has already been indicated, a solid beam of a single material will have the greatest intensity of tensile stress at the neutral surface, making an angle of 45 degrees with a horizontal line and sloping upward and away from the centre of span, as indicated in Fig. 4. Tests of plain and reinforced concrete beams show that in those parts of the span near the end, this diagonal tension is likely to cause failure of the concrete. Hence these inclined bars are run up from the main tension steel reinforcement to assist the concrete in taking up this diagonal tension and thus preventing its failure so far as possible. The concrete will be in compression in the diagonal direction at right angles to these inclined bars.

If a vertical section of a beam should cut two or more sets of them, the force or stress obtained by multiplying half the total transverse shear at such a section by the secant of the inclination of these bars to a vertical line will give the total stress to which those two or more sets will be subjected, the distribution being assumed to be uniform among them. The other half of the transverse shear at that section may be considered as giving compression to the concrete at right angles to the 45-degree tension in the inclined bars. It is clear also that the total bond shear on the surface of each one of the inclined bars must be at least equal to the tensile stress which the bar carries at an intensity not greater than that prescribed in the specifications for the work. If such a normal or vertical section of the beam cuts but one set of these inclined bars, the single set must take the stress due to half the total transverse shear, precisely as described above for two or more sets. The ends of these inclined bars should be bent at right angles or otherwise formed so as to prevent the possibility of slipping, and thus supplement effectively the bond shear.

It is clear that such bars must add to the carrying capacity of a beam, not only by taking up the inclined tensile stresses as described, but also as tending to bind the entire beam together as a unit.

The greatest transverse shear is that at the ends of the span where the sum of the vertical components of the stresses in the bent rods must be equal to that end shear or to so much of it as may be prescribed. In Fig. 3, for example, the sum of the vertical components of the stresses in the inclined rods *bK* (a set of reinforcing rods) must be equal to the transverse shear prescribed. All inclined rods like *bK*, *JD*, etc., lie in the direction of the diagonal tension (maximum at inclination of 45 degrees) and act directly to carry shear.

When beams are continuous over supports, as shown in Figs. 2 and 3, bending moments will be developed over those supports opposite in sign to those found at and near the centres of the spans, producing tension in the upper parts of the beams. For this reason tensile steel reinforcement formed either of the bent rods continued into the adjoining spans, as shown in Fig. 3, or of separate rods introduced for the purpose are required to take that tension.

The precise degree of constraint when beams or girders are " continuous " over points of support cannot be determined, but certain values of moments expressing the results of experience in modifications of formulæ for conditions of perfect continuity will be given in the next article.

In the practical consideration of provision for transverse shear in reinforced concrete beams, it is a matter of some uncertainty how much the concrete may be allowed to take, if any, and hence what corresponding steel must be introduced in the form of bent reinforcing rods or stirrups. As has been intimated, it is a serious question whether the concrete should be credited with any resistance to shear. It is frequently the practice to assume that one-third of the transverse shear will be carried by the concrete under suitable conditions and a prescribed working stress, but that the other two-thirds shall be taken by steel provided for the purpose as already described. Aside from the difficulty arising in the attempt to distribute by measure the discharge of an important function between two different methods, there is grave doubt about the propriety of assuming dependable resistance against shear by concrete, particularly if the moving load is of a character to produce vibrations or shock. In the latter case steel should certainly be provided to take all shear. That procedure is more prudent in all cases except, possibly, where the load is wholly dead or essentially so, when the concrete may be allowed to carry one-third of the total transverse shear.

Many tests of full-size reinforced concrete T-beams and beams of rectangular section have been made by Profs. Talbot, Withey, Hatt, and others in the United States and by Considére, Feret and other foreign investigators in Europe, and full descriptions of all results may be found in the Bulletins of the Universities of Illinois and Wisconsin and in many other publications, hence it would be superfluous to repeat them here. The working results of those tests bearing upon computations for design or other practical work will be given in the next article, chiefly in connection with the recommendations of the " Report of the Committee on Concrete and Reinforced Concrete " of the American Society for Testing Materials, Vol. XIII, 1913.

Art. 101.—Working Stresses and Other Conditions in Reinforced Concrete Design.*—Design of T-beams.

In the design of reinforced concrete beams there are some features of the work determined by experience and

---

* The report of the Committee on Concrete and Reinforced Concrete, Proc. of Am. Soc. for Testing Materials, Vol. XIII, 1913, has largely been used in the preparation of this article.

quite independent of analysis. Reinforcing bars or rods must be surrounded by enough concrete to receive the proper stress from the latter. This may be assumed to be done if the lateral spacing between the centres of parallel rods is not less than three diameters, or two diameters from the outer concrete surface to the centre of the nearest rod, the clear vertical space between two horizontal layers of rods being not less than 1 inch. It is seldom advisable to use more than two courses of such rods. In all cases scrupulous and effective care should be taken by the aid of blocking, ties and other devices to hold the reinforcing steel accurately in place until the concrete is set.

As a fire protection a thickness of at least 2 inches of concrete should be placed outside of the steel in all reinforced concrete beams and columns. In relatively small beams a least thickness of $1\frac{1}{2}$ inches may be allowed, and 1 inch may be permitted in floor slabs.

Floor slabs should be designed and reinforced as continuous over supports, and if the length in any case exceeds 1.5 the width transverse reinforcement should be provided to carry the entire load.

The continuity of beams and slabs may be recognized and expressed as follows, assuming the combined dead and moving loads equivalent to a uniform load of q (pounds) per linear foot on the effective span:

Floor slabs: moment at centre of span and at end of span. . . . . $\frac{ql^2}{12}$.

Beams: For exterior span of series, moment at centre of span and at outer fixed end of span. $\frac{ql^2}{10}$.

For interior spans moment at centre and at end of span. . . . . $\frac{ql^2}{12}$.

Beams and Slabs: continuous over two spans only, moment at central support . . . . $\frac{ql^2}{8}$.

Moment near middle of span . . . . $\frac{ql^2}{10}$.

At ends of continuous beams and girders where the degree of constraint is uncertain, the computation of the negative end bending moment must be controlled by the judgment of the responsible engineer.

Working Stresses.

The following working stresses are chiefly given as per cents. of the accompanying ultimate compressive resistances. They are for moving and dead loads considered as static loads, with the assumption that proper additions to moving loads must be made, when advisable, to provide for impact or vibrations.

ULTIMATE COMPRESSIVE RESISTANCES

Aggregate.	Proportions and Ult. Resistances, Pounds per Sq. In.				
	1 : 1 : 2	1 : 1½ : 3	1 : 2 : 4	1 : 2½ : 5	1 : 3 : 6
Granite, trap rock	3,300	2,800	2,200	1,800	1,400
Gravel, hard limestone, sandstone	3,000	2,500	2,000	1,600	1,300
Soft limestone and sandstone	2,200	1,800	1,500	1,200	1,000
Cinders	800	700	600	500	400

Working Compression in Extreme Fibre of Beam.

The working intensity in the extreme compression fibre of a beam may be taken at 32.5 per cent. of the ultimate compressive resistance as determined by testing concrete cylinders 8 inches in diameter and 16 inches high at the age of 28 days. If, for instance, the ultimate compressive resistance of a 1 : 2 : 4 concrete is 2200 pounds per square inch, then the extreme fibre working stress would be 2200 × .325 = 715 pounds per square inch. Adjacent to the support of continuous beams, these stresses may be increased 15 per cent.

Shear and Diagonal Tension.

For beams with horizontal reinforcing bars only, i.e., with no web reinforcement, 2 per cent. of the ultimate compressive resistance may be allowed. If the latter were 2200 pounds per square inch, as for the 1 : 2 : 4 concrete of the above table, the allowed shear would be .02 × 2200 = 44 pounds square inch. This shear would be taken wholly by the concrete.

For beams thoroughly reinforced in the web, 6 per cent. of the ultimate compressive resistance may be allowed. In this case, however, the web reinforcement, exclusive of bent-up reinforcing bars, must be designed to take two-thirds of the external vertical shear. Again, using the 1 : 2 : 4 concrete, the allowed shear would be 0.06 × 2200 = 132 pounds per square inch of total concrete section. In this case, however, the steel reinforcement would be designed to carry two-thirds of the total transverse shear, making the actual shear in the concrete 44 pounds per square inch on the basis of the exact division between the two methods of carrying the shear prescribed.

" For beams in which part of the longitudinal reinforcement is used in the form of bent-up bars distributed over a portion of the beam in a way covering the requirements for this type of web reinforcement, the limit of maximum vertical shearing stress " may be taken 3 per cent. of the ultimate compressive resistance.

Where what is termed " punching shear " occurs, i.e., pure shear without bending, a working shearing stress of 6 per cent. of the ultimate compressive resistance may be allowed.

Bond or Adhesive Shear.

The working intensity for this bond or shear between concrete and plain reinforcing rods may be taken at 4 per cent. of the compressive resistance, but 2 per cent. only for drawn wire. For 1 : $2\frac{1}{2}$: 5 concrete at 1600 pounds per square inch of ultimate compressive resistance, the two working stresses would be $.04 \times 1600 = 64$ pounds per square inch or half that for drawn wire.

Steel Reinforcement.

The tensile or compressive working stress in steel reinforcement should not exceed 16,000 pounds per square inch.

Modulus of Elasticity.

For computations locating the neutral axis and for computing the resisting moment of beams and for compression of concrete in columns, it is recommended that the ratio of the steel modulus divided by the concrete modulus be taken at 15 if the ultimate compressive resistance of the concrete is taken at 2200 pounds per square inch or less; and at 12 if the ultimate compressive resistance of the concrete is greater than 2200 pounds per square inch and less than 2900 pounds per square inch; and, finally, at 10 if the ultimate compressive resistance of the concrete is taken greater than 2900 pounds per square inch.

The preceding specifications express substantially the views of a Committee on Concrete and Reinforced Concrete of the American Society for Testing Materials, 1913. In that Report the transverse shear is computed as if uniformly distributed throughout the normal section of a bent beam, which, as has already been indicated, is incorrect. On the whole, however, the recommended values are judicious and may be commended for practical use.

Design of T-Beam for Heavy Uniform Load.

The given data are as follows: Effective length of span 32 feet (non-continuous); moving load on floor, 250 pounds per square foot. Floor slab 6 inches thick. Each T-beam carries 10 feet width of floor.

The steel reinforcement of the beam is on the tension side only.

As the floor slab will be reinforced (at right-angles to the beam) its weight will be taken at 155 pounds per cubic foot. The concrete will be considered a 1 : 2 : 4 mixture weighing about 150 pounds per cubic foot.

The floor slab being 6 inches thick, a little less than four times its thickness will be assumed as effective compression flange area on each side of the stem or web of the beam. Referring to Fig. 1 of Art. 96, the following dimensional data will be assumed for trial computations:

$b'=60$ inches; $f=6$ inches; $b=15$ inches. $h_1=29$ inches; thickness of concrete outside of steel, 2 inches. Trial value of steel ratio or per cent., $p=.8$ per cent. $=.008$.

The working stresses are:

Compression for concrete: $k_1=$ 650 lbs. per sq.in.

Tension for steel: $t=16{,}000$ lbs. per sq.in.

$e=15$.

Eq. (13) of Art. 96 then gives:

$$\frac{d_1}{h_1}=-1.11\pm1.524=+.414 \therefore d_1=.414h_1=12 \text{ ins.}$$

Eq. (18) of Art. 96 may now be used:

$$p = \frac{650}{2 \times 16,000} \times .414 = .0084 = .84 \text{ per cent.}$$

This last value of p agrees closely enough with the assumed value. Hence the computed values of d_1 and p may be accepted. Consequently: $d_3 = 29 - 12 = 17$ inches.

The required steel sectional area is:

$$A_2 = pb'h_1 = .0084 \times 60 \times 29 = 14.62 \text{ square inches.}$$

There may then be taken eight $1\frac{1}{2}$-inch round bars whose aggregate area is 14.16 square inches.

The cross-section of the effective beam may be made as shown in Fig. 1. Deformed rods of any suitable section of the aggregate computed area may obviously be used.

The dead load or own weight of the beam, including 10 feet width of floor slab, may be taken at 1225 pounds per linear foot of span. The uniformly distributed moving load will be $10 \times 250 = 2500$ pounds per linear foot of span. The bending moment produced by these two loads will be:

$$M = (2500 + 1225)\frac{30 \times 30}{8} \times 12 = 5,028,800 \text{ inch pounds.}$$

The resisting moment of the beam section must now be computed by the aid of eq. (8), Art. 97.

$\frac{b}{d_1} = 1.25$; $\frac{b'}{b} = 4$; $d_1 - f = 6$; $\frac{A_2}{b} = .944$; $d_3 = 17$; $d_1 = 12$ and $e = 15$.

Introducing these numerical quantities in eq. (8), Art. 97:

$$M = 5,024,611 \text{ inch-pounds.}$$

This result is substantially equal to the external bending moment found above and the tentative design may be accepted as satisfactory.

There still remains to be considered suitable provision for end and intermediate shears which will be made by bending upward the proper number of reinforcing rods supplemented by stirrups.

Fig. 1 shows to scale about 12½ feet of the T-beam, the effective cross-section, 60 inches wide, being shown in shaded outline. The dimensions are self-explanatory in connection with the computations already made. *NS* is the neutral axis. The eight 1½-inch round rods in two courses with their central line 4 inches from the bottom surface are shown both in section and in longitudinal broken lines. This latter dimension allows a fire-protecting shell of concrete 2 inches thick and 1 inch clear vertical distance between the two layers of four rods each.

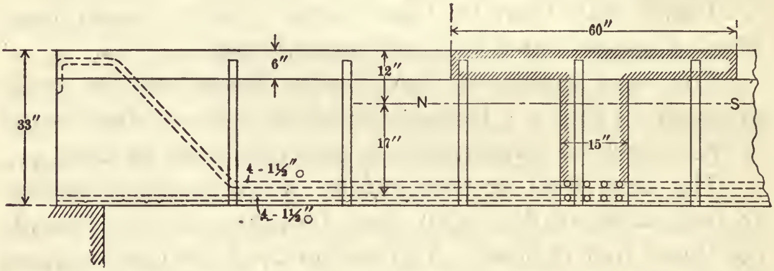


FIG. 1.

The combined dead and moving load on the beam has already been shown to be 3725 pounds per linear foot, making the end shear 3725 × 15 = 55,875 pounds. If bent rods inclined at an angle of 45° be supposed to take this whole shear, the total stress in those rods will be 55,875 × sec. 45 degrees = 79,007 pounds. If the steel be stressed at 16,000 pounds per square inch, a little less than

5 square inches of section will be required. Three $1\frac{1}{2}$-inch rounds, or their equivalent sectional area, will supply the desired section. It will be convenient to bend the upper set of four rods as shown in Fig. 1, thus reducing the actual stress in the inclined parts to about 12,000 pounds per square inch, the reduced unit stress not being objectionable. A greater vertical depth of concrete would have been available for shear if the lower set had been bent upward, but with the use of stirrups this is not important and the arrangement shown is a little more convenient in actual construction. If desired the lower set could be bent, but it would be necessary to slightly rearrange the position of all the rods so that the bent parts of the lower set may pass the upper set, all of which is quite feasible. The horizontal ends of the bent rods should also be bent at right angles so as to secure the firmest possible hold on the concrete at the end of the beam. The horizontal ends of the bent bars are about 12 inches long, making the lower bend of the same rods about 3.25 feet from the end of the beam.

Vertical stirrups, 24 inches apart, will be placed throughout the central part of the beam and they will be carried down so as to pass under the lower reinforcing rods. There will be four prongs to each stirrup, looped at top and bottom. By this arrangement of the stirrups the bond shear on their surfaces is greatly reinforced by the vertical bearing on the concrete and reinforcing rods at the bottom. The first stirrup, as shown, will be placed at the lower bend in the upper set of reinforcing rods, although the stress in it is indeterminate, as the inclined rod is supposed to take the total shear.

The total transverse shear in the second stirrup, 5.25 feet from the end of the beam, will be computed as carrying in tension $9.75 \times 3725 = 26,320$ pounds, requiring at 16,000 pounds per square inch, 2.25 square inches. Four $1\frac{1}{2}$-inch ×

$\frac{3}{8}$-inch flat bars will give the required area, each such flat bar constituting one member or prong of the stirrup. The shear at the next stirrup point, 2 feet farther from the end of the span, will be 28,870 pounds, and four $1\frac{1}{2}$-inch $\times \frac{5}{16}$-inch stirrup sections will give a little more than needed, and that section of bar will be adopted. Although smaller bars would be sufficient for the remaining sections, the $1\frac{1}{2}$-inch $\times \frac{5}{16}$-inch bars will be retained for the remaining stirrups.

The total available concrete section for resisting shear is 29 inches $\times$ 15 inches $= 435$ square inches which, under the specifications of the preceding article, may be taken at 44 pounds per square inch, making a total shear of 19,140 pounds to be provided for in this way if it should be considered permissible. If the latter procedure were followed it would leave but two-thirds of the total transverse shear at each stirrup section to be resisted by the steel stirrups. In the case of such a heavy beam, however, it is believed to be the better practice to take care of all the shear by steel reinforcement.

If 45-degree steel reinforcements attached to the main reinforcing rods were used, the length of such inclined bars would be about $27 \times \sec. 45$ degrees $= 38$ inches. Inasmuch as half the transverse shear at any section may be assumed to produce 45-degree compression at right angles to such inclined tension bars, the latter may be computed as being stressed by half the transverse shear multiplied by sec. 45 degrees. The 45-degree tension bars near the end of the span under such an assumption would take about 28,000 pounds only and if there were four of them, each $1\frac{1}{2}$ inch $\times \frac{7}{16}$ inch, they would be sufficient. At intermediate positions further removed from the ends, a correspondingly smaller section might be used. The bond shear at the surface of such inclined bars could be taken at a working stress of 88 pounds per square inch of surface. Such inclined tension bars should be placed not more than about 21 inches apart horizontally in order to secure effective action. Their upper ends should be bent at right angles or looped to secure a firmer hold on the concrete.

These computations illustrate clearly the simple procedures required in the design of a reinforced concrete T-beam. If the beam is of rectangular section, the procedures are precisely the same, as the actual rectangular section in that case would correspond precisely to the effective shear section taken for the T-beam.

Design of Continuous Floor Slab for 6 Feet Span between Steel Beams.

The slab is assumed to carry a warehouse load of 175 pounds per square foot in addition to own weight. It will also be assumed to be continuous over the steel beams 6 feet apart centres, the degree of continuity being that prescribed in Art. 100, making the centre and end bending moments each $\frac{wl^2}{12}$, w being the load per lineal foot of span.

A trial depth of slab of 4 inches will be assumed and the design will be made for a 12-inch width of slab. A depth of 1 inch of concrete will be taken outside of the steel reinforcement, which will be wholly on the tension side of the slab, and the tensile resistance of the concrete will be neglected. The data to be used will then be:

Span = 6 feet. Moving load = 175 pounds per square foot.
Dead load = 50 pounds per square foot.

Tension in steel, $t = 16{,}000$ pounds per square inch.
Compression in concrete, $k_1 = 500$ pounds per square inch.
$\frac{E_2}{E_1} = e = 15$; $h = 4$ inches; $h_1 = 2.75$ inches; $b = 12$ inches.

The external bending moment, $M = \frac{225 \times 6 \times 6}{12} \times 12 = 8100$ inch-pounds. The section to be designed must give a resisting moment at least equal to 8100 inch-pounds.

Eq. (8), Art. 98, gives the steel ratio:

$$p = .005 = .5 \text{ per cent.}$$

Hence,

$$A_2 = .005 \times 4 \times 12 = .24 \text{ square inch.}$$

Eq. (4), Art. 98, then gives the position of the neutral axis:

$$\frac{d_1}{h_1} = -.075 \pm .394 = +.319;$$

and

$$d_1 = 0.88 \text{ inch};$$

$$d_3 = h_1 - d_1 = 1.87 \text{ inches.}$$

The internal resisting moment will now be given eq. (12), Art. 98:

$$M = \frac{500}{.88}\left(\frac{12 \times \overline{.88}^3}{3} + 3.6 \times \overline{1.87}^2\right) = 8700 \text{ inch-pounds.}$$

By revising the design the excess above 8100 inch-pounds may be reduced if desired, but the difference is too small to be material.

Two $\frac{3}{8}$-inch square bars, placed 6 inches apart, having a combined area of .28 square inch, will afford satisfactory reinforcement, remembering that they must be carried from $1\frac{1}{4}$ inches above the lower surface of the slab at the centre of span to that distance below the upper surface at the ends of the span.

The end shear of $3 \times 225 = 675$ pounds is provided for by the bending up of the reinforcing rods, especially as the concrete section is $4 \times 12 = 48$ square inches.

Art. 102.—Reinforced Concrete Columns.

Reinforced concrete columns may be divided into two classes. The reinforcing steel in one of these classes is a wrapping or banding, usually as a spiral, of the concrete by coarse wire or thin flat bars, so that the lateral strains or enlargement due to axial compression will be prevented as much as possible with the intent to increase correspondingly the carrying capacity of the column. It is customary to use longitudinal steel rods spaced equidistantly around the column adjacent to and inside of the spiral banding, as shown in Fig. 2, the former being strongly fastened to the latter by clamps or wires. When the cylindrical cage thus formed is filled with concrete, usually a rich mixture such as 1 : 2 : 4, and encased with concrete about 2 inches thick, the complete column is formed.

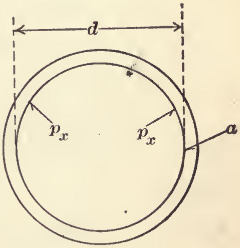


FIG. 1.

The steel reinforcement in the other class of columns is a load-carrying member, in fact a steel column in itself, filled with concrete and encased with the same exterior shell of concrete as in the banded column, as shown in Fig. 3. In the latter case the parts of the steel column reinforcement form the banding or wrapping around the concrete. The shape of cross-section of column for either class may be any desired, although the circular section is more convenient for the first class.

Lateral Reinforcement and Shrinkage

The analytic expression for the gain in carrying capacity arising from banding is easily written. Let Fig. 1 represent a band one unit (inch) in length, i.e., along the axis of the column, its interior diameter being d. When the column receives load its diameter d tends to increase in consequence of the lateral strains, thus pressing against the interior of the band and causing the latter to stretch accordingly. Let

$E_2 = 30{,}000{,}000 =$ modulus of elasticity of the steel;
$E_1 = 2{,}000{,}000 =$ modulus of elasticity of the concrete;
$p_x =$ uniform intensity of pressure between the ring or band and concrete;
$p_1 =$ intensity of column loading on a normal section;
$b_t =$ area of section of band;
$\Delta =$ stretch of steel ring due to internal pressure p_x.

Hence:

$$\Delta = \frac{p_x d}{2tE_2}\pi d. \quad \therefore \text{ New circumference} = \pi d\left(1 + \frac{p_x d}{2tE_2}\right) \quad (1)$$

The new diameter will be $d\left(1 + \frac{p_x d}{2tE_2}\right)$.

If r is the ratio between the direct compressive and lateral strains for concrete, the new diameter of the banded concrete will be:

$$\left(\frac{p_1 r}{E_1} - \frac{p_x(1-r)}{E_1}\right)d + d = d\left(\frac{p_1 r}{E_1} - \frac{p_x(1-r)}{E_1} + 1\right) \quad . \quad . \quad (2)$$

Equating the two values of the new diameter, if $\frac{E_2}{E_1} = e$,

$$\frac{p_x d}{2tE_2} = \frac{p_1 r}{E_1} - \frac{p_x(1-r)}{E_1} \quad \therefore \quad p_x = p_1\left(\frac{1}{\frac{d}{2ter} + \frac{1}{r} - 1}\right) \quad . \quad (3)$$

Eq. (3) gives the value of the intensity of pressure between the banding and the concrete. If $\frac{E_2}{E_1}=15$, and if $r=\frac{1}{5}$,

$$p_x=p_1\left(\frac{1}{\frac{d}{6t}+4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If there is no change in diameter, eq. 3 gives,

$$p_x=\frac{r}{1-r}p_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

With the above value of r, $p'_x=\frac{p_1}{4}$ would prevent all lateral strain, and as eq. (4) shows that p_x has real value, it is clear that the banding appears to be highly effective. Concrete, however, shrinks when it sets in air with a coefficient of shrinkage, according to such tests as have been made, of .0002 to .0005. If $E_1=2,000,000$ and if, for example, $p_1=600$ pounds per square inch, then by eqs. (2) and (4), if $t=\frac{1}{4}$ inch,

$$\frac{1}{2,000,000}\left(\frac{600}{5}-\frac{43\times4}{5}\right)=\frac{1}{23,400} \text{ (nearly)} \quad . \quad . \quad (6)$$

As both $\frac{1}{5000}$ and $\frac{1}{2000}$ are greater than $\frac{1}{23,400}$, these computations show that shrinkage of concrete setting in air will more than neutralize the advantage supposed to be due to banding, at least until the elastic limit of the concrete, and probably the yield point, is exceeded. This explains why banding shows no advantage in full-size column tests until the yield point is passed, as will be seen later.

Longitudinal Reinforcement

In considering the effect of longitudinal steel reinforcement, let

A = total available sectional area of the column (the outer 2-inch thickness is neglected in computations for carrying capacity);
A_2 = sectional area of steel;
A_1 = sectional area of concrete;
p = steel ratio $\frac{A_2}{A}$;
p_1 = intensity of compression in concrete;
c = intensity of compression in longitudinal steel;
$A = A_1 + A_2$ and $e = \frac{E_2}{E_1}$.
P = carrying capacity of reinforced column;
p_1 = carrying capacity of plain concrete column of section A.

Hence:

$$P = cA_2 + p_1(A - A_2) = (pe + 1 - p)p_1A. \tag{7}$$

Or

$$\frac{P}{P'} = p(e - 1) + 1. \tag{8}$$

Eq. (8) shows the gain of carrying capacity due to the longitudinal reinforcing steel. The fractional gain is:

$$\frac{P - P'}{P'} = p(e - 1). \tag{9}$$

Many tests of full-size columns of both types have been made by Professors Talbot, Withey, the author, and others, the results of which fully described may be found in the bulletins of the Universities of Illinois and Wisconsin, and those of the author in the " Proceedings of the Institution of Civil Engineers " of London.

The effect of a proper amount of spiral or other banding, either by itself or in connection with longitudinal steel rods firmly secured to it, or of a self-supporting load-carrying steel column, is, in all these types, to support the concrete to such a degree as to develop substantially its ultimate carrying power in short blocks, for such lengths of columns as have been tested.

In order to accomplish this result 1 per cent. of lateral steel reinforcement in spiral shape is sufficient. Furthermore it is preferable to use longitudinal steel rod reinforcement in connection with the lateral spiral reinforcement, the two being firmly attached to each other in all cases. The spiral cage firmly secured to the longitudinal rods constitutes practically an independent load-carrying steel column, particularly when filled with concrete. A properly designed reinforced column of this type may have its ultimate carrying capacity closely represented by eq. (7), in which p_1 is the ultimate compressive resistance of the concrete and c the ultimate compressive resistance of the steel. If longitudinal rods are used without the steel banding, they cripple or buckle under comparatively light loads, as would be expected, and make an unsatisfactory column in combination with the concrete of reduced carrying power.

As has already been shown in connection with eqs. (1) to (5) the shrinkage of the concrete in setting prevents the banding influence of the steel from being effective until the yield point of the concrete has been passed, and the results of tests have confirmed fully the indications of analysis. The same tests, however, have shown that in properly designed columns of both classes the concrete and the steel act together effectively except in the case of longitudinal rods without spiral or other banding. This latter type of column, however, is too indifferent in character to be used in practice.

Types of Columns.

Figs. 2 and 3 illustrate the two types or classes of reinforced concrete columns generally used. Fig. 2 shows a spiral reinforcement inside of which there are a suitable number of longitudinal round or other rods which must

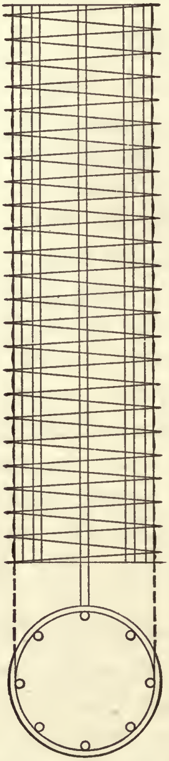


FIG. 2.

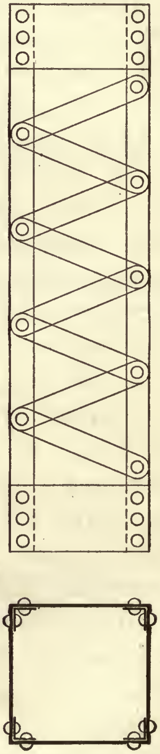


FIG. 3.

be firmly secured to the spiral reinforcement. The size of the latter may vary according to the size of the column from $\frac{1}{4}$ inch diameter to $\frac{5}{8}$ inch or more, and the pitch may vary from 1 to several inches, according to the size of the column. It has been found, as already indicated, that the amount of spiral or lateral reinforcement should be about 1 per cent., i.e., the volume of the spiral metal should be about 1 per cent. of the volume of the column, counting the diameter of the latter as the diameter of the cylinder formed by the centre line of the spiral. The amount of longitudinal steel rods may be $1\frac{1}{2}$ to 2 or 3 per cent. or more; although it has been found generally to be more economical to increase the richness of the concrete core and use less longitudinal steel than to use more of the latter with leaner and less expensive concrete. The exterior concrete shell, usually about 2 inches thick, is not considered as an available or load-carrying part of the complete column. It has been found by experiment that this exterior shell may carry from 40 to 50 per cent. as much load per square inch as the concrete core, and that it will not crack off under test until the yield point of the steel has been reached, but it is quite likely to be at least partially destroyed in a burning building. On the whole, therefore, it is considered better practice, and it is certainly safer to consider the core only of reinforced columns, i.e., only that part within the exterior enveloping volume of the steel as load carrying.

Fig. 3 is typical of the class of columns in which the steel is designedly a load-carrying member. The figure shows a column of four angles latticed in the usual manner with batten plates as well as lattice bars on all four sides, but a great variety of forms may obviously be used in this type of column. Many full-size tests have shown that the concrete filling of this type of column, no less than in the other type, may be considered as carrying load up to its full ultimate short block capacity before failure of the column for all lengths used in actual tests. The load carried by such a column may be computed by adding the carrying capacity of the concrete filling considered as a short block to the carrying capacity of the steel column computed as such.

Prof. Withey has concisely expressed the results of the tests of full-size columns of both these types in the Bulletin of the University of Wisconsin as follows:

"1. A small amount, 0.5 to 1 per cent., of closely spaced lateral reinforcement, such as the spirals used, will greatly increase the toughness and ultimate strength of a concrete column, but does not materially affect the yield point. More than 1 per cent. of lateral reinforcement does not appear to be necessary. The use of lateral reinforcement alone does not seem advisable.

"2. Vertical steel in combination with such lateral reinforcement raises the yield point and ultimate strength of the column and increases its stiffness. Columns reinforced with vertical steel only are brittle, and fail suddenly when the yield point of the steel is reached, but are considerably stronger than plain columns made from the same grade of concrete.

"3. Increasing the amount of cement in a spirally reinforced column increases the strength and stiffness of the column. A column made of rich concrete or mortar and containing small percentages of longitudinal and lateral reinforcement, is without doubt fully as stiff and strong and more economical than one made from a leaner mix reinforced with considerably more steel. In these tests, doubling the amount of cement increased the yield point and ultimate strength of the columns without vertical steel about 100 per cent., and added about 50 per cent. to the strength of those reinforced with 6.1 per cent. vertical steel.

"4. From the behavior under test of the columns reinforced with spirals and vertical steel and the results computed, it would seem that a static load equal to from 35 to 40 per cent. of the yield point would be a safe working load.

"5. The results obtained from tests of columns reinforced with structural steel indicate that such columns have considerable strength and toughness, and that the steel and concrete core act in unison up to the yield point of the former. The shell concrete will remain intact until the yield point of the steel is reached, but no allowance should be made for its strength or stiffness."

. . . .

"2. Although the yield point of a reinforced concrete column is practically independent of the percentage of spiral reinforcement, the ultimate strength and the toughness are directly affected by it. . . . Consequently, only enough lateral reinforcement is needed to prevent the longitudinal rods from bulging outward, and to provide an additional factor of safety against an overload by increasing the toughness and raising the ultimate strength somewhat above the yield point. From these tests 1 per cent. of a closely spaced spiral of high-carbon steel seems to be sufficient for this purpose.

"3. By the addition of longitudinal steel the yield point, ultimate strength and stiffness of a spirally reinforced column can be considerably increased. If maximum economy in floor space is desired, if a column is so long or is so eccentrically loaded that tension exists on a portion of the cross-section, or if a large dead load must be sustained by the column while the concrete is green, a high percentage of longutidinal reinforcement may often be advantageously employed. Such reinforcement is also a valuable safeguard against failure due to flaws in the concrete. If the cost of cement is extremely high, it may be economical to use a leaner mixture than suggested in (1) and considerable longitudinal steel to increase the stiffness and strength; columns like those of Series 1 may profitably be used. In general, however, cement is a more economical reinforcement than steel. Therefore, for ordinary constructions it does not seem advantageous to use in combination with a rich concrete more than 2 or 3 per cent. of longitudinal steel."

. . . .

"8. Briefly summarizing the foregoing, it seems economical to use for reinforced concrete columns a very rich mixture, and advantageous to employ about 1 per cent. of closely spaced high-carbon steel lateral reinforcement combined with 2 or 3 per cent of longitudinal reinforcement. From the test data presented it seems apparent that such columns, centrally loaded, may be subjected to a static working stress equal to one-third of the stress at yield point."

Working Stresses

The results of analysis and of the full-size tests to which reference has been made furnish a rational basis on which proper working stresses may be based. The concrete is so held and supported in both types of column, when properly designed, that the working stress in it may be prescribed as if it were a short block. In that class of columns in which the steel reinforcement is a steel column by itself, the working stress in the latter may be prescribed precisely as for any other steel column. Manifestly the fraction of the ultimate resistance represented by the working stresses for both materials must be the same. Actual tests of full-size columns enable the unit working stress for the longitudinal steel in the spiral-banded columns to be properly prescribed, the steel spiral banding being a 1 per cent. lateral reinforcement not to be credited as carrying any direct load.

The unit compressive working stress of the longitudinal steel reinforcing members in either type of column is taken, in the recommendations of the American Society for Testing Materials in their Proceedings for 1913, at 16,000 pounds per square inch, it being understood that the length of no column shall exceed fifteen times the least width, that width not including the protective shell, usually about 2 inches thick.

The same ratio of length to least width holding for both types of columns, the following compressive working stresses are recommended by the Committee on Concrete and Reinforced Concrete of the American Society of Civil Engineers, 1913, the per cents. stated to be applied to the ultimate resistances of the various grades of concrete given in Art. 101.

Structural steel in tension.......................... 16,000 lbs per sq.in.

	Per cent. of Ult. Compressive Resist.
Concrete in compression where resisting area is at least twice loaded area..	32.5
Concrete in plain concrete column or pier centrally loaded, length = 12 diameters or less..	22.5
Concrete in column with 1 to 4 per cent. longitudinal reinforcement only; length of column = 12 diameters or less.................	22.5
Concrete in column with lateral reinforcement of spirals, etc., at least 1 per cent. of volume of column, clear spacing of spirals or hooping, $\frac{1}{10}$ to $\frac{1}{6}$ of diameter of encased column, in no case exceeding $2\frac{1}{2}$ inches, the length of laterally unsupported column to be not more than 8 diameters of hooped core.................	27.
Concrete in column with 1 per cent. to 4 per cent. of longitudinal bars with spirals, hoops, etc., as specified above column, the length of laterally unsupported hooped core, not more than 8 diameters of core..	32.625

Reinforced columns with longitudinal steel rods, only, embedded in the concrete are highly unsatisfactory and they should not be used where the failure of the column would entail serious consequences.

The tests of such load-carrying columns for steel reinforcement of reinforced concrete as have been made by the author show that their ultimate resistances will be closely given for such lengths as have been tested by the simple straight-line formula

$$\frac{P}{A}=43,000-155\frac{l}{r}.$$

In this formula $\frac{l}{r}$ is the ratio of length of column to the radius of gyration of its cross-section about the neutral axis and A is the area of cross-section. Hence $\frac{P}{A}$ is the average unit compressive stress over a normal section of column.

This type of column is not limited in use to any ratio of length over least diameter, nor is the per cent. of steel section restricted. As the steel reinforcement is a perfectly designed load-carrying column, it may be treated like any other steel column, while the concrete filling is so banded and supported by the enclosing steel column that load may be imposed upon it as in the case of a short concrete block.

These columns have been used for tall buildings of eleven stories or more in height. They are well adapted to such a purpose, not only in consequence of the load-carrying capacity of the steel, but also on account of the facility with which floor beams and girders or other members may be detailed to them.

The spiral or otherwise banded column is not so well adapted to structural purposes. The design is such that they are available only for comparatively short lengths in connection with the prescribed working stresses. They may probably be used up to lengths of unsupported core equal to twelve times the least diameter under a reduction of working stresses to 80 per cent. of those prescribed.

The two following problems will illustrate the applications of the preceding results to actual design work:

PROBLEM I.

Design a reinforced-concrete column 13 feet 6 inches long, with spiral banding and longitudinal rod reinforcement to carry a load of 354,000 pounds.

As the column must not exceed 8 diameters in length, the diameter of the spiral banding will be taken as 20 inches, giving an effective area of 314.2 square inches. There will be assumed eight $1\frac{1}{8}$-inch longitudinal round rods arranged as shown in Fig. 2. The concrete will be taken as a 1 : 2 : 4 mixture with an ultimate resistance at twenty-eight days of 2250 pounds per square inch. Hence the working unit stress will be: $p = 2250 \times 32.625 = 734$ pounds per square inch. The working stress, *c*, of the steel, as has been shown by the specifications of the joint committee of the Am. Soc. C.E. and the Am. Soc. for Testing Materials, may be taken at 16,000 pounds per square inch. Hence the total carrying capacity of the column is:

Of the steel section.....$8 \times 16{,}000 = 128{,}000$ lbs.
Of the concrete section.$(314.2 - 8) \times 734 = 244{,}751$ lbs.
Total..........................——————
352,751 lbs.

This is sufficiently near 354,000 to be considered satisfactory and it will be accepted. It illustrates fully the procedure to be followed in the design of this type of column.

The 1 per cent of spiral lateral reinforcement may be determined as follows: The volume of spiral metal per inch of length of column is $0.01 \times 314 = 3.14$ cubic inches. If the pitch is 2 inches (one-tenth the diameter) the length of one complete turn of the spiral will be about 63 inches. Hence the sectional area of the spiral rod will be $\frac{6.28}{63} = .1$ square inch (nearly), requiring a $\frac{3}{8}$-inch round rod. This close wrapping or banding by a $\frac{3}{8}$-inch spiral with 2-inch pitch must be firmly fastened by coarse wire or clips to the eight $1\frac{1}{8}$-inch longitudinal round rods.

PROBLEM II.

Design a reinforced-concrete column 20 feet long to carry a load of 283,000 pounds, the steel reinforcement to be a load-carrying column.

Let the reinforcing column be composed of four $3 \times 3 \times \frac{7}{16}$-inch steel angles latticed to form a column like Fig. 3. The square formed by the angles will be 15 inches on a side, i.e., from back to back of angles. The radius of gyration r of such a section is 6.7 inches. Hence $\frac{l}{r} = \frac{240}{6.7} = 36$, and eq. (10) gives $\frac{P}{A} = 37{,}420$ pounds per square inch. If working stresses be taken at one-third the ultimate, the working stress for steel will be $c = \frac{37{,}420}{3} = 12{,}470$ pounds per square inch. The sectional area of a $3 \times 3 \times \frac{7}{16}$-inch angle is 2.43 square inches. Hence the effective area of the concrete section is $15 \times 15 - 4 \times 2.43 = 215.3$ square inches. The concrete will be assumed to be a 1 : 2 : 4 mixture, for which the ultimate resistance may be taken at 2250 pounds per square inch, and the working resistance, 750 pounds per square inch. The total carrying capacity of the column will then be:

Of the steel section.....	$12,470 \times 9.72 =$	121,240 lbs.
Of the concrete section.	$750 \times 215.3 =$	161,460 lbs.
Total........................		282,700 lbs.

This result shows that the design is satisfactory.

Art. 103.—Division of Loading Between the Concrete and Steel Under the Common Theory of Flexure.

It is occasionally desirable to determine the portion of the total loading of either a concrete-steel beam or arch carried by the steel and concrete parts of the member. In making this determination the formula established in the preceding articles in accordance with the common theory of flexure will be employed. It will be convenient also for this purpose to represent the intensity of stress in the extreme fibre of the steel, whether tension or compression, by k_2, the distance of that extreme fibre from the neutral axis of the composite section, established in Art. 97, being represented by d_2. It will further be supposed that the coefficients of elasticity for concrete in tension and compression are the same. Eqs. (4) of Arts. 96 and 98, representing the resisting moment of the internal stresses in a normal section of a composite member, may then be written

$$M = \frac{k_1 I_1}{d_1} + \frac{k_2 I_2}{d_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let the total load on the composite beam or arch be represented by W, while W_1 and W_2 represent the portions of W carried by the steel and concrete respectively. Also let q_1 and q_2 be so taken that $q_1 = \frac{W_1}{W}$ and $q_2 = \frac{W_2}{W}$. The remaining notation will be that given in Art. 96.

Since the bending moments in the portions A_1 and A_2 are proportional to the loads which those portions carry, remembering that $\frac{k_1}{d_1}$ and $\frac{k_2}{d_2}$ are equal to E_1u and E_2u respectively, there may be written, as indicated by eq. (1),

$$\left.\begin{array}{l} q_1 = \frac{W_1}{W} = \frac{E_1I_1}{E_1I_1 + E_2I_2} \quad \text{and} \quad q_2 = \frac{W_2}{W} = \frac{E_2I_2}{E_1I_1 + E_2I_2} \\ \text{Also} \\ q_1 = \frac{I_1}{I_1 + eI_2} \quad \text{and} \quad q_2 = \frac{eI_2}{I_1 + eI_2} \end{array}\right\} . \quad (2)$$

Also, if $n = \frac{W_1}{W_2}$,

$$q_1 = \frac{n}{n+1} \quad \text{and} \quad q_2 = \frac{1}{n+1}. \quad . \quad . \quad . \quad . \quad (3)$$

Then, since $M_1 = q_1M$ and $M_2 = q_2M$,

$$k_1 = \frac{q_1Md_1}{I_1} \quad \text{and} \quad k_2 = \frac{q_2Md_2}{I_2}. \quad . \quad . \quad . \quad (4)$$

Eqs. (2), (3), and (4) show the portions of loading carried by the two materials and the greatest intensities of stresses in their extreme fibres.

It is sometimes necessary to combine a bending moment with the direct compression (or tension) produced by a force P acting along or parallel to the axis of a beam or arch. Let p_1 and p_2 represent the intensities of stress produced in the two portions A_1 and A_2 by such a direct force. Since equal unit longitudinal strains exist in the two materials, the intensities of stress in the portions A_1 and A_2 will be proportional to their coefficients of elasticities. Hence

$$\frac{p_1}{p_2}=\frac{E_1}{E_2} \quad \text{and} \quad p_2=\frac{E_2}{E_1}p_1. \quad . \quad . \quad . \quad . \quad (5)$$

Hence

$$p_1A_1+ep_1A_2=P; \quad \therefore \ p_1=\frac{P}{A_1+eA_2}. \quad . \quad . \quad (6)$$

In the case of an elastic arch like those of combined concrete and steel, the thrust P is in general exerted along the axis of the arch ring but at some distance, l, from it. In such a case the bending moment is

$$M=Pl; \quad \text{hence} \quad M_1=q_1Pl \quad \text{and} \quad M_2=q_2Pl. \quad . \quad (7)$$

The values of the bending moments are to be placed in eq. (4), in order to determine the intensities k_1 and k_2.

In determining the resultant of stress for any section of an arch ring, if the conditions under which eqs. (2) were written be employed, the thrust on the portion A_1 will be q_1P, and q_2P on A_2, since the thrusts on the two portions will be proportional to the loads which they carry. Hence, if k_1 and k_2 again be used to represent the greatest intensities of stress in the two portions, there may at once be written

$$k_1=q_1\left(\frac{P}{A_1}\pm\frac{Md_1}{I_1}\right), \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$k_2=q_2\left(\frac{P}{A_2}\pm\frac{Md_2}{I_2}\right). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In eqs. (8) and (9), $M=Pl$.

If, again, the last members of eqs. (5) and (6) be used in connection with eqs. (2) and (4) the resultant values of k_1 and k_2 will be

$$k_1 = p_1 + \frac{q_1 M d_1}{I_1} = \frac{P}{A_1 + eA_2} + \frac{Md_1}{I_1 + eI_2}, \quad . \quad . \quad . \quad (10)$$

$$k_2 = p_2 + \frac{q_2 M d_2}{I_2} = e\left(\frac{P}{A_1 + eA_2} + \frac{Md_2}{I_1 + eI_2}\right). \quad . \quad . \quad (11)$$

In the use of all these equations, care must be taken to give the proper sign to the bending moment M.

These equations comprise all that are necessary in order to ascertain the distribution of the loading between the steel and the concrete, or any other two materials, whether the case may be one of pure bending or a combination of bending and direct stress.

CHAPTER XIV.

ROLLED AND CAST FLANGED BEAMS

Art. 104.—Flanged Beams in General.

ROLLED flanged beams as produced by steel mills and used in building or other construction have already been treated in cases of simple bending, using the moment of inertia either by itself or as part of the section modulus for steel beams where their moments of resistance take the usual form,

$$M=\frac{kI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation d_1 is the distance of the extreme fibre from the neutral axis in which the intensity of stress k exists, and I and $\frac{I}{d_1}$ are the moment of inertia and the section modulus, respectively, numerical values of which for all shapes are given in handbooks. In this treatment of rolled or other flanged beams the resistance of the web is included, but there are cases when it is permissible to neglect the bending resistance of the web or, again, in which the bending resistance of the two flanges is treated separately, as if the intensity of stress in each is uniform throughout the flange section, to which a closely approximate simple expression for the bending resistance of the web may or may not be added.

If the bending resistance of the flanges is to be computed by itself, it is evident that economy of design requires that the two flanges must fail concurrently if the beam be loaded to failure. If the ultimate tensile and compressive resistances of the material are not the same, it is equally clear that the two flanges should not be of equal section, the area of the flange in which the ultimate resistance is greater being less than that of the flange in which the ultimate resistance is less. This results from the fact that the total stress of compression in the compression flange must be equal to the total tensile stress in the tension flange, the beam being supposed to be horizontal and the load vertical. If the bending resistance of the web is recognized, the equality of the two total flange stresses no longer holds, since the tension and compression developed in the web is to be added to the corresponding stresses in the flanges in order to make equality.

Each total flange stress is evidently equal to the flange area multiplied by the intensity of assumed uniform stress in it. The centre of each flange stress will then be the centre of gravity of the section on which it acts. The vertical distance d between the centres of gravity or stress of the two flanges is called the effective depth of the beam, because if it be multiplied by either flange stress the product will be the resisting moment of the stresses acting in the section of the beam. In other words the effective depth d is the lever arm of the internal couple whose moment is equal to the external bending moment.

Let a be the sectional area of the tension flange and T the uniform intensity of stress in it, and let a' and C be the corresponding values for the compression flange, while d is the effective depth. Then, since $aT = a'C$, the moment of the internal stresses will be

$$M = aTd = a'Cd. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The use of both eqs. (1) and (2) will be illustrated by numerous practical applications.

It is clear from what has preceded that the chief function of the flanges is to resist the bending proper, while the main function of the web is to resist the transverse shear.

The direct stresses of tension and compression in a beam with solid rectangular section correspond to, i.e., perform the same function as, the flange stresses of tension and compression in the flange beam, while the web, supposed to take shear only, corresponds approximately to the zone of material in the vicinity of the neutral surface of the solid section in which the direct stresses of tension and compression are either zero or nearly zero.

Art. 105.—Flanged Beams with Unequal Flanges.

By the common theory of flexure, if the two coefficients of elasticity are equal, it has been shown that if *C*, Fig. 1, is the centre of gravity of the cross-section, the neutral axis of the section will pass through that point. Let it now be supposed that the lower flange is in tension, while the upper is in compression. Also let *T* represent the ultimate resistance to tension in bending, and let *C* represent the same quantity for compression in bending. Then since intensities vary directly as distances from the neutral axis,

$$\frac{h_{,}}{h} = \frac{T}{C}; \quad \therefore \; h_1 = h\frac{T}{C} = n'h. \quad . \quad . \quad . \quad . \qquad (1)$$

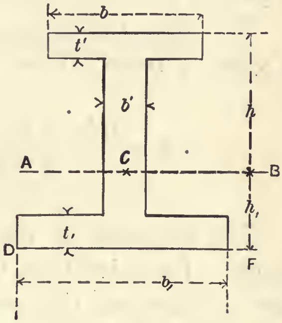


FIG. 1.

The ratio between ultimate intensities is represented by n'. If $d=h+h_1$ is the total depth of the beam, and hence $h=d-h_1$,

$$h_1 = n'(d-h_1) = \frac{dn'}{1+n} = \frac{\frac{T}{C}d}{1+\frac{T}{C}} \quad . \quad . \quad . \quad . \quad (2)$$

If, as an example, for cast iron there be taken

$$n' = \frac{T}{C} = 0.2, \quad h_1 = \frac{1}{6}d.$$

The relation between h and h_1 shown in eq. (2) is entirely independent of the form of cross-section. But according to the principles just given, in order that economy of material shall obtain, *the cross-section should be so designed that h and h_1 shall represent the distances of the centre of gravity from the exterior fibres.*

The analytical expression for the distance of the centre of gravity from DF is

$$x_1 = \frac{\frac{1}{2}b'a^2 + (b-b')t'(d-\frac{1}{2}t') + \frac{1}{2}(b_1-b')t_1{}^2}{b\,d + (b-b')t' + (b_1-b')t_1} \quad . \quad . \quad . \quad (3)$$

The meaning of the letters used is fully shown in the figure. In order that the beam shall be equally strong in the two flanges, the various dimensions of the beam must be so designed that

$$x_1 = h_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It would probably be found far more convenient to cut sections out of stiff manila paper and balance them upon a knife-edge.

The moment of inertia about the axis AB, thus determined, is

$$I = \tfrac{1}{3}[bh^3 + b_1h_1^3 - (b-b')(h-t')^3 - (b_1-b')(h_1-t_1)^3] \quad . \quad (4a)$$

This value is to be substituted in the formula $M = \frac{kI}{d_1}$, now changed to

$$M = \frac{CI}{h} = \frac{TI}{h_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

For various beams whose lengths are l and total load W the greatest value of M becomes:

Cantilever uniformly loaded,

$$M = \frac{Wl}{2}.$$

Cantilever loaded at end,

$$M = Wl.$$

Beam supported at each end and uniformly loaded,

$$M = \frac{W}{8} = \frac{pl^2}{8}.$$

Beam supported at each end and loaded at centre,

$$M = \frac{Wl}{4}.$$

The last two cases combined,

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right).$$

Sometimes the resistance of the web is omitted from consideration. In such a case the intensity of stress in each flange is assumed to be uniform and equal to either T or C. At the same time the lever-arms of the different fibres are taken to be uniform, and equal to h for one flange and h_1 for the other, *h and h_1 now representing the vertical distances from the neutral axis to the centres of gravity of the flanges, while $d = h + h_1$.*

On these assumptions, if f is the area of the upper flange and f' that of the lower, there will result

$$M = fC.h + f'T.h_1. \quad \dots \dots \quad (5)$$

But since the case is one of pure flexure,

$$fC = f'T. \quad \dots \dots \quad (6)$$

$$\therefore \; M = fC(h + h_1) = fCd = f'Td. \quad \dots \quad (7)$$

Also, from eq. (6),

$$\frac{f}{f'} = \frac{T}{C}. \quad \dots \dots \quad (8)$$

Or, *the areas of the flanges are inversely as the ultimate resistances.*

Frequently there is no compression flange, the section being like that shown in Fig. 2. In such case b is equal to b', or t' is equal to zero; hence $b = b'$ in eq. (4*a*), but no other change is to be made in the second member of that equation. Eq. (4*b*) may then be used precisely as it stands for the internal resisting moment of a beam with the section shown in Fig. 2.

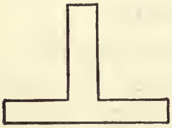


FIG. 2.

Prob. 1. It is required to design a cast-iron flanged beam of 5 feet effective span to carry a load of 1800 pounds applied at the centre of span, the section of the beam to be like that shown in Fig. 2, i.e., without upper flange. The greatest permitted working stress in compression will be 8000 pounds per square inch, and the total depth of the beam is to be taken at 9 inches.

Referring to eqs. (4*a*), (4*b*), and Fig. 1 for the notation, the given data and the dimensions to be assumed for trial will be as follows: $d = 9$ inches; $b = b' = \frac{7}{8}$ inch; $b_1 = 8$ inches; $t_1 = 1$ inch; $l = 5$ feet; and $C = 8000$. The introduction of these values into eq. (3) will give for the distance of the centre of gravity above the bottom surface of the beam

$$h_1 = 2.6 \text{ inches} \quad \text{and} \quad h = d - h_1 = 6.4 \text{ inches.}$$

The preceding trial dimensions will make the beam weigh about 50 pounds per lineal foot. If all the preceding values are substituted in eqs. (4*a*) and (4*b*), remembering that $M = \frac{Wl}{4}$, there will be found

$$W = 1994 - 125 = 1869 \text{ pounds.}$$

The trial dimensions, therefore, give the centre-load capacity of the beam 69 pounds greater than required, which may be considered sufficiently near to show that the assumed dimensions are satisfactory.

Art. 106.—Flanged Beams with Equal Flanges.

Nearly all the flanged beams used in engineering practice are composed of a web and two equal flanges. It has already been seen that the ultimate resistances, T and C, of structural steel and wrought iron to tension and compression are essentially equal to each other; the same may be said also of their coefficients of elasticity for tension and compression. These conditions require equal flanges for both steel and wrought-iron rolled beams.

In Fig. 1 is represented the normal cross-section of an equal-flanged beam. It also approximately represents what may be taken as the section of any wrought-iron or steel I beam, the exact forms with the corresponding moments of inertia being given in handbooks. Although the thickness t' of the flanges of such beams is not uniform, such a mean value may be taken as will cause the transformed section of Fig. 1 to be of the same area as the original section.

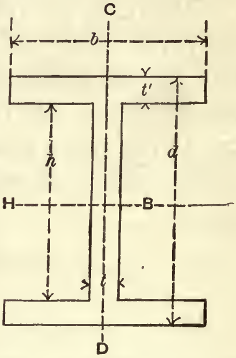


FIG. 1.

Unless in exceptional cases where local circumstances compel otherwise, the beam is always placed with the web vertical, since the resistance to bending is much greater in that position. The neutral axis HB will then be at half the depth of the beam. Taking the dimensions as shown in Fig. 1, the moment of inertia of the cross-section about the axis HB is

$$I=\frac{bd^3-(b-t)h^3}{12}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while the moment of inertia about CD has the value

$$I_1=\frac{2t'b^3+ht^3}{12}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

With these values of the moment of inertia, the general formula, $M=\frac{kI}{d_1}$, becomes $\left(\text{remembering that } d_1=\frac{d}{2} \text{ or } \frac{b}{2}\right)$

$$M=k\frac{bd^3-(b-t)h^3}{6d}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$M'=k\frac{2t'b^3+ht^3}{6b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

k is written for all extreme fibre stress.

Eq. (3) is the only formula of much real value. It will be found useful in making comparisons with the results of a simpler formula to be immediately developed.

Let $d_1 = \frac{1}{2}(d+h)$. Since t' is small compared with $\frac{d}{2}$, the intensity of stress may be considered constant in each flange without much error. In such a case the total stress in each flange will be $kbt' = Tbt'$, and each of those forces will act with the lever-arm $\frac{1}{2}d_1$. Hence the moment of resistance of both flanges will be

$$kbt' \cdot d_1.$$

The moment of inertia of the web will be $\frac{th^3}{12}$. Consequently its moment of resistance will have very nearly the value

$$\frac{kth^2}{6}.$$

The resisting moment of the whole beam will then be

$$M = k\left(bt'd_1 + \frac{th^2}{6}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A further approximation is frequently made by writing d_1h for h^2; then if each flange area $bt' = f$, eq. (5) takes the form

$$M = kd_1\left(f + \frac{th}{6}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Eq. (6) shows that *the resistance of the web is equivalent to that of one sixth the same amount concentrated in each flange.*

If the web is very thin, so that its resistance may be neglected,

$$M = kfd_1 = kbt'd_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$f = \frac{M}{kd_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Cases in which these formulæ are admissible will be given hereafter. It virtually involves the assumption that the web is used wholly in resisting the shear, while the flanges resist the whole bending and nothing else. In other words, the web is assumed to take the place of the neutral surface in the solid beam, while the direct resistance to tension and compression of the longitudinal fibres of the latter is entirely supplied by the flanges.

Again recapitulating the greatest moments in the more commonly occurring cases:

Cantilever uniformly loaded,

$$M = \frac{Wl}{2} = \frac{pl^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Cantilever loaded at the end,

$$M = Wl. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Beam supported at each end and uniformly loaded,

$$M = \frac{Wl}{8} = \frac{pl^2}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Beam supported at each end and loaded at centre,

$$M = \frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Beam supported at each end and loaded both uniformly and at centre,

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In all cases W is the total load or single load, while p, as usual, is the intensity of uniform load, and l the length of the beam.

Art. 107.—Rolled Steel Flanged Beams.

The resisting moments of all rolled steel beams subjected to bending are computed by the exact formula

$$M = \frac{KI}{d_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

k being the greatest intensity of stress (i.e., in the extreme fibres) at the distance d_1 from the neutral axis about which the moment of inertia I is taken. In all ordinary cases the webs of beams are vertical so that the axis for I is horizontal; but it sometimes is necessary to use the moment of inertia I computed about the axis passing through the centre of gravity of section and parallel to the web. The latter is frequently employed in considering the lateral bending effect of the compression in the upper flange.

The upper or compression flange of a rolled beam under transverse load, unless it is laterally supported, is somewhat in the condition of a long column and, hence, tends to bend or deflect in a lateral direction. This tendency depends to some extent on the ratio of the length of flange (l) to the radius of gyration (r) of the section about the axis parallel to the web, as will be shown in detail in a later article. It will be found there that the ultimate compression flange stress decreases as the ratio $l \div r$ increases. Hence in Table I there will be found values of $l \div r$ for the different beams tested.

The results of tests given in Table I were found by Mr. James Christie, Supt. of the Pencoyd Iron Co., and they are taken from a paper by him in the "Trans. Am. Soc. C. E." for 1884. All beams, both I and bulb, were loaded at the centre of span. Hence the moment of the centre load, W, and the uniform weight of the beam itself, pl, will be, as shown in eq. (13) of Art. 106,

$$M = \frac{l}{4}\left(W + \frac{pl}{2}\right) = \frac{kI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence

$$k = \frac{d_1 l}{4I}\left(W + \frac{pl}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The known data of each test will give all the quantities in the second member of eq. (3). The two columns of elastic and ultimate values of k in the table were computed by eq. (3). The positions of the bulb beams (i.e., the bulb either up or down) in the tests are shown by the skeleton sections in the second column.

The coefficients of elasticity E were computed from the data of the tests taken below the elastic limit by the aid of eq. (21), Art. 28:

$$w_1 = \frac{l^3}{48EI}\{W + \tfrac{5}{8}pl\}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

W being the centre load and pl the weight of the beam, the length of span l being given in inches.

All beams were rolled at the Pencoyd Iron Works. The "mild steel" contained from 0.11 to 0.15 per cent. of carbon, and the "high steel" about 0.36 per cent. of carbon. These steels are the same as those referred to in Art. 60.

No. 14 is the only test of a "high" steel beam; all the remaining tests being with mild-steel shapes. Tests 3 to 9 inclusive were of deck or bulb beams, as the skeleton sections show.

Beams 3 and 4 were rolled from the same ingot, as were also 6 and 7, as were also 10, 12, and 13, and as were also 16, 17, 18, and 19. All beams were unsupported laterally in either flange. The moments of inertia were computed from the actual beam sections. The length of span is represented by *l*, while *r* is the radius of gyration of each beam section about an axis through its centre of gravity and parallel to its web. The values of *r* were as follows:

5 inch I . . . . $r = 0.54$ inch.
6 " " . . . . $r = 0.63$ "
7 " " . . . . $r = 0.71$ "
9 " " . . . . $r = 0.83$ "

3 inch I . . . . $r = 0.59$ inch.
8 " " . . . . $r = 0.88$ "
10 " " . . . . $r = 0.95$ "
12 " " . . . . $r = 1.01$ "

TABLE I.

TRANSVERSE TESTS OF STEEL BEAMS.

No.	Kind of Beam.	Span in Ins.	$\frac{l}{r}$.	Moment of Inertia.	Final Centre Load in Pounds.	*k* in Pounds per Square Inch at		Coefficient of Elasticity *E*, in Pounds per Square Inch.
						Elastic.	Ultimate	
1	Mild 3″ I	59	100	2.76	5,500	41,100	45,200	30,890,000
2	" 3″ "	39	66	2.76	8,300	40,800	45,100	25,011,000
3	" 5″ I	108	200	12	8,800	50,000	55,000	27,718,000
4	" 5″ I	108	200	12	8,400	46,900	52,500	25,489,000
5	" 6″ I	96	152	22	14,860	51,200	54,300	23,692,000
6	" 7″ "	69	97	37.6	34,000	47,100	59,300	18,765,000
7	" 7″ I	69	97	37.6	34,000	47,100	59,300	23,040,000
8	" 9″ I	240	290	84.8	14,500	46,000	51,300	29,923,000
9	" 9″ I	240	290	82.9	13,500	39,800	48,800	30,209,000
10	" 8″ I	240	273	70.2	13,000	37,600	44,400	28,889,000
11	" 8″ "	240	273	70.3	12,930	37,500	44,100	29,055,000
12	" 8″ "	144	164	70.2	19,480	32,800	39,900	31,313,000
13	" 8″ "	96	109	70.2	31,300	40,300	42,800	23,689,000
14	High 3″ "	39	——	2.74	11,500	54,300	——	27,515,000
15	Mild 10″ "	156	164	150.5	22,500	35,000	——	28,414,000
16	" 10″ "	168	177	150.5	21,000	35,200	——	27,182,000
17	" 10″ "	180	189	150.5	19,500	35,000	——	29,160,000
18	" 10″ "	192	202	150.5	18,000	34,400	——	29,727,000
19	" 12″ "	240	238	264.7	24,500	33,400	——	30,749,000
20	" 12″ "	240	238	267.6	24,200	32,500	——	29,568,000
21	" 12″ "	228	226	273.8	22,000	27,500	——	29,164,000
22	" 12″ "	216	214	263.7	29,000	35,600	——	30,219,000
23	" 12″ "	204	202	256.7	27,000	32,100	——	30,030,000
24	" 12″ "	192	190	257.8	34,000	38,000	——	29,709,000
25	" 12′ "	192	190	262.6	34,000	37,300	——	28,234,000
26	" 12″ "	180	178	262.4	36,700	37,700	——	27,717,000
27	" 12″ "	168	166	264.0	38,000	36,300	——	28,784,000
28	" 12″ "	156	154	261.7	43,000	38,400	——	27,818,000

The values of k both for the elastic limit and the ultimate are erratic, and the range of results in the table is not sufficient to establish any law, but on the whole the small ratios $l \div r$ accompany the larger values of k. The bulb or deck beams also appear to give larger values of k than the I beams.

The results of these tests indicate that the greatest working intensities of stress in the flanges of rolled steel beams may be taken from 12,000 to 16,000 pounds per square inch if the length of unsupported compression flange does not exceed $150r$ to $200r$.

In the work of design, the quantity $I \div d_1$ used in eq. (2), called the "section modulus," is much employed, and it can be taken directly from the Cambria Steel Company's tables at the end of the book, as can the moment of inertia I. Eq. (2) shows that

$$\frac{I}{d_1} = \frac{M}{k}. \qquad \qquad (5)$$

Hence the moment of the loading in inch-pounds divided by the allowed greatest flange stress in pounds per square inch must be equal or approximately equal to the section modulus of the required beam.

There may be found in the Proceedings of the American Society for Testing Materials, 1909, the results of tests of rolled I beams and girders produced by the Bethlehem Steel Company and of standard rolled I beams by Professor Edgar Marburg. Also Professor H. F. Moore gives results of his testing of steel I beams of the regular or standard pattern in Bulletin No. 68 of the University of Illinois. Professor Marburg's main purpose appears to have been to make comparative tests of the ordinary I beam and of the wide-flange Bethlehem shapes, while the principal object of Professor Moore was to investigate the influence of lateral deflection on the capacity of the compressive flange without lateral support. Table II gives the results of these tests, each of professor Marburg's results except one being an average of three.

TABLE II.

TESTS OF ROLLED STEEL BEAMS.

Type.	Span, l Ft.	$\frac{l}{r'}$	Extreme Fibre Stress k, Lbs. per Sq.in.		Modulus of Elasticity.	Size.
			Elas. Limit.	Ultimate.		
Beth. I.	15	125	31,700	46,100	26,900,000	15″ 38 lb.
Std. I.	15	167	20,400	42,200	26,200,000	15″ 42 "
Girder.	15	75	26,700	53,900	26,900,000	15″ 73 "
Beth. I.	15	125	21,800	37,900	26,400,000	15″ 38 "
Std. I.	15	167	20,600	34,700	26,900,000	15″ 42 "
Girder.	15	75	22,500	41,100	27,200,000	15″ 73 "
Beth. I.	20	129	20,900	34,600	26,400,000	24″ 72 "
Std. I.	20	176	19,500	33,000	25,800,000	24″ 80 "
Girder.	20	90	15,400	34,300	25,600,000	24″ 120 "
Beth. I*. . . .	20	111	13,000	32,300	29,400,000	30″ 120 "
Girder.	20	84	11,800	31,000	24,800,000	30″ 175 "

* One beam only.

Prof. Moore's fifteen tests were with 8-inch, 18-pound and one 25-pound I beams, the spans being 5, 7.5, 7.92, 10, 15, 15.7 and 20 feet. The ratio $l \div r'$ varied from 71 to 286. The ultimate fibre stress k was *Max.* 36,600; *Mean* 32,300; *Min.* 28,100. The Modulus E was, *Max.* 32,300,000; *Mean* 28,400,000; *Min.* 25,100,000. The Max. E is to be regarded with doubt.

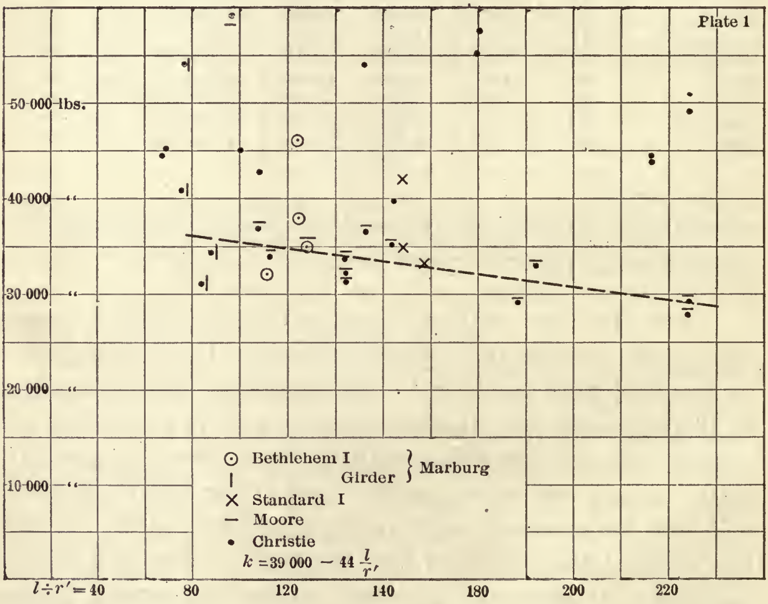
As is the case with all tests of full-size rolled beams, the results are seen to vary quite widely. This is largely due to the fact that such full-size members are seldom true in all their parts, i.e., the web may be a little twisted on the cooling bed and the flange will perhaps never be perfectly plane, consequently the applied load in the testing machine will not be received with presupposed exactness. Again, the work of the rolls and the effects of cooling will not be uniform. At any rate the most scrupulous care in testing will not prevent many erratic results, apparently unaccountable.

In order to show these results graphically they have been plotted on Plate I and the explanatory matter on the Plate will make clear the results belonging to each investigator. The horizontal ordinate is the ratio $\frac{l}{r'}$, r' being the radius of gyration of the normal section of the column about a vertical axis parallel to the web and passing through its centre. The vertical ordinate is the intensity k of the extreme fibre stress produced by the ultimate load on the beam as shown in Table I.

The equation

$$k=39,000-44\frac{l}{r'},$$

represents the broken line drawn on Plate I. It is a tentative expression, as there are not sufficient tests with the requisite variation of $\frac{l}{r'}$ to justify more than a trial value of k.



Professor Marburg made no effort to give lateral support to his beams under test, nor did he endeavor to give the compressive flange lateral freedom, as did Professor Moore for a part of his tests. As, however, the results appear to be about the same, whether the compressive flange has complete lateral freedom or not, under ordinary circumstances of testing, no distinction is made on this account between the various plottings on Plate I. The extremely high values on that Plate belong to the first nine tests by Mr. Christie, as given in Table I. They are abnormally high and whether such results are characteristic of bulb sections or due to some other reason is not clear.

Prob. 1. It is required to design a rolled steel beam for an effective span of 20 ft. to carry a uniform load of 725 lbs. per linear foot in addition to the weight of the beam itself, the circumstances being such that it is not advisable to use a greater total depth of beam than 12 ins. The greatest permitted extreme fibre stress *k* will be taken at 12,000 lbs. per sq. in. It will be assumed for trial purposes that the beam itself will weigh 35 lbs. per linear foot, so that the total uniform load will be 760 lbs. per linear foot. The centre moment in inch-pounds will, therefore, be

$$M = \frac{760 \times 20 \times 20 \times 12}{8} = 456,000 \text{ in.-lbs.}$$

By eq. (5) the section modulus will be 456,000 ÷ 12,000 = 38. By referring to the tables in almost any steel company's handbook it will be found that this section modulus belongs to a 12-inch, 35-pound steel rolled beam, and that beam fulfills the requirements of the problem.

Prob. 2. It is required to design a rolled-steel beam for a 32-ft. effective span to carry a load of 1280 pounds per linear foot in addition to the weight of the beam, and a concentrated load of 11,000 pounds at a point 11 feet distant from one end of the span. The greatest permitted working stress in the extreme fibres of the beam is 16,000 lbs. per sq. in.

It will be assumed for trial purposes that a 24-in. beam weighing 95 lbs. per linear foot will be required so that the total uniform load per linear foot will be 1375 pounds. It will then be necessary to ascertain at what point in the span the maximum bending moment occurs, i.e., at what point the transverse shear is equal to zero. Let a be the distance of the concentrated weight from the nearest end of the span, i.e., $a = 11$ ft. Then let P be the single weight, p the total uniform load per linear foot, and l the length of span. The following equation representing the condition that the transverse shear must be equal to zero may be written

$$\frac{pl}{2} - px + \frac{Pa}{l} = 0.$$

Hence

$$x = \frac{l}{2} + \frac{Pa}{pl}.$$

In the above equation x is obviously the distance from that end of the span farthest from P to the section of greatest bending moment. Substituting the above numerical values in the equation for x, there will result

$$x = 16 + 2.75 = 18.75 \text{ ft.}$$

Since $32 - 18.75 = 13.25$ the following will be the value of the greatest bending moment in inch-pounds:

$$M = \left(\frac{1375 \times 18.75}{2} \times 13.25 + \frac{11,000 \times 11}{32} \times 18.75 \right) 12$$
$$= 2,900,363 \text{ inch-pounds.}$$

The section modulus of the beam required is by eq. (5) $2,900,363 \div 16,000 = 181$. The section modulus of a 24.-in. steel beam weighing 85 lbs. per linear foot is 180.7, as will be found by referring to the tables at the end of the book. Hence that beam will be assumed for the correct solution of the problem. The fact that the beam weighs 10 lbs. per linear foot less than the assumed weight has too small an effect upon the greatest bending moment to call for any revision.

Prob. 3. A steel tee beam of 8 ft. span is to be used as a purlin to carry a uniform load of 125 lbs. per linear foot with the web of the tee in a vertical position. The greatest permitted intensity of stress in the extreme fibre of the tee is 14,000 lbs. per sq. in. It is required to find the dimensions of the tee. By referring to eq. (5) the section modulus will be written

$$S = \frac{1000 \times 96}{8 \times 14,000} = .86 \text{ in.}$$

By referring again to the steel handbook tables it will be found that a $3 \times 3 \times \frac{5}{16}$ in. steel tee weighing 6.6 lbs. per lin. ft. has just the section modulus required. That tee therefore fulfils the requirements of the problem.

Prob. 4. It is required to support a single weight of 12,000 lbs. at the centre of a span of 13 ft. 6 ins. on two rolled steel channels with their webs in a vertical position and separated back to back by a distance of 3 ins., the greatest permitted intensity of stress in the extreme fibre of the flanges being 15,000 lbs. Find the size of channels required.

Art. 108.—The Deflection of Rolled Steel Beams.

The deflections of rolled steel beams may readily be computed by the formulæ of Art. 28. The general procedure will be illustrated by using the equations for a non-continuous beam simply supported at each end and loaded by a weight at the centre of span, or uniformly, or in both ways concurrently. Eq. (20) will give the deflection at any point located by the coordinate x, while eq. (21) will give the centre deflection only. The tangent of the inclination of the neutral surface at any point located by x will be given by the value of $\frac{dw}{dx}$ found in eq. (19).

Prob. 1. Let the centre deflection of the rolled-steel beam of Prob. 1 of Art. 107 be required. Referring to eq. (21) of Art. 22,

$$W=0; \quad l=20 \text{ feet}=240 \text{ inches}; \quad p=760 \text{ pounds};$$
$$I=228.3; \quad \text{and} \quad E \text{ may be taken at } 29,000,000.$$

Hence the centre deflection is

$$w_1=\frac{240\times240\times240\times5\times760\times20}{48\times8\times29,000,000\times228.3}=.414 \text{ inch.}$$

If half the external uniform load of 725 pounds per linear foot had been concentrated at the centre of span,

$$W=\frac{725\times20}{2}=7250 \text{ pounds}; \quad p=35$$

and $l=20$ ft. $=240$ ins. Also $pl=700$ pounds.

Hence the centre deflection would be

$$w_1=\frac{240\times240\times240\times(7250+437\cdot5)}{48\times29,000,000\times228.3}=.333 \text{ inch.}$$

Prob. 2. In Prob. 2 of Art. 107 place the 11,000-pound weight at the centre of span, then find the inclination of the neutral surface and the deflection of the 24-inch 85-

pound steel beam at the centre and quarter points of the 32-foot span, taking $E=29,000,000$ pounds.

Art. 109.—Wrought-iron Rolled Beams.

Although wrought-iron rolled beams are not now manufactured, being completely displaced by steel beams, yet many are still in use. Hence it is advisable to exhibit the empirical quantities required to design them and to determine their safe carrying capacities as well as their deflections under loading.

It has been observed in Art. 107 that the upper or compression flange of a loaded flanged beam will deflect or tend to deflect laterally at a lower intensity of compressive stress as the unsupported length of such a flange is increased. The experimental results given in Table I exhibit the values of the intensity of stress K in the extreme fibres of the beam both at the elastic and ultimate limits, the usual formula for bending resistance being used,

$$M=\frac{KI}{d_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the autumn of 1883 an extensive series of tests of wrought-iron rolled beams, subjected to bending by centre loads, was made by the author, assisted by G. H. Elmore, C.E., at the mechanical laboratory of the Rensselaer Polytechnic Institute. The object of these tests was to discover, if possible, the law connecting the value of K for this class of beams with the length of span when the beam is *entirely without lateral support.* The means by which the latter end was accomplished, and a full detailed account of the tests will be found in Vol. I, No. 1, "Selected Papers of the Rensselaer Society of Engineers." The main results of the tests are given in Table I. All the tests were made on 6-inch I beams with the same area of normal cross-

TABLE I.

No	Span, Feet.	Final Centre Weight, Pounds.	$\frac{l}{r}$.	K		Perm'nent Vertical Deflection, Inches.	Perm'nent Lateral Deflection, Inches.	E Pounds per Square Inch.
				Elastic Limit, Pounds.	Ultimate, Pounds.			
1	20	4,060	400	27,726	31,094	0.14	——	24,170,000
2		4,200	400	29,623	32,885	0.30	——	26,374,000
3	18	4,390	360	28,264	30,791	0.2	0.5	24,520,000
4		4,570	360	28,264	32,020	0.18	0.4	24,313,000
5	16	4,770	320	26,564	29,579	0.28	1.00	25,771,000
6		5,270	320	29,596	32,632	0.48	1.25	25,003,000
7	14	6,130	280	31,191	33,049	0.30	1.20	26,082,000
8		6,125	280	31,164	33,023	0.30	1.10	23,373,000
9	12	7,161	240	30,221	32,907	0.35	1.08	25,287,000
10		7,350	240	31,314	33,817	0.33	1.09	24,022,000
11	10	9,255	200	33,082	35,358	0.39	1.08	25,115,000
12		9,655	200	33,082	37,064	0.50	1.50	24,218,000
13	8	11,485	160	29,736	35,010	0.30	0.90	21,611,000
14		11,980	160	31,936	36,527	0.29	1.05	21,987,000
15	6	18,300	120	35,497	41,737	0.605	1.53	23,040,000
16		18,145	120	36,617	41,396	0.67	1.88	20,935,000
17	5	22,870	100	34,136	43,434	0.67	1.75	22,023,000
18		23,065	100	34,136	43,813	0.67	1.75	25,272,000
19	4	29,985	80	32,619	45,532	0.96	1.70	24,315,000
20		28,585	80	32,619	44,744	0.60	1.86	21,275,000

section of 4.35 square inches. Actual measurement showed the depth d of the beams to be 6.16 inches. The moment of inertia of the beam section about a line through its centre and normal to the web was $I=24.336$. The radius of gyration of the same section in reference to a line through its centre and *parallel* to the web was $r=0.6$ inch. l was the length of span in inches.

If M is the bending moment in inch-pounds, W the total centre load (including weight of beam), and K the stress per square inch in extreme fibre, the following formulæ result:

$$k=\frac{Md}{2I} \quad \text{and} \quad M=\frac{Wl}{4}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \ k=\frac{Wld}{8I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The experimental values of W, l, d, and I inserted in the above formula give the values of k shown in the table. The coefficient of elasticity, E, was found by the usual formula,

$$E = \frac{Wl^3}{48wI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which w is the deflection caused by W.

The full line is the graphical representation of the values of k given in Table I. Since k must clearly decrease with the length of span, and increase with the radius of gyration of the section about an axis through its centre and parallel to the web (the latter, of course, being vertical), k has been plotted in reference to $l \div r$ as shown. No simple formula will closely represent this curve, but the broken line covers all lengths of span used in ordinary engineering practice, and is represented by the formula

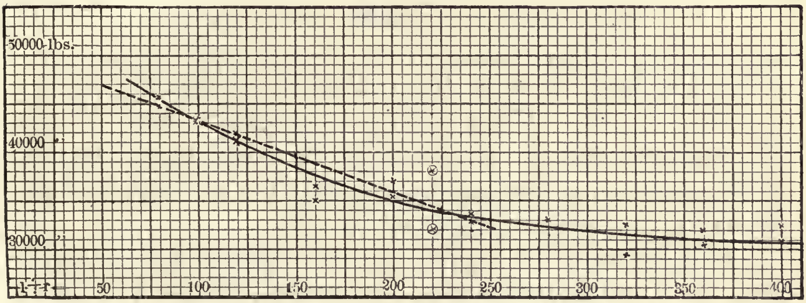


Plate I.

$$k = 51{,}000 - 75\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

For railway structures the greatest allowable stress per square inch in the extreme fibres of rolled beams may be taken at

$$k = 10{,}000 - 15\frac{l}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Values of k taken from a large scale plate, like Plate I, are, however, far preferable to those given by any formula.

The ultimate values of k given in Table I are fairly representative of the best wrought-iron I beams. The coefficients of elasticity E range from about 22,000,000 to about 25,000,000 pounds; the average may be taken about 24,000,000 pounds.

The deflection of wrought-iron beams may be computed by the formula

$$w=\frac{Wl^3}{48EI}, \quad \dots\dots\dots \quad (7)$$

when the load W is at the centre of the beam. In the general case of a beam carrying the centre load W and the uniform load pl, the quantity $(W+\frac{5}{8}pl)$ must displace W in eq. (7). If the beam carry only the uniform load pl, W in eq. (7) must be displaced by $\frac{5}{8}pl$.

If it is desired to apply the law expressed in eqs. (5) and (6) to mild-steel beams, the second members of those equations may be multiplied by $\frac{6}{5}$ to $\frac{5}{4}$ for close approximations.

CHAPTER XV.

PLATE GIRDERS.

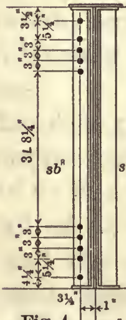
Art. 110.—The Design of a Plate Girder.

A PLATE girder is a flanged girder or beam built usually of plates and angles, the flanges being secured to the web by the proper number of rivets suitably distributed. The flanges, unlike those of rolled beams, are usually of varying sectional area, although occasionally either flange may be of uniform section throughout when formed of two angles, or two angles and a cover-plate. Fig. 1 is a general view of a plate girder, while Figs. 2, 3, 4, and 5 show some of the general features of design.

The total length of a plate girder is materially more than the length of clear span over which the girder is designed to carry load. Blocks or pedestals of masonry or metal, as the case may be, support the ends of the girders and rest on the masonry or other supporting masses or members carrying the girder and its load. The distance between the centres of these blocks or pedestals is called the effective span of the girder, as it is the span length which must be used in computing bending moments, shears, or reactions. Plate girders must evidently be somewhat longer than the effective span. In the Figs. the relations of the various parts at the end of the plate girder are shown in detail. The girder illustrated in Fig. 1 has an effective span of 68 ft. with the centre of the pedestal block 15 inches from the face of the masonry abutment and 12 inches from the extreme end of the girder. The effective depth of the girder is the vertical distance or depth between the centres of gravity of the two flanges. When the girder has cover-plates this effective depth may be greater than the depth of web plate at the centre of span and less than that at the ends, even when the web plate is of uniform depth. It is always customary, however, to take the effective depth of a plate girder with uniform depth of web as constant. Frequently that depth is taken equal to the depth of the web plate; or, again, it may be taken equal to the depth between the centres of gravity of the flanges at mid-span without sensible error. In case the web plate is not of uniform depth the effective depth might still be taken as the depth of web plate at the various sections of the girder, or it may be taken as the depth between centres of gravity of the flanges at the same sections.

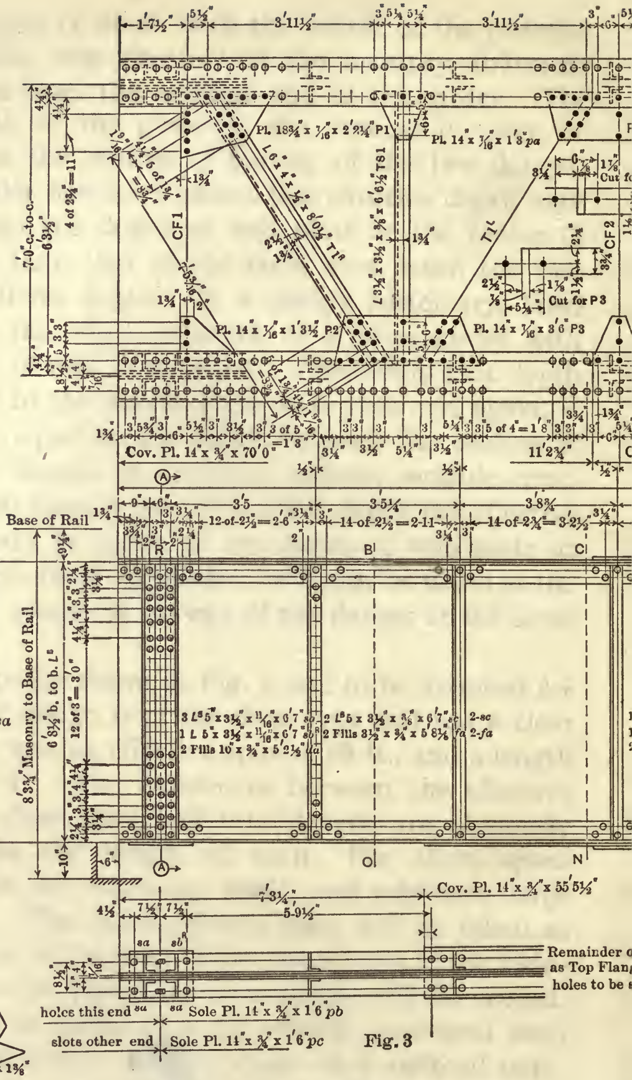
The plate girder shown in Fig. 1 and to be assumed for the purposes of design is of the deck type and has a clear span of 65 ft. 6 ins., an effective span of 68 ft., and a length over all of 70 ft. The differences between the effective span and the clear span and total length are obviously dependent upon the length of span. For short spans those differences are relatively small, and relatively large for long spans. The depth of web plate will be taken as 6 ft. 8 ins., and it will be found later that at and in the vicinity of the centre of span three cover-plates will be needed. The girder will be assumed to be of mild structural steel and will be supposed to carry a single-track railroad moving load with the concentrations and spacings shown in Table I, Art. 21.

The dead load or own weight of the girder and track will depend somewhat upon whether the girder is of the



3½"
5¼"
3 3
3L 8¼
sb^R
sa
3 3
5¼"
4¼
3¼"
1"
4¼"

Fig. 4



1'-7½"
5¼"
3'-11½"
3" 5¼ 5¼
3'-11½"
Pl. 18¾" x 7/16" x 2'2¼" P1
Pl. 14" x 7/16" x 1'3" pa
L 6" x 4" x ½" x 8'0¼" T1^R
L 3½" x 3½" x ⅜" x 6'6½" TS1
T1^L
CF1
CF2
Cut for P3
Pl. 14" x 7/16" x 1'3½" P2
Pl. 14" x 7/16" x 3'6" P3
7'-0" c.-to-c.
6'3½"
4 of 2¾"=11"
Cov. Pl. 14" x ¾" x 70'0"
3 of 5"=1'3"
5 of 4"=1'8"
11'2¾"
Base of Rail
12-of-2½"=2-6"
14-of-2½"=2-11"
14-of-2¾"=3-2½"
3'5
3'-5¼
3'-8¾
R
B
C
A
8'3¾" Masonry to Base of Rail
6'8½" b. to b. L^s
12 of 3"=3'0"
3 L^s 5" x 3½" x 11/16" x 6'7" sb
1 L 5" x 3½" x 11/16" x 6'7" sb^R
2 Fills 10" x ¾" x 5'2½" da
2 L^s 5 x 3½" x ⅜" x 6'7" sc
2 Fills 3½" x ¾" x 5'8⅜" fa
2-sc
2-fa
O
N
Cov. Pl. 14" x ¾" x 55'5½"
7'-3¼"
5'-9½"
sa
sb^R
Remainder o
as Top Flang
holes to be s
holes this end
slots other end
Sole Pl. 14" x ¾" x 1'6" pb
Sole Pl. 14" x ¾" x 1'6" pc
1⅜"
2⅝" x 1⅜"

Fig. 3

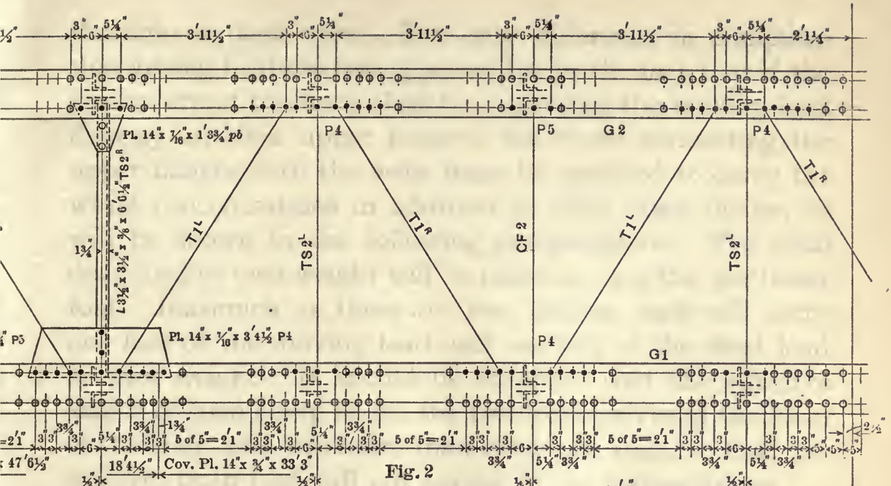


Fig. 2

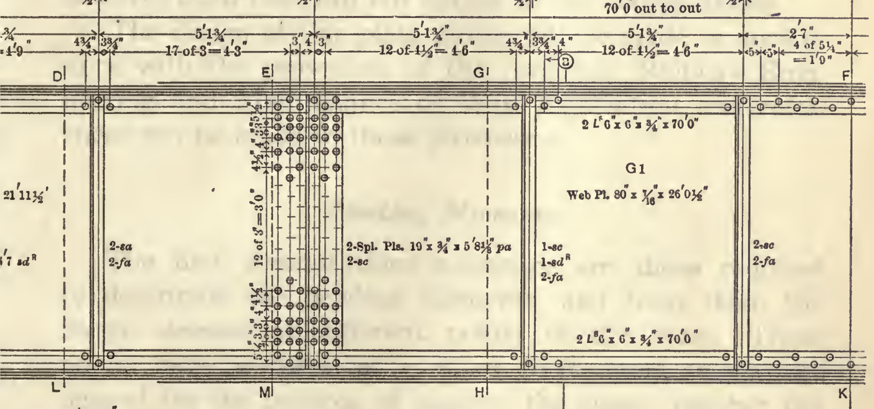


Fig. 1

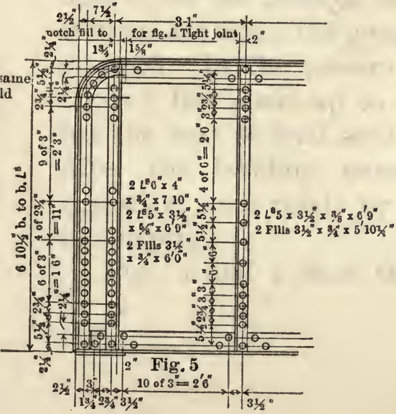


Fig. 5

(*To face page* 685.)

through or deck type. The only difference in computation arising in those two types is due to the fact that if the girders are of the deck class (i.e., carrying the moving load directly on their upper flanges) the rivets connecting the upper flanges with the webs must be assumed to carry the wheel concentrations in addition to their other duties, as will be shown in the following computations. The total dead load or own weight will be taken as 1400 lbs. per linear foot. Inasmuch as there are two girders, each will carry one half of the moving load and one half of the dead load or own weight. It should be observed that the effective length of span being 68 ft., the two locomotives at the head of the train load will more than cover the span, so that the uniform train load will not appear in the computations.

The design of this plate girder will be made in accordance with the provisions of the American Railway Engineering and Maintenance of Way Association and references will be made to those provisions.

Bending Moments.

The first computations necessary are those required to determine the bending moments, and from them the flange stresses at different points of the span. Those points may be taken at 5, 8, or 10 ft. apart as may be desired for the purpose of design; the closer together the sections are taken the greater will be the degree of accuracy attained. In the present instance those sections will be taken 5 feet apart up to 25 ft. from the end of the span, but the next or final section will be at the centre of span. After the bending moments are obtained, the flange stresses at once result by dividing the former by the effective depth.

Figs. 1 and 2 show the complete single-track railway deck-plate girder span consisting of two girders with the requisite bracing connections between them. The total dead load or own weight is a uniform load and consists of:

	Lbs. per Lin. Ft.
Track (ties, rails, etc.)	450
Two girders and bracing	1050
Total	1500
Or for one girder	$\frac{1500}{2} = 750$

As each girder will carry 750 lbs. of dead load per linear foot, and as the effective span is 68 ft., the expression for the dead-load bending moment in foot-pounds at any point will be as follows:

$$M = \frac{750}{2}(68x - x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The application of eq. (1) to the sections of the girder 5, 10, 15, 20, 25, and 34 ft. from the ends will give the following expressions for the bending moments in foot-pounds:

x	D. L. Moment. Ft. Lbs.
5	118,120
10	217,500
15	298,100
20	360,000
25	403,100
34	433,500

The moving-load bending moments are next to be found by using the concentrations shown in Table 1, Art. 21. For this purpose the criterion for the maximum bending moment, eq. (5), Art. 21, must be applied at the assumed sections in which l' (equal to x in the above dead-load computations) has the values 5, 10, 15, 20, 25, and 34 ft. The application of that criterion to the section BO, Fig. 1, 5 ft. from the end of the span shows that W_2, or the first driving wheel, must rest at the section in question for the maximum bending moment, the loads W_2 to W_{12} inclusive resting on the span. W_1 will be off the span. By the aid of Table 1, Art. 21, the greatest bending moment desired is:

$$M_5 = \frac{5}{68}(9{,}030{,}000 + 2 \times 273{,}000) = 704{,}000 \text{ ft.-lbs.}$$

Similarly for the section CN, 10 ft. from the end of the span, the criterion eq. (5) of Art. 21 shows that W_3 must be placed at C with W_{12} 2 ft. from the end of the span and W_1 off the span. By the aid of Table 1 the desired moment takes the value:

$$M_{10} = \frac{10}{68}(9{,}030{,}000 + 2 \times 273{,}000) - 150{,}000 = 1{,}260{,}000 \text{ ft.-lbs.}$$

Concisely stating the conditions and results for the remaining sections shown on Fig. 1: For DL, 15 feet from end of span, two positions of moving load, W_3 at D and W_{12} at D satisfy the criterion, but the latter with 13 feet of uniform train load on the span gives the greatest moment. Total load on the span is

$$(W_{10} + \ . \ . \ . \ + W_{18} + 3000 \times 13)$$

and the moment is:

$$M_{15} = \frac{15}{68}\left(6{,}310{,}000 + 213{,}000 \times 13 + 3000 \times \frac{13^2}{2}\right) - 345{,}000 = 1{,}715{,}000 \text{ ft.-lbs.}$$

For *EM*, 20 feet from end of span, place W_{12} at *E*;

$$M_{20}=\frac{20}{68}\left(6{,}310{,}000+8\left(213{,}000+\frac{8+3000}{2}\right)\right)-345{,}000=$$

$$2{,}040{,}000 \text{ ft.-lbs.}$$

For *GH*, 25 feet from end of span, place W_{12} at *G* and the moment is:

$$M_{25}=\frac{25}{68}\left(7{,}500{,}000+232{,}500\times3+3000\frac{3^2}{2}\right)-755{,}000=$$

$$2{,}265{,}000 \text{ ft.-lbs.}$$

The moment at the centre of the span can be computed in the same manner, but by referring to Table II of Art. 21, it will be seen to be:

$$M_{34}=2{,}435{,}400 \text{ ft.-lbs.}$$

A reference to the American Railway Engineering and Maintenance of Way Association specifications, Art. 9, will show that the required allowance for impact is represented by the factor *I*, in which L' is the length of load on the span:

$$I=\frac{300}{L'+300}.$$

The positions of loading already found for the greatest moving load moments give the lengths L' in feet in the following table:

Pt. Ft.	Loaded Length, L'. Ft.	Impact Factor I.	Moving Load Moment, Ft.-lbs.	Impact Moment Ft.-lbs.
5	63	.827	704,000	582,000
10	63	.827	1,260,000	1,040,000
15	66	.820	1,715,000	1,405,000
20	61	.897	2,040,000	1,830,000
25	64	.825	2,265,000	1,866,000
34	68	.815	2,435,000	1,985,000

By adding the dead load or own weight moments, already computed, to the moving load and impact moments in the preceding table, the total or resultant moments will be:

TABLE I.

Pt.	Total Moment Ft.-lbs.
5	1,404,000
10	2,518,000
15	3,418,000
20	4,230,000
25	4,534,000
34	4,855,000

Shears.

Both dead and moving load shears must be computed. As the dead load or own weight is a uniform load on the girder, the shear at any point is simply the load between that point and the centre of span. Hence indicating the transverse shear at any section by the figure showing its distance from the end of the span, there will result the following values, S_0 being the end shear or reaction:

$$S_0 = 34 \times 750 = 25{,}000 \text{ lbs.}$$
$$S_5 = 29 \times 750 = 21{,}750 \quad \text{``}$$
$$S_{10} = 24 \times 750 = 18{,}000 \quad \text{``}$$
$$S_{15} = 19 \times 750 = 14{,}250 \quad \text{``}$$
$$S_{20} = 14 \times 750 = 10{,}500 \quad \text{``}$$
$$S_{25} = 9 \times 750 = 6{,}750 \quad \text{``}$$
$$S_{34} = 0 \times 750 = 0 \quad \text{``}$$

The moving load shears will also be needed. Although there is no systematic criterion for such shears at different points in a span traversed by a train of concentrations, it is a simple matter to find the greatest moving load shears at the sections contemplated by inspection and trial. The greatest end shear, i.e., the greatest reaction, has been found in Art. 21 and is given in Table II of that Article:

End shear for 68-ft. span = 161,700 lbs.

End impact shear = 131,800 "

The impact factors for the shears are computed by the same formula already used for impact moments.

For a shear 5 feet from end of span: place W_2 at the 5-foot section, then the greatest shear is

$$S_5 = \frac{9{,}030{,}000 + 2 \times 273{,}000}{68} = 141{,}000 \text{ lbs.}$$

By trying other positions it will be found that this gives the greatest shear. W_1 is not on the girder and W_{12} is 2 feet from the end of the span.

For section 10 feet from end: place W_{11} at the section. Hence

$$S_{10} = \frac{6{,}310{,}000 + 213{,}000 \times 13 - \left(3000 \times \frac{13^2}{2}\right)}{68} - 150{,}000 = 122{,}000 \text{ lbs.}$$

For section 15 feet from end: place W_{11} at the section and there will result

$$S_{15} = \frac{6{,}310{,}000 + 213{,}000 \times 8 + 3000 \times \frac{8^2}{2}}{68} - 150{,}000 = 104{,}300 \text{ lbs.}$$

For 20-ft. section: place W_2 at the section and there will result

$$S_{20} = \frac{6,950,000}{68} - 150,000 = 87,200 \text{ lbs.}$$

For a 25-ft. section: place W_2 at the section and the greatest shear will be

$$S_{25} = \frac{5,240,000 + 213,000 \times 3}{68} - 150,000 = 71,500 \text{ lbs.}$$

For the centre of span: place W_2 at that point and the greatest shear will be:

$$S_{34} = \frac{3,230,000 + 174,000 \times 5}{68} - 150,000 = 45,300 \text{ lbs.}$$

The loaded lengths in each of these cases to be used in computing the impact factors are in the order of the sections beginning with that at 5 feet from the end, 63, 66, 61, 56, 51, and 42 feet, the latter belonging to the centre of span. The following tabular statement represents the elements of these moving load shears and the impact allowances:

SHEARS AND IMPACT ALLOWANCES

Section.	Loaded Length. Ft.	Impact Factor.	Moving Load Shear. Lbs.	Impact Shear. Lbs.
5	63	.827	141,000	116,500
10	66	.820	122,000	100,000
15	61	.831	104,300	86,600
20	56	.824	87,200	71,800
25	51	.855	71,500	61,100
34	42	.877	45,300	39,700

Adding together the dead load, moving load and impact shears as now determined, the following will be the resultant or total shears at sections under consideration:

TABLE II.

RESULTANT OR TOTAL SHEARS.

Section.	Total Shears. Lbs.
End	319,000
5	279,300
10	240,500
15	205,100
20	169,500
25	139,400
34	85,000

The preceding results or computations due to the dead and moving loads are the principal data required in the design of the girder.

Web Plate.

The effective depth of the girder will tentatively be taken as 6 feet 8 inches and the depth from the back of flange angles in the upper flange to the back of the lower flange angles will be taken as 6 feet 8½ inches. As the depth of the web plate must be taken a little less than the depth from back to back of angles, in order that the flange plates may not touch the edges of the web plates when the different parts of the girder are assembled, that depth should be taken as 6 feet 8 inches. In fact the effective depth of a plate girder is sometimes prescribed as the depth of the web plate. This depth of web plate will leave ¼ inch clear at the top and bottom flanges, which is sufficient to insure the flange plates freedom from hitting the edges of the web.

Art. 18 of the Specifications allows a working stress in shear of 10,000 pounds per square inch of gross cross-section of the web. As the total end shear has been found to be 319,000 pounds, the gross web plate section at the end of span should be 31.9 square inches. The minimum thickness must then be $\frac{31.9}{80}=.399$ inch.

A web plate $80\times\frac{7}{16}$ inch will be used, giving a gross sectional area of $80\times.4375=35$ square inches. The surplus area is small and it is judicious design to have it. This web plate thickness also satisfies Art. 29 of the Specifications which prescribes that "The thickness of web plates shall not be less than $\frac{1}{160}$ of the unsupported distance between flange angles," as 6×6 inch flange angles will be used.

Flanges.

Art. 29 of the Specifications provides that the design of the flanges may be based either on the moment of inertia of the net section of the girder or on the assumption that the flange stress is of constant intensity with its centre at the centre of gravity of the flange area, the latter including one-eighth of the gross section of the web, the difference between one-sixth and one-eighth of the web section being supposed to cover the material punched out in the tension side of the web plate. The latter method will be employed.

Art. 30 of the Specifications provides that "The gross section of the compression flanges of plate girders shall not be less than the gross section of the tension flanges." It will be best, therefore, to design the tension flange first.

Using the total or resultant bending moment at the centre of the span, the trial effective depth of 6 feet 8 inches will give the centre flange stress as follows:

$$\frac{4,855,000}{6.67}=728,000 \text{ lbs.}$$

The specifications permit a working tensile stress in the net section of the tension flange of 16,000 pounds per square inch. Hence the required net tension flange area is

$$\frac{728,000}{16,000}=45.5 \text{ sq.ins.}$$

The available flange section due to one-eighth the gross sectional area of the web is $\frac{35}{8}=4.375$ square inches. The amount of flange area to be supplied by the flange plates and angles is, therefore,

$$45.5-4.4=41.1 \text{ sq.ins.}$$

In providing 41.1 square inches it is necessary to know what rivet holes are to be deducted from each cover-plate and each flange angle. It is clear that two rivet holes only need be deducted from each cover-plate, and it is plain that at least two rivet holes must be deducted from each flange angle section. In designing cover-plates for flanges it must be remembered that no such plate must be thicker than the one under it, i.e., if these plates are not of the same thickness, the thickest one must lie on the angles, the remaining thicknesses to decrease or be the same in passing outward from the angles. As a trial section let the following be assumed:

Angles or Cover-plates.	Gross Area. Sq.Ins.	Less Rivet Holes. Sq.Ins.	Net Section. Sq.Ins.
2 6″×6″×$\frac{3}{4}$″......	16.88	4×1×$\frac{3}{4}$=3.0	13.88
3 covers 14″×$\frac{3}{4}$″...	31.5	6×1×$\frac{3}{4}$=4.5	27.00
	48.38		40.88

As 40.88 square inches is but 1¼ per cent. less than the desired area, 41.4 square inches, the former may be accepted subject to further confirmation.

If the centre of gravity of the gross section of the tentative flange area consisting of the three plates and two angles indicated above be determined, it will be found .11 inch above the back of the angles. This will make the effective depth

$$6 \text{ ft. } 8.5 \text{ ins.} + .22 \text{ in.} = 6 \text{ ft. } 8.72 \text{ ins.}$$

This increase in effective depth will correspondingly decrease the centre flange stress so as to make the total actual net area of 45.3 square inches a little larger than required. Hence the trial centre tension flange area as determined above will be accepted as the actual flange area to be used, i.e., three 14×$\frac{3}{4}$-inch cover-plates and two angles 6×6×$\frac{3}{4}$ inch.

Length of Cover-plates.

In the next Article there will be shown two methods of determining the lengths of cover-plates after the sections of those plates have been found for the greatest bending moment, usually taken as at the centre of span. These two methods are simply different forms of expression of the same thing. The following notation will be used:

l = length of span in feet;
L_1 = length of outside cover-plate in feet;
L_2 = length of second cover-plate in feet;
A = total net flange area, square inches;
a_1 = net area of outside cover-plate, square inches;
a_2 = net area of second cover-plate, square inches;
a_3 = net area of third cover-plate, square inches.

It has already been seen that if a beam simply supported at each end be loaded uniformly throughout the span, the bending moment at any point will be represented by the vertical ordinate of a parabola whose vertex is over the centre of span while the end of each branch is at one end of the span. It is assumed that the greatest bending moments in the plate girder, already computed, vary by the same parabolic law. This is not quite true, but sufficiently near for ordinary purposes.

Then, as will be shown in the next Article,

$$L_1 = l\sqrt{\frac{a_1}{A}}; \quad L_2 = l\sqrt{\frac{a_1+a_2}{A}}; \quad L_3 = l\sqrt{\frac{a_1+a_2+a_3}{A}}.$$

In this case $l = 68$ feet and $A = 45.3$ square inches.

$$a_1 = a_2 = a_3 = 9 \text{ sq. ins.}$$

Making these numerical substitutions, there will result $L_1 = 30.7$ feet; $L_2 = 42.9$ feet; $L_3 = 52.5$ feet. These lengths are clearly the minimum permissible. In actual construction it is desirable to have the end of the plate from 1 to 1.5 feet further from the centre, making the total length of the plate 2 to 2.5 feet greater than the length computed above. This lengthening of the cover-plate is essential in order that the cover-plate metal may be taking stress at the point where the plate is computed to begin. Also as will be seen a little further on, the pitch of rivets in these ends of the cover-plates is made less than in the body of the plate for greater effectiveness where the plate begins to take its stress. The lengths of cover-plates then, beginning with the shortest, will be 33.2, 45.4, and 55 feet.

Another method of procedure, more accurate than the preceding, is to draw a moment curve on the effective span, which can readily be done by laying down as vertical ordinates the resultant or total moments as given in Table I. These moment ordinates would be 5 feet apart except at the centre of span. The lengths of cover-plates must be such as to give resisting moments of the flange stresses at least equal to the external bending moments shown on such a diagram. The moments of the flange stresses will require the centres of gravity of parts of the flange sections to be computed at each moment point. The following tabulation shows the elements of this method of procedure for the centre section of the girder:

Section.	Sq. Ins.	Stress per Sq. In.	Lever Arm Ft.	Moment. Ft.-lbs.
One-eighth web plus flange angles . . .	18.3	16,000	6.41	1,875,000
First cover-plate	9	16,000	6.77	976,000
Second cover-plate	9	16,000	6.83	984,000
Top cover-plate	9	16,000	6.9	994,000

This operation must be repeated at each moment section of the girder, but the numerical work need not be repeated here, being precisely like that for the centre section.

The net lengths of plates found by this method are 32.8, 42.9 and 53.8 feet, a substantial agreement with the lengths found by the shorter procedure.

In the compression flange the cover-plate lying on the angles should run the entire length of the girder, especially if the girder be of the deck type, i.e., with ties resting upon the upper flange. That flange being under compression, it is advisable that the horizontal legs of the angles be supported throughout their entire length by riveting them to a cover-plate. This will add to the stiffness and carrying capacity of the flange. If ties rest directly upon the upper flange, their deflection tends to bend one side of it out of its horizontal position, but this tendency will be materially lessened by the added stiffness gained in riveting the horizontal angle legs of the flange to the cover-plate.

Although this process of design has been used in connection with the tension flange, under the specifications the compression flange is to be made like the tension flange, i.e., a duplicate of it.

Pitch of Rivets in Flanges.

Arts. 5 and 31 of the specifications relate to the rivets required to join the vertical legs of the flange angles to the web plate. Art. 31 requires that "The flanges of plate girders shall be connected to the web with a sufficient number of rivets to transfer the total shear at any point in a distance equal to the effective depth of the girder at that point combined with any load that is applied directly on the flange. The wheel loads where the ties rest on the flanges shall be assumed to be distributed over three ties."

The chief function of these rivets is to transfer horizontal shear from the web plate to the flanges, as it is in this way that the flanges receive their stresses. If the rivets take the direct load of the locomotive driving wheels, as in the case of a deck girder like that being designed, they must resist the resultant stress due to both vertical and horizontal loads.

Strictly speaking the number of rivets required between two moment sections, as shown in Fig. 1, should be just sufficient to give the increase of flange stress in passing from one section to the next one toward the centre of span. Art. 31 of the specifications, therefore, requires more rivets than are needed except at the end of the span. It is always necessary, however, to introduce more rivets near the centre of span than is required by actual computations, for the general stiffness of the girder. Indeed even more rivets are generally provided than those prescribed in Art. 31 of the specifications.

If d is the effective depth of the girder at the end of the span and if the end shear or reaction is R, and if tA is the flange stress at the distance d from the end of span, then will the following equation of moments be found, neglecting the negative moment of any load within the distance d from the end of the span:

$$Rd = tAd.$$

Hence

$$R = tA.$$

This shows that an amount of stress equal to the end shear must be given to each flange within the distance d from the end. The number of rivets required by this computation is a little more than necessary if any load rests upon the girder between the end and the section at the distance d from it. It will be clear that the general provision of Art. 31, quoted above, is based upon this end shear requirement, and it is analytically incorrect, but the excess of rivets which it calls for adds to the general stiffness and capacity of the girder.

The weight of one driving wheel is 30,000 pounds, and it is to be distributed over three ties or 42 inches. As the prescribed impact is 100 per cent., the vertical load per horizontal inch of girder will be:

$$V=\frac{2\times 30{,}000}{42}=1430 \text{ lbs.}$$

It is obvious that the flange stress taken by one-eighth of the sectional area of the web is received directly by the latter and does not affect the rivets through the vertical legs of the flange angles. If A_1 is the actual net flange section of cover-plates and angles and A_2 the total flange area, including one-eighth of the web section, and if S is the total shear at any moment section, while d is the effective depth of the girder, then the horizontal flange stress H to be taken up per linear inch by the rivets between two sections the distance d apart will be $H=\frac{S}{d}\frac{A_1}{A_2}$.

The values of A_1 and A_2, beginning at the end section of the girder, are as follows:

Section	A_1	A_2
End	22.88 sq.ins.	27.26 sq.ins.
5 ft.	22.88 "	27.26 "
10 "	22.88 "	27.26 "
15 "	31.88 "	36.26 "
25 "	40.88 "	45.26 "
Centre	40.88 "	45.26 "

The unit (inch) increments H of horizontal flange stress found for the various sections by the preceding formula are:

End $H=\frac{319{,}500}{\underline{80.5}}\times\frac{22.88}{27.26}=3330$ lbs.

5 ft. $H=$ $=2920$ "

10 " $H=$ $=2510$ "

15 ft. $H =$ $= 2170$ lbs.
25 " $H =$ $= 1560$ "
Centre $H =$ $= 954$ "

Each of the above results gives the horizontal stress H in pounds per linear inch, over each 80.5 inches of girder flange for each moment section and to be taken up by the rivets.

The rivet pitch p at any section will then be determined by the following formula if K is the working value of one rivet in shear or bearing:

$$p\sqrt{V^2+H^2}=K.$$

Each rivet bears against the web plate as well as against each vertical leg of the flange angle, and as the web plate is much thinner than the sum of the thickness of the two angle legs, the bearing value against the web plate will be much less than that against the angle legs. Furthermore each rivet is subjected to double shear, the two shearing sections of the rivets coinciding with the two faces of the web plate. K, therefore, must be taken as the least of the double shearing value and the bearing value against the web plate. The rivets to be used are $\frac{7}{8}$-inch diameter before being driven and the bearing value of such a rivet against a $\frac{7}{16}$-inch plate at 24,000 pounds per square inch is 9190 pounds and 14,430 pounds in double shear at 12,000 pounds per square inch, both of these working stresses being in accord with the specifications.

Applying the numerical results thus established to the formula for the pitch,

$$p=\frac{K}{\sqrt{V^2+H^2}},$$

there will result:

At end $p = \frac{9190}{\sqrt{1430^2 + 33\overset{3}{2}0^2}} = 2.55$ ins.

5 ft. point $p = \frac{9190}{\sqrt{1430^2 + 2920^2}} = 2.83$ "

10 " $p = \quad = 3.18$ "

15 " $p = \quad = 3.53$ "

25 " $p = \quad = 4.34$ "

Centre $p = \quad = 6.26$ "

If desired a curve can be drawn at the various points with the corresponding pitch as a vertical ordinate at each point. Such a curve will give the rivet pitch at any point in the span, but such detail is not usually required. The above values of the pitch may be used, with judgment, without further computations for any part of the girder. Fig. 1 shows the pitch used at the different girder points; it is frequently adjusted to the position of the intermediate stiffeners.

Pitch of Rivets in Cover-plates.

The number of rivets required in a cover-plate is at once determined from its net section. In the present case the net section of each cover-plate is 9 square inches, which, at 16,000 pounds, gives 144,000 pounds as the stress value of the plate. The rivets in the cover-plates are subjected to single shear and the single-shear value of one $\frac{7}{8}$-inch rivet is 7220 pounds. Hence the number of rivets required to develop the full value of one cover-plate is $\frac{144,000}{7220} = 20$ rivets. Between the end of the cover-plate, therefore, and the point at which the next cover-plate outside of it begins, there must be at least 20 rivets. As a matter of fact considerably more than that number will be found, as the pitch must not exceed 6 inches in any case and it should not be more than 3 inches for a distance of 12 to 18 inches from the end of the plate. It will be seen upon examining the drawing that these conditions are fulfilled.

Top Flange.

As this flange is in compression, gross areas may be used. If the provisions of Art. 30 and other Articles of the specifications be scrutinized, it will be found that they are fulfilled by the compression flange made up as shown in the figures, and they need no further detailed attention.

End Stiffeners.

The end stiffeners must be heavy members of their class and rigidly riveted to the girder, as they take the severe impact or pounding at the points of support due to rapidly moving heavy locomotives and trains. Art. 79 of the specifications provides that "There shall be web stiffeners generally in pairs, over bearings, at points of concentrated loading, and at other points where the thickness of the web is less than one-sixtieth of the unsupported distance between flange angles. . . . The stiffeners at the ends and at points of concentrated loads shall be proportioned by the formula of paragraph 16, the effective length being assumed as one-half the depth of girders. . . ." This provision makes it necessary to treat the end stiffeners as a column, the working stress to be:

$$p = 16{,}000 - 70\frac{l}{r}.$$

The column load in this case is the maximum end shear including impact allowance as given by Table II, i.e., 319,000 pounds.

If two pairs of $5 \times 3\frac{1}{2} \times \frac{11}{16}$-inch angles be assumed for trial with the $3\frac{1}{2}$-inch legs against the web plate, remembering that they will be separated by the thickness of the plate, the radius of gyration of their combined section about an axis lying in the centre of a horizontal web section and parallel to the web will be 3.13 inches. The length of the column is $\frac{80.5}{2} = 40.25$ inches $= l$. Hence the prescribed formula will give a working stress of 15,100 pounds per square inch. On this basis

$$\text{Area required} = \frac{319,000}{15,000} = 21 \text{ sq.ins.}$$

The actual sectional area of four of the assumed angles will be 23.24 square inches, which is sufficiently close to the area required to be accepted as satisfactory.

The entire load is carried to the end stiffeners by the $\frac{7}{8}$-inch rivets which bind them to the web plate. The rivets are in double shear and bear on the web plate. It has already been seen that the bearing value on the web plate, 9190 pounds per rivet, is much less than the double shear value. Hence the number of rivets required is $\frac{319,000}{9190} = 35$ rivets. This computed number of rivets distributed throughout the length of the $3\frac{1}{2}$-inch angle legs would make the pitch too great. The pitch should not exceed about 4 inches, which would make the number of rivets about 40. It is essential, as already indicated, that the end stiffeners be made exceptionally stiff and rigid.

End stiffeners are not bent, but are riveted onto filling plates having the same thickness as the flange angle legs. These filling plates enhance the stiffness and resisting capacity of the end stiffeners as they, in fact, form a part of the latter.

Intermediate Stiffeners.

By referring to Art. 79 of the specifications there will be found an empirical formula giving the maximum distance between intermediate stiffeners, providing, however, that that distance in no case shall exceed the clear depth of the web. Intermediate stiffeners are sometimes regarded as being equivalent to the vertical compression members of a Pratt truss, but as a matter of fact there is no rational system of basing their design on computations. They are almost invariably made of angles, but sectional areas are determined by experience. Inasmuch as the total transverse shear at the centre of span is small, they are sometimes omitted there. As a rule they are never placed farther apart than the depth of web plate.

As this girder is to carry a heavy railroad load presumably at high speed, $5 \times 3\frac{1}{2} \times \frac{3}{8}$-inch steel angles will be used with the $3\frac{1}{2}$ inch leg placed against the web plate. As the transverse shear increases toward the end of the span, the distance apart of these intermediate stiffeners will correspondingly be decreased. In the central part of the span this distance is seen to be 5 feet $1\frac{3}{4}$ inches, but near the ends it is reduced to 3 feet $5\frac{1}{4}$ inches. The pitch of the rivets in these intermediate stiffeners may vary from 3 inches to 5 or 6 inches, the greater pitch being near the mid depth of the web.

Splices in Flanges.

It will be found that cover-plates and flange angles may be purchased of full lengths required on this plate girder. When, in general, the girders are so long as to require splicing of the parts of flanges, those joints for the tension flange must be so designed as to leave the net section as large as practicable, as the entire stress must be carried by the net section. It is good practice and customary not to have two joints in adjacent parts concur, i.e., there should be breaking of joints so as to have a joint in one part only of the flange at the same section. In this manner the net section at each joint may attain its maximum value. In the splicing of angles both legs should be spliced. In compression, riveted joints can scarcely be expected to transfer stresses by abutting surfaces in those joints. They should be spliced about as effectively as tension joints, although the question of net section does not arise, the gross section being available.

Splices in Web Plates.

As one-eighth of the gross web-plate section is considered as resisting bending as a part of the flange area, the rivets at a web-plate splice must be sufficient to resist the corresponding bending moment. This web-plate moment is, therefore,

$$\frac{16,000}{8} \times \frac{7}{16} \times \overline{80.5}^2 = 5,670,000 \text{ in.-lbs.}$$

There must be two splice-plates, one on each side of the web, each of which need not be as thick as the main plate, but in this case $\frac{3}{4}$-inch splice-plates have been used so that the intermediate stiffener need not be bent. For this size of girder there should be three rows of rivets on each side of the joints. If it be assumed that the pitch be 4 inches in each row, there will be nine rivets in each of the three rows between the mid depth of the web and the back of the flange angles. If the loads carried by these rivets in resisting bending vary directly as the distance from the neutral axis at mid depth, their resultant will act at $\frac{2}{3} \times 40 = 26.7$ inches from that line. The bearing value of a $\frac{7}{8}$-inch rivet against the $\frac{7}{16}$-inch web is 9190 pounds. Hence the resisting moment of the 54 rivets on one side of the joint is:

$$M = \frac{27 \times 9190}{2} \times 2 \times 26.7 = 6,600,000 \text{ in.-lbs.}$$

As this is greater than 5,670,000 in.-lbs., the proposed arrangement of the joint is satisfactory. The two splice-plates will, therefore, each be 19 × $\frac{3}{4}$ inches by 5 feet 8$\frac{1}{8}$ inches, as shown in Fig. 1.

In general every joint splicing should be tested for the transverse shear which it must carry. In this instance it is clear that the splice-plates will carry more shear than the web.

General Considerations.

The girder proper with its flanges, web, and stiffeners has been designed in this article without indicating the manner of connecting such lateral or cross bracing as would be required in the complete design of a railroad plate-girder span. The design of such bracing would be supplementary to the actual design of the girder as made, and it is the purpose here to illustrate only those principles belonging to the design of the girder proper. The design of the bracing and the details of its connection with the girder belong rather to bridge construction than to the subject treated here. Fig. 2 has been introduced, however, as an illustration to indicate the general features of the complete structure.

Large plate girders are not always built complete in the shop, although girders nearly 100 feet in length are frequently and perhaps usually so completed at the present time. When it is necessary to build them in portions and rivet the portions together in the field, the general principles governing the construction of the necessary field-joints are precisely the same as those illustrated in this article. They are simply adjusted or adapted to the exigencies of each particular case.

The bill of material and estimated weight of a single girder as designed is as follows:

	Pounds.
Two $80'' \times \frac{7}{16}''$ web plates, $21'\ 11\frac{1}{2}''$ long	5,236
One $80'' \times \frac{7}{16}''$ web plate, $26'\ \frac{1}{2}''$ long	3,094
Four $6'' \times 6'' \times \frac{3}{4}''$ angles, $70'$ long	8,036
One $14'' \times \frac{3}{4}''$ cover-plate, $70'$ long	2,499
One $14'' \times \frac{3}{4}''$ cover-plate, $55'\ 5\frac{1}{2}''$ long	1,981
Two $14'' \times \frac{3}{4}''$ cover-plates, $47'\ 6\frac{1}{2}''$ long	1,696
Two $14'' \times \frac{3}{4}''$ cover-plates, $33'\ 3''$ long	1,190
Eight $5'' \times 3\frac{1}{2}'' \times \frac{11}{16}''$ angles, $6'\ 7''$ long	1,037
Twenty-eight $5'' \times 3\frac{1}{2}'' \times \frac{3}{8}''$ angles, $6'\ 7''$ long	1,917
Four $10'' \times \frac{3}{4}''$ filler-plates, $5'\ 8\frac{1}{8}''$ long	581
Four $19'' \times \frac{3}{4}''$ splice-plates, $5'\ 8\frac{1}{8}''$ long	1,106
Twenty-four $3\frac{1}{2}'' \times \frac{3}{4}''$ filler-plates, $5'\ 8\frac{1}{8}''$ long	1,222
Two $14'' \times \frac{3}{4}''$ sole-plates, $1'\ 6''$ long	107
Rivets	800
Total for one girder	30,502

The weight of girder per linear foot therefore is:

$$\frac{30,502}{70} = 436 \text{ lbs.}$$

If the plate girder were of the through type, there would be no change whatever in the procedures of design which have been followed, but in order to give a better appearance to the ends they would be formed as shown in Fig. 5. The latter figure shows the same end stiffness, depth of girder and the same flange angles as Fig. 1.

Art. 111.—Length of Cover-plates.

There are various methods of determining the lengths of cover-plates of plate girders involving simple computations only, which are well illustrated by the following procedures:

The first of these procedures is based on the assumption that the depth of the girder is uniform and that the bending moment throughout the length of girder varies as the ordinate of a parabola as in the case of uniform loading. The following notation is required:

l = effective length of span either in feet or inches;
L = length of cover-plate required in the same unit as l;
A = total net flange area;
a = net cover-plate area required.

Since the flange and cover-plate areas vary directly as the flange stresses, and as the latter vary as the ordinates of a parabola when the depth of girder is constant, the following equation will result:

$$\frac{L^2}{l^2}=\frac{a}{A},$$

or

$$L=l\sqrt{\frac{a}{A}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Eq. (1) will give the length of the cover-plate whose area of section is a. Any convenient unit may be taken for a and A, but the square inch is ordinarily employed.

If there are several cover-plates, a is to be taken successively the area of the first, second, third, etc., cover-plates in summation, i.e., it will first be taken as the net sectional area of the top cover, then as the net sectional area of the top cover added to that of the cover-plate below it, and so on.

The second method is the following, and is applicable to the case of a girder with varying depth, the notation being as follows:

Let w = uniform load per linear foot, or "equivalent uniform load" per linear foot;

d and d' represent the effective depths of girder in feet at the centre of span and at the end of the cover-plate respectively;

$A-a=a'$ = area of flange section at the end of cover-plate;

T = permissible flange stress per square inch;

the bending moment at the end of the cover-plate will then be

$$M=w\frac{l^2}{8}-\frac{w}{2}\left(\frac{L}{2}\right)^2=AdT-w\frac{L^2}{8}=d'a'T. \quad . \quad . \quad (2)$$

By solving the second and third members of the preceding equation there will result

$$L=2\sqrt{2}\sqrt{\frac{(Ad-a'd')T}{w}}=2.83\sqrt{\frac{(Ad-a'd')T}{w}}. \quad . \quad . \quad (3)$$

It must be remembered that the application of either of the two preceding methods will give the net length of the cover-plate. There must be added 12 to 18 ins. at each end with rivets closely pitched so that the cover-plate may certainly take its stress at the points where its effectiveness should begin.

Art. 112.—Pitch of Rivets.

A simple method of finding the pitch of rivets piercing the vertical legs of the flange angles and the web plate of a plate girder at any section of the beam may readily be found by using the general but elementary expression for the bending moment,

$$\Sigma Px = M.$$

By differentiating this equation,

$$\Sigma P . dx = Sdx = dM;$$

S representing the total transverse shear.

If dM is the change of bending moment for the distance along the flange represented by the pitch of rivets, p, the change of flange stress for the same distance will be found by dividing dM by the effective depth of the girder, d. If the pitch of rivets, p, be placed in the preceding equation in place of dx, the corresponding change of flange stress will represent the amount of stress transferred to the flange by one rivet. Representing that variation of flange stress by v, the last of the preceding equations may be written

$$Sp = dv; \quad \therefore \ p = \frac{dv}{S}.$$

In this equation v represents either the bearing capacity of one rivet against the web plate or against the two flange angles, or the double shearing value of the same rivet, i.e., the least of those three values. Ordinarily the bearing of the rivet against the web plate will be less than either of the two other quantities; hence that bearing value would then be substituted for v. In general the least of the three preceding values for one rivet is to be substituted for v in an actual computation. The total transverse shear S is always known at any section or may readily be determined. The preceding formula for the pitch, therefore, is a very simple one and is much employed.

CHAPTER XVI.

MISCELLANEOUS SUBJECTS.

Art. 113.—Curved Beams in Flexure.

If beams are sharply curved, i.e., if the radius of curvature of the neutral surface is comparatively small, the formulæ expressing the common theory of flexure for such beams will contain the radius of curvature and corresponding variations from the formulæ for straight beams.

Let Fig. 1 represent part of a curved beam subjected to flexure, AC representing the radius of curvature at the point A before flexure while $C'A'$ represents the radius of curvature of the same surface after flexure takes place. OAO' represents the neutral surface. $A'b''$ is the continuation of $C'A'$. Similarly $A'b$ is the continuation of CA'. Finally, $A'b'$ is drawn parallel to CA. def represents the normal section of the beam and AA' is supposed to be a differential of the length of the neutral surface.

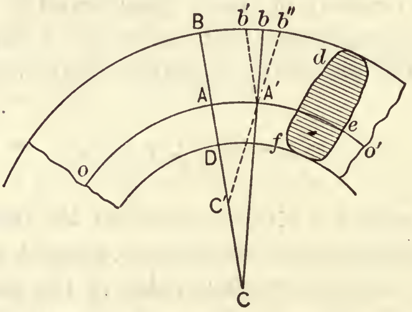


Fig. 1.

The ordinate $\pm y$ is measured from A as an origin toward B or D, respectively. z is the varying width of the normal section of the beam and hence it is measured normal to y and x, the latter being measured along OAO'. A differential of the section of the beam is zdy.

As the normal sections of the beam are assumed to remain plane after flexure, let the rate of strain, i.e., the strain per unit of length of fibre at any point distant y from the neutral surface be uy, u being the apparent rate of strain at unit distance from the neutral surface.

By referring to Fig. 1 there may at once be written:

$$b'b = dx\frac{y_1}{r}; \quad bb'' = \left(dx + dx\frac{y_1}{r}\right)uy_1.$$

By similarity of triangles,

$$\frac{dx\frac{y}{r} + dx\left(1 + \frac{y}{r}\right)uy}{y} = \frac{dx}{r'}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation gives at once:

$$u = \frac{r - r'}{(r+y)r'} = \frac{\frac{r}{r'} - 1}{r+y}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the beam were originally straight, in which case the radius of curvature $r = \infty$, eq. (2) would take the form $u = \frac{1}{r'}$, the usual expression for the rate of strain at unit distance from the neutral surface of a straight beam. If again the radius of curvature is sufficiently large, so that r may be written for $r+y$ without sensible error:

$$u = \frac{1}{r'} - \frac{1}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This expression for u may be used for curved beams if the curvature is not too sharp.

If the radius r is infinitely great, $u=\frac{1}{r'}$, which is the value for a straight beam.

Eq. (2) shows that the rate of strain u at unit distance from the neutral surface and corresponding to the rate of strain at any distance y is variable, as y appears in the denominator in such a way as to make u smaller the greater the distance of the fibre from the neutral surface. This is in consequence of the curvature of the beam and results from the assumption that normal sections plane before flexure remain plane after flexure. With the increase of length of fibre due to curvature as its distance from the neutral axis increases, a less rate of strain is required to keep the section plane after flexure. This assumption is not strictly true, and it may be a matter of doubt whether it is necessary or advisable even in the interests of correct analysis.

If k is the fibre stress of tension or compression at any distance y from the neutral axis, there may be at once written:

$$k=Euy=E\left(\frac{r}{r'}-1\right)\frac{y}{r+y}. \quad . \quad . \quad . \quad . \quad (4)$$

The stress on an element zdy of the section will then be:

$$kzdy=E\left(\frac{r}{r'}-1\right)\frac{zydy}{r+y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Let k' and k'' be the intensities of stress at the distances y' and $-y''$ from the neutral surface. Then by eq. (4):

$$\frac{k'}{k''}=\frac{y'}{r+y'}\,\frac{r-y''}{-y''}.$$

From this equation:

$$k'' = -k'\left(\frac{r+y'}{y'}\right)\frac{y''}{r-y''}. \quad . \quad . \quad . \quad . \quad (5a)$$

If $y' = y''$, eq. (5*a*) becomes:

$$k'' = -k'\frac{r+y'}{r-y''}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

Eq. (5*b*) shows that the intensity of stress at a given distance from the neutral axis will be greater on the concave side of the curve than on the convex, and that this relation holds until the radius of curvature becomes infinitely great.

In order to locate the neutral axis the integral of the two members of eq. (5) between the limits of y and $-y$ must be placed equal to zero, giving eq. (6):

$$\int_{-y_0}^{y_1} kzdy = E\left(\frac{r}{r'} - 1\right)\int_{-y_0}^{y_1} \frac{zydy}{r+y} = 0. \quad . \quad . \quad . \quad (6)$$

Again, the bending moment formed by the direct stresses of tension and compression in the section may be written in the usual manner as follows, M representing the moment:

$$dM = kzdy \cdot y = E\left(\frac{r}{r'} - 1\right)\frac{zy^2dy}{r+y}.$$

$$\therefore \quad M = E\left(\frac{r}{r'} - 1\right)\int_{-y_0}^{y_1} \frac{zy^2dy}{r+y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eq. (6) shows that the neutral axis will not pass through the centre of gravity of the section. As the intensity of stress on the convex side of the curve will be less than if the beam were straight, the neutral axis will be on that side of the centre of gravity of the section toward the concave surface of the beam. Eq. (7) shows, again, that the integral is not the moment of inertia of the section about the neutral axis, but it will reduce to that if the radius of curvature r be supposed infinitely great.

The integrations shown in the second members of eqs. (6) and (7) can at once be made when the form of cross-section is known. Inasmuch as this analysis for curved beams finds one of its important applications in connection with the design and carrying capacity of large hooks, a trapezoidal cross-section shown in Fig. 2 will be assumed by way of illustration, and from that the rectangle section at once results. In that figure the larger end CD of the trapezoid will be considered to lie in the concave or inner surface of the hook and at right angles to the plane of the hook. As the trapezoid is symmetrical, $a = \frac{1}{2}FH$, and the angle of inclination of a sloping side as HD to the centre line will be taken as α. Then z will represent one-half of the width of the trapezoid at any point:

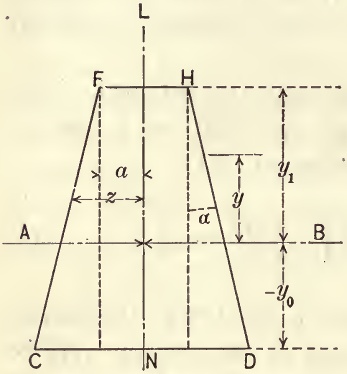


FIG. 2.

$$z = a + (y_1 - y) \tan \alpha. \quad . \quad . \quad . \quad . \quad (8)$$

If z be inserted in eqs. (6) and (7) there will be required the following integrations in which $y_1 + y_0 = d$:

$$\int_{-y_0}^{y_1} \frac{ydy}{r+y} = d - r \log \frac{r+y_1}{r-y_0}, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\int_{-y_0}^{y_1} \frac{y^2dy}{r+y} = d\left(\frac{y_1-y_0}{2}-r\right)+r^2 \log\frac{r+y_1}{r-y_0}, \quad . \quad . \quad . \quad (10)$$

$$\int_{-y_0}^{y_1} \frac{y^3dy}{r+y} = r^2d - \tfrac{1}{2}rd(y_1-y_0)+\tfrac{1}{3}d(d^2-3y_1y_0)$$
$$-r^3 \log\left(\frac{r+y_1}{r-y_0}\right). \quad . \quad . \quad . \quad (11)$$

If these values of z and the integrals given in eqs. (9), (10) and (11) be substituted in eq. (6), there will at once result:

$$\log\frac{r+y_1}{r-y_0} = \frac{d\left\{\left(r+\frac{d}{2}\right)\tan\alpha+a\right\}}{r\{(r+y_1)\tan\alpha+a\}}. \quad . \quad . \quad . \quad (12)$$

As known quantities let $r+y_1=R$ and $r-y_0=R_0$, then eq. (12) may take the form:

$$r\log\frac{R}{R_0}(R\tan\alpha+a) = rd\tan\alpha+d\left(\frac{d}{2}\tan\alpha+a\right).$$

Hence:

$$r = \frac{d\left(\frac{d}{2}\tan\alpha+a\right)}{\left(R\log\frac{R}{R_0}-d\right)\tan\alpha+a\log\frac{R}{R_0}}. \quad . \quad . \quad . \quad (13)$$

After r is determined by eq. (13) there will at once result:

$$y_1 = R-r \quad \text{and} \quad y_0 = d-y_1. \quad . \quad . \quad . \quad (14)$$

If the section is rectangular, $\alpha=\tan\alpha=0$, hence,

$$r = \frac{d}{\log\frac{R}{R_0}} \quad \text{and} \quad y_1 = R-\frac{d}{\log\frac{R}{R_0}}. \quad . \quad . \quad . \quad (14a)$$

If the section is triangular, $a=0$ and the second member of eq. (13) will be correspondingly simplified as follows:

$$r=\frac{d^2}{2\left(R\log\frac{R}{R_0}-d\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

As this expression is independent of α, y_1 and y_0 remain unchanged whatever may be the value of that angle.

Having thus found y_1 and y_0, the position of the neutral axis of the section is determined and the expression for the bending moment can now be written by the aid of eqs. (4) and (7), the latter being the general expression for the bending moment. By the aid of eq. (4) the intensity of stress in the extreme fibre at the distance y_0 from the neutral axis may be written as follows:

$$k_0=-E\left(\frac{r}{r'}-1\right)\frac{y_0}{r-y_0}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Hence,

$$E\left(\frac{r}{r'}-1\right)=-\frac{k_0(r-y)}{y_0}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

By introducing the second member of eq. (17) in eq. (7) as well as the value of z from eq. (8) and the integrals given in eqs. (10) and (11), the following value of the moment M will result:

$$-M=\frac{2k_0(r-y_0)}{y_0}\int_{-y_0}^{y_1}\left\{(a+y_1\tan\alpha)\frac{y^2dy}{r+y}-\tan\alpha\frac{y^3dy}{r+y}\right\} \quad (18)$$

$$=\frac{2k_0(r-y_0)}{y_0}\left\{(a+(r+y_1)\tan\alpha)\left(\frac{d}{2}(y_1-y_0)\right.\right.$$

$$\left.\left.-dr+r^2\log\frac{r+y_1}{r-y_0}\right)-\frac{d\tan\alpha}{3}(d^2-3y_1y_0)\right\}. \quad . \quad . \quad (19)$$

As is evident, the factor 2 appears in the second members of eqs. (18) and (19), for the reason that the section taken is symmetrical and the varying ordinate z is half the width of section at any point. If a were taken as the extreme width of section on the narrow side instead of half that width and if α were to be so taken that $(y_1 - y)\tan\alpha$ added to a represents the full width of the section at the point located by y, the factor 2 would be omitted from the second member of the value for M.

If the section is rectangular $\alpha = \tan\alpha = 0$ and the expression for the moment M then becomes:

$$-M = \frac{2k_0(r-y_0)}{y_0}\left\{a\left(\frac{d}{2}(y_1 - y_0) - dr + r^2 \log\frac{r+y_1}{r-y_0}\right)\right\}. \quad (20)$$

If the section were triangular $a = 0$ in the second member of eq. (19).

These equations may be employed in the design of curved beams of any form of cross-section or degree of curvature when those based on the common theory of flexure for straight beams are not applicable. As a general statement it may be said that the formulæ for straight beams may be used without essential error in all cases except those of such special character as hooks and other structural or machine members in which the curvature is sharp. The application of the preceding formulæ to the case of hooks will be illustrated in the next article.

Art. 114.—Stresses in Hooks.

The diagram of a hook shown in Fig. 1 illustrates the conditions of loading to which hooks in general are subjected. The material to the right of the point of application of the load is subjected to no stress whatever except in a secondary way near that point. On the left of the load, however, the arc of the hook, supposed to be circular in this case, is subjected to direct stress, shear and bending, the bending moment increasing as that part of the hook

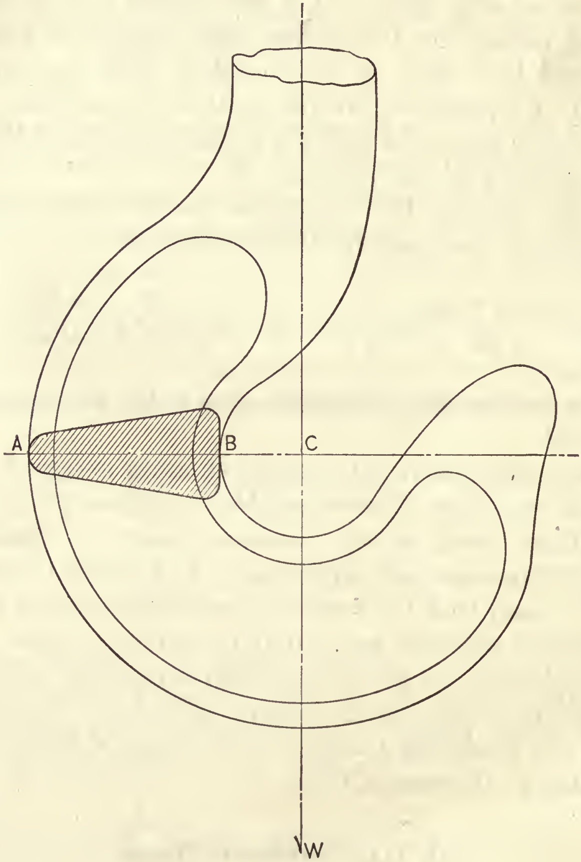


FIG. I.

parallel to the loading is approached, but it decreases in passing on to the shaft of the hook supposed to be in line with the load. The section of maximum bending *AB* is subjected to the combined direct pull of the load and the bending moment equal to the load multiplied by the normal distance from its line of action to the centre of gravity of the section. This cross-section of greatest bending moment will first be treated as if subjected to pure flexure. The necessary simple analysis required to determine the greatest intensity of stress in the section will then be made. In the section of greatest bending moment there is no shear.

The cross-section of the main part of a hook may be taken as approximately trapezoidal, as shown in Figs. 1 and 2. In the present instance the greatest dimension of this cross-section lying in the central plane of the hook will be taken as 5 inches and the corners will be rounded approximately as shown.

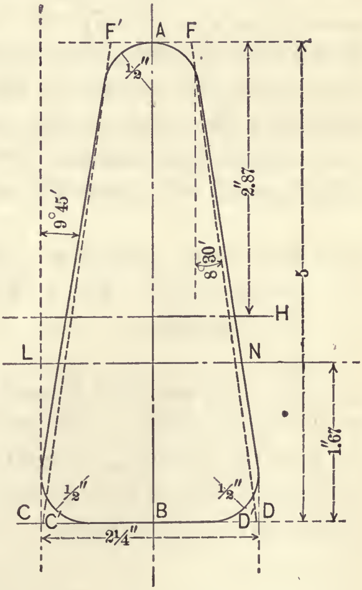


Fig. 2.

Obviously the integrations of eqs. (9), (10) and (11) of the preceding article do not represent accurately the approximate trapezoid of Fig. 2. This integration or its equivalent, however, may be accomplished with sufficient accuracy by a number of approximate processes, i.e., by transformed figures and by dividing the section into a sufficient number of small parts. A simpler method and one giving reasonably accurate results is to draw two lines *F′C′* and *FD* in such a way as to make a true trapezoid whose resisting moment will be essentially the same as the approximate trapezoid. This will be accomplished if the two lines indicated be drawn in such a way that each area between a broken line as $F'C'$ and the inclined full line of the actual section be three times the combined area between CB and the curved end of the section and between AF' and the other curved end of the section. This relation results from the fact that the bending stress between the two lines indicated varies in intensity from zero at the neutral axis to nearly the maximum in the extreme fibre of the section and has its centre at two-thirds of the distance from the neutral axis to the extreme fibre. The relation indicated, therefore, makes the bending moments of the two parts inside and outside of the actual section equal. This construction will give for one-half the modified figure:

$AF' = AF = .43$ inch $= a$; $BC = 1.1$ inches; $\alpha' = 8° \ 30'$; tan $\alpha' = .148$; $R = 5 + 2.2 = 7.2$ inches; $R = 2.2$ inches. $d = 5$ inches.

Fig. 1 shows that R and R_0 are the interior and exterior radii respectively of the arc of the hook where the section of greatest bending moment exists. By introducing these numerical quantities in eq. (13) of the preceding article there will at once result:

$$r = 3.87 \text{ inches.}$$

Hence,

$$y_1 = R - r = 3.33 \text{ inches;}$$

$$y_0 = d - y_1 = 1.67 \text{ inches.}$$

By inserting the same numerical values together with y_1 and y_0 in eq. (19) of the preceding articles, the value of the bending moment becomes:

$$M = 4.88 k_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This moment obviously can be expressed in terms of the intensity of stress in the extreme fibres on the opposite side of the section, i.e., 3.33 inches from the neutral surface. By eq. (4) of the preceding article:

$$k_1 = k_0 \frac{(r - y_0)y_1}{y_0(r + y_1)}.$$

After substituting the values of the quantities already determined there will be found $k_1 = .61k_0$. Or there may be written from the same eq. $k_0 = 1.64k_1$. The bending moment expressed in terms of the greatest intensity of stress in the extreme fibres is obviously the form desired for practical purposes.

Let the hook shown in Fig. 1 be supposed to carry a load of 20,000 pounds. The centre of gravity *G* of the actual cross-section is 2.13 inches from the side *CD* of the cross-section, Fig. 2. Hence the load assumed will cause a bending moment about the line *GH* equal to $20,000 \times (2.13 + 2.2 = 4.33) = 86,600$ inch-pounds. It is to be observed that inasmuch as the 20,000 pounds is taken as uniformly distributed over the cross-section the lever arm of the load is the normal distance from its line of action to the centre of gravity of that section, although the resisting moment of internal stresses has the axis determined by eq. (14) of the preceding article, the two axes being parallel to each other.

The greatest intensity of tensile bending stress in the section therefore takes the following value:

$$k_0 = \frac{86,600}{4.88} = 17,740 \text{ lbs. per sq.in.} \quad . \quad . \quad (2)$$

The uniformly distributed tensile stress equal to the load will act upon the entire actual area of section, which is 7.9 square inches. Hence, that tensile intensity will be $\frac{20,000}{7.9} = 2530$ pounds per square inch. The resultant greatest intensity of stress in the entire section will be:

$$17,740 + 2530 = 20,270 \text{ lbs. per sq.in.} \quad . \quad . \quad (3)$$

The resultant intensity on the opposite side of the section at A, Fig. 2, will be, since $k_1 = .61k_0$;

$$-17,740 \times .61 + 2530 = -8291 \text{ lbs. per sq.in.} \quad . \quad (4)$$

The minus sign is used because the bending stress is compression throughout that part of the section indicated by y_1.

It is commonly observed in actual experience that hooks or other similar bent members break at the inside of the section where the curvature is the sharpest. The eqs. (4) and (5*b*) of the preceding article indicate clearly the reason for such failures as the intensity of stress k in the extreme fibre is shown to vary inversely with the radius of curvature $r+y$. When, therefore, the curvature is sharp, i.e., the radius of curvature is small, the fibre stress k increases rapidly, especially on the inside of the curve where the radius of curvature is $r-y$.

This example shows the general method of treating the stresses in hooks by the common theory of flexure based on the assumption that normal sections plane before flexure remain plane after bending.

It is well known that this assumption is not strictly correct, and it is further known that the ordinary or common theory of flexure is not accurately applicable to such short beams as are contemplated in the theory of hooks.

Comparison with the Theory of Flexure for Straight Beams.

It is indicated above that the assumptions on which the preceding analyses are based are not strictly correct. If it be assumed that the intensity of stress varies directly as the distance from a neutral axis passing through the centre of gravity of the section, as for straight beams, and if k'_1 is the greatest intensity of stress in the extreme fibres (FF', Fig. 2) the bending moment will be:

$$M = \frac{k'_1 I}{y_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In this equation I is the moment of inertia about an axis through the centre of gravity G, Fig. 2, while y_c is the distance of that axis from the most remote fibre at A. The moment of inertia I of the actual section shown in Fig. 2 about a neutral axis through the centre of gravity G at the distance 2.87 inches from A is 14.9. Hence, the bending moment on the preceding assumption is:

$$M = \frac{14.9}{2.87} k'_1 = 5.2 k'_1. \quad . \quad . \quad . \quad . \quad (6)$$

As the fraction $\frac{5.2}{4.88} = 1.07$ this assumption is seen to give a result only 7 per cent. greater than that of the analysis for curved beams if the extreme fibre stress is the same in amount in both cases. It is true that the result has the apparent defect of placing the greatest intensity of stress on the wrong end of the section.

Art. 115.—Eccentric Loading.

The analysis of stresses produced in a column or other structural member by eccentric loading has already been discussed in preceding articles, but it is desirable to consider some further and more general features of that analysis.

A column or structural member is said to be eccentrically loaded when it carries a force or load acting parallel to its axis but not along that axis. The perpendicular distance between the axis of the piece and the line of action of the load is called the eccentricity of the latter.

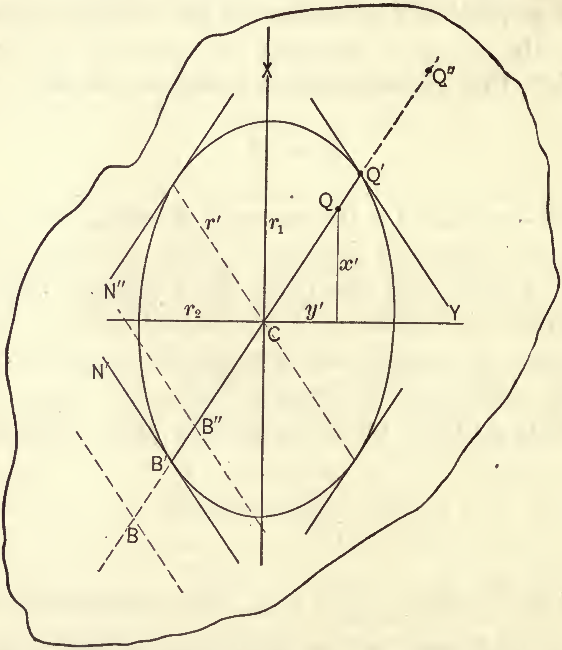


FIG. 1.

Let Fig. 1 represent the normal cross-section of such a member when the load *P* acts at any point *Q* in that cross-section. The load *P* will then act parallel to the axis of the piece, but at the distance *CQ* from it, *C* being supposed to be the centre of gravity of the section. The ellipse drawn with C as its centre is the ellipse of inertia, the semi-axes r_1 and r_2 being the principal radii of gyration of the normal section. Any semi-diameter as CQ' represents a radius of gyration r'.

If the force or load P acts at any point whatever, as Q, and parallel to the axis of the piece, it will create a bending moment equal to $P \times QC$. If x' and y' are the coordinates of Q the components of that moment will be Px' and Py', the former about the axis Y, the latter about the axis X. I_1 and I_2 being the principal moments of inertia, as already indicated, the intensities of bending stresses produced by these two component moments at any point, whose coordinates are x and y, will be $\frac{Px'}{I_1}x$ and $\frac{Py'}{I_2}y$, respectively. Furthermore, the load P will produce a uniform normal stress over the entire cross-section of the member, if A is the area of that cross-section, represented by $\frac{P}{A}$. The resultant intensity of stress k therefore at any point of the section will be:

$$k = \frac{P}{A} + \frac{Px'}{I_1}x + \frac{Py'}{I_2}y.$$

$$\therefore \quad k = \frac{P}{A}\left(1 + \frac{x'x}{r_1^2} + \frac{y'y}{r_2^2}\right). \quad . \quad . \quad . \quad . \quad (1)$$

At the neutral axis the intensity of stress is equal to zero, hence,

$$\frac{x'x}{r_1^2} + \frac{y'y}{r_2^2} = -1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eq. (2) is the equation of a straight line, i.e., the neutral axis, along which the intensity of stress is zero, x and y being the variable coordinates. It is obvious from eq. (2) in connection with the general considerations respecting the action of the load P that the position of the neutral axis will depend upon the magnitude of that load and the distance of its line of action from the axis of the member. If x and y are zero, there will be no bending, and the section of the member will be subjected to uniform compression only.

If the point of application Q of P is on the curve its coordinates x' and y' must satisfy the equation of the ellipse:

$$\frac{x'^2}{r_1^2}+\frac{y'^2}{r_2^2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The equation of a straight line tangent to the ellipse at a point whose coordinates are x' and y' is:

$$\frac{x'x}{r_1^2}+\frac{y'y}{r_2^2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

When the point of application of P is on the ellipse, x' and y' have the same values in eqs. (2) and (4). Hence in that case $\frac{dy}{dx}$ also has the same value in the two equations, showing that the neutral axis is parallel to the tangent to the ellipse at the point where P acts. If in eq. (4) $-x'$ and $-y'$ be substituted for $+x$ and $+y$, that equation will become identical with eq. (2), i.e., for this case the neutral axis is tangent to the ellipse at a point diametrically opposite to the point of application of the load P; in other words, the load is applied at one extremity of a diameter and the neutral axis is tangent to the curve at the other extremity of that diameter.

In Fig. 1 if the load P is applied at Q' (on the curve) the neutral axis $N'B'$ will be tangent to the ellipse at B', the other extremity of the diameter $Q'B'$.

If the point of application of the force P moves along the indefinite straight line BQ, the coordinates x' and y' will vary in the same proportion, making $\frac{x'}{y'}$ a constant.

From eq. (2):

$$\frac{dy}{dx} = -\frac{x'}{y'}\frac{r_2^2}{r_1^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence, as $\frac{x'}{y'}$ is constant, all neutral axes will be parallel while the point Q moves along a straight line.

Again, the coordinates x and y of the points of intersection of the line QB with the neutral axes must necessarily be opposite in sign from x' and y', as the origin C lies between them. If therefore $-x$ and $-y$ be inserted in eq. (2):

$$\frac{x'x}{r_1^2} + \frac{y'y}{r_2^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

By similarity of triangles, a being a constant:

$$\frac{x'}{y'} = \frac{x}{y} = a \quad \therefore \quad xx' = ay'x = ayy'. \quad . \quad . \quad . \quad (7)$$

Eq. (7) in connection with eq. (6) shows that each of the quantities $x'x$ and $y'y$ is constant, and that is equivalent to making the products of the segments of the line QB on either side of C constant:

$$QC \times CB = Q'C \times CB' = r'^2 = Q''C \times CB''. \quad . \quad . \quad (8)$$

As the point of application of P will always be given, the quantity to be found will be the distance from the centre C to the neutral axis, which may be called v. The semi-diameter $r' = CQ'$ at once becomes known after the ellipse of inertia is constructed. In general, therefore:

$$v = \frac{r'^2}{QC}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In some cases the reverse problem is given, i.e., v is known and the distance of the point of application of the load P is required. Hence,

$$QC = \frac{r'^2}{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Rotation of the Neutral Axis about a Fixed Point in It.

One feature of eq. (2) remains to be considered before the actual application of the preceding results can be made to form a complete graphical construction. If the coordinates x and y of the neutral axis be considered constant, while the coordinates x' and y' of the point of application of the load P vary, eq. (2) shows that the path of the movement of the point of application of P will be a straight line, since the equation is of the first degree in respect to x' and y'. This is equivalent to a movement of rotation of the neutral axis about the fixed point whose coordinates are x and y, while x' and y' determine the path through which the line of action of P moves. The same result can be shown by treating eq. (1) in precisely the same manner for a fixed or constant value of k, that constant being zero for the neutral axis.

The preceding procedures may be applied to a number of problems, one or two of which will be illustrated. It is sometimes desired to determine that part of the cross-section of a member of a structure, or sometimes of the structure itself, within which a resultant load may be applied anywhere without any change in the kind of stress induced, usually compression.

Application of Preceding Procedures to Z-bar and Rectangular Sections.

Let it be required to ascertain within what part of a Z-bar section an axial compressive force may be applied without any part of the section being subjected to tensile stress. The Z-bar section is shown in Fig. 2, the depth of bar being 6 inches and the thickness of metal $\frac{3}{4}$ inch. As this section is unsymmetrical the axes for the principal moments of inertia passing through the centre of gravity C of the section will be inclined to the central plane of the web. The ellipse of inertia $MVNU$ has MN for its major axis and UV for its minor axis, the former representing a moment of inertia of 52 and the latter a minimum moment of inertia of 5.7, the corresponding radii of gyration being $r_1 = 2.55$ inches and $r_2 = .81$ inch.

If no part of the cross-section of the bar is to be subjected to tension, the outer limits or lines of that section such as TS, SO, OL, etc., may be neutral axes for different positions of the load B, but in no case must the neutral axis lie in any part of the metal section, even to cut across a corner of it. This means that TS, SO, OL, LH, HE, and ET will be successively considered neutral axes. Let ET be the first neutral axis considered or, rather, ET and OL may be considered concurrently, as they are parallel to each other and at the same distance from the centre of the ellipse. First draw tangents to the ellipse parallel to ET and OL as shown in the figure. The points of tangency will fix the diameter DA, which is then extended to R and W in the assumed neutral axes. As shown in the preceding demonstration, the square of half the diameter represented by AD will be equal to CR multiplied by CA, the distance from the centre of the ellipse to the point of application A of the force P. The distance CR is the v of eq. (10), while CA is the distance QC desired, r' is half the diameter determined by the two points of tangency. Dividing the square of half the diameter by CR locates the point A, one of the points desired. In precisely the same manner D is located by dividing the square of half the same diameter by $CW = CR$.

Tangents to the ellipse parallel to *TS* and *HL* are then drawn as shown, one at *N*, as indicated, at the lower extremity of the ellipse and the other at the upper extremity, thus locating the diameter *NCF*. Squaring half the diameter so determined and then dividing by the distance from the centre of the ellipse to *TS* or *HL* along the diameter *NF*, the points *B* and *F* are found. In a precisely similar way the vertical tangents indicated are drawn parallel to *SO* and *EH*, determining the corresponding diameter. By the use of that diameter in the manner already indicated, the points *G* and *K* are located.

The points *A* and *B* are points of application of the force *P* when *ET* and *TS* respectively are neutral axes. In the preceding sections of this article it has been shown that if a neutral axis such as *ET* be revolved about a point in it, as *T*, to the position *TS*, the corresponding path of the point of application of the load will be a straight line, and in this case *AB* will be that straight line, since the two points *A* and *B* correspond to the neutral axes *ET* and *TS*. By similarly connecting the other points, the closed figure *ABKDFG* is found. So long as the force *P* acts within this area no part of the section can be subjected to tension, but if the point of application is outside of this figure some part of the section will be in tension

The closed figure thus established is called the " core section." Although it possesses much analytic interest, the ordinary operations of the engineer are such as to make it of comparatively little value in actual structural operations.

If any line such as *Z'L* parallel to a tangent to the ellipse at *g* be drawn through a corner *L* of the Z-bar section, and if a line *dgZ'* be drawn through the same point of tangency and the centre *C*, cutting the side of the core at *d*, it is shown in the preceding section of this article that the product of *dC* by *CZ′* is equal to the square of the semi-diameter *Cg* of the ellipse of inertia For any other position of a line *Z′L* the same general observation holds, the line always being parallel to a tangent to the ellipse

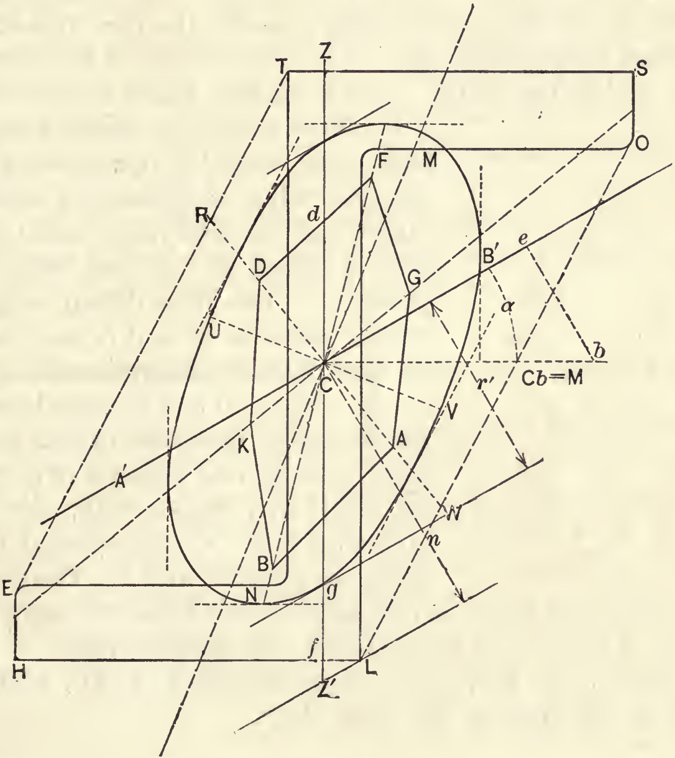


FIG. 2.*

at a point through which is drawn the line extending through the centre and cutting the side of the core.

Probably the most usual section to which the core

---

* A number of construction lines shown in this figure are drawn for use in the next article.

procedure may be applied is the simple rectangle. A masonry structure having such a horizontal section must be designed so that compression only may always be found in it. A simple diagram of pressures will show that the resultant force or load must act within the middle third of the section, but Fig. 3 shows the core procedure applied to the same axis. *AB* is the length of the section and *BD* is the width. *AB* is usually taken as one unit. The ellipse *OLMN* is drawn with its semi-major axis *LC* representing the greatest radius of gyration of the rectangle and the semi-minor axis *OC* is laid off equal to the least radius of gyration. Two lines drawn tangent to the ellipse at *M* and *N* parallel to *BD* and *ED* will determine the axes of the ellipse, in fact already known, then dividing the square of each semi-axis by the normal distance of *C* from *BD* and *ED*, respectively, the distances *CF* and *CK* will be found, thus fixing the points *F* and *K*. The points *H* and *G* are found in precisely the same manner, using the sides *AE* and *AB* respectively. As already indicated, the distance of *H* from *BD* will be one-third of *AB*, while *K* will be one-third of *BD* from *AB*.

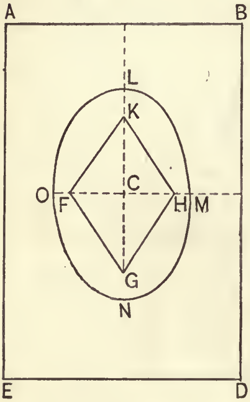


FIG. 3.

General Observations.

The preceding results show that bending combined with uniform stress induced by a load normal to the section will prevent the neutral axis from passing through the centre of gravity of the cross-section. Furthermore, in this general case the neutral axis or neutral surface will not be at right angles to the plane containing the axis of the piece and the line of action of the force unless that plane contains one of the principal axes of inertia.

Manifestly the neutral axis for any section will be on the opposite side of the centre of gravity of that section from the force *P*. Eq. (8) shows that if the force acts at *C*, making *QC* equal zero, *CB* will be infinitely great, which means that the stress will be uniformly distributed, i.e., there will be no bending. On the other hand, if the force *P* is at an indefinitely great distance from *C*, making *QC* infinity, then will *CB* be equal to zero, i.e., the neutral axis will pass through the centre of gravity. This is the ordinary case of flexure and it is equivalent to taking all load on the member at right angles to its axis.

Art. 116.—General Flexure Treated by the Core Method.

The procedures given in the preceding article may be used for the general problem of flexure for straight beams of any form of cross-section carrying any parallel loads at right angles to their axes, the loads supposed to be acting in a plane which contains the axis of the beam in each case. Under such conditions there will clearly be no direct uniform compression on any normal section of a beam. This is equivalent to assuming that the flexure is produced by an indefinitely small force acting parallel to the axis (or at right angles to a normal section) of the beam and at an infinite distance from the latter.

It is clear, since the product of the distance of the point of application of a force normal to the cross-section from the centre of gravity of the latter multiplied by the distance of the neutral axis from the same point, but on the opposite side from the point of application of the loading, must be equal to the square of half the diameter of the ellipse of inertia, that if that square be divided by infinity, the distance of the point of application of the load from the centre, the quotient will be zero, i.e., the neutral axis must pass through the centre of gravity of the section.

This condition is further equivalent to taking any finite loading at right angles to the axis of the beam, as in the ordinary cases of engineering practice. The stresses found in the normal section in such cases will be the direct tension and compression with intensity varying directly as the normal distance from the neutral axis with the accompanying shears, as in the common theory of flexure.

The preceding investigations show, however, that with unsymmetrical sections the neutral axis, while passing through the centre of gravity of the section, is not at right angles to the plane of loading, unless that plane happens to contain one of the two principal axes of inertia of the section.

Let the Z-bar section shown in Fig. 2 of the preceding article be considered and suppose that the loading acts in the vertical plane ZZ', the latter line passing through the centre of gravity C of the cross-section. It may be considered that the Z-bar is supported at each and on the lower surface HL of the lower flange. Inasmuch as the bending moment acts in the plane ZCZ' the neutral axis will be drawn through the centre C parallel to the tangents to the ellipse where the line ZZ' cuts the latter, as shown at g and at the opposite end, not lettered, of the vertical diameter. The diameter $A'B'$ is then the neutral axis desired. The line Cb drawn at right angles to ZZ' may be considered the axis of the external bending moment to which the beam is subjected. The angle between Cb and the neutral axis is α, as shown.

If the coordinate x be taken as at right angles to the neutral axis $A'B'$, and if dA represent an element of the normal section of the beam, then the distance of that element from the neutral axis measured parallel to ZZ' will be $x \sec \alpha$. If k is the maximum intensity of stress at any point of the section, that stress will occur at L or T, where the value of $x=n$ is the greatest for the entire section. The distance of that point parallel to ZZ' will be $n \sec \alpha$. If M is the value of the external bending moment acting in the plane ZZ', dM may be written:

$$dM = \frac{k}{n \sec \alpha} x \sec \alpha \cdot dA \cdot x \sec \alpha. \quad . \quad . \quad . \quad (1)$$

If I is the moment of inertia of the section about the neutral axis,

$$M = \int dM = \frac{k}{n} I \sec \alpha \cdot = \frac{k}{n} r' A \sec \alpha. \quad . \quad . \quad (2)$$

In Fig. 2 the line ZZ' cuts at d the side DF of the core. Let the distance dC be represented by j. Then, as shown in the preceding article,

$$jCZ' = Cg.$$

But the radius of gyration of the section about the axis $A'B'$ has been shown in Art. 81 to be equal to the normal distance r' between the neutral axis and the parallel tangent to the ellipse drawn at g.

$$\therefore \quad Cg = r' \sec \alpha.$$

It has already been seen that CZ' is equal to $n \sec \alpha$.

$$\therefore \quad r'^2 \sec \alpha = nj.$$

If this value of $r'^2 \sec \alpha$ be substituted in the third member of Eq. (2) there will result,

$$M = kAj. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eq. (3) is the expression for the external bending moment in terms of the greatest intensity of stress in the section, the area of that section, and the distance j from the centre of the section to the side of the core as constructed by the methods explained in the preceding section. Although the construction has been made with the Z section the method of procedure is precisely the same for any form of section whatever.

Component Moments.

By again referring to Fig. (2) of the preceding article it will be seen that $M \cos \alpha$ is that component of the external moment whose axis is parallel to the neutral axis, while the component $M \sin \alpha$ has an axis be at right angles to the neutral axis, but lying in the plane of the normal section of the beam. The former component produces the bending stresses about the neutral axis, the maximum intensity of which is k and a deflection normal to it; the latter component moment tends to produce an oblique movement of the beam in consequence of its unsymmetrical section.

This tendency in oblique flexure, especially marked with unsymmetrical sections, is always toward that position in which the least radius of gyration of the section (represented by the least semi-axis of the ellipse of inertia) is found in the plane of bending, i.e., that plane in which the bending moment acts.

In Fig. 2 of the preceding article ZZ' is the plane in which the vertical loading acts, and it is clear that the plane in which the resultant bending compression on one side of the neutral axis $A'B'$ and the resultant bending tension on the other side act is not the plane in which ZZ' lies, but inclined somewhat to the right of CZ. Inasmuch as these two planes are neither the same nor parallel, there must be combined with the couple producing pure flexure such a couple as to make the resultant external moment equal and opposite to the internal resisting moment, and the component of *M* represented by *be* is such a couple, *Ce* representing the couple producing pure flexure about *A'B'*.

These analytic considerations show how essential it is to give careful consideration to the principles governing oblique or general flexure for loads not in a plane of symmetry of a beam and for unsymmetrical sections.

The method of finding the location of the plane of resistance of the bending stress existing in any normal section of the beam will be given in the next article.

Art. 117.—Planes of Resistance in Oblique or General Flexure.

The preceding treatment of general flexure has shown that the plane of action of the external bending moment will not in general coincide with the plane in which the internal resisting couple acts. The plane of the external bending moment is supposed to pass through the axis of the beam assumed to be straight. If this external bending couple is to produce pure flexure it must be in equilibrium with the internal moment produced by the stresses in any normal section, and that requires that the two planes of action shall either coincide or be parallel.

Let it be supposed that the $6 \times 3\frac{1}{2} \times \frac{3}{4}$-inch steel angle section shown in Fig. 1 represent any unsymmetrical section, and let it also be supposed that *GY* is the neutral axis of the section, *G* being the centre of gravity; then let *GX* and *GY* be the axes of rectangular coordinates negative when measured to the left and downwards. The stresses above *GY* will be supposed compressive, and those below, tensile. The intensities will be assumed to vary directly as the normal distances from *GY* as in the ordinary theory of flexure. The centre of all the compressive stresses will be taken at C and at T for the tensile stresses. The plane whose trace is CT will be called the plane of resistance, while AB' will be taken as the plane of action of the external bending moment. In other words, if the angle were to carry vertical loading as a beam AB should be vertical with the lines of cross-section correspondingly inclined.

If x_1 and y_1 are the coordinates of the centre C of the compressive stresses in the section and if a is the intensity of stress at a unit's distance from the neutral axis GY, eqs. (1) and (2) will immediately result:

$$y_1 = \frac{\int\int yaxdxdy}{\int\int axdxdy} = \frac{\int\int xydxdy}{\int\int xdxdy} = \frac{J_1}{Q_1}. \quad . \quad . \quad (1)$$

$$x_1 = \frac{\int\int xaxdxdy}{\int\int axdxdy} = \frac{\int\int x^2dxdy}{\int\int xdxdy} = \frac{I'_1}{Q_1}. \quad . \quad . \quad (2)$$

The quantities J_1 and I'_1 are the so-called "product of inertia" and the moment of inertia of that part of the cross-section lying above GY, while Q_1 is obviously the statical moment of the same part of the cross-section in reference to the same axis.

If the subscript 2 be used for the corresponding quantities relating to that part of the section below GY, eqs. (3) and (4) will at once result, the negative sign being used in the second member because the coordinates are negative:

$$y_2 = -\frac{J_2}{Q_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$x_2 = -\frac{I'_2}{Q_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Q, I and J represent quantities belonging to the whole cross-section, then, since G is the centre of gravity of that section,

$$Q_1 = Q_2 = Q;$$
$$I'_1 + I'_2 = I;$$
$$J_1 + J_2 = J.$$

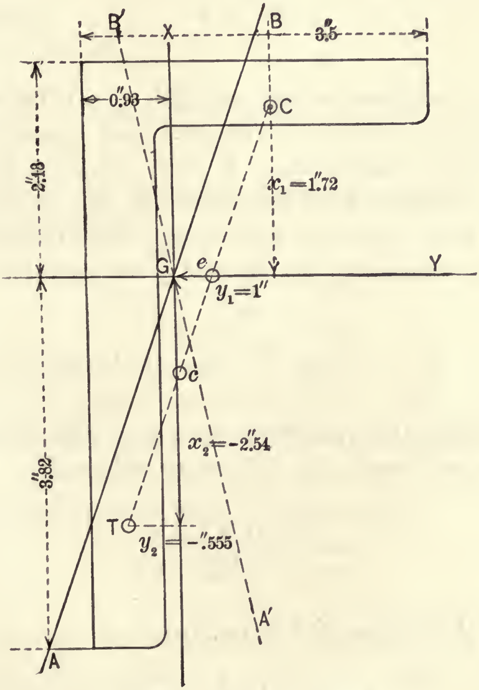


FIG. 1.

It is desired to find the straight line joining C and T, and in order to do that the general equation of a straight line may be written as follows:

$$x + by = c. \qquad . \ . \ . \ . \ . \ . \ . \quad (5)$$

If y_1 and x_1 taken from eqs. (1) and (2) be first written in eq. (5) and then y_2 and x_2 from eqs. (3) and (4), and if the second of the equations so formed be subtracted from the first, there will result: $b = -\frac{I}{J}$.

Then eq. (5) will take the form

$$x = \frac{I}{J} y + c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In Fig. 1 suppose a line parallel to CT drawn from G to B. If the ordinate x_1 be produced upward, the line $BC = Gc'$ will be determined. If in eq. (6) $y = 0$, $x = c = Gc' = BC$. The triangles with the bases y_1 and y_2 will then be similar and that similarity will be expressed by the following equation, remembering that x and y are negative:

$$\frac{x_1 - c}{y_1} = \frac{-x_2 + c}{-y_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting the values of x_1, y_1, x_2 and y_2 established above there will result the following value of c:

$$c = \frac{I'_1 J_2 - J_1 I'_2}{JQ}.$$

Placing this value of c in eq. (6),

$$x = \frac{I}{J} y + \frac{I'_1 J_2 - J_1 I'_2}{JQ}, \quad . \quad . \quad . \quad . \quad (8)$$

This is the equation of the line CT, Fig. 1, drawn through the centres of the tensile and compressive resisting stresses acting in the normal section, i.e., it is the trace of the plane in which the resisting couple acts. The tangent of the angle which it makes with the neutral axis GY is $\frac{dx}{dy}=\frac{I}{J}$. If GY is one of the principal axes of inertia of the section $J=0$ and $\frac{dx}{dy}$ becomes infinitely great, i.e., in that case the line CT is at right angles to GY and it will presently be shown that it will pass through G, the centre of gravity of the section.

If $y=0$ in eq. (8),

$$x_0=\frac{I'_1J_2-J_1I'_2}{JQ}=Gc'. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The distance Gc' is on the negative side of G. Again if $x=0$, there will result:

$$y=\frac{J_1I'_2-I'_1J_2}{IQ}=Ge. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

These coordinates Ge and Gc' shown in Fig. 1 give two points e and c' in the desired line CT, which must agree obviously with the points C and T as found by computations.

If Ge should be zero, eq. (11) will result:

$$J_1I'_2-I'_1J_2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Inasmuch as the moments of inertia I'_1 and I'_2 will always have real values for an actual section, in general if eq. (11) holds true, then must $J_1=J_2=0$. That condition will of course exist for the principal axes and for the case where at least one of the coordinate axes is an axis of symmetry of the section.

Although the figure used for the establishment of the preceding formulæ is the normal section of a steel angle, those formulæ are completely general and are applicable to any form of cross-section whatever, as indicated by eqs. (1) and (2) and all the equations following.

It is thus seen that if the plane of action *AB* of any external loading producing flexure of a beam with unsymmetrical cross-section is parallel to the plane whose trace is *CT*, there will be pure bending only as the external bending moment has the same axis as the couple formed by the internal stresses. The planes of the external bending moment and that of the internal resisting stresses may in some cases coincide.

If the steel angle shown in Fig. 1 is to act as a beam under vertical loading in pure flexure, the end supports should be so formed as to make the lines *AB* and *CT* vertical. In general, whatever may be the cross-section of a beam, the latter should be so held at its points of support that the loading will produce pure flexure. If the section of the beam has an axis of symmetry, the plane of loading may be taken through the axes of symmetry of the cross-section.

Example. The application of the preceding formulæ may be illustrated by using the $6 \times 3\frac{1}{2}$-inch, 22.4-lb. steel angle shown in Fig. 1. The thickness of each leg is .75 inch. By using eqs. (1) and (2) there will at once result:

$$I'_1 = 9.41; \qquad I'_2 = 13.94; \qquad J_1 = 5.47;$$
$$J_2 = 3.04; \qquad J = 8.51; \qquad Q = 5.484$$

Inserting these values in eqs. (1), (2), (3) and (4) there will result:

$$y_1 = 1 \text{ in.}; \qquad x_1 = 1.72 \text{ ins.}; \qquad y_2 = -.555 \text{ in.};$$
$$x_2 = -2.54 \text{ ins.}; \quad x_0 = -1.02 \text{ ins.}; \qquad y_0 = .372 \text{ in.}$$

These coordinates are laid off in Fig. 1, as shown, so as to locate the four points *C*, *e*, *c'* and *T*. In making these computations it should be remembered that I'_1 and I'_2 are moments of inertia of areas, one of whose sides coincides with the axis of y and that the same observation is also true of the quantities, J_1 and J_2, as well as Q.

Art. 118.—Deflection in Oblique Flexure.

The general case of deflection of a beam with unsymmetrical cross-section, or of a beam with symmetrical cross-section but loaded obliquely, may readily be found by the aid of the ordinary formulæ for flexure used in connection with the preceding investigations. The requisite treatment may be well illustrated by considering the case of a $6 \times 3\frac{1}{2} \times \frac{3}{4}$-inch steel angle, the section of which is shown in Fig. 1 to be same as that used in the preceding article. Such an angle may be considered to be used as a beam in roof work or for some other similar purpose with the 6-inch leg placed in a vertical position. It will be assumed that the span length is 15 feet = 180 inches and that the angle is to carry as a beam a uniform load of 200 pounds per linear foot. The data given in an ordinary handbook on steel sections will show the position of the centre of gravity G of the section and enable the ellipse of inertia to be constructed as in Fig. 1.

The maximum radius of gyration represented by the greater semi-axis of the ellipse is 1.97 inches, while the least radius of gyration at right angles to the preceding and represented by the smaller axis is .75 inch. The load acts in a vertical plane passing through the axis G. The various dimensions of the cross-section required in the computations are all shown in Fig. 1.

By drawing vertical tangents on opposite sides of the ellipse, the neutral axis $A'B'$ drawn through the points of tangency and the centre G of the ellipse is determined.

This neutral axis of the section makes the angle, 46° 30′, as carefully measured on the diagram, with the horizontal axis of Y. By drawing a tangent to the ellipse parallel to $A'B'$ the radius of gyration about the neutral axis is found to be 1.1 inches, i.e., the normal distance between the neutral axis and the parallel tangent to the ellipse.

The greatest deflection of the angle beam will be found at the centre of span at which the moment of the external forces is

$$M=\frac{200\times 225}{8}\times 12=67,500 \text{ in.-lbs.} \quad . \quad . \quad . \quad (1)$$

The component moment, as shown in the preceding article, with axis parallel to the neutral surface, is

$$M \cos \alpha=.6884M=46,467 \text{ in.-lbs.} \quad . \quad . \quad . \quad (2)$$

The component moment having an axis at right angles to the neutral axis is, similarly,

$$M \sin \alpha=.7254M=48,964 \text{ in.-lbs.} \quad . \quad . \quad . \quad (3)$$

The actual flexure is produced by the first of these components $M \cos \alpha$. The deflection produced by it will obviously be normal to the neutral axis, and it can be computed by the ordinary formula for the deflection at the centre of span of a beam simply supported at each end and loaded uniformly throughout its length, the uniform load to be taken in this case as $200 \cos \alpha=138$ pounds per linear foot. If g is the load per linear foot of span, the usual expression for the centre deflection is $w=\frac{5gl^4}{384\,EI}$. Substituting 138×15 for gl in the formula, $l=180$ inches, $E=30,000,000$, and $I = 7.94$ (moment of inertia of section about the neutral axis) there will result:

$$w = 0.66 \text{ inch.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As $\cos\alpha = .6884$ and $\sin\alpha = .7254$, the vertical deflection $= .66 \times .6884 = .454$ inch; and the horizontal deflection $= .66 \times .7254 = .479$ inch.

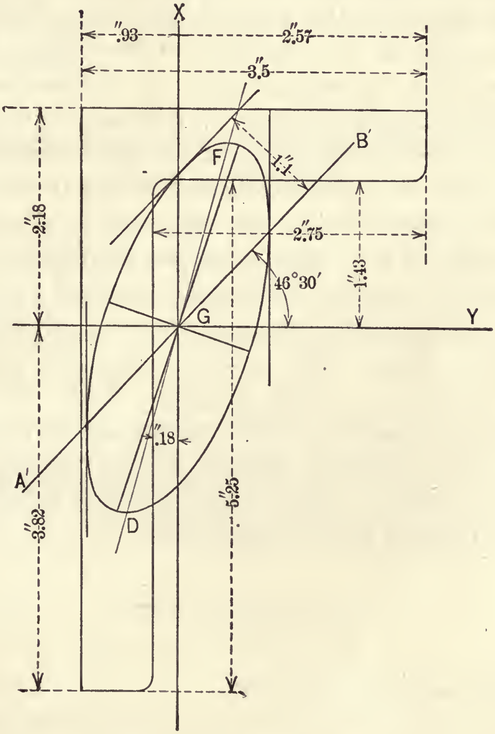


FIG. 1.

It is thus seen that the horizontal deflection slightly exceeds the vertical, in consequence of the major axis of the ellipse of inertia being slightly inclined to a vertical line, thus causing the inclination of the neutral axis of the section to be relatively large.

Precisely the same general treatment would be followed for any form of cross-section or any other amount or disposition of loading.

In the preceding article where the same angle was so held as to make the plane of loading parallel to that of the resisting couple, the horizontal diameter of the ellipse drawn through G is the neutral axis corresponding to the conjugate diameter DF, parallel to the trace of the plane of the resisting internal couple as determined in that article. The normal distance, 1.95 inches, between the horizontal diameter through G and the horizontal tangent at F is the radius of gyration corresponding to the horizontal neutral axis through G. As the area of cross-section of the steel angle is 6.56 square inches, the moment of inertia corresponding to the horizontal neutral axis through G is $I=6.56\times\overline{1.95}^2=24.93$, the moment of inertia of the cross-section about the neutral axis $A'B'$, Fig. 1, is $I=6.56\times\overline{1.1}^2=7.94$. The distance from the horizontal neutral axis through G to the extreme fibre is 3.82 inches, while the corresponding distance of the extreme fibre from $A'B'$ is 2.3 inches. Hence, the resisting moment for the horizontal neutral axis through G is

$$M=k\frac{24.93}{3.82}=6.53k.$$

For the neutral axis $A'B'$:

$$M'=k\frac{7.94}{2.3}=3.45k.$$

Hence $\frac{M}{M'}=1.9$. In other words, the same angle placed so as to take the vertical loading in a plane parallel to the resisting internal couple will offer nearly twice as much bending resistance with the same extreme fibre stress as when placed with the longer leg vertical. Economic use of the metal as well as avoidance of unnecessary deflection, therefore, requires that the beam of unsymmetrical section shall be so held at its supports as to make the plane of loading parallel to the resisting plane and as nearly parallel to the greater axis of the ellipse of inertia of the cross-section as possible.

Art. 119.—Elastic Action under Direct Loading of a Composite Piece of Material.

Let it be supposed that a combined straight or cylindrical piece of material with length L is subjected to the direct stress of either tension or compression. If the total area of cross-section is A, it may be assumed to be composed of the following parts:

A_1 = area of cross-section with modulus of elasticity E_1;

A_2 = area of cross-section with modulus of elasticity E_2;

A_3 = area of cross-section with modulus of elasticity E_3;
etc., etc.

Then will

$$A = A_1 + A_2 + A_3 + \text{etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let the total load P act parallel to L and let l be the strain per unit of length of the piece, i.e., the unit strain, then will lL be the total lengthening or shortening of the piece. Under these conditions every part of the piece will be subjected to the same rate of longitudinal strain and the following equation may be at once written:

$$E_1lA_1+E_2lA_2+E_3lA_3+\text{etc.}=P=ElA. \quad . \quad . \quad (2)$$

Hence,

$$l=\frac{P}{E_1A_1+E_2A_2+E_3A_3+\text{etc.}}. \quad . \quad . \quad (3)$$

Also the first and third members of eq. (2) will give eq. (4):

$$E=\frac{E_1A_1+E_2A_2+E_3A_3+\text{etc.}}{A}. \quad . \quad . \quad (4)$$

Eq. (3) will give the lengthening or shortening of each unit of length of the piece under any assigned load P, the moduli of elasticity of the areas of the different parts of the section being known.

The modulus of elasticity E given by eq. (4) may be considered a mean or average modulus or an equivalent value for the actual moduli, as the same longitudinal strain would be yielded by a piece of uniform material having that modulus of elasticity and the same area of cross-section as the composite piece.

Art. 120.—Helical Spiral Springs.

A spiral spring like that shown in Fig. 1 takes its load at the ends as indicated at A and B. In the general case there may be applied at each end a single load P and a couple, or either a force or a couple alone may act. The analysis will be so written as to include concurrent force and couple or either one separately. The following notation will be employed:

R = radius of spiral, Fig. 1;
ϕ = pitch angle of spiral, Fig. 1;
z = axial elongation or compression of spring under loading;

l = length of spiral;
r = radius of spiral wire;
P = axial load, Fig. 1;
M = moment of applied twisting couple or torque, assumed to be a right-hand moment;
u = unit strain at unit distance from the neutral axis in bending or flexure;
α = angle of torsion (unit strain at unit distance from axis of piece in torsion);
T = total twist or rotation of spring measured on central cylinder of spiral;
$\tau = \frac{T}{R}$ = angle of twist of spring in radians.

The force P will be considered positive when it stretches the spring as shown in Fig. 1. If the force P compresses the spring it must have the negative sign in all the following analysis.

The moment M will be considered a right-hand moment when it twists the spiral so as to bring the helical parts near together, i.e., tightens the spiral. It should be remembered that all parts of the spiral are uniformly stressed or bent. The cross-section of the spiral rod will be considered circular, although the general analysis is adapted to any form of cross-section.

The load P produces a moment M_1 about the centre of any section of the spiral rod given by

$$M_1 = PR. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The axis of this moment is a horizontal line through the centre of the section and tangent to the central cylinder of the spiral shown by a broken-line circle in the lower part of Fig. 1. If A, Fig. 2, be the centre of the section considered, KL may be taken as the axis of the moment PR. If AK, therefore, represent by a convenient scale, the moment $M_1 = PR$, AG and GK (drawn perpendicular to AG) will represent by the same scale the component moments of M_1 about those lines as axes passing through the centre of the section. As the axis AG is the axis of the spiral rod, it represents a torsion moment. Similarly GK represents a bending moment as it lies in the section and, in fact, is a neutral axis. Hence, if the subscripts t and b mean torsion and bending,

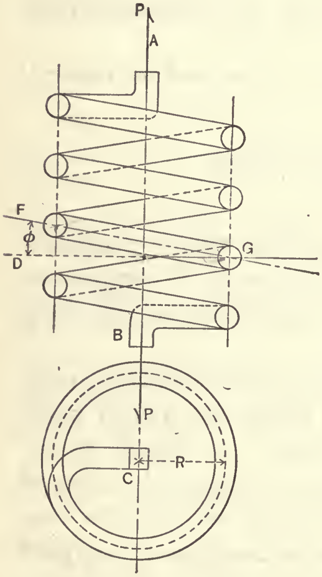


FIG. 1.

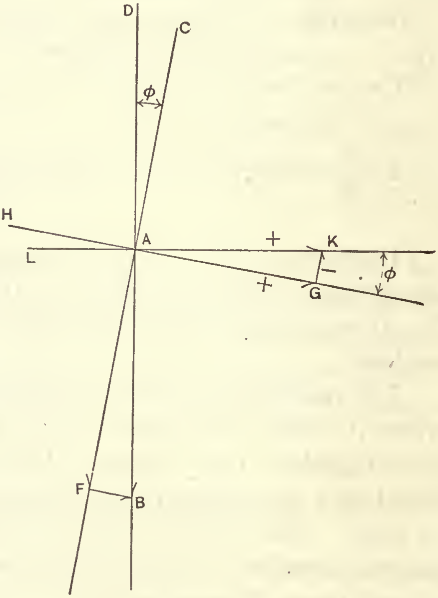


FIG. 2.

$$AG = M'_t = M_1 \cos \phi. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And,

$$GK = M'_b = -M \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The moment $-M \sin \phi$ has a negative sign because the triangle AKG, Fig. 2, shows that it will tend to untwist the spiral of Fig. 1, which is opposite to a positive effect.

The right-hand moment M will act at the centre of section of the spiral rod about an axis parallel to AC, Fig. 1, i.e., about BD, Fig. 2, and AB may represent that moment. Its two components will be:

$$BF = M''_t = M \sin \phi, \quad . \quad . \quad . \quad . \quad (4)$$

$$AF = M''_b = M \cos \phi. \quad . \quad . \quad . \quad . \quad (5)$$

The resultant moments of torsion and bending at the section considered will therefore be:

$$M_t = M_1 \cos \phi + M \sin \phi, \quad . \quad . \quad . \quad (6)$$

$$M_b = M \cos \phi - M_1 \sin \phi. \quad . \quad . \quad . \quad (7)$$

By the common theory of torsion (correct for a circular section only) if G is the modulus of shearing elasticity, the angle of torsion, or unit strain α, is

$$\alpha = \frac{\text{moment}}{G\frac{\pi r^4}{2}} = \frac{M_1 \cos \phi + M \sin \phi}{Q} . \quad . \quad . \quad . \quad (8)$$

Evidently, $Q = G\frac{\pi r^4}{2}$ (for circle); and $Q = G\frac{b^4}{6}$ (for square).

If the exact theory of torsion is used for other sections of the spiral rod than circular, the corresponding value of α must be introduced, but no other change is needed.

In the same manner, if E is the modulus of elasticity for direct stress, I the moment of inertia of the section about its neutral axis, and if $Q' = EI = E\frac{\pi r^4}{4}$ (for circular section) or $Q' = E\frac{b^4}{12}$ (for square section), the unit strain, u, for bending is,

$$u = \frac{\text{moment}}{Q'} = \frac{M \cos \phi - M_1 \sin \phi}{Q'} \quad . \quad . \quad . \quad (9)$$

The quantities α and u are unit motions giving to the spiral spring corresponding motions of rotation and axial lengthenings or shortenings.

The torsion moment M_t will cause one end of an indefinitely short length dl of the spiral rod to rotate through the angle αdl, inducing a movement of that end, relative to the axis of the spiral, perpendicular to the axis of the rod, equal to $R\alpha dl$, as shown by Fig. 3. The horizontal component of this movement tangent to the spiral cylinder is, $R\alpha dl \sin \phi$, or for each unit of length of the rod, $R\alpha \sin \phi$. As the state of stress is uniform throughout the spiral rod, the total circumferential twist of the spiral spring due to torsion is

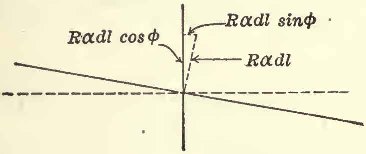


FIG. 3.

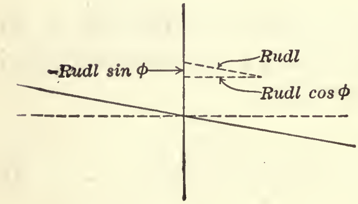


FIG. 4.

$$T' = Rl\alpha \sin \phi = Rl\frac{M_1 \cos \phi + M \sin \phi}{Q} \sin \phi . \quad . \quad (10)$$

And the angle of twist is

$$\tau' = \frac{T'}{R} = l\frac{M_1 \cos \phi + M \sin \phi}{Q} \sin \phi . \quad . \quad . \quad . \quad (10a)$$

The axial component of the same movement, as shown by Fig. 3 is, $R\alpha dl \cos \phi$. Hence the total axial movement due to torsion is

$$z' = Rl\frac{M_1 \cos \phi + M \sin \phi}{Q} \cos \phi. \quad . \quad . \quad (11)$$

The movement of a normal section of the spiral rod, relative to the axis of the spring, due to bending about its neutral axis parallel to GK and AF, Fig. 2, is illustrated by Fig. 4. That movement will be parallel to the axis of the rod and the broken-line triangle showing it and its components is moved vertically to clear it from the centre line of the rod. The horizontal component representing the tangential or rotating movement due to bending is seen to be

$$Rudl \cos \phi.$$

Or, for the entire length l of the spring,

$$T'' = Rl\frac{M \cos \phi - M_1 \sin \phi}{Q'} \cos \phi. \quad . \quad . \quad . \quad (12)$$

The angular twist is

$$\tau'' = \frac{T''}{R} = l\frac{M \cos \phi - M_1 \sin \phi}{Q'} \cos \phi. \quad . \quad . \quad (12a)$$

Similarly, the axial component of the movement due to bending, as shown in Fig. 4, is

$$-Rudl \sin \phi.$$

This value is negative, as the axial motion is downward and opposite to that due to torsion shown in Fig. 3.

Hence,

$$z'' = -Rl\frac{M \cos \phi - M_1 \sin \phi}{Q'} \sin \phi. \quad . \quad . \quad (13)$$

The angle of twist of the spring under loading will be the sum of the second members of eqs. (10a) and (12a):

$$\tau = M_1 l \sin\phi \cos\phi\left(\frac{1}{Q} - \frac{1}{Q'}\right) + Ml\left(\frac{\sin^2\phi}{Q} + \frac{\cos^2\phi}{Q'}\right). \quad (14)$$

The circumferential motion of the spring will be

$$T = R\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The axial extension or compression of the spring will be found by the aid of eqs. (11) and (13):

$$z = z' + z'' = Rl\left\{M_1\left(\frac{\cos^2\phi}{Q} + \frac{\sin^2\phi}{Q'}\right)\right.$$

$$\left. + M\sin\phi\cos\phi\left(\frac{1}{Q} - \frac{1}{Q'}\right)\right\}. \quad (16)$$

Eqs. (6) and (7) will enable any spiral spring to be designed to perform a given duty such as to carry a prescribed load or serve the purposes of a dynamometer, while eqs. (14) and (16) will give the distortions of the spring, either angular or axial.

If s is the greatest intensity of torsive shear in a normal section of the spiral rod at the distance r from the centre, while I_p is the polar moment of inertia of the section,

$$M_t = \frac{s}{r} I_p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For a circular section, $I_p = \frac{\pi r^4}{2}$.

For a square section, $I_p = \frac{b^4}{6}$ (b = side of square).

Eq. (17) gives:

$$s = \frac{M_t r}{I_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When s is given,

$$r = \sqrt[3]{\frac{2M_t}{\pi s}} \text{ (circular section).} \quad . \quad . \quad . \quad (18a)$$

In both eqs. (6) and (7), M_1 and M are known quantities, as they are the given loads.

Again if k is the intensity of stress in the most remote fibre at the distance d_1 from the neutral axis, and if I is the moment of inertia of the section about the neutral axis,

$$k = \frac{M_b d_1}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

When k is given,

$$d_1 = r = \sqrt[3]{\frac{4M_b}{\pi k}} \text{ (circular section).} \quad . \quad . \quad (19a)$$

The two intensities s and k exist at the same point, and they are to be used to determine the greatest intensities of stress in the cross-section of the spiral rod precisely as was done in Art. 10.

By eq. (2) of that article, the greatest and least intensities of stress (principal stresses of opposite kinds) will be:

$$\text{max. intensity} = \frac{k}{2} + \sqrt{s^2 + \frac{k^2}{4}} \text{ (tension);}$$

$$\text{min. intensity} = \frac{k}{2} - \sqrt{s^2 + \frac{k^2}{4}} \text{ (compression).}$$

At the opposite end of that diameter of section of the rod normal to the neutral axis where k is compression, the above "max. intensity" will be compression also, and the "min. intensity" will be tension.

The planes on which these principal stresses act are given by eq. (3) of Art. (10):

$$\tan 2\alpha = -\frac{2s}{k}.$$

The greatest shear at the same point is given by eq. (6) of Art. (9); i.e., its intensity is half the difference of the principal intensities, or,

$$p_t = \frac{\text{max.} - \text{min.}}{2}.$$

There are a number of special cases which may easily be developed from the preceding general analysis.

Small Pitch Angle.

If the pitch angle ϕ is so small that $\sin \phi$ may be considered zero without essential error,

$$\sin \phi = 0 \quad \text{and} \quad \cos \phi = 1.$$

Eqs. (6) and (7) then give:

$$M_t = M_1 = PR; \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$M_b = M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

From eqs. (14) and (16):

$$\tau = \frac{Ml}{Q'}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$z = \frac{PR^2l}{Q}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Rotation of Spring Prevented.

In this case twisting of the spring is prevented, or $\tau=0$. Eq. (14) then gives:

$$M=-M_1\frac{\sin\phi\cos\phi(Q'-Q)}{Q'\sin^2\phi+Q\cos^2\phi}. \quad . \quad . \quad . \quad (24)$$

Substituting this value of M in eq. (16):

$$z=PR^2l\left\{\frac{\cos^2\phi}{Q}+\frac{\sin^2\phi}{Q'}-\frac{(\sin\phi\cos\phi)^2(Q'-Q)^2}{(Q'\sin^2\phi+Q\cos^2\phi)QQ'}\right\}. \quad (25)$$

The torsion moment M_t, eq. (6), and bending moment M_b, eq. (7), are to be computed by using the value of M given in eq. (24).

Axial Extension or Compression Prevented.

By making $z=0$ in eq. (16),

$$M_1=-M\frac{\sin\phi\cos\phi(Q'-Q)}{Q'\cos^2\phi+Q\sin^2\phi}. \quad . \quad . \quad . \quad (26)$$

The angle of twist then becomes:

$$\tau=Ml\left\{\frac{\sin^2\phi}{Q}+\frac{\cos^2\phi}{Q'}-\frac{(\sin\phi\cos\phi)^2(Q'-Q)^2}{(Q'\cos^2\phi+Q\sin^2\phi)Q'Q}\right\}. \quad (27)$$

For circular or square sections $Q'-Q=\left(\frac{E}{2}-G\right)\left(\frac{\pi r^4}{2}\text{ or }\frac{b^4}{6}\right)$ and the square of the latter alternative factor is common to $(Q'-Q)^2$ and $Q'Q$ in the second number of eq. (27), thus canceling and simplifying the numerical application of that equation.

In computing M_t and M_b, eqs. (6) and (7), the value of M_1 given by eq. (26) is to be used.

This form of helical spring is employed in the transmission dynamometer.

Work Performed in Distorting the Spring.

The work performed in producing the angular and axial distortions τ and z by the moment M and force P is easily found by the aid of eqs. (14) and (16) or corresponding equations for special cases. The couple whose moment is M performs work in twisting the spring through the arc τ (measured at unit distance from the axis of the helix) expressed by

$$W_t = \frac{M\tau}{2}. \qquad \qquad (28)$$

The force P performs work in extending or compressing the spring the distance z given by the equation

$$W_z = \frac{Pz}{2}. \qquad \qquad (29)$$

The total work done in the general case will then be:

$$W = W_t + W_z = \tfrac{1}{2}(M\tau + Pz). \qquad \qquad (30)$$

For special cases, as already indicated, the corresponding value of τ and z must be used in eq. (30).

In writing the preceding equations it has been assumed that both M and P are gradually applied. If they were suddenly applied, the distortions would be 2τ and $2z$ and oscillations having those amplitudes would be set up. The periods of the amplitudes would depend upon the masses moved.

Art. 121.—Plane Spiral Springs.

A plane spiral spring may be represented by Fig. 1. The outer end is fastened at B, but the inner end is secured to a rotating post or small shaft at C. The spring or coil is " wound up " to an increasing number of turns by applying a couple to the shaft C, as in winding a clock.

As a couple only is applied at C, every section of the spring is subjected to bending by the same couple, i.e., there is a uniform bending moment throughout the entire spring. This uniform condition of stress makes the analysis of this spring exceedingly simple if the thickness of the metal is small. As the spring is a spiral beam subjected to uniform bending, the analysis, to be perfectly correct, should be based on that for curved beams. That procedure would introduce much complexity, and as the thickness of the strip of metal constituting the spring is small compared with its radius, no essential error is committed in neglecting the effects of curvature. The usual cross-section of this type of spring is a much elongated or narrow rectangle, the greater dimension of the rectangle being parallel to the axis of the couple or perpendicular to the plane of the spring.

If u is the unit strain at unit distance from the neutral axis of a section of the spring, I the moment of inertia of the same section about the neutral axis, and E the modulus of elasticity, while M is the moment applied at C, Fig. 1,

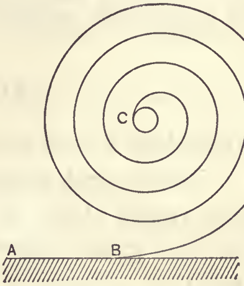


FIG. 1.

$$M = EIu = \text{constant}. \quad . \quad . \quad (1)$$

If l is the total length of the spring and β the total angular distortion for that length, then will udl be the change of direction or angular distortion for each element *dl*. Hence,

$$Mdl = EIudl = EId\beta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Integrating:

$$Ml = EI\beta; \quad \text{and} \quad \beta = \frac{Ml}{EI}. \quad . \quad . \quad . \quad (3)$$

With the thin metal used I is small and β may be a number of complete circles, perhaps sufficient to wrap the spring closely around the shaft C.

If the moment M is applied gradually, the work done in producing the total angular distortion β is

$$W = \frac{M}{2}\beta = \frac{M^2 l}{2EI}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This is the same as the expression for the work performed in bending a beam by a moment uniform throughout its length. In fact the plane spiral is simply a special case of flexure, the bending moment being uniform.

If the moment M should be applied suddenly, the total angular distortion would be 2β, and oscillations having that amplitude might be set up.

Art. 122.—Problems.

Problem 1.—A helical spring having a diameter of helix of 3 inches and composed of twelve complete turns of a $\frac{3}{8}$-inch round steel rod sustains an axial load of 45 pounds. Find the axial deflection of the spring and the greatest intensities of torsive shear and bending tension and compression in the rod.

$P = 45$ lbs.; $R = 1.5$ ins.; $\phi = 15°$; $l = 117$ ins.;

$E = 30{,}000{,}000$; $G = 12{,}000{,}000$; $r = \frac{3}{16}$ in.

$M_1 = PR = 68.5$ in.-lbs.; $\qquad M = 0$;

$$Q = G\frac{\pi r^4}{2} = 23{,}373; \qquad Q' = E\frac{\pi r^4}{4} = 29{,}217.$$

Substituting these quantities in eq. (16):

$$z = 3 \times 117\left(\frac{.933}{23{,}373} + \frac{.06708}{29{,}217}\right)68.5 = .746 \text{ in.}$$

By eqs. (6) and (7):

$$M_t = M_1 \cos\phi = 66.2 \text{ in.-lbs.};$$

and

$$M_b = -M_1 \sin\phi = -17.74 \text{ in.-lbs.}$$

Since $I_p = \frac{\pi r^4}{2}$ and $I = \frac{\pi r^4}{4}$, eqs. (18) and (19) give:

$s = 9460$ lbs. per sq.in. torsive shear;

$k = 3432$ lbs. per sq.in. greatest bending stress.

Problem 2.—Design a helical spring for a transmission dynamometer for 8 H.P. at 90 revolutions per minute. Axial distortion of the spring is prevented, or $z = 0$. Let low working stresses and other data be taken as follows·

$k = 16{,}000$ lbs. per sq.in.; $\qquad s = 12{,}000$ lbs. per sq.in.;

$R = 3$ ins.; $\qquad \phi = 11°$ $\quad \therefore \sin\phi = .191$ and $\cos\phi = .982$;

$G = 12{,}000{,}000$; $\qquad E = 30{,}000{,}000$.

$M \times 90 \times 2\pi = 8 \times 33{,}000 \quad \therefore M = 466.8$ ft.-lbs. $= 5602$ in.-lbs.

$$Q = G\frac{\pi r^4}{2} \quad \text{and} \quad Q' = E\frac{\pi r^4}{4}.$$

Eq. (26) then gives:

$$M_1 = -212 \text{ in.-lbs.}$$

By eqs. (6) and (7):

$$M_t = 862 \text{ in.-lbs.}; \quad \text{and} \quad M_b = 5541 \text{ in.-lbs.}$$

Solving eqs. (18) and (19) for the radius of the rod: By eq. (18a), $r = .36$ in.; and by eq. (19a), $r = .76$ in. Bending of the rod, therefore, requires the greater radius, and $r = .76$ in. will be taken.

Eq. (17) gives the greatest torsive shear in a section:

$$s = 1250 \text{ lbs. per sq.in.}$$

The equations following eq. (19) now give:

$$\text{max. intensity} = +16,097 \text{ lbs. per sq.in.};$$

$$\text{min. intensity} = -97 \text{ lbs. per sq.in.}$$

The spring will be assumed to have twelve complete turns, so that its length will be:

$$l = 2\pi 3 \times 12 \times \sec \phi = 230.5 \text{ ins.}$$

The twist τ at unit distance from the axis of the helix now becomes:

$$\tau = .159 \text{ in.}$$

At the distance of 10 inches from the axis the twist would be 1.59 inches, but the spring is too stiff to be very sensitive. A higher working stress k may properly be taken.

If in the same problem there be taken 120 revolutions per minute and an alloy steel for which the working stresses $k = 40,000$ pounds per square inch and $s = 30,000$ pounds per square inch may be prescribed, then by using the results already established:

$$M = \frac{90}{120} \times 5602 = 4200 \text{ in.-lbs.};$$

$$M_1 = -\tfrac{3}{4} \times 212 = -159 \text{ in.-lbs.};$$

$$M_t = \tfrac{3}{4} \times 862 = 647 \text{ in.-lbs.};$$

$$M_b = \tfrac{3}{4} \times 5541 = 4156 \text{ in.-lbs.}$$

For shear: $r = \sqrt[3]{\frac{3}{4} \times \frac{1}{2.5}} \times .36 = .67 \times .36 = .24$ in.

For bending: $r = \sqrt[3]{\frac{3}{4} \times \frac{1}{2.5}} \times .76 = .67 \times .76 = .51$ in.

$$\tau = \frac{.159}{(.67)^4} = .795.$$

At the distance of 10 inches from the axis of the helix the twist would be $10 \times .795 = 7.95$ inches.

Problem 3.—What will be the angular distortion β of a plane spiral spring 1 inch by $\frac{1}{50}$ inch in section and 20 inches long if the distorting moment is 10 inch-pounds. Eq. (3) of Art. 121 gives:

$$\beta = \frac{10 \times 20}{30{,}000{,}000 \times I} = \frac{10 \times 20 \times 12 \times 125{,}000}{30{,}000{,}000} = 10$$

(about $1\frac{1}{2}$ complete turns).

The fibre stress is

$$k = \frac{10 \times \frac{1}{100}}{\frac{1}{12 \times 125{,}000}} = 150{,}000 \text{ lbs. per sq.in.}$$

Art. 123.—Flat Plates.

The correct analysis of stresses in loaded flat plates even of the simplest form of outline has not yet been made sufficiently workable for ordinary engineering purposes, either for plates simply resting on edge supports or with edges of plates rigidly fixed to their supports. It is necessary, therefore, to combine simple, but approximate analysis based on reasonable assumptions, with experimental results so as to obtain workable formulæ. The following procedures, due chiefly to Bach and Grashof, are commonly employed in treating flat plates:

Square Plates—Uniform Load.

In Fig. 1 let *ABCD* represent a square plate simply resting on the edges of a square opening. Tests of such plates by Bach have shown that when increasingly loaded they will ultimately fail along a diagonal, as *AB*.

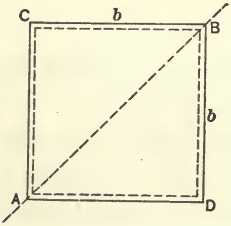


Fig. 1.

Let the plate be uniformly loaded with p pounds per square unit, then let moments be taken about the diagonal *AB*. If b is the side of the square, the load on the triangular half of the square is $\frac{pb^2}{2}$, and the distance of its centre from *AB* is $\frac{1}{3}b \sin 45^\circ = .236b$. The upward supporting forces or reaction on the sides *AD* and *DB* will also be half the load on the plate, $\frac{pb}{2}$, and its centre will be at the distance $\frac{b \sin 45^\circ}{2} = .354b$ from *AB*. Hence the moment about *AB* will be:

$$M = \frac{pb^2}{2}(.354b - .236b) = .059pb^3. \quad . \quad . \quad . \quad (1)$$

If h be the thickness of the plate, the moment of inertia I about its neutral axis will be:

$$I = \frac{b \sec 45^\circ \, h^3}{12} = .118bh^3. \quad . \quad . \quad . \quad . \quad (2)$$

The ordinary flexure formula then gives for the greatest intensity of bending stress k, assuming it to be uniform throughout the diagonal section,

$$k=\frac{M\frac{h}{2}}{I}=\frac{p}{4}\frac{b^2}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Or, if the thickness is desired,

$$h=b\sqrt{\frac{p}{4k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eq. (4) gives the thickness of plate required to carry the unit load p when the working stress is k.

Square Plates—Single Centre Load.

If a single load P rests at the centre of a square plate, using Fig. 1 and following the same method as in the preceding section, the moment about the diagonal AB will be:

$$M=\frac{P}{2}\frac{b \sin 45^\circ}{2}=.177Pb. \quad . \quad . \quad . \quad . \quad (5)$$

The moment of inertia I is the same as before and it is given by eq. (2). Hence, assuming a uniform intensity k throughout the extreme fibres:

$$k=\frac{.177Pbh}{2I}=\frac{3}{4}\frac{P}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Or,

$$h=.866\sqrt{\frac{P}{k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Rectangular Plates—Uniform Load.

Fig. 2 shows a rectangular plate with sides a and b. With a much oblong rectangle the indications of tests are not so well defined as to the section of failure, but tentatively the diagonal section AB may be taken as a close approximation for usual proportions. DF is a normal to AB drawn from D. The uniform load on the triangular half ABD of the plate is $\frac{pab}{2}$ and its centre of action is at the normal distance $\frac{n}{3}$ from AB. The centre of the supporting forces or reaction along the edges AD and DB is $\frac{n}{2}$ from AB. Hence the moment about AB is

$$M=\frac{pab}{2}\left(\frac{n}{2}-\frac{n}{3}\right)=\frac{pabn}{12}. \quad . \quad . \quad . \quad . \quad (8)$$

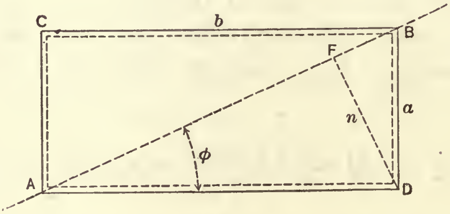


FIG. 2.

Referring to Fig. (2):

$$n=b \sin \phi \quad \text{and} \quad AB=b \sec \phi. \quad . \quad . \quad (9)$$

Therefore the moment of inertia of the diagonal section is

$$I=\frac{b \sec \phi h^3}{12}; \quad \text{and} \quad \frac{pab^2 \sin \phi}{12}=k\frac{b \sec \phi h^2}{6}.$$

Hence,

$$k = p\frac{ab \sin \phi \cos \phi}{2h^2}; \quad \text{or} \quad h = \sqrt{\frac{P}{2k}} \sin \phi \cos \phi. \qquad (10)$$

As is obvious, $P = pab$ is the total load on the plate.

Rectangular Plate—Centre Load.

If a single load P rests at the centre of the plate, the moment about the diagonal AB, Fig. 2, is produced by the reaction, only, of the supporting forces along the edges AB and BD, and its value is

$$M = p\frac{n}{2} = \frac{Pb \sin \phi}{2}. \quad . \quad . \quad . \quad . \qquad (11)$$

Consequently,

$$k = \frac{3P \sin \phi \cos \phi}{h^2}; \quad \text{or,} \quad h = \sqrt{\frac{3P}{k}} \sin \phi \cos \phi. \qquad (12)$$

Circular Plate—Uniform Load—Centre Load.

The circular plate with radius r is shown in Fig. 3. The same general assumptions are made as in the preceding cases, i.e., uniform condition of bending stress throughout the section of failure and uniform support along the edge of the plate. It is clear that the latter assumption is strictly correct for the circular outline. Any diameter, as AB, may be taken for the section of failure.

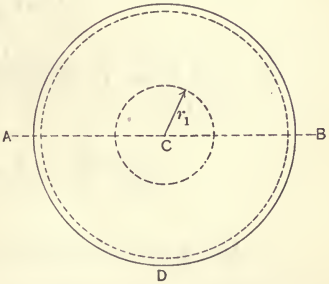


FIG. 3.

It will be convenient to suppose the uniform load to be applied on a circle of radius r_1, as shown in Fig. 3. Then the load on half of the plate is $p\frac{\pi r_1^2}{2}$ and its centre is at the distance $\frac{4r_1}{3\pi}$ from AB. The edge-supporting force or reaction, equal to the half load on the plate, has its centre at the distance of $\frac{2r}{\pi}$ from AB. The moment about the latter diameter is, therefore,

$$M = p\frac{\pi r_1^2}{2}\left(\frac{2r}{\pi} - \frac{4r}{3\pi}\right) = pr_1^2\left(r - \frac{2}{3}r_1\right). \quad . \quad . \quad . \quad (13)$$

If h is the thickness of the plate, as in the preceding cases, the moment of inertia I is

$$I = \frac{2rh^3}{12} = \frac{rh^3}{6}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$M = pr_1^2\left(r - \frac{2}{3}r_1\right) = k\frac{rh^2}{3}. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Hence,

$$k = pr_1^2\frac{\left(3 - 2\frac{r_1}{r}\right)}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Or,

$$h = r_1\sqrt{\frac{p}{k}\left(3 - 2\frac{r_1}{r}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If the load is uniform over the entire circular plate, $r = r_1$, and

$$M = \frac{pr^3}{3}; \quad k = p\frac{r^2}{h^2}; \quad \text{and,} \quad h = r\sqrt{\frac{p}{k}}. \quad . \quad . \quad (18)$$

If the load is concentrated at essentially a point, $r_1 = 0$, but $\frac{p\pi r_1^2}{2}$ must be displaced by $\frac{P}{2}$;

$$M = P\frac{r}{\pi}; \quad k = \frac{3P}{\pi h^2}; \quad \text{and,} \quad h = \sqrt{\frac{3P}{\pi k}}. \quad . \quad . \quad (19)$$

These formulæ for circular plates are more nearly correct in analysis and give results more nearly in agreement with tests than those derived for other cases.

Elliptical Plates—Centre Load—Uniform Load.

An elliptical plate is shown in Fig. 4. The approximate formulæ for this case may be conveniently established by first considering two axial strips of the same (unit) width, the length of AB being $2a$ and of CD, $2b$, a single load being placed at their intersection. The centre deflections of the two strips as parts of the plates must be the same. Let P_1 be the centre load for the strip AB, and P_2 the centre load for CD.

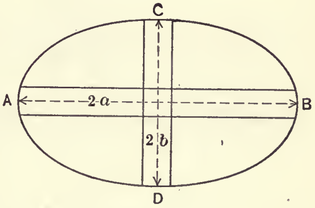


FIG. 4.

The desired centre deflection for each strip acting as a beam is given by eq. (28), Art. 28. The equality of the two deflections gives the equation, $2a$ being one span and $2b$ the other:

$$\frac{P_1 a^3}{6EI} = \frac{P_2 b^3}{6EI}; \quad \text{or} \quad \frac{P_1}{P_2} = \frac{b^3}{a^3}. \quad . \quad . \quad . \quad . \quad (20)$$

As each strip is of unit width, $I = \frac{h^3}{12}$, h being the thickness of plate. Hence the greatest fibre stresses are

$$k_1 = \frac{Mh}{2I} = 3P_1\frac{a}{h^2}; \quad \text{and,} \quad k_2 = 3P_2\frac{b}{h^2}. \quad . \quad . \quad (21)$$

Eqs. (21) and (20) then give:

$$\frac{k_1}{k_2} = \frac{P_1}{P_2}\frac{a}{b} = \frac{b^2}{a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Eq. (22) shows that k_2 is the greatest fibre stress and, hence, that the major axis of the ellipse will be the line of failure, as would be anticipated without the analysis.

If the ellipse of Fig. 4 be elongated by lengthening the major axis $2a$ to infinity, the result will be a correspondingly long rectangular plate of $2b$ width or span. Hence, the greatest fibre stress for this case of uniform load will be for a unit cross strip of plate:

$$k = \frac{Mh}{2I} = \frac{p(2b)^2}{8} \cdot \frac{h}{2} \frac{12}{h^3} = p\frac{3b^2}{h^2}. \quad . \quad . \quad . \quad (23)$$

This is the greatest intensity of stress for an ellipse whose major axis $2a$ is infinity. The other extreme is the circle for which the greatest intensity of stress is, eq. (18),

$$k = p\frac{r^2}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

For ellipses in general, in the absence of a satisfactory analysis, it is tentatively proposed to write:

$$k = \left(3 - 2\frac{b}{a}\right)\frac{pb^2}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

When $b = a$, eq. (25) gives the correct value for a circle, and when $b = 0$ the result is correct for the extreme ellipse.

The thickness of plate for a given uniform load p is

$$h = l\sqrt{\left(3 - 2\frac{b}{a}\right)\frac{p}{k}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Flat Plates Fixed at Edges.

Grashof and others have partly by analysis and partly empirically deduced a number of formulæ for plates fixed at their edges, i.e., *encastré*, instead of simply supported. The following have been used and may be considered fairly satisfactory, using the same notation as in the preceding parts of this article.

I. Circular plate with radius r and uniform load p. The greatest intensity of stress is, if h is the thickness,

$$k = p\frac{2}{3}\frac{r^2}{h^2}; \quad \text{and,} \quad h = r\sqrt{\frac{2}{3}\frac{p}{k}}. \quad . \quad . \quad . \quad (27)$$

II. Stayed flat surfaces, stay bolts being the distance c apart in two directions at right angles to each other. Each stay carries the uniform load pc^2. The greatest intensity of stress may be taken:

$$k = p\frac{2}{9}\frac{c^2}{h^2}; \quad \text{and,} \quad h = \frac{c}{3}\sqrt{\frac{2p}{k}}. \quad . \quad . \quad . \quad (28)$$

III. Rectangular plate a long, b wide, supporting uniform unit load p. The greatest intensity of stress may be taken:

$$k = p\frac{a^4}{2(a^4+b^4)}\frac{b^2}{h^2}; \quad \text{and,} \quad h = a^2b\sqrt{\frac{p}{k}\frac{1}{2(a^4+b^4)}}. \quad (29)$$

If the plate is square, $a = b$:

$$k = p\frac{b^2}{4h^2}; \quad \text{and,} \quad h = \frac{b}{2}\sqrt{\frac{p}{k}}. \quad . \quad . \quad . \quad (30)$$

All these plates with edges either fixed or simply supported are supposed to be truly flat, as any arching or dishing changes materially the conditions of stress.

Problem 1.—What thickness of steel plate is required to carry a load of 200 pounds per square inch over a rectangular opening 24 by 36 inches. Eq. (10) gives the expression for the thickness h of the plate when simply supported along its edges. The total load is $P = 200 \times 36 \times 24 = 168{,}800$ pounds.

$\tan\phi = \frac{24}{36} = .667 \quad \therefore \quad \phi = 33° \, 40'$ and $\sin\phi\cos\phi = .461$. If the working stress $k = 16{,}000$ pounds per square inch; $h = \sqrt{\frac{168{,}800}{2 \times 16{,}000}} \times .461 = 1.56$ inches. A plate $1\frac{9}{16}$ inches thick, therefore, meets the requirements.

Problem 2.—Design a circular steel plate, simply supported on its edge, for an opening 30 inches in diameter to carry a load of 100 pounds per square inch, if $k=15,000$ pounds per square inch. $r=15$ inches and $P=100\times\pi r^2=100\times 706.9=70,690$ pounds.

Eq. (18) then gives: $h=15\sqrt{\frac{100}{1500}}=1.22$ inches. A plate $1\frac{1}{4}$ inches thick will therefore be satisfactory.

If the plate were rigidly fixed along its edge, eq. (27) shows that the thickness would be: $h=1.22\sqrt{\frac{2}{3}}=1$ inch thick.

Art. 124. Resistance of Flues to Collapse.

If a circular tube or flue be subjected to external normal pressure, such as that of steam or water, the material of which it is made will be subjected to compression around the tube, in a plane normal to its axis. If the following notation be adopted,

l = length of tube;
d = diameter of tube;
t = thickness of wall of the tube;
p = intensity of excess of external pressure over internal;

then will any longitudinal section lt, of one side of the tube, be subjected to the pressure $\frac{pld}{2}$. But let a unit only of length of tube be considered. This portion of the tube is approximately in the condition of a column whose length and cross-section, respectively, are πd and t.

The ultimate resistance of such a column is (Art. 35)

$$P=\frac{\pi^2 EI}{\pi^2 d^2}.$$

As this ideal column is of rectangular section,

$$I=\frac{t^3}{12},$$

and

$$P=\frac{Et^3}{12d^2}.$$

But $P=pd$, hence

$$p=\frac{Et^3}{12d^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is the greatest intensity of external pressure which the tube can carry. But the formulæ of Art. 35 are not strictly applicable to this ideal column. The curvature on the one hand and the pressure on the other tend to keep it in position long after it would fail as a column without lateral support. *Hence p will vary inversely as some power of d much less than the third.*

Again, it is clear that a very long tube will be much more apt to collapse at its middle portion than a short one, as the latter will derive more support from the end attachments; and this result has been established by many experiments. Hence p must be considered as some inverse function of the length l.

Eq. (1), therefore, can only be taken as typical in form, and as showing in a general way, only, how the variable elements enter the value of p. If x, y, and z, therefore, are variable exponents to be determined by experiment, there may be written

$$p=c\frac{t^x}{l^y d^z}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which c is an empirical coefficient.

Sir Wm. Fairbairn ("Useful Information for Engineers, Second Series") made many experiments on wrought-iron tubes with lap- and butt-joints single riveted. He inferred from his tests that $y=z=1$. Two different experiments would then give

$$pld=ct^x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p'l'd'=ct'^x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence

$$\log (pld)=\log c+x \log t,$$
$$\log (p'l'd')=\log c+x \log t';$$

in which "log" means "logarithm." Subtracting one of these last equations from the other, the value of x becomes

$$x=\frac{\log (pld)-\log (p'l'd')}{\log t-\log t'}=\frac{\log\left(\frac{pld}{p'l'd'}\right)}{\log\left(\frac{t}{t'}\right)}. \quad . \quad . \quad . \quad (5)$$

As p, l, d, t, p', l', d', and t' are known numerical quantities in every pair of tests, x can at once be computed by eq. (5); c then immediately results from either eq. (3) or eq. (4). By the application of these equations to his experimental data, Fairbairn found for wrought-iron tubes:

$$p=9{,}675{,}600\frac{t^{2.19}}{ld}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which p is in pounds per square inch, while t, l, and d are in inches. *Eq.* (6) *is only to be applied to lengths between* 18 *and* 120 *inches.*

He also found that the following formula gave results agreeing more nearly with those of experiment, though it is less simple:

$$p=9{,}675{,}600\frac{t^{2.19}}{ld}-0.002\frac{d}{t}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Fairbairn found that by encircling the tubes with stiff rings he increased their resistance to collapse. *In cases where such rings exist, it is only necessary to take for l the distance between two adjacent ones.*

In 1875 Prof. Unwin, who was Fairbairn's assistant in his experimental work, established formulæ with other exponents and coefficients ("Proc. Inst. of Civ. Engrs.," Vol. XLVI). He considered x, y, and z variable, and found *for tubes with a longitudinal lap-joint:*

$$p = 7,363,000 \frac{t^{2.1}}{l^{0.9} d^{1.16}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

From *one tube with a longitudinal butt-joint*, he deduced:

$$p = 9,614,000 \frac{t^{2.21}}{l^{0.9} d^{1.16}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For *five tubes with longitudinal and circumferential joints*, he found:

$$p = 15,547,000 \frac{t^{2.35}}{l^{0.9} d^{1.16}}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

By using these same experiments of Fairbairn, other writers have deduced other formulæ, which, however, are of the same general form as those given above. It is probable that the following, which was deduced by J. W. Nystrom, will give more satisfactory results than any other:

$$p = 692,800 \frac{t^2}{d\sqrt{l}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

At the same time, it has the great merit of more simple application.

From one experiment on an elliptical tube, by Fairbairn, it would appear that the formulæ just given can be approximately applied to such tubes by substituting for d twice the radius of curvature of the elliptical section at either extremity of the smaller axis. If the greater diameter or axis of the ellipse is a and the less b, then, for d, there is to be substituted $\frac{a^2}{b}$.

Art. 125.—Approximate Treatment of Solid Metallic Rollers.

An approximate expression for the resistance of a roller may easily be written. The approximation may be considered a loose one, but it furnishes a basis for an accurate empirical formula.

The following investigation contains the improvements by Prof. J. B. Johnson and Prof. H. T. Eddy on the method originally given by the author.

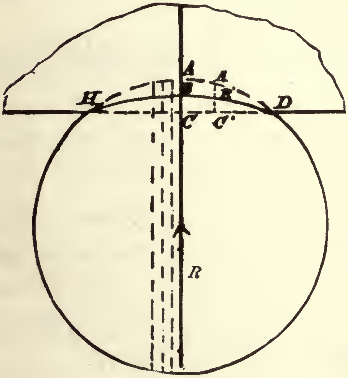


FIG. 1

The roller will be assumed to be composed of indefinitely thin vertical slices parallel to its axis. It will also be assumed that the layers or slices act independently of each other.

Let E' be the coefficient of elasticity of the metal over the roller.

Let E be the coefficient of elasticity of the metal of the roller.

Let R be the radius of the roller and R' the thickness of the metal above it.

Let w = intensity of pressure at A;

p = " " " " any other point;

Let P = total weight which the roller sustains per unit of length.

x be measured horizontally from A as the origin;

$d = AC$;

$e = DC$.

From Fig. 1:

$$AB = \frac{wR}{E}; \quad A'B' = \frac{pR}{E}.$$

$$BC = \frac{wR'}{E'}; \quad C'B' = \frac{pR'}{E'}.$$

$$\therefore \; d = AC = AB + BC = w\left(\frac{R}{E} + \frac{R'}{E'}\right) \quad . \quad . \quad . \quad (1)$$

and

$$A'C' = A'B' + B'C' = p\left(\frac{R}{E} + \frac{R'}{E'}\right). \quad . \quad . \quad . \quad (2)$$

Dividing eq. (2) by eq. (1),

$$p = A'C'\frac{w}{d}.$$

But

$$P = \int_{-e}^{+e} p dx = \frac{w}{d}\int_{-e}^{+e} A'C'dx.$$

If the curve DAH be assumed to be a parabola, as may be done without essential error, there will result:

$$\int_{-e}^{+e} A'C'dx = \frac{4}{3}ed.$$

Hence

$$P = \frac{4}{3}we. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But

$$e = \sqrt{2Rd - d^2} = \sqrt{2Rd}, \text{ nearly.}$$

By inserting the value of d from eq. (1) in the value of e, just determined, then placing the result in eq. (3),

$$P = \frac{4}{3}\sqrt{2w^3R\left(\frac{R}{E} + \frac{R'}{E'}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If $R = R'$,

$$P = \frac{4}{3}R\sqrt{2w^3\frac{E+E'}{EE'}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The preceding expressions are for one unit of length. If the length of the roller is l, its total resistance is

$$P' = Pl = \frac{4}{3}l\sqrt{2w^3R\left(\frac{R}{E} + \frac{R'}{E'}\right)}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Or if $R = R'$,

$$P' = \frac{4}{3}Rl\sqrt{2w^3\frac{E+E'}{EE'}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In ordinary bridge practice eq. (7) is sufficiently near for all cases.

A simple expression for conical rollers may be obtained by using eqs. (4) or (5).

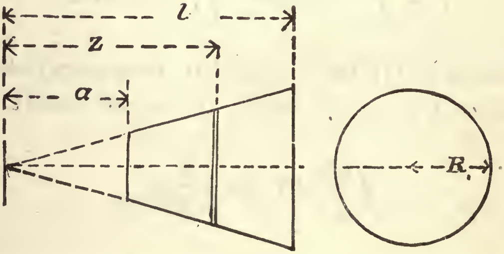


FIG. 2

As shown in Fig. 2, let z be the distance, parallel to the axis, of any section from the apex of the cone; then consider a portion of the conical roller whose length is dz. Let R_1 be the radius of the base. The radius of the section under consideration will then be

$$R = \frac{z}{l} R_1,$$

and the weight it will sustain, if $R_1 = R'$,

$$dP' = \frac{R_1}{l} \sqrt{2w^3 \frac{E+E'}{EE'}} \, . \, zdz.$$

Hence

$$P' = \int_a^l dP' = \frac{l^2 - a^2}{2l} R_1 \sqrt{2w^3 \frac{E+E'}{EE'}} \, . \quad . \quad . \quad . \qquad (8)$$

Eqs. (6), (7), and (8) give ultimate resistances if w is the ultimate intensity of resistance for the roller.

It is to be observed that the main assumptions on which the investigation is based lead to an error on the side of safety.

If for wrought iron, $w = 12,000$ pounds per square inch, and $E = E' = 28,000,000$ pounds, eq. (5) gives

$$P = \frac{8}{3} R \sqrt{\frac{w^3}{E}} = 664R.$$

Art. 126. —Resistance to Driving and Drawing Spikes.

Some very interesting experiments on driving and drawing rail spikes were made by Mr. A. M. Wellington, C.E., and reported by him in the "R. R. Gazette," Dec. 17, 1880. He experimented with wood both in the natural state and after it had been treated by the Thilmeny (sulphate of baryta) preserving process.

"The test-blocks were reduced to a uniform thickness of 4.5 inches, this thickness being just sufficient to give a full bearing surface to the parallel sides of the spikes when driven to the usual depth, and to allow the point of the spike to project outwards. It was considered that the bevelled point could add very little to the holding power of the spike, and it was desired to press the spike out again by direct pressure after turning the block over. . . ."

TABLE I.

SPIKES WERE STANDARD: 5.5 INCHES × $\frac{9}{16}$ INCH.

Kind of Wood.	Natural Wood.				Prepared Wood.			
	To Driving Spike, Pounds.		To Pulling Spike, Pounds.		To Driving Spike, Pounds.		To Pulling Spike, Pounds.	
		Mean.		Mean.		Mean.		Mean.
Beech	5,216 6,743	5,980	5,673 6,282	5,978	7,288 7,656	7,472	8,873 8,267	8,420
White oak, green	5,970 5,670	5,820	7,179 5,869	6,523	——		——	
Pin oak	5,216 5,521	5,368	6,638 6,469	6,553	6,117 4,589	5,353	6,135 6,267	6,201
White ash		5,953		4,560	6,588 5,978	6,283	(Split)	
White oak, well seasoned	6,433 6,433	6,433	5,128 3,435	4,281	——		——	
Black ash	3,996 4,202	4,090	4,408 4,868	4,638	4,453 4,453 4,301 3,380	4,147	3,340 3,028 3,300 3,493	3,290
Elm	4,453 4,758	4,606	3,536 3,843	3,690	4,453 4,148	4,300	4,148 4,202	4,175
Chestnut, green	3,996 3,386	3,691	2,730 3,790	3,260	——		——	
Soft maple	4,148 3,538	3,843	2,578 3,645	3,111	3,843 3,448	3,645	2,725 3,030	2,877
Sycamore	4,103 3,493	3,798	3,188 3,188	3,188	3,691 3,976	3,833		1,968
Hemlock		2,910		1,996	——		——	

The forces exerted in pulling and driving the spikes were produced by a lever. A few tests with a hydraulic press showed that the friction of the plunger varied from about 6 to 18 per cent. The experimental results are given in Table I.

Some very excellent tests of the holding power of railroad spikes and lag-screws were made by Mr. A. J. Cox, of the University of Iowa, during 1891, in the engineering laboratory of that institution, the results of which were

TABLE II.

RESISTANCE OF RAILROAD SPIKES TO PULLING OUT AND PRESSING IN.

Kind of Tie and Spike.	No. of Tests.	Greatest Resistance in Pounds.			Average Resistance in Pounds per Square Inch Surface of Spike†	Average Resistance per Ounce of Spike.
		Maximum.	Average.	Minimum.		
Seasoned White-oak Tie.						
Common spike	20	7,700	5,514	3,500	643	664
Common spike, ½-in. bored hole	9	6,660	4,936	3,950	575	595
Common spike, redrawn	1	—	5,120* 4,460	—	520	537
Common spike, ½-in bored hole, redrawn	2	—	4,040* 3,240	—	378	390
Hill curved spike	4	6,580	5,843	5,290	—	632
Bayonet spike	3	6,850	6,350	5,600	—	934
Unseasoned White Oak.						
Common spike	7	6,130	4,706	4,050	548	567
Common spike, ½-in. bored hole	3	5,950	5,807	5,720	716	740
Hill curved spike	5	5,680	5,130	4,400	—	555
Bayonet spike	3	6,930	5,334	4,030	—	784
Unseasoned White Cedar.						
Common spike	2	1,240	1,140	1,040	133	137
Common spike, ½-in. bored hole	2	1,460	1,400	1,340	162	169
Hill curved spike	2	1,830	1,775	1,720	—	192
Bayonet spike	2	980	955	930	—	140

PRESSING SPIKES INTO TIES UNDER STEADY PRESSURE OF TESTING MACHINE.

White-oak Ties.						
Curved spike, pressing in	2	7,430	7,375	7,320	—	—
Curved spike, pulling out	2	6,830	6,615	6,400	—	—
Bayonet spike, pressing in	2	6,660	6,530	6,400	—	—
Bayonet spike, pulling out	2	4,400	3,845	3,290	—	—

* These values are the first resistance to drawing out. The spikes were then redriven in the same holes and redrawn, with the results shown.

† Wedge surface not considered.

published in the technical journal ("The Transit") of the university for September, 1891; they will be found somewhat rearranged in Tables II and III. Three kinds of spikes were used, viz., the common spike (length 5.5 ins., 0.5625 in. square, weight 8.3 oz.), Hill's curved spike (length 5.875 ins., weight 9.25 oz.), and the bayonet or grooved spike (length 5.5 ins., weight 6.8 oz.). The timber of the ties is shown in the two tables. The spikes were forced into the wood by the pressure exerted by the 100,000-pound testing machine used in the tests, and by which they were pulled out of the ties.

TABLE III.

RESISTANCE OF LAG-SCREWS TO PULLING OUT.*

Kind of Wood.	Diameter of Screw, Inches.	Diameter of Bored Hole, Inches.	Length Screw in Hole, Ins.	Maximum Average Resistance in Pounds.	Resistance pounds per Square Inch.	No. of Tests.
Seasoned white oak. . . .	5/8	1/2	4 1/2	8,037	1,024	3
Seasoned white oak. . . .	9/16	7/16	3	6,480	1,223	1
Seasoned white oak. . . .	1/2	3/8	4 1/2	8,780	1,239	2
Yellow-pine stick.	5/8	1/2	4	3,800	484	2
White cedar unseasoned	5/8	1/2	4	3,405	434	2

The greatest pulling resistance of any spike is offered at the very beginning of motion, and it then rapidly decreases. A common spike which resisted 5120 pounds at the beginning of motion offered but 3050 pounds after having moved a half-inch, 2,440 pounds after 1 inch of motion, 1,300 pounds after 1.75 inches, 940 pounds after 2 inches, and 440 pounds after moving 3 inches; the original penetration of the spike was 4.375 inches in a seasoned white-oak tie. Similar results were reached with other timbers.

When spikes were pressed into the ties the timber offered an increasing resistance to penetration, but at a rate less rapid than that of the decrease in pulling out. A ½-inch penetration in a seasoned white-oak tie gave a resistance to a common spike of 2,320 pounds which increased to 3,340 pounds for 1-inch penetration, to 4550 pounds for

---

* The area of surface for these lag-screws used in finding the resistance per square inch was computed as that of a cylinder whose diameter was equal to the diameter of the screw considered. In pulling the first lag-screw of Table III, the resistance of 8037 pounds at the end of a ¼-inch movement decreased to 4550, 2476, 1475, and 410 pounds at the ends of movements of 0.5, 1, 2, and 2.75 inches respectively.

2 inches, to 5580 pounds for 3.5 inches, and to 6555 pounds for 4.5 inches.

The following results showing the relative holding power of common and screw railroad spikes were found by tests made by Prof. W. Kendrick Hatt for the U. S. Dept. of Agriculture and published in Forest Service Circular 46, 1906.

TABLE IV.

HOLDING FORCE OF COMMON AND SCREW SPIKES.

Species of Wood and Kind of Spike.	Number of Tests.	Condition of Wood.	Force Required to Pull Spike.		
			Average.	Max.	Min.
White oak:			*Pounds.*	*Pounds.*	*Pounds.*
Common spike	5	Partially seasoned	6,950	7,870	6,160
Screw spike	5	Partially seasoned	13,026	14,940	11,050
Ratio			1.88		
Oak (probably red):					
Common spike	5	Seasoned	4,342	5,300	3,490
Screw spike	8	Seasoned	11,240	13,530	8,900
Ratio			2.61		
Loblolly pine:					
Common spike	28	Seasoned	3,670	6,000	2,320
Screw spike	26	Seasoned	7,748	14,680	4,170
Ratio			2.11		
Hardy catalpa:					
Common spike	12	Green	3,224	4,000	2,190
Screw spike	14	Green	8,261	9,440	6,280
Ratio			2.56		
Common catalpa:					
Common spike	11	Green	2,887	4,500	2,240
Screw spike	11	Green	6,939	8,340	5,890
Ratio			2.42		
Chestnut:					
Common spike	4	Seasoned	2,980	3,220	2,600
Screw spike	5	Seasoned	9,418	11,150	7,470
Ratio			3.15		

Table V.

HOLDING FORCE OF COMMON AND SCREW SPIKES.

Seasoned Clear and Knotty Loblolly Pine Ties.

Position of Spike.	Kind of Spike.	Number of Tests.	Force Required to Pull Spike.		
			Average.	Max.	Min.
			Pounds.	*Pounds.*	*Pounds.*
In clear wood.........	Common........	36	3,466	6,250	1,880
In knotty wood.......	Common........	18	2,615	3,750	1,010
In clear wood.........	Screw...........	40	7,180	13,710	2,000
In knotty wood.......	Screw...........	20	9,763	17,200	4,890

Art. 127.—Shearing Resistance of Timber behind Bolt or Mortise Holes.

Col. T. T. S. Laidley, U.S.A., made some tests during 1881 at the United States Arsenal, Watertown, Mass., on the resistance offered by timber to the shearing out of bolts or keys when the force is exerted parallel to the fibres.

Fig. 1.

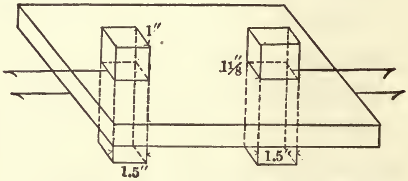


Fig. 2.

The test specimens are shown in Figs. 1 and 2. Wrought-iron bolts and square wrought-iron keys were used. All the timber specimens were six inches wide and two inches thick. The diameter of the bolts used (Fig. 1) was one inch for all the specimens. The keys were $1'' \times 1.''5$ and $1''.125 \times 1''.5$ as shown in Fig. 2. In all the latter specimens, failure took place in front of the smaller key where the pressure was greatest.

In many cases the specimen sheared and split simultaneously in front of the hole. By putting bolts through the pieces in a direction normal to the force exerted, so as to prevent splitting, the resistance was found (in most cases) to be considerably, though irregularly, increased.

Unless otherwise stated, the wood was thoroughly seasoned.

The accompanying table gives the results of Col. Laidley's tests.

Kind of Wood.	Centre of Hole from End of Specimen.	Total Area of Shearing.	Ultimate Shearing Resistance per Square Inch, in Pounds.
	Ins.	Sq. Ins.	
Spruce (bolts)	2	8	399
	4	16	359
	6	24	275
	8	32	202
White pine (bolts)	2	8	457
	4	16	611
	6	24	450
	8	32	327
Yellow pine (bolts)	2	8	607
	4	16	720
	6	24	456
	8	32	337.
Yellow pine (square keys)	2	8	599
	4	16	369
	6	24	572
	7	28	438
White pine (square keys)	2	8	550
	4	16	412
	6	24	332
	7	28	236
Spruce (square keys)	2	8	410 (Not thoroughly seasoned.)
	4	16	329 " " "
	6	24	242 (Wet timber.)
	7	28	279

Art. 128.—Method of Least Work—Stresses in a Bridge Portal.

In the consideration of stresses in structures or parts of structures where the equations of condition for statical equilibrium are not enough to determine all the unknown quantities, it is necessary to find other equations involving the elastic properties of the materials used. The Method of Least Work affords one procedure by which such extra equations may be found.

If a force P is gradually applied at a point in a structure it produces a deflection or distortion δ in its own direction and performs the work,

$$W=\frac{P\delta}{2}=\frac{a\delta^2}{2}=\frac{P^2}{2a}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As a consequence of Hook's law $P=a\delta$, a being a constant and a direct function of the modulus of elasticity E or G. Hence

$$\frac{dW}{dP}=\frac{P}{a}=\delta. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is called the first theorem of Castigliano, enunciated in his "Theorie des Gleichgewichtes elastischer Systeme." Eq. (2) is perfectly general and includes all elastic deformation or deflection. It shows that the first derivative of W, the work performed by the load P, in respect to that load as the independent variable, is the elastic distortion as well in the case of a force acting axially along a bar either in tension or compression as in that of a load producing deflection of a bridge at its point of application.

The third member of eq. (1) shows that

$$\frac{dW}{d\delta}=a\delta=P. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This equation may at times be useful.

If the point of application of the force or load P in eq. (2) be supposed unchanged in position while P acts, the other parts of the structure or piece moving in adjustment to that condition as may be required by the corresponding strains, then will $\delta=0$ and

$$\frac{dW}{dP}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If this equation be satisfied by solving it for P, the resulting value will make W, in general, either a maximum or minimum. In engineering structures, however, it is obvious that W will be a minimum, as the test by the second derivative will show in individual cases.

Eq. (4) expresses Castigliano's second theorem. If then the first derivative of a function W expressing the work performed in distorting a structure or structural member in terms of an indeterminate force or stress P, whose point of application may be supposed fixed, be taken in reference to that indeterminate force as the variable, a new equation of condition will result whose solution will yield a value of the force making the energy expended in the elastic distortions the least possible. Hence this procedure is called "the method of least work."

Stresses in a Bridge Portal.

The treatment of a bridge portal will illustrate the use of the method of least work in treating an important part of a bridge. Fig. 1 shows a skeleton diagram of the portal, AF and BG being the end posts in full length h in their own plane. $ABCD$ is the outline of the portal bracing which may be a plate girder or open bracing. The corner or gusset bracings at C and D are omitted. The equal end post stresses due to vertical dead and moving loads are indicated by *P* and *P*. The total horizontal wind load acting at the upper ends of the end posts is shown by *H*, and it is taken as applied wholly on the windward side. As is usual, the end posts are considered fixed in direction at both upper and lower ends. The lateral action of the wind will distort the portal in the manner shown exaggerated.

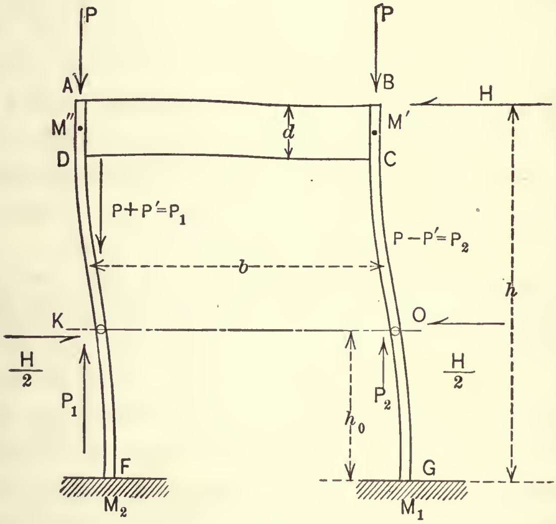


FIG. 1.

As both posts are supposed to be in the same condition and equally affected by the lateral wind pressure, the two points of contraflexure *K* and *O* must be at same distance h_0 (to be determined) from *FQ*. The points *M′* and *M″* are in the neutral surface of *ABCD*, i.e., at its mid-depth. The notation of Fig. 1 is self explanatory. The left arrow $\frac{H}{2}$ is below *K* and external to the upper part of Fig. 1, but the right arrow $\frac{H}{2}$ is above O and external to the lower part of the figure. Right-hand moments are positive and left-hand negative. Taking moments of forces acting on the upper part $ABOK$ of the portal and about O;

$$(P_1 - P = P')b - H(h - h_0) = 0 \quad \therefore \quad P'b = H(h - h_0). \quad (5)$$

Obviously, P' is the transferred load from the windward truss to the leeward due to the wind pressure H.

In order to find the work performed in distorting the members of the portal, it is necessary to determine the bending moments M', M'', M_2 and M_1 at the points indicated by these letters. Taking a section through K and moments about M', Fig. 1:

$$M' = -\frac{H}{2}\left(h - h_0 - \frac{d}{2}\right) - H\frac{d}{2} + (P'b = H(h - h_0))$$

$$\therefore \quad M' = \frac{H}{2}\left(h - h_0 - \frac{d}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Then moments about M'' will give

$$M'' = -\frac{H}{2}\left(h - h_0 - \frac{d}{2}\right) = -M'. \quad . \quad . \quad . \quad (7)$$

Obviously the signs of the moments M_2 and M_1 must be opposite to those of M'' and M' respectively. Hence,

$$M_2 = \frac{H}{2}h_0; \quad \text{and} \quad M_1 = -\frac{H}{2}h_0 \quad . \quad . \quad . \quad . \quad (8)$$

The moments throughout the parts of the portal will then be:

For girder AC,

$$M_g = M' - \frac{M' - M''}{b}x = H\left(h - h_0 - \frac{d}{2}\right)\left(\frac{1}{2} - \frac{x}{b}\right). \quad . \quad (9)$$

For left post FA,

$$M_p = M_2 - \frac{M_2 - M''}{h}x = \frac{H}{2}\left(h_0 - \left(h - \frac{d}{2}\right)\frac{x}{h}\right). \quad . \quad (10)$$

For right post BG,

$$M'_p = M_1 - \frac{M_1 - M'}{h}x = \frac{H}{2}\left(-h_0 + \left(h - \frac{d}{2}\right)\frac{x}{h}\right). \quad . \quad (11)$$

It has been shown in the chapter on resilience that the work done in bending a beam is $\frac{1}{2EI}\int M^2dx$, I being the moment of inertia of the normal section of the beam. Similarly the work performed by an axial force P on a straight member whose area of cross-section is A and length h is $\frac{P^2h}{2AE}$. If I_1 is the moment of inertia for the member AC, Fig. 1, and I_2 for each post AF and BG, while A_2 is the common area of cross-section for the latter, one carrying the axial load $P + \frac{H}{b}(h - h_0)$ and the other $P - \frac{H}{b}(h - h_0)$, the total work done on the entire portal is

$$W = \frac{1}{2EI_1}\int_0^b M_g^2dx + 2\frac{1}{2EI_2}\int_0^h M_p^2dx$$

$$+ \frac{h}{2A_2E}\left[\left(P + \frac{H}{b}(h - h_0)\right)^2 + \left(P - \frac{H}{b}(h - h_0)\right)^2\right].$$

If $n = h - \frac{d}{2}$ and $g = P^2 + \frac{H^2h^2}{b^2}$ there will result:

$$W = \frac{H^2b}{24EI_1}(n^2 - 2nh_0 + h_0^2) + \frac{H^2h}{4EI_2}\left(\frac{n^2}{3} - h_0n + h_0^2\right)$$
$$+ \frac{h}{A_2E}\left(g - \frac{2H^2h}{b^2}h_0 + \frac{H^2}{b^2}h_0^2\right). \quad (12)$$

Eq. (5) shows that h_0 may be replaced in this equation in terms of P'; hence $\frac{dW}{dh_0}$ corresponds to $\frac{dW}{dP'}$. Placing $\frac{dW}{dh_0} = 0$ and solving, therefore,

$$h_0 = \frac{\left(h - \frac{d}{2}\right)(bI_2 + 3hI_1)b^2A_2 + 24h^2I_1I_2}{b^3A_2I_2 + 6hb^2A_2I_1 + 24hI_1I_2}. \quad . \quad (13)$$

This locates the points of contraflexure and enables all computations to be made.

If the axial compression of the two end posts be neglected, the last term in both numerator and denominator of the second member of eq. (13) disappears, and

$$h_0 = \frac{\left(h - \frac{d}{2}\right)(bI_2 + 3hI_1)}{bI_2 + 6hI_1}. \quad . \quad . \quad . \quad . \quad (14)$$

If r_2 is the radius of gyration of A_2 and if $\frac{I_2}{I_1} = i$, eq. (13) may take a more convenient form for computation:

$$h_0 = \frac{\left(1 - \frac{d}{2h}\right)(bi + 3h)b + 24r_2^2\frac{h}{b}}{\frac{b^2}{h}i + 6b + 24\frac{r_2^2}{b}}. \quad . \quad . \quad . \quad (15)$$

In the same manner eq. (14) becomes:

$$h_0=\frac{\left(1-\frac{d}{2h}\right)(bi+3h)}{\frac{b}{h}i+6}. \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

CHAPTER XVII.

THE FATIGUE OF METALS.

Art. 129.—Woehler's Law.

In all the preceding pages, that force or stress which, by a single or gradual application, will cause the failure or rupture of a piece of material has been called its "ultimate resistance." It has long been known, however, that a stress less than the ultimate resistance *may* cause rupture if its application be repeated (without shock) a sufficient number of times. Preceding 1859 no experiments had been made for the purpose of establishing any law connecting the number of applications with the stress requisite for rupture, or with the variation between the greatest and least values of the applied stress.

During the interval between 1859 and 1870, A. Wöhler, under the auspices of the Prussian Government, undertook the execution of some experiments, at the completion of which he had established the following law:

Rupture may be caused not only by a force which exceeds the ultimate resistance, but by the repeated action of forces alternately rising and falling between certain limits, the greater of which is less than the ultimate resistance; the number of repetitions requisite for rupture being an inverse function both of this variation of the applied force and its upper limit.

This phenomenon of the decrease in value of the breaking load with an increase of repetitions is known as "*the fatigue of materials.*"

Although the experimental work requisite to give Wöhler's law complete quantitative expression in the various conditions of engineering constructions can scarcely be considered more than begun, yet enough has been done by Wöhler and Spangenberg to establish the *fact* of metallic fatigue, and a few simple formulæ, provisional though they may be. The importance of the subject in its relation to the durability of all iron and steel structures is of such a high character that a synopsis of some of the experimental results of Wöhler and Spangenberg will be given in the next article.

Art. 130.—Experimental Results.

The experiments of Wöhler are given in "Zeitschrift für Bauwesen," Vols. X., XIII., XVI., and XX., and those of Spangenberg may be consulted in "Fatigue of Metals," translated from the German of Prof. Ludwig Spangenberg, 1876.

These results show in a very marked manner the effect of repeated vibrations on the intensity of stress required to produce rupture.

Spangenberg states that "the experiments show that vibrations may take place between the following limits with equal security against rupture by tearing or crushing:

Wrought iron	+17,600 and −17,600	lbs.	per	sq.	in.
	+33,000 and— 0	"	"	"	"
	+48,400 and +26,400	"	"	"	"
Axle cast steel	+30,800 and −30,800	"	"	"	"
	+52,800 and 0	"	"	"	"
	+88,000 and +38,500	"	"	"	"
Spring-steel not hardened..	+55,000 and 0	"	"	"	"
	+77,000 and +27,500	"	"	"	"
	+88,000 and +44,000	"	"	"	"
	+99,000 and +66,000	"	"	"	"

And for axle cast steel in shearing:

+24,200 and −24,200 lbs. per sq. in.
+41,800 and 0 " " " "

PHŒNIX IRON IN TENSION.

Pounds Stress per Square Inch.	Number of Repetitions.	Pounds Stress per Square Inch.	Number of Repetitions.
0 to 52,800	800 rupture	0 to 39,600	480,852 rupture
0 to 48,400	106,910 rupture	0 to 35,200	10,141,645 rupture
0 to 44,000	340,853 rupture	22,000 to 48,400	2,373,424 rupture
0 to 39,600	409,481 rupture	26,400 to 48,400	4,000,000 not broken

WESTPHALIA IRON IN TENSION.

0 to 52,800	4,700 rupture	0 to 39,600	180,800 rupture
0 to 48,400	83,199 rupture	0 to 39,600	596,089 rupture
0 to 48,400	33,230 rupture	0 to 39,600	433,572 rupture
0 to 44,000	136,700 rupture	0 to 35,200	280,121 rupture
0 to 44,000	159,639 rupture	0 to 35,200	566,344 rupture

FIRTH & SONS' STEEL IN TENSION.

0 to 66,000	83,319 rupture	0 to 55,000	103,540 rupture
0 to 60,500	168,396 rupture	0 to 53,900	12,200,000 not broken
0 to 55,000	133,910 rupture	0 to 53,900	229,230 rupture
0 to 55,000	185,680 rupture	0 to 52,800	692,543 rupture
0 to 55,000	360,235 rupture	0 to 52,800	12,200,000 not broken
0 to 55,000	186,005 rupture	0 to 50,600	——

KRUPP'S AXLE-STEEL IN TENSION.

0 to 88,000	18,741 rupture	0 to 55,000	473,766 rupture
0 to 77,000	46,286 rupture	0 to 52,800	13,600,000 not broken
0 to 66,000	170,000 rupture	0 to 50,600	12,200,000 not broken
0 to 60,500	123,770 rupture	——	——

PHOSPHOR-BRONZE (UNWORKED) IN TENSION

0 to 27,500	147,850 rupture	0 to 13,750	1,548,920 rupture
0 to 22,000	408,350 rupture	0 to 13,750	2,340,000 rupture
0 to 16,500	2,731,161 rupture	——	——

PHOSPHOR-BRONZE (WROUGHT) IN TENSION.

0 to 22,000	53,900 rupture	0 to 13,750	1,621,300 rupture
0 to 16,500	2,600,000 not broken	——	——

COMMON BRONZE IN TENSION.

0 to 22,000	4 200 rupture	0 to 11,000	5,447,600 rupture
0 to 16,500	6,300 rupture	——	——

PHŒNIX IRON IN FLEXURE (ONE DIRECTION ONLY).

Pounds Stress per Square Inch.	Number of Repetitions.	Pounds Stress per Square Inch.	Number of Repetitions.
0 to 60.500	169,750 rupture	0 to 39,600	4,035,400 rupture
0 to 55,000	420,000 rupture	0 to 35,200	3,420,000 rupture
0 to 49,500	481,975 rupture	0 to 33,000	4,820,000 not broken
0 to 44,000	1,320.000 rupture	——	——

WESTPHALIA IRON IN FLEXURE (ONE DIRECTION ONLY).

0 to 52,250	612,065 rupture	0 to 44.000	1,493,511 rupture
0 to 49.500	457,229 rupture	0 to 39,600	3,587,509 rupture
0 to 46.750	799,543 rupture	——	——

HOMOGENEOUS IRON IN FLEXURE (ONE DIRECTION ONLY).

0 to 60,500	169,750 rupture	0 to 39,600	4,035,400 rupture
0 to 55,000	420,000 rupture	0 to 35,020	3,420,000 not broken
0 to 49,500	481,975 rupture	0 to 33,000	48,200,000 not broken
0 to 44 000	1,320,000 rupture	——	——

FIRTH & SONS' STEEL IN FLEXURE (ONE DIRECTION ONLY).

0 to 63,250	281,856 rupture	0 to 52,250	578,323 rupture
0 to 60,500	266,556 rupture	0 to 49,500	5,640,596* rupture
0 to 55,000	1,479,908 rupture	0 to 49,500	13,700,000 not broken

* Accidental.

KRUPP'S AXLE-STEEL IN FLEXURE (ONE DIRECTION ONLY).

0 to 77.000	104,300 rupture	0 to 55.000	729,400 rupture
0 to 66.000	317,275 rupture	0 to 55.000	1.499,600 rupture
0 to 60,500	612,500 rupture	0 to 49.500	43,000,000 not broken

KRUPP'S SPRING-STEEL IN FLEXURE (ONE DIRECTION ONLY).

0 to 110,000	39,950 rupture	72,600 to 110,000	19,673,300 not broken
0 to 88,000	117,000 rupture	66,000 to 99,000	33,600,000 not broken
0 to 66,000	468,200 rupture	44,000 to 88,000	35,800,000 not broken
0 to 55 000	40,600,000 not broken	44,000 to 88,000	38,000,000 not broken
0 to 49,500	32,942,000 not broken	61,600 to 88,000	36,000,000 not broken
88,000 to 132,000	35,600,000 not broken	27,500 to 77,000	36,600,000 not broken
99,000 to 132,000	33,478,700 not broken	33,000 to 77,000	31,152,000 not broken

PHOSPHOR-BRONZE IN FLEXURE (ONE DIRECTION ONLY).

Pounds Stress per Square Inch.	Number of Repetitions.	Pounds Stress per Square Inch.	Number of Repetitions.
0 to 22,000	862,980 rupture	0 to 16,500	5,075,169 rupture
0 to 19,800	8,151,811 rupture	0 to 13,200	10,000,000 not broken

COMMON BRONZE IN FLEXURE (ONE DIRECTION ONLY).

0 to 22,000	102,659 rupture	0 to 16,500	837,760 rupture
0 to 19,800	151,310 rupture	0 to 13,200	10,400,000 not broken

PHŒNIX IRON IN TORSION (BOTH DIRECTIONS).

− 35,200 to + 35,200	56,430 rupture	− 24,200 to + 24,200	3,632,588 rupture
− 33,000 to + 33 000	99,000 rupture	− 22,000 to + 22,000	4,917,992 rupture
− 28,600 to + 28 600	479,490 rupture	− 19,800 to + 19,800	19 186,791 rupture
− 26,400 to + 26,400	909,810 rupture	− 17,600 to + 17,600	132,250,000 not broken

ENGLISH SPINDLE-IRON IN TORSION (BOTH DIRECTIONS).

− 37,400 to + 37,400	204,400 rupture	− 30,800 to + 30,800	979,100 rupture
− 37,400 to + 37,400	147,800 rupture	− 28 600 to + 28,600	1,142,600 rupture
− 35,200 to + 35,200	911,100 rupture	− 28,600 to + 28,600	595,910 rupture
− 35,200 to + 35,200	402,900 rupture	− 26,400 to + 26,400	3,823,200 rupture
− 33,000 to + 33,000	1,064,700 rupture	− 26,400 to + 26,400	6 100,000 not broken
− 33 000 to + 33,000	384,800 rupture	− 22,000 to + 22,000	8,800 000 not broken
− 30 800 to + 30,800	1,337,700 rupture	− 22,000 to + 22,000	4,000,000 not broken

KRUPP'S AXLE-STEEL IN TORSION (BOTH DIRECTIONS)

− 44,000 to + 44,000	367,400 rupture	− 46,200 to + 46,200	55,100 rupture
− 39,600 to + 39 600	925,800 rupture	− 37,400 to + 37,200	797,525 rupture
− 37,400 to + 37,400	4,900,000 not broken	− 35 200 to + 35,200	1 665,580 rupture
− 35,200 to + 35,200	4,800,000 not broken	− 33,000 to + 33,000	4,163 375 rupture
− 33,000 to + 33 000	5,000 000 not broken	− 33,000 to + 33,000	45,050,640 rupture

The late Capt. Rodman, U.S A., made a considerable number of experiments on the fatigue of cast iron, but they were sufficient in number and character to show the general effect only, and gave no quantitative results.

The specimens used in all the preceding experiments were small.

During 1860, '61, and '62 Sir Wm. Fairbairn constructed a built beam of plates and angles with a depth of 16 inches, clear span of 20 feet, and estimated centre breaking load of 26,880 pounds.

This beam was subjected to the action of a centre load of 6643 pounds, alternately applied and relieved eight times per minute; 596,790 continuous applications produced no visible alterations.

The load was then increased from one fourth to two sevenths the breaking weight, and 403,210 more applications were made without apparent injury.

The load was next increased to two fifths the breaking weight, or to 10,486 pounds; 5175 changes then broke the beam in the tension flange near the centre.

The total number of applications was thus 1,005,175.

The beam was then repaired and loaded with 10,500 pounds at centre 158 times, then with 8025 pounds 25,900 times, and finally with 6643 pounds enough times to make a total of 3,150,000.

In these experiments the load was completely removed each time.

It is thus seen that vibrations (without shock) with one-fourth the calculated breaking centre load produced no apparent effect on the resistance of the beam, but that two fifths of that load caused failure after a comparatively small number of repetitions.

It is probable that the breaking centre load was calculated too high, in which case the ratios $\frac{1}{4}$ and $\frac{2}{5}$ should be somewhat increased.

Art. 131.—Formulæ of Launhardt and Weyrauch.

Let R represent the intensity (stress per square unit of section) of ultimate resistance for any material in tension, compression, shearing, torsion, or bending; R will cause rupture at a single, gradual application. But the material may also be ruptured if it is subjected a sufficient number of times, and alternately, to the intensities P and Q, Q being less than P and both less than R, *while all are of the same kind*. When $Q=0$ let $P=W$, and let $D=P-Q$. W is called the "primitive safe resistance," since the bar returns to its primitive unstressed condition at each application. In the general case P is called the "working ultimate resistance."

By the notation adopted:

$$P=Q+D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But by Wöhler's law, P is a function of D, or

$$P=f(D). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

A sufficient number of experiments have not yet been made in order to completely determine the form of the function $f(D)$.

It is known, however, that

$$\text{for } Q=0, \qquad P=D=W;$$
$$\text{and for } D=0, \qquad P=Q=R.$$

Provisionally, Launhardt satisfies these two extreme conditions by taking

$$P=\frac{R-W}{R-P}D=\frac{R-W}{R-P}(P-Q). \quad . \quad . \quad . \quad . \quad (3)$$

Even at these limits this is not thoroughly satisfactory, for when $D=0$, $P=\frac{0}{0}(R-W)$, or is indeterminate.

By solving eq. (3),

$$P=W\left(1+\frac{R-W}{W}\cdot\frac{Q}{P}\right). \quad . \quad . \quad . \quad . \quad (4)$$

But if the least value of the total stress to which any member of a structure is subjected is represented by *min. B*, and its greatest value by *max B*, there will result $\frac{min\ B}{max\ B}=\frac{Q}{P}$.

Hence

$$P=W\left(1+\frac{R-W}{W}\frac{min\ B}{max\ B}\right), \quad . \quad . \quad . \quad . \quad (5)$$

which is Launhardt's formula. In the preceding article some values of W are shown. In applying eq. (5) it is only necessary to take the primitive safe resistance, W, for the total number of times which the structure will be subjected to loads. Since bridges are expected to possess an indefinite duration of life, in such structures that number should be indefinitely large.

Eq. (5), it is to be borne in mind, is to be applied when the piece is *always subjected to stress of one kind, or in one direction only.* It agrees well with some experiments by Wöhler on Krupp's untempered cast spring steel.

If the stress in any piece varies from one kind to another, as from tension to compression, or *vice versa*, or from one direction to another, as in torsion on each side of a state of no stress, Weyrauch has established the following formula by a course of reasoning similar to that used by Launhardt.

If the opposite stresses, which will cause rupture by a certain number of applications, are equal in intensity, and if that intensity is represented by S, then will S be called the "vibration resistance"; this was established by Wöhler for some cases, and some of its values are given in the preceding article.

Let $+P$ and $-P'$ represent two intensities of opposite kinds or in opposite directions, of which P *is numerically the greater*. Then if $D=P+P'$,

$$P=D-P'.$$

The two following limiting conditions will hold:

$$\text{For } P'=0, \quad P=D=W;$$
$$\text{For } P'=S; \quad P=S=\tfrac{1}{2}D.$$

But by Wöhler's law $P=f(D)$, and the two limiting conditions just given will be found to be satisfied by the provisional formula

$$P=\frac{W-S}{2W-S-P}D=\frac{W-S}{2W-S-P}(P+P'). \quad . \quad . \quad . \qquad (6)$$

By the solution of eq. (6),

$$P=W\left(1-\frac{W-S}{W}\cdot\frac{P'}{P}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

If, without regard to kind or direction, *max* B is numerically the greatest total stress which the piece has to carry, while *max* B' is the greatest total stress of the other kind or direction, then will $\frac{P'}{P}=\frac{max\ B'}{max\ B}$. Hence there will result the following, which is the formula of Weyrauch:

$$P=W\left(1-\frac{W-S}{W}\,\frac{max\ B'}{max\ B}\right). \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Eqs. (5) and (8) give values of the intensity P which are to be used in determining the cross-section of pieces designed to carry given amounts of stress. If n is the safety factor and F the total stress to be carried, the area of section desired will be

$$A=\frac{nF}{P},$$

in which $\frac{P}{n}$ is the greatest working stress permitted.

If for wrought iron in tension $W=30,000$ and $R=50,000$, eq. (5) gives

$$P=30,000\left(1+\frac{2}{3}\frac{min\ B}{max\ B}\right).$$

Hence, if the total stress due to fixed and moving loads in the web member of a truss is $max\ B=80,000$ pounds, while that due to the fixed load alone is $min\ B=40,000$, there will result

$$P=30,000\left(1+\frac{2}{3}\cdot\frac{40,000}{80,000}\right)=40,000.$$

In such a case the greatest permissible working stress with a safety factor of 3 would be about 13,300 pounds.

For steel in tension, if $W=50,000$ and $R=75,000$,

$$P=50,000\left(1+\frac{1}{2}\frac{min\ B}{max\ B}\right).$$

For wrought iron in torsion, if $S=18,000$ and $W=24,000$, eq. (8) will give

$$P=24.000\left(1-\frac{1}{4}\frac{max\ B'}{max\ B}\right).$$

Other methods based on Wöhler's experiments have been deduced by Müller, Gerber, and Schäffer, of which synopses may be found in Du Bois' translation of Weyrauch's "Structures of Iron and Steel."

Art. 132.—Influence of Time on Strains.

In an earlier section of this book devoted to data of certain tests, the effect of prolonged tensile stress and subsequent rest between the elastic limit and ultimate resistance was shown to be the elevation of both those quantities. It is a matter of common observation, however, that if a piece of wrought iron be subjected to a tensile stress nearly equal to its ultimate resistance, and held in that condition, the stretch will increase as the time elapses.

Experiments are still lacking which may show that a piece of metal can be ruptured by a tensile stress much below its ultimate resistance. It may be indirectly inferred, however, from experiments on flexure, that such failure may be produced, as the following by Prof. Thurston will show.

A bar 10 parts tin and 90 parts copper, 1 × 1 × 22 inches and supported at each end, sustained about 65 per cent. of its breaking load at the centre for five minutes. During that time its deflection increased 0.021 inch. The same bar sustained 1485 pounds at centre for 13 minutes and then failed.

A second bar of the same size, but 90 parts tin and 10 parts copper, was loaded at the centre with 160 pounds, causing a deflection of 1.294 inches. After 10 minutes the deflection had increased 0.025 inch; after one day, 1.00 inch; after two days, 2.00 inches; and after three days, 3.00 inches, when the bar failed under the load of 160 pounds.

Another bar of the same size showed remarkable results;

it was composed of 90 parts zinc and 10 parts copper. It gave the same general increase of deflection with time, but eventually broke under a centre load which ran down from 1233 to 911 pounds, after holding the latter about three minutes.

A bar of the same size and 96 parts copper with 4 parts tin, after it had carried 700 pounds at centre for sixty minutes was loaded with 1000 pounds, with the following results:

After.	Deflection.
0 minute	3.118 inches.
5 minutes	3.540 "
15 minutes	3.660 "
45 minutes	4.102 "
75 minutes	7.634 "
Broke under 1000 pounds.	

A wrought-iron bar of the same size gave, under a centre load of 1600 pounds:

After.	Deflection.
0 minute	0.489 inch.
3 minutes	0.632 "
6 minutes	0.650 "
16 minutes	0.660 "
344 minutes	0.660 "

It subsequently carried 2589 pounds with a deflection of 4.67 inches.

During 1875 and 1876 Prof. Thurston made a number of other similar experiments with the same general results.

Metals like tin and many of its alloys showed an increasing rate of deflection and final failure, far below the so-called "ultimate resistance." The wrought-iron bars, however, showed a decreasing increment of deflection, which finally became zero, leaving the deflection constant.

Whether there may be a point for every metal, beyond which, with a given load, the increment of deflection may retain its value or go on increasing until failure, and below which this increment decreases as the time elapses, and finally becomes zero, is yet undetermined, but seems probable.

It does not follow, therefore, that the principle enunciated in the section named at the beginning of this article is to be taken without qualification. If "rest" under stress, too near the ultimate resistance, be sufficiently prolonged, it has been seen that it is possible that failure may take place.

In verifying some experimental results by Herman Haupt, determined over forty years ago, Prof. Thurston tested three seasoned pine beams about 1⅛ inches square and 40 inches length of span, and found that 60 per cent. of the ordinary "breaking load" caused failure at the end of 8, 12, and 15 months. In these cases the deflection slowly and steadily increased during the periods named.

Two other sets of three pine beams each broke under 80 and 95 per cent. of the usual "breaking load," after much shorter intervals of time.

In all these instances it is evident that the molecules under the greatest stress "flow" over each other to a greater or less extent. In the cases of decreasing increments of strain, the new positions afford capacity of increased resistance; in the others, those movements are so great that the distances between some of the molecules exceed the reach of molecular action, and failure follows.

In many cases strained portions of material recover partially or wholly from permanent set. In such cases a portion of the material has been subjected to intensities of stress high enough to produce true "flow" of the molecules, while the remaining portion has not. The internal elastic stresses in the latter portion, after the removal of the external forces, produce in time a reverse flow in consequence of the elastic endeavor to resume the original shape.

It is altogether probable that the phenomena of fatigue and flow of metals are very intimately associated. Some of the prominent characteristics of the latter will be given in the next chapter.

CHAPTER XVIII.

THE FLOW OF SOLIDS.

Art. 133.—General Statements.

Although there is no reason to suppose that true solids may not retain a definite shape for an indefinite length of time if subjected to no external force other than gravity,* many phenomena resulting both from direct experiment for the purpose, and incidentally from other experiments involving the application of external stress of considerable intensity, show that a proper intensity of internal stress (in many cases comparatively low) will cause the molecules of a solid to flow at ordinary temperatures like those of a liquid. And this flow, moreover, is entirely different from, and independent of, the elastic properties of the material; for it arises from a permanent and considerable relative displacement of the molecules. Nor is it to be confounded with that internal "friction" which, if an elastic body is subjected to oscillations, causes the amplitudes to gradually decrease and finally disappear, even in vacuo. This latter motion is typically elastic and the retarding cause may be considered a kind of elastic friction.

It is evident that if a mass of material be enclosed on all its faces, or outer surfaces, but one or a portion of one, and if external pressure be brought to bear on those faces, the

---

* This, perhaps, may be considered a definition of a true solid.

material will be forced to move to and through the free surface; in other words, *the flow of the material will take place in the direction of least resistance.*

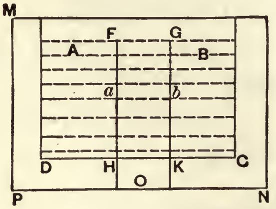


Fig. 1.

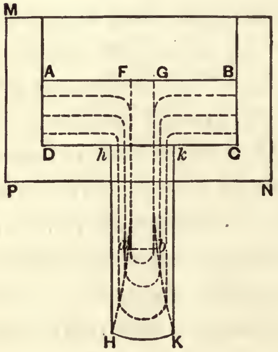


Fig. 2.

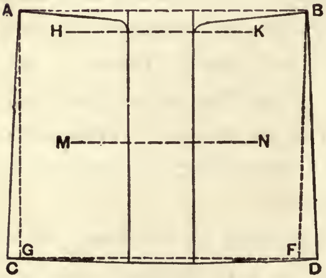


Fig. 3.

The theory of the flow of solids to be given is that developed by Mons. H. Tresca in his "Mémoire sur l'Écoulement des Corps Solides," 1865. He made a large number of experiments on hard and soft metals, ceramic pastes, sand, and shot.

These different materials all manifested the same characteristics of flow, which are well shown in Fig. 2. *ABCD*, Fig. 1, is supposed to be a cylindrical mass of lead with circular horizontal section, confined in a circular cylinder, *MN*, closed at one end with the exception of the orifice *O*.

This cylinder is supported on the base *PN*, while the face *AB* of the lead receives external pressure from a close-fitting piston. When the pressure is sufficiently increased, the face *AB* in Fig. 1 sinks to *AB* in Fig. 2, while the column *hkHK*, in the latter figure, is forced to flow through the orifice *O*.

In Tresca's experiments with lead, the diameter *AB* was about 3.9 inches; the diameter *HK* of the orifice, from 0.75 in. to 1.5 ins., while the length of the column or jet *hK* varied from 0.4 in. to about 24 ins.

The total pressure on the face *AB* varied from 119,000 to 198,000 pounds. The initial thickness *AD* varied from 0.24 inch to 2.4 inches.

Some experiments exhibiting in a remarkably clear manner the flow of metals in cold punching were made by David Townsend in 1878, and the results were given by him in the "Journal of the Franklin Institute" for March of that year. If the dotted rectangle *ABFG*, Fig. 3, shows the original outline of the middle section of a nut before punching, he found that the final outline of the same section would be represented by the full lines. The top and bottom faces were depressed by the punching, as shown; the upper width *AB* remained about the same, but the lower, *GF*, was increased to *CD*. Although the depth of the nut, *AC*, was 1.75 inches, the length of the core punched out was only 1.063 inches. The density of this core was then examined and found to be the same as that of the original nut. Hence a portion of the core equal in length to $1.75-1.063=0.687$ inch was forced, or flowed, back into the body of the nut. Subsequent experiments showed that this flow did not take place at the immediate upper surface *AB*, nor very much in the lower half of the nut, but that it was chiefly confined to a zone equal in depth to about half that of the nut, the upper surface of which lies a very short distance below the upper face of the nut. The location of this zone is shown by the lines *HK* and *MN* in Fig. 3.

Tresca's experiments on punching showed essentially the same result.

Art. 134.—Tresca's Hypotheses.

The central cylinder *FGKH*, Fig. 1 of Art. 133 was called by Tresca the "primitive central cylinder." As the metal flows, this cylinder will be drawn out into the volume of revolution, whose axis is that of the orifice and whose meridian section is *FGkKIHh*, Fig. 2, the diameter *FG* being gradually decreased.

It was found by experiment that if the original mass *AC*, Fig. 1, was composed of horizontal layers of uniform thickness, the reduced mass in Fig. 2 was also composed of the same number of layers of uniform thickness, except in the immediate vicinity of the central cylinder.

Tresca then assumed these three hypotheses:

1°.—*The density of the material remains the same whether in the cylinder or in the jet;* in other words, the volume of the material in the jet and in the cylinder remains constant.

Let R = radius of the cylinder;
R_1 = radius of the orifice;
y = variable length of the jet (i.e., hH);
D = original depth of material ($BC = AD$, Fig. 1) in the cylinder;
d = variable depth of material ($BC = AD$, Fig. 2) in the cylinder;

then by the hypothesis just stated

$$R^2d = R^2D - R_1^2y. \qquad (1)$$

2°.—*The rate of compression along any and all lines parallel to the axis of the primitive central cylinder, and taken outside of that limit, is constant.*

If, then, the material lying outside of the central cylinder be divided into horizontal layers of equal thickness, a very small decrease in the variable depth equal to d (a) will cause the same amount of material to move or flow from each of these layers into the space originally occupied by the central cylinder, thus causing a portion of the material previously resting over the orifice to flow through the latter. If $d(d)$ is the indefinitely small change of depth, and dR_1 the indefinitely small change in the radius of the cylindrical portion resting over the orifice, then the equality of volumes expressing this hypothesis is the following:

$$\pi(R^2 - R_1{}^2) . d(d) = 2\pi R_1 d . dR_1,$$

or

$$\frac{d(d)}{d} = \frac{2R_1 dR_1}{R^2 - R_1{}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

3°.—*The rate of decrease of the radius of the primitive central cylinder is constant throughout its length at any given instant during flow.*

Let r be any radius less than R_1, then if the latter is decreased by the very small amount dR_1, the former will be shortened by the amount dr; and by the last hypothesis there must result

$$\frac{dR_1}{R_1} = \frac{dr}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This is a perfectly general equation, in which r may or may not be the variable value of the radius of that portion of the primitive central cylinder remaining above the orifice at any instant during flow.

These are the three hypotheses on which Tresca based his theory of the flow of solids. It is thus seen to be put upon a purely geometrical basis, entirely independent of the elastic or other properties of the material.

Art. 135.—The Variable Meridian Section of the Primitive Central Cylinder.

The meridian curve *haH*, or *hbK*, Fig. 2 of Art. 133, may now easily be determined.

Eq. (1) of Art. 134 may take the first of the following forms, while its differential, considering d and y variable, may take the second:

$$d = D - \frac{R_1^2}{R^2} y,$$

$$d(d) = -\frac{R_1^2}{R^2} dy.$$

Dividing the second by the first,

$$\frac{d(d)}{d} = \frac{dy}{y - \frac{R^2}{R_1^2} D} = \frac{2 R_1 dR_1}{R^2 - R_1^2}.$$

The last member of this equation is simply eq. (2) of Art. 134; and if the value of dR_1, in eq. (3) of the same article, be inserted in the third member of this equation, there will result

$$\frac{2 R_1^2}{R^2 - R_1^2} \cdot \frac{dr}{r} = \frac{dy}{y - \frac{R^2}{R_1^2} D}.$$

Integrating between the limits of r and R_1, and remembering that r will be restricted to the representation of the radius of that portion of the primitive central cylinder which remains, at any instant, over the orifice, by taking $y = 0$ for $r = R_1$,

$$\frac{2 R_1^2}{R^2 - R_1^2} \log \frac{r}{R_1} = \log \left(\frac{y - \frac{R^2}{R_1^2} D}{- \frac{R^2}{R_1^2} D} \right);$$

"*log*" indicates a Napierian logarithm.

Passing from logarithms to the quantities themselves, and reducing,

$$y = \frac{R^2 D}{R_1^2}\left[1 - \left(\frac{r}{R_1}\right)^{\frac{2R_1^2}{R^2 - R_1^2}} \right] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the desired equation of the line, in which r is measured normal to the axis of the cylinder or jet, while y is measured along that axis from the extremity of the jet. When the material is wholly expelled,

$$y = \frac{R^2}{R_1^2} D, \text{ and } r = 0.$$

Eq. (2) is applicable to the jet only. For the line hF or Gk, resort will be had to the equation

$$\frac{d(d)}{d} = \frac{2R_1^2}{R^2 - R_1^2} \frac{dr}{r}.$$

Again integrating between the limits d and D, or r and R_1, and reducing,

$$r = R_1 \left(\frac{d}{D}\right)^{\frac{R^2 - R_1^2}{2R_1^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This value of r is the radius of that portion of the primitive central cylinder which remains over the orifice when D is reduced to d.

Art. 136.—Positions in the Jet of Horizontal Sections of the Primitive Central Cylinder.

That portion of the primitive central cylinder below *ab*, in Fig. 1 of Art. 133 will be changed to *abKH* in Fig. 2 of the same article.

If, in the latter Fig., y' is the distance from HK to ab, measured along the axis, then the volume of $HKab$ will have the value

$$\int_0^{y'} \pi r^2 dy.$$

If d' is the distance $aF = bG$, in Fig. 1, the equality of volumes will give

$$\int_0^{y'} r^2 dy = R_1^2(D - d').$$

Eq. (1) of Art. 125 gives

$$r_2 = R_1^2\left(1 - \frac{R_1^2 y}{R^2 D}\right)^{\frac{R^2 - R_1^2}{R_1^2}};$$

$$\therefore \int_r r^2 dy = R_1^2 D - R_1^2 D\left(1 - \frac{R_1^2 y'}{R^2 D}\right)^{\frac{R^2}{R_1^2}} = R_1^2(D - d');$$

$$\therefore y' = \frac{R^2}{R_1^2}\left[1 - \left(\frac{d'}{D}\right)^{\frac{R_1^2}{R^2}}\right]D. \quad . \quad . \quad . \quad . \quad (1)$$

If N is the number of horizontal layers required to compose the total thickness D, and n the number in the depth d',

$$\frac{d'}{D} = \frac{n}{N}.$$

Hence

$$y' = \frac{R^2}{R_1^2}\left[1 - \left(\frac{n}{N}\right)^{\frac{R_1}{R^2}}\right]D. \quad . \quad . \quad . \quad . \quad (2)$$

Tresca computed values of y' for some of his experiments and compared the results with actual measurements. The agreement, though not exact, was very satisfactory. Within limits not extreme, the longer the jet the more satisfactory was the agreement.

Art. 137.—Final Radius of a Horizontal Section of the Primitive Central Cylinder.

Let it be required to determine what radius the section situated at the distance d' from the upper surface of the primitive central cylinder will possess in the jet.

It will only be necessary to put for y in eq. (1) of Art. 135 the value of y' taken from eq. (1) of Art. 136. This operation gives

$$\left(\frac{d'}{D}\right)^{\frac{R_1^2}{R^2}} = \left(\frac{r'}{R_1}\right)^{\frac{2R_1^2}{R^2-R_1^2}}.$$

Hence

$$r' = R_1\left(\frac{d'}{D}\right)^{\frac{R^2-R_1^2}{2R^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If R_1 is small, as compared with R, there will result approximately

$$r' = R_1\left(\frac{d'}{D}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Art. 138.—Path of Any Molecule.

The hypotheses on which the theory of flow is based enable the hypothetical path of any molecule to be easily established.

In consequence of the nature of the motion there will be three portions of the path, each of which will be represented by its characteristic equation, as follows:

First, *let the molecule lie outside of the primitive central cylinder.*

Let R' and H be the original co-ordinates of the molecule considered, measured normal to and along the axis of the cylinder, respectively, from the centre of the orifice HK (Fig. 1, Art. 133) as an origin, while r and h are the variable co-ordinates.

The first hypothesis, by which the density remains constant, then gives the following equation:

$$\pi(R^2-R'^2)H=\pi(R^2-r^2)h,$$

or

$$hR^2-hr^2=(R^2-R'^2)H. \quad . \quad . \quad . \quad . \quad (1)$$

This is the equation to the path of the molecule, in which r must always exceed R_1.

As this equation is of the third degree, the curve cannot be one of the conic sections.

Second, *let the molecule move in the space originally occupied by the central cylinder.*

While h and r now vary, the volume $\pi r^2(D-h)$ must remain constant. When $r=R_1$ let $h=h_1$. Hence

$$r^2(D-h)=R_1{}^2(D-h_1). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But if $h=h_1$ and $r=R_1$ in eq. (1),

$$h_1=\left(\frac{R^2-R'^2}{R^2-R_1{}^2}\right)H.$$

Placing this value in eq. (2).

$$r^2(D-h)=R_1{}^2\left(D-H\frac{R^2-R'^2}{R^2-R_1{}^2}\right). \quad . \quad . \quad . \quad (3)$$

Third, *let the molecule move in the jet.*

After the molecule passes the orifice, its path will evidently be a straight line parallel to the axis of the jet. Its distance r_1 from that axis will be found by putting $h = 0$ in eq. (3). Hence

$$r_1 = R_1 \left(1 - \frac{H}{D} \frac{R^2 - R'^2}{R^2 - R_1^2} \right)^{\frac{1}{2}} . \quad . \quad . \quad . \quad . \quad (4)$$

APPENDIX I.

ELEMENTS OF THEORY OF ELASTICITY IN AMORPHOUS SOLID BODIES.

CHAPTER I.

GENERAL EQUATIONS.

Art. 1.—Expressions for Tangential and Direct Stresses in Terms of the Rates of Strains at Any Point of a Homogeneous Body.

LET any portion of material perfectly homogeneous be subjected to any state of stress whatever. At any point as O, Fig. 1, let there be assumed any three rectangular co-ordinate planes; then consider any small rectangular parallelopiped whose faces are parallel to those planes. Finally let the stresses on the three faces nearest the origin be resolved into components normal and parallel to their planes of action, whose directions are parallel to the co-ordinate axis.

The intensities of these tangential and normal components will be represented in the usual manner, i.e., p_{xy} signifies a tangential intensity on a plane normal to the axis of X (plane ZY), whose direction is parallel to the axis of Y, while p_{xx} signifies the intensity of a normal stress on a plane normal to the axis of X (plane ZY) and in the direction of the axis of X. Two unlike subscripts, therefore, indicate a tangential stress, while two of the same kind signify a normal stress.

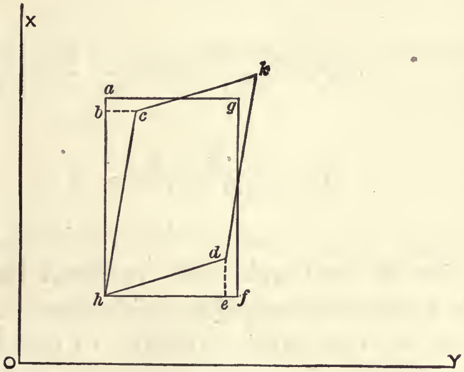


FIG. 1.

From eq. (3), Art. 2, and eq. (7), Art. 5, there is at once deduced

$$S = \frac{E}{2(1+r)}\phi = G\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now when the material is subjected to stress the lines bounding the faces of the parallelopiped will no longer be at right angles to each other. It has already been shown in Art. 2 that the angular changes of the lines from right angles are the characteristic shearing strains, which, multiplied by Gs give the shearing intensities.

Let ϕ_1 be the change of angle of the boundary lines parallel to X and Y.

Let ϕ_2 be the change of angle of the boundary lines parallel to Y and Z.

Let ϕ_3, be the change of angle of the boundary line parallel to Z and X.

Eq. (1) will then give the following three equations:

$$p_{xy} = \frac{E}{2(1+r)}\phi_1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$p_{yz} = \frac{E}{2(1+r)}\phi_2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p_{zx} = \frac{E}{2(1+r)}\phi_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In Fig. 1 let the rectangle *agfh* represent the right projection of the indefinitely small parallelopiped $dx\,dy\,dz$. If u, v, and w are the unit strains parallel to the axes of x, y, and z of the original point h, the rates of variation of strain $\frac{du}{dx}$, $\frac{dv}{dy}$, $\frac{dw}{dz}$, etc., may be considered constant throughout this parallelopiped; consequently the rectangular faces will change to oblique parallelograms. The oblique parallelogram *dhck*, whose diagonals may or may not coincide with those of *agfh*, therefore, may represent the strained condition of the latter figure.

Then, by Art. 2, the difference between *dhc* and the right angle at *h* will represent the strain ϕ_1. But, from Fig. 1, ϕ_1 has the following value:

$$\phi_1 = dhe + bhc. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But the limiting values of the angles in the second member are coincident with their tangents; hence

$$\phi_1 = \frac{de}{dy} + \frac{bc}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

But, again, de is the distortion parallel to OX *found by moving parallel to OY only;* hence it is a partial differential of u, or it has the value

$$de = \frac{du}{dy} dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In precisely the same manner bc is the partial differential of v in respect to x, or

$$bc = \frac{dv}{dx} dx.$$

By the aid of these considerations, eq. (6) takes the form

$$\phi_1 = \frac{du}{dy} + \frac{dv}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If XY be changed to YZ, and then to ZX, there may be at once written by the aid of eq. (8)

$$\phi_2 = \frac{dv}{dz} + \frac{dw}{dy}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\phi_3 = \frac{dw}{dx} + \frac{du}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eqs. (2), (3), and (4) now take the following form:

$$p_{xy} = G\left(\frac{du}{dy} + \frac{dv}{dx}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$p_{yz} = G\left(\frac{dv}{dz} + \frac{dw}{dy}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$p_{zx} = G\left(\frac{dw}{dx} + \frac{du}{dz}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The direct stresses are next to be given in terms of the displacements u, v, and w. Again, let the rectangular parallelopiped $dx\ dy\ dz$ be considered. Eq. (1), on page 3, shows that the strain per unit of length is found by dividing the intensity of stress by the coefficient of elasticity, *if a single stress only exists.* But in the present instance, any state of stress whatever is supposed. Consequently the strain caused by p_{xx}, for example, acting alone must be combined with the lateral strains induced by p_{yy} and p_{zz}. Denoting the actual rates of strain along the axes of X, Y, and Z by l_1, l_2, and l_3, therefore, the following equations may be at once written by the aid of the principles given on pages 9 and 10:

$$\frac{p_{xx}}{E}=l_1+(p_{yy}+p_{zz})\frac{r}{E}; \quad . \quad . \quad . \quad . \quad (14)$$

$$\frac{p_{yy}}{E}=l_2+(p_{xx}+p_{zz})\frac{r}{E}; \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{p_{zz}}{E}=l_3+(p_{yy}+p_{xx})\frac{r}{E}. \quad . \quad . \quad . \quad . \quad (16)$$

Eliminating between these three equations,

$$p_{xx}=\frac{E}{1+r}\left[l_1+\frac{r}{1-2r}(l_1+l_2+l_3)\right]; \quad . \quad . \quad (17)$$

$$p_{yy}=\frac{E}{1+r}\left[l_2+\frac{r}{1-2r}(l_1+l_2+l_3)\right]; \quad . \quad . \quad (18)$$

$$p_{zz}=\frac{E}{1+r}\left[l_3+\frac{r}{1-2r}(l_1+l_2+l_3)\right]. \quad . \quad . \quad (19)$$

But if u, v, and w are the actual strains at the point where these stresses exist, the rates of strain l_1, l_2, and l_3 will evidently be equal to $\frac{du}{dx}$, $\frac{dv}{dy}$, and $\frac{dw}{dz}$, respectively. The volume of the parallelopiped will be changed by those strains to

$$dx(1+l_1)dy(1+l_2)dz(1+l_3)=dx\,dy\,dz(1+l_1+l_2+l_3)$$

if powers of l_1, l_2, and l_3 above the first be omitted. The quantity $(l_1+l_2+l_3)$ is, then, *the rate of variation of volume*, or *the amount of variation of volume for a cubic unit.* If there be put

$$\theta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}, \quad \text{and} \quad G=\frac{E}{2(1+r)},$$

eqs. (17), (18), and (19) will take the forms

$$p_{xx}=\frac{2Gr}{1-2r}\theta+2G\frac{du}{dx}; \quad . \quad . \quad . \quad . \quad (20)$$

$$p_{yy}=\frac{2Gr}{1-2r}\theta+2G\frac{dv}{dy}; \quad . \quad . \quad . \quad . \quad (21)$$

$$p_{zz}=\frac{2Gr}{1-2r}\theta+2G\frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad (22)$$

The form in which eqs. (14), (15), and (16) are written shows that if p_{xx}, p_{yy}, or p_{zz} is positive, the stress is tension, and compression if it is negative. Consequently a positive value for any of the intensities in eqs. (20), (21), or (22) will indicate a tensile stress, while a negative value will show the stress to be compressive.

The eqs. (14) to (19), together with the elimination involved, also show that the coefficients of elasticity for tension and compression have been taken equal to each other, and that the ratio r is the same for tensile and compressive strains.

Further, in eqs. (11), (12), and (13), it has been assumed that G is the same for all planes.

Hence eqs. (11,) (12), (13), (20), (21), and (22) *apply only to bodies perfectly homogeneous in all directions.*

It is to be observed that the co-ordinate axes have been taken perfectly arbitrarily.

Art. 2.—General Equations of Internal Motion and Equilibrium.

In establishing the general equations of motion and equilibrium, the principles of dynamics and statics are to be applied to the forces which act upon the parallelopiped represented in Fig. 1, the edges of which are dx, dy, and dz. The notation to be used for the intensities of the stresses acting on the different faces will be the same as that used in the preceding article.

Let the stresses which act on the faces nearest the origin be considered negative, while those which act on the other three faces are taken as positive.

The stresses which act in the direction of the axis of X are the following:

On the face normal to X, nearest to O, $-p_{xx}\,dy\,dz$;

" " " " " " farthest from O, $\left(p_{xx}+\frac{dp_{xx}}{dx}dx\right)dy\,dz$;

" " " $dy\,dx$ nearest to O, $-p_{zx}\,dy\,dx$;

" " " " " farthest from O, $\left(p_{zx}+\frac{dp_{zx}}{dz}dz\right)dy\,dx$;

" " " $dz\,dx$ nearest to O, $-p_{yx}\,dz\,dx$;

" " " " " farthest from O. $\left(p_{yx}+\frac{dp_{yx}}{dy}dy\right)dz\,dx$.

The differential coefficients of the intensities are the rates of variation of those intensities for each unit of the variable, which, multiplied by the differentials of the variables, give the amounts of variation for the different edges of the parallelopiped.

Let X_0 be the external force acting in the direction of X on a unit of volume at the point considered; then $X_0\, dx\, dy\, dz$ will be the amount of external force acting on the parallelopiped.

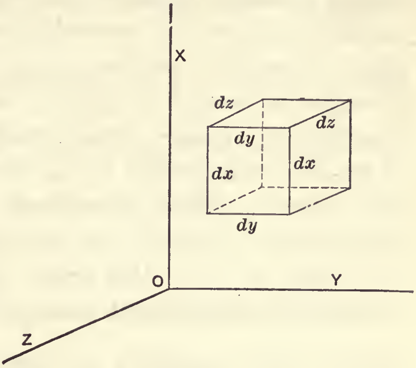


FIG. 1.

These constitute all the forces acting on the parallelopiped in the direction of the axis of X, and their sum, if unbalanced, must be equal to $m\frac{d^2u}{dt^2}dx\, dy\, dz$; in which m is the mass or inertia of a unit of volume, and dt the differential of the time. Forming such an equation, therefore, and dropping the common factor $dx\, dy\, dz$, there will result

$$\frac{dp_{xx}}{dx}+\frac{dp_{yx}}{dy}+\frac{dp_{zx}}{dz}+X_0=m\frac{d^2u}{dt^2}. \quad . \quad . \quad . \quad (1)$$

Changing x to y, y to z, and z to x, eq. (1) will become

$$\frac{dp_{xy}}{dx}+\frac{dp_{yy}}{dy}+\frac{dp_{zy}}{dz}+Y_0=m\frac{d^2v}{dt^2}. \quad . \quad . \quad . \quad (2)$$

Again, in eq. (1), changing x to z, z to y, and y to x,

$$\frac{dp_{xz}}{dx}+\frac{dp_{yz}}{dy}+\frac{dp_{zz}}{dz}+Z_0=m\frac{d^2w}{dt^2}. \quad . \quad . \quad . \quad (3)$$

The line of action of the resultant of all the forces which act on the indefinitely small parallelopiped, at its limit, passes through its centre of gravity, *consequently it is subjected to the action of no unbalanced moment.* The parallelopiped, therefore, can have no rotation about an axis passing through its centre of gravity, whether it be in motion or equilibrium. Hence, let an axis passing through its centre of gravity and parallel to the axis of X, be considered. The only stresses, which, from their direction can possibly have moments about that axis, are those with the subscripts (*yz*), (*zy*), (*yy*), or (*zz*). But those with the last two subscripts act directly through the centre of the parallelopiped, consequently their moments are zero. The stresses $\frac{dp_{yz}}{dy}dy \, . \, dx \, dz$ and $\frac{dp_{zy}}{dz} dz \, . \, dx \, dy$ are two of six forces whose resultant is directly opposed to the resultant of those three forces which represent the increase of the intensities of the normal, or direct, stresses on three of the faces of the parallelopiped; these, therefore, have no moments about the assumed axis. The only stresses remaining are those whose intensities are p_{zy} and p_{yz}. The resultant moment, which must be equal to zero, then, has the following value:

$$p_{yz} dx \, dz \, . \, dy + p_{zy} dx \, dy \, . \, dz = 0; \quad . \quad . \quad . \quad (4)$$

$$\therefore \; p_{yz} = - p_{zy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Hence the two intensities are equal to each other.

The negative sign in eq. (5) simply indicates that their *moments* have opposite signs or directions; consequently, that the shears themselves, on adjacent faces, act toward or from the edge between those faces. In eqs. (1), (2), and (3), the tangential stresses, or shears, are all to be affected by the same sign, since direct, or normal, stresses only can have different signs.

The eq. (5) is perfectly general, hence there may be written:

$$p_{xy}=p_{yx}, \text{ and } p_{zx}=p_{xz}. \quad . \quad . \quad . \quad . \quad (6)$$

Adopting the notation of Lamé, there may be written:

$$p_{xx}=N_1, \quad p_{yy}=N_2, \quad p_{zz}=N_3;$$
$$p_{zy}=T_1, \quad p_{xz}=T_2, \quad p_{xy}=T_3;$$

by which eqs. (1), (2), and (3) take the following forms:

$$\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}+X_0=m\frac{d^2u}{dt^2}; \quad . \quad . \quad . \quad (7)$$

$$\frac{dT_3}{dx}+\frac{dN_2}{dy}+\frac{dT_1}{dz}+Y_0=m\frac{d^2v}{dt^2}; \quad . \quad . \quad . \quad (8)$$

$$\frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dN_3}{dz}+Z_0=m\frac{d^2w}{dt^2}. \quad . \quad . \quad . \quad (9)$$

The equations (11), (12), (13), (20), (21), and (22) of the preceding article are really kinematical in nature; in order that the principles of dynamics may hold, they must satisfy eqs. (7), (8), and (9). As the latter stand, by themselves, they are applicable to rigid bodies as well as elastic ones; but when the values of N and T, in terms of the strains u, v, and w, have been inserted, they are restricted, in their use, to elastic bodies only. With those values so inserted, they form the equations on which are based the mathematical theory of sound and light vibrations, as well as those of elastic rods, membranes, etc. In general, they are the equations of motion which the different parts of the body can have in reference to each other, in consequence of the elastic nature of the material of which the body is composed.

If all parts of the body are in equilibrium under the action of the internal stresses, the rates of variation of the strains $\frac{d^2u}{dt^2}$, $\frac{d^2v}{dt^2}$, and $\frac{d^2w}{dt^2}$, will each be equal to zero. Hence, eqs. (7), (8), and (9) will take the forms

$$\frac{dN_1}{dx}+\frac{dT_3}{dy}+\frac{dT_2}{dz}+X_0=0; \quad . \quad . \quad . \quad . \quad (10)$$

$$\frac{dT_3}{dx}+\frac{dN_2}{dy}+\frac{dT_1}{dz}+Y_0=0; \quad . \quad . \quad . \quad . \quad (11)$$

$$\frac{dT_2}{dx}+\frac{dT_1}{dy}+\frac{dN_3}{dz}+Z_0=0. \quad . \quad . \quad . \quad . \quad (12)$$

These are the general equations of equilibrium. As they stand, they apply to a rigid body. For an elastic body, the values of N and T from the preceding article, in terms of the strains u, v, and w, must satisfy these equations.

The eqs. (10), (11), and (12) express the three conditions of equilibrium that the sums of the forces acting on the small parallelopiped, taken in three rectangular co-ordinate directions, must each be equal to zero. The other three conditions, indicating that the three component moments about the same co-ordinate axes must each be equal to zero, are fulfilled by eqs. (5) and (6). The latter conditions really eliminate three of the nine unknown stresses. The remaining six consequently appear in both the equations of motion and equilibrium.

The equations (7) to (12), inclusive, belong to the interior of the body. At the exterior surface, only a portion of the small parallelopiped will exist, and that portion will be a tetrahedron, the base of which forms a part of the exterior surface of the body, and is acted upon by external forces. Let $\frac{da}{2}$ be the area of the base of this tetrahedron, and let p, q, and r be the angles which a normal to it forms with the three axes of X, Y, Z, respectively. Then will

$$da \cos p = dy\, dz,\ da \cos q = dz\, dx, \text{ and } da \cos r = dx\, dy.$$

Let P be the known intensity of the external force acting on da, and let π, χ, and ρ be the angles which its direction makes with the co-ordinate axes. Then there will result:

$$X_0 = P\, da . \cos \pi,\ Y_0 = P\, da . \cos \chi, \text{ and } Z_0 = P\, da . \cos \rho.$$

The origin is now supposed to be so taken that the apex of the tetrahedron is located between it and the base; hence that part of the parallelopiped in which acted the stresses involving the derivatives, or differential coefficients, is wanting; consequently those stresses are also wanting.

The sums of the forces, then, which act on the tetrahedron, in the co-ordinate directions, are the following:

$$\begin{aligned} -(N_1\, dy\, dz + T_3\, dz\, dx + T_2\, dy\, dx) + Pda \cos \pi = 0; \\ -(T_3\, dz\, dy + N_2\, dz\, dx + T_1\, dy\, dx) + Pda \cos \chi = 0; \\ -(T_2\, dz\, dy + T_1\, dz\, dx + N_3\, dy\, dx) + Pda \cos \rho = 0. \end{aligned}$$

Substituting from above,

$$N_1 \cos p + T_3 \cos q + T_2 \cos r = P \cos \pi; \quad . \quad . \quad (13)$$

$$T_3 \cos p + N_2 \cos q + T_1 \cos r = P \cos \chi; \quad . \quad . \quad (14)$$

$$T_2 \cos p + T_1 \cos q + N_3 \cos r = P \cos \rho. \quad . \quad . \quad (15)$$

These equations must always be satisfied at the exterior surface of the body; and since the external forces must always be known, in order that a problem may be determinate, they will serve to determine constants which arise from the integration of the general equations of motion and equilibrium.

Art. 3.—Equations of Motion and Equilibrium in Semi-polar Co-ordinates.

For many purposes it is convenient to have the conditions of motion and equilibrium expressed in either semi-polar or polar co-ordinates; the first form of such expression will be given in this article.

The general analytical method of transformation of co-ordinates may be applied to the equations of the preceding article, but the direct treatment of an indefinitely small portion of the material, limited by co-ordinate surfaces, possesses many advantages. In Fig. 1 are shown both the small portion of material and the co-ordinates, semi-polar as well as rectangular. The angle made by a plane normal to ZY, and containing OX, with the plane XY is represented by ϕ; the distance of any point from OX, measured parallel to ZY, is called r; the third co-ordinate, normal to

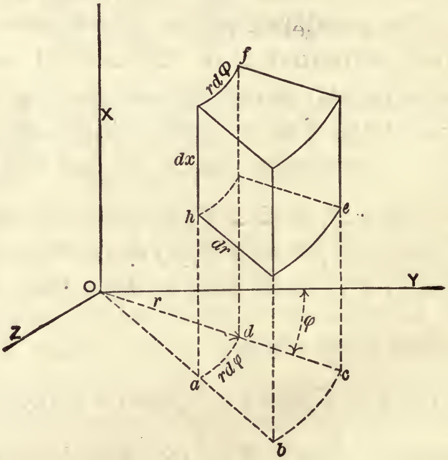


FIG. 1.

r and ϕ, is the co-ordinate x, as before. It is important to observe that the co-ordinates x, r, and ϕ, at any point, are *rectangular*.

The indefinitely small portion of material to be considered will, as shown in Fig. 1, be limited by the edges dx, dr, and $r\,d\phi$. The faces $dx\,dr$ are inclined to each other at the angle $d\phi$.

The intensities of the normal stresses in the directions of X and r will be indicated by N_1 and R, respectively. The remainder of the notation will be of the same general character as that in the preceding article; i.e., T_{xr} will represent a shear on the face $dr\,.\,r\,d\phi$ in the direction of r, while $N_{\phi\phi}$ is a normal stress, in the direction of ϕ, on the face $dx\,dr$.

The strains or displacements, in the directions of x, r, and ϕ, will be represented by u, ρ, and w; consequently the unbalanced forces in those directions, per unit of mass, will be

$$m\frac{d^2u}{dt^2}, \quad m\frac{d^2\rho}{dt^2}, \quad \text{and} \quad m\frac{d^2w}{dt^2}. \quad . \quad . \quad . \quad . \qquad (1)$$

Those forces acting on the faces hf, fe, and he, will be considered negative; those acting on the other faces, positive.

Forces Acting in the Direction of r.

$-R\,.\,r\,d\phi\,dx$, and

$$+Rr\,d\phi\,dx + \left(\frac{d(Rr)}{dr}dr = r\frac{dR}{dr}dr + R\,dr\right)d\phi\,dx.$$

$-T_{\phi r}dr\,dx$, and

$$+T_{\phi r}dr\,dx + \frac{dT_{\phi r}}{d\phi}d\phi\,.\,dr\,dx.$$

$-T_{xr}\,.\,r\,d\phi\,dr$, and

$$+T_{xr}\,.\,r\,d\phi\,dr + \frac{dT_{xr}}{dx}dx\,.\,r\,d\phi\,dr.$$

On the face $dr\,dx$, nearest to ZOX, there acts the normal stress $\left(N_{\phi\phi}dr\,dx+\frac{dN_{\phi\phi}}{d\phi}d\phi\,.\,dr\,dx\right)=N'$; and N' has a component acting parallel to the face *fe* and toward OX, equal to $N'\sin(d\phi)=N'\frac{r\,d\phi}{r}=N'd\phi$. But the second term of this product will hold $(d\phi)^2$, hence it will disappear, at the limit, in the first derivative of $N'd\phi$ $\therefore$ $N'd\phi=N_{\phi\phi}d\phi\,dr\,dx$. Since this force must be taken as acting toward OX, it acts with the normal forces on *hf*, and, consequently, must be given the negative sign.

If R_0 is the external force acting on a unit of volume, another force (external) acting along r will be $R_0\,.\,r\,d\phi\,dr\,dx$.

The sum of all these forces will be equal to

$$m\,.\,r\,d\phi\,dr\,dx\,.\,\frac{d^2\rho}{dt^2}.$$

Forces Acting in the Direction of ϕ.

$$-N_{\phi\phi}dr\,dx, \text{ and}$$

$$+N_{\phi\phi}dr\,dx+\frac{dN_{\phi\phi}}{d\phi}d\phi\,.\,dr\,dx.$$

$$-T_{r\phi}\,.\,r\,d\phi\,dx, \text{ and}$$

$$+T_{r\phi}\,.\,r\,d\phi\,dx+\left(\frac{d(rT_{r\phi})}{dr}dr=r\frac{dT_{r\phi}}{dr}dr+T_{r\phi}dr\right)d\phi\,dx.$$

$$-T_{x\phi}\,.\,r\,d\phi\,dr, \text{ and}$$

$$+T_{x\phi}\,.\,r\,d\phi\,dr+\frac{dT_{x\phi}}{dx}dx\,.\,r\,d\phi\,dr.$$

As in the case of $N_{\phi\phi}$, in connection with the forces along r, so the force $T_{\phi r}\,dr\,dx$ has a component along ϕ (normal to *fe*) equal to $T_{\phi r}\,dr\,dx\,.\,\sin(d\phi)=T_{\phi r}\,d\phi\,dr\,dx$. It will have a positive sign, because it acts from OX.

The external force is $\Phi_0\,.\,r\,d\phi\,dr\,dx$.

Forces Acting in the Direction of x.

$-N_1 . r\, d\phi\, dr$, and

$+N_1 r\, d\phi\, dr + \frac{dN_1}{dx} dx . r\, d\phi\, dr.$

$-T_{rx} . dx\, r\, d\phi$, and

$+T_{rx} . dx\, r\, d\phi + \left(\frac{d(rT_{rx})}{dr} dr = r\frac{dT_{rx}}{dr} dr + T_{rx}\, dr\right) dx\, d\phi.$

$-T_{\phi x} dx\, dr$, and

$+T_{\phi x} dx\, dr + \frac{dT_{\phi x}}{d\phi} d\phi . dx\, dr.$

The external force is $X_0 . r\, d\phi\, dx\, dr$.

Putting each of these three sums equal to the proper rates of variation of momentum, and dropping the common factor $r\, d\phi\, dx\, dr$:

$$\frac{dN_1}{dx} + \frac{dT_{rx}}{dr} + \frac{dT_{\phi x}}{r\, d\phi} + \frac{T_{rx}}{r} \qquad + X_0 = m\frac{d^2u}{dt^2}; \quad . \quad (2)$$

$$\frac{dT_{xr}}{dx} + \frac{dR}{dr} + \frac{dT_{\phi r}}{r\, d\phi} + \frac{R - N_{\phi\phi}}{r} + R_0 = m\frac{d^2\rho}{dt^2}; \quad . \quad (3)$$

$$\frac{dT_{x\phi}}{dx} + \frac{dT_{r\phi}}{dr} + \frac{dN_{\phi\phi}}{r\, d\phi} + \frac{T_{r\phi} + T_{r\phi}}{r} + \Phi_0 = m\frac{d^2w}{dt^2}. \quad . \quad (4)$$

These are the general equations of motion (vibration) in terms of semi-polar co-ordinates; if the second members are made equal to zero, they become equations of equilibrium. Eqs. (2), (3), and (4), are not dependent upon the nature of the body.

Since x, r, and ϕ are rectangular, it at once follows that

$$T_{rx} = T_{xr}, \ T_{r\phi} = T_{\phi r}, \text{ and } T_{x\phi} = T_{\phi x}. \quad . \quad . \quad (5)$$

In order that eqs. (2), (3), and (4) may be restricted to elastic bodies, it is necessary to express the six intensities of stresses involved, in terms of the rates of variation of the strains in the rectangular co-ordinate directions of x, r, and ϕ. Since these co-ordinates are rectangular, the eqs. (11), (12), (13), (20), (21), and (22) of Article 1, may be made applicable to the present case by some very simple changes dependent upon the nature of semi-polar co-ordinates.

For the present purpose the strains in the co-ordinate directions of x, y, and z will be represented by u', v', and w'. Since the axis of x remains the same in the two systems, evidently

$$\frac{du'}{dx}=\frac{du}{dx}.$$

From Fig. 1 it is clear that the axis of y corresponds exactly to the co-ordinate direction r; hence

$$\frac{dv'}{dy}=\frac{d\rho}{dr}.$$

From the same Fig. it is seen that the axis of z corresponds to ϕ, or $r\phi$. But the total differential, dw', must be considered as made up of two parts; consequently the rate of variation $\frac{dw'}{dz}$ will consist of two parts also. If there is no distortion in the direction of r, or if the distance of a molecule from the origin remains the same, one part will be $\frac{dw}{d(r\phi)}=\frac{dw}{r\,d\phi}$. If, however, a unit's length of material be removed from the distance r to $r+\rho$ from the centre O, Fig. 1, while ϕ remains constant, its length will be changed from 1 to $\left(1+\frac{\rho}{r}\right)$, in which ρ may be implicitly positive or negative. Consequently there will result

$$\frac{dw'}{dz} = \frac{dw}{r\,d\phi} + \frac{\rho}{r}.$$

For the reason already given, there follow

$$\frac{du'}{dy} = \frac{du}{dr} \quad \text{and} \quad \frac{dv'}{dx} = \frac{d\rho}{dx}.$$

In Fig. 2 let *dc* be the side of a distorted small portion of the material, the original position of which was *d′e*. *Od* is the distance *r* from the origin, $ad = dr$ and $ac = dw$, while $dd' = w$. The angular change in position of *dc* is $\frac{ac}{ad} = \frac{dw}{dr}$; but an amount equal to $\frac{ab}{ad} = \frac{w}{r}$ is due to the movement of *r*, and is not a movement of *dc* relatively to the material immediately adjacent to *d*.

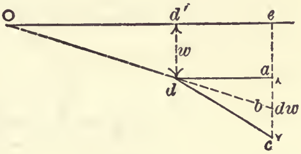


FIG. 2.

Hence

$$\frac{dw'}{dy} = \frac{dw}{dr} - \frac{w}{r}, \quad \text{also} \quad \frac{dv'}{dz} = \frac{d\rho}{r\,d\phi}.$$

There only remain the following two, which may be at once written

$$\frac{dw'}{dx} = \frac{dw}{dx} \quad \text{and} \quad \frac{du'}{dz} = \frac{du}{r\,d\phi}.$$

The rate of variation of volume takes the following form in terms of the new co-ordinates:

$$\theta = \frac{du'}{dx} + \frac{dv'}{dy} + \frac{dw'}{dz} = \frac{du}{dx} + \frac{d\rho}{dr} + \frac{dw}{r\,d\phi} + \frac{\rho}{r}. \quad . \quad . \quad (6)$$

Accenting the intensities which belong to the rectangular system x, y, z, the eqs. (11), (12), (13), (20), (21), and (22), of Art. 1, take the following form:

$$N_1 = N_1' = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{du}{dx}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$R = N_2' = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$N_{\phi\phi} = N_3' = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\left(\frac{dw}{r\,d\phi} + \frac{\rho}{r}\right); \quad . \quad . \quad (9)$$

$$T_{xr} = T_3' = G\left(\frac{du}{dr} + \frac{d\rho}{dx}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$T_{r\phi} = T_1' = G\left(\frac{d\rho}{r\,d\phi} + \frac{dw}{dr} - \frac{w}{r}\right); \quad . \quad . \quad . \quad . \quad (11)$$

$$T_{\phi x} = T_2' = G\left(\frac{dw}{dx} + \frac{du}{r\,d\phi}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

If these values are introduced in eqs. (2), (3), and (4), those equations will be restricted in application to bodies of homogeneous elasticity only.

The notation $\mathfrak{r}$ is used to indicate that the r involved is the ratio of lateral to direct strain, and that it has no relation whatever to the co-ordinate r.

The limiting equations of condition, (13), (14), and (15) of Art. 2, remain the same, except for the changes of notation, shown in eqs. (7) to (12), for the intensities N and T.

Art. 4.—Equations of Motion and Equilibrium in Polar Co-ordinates.

The relation, in space, existing between the polar and rectangular systems of co-ordinates is shown in Fig. 1. The angle ϕ is measured in the plane ZY and from that of XY; while ψ is measured normal to ZY in a plane which contains OX. The analytical relation existing between the two systems is, then, the following:

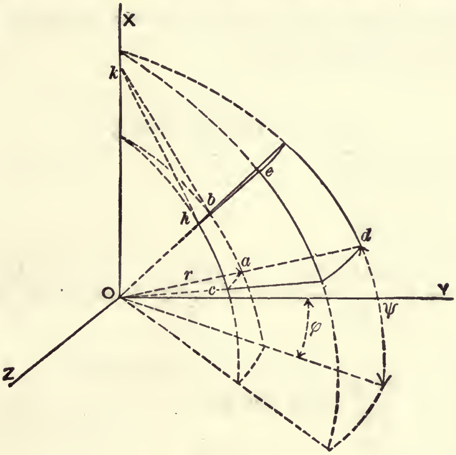


FIG. 1.

$$x = r \sin \psi, \quad y = r \cos \psi \cos \phi, \quad \text{and} \quad z = r \cos \psi \sin \phi.$$

The indefinitely small portion of material to be considered is $a\ h\ e\ d$. It is limited by the co-ordinate planes located by ϕ and ψ, and concentric spherical surfaces with radii r and $r + dr$. The directions r, ϕ, and ψ, at any point, are rectangular; hence the sums of the forces acting on the small portion of the material, taken in these directions, must be found and put equal to

$$m\frac{d^2\rho}{dt^2}, \quad m\frac{d^2\eta}{dt^2}, \quad \text{and} \quad m\frac{d^2\omega}{dt^2},$$

in which expressions, ρ, η, and ω represent the strains in the direction of r, ϕ, and ψ respectively.

Those forces which act on the faces *ah*, *bd*, and *cd* will be considered negative, and those which act on the other faces positive.

The notation will remain the same as in the preceding articles, except that the three normal stresses will be indicated by N_r, N_ϕ, and N_ψ.

Forces Acting Along r.

$$-N_r . r\,d\psi\,r\cos\psi\,d\phi.$$

$$+N_r . r^2\cos\psi\,d\psi\,d\phi$$

$$+\left(\frac{d(N_r r^2)}{dr}dr = r^2\frac{dN_r}{dr}dr + 2r\,N_r\,dr\right)\cos\psi\,d\psi\,d\phi.$$

$$-T_{\phi r} . r\,d\psi\,dr.$$

$$+T_{\phi r} . r\,d\psi\,dr + \frac{dT_{\phi r}}{d\phi}d\phi . r\,d\psi\,dr.$$

$$-T_{\psi r} . r\cos\psi\,d\phi\,dr.$$

$$+T_{\psi r} . r\cos\psi\,d\phi\,dr$$

$$+\left(\frac{d(T_{\psi r}\cos\psi)}{d\psi}d\psi = \cos\psi\frac{dT_{\psi r}}{d\psi}d\psi - T_{\psi r}\sin\psi\,d\psi\right)r\,d\phi\,dr.$$

$-N_\phi . r\,d\psi\,dr . \sin aOc = -N_\phi . r\,d\psi\,dr . \cos\psi\,d\phi$, on face *ce*.

$-N_\psi . r\cos\psi\,d\phi\,dr . \sin aOb = -N_\psi . r\cos\psi\,d\phi\,dr . d\psi$, on face *be*.

Forces Acting Along φ.

$$-T_{r\phi} . r\cos\psi\,d\phi\,r\,d\psi.$$

$$+T_{r\phi} . r^2\cos\psi\,d\phi\,d\psi$$

$$+\left(\frac{d(T_{r\phi}r^2)}{dr}dr = r^2\frac{dT_{r\phi}}{dr}dr + 2r\,T_{r\phi}\,dr\right)\cos\psi\,d\psi\,d\phi.$$

$-N_\phi . r\, d\psi\, dr.$

$+N_\phi . r\, d\psi\, dr + \dfrac{dN_\phi}{d\phi} d\phi\, r\, d\psi\, dr.$

$-T_{\psi\phi} . r \cos\, \psi\, d\phi\, dr.$

$+T_{\psi\phi} \cos\, \psi . r\, d\phi\, dr$

$$+\left(\frac{d(T_{\psi\phi} \cos\, \psi)}{d\psi} d\psi = \cos\, \psi \frac{dT_{\psi\phi}}{d\psi} d\psi - T_{\psi\phi} \sin\, \psi\, d\psi\right) r\, d\phi\, dr.$$

$+T_{\phi r}\, r\, d\psi\, dr . \cos\, \psi\, d\phi,$ on face *ce*.

$-T_{\phi\psi}\, r\, d\psi\, dr\left(\sin\, akc = \dfrac{r \cos\, \psi\, d\phi}{r \cot\, \psi}\right) = -T_{\phi\psi}\, r\, d\psi\, dr . \sin\, \psi\, d\phi,$

on face *ce*.

The lines *ak* and *ck* are drawn normal to *Oc* and *Oa*.

Forces Acting Along ψ.

$-T_{r\psi} . r \cos\, \psi\, d\phi . r\, d\psi.$

$+T_{r\psi}\, r^2 \cos\, \psi\, d\phi\, d\psi$

$$+\left(\frac{d(T_{r\psi} r^2)}{dr} dr = r^2 \frac{dT_{r\psi}}{dr} dr + 2r\, T_{r\psi}\, dr\right) \cos\, \psi\, d\psi\, d\phi.$$

$-T_{\phi\psi} . r\, d\psi\, dr.$

$+T_{\phi\psi}\, r\, d\psi\, dr + \dfrac{dT_{\phi\psi}}{d\phi} d\phi . r\, d\psi\, dr.$

$-N_\psi . r \cos\, \psi\, d\phi\, dr.$

$+N_\psi . r \cos\, \psi\, d\phi\, dr$

$$+\left(\frac{d\, (N_\psi \cos\, \psi)}{d\psi} d\psi = \cos\, \psi \frac{dN_\psi}{d\psi} d\psi - N_\psi \sin\, \psi\, d\psi\right) r\, d\phi\, dr.$$

$+T_{\psi r} . r \cos\, \psi\, d\phi\, dr . d\psi,$ on face *be*.

$+ N_\phi . r\, d\psi\, dr . \sin\, akc = + N_\phi\, r\, d\psi\, dr . \sin\, \psi\, d\phi,$ on face *ce*.

The volume of the indefinitely small portion of the material is (omitting second powers of indefinitely small quantities)

$$r \cos \psi \, d\phi . r \, d\psi . dr = \Delta V,$$

and its mass is m multiplied by this small volume. The latter may be made a common factor in each of the three sums to be taken.

The external forces acting in the directions R, ϕ, and ψ will be represented by

$$R_0 \Delta V, \quad \Phi_0 \Delta V, \quad \text{and} \quad \Psi_0 \Delta V,$$

respectively.

Taking each of the three sums, already mentioned, and dropping the common factor ΔV, there will result

$$\frac{dN_r}{dr} + \frac{dT_{\phi r}}{r \cos \psi . d\phi} + \frac{dT_{\psi r}}{r \, d\psi} + \frac{2N_r - N_\phi - N_\psi - T_{\psi r} \tan \psi}{r} + R_0 = m\frac{d^2 \rho}{dt^2}; \quad (1)$$

$$\frac{dT_{r\phi}}{dr} + \frac{dN_\phi}{r \cos \psi . d\phi} + \frac{dT_{\psi\phi}}{r \, d\psi} + \frac{2T_{r\phi} + T_{\phi r} - T_{\psi\phi} \tan \psi - T_{\phi\psi} \tan \psi}{r} + \Phi_0 = m\frac{d^2 \eta}{dt^2}; \quad (2)$$

$$\frac{dT_{r\psi}}{dr} + \frac{dT_{\phi\psi}}{r \cos \psi \, d\phi} + \frac{dN_\psi}{r \, d\psi} + \frac{2T_{r\psi} + T_{\psi r} - N_\psi \tan \psi + N_\phi \tan \psi}{r} + \Psi_0 = m\frac{d^2 \omega}{dt^2}. \quad (3)$$

Since r, ϕ, and ψ are rectangular at any point,

$$T_{\phi r} = T_{r\phi}, \quad T_{r\psi} = T_{\psi r}, \quad \text{and} \quad T_{\psi\phi} = T_{\phi\psi}.$$

Hence

$$\frac{2T_{r\phi}+T_{\phi r}-\tan\psi(T_{\psi\phi}+T_{\phi\psi})}{r}=\frac{3T_{r\phi}-2\tan\psi\,.\,T_{\psi\phi}}{r},$$

$$\frac{2T_{r\psi}+T_{\psi r}-\tan\psi(N_{\psi}-N_{\phi})}{r}=\frac{3T_{r\psi}-\tan\psi(N_{\psi}-N_{\phi})}{r},$$

These relations somewhat simplify the first members of eqs. (2) and (3).

Eqs. (1), (2), and (3) are entirely independent of the nature of the material; also, they apply to the case of equilibrium, if the second members are made equal to zero.

The rectangular rates of strain, at any point, in terms of r, ϕ, and ψ are next to be found. As in the preceding article, the rates of strain in the rectangular directions of r, ϕ, and ψ will be indicated by

$$\frac{dv'}{dy'},\ \frac{dw'}{dz'},\ \frac{du'}{dx'},\ \frac{dv'}{dx'},\ \frac{du'}{dy'},\ \text{etc.}$$

Remembering the reasoning in connection with the value of $\frac{dw'}{dz}$, in the preceding article, and attentively considering Fig. 1, there may at once be written,

$$\frac{du'}{dx'}=\frac{d\omega}{r\,d\psi}+\frac{\rho}{r}.$$

In Fig. 1, if $ac=1$ and $ab=\omega$, while $ak=r\cot.\psi$ (ak is perpendicular to aO), the difference in length between ac and bh will be

$$-\frac{\omega}{r\cot\psi}=-\frac{\omega\tan\psi}{r}.$$

This expression is negative because a *decrease* in length takes place in consequence of a movement in the *positive* direction of $r\psi$.

Again, a consideration of Fig. 1, and the reasoning connected with the equation above, will give

$$\frac{dw'}{dz'}=\frac{d\eta}{r\cos\psi\, d\phi}+\frac{\rho}{r}-\frac{\omega\tan\psi}{r}.$$

Without explanation there may at once be written:

$$\frac{dv'}{dy'}=\frac{d\rho}{dr}.$$

Fig. 1 of this, and Fig. 2 of the preceding article, give

$$\frac{du'}{dy'}=\frac{d\omega}{dr}-\frac{\omega}{r}\quad\text{and}\quad\frac{dv'}{dx'}=\frac{d\rho}{r\,d\psi}.$$

These are to be used in the expression for $T_{\psi r}$. Precisely the same figures and method give

$$\frac{dv'}{dz'}=\frac{d\rho}{r\cos\psi\, d\phi}\quad\text{and}\quad\frac{dw'}{dy'}=\frac{d\eta}{dr}-\frac{\eta}{r},$$

which are to be used in finding $T_{\phi r}$.

The expression for $\frac{dw'}{dx'}$ will be composed of the *sum* of two parts. In Fig. 2, *ab* is the original position of $r\,d\psi$, and after the strain η exists it takes the position *ec*. Consequently *ac* (equal and parallel to *bd* and perpendicular to *ak*) represents the strain η, while *ed* represents $d\eta$. Since, also, *fc* is perpendicular to *ck*, the strains of the kind η change the right angle *fck* to the angle *fce*; or the angle *eck* is equal to

$$\frac{dw'}{dx'}=ecd+dck=\frac{ed}{dc}+\frac{ca}{ak}$$

$$=\frac{d\eta}{r\,d\psi}+\frac{\eta}{r\cot\psi}.$$

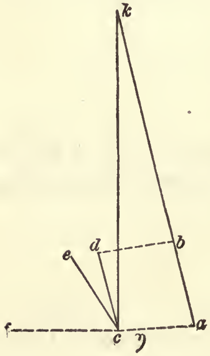


FIG. 2.

In Fig. 2, the points *a*, *b*, and *k* are identical with the points similarly lettered in Fig. 1. The expression for $\frac{du'}{dz'}$ may be at once written from Fig. 1. There may, then, finally be written,

$$\frac{dw'}{dx'}=\frac{d\eta}{r\,d\psi}+\frac{\eta\tan\psi}{r}\quad\text{and}\quad\frac{du'}{dz'}=\frac{d\omega}{r\cos\psi\,d\phi}.$$

These equations will give the expression for $T_{\phi\psi}$.

The value of

$$\theta=\frac{du'}{dx'}+\frac{dv'}{dy'}+\frac{dw'}{dz'}$$

now takes the following form:

$$\theta=\frac{d\rho}{dr}+\frac{d\eta}{r\cos\psi\,d\phi}+\frac{d\omega}{r\,d\psi}+\frac{2\rho}{r}-\frac{\omega\tan\psi}{r}.\quad .\quad .\quad .\quad (4)$$

The last two terms are characteristic of the spherical co-ordinates.

The eqs. (20), (21), (22), (11), (12), and (13), of Art. 1, take the forms

$$N_r=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta+2G\frac{d\rho}{dr};\quad .\ .\ .\ .\ .\ .\ .\ .\ .\quad (5)$$

$$N_\phi=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta+2G\left(\frac{d\eta}{r\cos\psi\,d\phi}+\frac{\rho}{r}-\frac{\omega\tan\psi}{r}\right);\quad .\ .\quad (6)$$

$$N_\psi=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta+2G\left(\frac{d\omega}{r\,d\psi}+\frac{\rho}{r}\right);\quad .\ .\ .\ .\ .\ .\quad (7)$$

$$T_{\phi\psi}=G\left(\frac{d\eta}{r\,d\psi}+\frac{d\omega}{r\cos\psi\,d\phi}+\frac{\eta\tan\psi}{r}\right);\quad .\ .\ .\ .\ .\quad (8)$$

$$T_{r\psi}=G\left(\frac{d\omega}{dr}-\frac{\omega}{r}+\frac{d\rho}{r\,d\phi}\right);\quad .\ .\ .\ .\ .\ .\ .\ .\quad (9)$$

$$T_{r\phi}=G\left(\frac{d\rho}{r\cos\psi\,d\phi}+\frac{d\eta}{dr}-\frac{\eta}{r}\right);\quad .\ .\ .\ .\ .\ .\ .\quad (10)$$

If these values are inserted in eqs. (1), (2), and (3), the resulting equations will be applicable to isotropic material only.

As in the preceding article, $\mathfrak{r}$ is used to express the ratio between direct and lateral strains, and has no relation whatever to the co-ordinate r.

It is interesting and important to observe that the equations of motion and equilibrium for elastic bodies are only special cases of equations which are entirely independent of the nature of the material, of equations, in fact, which express the most general conditions of motion or equilibrium.

CHAPTER II.

THICK, HOLLOW CYLINDERS AND SPHERES, AND TORSION.

Art. 5.—Thick, Hollow Cylinders.

In Fig. 1 is represented a section, taken normal to its axis, of a circular cylinder whose walls are of the appreciable thickness t. Let p and p_1 represent the interior and exterior intensities of pressures, respectively. The material will not be stressed with uniform intensity throughout the thickness t. Yet if that thickness, comparatively speaking, is small, the variation will also be small; or, in other words, the intensity of stress throughout the thickness t may be considered constant. This approximate case will first be considered.

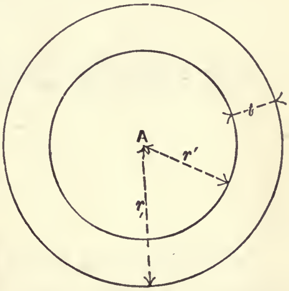


FIG. 1.

The interior intensity p will be considered greater than the exterior p_1, consequently the tendency will be toward rupture along a diametral plane. If, at the same time, the ends of the cylinder are taken as closed, as will be done, a tendency to rupture through the section shown in the figure will exist.

The force tending to produce rupture of the latter kind will be

$$F = \pi(pr'^2 - p_1 r_1^2). \quad \ldots \quad (1)$$

If N_1 represents the intensity of stress developed by this force,

$$N_1 = \frac{F}{\pi(r_1^2 - r'^2)} = \frac{pr'^2 - p_1 r_1^2}{r_1^2 - r^2}. \quad . \quad . \quad . \quad . \quad (2)$$

If the exterior pressure is zero, and if r' is nearly equal to

$$\frac{r_1 + r'}{2},$$

$$N_1 = \frac{pr'}{2(r_1 - r')} = \frac{pr'}{2t}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this same approximate case, the tendency to split the cylinder along a diametral plane, for unit of length, will be

$$F' = pr' - p_1 r_1.$$

If N' is the intensity of stress developed by F',

$$N' = \frac{F'}{t} = \frac{pr' - p_1 r_1}{t}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

N' is thus seen to be *twice* as great as N_1 when $p_1 = 0$. If, therefore, the material has the same ultimate resistance in both directions the cylinder will fail longitudinally when the interior intensity is only *half* great enough to produce transverse rupture, *the thickness being assumed to be very small and the exterior pressure zero.*

N_1 and N' are tensile stresses, because the interior pressure was assumed to be large compared with the exterior. If the opposite assumption were made, they would be found to be compression, while the general forms would remain exactly the same.

The preceding formulas are too loosely approximate for many cases. The exact treatment requires the use of the general equations of equilibrium, and the forms which they take in Art. 3 are particularly convenient. As in that article, the axis of x will be taken as the axis of the cylinder.

Since all external pressure is uniform in intensity and normal in direction, no shearing stresses will exist in the material of the cylinder. This condition is expressed in the notation of Art. 3 by putting

$$T_{\phi x} = T_{rx} = T_{r\phi} = 0.$$

Again the cylinder will be considered closed at the ends, and the force F, eq. (1), will be assumed to develop a stress of *uniform* intensity throughout the transverse section shown in Fig. 1. This condition, in fact, is involved in that of making all the tangential stresses equal to zero.

Since this case is that of equilibrium, the equations (2), (3), and (4) of Art. 3 take the following form, after neglecting X_0, R_0, and Φ_0:

$$\frac{dN_1}{dx} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{dR}{dr} + \frac{R - N_{\phi\phi}}{r} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{dN_{\phi\phi}}{r\,d\phi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

These equations are next to be expressed in terms of the strains u, ρ, and w.

In consequence of the manner of application of the external forces, all movements of indefinitely small portions of the material will be along the radii and axis of the cylinder. Hence

u will be independent of r and ϕ;
ρ " " " " ϕ " x;
$w = 0$.

The rate of change, therefore, of volume will be (eq. (6) of Art. 3)

$$\theta = \frac{du}{dx} + \frac{d\rho}{dr} + \frac{\rho}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

As ρ is independent of x, $\frac{d\theta}{dx} = \frac{d^2u}{dx^2}$; hence if the value of N_1 be taken from eq. (7) of Art. 3 and put in eq. (5) of this article,

$$\frac{dN_1}{dx} = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}} \frac{d^2u}{dx^2} + 2G\frac{d^2u}{dx^2} = 0;$$

$$\therefore \frac{d^2u}{dx^2} = 0 \quad \text{and} \quad u = ax + a'.$$

But the transverse section in which the origin is located may be considered fixed. Consequently if $x = 0$, $u = 0$ and thus $a' = 0$. The expression for u is then $u = ax$.

The ratio $u \div x$ is the l of eq. (1), on page 3, while the p of the same equation is simply N_1 of eq. (2), given above. Hence

$$a = \frac{u}{x} = \frac{N_1}{E} = \frac{pr'^2 - p_1r_1^2}{E(r_1^2 - r'^2)}. \quad . \quad . \quad . \quad . \quad (9)$$

Again, eq. (8) of Art. 3, in connection with eqs. (8) and (6) of this, gives

$$\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\left(\frac{d^2\rho}{dr^2}+\frac{d\rho}{r\,dr}-\frac{\rho}{r^2}\right)+2G\left(\frac{d^2\rho}{dr^2}+\frac{d\rho}{r\,dr}-\frac{\rho}{r^2}\right)=0.$$

$$\therefore\ \frac{d^2\rho}{dr^2}+\frac{d\rho}{r\,dr}-\frac{\rho}{r^2}=\frac{d^2\rho}{dr^2}+\frac{d\left(\frac{\rho}{r}\right)}{dr}=0.$$

$$\therefore\ \frac{d\rho}{dr}+\frac{\rho}{r}=c,\ \text{or}$$

$$r\,d\rho+\rho\,dr=d(\rho r)=cr\,dr.$$

$$\therefore\ \rho r=\frac{cr^2}{2}+b,\ \text{or}\ \ \rho=\frac{cr}{2}+\frac{b}{r}.\quad .\quad .\quad .\quad (10)$$

This value of ρ in eqs. (8) and (9) of Art. 3 will give

$$R=2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}}+\frac{c}{2}-\frac{b}{r^2}\right\};\quad .\quad .\quad .\quad .\quad (11)$$

$$N_{\phi\phi}=2G\left\{\frac{\mathfrak{r}(a+c)}{1-2\mathfrak{r}}+\frac{c}{2}+\frac{b}{r^2}\right\}.\quad .\quad .\quad .\quad .\quad (12)$$

At the interior surface R must be equal to the internal pressure, and at the exterior surface to the external pressure. Or since negative signs indicate compression,

If $r=r'$ . . . . . . $R=-p$.
If $r=r_1$ . . . . . . $R=-p_1$.

Either of these equations is the simple result of applying eqs. (13), (14), and (15) to the present case, for which

$$\cos p=\cos r=\cos\pi=\cos\rho=0,$$
$$\cos q=\cos\chi=1,\ \text{and}\ P=-p\ \text{or}\ -p_1.$$

Applying eq. (11) to the two surfaces,

$$-p = 2G\left\{\frac{\mathbf{r}(a+c)}{1-2\mathbf{r}} + \frac{c}{2} - \frac{b}{r'^2}\right\}; \quad . \quad . \quad . \quad (13)$$

$$-p_1 = 2G\left\{\frac{\mathbf{r}(a+c)}{1-2\mathbf{r}} + \frac{c}{2} - \frac{b}{r_1^2}\right\}. \quad . \quad . \quad . \quad (14)$$

Subtracting (14) from (13),

$$2Gb = \frac{(p_1 - p)\, r_1^2\, r'^2}{r'^2 - r_1^2}.$$

Inserting this value in eq. (13),

$$2G\left\{\frac{\mathbf{r}(a+c)}{1-2\mathbf{r}} + \frac{c}{2}\right\} = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2}.$$

The general expressions of R and $N_{\phi\phi}$, freed from the arbitrary constants of integration, can now be easily written by inserting these last two values in eqs. (11) and (12). By making the insertions there will result

$$R = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2} - \frac{(p_1 - p)\, r_1^2 r'^2}{r'^2 - r_1^2} \cdot \frac{1}{r^2}; \quad . \quad . \quad (15)$$

$$N_{\phi\phi} = \frac{p_1 r_1^2 - p r'^2}{r'^2 - r_1^2} + \frac{(p_1 - p)\, r_1^2 r'^2}{r'^2 - r_1^2} \cdot \frac{1}{r^2}. \quad . \quad . \quad (16)$$

The stress $N_{\phi\phi}$ is a tension directed *around* the cylinder, and has been called "hoop tension." Eq. (16) shows that the hoop tension will be greatest at the interior of the cylinder. An expression for the thickness, t, of the annulus in terms of the greatest hoop tension (which will be called h) can easily be obtained from eq. (16).

If $r = r'$ in that equation,

$$h = \frac{2p_1 r_1^2 - p(r'^2 + r_1^2)}{r'^2 - r_1^2}.$$

$$\therefore \frac{r_1}{r'} = \left(\frac{h+p}{2p_1 - p + h}\right)^{\frac{1}{2}}.$$

$$\therefore r_1 - r' = t = r'\left\{\left(\frac{h+p}{2p_1 - p + h}\right)^{\frac{1}{2}} - 1\right\}. \quad . \quad . \quad (17)$$

Eq. (17) will enable the thickness to be so determined that the hoop tension shall not exceed any assigned limit h. If p_1 is so small in comparison with p that it may be neglected, t will become

$$t = r'\left\{\left(\frac{h+p}{h-p}\right)^{\frac{1}{2}} - 1\right\}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If p_1 is greater than p, $N_{\phi\phi}$ becomes compression, but the equations are in no manner changed.

The values of the constants b and c may easily be found from the two equations immediately preceding eq. (15).

It is interesting to notice that the rate of change of volume, θ, is equal to $(a+c)$ and therefore constant for all points.

Art. 6.—Torsion in Equilibrium.

The formulas to be deduced in this article are those first given by Saint-Venant, and established in substantially the same manner.

It will in all cases, except that of the final result for a rectangular cross-section, be convenient to use those equations of Art. 3 which are given in terms of semi-polar coordinates.

Let Fig. 1 represent a cylindrical piece of material, with any cross-section, fixed in the plane ZY, and let the origin of co-ordinates be taken at O. Let it be twisted also by a couple

$$P.ab = Pl,$$

the plane of which is parallel to ZY. The material will thus be subjected to no bending, but to pure torsion.

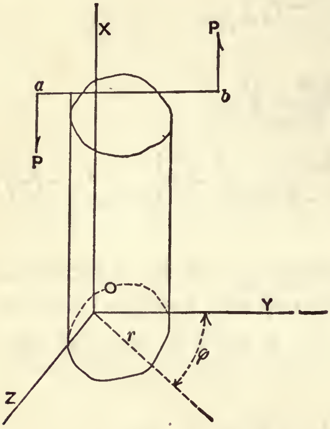


Fig. 1.

The axis of the piece is supposed to be parallel to the axis of X as well as the axis of the couple. Normal sections of the piece, originally parallel to ZOY, will not remain plane after torsion takes place. But the tendency to twist any elementary portion of the piece about an axis passing through its centre and parallel to the axis of X will be very small compared with the tendency to twist it about either the axis of r or ϕ; consequently the first will be neglected. In the notation of Art. 3, this condition is equivalent to making $T_{r\phi} = 0$.

As the piece is acted upon by a couple only, all normal stresses will be zero.

Eqs. (7), (8), (9), and (11) of Art. 3 then become

$$N_1 = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{du}{dx} = 0; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$R = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr} = 0; \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$N_{\phi\phi} = \frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\theta + 2G\left(\frac{dw}{r\,d\phi} + \frac{\rho}{r}\right) = 0; \quad . \quad . \quad (3)$$

$$T_{r\phi} = G\left(\frac{d\rho}{r\,d\phi} + \frac{dw}{dr} - \frac{w}{r}\right) = 0. \quad . \quad . \quad . \quad . \quad (4)$$

After introducing the values of T_{rx} and $T_{\phi x}$, from eqs. (10) and (12) of Art. 3, in eqs. (2), (3), and (4) of the same article, at the same time making the external forces and second members of those equations equal to zero, and bearing in mind the conditions given above, there will result

$$\frac{dT_{rx}}{dr}+\frac{dT_{\phi x}}{r\,d\phi}+\frac{T_{rx}}{r}$$
$$=G\left(\frac{d^2u}{dr^2}+\frac{d^2\rho}{dr\,dx}+\frac{d^2w}{r\,d\phi\,dx}+\frac{d^2u}{r^2d\phi^2}+\frac{du}{r\,dr}+\frac{d\rho}{r\,dx}\right)=0; \quad (5)$$

$$\frac{dT_{rx}}{dx}=G\left(\frac{d^2u}{dr\,dx}+\frac{d^2\rho}{dx^2}\right)=0; \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{dT_{x\phi}}{dx}=G\left(\frac{d^2w}{dx^2}+\frac{d^2u}{r\,d\phi\,dx}\right)=0. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Also by eq. (6) of Art. 3,

$$\theta=\frac{du}{dx}+\frac{d\rho}{dr}+\frac{dw}{r\,d\phi}+\frac{\rho}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The cylindrical piece of material is supposed to be of such length that the portion to which these equations apply is not affected by the manner of application of the couple. This portion is, therefore, twisted uniformly from end to end; consequently the strain u will not vary with any change in x. Hence

$$\frac{du}{dx}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Eq. (1) then shows that $\theta=0$. This was to be anticipated, since a pure shear cannot change the volume or density. Because $\theta=0$, eqs. (2) and (3) at once give

$$\frac{d\rho}{dr}=\frac{dw}{r\,d\phi}+\frac{\rho}{r}=0. \qquad (10)$$

As the torsion is uniform throughout the portion considered,

$$\frac{d\rho}{dx}=0=\frac{d\rho}{r\,dx}. \qquad (11)$$

Eq. (11), in connection with eq. (10), gives

$$\frac{d^2w}{r\,dx\,d\phi}=0. \qquad (12)$$

Eqs. (11) and (12), in connection with eq. (10), reduce eq. (5) to the following form:

$$\frac{d^2u}{r^2\,d\phi^2}+\frac{d^2u}{dr^2}+\frac{du}{r\,dr}=0=\frac{d^2u}{d\phi^2}+r\frac{d\left(r\frac{du}{dr}\right)}{dr}. \qquad (13)$$

Both terms of the second member of eq. (6) reduce to zero by eqs. (9) and (11), and give no new condition. The second term of the second member of eq. (7) is zero by eq. (9); the remaining term therefore gives

$$\frac{d^2w}{dx^2}=0. \qquad (14)$$

As the stress is all shearing, ρ will not vary with ϕ. Hence

$$\frac{d\rho}{r\,d\phi}=0. \qquad (15)$$

Eqs. (10), (11), and (15) show that $\rho=0$, and reduce eq. (4) to

$$\frac{dw}{dr}-\frac{w}{r}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Eq. (10) now becomes $\frac{dw}{r\,d\phi}=0$, and shows that w does not contain ϕ; while eq. (14) shows that w does not contain x^2 or any higher power of x. The strain w, in connection with these conditions, is to be so determined as to satisfy eq. (16).

If α is a constant, the following form fulfils all conditions:

$$w=\alpha rx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

Eq. (17) shows that *the strain w, in the direction of ϕ*, i.e., *the angular strain at any point, varies directly as the distance from the axis of X, and as the distance from the origin measured along that axis.* This is a direct consequence of making $T_{r\phi}=0$.

The quantity α is evidently the *angle of torsion*, or the angle through which one end of a unit of fibre, situated at unit's distance from the axis, is twisted; for if

$$r=x=1, \quad w=\alpha.$$

An equation of condition relative to the exterior surface of the twisted piece yet remains to be determined; and that is to be based on the supposition that no external force whatever acts on the outer surface of the piece. In eqs. (13), (14), and (15) of Art. 2, consequently, $P=0$. The conditions of the problem also make all the stresses except

$$T_3=T_{xr} \quad \text{and} \quad T_2=T_{\phi x}$$

equal to zero, while the cylindrical character of the piece makes

$$p=90^\circ; \quad \therefore \cos p=0.$$

If cos t be written for cos r,

$$\cos t=\sin q.$$

Eq. (13), just cited, then gives

$$T_{xr} \cos q+T_{\phi x} \sin q=0. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

But since $\rho=0$ and $w=\alpha r x$,

$$T_{xr}=G\frac{du}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and

$$T_{x\phi}=G\left(\frac{du}{r\,d\phi}+\alpha r\right). \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Eq. (18) now becomes

$$\frac{\frac{du}{dr}}{\frac{du}{r\,d\phi}+\alpha r}=-\tan q=-\frac{dr_0}{r_0\,d\phi}, \quad . \quad . \quad . \quad . \quad (21)$$

in which r_0 is the value of r for the perimeter of any normal section.

Eqs. (13) and (21) are all that are necessary and all that exist for the determination of the strain u. Eq. (13) must be fulfilled at all points in the interior of the twisted piece, while eq. (21) must at the same time hold true at all points of the exterior surface.

After u is determined, T_{xr} and $T_{x\phi}$ at once result from eqs. (19) and (20). The resisting moment of torsion then becomes

$$M=\int\int T_{x\phi}r^2\,d\phi\,.dr=G\int\int\frac{du}{d\phi}.r\,dr\,d\phi+G\alpha I_p. \quad (22)$$

In this equation $I_p=\int\int r^2.r\,d\phi\,dr$ is the polar moment of inertia of the normal section of the piece about the axis of X, and the double integral is to be extended over the whole section.

According to the old or common theory of torsion

$$M=G\alpha I_p.$$

The third member of eq. (22) shows, however, that such an expression is not correct unless u is equal to zero; i.e., unless all normal sections remain plane while the piece is subjected to torsion. It will be seen that this is true for a circular section only.

It may sometimes be convenient to put eq. (22) in the following form:

$$M=G\int\int r\,dr.\frac{du}{d\phi}d\phi+G\alpha I_p=G\int u.r\,dr+G\alpha I_p. \quad (23)$$

In this equation u is to be considered as

$$\int_0^\phi\frac{du}{d\phi}d\phi,$$

while the remaining integration in r is to be so made that the whole section shall be covered.

The preceding analysis shows that the old or common theory of torsion is correct in its expression for torsive strain, as it is identical with eq. (17) of Art. 6, i.e.,

$$w = \alpha r x;$$

but it will be seen later that the remaining formulæ of the common theory are incorrect for all shapes of cross-section except the circle. Fortunately the torsion members principally used in engineering practice are shafts of circular section.

Equations of Condition in Rectangular Co-ordinates.

In the case of a rectangular normal section, the analysis is somewhat simplified by taking some of the quantities used in terms of rectangular co-ordinates.

In the notation of Art. 2 all stresses will be zero except T_3 and T_2. Hence eqs. (10), (11), and (12) of that article reduce to

$$\frac{dT_3}{dy} + \frac{dT_2}{dz} = 0;$$

$$\frac{dT_3}{dx} = 0;$$

$$\frac{dT_2}{dx} = 0.$$

The strains in the directions of x, y, and z are, respectively, u, v, and w. Introducing the values of T_3 and T_2 in the equations above, in terms of these strains, from eqs. (11) and (13) of Art. 1, and then doing the same in reference to the conditions,

$$N_1 = N_2 = N_3 = T_1 = 0,$$

the following equations will result:

$$\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\frac{dv}{dz}+\frac{dw}{dy}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The operations by which these results are reached are identical with those used above in connection with semi-polar co-ordinates, and need not be repeated.

Eq. (27) is satisfied by taking

$$v = \quad \alpha xz;$$

$$w = -\alpha xy;$$

in which α is the angle of torsion, as before.

Eqs. (11) and (13) of Art. 5 then give

$$T_3=G\left(\frac{du}{dy}+\frac{dv}{dx}\right)=G\left(\frac{du}{dy}+\alpha z\right); \quad . \quad . \quad . \quad . \quad (28)$$

$$T_2=G\left(\frac{du}{dz}+\frac{dw}{dx}\right)=G\left(\frac{du}{dz}-\alpha y\right). \quad . \quad . \quad . \quad . \quad (29)$$

The element of a normal section is $dz\ dy$. Hence the moment of torsion is

$$M=\int\int(T_3z-T_2y)dy\ dz;$$

$$\therefore\ M=G\int\int\left(\frac{du}{dy}z-\frac{du}{dz}y\right)dy\ dz+G\alpha I_p; \quad . \quad (30)$$

$$\therefore\ M=G\int(zu\ dz-yu\ dy)+G\alpha I_p. \quad . \quad . \quad . \quad . \quad (31)$$

$$I_p=\int\int(z^2+y^2)dy\ dz$$

is the polar moment of inertia of any section about the axis of X.

The integrals are to be extended over the whole section; hence, in eq. (31), $zu\,dz$ is to be taken as

$$z\,dz\,.\int_{-y_0}^{+y_0}\frac{du}{dy}dy$$

and $yu\,dy$ as

$$y\,dy\int_{-z_0}^{+z_0}\frac{du}{dz}dz,$$

in which expressions y_0 and z_0 are general co-ordinates of the perimeter of the normal section.

Eq. (26) is identical with eq. (13), and can be derived from it, through a change in the independent variables, by the aid of the relations

$$z=r\cos\phi \quad \text{and} \quad y=r\sin\phi.$$

Solutions of Eqs. (13) *and* (21).

It has been shown that the function u, which represents the strain parallel to the axis of the piece, must satisfy eq. (13) [or eq. (26)] for all points of any normal section, and eq. (21) (or a corresponding one in rectangular co-ordinates) at all points of the perimeter; and those two are the only conditions to be satisfied.

It is shown by the ordinary operations of the calculus that an indefinite number of functions u, of r and ϕ, will satisfy eq. (13); and, of these, that some are algebraic and some transcendental.

It is further shown that the various functions u which satisfy both eqs. (13) and (21) differ only by constants.

If u is first supposed to be algebraic in character, and if c_1, c_2, c_3, etc., represent constant coefficients, the following general function will satisfy eq. (13):

$$u=\alpha\left\{\begin{array}{l} c_1r\ \sin\phi+c_2r^2\ \sin 2\phi+c_3r^3\ \sin 3\phi+\ldots \\ +c'_1r\cos\phi+c'_2r^2\cos 2\phi+c'_3r^3\cos 3\phi+\ldots \end{array}\right\} \quad (32)$$

and the following equation, which is supposed to belong to the perimeter of a normal section only, will be found to satisfy eq. (21):

$$\begin{array}{l} \frac{r^2}{2}+c_1r\ \cos\ \phi+c_2r^2\ \cos\ 2\phi+c_3r^3\ \cos\ 3\phi+\ldots \\ \qquad -c'_1r\sin\phi-c'_2r^2\sin 2\phi-c'_3r^3\sin 3\phi-\ldots=C. \end{array} \quad (33)$$

C is a constant which changes only with the form of section.

If $\frac{du}{dr}$ and $\frac{du}{r\,d\phi}$ be found from eq. (32), while $\frac{dr_0}{r_0\,d\phi}$ be taken from eq. (33), and if these quantities be then introduced in eq. (21), it will be found that that equation is satisfied.

The only form of transcendental function needed, among those to which the integration of eq. (13) or eq. (26) leads, will be given in connection with the consideration of pieces with rectangular section, where it will be used.

Elliptical Section about its Centre.

Let a cylindrical piece of material with elliptical normal section be taken, and let a be the semi-major and b the semi-minor axis, while the angle ϕ is measured from a with the centre of the ellipse as the origin of co-ordinates, since the cylinder will be twisted about its own axis. The polar equation of the elliptical perimeter may take the following shape:

$$\frac{r^2}{2}+\frac{r^2}{2}\cdot\frac{b^2-a^2}{a^2+b^2}\cos 2\phi=\frac{a^2b^2}{a^2+b^2}. \quad . \quad . \quad . \quad (34)$$

By a comparison of eqs. (33) and (34), it is seen that

$$c_2=\frac{b^2-a^2}{2(a^2+b^2)} \quad \text{and} \quad C=\frac{a^2b^2}{a^2+b^2},$$

and that all the other constants are zero. Hence eq. (32) gives

$$u=\alpha\frac{b^2-a^2}{2(a^2+b^2)}r^2\sin 2\phi=\frac{\alpha}{2}fr^2\sin 2\phi. \quad . \quad . \quad . \quad (35)$$

The quantity represented by f is evident.

By eqs. (19) and (20)

$$T_{xr}=G\alpha\frac{b^2-a^2}{a^2+b^2}r\sin 2\phi; \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$T_{x\phi}=G\alpha\left(\frac{b^2-a^2}{a^2+b^2}r\cos 2\phi+r\right). \quad . \quad . \quad . \quad . \quad (37)$$

Since $\frac{r_0 . r_0\, d\phi}{2}=dA$, A being the area of the ellipse, or πab, the second member of eq. (22), by the aid of eq. (37), may take the form

$$M=G\alpha\int d\phi\int_0^r\left(\frac{b^2-a^2}{a^2+b^2}r^3\cos 2\phi+r^3\right)dr;$$

$$\therefore\ M=G\alpha\int\left(\frac{b^2-a^2}{a^2+b^2}\frac{r^4}{4}\cos 2\phi+\frac{r^4}{4}\right)d\phi.$$

Then using eq. (34),

$$M = G\alpha\frac{a^2b^2}{a^2+b^2}\int dA = G\alpha\frac{\pi a^3b^3}{a^2+b^2}. \quad . \quad . \quad . \quad (38)$$

If I_p is the polar moment of inertia of the ellipse (i.e., about an axis normal to its plane and passing through its centre), so that

$$I_p = \frac{\pi ab(a^2+b^2)}{4};$$

then

$$M = G\alpha\frac{A^4}{4\pi^2 I_p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Using f in the manner shown in eq. (35), the resultant shear at any point becomes, by eq. (24),

$$T = G\alpha r\sqrt{f^2+2f\cos 2\phi+1}.$$

$$\therefore \frac{dT}{d\phi} = 0$$

gives

$$\sin 2\phi = 0, \quad \text{or} \quad \phi = 90° \quad \text{or} \quad 0°.$$

Since f is negative, T will evidently take its maximum when ϕ has such a value that $2f\cos 2\phi$ is positive, or ϕ must be 90°.

Hence the greatest intensity of shear will be found somewhere along the minor axis. But the preceding expression shows that T varies directly as the distance from the centre. Hence *the greatest intensity of shear is found at the extremities of the minor axis.*

Making $\phi = 90°$ and $r = b$ in the value of T,

$$T = T_m = G\alpha b(1 - f) = G\alpha \frac{2a^2 b}{a^2 + b^2}. \quad . \quad . \quad . \quad (40)$$

Taking $G\alpha$ from eq. (40) and inserting it in eq. (38),

$$M = T_m \frac{\pi a b^2}{2} = 2 T_m \frac{I_a}{b}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

in which

$$I_a = \frac{\pi a b^3}{4},$$

or the moment of inertia of the section about the major axis.

Equilateral Triangle about its Centre of Gravity.

This case is that of a cylindrical piece whose normal cross-section is an equilateral triangle, and the torsion will be supposed about an axis passing through the centres of gravity of the different normal sections. The cross-section is represented in Fig. 3, G being the centre of gravity as well as the origin of co-ordinates.

Let $GH = \frac{1}{2} GD = a$. Then from the known properties of such a triangle,

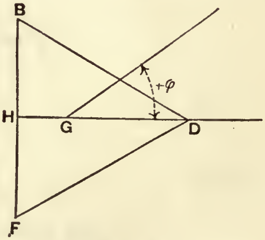


FIG. 3.

$$FD = DB = BF = 2a\sqrt{3}.$$

Hence the equation for DB is; $r \sin \phi - \frac{2a - r\cos\phi}{\sqrt{3}} = 0.$

Hence the equation for BF is; $r\cos\phi + a = 0.$

Hence the equation for FD is; $r \sin \phi + \frac{2a - r\cos\phi}{\sqrt{3}} = 0.$

Taking the product of these three equations and reducing, there will result for the equation to the perimeter

$$\frac{r^2}{2}-\frac{r^3}{6a}\cos 3\phi=\frac{2a^2}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (42)$$

Comparing this equation with eq. (33),

$$c_3=-\frac{1}{6a} \quad \text{and} \quad C=\frac{2a^2}{3}.$$

Hence

$$u=-\alpha\frac{r^3 \sin 3\phi}{6a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (43)$$

And by eqs. (19) and (20)

$$T_{xr}=-G\alpha\frac{r^2 \sin 3\phi}{2a}; \quad . \quad . \quad . \quad . \quad . \qquad (44)$$

$$T_{x\phi}=G\alpha\left(r-\frac{r^2 \cos 3\phi}{2a}\right). \quad . \quad . \quad . \quad . \qquad (45)$$

Eq. (22) then gives

$$M=G\alpha I_p \quad G\alpha\int\int\frac{r^4 \cos 3\phi}{2a}dr\,d\phi;$$

$$=G\alpha I_p-G\alpha\int\frac{r^4 \sin 3\phi}{6a}dr;$$

$$=G\alpha\left(I_p-\frac{6}{5}a^4\sqrt{3}\right)=0.6\ G\alpha I_p=1.8\ G\alpha a^4\sqrt{3}; \quad . \qquad (46)$$

since I_p = polar moment of inertia $=3a^4\sqrt{3}$.

By eq. (24)

$$T = G\alpha\sqrt{r^2 - \frac{r^3 \cos 3\phi}{a} + \frac{r^4}{4a^2}}; \quad . \quad . \quad . \quad (47)$$

$$\therefore \frac{dT}{d\phi} = 0 \quad \text{gives} \quad \sin 3\phi = 0,$$

or

$$\phi = 0°, 60°, 120°, 180°, 240°, 300°, \text{ or } 360°.$$

The values 0°, 120°, 240°, and 360° make

$$\cos 3\phi = +1;$$

hence, for a given value of r, these make T a minimum. The values 60°, 180°, and 300° make,

$$\cos 3\phi = -1;$$

hence, for a given value of r, these make T a maximum. Putting $\cos 3\phi = -1$ in eq. (47),

$$T = G\alpha\left(r + \frac{r^2}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

This value will be the greatest possible when r is the greatest. But $\phi = 60°$, 180°, and 300° correspond to the normal a dropped on each of the three sides of the triangle from G. Hence $r = a$, in eq. (48), gives the greatest intensity of shear T_m, or

$$T_m = \frac{3}{2} G\alpha a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Or *the greatest intensity of shear exists at the middle point of each side.* Those points are the nearest of all, in the perimeter, to the axis of torsion.

The value of $G\alpha$, from eq. (49), inserted in eq. (46), gives

$$M = 0.4 \frac{I_p}{a} T_m = \frac{l^3 T_m}{20}, \quad . \quad . \quad . \quad . \quad . \quad (50)$$

in which $l =$ side of section $= 2a\sqrt{3}.$

Rectangular Section about an Axis passing through its Centre of Gravity.

In this case it will be necessary to consider one of the transcendental forms to which the integration of eq. (13) [or (26)] leads; for if the polar equation to the perimeter be formed, as was done in the preceding case, it will be found to contain r^4, to which no term in eq. (33) corresponds.

If e is the base of the Napierian system of logarithms (numerically $e = 2.71828$, nearly) and A any constant whatever, it is known that the general integral of the partial differential eq. (13) may be expressed as follows:

$$u = Ae^{nr \cos \phi} \; e^{n'r \sin \phi}, \quad . \quad . \quad . \quad . \quad . \quad (51)$$

when $n^2 + n'^2 = 0$; for

$$\frac{d^2u}{dr^2} + \frac{d^2u}{r^2 \, d\phi^2} + \frac{du}{r \, dr} = A(n^2 + n'^2) \; e^{nr \cos \phi} \; e^{n'r \sin \phi}.$$

But the second member of this equation is evidently equal to zero if

$$(n^2 + n'^2) = 0 \quad \text{or} \quad n' = \sqrt{-n^2}.$$

These relations make it necessary that neither n or n' shall be imaginary.

It will hereafter be convenient to use the following notation for hyperbolic sines, cosines, and tangents:

$$\text{sih}\ t = \frac{e^t - e^{-t}}{2}; \quad \text{coh}\ t = \frac{e^t + e^{-t}}{2}; \quad \text{and tah}\ t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

By the use of Euler's exponential formula, as is well known, and remembering that $n'^2 = -n^2$, eq. (51) may be put in the following form:

$$u = \Sigma e^{nr \cos \phi}\ [A_n \sin (nr \sin \phi) + A'_n \cos (nr \sin \phi)],$$

in which the sign of summation is to be extended to all possible values of A_n and A'_n. At the centre of any section for which r is zero, u must be zero also, for the axis of the piece is not shortened. This condition requires that $A'_n = 0$; u then becomes

$$u = \Sigma e^{nr \cos \phi}\ A_n \sin (nr \sin \phi).$$

The subsequent analysis will be simplified by introducing the form of the hyperbolic sine, and this may be done by adding and subtracting the same quantity to that already under the sign of summation, in such a manner that

$$u = \Sigma[A_n \sin (nr \sin \phi) . \text{sih}\ (nr \cos \phi) + \tfrac{1}{2}A_n \sin (nr \sin \phi)\ e^{-nr \cos \phi}]. \qquad (52)$$

Now if the product

$$\sin (nr \sin \phi)\ e^{-nr \cos \phi}$$

be developed in a series and multiplied by A_n, one term will consist of the quantity

$$-r^2 \sin \phi \cos \phi$$

multiplied by a constant, and if

$$\Sigma A_n \sin (nr \sin \phi)\, e^{-nr \cos \phi}$$

be replaced by simply,

$$-\alpha r^2 \sin \phi \cos \phi,$$

all the conditions of the problem will be found to be satisfied. This is equivalent to putting

$$-\alpha r^2 \sin \phi \cos \phi$$

for a general function of $r \sin \phi$ and $r \cos \phi$. This change will give the following form to u, first used by Saint-Venant:

$$u = \Sigma A_n \sin (nr \sin \phi) \,.\, \mathrm{sih}\, (nr \cos \phi) - \alpha r^2 \sin \phi \cos \phi. \quad (53)$$

Fig. 4 represents the cross-section with C as the origin of co-ordinates and axis. The angle ϕ is measured positively from CN toward CH. At the points N, H, K, and L, in the

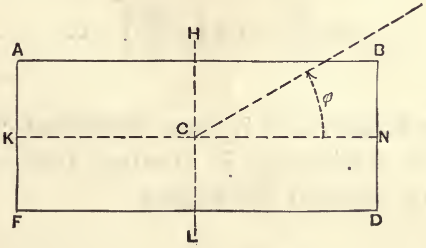


Fig. 4.

equation to the perimeter, dr_0 will be zero. Hence at those points, by eq. (21),

$$\frac{du}{dr} = \Sigma[A_n \sin(nr \sin\phi) . n \cos\phi . \text{coh}(nr \cos\phi)$$
$$+ A_n . n \sin\phi . \cos(nr \sin\phi) . \text{sih}(nr \cos\phi)]$$
$$- 2\alpha r \sin\phi \cos\phi = 0.$$

At the points under consideration ϕ has the values 0°, 90°, 180°, 270°, and 360°. At the points N and K, $\phi = 0°$ or 180°; hence $\sin\phi = 0$, and both terms of the second member of $\frac{du}{dr}$ reduce to zero, whatever may be the value of n. But at H and L, $\phi = 90°$ and 270°; hence $\sin\phi = +1$ or -1 and $\cos\phi = 0$.

In order, then, that $\frac{du}{dr} = 0$ at H and L, these must obtain:

$$\cos nr = \cos(-nr) = 0.$$

If $HL = c$ and $KN = b$, then

$$\cos\frac{nc}{2} = \cos\left(-\frac{nc}{2}\right) = 0. \quad . \quad . \quad . \quad . \quad (54)$$

If the signification of n be now somewhat changed so as to represent all possible whole numbers between 0 and ∞, eq. (54) will be satisfied by writing

$$\frac{2n - 1}{c}\pi$$

for n in that equation. Eq. (53) will then become

$$u = \sum_{1}^{\infty} A_n \sin\left(\frac{2n-1}{c}\pi r \sin\phi\right) . \text{sih}\left(\frac{2n-1}{c}\pi r \cos\phi\right) - \alpha r^2 \sin\phi \cos\phi. \quad . \quad . \quad . \quad . \quad (55)$$

The quantity A_n yet remains to be determined by the aid of eq. (21), which expresses the condition existing at the perimeter of any section.

Now, for the portion BN of the perimeter,

$$r \cos\phi = \frac{b}{2},$$

and $\frac{dr_0}{r_0 d\phi}$ will be the tangent of $(-\phi)$, or

$$-\frac{dr_0}{r_0 d\phi} = -\tan(-\phi) = \tan\phi.$$

Hence eq. (21) becomes

$$\frac{\frac{du}{dr}}{\frac{du}{r\,d\phi} + \alpha r} = \tan\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

or

$$\alpha r \sin\phi = \frac{du}{dr}\cos\phi - \frac{du}{r\,d\phi}\sin\phi.$$

Substituting from eq. (55), then making

$$r \cos\phi = \frac{b}{2},$$

$$r \sin\phi = \sum_{1}^{\infty} A_n . \frac{2n-1}{2\alpha c}\pi . \text{coh}\left(\frac{2n-1}{2c}\pi b\right) . \sin\left(\frac{2n-1}{c}\pi r \sin\phi\right).$$

If $r \sin \phi$ be represented by the rectangular co-ordinate y, and another quantity by H, the above equation may be written

$$y = H_1 \sin \frac{\pi y}{c} + H_2 \sin \frac{3\pi y}{c} + H_3 \sin \frac{5\pi y}{c}$$

$$+ \ . \ . \ . \ - H_n \sin \left(\frac{2n-1}{c}\pi\right) y + \ . \ . \ .$$

If both sides of this equation be multiplied by

$$\sin \left(\frac{2n-1}{c}\pi y\right) . dy,$$

and if the integral then be taken between the limits 0 and $\frac{c}{2}$, it is known from the integral calculus that all terms except the n^{th} will disappear, and that

$$H_n = \int_0^{\frac{c}{2}} y . \sin \left(\frac{2n-1}{c}\pi y\right) . dy \div \int_0^{\frac{c}{2}} \sin^2 \left(\frac{2n-1}{c}\pi y\right) . dy.$$

Completing these simple integrations,

$$H_n = \left(\frac{c}{(2n-1)\pi}\right)^2 (-1)^{n-1} . \frac{4}{c}.$$

Hence

$$A_n = \frac{(-1)^{n-1} c^2}{(2n-1)^2 \pi^2} . \frac{4}{c} . \frac{2\alpha c}{(2n-1)\pi} . \frac{1}{\operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)}.$$

If this value of A_n be put in eq. (55), and if rectangular co-ordinates

$$y = r \sin \phi \quad \text{and} \quad z = r \cos \phi$$

be introduced, that equation will become

$$u = -\alpha z y + \left(\frac{2}{\pi}\right)^3 . \alpha c^2 \sum_1^\infty \frac{(-1)^{n-1} \sin\left(\frac{2n-1}{c}\pi y\right) . \operatorname{sih}\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^3 \operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)}. \quad (57)$$

This value of u placed in eq. (31) will enable the moment of torsion to be at once written.

The limits $+y_0$ and $-y_0$ are $+\frac{c}{2}$ and $-\frac{c}{2}$, and the limits $+z_0$ and $-z_0$ are $+\frac{b}{2}$ and $-\frac{b}{2}$. Hence

$$\Big[u\Big]_{-\frac{c}{2}}^{+\frac{c}{2}} = \alpha bc \left[-\frac{z}{b} + \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{2 \operatorname{sih}\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^3 \operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)} \right]$$

$= Q$, for brevity;

$$-\Big[u\Big]_{-\frac{b}{2}}^{+\frac{b}{2}} = \alpha bc \left[\frac{y}{c} - \left(\frac{2}{\pi}\right)^3 \frac{c}{b} \sum_1^\infty \frac{(-1)^{n-1} . 2 \operatorname{sih}\left(\frac{2n-1}{2c}\pi b\right) . \sin\left(\frac{2n-1}{c}\pi y\right)}{(2n-1)^3 \operatorname{coh}\left(\frac{2n-1}{2c}\pi b\right)} \right] = R.$$

For the next integration

$$\int_{-\frac{b}{2}}^{+\frac{b}{2}} Qz\,dz = \alpha bc\left[-\frac{b^2}{12}\right.$$

$$\left.+\left(\frac{2}{\pi}\right)^3\frac{c}{b}\sum_1^\infty\frac{\frac{2bc}{(2n-1)\pi}.\mathrm{coh}\frac{2n-1}{2c}\pi b-\frac{4c^2}{(2n-1)^2\pi^2}\mathrm{sih}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^3\,\mathrm{coh}\left(\frac{2n-1}{2c}\pi b\right)}\right]$$

$$\int_{-\frac{c}{2}}^{+\frac{c}{2}} Ry\,dy = \alpha bc\left[\frac{c^2}{12}-\left(\frac{2}{\pi}\right)^3\frac{c}{b}\sum_1^\infty\frac{\frac{4c^2}{(2n-1)^2\pi^2}\,\mathrm{sih}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^3\,\mathrm{coh}\left(\frac{2n-1}{2c}\pi b\right)}\right]$$

Thus the integrations indicated in eq. (31) are completed. Hence

$$M = G\left\{\int Qz\,dz+\int Ry\,dy+\alpha I_p\right\}.$$

Remembering that

$$I_p = bc\left(\frac{c^2+b^2}{12}\right),$$

$$M = G\alpha\left[\frac{bc^3}{6}+\frac{16bc^3}{\pi^4}\sum_1^\infty\frac{1}{(2n-1)^4}-\frac{64c^4}{\pi^5}\sum_1^\infty\frac{\mathrm{tah}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^5}\right]. \quad (58)$$

But it is known that

$$\sum_1^\infty\frac{1}{(2n-1)^4}=\frac{2}{1.2.3}\cdot\frac{\pi^4}{2^5}.$$

Hence eq. (58) becomes

$$M = G\alpha bc^3\left[\frac{1}{3} - \frac{64}{\pi^5}\frac{c}{b}\sum_1^\infty \frac{\text{tah}\left(\frac{2n-1}{2c}\pi b\right)}{(2n-1)^5}\right]. \quad . \quad (59)$$

Since

$$\left(\frac{1}{1}+\frac{1}{3^5}+\frac{1}{5^5}+\ .\ .\ .\right)$$

$$-\left(\frac{1-\text{tah}\ \pi}{1}+\frac{1-\text{tah}\ 3\pi}{3^5}+\frac{1-\text{tah}\ 5\pi}{5^5}+\ .\ .\ .\right)$$

$$=\frac{\text{tah}\ \pi}{1}+\frac{\text{tah}\ 3\pi}{3^5}+\frac{\text{tah}\ 5\pi}{5^5}+\ .\ .\ .\ ,$$

and since

$$\frac{64}{\pi^5}=0.209137,$$

and remembering that

$$\sum_1^\infty\left(\frac{1}{2n-1}\right)^5=1+\frac{1}{3^5}+\frac{1}{5^5}+\ .\ .\ .\ =\left(1-\frac{1}{2^5}\right)\frac{\pi^5}{295.1215},$$

eq. (59) becomes

$$M = G\alpha bc^3\left[\frac{1}{3}-0.210083\frac{c}{b}\right.$$

$$\left.+0.209137\frac{c}{b}\left(\frac{1-\text{tah}\ \frac{\pi b}{2c}}{1}+\frac{1-\text{tah}\ \frac{3\pi b}{2c}}{3^5}+\ .\ .\ .\right)\right]. \quad (60)$$

Eq. (60) gives the value of the moment of torsion of a rectangular bar of material.

If z had been taken parallel to b, and y parallel to c, a moment of equal value would have been found, which can be at once written from eq. (60) by writing b for c and c for b.

That moment will be

$$M = G\alpha cb^3 \left[\frac{1}{3} - 0.210083 \frac{b}{c} + 0.209137 \frac{b}{c} \left(\frac{1 - \tanh\frac{\pi c}{2b}}{1} + \frac{1 - \tanh\frac{3\pi c}{2b}}{3^5} + \dots \right) \right]. \quad (61)$$

Eq. (60) *should be used when b is greater than c, and* eq. (61) *when c is greater than b*, because the series in the parentheses are then very rapidly converging, and not diverging. It will never be necessary to take more than three or four terms and one, only, will ordinarily be sufficient. The following are the values of

$$\left(1 - \tanh \frac{n\pi}{2}\right)$$

for a few values of n:

$$\left(1 - \tanh \frac{n\pi}{2}\right) = 0.083 : 0.00373 : 0.000162 : 0.000007;$$

$$n = 1 : 2 : 3 : 4$$

Square Section.

If $c = b$ either eq. (60) or eq. (61) gives

$$M = G\alpha b^4 \left[\frac{1}{3} - 0.2101 + 0.209 \left(1 - \tanh \frac{\pi}{2} \right) \right];$$

$$\therefore \ M = 0.1406\, G\alpha b^4 = G\alpha \frac{A^4}{42.7\, I_p}, \quad . \quad . \quad (62)$$

in which A is the area $(=b_2)$ and I_p is the polar moment of inertia $\left(=\frac{b^4}{6}\right)$.

Rectangle in which $b=2c$.

If $b=2c$, eq. (60) gives

$$M=G\alpha\,.\,2c^4\left(\frac{1}{3}-0.105+0.1046\ (1-\text{tah}\ \pi)\right);$$

$$\therefore\ M=0.457\ G\alpha c^4=G\alpha\frac{A^4}{42\ I_p}, \quad . \quad . \quad . \quad (63)$$

in which A is the area $(=2c^2)$ and $I_p=$polar moment of inertia

$$=\frac{bc^3+b^3c}{12}=\frac{5c^4}{6}.$$

Rectangle in which $b=4c$.

If $b=4c$, eq. (60) then gives

$$M=G\alpha bc^3\left(\frac{1}{3}-0.0525\right)=1.123\ G\alpha c^4;$$

$$\therefore\ M=G\alpha\frac{A^4}{40.2\ I_p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

in which $A=$area$=4c^2$ and $I_p=$polar moment of inertia

$$=\frac{bc^3+b^3c}{12}=\frac{17c^4}{3}.$$

If b is greater than $2c$, it will be sufficiently near for all ordinary purposes to write

$$M = G\alpha\frac{bc^3}{3}\left(1 - 0.63\frac{c}{b}\right). \quad . \quad . \quad . \quad . \quad (65)$$

Greatest Intensity of Shear.

There yet remains to be determined the greatest intensity of shear at any point in a section, and in searching for this quantity it will be convenient to use eqs. (28) and (29).

It will also be well to observe that by changing z to y, y to $-z$, c to b, and b to c, in eq. (57), there may be at once written

$$u = \alpha zy - \left(\frac{2}{\pi}\right)^3 . \alpha b^2 \sum_1^\infty \frac{(-1)^{n-1}\sin\left(\frac{2n-1}{b}\pi z\right).\text{sih}\left(\frac{2n-1}{b}\pi y\right)}{(2n-1)^3\text{coh}\left(\frac{2n-1}{2b}\pi c\right)}. \quad (66)$$

This amounts to turning the co-ordinate axes 90°.

Since the resultant shear at any point is

$$T = \sqrt{T_2{}^2 + T_3{}^2},$$

it will be necessary to seek the maximum of

$$\left(\frac{du}{dy} + \alpha z\right)^2 + \left(\frac{du}{dz} - \alpha y\right)^2 = \frac{T^2}{G^2}.$$

The two following equations will then give the points desired:

$$\frac{d\left(\frac{T^2}{G^2}\right)}{dy}=\left(\frac{du}{dy}+\alpha z\right)\frac{d^2u}{dy^2}+\left(\frac{du}{dz}-\alpha y\right)\left(\frac{d^2u}{dz\,dy}-\alpha\right)=0; \quad (67)$$

$$\frac{d\left(\frac{T^2}{G^2}\right)}{dz}=\left(\frac{du}{dy}+\alpha z\right)\left(\frac{d^2u}{dz\,dy}+\alpha\right)+\left(\frac{du}{dz}-\alpha y\right)\frac{d^2u}{dz^2}=0. \quad (68)$$

It is unnecessary to reproduce the complete substitutions in these two equations, but such operations show that *the points of maximum values of T are at the middle points of the sides of the rectangular sections*, omitting the evident fact that $T=0$ at the centre. It will also be found that *the greatest intensity of shear will exist at the middle points of the greater sides.*

This result may be reached independent of any analytical test, by bearing in mind that an elongated ellipse closely approximates a rectangular section, and it has already been shown that the greatest intensity in an elliptical section is found at the extremities of the smaller axis.

By the aid of eqs. (28), (29), (57), and (66), it will also be found that $T_3=0$ at the extremities of the diameter c, and $T_2=0$ at the extremities of the diameter b. The maximum value of T will then be

$$T_m=-T_2=-G\left(\frac{du}{dz}-\alpha y\right)_{\substack{z=0\\ y=\frac{c}{2}}} \quad . \quad . \quad . \quad . \quad (69)$$

By the use of eq. (57)

$$\frac{du}{dz}-\alpha y=$$

$$-2\alpha y+\left(\frac{2}{\pi}\right)^3.\alpha c^2\sum_1^\infty\frac{(-1)^{n-1}.\frac{\pi}{c}.\sin\left(\frac{2n-1}{c}\pi y\right).\mathrm{coh}\left(\frac{2n-1}{c}\pi z\right)}{(2n-1)^2\,\mathrm{coh}\left(\frac{2n-1}{2c}\pi b\right)}.$$

Putting $z=0$ and $y=\frac{c}{2}$ in this equation, there will result

$$T_m=G\alpha c\left[1-\frac{8}{\pi^2}\sum_1^\infty\frac{1}{(2n-1)^2\operatorname{coh}\frac{2n-1}{2c}\pi b)}\right]. . \quad (70)$$

If b is greater than c the series appearing in this equation is very rapidly convergent, and it will never be necessary to use more than two or three terms if the section is square, and if b is four or five times c there may be written

$$T_m=G\alpha c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Square Section.

Making $b=c$ in eq. (70), and making $n=1$, 2, and 3 (i.e., taking three terms of the series), there will result

$$T_m=0.676\,G\alpha c; \quad \therefore G\alpha=1.48\frac{T_m}{c}.$$

Inserting this value in eq. (62),

$$M=0.21\,b^3T_m=\frac{1.26IT_m}{a}. \quad . \quad . \quad . \quad . \quad . \quad (72)$$

$$\therefore\ T_m=0.8\frac{M}{I}a=5\frac{M}{b^3}, \quad . \quad . \quad . \quad . \quad . \quad (73)$$

in which

$$I=\frac{b^4}{12} \quad \text{and} \quad a=\frac{b}{2}=\frac{c}{2}.$$

Rectangular Section; $b=2c$.

Making $b=2c$ in eq. (70), and making $n=1$, only, there will result

$$T_m = 0.93\,G\alpha c; \quad \therefore\ G\alpha = 1.08\frac{T_m}{c}.$$

Inserting this value in eq. (63),

$$M = 0.49\,c^3T_m = 1.47\frac{IT_m}{a}. \quad . \quad . \quad . \quad . \quad (74)$$

$$\therefore\ T_m = 0.68\frac{M}{I}a = 2\frac{M}{c^3}, \quad . \quad . \quad . \quad . \quad (75)$$

in which

$$I = \frac{bc^3}{12} = \frac{c^4}{6} \quad \text{and} \quad a = \frac{c}{2}.$$

Rectangular Section; $b=4c$.

Making $b=4c$ in eq. (70), and making $n=1$, only,

$$T_m = 0.997\,G\alpha c; \quad \therefore\ G\alpha = 1.003\frac{T_m}{c}.$$

Inserting this value in eq. (64),

$$M = 1.126\,c^3T_m = 1.69\frac{IT_m}{a}. \quad . \quad . \quad . \quad (76)$$

$$\therefore\ T_m = 0.6\frac{M}{I}a = 0.9\frac{M}{c^3}, \quad . \quad . \quad . \quad . \quad . \quad (77)$$

in which

$$I = \frac{bc^3}{12} = \frac{c^4}{3} \quad \text{and} \quad a = \frac{c}{2}.$$

Circular Section about its Centre.

The torsion of a circular cylinder furnishes the simplest example of all.

If r_0 is the radius of the circular section, the polar equation of that section is

$$\frac{r_0^2}{2}=C \text{ (constant).}$$

Comparing this equation with eq. (33), it is seen that

$$c_1=c_2=c_3= \dots \quad =c_1'=c_2'= \dots \quad =0.$$

By eq. (32) this gives $u=0$. Hence *all sections remain plane during torsion.*

Eqs. (19) and (20) then give

$$T_{xr}=0 \quad \text{and} \quad T_{x\phi}=G\alpha r. \quad . \quad . \quad . \quad . \quad . \quad (78)$$

Eq. (23) gives for the moment of torsion

$$M=G\alpha I_p, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

or

$$M=0.5\ \pi r_0^4 . G\alpha=\frac{A^4G}{4\pi^2 I_p}\alpha; \quad . \quad . \quad . \quad . \quad . \quad (80)$$

in which equation A is the area of the section and

$$I_p=\frac{\pi r_0^4}{2}.$$

The greatest intensity of shear in the section will be obtained by making $r = r_0$ in eq. (78), or

$$T_m = G\alpha r_0; \quad \therefore G\alpha = \frac{T_m}{r_0}. \quad . \quad . \quad . \quad . \quad . \quad (81)$$

Eq. (80) then becomes

$$M = 0.5\,\pi r_0^3 T_m = 2\frac{IT_m}{r_0}. \quad . \quad . \quad . \quad . \quad . \quad (82)$$

$$\therefore T_m = 0.64\frac{M}{r_0^3} = 0.5\frac{M}{I}r_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

in which $$I = \frac{\pi r_0^4}{4}.$$

It is thus seen that the circular section is the only one treated which remains plane during torsion.

General Observations.

The preceding examples will sufficiently exemplify the method to be followed in any case. Some general conclusions, however, may be drawn from a consideration of eq. (33).

If the perimeter is symmetrical about the line from which ϕ is measured, then r must be the same for $+\phi$ and $-\phi$; hence

$$c_1' = c_2' = c_3' = \ldots = 0.$$

If the perimeter is symmetrical about a line at right angles to the zero position of r, then r must be the same for

$$\phi = 90° + \phi' \quad \text{and} \quad 90° - \phi';$$

hence

$$c_1 = c_3 = c_5 \ldots \quad = c_2' = c_4' = c_6' = \ldots = 0.$$

In connection with the first of these sets of results, eq. (32) shows that *every axis of symmetry of sections represented by* eq. (33) *will not be moved from its original position by torsion.*

If the section has two axes of symmetry passing through the origin of co-ordinates, then will all the above constants be zero, and its equation will become

$$\frac{r^2}{2} + c_2 r^2 \cos 2\phi + c_4 r^4 \cos 4\phi + c_6 r^6 \cos 6\phi + \ldots \quad = K.$$

Art. 7.—Torsional Oscillations of Circular Cylinders.

Two cases of torsional oscillations will be considered, in the first of which the cylindrical body twisted is supposed to be the only one in motion. In the second case, however, the mass of the twisted body will be neglected, and the motion of a heavy body, attached to its free end, will be considered. In both cases the section of the cylinder will be considered circular.

Since these cases are those of motion, the internal stresses are not, in general, in equilibrium; hence equations of motion must be used, and those of Art. 3 are most convenient. Of these last, the investigations of the preceding article show that eq. (4) is the only one which gives any conditions of motion in the problem under consideration.

Putting the value of

$$T = T_{\phi x} = G\frac{dw}{dx}$$

in eq. (4) of Art. 3, that equation may take the form

$$\frac{d^2w}{dt^2}=\frac{G}{m}\frac{d^2w}{dx^2}, \quad \text{or} \quad \frac{d^2w}{dt^2}-b^2\frac{d^2w}{dx^2}=0. \quad . \quad . \quad . \quad (1)$$

For brevity, b^2 is written for $\frac{G}{m}$.

That dimension of the cross-section of the body which lies in the direction of the radius will be assumed so small that w may be considered a function of x and t only. The results will then apply to small solid cylinders and all hollow ones with thin walls.

The general integral of eq. (1), on the assumption just made, is (Booles' "Differential Equations," Chap. XV, Ex. 1)

$$w=f(x+bt)+F(x-bt),$$

in which f and F signify *any arbitrary functions whatever.* Now it is evident that all oscillations are of a periodic character, i.e., at the end of certain equal intervals of time, w will have the same value. Hence since f and F are arbitrary forms, and since circular functions are periodic, there may be written

$$w=A_n\{\sin(\alpha_n x+\alpha_n bt)+\sin(\alpha_n x-\alpha_n bt)\} \\ -B_n\{\cos(\alpha_n x+\alpha_n bt)-\cos(\alpha_n x-\alpha_n bt)\}, \quad (2)$$

in which α_n, A_n, and B_n are coefficients to be determined.

Substituting for the sines and cosines of sums and differences of angles,

$$w=2\sin\alpha_n x(A_n\cos\alpha_n bt+B_n\sin\alpha_n bt). \quad . \quad . \quad (3)$$

Let the origin of co-ordinates be taken at the fixed end of the piece, w must then be equal to zero, as is shown by eq. (3). But there may be other points at which w is always equal to zero, whatever value the time t may have. These points, called *nodes*, are found by putting $w=0$, or

$$\sin \alpha x = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation is satisfied by taking

$$\alpha_n = \frac{\pi}{a}, \quad \frac{2\pi}{a}, \quad \frac{3\pi}{a}, \ldots, \frac{n\pi}{a},$$

and $x=a$; in which a is the length of the piece.

Hence *at the distances*

$$a, \frac{a}{2}, \frac{a}{3}, \ldots, \frac{a}{n}$$

from the fixed end of the piece, there will exist sections which are never distorted or moved from their positions of rest. These are called *nodes*, and one is assumed at the free end, although such an assumption is not necessary, since a is really the distance from the fixed end to the farthest *node* and not necessarily to the free end.

If, as is permissible, A_n and B_n be written for twice those quantities, the general value of w now becomes

$$\begin{aligned} w = {} & \sin \frac{\pi x}{a}\left(A_1 \cos \frac{\pi bt}{a} + B_1 \sin \frac{\pi bt}{a}\right) \\ & + \sin \frac{2\pi x}{a}\left(A_2 \cos \frac{2\pi bt}{a} + B_2 \sin \frac{2\pi bt}{a}\right) \\ & + \sin \frac{3\pi x}{a}\left(A_3 \cos \frac{3\pi bt}{a} + B_3 \sin \frac{3\pi bt}{a}\right) \\ & \quad . \quad . \quad . \quad . \quad . \\ & + \sin \frac{n\pi x}{a}\left(A_n \cos \frac{n\pi bt}{a} + B_n \sin \frac{n\pi bt}{a}\right). \quad . \quad . \quad (5) \end{aligned}$$

The coefficients A and B are to be determined by the ordinary procedure for such cases. Let

$$w_1 = \phi(x)$$

be the expression for the initial or known strain at any point, for which the time t is zero. Then if A_n is any one of the coefficients A,

$$A_n = \frac{2}{a}\int_0^a \phi(x) \sin\frac{n\pi x}{a}dx. \quad . \quad . \quad . \quad . \quad (6)$$

The velocity at any point, or at any time, will be given by

$$\frac{dw}{dt} = -\sin\frac{\pi x}{a}\left(A_1 \sin\frac{\pi bt}{a} - B_1 \cos\frac{\pi bt}{a}\right)\frac{\pi b}{a}. \quad . \quad . \quad (7)$$

In the initial condition, when the time is zero, or $t=0$, it has the given, or known, value

$$\frac{dw_1}{dt} = \Phi(x) = \frac{\pi b}{a}\left(B_1 \sin\frac{\pi x}{a} + 2B_2 \sin\frac{2\pi x}{a} + 3B_3 \sin\frac{3\pi x}{a} + \ldots\right).$$

Then, as before,

$$B_n = \frac{2}{n\pi b}\int_0^a \Phi(x) \sin\frac{n\pi x}{a}dx. \quad . \quad . \quad . \quad . \quad (8)$$

Thus the most general value of w is completely determined.

The intensity of shear at any place or time is given by

$$T = G\frac{dw}{dx}$$

w being taken from eq. (5).

The second case to be treated is that of the torsion pendulum, in which the mass of the twisted body is so inconsiderable in comparison with that of the heavy body, or bob, attached to its free end that it may be neglected.

Let M represent the mass of the pendulum bob, and k *its radius of gyration in reference to the axis about which it is to vibrate*, then will Mk^2 be its moment of inertia about the same axis.

The unbalanced moment of torsion, with the angle of torsion α, is, by eq. (9) of Art. 6,

$$G\alpha I_p.$$

The elementary quantity of work performed by this unbalanced couple, if β is the general expression for the angular velocity of the vibrating body, is

$$G\alpha I_p . \beta \, dt.$$

This quantity of energy is equal in amount but opposite in sign to the indefinitely small variation of actual energy in the bob; hence

$$G\alpha I_p \beta dt = -d\left(\frac{Mk_2\beta^2}{2}\right) = -Mk^2\beta d\beta.$$

But if a is the length of the piece twisted,

$$\beta = \frac{d(\alpha a)}{dt} \quad \text{and} \quad d\beta = \frac{d^2(\alpha a)}{dt};$$

$$\therefore \quad \left(\frac{GI_p}{a}\right)(\alpha a) = -Mk^2\frac{d^2(\alpha a)}{dt^2}.$$

Multiplying this equation by $2d(a\alpha)$, and for brevity putting

$$\left(\frac{GI_p}{a}\right)=H, \quad (Mk^2)=K;$$

then integrating and dropping the common factor a^2,

$$H\alpha^2=-K\left(\frac{d\alpha}{dt}\right)^2+C.$$

When $\alpha=\alpha_1$, the value of the angle of torsion at the extremity of an oscillation, the bob will come to rest and $\frac{d\alpha}{dt}$ will be zero. Hence

$$C=H\alpha_1{}^2$$

and

$$K\left(\frac{d\alpha}{dt}\right)^2=H\ (\alpha_1{}^2-\alpha^2).$$

$$\therefore \quad \frac{da}{\sqrt{\alpha_1{}^2-\alpha^2}}=\sqrt{\frac{H}{K}}.dt;$$

$$\therefore \quad \sin^{-1}\frac{\alpha}{\alpha_1}=t\sqrt{\frac{H}{K}}+(C'=0). \quad . \quad . \quad . \quad (9)$$

$C'=0$ because α and t can be put equal to zero together.

At the opposite extremities of a complete oscillation α will have the values

$$(+\alpha_1) \quad \text{and} \quad (-\alpha_1).$$

Putting these values in the expression

$$t=\sqrt{\frac{K}{H}}.\sin^{-1}\frac{\alpha}{\alpha_1}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and taking the difference between the results thus obtained, the following interval of time for a complete oscillation will be found:

$$\tau = \pi \sqrt{\frac{K}{H}} = \pi \sqrt{\frac{Mk^2a}{GI_p}}. \quad . \quad . \quad . \quad . \quad (11)$$

The time required for an oscillation is thus seen to *vary directly as the square root of the moment of inertia of the bob and the length of the piece, and inversely as the square root of the coefficient of elasticity for shearing and the polar moment of inertia of the normal section of the piece twisted.*

The number of complete oscillations per second is $\frac{1}{\tau}$. If this number is the observed quantity, the following equation will give G:

$$G = \left(\frac{1}{\tau}\right)^2 \frac{\pi^2 Mk^2a}{I_p}.$$

The formulas for this case should only be used when the mass of the cylindrical piece twisted is exceedingly small in comparison with M.

Art. 8.—Thick, Hollow Spheres.

In order to investigate the conditions of equilibrium of stress at any point within the material which forms a thick hollow sphere, it will be most convenient to use the equations of Art. 4. As in the case of a thick hollow cylinder, the interior and exterior surfaces of the sphere are supposed to be subjected to fluid pressure.

Let r' and r_1 be the interior and exterior radii, respectively.

Let $-p$ and $-p_1$ be the interior and exterior intensities, respectively.

Since each surface is subjected to normal pressure of uniform intensity *no tangential internal stress can exist*, but normal stresses in three rectangular co-ordinate directions may and do exist. Consequently, in the notation of Art. 4,

$$T_{\phi r} = T_{\psi r} = T_{\psi \phi} = 0.$$

With a given value of r, also, a uniform state of stress will exist. *Neither N_ψ nor N_ϕ can, then, vary with ϕ or ψ.* By the aid of these considerations, and after omitting R_0, Φ_0, Ψ_0, and the second members, the eqs. (1), (2), and (3) of Art. 4 reduce to

$$\frac{dN_r}{dr} + \frac{2N_r - N_\phi - N_\psi}{r} = 0; \quad . \quad . \quad . \quad . \quad (1)$$

$$-N_\psi + N_\phi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

By eq. (2)

$$N_\psi = N_\phi.$$

Eq. (1) then becomes

$$\frac{dN_r}{dr} + 2\frac{N_r - N_\phi}{r} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

On account of the existing condition of stress which has just been indicated it at once results that

$$\eta = \omega = 0,$$

and that ρ is a function of r only.

Eqs. (4) to (10) of Art. 4 then reduce to

$$\theta = \frac{d\rho}{dr} + \frac{2\rho}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$N_r = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{d\rho}{dr}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$N_\psi = N_\phi = \frac{2G\mathfrak{r}}{1 - 2\mathfrak{r}}\theta + 2G\frac{\rho}{r}. \quad . \quad . \quad . \quad . \quad (6)$$

After substitution of these quantities, eq. (3) becomes

$$\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}\left(\frac{d^2\rho}{dr^2}+\frac{2r\,d\rho-2\rho\,dr}{r^2\,dr}\right)+2G\frac{d^2\rho}{dr^2}+4G\frac{d\rho}{r\,dr}-4G\frac{\rho}{r^2}=0,$$

or

$$\frac{d^2\rho}{dr^2}+\frac{\left(d\frac{2\rho}{r}\right)}{dr}=0.$$

One integration gives

$$\frac{d\rho}{dr}+\frac{2\rho}{r}=c=\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Hence θ, the rate of variation of volume, is a constant quantity. Eq. (7) may take the form

$$r\,d\rho+2\rho\,dr=cr\,dr.$$

As it stands, this equation is not integrable, but, by inspecting its form, it is seen that r is an integrating factor. Multiplying both sides of the equation, then, by r,

$$r^2\,d\rho+2r\rho\,dr=d(r^2\rho)=cr^2\,dr;$$

$$\therefore\ r^2\rho=c\frac{r^3}{3}+b;\quad \therefore\ \rho=\frac{cr}{3}+\frac{b}{r^2}. \quad . \quad . \quad . \qquad (8)$$

Substituting from eqs. (7) and (8) in eq. (5),

$$N_r=\frac{2G\mathfrak{r}}{1-2\mathfrak{r}}c+\frac{2Gc}{3}-\frac{4bG}{r^3}; \quad . \quad . \quad . \quad . \qquad (9)$$

$$=A-\frac{4bG}{r^3}.$$

It is obvious what A represents.

When r' and r_1 are put for r, N_r becomes $-p$ and $-p_1$. Hence

$$A - \frac{4bG}{r^3} = -p$$

and

$$A - \frac{4bG}{r_1^{\ 3}} = -p_1.$$

These equations express the conditions involved in eqs. (13), (14), and (15) of Art. 2.

The last equations give

$$4Gb = \frac{(p_1 - p)\, r_1^{\ 3} r'^3}{r'^3 - r_1^{\ 3}};$$

$$\therefore\ A = \frac{p_1 r_1^{\ 3} - pr'^3}{r'^3 - r_1^{\ 3}}.$$

These quantities make it possible to express N_r and N_ϕ independently of the constants of integration, c and b, for those intensities become

$$N_r = \frac{p_1 r_1^{\ 3} - pr'^3}{r'^3 - r_1^{\ 3}} - \frac{(p_1 - p)\, r_1^{\ 3} r'^3}{r'^3 - r_1^{\ 3}} \cdot \frac{1}{r^3}; \quad . \quad (10)$$

$$N_\phi = N_\psi = \frac{p_1 r_1^{\ 3} - pr'^3}{r'^3 - r_1^{\ 3}} + \frac{(p_1 - p)\, r_1^{\ 3} r'^3}{2(r'^3 - r_1^{\ 3})} \cdot \frac{1}{r^2}. \quad . \quad (11)$$

Thus it is seen that $N_\phi = N_\psi$ has its greatest value for the interior surface; that intensity will be called h.

It is now required to find $r_1 - r' = t$ in terms of h, p, and p_1.

If $r = r'$ in eq. (11),

$$2h(r'^3 - r_1^{\ 3}) = 3p_1 r_1^{\ 3} - p(2r'^3 + r_1^{\ 3}).$$

Dividing this equation by r'^3 and solving,

$$\frac{r_1^3}{r'^3} = \frac{2(h+p)}{2h-p+3p_1};$$

$$\therefore\ r_1 - r' = t = r'\sqrt[3]{\frac{2(h+p)}{2h-p+3p_1}} - r'. \quad . \quad . \quad (12)$$

If the intensities p and p_1 are given for any case, eq. (12) will give such a thickness that the greatest tension h (supposing p_1 considerably less than p) shall not exceed any assigned value. If the external pressure is very small compared with the internal, p_1 may be omitted.

The values of A and $4Gb$ allow the expressions for c and b to be at once written.

If p_1 is greater than p, nothing is changed except that $N_\phi = N_\psi$ becomes negative, or compression.

CHAPTER III.

THEORY OF FLEXURE.

Art. 9.—General Formulæ.

If a prismatic portion of material is either supported at both ends, or fixed at one or both ends, and subjected to the action of external forces whose directions are normal to, *and cut*, the axis of the prismatic piece, that piece is said to be subjected to "flexure." If these external forces have lines of action which are oblique to the axis of the piece, it is subjected to combined flexure and direct stress.

Again, if the piece of material is acted upon by a couple having the same axis with itself, it will be subjected to "torsion."

The most general case possible is that which combines these three, and some general equations relating to it will first be established.

The co-ordinates axis of X will be taken to coincide with the axis of the prism, and *it will be assumed that all external forces act upon its ends only.* Since no external forces act upon its lateral surface, there will be taken

$$T_1 = N_2 = N_3 = 0,$$

retaining the notation of Art. 2. These conditions are not strictly true for the general case, but the errors are, at most, excessively small for the cases of direct stress or flexure, or for a combination of the two. By the use of eqs. (12), (21), and (22) of Art. 1 the conditions just given become

$$\frac{r}{1-2r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{dv}{dy}=0; \quad . \quad . \quad . \quad (1)$$

$$\frac{r}{1-2r}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{dw}{dz}=0; \quad . \quad . \quad . \quad (2)$$

$$\frac{dv}{dz}+\frac{dw}{dy}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Eqs. (1) and (2) then give

$$\frac{dv}{dy}-\frac{dw}{dz}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In consequence of eq. (4) eqs. (1) and (2) give

$$\frac{dv}{dy}=\frac{dw}{dz}=-r\frac{du}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

By the aid of eq. (5) and the use of eqs. (11), (13), and (20) of Art. 1, in eqs. (10), (11), and (12) of Art. 2 (in this case $X_0=Y_0=Z_0=0$), there will result

$$2\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{d^2u}{dx\,dy}+\frac{d^2v}{dx^2}=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{d^2u}{dx\,dz}+\frac{d^2w}{dx^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Eqs. (3), (5), (6), (7), and (8) are five equations of condition by which the strains *u*, *v*, and *w* are to be determined.

Let eq. (6) be differentiated in respect to x:

$$2\frac{d^3u}{dx^3}+\frac{d^3u}{dy^2\,dx}+\frac{d^3u}{dz^2\,dx}=0.$$

From this equation let there be subtracted the sum of the results obtained by differentiating eq. (7) in respect to y and (8) in respect to z:

$$2\frac{d^3u}{dx^3}-\frac{d^3v}{dx^2\,dy}-\frac{d^3w}{dx^2\,dz}=0.$$

In this equation substitute the results obtained by differentiating eq. (5) twice in respect to x, there will result

$$\frac{d^3u}{dx^3}=\frac{d^2\left(\frac{du}{dx}\right)}{dx^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This result, in the equation immediately preceding eq. (9), by the aid of eq. (5) will give

$$\frac{d^3v}{dx^2\,dy}=0.$$

After differentiating eq. (7) in respect to y, and substituting the value immediately above,

$$\frac{d^3u}{dy^2\,dx}=\frac{d^2\left(\frac{du}{dx}\right)}{dy^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Eqs. (9) and (10) enable the second equation preceding eq. (9) to give

$$\frac{d^3u}{dz^2\,dx}=\frac{d^2\left(\frac{du}{dx}\right)}{dz^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Let the results obtained by differentiating eq. (7) in respect to z and (8) in respect to y be added:

$$2\frac{d^3u}{dx\,dy\,dz}+\frac{d^3v}{dx^2\,dz}+\frac{d^3w}{dx^2\,dy}=0.$$

The sum of the second and third terms of the first member of this equation is zero, as is shown by twice differentiating eq. (3) in respect to x. Hence

$$\frac{d^3u}{dy\,dz\,dx}=\frac{d^2\left(\frac{du}{dx}\right)}{dy\,dz}=0. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Eqs. (9), (10), (11), and (12) are sufficient for the determination of the form of the function $\frac{du}{dx}$, if it be assumed to be algebraic, for

Eq. (9) shows that x^2 does not appear in it;
" (10) " " y^2 " " " "
" (11) " " z^2 " " " "
" (12) " " yz " " " "

The products xz and xy may, however, be found in the function. Hence if a, a_1, a_2, b, b_1, and b_2 are constants, there may be written

$$\frac{du}{dx}=a+a_1z+a_2y+x(b+b_1z+b_2y). \quad . \quad . \quad . \quad (13)$$

Eq. (5) then gives

$$\frac{dv}{dy}=\frac{dw}{dz}=-r\{a+a_1z+a_2y+x(b+b_1z+b_2y)\}. \quad . \quad (14)$$

Substituting from eq. (13) in eqs. (7) and (8),

$$\frac{d^2v}{dx^2} = -a_2 - b_2x; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$\frac{d^2w}{dx^2} = -a_1 - b_1x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The method of treatment of the various partial derivatives in the search for eqs. (13) and (14) is identical with that given by Clebsch in his "*Theorie der Elasticitat Fester Korper.*"

It is to be noticed that the preceding treatment has been entirely independent of the *form of cross-section* or *direction of external forces.*

It is evident from eqs. (13) and (14) that the constant a depends upon that component of the external force which acts parallel to the axis of the piece and produces tension or compression only. For (pages 9, 10) it is known that if a piece of material be subjected to direct stress only,

$$\frac{du}{dx} = a \quad \text{and} \quad \frac{dv}{dy} = \frac{dw}{dz} = -ra;$$

the negative sign showing that ra is opposite in kind to a, both being constant.

Again, if z and y are each equal to zero, eq. (13) shows that

$$\frac{du}{dx} = a + bx.$$

Hence bx is a part of the rate of strain in the direction of x which is *uniform over the whole of any normal section of the piece of material*, and it varies directly with x. But such a portion of the rate of strain can only be produced by an external force acting parallel to the axis of X, and whose intensity varies directly as x. But in the present case such a force does not exist. Hence b must equal zero.

The eqs. (13), (14), (15), and (16) show that a_1, b_1 and a_2, b_2 are symmetrical, so to speak, in reference to the coordinates z and y, while eqs. (13) and (14) show that the normal intensity N_1 is dependent on those, and no other, constants in pure flexure in which $a=0$. It follows, therefore, that those two pairs of constants belong to the two cases of flexure about the two axes of Z and Y.

No direct stress N_1 can exist in torsion, which is simply a twisting or turning about the axis of X.

Since the generality of the deductions will be in no manner affected, pure flexure about the axis of Y will be considered. For this case

$$a=a_2=b_2=0=b.$$

Making these changes in (13) and (14),

$$\frac{du}{dx}=a_1z+b_1xz; \quad \ldots \ldots \quad (17)$$

$$\frac{dv}{dy}=\frac{dw}{dz}=-r\frac{du}{dx}=-r(a_1z+b_1xz). \quad \ldots \quad (18)$$

$$\therefore \ \theta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=z(a_1+b_1x)(1-2r). \quad \ldots \quad (19)$$

Also,

$$N_1=\frac{2Gr}{1-2r}\theta+2G\frac{du}{dx}.$$

$$\therefore \ N_1=2G(r+1)(a_1+b_1x)z=E(a_1+b_1x)z, \quad . \quad (20)$$

since

$$2G(r+1)=E.$$

Taking the first derivative of N_1,

$$\frac{dN_1}{dz}=E(a_1+b_1x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This important equation gives the law of variation of the intensity of stress acting parallel to the axis of a bent beam, in the case of pure flexure produced by forces exerted at its extremity. That equation proves that in a given normal section of the beam, whatever may be the form of the section, *the rate of variation of the normal intensity of stress is constant; the rate being taken along the direction of the external forces.*

It follows from this that N_1 must vary directly as the distance from some particular line in the normal section considered in which its value is zero. Since the external forces F are normal to the axis of the beam and direction of N_1, and because it is necessary for equilibrium that the sum of all the forces $N_1 dy\, dz$, for a given section, must be equal to zero, it follows that on one side of this line tension must exist, and on the other compression.

Let N represent the normal intensity of stress at the distance unity from the line, b the variable width of the section parallel to y, and let $\varDelta = bdz$. The sum of all the tensile stress in the section will be

$$\int_0^{z'} Nz\varDelta = N\int_0^{z'} z\varDelta.$$

The total compressive stress will be

$$N\int_{-z_1}^{0} z\varDelta.$$

The integrals are taken between the limits 0 and the greatest value of z in each direction, so as to extend over the entire section. In order that equilibrium may exist, therefore,

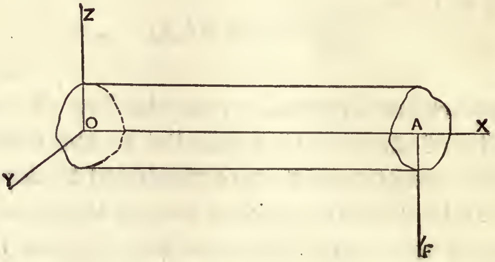


FIG. 1.

$$N\left\{\int_0^{z'} z\Delta + \int_{-z_1}^{0} z\Delta\right\} = 0.$$

$$\therefore \int_{-z_1}^{z'} z\Delta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Eq. (22) *shows that the line of no stress must pass through the centre of gravity of the normal section.*

This line of no stress is called the *neutral axis* of the section. Regarding the whole beam, there will be a surface which will contain all the neutral axes of the different sections, and it is called the *neutral surface* of the bent beam. The neutral axis of any section, therefore, is the line of intersection of the plane of section and neutral surface.

Hereafter the axis of X will be so taken as to traverse the centres of gravity of the different normal sections before flexure. The origin of co-ordinates will then be taken at the centre of gravity of the fixed end of the beam, as shown in Fig. 1.

The value of the expression (a_1+b_1x), in terms of the external bending moment, is yet to be determined. Consider any normal section of the beam located at the distance x from O, Fig. 1, and let $OA=l$. Also, let Fig. 2 represent the section considered, in which BC is the neutral axis and d' and d_1 the distances of the most remote fibres from BC. Let moments of all the forces acting upon the portion $(l-x)$ of the beam be taken about the neutral axis BC. If, again, b is the variable width of the beam, the internal resisting moment will be

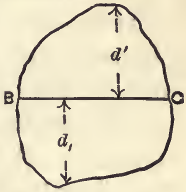


FIG. 2.

$$\int_{-d_1}^{d'} N_1 bz\,dz = E(a_1+b_1x)\int_{-d_1}^{d'} z^2 . bdz.$$

But the integral expression in this equation is *the moment of inertia of the normal section about the neutral axis*, which will hereafter be represented by I. The moment of the external force, or forces, F, will be $F(l-x)$, and it will be equal, but opposite in sign, to the internal resisting moment. Hence

$$F\ l-x) = M = -E(a_1+b_1x)I. \quad . \quad . \quad . \quad (23)$$

$$\therefore\ -(a_1+b_1x) = \frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Substituting this quantity in eq. (16),

$$\frac{d^2w}{dx^2} = \frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

It has already been seen (page 38) that eq. (25) is one of the most important equations in the whole subject of the "Resistance of Materials."

An equation exactly similar to (25) may of course be written from eq. (15); but in such an expression M will represent the external bending moment about an axis parallel to the axis of Z.

No attempt has hitherto been made to determine the complete values of u, v, and w, for the mathematical operations involved are very extended. If, however, a beam be considered whose width, parallel to the axis of Y, is indefinitely small, u and w may be determined without difficulty. The conclusions reached in this manner will be applicable to any long rectangular beam without essential error.

If y is indefinitely small, all terms involving it as a factor will disappear in u and w; or, *the expressions for the strains u and w will be functions of z and x only.* But making u and w functions of z and x *only* is equivalent to a restriction of lateral strains to the direction of z *only*, or to the reduction of the direct strains one half, since direct strains and lateral strains in two directions accompany each other in the unrestricted case. Now as the lateral strain in one direction is supposed to retain the same amount as before, while the direct strain is considered only half as great, the value of their ratio for the present case will be twice as great as that used on pages 9 to 12. Hence $2r$ must be written for r, in order that that letter may represent the ratio for the unrestricted case, and this will be done in the following equations.

Since w and u are independent of y,

$$\frac{dw}{dy}=\frac{du}{dy}=0, \quad \text{and} \quad T_3=G\frac{dv}{dx}.$$

But, by eq. (14),

$$v=-2r(a_1+b_1x)zy+f(x, z).$$

By eq. (3), since

$$\frac{dw}{dy} = 0,$$

$$\frac{dv}{dz} = -2r(a_1 + b_1 x)y + \frac{d}{dz}f(x, z) = 0.$$

This equation, however, involves a contradiction, for it makes $f(x, z)$ equal to a function which involves y, which is impossible. Hence

$$f(x, z) = 0.$$

Consequently

$$\frac{dv}{dz} = -2r(a_1 + b_1 x)y,$$

which is indefinitely small compared with

$$\frac{dv}{dy} = -2r(a_1 + b_1 x)z,$$

and is to be considered zero

Because $f(x, z) = 0$,

$$\frac{dv}{dx} = -2rb_1 zy.$$

This quantity is indefinitely small; hence

$$T_3 = -2Grb_1 zy$$

is of the same magnitude.

Under the assumption made in reference to y, there may be written, from eqs. (17) and (18),

$$u = a_1 xz + b_1 \frac{x^2}{2} z + f'(z); \quad . \quad . \quad . \quad . \quad (26)$$

$$w = -r(a_1 z^2 + b_1 xz^2) + f(x). \quad . \quad . \quad . \quad (27)$$

Using eq. (26) in connection with eq. (6),

$$2b_1z = -\frac{d^2f'(z)}{dz^2}.$$

By two integrations,

$$f'(z) = -\frac{b_1z^3}{3} - c'z + c''. \quad . \quad . \quad . \quad . \quad (28)$$

Using eq. (27) in connection with eq. (8),

$$\frac{d^2f(x)}{dx^2} = -b_1x - a_1.$$

By two integrations,

$$f(x) = -b_1\frac{x^3}{6} - \frac{a_1x^2}{2} + c_1x + c_{11}.$$

The functions u and w now become

$$u = a_1xz + b_1\frac{x^2}{2}z - \frac{b_1z^3}{3} - c'z + c''; \quad . \quad . \quad . \quad (29)$$

$$w = -ra_1z^2 - rb_1xz^2 - b_1\frac{x^3}{6} - \frac{a_1x^2}{2} + c_1x + c_{11}. \quad . \quad (30)$$

The constants of integration c', c'', etc., depend upon the values of u and w, and their derivatives, for certain reference values of the co-ordinates x and z, and also upon the manner of application of the external forces, F, at the end of the beam, Fig. 1. The last condition is involved in the application of eqs. (13), (14), and (15) of Art. 2.

In Fig. 1 let the beam be fixed at O. There will then result, for $x=0$ and $z=0$,

$$\left(\frac{du}{dz}=0\right)_{\substack{x=0\\z=0}};$$

$$(u=0, \quad \text{and} \quad w=0)_{\substack{z=0\\x=0}}.$$

In virtue of the last condition,

$$c''=c_{11}=0.$$

In consequence of the first,

$$c'=0.$$

After inserting these values in eqs. (29) and (30),

$$\frac{du}{dz}=a_1x+b_1\frac{x^2}{2}-b_1z^2;$$

$$\frac{dw}{dx}=-rb_1z^2-b_1\frac{x^2}{2}-a_1x+c_1.$$

$$\therefore \quad T_2=G\left(\frac{du}{dz}+\frac{dw}{dx}\right)=-Gb_1(1+r)z^2+Gc_1. \quad . \quad (31)$$

The surface of the end of the beam, on which F is applied, is at the distance l from the origin O and parallel to the plane ZY. Also, the force F has a direction parallel to the axis of Z. Using the notation of eqs. (13), (14), and (15) of Art. 2, these conditions give

$$\cos p=1, \quad \cos q=0, \quad \cos r=0,$$

$$\cos \pi=0, \quad \cos \chi=0, \quad \cos \rho=1.$$

Since, for $x=l$,

$$M=F(l-x)=0,$$

eqs. (24) and (20) give $N_1=0$ for all points of the end surface. Eq. (15) is, then, the only one of those equations which is available for the determination of c_1.

That equation becomes simply

$$T_2=P.$$

For a given value of z, therefore, any value may be assumed for T_2. For the upper and lower surfaces of the beam let the intensity of shear be zero; or for $z=\pm d$ let $T_2=0$. Hence, by eq. (31),

$$c_1=b_1(1+r)d^2.$$

$$\therefore\ T_2=Gb_1(1+r)(d^2-z^2);$$

$$\therefore\ T_2=\frac{Eb_1}{2}(d^2-z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The constants a_1 and b_1 still remain to be found. The only forces acting upon the portion $(l-x)$ of the beam are F and the sum of all the shears T_2 which act in the section x. Let Δy be the indefinitely small width of the beam, which, since z is finite, is thus really made constant. The principles of equilibrium require that

$$\int_{-d}^{+d}T_2 . \Delta y . dz=Gb_1(1+r)\int_{-d}^{+d}(d^2 . \Delta y . dz-z^2 . \Delta y . dz)=F.$$

The first part of the integral will be $2\Delta yd^3$, and the second part will be the moment of inertia of the cross-section (made rectangular by taking Δy constant) about the neutral axis. Hence

$$2Gb_1(1+r)I=F, \quad \text{or} \quad b_1=\frac{F}{2G(1+r)I}=\frac{F}{EI}. \quad . \quad (33)$$

$$\therefore T_2=\frac{F}{2I}(d^2-z^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

If $x=0$ in eq. (24),

$$a_1=-\frac{Fl}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Thus the two conditions of equilibrium are involved in the determination of a_1 and b_1. The complete values of the strains u and w are, finally,

$$u=\frac{F}{EI}\left(z\frac{x^2}{2}-\frac{z^3}{3}-xzl\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$w=\frac{F}{EI}\left(lrz^2-rxz^2-\frac{x^3}{6}+\frac{lx^2}{2}\right)+\frac{Fd^2x}{2GI}. \quad . \quad . \quad (37)$$

These results are strictly true for rectangular beams of indefinitely small width, but they may be applied to any rectangular beam fixed at one end and loaded at the other, with sufficient accuracy for the ordinary purposes of the civil engineer. It is to be remembered that the load at the end is supposed to be applied according to the law given by eq. (34), a condition which is never realized. Hence these formulæ are better applicable to long than short beams.

The greatest value of T_2, in eq. (34), is found at the neutral axis by making $z=0$; for which it becomes

$$T_2=\frac{Fd^2}{2I}=\frac{3}{2}\cdot\frac{F}{2d}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

$\frac{F}{2d}$ is the *mean intensity* of shear in the cross-section; hence *the greatest intensity of shear is once and a half as great as the mean.*

In eq. (36), if $z=0$, $u=0$. Hence no point of the neutral surface suffers longitudinal displacement.

In eq. (37) the last term of the second member is that part of the vertical deflection due to the shear at the neutral surface, as is shown by eq. (38). The first term of the second member, being independent of x, is that part of the deflection which arises wholly from the deformation of the normal cross-section.

The usual modification of this treatment, designed to supply formulæ for the ordinary experience of the engineer, has already been given in preceding articles.

APPENDIX II.

CLAVARINO'S FORMULA.

In Art. 13 reference is made to Clavarino's formula for thick cylinders. It will be sufficient here to establish the equation for the circumferential or hoop tension in a thick cylinder to illustrate Clavarino's fundamental idea.

If l represents the unit strain in the direction of a tensile force acting alone and whose intensity is T, and if l' is the unit longitudinal strain in the same direction under the same stress T but with two intensities of compressive stress R and S acting at right angles to each other and to the stress T with corresponding direct unit strains l_1 and l_2, and finally if r is the ratio of the lateral strain divided by the direct or longitudinal strain, then will

$$l' = l + rl_1 + rl_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1$$

According to Clavarino's view a lateral strain represents the action or an actual force or stress with an intensity equal to the modulus of elasticity E multiplied by the lateral unit strain. Consequently he considered

$$El' = T' = T + rR + rS. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the case of the thick cylinder T is the intensity of stress originally established by Lamé and given by eq. (16) Art. 5 of Appendix I, while R is the radial compression given by eq. (15) of the same Art., and S is the intensity of longitudinal tensile stress existing if the cylinder has closed ends and it is found by eq. (3);

$$S=\frac{pr'^2-p_1r_1^2}{r_1^2-r'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As S is a tensile stress and causes a negative lateral strain the term rS in eq. (2) must have the negative sign. Again, eq. (15), Art. 5, of Appendix I is so written as to make R negative. Hence, for the present purpose, eq. (2) must be written:

$$El'=T'=T-rR-rS. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting the values of R and T from eqs. (15) and (16), Art. 5, Appendix I, and the value of S from eq. (3), in eq. (4) and taking $r=\frac{1}{3}$,

$$T'=\left(p_1r_1^2-pr'^2+4(p_1-p)\frac{r_1^2r'^2}{r^2}\right)\frac{1}{3(r'^2-r_1^2)}. \quad . \quad (5)$$

If $r=r'$ in eq. (5), the greatest value of T' becomes:

$$T'=\{(r'^2+4r_1^2)p-5p_1r_1^2\}\frac{1}{3(r_1^2-r'^2)}. \quad . \quad . \quad (6)$$

Finally, if $p_1=0$,

$$T'=\frac{r'^2+4r_1^2}{3(r_1^2-r'^2)}p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If the stress $S=0$ the corresponding modifications of the formulæ are obvious.

Eq. (6) gives for the exterior radius;

$$r_1=r'\sqrt{\frac{3T'+p}{3T'-4p+5p_1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

These equations illustrate Clavarino's formulæ. For the reasons given fully in Art. 13, they can be considered approximate only.

Related closely to Clavarino's method is that procedure of arbitrarily assuming $T+\frac{R}{3}=$constant in an analysis of the stresses in the wall of a thick cylinder. At best the results are but approximate.

APPENDIX III.

RESISTING CAPACITY OF NATURAL AND ARTIFICIAL ICE.

In the early part of 1913 two graduating students in Civil Engineering, Messrs. A. F. Lipari and R. M. Marx, at Columbia University, acting under the immediate direction of Mr. J. S. Macgregor, in charge of the testing laboratory of the Department of Civil Engineering, conducted a series of physical tests of natural and artificial ice, both in compression and in flexure. These tests were made with scrupulous care as to the application of loads to test pieces and in the quantitative determination of results. The test pieces in compression were subjected to their loads in the cooling apparatus employed. The compression tests of the natural ice were made with the load applied in some cases normal to its natural surface and in other tests parallel to that surface, in other words normal to its bed and parallel to its bed.

The behavior of the two kinds of ice in the tests was quite different in some respects. A block of clear artificial ice would soon be clouded under a gradual application of loading by the formation of crystals, which finally would determine the lines of compressive failure; while the tendency of the natural ice was to separate and fail in columns. In both cases, however, there was a distinct tendency to shear on oblique planes, making an angle of about 45° with the direction of loading. The separation along these shear planes was distinctly marked in many specimens.

In general the height of the compression test specimens was about twice the greatest cross dimensions, but the largest specimens tested were exceptions to this observation. The accompanying table gives a concise statement of the results of the fifty-seven tests of natural ice in compression and of the thirty-one compressive tests of the artificial ice.

TABLE I.

NATURAL ICE IN COMPRESSION.

Size of Test Pieces.	Number of Tests.	Ult. Comp. Resistances Pounds per Sq. In.		
		Max.	Mean.	Min.
3.25 ins. by 3.75 ins. to 9.8 ins. by 13.9 ins.	57	1132	543	100
ARTIFICIAL ICE IN COMPRESSION				
3 ins. by 3.2 ins. to 10.5 ins. by 10.2 ins.	31	368	185	

The dimensions of the cross-sections of the test pieces are seen to vary greatly. The number of pieces tested with the larger cross-sections was not enough to establish any definite relation between the ultimate compressive resistances per square inch and the areas of the cross-sections of the test pieces. Within the limits of these tests there appears to be little, if any, material variation of ultimate resistance with the increase of cross-section.

It is important to observe that the ultimate resistance of the artificial ice is much less than that of the natural. In fact, the mean ultimate resistance of the natural ice is nearly three times as great as the mean ultimate resistance of the artificial, and about the same relation holds for the maximum intensities.

The temperature of the test pieces as determined by thermo-couples during the actual procedure of testing ranged generally from about +28° Fahr. to about freezing. It is probable that the temperature of the ice was considerably lower than indicated by the apparatus.

The test pieces were not selected with any special care, but were fair averages of natural and artificial ice as ordinarily sold in quantities for the usual purpose of city consumption. Naturally the quality varied materially in many blocks as bought, causing correspondingly wide variations in the ultimate resistances determined. The results of these compressive tests show that sound natural ice at about the temperatures indicated may be expected to give on the average an ultimate resistance of about 500 lbs. per sq. in., with a range of perhaps 100 to 1000 lbs. per sq. in. The artificial ice tested appears to have had about one-third the ultimate resistance only of the natural ice.

In some cases the test pieces of natural ice appeared to give somewhat greater ultimate resistances when tested on their beds than when tested on edge. In scrutinizing the whole list, however, there appears to be but little, if any, difference. Hence no distinction of this kind has been made in Table I, but all the tests have been treated as of one group.

Table II shows the results of testing beams of both natural and artificial ice with loads applied at the centre of span. The effective span in all cases was 18 inches. The normal cross-sections of the beams were square and varied but little from 3.5 inches by 3.5 inches. There were nine such tests of beams of natural ice and twelve of beams of artificial ice. The modulus of rupture is the usual so-called intensity of stress in the extreme fibre. It is difficult to state whether the ice failed by tension or compression. In some cases there was evidence of partial failure at least by internal shear. Some of these beams were placed so as to be loaded on their beds, so to speak, and some on edge, but on the whole there appeared to be little difference in the results. Occasionally there appeared to be a tendency to fail in such manner as to exhibit the "bedding" planes.

Table II

BEAMS OF NATURAL ICE

Load at Centre of Span

Span.	Number of Tests.	Modulus of Rupture. Pounds per Sq. In.		
		Max.	Mean.	Min.
18 ins.	9	351	247	140
BEAMS OF ARTIFICIAL ICE				
18 ins.	12	138	122	85

There is the same inferiority of ultimate resistance of the artificial ice beams as in compression, but the artificial ice beams show a little less than half the modulus of rupture given by the natural ice beams.

INDEX.

m. ske net c
e.l.s