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ELEMENTARY ALGEBRA

BY

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In Memoriam
Irving Stringham

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PREFACE.

THIS book is designed to be at once simple enough for the beginner and complete enough for the most advanced classes of academies and preparatory schools. The first three quarters of it constitute an elementary algebra in the strictest sense of the term; the remainder may be regarded as an intermediate step between elementary and higher algebra, and includes the topics of the most advanced requirements in this subject for admission to American colleges and technical schools.

One of the main differences between this book and its American predecessors lies in the prominence given to problems and the consequent early introduction of the equation. The statement of problems in the form of equations calls forth the pupil's intellectual resources and develops in him the power of concentrated thought. It is an invaluable mental exercise, and one, moreover, in which as a rule pupils take pleasure. Drill in algebraic operations, on the other hand, tends rather to strengthen the memory, to quicken the apprehension, and to cultivate habits of accuracy. Though absolutely necessary to secure facility in manipulating algebraic expressions, this drill is apt not to be interesting. For the sake, therefore, both of giving varied employment to the mental activities and of maintaining an equilibrium of interest, it seems desirable that problems and exercises should proceed together from the very outset. Problems are accordingly introduced at a much earlier stage than usual, and occur with uncommon frequency in every chapter. At first they are so simple that the resulting equations can be solved by elementary arithmetical processes, and they gradually increase in complication with the pupil's increasing knowledge of algebraic methods. The majority of them are either new or else the old ones with new data; the remainder have been selected from a great variety of sources.

The book further differs from its predecessors (1) in the attention given to negative quantities and to the formal laws of algebra, known as the Commutative, the Associative, the Distributive, and the Index laws. In presenting these laws the author has endeavored to be rigorous without sacrificing simplicity. (2) In the fuller development of factoring and in its more extensive application to the solution of equations. The method of solving quadratic equations has been based entirely on the principles of factoring. Certainly this method is more in harmony with the processes of advanced algebra, and it is the author's experience that, even for the beginner, it is quite as simple as the method of completing the square.

The first steps in the book have been simplified for the pupil by building upon his knowledge of arithmetic and adding, one by one, the distinguishing features of algebra; —the use of letters as well as figures to express numbers, the use of equations in the solution of problems, the more extended and systematic use of signs, the meaning and use of negative numbers, and the general proof of theorems. In further recognition of practical requirements, the exercises in Part I have been divided usually into two sets, the first set being as a rule easier than the second. Careful provision is made in both sets for frequent review of topics already studied.

As the author and publisher cannot hope to have been entirely successful in their efforts to keep the text free from typographical and other errors, they will esteem it a favor to have their attention called to any that may have escaped their vigilance.

J. A. G.

NORMAL COLLEGE, NEW YORK,
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PART I

FUNDAMENTAL PRINCIPLES AND OPERATIONS

ELEMENTARY ALGEBRA.

CHAPTER I.

ALGEBRAIC NOTATION AND SYMBOLS.

1. Symbols of Operation.—Algebra treats of the properties and relations of numbers. In this respect algebra agrees with arithmetic.

The fundamental operations of algebra are the same as those of arithmetic. These are addition, subtraction, multiplication, division, involution, and evolution.

These operations are also indicated by the same signs in algebra as in arithmetic. These are $+$ (plus) for addition, $-$ (minus) for subtraction, $\times$ for multiplication, $\div$ for division, a figure placed above at the right (called an exponent) for involution, and $\sqrt{\ }$ (radical) for evolution. These are called *operative* symbols, or symbols of operation.

Multiplication is also indicated by a dot between the factors. Thus, $4 . 5$ means that 4 is to be multiplied by 5.

2. Algebraic Expressions. — Numbers are denoted in algebra by letters as well as by figures. This is one respect in which algebra differs from arithmetic.

When figures are written one after another in arithmetic, the expression denotes the *sum* of the different orders of units denoted by the figures separately. Thus, $324 = 300 + 20 + 4$.

When letters are written one after another in algebra, the expression formed denotes the *product* of the numbers denoted by the individual letters. Thus, $abc = a \times b \times c$.

When figures are used in algebra, they are combined to form numbers in the same way as in arithmetic.

When figures and letters are written one after another, the expression denotes a product of which the numeral and literal parts are factors. Thus, $12bc = 12 \times b \times c$.

Literal expressions are more comprehensive than numeral expressions. Thus, 324 means one number only, while abc represents every product that is composed of three factors, and these factors may be integral, fractional, or surd. Owing to this comprehensiveness of its expressions, algebra is sometimes called *generalized* arithmetic.

To find the value of an algebraic expression is to find the number which it represents on the supposition that its letters stand for particular numbers.

3. Exponents.—When the same letter enters more than once as a factor in a product, the number of times that it enters as a factor is indicated by writing a figure after it at the top. Thus, $ab^2c^3 = a \times b \times b \times c \times c \times c$. The expression is read "a, b square, c cube," or "a, b second, c third."

The number used to denote how many times the same factor occurs in a product is called an *exponent*.

4. Coefficients.—The number used to denote how many times a single letter or a product of two or more letters is taken is written before the letter or product and on a line with it.

The number thus used is called a *coefficient*. Thus, $5x$ denotes that the number x is taken 5 times. That is, $5x = x + x + x + x + x$; while x^5 (x fifth) $= x \times x \times x \times x \times x$. $7abc = abc + abc + abc + abc + abc + abc + abc$.

When no coefficient or exponent is expressed, the number one is to be assumed.

EXERCISE I.

Find the value of the following expressions when $a = 3$, $b = 5$, and $c = 7$:

1. abc.
2. $5abc$.
3. ab^3c.
4. $4a^3bc^2$.
5. $6a^4b^2c$.
6. $12a^3b^2c^2$.
7. $25ab^2c^2$.
8. $40a^5b^2c$.
9. $75a^2bc^2$.
10. $250a^3b^2c$.

11. Find the cost of a oranges at 5 cents a piece.

12. Find the surface of a rectangular board 10 ft. long and a inches wide.

13. There are twenty pages in a book, and on each page there are m lines, and in each line n words. How many words in the book?

14. There are a drawers in a case, a compartments in each drawer and c specimens in each compartment, and there are 25 cases in a room How many specimens in all the cases?

5. **Numeric Values.**—A *magnitude* is any thing which has size or extent, and which is doubled when added to itself. Thus, lengths and distances are magnitudes.

Magnitudes are measured by comparing them with some other magnitude of the same kind, to see how many times they contain it.

The magnitude with which other magnitudes are compared in measurement is called the *unit of measurement*, or the *unit magnitude*.

When the magnitude contains the unit an exact number of times the number which expresses how many times a magnitude contains the unit is called the *numeric value* of the magnitude. This term is also extended to the cases in which the value can be expressed only by a fraction or a surd.

Numerical expressions, whether composed of figures or letters or of both, are called *quantities.* Every algebraic expression is numerical; that is, it represents some number. Hence every algebraic expression is a quantity.

6. Quantitative Symbols.—The symbols which express number are called *quantitative* symbols. In algebra they are both numeral and literal.

7. Terms.—When an algebraic expression is made up of parts separated by signs of operation, the parts separated by the consecutive signs are called *terms.*

Thus, in the expression $5a^3b + c - 12 + ab^2c$, $5a^3b$, c, 12, and ab^2c are terms.

It will be noticed that a term may be a single letter, a number expressed by one or more figures, or a product composed of literal or of literal and numeral factors. The numeral factor of a term is commonly called its *coefficient,* and when no numeral factor is expressed the coefficient is to be regarded as one.

Thus, in the expression $7x^2y - 5a + bc^3$, the coefficient of the first term is 7, of the second 5, of the third 1.

8. Monomials and Polynomials.—An algebraic expression which contains no signs of operation is called a *monomial,* or a one-term expression; one composed of two terms separated by a sign of addition or subtraction, a *binomial,* or a two-term expression; one composed of three terms separated by signs of addition or subtraction, a *trinomial,* or a three-term expression. Expressions which contain more than three terms are sometimes called *multi-*

nomials, and all expressions which contain more than one term are usually classed together as *polynomials*.

To find the value of a polynomial, we must find the value of each of its terms and then add or subtract these values according to the signs before the terms. Every minus term of a polynomial must be subtracted from the sum of the plus terms or from some individual plus term.

When no sign is placed before the first term of a polynomial it is understood to be a plus term.

EXERCISE II.

Find the value of each of the following polynomials, when $a = 3$, $b = 1/2$, and $c = 2/3$:

1. $5 + a^2c - 2abc - 8b^2c + 10a^3bc^2$.

2. $9ac^2 - 24b^3c - 6ab^2 - 18abc^2 + 7a^2c$.

Find the values of the following polynomials when $a = 2$, $b = 3$, $c = 4$, and $d = 5$:

3. $6abcd - 5a^3c - 7ab^2d + 3a^2cd^2$.

4. $3a^2cd^2 + 6abcd - 7ab^2d - 5a^3c$.

5. $3a^2cd^2 - 7ab^2d + 6abcd - 5a^3c$.

6. $-5a^3c + 3a^2cd^2 + 6abcd - 7ab^2d$.

Note that the value of a polynomial remains the same in whatever order its terms are written.

Note also that the value of a polynomial may be found by first adding together the values of its plus terms, and also of its minus terms, and then subtracting the latter sum from the former.

9. Similar Terms. — *Similar terms* are those which agree both in their letters and in their exponents. They need not, however, agree either in their signs or in their coefficients. Thus a^2xy^3, $5a^2xy^3$, $-3a^2xy^3$, are all similar terms.

The similar terms of a polynomial may be combined into one term by performing upon their coefficients the operations indicated by the signs of the term, and using the resulting number as the coefficient of the common literal factors of the terms.

Dissimilar terms cannot thus be combined into one.

Similar plus terms are combined into one plus term by adding their coefficients, similar minus terms are combined into one minus term by adding their coefficients, and a plus and a minus term, when similar, are combined into one by subtracting their coefficients.

EXERCISE III.

Reduce the following polynomials to simpler forms by combining their similar terms:

1. $9a^2b^3 + 10a^3b^2 - 4a^2b^3 - 3a^2b^3 + 12.$
2. $12a - 5b^2 - 6a - 7b^2 - 2a - 3a + 6 - 3.$
3. $-6x^2y + 8 - 3x^2y + 15x^2y - 10 + 7 - 5b^3.$
4. $7a^2y - 12a^2y + 9ay^2 + 9a^2y - a^2y - 7.$
5. $-7a^3x + 12a^3x - 5a^2x^3 - 6a^3x + 8a^3x + 15 - 9.$

CHAPTER II.

EQUATIONS AND PARENTHESES.

A. EQUATIONS.

10. Members of an Equation.—An algebraic expression of equality is called an *equation.* It is composed of two members separated by the sign of equality. The part before the sign of equality is called the *first* member, and the part after the sign, the *second* member.

Thus, $7x - 2x + 6 = 26 + x$ is an equation. $7x - 2x + 6$ is its first member, and $26 + x$ is its second member.

11. Verbal Symbols.—The signs $=$, $>$, $<$, $\therefore$ stand for the phrases "equal to," "greater than," "less than," "therefore" or "then," and are hence called *verbal* signs.

12. Axioms.—A mathematical truth so evident as to be generally accepted without proof is called an *axiom.* The following are important axioms about equations.

1°. If the same quantity or equal quantities be added to equals, the sums will be equal.

2°. If the same quantity or equal quantities be subtracted from equals, the remainders will be equal.

3°. If equals be multiplied by the same quantity or by equal quantities, the products will be equal.

4°. If equals be divided by the same quantity or by equal quantities, the quotients will be equal.

5°. The same powers of equals are equal.

6°. The same roots of equal quantities are equal.

The two following axioms are applicable to all algebraic expressions.

7°. The subtraction of any quantity from an algebraic expression neutralizes the effect of its addition to the expression.

8°. The division of an algebraic expression by any quantity neutralizes the effect of multiplying the expression by the same quantity.

13. Transposition of Terms.—It follows from axioms 1° and 2° that a term may be omitted from one member of an equation and written with the opposite sign in the other without destroying the equality of the members.

Thus, if $7x - 2x + 6 = 26 + x$, then, by axiom 2°,

$7x - 2x - x + 6 = 26 + x - x$, and, by ax. 7°,

$7x - 2x - x + 6 = 26$. Again, by axiom 2°,

$7x - 2x - x + 6 - 6 = 26 - 6$, and, by ax. 7°,

$7x - 2x - x = 26 - 6$.

When a term is omitted in one member and placed with the opposite sign in the other it is said to be *transposed.* A plus term is transposed by subtracting it from each member, and a minus term by adding it to each member.

Combining the similar terms in the last equation, we get

$$4x = 20.$$

14. Collection of Terms.—The combining of the similar terms in an equation is called *collecting* the terms.

15. Division by the Coefficient of x.—Dividing each member of the equation $4x = 20$ by 4 we get, by axiom 4°, $x = 5$.

16. Solution of an Equation.—To *solve* an equation is to find the value in terms of known quantities of the letter in it which represents an unknown quantity.

It is customary to represent known quantities by the first letters of the alphabet and unknown quantities by the last letters, x, y, z, etc.

Among the steps necessary to the solution of an equation are transposition, collection, and division by the coefficient of the unknown quantity.

EXERCISE IV.

Solve each of the following equations, and name and explain each step taken:

1. $9x + 3x - 12 = 5x + 72$.
2. $14y + 8 - 2y = 99 - y$.
3. $8z - 5 + 6 + 2z = 3z + 53$.
4. $1/2x + 3/2x - x + 7 = 27 - 1/3x$.
5. $7/5x - 1/3x - 18 = 72 + 3/4x$.
6. $3x + a = b + 5a$.
7. $ax + b + 3ax = c - 5ax$.

17. Literal Coefficients.—In the seventh example, a may be considered as the coefficient of x in the first term, $3a$ as the coefficient of x in the third term, and $5a$ as the coefficient of x in the last term. Coefficient means *fellow factor*, and in any literal product all the factors but one may be taken as the coefficient of that factor.

18. Algebraic Solution of Problems.—To solve a problem algebraically, we must first obtain an equation in terms of the known and unknown quantities of the problem, and then solve the equation to find the value of the unknown quantities in terms of the known.

e.g. 1. Divide the number 105 into two parts, one of which shall be six times the other.

Let $x =$ the number in the smaller part;

$\therefore\ 6x =$ the number in the larger part, and $6x + x =$ the whole number.

Also $105 =$ the whole number;

$$\therefore\ 6x + x = 105.$$

Collecting, $7x = 105.$

Dividing by 7, $x = 15.$

$$\therefore\ 6x = 90.$$

The numbers are 15 and 90.

e.g. 2. Eight times the smaller of two numbers is equal to 143 minus the larger, and the larger is three times the smaller. Find the numbers.

Let $x =$ the smaller number;

$\therefore\ 3x =$ the larger number.

$$\therefore\ 8x = 143 - 3x.$$

Transposing, $8x + 3x = 143.$

Collecting, $11x = 143.$

Dividing by 11, $x = 13.$

$$\therefore\ 3x = 39.$$

The numbers are 13 and 39.

EXERCISE V.

I.

1. Find two numbers whose difference is 9 and whose sum is 63.

2. Divide 103 into two parts whose difference shall be 13.

3. Find two numbers such that the larger shall be 4 times the smaller, and that 6 times the smaller shall equal 60 plus the larger.

4. Find two numbers such that the larger shall be 5 times the smaller, and that 7 times the smaller shall equal 374 minus 3 times the larger.

5. Divide 450 into three parts such that the second shall

contain twice as many as the third, and the first three times as many as the third.

II.

6. 120 marbles are arranged in 3 piles so that there are twice as many marbles in the first pile as in the second and three times as many in the second as in the third. How many marbles in each pile?

7. In a school there are three grades, and there are three times as many scholars in the lowest grade as in the middle grade and five times as many in the middle grade as in the highest. The whole school numbers 735. How many scholars are there in each grade?

8. A man bought a horse, a carriage, and a harness for 450 dollars. He paid three times as much for the horse as for the harness, and twice as much for the carriage as for the horse. What was the cost of each?

9. A boy bought a speller, an arithmetic, and a history for $2.30. He gave twice as much for the history as for the arithmetic, and three times as much for the arithmetic as for the speller. How much did he pay for each?

10. A boy is three years older than his sister, and has a brother who is five years older than himself. Their united ages are 41 years. How old is he?

19. Clearing Equations of Fractions.—Since a fraction is reduced to its numerator when it is multiplied by its denominator, and since both members of an equation may be multiplied by the same number without destroying their equality, an equation may be freed of a fraction by multiplying both its members by the denominator of the fraction.

e.g. Free the equation $\frac{3x}{5} = 6$ of its fraction.

$$\frac{3x}{5} \times 5 = 6 \times 5. \quad \text{(Why?)}$$

$$\therefore \quad 3x = 30;$$

$$\therefore \quad x = 10.$$

Note that $8 + 4$ multiplied by $2 =$ either $12 \times 2 = 24$ or $8 \times 2 + 4 \times 2 = 16 + 8 = 24$. Also that $8 - 4$ multiplied by $2 =$ either $4 \times 2 = 8$ or $8 \times 2 - 4 \times 2 = 16 - 8 = 8$.

So in general $a + b$ multiplied by $2 = 2a + 2b$, and $a - b$ multiplied by $2 = 2a - 2b$. That is, to multiply any algebraic expression by a number, we must multiply each term of the expression by the number.

If an equation contains two or more fractions it may be freed of all of them by multiplying both its members by the product of all the denominators at once.

e.g. Free the equation $\frac{2x}{3} + \frac{3x}{4} = 8$ of fractions.

Multiplying both members by 12, we get

$$\frac{24x}{3} + \frac{36x}{4} = 96,$$

$$\text{or} \qquad 8x + 9x = 96.$$

Instead of multiplying both members by the product of all the denominators, we may multiply by the least common multiple of the denominators.

e.g. Free the equation $\frac{2x}{3} + \frac{3x}{5} + \frac{4x}{12} = 2$ of fractions.

The L. C. M. of 3, 5, and 12 is 60. Multiplying both members by this, we obtain

$$\frac{120x}{3} + \frac{180x}{5} + \frac{240x}{12} = 120,$$

$$\text{or} \qquad 40x + 36x + 20x = 120.$$

Ex. 1. Divide 150 into two parts such that the first shall be 2/3 of the second.

Let $x =$ the number in the second part;

$\therefore \frac{2x}{3} =$ " " " " first "

Hence $x + \frac{2x}{3} = 150.$

$$\therefore 3x + 2x = 450,$$

or $5x = 450.$

$$\therefore x = 90.$$

$$\therefore \frac{2x}{3} = 60.$$

Hence the parts are 60 and 90.

Ex. 2. Divide $37.20 among four men so that the second shall have 2/3 as much as the first, the third 3/4 as much as the second, and the fourth 5/6 as much as the third.

Let $x =$ number of dollars received by the first,

$\therefore \frac{2x}{3} =$ " " " " " " second,

$\frac{6x}{12} = \frac{x}{2} =$ " " " " " third,

and $\frac{30x}{72} = \frac{5x}{12} =$ " " " " " fourth.

$$\therefore x + \frac{2x}{3} + \frac{x}{2} + \frac{5x}{12} = 37.20,$$

$$\therefore 12x + 8x + 6x + 5x = 446.40,$$

$$\therefore 31x = 446.40,$$

$$\therefore x = 14.40,$$

$$\frac{2x}{3} = 9.60,$$

$$\frac{x}{2} = 7.20,$$

and $$\frac{5x}{12} = 6.00.$$

Hence the first receives \$14.40, the second \$9.60, the third \$7.20, and the fourth \$6.00.

EXERCISE VI.

1. Divide 175 into two parts, so that the first shall be 2/3 of the second.

2. Two men in comparing their ages found that the first was 3/5 as old as the second, and that their united ages were 72 years. How old was each?

3. Divide \$4.89 among four boys so that the second shall receive 3/2 as much as the first, the third 3/4 as much as the second, and the fourth 2/5 as much as the third.

4. A man bought four houses for \$117,000.00. He paid 2/3 as much for the second as for the first, 4/5 as much for the third as for the second, and 3/4 as much for the fourth as for the third. How much did he pay for each?

5. A man buys three horses for \$325.00, and pays four times as much for the first as for the second, and twice as much for the third as for the first. How much does he pay for each?

B. PARENTHESES.

20. Symbols of Aggregation.—To indicate that any portion of an algebraic expression which lies between nonconsecutive signs is to be taken together as a complex term, we enclose the portion within parentheses or brackets.

Thus, in the expression $5 + 4ac - 3(4a + 2b)$, 5 and $4ac$ are simple terms, $3(4a + 2b)$ is a complex term. The 3 may be considered as the coefficient of the parenthesis, and the minus sign means that three times the quantity within the parenthesis is to be subtracted from what precedes it.

The parenthesis does not indicate an operation, but that certain parts of an algebraic expression are to be taken together in an operation. Hence it is called a sign of *aggregation.*

A bar or vinculum, drawn over or under the parts of the expression which are to be taken together in an operation, is often used instead of a parenthesis as a sign of aggregation.

Thus, $$5 + 4ac - 3 \,.\, \overline{4a + 2b}.$$

21. Signs of Parenthetic Terms.—When two or more minus terms occur in an expression, they are to be subtracted from the remaining terms.

Thus, $16 - 6 - 4$ means that both the 6 and the 4 are to be subtracted from 16. The final result will be 6. This is the same result that would be obtained by subtracting 10, the sum of 4 and 6, from 16. That is,

$$16 - 6 - 4 = 16 - (6 + 4).$$

In general,

$$a - b - c = a - (b + c).$$

Again, $16 - 6 + 4$ or $16 + 4 - 6$ means that 6 is to be subtracted from the sum of 16 and 4. We may first take 6 from 16 and add 4 to the result, or we may first add 4 to the 16 and then take 6 from the result. In either case the final result will be 14. This is the same result that would be obtained by taking the difference between 4 and 6 from 16. That is,

$$16 - 6 + 4 = 16 - (6 - 4).$$

In general,

$$a - b + c = a - (b - c).$$

That is, if a parenthesis have a minus sign before it, the sign of every term within the parenthesis must be changed both on putting on and on taking off the parenthesis. This is a very important rule and should be carefully borne in mind.

The expression $16 + (6 - 4)$ means that the difference between 4 and 6 is to be added to 16. The result is 18. This is the same result that would be obtained by first adding 6 to 16 and then taking 4 from the result. That is,

$$16 + (6 - 4) = 16 + 6 - 4.$$

In general,

$$a + (b - c) = a + b - c.$$

That is, if a parenthesis have a plus sign before it, the signs of the terms within it are not to be changed either on putting on or on taking off the parenthesis.

22. Parenthetic Factors.—

$$4(6 - 4) = 4 \times 2 = 8 = 4 \times 6 - 4 \times 4.$$

In general,

$$4(b - c) = 4b - 4c,$$

and $$4a(b - c) = 4ab - 4ac.$$

That is, in removing a parenthesis, every term within the parenthesis must be multiplied by the factors without the parenthesis, and on putting on a parenthesis all factors common to all the terms within the parenthesis may be placed without the parenthesis.

EXERCISE VII.

Remove the parenthesis from each of the following expressions:

1. $3a - 4b - 2a(3b - 4d) + 6.$
2. $3m + 4n - 5c(4x - 5y + c).$
3. $7 + 8(3c - 4b) - 12x.$
4. $5x - a(b + c) + 7a.$
5. $18m + 8(2a - 3b + 4c).$
6. $2x + 3(2x + 7).$

Place the three terms after the first of each of the following expressions within a parenthesis,—first with a minus and then with a plus sign before the parenthesis.

7. $5x - 3a - 6b + 9c + 9.$
8. $7ab - 8bc + 16cd + 24c^2 + 3.$
9. $27 + 6a^2c - 10a^2b + 12a^2.$
10. $10x + 20x^2 + 25a^2x - 35.$

EXERCISE VIII.

I.

1. Find two numbers whose difference is 4, and such that three times the less plus four times the greater shall equal 232 minus eight times the sum of the numbers.

2. Find two numbers whose difference is 6, and such that seven times the greater minus five times the less shall equal 156 minus nine times the sum of the numbers.

3. A man bought a carriage, a horse, and a harness for 720 dollars. He paid three times as much for the horse as

for the harness, and twice as much for the carriage as for the horse and harness together. How much did he pay for each?

4. A merchant received $31,640.00 in three months. The second month he received 80 dollars less than three times as much as he received the first month, and the third month he received 40 dollars less than three times as much as he received the first two months. How much did he receive each month?

5. What number increased by one-half and one-fifth of itself will equal 34?

II.

6. What number increased by two-thirds and three-fourths of itself, and 21 more, will equal three times itself?

7. What number increased by one-half and one-third of itself, and 17 more, will equal 50?

8. What number diminished by three-fourths and one-sixth of itself, and 6 more, will equal 5?

9. What number diminished by two-thirds and one-ninth of itself, and 11 more, will equal one-ninth of itself?

10. Divide 119 into three parts such that the second shall be three times the remainder obtained by subtracting 9 from the first, and the third shall be twice the remainder obtained by subtracting the first from the second.

23. NOTE.—For the present it will be necessary to transpose the terms of an equation in such a way that, after the terms have been collected, the term containing the unknown quantity will be plus.

It makes no difference whether the unknown quantity is finally in the first or the second member of the equation.

e.g. In a school of three grades, one-half the scholars are in the lowest grade, one-third in the middle grade, and 60 in the highest grade. How many scholars in each grade, and in the whole school?

Let $x =$ the number of scholars in the whole school.

$\therefore\ 1/2x =$ the number of scholars in the lowest grade,

$1/3x =$ the number of scholars in the middle grade,

and $60 =$ the number of scholars in the highest grade.

$$\therefore\ 1/2x + 1/3x + 60 = x,$$

or

$$3x + 2x + 360 = 6x,$$

$$\therefore\ 360 = 6x - 3x - 2x,$$

$$\therefore\ 360 = x = \text{whole school}.$$

$$1/2x = 180;\ 1/3x = 120.$$

The equation might have been written

$$x = 1/2x + 1/3x + 60,$$

and all the terms containing x might then have been transferred to the first member.

EXERCISE IX.

I.

1. A bin contains a mixture of rye, barley, and wheat. 2/5 of the grain are rye, 2/7 barley, and 77 bushels are wheat. How many bushels of grain are there in all, and how many of each kind?

2. In an orchard there are three kinds of apple-trees. 2/3 of the trees are baldwins, 2/11 greenings, and 35 are pippins. How many trees are there in all, and how many of each kind?

3. There are four villages on a straight road. The distance from the first to the second is 3/8 of the distance from the first to the fourth, the distance from the second

to the third is 2/5 of that distance, and the distance from the third to the fourth is 18 miles. How far are the villages apart?

4. Louis had four times as many stamps as Howard, and after Louis had bought 80 and Howard had sold 30 they had together 450. How many had each at first?

II.

5. Divide 226 into three parts, such that the first shall be four less than the second and nine greater than the third.

6. In an election 70,524 votes are cast for four candidates. The losing candidates received respectively 812, 532, and 756 votes less than the winning candidate. How many votes did each candidate receive?

7. Four towns *M*, *N*, *S*, and *T* are on a straight road. The distance from *M* to *T* is 108 miles, the distance from *N* to *S* is 2/7 of the distance from *M* to *N*, and the distance from *S* to *T* is three times the distance from *M* to *S*. Find the distance from *M* to *N*, from *N* to *S*, and from *S* to *T*.

CHAPTER III.

NEGATIVE QUANTITIES.

24. Counting.—The fundamental relations of numbers are determined by counting, and the fundamental operations of arithmetic and algebra, when they are performed on integers and result in integers, are simply abbreviated methods of counting.

Numbers may be counted forward or backward. In the former case the numbers obtained are always increasing and in the latter case decreasing. In arithmetic we may count forward indefinitely, but backward only to zero.

Counting forward is counting on, or addition; counting backward is counting off, or subtraction. In arithmetic subtraction is impossible when the number to be subtracted, or counted off, contains more units than the number from which it is to be subtracted, or counted off. $8 - 12$ represents an operation which is arithmetically impossible.

In algebra the operation is generalized, and counting off is considered to be as unlimited as counting on. Numbers, instead of running only forward from zero as in arithmetic, are considered as running backward from zero as well.

25. Signs of Quality.—In arithmetic the scale of numbers begins at zero and runs forward only, while in algebra it runs both ways from zero at the centre. To indicate in which part of the algebraic scale a number belongs, the forward part of the scale is called the *positive* part, and the numbers in this part of the scale are either written without a sign or are preceded by a plus sign. The numbers are called *positive* numbers, and the plus sign so used is called the *positive* sign. The backward part of the scale is called the *negative* part, and numbers in this part of the scale are written with a minus sign before them. These numbers are called *negative* numbers, and the minus sign so used is called the *negative* sign.

The signs + and − perform a double office in algebra. They indicate the operations of addition and subtraction, and also whether a quantity is to be taken in the positive or the negative sense. In the former case they are properly called plus and minus, and are symbols of operation and in the latter, positive and negative, and are symbols of quality or sense. When a term stands alone the sign before it is to be regarded as positive or negative.

A term standing alone without a sign is understood to be positive.

26. The Algebraic Scale of Numbers.—Counting along the algebraic scale towards the positive end is counting on, or in the positive direction, and counting along the scale towards the negative end is counting off, or in the negative direction.

The algebraic scale may be represented by a horizontal line of numbers with zero at the centre and the consecutive numbers differing by a single unit, those to the right of zero being distinguished by the positive sign, and those to the left of zero by the negative sign. Thus,

$$\overset{-}{13},\ \overset{-}{12},\ \overset{-}{11},\ \overset{-}{10},\ \overset{-}{9},\ \overset{-}{8},\ \overset{-}{7},\ \overset{-}{6},\ \overset{-}{5},\ \overset{-}{4},\ \overset{-}{3},\ \overset{-}{2},\ \overset{-}{1},\ 0,$$

$$\overset{+}{1},\ \overset{+}{2},\ \overset{+}{3},\ \overset{+}{4},\ \overset{+}{5},\ \overset{+}{6},\ \overset{+}{7},\ \overset{+}{8},\ \overset{+}{9},\ \overset{+}{10},\ \overset{+}{11},\ \overset{+}{12},\ \overset{+}{13}.$$

Counting along this line from any point towards the right is counting forward, or *positively*, and from any point towards the left is counting backward, or *negatively*.

e.g. Beginning at minus five and counting positively, we have minus five, minus four, minus three, minus two, minus one, zero, one, two, three, four, five, etc. In this case each new number mentioned is one greater than the last, minus four being one greater than minus five.

Beginning at five and counting negatively, we have five, four, three, two, one, zero, minus one, minus two, minus three, minus four, minus five, etc. In this case each new number mentioned is one less than the last.

Whatever a positive unit may be, the corresponding negative unit is something just the opposite.

27. Absolute and Actual Values of Numbers. — The absolute value of a number is the number of units in it irrespective of their sign, while its actual value is its value due to the number and sign of its unit. As the absolute value of a positive number increases, its actual value also increases, but as the absolute value of a negative number *increases*, its actual value *decreases*.

28. Algebraic Addition and Subtraction of Integers.— $+4$ or simply 4 means the number obtained by beginning at zero and counting four steps forward, and -4 means the number obtained by beginning at zero and counting four steps backward.

In general $+a$ or a means the number obtained by beginning at zero and counting a steps forward, and $-a$ means the number obtained by beginning at zero and counting a steps backward.

$6+(+4)$ means the operation of beginning at plus 6 on the scale and counting four steps forward, or in the direction indicated by the sign of the number to be added.

$6+(-4)$ means the operation of beginning at plus 6 on the scale and counting four steps backward.

$6 - (+4)$ means the operation of beginning at plus 6 on the scale and counting four steps backward, or in the opposite direction to that indicated by the sign of the number to be subtracted.

$6 - (-4)$ means the operation of beginning at plus 6 on the scale and counting four steps forward, or in the opposite direction to that indicated by the sign of the number to be subtracted.

NOTE. $6 + (+4)$ and $6 + (-4)$ having the meanings given, which are really definitions of addition of a positive and a negative quantity, $6 - (+4)$ and $6 - (-4)$ *must* have the meanings given them because of subtraction being the *inverse,* or opposite, of addition.

In general, the placing of one number after another with a plus sign between indicates the operation of beginning on the scale at the first of the two numbers and counting as many steps as there are units in the number to be added and in the direction indicated by the sign of that number.

The placing of one number after another with a minus sign between indicates the operation of beginning on the scale at the first of the two numbers and counting as many steps as there are units in the number to be subtracted, and in the opposite direction to that indicated by the sign of that number.

EXERCISE X.

Find by actual counting on the scale the values of the following expressions:

I.

1. $12 + (+6)$.
2. $12 + (-6)$.
3. $6 + (+12)$.
4. $-6 + (+12)$.
5. $6 + (-12)$.
6. $-12 + (+6)$.

7. $-6 + (-12)$.

8. $-12 + (-6)$.

9. $12 - (-6)$.

10. $-12 - (-6)$.

11. $4 - (+4)$.

12. $4 + (-4)$.

13. $a - (+a)$.

14. $a + (-a)$.

15. $-6 - (+12)$.

16. $-6 - (-12)$.

17. $a - (-a)$.

18. $-a - (+a)$.

19. Designate the pairs of operations above which give precisely the same result.

29. Corresponding Positive and Negative Numbers.—Every positive number in algebra has a corresponding negative number, that is, a number the same distance from zero on the opposite side.

The sum of a positive number and its corresponding negative number is zero. Thus,

$$6 + (-6) = 0, \quad a + (-a) = 0.$$

30. Special Signs of Quality.—To indicate whether the number to be added or subtracted is positive or negative, instead of enclosing the number with an ordinary plus or minus sign before it within a parenthesis, we may simply put a small plus or minus sign before the number at the top, and when the number is positive the small plus sign may be omitted. Thus,

$a + (+b)$ may be written $a + {}^{+}b$ or $a + b$.

$a + (-b)$ may be written $a + {}^{-}b$.

$a - (+b)$ may be written $a - {}^{+}b$ or $a - b$.

$a - (-b)$ may be written $a - {}^{-}b$.

$-a - (-b)$ may be written ${}^{-}a - {}^{-}b$,

etc.

To indicate that the a and b may represent either positive or negative numbers we may write ${}^{\pm}a + {}^{\pm}b$.

31. Commutative Law of Addition.—From examples 1 and 3 in Exercise X we see that $a + b = b + a$; from examples 7 and 8, that ${}^{-}a + {}^{-}b = {}^{-}b + {}^{-}a$; from examples 5 and 6, that ${}^{-}a + b = b + {}^{-}a$; and from examples 2 and 4, that $a + {}^{-}b = {}^{-}b + a$.

Whence we have the following general law:

$$ {}^{\pm}a + {}^{\pm}b = {}^{\pm}b + {}^{\pm}a. $$

In words, the algebraic sum of two numbers is the same no matter in what order the numbers are taken.

This is known as the *Commutative Law of Addition.*

32. Addition and Subtraction of Corresponding Numbers.—Show by actual counting on the algebraic scale that

$$ 8 + {}^{-}4 = 8 - {}^{+}4, \quad \text{or} \quad 8 - 4 = 4 $$

$$ {}^{-}8 + {}^{+}4 = {}^{-}8 - {}^{-}4 = {}^{-}4. $$

Also that

$$ 8 + {}^{+}4 = 8 - {}^{-}4 = 12 $$

and $$ {}^{-}8 + {}^{+}4 = {}^{-}8 - {}^{-}4 = -4. $$

In general,

$$ {}^{\pm}a + {}^{-}b = {}^{\pm}a - {}^{+}b, \quad \text{or} \quad {}^{\pm}a - b $$

and $$ {}^{\pm}a + {}^{+}b = {}^{\pm}a - {}^{-}b. $$

Whence $$ {}^{\pm}a + {}^{\pm}b = {}^{\pm}a - {}^{\mp}b. $$

In words, *the addition of any number has precisely the same effect as the subtraction of the corresponding number* *with the reverse sign. And the subtraction of any number has precisely the same effect as the addition of the corresponding number with the reverse sign.* This is one of the most important theorems of algebra.

33. Associative Law of Addition. — Show by actual counting on the algebraic scale that

$$
\begin{aligned}
8 + 5 - 3 - 4 &= (8 + 5) - 3 - 4, \\
&= 8 + (5 - 3) - 4, \\
&= (8 + 5 - 3) - 4, \\
&= 8 + (5 - 3 - 4), \\
&= 8 + 5 - (3 + 4) = 6.
\end{aligned}
$$

In general,

$$
\begin{aligned}
{}^{\pm}a + {}^{\pm}b - {}^{\pm}c - {}^{\pm}d &= ({}^{\pm}a + {}^{\pm}b) - {}^{\pm}c - {}^{\pm}d, \\
&= {}^{\pm}a + ({}^{\pm}b - {}^{\pm}c) - {}^{\pm}d, \\
&= ({}^{\pm}a + {}^{\pm}b - {}^{\pm}c) - {}^{\pm}d, \\
&= {}^{\pm}a + ({}^{\pm}b - {}^{\pm}c - {}^{\pm}d), \\
&= {}^{\pm}a + {}^{\pm}b - ({}^{\pm}c + {}^{\pm}d).
\end{aligned}
$$

In words, the sum of three or more numbers is the same in whatever way the numbers may be aggregated. This is known as the *Associative Law of Addition.*

N.B.—When terms are associated with a negative sign before the sign of aggregation, the signs of all the terms within the sign of aggregation must be reversed. **(21.)**

34. Oppositeness of Positive and Negative Numbers.—Positive and negative signs always imply oppositeness. In case of abstract numbers, a negative number is simply the opposite of a positive number; that is, a number which would produce zero when added to its corresponding positive number. Positive and negative numbers always tend to cancel each other.

In the case of concrete numbers, a negative number is the result of a measurement in the opposite direction to that which gives a positive number.

Thus, distances measured to the right or upward are usually regarded as positive, and those measured to the left or downward as negative. Dates after a certain era are regarded as positive, and those before the era as negative. Degrees of temperature above zero are positive, while those below zero are negative.

Assets are usually regarded as positive, and debts as negative.

A surplus is positive, and a deficiency negative.

The following quotation is from Dupuis' *Principles of Elementary Algebra:*

"If an idea which can be denoted by a quantitative symbol has an opposite so related to it that one of these ideas tends to destroy the other or to render its effects nugatory, these two ideas can be algebraically and properly represented only by the opposite signs of algebra.

"If a man buys an article for b dollars and sells it for s dollars, his gain is expressed by $s - b$ dollars. So long as $s > b$, this expression is $+$, and gives the man's gain.

"But if $s < b$, the expression is $-$. It denotes that whatever his gain is now, it is something exactly opposite in character to what it was before. And as he now sells for less than he buys for, he loses. In other words, a negative gain means loss.

"Thus, *gain and loss* are ideas which have that kind of oppositeness which is expressed by oppositeness in sign. If a man gains $+ a$ dollars, he is so much the wealthier: if he gains $- a$ dollars, he is so much the poorer.

"Whether gain or loss is to be considered positive must be a matter of convenience, but only opposite signs can denote the opposite ideas.

"Among the ideas which possess this oppositeness of character are the following:

"(1) To receive and to give out; and hence, to buy and to sell, to gain and to lose, to save and to spend, etc.

"(2) To move in any direction and in the opposite direction; and hence, measures or distances in any direction and in the opposite direction, as east and west, north and south, up and down, above and below, before and behind, etc.

"(3) Ideas involving time past and time to come; as, the past and the future, to be older and to be younger than, since and before, etc.

"(4) To exceed and to fall short off; as, to be greater than and to be less than, etc."

EXERCISE XI.

Give the meaning of the following expressions:

I.

1. -6 A.D.
2. ^{-}n A.D.
3. $^{-}40$ B.C.
4. ^{-}a B.C.
5. $-(-30)$ B.C.
6. $-\,^{-}b$ B.C.
7. $-\,^{-}50$ A.D.
8. $-(-c)$ A.D.
9. The temperature is $-20°$.
10. The temperature has risen $-12°$.
11. The temperature has fallen $-16°$.
12. The temperature has fallen $-(-7°)$.
13. The temperature has fallen $-\,^{-}8°$.
14. The temperature has risen $-\,^{-}a°$.

15. It is – 17° colder to-day than yesterday.

16. It is – 8° warmer to-day than yesterday.

17. It is $-\,{}^{-}12^\circ$ warmer to-day than yesterday.

18. Howard lives – 3 miles east of Albert.

II.

19. Louis lives – 5 miles north of Horace.

20. Ethel is – 4 years older than Edith.

21. Mabel is – 6 years younger than Florence.

22. Hilda is – (– 2) years younger than Margaret.

23. Hermon owes the grocer – 3 dollars.

24. Hilda weighs – 7 pounds more than Louis.

25. Mr. Crane is – 20,000 dollars richer than Mr. Weston.

EXERCISE XII.

1. A man having c dollars paid out a dollars to one person and b dollars to another. Express in two ways what he had left.

2. A man bought at a market tomatoes at a cents a peck and potatoes at b cents a peck, and paid m cents for an equal number of pecks of each. How many pecks did he buy?

3. Two cities are 42 miles apart. Two men start at the same time from the two cities and walk towards each other. The first travels four miles an hour and the second three miles an hour. In how many hours will they meet and how far will each have travelled?

4. Two cities are a miles apart. Two men start at the same time from the two cities and travel towards each

other, the first at the rate of m miles an hour, and the second at the rate of n miles an hour. In how many hours will they meet, and how far will each have travelled?

5. Find two numbers whose sum is 108 and such that 10 times the greater minus 5 times the less shall be less than 762 by 4 times the sum of the numbers.

CHAPTER IV.

ADDITION OF INTEGRAL ALGEBRAIC EXPRESSIONS.

35. Arithmetical and Algebraic Sums.—The sum, or amount, of two or more integral numbers is the number obtained by counting all the numbers together. The operation of finding the sum of two or more numbers is called *addition.*

Since the numbers of arithmetic are all positive, the addition of a number in arithmetic will always increase the number of units in the number to which the addition is made, and the sum of two or more numbers will contain as many units as all the numbers together. The *arithmetic sum* of two or more numbers is the sum of the numbers without regard to their signs. That is, it is the sum of the absolute values of the numbers.

In algebra, the addition of a positive and a negative number will tend to diminish the number of units in the number which has the greater absolute value. The algebraic sum of two such numbers is the arithmetical difference of the numbers with the sign of the one which has the larger absolute value.

The algebraic sum of two numbers both positive or both negative is the arithmetic sum of the numbers with their common sign. Thus,

$$8 + 10 = 18, \qquad {}^-8 + {}^-10 = -18,$$

$$8 + {}^-10 = -2, \qquad {}^-8 + 10 = +2.$$

The *algebraic sum* of two or more numbers is the sum of the numbers regard being had to their signs. That is, it is the sum of the actual values of the numbers.

36. Signs of Coefficients.—The sign of a term may be regarded as belonging to its coefficient only. That is, plus terms may be regarded as those whose coefficients are positive. The reason for this will appear farther on, under Multiplication.

37. Integral Algebraic Expressions. — It has been learned in arithmetic that numbers are not only *integral*, but also *fractional* and *surd*. In any algebraic expression the letters may stand for any kind of number.

An algebraic expression such as

$$x^5 + 5x^4 - 4x^3 - 3x^2 + 2x + 1,$$

or

$$1 + 2x - 3x^2 - 4x^3 + 5x^4 + x^5,$$

in which the exponents of the letters are all positive integers, and in which none of the letters occur in the denominators of fractions, or in the divisors of an indicated division, are called *integral algebraic expressions.* The coefficients of the various terms may be fractional.

38. Extension of the Application of the Formal Laws of Addition.—In the addition of integral algebraic expressions it is assumed that the commutative and associative laws already established for integral numbers apply equally to fractional and surd numbers. This is in accordance with the *generalizing spirit* of algebra.

39. Definition of Addition of Algebraic Expressions.—*To add integral algebraic expressions is to combine their various terms into a single algebraic expression, each term to be preceded by its own proper sign.* The resulting expression should be given in its simplest form.

40. Addition of Monomials and Polynomials.—Similar terms are analogous to concrete numbers of like denominations, and dissimilar terms are analogous to concrete numbers of unlike denominations.

Similar terms may be added by finding the algebraic sum of their coefficients and writing after this the common literal factors of the terms. Thus, the sum of $5a^2b$, $7a^2b$, and $-8a^2b$ is $4a^2b$.

Dissimilar terms can be added only by placing them one after another in a polynomial expression each with its own sign. Thus, the sum of $3a^2b$, $-4ab$, and $5c$ is $3a^2b - 4ab + 5c$. The sum of these dissimilar terms is really $3a^2b + {}^-4ab + 5c$, but, as we have seen, to add ${}^-4ab$ is the same as to subtract ${}^+4ab$, or $+ {}^-4ab = -4ab$.

The following examples will illustrate the working rules of addition:

Ex. **1.**

$$\begin{array}{r} 3a^2x \\ 7a^2x \\ 5a^2x \\ \hline 15a^2x \end{array} \qquad\qquad \begin{array}{r} -7b^3y \\ -9b^3y \\ -5b^3y \\ \hline -21b^3y \end{array}$$

To add similar terms with like signs, annex the common literal factors to the arithmetical sum of the coefficients, and prefix the common sign.

Ex. **2.**

$$\begin{array}{r} 7x^2y^2 \\ 4x^2y^2 \\ -6x^2y^2 \\ 3x^2y^2 \\ -9x^2y^2 \\ \hline -x^2y^2 \end{array} \qquad\qquad \begin{array}{r} 9abx \\ -4abx \\ -2abx \\ 16abx \\ -7abx \\ \hline 12abx \end{array}$$

To add similar terms with unlike signs, find the arithmetical sum of the coefficients of the plus terms, and of the coefficients of the minus terms, and the arithmetical difference of these two sums, annex to this difference the common literal factors, and prefix the common sign of the terms whose coefficients produce the larger arithmetical sum.

Ex. **3.**

$$\begin{array}{r} a \\ b \\ -c \\ \hline a+b-c \end{array} \qquad\qquad \begin{array}{r} 3ax \\ -4by \\ -5d \\ \hline 3ax-4by-5d \end{array}$$

To add dissimilar terms, write them one after another, each with its own sign.

Ex. **4.**

$$\begin{array}{r} -a \\ -b \\ 3a \\ -2b \\ -5 \\ \hline 2a-3b-5 \end{array} \qquad\qquad \begin{array}{r} 3x^2y \\ -7x^2y \\ -6xy^2 \\ -8b \\ -3xy^2 \\ \hline -4x^2y-9xy^2-8b \end{array}$$

To add terms some of which are similar and some dissimilar, combine the different sets of similar terms into single terms, and write the resulting terms together with the remaining terms one after another in a polynomial expression each with its own sign.

Ex. **5.**

$$\begin{array}{rrrrr} 2cd & -\ 3cx^2 & +\ 2c^2x & & \\ -8cd & -\ cx^2 & -\ 5c^2x & +\ cx^3 & \\ 12cd & +\ 10cx^2 & -\ 6c^2x & & -\ 11 \\ \hline 6cd & +\ 6cx^2 & -\ 9c^2x & +\ cx^3 & -\ 11 \end{array}$$

To add polynomials, combine the different sets of similar *terms in the polynomials into single terms, and write these and the remaining terms as a polynomial.*

In the addition of polynomials, it is convenient to arrange the terms so that the similar terms will fall in vertical columns.

41. Simplification of Polynomials.—When any polynomial contains one or more sets of similar terms, it may be simplified by combining these sets into single terms.

EXERCISE XIII.

Find the sum of the following terms:

I.

1. $3a,\ 7a,\ 2a,\ a,\ 12a.$
2. $7a^2x,\ 9a^2x,\ a^2x,\ 20a^2x.$
3. $-5ab^2,\ -ab^2,\ -7ab^2,\ -11ab^2,\ -4ab^2,\ -8ab^2.$
4. $-7x,\ -2x,\ -8x,\ -x,\ -12x,\ -11x,\ -15x.$
5. $3x^2,\ -5x^2,\ 8x^2,\ -12x^2.$
6. $-5ac^2x,\ ac^2x,\ -8ac^2x,\ 14ac^2x.$
7. $5y^2,\ 4ac,\ -ac,\ -7y^2,\ -5ac,\ 4y^2,\ -5.$
8. $7a^2x.\ -4ab,\ -ax^2,\ -3a^2x,\ -8,\ -5ab.$

Simplify the following polynomials:

I.

9. $4x - 5ab + 7x + c + 11ab - 20x.$
10. $3a^2b^3 - 7x^2 - 5 + 12x^2 - 4a^2b^3 + 12 - c.$
11. $1/3x - 1/2x + 3/4x + x.$
12. $2/3y - 3/4y - 2y - 1/3y + 5/6y + y.$
13. $9(a+b) + 10(a+b) - (a+b) - 2(a+b).$

II.

14. $7a - 3(x + y) + 8a - (x + y) + 3(x + y) - 16a$.

15. $2(m + n) + 3(a + b) + (a + b) - (m + n) + (a + b) - 6(m + n)$.

16. $3a(b + x) + 5a(b + x) + 7a(b + x) - 11a(b + x)$.

17. $2c(a^2 - b^2) - 3c(a^2 - b^2) + 6c(a^2 - b^2) - 4c(a^2 - b^2)$.

Add the following polynomials:

I.

18. $3az - 4by - 8$, $-2az + 5by + 6$, $5az + 6by - 7$, and $-8az - 7by + 5$.

19. $8ax - 3cz^2$, $-5ax + 5cz^2$, $ax + 2cz^2$, and $-4ax - 4cz^2$.

20. $8a + b$, $2a - b + c$, $-3a + 5b + 2d$, $-6b - 3c + 3d$, and $-5a + 7c - 2d$.

II.

21. $7x - 6y + 5z + 3 - g$, $-x - 3y - 8 - g$, $-x + y - 3z - 1 + 7g$, $-2x + 3y + 3z - 1 - g$, and $x + 8y - 5z + 9 + g$.

22. $2a^2 + 5ab - xy$, $-7a^2 + 3ab - 3xy$, $-3a^2 - 7ab + 5xy$, and $9a^2 - ab - 2xy$.

23. $5a^3b^2 - 8a^2b^3 + x^2y + xy^2$, $4a^2b^3 - 7a^3b^2 - 3xy^2 + 6x^2y$, $3a^3b^2 + 3a^2b^3 - 3x^2y + 5xy^2$, and $2a^2b^3 - a^3b^2 - 3x^2y - 3xy^2$.

I.

24. A lady bought three yards of ribbon at a cents a yard, 10 yards of tape at c cents a yard, and five spools of thread at d cents a spool. She paid x cents on the bill. How much remains due?

25. One morning the mercury in the thermometer stood at x degrees. During the next 24 hours it rose b degrees and fell c degrees. The following day it rose d degrees. What was its height then?

26. A father divided his property of 27,000 dollars among his four children, giving 500 dollars less to each in succession from the eldest to the youngest. How much did he give to each?

II.

27. A father gave his eldest son x dollars, his second son 7 dollars less, his third son 9 dollars less than the second, and his fourth son 11 dollars less than the third. How much did he give to all?

28. A father divided his property among his four children. To each of the first three he gave 1/4 of his property plus 200 dollars, and to the fourth he gave 1400 dollars. What was the value of his property?

29. A man left his five children x bonds worth a dollars each, and x acres of land worth b dollars each; but he owed m dollars to each of q creditors. What was each child's share of the estate?

42. Aggregation of Coefficients.—When two or more terms of a polynomial contain one or more common factors, whether numeral or literal, the terms may be collected into one by enclosing the terms within a parenthesis and placing the common factors outside.

When the common factors are numeral and literal, it is customary to place the numeral factor and the letters which belong to the first part of the alphabet before the parenthesis, and the letters which belong to the last part of the alphabet after the parenthesis.

e.g. $5acx + 5bcx - 5cdx = 5c(a + b - d)x.$

EXERCISE XIV.

Collect the coefficients of x and y in the following expressions:

I.

1. $ax + by + mx + ny$.

2. $mnx + 2by + pqx - 4by$.

3. $3x - 2y + 6bx - 4y + 7ax + m + n$.

4. $8ax + 8bx + by + 7x - 5y + x - 5y$.

5. Howard is twice as old as Albert. If x represents Albert's age now, what would represent their respective ages eight years hence?

6. Howard is now twice as old as Albert, but 12 years from now he will be only 3/2 as old. How old is each?

7. Two cities, A and B, are on a straight road and 18 miles apart. Two couriers, P and Q, start at the same time from the respective cities and travel in the same direction, P from A towards B at the rate of eight miles an hour, and Q from B at the rate of six miles an hour. In how many hours will P overtake Q, and how far will each have travelled?

8. Divide the number a into two parts, one of which shall exceed the other by b.

II.

9. $ax + by + cz - mx - ny - pz$.

10. $2dx + 3ey + 4fz - 2fx - 3dy + 4ez$.

11. $2/3ay - 2x + 3/4by + 6ax$.

12. $2ax - by - 3bx - 4ay$.

13. Horace is now twice as old as Herbert, but a years from now he will be only 4/3 as old. How old is each ?

14. Two towns, A and B, are a miles apart. Two couriers, P and Q, set out at the same time from the respective towns, and travel in the same direction. P travels from A towards B at the rate of b miles an hour, and Q from B at the rate of c miles an hour. In how many hours will P overtake Q, and how far will each have travelled?

CHAPTER V.

SUBTRACTION OF INTEGRAL ALGEBRAIC EXPRESSIONS.

43. Definition of Subtraction.—Subtraction is the inverse of addition, or the process of undoing the operation of addition. In addition, two numbers are given and their sum or amount required. In subtraction, the sum of two numbers and one of the numbers are given, and the other is required.

The given sum is called the *minuend,* the given number the *subtrahend,* and the required number the *difference* or *remainder.*

Since the minuend is the sum of the subtrahend and difference, we may prove our subtraction by adding the subtrahend and difference to see if their sum agrees with the minuend.

44. Rule for Subtraction of Integral Algebraic Expressions.—We have already seen in section 15 that the addition of any number produces the same effect as the subtraction of the corresponding number with the reverse sign, or, conversely, the subtraction of any number is equivalent to the addition of the corresponding number with the reverse sign. Hence we have the following rule for algebraic subtraction:

Add the subtrahend with its signs reversed to the minuend.

In the operation of subtraction it is better not actually to change the old signs, but merely to think of them as changed in the addition. If the new signs are written, it is better not to change the old into the new, but to write the new as small signs before the terms at the top.

EXERCISE XV.

I.

1. From $2x + y + 7z$ take $5x + 2y - 7z$.

2. From $9a - 4b + 3c$ take $5a - 3b + c$.

3. Subtract $3a^4 - a^2 + 7a - 14$ from $11a^4 - 2a^3 + 3a^2 - 8a$.

4. From $10a^2x^2 + 15ax^2 + 8a^2x$ take $-10a^2x^2 + 15ax^2 - 8a^2x$.

5. Subtract $1 - a + a^2 - 3a^3$ from $a^3 - 1 + a^2 - a$.

6. From $2/3x^2 - 5/2x - 1$ take $-2/3x^2 + x - 1/2$.

7. From a take $b - c$.

8. What must be taken from $6a + 5 - 3b$ to produce $8a + 6b + 13$?

9. What must be taken from $2x^2 - 3a^2x^2 + 9$ to produce $x^2 + 5a^2x^2 - 3$?

10. What must be added to $a + 5b + 9$ to produce $3a - 2b + 6$?

11. Ethel is twice as old as Edith, and six years ago she was four times as old. What is the age of each?

12. A and B have together 150 dollars. If A were to give B 35 dollars, B would have three times as much as A. How much has each?

II.

13. What must be added to x to produce y?

14. By how much does $5x - 7$ exceed $3x + 4$?

15. From what must $5x + 4y + 7a - 12$ be subtracted to produce unity?

16. From what must $x^3 - x^2 + x - 1$ be subtracted to produce $2x^2 + 2$?

17. From $7(a + b)$ take $3(a + b)$.

18. From $3a(c - x)$ take $a(c - x)$.

19. From $7a^2(b - x) - ab(a - b)$ take $5a^2(b - x) - 5ab(a - b)$.

20. Howard is x years old. How old was he eight years ago?

21. Divide the number m into two parts such that, when a is taken from the first and given to the second, the second will be five times the first.

PARENTHESES.

45. Operation upon Aggregates.—Every algebraic expression, however complex, represents a quantity, and may be operated upon as if it were a single symbol of that quantity.

When an expression is to be operated upon as a single quantity it is enclosed within parentheses or brackets, but the parenthesis may be omitted when no ambiguity or error will result from the omission.

Thus, one polynomial may be added to another or to a monomial by writing it, enclosed within a parenthesis and preceded by a plus sign, after the expression to which it is to be added; and a polynomial may be subtracted from a polynomial or monomial expression by writing it, enclosed within a parenthesis and preceded by a minus sign, after the expression from which it is to be subtracted.

Since terms written after one another each with its own sign in a polynomial expression are to be considered as added, and since in addition there is no change of signs, a parenthesis preceded by a plus sign may be omitted without any change of signs; and since the subtraction of any quantity produces the same effect as the addition of the corresponding quantity with the reverse sign, a parenthesis preceded by a minus sign may be omitted if the sign of every term be changed.

N.B.—It must be carefully borne in mind that the sign before the parenthesis is not the sign of the first term within it, but of the parenthesis as a whole. This sign really goes with the parenthesis when the latter is removed. When no sign is expressed with the first term within the parenthesis, the term is understood to be plus, and its sign must be written on the removal of the parenthesis, as plus when the parenthesis is plus, and as minus when the parenthesis is minus.

EXERCISE XVI.

Clear the following expressions of parentheses and reduce the results to the simplest form:

I.

1. $ab - (m - 3ab + 2ax) - 7ab.$
2. $x - (a - x) + (x - a).$
3. $2b + (b - 2c) - (b + 2c).$
4. $4x - 3y + 2z - (-7x + 5y - 3z) - (x - y).$

II.

5. $7ax - 2by - (8ax + 3by) - (8ax - 3by)$
6. $(a - x) - (a + x) + 2x.$
7. $-(a - b) - (b - c) - (c - a).$
8. $-(3m + 2n) - (3m - 2n) + 9m.$

22. Of course in forming aggregates preceded by a minus sign, the sign of every term enclosed within the parenthesis must be changed.

EXERCISE XVII.

Reduce the following expressions to the form $x -$ (an aggregate):

I.

1. $x - a - b$.
2. $x - m - n$.
3. $a + x - 3x + 2y$.
4. $-3b + x + 2c + 5d$.
5. $2x - 2a + 2b$.
6. $x + 3 - (a + b)$.
7. $x + a - (b - c) + (m - n)$.

II.

8. $2x + a - b$.
9. $3x - 2m + 2n$.
10. $3x + ab - m - 3ab + 2m$.
11. $x - 2m - (3a - 2b)$.
12. $x - (am + b) - (p - q) - (am - n)$.
13. $x - (a + b) - (p - q) - (m - n)$.

46. Compound Parentheses.—An algebraic expression having parentheses as a part of it may be itself enclosed in parentheses with other expressions, and this may be repeated to any extent. Each order of parentheses must then be made larger or thicker, or different in shape, to distinguish it.

e.g. Suppose we have to subtract a from b, the remainder from c, that remainder from d, and so on. We shall have:

First remainder, ... $b - a$.
Second remainder, ... $c - (b - a)$.
Third remainder, ... $d - [c - (b - a)]$.
Fourth remainder, . $e - \{d - [c - (b - a)]\}$.
Fifth remainder, $f - [e - \{d - [c - (b - a)]\}]$.

Such parentheses are called *compound parentheses.*

Compound parentheses of addition and subtraction may be removed by removing separately the individual parentheses of which they are composed. We may begin either with the outer ones and go inward, or with the inner ones and go outward. It is customary to begin with the inmost.

e.g. Clear of parentheses:

$$x - [a - \{b - [c - (d - e)]\}].$$

Beginning with the inmost, the expression takes, in succession, the following forms:

$$x - [a - \{b - [c - d + e]\}] =$$
$$x - [a - \{b - c + d - e\}] =$$
$$x - [a - b + c - d + e] =$$
$$x - a + b - c + d - e.$$

Beginning with the outmost, we have

$$x - [a - \{b - [c - (d - e)]\}] =$$
$$x - a + \{b - [c - (d - e)]\} =$$
$$x - a + b - [c - (d - e)] =$$
$$x - a + b - c + (d - e) =$$
$$x - a + b - c + d - e.$$

Again, $x - [-(a + b) + (c + d) - (e - z)]$

gives, when we begin with the inner parentheses,

$$x - [-a - b + c + d - e + z] =$$
$$x + a + b - c - d + e - z;$$

and when we begin with the outer parentheses,

$$x + (a + b) - (c + d) + (e - z) =$$
$$x + a + b - c - d + e - z.$$

EXERCISE XVIII.

Remove the parentheses in the following expressions, and combine the terms containing x, y, and z:

I.

1. $m + [-(p - q) + (a - b) + (-c + d)]$.

2. $m - [-(a - b) - (p + q) + (n - k)]$.

3. $7ax - [(2ax + by) - (3ax - by) + (-7ax + 2by)]$.

4. $a - [a - \{a - [a - (a - a)]\}]$.

5. $p - [a - b - (s + t + a) + (-m - n)]$.

6. A father left 80,000 dollars to his four children. The eldest was to receive four times as much as the youngest less 1800 dollars, the second was to receive three times as much as the youngest less 1200 dollars, and the third was to receive twice as much as the youngest less 600 dollars. How much did each receive?

7. Divide a into three parts such that the second shall equal the first minus b and the third shall be c less than twice the first.

II.

8. $2ax - [3ax - by - (7ax + 2by) - (5ax - 3by)]$.

9. $ax + by + cz + [2ax - 3cz - (2cz + 5ax) - (7by - 3cz)]$.

10. $x - \{2x - y - [3x - 2y - (4x - 3y)]\}$.

11. $ax - bz - \{ax + bz - [ax - bz - (ax + bz)]\}$.

12. $my - \{x + 3y + [2my - 3(x - y) - 4ab] + 5\}$.

13. Divide 186 into five parts such that the second shall exceed the first by 12, the third shall exceed twice the first by 24, the fourth shall exceed three times the first by 36, and the fifth shall exceed four times the first by 48.

CHAPTER VI.

MULTIPLICATION OF INTEGRAL ALGEBRAIC EXPRESSIONS.

A. LAW OF SIGNS, OF COMMUTATION, AND OF ASSOCIATION.

47. Multiplication of Integers.—Multiplication is the operation of finding what number is obtained by counting a number over a given number of times.

The number to be counted over is called the *multiplicand*, the number which indicates how many times the multiplicand is to be counted over is called the *multiplier*, and the number obtained as the result of the operation is called the *product*.

The multiplier and the multiplicand are called *factors* of the product.

48. Two Cases of Multiplication.—As there are two directions of counting from zero in algebra, so there are two cases of multiplication. In addition, as we have seen, the numbers to be added are counted in the direction indicated by their signs, while in subtraction the numbers to be subtracted are counted in the opposite direction to those indicated by their signs. The direction in which the multiplicand is to be counted is indicated by the sign of the multiplier. When this sign is positive the multiplicand is counted in the direction indicated by its sign. Hence the sign in the product will be the same as the sign in the multiplicand. When the multiplier is negative the multiplicand is counted in the opposite direction to that indicated by its sign. Hence the sign is the reverse of the sign in the multiplicand. The former case corresponds to addition and the latter to subtraction. In multiplication the counting is always understood to begin at zero.

49. Law of Signs in Multiplication.—

Ex. $12 \times 4 = 48.$ $\quad {}^{-}12 \times 4 = {}^{-}48.$

$12 \times {}^{-}4 = {}^{-}48.$ $\quad {}^{-}12 \times {}^{-}4 = 48.$

$4 \times 12 = 48.$ $\quad 4 \times {}^{-}12 = {}^{-}48.$

${}^{-}4 \times 12 = {}^{-}48.$ $\quad {}^{-}4 \times {}^{-}12 = 48.$

In general,

$a \times b = ab.$ $\quad {}^{-}a \times b = -ab.$

$a \times {}^{-}b = -ab.$ $\quad {}^{-}a \times {}^{-}b = ab.$

$b \times a = ab.$ $\quad b \times {}^{-}a = -ab.$

${}^{-}b \times a = -ab.$ $\quad {}^{-}b \times {}^{-}a = ab.$

From the above we see:

1°. That *like* signs in multiplication produce *plus*, and *unlike* signs *minus*.

2°. That interchanging the signs of the factors does not alter the sign of the product. $a \times {}^{-}b = -ab = {}^{-}a \times b.$

3°. That interchanging the multiplier and multiplicand does not alter the product. $a \times {}^{-}b = -ab = {}^{-}b \times a.$

50. Commutative Law of Multiplication.—From 2° and 3° we see that multiplication is *commutative* both as regards its signs and its factors. Addition is commutative only as regards its terms and not as regards its signs.

$12 + {}^{-}4 = {}^{-}4 + 12$, but $12 + {}^{-}4$ does not equal ${}^{-}12 + 4$.

That multiplication is commutative as regards its factors, that is, that the same result will be obtained by counting m things over n times as by counting n things over m times, may be shown as follows.

Place m squares in a horizontal row and repeat the row vertically n times as in Fig. 1. Evidently we would get the number of squares in the figure either by counting the m squares of the lower row over n times, or by counting the n squares of the left-hand column over m times. Hence $m \times n = n \times m$. Thus the commutative law of multiplication is seen to be a consequence of the associative and commutative laws of addition.

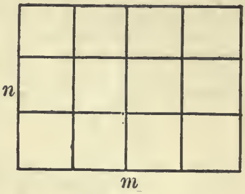

FIG. 1.

51. Associative Law of Multiplication.—In the operation of multiplication we combine only two factors at a time into a product. If there are more than two factors to combine, we first combine two of the factors into a product, and then use the product obtained and a third factor as two factors to form a new product, and so on, till the factors are all used.

e.g. $\quad 9 \times 3 \times 2 = 27 \times 2 = 54,$

or $\quad 9 \times 3 \times 2 = 9 \times 6 = 54.$

In general,

$$a \, . \, b \, . \, c = (ab) \, . \, c, \quad \text{or} \quad a \, . \, (bc).$$

That is, the result of multiplying a by b and the product by c is the same as multiplying a by the product of b and c.

The fact that the factors may be grouped or *associated* in any way is known as the *Associative* Law of Multiplication.

The associative law of multiplication may be shown to be true for integers as follows:

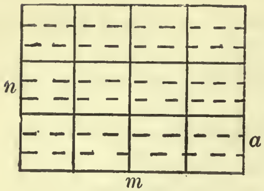

FIG. 2.

Use the diagram of the last section, and suppose each of the small squares to be divided into a rectangles by horizontal lines (Fig. 2). There will evidently be ma of these small rectangles in the lower row of squares, and na in the left-hand column, and we would get the whole number of rectangles by counting the lower set of ma rectangles over n times, or by counting the lowest row of m rectangles over na times. Hence

$$(ma) \,.\, n = m \,.\, (na).$$

From the commutative law of multiplication we see that it makes no difference in what order the factors of a product are written.

Hence the factors of a term may be written in any order. It is, however, customary to write the numerical factor first and the literal factors in their alphabetic order.

If there are more than two factors, the product will be *plus* when all the factors are positive, or when the number of negative factors is even. The product will be *minus* when the number of negative factors is odd.

e.g. $$^{-}a \,.\, b \,.\, {}^{-}c = -\,ab \,.\, {}^{-}c = abc.$$

$$^{-}a \,.\, {}^{-}b \,.\, {}^{-}c \,.\, d = ab \,.\, {}^{-}cd = -\,abcd.$$

52. Multiplication of Monomials. — In the multiplication of integral algebraic expressions we assume that the laws of commutation and association which we have demonstrated for integers also apply to all numbers which may be represented by letters, fractional and surd as well as integral.

Hence we multiply two integral monomial algebraic expressions together by grouping all their factors together in a single term.

This term must therefore contain every factor contained in the terms multiplied together, and each factor as many times as in all the terms together.

e.g. $3a^2b^3c \times 4a^3b^2x =$

$$3a \,.\, a \,.\, b \,.\, b \,.\, b \,.\, c \,.\, 4 \,.\, a \,.\, a \,.\, a \,.\, b \,.\, b \,.\, x =$$

$$3 \,.\, 4 \,.\, a \,.\, a \,.\, a \,.\, a \,.\, a \,.\, b \,.\, b \,.\, b \,.\, b \,.\, b \,.\, c \,.\, x =$$

$$12a^5b^5cx.$$

To multiply one monomial by another, multiply together their numeral coefficients and write after the product obtained each letter of both monomials with an exponent equal to the sum of its exponents in the two terms. Briefly, *multiply* coefficients and *add* exponents.

The sign of the product must be determined by the law of signs in multiplication.

EXERCISE XIX.

Find the product of the following factors:

I.

1. $3a$ and $7b$.
2. $5a$ and $6a^2b$.
3. $4a^2x$ and $-8x^3y^2$.
4. a^2bx and $-a^3b^4y^2$.
5. $-3a^2x^3$ and $-b^2x^5z^2$.
6. $-7m^2ny^3$ and $6m^3n^2x^4$.
7. $am \times ab \times ac \times ad$.
8. $ax \times -bx \times cx \times dx$.

9. $x \times -ax \times -abx \times -abcx$.

10. $3ax \times -2a^2b^3 \times -5a^3mx$.

11. $-7m^2y \times -3a^2y^2 \times 5ax$.

12. $2m \times n \times -a \times -2b$.

13. $-3ax \times -2km \times -7x \times -4bmx$.

14. $-ny \times gy \times -2y \times 3bm$.

15. $xy \times 2y^2 \times y^2x \times 2ayx^2$.

16. $5y^2 \times -3gy \times -2x^2 \times -ax^2z$.

II.

17. $5ax \times anx \times 3z \times b^2xy$.

18. $-4bz \times -xz \times -yz \times agz$.

19. $2c^2n \times 2x^2z \times -z^2 \times -bgz^2$.

20. $-c^2x \times 3x \times cb^2 \times ay$.

21. $-2e \times -2y \times a \times bx$.

22. $-4ax \times 3ay \times -2a^2y \times -xy$.

23. $ax^2 \times -y^2 \times -1 \times 3ax \times -a^2y$.

24. $m^2x \times -n^2x \times -mn^2 \times -m^3$.

25. $-abx \times -ay^2 \times ax \times a^2x^2$.

26. $px^2 \times qy^2 \times xy \times -ax$.

27. $abc \times -d^2 \times ax \times -1 \times 3ax$.

28. $1/4ax \times 3cx \times -1/2mx \times -4y^2 \times 6m$.

29. $-6mx \times -2n^2x \times 1/6ac \times -1/5m^2$.

30. $-a \times bc \times -1 \times 1/4 \times 3a^2 \times 4xy \times y$.

53. Changing the Signs of an Equation.—If an algebraic expression be multiplied by -1 its signs will all be reversed, and, of course, the value of the expression will be changed. To multiply any number by -1 will change it into the corresponding number with the reverse sign.

If both members of an equation be multiplied by -1, the value of each member will be changed, but their equality will not be destroyed. (Why not?)

Hence in working with equations, it is legitimate to change the signs at any stage of the operation, provided that the sign of every term, simple and complex, on both sides of the equation be changed.

EXERCISE XX.

1. $x = 80 - (x - 20) + (3x - 120)$. Find the value of x.

2. $240 - (x + 40) = 20 + (5x - 60) - (2x - 80)$. Find the value of x.

3. A father left his property of 47,000 dollars to his four children, giving the eldest four times what he gave the youngest less as much as he gave the second, to the second three times as much as he gave the youngest less as much as he gave the third, and to the third twice as much as he gave the youngest less 2000 dollars. What did he give each?

4. Divide 81 into five parts such that the second shall be twice the first less eight, the third shall be three times the first less the second, the fourth shall be four times the first less the third, and the fifth shall be five times the first less the fourth.

54. Distributive Law of Multiplication of Integers.—

Ex. 1. $(12 + 8) \times 4 = 20 \times 4 = 80,$

and $12 . 4 + 8 . 4 = 48 + 32 = 80.$

$(12 - 8) \times 4 = 4 \times 4 = 16,$

and $$12 \,.\, 4 - 8 \,.\, 4 = 48 - 32 = 16.$$

$$(-12 + 8) \times 4 = -4 \times 4 = -16,$$

and $$-12 \times 4 + 8 \times 4 = -48 + 32 = -16.$$

$$(-12 - 8) \times 4 = -20 \times 4 = -80,$$

and $$-12 \times 4 - 8 \times 4 = -48 - 32 = -80.$$

$$(12 + 8) \times -4 = 20 \times -4 = -80,$$

and $$12 \times -4 + 8 \times -4 = -48 - 32 = -80.$$

In general,

$$({}^{\pm}a + {}^{\pm}b) \times {}^{\pm}c = {}^{\pm}a \times {}^{\pm}c + {}^{\pm}b \times {}^{\pm}c.$$

The product of a polynomial and a monomial factor is the sum of the products of its several terms and that factor. This is known as the *Distributive Law of Multiplication.*

It is a law controlling the combination of multiplication with addition and subtraction.

The truth of the Distributive Law may be shown by the following conventional arrangement of units on a plane surface.

If a vertical and a horizontal line intersect each other on a plane, they will divide the plane into four quarters, or *quadrants.* These quadrants are numbered as shown in Fig. 3.

By general agreement, units counted to the right of the vertical line, whether above or below the horizontal line, are regarded as positive; while those counted to the left of the vertical line are regarded as negative. Also units counted upward from the horizontal line, whether at the right or left of the vertical line, are regarded as positive, while those counted

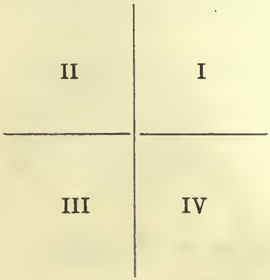

FIG. 3.

downward from the horizontal line are regarded as negative.

The quality of the units arranged in the four quadrants is shown in Fig. 4, the units being represented by the small circles.

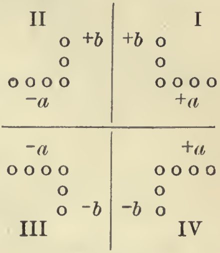

FIG. 4.

A rectangle of units in any quadrant, as shown in Fig. 5, represents a product of two factors. A rectangle in the first quadrant represents a positive product, since it is composed of two positive factors; a rectangle in the second quadrant represents a negative product (why?); a rectangle in the third quadrant represents a positive product (why?); and a rectangle in the fourth quadrant represents a negative product (why?).

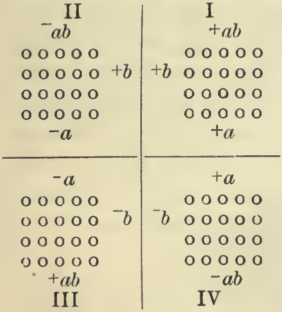

FIG. 5.

To represent the case of $(a + b) \times c$, mark $a + b$ units in a horizontal row in the first quadrant, and repeat the row c times one above the other (Fig. 6). These rows represent the product of $a + b$ and c, and the vertical dotted line between the a units and the b units shows that this product is the sum of the two products ac and bc.

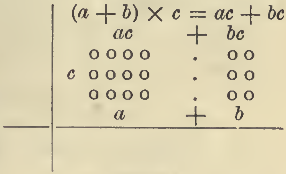

FIG. 6.

To represent the case of $(a + {}^{-}b) \cdot c$ or $(a - b) \cdot c$, arrange c rows of a units each in the first quadrant and c rows of ${}^{-}b$ units each in the second quadrant (Fig. 7). Each complete horizontal row will be composed of $a + {}^{-}b$, or $a - b$ units, and the c rows together represent the product of $(a + {}^{-}b)$ and c, or $(a-b)c$. This product is evidently the sum of the two products ac and $-bc$, and is equal to $ac + (-bc)$, or $ac - bc$.

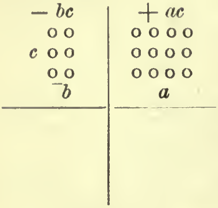

Fig. 7.

The two expressions $ac + (-bc)$ and $ac - bc$ are not identical in meaning. The former represents two sets of units, one positive and one negative, and indicates that they are to be combined into one; the latter represents one set of units and indicates that it has been obtained by uniting two sets of units, one positive and one negative.

Of course the products ac and $-bc$ tend to cancel each other wholly or in part, but the actual cancellation can be expressed only when the products are numerals or similar terms with numeral coefficients. In the actual illustration ac represents 12 positive units and $-bc$ 6 negative units, and $ac - bc$ represents 6 positive units obtained by cancelling 6 of 12 positive units by 6 negative units. Were $bc > ac$, the result of the cancellation would have been a number of negative units equal to the arithmetical difference of the two products.

So long as $ac > bc$, the expression $ac - bc$, as a whole, is positive, and denotes that the operation produces a surplusage of the kind of units employed; and when $ac < bc$, the expression $ac - bc$, as a whole, is negative and indicates that the operation produces a deficiency of the kind of units employed.

EXERCISE XXI.

1. Arrange the units to represent the case $(a + b) \times {}^{-}c$ and show that it equals $-ac - bc$.

2. Arrange the units to represent $({}^{-}a + {}^{-}b) \times c$, or $(-a - b) \times c$, and show that it equals $-ac - bc$.

3. Arrange the units to represent $({}^-a + {}^-b) \times {}^-c$, or $(-a-b) \times -c$, and show that it equals $ac + bc$.

Ex. **2.** $(6+4)(3+2) = 10 \times 5 = 50,$

and

$$6 \,.\, 3 + 4 \,.\, 3 + 6 \,.\, 2 + 4 \,.\, 2 = 18 + 12 + 12 + 8 = 50.$$

In general,

$$({}^{\pm}a + {}^{\pm}b)({}^{\pm}c + {}^{\pm}d) = {}^{\pm}ac + {}^{\pm}bc + {}^{\pm}ab + {}^{\pm}bd.$$

To represent the case $(a+b)(c+d)$, arrange $c+d$ rows containing $a+b$ units each in the first quadrant (Fig. 8). The $c+d$ rows will represent the product of $a+b$ and $c+d$. This product is evidently equal to

$$ac + bc + ad + bd.$$

The product of a polynomial and a polynomial is the sum of the products of the first polynomial and each term of the second.

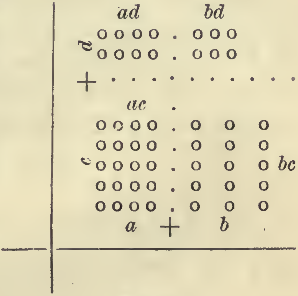

FIG. 8.

55. Extension of the Application of the Distributive Law.—The distributive law of multiplication which we have demonstrated for integers is assumed to hold for all kinds of numbers which can be expressed by letters. Hence the last two definitions hold for all integral algebraic expression in which the multiplicand is an integral polynomial.

EXERCISE XXII.

I.

1. Arrange the units to represent the case

$$(a+b)(c+{}^{-}d), \text{ or } (a+b)(c-d),$$

and show that it equals

$$ac+bc-ad-bd.$$

Show by a similar arrangement that

2. $(a+b)({}^{-}c+d)$, or $(a+b)(-c+d)=-ac-bc+ad+bd$.

3. $(a+b)({}^{-}c+{}^{-}d)$, or $(a+b)(-c-d)=-ac-bc-ad-bd$.

4. $(a+{}^{-}b)(c+d)$, or $(a-b)(c+d)=ac-bc+ad-bd$.

5. $(a+{}^{-}b)(c+{}^{-}d)$, or $(a-b)(c-d)=ac-bc-ad+bd$.

6. $(a+{}^{-}b)({}^{-}c+d)$, or $(a-b)(-c+d)=-ac+bc+ad-bd$.

II.

7. $(a+{}^{-}b)({}^{-}c+{}^{-}d)$, or $(a-b)(-c-d)=-ac+bc-ad+bd$.

8. $({}^{-}a+b)(c+d)$, or $(-a+b)(c+d)=-ac+bc-ad+bd$.

9. $({}^{-}a+b)(c+{}^{-}d)$, or $(-a+b)(c-d)=-ac+bc+ad-bd$.

10. $({}^{-}a+b)({}^{-}c+d)$, or $(-a+b)(-c+d)=ac-bc-ad+bd$.

11. $({}^{-}a+b)({}^{-}c+{}^{-}d)$, or $(-a+b)(-c-d)=ac-bc+ad-bd$.

12 $({}^{-}a+{}^{-}b)(c+d)$, or $(-a-b)(c+d)=-ac-bc-ad-bd$.

13. $({}^{-}a+{}^{-}b)(c+{}^{-}d)$, or $(-a-b)(c-d)=-ac-bc+ad+bd$.

14. $({}^{-}a+{}^{-}b)({}^{-}c+d)$, or $(-a-b)(-c+d)=ac+bc-ad-bd$.

15. $({}^{-}a+{}^{-}b)({}^{-}c+{}^{-}d)$, or $(-a-b)(-c-d)=ac+bc+ad+bd$.

Note that the numbers in the adjacent quadrants tend to cancel each other, while those in the opposite quadrants tend to augment each other. The expression finally obtained will be positive or negative according as the sum of the units in the first and third quadrants is greater or less than the sum of those in the second and fourth quadrants.

56. Arrangement of Terms according to the Powers of a Letter.—A polynomial is said to be arranged accord-to the powers of some letter when the exponents of that letter either ascend or descend in magnitude in regular order. Thus, $5a - 6bx + 3cx^2 - 4a^2x^3$ is arranged according to the ascending powers of x; and $3x^3 - 4ax^2 + cx - 7$ is arranged according to the descending powers of x.

57. Multiplication of Polynomials.—(a) *To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial, and write the result as a polynomial reduced to its simplest form.*

EXERCISE XXIII.

Multiply together:

I.

1. $3xy + 4yz$ and $-12xyz$.
2. $ab - bc$ and a^2bc^3.
3. $-x - y - z$ and $-3x$.
4. $a^2 - b^2 + c^2$ and abc.
5. $-ab + bc - ca$ and $-abc$.
6. $-2a^2b - 4ab^2$ and $-7a^2b^2$.
7. $5x^2y - 6xy^2 + 8x^2y^2$ and $3xy$.
8. $-7x^3y - 5xy^3$ and $-8x^3y^3$.
9. $-5xy^2z + 3xyz^2 - 8x^2yz$ and xyz.
10. $4x^2y^2z^2 - 8xyz$ and $-12x^3yz^3$.
11. $-13xy^2 - 15x^2y$ and $-7x^3y^3$.

II.

12. $8xyz - 10x^3yz^3$ and $-xyz$.

13. $abc - a^2bc - ab^2c$ and $-abc$.

14. $-a^2bc + b^2ca - c^2ab$ and $-ab$.

Find the product of

15. $2a - 3b + 4c$ and $-3/2a$.

16. $3x - 2y - 4$ and $-5/6x$.

17. $2/3a - 1/6b - c$ and $3/8ax$.

18. $6/7a^2x^2 - 3/2ax^3$ and $-7/3a^3x$.

19. $-5/3a^2x^2$ and $-3/2a^2 + ax - 3/5x^2$.

20. $-7/2xy$ and $-3x^2 + 2/7xy$.

21. $-3/2x^3y^2$ and $-1/3x^2 + 2y^2$.

22. $-4/7x^5y^3$ and $7/4x^3 - 4/7y^3$.

(*b*) *To multiply a polynomial by a polynomial, multiply the first polynomial by each term of the second, and add the partial products thus obtained.*

In multiplying polynomials it is convenient to arrange the terms of both factors in the same order according to the powers of some letter, to write the multiplier under the multiplicand, and to place like terms of the partial products in columns.

e.g. (1) Multiply $4x + 3 + 5x^2 - 6x^3$ by $4 - 6x^2 - 5x$.

Arrange both multiplicand and multiplier according to the ascending powers of x.

$$\begin{array}{rrrrrr}
3 + & 4x + & 5x^2 - & 6x^3 & & \\
4 - & 5x - & 6x^2 & & & \\
\hline
12 + & 16x + & 20x^2 - & 24x^3 & & \\
 & -15x - & 20x^2 - & 25x^3 + & 30x^4 & \\
 & & -18x^2 - & 24x^3 - & 30x^4 + & 36x^5 \\
\hline
12 + & x - & 18x^2 - & 73x^3 & & +36x^5
\end{array}$$

(2) Multiply $1 + 2x + x^4 - 3x^2$ by $x^3 - 2 - 2x$.

Arrange according to the descending powers of x.

$$\begin{array}{lllllll}
\multicolumn{7}{l}{x^4 - 3x^2 + 2x + 1} \\
\multicolumn{7}{l}{x^3 - 2x \;\; - 2} \\
\hline
x^7 & -\,3x^5 & +\,2x^4 & +\;\;x^3 & & & \\
 & -\,2x^5 & & +\,6x^3 & -\,4x^2 & -\,2x & \\
 & & -\,2x^4 & & +\,6x^2 & -\,4x & -\,2 \\
\hline
x^7 & -\,5x^5 & & +\,7x^3 & +\,2x^2 & -\,6x & -\,2
\end{array}$$

EXERCISE XXIV.

Multiply together:

I.

1. $x + 1$ and $x - 1$.

2. $x^2 + xy + y^2$ and xy.

3. $x^3 - 3x^2 + x - 4$ and $-3x^2$.

4. $x^2 + x + 1$ and $x^2 - 1$.

5. $x^2 + 2x + 3$ and $x^2 - x + 1$.

6. $x^2 - 5x + 6$ and $x^2 + 5x + 6$.

7. $x^2 + xy + y^2$ and $x - y$.

8. $x^2 - xy + y^2$ and $x + y$.

9. $x^2 + xy + y^2$ and $x^2 - xy + y^2$.

10. $x^3 + 3x^2 + 3x + 1$ and $x^2 + 2x + 1$.

11. $3(x - 4) = 361 + 8(2x - 12) - 5(4x + 40)$.

Clear of parentheses and find the value of x.

12. A man bought three houses. He paid for the second 8000 dollars less than three times as much as he paid for the first, and for the third five times what he paid for the first less the cost of the second. Five times the cost of

the first minus the cost of the second is equal to 192,000 dollars minus three times the cost of the third. What was the cost of each house?

13. A man started to give 50 cents apiece to some beggars and found he had not money enough within 7 cents. He then gave them 45 cents apiece and had 18 cents left. How many beggars were there?

II.

Multiply together:

14. $x^3 - 2ax^2 + 2a^2x - 3a^3$ and $x^2 - 3ax + 2a^2$.

15. $x^3 - ax^2 - 2a^2x + a^3$ and $x^2 + ax - a^2$.

16. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ and $x^2 - 2xy + y^2$.

17. $x - a$, $x + a$, and $x^2 + a^2$.

18. $x - a$, $x + b$, and $x - c$.

19. $1 + x + x^2$, $1 - x + x^2$, and $1 - x + x^4$.

20. $a - b$, $a + b$, $a^2 - ab + b^2$, and $a^2 + ab + b^2$.

21. $9x^2 + 12xy + 16y^2$ and $3x - 4y$.

22. $25a^4x^2 - 15a^2b^2xy^2 + 9b^4y^4$ and $5a^2x + 3b^2y^2$.

23. $16a^2z^4 + 20ab^3xz^2 + 25b^6x^2$ and $4az^2 - 5b^3x$.

24. A man bought three horses. He paid 50 dollars less than twice as much for the second as for the first, and for the third three times the cost of the first less the cost of the second. Seven times the cost of the first minus twice the cost of the second is equal to 1700 dollars minus twice the cost of the third. What was the cost of each?

25. A man gave some beggars 30 cents apiece and had 12 cents left. He found that he needed four cents more to enable him to give them 32 cents apiece. How many beggars were there?

58. Multiplication by Detached Coefficients. — When two expressions contain one and the same letter and both are arranged according to the ascending or descending powers of that letter, much labor of multiplication can be saved by writing down the coefficients only.

Thus, to multiply $x^2 - 5x + 6$ by $x^2 + 5x + 6$, we write

$$\begin{array}{rrrrr}
1 & -\,5 & +\;\;6 & & \\
1 & +\,5 & +\;\;6 & & \\
\hline
1 & -\,5 & +\;\;6 & & \\
 & 5 & -\,25 & +\,30 & \\
 & & 6 & -\,30 & +\,36 \\
\hline
1 & +\,0 & -\,13 & +\;\;0 & +\,36
\end{array}$$

The highest power of x in the result is x^4, and the rest follow in order. Hence the required product is

$$x^4 + 0x^3 - 13x^2 + 0x + 36,$$

or $$x^4 - 13x^2 + 36.$$

When some of the powers of the letter are wanting, the coefficients must be written down as zeros in their proper places. Thus, to multiply $x^4 + 3x^2 + 3x + 1$ by $x^3 + 2x^2 + 1$, we write

$$\begin{array}{rrrrrrrr}
1 & +\,0 & +\,3 & +\;\;1 & & & & \\
 & 1 & +\,2 & +\;\;0 & +\,1 & & & \\
\hline
1 & +\,0 & +\,3 & +\;\;3 & +\,1 & & & \\
 & 2 & +\,0 & +\;\;6 & +\,6 & +\,2 & & \\
 & & 0 & +\;\;0 & +\,0 & +\,0 & +\,0 & \\
 & & & 1 & +\,0 & +\,3 & +\,3 & +\,1 \\
\hline
1 & +\,2 & +\,3 & +\,10 & +\,7 & +\,5 & +\,3 & +\,1
\end{array}$$

Hence the product is

$$x^7 + 2x^6 + 3x^5 + 10x^4 + 7x^3 + 5x^2 + 3x + 1.$$

The method illustrated above is known as the *method of detached coefficients.*

EXERCISE XXV.

Do the following multiplications by the method of detached coefficients.

Multiply:

I.

1. $3x^2 - x + 2$ by $3x^2 + 2x - 2$.
2. $x^4 - 2x^2 + x - 3$ by $x^4 + x^3 - x - 3$.
3. $x^3 - 5x^2 + 1$ by $2x^3 + 5x + 1$.
4. $2x^3 - 3x^2 + x - 2$ by $x^3 - 2x^2 - x + 2$.
5. $1 - 2x + x^2$ by $1 + 2x + 3x^2 + 4x^3 + 5x^4$.
6. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$ by $1 - 2x + x^2$.
7. $1 - 2x + 3x^2$ by $1 + 3x - 5x^2$.
8. $2 + 3x - 2x^2$ by $2 - 3x + 2x^2$.
9. $x^6 - 2x^3 + x + 1$ by $x^2 + 1$.
10. $x^5 - 2x^3 + 3$ by $2x^4 - x^2$.

II.

Examples 1–10 of Exercise XXIV.

59. Degree of an Integral Expression.—The *degree* of an integral term in any letter is the number of times that letter is contained as a factor in the term, and is equal to the exponent of the letter.

The degree of an integral term in two or more letters is the number of times all together that these letters occur as factors in the term, and is equal to the sum of the exponents of the letters in the term.

The degree of a term in any letter or letters is often called the *dimension* of the term in that letter or those letters.

The degree of any integral algebraic expression in any letter or letters is the degree of the term in it which is of the highest dimensions in that letter or those letters.

e.g. The term $5a^3b^4x^5$ is of the fifth degree in x, of the ninth degree in bx, and of the twelfth degree in abx.

The expression $5a^3x^4 + 6a^2x^5 - 11ax^6$ is of the sixth degree in x and of the seventh degree in ax.

It will be noticed that in the last example every term is of the same degree in ax. When all the terms of an expression are of the same degree in any letters, the expression is said to be *homogeneous* in these letters.

60. Product of Homogeneous Expressions.—*The product of two homogeneous expressions must be homogeneous.*—For each the terms of the product is obtained by multiplying some one term of the multiplicand by some one term of the multiplier, and the number of dimensions of the product of two terms is clearly the sum of the number of dimensions of the separate terms. Hence, if all the terms of the multiplicand are of the same degree, and all the terms of the multiplier are also of the same degree, it follows that all the terms of the product must be of the same degree.

It also follows from the above consideration that the degree of the product is the *sum* of the degrees of the factors.

When the two factors to be multiplied are homogeneous, there must be some error if the products obtained are not homogeneous.

61. Highest and Lowest Terms of a Product.—It is important to notice that, in the product of two algebraic expressions, the term which is of the *highest* degree in any particular letter is the product of the terms in the factors which are of the highest degree in that letter, and the term which is of the *lowest* degree in that letter is the product of the terms which are of the lowest degree in that letter in the factors. Thus there can be obtained only *one* highest-degree term and *one* lowest-degree term.

62. Complete and Incomplete Integral Expressions.—It is also important to notice that if each factor in multiplication is complete in any letter, that is, contains every degree of that letter from the highest one given down to zero, the product will be complete in that letter.

Thus the product of $x^4 + x^3 + x^2 + 1$ and $x^2 + x + 1$

is $$x^6 + 2x^5 + 3x^4 + 3x^3 + 3x^2 + 2x + 1.$$

If an expression is incomplete in any letter it may be completed by filling in the blank spaces with terms of the proper degree having zero as their coefficients. Thus $x^5 + x^2 + 1$ may be written $x^5 + 0x^4 + 0x^3 + x^2 + 0x + 1$.

CHAPTER VII.

DIVISION OF INTEGRAL ALGEBRAIC EXPRESSIONS.

63. Definition of Division.—Division is the inverse of multiplication, or the process of undoing multiplication. In multiplication two factors are given and their product is required. In division the product and one of the factors are given and the other factor is required.

The product of the two factors is called the *dividend*, the given factor the *divisor*, and the required factor the *quotient*.

Since the dividend is the product of the divisor and quotient, we may prove our division by multiplying together the divisor and quotient to see if their product agrees with the dividend.

64. Division of Monomials.—The rules for division are obtained by studying the corresponding cases of multiplication.

Take the following cases of the multiplication of monomials:

$$\begin{array}{cccc} 3a^2b^3 & -3a^2b^3 & 3a^2b^3 & -3a^2b^3 \\ 2a^3c & 2a^3c & -2a^3c & -2a^3c \\ \hline 6a^5b^3c & -6a^5b^3c & -6a^5b^3c & 6a^5b^3c \end{array}$$

Note: 1°. That the sign of one factor is + when the signs of the product and of the other factor are alike, and

— when the signs of the product and of the other factor are unlike.

2°. That the coefficient of one factor is the quotient obtained by dividing the coefficient of the product by the coefficient of the other factor.

3°. That the exponent of any letter in one factor is the difference between its exponent in the product and in the other factor, and that when this difference is zero the letter does not appear in the other factor. When any letter which appears in the product does not appear in one factor, its exponent in that factor is to be regarded as zero.

From these observations we obtain the following rule for the division of a monomial by a monomial:

Divide the coefficient of the dividend by that of the divisor for the coefficient of the quotient, subtract the exponent of each letter in the divisor from its exponent in the dividend for its exponent in the quotient, and place before the term in the quotient the plus sign when the signs of the divisor and dividend are alike, and the minus sign when the signs of the divisor and dividend are unlike.

EXERCISE XXVI.

Divide:

I.

1. $20x^3y$ by $4x^2$.

2. $21a^2b$ by $7b$.

3. $54a^4b^2c$ by $6a^2b^2c$. $= 9a^2$

4. $49x^3y^3z$ by $7xy^2z$.

5. $51ax^3z$ by $-3azx^2$.

6. $-132x^3y^3z$ by $12y^2z$.

II.

7. $-35x^3y^2z^2$ by $-7x^3y^2$

8. $-27a^2bc^4$ by $-3abc^2$.

9. $1/5x^4y^5$ by $1/10x^2y^2$.

10. $1/4a^3b^4$ by $-1/12ab^3$.

11. $-2/3a^7y^2$ by $-5/6a^5y$.

12. $-6x^3y^2z^8$ by $2/3xz^5$.

Multiply:

I.

13. $5(x+y)^2z$ by $3(x+y)^3z^2$.

14. $13(a-b)^3x$ by $-3(a-b)^2x^3$.

15. $-5c(a+b)^4x^2y^3$ by $6d(a+b)^3x^2$.

II.

16. $-7a^2b(c-d)y^2$ by $-8ab^3(c-d)^3x$.

Divide:

17. $45(a+b)^3x^3$ by $9(a+b)x^2$.

18. $63ac^2(b-d)^4xy^2$ by $-7c(b-d)^2xy$.

19. $-42c^2d(b+c)^2x^2$ by $-3c^2(b+c)x$.

Simplify:

I.

20. $a^5b^2c\times(-8a^3b^4c^5)\div-4a^6b^6c^4$.

21. $-7x^2y^3\times(-12x^4y^6)\div-4x^4y^8$.

22. $260-3(x-2)=14+4(x+3)-12x^2\div 4x$.

23. Divide 180 into two parts such that 80 minus three times the sum of the smaller part and 12 shall be equal to the larger part minus 8 less than five times the smaller part.

65. Division of Polynomials.—*a.* We have seen in multiplication that, when one of the factors is a monomial and the other a polynomial, the product will be a polynomial, and that this product is obtained by multiplying each term of the polynomial factor by the monomial factor. Hence in division, when the dividend is a polynomial and the divisor is a monomial, the quotient will be a polynomial, and this quotient will be obtained by dividing each term of the dividend by the divisor. Of course, the law of signs must be carefully observed.

EXERCISE XXVII.

Divide:

I.

1. $x^4y^2 + x^3y^3 + x^2y^4$ by x^2y^2.
2. $a^4b - a^3b^2 + a^2b^2$ by a^2b.
3. $- 2a^4b + 6a^3b^2 - 2ab^4$ by $- 2ab$.
4. $24x^4y^2 + 108x^3y^3 + 81xy^5$ by $3xy^2$.
5. $a^5b^2 - 6/25a^4b^3 - 2/5a^3b^4$ by $6/5ab^2$.

II.

6. $14a^4b^3 + 28a^3b^4$ by $- 7a^2b^2$.
7. $15x^3y^2 - 18x^2y^3 + 24x^3y^3$ by $3xy$.
8. $- 3a^2 + 9/2ab - 6ac$ by $- 3/2a$.
9. $- 5/2x^2 + 5/3xy + 10/3x$ by $- 5/6x$.
10. $1/4a^2x - 1/16abx - 3/8acx$ by $3/8ax$.

66. *b.* In multiplication, we have seen that, when each factor is a polynomial, their product is the sum of the partial product obtained by multiplying the whole multiplicand by each term of the multiplier. In this case the product is a polynomial.

Hence in division, when the divisor is a polynomial, we obtain a set of partial subtrahends by multiplying the whole divisor (the multiplicand) by each term of the quotient, as it is found. These partial subtrahends are subtracted in succession from the dividend. The operation is continued until there is no remainder, or, in case the divisor is not an aliquot part of the dividend, until the remainder is of a lower degree than the divisor.

The method of procedure in division will be readily understood by examining a case in multiplication of polynomials, and the corresponding case in division.

e.g.

$$\begin{array}{rrrrr}
x^4 & -3x^3 & +4x^2 & & \\
3x^2 & -2x & -7 & & \\
\hline
3x^6 & -9x^5 & +12x^4 & & \\
 & -2x^5 & +6x^4 & -8x^3 & \\
 & & -7x^4 & +21x^3 & -28x^2 \\
\hline
3x^6 & -11x^5 & +11x^4 & +13x^3 & -28x^2
\end{array}$$

Note that the first term of the first partial product is also the first term of the complete product, and that it is the product of the first term of the multiplier and multiplicand. Hence, in dividing the product by one factor, the first term of the other factor will be the quotient obtained by dividing the first term of the dividend by the first term of the divisor, and the first partial subtrahend (partial product) will be obtained by multiplying the whole divisor by this first term of the quotient. Thus:

$$\begin{array}{rrrrr|l}
3x^6 & -11x^5 & +11x^4 & +13x^3 & -14x^2 & x^4-3x^3+4x^2 \\
\cline{6-6}
3x^6 & -9x^5 & +12x^4 & & & 3x^2 \\
\cline{1-3}
 & -2x^5 & -x^4 & +13x^3 & -14x^2 &
\end{array}$$

Note again that the first term of the remainder just obtained is also the first term of the second partial product in the corresponding multiplication, and that it is the product of the first term of the factor used as a divisor and the second term of the other factor or quotient. Hence in division the second term of the quotient will be obtained by dividing the first term of the first remainder by the first term of the divisor, and the second partial subtrahend (partial product) will be obtained by multiplying the whole divisor by this second term of the quotient. Thus:

$$\begin{array}{rrrrr|l}
3x^6 & -11x^5 & +11x^4 & +13x^3 & -28x^2 & x^4-3x^3+4x^2 \\ \cline{6-6}
3x^6 & -9x^5 & +12x^4 & & & 3x^2-2x-7 \\ \cline{1-5}
 & -2x^5 & -x^4 & +13x^3 & -28x^2 & \\
 & -2x^5 & +6x^4 & -8x^3 & & \\ \cline{2-5}
 & & -7x^4 & +21x^3 & -28x^2 & \\
 & & -7x^4 & +21x^3 & -28x^2 & \\ \cline{3-5}
\end{array}$$

Note as before that the first term of the second remainder is the same as the first term of the third partial product, and that it is the product of the first term of the factor used as the divisor and the third term of the other factor or quotient. Hence in division the third term of the quotient will be obtained by dividing the first term of the second remainder by the first term of the divisor, and the third partial subtrahend (partial product) will be obtained by multiplying the whole divisor by this third term of the quotient.

Should there be another remainder, the next term of the quotient will be obtained in a similar way.

Use the second factor in the preceding case of multiplication as a divisor, and go through the work in the same way, and note the same points.

Also go through the same case, arranging the terms of divisor and dividend according to the ascending powers of x.

It is customary to bring down only one term at a time, and, in case the dividend is not exactly divisible by the divisor, to express the remainder in the form of a fraction as in arithmetic.

When some of the powers of the letter according to which the terms are arranged are wanting, their places may be supplied by terms with zero coefficients. Thus, suppose the dividend to be $x^6 - 27$: it may be written

$$x^6 + 0x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 27.$$

This is not absolutely necessary, but will be found convenient.

The rule for the division of a polynomial by a polynomial may be stated as follows:

Arrange the terms of the divisor and dividend similarly; divide the first term of the dividend by the first term of the divisor for the first term of the quotient, and multiply the divisor by this term for the first partial subtrahend; divide the first term of the remainder by the first term of the divisor for the second term of the quotient, and multiply the divisor by this term for the second partial subtrahend; and continue the process until there is no remainder, or until the first term of the remainder does not contain the first term of the divisor.

EXERCISE XXVIII.

Divide:

I.

1. $x^2 - x - 6$ by $x + 2$.

2. $x^2 - 4x - 21$ by $x - 7$.

3. $x^2 - 12x + 35$ by $x - 5$.

4. $2x^2 - x - 6$ by $2x + 3$.

5. $6x^2 - 13x + 6$ by $3x - 2$.

6. $12x^2 + 11x - 56$ by $4x - 7$.

7. $16x^2 - 24x + 9$ by $4x - 3$.

8. $25x^2 - 16$ by $5x - 4$.

9. $49x^2 + 70x + 25$ by $7x + 5$.

10. $x^3 - y^3$ by $x - y$.

11. $x^3 + y^3$ by $x^2 - xy + y^2$.

12. $27a^3x^3 - 64b^3$ by $3ax - 4b$.

II.

13. $8a^9x^6 - 27c^6b^9$ by $4a^6x^4 + 6a^3b^3c^2x^2 + 9c^4b^6$.

14. $14x^4 + 45x^3y + 78x^2y^2 + 45xy^3 + 14y^4$ by $2x^2 + 5xy + 7y^2$.

15. $x^5 - 5x^4 + 9x^3 - 6x^2 - x + 2$ by $x^2 - 3x + 2$.

16. $x^5 - 4x^4 + 3x^3 + 3x^2 - 3x + 2$ by $x^2 - x - 2$.

17. $x^5 - x^4y + x^3y^2 - x^3 - y^3$ by $x^3 - x - y$.

18. $x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3$ by $x^2 + xy - y^2$.

19. $x^5 - 2x^4 - 4x^3 + 19x^2 - 31x + 15$ by $x^3 - 7x + 5$.

20. $2x^3 - 8x + x^4 + 12 - 7x^2$ by $x^2 + 2 - 3x$.

21. $14a^4 - 45a^3b + 78a^2b^2 - 45ab^3 + 14b^4$ by $2a^2 - 5ab + 7b^2$.

Find the remainder in each of the following examples:

I.

22. $x^3 - 6x^2 + 11x + 2$ divided by $x - 2$.
23. $x^3 - 6x^2 + 12x - 17$ " " $x - 3$.
24. $2x^3 + 5x^2 - 4x - 7$ " " $x + 2$.
25. $3x^3 - 7x - 9$ " " $x + 1$.
26. $4x^3 + 7x^2 - 3x - 33$ " " $4x - 5$.
27. $27x^3 + 9x^2 - 3x - 5$ " " $3x - 2$.
28. $16x^3 - 19 + 39x - 46x^2$ " " $8x - 3$.
29. $8x - 8x^2 + 5x^3 + 7$ " " $5x - 3$.
30. $21a^3 - 27a + 15 - 26a^2$ " " $3a - 5$.

II.

31. $30x^4 + 11x^3 - 82x^2 - 5x + 3$ divided by $2x - 4 + 3x^2$.

32. $6x - 5x^3 + 12x^4 + 20 - 33x^2$ divided by $x + 4x^2 - 5$.

33. $30x + 9 - 71x^3 + 28x^4 - 35x^2$ divided by $4x^2 - 13x + 6$.

Divide:

34. $2x^2 + 7/6x + 1/6$ by $2x + 1/2$.

35. $1/3x^3 + 17/6x^2 - 5/4x + 9/4$ by $1/3x + 3$.

36. 1 by $1 + x$.

37. $1 + x$ by $1 - x$.

38. $4(x - y)^5 - 16(x - y)^3 - 8(x - y)^2 - (x - y)$ by $2(x - y)^2 + 4(x - y) + 1$.

The division of a polynomial by a polynomial may be indicated by writing the divisor after the dividend, each enclosed within a parenthesis, with the sign of division between. Thus, $(x^2 + 12x + 35) \div (x + 7) = x + 5$.

67. To Free an Equation from Expressions of Division. — Since multiplication by any quantity neutralizes the effect of division by the same quantity, and since to multiply both members of an equation by the same quantity does not destroy their equality, an equation may be freed from an expression of division in either member by multiplying both members by the indicated divisor.

e.g. $$4 + (5x^2 - 40) \div (x - 3) = 5x,$$

$$4(x - 3) + 5x^2 - 40 = 5x^2 - 15x,$$

or $$4x - 12 + 5x^2 - 40 = 5x^2 - 15x,$$

or $$4x + 15x + 5x^2 - 5x^2 = 52,$$

or $$19x = 52,$$

$$x = 2\tfrac{14}{19}.$$

The above example might have been written

$$4 + \frac{5x^2 - 40}{x - 3} = 5x.$$

N.B.—In clearing an equation of a fraction it must be borne in mind that the bar of the fraction is a sign of aggregation, and requires a change of sign when there is a minus sign before the fraction.

EXERCISE XXIX.

Free the following equations of expressions of division or fractions:

I.

1. $\dfrac{50x - 35x^2}{x - 7x} = 5x - 4.$

2. $6x + 1 - (30x - 60) \div (7x - 16) = 6x - 3.$

3. $6x + 7 - 5(2x - 2) \div (7x - 16) = 3(2x + 1).$

4. $6x + 13 - \dfrac{15(3x + 5)}{5x - 25} = 6x.$

5. $7x - 6 - \dfrac{35(x + 5)}{6x - 101} = 7x.$

6. A woman buys eggs at 18 cents a dozen. Had she bought five dozen more for the same money, the eggs would have cost her $2\frac{1}{2}$ cents a dozen less. How many dozen did she buy?

7. A man bought some sheep at three dollars a head. Had he bought two less for the same money, they would have cost him one dollar more a head. How many did he buy?

68. Division by Detached Coefficients.—It is evident if the dividend and divisor are both homogeneous, the degree of the quotient will be that of the dividend minus that of the divisor.

Also if the dividend and divisor are complete in any letter, the quotient will also be complete in that letter.

In finding the quotient of two integral algebraic expressions which are arranged in the same order according to the powers of some letter, much labor may be saved by the method of detached coefficients.

e.g. Divide $12x^6 + 6x^5 - 16x^4 + 4x^3 + 12x^2 + 16x - 24$ by $4x^3 + 2x^2 - 4$.

$$\begin{array}{rrrrrrr|l}
12 & +6 & -16 & +4 & +12 & +16 & -24 & 4+2+0-4 \\ \cline{8-8}
12 & +6 & +0 & -12 & & & & 3+0-4+6 \\ \cline{1-4}
 & 0 & -16 & +16 & +12 & & & \\
 & 0 & +0 & +0 & +0 & & & \\ \cline{2-5}
 & & -16 & +16 & +12 & +16 & & \\
 & & -16 & -8 & +0 & +16 & & \\ \cline{3-6}
 & & & 24 & +12 & +0 & -24 & \\
 & & & 24 & +12 & +0 & -24 & \\ \cline{4-7}
\end{array}$$

The required quotient is $3x^3 - 4x + 6$.

EXERCISE XXX.

II.

Exercise XXVIII, Examples 15–20.

SYNTHETIC DIVISION.

N.B.—This section may be omitted; but if mastered, it will lead to an immense saving of labor in the end, even in Elementary Algebra.

69. Synthetic Multiplication.—In the first place let us examine some cases of what may be called *synthetic multiplication;* that is, multiplication of complete integral algebraic expressions in which the coefficients of the several powers of the letter are built up one after another. This is effected by a kind of cross-multiplication, with which one may be made familiar by a little practice.

e.g. 1. Multiply $px^3 + qx^2 + rx + s$ by $ax^2 + bx + c$.

$$px^3 + qx^2 + rx + s$$

$$ax^2 + bx \ + c$$

$$\begin{array}{r|r|r|r|r}
\hline
apx^5 \ + aq & x^4 + ar & x^3 + as & x^2 + bs & x + cs \\
+ bp & + bq & + br & + cr & \\
 & + cp & + cq & & \\
\hline
\end{array}$$

$$Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F.$$

The first coefficient of the product is formed of the first coefficients of the multiplicand and multiplier ($a \times p$); the second coefficient is formed out of the first two coefficients of the multiplier and multiplicand, combined two by two crosswise ($a \times q$ and $b \times p$); the third coefficient is formed out of the first three coefficients of the multiplicand and of the multiplier, combined two by two crosswise ($a \times r$, $b \times q$, $c \times p$); and so on, the number of factors of the multiplicand and of the multiplier increasing by one at each step till the last coefficient of the multiplier has been reached.

Then, if there are more coefficients in the multiplicand than in the multiplier, all the coefficients of the multiplier being retained, the initial coefficients of the multiplicand are dropped one by one, and a new one taken on at the end, till the last coefficient of the multiplicand has been reached.

Then one initial coefficient is dropped from both multiplicand and multiplier till none are left. In every case, the partial products are formed out of the coefficients employed by cross-multiplication.

When there are more coefficients in the multiplier than in the multiplicand, proceed as above till you reach the last coefficient of the multiplicand, then, retaining all the coefficients of the multiplicand, drop the initial coefficients of the multiplier, one by one, and take in one at the end, till you reach the last, and then drop one initial coefficient from both multiplier and multiplicand till none are left. The partial products are formed as before by cross-multiplication.

e.g. 2.

$px^2 + qx + r$

$ax^4 + bx^3 + cx^2 + dx + e.$

$apx^6 +$	aq	$x^5 + ar$	$x^4 + br$	$x^3 + cr$	$x^2 + dr$	$x + er$
$+$	bp	$+ bq$	$+ cq$	$+ dq$	$+ eq$	
		$+ cp$	$+ dp$	$+ ep$		

$Ax^6 + Bx^5 + Cx^4 + Dx^3 + Ex^2 + Fx + G.$

Note that, in each of the examples just worked out, the partial products cut off by the dotted line are the only ones that contain the first coefficient of the multiplicand as a factor, and that these partial products contain this factor combined with each of the coefficients of the multiplier in turn.

70. The Coefficients of the Quotient.—Hence if the product be taken as a dividend and the multiplicand as the divisor, the coefficients of the quotient may be found by the following process.

1°. In Example 1:

$$\begin{aligned} ap &= A, & \therefore\ a &= A \div p. \\ bp &= B - aq, & \therefore\ b &= (B - aq) \div p. \\ cp &= C - (ar + bq), & \therefore\ c &= [C - (ar + bq)] \div p. \end{aligned}$$

Now since A and p are known at starting, a can be found; then B, p, a, and q being known, b can be found; and finally, C, p, a, b, r, and q being known, c can be found.

2°. In Example 2:

$$\begin{aligned} ap &= A, & \therefore\ a &= A \div p. \\ bp &= B - aq, & \therefore\ b &= (B - aq) \div p. \\ cp &= C - (ar + bq), & \therefore\ c &= [C - (ar + bq)] \div p. \\ dp &= D - (br + cq), & \therefore\ d &= [D - (br + cq)] \div p. \\ ep &= E - (cr + dq), & \therefore\ e &= [E - (cr + cq)] \div p. \end{aligned}$$

In this case, a, b, c, d, and e can be found in the same manner as in the first.

Observe that the first coefficient of the quotient is obtained by dividing the first coefficient of the dividend by the first coefficient of the divisor, and that the remaining coefficients of the quotient are obtained by subtracting certain partial products from the coefficients of the dividend which follow the first, and then dividing the remainders by the first coefficient of the divisor.

Observe also that the partial products to be subtracted from the coefficients of the dividend are those above the dotted line in the two examples worked out, and that they are obtained by a cross-multiplication in the way already described. In this process the coefficients of the quotient (multiplicand), are used as found, and only those coefficients of the divisor (multiplier) which follow the first are employed.

If the signs of all the terms of the divisor which follow the first are reversed, the signs of the partial products to be subtracted would be reversed, and the partial products would become additive.

This process of finding the coefficients of the quotient from those of the dividend and divisor is known as *synthetic division,* because we build up the coefficient of the dividend by getting the partial products which enter into their composition, and through this synthesis we obtain the coefficients of the quotient.

The following example will serve to show how this process may be carried out systematically.

Divide $6x^{10} - x^8 - 12x^7 - 28x^6 + 18x^5 - 16x^4 + 24x^3 + 12x + 4$ by $2x^6 - 3x^4 - 4x^2 - 2$.

First, write down the coefficients of the divisor with the signs of all the terms after the first changed, the coefficients of the missing terms being represented by zeros. Under this write the coefficients of the completed dividend, so arranged that each coefficient may fall under the coefficient of the term of the same degree in the divisor, and as a matter of convenience draw a vertical line after the first coefficient of the divisor. Then obtain the coefficients of the quotient by gradually filling in the partial products to be added to the coefficients of the dividend. The coefficients of the quotient are written in the bottom line to the left of the vertical line, thus:

$$
\begin{array}{r|l}
2 & + 0 + 3 + 0 + 4 + 0 + 2 \\
\hline
6 + 0 - 1 - 12 - 28 & + 18 - 16 + 24 + 0 + 12 + 4 \\
0 + 9 + 0 + 12 & + 0 + 6 + 0 + 8 - 12 - 4 \\
0 + 0 + 0 & + 0 + 0 + 0 + 0 + 0 \\
0 + 12 & + 0 + 16 - 24 - 8 \\
0 & - 18 + 0 + 0 \\
 & \quad\; 0 - 6 \\
\hline
3 + 0 + 4 - 6 - 2 & + 0 + 0 + 0 + 0 + 0 + 0
\end{array}
$$

The coefficients in the last line are obtained as follows:

1°. Divide 6 by 2 and write the quotient in the bottom line under the first coefficient of the dividend.

2°. Multiply 3 by 0 (the second coefficient of the divisor) for the first partial product, write the result under the second coefficient of the dividend, add, divide by 2, and place the quotient underneath in the bottom line.

3°. Form the next set of partial products by using the two coefficients of the quotient already obtained and the two of the divisor immediately after the vertical line, and multiplying crosswise, thus: $3 \times 3 = 9$ and $0 \times 0 = 0$. Write these under the third coefficient of the dividend, add, divide the sum by 2, and write the quotient beneath in the bottom line.

4°. Form the next set of partial products by using the three coefficients of the quotient already obtained and the three of the divisor immediately following the vertical line, and multiplying crosswise, thus: $3 \times 0 = 0$, $0 \times 3 = 0$, and $4 \times 0 = 0$. Write these under the fourth coefficient of the dividend, add, divide the sum by 2, and write the quotient beneath.

5°. Form the next set of partial products by using the four coefficients of the quotient already obtained and the four of the divisor immediately after the vertical line, and multiplying crosswise, thus: $3 \times 4 = 12$, $0 \times 0 = 0$, $4 \times 3 = 12$, and $-6 \times 0 = 0$. Write these under the fifth coefficient of the dividend, add, divide the sum by 2, and write the result underneath.

We have now reached the vertical line and have obtained the coefficients of the integral part of of the quotient. The remaining part of the work is merely to ascertain whether or not there is a remainder, and in case there be a remainder, to obtain its coefficients.

If, on filling in the remaining partial products and adding, we find the sum to be zero in each case, there is no remainder. If, however, on filling in and adding, we find the sums are not all zeros, there is a remainder, and the sums obtained are the coefficients of the corresponding terms of the remainder. For the addition of these partial products will subtract from the portion of the dividend which comes after the vertical line the corresponding portion of the product of the divisor and the quotient obtained. Hence, if the result is zero, there is no difference between the dividend and the product of the divisor and the quotient obtained; and if the result obtained is not zero, it must be the difference between the dividend and the product of the divisor and the quotient obtained.

6°. To obtain the first set of partial products after the vertical line, use the five coefficients of the quotient already obtained and the five of those of the divisor immediately after the vertical line, multiplying crosswise, thus: $3 \times 0 = 0$, $0 \times 4 = 0$, $4 \times 0 = 0$, $-6 \times 3 = -18$, and $-2 \times 0 = 0$.

7°. To obtain the next set, use the five coefficients of the quotient and the five of the divisor which follow the first after the vertical line, thus: $3 \times 2 = 6$, $0 \times 0 = 0$, $4 \times 4 = 16$, $-6 \times 0 = 0$, $-2 \times 3 = -6$.

8°. To obtain the next set, omit the initial coefficient from each set used last, and multiply crosswise, thus: $0 \times 2 = 0$, $4 \times 0 = 0$, $-6 \times 4 = -24$, $-2 \times 0 = 0$.

9°. To obtain the next set, omit the initial coefficient from each set used last time. Thus: $4 \times 2 = 8$, $-6 \times 0 = 0$, $-2 \times 4 = -8$.

10°. To obtain the next set, omit again the initial coefficient, and use the remainder. Thus: $-6 \times 2 = -12$, $-2 \times 0 = 0$.

11°. To obtain the last, omit again the initial coefficient, and use the one remaining in each set. Thus: $-2 \times 2 = -4$.

The degree of the first term of the quotient will be the difference between the degrees of the first terms of the dividend and of the divisor, or 4 in this example. Hence the quotient is $3x^4 + 4x^2 - 6x - 2$.

With a little practice the coefficients of the quotient can be obtained with great ease and rapidity by this method.

As a second example let it be required to divide

$$x^8 + x^4 + 3x^3 - 2x^2 + 3 \text{ by } x^4 - x^2 + 1.$$

$$\begin{array}{r|l}
1 & +0+1+0-1 \\
1+0+0+0+1 & +3-2+0+3 \\
+0+1+0-1 & +0-1+0-1 \\
+0+0+0 & +0+0+0 \\
+0+1 & +0+1 \\
+0 & +0 \\
\hline
1+0+1+0+1 & +3-2+0+2
\end{array}$$

Quotient.	Remainder.
$x^4 + x^2 + 1$	$3x^3 - 2x + 2$

The above method of synthetic division is applicable to all cases of integral algebraic expressions which contain only one letter.

EXERCISE XXXI.

II.

Exercise XXVIII, Examples 1–9, 15, 16, 19, 20, 22–33.

CHAPTER VIII.

INVOLUTION OF INTEGRAL ALGEBRAIC EXPRESSIONS.

71. Definition of Involution.—*Involution* is a case of multiplication in which the factors are all alike. The product obtained by using the same factor a number of times is called a *power* of the factor. When the factor is used twice the product is called the second power, or *square*; when three times, the third power, or *cube*; when four times, the fourth power; when five times, the fifth power; etc.

Involution may be defined as the operation of finding powers of numbers, or quantities.

The operation is indicated by placing the quantity within a parenthesis with an exponent after it.

Thus, $(3a^2b^3)^3$ indicates that $3a^2b^3$ is to be cubed, or raised to the third power.

72. Involution of Monomials.—Since a product contains every one of its factors as many times as each of these factors is contained in the several factors counted together, a monomial is raised to a given power by raising its numeral coefficient to that power and multiplying the exponent of each letter by the exponent of the given power. Thus:

$$(3a^2b^3)^3 = 3a^2b^3 \times 3a^2b^3 \times 3a^2b^3 =$$

$$3 . 3 . 3 . a . a . a . a . a . a . b . b . b . b . b . b . b . b . b = 27a^6b^9.$$

When the quantity which is to be raised to any power is positive, it must be borne in mind that every power of it will be positive, and that, if the quantity to be operated upon is negative, every even power of it will be positive and every odd power negative. The raising of an expression to a power is called expanding the expression.

EXERCISE XXXII.

Expand:

1. $(ab^3)^2$. **2.** $(x^3y^2)^3$. **3.** $(3x^2yz)^4$.

4. $(-12c^5dx^2)^2$. **5.** $(-5x^3y^2)^3$. **6.** $(-2x^3y^4)^5$.

Write down the square of each of the following expressions:

7. $3a^3b$. **8.** ac^3. **9.** $5a^2b^2$.

10. $-9a^2x^3$. **11.** $-7a^5b^4x^3$. **12.** $-2/3a^2x^5$.

Write down the cube of each of the following expressions:

13. $3a^3b^4$. **14.** $-3a^3x$. **15.** $-a^2b^3x$. **16.** $-3/4x^7$.

73. Squaring of Binomials.—Any polynomial may be squared by multiplying it by itself; but it is easy to learn to square any polynomial at sight.

e.g. $(a+b)^2=(a+b).(a+b)=a^2+2ab+b^2$.

$(a-b)^2=(a-b).(a-b)=a^2-2ab+b^2$.

$(x+3)^2=(x+3).(x+3)=x^2+6x+9$.

$(x-3)^2=(x-3).(x-3)=x^2-6x+9$.

$(-a+b)^2=(-a+b).(-a+b)=a^2-2ab+b^2$.

$(-a-b)^2=(-a-b).(-a-b)=a^2+2ab+b^2$.

Note that in every case the square of a binomial is a trinomial, and that two of the three terms of this trinomial are the squares of the two terms of the binomial which we are squaring, and that the third term is twice the product of the two terms of the binomial, regard being had to the signs of the terms. Hence the following rule for squaring a binomial at sight:

Square each term of the binomial and take twice the product of the two terms, and write the three terms thus obtained as a polynomial, each with its own sign.

It is customary to write the double product as the middle term in the result, but this is not necessary.

EXERCISE XXXIII.

Write down the square of each of the following expressions:

I.

1. $a + 3b.$
2. $a - 3b.$
3. $x - 5y.$
4. $2x + 3y.$
5. $3x - y.$
6. $3x + 5y.$
7. $9x - 2y.$
8. $5ab - c.$
9. $pq - r.$

II.

10. $x - abc.$
11. $ax + 2by.$
12. $x^2 - 1.$
13. $-4 + x.$
14. $x + 2/3a.$
15. $x - 2/6b.$
16. $x - \frac{3a}{2}.$
17. $-x - a.$
18. $-4 - x.$

74. Squaring of Polynomials.—

Ex.
$$\begin{aligned}(a+b+c)^2 &= (a+b+c)(a+b+c)\\ &= a^2+b^2+c^2+2ab+2ac+2bc.\\ (a-b+c)^2 &= (a-b+c)(a-b+c)\\ &= a^2+b^2+c^2-2ab+2ac-2bc.\\ (a-b-c)^2 &= (a-b-c)(a-b-c)\\ &= a^2+b^2+c^2-2ab-2ac+2bc.\\ (-a-b-c)^2 &= (-a-b-c)(-a-b-c)\\ &= a^2+b^2+c^2+2ab+2ac+2bc.\end{aligned}$$

Note that in each of these cases the square consists of the square of each term of the polynomial and, in addition, twice the product of the terms of the polynomial taken two by two in every possible way, regard being had to the signs of the terms.

The surest way to get every possible combination of the terms two by two is to combine each term of the polynomial with each term which follows it.

The law stated above holds whatever be the number of the terms in the polynomial to be squared. Hence we have the following rule for squaring a polynomial:

Square each term of the polynomial, and take twice the sum of the products of each term and the terms which follow it, and write the terms thus obtained as a polynomial, each with its own sign.

EXERCISE XXXIV.

Form the squares of:

1. $1 + 2x + 3x^2$.
2. $1 + 2x + 3x^2 + 4x^3$.
3. $1 + 2x + 3x^2 + 4x^3 + 5x^5$.
4. $a - b + c - d$.
5. $3a + 2b - c + d$.

75. Cubing of Binomials.—

Ex.
$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= a^3 + 3a^2b + 3ab^2 + b^3.\\ (a-b)^3 &= (a-b)(a-b)(a-b)\\ &= a^3 - 3a^2b + 3ab^2 - b^3.\\ (-a+b)^3 &= (-a+b)(-a+b)(-a+b)\\ &= -a^3 + 3a^2b - 3ab^2 + b^3.\\ (-a-b)^3 &= (-a-b)(-a-b)(-a-b)\\ &= -a^3 - 3a^2b - 3ab^2 - b^3.\end{aligned}$$

Note that in each case the cube of a binomial is a quadrinomial, and that two of its four terms are cubes of the two terms of the binomial, and each of the other two terms is three times the product of one of the terms of the binomial and the square of the other. Hence we have the following rule for cubing a binomial:

Cube the first term, take three times the product of the square of the first term and the second term, also three times the product of the first term and the square of the second, and the cube of the second term, and write the terms obtained as a polynomial, each with its own sign.

e.g. $$(3x - 2a^2)^3 = (3x)^3 - 3(3x)^2 . 2a^2 + 3(3x)(2a^2)^2 - (2a^2)^3$$
$$= 27x^3 - 54a^2x^2 + 36a^4x - 8a^6.$$

EXERCISE XXXV.

Write down the cube of each of the following expressions:

I.

1. $x + a.$ **2.** $x - a.$ **3.** $x - 2y.$

4. $2x + y.$ **5.** $3x - 5y.$ **6.** $ab + c.$

II.

7. $2ab - 3c.$ **8.** $5a - bc.$ **9.** $x^2 + 4y^2.$

10. $4x^2 - 5y^2.$

EXERCISE XXXVI.

I.

1. Divide $9x^3 - 6x^2 - 5x^4 + x^5 - x + 2$ by $x^2 - 3x + 2$.

2. Divide $1/4x^3 + 1/72xy^2 + 1/12y^3$ by $1/2x + 1/3y$.

3. Find two numbers whose difference is 5, and such that the square of the smaller plus 9 will equal the square of the larger minus 56.

4. Find two numbers which shall differ by 3, and such that the square of the smaller plus 15 shall equal the square of the larger minus 24.

5. Find two numbers that shall differ by 2, and such that the cube of the smaller increased by six times its square shall be 44 less than the cube of the larger.

6. A farmer bought some cattle at 30 dollars a head. Had he bought three more for the same money, they would have cost him 2 dollars less a head. How many did he buy ?

CHAPTER IX.

EVOLUTION OF INTEGRAL ALGEBRAIC EXPRESSIONS.

76. Definition of Evolution.—*Evolution* is the inverse of involution. In involution we have given the factor and the number of times it is employed, and are required to find the product, or the power, of the factor. In evolution we have given the power, or product, and the number of times a factor must be employed to produce it, and are required to find the factor.

The factor whose involution will produce a power or number is called the *root* of the number, and the number of times the factor is to be employed is called the *index* of the root. The operation of finding the required factor is called *extracting* the root of the number.

The operation of evolution is indicated by the radical sign, $\sqrt{}$, with a bar extending over the expression whose root is to be extracted, unless that expression be a numeral or single literal factor. The index of the root is written in front of the radical at the top. Thus: $\sqrt[3]{a^4}$, $\sqrt[4]{6b}$. When the index is 2 it is ordinarily omitted. A parenthesis may be used in any case instead of the bar.

77. Inverse of Involution.—Involution is not commutative, that is, 2^5 does not equal 5^2. In subtraction, the inverse of addition, there are two questions that may be asked. For example, we may ask what number must be added to 5 to make 9, or to what number must 5 be added to make 9; but as addition is commutative, there is only one inverse operation. Each of the above questions is answered by subtraction.

Also in division, the inverse of multiplication, two questions may be asked. For example, we may ask how many times is 4 contained in 20, or what number is contained 4 times in 20. This is equivalent to asking "20 is how many times 4, or 20 is 4 times what number." But since multiplication is commutative, there is only one inverse operation. Each of the above questions is answered by division.

In evolution, the inverse of involution, two questions may likewise be asked. For example, we may ask what is the fifth root of 32, or what root of 32 is 2. As involution is not commutative, these questions cannot be answered by one and the same operation. The former is answered by *evolution,* and the latter by *logarithms.* The former is the only inverse operation that we shall consider here.

78. Corresponding Direct and Inverse Operations do not always Cancel each Other. — Corresponding inverse and direct operations usually cancel each other. Thus the addition and subtraction of the same number cancel each other, the multiplication and division by the same number cancel each other, also the extraction of a root and raising to the corresponding power cancel each other. Thus:

$$(\sqrt[3]{a+b})^3 = a + b.$$

It must, however, be borne in mind that roots are more than one-valued, and hence the statement with reference to the inverse operations of extracting roots and raising to powers need restriction. It is true, necessarily and universally, that $(\sqrt[n]{a})^n = a$, but not that $\sqrt[n]{a^n} = a$. For instance, $\sqrt{a^2} = \pm a$. While the statement that the extraction of a root is cancelled by raising the result to the corresponding power is true necessarily and universally, the inverse statement that the raising an expression to a power is cancelled by the extraction of the corresponding root of the result is not necessarily true.

79. Extraction of Roots of Monomials.—Since evolution is the inverse of involution, we extract the root of an expression by doing just the opposite to what we do in finding a power.

Thus, we find the power of a monomial by raising its numeral factor to the power indicated by the exponent, and multiply the exponent of each literal factor by the exponent of the power.

e.g. $$(4x^2z^3)^3 = 64x^6z^9.$$

Hence we extract the root of a monomial by extracting the indicated root of the numeral factor and dividing the exponent of each letter by the index of the root.

e.g. $$\sqrt[3]{64x^6z^9} = 4x^2x^3.$$

N.B.—Since $(\pm a)^2 = a^2, \therefore \sqrt{a^2} = \pm a.$

That is, the square root of a positive quantity is either $+$ or $-$, and the square root of a negative quantity is impossible, or *imaginary*. The same is true of any even root.

The odd root of a positive quantity is $+$, and of a negative quantity $-$.

EXERCISE XXXVII.

I.

Find the indicated roots of the following monomials:

1. $\sqrt{a^8b^2c^{12}}$.
2. $\sqrt{64x^6y^{18}}$.
3. $\sqrt[3]{27a^6b^3c^9}$.
4. $\sqrt[3]{-343a^{12}b^{18}}$.
5. $\sqrt[7]{x^{14}y^{21}z^7}$.
6. $\sqrt[5]{-x^{10}y^{15}}$.

80. Extraction of the Square Root of Polynomials.—To obtain a rule for extracting the root of a polynomial, let us examine the square of a polynomial.

e.g. $(a+b+c+d)^2$

$$= a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$$

$$= a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2$$

$$= a^2+(2a+b)b+(2a+2b+c)c+(2a+2b+2c+d)d$$

$$= a^2+(2a+b)b+[2(a+b)+c]c+[2(a+b+c)+d]d.$$

From the last of the above equations we may derive the following rule for writing at sight the square of any polynomial:

Write the square of the first term, then the product of twice the first term plus the second multiplied by the second, then the product of twice the first two terms plus the third multiplied by the third, then the product of twice the first three terms plus the fourth multiplied by the fourth, etc.

If now we take the second member of the second equation and compare it with the second member of the last, we may readily obtain a rule for extracting the root of a polynomial.

$$\begin{array}{r|l|l}
 & a^2+2ab+b^2+2ac+2bc+c^2+2ad+2bd+2cd+d^2 & \underline{a+b+c+d} \\
 & a^2 & \\
\cline{2-2}
2a+b & 2ab+b^2 & \\
 & 2ab+b^2 & \\
\cline{2-2}
2a+2b+c & 2ac+2bc+c^2 & \\
 & 2ac+2bc+c^2 & \\
\cline{2-2}
2a+2b+2c+d & 2ad+2bd+2cd+d^2 & \\
 & 2cd+2bd+2cd+d^2 &
\end{array}$$

First arrange the terms of the polynomial according to the powers of some letter; then take the square root of the first term, place it in the root or quotient, square, subtract, and bring down one or more terms; then double the root *already found and place the result in the divisor, find how many times this is contained in the first term of the remainder, place the result in both the root and in the divisor, multiply, subtract, and bring down; then double the root already found and proceed as before; and so on to the end.*

EXERCISE XXXVIII.

Extract the square roots of:

I.

1. $a^4 + 4a^3 + 2a^2 - 4a + 1.$
2. $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4.$
3. $4a^6 - 12a^5x + 5a^4x^2 + 6a^3x^3 + a^2x^4.$
4. $9x^6 - 12x^3y^3 + 16x^2y^4 - 24x^4y^2 + 4y^6 + 16xy^5.$
5. $4a^8 + 16c^8 + 16a^6c^2 - 32a^2c^6.$
6. $4x^4 + 9 - 30x - 20x^3 + 37x^2.$
7. $16x^4 - 16abx^2 + 16b^2x^2 + 4a^2b^2 - 8ab^3 + 4b^4.$

II.

8. $x^6 + 25x^2 + 10x^4 - 4x^5 - 20x^3 + 16 - 24x.$
9. $x^6 + 8x^4y^2 - 4x^5y - 4xy^5 + 8x^2y^4 - 10x^3y^3 + y^6.$
10. $4 - 12a - 11a^4 + 5a^2 - 4a^5 + 4a^6 + 14a^3.$
11. $25x^6 - 31x^4y^2 + 34x^3y^3 - 30x^5y + y^6 - 8xy^5 + 10x^2y^4.$
12. $x^4 - x^3y - 7/4x^2y^2 + xy^3 + y^4.$
13. $x^4 - 4x^3y + 6x^2y^2 - 6xy^3 + 5y^4 - \dfrac{2y^5}{x} + \dfrac{y^6}{x^2}.$

81. Squaring Numbers as Polynomials.—Every number composed of two or more digits may be written as a polynomial. Thus: $25 = 20 + 5$, $234 = 200 + 30 + 4$, etc.

Hence $(234)^2 = (200 + 30 + 4)^2$

$= (200)^2 + (2.200 + 30) \times 30 + (2.230 + 4)4.$

$40000 + 12900 + 1856 = 54756.$

EXERCISE XXXIX.

In a similar way find the squares of the following numbers:

I.

1. 327. **2.** 3789. **3.** 845.

II.

4. 5006. **5.** 19683. **6.** 5083.

Observe that the square of a number contains either twice as many or one less than twice as many places as the number itself.

Ex. $.234 = .2 + .03 + .004.$

$(.234)^2 = (.2 + .03 + .004)^2$

$= (.2)^2 + (2 \times .2 + .03).03 + (2 \times .23 + .004).004$

$= .054756.$

In a similar way find the square of:

I.

7. .0304. **8.** .0028.

Observe that when a number is a decimal, its square is a decimal and contains twice as many places as the number.

Ex. $23.4 = 20 + 3 + .4.$

$(23.4)^2 = (20 + 3 + .4)^2 = (20)^2 + (2 \times 20 + 3)3 + (2 \times 23 + .4).4$

$= 547.56.$

In a similar way find the squares of:

I.

9. 69.4. **10.** 43.21.

II.

11. 37.89. **12.** 8.008.

Observe that when the number is composed of an integer and a decimal, its square is composed of an integer and a decimal, and that the number of places in the integral part of the square is either twice as great or one less than twice as great as that in the integral part of the number, and in the decimal part of the square twice as great as in the decimal part of the number.

82. Extracting the Square Root of Numbers.—Observe, in all the cases of the last section, that if we begin at the decimal point and divide the square into periods of two places each, the square root of the largest square in the left-hand period will be the left-hand figure of the number squared, and the number of this left-hand period, counting from the decimal point, will be the order, or place, of the figure in the root, or in the number squared.

Hence the first step in finding the root of a number is to divide the number into periods of two figures each, beginning at the decimal point.

The periods thus obtained correspond to the terms of a polynomial whose square root is to be found, and the process of finding the square root of a number is precisely analogous to that of finding the square root of a polynomial.

e.g. $\sqrt{387420489}$.

	3 –	87 –	42 –	04 –	89	10000 + 9000 + 600 + 80 + 3 =
	1	00	00	00	00	
20000 + 9000	2	87	42	04	89	19683.
29000	2	61	00	00	00	
38000 + 600		26	42	04	89	
38600		23	16	00	00	
39200 + 80		3	26	04	89	
39280		3	14	24	00	
39360 + 3			11	80	89	
39363			11	80	89	

It appears from the above example that, after the first step, the extraction of the square is a case of division, in which the divisor varies with each remainder, and in which the exact or *complete* divisor is unknown. It also appears that the incomplete or *trial* divisor in each case is double the part of the root already found.

Evidently the work in the above example might be made more compact by omitting the ciphers, and writing the root at once in the usual form, instead of in the form of a polynomial. Thus:

	3 –	87 –	42 –	04 –	89	19683
	1					
29	2	87				
	2	61				
386		26	42			
		23	16			
3928		3	26	04		
		3	14	24		
39263			11	80	89	
			11	80	89	

From the above considerations we may deduce the following rule for extracting the square root of a number:

Divide the number into periods of two places each, beginning at the decimal point; find the largest perfect square in the left-hand period, subtract it from this period and place its root in the quotient, and bring down the next period; double the root already found for a trial divisor, and seek how many times this is contained in the remainder exclusive of the last figure, and place the result in both the divisor and the quotient; multiply, subtract, bring down, and proceed as before.

As the trial divisor is smaller than the real divisor, we must guard against taking too large a figure for the quotient. Of course this figure can never exceed 9.

Should the trial divisor not be contained in the remainder after the last figure has been excluded, place a cipher in the divisor and quotient, and bring down the next period and try again, and so on till a significant figure is obtained.

In the actual work, after the number has been separated into periods, the decimal points may be disregarded. It should be placed in the quotient, or root, when its position has been reached, but farther than this it may be entirely neglected.

When the number is not an exact square, its root may be obtained to any required degree of approximation by bringing down two ciphers for each new period. Of course care must be taken to place the decimal point in the right position in the quotient.

EXERCISE XL.

Find the square roots of:

I.

1. 14356521.
2. 25060036.
3. 25836889.
4. 16803.9369.

II.

5. 4.54499761. 6. .9.

7. 6.21. 8. .00852.

83. Cubing of Polynomials.—

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$= a^3 + (3a^2 + 3ab + b^2)b.$$

$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2c + 3ab^2 + 3ac^2 + 3bc^2 + 6abc$$
$$= a^3 + (3a^2 + 3ab + b^2)b + [3(a+b)^2 + 3(a+b)c + c^2]c.$$

By means of the above formulæ the cube of any polynomial may be written at sight. First, write the cube of the first term; then the product of three times the square of the first term plus three times the product of the first and second terms plus the square of the second term multiplied by the second; then the product of three times the square of the first two terms plus three times the product of the first two terms and the third plus the square of the third multiplied by the third; etc.

EXERCISE XLI.

Cube the following polynomials by the above method:

I.

1. $a+1$. 2. $x+2$. 3. $ax-y^2$.

4. $2m-1$. 5. $4a-3b$. 6. $1+x+x^2$.

II.

7. $1-2x+3x^2$. 8. $a+2b-c$.

9. $2a^2-3a+1$. 10. $1-x+x^2-x^3$.

84. Extracting the Cube Root of Polynomials.—If we arrange the terms of $(a+b+c)^3$ according to the descending powers of a and the ascending powers of c, we have

$$a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$$

Comparing this with

$$a^3+(3a^2+3ab+b^2)b+[3(a+b)^2+3(a+b)c+c^2]c,$$

we may readily extract the cube root of the first expression. Thus:

$$\begin{array}{l|l}
 & a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \,\underline{|\,a+b+c} \\
 & a^3 \\ \cline{2-2}
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\ \hline
3a^2+6ab+3b^2+3ac+3bc+c^2 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\ \hline
\end{array}$$

The rule for extracting the cube root of a polynomial may be stated as follows:

Arrange the terms according to the powers of some letter or letters; extract the cube root of the first term and place the root in the quotient and subtract the cube from the polynomial, and bring down a part of the remainder; use three times the square of the root already found as a trial divisor, and seek how many times this is contained in the first term of the remainder, place the result as a new term in the quotient, and place three times the product of this term and the root already found, and also the square of this term, as a new term in the divisor, multiply, subtract, and bring down; and so on till there is no remainder, or until the desired degree of approximation has been reached.

EXERCISE XLII.

Find the cube roots of:

I.

1. $1 - 3x + 3x^2 - x^3$.
2. $1 + 6x + 12x^2 + 8x^3$.
3. $8x^3 - 36x^2y + 54xy^2 - 27y^3$.
4. $27x^3y^3 - 27x^2y^2z^2 + 9xyz^4 - z^6$.

II.

5. $24a^2b + a^3 + 512b^3 + 192ab^2$.
6. $108x^5 - 144x^4 - 27x^6 + 64x^3$.
7. $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$.
8. $1 - x$ to four terms.

85. Cubing Numbers as Polynomials.—Any number may be written as a polynomial and then cubed by the method of 83. Thus:

$$1854 = 1000 + 800 + 50 + 4,$$

and $(1854)^3 = (1000 + 800 + 50 + 4)^3$

$$= 1000^3 + [3(1000^2 + 1000 \times 800) + 800^2]800 +$$
$$[3(1800^2 + 1800 \times 50) + 50^2]50 + [3(1850^2 + 1850 \times 4) + 4^2]4$$
$$= 1000000000 + 4\,832\,000\,000 + 499\,625\,000 + 41\,158\,864$$
$$= 6\,372\,783\,864.$$

EXERCISE XLIII.

Cube the following numbers by the process of 46:

I.

1. 135.
2. 223.
3. 106.
4. 258.
5. 478.
6. 46.8.

II.

7. 9.36.
8. 27.55.
9. .384.

86. Extracting the Cube Root of Numbers.—Observe in each of the above cases that the cube of a number contains three times as many figures as the number cubed, or one or two less than three times as many; that when the number cubed is an integer, the cube is an integer; that when the number cubed is a decimal, the cube is a decimal; that when the number cubed is composed of an integer and a decimal, the cube is also composed of an integer and a decimal.

Observe also that if we divide the cube into periods of three places each, beginning at the decimal point, the number of periods in the cube will equal the number of figures in the number cubed; and that the cube root of the largest cube in the left-hand period will be the left-hand figure of the number cubed.

Hence the first step in finding the cube root of a number is to divide the number into periods of three figures each, beginning at the decimal point.

The periods thus obtained correspond to the terms of a polynomial whose cube root is to be found, and the process of finding the cube root of a number is precisely analogous to that of finding the cube root of a polynomial.

e.g. Extract the cube root of 12 977 875.

$$
\begin{array}{rr|r|l}
 & & 12\text{–}977\text{–}875 & 200+30+5=235 \\ \cline{4-4}
 & 200^3 = & 8\ 000\ 000 & \\ \cline{3-3}
3\times 200^2 = & 120000 & 4\ 977\ 875 & \\
3\times 200\times 30 = & 18000 & 4\ 167\ 000 & \\ \cline{3-3}
30^2 = & 900 & & \\ \cline{1-2}
 & 138900 & & \\
3\times 230^2 = & 158700 & 810\ 875 & \\
3\times 230\times 5 = & 3450 & 810\ 875 & \\ \cline{3-3}
5^2 = & 25 & & \\ \cline{1-2}
 & 162175 & &
\end{array}
$$

It appears from the above example that, after the first step, the extraction of the cube root is a case of division, in which the exact or complete divisor is unknown. It also appears that the incomplete or trial divisor in each case is three times the square of the part of the root already found.

As in square root, the process may be made more compact, by omitting the ciphers in the root, and writing it at once in the usual form. The ciphers may also be omitted from the partial subtrahends, and only one period need be brought down at a time. One cipher must, however, be employed for the next place in finding the trial and complete divisors. This is necessary because the significant figures in the additions to the trial divisor often overlap those of the trial divisor.

As regards decimal points and imperfect cubes, the same remarks apply as to square root.

As the trial divisor in cube root is considerably smaller than the real divisor, there is great liability to make the next figure too large, and the right figure often can be ascertained only after two or three trials.

EXERCISE XLIV.

Find the cube roots of:

I.

1. 109 215 352.
2. 56 623 104.
3. 102.503 232.
4. 820.025 856.
5. 20 910.518 875.
6. 2.5.

II.

7. .2.
8. .01.
9. 4.
10. .4.
11. 28.25.
12. $15\frac{2}{3}$.

I.

13. Divide $27a^6x^3 - 8b^9y^3$ by $3a^2x - 2b^3y$.

14. $(x+1)^2 - (x^2-1) = x(2x+1) - 2(x+2)(x+1) + 20$.

15. The length of a room exceeds its breadth by 3 ft. Were its length increased by 3 feet and the breadth diminished by 2 feet, the area of the room would remain the same. Find the dimensions of the room.

II.

16. Divide $8a^9 + 64c^6$ by $4a^6 - 8a^3c^2 + 16c^4$.

17. $25x - 19 - [3 - (4x - 5)] = 3x - (6x - 5)$.

18. The length of a room exceeds its breadth by 8 ft. Were each increased by 2 feet, it would take $26\frac{2}{3}$ yards more of carpeting 3/4 of a yard wide to cover the floor. Find the dimensions of the room.

19. In a cellar one fifth of the wine is port and one third claret. Besides this it contains 15 dozen bottles of sherry and 30 bottles of spirits. How many bottles of port and of claret does it contain ?

20. A boy bought some apples at three a cent and 5/6 as many at four a cent. He sells them at 16 for 6 cents and gains $3\frac{1}{2}$ cents. How many apples did he buy ?

CHAPTER X.

MULTIPLICATION AT SIGHT.

87. Complete Algebraic Expressions.—A complete algebraic expression of the first degree in any one letter is a binomial, one of whose terms contains the first power of the letter and the other does not contain the letter at all. Thus, $x + 5$, $3x - a$ are complete expressions of the first degree in x.

The term of an expression which does not contain the letter or unknown quantity is called the *constant* or *absolute* term.

A complete algebraic expression of the second degree in any one letter is a trinomial, one of whose terms contains the second power of the letter, another the first power of the letter, and the third does not contain the letter at all. Thus, $x^2 + 5x - 6$, $3x^2 - 4x + a$ are complete expressions of the second degree in x.

88. Product of Two Binomials of the First Degree.—The product of two binomial expressions of the first degree in any letter is generally a trinomial of the second degree in that letter, though it is in one case a quadratic binomial. The student should be able to write with facility at sight the product of any two first-degree binomials in the same letter.

Suppose we are required to obtain the product of $3x+4$ and $5x - 7$. The literal factor of the first term will be x^2, of the second term x, and the third term will not contain x.

The annexed diagrammatic arrangement will enable us to obtain the coefficients.

The coefficients are to be multiplied together as indicated by the connecting lines. The product of the left-hand coefficients will be the coefficient of x^2, the sum of the two cross-products will be the coefficient of x, and the product of the right-hand factors will be the absolute term. Care must be taken to use the right sign with each coefficient of x and with the absolute term, and also with each product.

The product of the above binomials will be found to be $15x^2 - x - 28$.

We would advise using the diagrammatic arrangement in all cases at first till the pupil has acquired facility in obtaining the new coefficients. The diagram may then be discarded, and the product written down at once, the work of obtaining the result being entirely mental.

EXERCISE XLV.

Find by the above method the products of the following pairs of first-degree binomials:

I.

1. $2x - 5$ and $7x - 4$.
2. $5x + 8$ and $4x + 6$.
3. $4 - 5x$ and $7 - 3x$.
4. $6 + 8x$ and $5 - 10x$.
5. $x + 7$ and $x + 9$.
6. $x - 5$ and $x - 3$.
7. $x - 6$ and $x + 9$.
8. $x - 11$ and $x + 7$.
9. $x + 5$ and $x - 6$.
10. $x + 7$ and $x - 4$.
11. $x + 3$ and $x + 3$.
12. $x - 4$ and $x - 4$.
13. $x + 8$ and $x - 8$.
14. $x - 6$ and $x + 6$.

II.

15. $7x - 9$ and $5x + 12$.

16. $6x - 3$ and $12x + 8$.

17. $3x + 7$ and $8x - 25$.

18. $2x + 6$ and $9x - 30$.

19. $ax - b$ and $4x - 5$.

20. $3ax + c$ and $6x + 8$.

21. $ax - c$ and $5ax + b$.

22. $(a + b)x + c$ and $2ax - b$.

23. $3 - 9x$ and $8 + 12x$.

24. $9 + 4x$ and $7 - 8x$.

89. Product of $x + a$ and $x + b$.—Observe in examples 5–10 that when the coefficient of x in the factors is unity, the coefficient of x^2 in the product will be unity, that the coefficient of x in the product will be the algebraic sum of the constant terms of the factors, and that the constant term in the product will be the algebraic product of the constant terms of the factors. Also that the constant term of the product will be positive when the constant terms of the factors have like signs, and negative when the constant terms of the factors have unlike signs, and that the sign of the term in x in the product is that of the constant term of the factors which is numerically the larger.

The cases illustrated by these six examples are of very common occurrence, and careful attention should be given them.

90. Product of $x + a$ and $x + a$.—Observe in examples 11 and 12 that when the two factors are alike, the result is the same as that obtained by the formula for squaring a binomial.

91. Product of $x + a$ and $x - a$.—Observe in examples 13 and 14 that, when the corresponding terms of the two binomial factors are alike in absolute value but different in their connecting sign, the product is a *binomial*, and that the two terms of the product are the squares of the corresponding terms of the factors, and that the sign between the terms of the product is *minus.*

This is the only case in which the product of two binomials is a binomial. In all other cases it is a trinomial. This case is particularly important, and is known as the "product of the sum and difference of two quantities," and is usually stated thus:

The product of the sum and difference of two quantities is equal to the difference of their squares.

92. Product of any Two Binomial Factors of the same Degree.—Any two binomial factors which are of the same degree in the same letter, and each of which has a constant term, may be multiplied at sight by the method of section 88. The literal factor in one term of the product will be the square of the factor in the given binomials, in another term of the product it will be the same as in the given binomials, and in the third term of the product it will not occur at all. The coefficients and constant term of the product may be found by the diagrammatic arrangement given in section 88. 4,3-

Ex. Find the product of $3x^3 + 5$ and $4x^3 - 8$.

Ans. $12x^6 - 4x^3 - 40$.

EXERCISE XLVI.

Write at sight the products of the following pairs of binomials:

I.

1. $4x^2 - 7$ and $5x^2 - 3$.
2. $7x^4 + 4$ and $3x^4 + 5$.
3. $5x^3 + 4$ and $6x^3 - 8$.
4. $6x^5 - 2$ and $7x^5 + 3$.
5. $3z^2 - 8$ and $7z^2 + 12$.
6. $9y^6 + 11$ and $6y^6 - 7$.
7. $\sqrt{x} + 5$ and $\sqrt{x} + 7$.
8. $2\sqrt{x} - 6$ and $3\sqrt{x} + 8$.

9. $\sqrt{x} - 7$ and $\sqrt{x} + 7$.
10. $3\sqrt{x} + 4$ and $3\sqrt{x} + 4$.
11. $x + \sqrt{5}$ and $x - \sqrt{5}$.
12. $\sqrt{m} + \sqrt{5}$ and $\sqrt{m} - \sqrt{5}$.

II.

13. $m^4 - 2$ and $5m^4 - 8$.
14. $n^3 + 12$ and $3n^3 - 15$.
15. $s^5 - 7$ and $s^5 + 8$.
16. $a^7 + 9$ and $a^7 - 11$.
17. $x^4 - 7$ and $x^4 + 7$.
18. $m^3 + 6$ and $m^3 - 6$.
19. $2x^2 - 4$ and $2x^2 + 4$.
20. $5a^2x^3 - 3$ and $5a^2x^3 + 3$.
21. $3\sqrt{x} + 5\sqrt{7}$ and $3\sqrt{x} - 5\sqrt{7}$.
22. $6\sqrt{x} - 7\sqrt{3}$ and $6\sqrt{x} + 7\sqrt{3}$.
23. $x + \sqrt{-4}$ and $x - \sqrt{-4}$.
24. $2x^2 + 3\sqrt{-5}$ and $2x^2 - 3\sqrt{-5}$.

93. Products of Binomial Aggregates.—Any aggregate may take the place of the literal factor in the preceding binomials and the product obtained by the same methods. Of course the aggregate must have the same exponent or radical index in the two binomial factors.

EXERCISE XLVII.

Write at sight the product of the following pairs of binomials:

I.

1. $(a + x) + 4$ and $(a + x) - 7$.
2. $(m + x) - 8$ and $(m + x) + 9$.
3. $(x - b) - 5$ and $(x - b) + 9$.
4. $(x - m) - 12$ and $(x - m) + 7$.
5. $x - \sqrt{(m - 5)}$ and $x + \sqrt{(m - 5)}$.
6. $x + \sqrt{(3 - a)}$ and $x - \sqrt{(3 - a)}$.

7. $(x - 4) + (x - a)$ and $(x - 4) - (x - a)$.

8. $\sqrt{(x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1)} = ?$

9. $\sqrt[3]{(1 - 9x^2 + 33x^4 - 63x^6 + 66x^8 - 36x^{10} + 8x^{12})} = ?$

10. $\frac{\sqrt{(x - 6)}}{7} - \frac{x + 8}{3\sqrt{(x - 6)}} = 5\sqrt{(x - 6)}$.

11. *A* alone can do a piece of work in nine days, and *B* alone in 12 days. How many days will it take them to do it together?

12. A cistern could be filled in 12 minutes by two pipes which empty into it, and it could be filled in 20 minutes by one of these pipes alone. How many minutes would it take the other pipe alone to fill it?

II.*

13. $(x - 5) + (x + 6)$ and $(x - 5) - (x + 6)$.

14. $(x + 7) - (x - 5)$ and $(x + 7) + (x - 5)$.

15. $\sqrt{(x + 2)} + 5$ and $\sqrt{(x + 2)} - 5$.

16. $\sqrt{(x - 7)} + 4$ and $\sqrt{(x - 7)} - 4$.

17. $x + \sqrt{(x - 5)}$ and $x - \sqrt{(x - 5)}$.

18. $\sqrt{(x + 4)} + \sqrt{(x - 7)}$ and $\sqrt{(x + 4)} - \sqrt{(x - 7)}$.

19. $\sqrt{(x + 8)} + \sqrt{(x + 5)}$ and $\sqrt{(x + 8)} - \sqrt{(x + 5)}$.

20. $3\sqrt{(5 + x)} + 5\sqrt{(x - 7)}$ and $3\sqrt{(5 + x)} - 5\sqrt{(x - 7)}$.

21. $4\sqrt{(7 + x)} - 3\sqrt{(x - 4)}$ and $4\sqrt{(7 + x)} + 3\sqrt{(x - 4)}$.

22. $\frac{3\sqrt{(2x + 4)}}{8} - \frac{3x - 10}{4\sqrt{(2x + 4)}} = 6\sqrt{(2x + 4)}$.

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* Unless otherwise stated, directions for I apply to II also.

23. A cistern could be filled by one pipe alone in six hours, and by another pipe alone in eight hours; and it could be emptied by an outlet pipe in twelve hours. In how many hours would the cistern be filled were all three pipes opened together when the cistern was empty?

94. Product of $x + y$ and $x^2 - xy + y^2$.—The product of $x + y$ and $x^2 - xy + y^2$ is $x^3 + y^3$, and of $x - y$ and $x^2 + xy + y^2$ is $x^3 - y^3$. (Show these by actual multiplication.)

In words, the product of the sum of two terms and the sum of the squares of the terms minus their product is the sum of the cubes of the terms, and the product of the difference of two terms and the sum of the squares of the terms plus their product is the difference of the cubes of the terms.

EXERCISE XLVIII.

Write at sight the product of the following pairs of factors:

I.

1. $x + a$ and $x^2 - ax + a^2$.
2. $x + 3$ and $x^2 - 3x + 9$.
3. $x - 7$ and $x^3 + 7x + 49$.
4. $x - c$ and $x^2 + cx + c^2$.
5. $2x^2 - 3a$ and $4x^4 + 6ax^2 + 9a^2$.

Write at sight the missing factor of the two following examples:

6. $(x - 4)(\qquad\qquad) = x^3 - 64$.
7. $(2ax^2 + 7)(\qquad\qquad) = 8a^3x^6 + 343$.
8. Square $x^3 + x^2 + x + 1$ by the method of section 73.
9. Cube $1 - 3x^2 + 2x^4$ by the method of section 75.

II.

10. $a^2 + 1/3b$ and $a^4 - 1/3a^2b + 1/9b^2$.

11. $1/2a^2x^3 - 2/3b^3x^2$ and $1/4a^4x^6 + 1/3a^2b^3x^5 + 4/9b^6x^4$.

12. $1/5a^3x^4 + 1/6b^4x^5$ and $1/25a^6x^8 - 1/30a^3b^4x^9 + 1/36b^8x^{10}$.

Write at sight the missing factor of the following examples:

13. $(3a^2x^3 - 1/3ax)(\qquad\qquad) = 27a^6x^9 - 1/27a^3x^3$.

14. $(1/4a^3x^5 + 1/6b^2x^7)(\qquad\qquad) = 1/64a^9x^{15} + 1/216b^6x^{21}$.

95. To Convert $x^2 + bx$ into a Perfect Square.—The square of a binomial of the first degree of the form $x + a$, that is, of one having a constant term and unity as the coefficient of its first-degree term, is a complete quadratic trinomial. The first-degree term of this trinomial is twice the product of the two terms of the binomial, and the constant term of the trinomial is the square of the constant term of the binomial, or the square of half the coefficient of the first-degree term of the trinomial.

e.g. $(x + 4)^2 = x^2 + 8x + 16$. Here $16 = \left(\frac{8}{2}\right)^2$.

$(x - 1/2)^2 = x^2 - x + 1/4$. Here $1/4 = (1/2)^2$.

Hence a quadratic binomial of the form $x^2 + bx$, that is, one having a first- and a second-degree term in a letter and unity as the coefficient of its second-degree term, may be converted into a perfect square by adding as a constant term the square of half the coefficient of its first-degree term.

e.g. The quadratic binomial $x^2 - 6x$ becomes a perfect square on the addition of $(3)^2$ to it as a constant term. When thus completed it becomes the trinomial $x^2 - 6x + 9$.

EXERCISE XLIX.

Convert the following quadratic binomials into perfect squares:

I.

1. $x^2 + 8x$.
2. $m^2 - 10m$.
3. $x^2 - 3x$.
4. $n^2 - 5n$.
5. $x^2 + 7x$.
6. $y^2 - 9y$.

II.

7. $x^2 - 3/4x$.
8. $z^2 + 5/6z$.
9. $x^2 + bx$.
10. $x^2 - 5bx$.
11. $x^2 + x$.
12. $y^2 - y$.

96. To Convert $x^{2n} + bx^n$ into a Perfect Square.—Binomials of a similar form but of a higher degree may be converted into perfect squares in the same way. The form of the expression will be similar when the degree of one term in any letter is twice that of the other term in the same letter, and the coefficient of the term of the higher degree is unity.

e.g. $x^4 - 8x^2$ becomes a perfect square on the addition of $(4)^2$. It will then be $x^4 - 8x^2 + 16$. This is the square of $x^2 - 4$.

Of course in any of these cases an aggregate may take the place of a single literal factor.

EXERCISE L.

Convert the following binomial expressions into complete squares:

I.

1. $x^6 + 6x^3$.
2. $m^4 - 12m^2$.
3. $x^4 - 5x^2$.
4. $a^8 + 7a^4$.
5. $x^6 + bx^3$.

II.

6. $z^4 - z^2$.
7. $x^{10} - 2/3x^5$.
8. $n^6 - 3/4n^3$.
9. $(x + 2)^2 + 6(x + 2)$.
10. $(x - 5)^2 - 3(x - 5)$.

97. To Convert $x^2 + bx + c$ into a Perfect Square.— Quadratic trinomials of the form $x^2 + bx + c$ may be converted, without change of value, into perfect squares plus or minus a term which may be either simple or complex, by the addition and subtraction of the square of half the coefficient of x. It is best to make the addition and subtraction immediately after the second term, and then to combine the last two terms into one.

$$\text{e.g.}\quad x^2 + 4x - 8 = x^2 + 4x + 4 - 4 - 8$$
$$= x^2 + 4x + 4 - 12.$$

The first three terms of the last polynomial are a perfect square.

$$x^2 + 6x + 10 = x^2 + 6x + 9 - 9 + 10 = x^2 + 6x + 9 + 1.$$

$$x^2 + 5x - 7 = x^2 + 5x + \frac{25}{4} - \frac{25}{4} - 7$$
$$= x^2 + 5x + \frac{25}{4} - \frac{53}{4}.$$

EXERCISE LI.

Convert each of the following trinomials into a perfect square plus or minus a constant term, without change of value:

I.

1. $x^2 - 8x - 2$.
2. $x^2 - 12x + 30$.
3. $x^2 + 7x - 3/4$.
4. $x^2 - 7x + 3/5$.
5. Divide $1/32x^5 - 1024$ by $1/2x - 4$.

6. A workman was employed for 60 days, on condition that he should receive 3 dollars for every day he worked, and forfeit 1 dollar for every day he was absent. At the end of the time he received 48 dollars. How many days did he work?

7. A can do a piece of work in 10 days, and B can do it in eight days. After A has been at work on it for three days, B comes to help him. In how many days will they finish?

II.

8. $y^2 - 9y + 3.$ 9. $z^2 + 11z - 7.$

10. $x^2 + bx + c.$ 11. $y^2 - by - c.$

12. Divide $32/243x^5 + 3125$ by $2/3x + 5.$

13. A privateer, running at the rate of 10 miles an hour, discovers a ship 18 miles off running at the rate of 8 miles an hour. How many miles can the ship run before she is overtaken?

14. A cistern has two supply-pipes respectively capable of filling it in $4\frac{1}{2}$ and 6 hours. It also has a leak capable of emptying it in 5 hours. In how many hours would it be filled when both pipes are on?

98. To Convert $ax^2 + bx$ into a Perfect Square. — Quadratic binomials of the form $ax^2 + bx$ may be converted into perfect squares by first dividing them by the coefficient of x^2 and then adding the square of half of the resulting coefficient of x.

e.g. $3x^2 + 12x$ becomes, on division by 3, $x^2 + 4x$, and then, on addition of the square of half of 4, $x^2 + 4x + 4$, which is a perfect square.

Similarly, $3x^2 - 5x$ becomes $x^2 - 5/3x$, and then $x^2 - 5/3x + 25/36$, which last is a perfect square.

EXERCISE LII.

Convert the following quadratic binomials into perfect squares, and solve the given equations:

I.

1. $6x^2 + 18x.$
2 $3x^2 - 15x.$
3. $5x^2 - 15x.$
4. $7x^2 + 63x.$
5. $3(x + a)^2 - 5(x + a).$
6. $\frac{x + 3}{x - 2} = \frac{x - 5}{x - 3}.$

II.

7. $ax^2 + bx.$
8. $my^2 - ny.$
9. $2x^4 + 3x^2.$
10. $3z^6 - 9z^3.$
11. $7(z - 5)^4 + 3(z - 5)^2.$
12. $\frac{x + 5}{x - 5} = \frac{x - 5}{x - 4}.$

EXERCISE LIII.

Convert the following quadratic trinomials, without change of value, into expressions which shall be a perfect square plus or minus a constant term:

I.

1. $2x^2 + 3x + 6.$
2. $3x^2 - 18x - 12.$
3. $4x^2 - 6x + 7.$
4. $5x^2 + 25x - 20.$
5. $6x^2 + 42x + 50.$
6. Find the square root of 2 to four places of decimals.
7. Find the cube root of 3 to three places of decimals.

II.

8. $7x^2 - 63x + 49.$
9. $8x^2 - 40x - 12.$
10. $9x^2 - 81x + 63.$
11. $10x^2 + 70x - 80.$
12. $11x^2 - 2x + 3.$
13. $ax^2 + bx + c.$
14. $mz^2 - nz + p.$

CHAPTER XI.

FACTORING.

99. Resolution into Factors.—To factor an expression is to resolve it into its component factors. To be able to factor algebraic expressions readily and accurately is a matter of very great importance. Other things being equal, the one most skilful at factoring is the best algebraist.

1°. *To Resolve an Expression into a Monomial and a Polynomial Factor.*—When every term of a polynomial contains a common factor, it may be resolved into a monomial and a polynomial factor.

The factor common to all the terms will be the monomial factor, and the quotient obtained by dividing the expression by this factor will be the polynomial factor.

e.g. $$6x^2 + 12x - 18 = 6(x^2 + 2x - 3).$$
$$a^3x - a^3 = a^3(x - 1).$$

EXERCISE LIV.

Resolve each of the following expressions into a monomial and a polynomial factor:

I.

1. $6ab + 2ac$. **2.** $2a^2bx^2 - 8a^2bx + 2a^3b^3$.

3. $5b^4c^2x + 5b^3c^4y - 5b^3c^2$. **4.** $7a - 7a^3 + 14a^4$.

5. $6x^3 + 2x^4 + 4x^5$.

II.

6. $15a^2 - 225a^4$.

7. $5x^5 - 10a^2x^3 - 15a^3x^3$.

8. $38a^3x^5 + 57a^4x^2$.

9. $3x^3 - x^2 + x$.

10. $2x^2y^3 - 3x^2y^2 + 2xy^3$.

2°. *To Factor the Difference of Two Squares.*—The difference of two squares is equal to the product of the sum and difference of their roots.

EXERCISE LV.

Factor each of the following expressions:

I.

1. $x^2 - a^2$.

2. $x^2 - 9$.

3. $4a^2 - 64$.

4. $9a^2x^2 - 25b^2$.

5. $81 - 16a^2x^4$.

6. $49a^4x^2 - 16a^6z^8$.

7. $(x^2 + 12x + 36) - 49$.

8. $y^2 - 8y + 16 - 81$.

9. $(a^2 - 4a + 4) - 16$.

10. $(b^2 + 24b + 144) - 121$.

11. The head of a fish is 9 inches long, the tail is as long as the head and half the body, and the body is as long as the head and tail together. What is the length of the fish?

NOTE.—In solving problems concerning numbers composed of digits, the student must bear in mind that a number composed of two digits is equal to 10 times the left-hand digit plus the right-hand digit; that a number composed of three digits is equal to 100 times the left-hand digit plus 10 times the middle digit plus the right-hand digit. Thus, $46 = 10 \times 4 + 6$, and $387 = 100 \times 3 + 10 \times 8 + 7$.

12. A number is composed of two digits, and the left digit is 4/3 of the right. If 18 be subtracted from the number, its digits will be reversed. What is the number?

II.

Factor:

13. $12 - 3a^2$.

14. $48a^3 - 108ab^2$.

15. $27a^5 - 75ax^4$. **16.** $125a^2x^7 - 45x^3y^4$.

Convert the following trinomials into the difference of two squares and then factor:

17. $x^2 + 14x + 40$. **18.** $x^2 - 16x - 17$.

19. $x^2 - 10x - 11$. **20.** $x^2 + 30x + 29$.

21. A and B together can do a piece of work in 12 hours, A and C together can do it in 16 hours, and A alone can do it in 20 hours. In what time can they all do it together, and in what time could B and C together do it?

22. A number is composed of two digits whose sum is 13, and if 9 be added to the number its digits will be reversed. What is the number?

3°. *Special Cases of Factoring Quadratic Trinomials.* —We have seen that the product of two binomials of the first degree in any letter is, in general, a quadratic trinomial in the same letter, and that the coefficient of the second-degree term of the letter is the product of the coefficient of the first-degree terms of the letter in the binomials, the coefficient of the first-degree term of the letter in the product is the sum of the products of the coefficient of the first-degree term of the letter in each binomial multiplied by the constant term of the other binomial, and the constant term of the product is the product of the constant terms of the binomials.

Hence a quadratic trinomial in any letter may be resolved into two binomial factors of the first degree in that letter whenever we can discover four numbers such that the product of the first two will be the coefficient of the second-degree term of the trinomial, the product of the last two will be the constant term of the trinomial, and the algebraic sum of the cross-products of the numbers will be the coefficient of the first-degree term of the trinomials. The first two numbers will then be the coefficients of the first-degree terms of the factors, and the last two numbers will be the constant terms of the factors.

It is best to arrange diagrammatically the four numbers selected for trial, as in the corresponding case of sight multiplication.

e.g. Resolve $6x^2 + 7x - 20$ into binomial factors.

$3 \times 2 = 6$, the coefficient of x^2;

$2 \times -4 = -8$;

$3 \times 5 = 15$;

$15 + (-8) = 7$, the coefficient of x;

$5 \times (-4) = -20$, the constant term.

Hence $6x^2 + 7x - 20 = (2x + 5)(3x - 4)$.

Notice that the complete test involves two trials, if first be unsuccessful: e.g. 3 above and 2 below as well as 2 above and 3 below.

Again, resolve $3x^2 - 12x - 63$ into binomial factors.

The required factors are $(x - 7)$ and $(3x + 9)$.

Resolve $x^2 - 2x - 63$ into binomial factors.

The factors are $(x + 7)$ and $(x - 9)$.

The case in which the coefficient of the second-degree term of the trinomial is unity is of frequent occurrence and of great importance.

EXERCISE LVI.

Resolve the following quadratic trinomials into binomial factors:

I.

1. $x^2 + 12x + 35.$ **2.** $x^2 - 12x + 27.$

3. $x^2 - 4x - 32.$ **4.** $x^2 + 7x - 30.$

5. $x^2 - x - 42.$ **6.** $x^2 + x - 20.$

7. $2x^2 - 10x - 48.$ **8.** $3x^2 + 26x + 55.$

9. $6x^2 - 17x + 7.$ **10.** $20x^2 + 37x + 8.$

11. $35x^2 + 39x - 36.$ **12.** $56x^2 - 100x - 100.$

13. A, B, and C together can do a piece of work in 5 days, A and B together can do it in 8 days, and B and C together in 7 days. In what time can each do it alone?

II.

Factor the following expressions:

14. $12 + 10x - 8x^2.$ **15.** $48 - 128x + 84x^2.$

16. $35 + 41x + 12x^2.$ **17.** $6x^2 + (21 - 2a)x - 7a.$

18. $abx^2 + (7a - 5b)x - 35.$

19. $acx^2 + (bc - ad)x - bd.$

20. $x^2 + 2bx - a^2 + b^2.$

21. $(a^2 - b^2)x^2 - 2(a + 3b)x - 8.$

22. $3x^2 + 9x - 54.$ **23.** $7x^2 - 7x - 210.$

24. $10x^2 + 50x - 140.$ **25.** $75a^2x^2 - 5a^2x - 30a^2.$

26. Find a number composed of two digits whose sum is twelve and which will have its digits reversed by adding 63 to the number and dividing the sum by 4.

100. Functions.—In mathematics, one quantity is said to be a function of another when its value depends upon the value of the other and changes with it.

e.g. The value of the expression $x^2 + 6x - 5$ depends upon the value of x and changes with the value of x. Hence the expression $x^2 + 6x - 6$ is a function of x.

The symbol $f(x)$ means any algebraic expression containing x. This is a very convenient notation when we wish to indicate any expression containing x without designating any particular expression. $f(a)$ indicates the algebraic expression obtained by substituting a for x in $f(x)$. Thus if $f(x) = x^2 + 3x + 6$, then

$$f(a) = a^2 + 3a + 6.$$

EXERCISE LVII.

I.

1. If $f(x) = x^3 + 3x^2 - 10$, find $f(3)$.
2. If $f(x) = x^3 + 3x^2 - 10$, find $f(-3)$.
3. If $f(x) = x^2 - 5x + 6$ and $y = 3 - x$, find $f(y)$ in terms of x.
4. If $f(x) = x^2 + x + 1$, find $f(x - 1)$.
5. If $f(x) = x^2 + 2x - 7$, find $f(5)$.
6. If $f(a) = (a + b + c)^3 - a^3 - b^3 - c^3$, find $f(-b)$.
7. If $f(x) = x^5 - y^5$, find $f(y)$.
8. If $f(x) = x^6 - y^6$, find $f(y)$.

II.

9. If $f(x) = x^n - y^n$, find $f(y)$.
10. If $f(x) = x^5 + y^5$, find $f(-y)$.
11. If $f(x) = x^5 + y^5$, find $f(y)$.
12. If $f(x) = x^4 + y^4$, find $f(-y)$.
13. If $f(x) = x^4 + y^4$, find $f(y)$.

14. If $f(x) = x^n + y^n$ and n is odd, find $f(-y)$.

15. If $f(x) = x^n + y^n$ and n is even, find $f(-y)$.

16. If $f(x) = x^n + y^n$, find $f(y)$.

101. Remainder Theorem.—When $f(x)$ is divided by $x - a$, the process of division being continued till the remainder, if there be one, does not contain x, the remainder will $= f(a)$.

Proof.—Denote the remainder, which is supposed not to contain x, by R and the quotient by Q. Then we have

$$\frac{f(x)}{x-a} = Q + \frac{R}{x-a},$$

or

$$f(x) = Q(x-a) + R.$$

If now we substitute a for x in each member, R must remain unaltered since it does not contain x, and $x - a$ will become $a - a = 0$. Hence $f(a) = R$.

e.g. Let $f(x) = x^3 + 2x^2 - 5x - 6$, and let $a = 4$. Then

$$f(a) = 4^3 + 2 \times 4^2 - 5 \times 4 - 6 = 64 + 32 - 20 - 6 = 70.$$

By division,

$$\begin{array}{llll|l}
x^3 & +\,2x^2 & -\ \ 5x & -\ \ 6 & x\ -\ 4 \\ \cline{5-5}
x^3 & -\,4x^2 & & & x^2 + 6x + 19 \\ \cline{1-4}
 & 6x^2 & -\ \ 5x & & \\
 & 6x^2 & -\,24x & & \\ \cline{2-4}
 & & 19x & -\ \ 6 & \\
 & & 19x & -\,76 & \\ \cline{3-4}
 & & & 70 &
\end{array}$$

Again, let $f(x) = x^5 + 32$, and let $a = 2$.

Then $\qquad f(a) = 32 + 32 = 64.$

By division,

$$\begin{array}{l|l}
x^5+32 & x-2 \\ \cline{2-2}
x^5-2x^4 & x^4+2x^3+4x^2+8x+16 \\ \cline{1-1}
\quad 2x^4+32 & \\
\quad 2x^4-4x^3 & \\ \cline{1-1}
\qquad 4x^3+32 & \\
\qquad 4x^3-8x^2 & \\ \cline{1-1}
\qquad\quad 8x^2+32 & \\
\qquad\quad 8x^2-16x & \\ \cline{1-1}
\qquad\qquad 16x+32 & \\
\qquad\qquad 16x-32 & \\ \cline{1-1}
\qquad\qquad\qquad 64 &
\end{array}$$

Again, let $f(x) = x^5 - 32$, and let $a = 2$.

Then $f(a) = 32 - 32 = 0$.

By division,

$$\begin{array}{l|l}
x^5-32 & x-2 \\ \cline{2-2}
x^5-2x^4 & x^4+2x^3+4x^2+8x+16 \\ \cline{1-1}
\quad 2x^4-32 & \\
\quad 2x^4-4x^3 & \\ \cline{1-1}
\qquad 4x^3-32 & \\
\qquad 4x^3-8x^2 & \\ \cline{1-1}
\qquad\quad 8x^2-32 & \\
\qquad\quad 8x^2-16x & \\ \cline{1-1}
\qquad\qquad 16x-32 & \\
\qquad\qquad 16x-32 & \\ \cline{1-1}
\end{array}$$

The theorem proved and illustrated above is a fundamental theorem in factoring. By it we can readily determine whether $x - a$ is a factor of $f(x)$. We have merely to substitute a for x in the given expression, and see whether it reduces to zero or not. In the former case the expression is divisible by $x - a$ without remainder, and therefore $x - a$ is a factor of it. In the latter case the expression is not divisible by $x - a$ without remainder, and therefore $x - a$ is not a factor of it.

EXERCISE LVIII.

Find in each of the following examples whether or not the given binomial is a factor of the given expression:

1. $x - 5$ of $x^3 - 7x^2 + 7x + 15$.
2. $x + 1$ of $2x^2 + x - 1$.
3. $x - 1$ of $x^6 + x^3 - 2$.
4. $x - 3$ of $2x^3 + 10x^2 - 8x - 40$.
5. $x - b$ of $x^5 - b^5$.
6. $x + b$ of $x^7 + b^7$.
7. $x + b$ of $x^3 - b^3$.
8. $x - b$ of $x^9 + b^9$.
9. $x - b$ of $x^6 - b^6$.
10. $x + b$ of $x^4 - b^4$.
11. $x - b$ of $x^8 + b^8$.
12. $x + b$ of $x^6 + b^6$.
13. $x - b$ of $x^n + b^n$ when n is odd.
14. $x - b$ of $x^n + b^n$ when n is even.
15. $x + b$ of $x^n + b^n$ when n is odd.
16. $x + b$ of $x^n + b^n$ when n is even.
17. $x - b$ of $x^n - b^n$ when n is odd.
18. $x - b$ of $x^n - b^n$ when n is even
19. $x + b$ of $x^n - b^n$ when n is odd.
20. $x + b$ of $x^n - b^n$ when n is even.
21. Divide $x^7 - b^7$ by $x - b$.
22. Divide $x^5 - b^5$ by $x + b$.
23. Divide $x^6 + b^6$ by $x - b$.
24. Divide $x^8 + b^8$ by $x + b$.

25. Divide $x^5 + b^5$ by $x - b$.

26. Divide $x^7 + b^7$ by $x + b$.

27. Divide $x^4 - b^4$ by $x - b$.

28. Divide $x^6 - b^6$ by $x + b$.

102. Factors of the Sum and Difference of the Same Powers of Two Quantities.—From examples 13–20 it appears:

1°. That the sum of the same odd powers of two quantities is divisible by the sum of their roots, but not by the difference of their roots.

2°. That the sum of the same even powers of two quantities is divisible by neither the sum nor the difference of their roots.

3°. That the difference of the same odd powers of two quantities is divisible by the difference of their roots, but not by the sum of their roots.

4°. That the difference of the same even powers of two quantities is divisible by both the sum and difference of their roots.

From examples 21, 26, 27, 28, it appears:

1°. That when the difference of the same powers is divided by the difference of the roots, the terms of the quotient are all positive; and that when the sum or difference of the same powers is divided by the sum of the roots, the terms of the quotient are alternately positive and negative.

2°. That in any case the first term of the quotient is the letter of the first term of the dividend with its exponent diminished by one, and that the exponent of this letter decreases by one in each of the succeeding terms of the quotient; and that the letter of the second term of the dividend occurs in the second term of the quotient with unity for its exponent, and that the exponent of this letter increases by one in each subsequent term till it becomes one less than its exponent in the dividend.

These two laws enable us in these cases of division to write the quotient at sight.

EXERCISE LIX.

Write at sight the quotient in each of the following cases:

1. $(x^5 - y^5) \div (x - y)$. **2.** $(x^8 - y^8) \div (x - y)$.

3. $(x^6 - y^6) \div (x + y)$. **4.** $(x^7 + y^7) \div (x + y)$.

5. $(x^3 - 27) \div (x - 3)$. **6.** $(x^4 - 81) \div (x - 3)$.

7. $(x^4 - 16) \div (x + 2)$. **8.** $(x^5 + 32) \div (x + 2)$.

Find the remainder when—

I.

9. $(x - 2a)^3 + (2x - a)^3$ is divided by $x - a$.

10. $(x + a + b)^3 + x^3$ is divided by $x + a$.

11. $(x + 2a)^{2n} + (2x + a)^{2n} - 2a^{2n}$ is divided by $x + a$.

12. $(a + b + c)^5 - a^5 - b^5 - c^5$ is divided by $a + b$.

II.

13. $(a + b + c)^7 - a^7 - b^7 - c^7$ is divided by $a + b$.

14. $(a + b + c)^4 - (b + c)^4 - (c + a)^4 - (a + b)^4 + a^4 + b^4 + c^4$ is divided by $a + b$.

15. $a^n(b - c) + b^n(c - a) + c^n(a - b)$ is divided by $b - c$.

Show that the given binomial is a factor of each of the following expressions, and find the other two factors:

I.

16. $3x^3 + x^2 - 22x - 24;\ x - 3$.

17. $x^3 + 2x^2 - 13x + 10;\ x - 2$.

18. $x^3 + 2x^2 - 11x - 12;\ x + 1$.

II.

19. $3x^3 - 20x^2 + 36x - 16;\ x - 4.$

20. $4x^3 + 13x^2 - 32x + 15;\ x + 5.$

EXERCISE LX.

I.

1. A cistern can be filled by one pipe in five hours and by another in eight hours, and it can be emptied by a third pipe in four hours. Were the cistern empty and all three pipes opened together, in what time would it be filled?

2. Suppose the cistern in the last example could be emptied by the third pipe in three hours. Were the cistern full and all three pipes opened together, in what time would it be emptied?

3. A man does 3/5 of a piece of work in 30 days and then calls in another man and they together finish it in 6 days. In what time can they do it separately?

II.

4. A marketwoman bought a number of eggs at the rate of two for a penny, and as many more at the rate of three for a penny, and sold the whole at the rate of four for 3 cents, and found she had made 24 cents. How many of each kind did she buy?

5. A person hired a laborer on condition that he was to receive 2 dollars for every day he worked and forfeit 75 cents for every day he was absent. He worked three times as many days as he was absent, and received $47.25. How many days did he work?

6. A sum of money was divided between A and B, so that the share of A was to that of B as 5 to 4. The share of A exceeded 5/11 of the whole by 300 dollars. What was each man's share?

CHAPTER XII.

HIGHEST COMMON FACTORS.

103. Highest Common Factor. — A *common factor* of two or more expressions is a factor which is contained in each of them, and the *highest common factor* of the expressions is the product of all their common factors. Thus, $2a^3b^2c^4$ and $6a^4b^4c$ have 2, a^3, b^2, and c as common factors, and $2a^3b^2c$ as their highest common factor.

The abbreviation H. C. F. stands for highest common factor.

The highest common factor is sometimes called the *greatest* common measure, and denoted by G. C. M.

104. The H. C. F. of monomials may be found by inspection. It is necessary merely to factor the expression, select the common factors and find their product, using each of these factors the least number of times that it occurs in any of the expressions.

e.g. Find the H.C.F. of $18a^2b^3c^4d$, $9a^3b^2c^5$, and $12a^4b^5d^3$.

Factoring, we have

$$3 \cdot 3 \cdot 2 \cdot a \cdot a \cdot b \cdot b \cdot b \cdot c \cdot c \cdot c \cdot c \cdot d,$$

$$3 \cdot 3 \cdot a \cdot a \cdot a \cdot b \cdot b \cdot c \cdot c \cdot c \cdot c \cdot c,$$

and $$3 \cdot 2 \cdot 2 \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b \cdot b \cdot d \cdot d \cdot d.$$

The factors common to all the expressions are, 3, a, and b. The least number of times that 3 occurs in any of the expressions is once; that a occurs in any of the expressions is twice; and that b occurs in any of the expressions is twice. Now $3 . a . a . b . b = 3a^2b^2$, and this is the highest common factor of the expressions. Of course we might have seen at once that the highest common factor of the coefficients is 3, that the common letters are a and b, and that the lowest dimension of these letters in any of the expressions is 2. Hence the H. C. F. would be $3a^2b^2$.

EXERCISE LXI.

Find the H. C. F. of the following expressions:

I.

1. $5x^3y$, $15x^3y^2z$. **2.** $7x^2y^3z$, $28x^5y^2z^3$.
3. $18ab^2c^2d$, $36a^2bcd^3$. **4.** $2x^3y^2$, $3x^2y^3$, $4x^4y^4z$.
5. $17a^5b^2c^3$, $51a^4b^3c^4$, $68a^6b^3c^4$.

II.

6. $14x^{m+n}y^{p+q}$, $70x^{m+1}y^{p+q-1}$, $140x^{m+2}y^{p+q-2}$.

7. Multiply $2x^m + 6x^n - 5x^py^q$ by $3x^n - 4x^2 + 6xy^2$.

8. Divide $6x^{m+2} + 9x^{m+1} + 12x^{n+2} + 18x^{n+1} - 8x^5 - 12x^4$ by $2x^2 + 3x$.

105. To Find Highest Common Polynomial Factor by Inspection.—In a similar way we may find the H. C. F. of two or more polynomial expressions by inspection when we are able to resolve them into polynomial factors. We have simply to resolve the expressions into their polynomial factors, select the factors common to all the expressions, and combine them into a product, using each factor the least number of times that it occurs in any of the expressions.

e.g. Find the H. C. F. of $x^2 + x - 6$, $x^2 + 6x + 9$, and $x^2 - x - 12$.

Factoring, we obtain

$$(x + 3)(x - 2),\ (x + 3)(x + 3), \text{ and } (x + 3)(x - 4).$$

The only common factor is $x + 3$, and the least number of times that this occurs in any of these expressions is once. Hence the H. C. F. of these three expressions is $x + 3$.

When any of the polynomial expressions contains a monomial factor, this factor should be removed before searching for polynomial factors; and if this factor is common to all the expressions, or contains a factor common to them, the common factor should be set aside to be made a factor of the H. C. F.

e.g. Find the H. C. F. of $3a^2x^2 + 3a^2x - 60a^2$, $6a^3x^2 - 96a^3$, and $12a^2bx^2 - 108a^2bx + 240a^2b$.

Removing the monomial factors, we have

$$3a^2(x^2 + x - 20),\ 6a^3(x^2 - 16),\ \text{and}\ 12a^2b(x^2 - 9x + 20).$$

$3a^2$ is the H. C. F. of the monomial factors thus removed. Factoring now the three polynomial expressions, we have

$$(x - 4)(x + 5),\ (x - 4)(x + 4),\ \text{and}\ (x - 4)(x - 5),$$

the highest common factor of which is $x - 4$. Therefore the H. C. F. of the three given expressions is

$$3a^2(x - 4) = 3a^2x - 12a^2.$$

EXERCISE LXII.

Find the H. C. F. of the following expressions:

I.

1. $x^2 - 1,\ x^2 + 3x + 2.$
2. $x^2 + 5x + 6,\ x^2 + 7x + 12.$
3. $x^2 - 9x - 10,\ x^2 + 2x - 120.$
4. $x^2 + 7x - 18,\ x^3 - 8.$
5. $x^2 + (a + b)x + ab,\ x^2 + (a - b)x - ab.$
6. $x^2 - 7xy + 6y^2,\ x^3 - xy^2.$
7. $x^3 - x,\ 2x^2 - 4x + 2,\ x^3 + x^2 - 2x.$

II.

8. $x^3 + y^3$, $(x + y)^3$, $x^3 + 2x^2y + 2xy^2 + y^3$.

9. $12(x + 1)^2$, $6(x^2 - 1)^3$, $18(x + 1)^4$.

10. $x^2 - y^2$, $3(x^4 - y^4)$, $7(x^6 - y^6)$.

11. $x^3 - 3a^2x - 2a^3$, $x^3 - 3ax^2 + 4a^3$, $x^2 - ax - 2a^2$.

12. $x^2 + xy - 2y^2$, $x^3 - 3xy^2 + 2y^3$, $x^3 + 3x^2y - 4y^3$.

106. The method of finding the highest common factor of two or more expressions which cannot readily be resolved into factors is based on the three following theorems:

1°. *If two expressions have a common factor, any multiples of these expressions will contain this factor.*

Let A and B represent any two expressions which have a common factor, and let this factor be represented by f; let p denote the quotient resulting from dividing A by f, and q the quotient obtained by dividing B by f. Then $A = pf$ and $B = qf$. Let m and n be any integral expressions whatever. Then mA will represent any multiple whatever of A, and nB any multiple of B.

But $mA = mpf$ and $nB = nqf$.

Hence f is a factor of both mA and nB.

2°. *If two expressions have a common factor, the sum and difference of the expressions or of any multiples of the expressions will contain this factor.*

Use the letters as in 1°. Then

$A - B = pf - qf = (p - q)f$, which contains the factor f.

Also

$A + B = pf + qf = (p + q)f$, which contains the factor f.

Again, $mA - nB = mpf - nqf = (mp - nq)f$, which contains the factor f.

Also $mA + nB = mpf + nqf = (mp + nq)f$, which contains the factor f.

3°. *If two expressions have a common factor, and one of them be divided by the other and there be a remainder, this remainder will contain the common factor.*

Let A and B represent the two expressions which have a common factor, Q the quotient obtained by dividing B by A, and R the remainder. Then

$$B = QA + R.$$

By hypothesis B and A have a common factor f, and by 1°, QA contains f as a factor. But since B is divisible by f, and one term of its equivalent expression $(QA + R)$ is divisible by f, the other must be also. Hence the remainder R must contain f as a factor.

Cor.—If now we divide A by R and denote the remainder by S, then the common factor of R and S will be the same as that of A and R and, therefore, of A and B.

If this process be continued to any extent, the common factor of any divisor and the corresponding dividend will be a common factor of the original expressions. In other words, *the remainder will always contain the common factors of the original expressions.*

If at any stage there is no remainder, the divisor must be a factor of the corresponding dividend, and therefore, since it is evidently the highest factor of itself, it must be the H. C. F. of the original expressions.

By the nature of division the remainders are necessarily of lower and lower dimensions, and hence, unless at some stage the division leaves no remainder, we must ultimately reach a remainder which does not contain the common letter. In this case the given expressions have no H. C. F.

As the process we are considering is to be used only to find the highest common polynomial factor, it is evident that any dividend or divisor which may occur in the process may be multiplied or divided by any monomial factor without destroying the validity of the operation; for such multiplication or division will not affect the polynomial factors.

Ex. 1. Find the H. C. F. of

$$x^3+x^2-2 \text{ and } x^3+2x^2-3.$$

$$\begin{array}{lll}
 & x^3+2x^2-3 & \big|\,\underline{x^3+x^2-2} \\
 & \underline{x^3+\ \ x^2-2} & \ 1 \\
x^3+x^2-2 & \big|\,\underline{\ \ x^2-1\ \ } & \\
\underline{x^3-x\qquad\qquad} & \ \ x+1 & \\
\quad x^2+x-2 & & \\
\quad \underline{x^2\quad -\quad 1} & & \\
x^2-1\ \big|\,\underline{\ x-1\ } & & \\
\underline{x^2-x}\qquad x+1 & & \\
x-1 & & \\
\underline{x-1} & &
\end{array}$$

The H. C. F. is $x-1$.

The work might be shortened by noticing that the factors of the first remainder, x^2-1, are $x-1$ and $x+1$, and that of these only $x-1$ is a factor of x^3+x^2-2.

Ex. 2. Find the H. C. F. of

$$x^3+4x^2y-8xy^2+24y^3 \text{ and } 4x^6-4x^5y+32x^3y^3-32x^2y^4.$$

The second expression is divisible by $4x^2$, which is evidently not a common factor. We have therefore to find the H. C. F. of $x^4-x^3y+8xy^3-8y^4$ and the first expression.

$$\begin{array}{ll}
x^4-\ \ x^3y+8xy^3-\ \ 8y^4 & \big|\,\underline{x^3+4x^2y-8xy^2+24y^3} \\
\underline{x^4+4x^3y-8x^2y^2+24xy^3} & \qquad x-5y \\
\qquad -5x^3y+\ \ 8x^2y^2-16xy^3-\ \ \ 8y^4 & \\
\qquad \underline{-5x^3y-20x^2y^2+40xy^3-120y^4} & \\
\qquad\qquad\qquad \underline{28x^2y^2-56xy^3+112y^4} &
\end{array}$$

Rejecting the factor $28y^2$, we have

$$\begin{array}{l|l}
x^3 + 4x^2y - 8xy^2 + 24y^3 & x^2 - 2xy + 4y^2 \\ \cline{2-2}
\underline{x^3 - 2x^2y + 4xy^2} & x + 6y \\
\quad 6x^2y - 12xy^2 + 24y^3 & \\
\quad \underline{6x^2y - 12xy^2 + 24y} &
\end{array}$$

Hence the H. C. F. is $x^2 - 2xy + 4y^2$.

Ex. 3. Find the H. C. F. of

$$2x^4 + 9x^3 + 14x + 3 \text{ and } 3x^4 + 15x^3 + 5x^2 + 10x + 2.$$

To avoid fractional coefficients, the second expression may be multiplied by 2 and then divided by the first.

$$\begin{array}{r|l}
3x^4 + 15x^3 + 5x^2 + 10x + 2 & 2x^4 + 9x^3 + 14x + 3 \\ \cline{2-2}
2 & \quad 3 \\ \hline
6x^4 + 30x^3 + 10x^2 + 20x + 4 & \\
6x^4 + 27x^3 \qquad\quad + 42x + 9 & \\ \hline
\end{array}$$

$$\begin{array}{l|l}
2x^4 + 9x^3 + 14x + 3 & 3x^3 + 10x^2 - 22x - 5 \\ \cline{2-2}
\qquad\qquad\qquad\quad 3 & 2x + 7 \\ \cline{1-1}
6x^4 + 27x^3 + 42x + 9 & \\
6x^4 + 20x^3 - 44x^2 - 10x & \\ \cline{1-1}
\qquad 7x^3 + 44x^2 + 52x + 9 & \\
\qquad\qquad\qquad\qquad\quad 3 & \\ \cline{1-1}
\qquad 21x^3 + 132x^2 + 156x + 27 & \\
\qquad 21x^3 + 70x^2 - 154x - 35 & \\ \cline{1-1}
\qquad\qquad 62x^2 + 310x + 62 & 62 \\ \cline{2-2}
\end{array}$$

$$\begin{array}{l|l}
3x^3 + 10x^2 - 22x - 5 & x^2 + 5x + 1 \\ \cline{2-2}
3x^3 + 15x^2 + 3x & 3x - 5 \\ \cline{1-1}
\quad - 5x^2 - 25x - 5 & \\
\quad - 5x^2 - 25x - 5 & \\ \cline{1-1}
\end{array}$$

The H. C. F. is $x^2 + 5x + 1$.

From the above theorems and examples we may derive the following rule for finding the H. C. F. of two expressions:

Arrange the two expressions according to the descending powers of some common letter and, if the expressions are of the same degree in that letter, divide either by the other, but if they are of different degrees in that letter, divide the one which is of the higher degree by the other. Take the remainder after division, if any, for a new divisor, and the former divisor as dividend; and continue the process till there is no remainder. The last divisor will be the H. C. F. required.

If the two expressions contain common monomial factors, their H. C. F. must be obtained by inspection, and this must be multiplied by the last divisor found by the above rule.

Any divisor, dividend, or remainder which occurs may be multiplied or divided by any monomial factor.

107. To find the H. C. F. of three or more polynomial expressions, we first find the H. C. F. of any two of them, and then of this and a third, and so on.

Let the expressions be A, B, C, D, etc.

First find the H. C. F. of A and B, and denote it by E. Then since the required H. C. F. is a common factor of A and B, it must be a factor of E, which contains every common factor of A and B, and so on.

108. NOTE.—The *highest common factor* of algebraic expressions is not necessarily their *greatest common measure.* For if one expression is of higher dimensions than another in a particular letter, it does not follow that it is numerically greater. In fact, if a be a positive fraction, a^2 is less than a.

EXERCISE LXIII.

Find the H. C. F. of—

I.

1. $x^2 + 2x + 1$ and $x^3 + 2x^2 + 2x + 1$.

2. $x^3 - 8x^2 + 7x + 24$ and $x^3 - 5x^2 + 8x - 6$.

3. $x^3 - 5x^2 + 3x + 6$ and $x^3 - 3x^2 + 4x - 4$.

4. $2x^3 - 7x - 2$ and $6x^4 - 3x^3 - 18x^2$.

5. $4x^6 + 8x^5 - 56x^4 - 12x^3$ and $6x^3 - 6x^2 - 36x$.

6. $12a^2x^4 + 120a^4x^2 - 132a^5x$ and $3a^2x^8 - 27a^3x^7 + 39a^4x^6 - 15a^7x^3$.

7. $7x^4 - 10ax^3 + 3a^2x^2 - 4a^3x + 4a^4$ and $8x^4 - 13ax^3 + 5a^2x^2 - 3a^3x + 3a^4$.

8. $25x^4 + 5x^3 - x - 1$ and $20x^4 + x^2 - 1$.

9. $1 - 4x^3 + 3x^4$ and $1 + x - x^2 - 5x^3 + 4x^4$.

II.

Work the last nine and also the following examples by synthetic division:

10. $11x^4 + 24x^3 + 125$ and $x^4 + 24x + 55$.

11. $2x^5 - 11x^2 - 9$ and $4x^5 + 11x^4 + 81$.

12. $x^5 + 11x^3 - 54$ and $x^5 + 11x + 12$.

13. $x^3 - 2x^2 - x + 2$, $x^3 - x^2 - 4x + 4$, and $x^3 - 7x + 6$.

14. $x^4 - 6x^2 + 8x - 3$, $x^4 - 2x^3 - 7x^2 + 20x - 12$, and $x^4 - 4x^2 + 12x - 9$.

15. Multiply $3x^m - 4x^{m-2} + 5x^{n+1}$ by $6x^3 + 7x^{m+2}$.

16. Multiply $x^a - 3x^{3a} + 5x^3$ by $4x^4 - 6x^{a-2}$.

EXERCISE LXIV.

I.

Ex. At what time after 5 o'clock will the minute-hand of the clock be ten minutes ahead of the hour-hand?

In examples about the position of the hands of a clock, it is best to draw a circle to represent the clock-dial, and to mark on it the positions of the hands at the beginning of the hour specified. Then note the number of minute-spaces between the hands at this time, and let x denote the number of minute-spaces that the minute-hand must pass over before it comes into the required position. Then, since the minute-hand goes 12 times around the dial while the hour-hand is going once around it, $x/12$ will denote the number of minute-spaces passed over by the hour-hand in the same time.

Then x will equal the number of minute-spaces between the hands at the beginning of the hour plus $x/12$ minus the number of spaces the hands are required to be apart when the minute-hand is required to be behind the hour-hand; and x will equal the number of minute-spaces between the hands at the beginning of the hour plus $x/12$ plus the number of spaces the hands are required to be apart when the minute hand is required to be ahead of the hour-hand.

Thus, in the example, the minute-hand will be at XII at the beginning of the hour specified, and the hour-hand at V, and there would be 25 minute-spaces between them. While the former is moving over the x spaces to its required position of 10 minute-spaces ahead of the hour-hand, the hour-hand will move over $x/12$ spaces. Therefore

$$x = 25 + x/12 + 10;$$

$$\therefore \quad 11/12x = 35,$$

$$x = 38\tfrac{2}{11}.$$

That is, the minute-hand would be in the required position at $38\frac{2}{11}$ minutes past five.

Had the question been, at what time after 5 o'clock will the minute-hand of the clock be ten minutes behind the hour-hand, we would have had

$$x = 25 + x/12 - 10;$$

$$\therefore\ 11/12x = 15,$$

$$x = 16\tfrac{4}{11}.$$

1. At what time after 3 o'clock is the minute-hand of the clock 18 minutes ahead of the hour-hand?

2. At what time after 7 o'clock is the hour-hand 20 minutes behind the minute-hand?

3. At what time after 9 o'clock is the hour-hand 15 minutes behind the minute-hand?

4. At what time nearest to 2 o'clock is the minute-hand 15 minutes behind the hour-hand?

5. At what time between 4 and 5 o'clock are the hour and minute hands at right angles?

6. The sum of the two digits of a number is 8, and if 36 be added to the number the digits will be interchanged. What is the number?

7. If the first of the two digits of a number be doubled it will be 3 more than the second, and the number itself is 6 less than five times the sum of its digits. What is the number?

8. A courier who goes at the rate of 40 miles in eight hours is followed after 10 hours by a second courier who goes at the rate of 72 miles in 9 hours. In how many hours will the second overtake the first?

II.

9. A courier who goes at the rate of $31\frac{1}{2}$ miles in five hours is followed, after eight hours, by a second courier

who goes at the rate of $22\frac{1}{2}$ miles in three hours. In how many hours will the second overtake the first?

10. Ten years hence a boy will be four times as old as he was ten years ago. How old is the boy?

11. One man is 60 years old, and another man is 2/3 as old. How long since the first man was five times as old as the second?

12. A father is four times as old as his son, and four years ago the father was six times as old as his son. What is the age of each?

CHAPTER XIII.

LOWEST COMMON MULTIPLE.

109. Lowest Common Multiple.—A *common multiple* of two or more expressions is an expression which is exactly divisible by each of them.

The *lowest common multiple* of two or more expressions is the expression of the lowest dimensions which is exactly divisible by each of them. The lowest common multiple is usually denoted by the letters L. C. M.

110. To Find L. C. M. by Inspection. — The lowest common multiple of two or more expressions must evidently contain every factor of each, and each of these factors the greatest number of times that it occurs in any one of them, otherwise it would not be divisible by each expression.

e.g. Let $3a^4b^2c$, $6a^3b^4c^2d$, and $9b^2c^3e$ be the numbers whose L. C. M. is required. To be divisible by each of these expressions the required expression must contain the factors 2, 3, a, b, c, d, and e, and it must also contain the first of these once, the second twice, the third four times, the fourth four times, the fifth three times, the sixth once, and the seventh once. The L. C. M. is $18a^4b^4c^4de$.

Hence we have the following rule for finding the lowest common multiple of two or more expressions which may be factored by inspection:

Find all the different factors of each expression, and take each of these factors the greatest number of times which it occurs in any of the expressions, or to the highest degree that it has in any of the expressions, and find the product of these factors.

EXERCISE LXV.

Find the L. C. M. of the following expressions:

I.

1. $18a^3b^2c$, $6a^2b^3d^4$, and $7a^4d^5$.

2. $3x^3yz^3$, $5xy^3z^2$, $15x^2y^2z$, and $20x^3y^3z^3$.

3. $x^2 - y^2$, $xy - y^2$, and $xy + y^2$.

4. $x^2 - 2x - 15$, $x^2 - 9$, and $x^2 - 8x + 15$.

5. $5x + 35$, $x^2 - 49$, and $x^2 + 14x + 49$.

II.

6. $x^2 - x - 20$, $x^2 + 3x - 40$, and $x^2 + 12x + 32$.

7. $2x^2 - x - 1$, $2x^2 + 3x + 1$, $x^2 - 1$, $4x^4 - 5x^2 + 1$.

8. $12x - 36$, $x^2 - 9$, $x^2 - 5x + 6$.

9. $x^2 - 3x + 2$, $x^2 - 5x + 6$, and $x^2 - 4x + 3$.

10. $x^2 - 6ax + 9a^2$, $x^2 - ax - 6a^2$, and $3x^2 - 12a^2$.

111. To Find L. C. M. by Division.—Since the highest common factor of two expressions contains every factor common to the expressions, if two expressions be each divided by their highest common factor, the quotients obtained will contain no common factors. Hence the L.C.M. of the two expressions will be the product of these quotients and their H. C. F.

e.g. Find the L. C. M. of

$$x^3 + x^2 - 2 \text{ and } x^3 + 2x^2 - 3.$$

The H. C. F. of these two expressions is $x - 1$.

$$(x^3 + x^2 - 2) \div (x - 1) = x^2 + 2x + 2,$$

and $$(x^3 + 2x^2 - 3) \div (x - 1) = x^2 + 3x + 3.$$

$$x^3 + x^2 - 2 = (x - 1)(x^2 + 2x + 2),$$

and $$x^3 + 2x^2 - 3 = (x - 1)(x^2 + 3x + 3).$$

Since $x^2 + 2x + 2$ and $x^2 + 3x + 3$ have no common factor, $(x - 1)(x^2 + 2x + 2)(x^2 + 3x + 3)$ must be the L. C. M. of $x^3 + x^2 - 2$ and $x^3 + 2x^2 - 3$.

In general, let A and B stand for any two expressions, and let h stand for their H. C. F. and l stand for their L. C. M., and let P and Q be the quotients when A and B respectively are divided by h; so that

$$A = P \cdot h \quad \text{and} \quad B = Q \cdot h.$$

Since h is the H. C. F. of A and B, P and Q can have no common factors. Hence the L. C. M. of A and B must be $P \times Q \times h$, or

$$l = PQh;$$

or

$$l = Ph \times \frac{Qh}{h} = A \times \frac{B}{h}.$$

Hence *the L. C. M. of two expressions may be found by dividing either one of the expressions by their H. C. F., and multiplying the quotient by the other expression.*

Also, since

$$l = \frac{A \times B}{h},$$

$$l \times h = A \times B.$$

That is, *the product of any two expressions is equal to the product of their H. C. F. and L. C. M.*

EXERCISE LXVI.

Find the L. C. M. of the following expressions:

I.

1. $6x^2 - 5ax - 6a^2$ and $4x^3 - 2ax^2 - 9a^3$.
2. $4a^2 - 5ab + b^2$ and $3a^3 - 3a^2b + ab^2 - b^3$.
3. $3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21$.
4. $x^4 - 11x^2 + 49$ and $7x^4 - 40x^3 + 75x^2 - 40x + 7$.

5. $x^3 + 6x^2 + 11x + 6$ and $x^4 + x^3 - 4x^2 - 4x$.

6. $x^4 - x^3 + 8x - 8$ and $x^4 + 4x^3 - 8x^2 + 24x$.

II.

7. $8a^3 - 18ab^2$, $8a^3 + 8a^2b - 6ab^2$, and $4a^2 - 8ab + 3b^2$.

8. $x^2 - 7x + 12$, $3x^2 - 6x - 9$, and $2x^3 - 6x^2 - 8x$.

9. $8x^3 + 27$, $16x^4 + 36x^2 + 81$, and $6x^2 - 5x - 6$.

10. $x^2 - 6xy + 9y^2$, $x^2 - xy - 6y^2$, and $3x^2 - 12y^2$.

11. Multiply $x^m + x^n$ by $x^m - x^n$.

12. Multiply $3a^nx^m - 4a^mx^n$ by $3a^nx^m + 4a^mx^n$.

13. Divide x^{-m+2} by x^{m+1}.

14. Divide $4a^mx^{-3m+p}$ by $2a^nx^{-m-p}$

EXERCISE LXVII.

I.

1. At what time after 10 o'clock will the minute-hand of a clock first be 20 minute-spaces ahead of the hour-hand?

2. A courier sets out from a city and travels at the rate of 8 miles an hour, and 3 hours later a second courier sets out from the same city and follows the first along the same road, travelling at the rate of 10 miles an hour. In how many hours will the second courier overtake the first, and how far will each have travelled?

3. A courier sets out from a city and travels 10 miles an hour. Four hours later a second courier sets out from the same place and travels along the same road and over-

takes the first courier in 20 hours. How fast does the second courier ride, and how far does each go?

4. Two bodies, A and B, are moving around concentric circles in the same direction, and, as seen from the common centre of the circles, they are together every 50 days. A is on the outer circle, and is longer in going around than B, which is on the inner circle. A goes around his circle in 20 days. How long does it take B to go around his circle?

Let $x =$ number days it takes B to go around.

$$\therefore \quad \frac{360}{x} = \text{number of degrees B goes over in a day.}$$

$$\text{Also } \frac{360}{20} = \text{number of degrees A goes over in a day,}$$

and $\dfrac{360}{x} - \dfrac{360}{20} =$ number of degrees gained by B in one day.

In 50 days B must evidently gain 360° on A.

$$\therefore \quad \frac{360}{50} = \text{number of degrees gained by B in one day.}$$

$$\therefore \quad \frac{360}{x} - \frac{360}{20} = \frac{360}{50}, \text{ or } \frac{1}{x} - \frac{1}{20} = \frac{1}{50}.$$

$$\therefore \quad \frac{1}{x} = \frac{7}{100}, \text{ or } 7x = 100, \text{ and } x = 14\tfrac{2}{7}.$$

II.

5. Suppose, in the last example, A went around his circle in the shorter time, then in what time would B go around?

6. Two bodies, A and B, move around two concentric circles in the same direction and are together every 60 days. A is on the outer circle and B on the inner, and A goes around its circle in 40 days. If B moves over more degrees

a day than A does, how long will it take B to go around its circle?

7. If in the last example A goes over more degrees a day than B does, how long will it take B to go around?

8. Divide $x^{3m} + y^{3n}$ by $x^m + y^n$.

9. Divide $x^{6m} - y^{6n}$ by $x^m - y^n$.

CHAPTER XIV.

FRACTIONS.

112. The Symbol $\frac{a}{b}$.—When the operation of division is indicated by placing the dividend over the divisor with a horizontal line between, the symbol is called a *fraction*, the dividend being called the *numerator* and the divisor the *denominator*. Thus, $\frac{a}{b}$ is a fraction, a is its numerator and b is its denominator. The quotient which results from the division is the value of the fraction. In the type, $\frac{a}{b}$, a and b stand for any integral expression, however complicated.

By definition, $\frac{a}{b} = a \div b$. Therefore

$$\frac{a}{b} \times b = a \div b \times b = a.$$

That is, the multiplication of a fraction by its denominator produces its numerator.

When the numerator is a polynomial, the horizontal line or bar of the fraction must be considered as a sign of aggregation, showing that the numerator as a whole is to be divided by the denominator.

In the various operations on fractions we assume that the associative, distributive, and commutative laws which have been demonstrated for integers apply also to the symbol $\frac{a}{b}$. Having made this assumption, we proceed to enquire what addition, multiplication, and other operations on fractions mean if they obey the same laws as the corresponding operations on numbers.

113. Theorem I. *The denominator of a fraction is distributive among the terms of its numerator.*

It is required to prove $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

By definition $\frac{a+b}{c} \times c = a + b$.

By the distributive law

$$\left(\frac{a}{c} + \frac{b}{c}\right)c = \frac{a}{c} \times c + \frac{b}{c} \times c = a + b.$$

$$\therefore \qquad \frac{a+b}{c} \times c = \left(\frac{a}{c} + \frac{b}{c}\right)c.$$

$$\therefore \qquad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$$

It is thus seen that this theorem is a consequence of the assumption that the distributive law of multiplication holds for fractional symbols.

Hence the denominator of a fraction is distributive throughout the terms of the numerator.

And, conversely, the algebraic sum of any number of fractions with the same denominator is the fraction whose numerator is the algebraic sum of the numerators of several fractions, and whose denominator is their common denominator.

The sign before a fraction may always be regarded as belonging to the numerator as a whole, and it must be so regarded in finding the algebraic sum of the numerators of fractions which have the same denominator. Thus,

$+\frac{a}{b}=\frac{+a}{b}$, and $-\frac{a}{b}=\frac{-a}{b}$. The value of a fraction is to be regarded as a quotient, and when the divisor is positive the sign of the quotient is the same as that of the dividend. Hence, if a and b both represent positive quantities, $-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$, but is not $=\frac{-a}{-b}$. That is, the minus sign before a fraction may be regarded as belonging to either the numerator or denominator as a whole, but not to both.

The same is evidently true when both a and b represent negative quantities, or when one of them represents a negative quantity and the other a positive quantity.

For $-\frac{{}^-a}{{}^-b}$, $\frac{-{}^-a}{{}^-b}$, and $\frac{{}^-a}{-{}^-b}$ each evidently represent the same negative quantity; and $-\frac{{}^-a}{{}^+b}$, $\frac{-{}^-a}{{}^+b}$, and $\frac{{}^-a}{-{}^+b}$ each evidently represent the same positive quantity, as do also $-\frac{{}^+a}{{}^-b}$, $\frac{-{}^+a}{{}^-b}$, and $\frac{{}^+a}{-{}^-b}$.

To illustrate by numerals:

$$-\frac{8}{4}=-2,\quad \frac{-8}{4}=-2,\quad \text{and}\quad \frac{8}{-4}=-2;$$

$$-\frac{-8}{-4}=-2,\quad \frac{-(-8)}{-4}=-2,\quad \text{and}\quad \frac{-8}{-(-4)}=-2.$$

$$-\frac{-8}{4}=2,\quad -\frac{-(-8)}{4}=2,\quad \text{and}\quad \frac{-8}{-4}=2;$$

$$-\frac{8}{-4}=2,\quad \frac{-8}{-4}=2,\quad \text{and}\quad \frac{8}{-(-4)}=2.$$

It must be borne in mind carefully that, in finding the algebraic sum of the numerators of fractions which have the same denominator, all the signs of the numerator of every fraction which has a minus sign must be changed.

EXERCISE LXVIII.

I.

1. Write $\frac{a - 4b + 5c}{9}$ as the sum of three separate fractions.

2. Write $\frac{2x - 5y - 8ac + 9}{a + b}$ as the sum of four fractions.

3. Write $\frac{3a}{2a + b} - \frac{5b}{2a + b} + \frac{4}{2a + b}$ as one fraction.

4. Write $\frac{3x + 5}{4c} - \frac{4x + 6}{4c} - \frac{5x - a}{4c} + \frac{7x - c}{4c}$ as one fraction.

II.

5. Write

$$\frac{2a + 3b - c}{x^2 - 3} + \frac{5a - 7b + 11}{x^2 - 3} - \frac{3a + 5b - 7}{x^2 - 3} - \frac{11a - d}{x^2 - 3}$$

as one fraction.

6. Write

$$\frac{3x - 4(a + b)}{a^2 - b^2} - \frac{5x + 7(a + b)}{a^2 - b^2} - \frac{7x - 5(a - c)}{a^2 - b^2}$$

as one fraction.

114. THEOREM II. *The value of a fraction is not altered by multiplying its numerator and denominator by the same quantity.*

It is required to prove $\frac{a}{b} = \frac{ma}{mb}$.

By the commutative law $\frac{a}{b} \cdot mb = \frac{a}{b} \cdot b \times m = am = ma.$

By definition $$\frac{ma}{mb} \cdot mb = ma.$$

$$\therefore \qquad \frac{a}{b} \cdot mb = \frac{ma}{mb} \cdot mb.$$

$$\therefore \qquad \frac{a}{b} = \frac{ma}{mb}.$$

115. Theorem III. *The value of a fraction is not altered by dividing its numerator and denominator by the same quantity.*

It is required to prove $\frac{a \div m}{b \div m} = \frac{a}{b}$.

By the last theorem $$\frac{a \div m}{b \div m} = \frac{(a \div m)m}{(b \div m)m}.$$

But by definition $$\frac{(a \div m)m}{(b \div m)m} = \frac{a}{b}.$$

116. It follows from Theorem III that a fraction may be simplified without altering its value by the rejection of any common factor from its numerator and denominator. Thus the fraction $\frac{3a^3x}{5b^3x}$ takes the simpler form $\frac{3a^3}{5b^3}$, when the factor x, which is common to its numerator and denominator, is rejected.

A fraction is said to be in its *lowest terms* when its numerator and denominator have no common factors.

A fraction may be reduced to its lowest terms by removing, or cancelling, the common factors one after another from the numerator and denominator by inspection, or by dividing the numerator and denominator by their H. C. F.

When the numerator and denominator of a fraction are polynomials which can be factored by inspection, it is best to write them as factored, and then to cancel their common factors.

e.g. $$\frac{3x^2+x-2}{2x^2-x-3}=\frac{(3x-2)(x+1)}{(2x-3)(x+1)}=\frac{3x-2}{2x-3}.$$

It is not worth while to divide the numerator and denominator by their H. C. F. except in cases where their common factors cannot be discovered by inspection.

EXERCISE LXIX.

Reduce the following fractions to their lowest terms.

I.

1. $\frac{12a^3x}{6a^2x^2}$.
2. $\frac{15a^3b^2c}{20a^2bc^2}$.
3. $\frac{x-a}{ax-a^2}$.
4. $\frac{x^2-a^2}{x^2+ax}$.
5. $\frac{3x^5-9x^3y}{7x^4-21x^2y}$.
6. $\frac{8a^4x^6-16a^3x^6}{8ab^2x^3-16b^2x^3}$.
7. $\frac{x^2+x-20}{x^2-11x+28}$.
8. $\frac{x^2-36}{x^2-3x-18}$.

II.

9. $\frac{4x^2-16}{2x^2-2x-12}$.
10. $\frac{6x^2-7x-3}{2x^2+x-6}$.
11. $\frac{x^3-64}{x^2+3x-28}$.
12. $\frac{x^3+27}{x^2-9}$.
13. $\frac{6x^2+xy-y^2}{8x^2+2xy-y^2}$.
14. $\frac{4x^2-8x+3}{4x^2+4x-3}$.

117. Reduction of Fractions to a Common Denominator.—Two or more fractions may be reduced to equivalent fractions with a common denominator by finding the L. C.

M. of the denominators for the common denominator, and dividing this by each of the old denominators in turn, and multiplying each numerator by the corresponding quotient for the numerator.

N.B.—This is equivalent to multiplying the numerator and denominator of each fraction by the quotient obtained by dividing the L. C. M. of all the denominators by its own denominator; and hence the value of the fractions will not be altered. (Why?)

An integer may be regarded as a fraction whose denominator is one. Hence an integral term may be reduced to a fraction with any denominator by multiplying it by the required denominator and placing the product obtained over the denominator.

Of course any fraction may be reduced to an equivalent fraction with any required denominator (which is a multiple of its own) by multiplying the denominator of the fraction by the factor which will produce the required denominator, and the numerator by the same factor. Such a factor may be obtained by dividing the required denominator by the old one, or, often, by simple inspection.

EXERCISE LXX.

I.

1. Reduce $\frac{4ac}{3}$ to an equivalent fraction whose denominator is $9ac^2$.

2. Reduce $\frac{1 - 6x^2}{5x}$ to an equivalent fraction whose denominator is $20a^2x^3$.

3. Reduce $\frac{x - 3}{x + 7}$ to an equivalent fraction whose denominator is $x^2 + x - 42$.

4. Reduce $\frac{2x+4}{3x-2}$ to an equivalent fraction whose denominator is $12x^2+x-6$.

5. Reduce $\frac{5x-7}{2x-6}$ to an equivalent fraction whose denominator is $8x^2-34x+30$.

6. Reduce $3a^2x$ to an equivalent fraction whose denominator is $5a^2x^3$.

II.

7. Reduce $2b^2x^2$ to an equivalent fraction whose denominator is $3-7ab^2x^3$.

8. Reduce $3x-5$ to an equivalent fraction whose denominator is $7x+8$.

9. Reduce $5x-7$ to an equivalent fraction whose denominator is $6-3x$.

10. Reduce $3x+8$ to an equivalent fraction whose denominator is $9-5x$.

I.

11. Reduce $\frac{5}{3ab}$ and $\frac{7x}{9a^3x}$ to equivalent fractions with a common denominator.

12. Reduce $\frac{x-5}{x+4}$ and $\frac{x+6}{x-4}$ to equivalent fractions with a common denominator, and find their sum.

Reduce the following terms to equivalent fractions with a common denominator, and then the whole to a single fraction:

13. $$1+\frac{x+7}{x-6}-\frac{x-8}{x+5}.$$

14. $\frac{3}{2a} - \frac{5x+6}{4a^2}$.

15. $3x - \frac{2x-3}{3x+4} - \frac{4x-6}{5x-2}$.

16. Reduce $-\frac{b^2}{4a^2} + \frac{c}{a}$ to a single negative fraction.

II.

17. Reduce $-\frac{25b^2}{36a^2} + \frac{7c}{3a}$ to a single negative fraction.

18. Reduce $1 - \frac{a^2+b^2-2ab}{-4ab}$ to a single positive fraction.

19. Divide $x^{4m} - x^{4n}$ by $x^m + x^n$.

20. Divide $x^{5m} + x^{5n}$ by $x^m + x^n$.

118. Theorem III. *The product of two fractions is the product of their numerators divided by the product of their denominators.*

It is required to prove $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

By the commutative law $\frac{a}{b} \times \frac{c}{d} \cdot bd = \frac{a}{b} \cdot b \times \frac{c}{d} \cdot d = ac$.

By definition $\frac{ac}{bd} \cdot bd = ac$.

$$\therefore \qquad \frac{a}{b} \times \frac{c}{d} \cdot bd = \frac{ac}{bd} \cdot bd.$$

$$\therefore \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Hence the product of two fractions is another fraction whose numerator is the product of their numerators, and whose denominator is the product of their denominators.

The product of any number of fractions may be found by first finding the product of any two of them, and then of the resulting fraction and a third, and so on to the end. The resulting product evidently will be the fraction whose numerator is the product of the numerators of all the given fractions and whose denominator is the product of their denominators. Thus,

$$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}\times\frac{g}{h}=\frac{ac}{bd}\times\frac{e}{f}\times\frac{g}{h}=\frac{ace}{bdf}\times\frac{g}{h}=\frac{aceg}{bdfh}.$$

Hence

$$\left(\frac{a}{b}\right)^2=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2},\quad\text{and}\quad\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}.$$

COR. 1. *A fraction may be multiplied by a quantity by multiplying its numerator by that quantity.*

For let $\frac{a}{b}$ be a fraction and c be the quantity by which it is to be multiplied. c may be written as the fraction $\frac{c}{1}$.

$$\therefore\qquad \frac{a}{b}\times c=\frac{a}{b}\times\frac{c}{1}=\frac{ac}{b}.$$

Also, by the Commutative Law, $\frac{a}{b}\times c=c\times\frac{a}{b}$.

$$\therefore\qquad c\times\frac{a}{b}=\frac{ac}{b}.$$

COR. 2. *A fraction may be multiplied by a quantity by dividing its denominator by that quantity.*

For let $\frac{a}{b}$ be a fraction, and c be the quantity by which it is to be multiplied.

Then $\frac{a}{b} \times c = \frac{ac}{b}$. Multiplying both the numerator and denominator by $\frac{1}{c}$, we have

$$\frac{ac \cdot \frac{1}{c}}{b \cdot \frac{1}{c}} = \frac{a}{\frac{b}{c}}, \quad \text{or} \quad \frac{a}{b \div c}.$$

EXERCISE LXXI.

N.B.—In multiplying fractions by integers or fractions it is best to cancel common terms as in arithmetic.

Find the following products:

I.

1. $\frac{2xy}{3z} \times \frac{6z^2}{4x^2y^2} \times \frac{xy}{z}$.

2. $\frac{a^2 - x^2}{2ax} \times \frac{a^2x + ax^2}{a^2 - 2ax + x^2} \times \frac{2(a-x)}{a^2 + ax}$.

3. $\frac{a^2 + ax}{a^2 - x^2} \times \frac{a^3 - x^3}{ax(a^2 + ax + x^2)}$.

4. $\frac{a^2 - x^2}{a + y} \times \frac{a^2 - y^2}{ax + x^2} \times \left(a + \frac{ax}{a - x}\right)$.

5. $\frac{x - 7}{x^2 + 12x + 32} \times (x^2 + 2x - 48)$.

II.

6. $\frac{3x - 4}{21 - 11x - 2x^2} \times (6 - 19x + 10x^2)$.

7. $\left(a + \frac{ab}{a - b}\right)\left(b - \frac{ab}{a + b}\right)$.

8. $\frac{ax - x^2}{a^2 - 2ax + x^2} \times \frac{a^2 + ax}{a^2 + 2ax + x^2}$.

9. $\left(\frac{a}{b}+\frac{b}{a}+1\right)\left(\frac{a}{b}+\frac{b}{a}-1\right).$

10.
$$\frac{4x^2-16x+15}{2x^2+3x+1}\times\frac{x^2-6x-7}{2x^2-17x+21}\times\frac{4x^2-1}{4x^2-20x+25}.$$

119. Reciprocals.—The reciprocal of a fraction is the fraction inverted. Thus, the reciprocal of $\frac{c}{d}$ is $\frac{d}{c}$.

120. THEOREM IV. *To divide one fraction by another is equivalent to multiplying the first fraction by the reciprocal of the second.*

It is required to prove that $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}.$

By definition of division, $\frac{a}{b}\div\frac{c}{d}\times\frac{c}{d}=\frac{a}{b}.$

By Theorem III and the associative law of multiplication,

$$\frac{a}{b}\times\frac{d}{c}\times\frac{c}{d}=\frac{a}{b}\cdot\frac{cd}{cd}=\frac{a}{b}.$$

$$\therefore \qquad \left(\frac{a}{b}\div\frac{c}{d}\right)\frac{c}{d}=\left(\frac{a}{b}\times\frac{d}{c}\right)\frac{c}{d}.$$

$$\therefore \qquad \frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}.$$

Hence to divide one fraction by another, we invert the divisor, and then proceed as in multiplication.

COR. 1. *A fraction may be divided by a quantity by multiplying its denominator by the quantity.*

For, let $\frac{a}{b}$ be a fraction and c be the given quantity.

Then will $\frac{a}{b}\div c=\frac{a}{b\times c}.$

Since $c = \frac{c}{1}$, we have $\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c} = \frac{a}{bc}$.

COR. 2. *A fraction may be divided by a quantity by dividing its numerator by the quantity.*

For, let $\frac{a}{b}$ be a fraction and c be the given quantity. Then $\frac{a}{b} \div c = \frac{a}{bc}$, and multiplying the numerator and denominator by $\frac{1}{c}$, we have

$$\frac{a}{b} \div c = \frac{a \cdot \frac{1}{c}}{bc \cdot \frac{1}{c}} = \frac{\frac{a}{c}}{b} \quad \text{or} \quad \frac{a \div c}{b}.$$

COR. 3. *To divide a quantity by a fraction we multiply the quantity by the reciprocal of the fraction.*

Let a be a quantity, and $\frac{c}{d}$ be a fraction. Then will

$$a \div \frac{c}{d} = a \times \frac{d}{c}.$$

$$a = \frac{a}{1}, \therefore a \div \frac{c}{d} = \frac{a}{1} \div \frac{c}{d} = \frac{a}{1} \times \frac{d}{c} = \frac{ad}{c} = a \times \frac{d}{c}.$$

EXERCISE LXXII.

Perform the operations indicated in the following examples:

I.

1. $\frac{14x^2 - 7x}{12x^3 + 24x^2} \div \frac{2x - 1}{x^2 + 2x}$.

2. $\frac{a^2b^2 + 3ab}{4a^2 - 1} \div \frac{ab + 3}{2a + 1}$.

3. $\dfrac{a^2 - 121}{a^2 - 4} \div \dfrac{a + 11}{a + 2}$.

4. $\dfrac{2x^2 + 13x + 15}{4x^2 - 9} \div \dfrac{2x^2 + 11x + 5}{4x^2 - 1}$.

5. $\dfrac{x^2 - 14x - 15}{x^2 - 4x - 45} \div \dfrac{x^2 - 12x - 45}{x^2 - 6x - 27}$.

6. $(10 + 11x - 6x^2) \div \dfrac{9x^2 - 4}{4 - 3x}$.

7. $(15x^2 - 19x + 6) \div \dfrac{18 - 18x - 20x^2}{2x + 7}$.

8. $(x^2 - 2x - 63) \div \dfrac{x^2 + 2x - 35}{2x - 3}$.

121. To Multiply Several Fractions by a Factor which will Cancel all their Denominators.—If each of several fractions be multiplied by the L. C. M. of their denominators, there will be introduced into the numerator of each fraction a factor which will cancel its denominator, and the resulting products will be the product of the numerator of each fraction and all the factors of the L. C. M. of the denominators except the denominator of the fraction. We may therefore obtain these products by dividing the L. C. M. of the denominators by each denominator and multiplying the numerator of each fraction by the resulting quotient.

e.g. Find the product which would result from multiplying each of the following fractions by the L.C.M.D.:

$$\frac{x + 7}{x^2 + 3x - 10}, \quad \frac{x - 8}{x^2 - 8x + 12} \quad \text{and} \quad \frac{x + 9}{x^2 - x - 30}.$$

Factoring the denominators, we get

$$\frac{x + 7}{(x - 2)(x + 5)}, \quad \frac{x - 8}{(x - 2)(x - 6)}, \quad \frac{x + 9}{(x + 5)(x - 6)}.$$

Hence the L. C. M. of the denominator is

$$(x-2)(x+5)(x-6).$$

Multiplying each fraction by this L. C. M., and cancelling the common factors, we obtain

$$\frac{(x+7)(x-2)(x+5)(x-6)}{(x-2)(x+5)},$$

$$\frac{(x-8)(x-2)(x+5)(x-6)}{(x-2)(x-6)},$$

and $$\frac{(x+9)(x-2)(x+5)(x-6)}{(x+5)(x-6)},$$

or x^2+x-42, $x^2-3x-40$, and $x^2+7x-18$.

EXERCISE LXXIII.

Find the products obtained by multiplying each fraction of the following sets by the L. C. M. of the denominators:

I.

1. $\frac{x-4}{x^2+x-56}$, $\frac{x-8}{x^2+11x+24}$, $\frac{6}{x^2-4x-21}$.

2. $\frac{3x-7}{10x^2-43x+28}$, $\frac{5x-4}{15x^2+8x-16}$, $\frac{x+11}{6x^2-13x-28}$.

3. $\frac{5x-8}{66x-15x^2-63}$, $\frac{6-7x}{24x^2-90x+54}$, $\frac{3-x}{40x^2-86x+42}$.

II.

4. $\frac{x-8}{x^3-64}$, $\frac{x-4}{x^2+4x-32}$, $\frac{7}{x^2+4x+16}$.

5. $\frac{x+7}{x^3+216}$, $\frac{x+6}{x^2-36}$, $\frac{15}{3x^2-108}$.

6. Divide $x^{7m}-x^{7n}$ by x^m-x^n.

EXERCISE LXXIV.

I.

1. A is four times as old as B and 6 years ago he was seven times as old. What is the age of each?

2. At what time after 3 o'clock are the hands of a watch opposite each other for the first time?

3. Divide 45 into two parts such that one of them shall be four times as much above 20 as the other is below 19.

4. A man had $13.55 in dollars, dimes, and cents. He had 1/7 as many cents as dimes, and twice as many dollars as cents. How many of each kind had he?

5. Divide 313 into two such parts that one divided by the other may give 2 as a quotient and 19 as a remainder.

II.

6. A is m times as old as B, and in c years he will be n times as old. What is the age of each?

7. At what rate of simple interest will a dollars amount to b dollars in c years?

8. The denominator of a fraction is equal to four times the numerator, diminished by 41, and if the numerator be diminished by 6 and the denominator be increased by 9, the value of the fraction will be 5/12. What is the fraction?

9. At what time after 5 o'clock are the hands of a watch together for the first time?

10. Divide n into two parts such that one divided by the other will give q as a quotient and r as a remainder.

CHAPTER XV.

CLEARING EQUATIONS OF FRACTIONS.

122. Three Classes of Equations Involving Fractions.—As we have seen, an equation may be cleared of fractions by multiplying both members by the least common multiple of the denominators of all the fractions in the equation.

Equations involving fractions may be divided into three classes:

1°. *Those in which we should clear of fractions at once or after making some slight reductions.*

2°. *Those which might be cleared of fractions partially and then simplified.*

3°. *Those in which some or all of the fractions had better be reduced to a mixed form.*

Case 1°.

In clearing equations of fractions, it must be borne in mind that every term of both members, integral as well as fractional, must be multiplied by the L. C. M. D.

In clearing equations of fractions, it is best to express the L. C. M. of the denominators as factors, and also to indicate the work of multiplication before actually performing it. In this way like factors in the numerators and denominators may be cancelled, and the work much shortened.

e.g. Solve $\frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+4} = 0.$

L. C. M. D. $(x+1)(x+2)(x+4)$.

$$\therefore \frac{(x+1)(x+2)(x+4)}{x+1} - \frac{2(x+1)(x+2)(x+4)}{x+2} + \frac{(x+1)(x+2)(x+4)}{x+4} = 0.$$

$$\therefore (x+2)(x+4) - 2(x+1)(x+4) + (x+1)(x+2) = 0,$$

or $$x^2 + 6x + 8 - 2x^2 - 10x - 8 + x^2 + 3x + 2 = 0.$$

$$\therefore \quad -x + 2 = 0.$$

$$\therefore \quad x = 2.$$

When all of the fractions are written as decimals, it is best first of all to reduce these to the form of vulgar fractions.

e.g. Solve $\dfrac{.05 - .01x}{.1} - (.03 - .02x) = .03.$

Reducing the decimals to vulgar fractions, we have

$$\frac{\frac{5}{100} - \frac{x}{100}}{\frac{1}{10}} - \left(\frac{3}{100} - \frac{2x}{100}\right) = \frac{3}{100},$$

or $\dfrac{5}{10} - \dfrac{x}{10} - \dfrac{3}{100} + \dfrac{2x}{100} = \dfrac{3}{100}$. L. C. M. D. = 100.

$$\therefore \frac{5 \times 100}{10} - \frac{x \times 100}{10} - \frac{3 \times 100}{100} + \frac{2x \times 100}{100} = \frac{3 \times 100}{100},$$

or $$50 - 10x - 3 + 2x = 3.$$

$$\therefore \quad -8x = -44;$$

$$\therefore \quad x = 5\tfrac{1}{2}.$$

EXERCISE LXXV.

I.

1. $\frac{1}{3}x - 10\tfrac{2}{3} = 2\left(\frac{6}{5}x - 1\right) - \frac{2}{3}(x + 4) - 1\tfrac{3}{5}.$

2. $\frac{3}{5}(2x - 7) - \frac{2}{3}(x - 8) = \frac{2x + 1/2}{7\frac{1}{2}} + 4.$

3. $\frac{1}{29}(4x + 1) - \frac{1}{6}(217 - x) = 45 - \frac{479 - 12x}{5}.$

4. $.03x + .02 = .02x - .06.$

5. $.07(x - 10) + .54x = .2(.1 - .1x) - 3(.05 - .02).$

6. $\frac{3 - x}{1 - x} - \frac{5 - x}{7 - x} = 1 - \frac{x^2 - 2}{7 - 8x + x^2}.$

7. $\frac{3}{2x - 4} - \frac{1}{x + 2} + \frac{x + 10}{2x^2 - 8} = 0.$

8. $\frac{2}{1 - 2x} - \frac{2}{2x - 7} = 1 - \frac{4x^2 - 1}{4x^2 - 16x + 7}.$

II.

9. $(1 - 2x)(.01 - .03x) - .23$
$= (.6x + .1)(.1x - .1) - .03x.$

10. $\frac{.01x}{.02} - \frac{x}{30} = \frac{.01x}{.5} + 1.34.$

11. $\frac{.03x - .01}{.02} - \frac{.02(x - 1)}{.03} = \frac{.01x - .03}{.4} + \frac{.21}{.2}.$

12. $(.1x + .2)^2 + .7(.3x - .1)$
$= .06(2x + 4) + (.1x - .2)^2 - .65.$

13. $\frac{4 - x}{2 - x} - \frac{6 - x}{8 - x} = 2 - \frac{2x^2 + 8}{16 - 10x + x^2}.$

14. $\frac{5}{3x - 9} - \frac{2}{x + 3} + \frac{11x}{3x^2 - 27} = 0.$

15. $-\dfrac{3}{1-3x}-\dfrac{3}{3x-7}=3-\dfrac{3x(9x-17)}{9x^2-24x+7}.$

CASE 2°.

123. When the L.C.M. of the denominators of all the fractions which occur in the equations is inconveniently large, it is easier to multiply both members by the L.C.M. of two or more of the denominators, and then reduce as much as possible before proceeding farther.

e.g. Solve $\dfrac{x}{x-1}-\dfrac{2+x^2}{x^2-2x+1}=\dfrac{3-2x}{2x-2}+1.$

Multiplying by $2(x-1)$, we get

$$2x-\frac{2(2+x^2)}{x-1}=3-2x+2x-2,$$

or

$$2x-\frac{2(2+x^2)}{x-1}=1.$$

$$\therefore \quad 2x(x-1)-2(2+x^2)=x-1.$$

$$\therefore \quad 2x^2-2x-4-2x^2=x-1.$$

$$\therefore \quad 3x=-3.$$

$$\therefore \quad x=-1.$$

EXERCISE LXXVI.

Solve the following equations:

1. $\dfrac{x}{5}+\dfrac{3-2x}{10}-\dfrac{x-3}{22}=\dfrac{1}{5}.$

2. $\dfrac{4x+3}{9}-\dfrac{8x+19}{18}+\dfrac{7x-29}{5x-12}=0.$

3. $\dfrac{1}{15}(3x+13)-\dfrac{3x+10}{10x-50}=\dfrac{x}{5}.$

II.

4. $\dfrac{8x+5}{14} - \dfrac{3-7x}{6x+2} - \dfrac{16x+15}{28} = \dfrac{2\frac{1}{4}}{7}.$

5. $\dfrac{6x-7\frac{1}{3}}{13-2x} + \dfrac{1+16x}{24} = \dfrac{53-24x}{12} - \dfrac{12\frac{5}{8}-8x}{3}.$

CASE 3°.

124. When the degree of the numerator of any of the fractions equals or exceeds that of the denominator, it is best in most cases to write the fraction in the mixed form obtained by dividing the numerator by the denominator and writing the remainder in the form of a fraction after the integral quotient; thus:

$$\frac{x-1}{x+1} = 1 - \frac{2}{x+1};$$

$$\frac{x^2-5x+4}{x-3} = x - 2 - \frac{2}{x-3}.$$

After writing the fractions as mixed numbers, the equation may generally be considerably reduced before finally clearing of fractions.

e.g. 1. Solve $x + \dfrac{x+9}{3(x-1)} = \dfrac{3x^2+6}{3x-1}.$

Writing the second fraction in the mixed form, we have

$$x + \frac{x+9}{3(x-1)} = x + \frac{x+6}{3x-1}.$$

$$\therefore \quad \frac{x+9}{3(x-1)} = \frac{x+6}{3x-1}.$$

$$\therefore \quad (x+9)(3x-1) = 3(x+6)(x-1).$$

$$\therefore \quad 3x^2+26x-9 = 3(x^2+5x-6).$$

$$\therefore \quad 11x = -9.$$

$$\therefore \quad x = -\frac{9}{11}.$$

2. Solve $\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-4}{x-5} - \frac{x-5}{x-6}$.

Writing each fraction as a mixed quantity, we have

$$1 + \frac{1}{x-2} - \left(1 + \frac{1}{x-3}\right) = 1 + \frac{1}{x-5} - \left(1 + \frac{1}{x-6}\right).$$

$$\therefore \quad \frac{1}{x-2} - \frac{1}{x-3} = \frac{1}{x-5} - \frac{1}{x-6}.$$

We may now write each member as one fraction and get

$$\frac{x-3-x+2}{(x-2)(x-3)} = \frac{x-6-x+5}{(x-5)(x-6)},$$

or
$$\frac{-1}{(x-2)(x-3)} = \frac{-1}{(x-5)(x-6)}.$$

$$\therefore \quad (x-2)(x-3) = (x-5)(x-6).$$

$$\therefore \quad x^2 - 5x + 6 = x^2 - 11x + 30.$$

$$\therefore \quad 6x = 24.$$

$$\therefore \quad x = 4.$$

EXERCISE LXXVII.

Solve the following equations:

I.

1. $\frac{x-1}{x-2} = \frac{x-2}{x-5}$.

2. $\frac{x+2}{x-3} + \frac{x-2}{x-6} = 2$.

3. $\frac{x+3}{x-1} + \frac{x-4}{x-6} = 2$.

4. $\frac{x}{x+1} - \frac{3x}{x-2} = -2$.

5. $\frac{3x+5}{3x-5} + \frac{2x+4}{x-2} = 3$.

6. $\frac{2x}{2x+1} + \frac{5}{2x-1} + \frac{2x-5}{2x+1} = 2$.

II.

7. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}$.

8. $\frac{x+1}{x+2}+\frac{x+6}{x+7}=\frac{x+2}{x+3}+\frac{x+5}{x+6}$.

9. $\frac{8-5x}{2x-1}+\frac{4x+3}{x+3}=1\frac{1}{2}$.

10. $\frac{x+a}{x-a}=\frac{x+b}{x-b}$.

11. $\frac{x}{x-a}-\frac{x+a-b}{x-b}=\frac{a(a-b)}{(x-c)(x-d)}$.

12. $\frac{x-7a}{x-9a}-\frac{x-a}{x-3a}+\frac{x-5a}{x-2a}=\frac{x-a}{x+2a}$.

EXERCISE LXXVIII.

I.

1. A vessel can be emptied by three taps: by the first alone in 3 hours and 40 minutes, by the second alone in 2 hours and 45 minutes, and by the third alone in 2 hours and 12 minutes. In what time would it be emptied were it full and all three taps were opened together?

2. A cistern can be filled in 15 minutes by two pipes, A and B, together. After A has been opened for 5 minutes B is also turned on, and the cistern is filled in 13 minutes more. In what time would it be filled by each pipe separately?

3. A man invests one third of his money in 3-per-cent bonds, two fifths of it in 4-per-cent bonds, and the remainder of it in 5-per-cent bonds. His income from his investment is 1180 dollars. How much had he invested?

4. A man invested one quarter of his money in 3-per-cent bonds, two sevenths of it in 4-per-cent bonds, and the remainder of it in $4\frac{1}{2}$-per-cent bonds. His income from his investment was 3450 dollars. How much had he invested?

5. Two men, A and B, 66 miles apart, set out, B 45 minutes after A, and travel towards each other, A at the rate of 4 miles an hour and B at the rate of 3 miles an hour. How far will each have travelled when they meet?

6. The second figure of a number composed of three figures exceeds the third by 5, and the first digit is one fourth of the second. If the number increased by 3 be divided by the sum of its digits, the quotient will be 22. What is the number?

7. A number is composed of three digits. The second digit is one half of the third and 2 smaller than the first. If the number be diminished by 18 and then divided by the sum of its digits, the quotient will be 37. What is the number?

8. A banker has two kinds of money. It takes a pieces of the first to make a dollar and b pieces of the second to make a dollar. He was offered d dollars for c pieces. How many of each kind would he give?

9. A and B start in business at the same time, A putting in 3/2 as much capital as B. The first year A gains 150 dollars and B loses 1/4 of his money. The next year A loses 1/4 of his money and B gains 300 dollars; and they now have equal amounts. How much had each at first?

II.

10. Two couriers, A and B, set out from the same place and travel along the same road in the same direc-

tion, A starting 8 hours before B. B rides at the rate of 8 miles an hour, and A at the rate of 6 miles. How far will each have travelled when B has overtaken A?

11. A and B find a sum of money. A takes $2.40 and 1/6 of what is left; then B takes $3.52 and 1/7 of what is left; and they find they have taken equal amounts. What was the sum found and what did each take?

12. A fox is pursued by a greyhound, and has 60 of her own leaps the start. The fox leaps three times while the greyhound leaps twice, but the hound goes as far in 3 leaps as the fox does in 7. How many leaps does each make before the hound catches the fox?

13. A hare takes 4 leaps to a greyhound's 3, but two of the hound's leaps are equivalent to three of the hare's. The hare has a start of 50 of her leaps. How many leaps must the hound make to catch the hare?

14. A man and a boy agreed to do a piece of work for $5.25, the boy to receive 1/2 as much per day as the man. When 2/5 of the work was done the boy left, and, in consequence, it took the man 1¼ days longer to complete the work than it would otherwise have done. How much did each receive per day?

15. In a mixture of spirits and water, half of the whole plus 25 gallons is spirits, and a third of the whole minus 5 gallons is water. How many gallons are there of each?

16. A garrison of 1000 men was provisioned for 60 days. After 10 days it was reinforced, and from that time the provisions lasted only 20 days. What was the number of the reinforcement?

17. A laborer was engaged for 36 days on condition that he should receive 2s. 6d. for every day he worked and should forfeit 1s. 6d. for every day he was idle. At the end of the time he received 58 shillings. How many days did he work?

18. At a cricket match the contractor provided dinner for 24 persons, and fixed the price per plate so as to gain $12\frac{1}{2}$ per cent upon his outlay. Three of the cricketers were absent. The remaining 21 paid the fixed price for their dinner, and the contractor lost 1 shilling. What was the price per plate?

CHAPTER XVI.

RADICALS AND SURDS.

125. Rational and Irrational Numbers.—A numerical quantity which can be exactly expressed as an integer or a fraction whose numerator and denominator are integers is called a *commensurable* or a *rational* number, and one which cannot be so expressed, an *incommensurable* or an *irrational* number.

126. Radicals.—Any algebraic expression which contains a factor under a radical or other root sign is called a *radical* expression, or simply a *radical*, and the factor under the root sign is called the *radical* factor.

Any algebraic expression which contains no radical factor is called a *rational* quantity.

To rationalize an expression is to free it of radical or other root symbols.

127. Surds.—A *surd* is an *incommensurable* root of a commensurable number. In other words, it is the root of an arithmetical number which can be found only approximately.

While every surd is an incommensurable number, there are many incommensurable numbers which are not surds, or due to any finite combinations of surds. As examples of these we have 3.1415926 . . . , the ratio of the circumference to the diameter of a circle, and 2.7182818 . . . , the base of the natural or Napierean system of logarithms.

A radical expression which cannot be freed from root symbols is called an *irrational* or *surd* expression, or simply a *surd*. The symbol of a surd is $\sqrt[n]{a}$, in which n denotes any positive integer, and a any integral algebraic expression.

A surd may be expressed as a radical quantity, but every radical quantity is not a surd. Thus, $\sqrt{3}$, $\sqrt[3]{5}$ are surds, but $\sqrt{4}$, $\sqrt[3]{8}$ are not surds. The expression $\sqrt{2+\sqrt{2}}$ is not a surd according to definition.

128. Imaginary Quantities.—Since no even combination of negative factors can produce a negative product, an even root of a negative quantity is called an *imaginary* quantity. Thus, $\sqrt{-2}$, $\sqrt{-a}$, $\sqrt[4]{-a}$ are imaginary quantities.

The value of the expression $\sqrt[n]{a}$ will be real or imaginary according to the values assigned to n and a. It will be imaginary when n is even and a is negative. In all other cases the value will be real.

When a is a perfect nth power, $\sqrt[n]{a}$ is rational and in all other cases irrational or surd.

129. To Express a Rational Quantity as a Radical.—Any rational quantity may be expressed as a radical by first raising it to the power indicated by the index of the radical and then placing it under the radical sign.

e.g. $4 = \sqrt{16}, \quad 3 = \sqrt[3]{27}.$

130. Orders of Radicals.—A radical is said to be of the first, second, or nth *orders* according as its index is 1, 2, or n.

EXERCISE LXXIX.

Express the following quantities as radicals of the second order:

I.

1. m.
2. n.
3. $3a$.
4. $5ab$.
5. $7a^3$.
6. $6x^2y^3$.
7. $1/4a^2x$.

II.

8. $1/3a^3y^2$.
9. $\frac{5a^3x^4}{3b}$.
10. $a + b$.
11. $x - y$.
12. $3a^2 + 7$.

Write the following as radicals of the third order:

I.

13. x.
14. $3a^2x$.
15. $1/3a^3y$.

II.

16. $x + 5$.
17. $a - 3$.
18. $\frac{3a^3x^2}{4c}$.

131. Arithmetical Roots.—We have already seen that $\sqrt{a^2}$ has two values, $+a$ and $-a$; also, that $\sqrt{a}$ has two values which differ only in sign, one being positive and the other negative. In higher algebra it is shown that $\sqrt[3]{a}$ has three values, one of which is real and the other two imaginary; also, that $\sqrt[n]{a}$ has n values, one, or at most two, of which may be real, and the others imaginary, and that when there are two real roots they will differ only in sign.

When a root symbol is placed before a number it denotes the arithmetical root only, but when placed before an algebraic expression it denotes *one* of the roots. Thus $\sqrt{a}$ has two values either of which is denoted by the symbol, but $\sqrt{2}$ is supposed to denote only the arithmetical root, unless it is written $\pm\sqrt{2}$.

In the demonstrations in the present chapter the symbol $\sqrt[n]{a}$ in all cases must be taken in a restricted sense,—to mean the real root of a whose sign is the same as the essential sign of a. Thus $\sqrt[n]{a^n}$ must be taken to mean a, and $\sqrt[n]{a}$ to mean the one real root of a which has the same sign as a. The theorems established in this chapter do not necessarily apply to other real roots than the one specified above, or to imaginary roots.

In this chapter it is assumed that the associative, distributive, commutative, and index laws, which have been established for integers, and applied to rational algebraic expressions, also apply to surds.

132. THEOREM I. *The product of the same roots of two factors is equal to that root of the product of the factors.*

By definition $\sqrt[n]{a}$ used n times as a factor will give a as a product.

$$\therefore\ (\sqrt[n]{a})^n = a.$$

Similarly, $(\sqrt[n]{b})^n = b$, and $(\sqrt[n]{ab})^n = ab$.

But $(\sqrt[n]{a} \times \sqrt[n]{b})^n = (\sqrt[n]{a})^n \times (\sqrt[n]{b})^n = ab,$

and $$(\sqrt[n]{ab})^n = ab.$$

$$\therefore\ (\sqrt[n]{a} \times \sqrt[n]{b})^n = (\sqrt[n]{ab})^n. \quad \text{(Why?)}$$

$$\therefore\ \sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}. \quad \text{(Why?)}$$

COR. *The product of the same roots of any number of factors is equal to that root of the product of those factors.*

NOTE.—It should be borne in mind that $\sqrt[n]{a}$, taken arbitrarily, $\times \sqrt[n]{b}$, taken arbitrarily, does not $= \sqrt[n]{ab}$, taken arbitrarily. Thus the negative root of 2 multiplied by the positive root of 3 does not equal the positive root of 6.

The equation $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ is true when the meaning of the symbols is restricted as in 131. It is also true that any one of the n roots of a multiplied by any one of the n roots of b will be equal to some one of the n roots of ab.

133. It follows from Theorem I that, when the quantity under the radical sign can be separated into factors one or more of which is an exact power of the order of the root indicated, the product of the indicated roots of these factors may be placed as a factor outside the radical.

e.g. $\sqrt{192} = \sqrt{16 \times 4 \times 3} = \overset{4}{\sqrt{16}} \times \overset{2}{\sqrt{4}} \times \sqrt{3} = 8\sqrt{3}.$

$\sqrt[3]{864} = \sqrt[3]{27 \times 8 \times 4} = \sqrt[3]{27} \times \overset{2}{\sqrt[3]{8}} \times \sqrt[3]{4} = 6\sqrt[3]{4}.$

134. Pure and Mixed Surds.—The factor without the radical sign may be regarded as the coefficient of the radical.

A *pure* surd is one that has no rational coefficient except unity.

A *mixed* surd is one that has a rational factor other than unity.

A surd is said to be in its simplest form when it has no rational factor under the radical sign.

EXERCISE LXXX.

Write the following as mixed surds in their simplest forms:

I.

1. $\sqrt{12}.$
2. $\sqrt{75}.$
3. $\sqrt{180}.$
4. $\sqrt{735}.$
5. $\sqrt{512}.$
6. $\sqrt{567}.$
7. $\sqrt[3]{135}.$

II.

8. $\sqrt[3]{448}$. **9.** $\sqrt[3]{5632}$. **10.** $\sqrt{48a^2b}$.

11. $\sqrt{125a^3x^4}$. **12.** $\sqrt{147a^5x^7}$.

13. $\sqrt{4a^5+8a^4b+4a^3b^2}$. **14.** $\sqrt{12x^7y^3-24x^6y^4+12x^5y^5}$.

A mixed surd may be reduced to the form of a pure surd by raising its coefficient to the power indicated by the order of the surd and placing it as a factor under the radical sign.

e.g. $7\sqrt{5}=\sqrt{7^2\times 5}=\sqrt{245}$.

EXERCISE LXXXI.

Express the following as pure surds:

I.

1. $3\sqrt{11}$. **2.** $4\sqrt{13}$. **3.** $6\sqrt{7}$.

4. $2\sqrt[3]{9}$. **5.** $4\sqrt[3]{5}$. **6.** $6\sqrt[3]{4}$.

7. $3a\sqrt{a-b}$. **8.** $(x+y)\sqrt{3x}$. **9.** $3a(a-b)\sqrt{5ab}$.

135. Theorem II. *The quotient of the same roots of two quantities is equal to that root of the quotient of the two quantities.*

Expressed algebraically, $\sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{a\div b}$.

$$(\sqrt[n]{a}\div\sqrt[n]{b})^n=(\sqrt[n]{a})^n\div(\sqrt[n]{b})^n=a\div b.$$

But $(\sqrt[n]{a\div b})^n=a\div b$.

$$\therefore\ (\sqrt[n]{a}\div\sqrt[n]{b})^n=(\sqrt[n]{a\div b})^n. \quad \text{(Why?)}$$

$$\therefore\ \sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{a\div b}. \quad \text{(Why?)}$$

Cor. $\sqrt[n]{\dfrac{a}{b}}=\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$. That is, any root of a fraction may be indicated by placing the corresponding radical over the fraction as a whole, or over its numerator and denominator separately.

136. Similar and Quadratic Surds.—*Similar* surds are those whose radical factors are identical. e.g. $\sqrt{5}$, $3\sqrt{5}$, are similar surds. So also are $a\sqrt[3]{x}$ and $c\sqrt[3]{x}$.

Surds of the second order are called *quadratic* surds.

137. THEOREM III. *The product of two similar quadratic surds is a rational quantity.*

$$m\sqrt{a} \times n\sqrt{a} = mn\sqrt{a^2} = mna.$$

The product of the coefficients is necessarily a rational quantity, and the product of the similar radical factors is necessarily the square root of a perfect square, and, therefore, rational.

138. THEOREM IV. *The product of two dissimilar quadratic surds cannot be rational.*

Let $\sqrt{a}$ and $\sqrt{b}$ be the surd factors. Since the surds are dissimilar, a and b cannot be composed of the same prime factors, and hence their product ab cannot be composed of square factors only. Therefore $\sqrt{ab}$ cannot be rational.

139. Rationalizing Factor. — Any factor which will convert a radical expression into a rational one is called a *rationalizing* factor.

It follows from Theorem II that the surd factor of a pure or mixed surd is a rationalizing factor.

e.g. $\sqrt{5} \times \sqrt{5} = 5.$ $\quad 3\sqrt{3} \times \sqrt{3} = 3 \times 3 = 9.$

$$5\sqrt{a-b} \times \sqrt{a-b} = 5(a-b).$$

140. To Reduce a Fractional Radical to an Integral Radical.—A fractional radical may be reduced to an integral radical with a fractional coefficient, by writing its numerator and denominator each as a separate radical, and then multiplying each by the rationalizing factor of the denominator.

$$\text{e.g.} \qquad \sqrt{1/5} = \frac{\sqrt{1}}{\sqrt{5}} = \frac{\sqrt{1} \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = 1/5\,\sqrt{5}.$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a} \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}} = \frac{\sqrt{ab}}{b} = \frac{1}{b}\sqrt{ab}.$$

EXERCISE LXXXII.

Reduce the following to integral radicals:

I.

1. $\sqrt{1/2}.$
2. $\sqrt{1/5}.$
3. $\sqrt{2/3}.$
4. $\sqrt{5/12}.$
5. $\sqrt{\dfrac{3a}{7x}}.$
6. $\sqrt{\dfrac{a+b}{a-b}}.$

II.

7. $\sqrt{\dfrac{x+4}{x+6}}.$
8. $\sqrt{\dfrac{x-5}{x+7}}.$
9. $\sqrt{\dfrac{5x-2}{2x+1}}.$
10. $\sqrt{\dfrac{4x-6}{3x-7}}.$
11. $\sqrt{\dfrac{5-2x}{3x+4}}.$
12. $\sqrt{\dfrac{4+3x}{4x-3}}.$

141. Addition and Subtraction of Radicals.—Similar radicals may be added and subtracted by combining their coefficients in the same way as similar rational terms. The common surd factor must be written after the coefficient resulting from the combination.

e.g. The sum of $3\sqrt{5}$, $9\sqrt{5}$, and $-7\sqrt{5}$ is $5\sqrt{5}$.

The difference of $3\sqrt{2}$ and $9\sqrt{2}$ is $-6\sqrt{2}$.

Dissimilar radicals can be added and subtracted only by writing them one after another, each with its proper sign, as in the case of dissimilar rational terms.

Thus, $\sqrt{y}$ added to $\sqrt{x} = \sqrt{x} + \sqrt{y}$, and never $\sqrt{x+y}$, unless either x or y is zero.

142. Rule for Addition of Radicals.—To add surds of the same order, reduce them to their simplest forms and add the coefficients of the resulting surds which are similar, and write those which are dissimilar after one another.

e.g. $3\sqrt{2} + \sqrt{18} + 2\sqrt{12} - \sqrt{48} + 3\sqrt{5}$

$6\sqrt{2}$

$= 3\sqrt{2} + 3\sqrt{2} + 4\sqrt{3} - 4\sqrt{3} + 3\sqrt{5} = 6\sqrt{2} + 3\sqrt{5}.$

143. Rule for Subtraction of Radicals.—To subtract two radicals of the same order, reduce them to their simplest form, and then, if they are similar, subtract their coefficients, and if they are dissimilar, write them one after the other with the proper sign between.

e.g. From $3\sqrt{5}$ take $2\sqrt{125}$.

$= 3\sqrt{5} - 10\sqrt{5} = -7\sqrt{5}.$

From $3\sqrt{3}$ take $2\sqrt{80}$.

$= 3\sqrt{3} - 8\sqrt{5}.$

144. Addition and Subtraction of Radicals of Different Orders.—Radicals of different orders can be added and subtracted only by writing them one after another with the proper signs between.

EXERCISE LXXXIII.

Find the sum of the following sets of radicals :

I.

1. $\sqrt{18}$, $\sqrt{32}$, $\sqrt{50}$, and $\sqrt{72}$.

2. $2\sqrt{8}$, $3\sqrt{50}$, and $6\sqrt{18}$.

3. $\sqrt{3/5}$, $\sqrt{1/15}$, and $\sqrt{15/49}$.

4. $2/3\sqrt[3]{2/9}$, $1/6\sqrt[3]{1/36}$, and $3/5\sqrt[3]{3/32}$.

5. $x\sqrt{12a^4x}$, $2a^2\sqrt{27x^3}$, $3a\sqrt{48a^2x^3}$, and $\sqrt{75a^4x^3}$.

II.

6. $2\sqrt{3}$, $1/2\sqrt{12}$, $4\sqrt{27}$, and $\sqrt{12/16}$.

7. $\sqrt[3]{54a^5b^5}$, $7a\sqrt[3]{2a^2b^5}$, and $8b\sqrt[3]{2a^5b^2}$.

8. $\dfrac{mn}{n-s}$ and $\sqrt{\dfrac{m^2n^2}{(n-s)^2}-\dfrac{m^2n}{n-s}}$.

EXERCISE LXXXIV.

I.

1. From $2\sqrt{320}$ subtract $3\sqrt{80}$.

2. From $a\sqrt[3]{64a^3b^4}$ subtract $b\sqrt[3]{343a^6b}$.

3. From $\sqrt{a^2b+2ab^2+b^3}$ subtract $\sqrt{a^2b-2ab^2+b^3}$.

4. From $\sqrt{2a^3+4a^2b+2ab^2}$ subtract $\sqrt{2a^3-4a^2b+2ab^2}$.

II.

5. From $2/3\sqrt[3]{2/9}+3/5\sqrt[3]{3/32}$ subtract $1/6\sqrt[3]{1/36}$.

6. From $\sqrt{289a^3b}$ subtract $3\sqrt{144a^3b}$.

7. From $2\sqrt{8c^3}+5\sqrt{72c^3}$ subtract $7c\sqrt{18c}+\sqrt{50cd^2}$.

8. From $(c-x)\sqrt{c^2-x^2}$ subtract $\sqrt{\dfrac{c+x}{c-x}}$.

145. Multiplication of Radicals of the Same Order.—To multiply together two radicals of the same order, multiply together their coefficients for the new coefficient, and the quantities under the radical sign for the new radical.

EXERCISE LXXXV.

Perform the following multiplications and reduce the results to the simplest form:

I.

1. $3\sqrt{8} \times 2\sqrt{96}$.
2. $7\sqrt[3]{2/81} \times 3/2\sqrt[3]{3/32}$.
3. $4\sqrt{12} \times 3\sqrt{2}$.
4. $\sqrt[3]{4/27} \times 3/4\sqrt[3]{12}$.
5. $5\sqrt{a^2x} \times 1/2\sqrt{25bx}$.
6. $b\sqrt{2ab} \times a\sqrt{8ab}$.
7. $(2\sqrt{2} - 3\sqrt{3} + 4\sqrt{5}) \times (3\sqrt{5} + 4\sqrt{3})$.
8. $(3\sqrt{5} - 4\sqrt{2})(2\sqrt{5} + 3\sqrt{2})$.
9. $(\sqrt{7} + 5\sqrt{3})(2\sqrt{7} - 4\sqrt{3})$.
10. $(\sqrt{2} + \sqrt{3} - \sqrt{5})(\sqrt{2} + \sqrt{3} + \sqrt{5})$.
11. $(3\sqrt{a} - 2\sqrt{x})(2\sqrt{a} + 3\sqrt{x})$.
12. Multiply $\sqrt{7} + 9$ by $\sqrt{7} - 6$.
13. Square $\sqrt{5} + 3$.
14. Multiply $\sqrt{5} - 6$ by $\sqrt{5} - 8$.
15. Square $\sqrt{7} - 5$.
16. Multiply $\sqrt{x} + 4$ by $\sqrt{x} + 3$.
17. Square $\sqrt{x} + 9$.
18. Multiply $\sqrt{x} + 6$ by $\sqrt{x} - 5$.
19. Square $\sqrt{x} - \sqrt{3}$.
20. Multiply $\sqrt{5} + \sqrt{7}$ by $\sqrt{5} - \sqrt{7}$.
21. Square $\sqrt{7} + \sqrt{8}$.
22. Multiply $\sqrt{x + 5}$ by $\sqrt{x - 8}$.

23. Square $\sqrt{x-4}+\sqrt{x+6}$.

24. Multiply $\sqrt{x+7}$ by $\sqrt{x-7}$.

25. Square $\sqrt{x-3}+\sqrt{x+3}$.

II.

26. Multiply $3x\sqrt{a-6}$ by $5x\sqrt{a-7}$.

27. Square $2a\sqrt{6x}+2\sqrt{b}$.

28. Multiply $5a\sqrt{x+7}$ by $7b\sqrt{x+7}$.

29. Square $3\sqrt{x+6}-4\sqrt{x-7}$.

30. Multiply $7a^2\sqrt{x-4}$ by $9a\sqrt{x+4}$.

31. Square $3a\sqrt{a+3}+5a\sqrt{a-5}$.

32. Multiply $\sqrt{x-4}-5$ by $\sqrt{x-4}+5$.

33. Multiply $\sqrt{x+8}+\sqrt{6}$ by $\sqrt{x+8}-\sqrt{6}$.

34. Multiply $\sqrt{x-5}+\sqrt{x+8}$ by $\sqrt{x-5}-\sqrt{x+8}$.

35. Multiply

$3\sqrt{x+6}+4\sqrt{x+5}$ by $3\sqrt{x+6}-4\sqrt{x+5}$.

36. Multiply

$3a^2x\sqrt{x-8}-5x^2\sqrt{x+7}$ by $3a^2x\sqrt{x-8}+5x^2\sqrt{x+7}$.

146. Simple, Compound, and Conjugate Radicals.—A *simple* radical expression is one which contains only one term, and a *compound* radical expression is one which contains more than one term.

Thus, $\sqrt{x}$, $\sqrt{a+x}$, $a\sqrt{ab}$, $(a+b)\sqrt{x+4}$, are simple radicals. $a+\sqrt{x}$, $\sqrt{a}+\sqrt{x+b}$, are compound radicals.

Two binomial quadratic radicals which have the same

3. $\dfrac{2\sqrt{7}+3\sqrt{2}}{9+2\sqrt{14}}$.

4. $\dfrac{2\sqrt{3}+4\sqrt{2}}{5+2\sqrt{6}}$.

5. $\dfrac{b^2}{a+\sqrt{a^2-b^2}}$.

6. $\dfrac{a^2}{\sqrt{a^2+y^2}-y}$.

II.

7. $\dfrac{\sqrt{3+a^2}-\sqrt{3-a^2}}{\sqrt{3+a^2}+\sqrt{3-a^2}}$.

8. $\dfrac{2\sqrt{x+3}+3\sqrt{x-3}}{2\sqrt{x+3}-3\sqrt{x-3}}$.

9. $\dfrac{\sqrt{5+x^2}-2}{\sqrt{5+x^2}+2}$.

10. $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{12}-\sqrt{32}+\sqrt{50}}$.

Divide the following radicals at sight:

I.

11. $\sqrt{18}$ by $\sqrt{6}$.

12. $\sqrt[3]{24}$ by $\sqrt[3]{8}$.

13. $12\sqrt{35}$ by $3\sqrt{7}$.

14. $a^3\sqrt{bx}$ by $a^2\sqrt{b}$.

15. $\sqrt{x^2-49}$ by $\sqrt{x+7}$.

16. $\sqrt{x^3-8}$ by $\sqrt{x-2}$.

17. $\sqrt[3]{x^3+27}$ by $\sqrt[3]{x^2-3x+9}$.

II.

18. $\sqrt[5]{x^2+2x-15}$ by $\sqrt[5]{x+5}$.

19. $\sqrt{x^2-13x+42}$ by $\sqrt{x-6}$.

20. $\sqrt{x^2-x-72}$ by $\sqrt{x+8}$.

21. $\sqrt{6x^2+17x-14}$ by $\sqrt{2x+7}$.

22. $\sqrt{5x\ \ -2x-7}$ by $\sqrt{x+1}$.

Divide the following radicals by first expressing the division in the form of a fraction and then rationalizing the denominator.

I.

23. 29 by $11 + 3\sqrt{7}$.

24. 17 by $3\sqrt{7} + 2\sqrt{3}$.

25. $3\sqrt{2} - 1$ by $3\sqrt{2} + 1$.

26. $2\sqrt{3} + 7\sqrt{2}$ by $5\sqrt{3} - 4\sqrt{2}$.

II.

27. $2x - \sqrt{xy}$ by $2\sqrt{xy} - y$.

28. $(3 + \sqrt{5})(\sqrt{5} - 2)$ by $5 - \sqrt{5}$.

29. $\dfrac{\sqrt{a}}{\sqrt{a} - \sqrt{x}}$ by $\dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{x}}$.

30. $\dfrac{2\sqrt{15} + 8}{5 + \sqrt{15}}$ by $\dfrac{8\sqrt{3} - 6\sqrt{5}}{5\sqrt{3} - 3\sqrt{5}}$.

150. THEOREM IV. *The nth power of the root of any quantity is the same root of the nth power of the quantity,* n and the index of the root both being positive integers.

1°. When the index of the root is the same as the exponent of the power.

By definition, $$(\sqrt[n]{a})^n = a,$$

and $$\sqrt[n]{a^n} = a.$$

$$\therefore\ (\sqrt[n]{a})^n = \sqrt[n]{a^n}.$$

2°. When the index of the root is not the same as the exponent of the power.

By definition, $$(\sqrt[m]{a^n})^m = a^n.$$

Also, $\left((\sqrt[m]{a})^m\right)^n$ means that $\sqrt[m]{a}$ is to be used mn times as a factor, and

$\left((\sqrt[m]{a})^n\right)^m$ means that $\sqrt[m]{a}$ is to be used mn times as a factor.

$$\therefore \quad \left((\sqrt[m]{a})^m\right)^n = \left((\sqrt[m]{a})^n\right)^m.$$

But
$$\left((\sqrt[m]{a})^m\right)^n = a^n.$$

$$\therefore \quad \left((\sqrt[m]{a})^n\right)^m = a^n.$$

$$\therefore \quad \left((\sqrt[m]{a})^n\right)^m = (\sqrt[m]{a^n})^m.$$

$$\therefore \quad (\sqrt[m]{a})^n = \sqrt[m]{a^n}.$$

151. THEOREM V. *The mth root of the nth root of a quantity is equal to the mnth root of the quantity.*

By definition,
$$\left(\sqrt[m]{\sqrt[n]{a}}\right)^m = \sqrt[n]{a}.$$

$$\therefore \quad \left(\left(\sqrt[m]{\sqrt[n]{a}}\right)^m\right)^n = (\sqrt[n]{a})^n = a.$$

Also,
$$(\sqrt[mn]{a})^{mn} = a,$$

and
$$\left(\left(\sqrt[m]{\sqrt[n]{a}}\right)^m\right)^n = \left(\sqrt[m]{\sqrt[n]{a}}\right)^{mn}.$$

$$\therefore \quad \left(\sqrt[m]{\sqrt[n]{a}}\right)^{mn} = (\sqrt[mn]{a})^{mn}.$$

$$\therefore \quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

152. To Change Radicals from One Index to Another.—It follows from Theorems IV and V that a radical may be changed from one index to another by multiplying both the index of the radical and the exponent of the quantity under the radical by the number which will produce the required index. For the former of these operations would extract a root of the radical quantity, and the latter would raise it to the corresponding power, and these two operations would neutralize each other.

e.g. $$\sqrt[3]{a^2} = \sqrt[3\times4]{a^{2\times4}} = \sqrt[12]{a^8}.$$

To change radicals of different orders to those of the same order with the smallest possible indices, multiply each index by the quotient obtained by dividing the least common multiple of all the indices by that index and raise the quantity under the radical sign to the corresponding power. This will, of course, make the index of each radical the least common multiple of all the indices.

e.g. Reduce $\sqrt{5}$, $\sqrt[3]{3}$, and $\sqrt[4]{2}$ to radicals of the same order with the smallest possible index.

The L. C. M. of 2, 3, and 4 is 12.

$$\sqrt{5} = \sqrt[2\times6]{5^6} = \sqrt[12]{15625}.$$

$$\sqrt[3]{3} = \sqrt[3\times4]{3^4} = \sqrt[12]{81}.$$

$$\sqrt[4]{2} = \sqrt[4\times3]{2^3} = \sqrt[12]{8}.$$

153. Multiplication and Division of Radicals of Different Orders.—Radicals of different orders may be multiplied together by first reducing them to the same order and then multiplying together their rational and their irrational factors.

Similarly, radicals of different orders may be divided by each other, by first reducing them to radicals of the same order and then dividing their integral and radical factors.

EXERCISE LXXXIX.

1. Reduce $\sqrt[3]{10}$, $\sqrt{5}$, and $\sqrt[6]{11/12}$ to a common index.

2. Reduce $\sqrt[3]{a+b}$, $\sqrt{a-b}$, and $\sqrt[4]{a^2+x^2}$ to a common index.

3. Multiply $\sqrt[3]{9}$ by $\sqrt{5}$.

4. Multiply $\sqrt[3]{1/2}$ by $\sqrt{3/4}$.

5. Divide $\sqrt{a^3}$ by $\sqrt[3]{a^2}$.

6. Divide $2\sqrt{2ac}$ by $\sqrt[3]{4bc^2}$.

7. Divide $1/2\sqrt{2/3}$ by $1/3\sqrt[3]{1/3}$.

154. Radical Equations.—An equation which contains radicals is called a *radical* equation. Such equations are solved by first clearing them of radicals, or *rationalizing* them. If the equation contains fractions it should be cleared of them first of all.

In the case of a quadratic radical equation, after it has been cleared of fractions, it is best to transpose all the terms into the left-hand member and place this equal to zero. Each member should then be multiplied by the conjugate of the first.

If the first member contains more than two terms, they should first be collected into a term and an aggregate, or into two aggregates, and the terms arranged, if possible, so that the aggregate shall contain no radical. Multiplying then by the conjugate expression will square each of the terms or aggregates, and place the minus sign between the squares obtained, and the result will be rational. If either aggregate contains a radical, the result of the first squaring will be irrational. In this case a new pair of aggregates must be formed and the operation must be repeated.

e.g. 1. $$\sqrt{x-6} - 4 = 9.$$

Transposing, we get

$$\sqrt{x-6} - 13 = 0.$$

Multiplying by the conjugate expression $\sqrt{x-6} + 13$, we get

$$x - 6 - 169 = 0,$$

$$\therefore \quad x = 175.$$

2. $$\sqrt{x-3} + 2\sqrt{x} - 5 = 3.$$

Transposing, we get

$$\sqrt{x-3} + 2\sqrt{x} - 8 = 0.$$

Writing this as the sum of two aggregates, thus,

$$\sqrt{4x-3} + (2\sqrt{x} - 8) = 0,$$

and multiplying this by the conjugate expression $\sqrt{4x-3} - (2\sqrt{x} - 8)$, we get

$$4x - 3 - 4x + 32\sqrt{x} - 64 = 0.$$

Collecting, we get

$$32\sqrt{x} - 67 = 0.$$

Multiplying again by the conjugate $32\sqrt{x} + 67$, we get

$$1024x - 4489 = 0.$$

$$\therefore \quad x = 4\tfrac{393}{1024}.$$

EXERCISE XC.

Solve the following radical equations:

I.

1. $\sqrt{x-5} = 3$. **2.** $\sqrt{4x-7} = 5$.

3. $7 - \sqrt{x-4} = 3$. **4.** $2\sqrt{5x+4} = 8$.

5. $\sqrt{5x-1} = 2\sqrt{x+3}$.

6. $2\sqrt{3-7x} - 3\sqrt{8x-12} = 0$.

7. $\sqrt{x+25} = 1 + \sqrt{x}$.

8. $\sqrt{8x+33} - 3 = 2\sqrt{2x}$.

9. $\sqrt{x+3} + \sqrt{x} = 5.$

10. $10 - \sqrt{25 + 9x} = 3\sqrt{x}.$

11. $\sqrt{x-4} + 3 = \sqrt{x+11}.$

12. $\sqrt{9x-8} = 3\sqrt{x+4} - 2.$

II.

13. $\sqrt{x + 4ab} = 2a + \sqrt{x}.$

14. $\sqrt{x} + \sqrt{4a + x} = 2\sqrt{b + x}.$

15. $\sqrt{x^2 + \sqrt{4x^2 + x + \sqrt{9x^2 + 12x}}} = 1 + x.$

16. $\sqrt{a + \sqrt{ax}} = \sqrt{a} - \sqrt{a - \sqrt{ax}}.$

17. $\sqrt{x} + \sqrt{ax} = a - 1.$

18. $\sqrt{5x} + \dfrac{12}{\sqrt{5x+6}} = \sqrt{5x+6}.$

19. $\sqrt{x} - 4 = \dfrac{237 - 10x}{4 + \sqrt{x}}.$

20. $\sqrt{a-x} = \dfrac{a}{\sqrt{a-x}} - x.$

21. $\dfrac{\sqrt{x}-2}{3} + 3 = \dfrac{x-4}{\sqrt{x}+2}.$

22. $\sqrt{x} - \sqrt{a-x} = \dfrac{\sqrt{x} + \sqrt{a-x}.}{2}$

155. Reduction of Radical Equations by Rationalization.—When a radical equation contains but one radical fraction, it is often best to rationalize the denominator of that fraction before clearing of fractions.

e.g. $$\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=b.$$

Rationalizing the fraction, we get

$$\frac{2a+2\sqrt{a^2-x^2}}{2x}=b, \quad \text{or} \quad \frac{a+\sqrt{a^2-x^2}}{x}=b.$$

Clearing of fractions and transposing, we get

$$a-bx+\sqrt{a^2-x^2}=0.$$

Multiplying by the conjugate, we have

$$a^2-2abx+b^2x^2-a^2+x^2=0,$$

or $$(b^2+1)x^2=2abx.$$

$$\therefore \quad (b^2+1)x=2ab.$$

$$\therefore \quad x=\frac{2ab}{b^2+1}.$$

EXERCISE XCI.

Solve the first four of the following equations by rationalizing the denominator:

I.

1. $\frac{\sqrt{3+x}+\sqrt{3-x}}{\sqrt{3+x}-\sqrt{3-x}}=4.$ 2. $\frac{\sqrt{6+x}+\sqrt{6-x}}{\sqrt{6+x}-\sqrt{6-x}}=8.$

3. $\frac{\sqrt{x+4}+\sqrt{x}}{\sqrt{x+4}-\sqrt{x}}=5.$ 4. $\frac{\sqrt{x+6}+\sqrt{x}}{\sqrt{x+6}-\sqrt{x}}=10.$

II.

5. $\frac{\sqrt{x+a}+\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}=c.$ 6. $\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}+\sqrt{x}}=\frac{5}{9}.$

7. $\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}+\sqrt{x}}=\frac{7}{12}.$ 8. $\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}+\sqrt{x}}=\frac{b}{a}.$

CHAPTER XVII.

THE INDEX LAW.

156. Meaning of Fractional Exponents.—It has been shown that, when m and n are positive integers,

$$a^m \times a^n = a^{m+n}. \qquad (1)$$

Also as a corollary to this, when $m > n$,

$$a^m \div a^n = a^{m-n}.$$

And as a consequence of (1) it has been shown that

$$(a^m)^n = a^{mn} = (a^n)^m, \qquad (2)$$

and $$(ab)^n = a^n b^n. \qquad (3)$$

These three laws follow from the definition that an exponent denotes the number of times a quantity is employed as a factor.

The law expressed by equation (1) is known as the *Index Law.*

The definition of an exponent becomes meaningless if the exponent, or index, be other than a positive integer.

The spirit of algebra is to generalize, and the use of indices cannot be restricted to the particular case of integers, but it must be extended to the case of fractional, zero, and negative indices. All of these indices must be governed by the index law, and they must be interpreted in accordance with this law.

We will proceed first to find the meaning of a fractional index in which both numerator and denominator are positive integers.

Let this index be denoted by $\frac{p}{q}$.

Since the equation $a^m \,.\, a^n = a^{m+n}$ is to be true for all values of m and n, we may replace each by $\frac{p}{q}$. We then have

$$a^{\frac{p}{q}} \,.\, a^{\frac{p}{q}} = a^{\frac{2p}{q}},$$

and multiplying each member by $a^{\frac{p}{q}}$, we get

$$a^{\frac{p}{q}} \,.\, a^{\frac{p}{q}} \,.\, a^{\frac{p}{q}} = a^{\frac{3p}{q}},$$

and so on up to q factors, when we should have

$$a^{\frac{p}{q}} \,.\, a^{\frac{p}{q}} \,.\, a^{\frac{p}{q}} \ldots q \text{ factors} = a^{\frac{qp}{q}} = a^p.$$

$$\therefore \quad \left(a^{\frac{p}{q}}\right)^q = a^p.$$

Therefore, by taking the qth root of each member, we have

$$a^{\frac{p}{q}} = \sqrt[q]{a^p},$$

or, in words, $a^{\frac{p}{q}}$ is equal to "the qth root of a to the pth power."

If $p = 1$, we should have

$$a^{\frac{1}{q}} = \sqrt[q]{a},$$

or $a^{\frac{1}{n}}$ is equal to the nth root of a.

For the present the meaning of the symbol $a^{\frac{1}{n}}$ must be restricted to the real nth root of a whose sign is the same as the essential sign of a, or to what may be called the arithmetical root of a. If this strict limitation is departed from, we are led to various paradoxes.

e.g. By the interpretation of fractional indices

$$(x)^{4/2} = \sqrt{x^4} = \pm x^2.$$

But $$x^{4/2} = x^2,$$

which is right if we take $x^{4/2}$ to stand for the positive value of $\sqrt{x^4}$; but leads to the paradox $x^2 = -x^2$ if we admit the negative value.

Again, according to the index law,

$$(x^m)^n = x^{mn} = (x^n)^m,$$

and $$(9^{1/2})^2 = (9^2)^{1/2},$$

or $$(\pm 3)^2 = \pm 9,$$

or $$9 = \pm 9,$$

if both values are admitted.

157. Meaning of Zero Exponent.—Since $a^m . a^n = a^{m+n}$ is to hold for all values of m and n, we may replace m by zero. We then have

$$a^0 . a^n = a^{0+n} = a^n.$$

Therefore, by dividing each member by a^n, we get

$$a^0 = \frac{a^n}{a^n} = 1.$$

Therefore a quantity with zero index is equal to 1.

158. Meaning of Negative Exponents.—Since $a^m . a^n = a^{m+n}$ is to hold for all values of m and n, we may replace m by $-n$. We then have

$$a^{-n} . a^n = a^{-n+n} = a^0.$$

Therefore by dividing each member by a^n we get

$$a^{-n} = \frac{a^0}{a^n} = \frac{1}{a^n}.$$

Also, dividing each member by a^{-n} we get

$$a^n = \frac{a^0}{a^{-n}} = \frac{1}{a^{-n}}.$$

Hence a quantity with a negative exponent is equal to the reciprocal of the same quantity with the corresponding positive exponent.

COR. *Any factor may be transposed from the denominator to the numerator of an expression, and the reverse, by simply changing the sign of its exponent.*

159. The Index Law holds for all Rational Values of m **and** n.—Now that we have found what, in accordance with the index law, indices must mean for all rational values of m and n, we must show that, with these meanings, the three laws

$$a^m \,.\, a^n = a^{m+n}, \qquad (1)$$

$$(a^m)^n = a^{mn}, \qquad (2)$$

and

$$(ab)^n = a^n b^n \qquad (3)$$

must hold for all rational values of m and n.

I. To show that $a^m \,.\, a^n = a^{m+n}$ for all rational values of m and n.

1°. Let m and n be any fractions $\frac{p}{q}$ and $\frac{r}{s}$, in which p, q, r, and s are positive integers.

Then $a^{p/q} \,.\, a^{r/s} = \sqrt[q]{a^p} \cdot \sqrt[s]{a^r}$, by definition;

$$= \sqrt[qs]{a^{ps}} \,.\, \sqrt[qs]{a^{rq}}, \text{ by 152;}$$

$$= \sqrt[qs]{a^{ps+rq}}, \text{ by 132;}$$

$$= a^{\frac{ps+rq}{qs}}, \text{ by definition}$$

$$= a^{p/q+r/s} = a^{m+n}.$$

If either m or n is a positive integer while the other is a fraction with positive integers for its numerator and denominator, the integer may be expressed in a fractional form, and the demonstration just given will hold.

We know already that the law holds when m and n are positive integers. Therefore

$$a^m \, . \, a^n = a^{m+n}$$

for all positive rational values of m and n.

2°. Let m and n be essentially positive, either fractions or integers.

Then $$a^{-m} \, . \, a^{-n} = \frac{1}{a^m} \cdot \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n},$$

by definition.

And if $m - n$ be positive,

$$a^{(m-n)} \, . \, a^n = a^{(m-n)+n} = a^m,$$

and $$a^m \, . \, a^{-n} \, . \, a^n = a^m \, . \, \frac{1}{a^n} \cdot a^n = a^m.$$

$$\therefore \quad a \quad . \, a^{-n} \, . \, a^n = a^{m-n} \, . \, a^n.$$

$$\therefore \quad a^m \, . \, a^{-n} = a^{m-n}.$$

Hence if $m - n$ be negative, that is $n - m$ be positive,

$$a^m \, . \, a^{-n} = \frac{1}{a^n} \cdot \frac{1}{a^{-m}} = \frac{1}{a^{n-m}} = a^{m-n}.$$

$$\therefore \quad a^m \, . \, a^{-n} = a^{m-n}.$$

Therefore for all rational values of m and n

$$a^m \, . \, a^n = a^{m+n}.$$

Cor. Since $a^{m-n} \, . \, a^n = a^m$ for all rational values of m and n, it follows, by dividing both sides by a^n, that

$$a^m \div a^n = a^{m-n} \text{ for all rational values of } m \text{ and } n.$$

II. To prove that $(a^m)^n = a^{mn}$ for all rational values of m and n.

1°. Let m have any value whatever, and let n be a positive integer.

Then, by definition,

$$(a^m)^n = a^m \,.\, a^m \,.\, a^m \ldots \text{to } n \text{ factors}$$
$$= a^{+m+m+m \ldots \text{to } n \text{ terms}}$$
$$= a^{mn}.$$

2°. Let m have any value whatever, and let n be a fraction $\frac{p}{q}$, in which p and q are positive integers.

Then $(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$, by definition;

$$= \sqrt[q]{a^{mp}}, \text{ by II, 1°;}$$
$$= a^{\frac{mp}{q}}, \text{ by definition;}$$
$$= a^{m \,.\, \frac{p}{q}} = a^{mn}.$$

3°. Let n be any rational negative quantity and $= -p$.

Then $(a^m)^{-p} = \dfrac{1}{(a^m)^p} = \dfrac{1}{a^{mp}} = a^{-mp}.$

$$= a^{m \,.\, -p} = a^{mn}.$$

We know already that the law holds when m and n are positive integers.

Hence for all rational values of m and n

$$(a^m)^n = a^{mn}.$$

III. To prove $(ab)^n = a^nb^n$ for all rational values of n.

1°. Let n be any positive rational quantity which may be denoted by a fraction $\frac{p}{q}$, in which p and q are positive integers.

Then $(ab)^n = (ab)^{\frac{p}{q}} = \sqrt[q]{(ab)^p}$, by definition;

$$= \sqrt[q]{a^pb^p}, \text{ by (2).}$$

We know already that the law holds when m and n are positive integers. Therefore

$$a^m \,.\, a^n = a^{m+n}$$

for all positive rational values of m and n.

2°. Let m and n be essentially positive, either fractions or integers.

Then $$a^{-m} \,.\, a^{-n} = \frac{1}{a^m} \cdot \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n},$$

by definition.

And if $m - n$ be positive,

$$a^{(m-n)} \,.\, a^n = a^{(m-n)+n} = a^m,$$

and $$a^m \,.\, a^{-n} \,.\, a^n = a^m \,.\, \frac{1}{a^n} \cdot a^n = a^m.$$

$$\therefore \quad a \quad .\, a^{-n} \,.\, a^n = a^{m-n} \,.\, a^n.$$

$$\therefore \quad a^m \,.\, a^{-n} = a^{m-n}.$$

Hence if $m - n$ be negative, that is $n - m$ be positive,

$$a^m \,.\, a^{-n} = \frac{1}{a^n} \cdot \frac{1}{a^{-m}} = \frac{1}{a^{n-m}} = a^{m-n}.$$

$$\therefore \quad a^m \,.\, a^{-n} = a^{m-n}.$$

Therefore for all rational values of m and n

$$a^m \,.\, a^n = a^{m+n}.$$

Cor. Since $a^{m-n} \,.\, a^n = a^m$ for all rational values of m and n, it follows, by dividing both sides by a^n, that

$$a^m \div a^n = a^{m-n} \text{ for all rational values of } m \text{ and } n.$$

II. To prove that $(a^m)^n = a^{mn}$ for all rational values of m and n.

1°. Let m have any value whatever, and let n be a positive integer.

Then, by definition,

$$(a^m)^n = a^m \,.\, a^m \,.\, a^m \ldots \text{to } n \text{ factors}$$
$$= a^{+m+m+m \ldots \text{to } n \text{ terms}}$$
$$= a^{mn}.$$

2°. Let m have any value whatever, and let n be a fraction $\frac{p}{q}$, in which p and q are positive integers.

Then $(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$, by definition;

$$= \sqrt[q]{a^{mp}}, \text{ by II, 1°;}$$
$$= a^{\frac{mp}{q}}, \text{ by definition;}$$
$$= a^{m \cdot \frac{p}{q}} = a^{mn}.$$

3°. Let n be any rational negative quantity and $= -p$.

Then $(a^m)^{-p} = \dfrac{1}{(a^m)^p} = \dfrac{1}{a^{mp}} = a^{-mp}$.

$$= a^{m \cdot -p} = a^{mn}.$$

We know already that the law holds when m and n are positive integers.

Hence for all rational values of m and n

$$(a^m)^n = a^{mn}.$$

III. To prove $(ab)^n = a^n b^n$ for all rational values of n.

1°. Let n be any positive rational quantity which may be denoted by a fraction $\frac{p}{q}$, in which p and q are positive integers.

Then $(ab)^n = (ab)^{\frac{p}{q}} = \sqrt[q]{(ab)^p}$, by definition;

$$= \sqrt[q]{a^p b^p}, \text{ by (2).}$$

Also, $(a^n b^n)^q$ for all values of n

$$= a^n b^n \,.\, a^n b^n \,.\, a^n b^n \,.\,.\,.\,. \text{ to } q \text{ factors}$$

$$= a^n \,.\, a^n \,.\, a^n \,.\,.\,. \text{ to } q \text{ factors} \times b^n \,.\, b^n \,.\, b^n \,.\,.\,. \text{ to } q \text{ factors}$$

$$= a^{nq} \,.\, b^{nq}.$$

$$\therefore \quad a^n b^n = \sqrt[q]{a^{nq} b^{nq}}.$$

But, since $n = \frac{p}{q}$ or $nq = p$,

$$\therefore \quad a^n b^n = \sqrt[q]{a^p b^p} = (ab)^n.$$

Therefore for all positive rational values of n

$$(ab)^n = a^n b^n.$$

2°. Let n be any rational negative quantity and $= -p$, p being a positive integer. Then

$$(ab)^n = (ab)^{-p} = \frac{1}{(ab)^p} = \frac{1}{a^p b^p} = a^{-p} \,.\, b^{-p} = a^n b^n.$$

We know already that the law holds when m and n are positive integers.

Hence for all rational values of n

$$(ab)^n = a^n b^n.$$

EXERCISE XCII.

I.

Find the values of:

1. $64^{2/3}$. **2.** $16^{-3/2}$. **3.** $25^{-1/2}$.

4. $\left(\frac{1}{36}\right)^{1/2}$. **5.** $(100000)^{-3/5}$. **6.** $\left(\frac{1}{216}\right)^{-2/3}$.

Simplify:

7. $a^{3/2}b^{2/3} \times a^{1/2}b^{5/3}$. **8.** $(a^{-4/5}b^{1/4})^{2/3}$.

9. $(a^{-1/3}b^{-3})^{-4}$. **10.** $(a^3b^{3/4})^{-2/3}$.

Express with fractional or negative indices:

11. $\sqrt[5]{a} + \sqrt{b} + \sqrt[3]{x^4}$. **12.** $\sqrt{x^2y^3} + \sqrt[3]{ay^6}$.

13. $\sqrt[5]{a^3x^4} + \sqrt[3]{a^5y^2}$. **14.** $\sqrt[3]{x^2y^3z^5} + \sqrt[4]{a^3x^2}$.

Express without fractional or negative indices:

15. $x^{2/3} - z^{-3}$. **16.** $a^{-7}b^{-1/5}$.

17. $a^{3/4}b^{-2} - a^{-3/4}b^2$. **18.** $a^{-5}b^{-2/3} + 3a^{1/3}b^{-3/4}$.

Multiply:

I.

19. $x^{2/5} + y^{2/5}$ by $x^{2/5} - y^{2/5}$.

20. $1 + x^{1/5} + x^{2/5}$ by $1 - x^{1/5}$.

21. $a^{1/2} - a^{1/4}b^{1/4} + b^{1/2}$ by $a^{1/4} + b^{1/4}$.

II.

22. $x^{7/6} - x^{5/6} + x^{1/2} - x^{1/6} + x^{-1/6} - x^{-3/6}$ by $x^{5/6} + x^{3/6}$.

23. $x^3 + x^{3/2} + 1$ by $x^{-3} + x^{-3/2} + 1$.

24. $\frac{1}{27}a - \frac{1}{36}a^{2/3}b^{1/3} + \frac{1}{48}a^{1/3}b^{2/3} - \frac{1}{64}b$ by $\frac{1}{3}a^{1/3} - \frac{1}{4}b^{1/3}$.

Divide:

I.

25. $x^5 - y^5$ by $x^{5/2} - y^{5\ 2}$.

26. $x^{\frac{6n}{5}} - y^{\frac{6n}{5}}$ by $x^{\frac{2n}{5}} - y^{\frac{2n}{5}}$.

27. $x^4 + y^4$ by $x^{4/3} + y^{4/3}$.

28. $x^2 + 32y^{5/4}$ by $x^{2/5} + 2y^{1/4}$.

II.

29. $x^{4/3} - 2 + x^{-4/3}$ by $x^{2/3} - x^{-2/3}$.

30. $a^{1/2} - x$ by $a^{1/10} - x^{1/5}$.

31. $x^{3/2} - xy^{1/2} + x^{1/2}y - y^{3/2}$ by $x^{1/2} - y^{1/2}$.

CHAPTER XVIII.

ELIMINATION.

160. Simultaneous and Independent Equations.—Two or more equations are said to be *simultaneous* when they are satisfied by the same values of their unknown quantities.

The equations are *independent* when one cannot be derived from the other.

When an equation contains two or more unknown quantities, an indefinite number of values of their quantities may be found which will satisfy the equation.

e.g. Let $$3x + 4y = 18.$$

Transpose the term containing y and solve for x, and we have

$$x = \frac{18 - 4y}{3}.$$

If in this result we put $y = 3$, we get

$$x = \frac{18 - 12}{3} = 2,$$

and if we put $y = 4$, we get

$$x = \frac{18 - 16}{3} = \frac{2}{3}.$$

From this it appears that when an equation contains two unknown quantities, it can be satisfied by an unlimited number of pairs of values of these quantities, for by assign-ing any value whatsoever to one of these quantities we obtain an equation from which the other may be found.

In general terms, if

$$ax + by + c = 0,$$

we may give y any value m. Then

$$ax + bm + c = 0,$$

$$x = -\frac{bm + c}{a}.$$

The values $y = m$, and $x = -\frac{bm + c}{a}$, evidently satisfy the given equation. That is, in an equation of the first degree in x and y, to every value of y there is a corresponding value of x which will satisfy the equation.

161. Two Unknown Quantities require two Independent Equations for their Solution. — If, however, we have two independent equations in x and y, of the indefinite number of pairs of values of x and y which will satisfy either equation alone, there is only one pair which will satisfy both.

To obtain this pair of values, we may solve each equation for the same letter, and put the resulting values equal.

e.g. Let $$3x + 4y = 18, \qquad (1)$$

and $$2x + 5y = 19. \qquad (2)$$

From (1), we have $x = \frac{18 - 4y}{3}$,

and from (2), $$x = \frac{19 - 5y}{2}.$$

Now as we are seeking the value of x, which is the same in both equations, we may put

$$\frac{18 - 4y}{3} = \frac{19 - 5y}{2}. \qquad (3)$$

As this is a simple equation of the first degree in y, we may solve it for y, and then find the value of y which will give the same value of x in the two equations.

Solving (3) for y, we obtain $y = 3$.

Substituting this value of y in (1), we get

$$3x + 12 = 18.$$

$$\therefore \quad 3x = 6,$$

and $$x = 2.$$

The same value of x would have been obtained had we substituted the value of y in (2).

162. Elimination.—The general method of solving simultaneous equations of two or more unknown quantities is to get rid one after another of all the unknown quantities but one, so as to obtain an equation containing that unknown quantity alone; then to find the value of this quantity from the resulting equation, and afterwards of the remaining unknown quantities by substitution.

The process of getting rid of the unknown quantities is called *elimination.*

163. Three Methods of Elimination.—There are three general methods of elimination, known respectively as the methods by *comparison,* by *substitution,* and by *addition* or *subtraction.*

The first has been illustrated already. It consists in finding the value of the same unknown quantity from each of the two equations, and putting their values equal to each other.

e.g. $$2x + 3y = 19; \qquad (1)$$

$$3x + 2y = 16. \qquad (2)$$

From (1), $$x = \frac{19 - 3y}{2},$$

and from (2), $$x = \frac{16-2y}{3}.$$

$$\therefore \quad \frac{19-3y}{2} = \frac{16-2y}{3},$$

or $$57 - 9y = 32 - 4y,$$

or $$5y = 25.$$

$$\therefore \quad y = 5.$$

Substituting this value of y in (1), we get

$$2x + 15 = 19;$$

$$2x = 4;$$

$$x = 2.$$

The second method consists in finding the value of one of the unknown quantities from one of the equations, and substituting that value in the other.

e.g. $$2x + 3y = 19; \qquad (1)$$

$$3x + 2y = 16. \qquad (2)$$

From (1), we obtain $x = \frac{19-3y}{2}$.

Substituting this value for x in (2), we get

$$\frac{3(19-3y)}{2} + 2y = 16,$$

or $$57 - 9y + 4y = 32,$$

or $$5y = 25.$$

$$\therefore \quad y = 5.$$

Substitute this value in (1) or (2), and we find

$$x = 2.$$

The third consists in multiplying each of the equations by some number which will make the coefficients of one of the unknown quantities the same in both, and adding the equations when these coefficients have opposite signs in the two equations, and subtracting the equations when the coefficients have the same signs in both.

e.g.
$$2x + 3y = 19, \qquad (1)$$
$$3x + 2y = 16. \qquad (2)$$

Multiplying the first equation by 3 and the second by 2, we have
$$6x + 9y = 57, \qquad (3)$$
and
$$6x + 4y = 32. \qquad (4)$$

Subtracting (4) from (3), we get
$$5y = 25.$$
$$\therefore \quad y = 5,$$
and
$$x = 2.$$

The third method is the one usually employed, and the first is least used. The student should, however, be familiar with the use of all three.

EXERCISE XCIII.

Solve the following equations by each of the three methods:

I.

1. $3x + y = 9,$
   $4x - 2y = 2.$

2. $5x - 2y = 5,$
   $2x + y = 11.$

3. $8x - 6y = 10,$
   $2x - 7y = -3.$

4. $7x + 11y = 17,$
   $2x - 5y = 13.$

5. $9x - 5y = -1,$
   $3x + 6y = 15.$

6. $11x + 7y = -5,$
   $4x - 5y = -32.$

7. $2x + y = 4,$
   $7x + 8y = -13.$

8. $8x - y = -6,$
   $x + 8y = -17.$

9. $$\begin{aligned} 14x - 3y &= 45, \\ 6x + 17y &= 1. \end{aligned}$$

10. $$\begin{aligned} 5x - 7y &= 0, \\ 7x + 5y &= 74. \end{aligned}$$

Solve the following by any method of elimination:

11. $$\begin{aligned} \frac{2x}{3} + y &= 10, \\ x + \frac{y}{4} &= 5. \end{aligned}$$

12. $$\begin{aligned} 2x - \frac{y-3}{5} &= 4, \\ 3y &= 9 - \frac{x-2}{3}. \end{aligned}$$

13. Find the first four terms of the square root of $1 - x$.

14. Find the cube root of $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

II.

15. $$\begin{aligned} \frac{3x}{5} + \frac{y}{4} &= 13, \\ \frac{x}{3} - \frac{y}{8} &= 3. \end{aligned}$$

16. $$\begin{aligned} \frac{2x-3y}{13} - \frac{3x-5y}{10} &= \frac{1}{10}, \\ \frac{x+4y}{11} + \frac{5x-4y}{7} &= 2. \end{aligned}$$

17. $$\begin{aligned} \frac{1}{3}(x+y) &= \frac{1}{5}(x-y), \\ \frac{1}{7}(x+y) &= \frac{5}{35}(x-y) - \frac{1}{7}. \end{aligned}$$

18. $$\begin{aligned} x(y+7) &= y(x+1), \\ 2x &= 3y - 19. \end{aligned}$$

19. $$\begin{aligned} 2x - \frac{y+15}{4} &= 4 + \frac{3y-2x}{5}, \\ \frac{8y-1}{2} - \frac{8-x}{3} &= 20 - \frac{2y-7}{2}. \end{aligned}$$

20. $$\begin{aligned} ax &= by, \\ x + y &= c. \end{aligned}$$

21. $$\begin{aligned} x + y &= 1, \\ ax + by &= c. \end{aligned}$$

22. $$\begin{aligned} x + y &= a + b, \\ \frac{x+a}{y+b} &= \frac{b}{a}. \end{aligned}$$

23. $$\begin{aligned} x + y &= a + b, \\ ax + by &= a^2 + b^2. \end{aligned}$$

24. $$\frac{a}{x+a} = \frac{b}{y+b},$$
$$ax + by = c.$$

25. $$y = \frac{x+a}{2} + \frac{b}{3},$$
$$x = \frac{y+b}{2} + \frac{a}{3}.$$

EXERCISE XCIV.

Solve the following problems by two unknown quantities:

Ex. 1. Find two numbers whose sum is 17 and whose difference is 3.

Let $x =$ the larger number,

and $y =$ the smaller number.

Then $$x + y = 15, \qquad (1)$$

and $$x - y = 3. \qquad (2)$$

Add equation (2) to equation (1), and we get

$$2x = 18.$$
$$\therefore \quad x = 9.$$

Subtract equation (2) from equation (1), and we get

$$2y = 12.$$
$$\therefore \quad y = 6.$$

Hence the numbers are 9 and 6.

2. Find a fraction such that when 5 is added to its numerator and 2 is added to its denominator, its value is 3/4; and if 1 be subtracted from its numerator and 5 be subtracted from its denominator, its value is 3/5.

Let $x =$ the numerator,

and $y =$ the denominator.

Then $$\frac{x+5}{y+2} = \frac{3}{4},$$

and
$$\frac{x-1}{y-5}=\frac{3}{5}.$$

Clearing of fractions, we have

$$4x+20=3y+\ 6,\quad \text{or}\quad 4x-3y=-14, \quad (1)$$

and

$$5x-\ 5=3y-15,\quad \text{or}\quad 5x-3y=-10. \quad (2)$$

Subtracting (1) from (2), we get

$$x=4.$$

$$\therefore\ 16-3y=-14,$$

or
$$3y=30.$$

$$\therefore\ y=10.$$

Hence the fraction is 4/10.

3. There is a number composed of two digits. The sum of the digits is 7, and if 9 be added to the number the digits will be reversed.

Let x = digit in the tens' place,

and y = digit in the units' place.

Then the number is $10x+y$. When the digits are reversed the number is $10y+x$.

Then
$$x+y=7, \quad (1)$$

$$10x+y+9=10y+x,$$

or
$$9x-9y=-9,$$

or
$$x-y=-1. \quad (2)$$

Adding (1) and (2), we get

$$2x=6.$$

$$\therefore\ x=3.$$

Subtracting (2) from (1), we get

$$2y = 8.$$

$$\therefore \quad y = 4.$$

Hence the number is 34.

I.

1. The sum of two numbers is 8 and their difference is 6. What are the numbers?

2. There is a certain fraction, such that if its numerator be increased by 4, its value is 4/5; and if its denominator be increased by one, its value is 1/2. What is the fraction?

3. A certain number of two digits is equal to five times the sum of its digits, and if 9 be added to the number, its digits will be reversed.

4. A number consists of two digits whose difference is 1; if it be diminished by the sum of its digits, the digits will be reversed. What is the number?

5. Eight years ago A was five times as old as B, and in two years he will be three times as old. What are their present ages?

6. A alone does 3/5 of a piece of work in 30 days, and then with B's help finishes it in 10 days. In what time could each do it alone?

II.

7. A man buys 8 lbs. of tea and 5 lbs. of sugar for \$2.39; and at another time 5 lbs. of tea and 8 lbs. of sugar for \$1.64, the price being the same as before. What were the prices?

8. Two vessels contain mixtures of wine and water. In the first there are three times as much wine as water, and in the second five times as much water as wine. How many gallons must be drawn from each vessel to fill a third, which

holds 7 gallons, with a mixture which shall be half wine and half water?

9. Two vessels contain mixtures of wine and water. In the first there are 4 gallons of wine to 3 gallons of water, and in the second there are 5 gallons of water to 2 gallons of wine. How many gallons must be drawn from each vessel to fill a third, which holds 12 gallons, with a mixture which shall be 1/3 wine?

10. A man buys 2 lbs. of tea and 6 lbs. of sugar for 81 cents, and at another time 4 lbs. of tea and 9 lbs. of sugar for \$1.51½, the price being the same as before. What were the prices?

164. To Solve for n Unknown Quantities requires n Independent Equations.—We have seen that we need two simultaneous equations in order to find the value of two unknown quantities. Similarly, we need three independent simultaneous equations in order to find the value of three unknown quantities, and n independent simultaneous equations in order to find the value of n unknown quantities.

With three unknown quantities, we first combine any pair of the three equations so as to eliminate one of the unknown quantities, and then another pair so as to eliminate the same unknown quantity. We shall then have two equations with two unknown quantities. Then we combine these two equations so as to eliminate one of the remaining unknown quantities, and thus obtain one equation with a single unknown quantity. From this we obtain the value of this quantity, and then, by successive substitution, the values of the other two.

e.g.
$$6x + 2y - 5z = 13, \qquad (1)$$
$$3x + 3y - 2z = 13, \qquad (2)$$
$$7x + 5y - 3z = 26. \qquad (3)$$

Eliminate y from (1) and (2) by subtraction, multiplying (1) by 3 and (2) by 2.

$$18x + 6y - 15z = 39,$$
$$6x + 6y - 4z = 26.$$
$$\therefore \quad 12x - 11z = 13. \qquad (4)$$

Next eliminate y from (1) and (3) by subtraction, multiplying (1) by 5 and (3) by 2.

$$30x + 10y - 25z = 65,$$
$$14x + 10y - 6z = 52,$$
$$\therefore \quad 16x - 19z = 13. \qquad (5)$$

Next eliminate x from (4) and (5) by subtraction, multiplying (4) by 4 and (5) by 3.

$$48x - 44z = 52,$$
$$48x - 57z = 39.$$
$$\therefore \quad 13z = 13,$$

and $$z = 1.$$

Remember that the equations may be combined in any order, and that those combinations are best which will produce the required result in the simplest and most direct way.

EXERCISE XCV.

I.

1. $x + 2y + 2z = 16,$
$2x + y + z = 11,$
$3x + 4y + z = 22.$

2. $x + 3y + 4z = 7,$
$x + 2y + z = 0,$
$2x + y + 2z = 6.$

3. $x + 4y + 3z = 14,$
$3x + 3y + z = 21,$
$2x + 2y + z = 13.$

4. $3x - 2y + z = 10,$
$2x + 3y + z = 18,$
$x + y + z = 5.$

5. $3x + 4y = 0,$
$2y - 4z = -14,$
$x + 3y + 2z = -1.$

6. $5x + 2y = 8\frac{5}{6},$
$3z - y = 1\frac{5}{6},$
$8x - 10z = 3\frac{2}{3}.$

II.

7. $x - \frac{y}{6} = 12,$
$y - \frac{z}{5} = 14,$
$z - \frac{x}{3} = 15.$

8. $\frac{y+z}{5} = \frac{z+x}{4} = \frac{x+y}{3},$
$x + y + z = 18.$

9. $\frac{y-z}{4} = \frac{y-x}{3} = 5z - 3x,$
$y + z = 3x - 3.$

10. $x + 16 = \frac{5y}{3} + 14$
$= 3z + 9$
$= 46 - 2(y + z).$

11. Multiply $3x^{\frac{n}{2}} + 4x^{\frac{n}{3}} - 5x^{\frac{n}{4}}$ by $x^{\frac{n}{3}} - 2x^{\frac{n}{4}}.$

12. Divide $x^{\frac{5n}{2}} - x^{\frac{5m}{2}}$ by $x^{\frac{n}{2}} - x^{\frac{m}{2}}.$

13. Square $2x^{1/3} - 3x^{2/3} + 4x.$

NOTE. — When there are more than three unknown quantities, the process of elimination is similar.

EXERCISE XCVI.

Work the following examples by three unknown quantities:

I.

1. The sums of three numbers, taken two by two, are 20, 29, and 27. What are the numbers?

2. The sum of three numbers is 78, 1/3 the difference of the first and second is 4, and 1/3 the difference of the first and third is 7. What are the numbers?

3. A person bought three silver watches. The price of

the first, with 1/3 the price of the other two, was 40 dollars, the price of the second, with 1/4 the price of the other two, was 42 dollars, and the price of the third, with 1/2 the price of the other two, was 44 dollars. What was the price of each watch?

4. A, B, and C together have \$2100. Were B to give A 300 dollars, A would have 380 dollars more than B, and if B received 200 dollars from C, they would both have the same sum. How many dollars has each?

5. A, B, and C can perform a piece of work in 20 days, A and B in 30 days, and B and C in 40 days. How long would it take each to do it alone?

6. A and B together can do a piece of work in 6 days, B and C in $6\frac{2}{3}$ days, and A and C in $5\frac{5}{11}$ days. How long would it take each to do it alone?

7. A number is composed of three digits whose sum is 9. The digit in the units' place is twice the digit in the hundreds' place, and if 198 be added to the number, the digits will be reversed. What is the number?

8. A number is composed of three digits whose sum is 10. The middle digit is equal to the sum of the other two, and if 99 be added to the number its digits will be reversed. What is the number?

9. A number is composed of three digits whose sum is 14. Seven times the second digit exceeds the sum of the other two by 2, and if the first and second digit be interchanged the resulting number will be less than the given number by 180. What is the number?

II.

10. A and B can do a piece of work in r days; B and C in s days; and A and C in t days. In how many days can each do it alone?

Do the following by two unknown quantities:

24. A crew can row 10 miles in 50 minutes down stream and 12 miles in an hour and a half up stream. What is the rate in miles per hour of the stream, and of the crew in still water?

Let $x =$ the rate in miles per hour of the crew in still water,

and $y =$ the rate in miles per hour of the current.

$\therefore\ x + y =$ the rate in miles per hour of the crew down stream,

and $x - y =$ the rate in miles per hour of the crew up stream.

Since the number of miles rowed, divided by the rate in miles per hour, is equal to the time in hours, we have

$$\frac{10}{x+y} = \frac{5}{6},$$

and

$$\frac{12}{x-y} = \frac{3}{2}.$$

$$\therefore\quad x = 10, \quad \text{and} \quad y = 2.$$

25 A crew can row 20 miles down stream in an hour and 20 minutes, and 18 miles up stream in 2 hours. What is the rate of the current in miles per hour, and what is the rate of the crew in still water?

26. Two trains start from two stations at the same time, and each proceeds at a uniform rate towards the other station. They meet in twelve hours, and one has gone 108 miles farther than the other, and then if they continue to travel at the same rate they will finish their journey in 9 hours and 16 hours respectively. What is the rate of the trains, and the distance between the towns?

27. Two trains start from two stations at the same time, and each proceeds at a uniform rate towards the other station. They meet in six hours, and one has gone 30 miles farther than the other, and then if they continue to travel at the same rate, they will finish the journey in 7 hours and 12 minutes, and in 5 hours, respectively. What is the rate of the trains, and what is the distance between the towns?

28. A certain number of persons paid a bill. Had there been 10 more, each would have paid $2 less, and had there been 5 less, each would have paid $2.50 more. How many were there, and how much did each pay?

29. A sum of money is divided equally between a certain number of persons. Had there been m more, each would have received a dollars less; if n less, each would have received b dollars more. How many persons were there, and how much did each receive?

CHAPTER XIX.

QUADRATIC EQUATIONS.

A. SURD AND IMAGINARY FACTORS.

165. Trinomial and Binomial Quadratics.—A *complete quadratic expression* in one unknown quantity contains three terms, one containing the square of the unknown quantity, one containing the first power of the unknown quantity, and the third without the unknown quantity. The most general form of such an expression is

$$ax^2 + bx + c.$$

The term which does not contain the unknown quantity is called the *constant term* of the expression, and the complete expression is called a *trinomial* quadratic.

When the term containing the first power of the unknown quantity is wanting, the expression becomes a binomial, and is called an *incomplete* or a *binomial quadratic* expression.

166. Factors of $x^2 + c$.—Every binomial quadratic of the form

$$x^2 + c$$

may be factored as the difference of two squares, since it may be written in the form

$$x^2 -- (-c).$$

The factors will be

$$x + \sqrt{-c} \text{ and } x - \sqrt{-c}.$$

1°. When c represents a positive number, these factors are imaginary.

2°. When c represents a negative number which is not a perfect square, the factors are surd.

3°. When c represents a negative number which is a perfect square, the factors are rational.

e.g. 1. $x^2+5=x^2-(-5)=(x-\sqrt{-5})(x+\sqrt{-5}),$

$$x^2+4=x^2-(-4)=(x-\sqrt{-4})(x+\sqrt{-4})$$
$$=(x-2\sqrt{-1})(x+2\sqrt{-1}).$$

2. $x^2+(-3)=x^2-3=(x-\sqrt{3})(x+\sqrt{3}).$

3. $x^2+(-9)=x^2-9=(x-3)(x+3).$

When the expression is in the form

$$ax^2+c,$$

a may be taken out as a factor first, and then the remaining factor may be factored as the difference of two squares. Thus,

$$ax^2+c=a\left(x^2+\frac{c}{a}\right)=a\left(x^2-\left(-\frac{c}{a}\right)\right)$$
$$=a\left(x-\sqrt{-\frac{c}{a}}\right)\left(x+\sqrt{-\frac{c}{a}}\right).$$

e.g. 1°. $3x^2+6=3(x^2+2)=3(x^2-(-2))$

$$=3(x-\sqrt{-2})(x+\sqrt{-2}).$$

2°. $4x^2+(-20)=4(x^2-5)$

$$=4(x+\sqrt{5})(x-\sqrt{5}).$$

3°.

$$\begin{aligned} 5x^2 + (-20) &= 5(x^2 - 4) \\ &= 5(x-2)(x+2). \end{aligned}$$

$$\begin{aligned} 3x^2 + (-5) &= 3\left(x^2 - \frac{5}{3}\right) \\ &= 3(x - \sqrt{5/3})(x + \sqrt{5/3}). \end{aligned}$$

$$\begin{aligned} 4x^2 + (-3) &= 4(x - \sqrt{3/4})(x + \sqrt{3/4}) \\ &= 4\left(x - \frac{\sqrt{3}}{2}\right)\left(x + \frac{\sqrt{3}}{2}\right) \\ &= 4\left(x - \frac{1}{2}\sqrt{3}\right)\left(x + \frac{1}{2}\sqrt{3}\right). \end{aligned}$$

EXERCISE XCVII.

Factor the following quadratic expressions:

1. $x^2 + 5$.
2. $x^2 - 7$.
3. $x^2 + 16$.
4. $3x^2 - 9$.
5. $5x^2 - 25$.
6. $7x^2 + 14$.
7. $2x^2 - 3$.
8. $3x^2 + 5$.
9. $5x^2 - 2$.
10. $4x^2 + 3$.
11. $3x^2 - 4$.
12. $7x^2 + 5$.

167. Factors of a Trinomial Quadratic. — Every trinomial quadratic expression may be factored as the difference of two squares.

We first take out the coefficient of the square of the unknown quantity, and after the second term of the expression we add and subtract the square of half the coefficient of the first power of the unknown quantity. This will give a polynomial of five terms, the first three of which will be a perfect square. The last two terms must be combined into one with a minus sign before it. The factors will both be real when this term is essentially positive, rational when it is an exact square, and surd when it is not an exact square. The factors will both be imaginary when this last term is essentially negative.

e.g. 1°. Factor $3x^2 + 15x + 18$.

First, we have $3x^2 + 15x + 18 = 3(x^2 + 5x + 6)$.

Then, after the second term of the second factor, add and subtract $(5/2)^2$, and we get

$$x^2 + 5x + \frac{25}{4} - \frac{25}{4} + 6 = x^2 + 5x + \frac{25}{4} - \frac{1}{4}$$

$$= \left(x + \frac{5}{2} + \frac{1}{2}\right)\left(x + \frac{5}{2} - \frac{1}{2}\right)$$

$$= (x + 3)(x + 2).$$

$$\therefore \quad 3x^2 + 15x + 18 = 3(x + 3)(x + 2).$$

2°. Factor $ax^2 + bx + c$.

First, $ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.

Then $$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}$$

$$= x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2}$$

$$= \left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right).$$

$$\therefore \quad ax^2 + bx + c$$

$$= a\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right).$$

Whether these factors be rational, surd, or imaginary depends upon the radical $\sqrt{b^2 - 4ac}$.

If the quantity under the radical be positive, the factors will be real.

If also the quantity under the radical be a perfect square, the factors will be rational; and if this quantity be not a perfect square, the factors will be surd.

If the quantity under the radical be 0, the factors will be equal.

If the quantity under the radical sign be negative, the factors will be imaginary.

Since $ax^2 + bx + c$ is the general form of a trinomial quadratic expression,

$$a\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right)$$

may serve as a formula by which all such expressions may be factored.

e.g. Factor $3x^2 + 4x + 5$.

Comparing this with $ax^2 + bx + c$, we see that $a = 3$, $b = 4$, and $c = 5$.

Substituting these values in the formula, we get

$$3\left(x + \frac{4 + \sqrt{16 - 60}}{6}\right)\left(x + \frac{4 - \sqrt{16 - 60}}{6}\right),$$

or $$3\left(x + \frac{4 + \sqrt{-44}}{6}\right)\left(x + \frac{4 - \sqrt{-44}}{6}\right),$$

or $$3\left(x + \frac{2 + \sqrt{-11}}{3}\right)\left(x + \frac{2 - \sqrt{-11}}{3}\right).$$

In this case the binomial factors are imaginary.

EXERCISE XCVIII.

Factor the following trinomial quadratic expressions by the formula:

I.

1. $4x^2 + 7x - 5$.
2. $2x^2 + 5x + 2$.

3. $5x^2 - 6x - 7$. 4. $6x^2 - 4x - 3$.

5. $4x^2 + 3x + 6$. 6. $2x^2 + 10x + 8$.

7. A man bought 175 acres of land for 6000 dollars. For a part of it he paid 40 dollars an acre, and for the remainder 25 dollars an acre. How many acres in each part?

II.

8. $7x^2 + 9x + 2$. 9. $7x^2 + 28x - 7$.

10. $3x^2 + 7x - 6$. 11. $4x^2 - 24x + 12$.

12. $15x^2 + x - 6$. 13. $3x^2 - 10x + 6$.

14. A man bought m acres of land for s dollars. For a part of it he paid a dollars an acre, and for the remainder b dollars an acre. How many acres were there in each part?

15 Solve $\dfrac{\sqrt{4x+1} + 2\sqrt{x}}{\sqrt{4x+1} - 2\sqrt{x}} = 9$.

16. Solve $\dfrac{\sqrt{x+a} + \sqrt{x}}{\sqrt{x+a} - \sqrt{x}} = c$.

B. ROOTS OF AN EQUATION.

168. Quadratic Equations.—A *quadratic equation* of one unknown quantity is an equation whose first member is a complete or an incomplete quadratic expression in that letter after the equation has been reduced to its simplest form and all its terms have been transposed into its first member. After reduction and transposition the equation takes either the form

$$ax^2 + bx + c = 0 \qquad (1)$$

or

$$ax^2 + c = 0. \qquad (2)$$

169. Roots of an Equation.—A *root* of an equation is a value of its unknown quantity which reduces its first member to zero, after it has been reduced to the form of (1) or (2).

170. Solution of a Quadratic Equation.—To *solve* a quadratic equation is to find its roots, or the values of its unknown quantity which will reduce to zero the first member of the equation after it has been brought into its type form.

Since a product is zero when any one of its factors is zero, the values of its unknown quantity which will reduce to zero the factors of the first member after it has been brought into its type form are the roots of the equation. Hence, to solve a quadratic equation, reduce it to the type form, factor its first member, equate each factor to zero, and solve for its unknown quantity.

e.g. Solve $x^2 - 6x = -8$.

Reduced to the type form this becomes

$$x^2 - 6x + 8 = 0,$$

or $$(x - 2)(x - 4) = 0.$$

Put $$x - 2 = 0,$$

and we have $$x = 2.$$

Put $$x - 4 = 0,$$

and we have $$x = 4.$$

Hence 2 and 4 are the roots of the equation, for either of these values of x will reduce the first member of the type form to zero.

We have seen that every quadratic expression in one letter may be resolved into two factors of the first degree in that letter. Hence every quadratic equation has two roots. Moreover a product cannot vanish unless one of its factors vanishes. Therefore a quadratic equation has only two roots. These roots will be rational when the factors of the first member of the reduced form are rational, and equal when the factors are identical; surd when the factors are surds; and imaginary when the factors are imaginary.

e.g. 1. Solve $x^2 - 6x = -9$.

When reduced to the type form this becomes

$$x^2 - 6x + 9 = 0.$$

$$\therefore \quad (x - 3)(x - 3) = 0.$$

Therefore the roots are 3 and 3, and are rational and equal.

The roots of a quadratic equation are equal when the first member of the reduced form is a perfect square.

2. Solve $x^2 - 11x = -28$.

Transposing, we have

$$x^2 - 11x + 28 =$$

$$\therefore \quad (x - 4)(x - 7) = 0.$$

$$\therefore \quad x = 4 \text{ or } 7.$$

Therefore the roots of the equation are 4 and 7, and are rational and unequal.

3. Solve $x^2 - 4x + 1 = 0$.

Bring the first member of this equation under the case of the difference of two squares by adding and subtracting the square of half the coefficient of x, and we have

$$x^2 - 4x + 4 - 3 = 0.$$

$$\therefore \quad (x - 2 + \sqrt{3})(x - 2 - \sqrt{3}) = 0.$$

$$\therefore \quad x = 2 - \sqrt{3} \text{ and } 2 + \sqrt{3}.$$

Therefore the roots of the equation are $2 - \sqrt{3}$ and $2 + \sqrt{3}$, and are surd and unequal.

4. Solve $x^2 - 6x + 11 = 0$.

Bring the first member under the case of the difference of two squares by adding and subtracting the square of half the coefficient of x, and we have

$$x^2 - 6x + 9 - (-2) = 0.$$

$$\therefore \quad (x - 3 + \sqrt{-2})(x - 3 - \sqrt{-2}) = 0.$$

$$\therefore \quad x = 3 - \sqrt{-2} \text{ and } 3 + \sqrt{-2}.$$

Therefore the roots of the equation are $3 - \sqrt{-2}$ and $3 + \sqrt{-2}$, and are imaginary.

EXERCISE XCIX.

Solve the following quadratic equations by factoring:

I.

1. $x^2 - 3x - 18 = 0.$

2. $x^2 + 4x = 45.$

3. $x^2 + 13x + 25 = -15.$

4. $x^2 - 12x - 5 = -40.$

5. $x^2 + 4x + 20 = 4 - 4x.$

6. $x^2 - 5x = 5x - 25.$

7. $x^2 - 3 = 6.$

8. $x^2 - 2a^2 = -a^2.$

II.

9. $x^2 + (a+b)x + ab = 0.$

10. $x^2 + (a-b)x - ab = 0.$

11. $2x^2 + x - 3 = 0.$

12. $3x^2 + 5x = 12.$

13. $15x^2 + 14x = 8.$

14. $7x^2 + 15x = -8.$

15. $12 + 2x^2 = 11x.$

16. $-3x^2 + 17x = 20.$

171. Formation of Quadratic Equations.—Since we obtain the roots of a quadratic equation by equating to zero each factor of the first member of its type form, it follows that these factors are the unknown quantity of the equation minus each of its roots in turn.

Hence we may obtain a quadratic equation in x whose roots shall have given values by using as factors x minus each of the given roots in turn, finding the product of these factors, and equating this product to zero.

e.g. 1. Form the quadratic equation in x whose roots are 4 and -7. The factors of the first member of its type form will be

$$(x-4) \quad \text{and} \quad (x+7).$$

$$\therefore \quad (x-4)(x+7)=0,$$

or

$$x^2+3x-28=0,$$

which is the required equation.

2. Form the quadratic equation in x whose roots are

$$3+\sqrt{5} \text{ and } (3-\sqrt{5}).$$

Here the factors are $x-(3+\sqrt{5})$ and $x-(3-\sqrt{5})$.

$$\therefore \quad \left(x-(3+\sqrt{5})\right)\left(x-(3-\sqrt{5})\right)=0,$$

or

$$x^2-6x+4=0.$$

EXERCISE C.

Form the quadratic equations in x whose roots have the following values:

I.

1. 3 and 7. **2.** 4 and -6. **3.** -7 and -1.

4. 0 and 2. **5.** -9 and 0. **6.** 7 and -7.

7. -8 and -8. **8.** 11 and 11. **9.** 3 and 3/4.

10. $\dfrac{3+\sqrt{7}}{6}$ and $\dfrac{3-\sqrt{7}}{6}$. **11.** $\dfrac{7+\sqrt{5}}{8}$ and $\dfrac{7-\sqrt{5}}{8}$.

12. $4+\sqrt{-6}$ and $4-\sqrt{-6}$.

II.

13. $-2/3$ and $-5/6$. **14.** 3/2 and -1.

15. 7 and $-2/5$. **16.** $3+\sqrt{5}$ and $3-\sqrt{5}$.

17. $2+\sqrt{8}$ and $2-\sqrt{8}$. **18.** $5+\sqrt{3}$ and $5-\sqrt{3}$.

19. $9+\sqrt{-4}$ and $9-\sqrt{-4}$.

20. $\dfrac{7+\sqrt{-3}}{10}$ and $\dfrac{7-\sqrt{-3}}{10}$.

21. $\dfrac{11+\sqrt{-5}}{12}$ and $\dfrac{11-\sqrt{-5}}{12}$.

I.

22. Reduce $-\dfrac{b^2}{4a^2}+\dfrac{c}{a}$ to a single negative fraction.

23. Reduce $\dfrac{2x^2}{x+2}-x$ to a single fraction.

24. Reduce $2x-\dfrac{2x^2-3x}{x-2}$ to a single fraction.

EXERCISE CI.

I.

1. $(x-2)^2-1=\dfrac{8}{3}(x+2)$.

2. $2x^2+2(x+1)^2=\dfrac{13}{3}x(x+1)$.

3. $(2-x)^2-(2-x)(x-3)+(x-3)^2=1$.

4. $x+\dfrac{1}{x}=4\frac{1}{4}$. 5. $\dfrac{x-1}{x}+\dfrac{x}{x-1}=2\frac{1}{2}$.

6. $\dfrac{x+2}{x+1}+\dfrac{x+1}{x+2}=\dfrac{26}{5}$. 7. $\dfrac{4}{x-3}-\dfrac{3}{x+5}=\dfrac{17}{10}$.

II.

8. $\dfrac{x}{2x+1}+\dfrac{12-x}{x+5}=\dfrac{13}{6}$. 9. $ax^2-(a^2+1)x+a=0$.

10. $\frac{4x-3}{3x-7} = 3 + \frac{2x-3}{x-1}$. **11.** $\frac{x-1}{x+1} + \frac{x+1}{x-1} = \frac{5x}{x^2-1}$.

12. $3\frac{x-1}{x+1} - 2\frac{x+1}{x-1} = 5$. **13.** $\frac{2x-1}{2x+1} + \frac{13}{11} = \frac{3x+5}{3x-5}$.

EXERCISE CII.

1. Solve $\sqrt{(3-4x)} + \sqrt{(2+5x)} = \sqrt{(5+x)}$.

Transposing, we have

$$\sqrt{(3-4x)} + \sqrt{(2+5x)} - \sqrt{(5+x)} = 0.$$

Multiplying by the conjugate, we have

$$3 - 4x + 2\sqrt{(3-4x)}\sqrt{(2+5x)} + 2 + 5x - 5 - x = 0,$$

or $$2\sqrt{(3-4)}\sqrt{(2+5x)} = 0.$$

$$\therefore \quad \sqrt{(3-4x)}\sqrt{(2+5x)} = 0.$$

$$\therefore \quad (3-4x)(2+5x) = 0.$$

$$\therefore \quad x = 3/4 \text{ and } -2/5.$$

I.

2. $\sqrt{(5-7x)} + \sqrt{(4x-3)} = \sqrt{(2-3x)}$.

3. $\sqrt{(x+a)} + \sqrt{(x-b)} = \sqrt{(2x+a-b)}$.

4. $\sqrt{(3+4x)} - \sqrt{(4+2x)} = \sqrt{(7+6x)}$.

5. $\sqrt{(2-3x)} - \sqrt{(7+x)} = \sqrt{(5+4x)}$.

6. $\sqrt{(x^2+3x-54)} - \sqrt{(x^2-3x-54)} = \sqrt{(2x^2-108)}$.

II.

7. $\sqrt{(x^2+4x-60)} - \sqrt{(x^2-4x-60)} = \sqrt{(2x^2-120)}$.

8 $\sqrt{(12x^2-x-6)} - \sqrt{(12x^2+x-6)} = \sqrt{(24x^2-12)}$.

9. $\sqrt{(36x^2+24x+1)} + \sqrt{(36x^2-24x+1)} = \sqrt{(72x^2+2)}$.

172. Interpretation of Solutions.—

Ex. 1. A man sold a watch for 24 dollars and lost as many per cent as there were dollars in the cost of the watch. What was the cost of the watch ?

Let $x =$ the cost in dollars.

Then $x =$ the lost per cent,

and $x \cdot \frac{x}{100} = \frac{x^2}{100} =$ loss in dollars.

Also, $x - 24 =$ loss in dollars.

$$\therefore \quad \frac{x^2}{100} = x - 24.$$

Solving this, we get

$$x = 60 \text{ or } 40.$$

That is, the cost was either 60 dollars or 40 dollars; for either of these values satisfies the conditions of the problem.

2. A farmer bought a number of sheep for 80 dollars. Had he bought 4 less for the same money, they would have cost him 1 dollar apiece more. How many did he buy ?

Let $x =$ the number bought.

Then $\frac{80}{x} =$ the price per head in dollars,

and $\frac{80}{x-4} =$ the price per head, if there had been 4 more.

$$\therefore \quad \frac{80}{x} = \frac{80}{x-4} - 1.$$

Solving this equation, we get $x = -16$ or $+20$.

Only the positive value will satisfy the condition of the problem. Therefore the number of sheep was 20.

In solving problems which involve quadratics, there will be, in general, two values of the unknown quantity, both of which may not answer to the conditions of the problem. This is due to the fact that the symbolic language of algebra is more general than ordinary language. So that the equations which correctly represent the conditions of the oral problems may represent other allied conditions also. The equation is entirely general, while the verbal statement is more or less restricted. Verbal statements are supposed generally to be restricted to an arithmetical sense which admits only of positive numbers; while there is no restriction on the numerical symbols of an algebraic equation.

A little consideration will enable the pupil to determine whether or not both values of the unknown quantity will fit the conditions of the verbal problem, and which one to select in case both will not answer. It will be found also a valuable exercise to interpret negative results when possible.

Thus in the last example, to buy − 16 sheep has no meaning in the arithmetical sense, but algebraically it means to sell 16 sheep.

To buy 4 less than − 16 would mean to sell 20.

In the first case he would have paid − $5 a head for the sheep; that is, he would have sold them for $5 a head. In the second case he would have bought them for 1 dollar more a head, or for − 4 dollars; that is, he would have sold them for 4 dollars a head.

When one of the solutions is negative the wording of the problem may be changed, in general, so as to make that solution positive and arithmetically true.

Thus, a farmer sold a number of sheep for 80 dollars. Had he sold 4 more for the same money he would have received 1 dollar a head less for the sheep. How many did he sell?

e.g. 1. The length of a field is 12 rods and its breadth is

10 rods. How many rods must be added to the length of the field that the area may be 100 square rods?

Let x = number of rods to be added.

Then $$(12 + x)10 = 100.$$
$$10x = 100 - 120.$$
$$x = -2.$$

Hence the number of rods to be added to the length is -2. This is possible algebraically, but impossible arithmetically.

In the arithmetical sense, to add means to increase; and as the area of the field at first was 120 square rods, no increase in its length could make its area 100 square rods.

But algebraically, to add -2 means to subtract 2 arithmetically; and were the statement, "How many rods must be subtracted from the length of the field to make its area 100 square rods?" we should find the 2 to be positive and, therefore, true in the arithmetical sense.

e.g. 2. A's age is 40, and B's 35. How many years hence will A's age be twice B's?

Let x = number of years hence.

Then $$40 + x = 2(35 + x).$$
$$x = -30.$$

This is impossible arithmetically, but perfectly true algebraically, since -30 years hence means 30 years ago.

Had the question been worded, "How many years *ago* would A's age have been twice B's?" the solution would have been positive and the problem would have been possible arithmetically.

When imaginary results are obtained in the solution of a problem, there is either an impossibility in the conditions of the problem or an error in the formation of the equation.

e.g. Divide 12 into two parts whose product shall be 37.

Let x denote one part.

$$
\begin{aligned}
\text{Then}\quad & x(12 - x) = 37. \\
& 12x - x^2 = 37. \\
& x^2 - 12x + 37 = 0. \\
& x^2 - 12x + 36 - 1 = 0. \\
& x - 6 \pm \sqrt{-1} = 0. \\
& x = 6 - \sqrt{-1}, \text{ or } 6 + \sqrt{-1}. \\
& 12 - x = 6 + \sqrt{-1}, \text{ or } 6 - \sqrt{-1}.
\end{aligned}
$$

That is, 12 cannot be divided into two parts whose product is 37.

EXERCISE CIII.

I.

1. Find two numbers whose difference is 7 and whose sum multiplied by the greater is 345.

2. Find three consecutive numbers whose sum is equal to 3/5 the product of the last two.

3. Find two numbers whose difference is 12 and whose sum multiplied by the greater is 560.

4. Find three consecutive numbers whose sum is equal to 3/7 the product of the last two.

5. Find two numbers whose sum is 6 and the sum of whose cubes is 72.

6. Find four consecutive numbers such that the product of the last two shall be equal to the number composed of the first two used as digits.

7. Find four consecutive numbers such that the prod-

uct of the last two shall be $2\frac{1}{2}$ times the product of the first two.

II.

8. A merchant bought a quantity of flour for 120 dollars. Had he bought 10 barrels more for the same money, the cost would have been 2 dollars a barrel less. How many barrels did he buy, and at what price?

9. A merchant sold a quantity of wheat for 16 dollars, and the loss per cent was equal to the cost in dollars. What was the cost of the wheat?

10. A merchant sold a quantity of cloth for 96 dollars, and the gain per cent was equal to the cost in dollars. What was the cost of the cloth?

11. A crew can row 10 miles down stream and back again in 2 hours and 40 minutes; and the rate of the stream is 2 miles an hour. What is the rate of the crew in still water?

12. A crew can row 20 miles down stream and back again in 7 hours, and the rate of the stream is 3 miles an hour. What is the rate of the crew in still water?

173. Solution of the General Quadratic Equation.—The most general type of a quadratic equation of one unknown quantity is

$$ax^2 + bx + c = 0. \qquad \text{(A)}$$

If we divide through by a, then

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0;$$

and if we substitute p for $\frac{b}{a}$, and q for $\frac{c}{a}$, the equation becomes

$$x^2 + px + q = 0, \qquad \text{(B)}$$

which is the quadratic equation reduced to its *simplest form.*

If in equation (B) we add and subtract the square of $\frac{p}{2}$, we get

$$x^2 + px + \frac{p^2}{4} - \frac{p^2}{4} + q \ = 0,$$

or

$$x^2 + px + \frac{p^2}{4} - \frac{p^2 - 4q}{4} = 0,$$

which factors into

$$\left(x + \frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2}\right)\left(x + \frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}\right) = 0.$$

Therefore $x = 1/2(-p + \sqrt{p^2 - 4q})$,

and $\qquad 1/2(-p - \sqrt{p^2 - 4q})$.

On account of the double sign of the root symbol, $\sqrt{}$, both values are included in the one expression

$$x = 1/2(-p \pm \sqrt{p^2 - 4q}), \qquad (1)$$

which is the solution of (B).

If in this equation we write $\frac{b}{a}$ for p and $\frac{c}{a}$ for q, we have

$$x = \frac{1}{2}\left(-\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}\right),$$

or

$$x = \frac{1}{2}\left(-\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - \frac{4ac}{a^2}}\right),$$

or

$$x = \frac{1}{2}\left(-\frac{b}{a} \pm \frac{\sqrt{b^2 - 4ac}}{a}\right),$$

or

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}),$$

which is the solution of (A). (2)

Formulæ (1) and (2), for the solution of quadratic equations, should be so thoroughly memorized that the roots of any quadratic equation may be written down at sight. Formula (1) is most convenient for use when the coefficient of x^2 is unity, and formula (2) when the coefficient of x^2 is not unity.

e.g. 1°. Find the roots of $x^2 + 2x - 35$.

$$1/2(-2 \pm \sqrt{4 + 140}),$$

or
$$1/2(-2 \pm 12).$$

Hence
$$x_1 = 5, \text{ and } x_2 = -7.$$

2°. Find the roots of $2x^2 + 5x - 12$.

$$1/4(-5 \pm \sqrt{25 + 96}),$$

or
$$1/4(-5 \pm \sqrt{121}),$$

or
$$1/4(-5 \pm 11).$$

Hence
$$x_1 = 3/2, \text{ and } x_2 = -4.$$

3°. Find the roots of $3x^2 + 7x - 25$.

$$1/6(-7 \pm \sqrt{49 + 300}),$$

or
$$1/6(-7 \pm \sqrt{349}).$$

Hence $x = \dfrac{-7 + \sqrt{349}}{6}$, and $x_2 = \dfrac{-7 - \sqrt{349}}{6}$.

Whether the roots be rational, surd, or imaginary depends upon the radicals $\sqrt{p^2 - 4q}$ and $\sqrt{b^2 - 4ac}$.

When $p^2 = 4q$ or $b^2 = 4ac$, the roots are equal, since the radical then becomes zero.

EXERCISE CIV.

I.

1. $x^2 + 6x + 8 = 0$.
2. $x^2 - 14x - 120 = 0$.
3. $2x^2 - 5x = 25$.
4. $3x^2 - 17x + 14 = 0$.

5. $7x^2 = 22x - 15.$ **6.** $(2x - 3)^2 = 2x + 3.$

7. $x^2 + \frac{x}{2} = 18.$ **8.** $x^2 - \frac{x - 1}{4} = 1.$

9. $\frac{3x^2}{4} - \frac{4x}{3} - \frac{1}{3} = 0.$

10. $(x - 2)^2 = 1 + \frac{8}{3}(x + 2).$

11. $\frac{x}{x + 1} + \frac{x + 1}{x + 2} + \frac{x + 2}{x + 3} = 3.$

12. $x^2 + 2ax = b^2 - a^2.$

13. $x^2 + a(1 + 3b)x + 3a^2b = 0.$

14. $ax^2 + b(1 - a^2)x = ab^2.$

II.

15. $(a - x)^2 - (a - x)(b - x) + (x - b)^2 = (a - b)^2.$

16. $a^2(x - b)^2 = b^2(x - a)^2.$

17. $(2a - b - x)^2 + 9(a - b)^2 = \left((a + b) - 2x\right)^2.$

18. $x + \frac{1}{x} = a + \frac{1}{a}.$ **19.** $\frac{x}{a} + \frac{a}{x} = \frac{a}{b} + \frac{b}{a}.$

20. $\frac{1}{a} + \frac{1}{a + x} + \frac{1}{a + 2x} = 0.$

21. $\frac{x + a + 2b}{x + a - 2b} = \frac{b - 2a + 2x}{b + 2a - 2x}.$

22. $\frac{x + 1}{x + 2} + \frac{x + 2}{x + 3} + \frac{x + 3}{x + 5} = 3.$

23. $\frac{5x + 2}{5x - 2} + \frac{5x - 2}{5x + 2} = \frac{25x + 11}{5x + 2}.$

24. $\frac{3x + 1}{3x - 1} + \frac{3x - 1}{x + 1} = \frac{9x - 13/2}{3x + 1}.$

EXERCISE CV.

I.

1. Two trains run over the same 120 miles of rail without stopping. One of them goes 10 miles an hour faster than the other and passes over the distance in 1 hour less time. What is the speed of the trains?

2. Two trains run, without stopping, over the same 90 miles of rail. One of them goes 5 miles an hour faster than the other, and passes over the distance in 15 minutes less time. What is the speed of the trains?

3. A crew can row a certain course up stream in 5 hours, and in still water they could row it in $4\frac{1}{2}$ hours less time than it would take them to drift down stream to the starting-point. How long would it take them to row back with the current?

4. A crew can row a certain course up stream in $5\frac{1}{4}$ hours, and in still water they could have rowed it in 4 hours less time than it would take them to drift down to the starting-point. How long would it take them to row back with the current?

II.

5. Simplify $(a^4b^{2/3})^{-3/2}(a^{-2}b^{3/4})^{1/3}(a^{2/3}b^{-1/4})^{-3}$.

6 Express $a^{-4}b^{-3/4} + 2a^{3/2}b^{-4/3}$ without negative or fractional exponents.

7. Find the value of $(64)^{-3/2}$.

8. Divide $a^{m+x/b}$ by $a^{n+c/d}$ and reduce the resulting exponents to a single fraction.

9. Multiply $\left(c + \frac{cb}{c-b}\right)$ by $\left(c - \frac{cb}{c+b}\right)$.

10. Factor $7x^2 - 14xy - 11x + 22y$.

174. Solution of Equations of the Form of Trinomial Quadratics.—Whenever an equation of one unknown quantity can be reduced to a trinomial the first term of which contains the unknown quantity only in the square of a factor, the second term only in the first degree of the same factor, and the third term not at all, it may be first solved as an ordinary quadratic for that factor, and then the values of the unknown quantity may be found from values of the factor.

e.g. 1°. Solve $3(x-3)^2+5(x-3)-2=0.$

Factoring, we obtain

$$\big((x-3)+2\big)\big(3(x-3)-1\big)=0;$$

and equating each factor to zero, we have

$$x-3+2=0, \qquad \text{or} \quad x=1;$$

and $\quad 3(x-3)-3=0, \quad \text{or} \quad x=4.$

2°. Solve $\quad 6x^4-5x^2-6=0.$

Factoring, we obtain

$$(3x^2+2)(2x^2-3)=0.$$

$$\therefore \quad 3x^2+2=0, \quad \text{or} \quad x^2=-2/3;$$

and $\quad 2x^2-3=0, \quad \text{or} \quad x^2=3/2.$

$$\therefore \quad x=\pm\sqrt{-2/3}=\pm 1/3\sqrt{-6},$$

and $\quad x=\pm\sqrt{3/2}=\pm 1/2\sqrt{6}.$

EXERCISE CVI.

Solve the following equations as quadratics:

I.

1. $6(2x-3)^2-12(2x-3)=0.$
2. $3x^4-19x^2+20=0.$

3. $3(x-4)^2 - 11(x-4) + 10 = 0.$

4. $(2x^2)^2 - 7(2x^2) + 12 = 0.$

5. $(x-3)^2 - 5(x-3) + 6 = 0.$

6. $9x^4 - 33x^2 + 28 = 0.$

II.

7. $24x^4 - 34x^2 + 12 = 0.$

8. $54x^4 - 21x^2 + 2 = 0.$

9. $8x^6 + 37x^3 = 216.$

10. $12x^{-2} + x^{-1} = 35.$

11. $69 - 20x^{-3} - x^{-6} = 0.$

12. $x^{-4} - 21x^{-2} = -108.$

13. $32x^5 + 1/x^5 = -33.$

14. $x^3 - 3x^{3/2} = 88.$

EXERCISE CVII.

I.

1. A person has 12 miles to walk. After he has been on the road one hour he increases his speed $\frac{1}{2}$ mile an hour and finishes his journey in $\frac{3}{7}$ of an hour less time than he would have accomplished it had he not altered his speed. How fast did he walk at first, and how long was he on the road?

2. A man has to drive 25 miles. After he has been on the road two hours he slackens the speed of his horses 1 mile an hour, and is $\frac{3}{4}$ of an hour longer than he would have been had he not changed the rate of driving. At what rate did he drive at first, and how long was he on the road?

3. Reduce $3x^2 - \dfrac{3b^2}{25a^2}$ to a single negative fraction.

4. Rationalize $\dfrac{5 + 7\sqrt{5}}{3 - 2\sqrt{5}}$.

II.

5. A and B together can do a piece of work in a certain time. Were each to do half of it alone, A would have to work 2 days less and B 4 days more than when they work together. In what time can they do it together?

6. A and B can do a piece of work in a certain time. Were each to do half of it alone, A would have to work 4 days less and B 8 days more than when they worked together. In what time can they do it together?

CHAPTER XX.

QUADRATIC EQUATIONS OF TWO UNKNOWN QUANTITIES.

175. Special Cases of Elimination. — Generally, by elimination, two equations of the second degree with two unknown quantities will produce an equation of the fourth degree, which are usually insolvable by any of the methods yet given.

e.g. $$x^2 + y = a. \qquad (1)$$

$$x + y^2 = b. \qquad (2)$$

From (1) we get $y = a - x^2$.

Substituting this in (2), we get

$$x + (a - x^2)^2 = b,$$

or $$x + a^2 - 2ax^2 + x^4 = b^2,$$

which is an equation of the fourth degree, and insolvable by any of the methods yet employed.

There are, however, several cases in which simultaneous quadratics with two unknown quantities may be solved by the rules of quadratics.

CASE 1°.

When each of the equations is of the form

$$ax^2 + by^2 = c.$$

In this case one of the unknown quantities may be eliminated by addition or subtraction, and then the value of the other be found by substitution.

e.g. Solve the equations $2x^2 + 3y^2 = 56, \quad (1)$

$$4y^2 - 13x^2 = 12. \quad (2)$$

Multiplying (1) by 4, $8x^2 + 12y^2 = 224.$

Multiplying (2) by 3, $-39x^2 + 12y^2 = 36.$

Subtracting, $47x^2 = 188.$

$$\therefore \quad x^2 = 4,$$

and $x = \pm 2. \quad (3)$

Substituting (3) in (1), we obtain

$$8 + 3y^2 = 56.$$

$$\therefore \quad 3y^2 = 48,$$

and $y^2 = 16.$

$$\therefore \quad y = \pm 4.$$

Therefore $x = 2$, $y = \pm 4$; or $x = -2$, $y = \pm 4$.

In this case there are four possible sets of values of x and y which satisfy the given equations:

1. $x = 2$, $y = 4$. **2.** $x = 2$, $y = -4$.

3. $x = -2$, $y = 4$. **4.** $x = -2$, $y = -4$.

It would not be correct to leave the results in the form $x = \pm 2$, $y = \pm 4$; for this would indicate only the first and fourth of the above sets of values.

EXERCISE CVIII.

Solve the following equations:

I.

1. $3x^2 + 2y^2 = 77,$
$3y^2 - 6x^2 = 21.$

2. $4x^2 + 8y^2 = 99,$
$8x^2 - 12y^2 = 23.$

3. $5x^2 + 4y^2 = 170,$
$3x^2 - 7y^2 = -86.$

4. $x^2 + y^2 = 10(m^2 + n^2),$
$x^2 - 9y^2 = -20n(3m + 4n).$

II.

5. $4x - 15 = 17\sqrt{x}$. 6. $x^{6/5} + x^{3/5} = 702$.

7. Multiply at sight $\frac{a}{b} + \frac{b}{a} + c$ by $\frac{a}{b} + \frac{b}{a} - c$, and express the result without fractions.

8. Factor $5x^2 - 10ax + 8bx - 16ab$.

CASE 2°.

When one equation is of the second degree and the other of the first.

All equations of this kind may be solved by finding the value of one of the unknown quantities from the first-degree equation, and then substituting that value in the second-degree equation.

The resulting equation will be a quadratic of one unknown quantity which may be solved. When the value of one unknown quantity has been found thus, the values of the second must be found by substituting the values of the one already found in the first-degree equation.

e.g. 1. Solve the equations $3x^2 - xy = 2y$. (1)

$$2x + y = 7. \qquad (2)$$

From (2), we have $y = 7 - 2x$. (3)

Substituting this value in (1), we get

$$3x^2 - x(7 - 2x) = 2(7 - 2x),$$

or

$$3x^2 - 7x + 2x^2 = 14 - 4x.$$

$$\therefore\ 5x^2 - 3x - 14 = 0.$$

$$\therefore\ (x - 2)(5x + 7) = 0.$$

Whence $x = 2$, or $x = -7/5$.

Substituting these values in (3), we get

$$y = 3, \text{ or } y = +49/5.$$

Therefore: 1. $x = 2, \qquad y = 3.$

2. $x = -7/5, \quad y = 49/5.$

Certain examples in which one equation is of the third degree and the other of the second degree may be solved in a similar way.

e.g. 2. Solve the equations

$$x^3 + y^3 = 152, \tag{1}$$

$$x + y = 8. \tag{2}$$

From (2), we obtain $y = 8 - x.$ (3)

By substituting this value of y in (1), we get

$$x^3 + (8 - x)^3 = 152,$$

or $$x^3 + 512 - 192x + 24x^2 - x^3 = 152,$$

or $$24x^2 - 192x + 360 = 0,$$

or $$x^2 - 8x + 15 = 0.$$

$$\therefore \quad (x - 5)(x - 3) = 0.$$

$$\therefore \quad x = 5, \text{ or } x = 3.$$

Substituting these values of x in (2), we get

$$5 + y = 8, \tag{4}$$

and $$3 + y = 8. \tag{5}$$

From (4), we have $y = 3,$

and from (5), $y = 5.$

Therefore $x = 5$ or 3, and $y = 3$ or 5.

1. $x = 5, \ y = 3.$

2. $x = 3, \ y = 5.$

EXERCISE CIX.

Solve the following equations:

I.

1. $3x^2 - xy = 2y,$
$2x + y = 7.$

2. $x + y = -2,$
$xy = -24.$

3. $x - y = 2,$
$x^2 + y^2 = 34.$

4. $x^2 + xy - y^2 = -11,$
$x - y = -4.$

5. $x^3 - y^3 = -296,$
$x - y = -2.$

6. $x^3 + y^3 = 152,$
$x + y = 8.$

II.

7. $x - y = 1,$
$xy = a^2 + a.$

8. $x/2 + y/3 = 4,$
$2/x + 3/y = 1.$

9. $8x^3 - y^3 = -7,$
$2x - y = -1.$

10. $x/y + y/x = 10/3,$
$3x - 2y = -12.$

11. $2x^{2/n} + 3x^{4/n} - 56 = 0.$

12. Factor $15ax - 10x + 6ab - 4b.$

CASE 3°.

An expression is said to be *symmetrical* with respect to any of its letters when any two of them can be interchanged without altering the value of the expression.

e.g. The expression $ab + bc + ca$ is symmetrical with respect to the letters a, b, and c; for if any two of them, as a and b, be interchanged, the expression becomes $ba + ac + cb$, which is the same as the original expression in meaning.

The equations
$$x + y = 2,$$
$$xy = 3,$$
are symmetrical in x and y.

The equations
$$x - y = a,$$
$$xy = b,$$
are symmetrical except in their signs.

When the given equations are symmetrical in x and y, and one of them is of the second degree and the other of the first, they may be solved by combining them in such a way as to obtain the values of $x + y$ and $x - y$.

e.g. Solve the equations
$$x + y = 1, \qquad (1)$$
$$xy = -6. \qquad (2)$$

Squaring (1), we have
$$x^2 + 2xy + y^2 = 1. \qquad (3)$$

Subtracting 4 times (2) from (3), we get
$$x^2 - 2xy + y^2 = 25,$$
which is the square of $x - y$.

Extracting the square root of each member,
$$x - y = \pm 5. \qquad (4)$$

Adding (4) to (1), we have
$$2x = 6 \text{ or } -4.$$
$$\therefore \quad x = 3 \text{ or } -2.$$

Subtracting (4) from (1), we have
$$2y = -4 \text{ or } 6.$$
$$\therefore \quad y = -2 \text{ or } 3.$$

$$\therefore \begin{cases} 1. & x = 3, \quad y = -2. \\ 2. & x = -2, \quad y = 3. \end{cases}$$

This method may be used in many cases when the equations are symmetrical except with respect to the signs of the terms.

e.g. Solve the equations
$$x^2 + y^2 = 65, \qquad (1)$$
$$x - y = -3. \qquad (2)$$

Multiply (1) by 2, and subtract the square of (2) from the result:

$$\begin{array}{rl} & 2x^2 + 2y^2 = 130 \\ & \underline{x^2 - 2xy + y^2 = 9} \\ & x^2 + 2xy + y^2 = 121 \\ \therefore & x + y = \pm 11. \end{array} \qquad (3)$$

Add (3) to (2), and we get

$$2x = 8 \quad \text{or} \quad -14.$$

$$\therefore \quad x = 4 \quad \text{or} \quad -7.$$

Subtract (3) from (2), and we get

$$-2y = -14 \quad \text{or} \quad 8.$$

$$\therefore \quad y = 7 \quad \text{or} \quad -4.$$

$$\begin{array}{rl} & 1. \quad x = 4, \quad y = 7. \\ \therefore & 2. \quad x = -7, \quad y = -4. \end{array}$$

Certain examples in which one equation is of the third degree and the other is of the first or second may be solved by the methods of this case.

e.g. Solve the equations $x^3 + y^3 = 189$, (1)

$$x^2 - xy + y^2 = 21. \qquad (2)$$

Divide (1) by (2), and we get

$$x + y = 9. \qquad (3)$$

Square (3) and subtract (2) from the result:

$$\begin{array}{rl} & x^2 + 2xy + y^2 = 81 \\ & \underline{x^2 - \ \ xy + y^2 = 21} \\ & 3xy \qquad = 60 \\ \therefore & -xy = -20. \end{array} \qquad (4)$$

Add (4) to (2), and we get

$$x^2 - 2xy + y^2 = 1.$$

$$\therefore \quad x - y = \pm 1. \qquad (5)$$

Add (5) to (3), and we get

$$2x = 10 \quad \text{or} \quad 8.$$

$$\therefore \quad x = 5 \quad \text{or} \quad 4.$$

Subtract (5) from (3), and we get

$$2y = 8 \quad \text{or} \quad 10.$$

$$\therefore \quad y = 4 \quad \text{or} \quad 5.$$

$$1. \quad x = 5, \quad y = 4.$$

$$\therefore \quad 2. \quad x = 4, \quad y = 5.$$

In solving examples under this case, it must be borne in mind that, in every instance, we must combine the given equations in such a way as to obtain the values of $x + y$ and $x - y$.

e.g. Solve the equations $x^2 + y^2 = 13,$ (1)

$$xy = 6. \qquad (2)$$

Multiply (2) by 2, and add the result to (1), and also subtract it from (1), and we get

$$x^2 + 2xy + y^2 = 25,$$

and

$$x^2 - 2xy + y^2 = 1.$$

$$\therefore \quad x + y = \pm 5,$$

and

$$x - y = \pm 1.$$

$$\therefore 2x = 5 \pm 1 \quad \text{or} \quad -5 \pm 1.$$

$$\therefore \quad x = 3 \quad \text{or} \quad 2, \quad \text{or} \quad -2, \quad \text{or} \quad -3.$$

And

$$2y = 5 \mp 1 \quad \text{or} \quad -5 \mp 1.$$

$$\therefore \quad y = 2 \quad \text{or} \quad 3, \quad \text{or} \quad -3, \quad \text{or} \quad -2.$$

Therefore: 1. $x = 3, \quad y = 2.$

2. $x = 2, \quad y = 3.$

3. $x = -2, \quad y = -3.$

4. $x = -3, \quad y = -2.$

A few examples in which both equations are of the third degree may be solved by the methods of this case.

e.g. 1. Solve the equations $x^3 - y^3 = 26,$ (1)

$$x^2y - xy^2 = 6. \qquad (2)$$

Multiply (2) by 3 and subtract the result from (1), and we get

$$x^3 - 3x^2y + 3xy^2 - y^3 = 8. \qquad (3)$$

Extract the cube root of (3), and we get

$$x - y = 2. \qquad (4)$$

Divide (2) by (4), and we get

$$xy = 3. \qquad (5)$$

From (4) and (5), we get $x + y = \pm 4.$ (6)

$$\therefore \quad 2x = 6 \quad \text{or} \quad -2,$$

and $x = 3$ or $-1.$

Also, $2y = 2$ or $-6.$

$$\therefore \quad y = 1 \quad \text{or} \quad -3.$$

Therefore: 1. $x = 3, \quad y = 1.$

2. $x = -1, \quad y = -3.$

EXERCISE CX.

Solve the following equations:

I.

1. $xy = 42,$
$x + y = 13.$

2. $xy = 24,$
$x + y = 11.$

3. $x^2 + y^2 = 29,$
$x + y = 7.$

4. $x^2 + y^2 = 58,$
$x + y = 10.$

5. $x^2 + y^2 = 26,$
$x - y = -4.$

6. $x^2 + y^2 = 68,$
$x - y = -6.$

7. $xy = -18,$
$x - y = 11.$

8. $xy = -72,$
$x - y = -18.$

9. $x^3 - y^3 = 279,$
$x^2 + xy + y^2 = 93.$

10. $x^3 + y^3 = 152,$
$x^2 - xy + y^2 = 19.$

11. $x^3 - y^3 = 152,$
$x - y = 2.$

12. $x^3 + y^3 = 637,$
$x + y = 13.$

II.

13. $x^3 + y^3 = 243,$
$x^2y + xy^2 = 162.$

14. $x^3 - y^3 = 386,$
$x^2y - xy^2 = 126.$

15. $x^3 - y^3 = 6a^2b + 2b^3,$
$xy(x - y) = 2b(a^2 - b^2).$

16. $x^2 + xy + y^2 = 7a^2 - 13ab + 7b^2,$
$x^2 - xy + y^2 = 3a^2 - 3ab + 3b^2.$

CASE 4°.

An expression is said to be *homogeneous* when each of its terms is of the same degree.

Certain equations which are of the form: a homogeneous expression in x and y of the second degree equals a constant, may be solved by the methods of cases 1° *and* 3°. When such equations can be solved by neither of these methods, they may be solved by putting $y = mx$, and solving, first for m, then for x, and finally for y.

e.g. Solve the equations
$$x^2 - 2xy = -8. \qquad (1)$$
$$x^2 + y^2 = 13. \qquad (2)$$

Putting $y = mx$, we have

$$x^2 - 2mx^2 = -8, \quad \text{or} \quad x^2 = \frac{8}{2m - 1}, \qquad (3)$$

and $$x^2 + m^2x^2 = 13, \quad \text{or} \quad x^2 = \frac{13}{1+m^2}.$$

$$\therefore \quad \frac{8}{2m-1} = \frac{13}{1+m^2}.$$

$$\therefore \quad 8 + 8m^2 = 26m - 13,$$

or $$8m^2 - 26m + 21 = 0,$$

or $$(2m-3)(4m-7) = 0.$$

$$\therefore \quad m = 3/2 \quad \text{or} \quad 7/4.$$

Substituting the first of these values in (3), we get

$$x^2 = \frac{8}{3-1} = 4.$$

$$\therefore \quad x = \pm 2.$$

Substitute these values of x in (2), and we get

$$y = \pm 3.$$

Substituting the second value of m in (3), we get

$$x^2 = \frac{8}{7/2 - 1} = \frac{16}{5}.$$

$$\therefore \quad x = \pm 4/5\sqrt{5}.$$

Substitute these values in (2), and we get

$$y = \pm 7/5\sqrt{5}.$$

Then:

1. $x = 4/5\sqrt{5}, \quad y = 7/5\sqrt{5}.$
2. $x = 4/5\sqrt{5}, \quad y = -7/5\sqrt{5}.$
3. $x = -4/5\sqrt{5}, \quad y = 7/5\sqrt{5}.$
4. $x = -4/5\sqrt{5}, \quad y = -7/5\sqrt{5}.$

In each case the value of y might have been obtained by substituting the values of m and x in $y = mx$.

EXERCISE CXI.

Solve the following equations:

I.

1. $x^2 + 3xy = 28,$ $xy + 4y^2 = 8.$

2. $x^2 + xy + 2y^2 = 74,$ $2x^2 + 2xy + y^2 = 73.$

3. $x^2 + xy - 6y^2 = 24,$ $x^2 + 3xy - 10y^2 = 32.$

II.

4. $x^2 + xy - 6y^2 = 21,$ $xy - 2y^2 = 4.$

5. $x^2 - xy + y^2 = 21,$ $y^2 - 2xy = -15.$

6. $x^2 + xy + 2y^2 = 44,$ $2x^2 - xy + y^2 = 16.$

EXERCISE CXII.

I.

Solve the following problems by using two unknown quantities:

1. The sum of two numbers is 8, and the sum of their squares is 34. What are the numbers?

2. The difference of two numbers is 3, and the difference of their squares is 33. What are the numbers?

3. The sum of the squares of two numbers is 106, and the product of the numbers is 45. What are the numbers?

4. The difference of two numbers is 6, and their product is 40. What are the numbers?

5. The sum of two numbers is 7, and the sum of their cubes is 91. What are the numbers?

6. The difference of two numbers is 4, and the difference of their cubes is 316. What are the numbers?

7. Find two numbers such that the square of the first

and twice the square of the second shall together equal 22, and the square of the second and three times the product of the two shall equal 27.

II.

8. Find two numbers such that three times the square of the smaller and the square of the larger shall together equal 7, and the square of the smaller shall be 7 less than four times the product of the two.

9. A man bought 8 cows and 5 sheep for 255 dollars. He bought 3 more sheep for 39 dollars than cows for 300 dollars. What was the price of each?

10. A number is composed of two digits. If its digits be inverted, the sum of the new and original numbers will be 44, and their product 403. What are the numbers?

11. Multiply $a + \frac{ab}{a - b}$ by $b - \frac{ab}{a + b}$.

12. Factor $12x^3 - 8xy - 9x^2y^2 + 6y^3$.

13. Reduce $-\frac{25b^2}{36a^2} + \frac{7c}{3a}$ to a single negative fraction.

14. Simplify $(1/125)^{-2/3}$.

15. Multiply $\sqrt{4}$ by $\sqrt{8}$.

16. Express the following without fractional or negative indices:

$$a^{2/3}b^{-1} - a^{-2/3}b.$$

17. Rationalize the denominator of $\frac{8 - 5\sqrt{2}}{3 - 2\sqrt{2}}$.

CHAPTER XXI.

INDETERMINATE EQUATIONS OF THE FIRST DEGREE.

176. Indeterminate Equations.—Equations are *indeterminate* when the number of independent equations given is less than that of the unknown quantities which they contain. For when such equations are solved for any one of their letters, the value obtained will contain constants and one or more of the letters which represent the other unknown quantities. Hence the value of the letter found will vary with the value assigned to the other letters.

Thus, if $2x + 5y = 8$, $x = 4 - 5/2y$, and y may take as many values as we please, and to every value of y will correspond a single value of x; and, conversely, to every value of x will correspond a single value of y. Unless some restrictions be placed on the values of the unknown quantities, the equation may be satisfied in an indefinite number of ways.

If, however, the values of the unknown quantities are subject to any restriction, n equations may suffice to determine the values of more than n unknown quantities.

In the present chapter we shall consider only indeterminate equations of the first degree in which the values of the unknown quantities are restricted to positive integers.

177. Solution of Indeterminate Equations of the First Degree in x and y.—Every equation of the first degree in x and y may be reduced to the form $ax \pm by = \pm c$, in which a, b, and c are positive integers, and have no common factor.

The form $ax + by = -c$ cannot be solved for positive integers; for if a, b, x, and y are positive integers, $ax + by$ must also be a positive integer.

The remaining forms, $ax \pm by = c$ and $ax - by = -c$, cannot be solved for positive integers when a and b are commensurable. For if x and y are positive integers, the common factor of a and b must also be a factor of $ax + by$, and therefore of c, which contradicts the hypothesis that a, b, and c have no common factor.

The form $ax - by = -c$ becomes by changing its signs $by - ax = c$, which is essentially the same as $ax - by = c$, a and b and x and y being interchanged.

Hence the two type forms $ax + by = c$ and $ax - by = c$ are the only ones that need be considered, and those only in the cases in which a and b are prime to each other.

Ex. Solve $5x + 12y = 263$ in positive integers.

Divide through by 5, the smaller coefficient, and we get

$$x + 2y + \frac{2y}{5} = 52 + \frac{3}{5}.$$

$$\therefore \quad x + 2y + \frac{2y - 3}{5} = 52. \qquad (1)$$

Since x and y are both integers, and the whole of the first member is an integer, therefore

$$\frac{2y - 3}{5} = \text{an integer}.$$

Multiplying this fraction by the integer which will make the coefficient of y one more than the denominator (5), or than a multiple of the denominator, we get

$$\frac{6y - 9}{5} = \text{an integer};$$

that is,
$$y - 1 + \frac{y-4}{5} = \text{an integer}.$$

$$\therefore \quad \frac{y-4}{5} = \text{an integer} \equiv p.$$

$$\therefore \quad y - 4 = 5p,$$

or
$$y = 5p + 4. \qquad (2)$$

Substituting this value of y in (1), we get

$$x + 10p + 8 + \frac{10p + 8 - 3}{5} = 52,$$

or
$$x + 10p + 8 + 2p + 1 = 52,$$

or
$$x + 12p = 43.$$

$$\therefore \quad x = 43 - 12p. \qquad (3)$$

From (2) and (3) it is evident that x and y will be integral when p is an integer and only when p is an integer; for they will both be integers when $5p$ and $12p$ are both integers and in no other case, and $5p$ and $12p$ will be integral when p is integral and in no other case.

From (3) it is evident that x will be negative when p exceeds 3, and y will be negative when p is negative. Hence p must be a positive integer less than 4. Hence the only possible values of p are 0, 1, 2, 3. Thus the only positive integral values of x and y are obtained by putting in (2) and (3) $p = 0$, 1, 2, and 3.

The corresponding values of x and y are shown in the following table:

$$p = 0,\ 1,\ 2,\ 3,$$

$$x = 43,\ 31,\ 19,\ 7,$$

$$y = 4,\ 9,\ 14,\ 19.$$

Note that the coefficients of p in the values of x and y in (2) and (3) are the coefficients of y and x respectively in the given equation, and that one of the signs is changed.

Hence when the given equation has the type form $ax + by = c$, the term in p in the value of x or y must be negative, and the integral values of p and therefore of x and y must be limited.

Ex. 2. Solve $8x - 3y = 28$ in positive units.

Dividing by 3, the smaller coefficient, we get

$$2x + \frac{2x}{3} - y = 9 + \frac{1}{3}.$$

$$\therefore \quad 2x - y + \frac{2x-1}{3} = 9. \qquad (1)$$

$$\therefore \quad \frac{2x-1}{3} = \text{an integer.}$$

Multiplying by 2 so as to make the coefficient of x greater by one than 3,

$$\frac{4x-2}{3} = \text{an integer.}$$

$$\therefore \quad x + \frac{x-2}{3} = \text{an integer.}$$

$$\therefore \quad \frac{x-2}{3} = \text{an integer} \equiv p.$$

$$\therefore \quad x - 2 = 3p,$$

or
$$x = 3p + 2. \qquad (2)$$

Substituting this value of x in (1), we get

$$4 + 6p - y + \frac{4 + 6p - 1}{3} = 9,$$

or
$$4 + 6p - y + 1 + 2p = 9,$$

or $$8p - y = 4.$$

$$\therefore \quad y = 8p - 4. \qquad (3)$$

From (2) and (3) we see that p may be any positive integer except zero.

When $$p = 1, 2, 3, \text{etc.},$$

$$x = 5, 8, 11, \text{etc.},$$

and $$y = 4, 12, 20, \text{etc.}$$

In this case the term in p is positive in both (2) and (3), and the number of solutions is unlimited. This will be the case always when the equation has the type form $ax - by = c$.

178. Solution of Indeterminate Equations of the First Degree in x, y, and z.—To solve two equations in three unknown quantities for positive integers: first eliminate one of the unknown quantities so as to get one equation in two unknown quantities; then solve this for positive integers and obtain the value of each of the two unknown quantities in terms of p and constants; and finally substitute these two values in one of the original equations to find the value of the third unknown quantity in terms of m and a constant, observe what values of p will make each of these three positive integers, and find the corresponding values of each of the unknown quantities.

e.g. Solve $$2x + 3y - 5z = -8,$$

$$5x - y + 4z = 21, \qquad (1)$$

for positive integers.

Eliminating y by addition, we get

$$17x + 7z = 55. \qquad (2)$$

$$\therefore \quad 2x + z + \frac{3x}{7} = 7 + \frac{6}{7},$$

or
$$2x + z + \frac{3x - 6}{7} = 7.$$

$$\therefore \quad \frac{3x - 6}{7} = \text{integer.}$$

$$\therefore \quad \frac{15x - 30}{7} = \text{integer.}$$

$$\therefore \quad 2x - 4 + \frac{x - 2}{7} = \text{integer.}$$

$$\therefore \quad \frac{x - 2}{7} = \text{integer} \equiv p.$$

$$\therefore \quad x - 2 = 7p,$$

or
$$x = 7p + 2. \qquad (3)$$

Substituting this value of x in (2), we get

$$119p + 34 + 7z = 55,$$

or
$$119p + 7z = 21.$$

$$\therefore \quad 17p + z = 3.$$

$$\therefore \quad z = 3 - 17p. \qquad (4)$$

Substituting (3) and (4) in (1), we get

$$35p + 10 - y + 12 - 68p = 21,$$

or
$$-33p - y = -1,$$

$$y = 1 - 33p. \qquad (5)$$

The only value of p that can make z a positive integer is 0. Substitute this value in (3), (4), and (5), and we get

$$x = 2,$$

$$y = 1,$$

and
$$z = 3.$$

EXERCISE CXIII.

Solve the following equations in positive integers:

I.

1. $7x + 15y = 59.$ 2. $8x + 13y = 138.$

3. $7x + 9y = 100.$ 4. $13x + 17y = 200.$

5 Find the number of solutions in positive integers of

$$11x + 15y = 1031.$$

Solve the following equations in positive integers:

6. $\begin{aligned} 6x + 7y + 4z &= 122, \\ 11x + 8y - 6z &= 145. \end{aligned}$ 7. $\begin{aligned} 12x - 11y + 4z &= 22, \\ -4x + 5y + z &= 17. \end{aligned}$

II.

8. $\begin{aligned} 20x - 21y &= 38, \\ 3y + 4z &= 34. \end{aligned}$ 9. $7x + 4y + 19z = 84.$

10. $23x + 17y + 11z = 130.$

Find the general integral solutions of the following equations:

11. $7x - 13y = 15.$ 12. $9x - 11y = 4.$

Solve in least positive integers:

13. $119x - 105y = 217.$ 14. $49x - 69y = 100.$

15. How can a length of 4 feet be measured by means of two measures, one 7 inches long and the other 13 inches long?

16. How can 45 pounds be exactly measured by means of 4-pound and 7-pound weights?

17. In how many different ways can the sum of $3.90 be paid with fifty- and twenty-cent pieces?

18. In how many different ways can the sum of $5.10 be paid with half-dollars, quarter-dollars, and dimes, the whole payment to be made with twenty pieces?

19. A farmer purchased a number of pigs, sheep, and calves for 160 dollars. The pigs cost 3 dollars each, the sheep 4 dollars each, and the calves 7 dollars each; and the number of calves was equal to the number of pigs and sheep together. How many of each did he buy?

20. Find the least multiples of 23 and 15 which differ by 2.

21. Find two fractions whose denominators shall be respectively 9 and 5 and whose sum shall be $\frac{113}{45}$.

CHAPTER XXII.

INEQUALITIES.

179. Definition of Greater and Less Quantities.—One quantity is said to be greater than another when the remainder obtained by subtracting the second from the first is *positive;* and one quantity is said to be less than another when the remainder obtained by subtracting the second from the first is *negative.*

N.B.—Throughout the present chapter every letter is supposed to denote a real positive quantity, unless the contrary is stated.

In accordance with the definition just given a is greater than b when $a - b$ is positive, and, conversely, when a is greater than b, $a - b$ is positive. Also, a is less than b when $a - b$ is negative, and, conversely, when a is less than b, $a - b$ is negative. Thus 2 is greater than -3 because $2 - (-3)$, or 5, is positive; also -2 is greater than -3 because $-2 - (-3)$, or 1, is positive. Again, -2 is less than 1 because $-2 - 1$, or -3, is negative; and -4 is less than -2 because $-4 - (-2)$, or -2, is negative.

According to this definition zero must also be regarded as greater than any negative quantity.

180. Inequalities.—An *inequality* is an algebraic statement of the fact that one of two expressions is greater than the other. The two expressions compared are connected together by the sign $>$, "greater than," or $<$, "less than," the open end of the symbol always being directed towards the larger member of the inequality.

Two or more inequalities are said to be in the same sense, or of the same species, when the first member of each is the greater or the less, and two inequalities are said to be in the opposite sense, or of the opposite species, when the first member of the one is the greater, and of the other is the less.

Thus $a > b$ and $c > d$ are two inequalities in the same sense, or of the same species. So also are $m<n$ and $p<q$. But $a > b$ and $c < d$, or $m < n$ and $p > q$ are inequalities in the opposite sense, or of opposite species.

The working rules for inequalities differ in some respects from those for equations. They are based upon certain elementary theorems of inequality which are readily deduced from the axioms of equality.

THEOREM I. *If equals be added to unequals, the sum will be unequal in the same sense.*

Let $a > b$, and let their difference be denoted by r. Then

$$a = b + r.$$

Adding x to each member of this equation, we get

$$a + x = b + x + r.$$

$$\therefore \quad a + x > b + x.$$

THEOREM II. *If equals be taken from unequals, the remainders will be unequal in the same sense.*

Let $a > b$, and let their difference be denoted by r. Then

$$a = b + r.$$

Subtracting x from each member of this equation, we get

$$a - x = (b - x) + r.$$

$$\therefore \quad a - x > b - x.$$

Cor. From these two theorems it follows that we have the right to add equals to the members of an inequality, and to subtract equals from the members of an inequality, without altering the sign of inequality.

Also, that we have the right to transfer a term from one member of an inequality to the other by changing its signs. without altering the sign of inequality.

Theorem III. *If unequals be subtracted from equals, the remainders will be unequal in the reverse sense.*

Let $a > b$, and let their difference be denoted by r. Then

$$a = b + r.$$

Subtracting each member of this equation from x, we get

$$x - a = x - (b + r) = (x - b) - r.$$

$$\therefore \quad x - a < x - b.$$

Cor. If $x = 0$, we would have $-a < -b$. Hence when we reverse the signs of an inequality, we must also reverse the sign of inequality.

Theorem IV. *If unequals be multiplied by equals, the products will be unequal in the same sense.*

Let $a > b$, and let their difference be denoted by r. Then

$$a = b + r.$$

Multiplying both members of this equation by x, we get

$$ax = bx + rx.$$

$$\therefore \quad ax > bx.$$

Theorem V. *If unequals be divided by equals, the quotients will be unequal in the same sense.*

Put $\quad a = b + r$ as heretofore.

Dividing each member of this equation by x, we get

$$\frac{a}{x}=\frac{b}{x}+\frac{r}{x}.$$

$$\therefore \quad \frac{a}{x}>\frac{b}{x}.$$

Cor. From Theorems IV and V it follows that we have the right to multiply or divide both members of an inequality by the same positive quantity without altering the sign of inequality.

If, however, both members of an inequality be multiplied or divided by a negative quantity, the signs of both members will be reversed. This reversal of signs is equivalent to an interchange of the members, and therefore it reverses the character of the inequality. Hence, on such multiplication, the sign of inequality must be reversed.

Theorem VI. *If equals be divided by unequals, the quotients will be unequal in the opposite sense.*

Put as before $a=b+r.$

Dividing x by each member of this equation, we get

$$\begin{aligned}\frac{x}{a}=\frac{x}{b+r}=\frac{bx}{b(b+r)}&=\frac{bx+rx-rx}{b(b+r)}\\ &=\frac{x(b+r)}{b(b+r)}-\frac{rx}{b(b+r)}\\ &=\frac{x}{b}-\frac{rx}{b(b+r)}.\end{aligned}$$

$$\therefore \quad \frac{x}{a}<\frac{x}{b}.$$

Theorem VII. *If two inequalities of the same species be added together, the results will be unequal in the same sense.*

Let $a>b$ and $c>d$.

Put $a = b + r$, and $c = d + s$.

Then, by addition of equals,

$$a + c = b + d + r + s.$$

$$\therefore \quad a + c > b + d.$$

NOTE.—By subtraction we would get

$$a - c = b - d + r - s;$$

from which we cannot infer whether $a - c > b - d$, or $a - c < b - d$.

If $r > s$, $a - c > b - d$; but if $r < s$, $a - c < b - d$.

Hence addition of corresponding members of inequalities of the same species without changing the sign of inequality is always admissible, but not subtraction.

COR. If $a > b$, $c > d$, $e > f$, etc., then

$$a + c + e + \text{etc.} > b + d + f + \text{etc.}$$

THEOREM VIII. *If two inequalities of the same species be multiplied together, the results will be unequal in the same sense.*

Let $a > b$, and $c > d$.

Put $a = b + r$, and $c = d + s$.

Then, by the multiplication of equals,

$$ac = (b + r)(d + s) = bd + bs + dr + rs.$$

$$\therefore \quad ac > bd.$$

COR. 1. If $a > b$, $c > d$, $e > f$, etc., then

$$a \,.\, c \,.\, e \,.\, \text{etc.} > b \,.\, d \,.\, f \,.\, \text{etc.}$$

COR. 2. If $a > b$, then $a^m > b^m$.

COR. 3. If $a > b$, then $a^{-m} < b^{-m}$.

EXERCISE CXIV.

1. For what values of x is

$$5x - \frac{16}{5} < \frac{10x}{5} + 6\,?$$

Multiplying both members by 5, we get

$$25x - 16 < 10x + 30.$$

By transposition, $15x < 46.$

$$\therefore \quad x < 3\tfrac{1}{15}.$$

This inequality holds for all values of x less than $3\tfrac{1}{15}$

2. For what values of x and y are

$$4x + 3y > 27,$$
$$3x + 4y = 29\,?$$

Multiplying both members of the inequality by 4, and of the equation by 3, we get

$$\begin{aligned} 16x + 12y &> 108; \\ 9x + 12y &= \ \ 87; \\ \therefore \quad 7x &> \ \ 21; \\ \therefore \quad x &> \ \ \ 3. \end{aligned}$$

Multiplying both members of the inequality by 3, and of the equation by 4, we get

$$\begin{aligned} 12x + \ \ 9y &> \ \ 81; \\ 12x + 16y &= 116; \\ -\ \ 7y &> -35. \\ \therefore \quad 7y &< 35. \\ \therefore \quad y &< 5. \end{aligned}$$

Hence the values are all of those of x greater than 3, and of y less than 5, which make $3x + 4y = 29$.

N.B.—The values of x and y obtained as above are called the limits of x and y. That is, they are the values which bound the possible values which x and y can have under the given conditions.

Find the limits of x in the following cases:

3. $(4x+2)^2-29>(2x+2)(8x-4)$.

4. $(3x-2)(4x+3)>(2x-4)(6x+5)+58$.

5. When $3x-12>35-5x$, and $4x-12>6x-31$.

Find the limits of x and y in the following case:

6. $3x+7y>46$,
$x-y=-1$.

181. Type Forms.—Inequalities among algebraic quantities are usually established by reference to certain standard forms.

The following is a very important standard form:

For all values of x and y except equality,

$$x^2+y^2>2xy. \qquad \text{(A)}$$

Proof.—$(x-y)^2$ is essentially positive and hence >0.

$$\therefore \quad x^2+y^2-2xy>0.$$

$$\therefore \quad x^2+y^2>2xy.$$

e.g. The sum of a number and its reciprocal is >2.

Let x denote the number. Then will

$$x+\frac{1}{x}>2.$$

Multiplying both members by x, we get

$$x^2+1>2x,$$

or

$$x^2+1^2>2x\,.\,1.$$

That is, the first inequality is true if the last is. But we know that the last is true by reference to standard (A). Hence we infer that the first is also true.

Theorem I. *The product of two positive quantities whose sum is constant is greatest when the quantities are equal.*

Denote the two quantities by $a + x$ and $a - x$. Then, whatever value be assigned to x, the sum of the quantities will be $2a$, and their product $a^2 - x^2$. Evidently the product will be greatest when $x = 0$; that is, when the quantities are equal.

If a and b be two unequal quantities, the two halves of their sum would be two equal quantities whose sum would be the same as that of a and b. Hence

$$\frac{a+b}{2} \cdot \frac{a+b}{2} > ab,$$

or

$$\left(\frac{a+b}{2}\right)^2 > ab,$$

or

$$\frac{a+b}{2} > \sqrt{ab}.$$

$$\therefore \quad a + b > 2\sqrt{ab}. \qquad \text{(B)}$$

THEOREM II. *The product of any number of positive quantities whose sum is constant is greatest when the quantities are all equal.*

For, if any two of the factors are unequal, their product would be increased by making them equal without changing their sum. This would necessarily increase the whole product without altering the sum of the factors.

If $a, b, c, \ldots.$ up to n quantities be unequal, by taking the nth parts of their sum we should obtain n equal quantities whose sum would be the same as that of the n unequal quantities. Hence

$$\left(\frac{a+b+c+\ldots}{n}\right)^n > abc\ldots,$$

or

$$\frac{a+b+c+\ldots}{n} > \sqrt[n]{abc\ldots}$$

$$\therefore \quad a+b+c+\ldots > n\sqrt[n]{abc\ldots} \qquad \text{(C)}$$

e.g. $a^2 + b^2 > 2ab,$

and $a^3 + b^3 + c^3 > 3abc,$

in all cases when a, b, and c are positive and unequal.

EXERCISE CXV.

I.

1. For what value of x would $16x^2 + 25 = 40x$? Show that for all other values of x, $16x^2 + 25 > 40x$.

2. Show that for no positive integral value of x is

$$x^2 + \frac{3x}{5} < 3x - \frac{36}{5}.$$

3. Show that for no positive value of a can

$$(3a + 2b)(3a - 2b) < 4b(6a - 5b).$$

4. Show that $(ab + xy)(ax + by) > 4abxy$.

5. Show that $(b + c)(c + a)(a + b) > 8abc$.

II.

6. If $a^2+b^2=1$, and $x^2+y^2=1$, show that $ax+by<1$.

7. If $a^2 + b^2 + c^2 = 1$, and $x^2 + y^2 + z^2 = 1$, show that $ax + by + cz < 1$.

8. Show that $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)>9x^2y^2z^2$.

9. Show that $a^4 + b^4 > a^3b + ab^3$, except when a and b are equal.

10. Show that $a^6 + b^6 > a^5b + ab^5$, except when a and b are equal.

CHAPTER XXIII.

RATIO AND PROPORTION.

A. RATIO.

182. Definition of Ratio.—The term *ratio* denotes the relation which one quantity bears to another of the same kind in magnitude.

The magnitude of one number compared with another is ascertained by dividing the number by the one with which it is compared.

When the number is a multiple of the one with which it is compared its ratio to it may be expressed by an integer, otherwise the ratio may be expressed by a mixed number or a fraction.

e.g. The ratio of 12 to 4 $= 12 \div 4 = 3$; the ratio of 3 to 5 $= 3 \div 5 = 3/5$; the ratio of 13 to 4 $= 13/4$ or $3\frac{1}{4}$.

The ratio of one number to another might be defined as the number by which the second must be multiplied to produce the first.

e.g. 5 must be multiplied by 4 to produce 20. Therefore the ratio of 20 to 5 is 4.

Again, 5 must be multiplied by 3/5 to produce 3. Therefore the ratio of 3 to 5 is 3/5.

183. Expression of a Ratio.—The ratio of one number to a second may be expressed either by writing the numbers in the form of a fraction with the first number as the numerator, or by writing the second number after the first with a colon between.

e.g. The ratio of 2 to 3 may be expressed thus:

$$\frac{2}{3}, \quad \text{or} \quad 2 : 3.$$

184. The Terms of a Ratio.—The first term of a ratio is usually called the *antecedent*, and the second term the *consequent*.

When either term of a ratio is a surd the ratio cannot be expressed exactly either by an integer or by a rational fraction, though it may be expressed to any required degree of approximation, by carrying out the extraction of the indicated root to a sufficient number of places.

e.g. The ratio of the $\sqrt{5}$ to 4 cannot be expressed exactly by any rational integer or fraction. Thus,

$$\frac{\sqrt{5}}{4} = \frac{2.236068\ldots}{4} = .559017\ldots$$

By carrying the decimals further a closer approximation may be obtained.

185. Kinds of Ratios.—When the antecedent of a ratio is equal to its consequent, the value of the ratio is one, and the ratio is said to be a ratio of *equality;* when the antecedent is greater than the consequent, the value of the ratio is greater than one, and the ratio is said to be a ratio of *greater inequality;* and when the antecedent is less than the consequent, the value of the ratio is less than one, and the ratio is said to be a ratio of *less inequality.*

186. Ratio of Equimultiples and Submultiples.—Since $\frac{a}{b} = \frac{ma}{mb}$, two numbers have the same ratio as their equimultiples.

Also, since $\frac{a}{b} = \frac{a \div m}{b \div m}$, two numbers have the same ratio as their equi-submultiples, equi-submultiples being the quotients obtained by dividing two or more numbers by the same number.

187. THEOREM I. *If the consequent of a ratio of greater inequality be positive, the ratio will be diminished by adding the same positive quantity to both of its terms, and increased by subtracting the same positive quantity (less than the consequent) from both of its terms.*

Let b be positive and $a > b$, then will $\frac{a+x}{b+x} < \frac{a}{b}$.

For $\frac{a+x}{b+x} - \frac{a}{b} = \frac{b(a+x) - a(b+x)}{b(b+x)} = \frac{x(b-a)}{b(b+x)}$.

Now since a, b, and x are positive by hypothesis and $b < a$, the fraction $\frac{x(b-a)}{b(b+x)}$ is negative. $\therefore$ $\frac{a+x}{b+x} < \frac{a}{b}$.

Again, $$\frac{a-x}{b-x} - \frac{a}{b} = \frac{x(a-b)}{b(b-x)}.$$

But, since $a > b$, $a - b$ is positive, and since $x < b$, $b - x$ is positive.

Hence the fraction $\frac{x(a-b)}{b(b-x)}$ is positive. $\therefore$ $\frac{a-x}{b-x} > \frac{a}{b}$.

188. THEOREM II. *If the consequent of a ratio of less inequality be positive, the ratio will be increased by adding the same positive quantity to both of its terms, and diminished by subtracting the same positive quantity (less than the consequent) from both of its terms.*

Let b be positive and $a < b$, then will $\frac{a+x}{b+x} > \frac{a}{b}$, and $\frac{a-x}{b-x} < \frac{a}{b}$.

Prove these cases in the same manner as those of the last section.

189. Compound Ratios. — When the antecedents and also the consequents of two or more ratios are multiplied together the ratios are said to be *compounded*, and the ratio of the products is called the *compound* ratio of its components. Thus, $ac : bd$ is the compound ratio of $a : b$ and $c : d$.

When a ratio is compounded with itself its terms are squared, and the result is called the *duplicate* ratio of the original. Thus, $a^2 : b^2$ is the duplicate of $a : b$.

Similarly $a^3 : b^3$ is called the *triplicate* ratio of $a : b$.

B. PROPORTION.

190. Definition of Proportion.—Four abstract numbers are said to be *proportional*, or to form a *proportion*, when the ratio of the first to the second is equal to that of the third to the fourth. Thus, if $a : b = c : d$, the four quantities a, b, c, and d form a proportion, which may be written in any one of the following ways:

$$a : b = c : d, \quad \frac{a}{b} = \frac{c}{d}, \quad \text{or} \quad a : b :: c : d.$$

The first and last terms of a proportion are called the *extremes*, and the second and third terms, the *means*. Thus, in the above proportion a and d are the extremes, and b and c the means.

If a, b, c, d, e, etc., are such that $a : b = b : c = c : d = d : e$, then a, b, c, d, e are said to be in *continued proportion*.

If three quantities, a, b, c, are in continued proportion, so that $a : b = b : c$, then b is said to be a *mean* proportional between a and c.

If $a : b = b : c = c : d$, then b and c are said to be two mean proportionals between a and d, and so on.

191. Test of the Equality of Two Ratios. — Since a ratio is virtually a fraction, we test the equality of two ratios in the same way that we test the equality of two fractions.

Two fractions are equal if, on reduction to a common denominator, the resulting numerators are equal. Thus, take the two fractions $\frac{a}{b}$ and $\frac{c}{d}$, reduce them to a common donominator, and we have $\frac{ad}{bd}$ and $\frac{bc}{bd}$. These resulting fractions will be equal when $ad = bc$. Hence the four quantities a, b, c, d are proportional when the product of the first and fourth is equal to the product of the second and third; and, conversely, if $a : b = c : d$, then $ad = bc$.

In any proportion the product of the extremes is equal to the product of the means. This is the great numerical law of proportions.

192. Permutations of Proportions.—Any interchange of the terms of a proportion is permissible which does not destroy the equality of the product of the extremes and means. The various interchanges of the terms of a proportion are called *permutations*.

If we write the four terms of a proportion in the four corners formed by two lines which cross at right angles, so that the first ratio shall be at the left and the second at the right, the two antecedents will be at the top and the two consequents at the bottom, and the extremes will be in one pair of opposite corners and the means in the other. Thus in the form $\dfrac{a}{b}\Bigg|\dfrac{c}{d}$, $a : b$ is the first ratio and $c : d$ the second; a and c are the antecedents and b and d the consequents; a and d are the extremes and b and c the means.

The letters a and d and b and c, which stand in opposite corners in the above form, may be called the *opposites* of a proportion, and we may make the general statement that

The terms of a proportion may be written in any order, provided the opposites remain the same.

An interchange of antecedent and consequent in each ratio is called an *inversion,* an interchange of an antecedent of one ratio with the consequent of another is called an *alternation,* and an interchange of one ratio with another a *transposition.*

There are seven permutations of an ordinary proportion, so that when four quantities are proportional they may be arranged in eight different ways.

Thus, by inversion $\dfrac{a}{b}\Bigg|\dfrac{c}{d}$ becomes $\dfrac{b}{a}\Bigg|\dfrac{d}{c}$, and by moving the terms of each of these successively around to the right each of the above may be changed three times by alternation.

$$\text{Thus}\quad \frac{a}{b}\Bigg|\frac{c}{d}\quad\text{becomes}\quad \frac{b}{d}\Bigg|\frac{a}{c},\quad \frac{d}{c}\Bigg|\frac{b}{a},\quad\text{and}\quad \frac{c}{a}\Bigg|\frac{d}{b}.$$

$$\text{And}\quad \frac{b}{a}\Bigg|\frac{d}{c}\quad\text{becomes}\quad \frac{a}{c}\Bigg|\frac{b}{d},\quad \frac{c}{d}\Bigg|\frac{a}{b},\quad\text{and}\quad \frac{d}{b}\Bigg|\frac{c}{a}.$$

Write out in the ordinary form each of the proportions given above, and state by what change each proportion is obtained from the last.

193. Transformations of Proportions. — Besides these eight permutations there are other transformations which a proportion may undergo.

If $a : b = c : d$, then $a + b : b = c + d : d$.

Let $\frac{a}{b} = x$. Therefore $a = bx$.

Then, also, $\frac{c}{d} = x$. (Why ?) Therefore $c = dx$.

Then $\frac{a+b}{b}$ becomes, by substitution,

$$\frac{bx+b}{b}=\frac{b(x+1)}{b}=x+1.$$

Also, $\frac{c+d}{d}$ will become $\frac{dx+d}{d}=(x+1)$.

Therefore $$\frac{a+b}{b}=x+1=\frac{c+d}{d}.$$

Hence $$a+b:b=c+d:d.$$

This change is called *composition.*

EXERCISE CXVI.

Prove the following cases by methods similar to the above:

1. $a-b:b=c-d:d$.

This change is called *division.*

2. $a+b:a-b=c+d:c-d$.

This change is called *composition and division.*

3. $a+b:a=c+d:c$.

4. $a-b:a=c-d:c$.

5. If $a:b=c:d=e:f=$ etc.,

then $$a+c+e:b+d+f=a:b.$$

This change is called *addition.*

6. If $a:b=c:d$, then $ma:mb=nc:nd$.

7. Write the last proportion in eight different ways.

8. If $a:b=c:d$, then $a^n:b^n=c^n:d^n$.

9. If $a:b=c:d$, and $m:n=r:s$,

then $$am:bn=cr:ds.$$

10. If $a : b = c : d$, then

$$la + mb : pa + qb = lc + md : pc + qd.$$

11. If $a : b = c : d$, and $m : n = r : s$, then

$$a\sqrt{m} - b\sqrt{n} : c\sqrt{r} - d\sqrt{s} = a\sqrt{m} + b\sqrt{n} : c\sqrt{r} + d\sqrt{s}.$$

EXERCISE CXVII.

Ex. Which is the greater ratio, $a^4 + b^4 : a + b$ or $a^4 - b^4 : a - b$, a and b each being positive?

Write each ratio in the form of a fraction, and subtract the second from the first, and show that the result is essentially negative. Hence the second ratio must be the greater. Thus,

$$\frac{a^4+b^4}{a+b} - \frac{a^4-b^4}{a-b} = \frac{(a^4+b^4)(a-b)-(a^4-b^4)(a+b)}{(a+b)(a-b)}$$

$$= \frac{2ab^4 - 2a^4b}{(a+b)(a-b)}$$

$$= -\frac{2a^4b - 2ab^4}{(a+b)(a-b)}$$

$$= -\frac{2ab(a^3-b^3)}{(a+b)(a-b)}$$

$$= -\frac{2ab(a^2+ab+b^2)}{a+b}.$$

Now since a and b are both positive, both the numerator and the denominator of this fraction must be positive. Hence the result obtained by subtracting $\frac{a^4-b^4}{a-b}$ from $\frac{a^4+b^4}{a+b}$ is negative. Therefore $\frac{a^4-b^4}{a-b}$ must be larger than $\frac{a^4+b^4}{a+b}$.

I.

1. Which is the greater ratio, $5 : 7$ or $151 : 208$?

2. Which is the greater ratio, $6 : 11$ or $575 : 1056$?

3. Which is the greater ratio, $7 : 12$ or $589 : 1008$?

4. Which is the greater ratio, $x^2 + y^2 : x + y$ or $x^2 - y^2 : x - y$, x and y both being positive?

5. Which is the greater ratio, $x^3 + y^3 : x + y$ or $x^3 - y^3 : x - y$, x and y both being positive?

6. Which is the greater ratio, $x^n + y^n : x + y$ or $x^n - y^n : x - y$, x and y both being positive?

7. In one city a man assessed for \$10,000 pays \$72 tax, and in another city a man assessed for \$720 pays \$4.50 tax. Compare the rate of taxation in the two cities.

8. Two men can do in 4 days what three boys can do in 5 days. Compare a man's working capacity with that of a boy.

9. For what value of x will the ratio $5 + x : 8 + x$ become $5 : 8$, $6 : 8$, $7 : 8$, $8 : 8$, $9 : 8$?

10. What number added to both antecedent and consequent will duplicate the ratio $3 : 4$?

11. If $x + 7$ is to $2(x + 14)$ in the duplicate ratio of $5 : 8$, what is the value of x?

II.

12. Find two numbers in the ratio of $7 : 12$ such that the greater exceeds the less by 275.

13. What number must be added to each term of the ratio $5 : 37$ to make it equal to $1 : 3$?

14. If $x : y = 3 : 4$, what is the ratio of $7x - 4y : 3x + 7y$?

15. If $15(2x^2 - y^2) = 7xy$, what is the ratio of $x : y$?

16. If $3(7x^2 - 24y^2) = -29xy$, what is the ratio of $x : y$?

17. What is the least integer which added to both terms of the ratio 5 : 9 will make a ratio greater than 7 : 10?

194. Solution of Fractional Equations.—When an equation consists of two fractions only, or can be expressed in the form of two fractions, its solution may be simplified by a judicious application of one or more of the following principles of composition and division.

Let $\frac{a}{b} = \frac{c}{d}$. Then

1°. $\frac{a-c}{b-d} = \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$.

2°. $\frac{a+b}{b} = \frac{c+d}{d}$.

3°. $\frac{a-b}{b} = \frac{c-d}{d}$.

4°. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

Prove the first of these cases by letting $\frac{a}{b} = \frac{c}{d} = x$. The remaining three have already been proved.

e.g. 1. Solve the equation $\frac{x-4}{x+4} = \frac{a-5}{a+5}$.

Applying 4°, $$\frac{(x-4)+(x+4)}{(x-4)-(x+4)} = \frac{(a-5)+(a+5)}{(a-5)-(a+5)},$$

or $$\frac{x-4+x+4}{x-4-x-4} = \frac{a-5+a+5}{a-5-a-5},$$

or $$\frac{2x}{-8} = \frac{2a}{-10},$$

or
$$\frac{x}{4} = \frac{a}{5}.$$

$$\therefore \quad 5x = 4a,$$

$$x = \frac{4a}{5},$$

e.g. 2. Solve the equation $\dfrac{x - 4 + b}{x + 4 - b} = \dfrac{x - 4}{x + 4}$.

Applying 1°, $\dfrac{(x - 4 + b) - (x - 4)}{(x + 4 - b) - (x + 4)} = \dfrac{x - 4}{x + 4},$

or
$$\frac{b}{-b} = \frac{x - 4}{x + 4}.$$

$$\therefore \quad \frac{x - 4}{x + 4} = \frac{-1}{1}.$$

Applying 4°, $\qquad \dfrac{2x}{8} = \dfrac{0}{2} = 0.$

$$\therefore \quad x = 0.$$

e.g. 3. Solve

$$(x+2)(x+5)(x+3)(x+8)=(x+1)(x+6)(x+4)(x+7).$$

Dividing both sides by $(x + 3)(x + 8)(x + 4)(x + 7)$, we have

$$\frac{(x + 2)(x + 5)}{(x + 4)(x + 7)} = \frac{(x + 1)(x + 6)}{(x + 3)(x + 8)}.$$

$$\therefore \quad \frac{x^2 + 7x + 10}{x^2 + 11x + 28} = \frac{x^2 + 7x + 6}{x^2 + 11x + 24}.$$

Applying 1°, we have

$$\frac{(x^2 + 7x + 10) - (x^2 + 7x + 6)}{(x^2 + 11x + 28) - (x^2 + 11x + 24)} = \frac{x^2 + 7x + 10}{x^2 + 11x + 28}.$$

or
$$\frac{4}{4} = \frac{x^2 + 7x + 10}{x^2 + 11x + 28} = \frac{1}{1}.$$

$$\therefore \quad x^2 + 7x + 10 = x^2 + 11x + 28,$$

or $$-4x = 18.$$

$$\therefore \quad x = -4\tfrac{1}{2}.$$

e.g. 4. Solve $(x-1)(2x-3)^2 = (x-3)(2x-1)^2$.

Dividing both sides by $(2x-3)^2(2x-1)^2$, we have

$$\frac{x-1}{(2x-1)^2} = \frac{x-3}{(2x-3)^2},$$

or $$\frac{x-1}{4x^2-4x+1} = \frac{x-3}{4x^2-12x+9}.$$

Applying 1°, we have

$$\frac{(x-1)-(x-3)}{(4x^2-4x+1)-(4x^2-12x+9)} = \frac{x-1}{4x^2-4x+1},$$

or $$\frac{2}{8(x-1)} = \frac{x-1}{4x^2-4x+1} = \frac{1}{4(x-1)}.$$

$$\therefore \quad 4(x-1)^2 = 4x^2-4x+1,$$

or $$4x^2-8x+4 = 4x^2-4x+1.$$

$$\therefore \quad -4x = -3.$$

$$\therefore \quad x = 3/4.$$

EXERCISE CXVIII.

Solve the following equations:

I.

1. $\dfrac{x-a}{a} = \dfrac{b-c}{c}$.

2. $\dfrac{x-5}{5} = \dfrac{b-7}{7}$.

3. $\dfrac{x-1}{x+1} = \dfrac{1-a}{1+a}$.

4. $\dfrac{x-3}{x+3} = \dfrac{3-c}{3+c}$.

5. $\dfrac{2x+3}{2x-3} = \dfrac{5}{2}$.

6. $\dfrac{3x-7}{3x+7} = \dfrac{7}{3}$.

7. $\dfrac{mx+n}{mx-n} = \dfrac{b+c-a}{c+a-b}$.

8. $\dfrac{3x+4}{3x-4} = \dfrac{c+a-b}{a+b-c}$.

9. $\dfrac{2x+1}{2x^2+2x+3}=\dfrac{1}{x+1}$. 10. $\dfrac{3x-1}{3x^2-3x+5}=\dfrac{1}{x-1}$.

II.

11. $(x+1)(2x+5)^2=4(x+2)^3$.

12. $(x-1)(x-2)(x+6)=(x+2)(x-3)(x+4)$.

13. $(x-1)(x-2)^2(x-5)=x(x-3)^2(x-4)$.

14. $\dfrac{6x^3+5x^2+6x+2}{6x^2+5x+3}=\dfrac{2x^2+x+1}{2x+1}$.

15. $\dfrac{9x^3+4x^2+8x+4}{9x^2+4x+5}=\dfrac{3x^2+2x+1}{3x+2}$.

C. VARIATION.

195. Direct Variation.—Suppose x and y to represent two variable quantities which depend upon each other in such a way that when one changes its value, the other must also change its value; and let x and y be so related that $y=mx$ (m being a constant), whatever be the value of x; and let x_1, x_2, x_3, etc., and y_1, y_2, y_3, etc., be corresponding values of x and y, so that $y_1=mx_1$, $y_2=mx_2$, etc.

Since $y=mx$ and $y_1=mx_1$,

$$\frac{y}{y_1}=\frac{mx}{mx_1}=\frac{x}{x_1}.$$

Whence $y:y_1=x:x_1$, or $x:y=x_1:y_1$.

When two quantities are thus related, one is said to *vary* as the other. Since the relation is mutual, we may say that y varies as x, or that x varies as y. The symbol $\propto$ denotes this relation, and is read "varies as" or "varies directly as." Thus $y \propto x$ is read "y varies as x"; and $x \propto y$, "x varies as y."

To say that y varies as x is to say that one is a constant multiple of the other, or that they so vary that their ratio remains constant, or that any two values of x and the corresponding values of y are in proportion.

It is a law of Optics that the intensity of the illumination upon a surface varies directly as the sine of the angle which the rays from the light make with the surface. That is, the larger the sine of this angle, or the more nearly perpendicular the rays are to the surface, the more intense is the illumination. If two surfaces are held at the same distance from the light, but one so as to make the angle-sine for the rays twice as great as for the other, the illumination of the former will be twice as intense as that of the latter; if the surface were held so as to make the angle-sine three times as great, the illumination would be three times as intense; and so on. While the illumination increases with the size of the angle, it does not increase in the same ratio. Hence the illumination does not vary as the angle.

196. Inverse Variation.—When y varies as x, or directly as x, as we have already seen, $y = mx$, m being a constant.

When $y = m\frac{1}{z}$, y is said to vary inversely as z. That is, y increases as z decreases, and *vice versa*, and both change at the same rate.

In the case of the light, the intensity of the illumination on a surface varies with the distance of the surface from the light, the intensity becoming less as the distance becomes greater, and the intensity changes at the same rate as the square of the distance changes. Hence we say that the intensity of the illumination varies inversely as the square of the distance from the light. If y denote the intensity of the illumination, z the distance from the light, and m the intensity of the illumination at a unit distance from the source, then $y = m\frac{1}{z^2}$, and $y \propto \frac{1}{z^2}$.

When $y = mx \cdot \frac{1}{z}$ or $m \cdot \frac{x}{z}$, y varies directly as x and inversely as z. In the case of the light already considered, if y denote the intensity of the illumination, x the sine of the angle which the rays make with the surface, and z the distance from the light, then $y = m\frac{x}{z^2}$. That is, the intensity of the illumination varies directly as the angle-sine and inversely as the square of the distance.

When $y = mwx$, y varies jointly as w and x.

If w denotes the intensity of the source of light, y the intensity of the illumination on the surface, x the angle-sine, and z the distance from the source, then $y = \frac{wx}{z^2}$. Express this relation in words.

197. The Constant of Variation.—In all the cases of variation, the *constant* (m) may be determined when any set of corresponding values is given; and when the constant and all but one of a set of corresponding values are known, the remaining one can be calculated.

e.g. 1. $A \propto B$, and when $A = 8$, $B = 6$. What will A equal when $B = 24$?

$$A = mB.$$

$$\therefore \quad 8 = 6m.$$

$$\therefore \quad m = 3/4.$$

$$\therefore \quad A = 3/4 \times 24 = 18.$$

2. $A \propto \frac{1}{B}$, and when $A = 8$, $B = 6$. What will A equal when $B = 24$?

$$A = m \cdot \frac{1}{B}.$$

$$\therefore \quad 8 = m \cdot \frac{1}{6}.$$

$$\therefore \quad 48 = m.$$

$$\therefore \quad A = 48 \,.\, \frac{1}{24} = 2.$$

3. $A \propto BC$, and when $A = 2$, $B = 6$ and $C = 4$. What will A equal when $B = 18$ and $C = 6$?

$$A = m \,.\, B \,.\, C.$$

$$\therefore \quad 2 = m \times 6 \times 4.$$

$$\therefore \quad m = 1/12.$$

$$\therefore \quad A = 1/12 \times 18 \times 6 = 9.$$

4. $A \propto B \,.\, \frac{1}{C}$, and when $A = 2$, $B = 6$ and $C = 4$. What will A equal when $B = 18$ and $C = 6$?

$$A = m \,.\, B \,.\, \frac{1}{C}.$$

$$\therefore \quad 2 = m \,.\, 6 \,.\, \frac{1}{4}.$$

$$\therefore \quad m = 4/3.$$

$$\therefore \quad A = 4/3 \,.\, 18 \,.\, 1/6 = 4.$$

EXERCISE CXIX.

I.

1. A varies as B, and when A is 6, B is 4. What is A when B is 9?

2. M varies inversely as N, and when M is 4, N is 13. What is M when N is 20?

3. A varies as B and C jointly, and $A = 3$ when $B = 5$ and $C = 4$. What is A when B is 8 and C is 3?

4. A varies as B and inversely as C, and $A = 4$ when $B = 6$ and $C = 8$. What is the value of A when $B = 18$ and $C = 6$?

5. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 10 is 314.16. What is the area of a circle whose radius is 20?

II.

6. The volume of a sphere varies as the cube of its radius, and the volume of a sphere whose radius is 1 foot is 4.188 cubic feet. What is the volume of a sphere whose radius is 5 feet?

7. The volume of a cone of revolution varies as its height and as the square of the radius of its base jointly, and the volume of a cone 7 feet high with a base whose radius is 3 feet is 66 cubic feet. Find the volume of a cone 14 feet high with a base whose radius is 18 feet.

8. The volume of a gas varies as the absolute temperature and inversely as the pressure upon it, and when the temperature is 280 and the pressure 15 the volume of a certain mass of a gas is one cubic foot. What would be its volume were the pressure 12 and the temperature 600?

9. The distance of the offing at sea varies as the square root of the eye above sea-level, and the distance is 3 miles when the height of the eye is 6 feet. What is the distance when the height is 72 yards?

10. The intensity of illumination varies as the sine of the angle which the rays make with the surface and inversely as the square of the distance from the source, and when the sine and distance are each unity the illumination is 40. What will be the illumination when the sine is 3/4 and the distance 8 units?

CHAPTER XXIV.

LOGARITHMS.

198. Definition of a Logarithm. — In the expression $a^x = y$, x is called the logarithm of y to the base a. This relation is indicated also by writing $x = \log_a y$.

The base a being some fixed positive number, to every value of y there is a corresponding value of x, and to every value of x there is a corresponding value of y, but these values are often incommensurable, so that they can be expressed only approximately.

The logarithm of a number may be defined in words as *the index of the power to which a given base must be raised to produce the number.*

A. GENERAL PROPERTIES OF LOGARITHMS.

199. The Working Rules of Logarithms.—

Let $\quad a^x = m, \quad$ and $\quad a^y = n.$

Then $\quad x = \log_a m, \quad$ and $\quad y = \log_a n.$

From these two equations we may deduce four important theorems:

1°. $\quad mn = a^x \,.\, a^y = a^{x+y};$

and $\quad \log_a(mn) = x + y;$

or $\quad \log_a(mn) = \log_a m + \log_a n.$

That is, *the logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

Of course this theorem may be extended readily to the product of any number of factors, and in its general form it would be:

The logarithm of any product is the sum of the logarithms of its factors.

2°. $$m \div n = a^x \div a^y = a^{x-y},$$

and $$\log_a(m \div n) = x - y,$$

or $$\log_a(m \div n) = \log_a m - \log_a n.$$

That is, *the logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

3°. $$m^p = (a^x)^p = a^{px},$$

and $$\log_a(m^p) = px,$$

or $$\log_a(m^p) = p \log_a m.$$

That is, *the logarithm of a power of a number is the logarithm of the number multiplied by the index of the power.*

4°. $$m^{1/p} = (a^x)^{1/p} = a^{x/p},$$

and $$\log_a(m^{1/p}) = 1/p \, . \, x,$$

or $$\log_a(m^{1/p}) = 1/p \log_a m.$$

That is, *the logarithm of a root of a number is the logarithm of the number divided by the index of the root.*

These four theorems are the working rules of logarithms as applied to numbers.

From these four theorems we see that addition of logarithms corresponds to multiplication of numbers, subtraction of logarithms to division of numbers, the multiplication of logarithms by numbers to the raising of numbers to powers, and the division of logarithms by numbers to the extraction of roots of numbers. There are no operations on logarithms which correspond to the addition and subtraction of numbers, and there is no operation on numbers in ordinary arithmetic which corresponds to the raising of logarithms to powers or to the extraction of their roots.

200. Systems of Logarithms.—The general properties of logarithms are the same for all bases, and any positive number, rational or irrational, may be taken as a base. Certain numbers, however, offer special advantages as bases in working with logarithms and in calculating them. The base which is most advantageous for numerical computation is 10, and the one most advantageous for theoretical investigation is the incommensurable 2.7182818 . . . . The former is the base of the system of logarithms in common use, and the latter of the *Napierean,* or *natural,* system of logarithms.

201. Common Logarithms.—When the base of the system is 10, the 10 is omitted after the abbreviation "log." Thus, $\log 100 = 2$, means that 10 must be raised to the second power to produce 100. Written in full the expression would be $\log_{10} 100 = 2.$

$$1 = 10^0, \quad \therefore \log 1 = 0.$$
$$10 = 10^1, \quad \therefore \log 10 = 1.$$
$$100 = 10^2, \quad \therefore \log 100 = 2.$$
$$1000 = 10^3, \quad \therefore \log 1000 = 3.$$
$$\text{etc.}$$

Whenever a number is an integral power of ten, its logarithm is a positive integer, and is equal to one less than the number of places in the number to the left of the decimal point.

$$.1 = \frac{1}{10^1} = 10^{-1}, \quad \therefore \log .1 = -1.$$
$$.01 = \frac{1}{10^2} = 10^{-2}, \quad \therefore \log .01 = -2.$$
$$.001 = \frac{1}{10^3} = 10^{-3}, \quad \therefore \log .001 = -3.$$
$$0 = \frac{1}{10^\infty} = 10^{-\infty}, \quad \therefore \log. \ 0 = -\infty.$$

The logarithm of 0 is negative infinity. The logarithm of a negative number is imaginary. Whenever a number is a decimal and equal to 1 divided by an integral power of 10, its logarithm is a negative integer and is equal to one more than the zeros to the right of the decimal point.

Inasmuch as the logarithm of any number to base 10 or any base greater than 1 increases with the number, it is evident from the above that the logarithm of any number greater than one is positive, and the logarithm of any numless than one is negative; also that the logarithm of any number between 1 and 10 lies between 0 and 1, and is a positive decimal; that the logarithm of any number between 10 and 100 lies between 1 and 2, and is 1 plus a positive decimal; and so on. It is further evident that the logarithm of any number between 1 and .1 lies between 0 and − 1, and is − 1 plus a decimal; that the logarithm of any number between .1 and .01 lies between − 1 and − 2, and is − 2 plus a decimal; and so on.

202. The Characteristic and Mantissa of a Logarithm.—In general, the logarithm of a number is composed of two parts, an integer and a decimal. The decimal part of a logarithm is incommensurable, and therefore cannot be expressed exactly. It is called the *mantissa* of the logarithm, and is always taken as positive.

The integral part of a logarithm is positive or negative according as the number is greater or less than one. It is called the *characteristic* of the logarithm.

The method of computing logarithms cannot be considered here. Its discussion is a matter of Higher Algebra. It has been found that

$$6742 = 10^{3.8276+}, \quad \therefore \quad \log 6742 = 3.8276 +.$$

Now $$67420 = 6742 \times 10 = 10^{3.8276} \times 10^{1} = 10^{4.8276},$$

$$\therefore \quad \log 67420 = 4.8276;$$

and $$674200 = 6742 \times 100 = 10^{3.8276} \times 10^2 = 10^{5.8276},$$

$$\therefore \quad \log 674200 = 5.8276.$$

Also, $$674.2 = 6742 \div 10 = 10^{3.8276} \div 10^1 = 10^{2.8276},$$

$$\therefore \quad \log 674.2 = 2.8276;$$

and $$67.42 = 6742 \div 100 = 10^{3.8276} \div 10^2 = 10^{1.8276},$$

$$\therefore \quad \log 67.42 = 1.8276,$$

etc.

We see from the above that so long as the figures of a number and their arrangement are the same, the mantissa of the logarithm is the same no matter what position the group of figures may occupy in the scale of enumeration. The shifting of the group of figures one place to the left increases the logarithm by unity, because it multiplies the number by 10, and the shifting the group of figures one place to the right diminishes the logarithm by unity, because it divides the number by 10.

This property of logarithms is peculiar to the system whose base is 10, and is of very great practical importance.

203. Logarithmic Tables.—The mantissæ of the logarithms of all numbers from 1 to 99999 have been calculated, and published in the form of tables. In these tables the approximation in the mantissæ is carried sometimes to four, sometimes to five, sometimes to six, and sometimes to seven decimal places, giving rise to tables of four-place, five-place, six-place, and seven-place logarithms. The characteristics of the logarithms are not given in these tables, because these can be found by inspection of the numbers.

The following table contains the mantissæ of the logarithms of all integers from 100 to 1000, calculated to four places of decimals, and from it can be found approximately the logarithms of all numbers.

COMMON LOGARITHMS.

n	0	1	2	3	4	5	6	7	8	9	d
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	40
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	37
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	33
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	31
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	29
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	27
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	25
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	24
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	23
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	21
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	17
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	14
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	13
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	12
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	8
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	8

COMMON LOGARITHMS.

n	0	1	2	3	4	5	6	7	8	9	d
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	7
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	6
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	6
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

204. Method of Using Logarithmic Tables.—In using a table of logarithms there are two operations, one of which is the inverse of the other: 1°. To find the logarithm of a given number; 2°. To find the number which has a given logarithm.

B. TO FIND THE LOGARITHM OF A NUMBER.

1°. *When the Number has not more than Three Figures.*—First determine the characteristic by inspection and write it down. Then look in the column headed n for the first two figures of the number, and at the top of the columns for the third figure. The required mantissa will be in the horizontal line of the first two figures and in the column which has the third figure at the top. This mantissa should be written after the characteristic already found.

e.g. Find the logarithm of 687.

The characteristic is 2, and the mantissa found in the horizontal line of 68 in the left-hand column and in the column of 7 at the top is 8370. Therefore

$$\log 687 = 2.8370.$$

When the characteristic is negative, the minus sign should be written above it, to indicate that it is the characteristic alone which is negative. The mantissæ of the tables are always positive. Thus

$$\log .0687 = \bar{2}.8370.$$

When the number consists of two figures only, the mantissa is found in the column headed 0. Thus,

$$\log 68 = 1.8325.$$

When the number consists of one figure only, consider the second figure as zero, and take the mantissa from the column headed 0. Thus the mantissa of 6 is found in the horizontal line of 60 in the column headed 0.

$$\log 6 = 0.7782.$$

2°. *When the Number has more than Three Figures.*—When a number has more than three figures, use must be made of the principle that when the difference of two numbers is small compared with either of them, these differences are approximately proportional to the differences of their logarithms. This principle is called the *Principle of Proportional Differences.*

e.g. Find the logarithm of 34567.

$$\begin{array}{rl} \log 34500 = & 4.5378 \\ \log 34600 = & 4.5391 \\ \hline \text{Difference of the mantissæ} = & 13 \end{array}$$

Thus a difference of one unit in the third place corresponds to a difference of 13 in the logarithms. But the given number differs from 34500, not by a whole unit in the third place, but only by .67 of that unit. Therefore the difference between the logs of 34500 and 34567 will be only .67 of 13 = 8.71, which we take as 9, the nearest integer.

$$\text{Therefore} \qquad \log 34567 = \begin{array}{r} 4.5378 \\ 9 \\ \hline 4.5387 \end{array}$$

The difference between one mantissa and the next following in the tables is called the *tabular* difference, and the result obtained by multiplying this by the following figures of the number considered as a decimal is called the *real* difference.

It is never necessary to use more than three of the following figures for a multiplier, and seldom more than two.

From the above we have the following rule for finding the logarithm of a number of more than three figures:

Find the mantissa of the first three figures of the number, and the tabular difference.

Multiply this tabular difference by the next two or three figures of the number, considered as a decimal, and add the result to the mantissa already found.

The tabular difference should be taken from the table at sight. To facilitate this operation, the difference between the last mantissa in one horizontal line and the first of the next is given in the last column, headed D.

EXERCISE CXX.

Find the logarithms of the following numbers:

1. 956. **2.** 58.7. **3.** 2.38.
4. .0325. **5.** 50. **6.** .003.
7. 40000. **8.** 2. **9.** .000007.
10. 28645. **11.** 16.327. **12.** .003579.
13. 2.468. **14.** 8.006.

C. TO FIND A NUMBER WHICH HAS A GIVEN LOGARITHM.

1°. *When the Exact Mantissa is found in the Tables.*— Find the mantissa in the table, and take out as the first two figures of the number the two figures of the column headed N which are on the horizontal line of the mantissa, and as the third figure of the number the one at the top of the column in which the mantissa is found, and point off according to the characteristic.

e.g. Find the number whose logarithm is $\overline{1}.9112$.

Find 9112 in the table and take 81 from the left-hand end of its horizontal line and 5 from the top of its column, and place the decimal point before the 8.

$$\log^{-1}\overline{1}.9112 = .815.$$

The symbol $\log^{-1}$ means *the number whose log is.*

2°. *When the Exact Mantissa is not found in the Table.*—Take out from the table the next smaller mantissa, the first three figures of the corresponding number, and the tabular difference, and find the real difference between this mantissa and the one given. Divide the real difference by the tabular difference to two or, at most, three places in the quotient, annex these figures to the three already taken out, and point off according to the characteristic. The result is seldom trustworthy to even two places.

It will be seen at once that this process is the reverse of that for finding the correction of the mantissa when the number has more than three figures.

EXERCISE CXXI.

Find the numbers which have the following logarithms:

1. 2.9355.
2. $\bar{1}.5635$.
3. $\bar{2}.9948$.
4. 3.8845.
5. 0.5982.
6. $\bar{3}.8340$.
7. $\bar{1}.4570$.
8. $\bar{2}.9559$.
9. 0.8077.

205. Cologarithms.—The cologarithm of a number is the logarithm of the reciprocal of the number.

$$\text{Thus,}\quad \text{colog } 987 = \log \frac{1}{987} = \log 1 - \log 987$$
$$= 0 - 2.9943$$
$$= -2.9943.$$

To avoid the negative mantissa, the logarithm of the number is usually subtracted from 10 instead of 0.

$$\text{Thus,}\qquad \text{colog } 987 = 10 - \log 987,$$
$$\text{or}\qquad 10 - 2.9943 = .0057.$$

Of course this logarithm is 10 too large. Such a logarithm is called an *augmented* logarithm.

The colog should be taken from the table at sight. We may begin at the left hand and take each figure from 9 till we come to the last, which should be taken from 10.

EXERCISE CXXII.

Find the cologarithms of the following numbers:

1. 3784.
2. 3959.
3. 2895.
4. .4265.
5. .078976.
6. .008.
7. 50.
8. .0008.
9. .00009.

D. ARITHMETICAL OPERATIONS.

206. Multiplication by Logarithms.—To multiply two or more factors together by means of logarithms, find the logarithm of each factor, add these logarithms and then find the number which corresponds to this resulting logarithm.

e.g. Find the product of 897, 564, and .0078.

$$\begin{aligned} \log\ 897 &= 2.9528 \\ \log\ 564 &= 2.7513 \\ \log .0078 &= \bar{3}.8921 \\ \hline & \ 3.5962 \end{aligned}$$

$$\log^{-1} 3.5962 = 3946.4$$

207. Division by Logarithms.—To divide one factor by another by means of logarithms, find the logarithm of each factor, subtract the logarithm of the divisor from that of the dividend, and then find the number which corresponds to the logarithm thus obtained.

As in many practical applications it is necessary to perform both multiplication and division in the same example, it is preferable in all cases to use the cologarithms of the factors of the divisor and add these to the logarithms of the multiplication factors.

This method is based upon the principle that to divide by a factor is equivalent to multiplying by its reciprocal. In using cologarithms it must be borne in mind that each colog is augmented, and, therefore, that as many 10's must be rejected from the result as there are cologs used.

e.g. Find the value of $\frac{526 \times 862}{232 \times 683}$.

$$\begin{aligned} \log 526 &= 2.7210 \\ \log 862 &= 2.9355 \\ \text{colog } 232 &= 7.6345 \\ \text{colog } 683 &= 7.1656 \\ \hline &\ 20.4566 \end{aligned}$$

$$\log^{-1} 0.4566 = 2.8613.$$

208. Involution by Logarithms.—To raise a number to a power by means of logarithms, find the logarithm of the number, multiply it by the index of the power, and find the number which corresponds to the resulting logarithm.

e.g. Raise 249 to the sixth power.

$$\begin{aligned} \log (294)^6 &= 2.4683 \times 6 \\ &= 14.8098. \end{aligned}$$

$$\log^{-1} 16.8098 = 645330000000000 \text{ approximately.}$$

209. Evolution by Logarithms.—To find the root of a number by means of logarithms, take out the logarithm of the number, divide it by the index of the root, and find the number which corresponds to the resulting logarithm.

If the characteristic of the logarithm is negative, before dividing by the index, add as many tens to it as there are units in the index of the root, and reject ten from the resulting logarithm, which would be augmented by 10. For this process consists in adding and subtracting the same multiple of 10 and then dividing by the index of the root.

e.g. Find the fifth root of .086.

$$\begin{aligned}\log (.086)^{1/5} &= \bar{2}.9345 \div 5\\ &= (48.9345 - 50) \div 5\\ &= (48.9345 \div 5) - 10\\ &= \bar{1}.7869.\end{aligned}$$

$$\log^{-1} \bar{1}.7869 = .6121, \text{ approximately.}$$

EXERCISE CXXIII.

NOTE.—A negative quantity has no real logarithm. If such quantities occur in computation, they may be treated as if they were positive and then the sign of the result determined by the number of negative factors. If this number be even, the result will be positive, and if odd, negative. In arranging the logarithms and cologarithms for addition, it is best to place an n after each one which has been found for a negative factor, and then a glance will show whether the resulting number should be positive or negative.

e.g. Find the value of $\dfrac{23 \times -8 \times -6}{5 \times -60}$.

$$\begin{aligned}\log 23 &= 1.3617\\ \log 8 &= 0.9031n\\ \log 6 &= 0.7782n\\ \text{colog } 5 &= 9.3010\\ \text{colog } 60 &= 8.2218n\\ \hline &\ 20.5658n\end{aligned}$$

$$\log^{-1} 0.5658n = -3.68.$$

Find by logarithms the values of the following:

I.

1. $250.42 \times .00687$. 2. $-7.8346 \times -.086427$.

3. -9.896×12.857. 4. $.04632 \times .008764$.

5. $\frac{.08}{7}$

6. $\frac{-9.876}{.0076}$.

7. $\frac{18.009 \times -.004}{.007695 \times .004}$.

8. $\frac{27 \times -82}{3.8 \times -4.9}$.

9. $(86.42)^3$.

10. $(.0086)^3$.

II.

11. $9^{2/3}$.

12. $\sqrt[3]{5}$.

13. $(-3.278)^5$.

14. $19^{2/3}$.

15. $(.12)^{6/5}$.

16. $\sqrt[5]{70}$.

17. $(-.000874)^{5/7}$.

18. $\sqrt[5]{.0009286}$.

19. $5^{3/2} \times 3^{2/3}$.

20. $\frac{4^{3/8}}{5^{2/5}}$.

21. $\frac{564^{3/5}}{283}$.

22. $\sqrt[5]{\frac{2}{5}} \div \sqrt[3]{-\frac{5}{3}}$.

210. THEOREM. *The logarithm of any number to base b is equal to the product of the logarithm of the number to the base a by logarithm of a to base b.*

It is required to prove $\log_b m = \log_a m \,.\, \log_b a$.

Let $\log_a m = x$, and $\log_b m = y$.

Then $m = a^x$,

and $m = b^y$.

$$\therefore \quad a^x = b^y.$$

Hence $a = b^{y/x}$.

And $a^{x/y} = b$.

$$\therefore \quad \frac{x}{y} = \log_a b,$$

and, similarly, $$\log_b a = \frac{y}{x}.$$

$$\therefore\quad y = x \log_b a.$$

$$\therefore\quad \log_b m = \log_a m \,.\, \log_b a,$$

or $$\frac{\log_b m}{\log_b a} = \log_a m.$$

It follows from the above theorem that if the logarithm of any number to base b is known, its logarithm to any other base a may be found by dividing the logarithm of the number to base b by the logarithm of a to base b.

e.g. Find log 3 to base 7.

$$\log_{10} 3 = 0.4771.$$

$$\log_{10} 7 = 0.8451.$$

$$\log_7 3 = \frac{0.4771}{0.8451} = .5643.$$

EXERCISE CXXIV.

Find the following logarithms:

1. $\log_2 15$. **2.** $\log_3 42$. **3.** $\log_4 8$.

4. $\log_8 .0803$. **5.** $\log_{15} .007008$. **6.** $\log_9 56.31$.

When the number can be expressed as an exact power of the base, examples like the above may be solved by inspection.

e.g. Find the value of $\log_{16} 128$.

$$128 = 16^{7/4}.$$

$$\therefore\quad \log_{16} 128 = 7/4.$$

7. $\log_3 729$. **8.** $\log_{25} 3125$. **9.** $\log_{64} 1/4$.

PART II

ELEMENTARY SERIES

CHAPTER XXV.

VARIABLES AND LIMITS.

211. Constants and Variables. — A number which, under the conditions of the problem into which it enters, may assume any one of an unlimited number of values is called a *variable.*

A number which, under the conditions of the problem into which it enters, has a fixed value is called a *constant.*

Variables are usually represented by the last letters, x, y, z, etc., of the alphabet, and constants either by the first letters, a, b, c, etc., or by Arabic numerals.

212. Functions.—Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case one variable is said to be a *function* of the other.

When one of two variables is a function of the other the relation between them may be expressed by an equation. Thus, if x and y are functions of each other, we may say that $\frac{x}{y} = a$, or $x = ay$, or $y = \frac{x}{a}$.

Hence, if the value of one variable be assumed, the corresponding value of the other variable may be computed. The variable for which values are assumed is called the *independent* variable; and the one whose value is found by computation, the *dependent* variable.

When an equation containing two variables is solved for one of them, the variable involved in the answer is regarded as the independent variable.

Thus, in equation $x = ay$, y is regarded as the independent variable; and in the equation $y = \frac{x}{a}$, x is regarded as the independent variable.

213. Limit of a Variable.—As a variable changes, its value may approach some constant. If the variable can be made to approach a constant as near as we please without ever becoming absolutely equal to it, the constant is called the *limit* of the variable.

214. Axioms.—Any quantity, however small, may be taken times enough to exceed any other fixed quantity, however great.

Conversely, any quantity, however great, may be divided into so many parts that each part shall be less than any other fixed quantity, however small.

215. THEOREM I. *If a fraction have a finite numerator and an independent variable for its denominator, we may assign to this denominator a value so great that the value of the fraction shall be less than any assignable value.*

Let a be the numerator of the fraction, x its denominator, and c any finite value, however small, which we may choose to assign. And let n be the number of times that we must take c to make it greater than a. Then

$$a < nc.$$

$$\therefore \quad \frac{a}{n} < c.$$

Hence, by taking x greater than n, we shall have

$$\frac{a}{x} < c.$$

216. THEOREM II. *If a fraction have a finite numerator and an independent variable for its denominator, we* *may assign to this denominator a value so small that the value of the fraction shall exceed any assignable value.*

Let a be the numerator of the fraction, x its denominator, and c any finite value, however large, which we may choose to assign.

Let n be a number greater than c. Divide a into n parts, and let b be one of them. Then

$$a = nb.$$

$$\therefore \quad \frac{a}{b} = n > c.$$

Hence, if we take x less than b,

$$\frac{a}{x} > n > c.$$

217. Infinites.—If a variable can become greater than any assigned value, however great that value may be, the variable is said to *increase indefinitely*, or to *increase without limit.*

When a variable is conceived to have a value greater than any assigned value however great, the variable is said to become *infinite.* Such a variable is called an *infinite* number, or simply an *infinite.* An infinite is usually denoted by the symbol ∞.

It must be borne in mind that this symbol denotes, not a constant, but a variable, which has already increased beyond any assignable limit, but which is still capable of an indefinite increase.

218. Infinitesimals.—If a variable can become less than any assignable value, however small that value may be, the variable is said to *decrease indefinitely*, or *decrease without limit.*

In this case the variable approaches zero as a limit.

When a variable which approaches zero as a limit is conceived to have a value less than any assigned value, however small this value may be, the variable is said to become *infinitesimal.* Such a variable is called an *infinitesimal number*, or simply an *infinitesimal.* An infinitesimal is often denoted by the symbol 0, which in this case must be understood to represent an exceedingly small variable.

We often express the relation between finite quantities and infinite and infinitesimal quantities as follows:

$$\frac{a}{0} = \infty, \ \frac{a}{\infty} = 0.$$

The expression $\frac{a}{0} = \infty$ cannot be interpreted literally, since we cannot divide by absolute 0; nor can the expression $\frac{a}{\infty} = 0$ be interpreted literally, since we cannot find a number so large that the quotient obtained by dividing a by it shall be absolute zero.

The expression $\frac{a}{0} = \infty$ is simply an abbreviated way of writing: when x approaches zero as its limit, then $\frac{a}{x}$ increases without limit.

$\frac{a}{\infty} = 0$ is simply an abbreviated way of writing: when x increases without limit, then $\frac{a}{x}$ approaches zero as its limit.

219. Approach to a Limit. — When a variable approaches a limit, it may approach it in one of three ways:

1°. The variable may be always less than its limit;

2°. The variable may be always greater than its limit;

3°. The variable may be alternately greater and less than its limit.

If x represent the sum of n terms of the series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots,$$

x is always less than its limit 2.

If x represent the sum of n terms of the series

$$3-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\cdots,$$

x is always greater than its limit 2.

If x represent the sum of n terms of the series

$$3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\cdots,$$

x is alternately less and greater than its limit 2.

220. THEOREM III. *If k be any fixed quantity however great, and x be a variable which we may make as small as we please, we may make the product kx less than any assignable quantity.*

If there be any smaller value of kx, let it be denoted by s. Since we may make x as small as we please, let us put

$$x<\frac{s}{k}.$$

$$\therefore \quad kx<s,$$

so that s cannot be the smallest value of the product. Hence the product cannot have a smallest value.

221. THEOREM IV. *If two functions are equal they must have the same limit.*

Assume it possible for the two functions to have different limits, and denote these limits by L and L'. Put

$$s=\frac{1}{2}(L-L'),$$

so that L and L' differ by $2s$.

Now since L is the limit of one function, that function may be made to approach L so as to differ from it by less than s, and since L' is the limit of the other function, this function may be made to approach L' so as to differ from it by less than s. And as the difference between L and $L' = 2s$, the functions in the above case must be unequal. But this is contrary to the hypothesis. Hence it is impossible for the functions to have different limits.

222. THEOREM V. *The limit of the sum of several functions is equal to the sum of their separate limits.*

Let the functions be denoted by $f(x)$, $f(x')$, $f(x'')$, etc., and their limits by L, L', L'', etc.; and let the differences from their limits be denoted by i, i', i'', etc. Then

$$f(x) = L - i,$$
$$f(x') = L' - i',$$
$$f(x'') = L'' - i'',$$
$$\text{etc.} \quad \text{etc.}$$
$$\therefore \quad f(x) + f(x') + f(x'') + \text{etc.}$$
$$= L + L' + L'' + \text{etc.} - (i + i' + i'' + \text{etc.}).$$

We must now prove that $i + i' + i'' +$ etc. can be made less than any quantity we can assign.

Let h denote this quantity, which may be as small as we please;

n denote the number of the quantities i, i', i'', etc.;

and i denote the largest of them.

Since the difference between a function and its limit may be made as small as we please, we may make

$$i < \frac{h}{n}, \quad \text{or} \quad ni < h.$$

But $i + i' + i'' + \text{etc.} < ni$, ($i$ being the largest.)

$$\therefore \quad i + i' + i'' + \text{etc.} < h.$$

Therefore $L + L' + L'' +$ etc. is the limit of

$$f(x) + f(x') + f(x'') + \text{etc.}$$

223. Theorem VI. *The limit of the product of two functions is equal to the product of their separate limits.*

Using the notation of Theorem V, we have

$$\begin{aligned} f(x) \times f(x') &= (L - i)(L' - i') \\ &= L \, . \, L' - (Li + L'i - ii'). \end{aligned}$$

Now as L and L' are finite, $Li' + L\,i$ can be made as small as we please, and therefore the quantity within the parenthesis may be made as small as we please. Hence

$$L \, . \, L' \text{ is the limit of } f(x) \times f(x').$$

Cor. 1. *The limit of the product of any number of functions is equal to the product of their limits.*

Cor. 2. *The limit of any power of a function is equal to the power of its limits when these limits are not both zero.*

224. Theorem VII. *The limit of the quotient of two functions is equal to the quotient of their limits when their limits are not both zero.*

Using the same notation as before, we have

$$\frac{f(x)}{f(x')} = \frac{L - i}{L' - i'}.$$

Now the difference between $\frac{L}{L'}$ and $\frac{L - i}{L' - i'}$, is

$$\frac{L'i - Li'}{L'(L' - i')}.$$

The numerator of this expression evidently approaches zero as its limit, and the denominator approaches L'^2 as its limit.

Hence the expression as a whole has zero for its limit when L' is not itself zero.

225. Definition.—The expressions

$$f[x]_{=a}, \quad \left.\frac{x^n - a^n}{x - a}\right]_{=a},$$

denote the value of these expressions when x becomes equal to a.

226. THEOREM VIII. The formula

$$\text{Lim.} \left.\frac{x^n - a^n}{x - a}\right]_{=a} = na^{n-1}$$

is true for all rational values of n.

CASE I. *When n is a positive integer.*

We have, when x is different from a,

$$\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1}.$$

Now suppose x to approach the limit a. Then x^{n-1} will approach the limit a^{n-1}, x^{n-2} the limit a^{n-2}, etc. Hence ax^{n-2}, a^2x^{n-3}, etc., will each approach the limit a^{n-1}. That is, each term of the second member approaches the limit a^{n-1}. Because there are n such terms, we have

$$\text{Lim.} \left.\frac{x^n - a^n}{x - a}\right]_{=a} = na^{n-1}.$$

CASE II. *When n is a positive fraction.*

Suppose $n = \frac{p}{q}$, p and q being whole numbers. Then

$$\frac{x^n - a^n}{x - a} = \frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x - a}.$$

Let us put, for convenience in writing,

$$x^{\frac{1}{q}} = y, \qquad a^{\frac{1}{q}} = b;$$

then $$x = y^q, \qquad a = b^q,$$

and $$\frac{x^n - a^n}{x - a} = \frac{y^p - b^p}{y^q - b^q} = \frac{\dfrac{y^p - b^p}{y - b}}{\dfrac{y^q - b^q}{y - b}}.$$

As x approaches indefinitely near to a, and consequently y to b, the numerator of this fraction (Case I) approaches to pb^{p-1} as its limit, and the denominator to qb^{q-1}. Hence the fraction itself approaches to

$$\frac{pb^{p-1}}{qb^{q-1}} = \frac{p}{q} b^{p-q}.$$

Substituting for b its value $a^{\frac{1}{q}}$, we have

$$\text{Lim.} \left.\frac{x^n - a^n}{x - a}\right]_{=a} = \frac{p}{q} b^{p-q} = \frac{p}{q} a^{\frac{p-q}{q}} = \frac{p}{q} a^{\frac{p}{q}-1}.$$

Hence the same formula holds when n is a positive fraction.

CASE III. *When n is negative.*

Suppose $n = -p$, p itself (without the minus sign) being supposed positive. Then

$$\frac{x^n - a^n}{x - a} = \frac{x^{-p} - a^{-p}}{x - a} = x^{-p}a^{-p}\left(\frac{a^p - x^p}{x - a}\right)$$

$$= -x^{-p}a^{-p}\left(\frac{x^p - a^p}{x - a}\right).$$

When x approaches a, then x^{-p} approaches a^{-p}, and $\dfrac{x^p - a^p}{x - a}$ approaches pa^{p-1}. Substituting these limiting values, we have

$$\text{Lim.} \left.\frac{x^n - a^n}{x - a}\right]_{=a} = -a^{-2p}pa^{p-1} = -pa^{-p-1}.$$

Substituting for $-p$ its value n, we have

$$\text{Lim.} \left.\frac{x^n - a^n}{x - a}\right]_{= a} = na^{n-1}.$$

Hence the formula

$$\text{Lim.} \left.\frac{x^n - a^n}{x - a}\right]_{= a} = na^{n-1}$$

is true for all values of n, whether entire or fractional, positive or negative.

227. Definition of Series.—A *series* is a succession of terms formed in order according to some definite law.

228. THEOREM IX. *The limit of the series*

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \dots$$

when x is indefinitely diminished is A_0, *provided all the coefficients are finite and the coefficient of the nth term approaches a finite limit as n is indefinitely increased.*

1°. Suppose the number of terms of the series to be infinite.

Let k denote the greatest of the coefficients A_1, A_2, etc., and denote the series by $A_0 + S$.

Since k is the largest of the coefficients A_1, A_2, etc.,

$$\therefore \quad kx + kx^2 + kx^3 + \text{etc.} > A_1x + A_2x^2 + A_3x^3 + \text{etc}$$

$$\therefore \quad S < kx + kx^2 + kx^3 + \text{etc.}$$

But $kx + kx^2 + kx^3 + \text{etc.}$ may be written in the form of the fraction $\frac{kx}{1-x}$, as may be shown by actual division of the numerator by the denominator.

$$\therefore \quad S < \frac{kx}{1 - x},$$

which, when x is indefinitely diminished, can be made as small as we please.

Hence by indefinitely diminishing x, A_0 can be made to differ from the series by less than any assignable quantity. Hence A_0 becomes the limit of the series.

2°. If the number of terms in the series is finite, S must be less than in case 1°; hence, *a fortiori*, the theorem is true.

229. THEOREM X. *In the series*

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \dots$$

by taking x small enough we may make any term as large as we please compared with the sum of all that follow it, and, by taking x large enough, we can make any term as large as we please compared with the sum of all that precede it.

1°. The rth term of the series will be A_rx^r, and the ratio of this to the sum of all the terms that follow will be

$$\frac{A_rx^r}{A_{r+1}x^{r+1} + A_{r+2}x^{r+2} + \dots} = \frac{A_r}{A_{r+1}x + A_{r+2}x^2 + \dots}.$$

By taking x small enough we can make the denominator of this last fraction as small as we please, and therefore the fraction itself as large as we please.

2°. The ratio of the rth term to the sum of all that precede it will be

$$\frac{A_rx^r}{A_{r-1}x^{r-1} + A_{r-2}x^{r-2} + \dots} = \frac{A_r}{A_{r-1}\frac{1}{x} + A_{r-2}\frac{1}{x^2} + \dots}.$$

By taking x large enough we may make the denominator of this last fraction as small as we please, and therefore the fraction as large as we please.

COR. In an expression of the form

$$A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0,$$

consisting of a finite number of terms in descending powers of x, by taking x small enough we may disregard all the terms but the last, and by taking x large enough we may disregard all the terms but the first.

230. Vanishing Fractions.—A fraction which assumes the form $\frac{0}{0}$ for some particular value of x is called a *vanishing fraction.*

The fraction, though indeterminate in form when x has this critical value, has a real value. To determine this value is to *evaluate* the fraction.

Sometimes for a particular value of x the fraction assumes the form $\frac{\infty}{\infty}$, which is also indeterminate in form.

The values of the fractions when they assume these indeterminate forms are really the limiting values of the fractions as x is indefinitely increased or diminished.

The limiting value of a fraction when x in both numerator and denominator is indefinitely increased or diminished may be found by Theorem X, cor.

e.g. Find the limiting value of $\frac{4x^3 - 6x^2 + 7}{3x^3 + 7x^2 - 4}$ when x is infinite and when x is zero.

1°. When $x = \infty$, every term except the first of the numerator may be disregarded, and we have as the limiting value

$$\frac{4x^3}{3x^3} = \frac{4}{3}.$$

2°. When $x = 0$, every term except the last of the numerator and denominator may be disregarded, and we have as the limiting value $-\frac{7}{4}$.

The limiting value of a fraction which assumes an indeterminate form for a critical value of x may be found by first removing from the numerator and denominator all common factors in x, and substituting the critical value of x in the result.

e.g. Find the limiting value of

$$\frac{x^2 - 4ax + 3a^2}{x^2 - a^2}$$

when $x = a$.

$$\frac{x^2 - 4ax + 3a}{x^2 - a^2} = \frac{(x-a)(x-3a)}{(x-a)(x+a)} = \frac{x-3a}{x+a}.$$

Put $x = a$ in this result, and we have

$$-\frac{2a}{2a} = -1.$$

EXERCISE CXXV.

Find the limiting values of the following:

1. $\left.\frac{x-a}{x}\right]_{=\infty}.$ 2. $\left.\frac{x-a}{x}\right]_{=0}.$

3. $\left.\frac{ax+b}{bx+a}\right]_{=\infty}.$ 4. $\left.\frac{ax+b}{bx+a}\right]_{=0}.$

5. $\left.\frac{mx^2}{px^2-ax}\right]_{=\infty}.$ 6. $\left.\frac{mx^2}{px^2-ax}\right]_{=0}.$

7. $\left.\frac{(2x-3)(3-5x)}{7x^2-6x+4}\right]_{=\infty}.$ 8. $\left.\frac{(2x-3)(3-5x)}{7x^2-6x+4}\right]_{=0}.$

9. $\left.\frac{x^2-a^2}{x-a}\right]_{=a}.$ 10. $\left.\frac{x^5-z^5}{x-z}\right]_{=z}.$

11. $\left.\frac{x^3+1}{x^2-1}\right]_{=-1}.$ 12. $\left.\frac{x^2-8x+15}{x^2-7x+12}\right]_{=3}.$

231. Discussion of Problems.—To discuss the solution of a problem when the answer is literal is to observe between what limiting numerical values of the known elements the problem is possible, and whether any singularities or remarkable circumstances occur within these limits.

The following discussions will serve to illustrate the significance of indeterminate forms of expression, and of 0 and ∞ as limiting values.

a. The Product of Two Quantities whose Sum is Constant.

Divide a into two parts whose product shall equal b.

Let x and y denote the parts. Then, by the conditions,

$$x + y = a. \qquad (1)$$

$$xy = b. \qquad (2)$$

From (1), $$y = a - x.$$

By substitution in (2),

$$x(a - x) = b.$$

$$\therefore \quad x^2 - ax + b = 0;$$

whence $$x = \frac{1}{2}a \pm \sqrt{\frac{a^2}{4} - b},$$

and the two parts are $\frac{1}{2}a + \sqrt{\frac{a^2}{4} - b}$,

and $$\frac{1}{2}a - \sqrt{\frac{a^2}{4} - b}.$$

Now these values are imaginary if $b > \frac{a^2}{4}$; that is, if the product of the two parts is greater than the square of half their sum.

COR. *The product of two quantities cannot be greater than the square of half their sum.*

Or, *the product of two parts of a given quantity is greatest when those parts are equal.*

The two parts will be incommensurable when the difference between their product and the square of half their sum is not a perfect square.

b. The General Quadratic Equation.

The equation

$$ax^2 + bx + c = 0$$

has been discussed already in so far as to observe when the values of x become imaginary, when they are real and rational, when real and irrational, and when equal.

We will now discuss some peculiarities which may arise by the vanishing of each of the coefficients in turn.

Note that c is really the coefficient of x^0.

If $c = 0$, then

$$ax^2 + bx = 0; \tag{1}$$

whence $$x = 0, \text{ or } -\frac{b}{a}.$$

That is, one of the roots is zero and the other is finite.

If $b = 0$, then

$$ax^2 + c = 0; \tag{2}$$

whence $$x = \pm\sqrt{-\frac{c}{a}}.$$

In this case the roots are equal in value and opposite in sign.

They will be real or imaginary according as a and c have opposite signs or the same sign.

If $a = 0$, then

$$bx + c = 0; \qquad (3)$$

and apparently in this case the quadratic has but one root, namely, $-\frac{c}{b}$. But every quadratic equation has two roots, and in order to discuss the values of these roots we may proceed as follows:

Put $\frac{1}{y}$ for x in the original equation, and clear of fractions. Then

$$cy^2 + by + a = 0.$$

Now put $a = 0$, and we have

$$cy^2 + by = 0;$$

whence $$y = 0, \quad \text{or} \quad -\frac{b}{c}.$$

$$\therefore \quad x = \frac{1}{0}, \quad \text{or} \quad \frac{1}{-\frac{b}{c}},$$

$$= \infty, \quad \text{or} \quad -\frac{c}{b}.$$

Hence, *in any quadratic equation one root becomes infinite when the coefficient of* x^2 *becomes zero.*

This is merely a convenient abbreviation of the following fuller statement:

In the equation $ax^2 + bx + c = 0$, if a is very small one root is very large, and becomes indefinitely great as a is indefinitely diminished. In this case the finite root approaches $-\frac{c}{b}$ as its limit.

c. The Problem of the Couriers.

Two couriers, A and B, are travelling along the same road in the same direction, RR', at the respective rates of m and n miles an hour. At a given hour A is at P, and B is a miles beyond him at Q. After how many hours, and how many miles beyond P, will the couriers be together?

Let x denote the number of hours after the given time, and y the number of miles beyond P. Then

$$y - a = \text{number of miles beyond } Q;$$

$$y = mx; \qquad (1)$$

$$y - a = nx. \qquad (2)$$

From (1) and (2),

$$x = \frac{a}{m-n}, \quad \text{and} \quad y = \frac{am}{m-n}.$$

I.

Suppose a to be positive.

1°. Let $\qquad m > n.$

In this case both x and y will be positive, and A will overtake B to the right of P.

This corresponds with the hypothesis; for since a is positive, B is ahead of A, and since m is greater than n, A is travelling faster than B.

2°. Let $m = n$.

In this case the values of x and y take $\frac{a}{0}$ and $\frac{am}{0}$, and each becomes ∞.

This result indicates that one never would overtake the other.

This interpretation corresponds with the hypothesis made. For B is a miles ahead of A, and both are travelling at the same rate.

3°. Let $m < n$.

In this case the values of x and y both become negative. This indicates that the couriers were together before the given time and before they reached the point P.

This corresponds with the supposition; for B travels faster and is ahead of A at the given time. He therefore must have overtaken A and have passed him before the given time.

II.

Suppose $a = 0$.

1°. Let $m > n$.

In this case the values of x and y both assume the form $\frac{0}{m - n} = 0$.

This is as it should be; for since the couriers travel at unequal rates and are together at the given hour, they never could have been together before, nor can they be together again afterward. As A travels faster than B, he must have overtaken B just at the given time.

2°. Let $m = n$.

In this case the values of x and y both assume the form $\frac{0}{0}$, and the problem becomes indeterminate.

This corresponds with the given conditions; for the couriers are together and travelling at the same rate. Hence they must have been together during all their past journey, and they must continue together for the future.

3°. Let $m < n$.

This gives the same results as 1°, the only difference being that B must have overtaken A at the given time.

III.

Suppose a to be negative.

1°. Let $m > n$.

In this case x and y are both negative, and the couriers must have been together on the road some time before the given hour.

This corresponds with the supposition; for A, being now ahead and travelling faster, must have passed B at some previous point.

2°. Let $m = n$.

This will again give ∞ for both x and y, and the problem is impossible.

These results evidently suit the conditions of the problem; for A is now ahead, and both are travelling at the same rate. Hence the couriers never could have been together in the past, and never can be in the future.

3°. Let $m < n$.

In this case x and y must both be positive, and the couriers must be together at some point farther along the road.

This also answers to the given conditions; for B is now behind at the given time, and travelling faster. Hence he must overtake A at some future point.

d. The Problem of the Lights.

Two lights, A and B, of given intensities, are situated at a given distance apart. Find the point on the line AB where the lights give equal illumination.

Let $m =$ illumination of A at a unit's distance,
$n =$ " " B " " " "
$a =$ distance from A to B,
and $x =$ distance from A to P, the point of equal illumination.

Then $a - x$ will be the distance from B to P.

Since the illumination at P varies directly as the intensity of the source and inversely as the square of its distance, the illumination of A at P will be $\frac{m}{x^2}$, and of B at P $\frac{n}{(a-x)^2}$.

By hypothesis these two illuminations are to be equal.

$$\therefore \quad \frac{m}{x^2} = \frac{n}{(a-x)^2}.$$

Whence
$$x = \frac{a\sqrt{m}}{\sqrt{m} \pm \sqrt{n}}.$$

The double sign of the denominator gives two values for x, and shows that there must be two points of equal illumination.

I.

Suppose a to be positive.

1°. Let $m > n$.

In this case both values of x will be positive, one less and the other greater than a, and the one which is less than a will be greater than $\frac{a}{2}$, since the denominator of the fraction is less than $2\sqrt{m}$. Hence the two points of equal illumination will both be on the same side of A, one between A and B and the other beyond B; and the one between A and B will be nearer to B than to A.

Evidently these results are what we ought to expect. The point of equal illumination between the lights ought to be nearer the less intense light, and the second point of illumination ought to be beyond the less intense light, so as to be nearer to it than to the more intense light.

2°. Let $m = n$.

In this case the first value of x will be positive and equal to $\frac{a}{2}$, and the second value of x will be ∞.

That is, one of the points of equal illumination will be midway between the lights, and the other must be at infinity.

The lights being of equal intensity, the points of equal illumination ought to be equally distant from them, and the only such points are the one half way between the two lights and the point at infinity, or nowhere.

3°. Let $m < n$.

In this case the first value of x will be positive and less than $\frac{a}{2}$, and the second value will be negative and greater than a.

That is, one of the points of equal illumination will be between A and B and nearer the less intense light, and the other is on the opposite side of A to B, so as also to be nearer the less intense light, A.

II.

Suppose a to be zero.

1°. Let $m > n$.

In this case both values of x become zero, and both illuminations become ∞.

These results are on the supposition that each light is a mathematical point, which is physically impossible.

Mathematical analysis does not concern itself with physical impossibilities. Could each light be reduced to a mathematical point, the intensity of the light would become infinite at that point, and were the two lights together at that point, both illuminations would be equal there and nowhere else.

3°. Let $m < n$.

The result in this case would be the same as in 1°.

III.

Suppose a to be negative.

The student may discuss this case when $m > n$, $m = n$, and $m < n$. The conclusions will be similar to those of I, though not identically the same.

CHAPTER XXVI.

THE PROGRESSIONS.

A. ARITHMETICAL PROGRESSION.

232. Arithmetical Series.—When the terms of a series *increase* or *decrease* by a common difference, it is called an *arithmetical series* or an *arithmetical progression.* This series is denoted by the letters A.P.

Each of the following series represents an arithmetical progression:

$$1,\ 4,\ 7,\ 10, \text{ etc.}$$

$$3,\ -1,\ -5,\ -9, \text{ etc.}$$
$$a-4d,\ a-d,\ a+2d, \text{ etc.}$$

In the first, the common difference is 3; in the second, -4; and in the third, $3d$.

The general type of an A.P. is

$$a,\ a+d,\ a+2d,\ a+3d, \text{ etc.},$$

in which a is the first term, and d the common difference.

233. The *n*th Term of an Arithmetical Progression.—Observe that the coefficient of d in any term of the type is one less than the number of the term, it being 1 in the second term, 2 in the third term, 3 in the fourth term, etc.

Hence the nth term of an arithmetical progression will be

$$a+(n-1)d.$$

Thus the fifteenth term of an arithmetical progression whose first term is 5 and whose common difference is 3 will be

$$5 + (15 - 1)3 = 47.$$

When any two terms of an arithmetical progression are given, the common difference, and any other term, may be found by the formula for the nth term.

e.g. Suppose the twelfth term of an arithmetical progression to be 36, and the eighteenth term to be 12. Find the first term, the common difference, and the sixth term of the progression.

Let a denote the first term, and d the common difference.

The twelfth term will be $a + 11d$, and the eighteenth term, $a + 17d$.

$$\therefore \quad a + 17d = 12,$$

and

$$a + 11d = 36.$$

$$\therefore \quad 6d = -24$$

and

$$d = -4.$$

Also, $a = 36 - 11(-4) = 80.$

Therefore the sixth term will be

$$80 + 5(-4) = 60.$$

234. Arithmetical Means.—When three quantities are in arithmetical progression, the second is called the *arithmetical mean* of the other two.

Thus, if a, b, and c are in A.P., b is the arithmetical mean of a and c.

By definition, $b - a = c - b,$

or

$$2b = a + c.$$

$$b = 1/2(a + c).$$

Hence, *the arithmetical mean of two quantities is half their sum.*

When any number of quantities are in arithmetical progression, all the intermediate terms are called *arithmetic means* of the two extreme terms.

Any number of arithmetical means may be inserted between any two given quantities.

e.g. Insert five arithmetical means between 12 and 36.

We must find an arithmetical progression with five terms between 12 and 36. Therefore 36 must be the seventh term.

$$\therefore \quad 12 + 6d = 36.$$

$$\therefore \quad d = 4.$$

Therefore the progression will be

12, 16, 20, 24, 28, 32, 36.

In general, to insert n terms in A.P. between a and b proceed as follows:

Denote the common difference by d.

Then b, or the $(n + 2)$th term, is $a + (n + 1)d$.

$$\therefore \quad a + (n + 1)d = b.$$

$$\therefore \quad (n + 1)d = b - a.$$

$$\therefore \quad d = \frac{b - a}{n + 1}.$$

Therefore the series is

$$a,\ a + \frac{b - a}{n + 1},\ a + 2\frac{b - a}{n + 1},\ a + 3\frac{b - a}{n + 1},\ \ldots\ldots,\ a + n\frac{b - a}{n + 1},$$

and the required means are

$$a + \frac{b - a}{n + 1},\ a + 2\frac{b - a}{n + 1},\ a + 3\frac{b - a}{n + 1},\ \ldots\ldots,\ a + n\frac{b - a}{n + 1},$$

$$\text{or } \frac{na + b}{n + 1},\ \frac{(n - 1)a + 2b}{n + 1},\ \frac{(n - 2)a + 3b}{n + 1},\ \ldots\ldots\ \frac{a + nb}{n + 1}.$$

EXERCISE CXXVI.

1. Find the twentieth term of each of the following arithmetical progressions:

1°. 7, 10, 13, etc. 2°. 2, 6, 10, etc.

3°. 20, 15, 10, etc. 4°. 1/12, 1/2, 11/12, etc.

2. Find the last term of each of the following series:

1°. 4, 7, 10, to 17 terms. 2°. 3, 7, 11, to 21 terms.

3°. 8, 6, 4, to 12 terms. 4°. 5, $8\frac{1}{3}$, $11\frac{2}{3}$, to 16 terms.

5°. 1/3, – 1/2, – 4/3, to 25 terms.

3. The eleventh term of an A.P. is 51 and the sixth is 31. What is the first term?

4. The seventh term of an A.P. is 37 and the twelfth term is 62. What is the first term?

5. The fourth term of an A.P. is 10 and the tenth term 24. What is the common difference?

6. The sixth term of an A.P. is 5/4 and the fifteenth term 11/4. What is the common difference?

7. The third term of an A.P. is 1/2 and the thirteenth is 2. What is the twenty-third term?

8. The seventh term of an A.P. is 5 and the fifth term is 7. What is the twelfth term?

9. Which term of the series 6, 11, 16, etc., is 96?

10. Which term of the series 7, 3, – 1, etc., is – 53?

11. Which term of the A.P. $16a - 8b$, $15a - 7b$, $14a - 6b$ is $8a$?

12. Insert twenty-two arrithmetical means between 8 and 54.

13. Insert eight arithmetical means between 1 and 0.

14. Insert ten arithmetical means between $5a-6b$ and $5b-6a$.

15. The sum of the fourth and seventh terms of an A.P. is 40, and the sum of the sixth and tenth is 60. Find the common difference and the first term.

16. The sum of the fifth and eleventh terms of an A.P. is 0, and the sum of the third and eighth terms is 15. Find the common difference and the first term.

17. The sum of the fourth and thirteenth terms of an A.P. is -22, and of the second and eighth is 24. What is the sum of the sixth and twelfth terms?

235. PROBLEM. *To find the sum of any number of terms of an arithmetical progression.*

Let a be the first term, d the common difference, n the number of terms whose sum is required, l the last term, and S the required sum.

Then, since l is the nth term, we have

$$l=a+(n-1)d.$$

$$\therefore \quad S=a+(a+d)+(a+2d)+\ldots+(l-2d)+(l-d)+l,$$

or in reverse order,

$$S=l+(l-d)+(l-2d)+\ldots+(a+2d)+(a+d)+a.$$

Adding these two equations, we obtain

$$2S=(a+l)+(a+l)+(a+l)+\ldots \text{ to } n \text{ terms}$$
$$=n(a+l).$$

$$\therefore \quad S=\frac{n}{2}(a+l). \qquad (1)$$

But $\qquad l=a+(n-1)d.$

Substitute this value of l in (1), and we get

$$S=\frac{n}{2}(2a+(n-1)d). \qquad (2)$$

Both these formulæ are important. By means of the second, when any three of the four quantities S, a, d, and n are given, the fourth may be computed.

e.g. 1. Find the sum of the first thirty terms of the series

$$3 + 6 + 9, \text{ etc.}$$

Here $a = 3, \quad d = 3, \quad \text{and} \quad n = 30.$

$$\therefore \quad S = \frac{30}{2}[6 + 29 \times 3] = 1395.$$

e.g. 2. The sum of twelve terms of an A.P. is 260 and the first term is 20. What is the common difference?

Here $S = 260, \quad n = 12, \quad \text{and} \quad a = 20.$

$$\therefore \quad 260 = \frac{12}{2}(40 + 11d),$$

or $$260 = 240 + 66d.$$

$$\therefore \quad 66d = -20,$$

and $$d = -\frac{10}{33}.$$

e.g. 3. How many terms of the series $40 + 36 + 32 +$ etc. must be taken that their sum may be 216?

Here $S = 216, \quad a = 40, \quad \text{and} \quad d = -4$

$$\therefore \quad 216 = \frac{n}{2}[80 + (n - 1) \times -4],$$

or $$432 = 80n - 4n^2 + 4n.$$

$$\therefore \quad n^2 - 21n + 108 = 0.$$

$$\therefore \quad (n - 9)(n - 12) = 0$$

$$\therefore \quad n = 9 \text{ or } 12.$$

The finding of the number of terms by this formula involves the solution of a quadratic equation in n, and one or both of the values of n may be negative, fractional, surd, or imaginary. In these cases all the values except the positive integral ones must be rejected. When the two values of n are positive and integral, the sum of the additional terms for the greater value must be zero. In the above case the tenth, eleventh, and twelfth terms are $4, 0, -4$.

EXERCISE CXXVII.

Find the sum of the following series:

1. $3 + 5 + 7 + \ldots$ to twenty-four terms.

2. $12 + 11\frac{2}{3} + 11\frac{1}{3} + \ldots$ to twenty-two terms.

3. $3 + 4\frac{1}{2} + 6 + \ldots$ to seventeen terms.

4. $-7 - 2 + 3 + \ldots$ to twenty terms.

5. $1/2 + 1/3 + 1/6 + \ldots$ to seven terms.

6. $5 + 6.2 + 7.4 + \ldots$ to twenty-one terms.

7. $(n + 1) + (2n + 3) + (3n + 5) + \ldots$ to n terms.

8. $(a + b)^2 + (a^2 + b^2) + (a - b)^2 + \ldots$ to n terms.

9. The fourth and thirteenth terms of an A.P. are -9 and $+9$. What is the sum of the first twenty terms?

10. The seventh term of an A.P. is $43\frac{3}{4}$ and the twelfth is $77\frac{1}{2}$. What is the sum of the first twenty-four terms?

11. Find the sum of thirty consecutive odd numbers of which the least is 7.

12. Find the sum of twenty consecutive odd numbers of which the greatest is 77.

13. Insert seventeen arithmetical means between 4 and 76, and find their sum.

14. Insert forty arithmetical means between 10 and 100, and find their sum.

15. Find the sum of all the multiples of 7 lying between 200 and 400.

16. Find the sum of all the positive multiples of 12 of less than four digits.

236. The Average Term.—An A.P. of an odd number of terms must contain a middle term, and the number of terms between the first term and this middle term must be the same as that between it and the last term. Hence the first, middle, and last terms must form an A.P., and the middle term must be half the sum of the two extreme terms.

Since the formula $S = \frac{n}{2}(a + l)$ may be written $S = n\left(\frac{a + l}{2}\right)$, the sum of an A.P. of an odd number of terms is equal to the product of the middle term and the number of terms. The middle term therefore must be the average of all the terms, or the arithmetical mean of any pair of terms equally removed from it.

The average of all the terms of an A.P. evidently must be half the sum of the extreme terms or their arithmetical mean. For the average of the first and last is $\frac{a + l}{2}$, of the second and next to the last $\frac{(a + d) + (l - d)}{2} = \frac{a + l}{2}$, and so on.

Hence, if the number of terms be odd, the average of all the terms will be the middle term, and if the number of terms be even, the average of all the terms will be the arithmetical mean of the two middle terms.

e.g. 1°. The first term of an A.P. of seventeen terms is 3 and the last term is 27. What is the sum of the terms?

Here the middle term $= \frac{3 + 27}{2} = 15$,

and the sum $= 17 \times 15 = 255$.

e.g. 2°. The first term of an A.P. is 17, the common difference is -3, and the middle term is -4. Find the number of terms and their sum.

Here $$-4 = 17 + (n-1) \times -3,$$
$$-4 = 17 - 3n + 3.$$
$$\therefore \quad 3n = 24.$$
$$\therefore \quad n = 8.$$

Since 8 is the number of the middle term, the whole number of terms must be 15, and their sum -60.

EXERCISE CXXVIII.

1. Find the sum of the twenty-one terms of an A.P. of which the middle term is 33.

2. Find the sum of forty-five terms of an A.P. of which the twenty-third is 75.

3. The first term of an A.P. is 3, the last term is 77, and the sum of the terms is 520. What is the number of terms?

4. The first term of an A.P. is 12, the last term is -198, and the sum of the terms is -3069. What is the number of the terms?

5. A man travels 5 miles the first day, 8 miles the second, 11 miles the third, and so on. At the expiration of a certain time he finds he has travelled at the average rate of $18\frac{1}{2}$ miles a day. How many days did he travel?

6. A pedestrian having to go 184 miles walks 30 miles the first day, and two miles less each subsequent day till his journey was completed. How many days did it take him?

7. In an A.P. the product of the sixth and eighth terms exceeds the product of the fourth and tenth by 200. What is the common difference?

8. In an A.P. the product of the eighth and thirteenth terms is less than the product of the ninth and twelfth terms by 25. What is the common difference?

9. Two travellers start together on the same road. One of them travels uniformly at the rate of 10 miles a day. The other goes 8 miles the first day, and increases his speed half a mile each subsequent day. In how many days will the latter overtake the former?

10. One hundred stones are placed on the ground in a straight line at intervals of 5 yards. A runner has to start from a basket 5 yards from the first stone, pick up the stones, and bring them back to the basket one by one. How far will he be obliged to travel?

11. An author wished to buy up the whole edition of 1000 copies of a book which he had published. He paid 20 cents for the first copy, but the price rose so that he was obliged to pay 1 cent more for each subsequent copy than for the last. What was he obliged to pay for the whole?

12. Find three numbers in A.P. the sum of whose squares is 2900, and the square of whose means exceeds the product of the extremes by 100.

13. Find four numbers in A.P. such that the sum of the squares of the extremes equals 464, and the sum of the squares of the means equals 400.

14. Find four numbers in A.P. such that the product of the means shall exceed the product of the extremes by 72, and the sum of their squares shall equal 280.

237. Two Important Series Allied to the Arithmetical Series.—Let S_1 denote the sum of the first powers of the natural numbers from 1 to n, S_2 denote the sum of their squares, and S_3 the sum of their cubes. Then—

1°. $$S_1 = \frac{n(n+1)}{2}.$$

For this is an A.P. in which the first term is 1 and the last term is n and the number of terms is also n.

2°. $$S_2 = \frac{n(n+1)(2n+1)}{6}.$$

This may be proved as follows:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1.$$

Writing 1, 2, 3, etc., in turn for n in this identity, we get

$$\begin{aligned}
2^3 &= 1^3 + 3 \,.\, 1^2 + 3 \,.\, 1 + 1;\\
3^3 &= 2^3 + 3 \,.\, 2^2 + 3 \,.\, 2 + 1;\\
4^3 &= 3^3 + 3 \,.\, 3^2 + 3 \,.\, 3 + 1;\\
&\text{etc.} \qquad\qquad \text{etc.};\\
(n+1)^3 &= n^3 + 3 \,.\, n^2 + 3 \,.\, n + 1.
\end{aligned}$$

Note that we have on each side of these equations 2^3, 3^3, 4^3 . . . to n^3. Adding these equations and cancelling their common terms, we get

$$\begin{aligned}
(n+1)^3 &= 1^3 + 3(1^2+2^2+3^2 \ldots + n^2) + 3(1+2+3 \ldots + n) + n\\
&= 1 + 3S_2 + 3\frac{n(n+1)}{2} + n\\
&= 3S_2 + \frac{3n(n+1)}{2} + \frac{2n+2}{2}\\
&= 3S_2 + \frac{3n^2+5n+2}{2}.
\end{aligned}$$

$$\therefore\quad 3S_2 = \frac{2(n+1)^3}{2} - \frac{3n^2+5n+2}{2} = \frac{2n^3+3n^2+n}{2}$$

$$= \frac{n(n+1)(2n+1)}{2}.$$

$$\therefore\quad S_2 = \frac{n(n+1)(2n+1)}{6}.$$

3°. $$S_3 = S_1^2.$$

Here $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1.$

Writing 1, 2, 3, etc., in turn for n in this identity, we get

$$2^4 = 1^4 + 4 \,.\, 1^3 + 6 \,.\, 1^2 + 4 \,.\, 1 + 1;$$
$$3^4 = 2^4 + 4 \,.\, 2^3 + 6 \,.\, 2^2 + 4 \,.\, 2 + 1;$$
$$4^4 = 3^4 + 4 \,.\, 3^3 + 6 \,.\, 3^2 + 4 \,.\, 3 + 1;$$
$$\text{etc.} \qquad\qquad \text{etc.};$$
$$(n+1)^4 = n^4 + 4 \,.\, n^3 + 6 \,.\, n^2 + 4 \,.\, n + 1.$$

Adding and cancelling as before, we get

$$(n+1)^4 = 1^4 + 4.S_3 + 6.S_2 + 4.S_1 + n$$
$$= 1 + 4.S_3 + n(n+1)(2n+1) + 2n(n+1) + n,$$
$$\therefore \quad 4.S_3 = (n+1)^4 - [n(n+1)(2n+1) + 2n(n+1) + n + 1]$$
$$= n^4 + 4n^3 + 6n^2 + 4n + 1 - [2n^3 + 3n^2 + n + 2n^2 + 2n + n + 1]$$
$$= n^4 + 2n^3 + n^2 = n^2(n+1)^2 = [n(n+1)]^2.$$
$$\therefore \quad S_3 = \left[\frac{n(n+1)}{2}\right]^2 = S_1^2.$$

B. GEOMETRICAL PROGRESSION.

238. Geometrical Series.—Quantities are said to be in *geometrical progression* (G.P.) when the ratio of any term to that which immediately precedes it is the same throughout the series.

Thus each of the following series forms a geometrical progression:

$2, 4, 8, 16,$ etc.

$1, -1/4, 1/16, -1/64,$ etc.

$a, ar, ar^2, ar^3,$ etc.

The constant ratio is called the common ratio, and is found by dividing any term by the one which immediately precedes it.

Thus, in the first of the above series, 2 is the common ratio; in the second, $-1/4$; and in the third, r.

239. Type Form of the Series.—The type form of a geometric series is

$$a + ar + ar^2 + ar^3 + ar^4 \ldots + ar^{n-1}.$$

It will be noticed that in this series the exponent of r in each term is one less than the number of the term.

If n denote the number of terms, and l the last or nth term, then $l = ar^{n-1}$.

240. Geometrical Means.—When three quantities are in geometrical progression, the middle one is called the *geometrical mean* between the other two.

Let a, b, and c be three quantities in G.P. By definition,

$$\frac{b}{a} = \frac{c}{b}, \quad b^2 = ac, \quad \text{and} \quad b = \sqrt{ac}.$$

That is, the geometrical mean between two quantities is equal to the square root of their product.

All the terms in a G.P. between the extremes may be called geometrical means, and any number of such means may be inserted between two terms.

Let a and b be the two terms between which n geometrical means are to be inserted.

The whole number of terms will be $n + 2$, and b will be the $(n + 2)$th term.

Let r be the common ratio. Then

$$b = ar^{n+1}.$$

$$\therefore \quad r^{n+1} = \frac{b}{a}.$$

$$r = \sqrt[n+1]{\frac{b}{a}}.$$

e.g. Insert four geometrical means between 224 and 7.

In this case we must find six terms in G.P. of which the first is 224 and the sixth is 7. Therefore

$$7 = 224r^5.$$

$$\therefore \quad r^5 = 1/32,$$

and $$r = \sqrt[5]{1/32} = 1/2.$$

Hence the means are 112, 56, 28, 14.

EXERCISE CXXIX.

In finding the common ratio in a G.P. it is often necessary to extract a root of a high index, which is tedious without the use of logarithms. In the following examples it will be easy to extract the required roots by inspection. Remember that the fourth root is the square root of the square root, that the sixth root is the cube root of the square root, and that the eighth root is the square root of the square root of the square root.

1. Insert two geometrical means between 2 and 250.

2. Insert three geometrical means between -3 and -768.

3. Insert four geometrical means between 5 and -1215.

4. Insert five geometrical means between 3 and .000192.

5. Insert four geometrical means between 1/6 and 64/3.

241. PROBLEM. *To find the sum of n terms of a geometrical progression.*

Let S denote the sum, and let the series be

$$a + ar + ar^2 + \ldots + ar^{n-1}.$$

Then $$S = a + ar + ar^2 + \ldots + ar^{n-1}. \qquad (1)$$

Multiply each side by r:

$$rS = ar + ar^2 + ar^3 + \ldots + ar^n. \qquad (2)$$

Subtract (2) from (1), and we get

$$S - rS = a - ar^n,$$

or $$(1 - r)S = a(1 - r^n).$$

$$\therefore \quad S = a\frac{1 - r^n}{1 - r}.$$

e.g. Find the sum of ten terms of the series $2 + 4 + 8 +$ etc.

Here $a = 2, \; r = 2,$ and $n = 10.$

Therefore

$$S = 2\,\frac{1 - 2^{10}}{1 - 2} = 2(2^{10} - 1) = 2(1023) = 2046.$$

242. Divergent and Convergent Series.—The formula $a\frac{1 - r^n}{1 - r}$, or $a\frac{r^n - 1}{r - 1}$, may be written $\frac{ar^n}{r - 1} - \frac{a}{r - 1}$.

In the series $a + ar + ar^2 + ar^3 + \ldots ar^{n-1}$, if r be made 1, the series becomes $a + a + a + \ldots n$ terms $= na$.

Hence, by sufficiently increasing n, we may cause S to surpass any value however great. When n becomes ∞, S becomes ∞.

If r be greater than 1, r^n increases with n, and, by sufficiently increasing n, r^n may be made as great as we please. When n becomes ∞, $a\frac{r^n - 1}{r - 1}$ becomes ∞.

Hence, by sufficiently increasing the value of n, we may cause S to exceed any value however great, and when $n = \infty$, $S = \infty$.

In these two cases the geometric series, if supposed continued to an infinite number of terms, is said to be *divergent.*

If r be numerically less than 1, that is a proper fraction either positive or negative, r^n decreases as n increases. By making n sufficiently large r^n may be made as small as we please. When $n = \infty$, $r^n = 0$, and $a\frac{1 - r^n}{1 - r}$ becomes $\frac{a}{1 - r}$.

Hence $\frac{a}{1 - r}$ is the value which S approaches as a limit as n is indefinitely increased.

In this case the series is said to be *convergent.*

The *sum* of an infinite series is the limit to which the sum of its first n terms approaches as n is indefinitely increased.

If $r = -1$, the series becomes

$$S = a - a + a - a + \dots.$$

In this case the sum of any odd number of terms is a, and of any even number of terms 0. The sum, therefore, does not become infinite when an infinite number of terms are taken, nor does it converge to one definite value. A series which has this property is said to *oscillate,* and is called an oscillating series.

If a series is composed of an infinite number of terms, its sum can be found only when the series is converging.

e.g. 1°. Find the sum of the series

$$1/2 + 1/3 + 2/9 + \dots \text{ to six terms.}$$

$$S = \frac{1/2(1-(2/3)^6)}{1-2/3} = \frac{1/2(1-64/729)}{1/3}$$

$$= \frac{1/2(665/729)}{1/3} = \frac{665}{486}.$$

EXERCISE CXXX.

1. Find the sum of the G.P. $6 + 18 + 54 + \ldots$ to eight terms.

2. Find the sum of the G.P. $6 - 18 + 54 + \ldots$ to eight terms.

3. Sum $-2 + 2\frac{1}{2} - 3\frac{1}{8} + \ldots$ to six terms.

4. Sum $3/4 + 1\frac{1}{2} + 3 + \ldots$ to eight terms.

5. Sum $2 - 4 + 8 - \ldots$ to ten terms.

6. Sum $16.2 + 5.4 + 1.8 + \ldots$ to twelve terms.

7. Sum $-1/3 + 1/2 - 3/4 + \ldots$ to seven terms.

8. Sum $8/5 - 1 + 5/8 - \ldots$ to infinity.

9. Sum $.45 + .015 + .0005 + \ldots$ to infinity.

10. Sum $1.665 - 1.11 + .74 - \ldots$ to infinity.

11. Sum $3^{-1} + 3^{-2} + 3^{-3} + \ldots$ to infinity.

12. The fifth term of a G.P. is 324 and the eighth term is -8748. What is the first term?

13. There are five terms in G.P. The sum of the first and second is 30, and the sum of the fourth and fifth is 1920. What are the numbers?

14. There are three numbers in G.P. The sum of the first and second is 24, and of the second and third is -72. What are the numbers?

15. There are three numbers in G.P. The second minus the first equals 36, and the third plus the second equals 210. What are the numbers?

243. The Value of Repeating Decimals.—The value of a repeating or a recurring decimal may be found by summing a G.P. to infinity.

e.g. 1°. Find the value of the repeating decimal .333+.

$$.333 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \ldots \text{ to infinity}$$

$$= \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots \text{ to infinity}\right)$$

$$= \frac{3}{10} \cdot \frac{1}{1 - 1/10} = \frac{3}{10} \cdot \frac{10}{9} = \frac{1}{3}.$$

2°. Find the value of the circulating decimal $.2\dot{4}\dot{1}$.

$$2\dot{4}\dot{1} = \frac{2}{10} + \frac{41}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right)$$

$$= \frac{2}{10} + \frac{41}{10^3} \cdot \left[\frac{1}{1 - 1/10^2} = \frac{10^2}{99}\right]$$

$$= \frac{2}{10} + \frac{41}{10^3} \times \frac{10^2}{99} = \frac{2}{10} + \frac{41}{990}$$

$$= \frac{2 \times 99 + 41}{990} = \frac{239}{990}.$$

244. Rule for Values of Recurring Decimals.—Note that the last answer $= \dfrac{241 - 2}{990}$.

Hence we obtain the following arithmetical rule for finding the value of a mixed circulating decimal:

Subtract the non-repeating figures from all the digits down to the end of the first period, and write as a denominator as many 9's as there are digits in the repeating part, followed by as many ciphers as there are digits in the non-repeating part.

Note also that the answer to the previous example = 3/9. Hence we obtain the following rule for finding the value of a pure recurring decimal:

Write as a denominator to the recurring digits as many 9's as there are digits in the period.

EXERCISE CXXXI.

Sum the following recurring decimals as geometrical progressions, and show in each case that the result is in agreement with the rules just given:

1. $.\dot{1}\dot{5}$.
2. $.\dot{1}8\dot{5}$.
3. $.\dot{3}9\dot{6}$.
4. $.\dot{4}2857\dot{1}$.
5. $.\dot{0}1298\dot{7}$.
6. $.7\dot{9}$.
7. $.3\dot{1}\dot{5}$.
8. $.11\dot{6}$.
9. $.19\dot{3}2\dot{4}$.

C. COMPOUND INTEREST AND ANNUITIES.

245. Compound Interest.—There are many problems in Geometrical Progression of which an approximate solution can be obtained readily by means of logarithms. Among these the different cases of compound interest and annuities are of especial importance.

Money is said to be invested at compound interest when at stated intervals the interest which has accrued is added to the principal, so as itself to draw interest. These additions are made usually annually, semi-annually, or quarterly.

246. PROBLEM I. *To find the amount at the end of a given time of a sum of money invested at compound interest at a given rate.*

Let P denote the given sum,
n denote the number of years,
r denote the interest of one dollar for one year,
and A denote the required amount.

1°. Suppose the interest to be computed annually. At the end of the first year the amount will be

$$P + rP = P(1 + r);$$

at the end of the second year the amount will be

$$P(1+r)+rP(1+r)=P(1+r)(1+r)=P(1+r)^2;$$

at the end of the third year the amount will be

$$P(1+r)^2+rP(1+r)^2=P(1+r)^2(1+r)=P(1+r)^3;$$

and at the end of the nth year the amount will be

$$P(1+r)^{n-1}+rP(1+r)^{n-1}=P(1+r)^{n-1}(1+r)$$
$$=P(1+r)^n.$$

The amounts

$$P(1+r),\quad P(1+r)^2,\quad P(1+r)^3,\ \ldots\ P(1+r)^n,$$

are in geometrical progression, the first term being $P(1+r)$, the last term $P(1+r)^n$, and the common ratio $1+r$.

$$A=P(1+r)^n. \tag{1}$$

To solve this by logarithms, it is necessary to take out $\log P$, $\log\,(1+r)^n$, and the antilog of the sum of these two logs.

2°. If the interest be computed semi-annually, the formula for the amount becomes

$$A=P\left(1+\frac{r}{2}\right)^{2n}; \tag{2}$$

and if the interest be computed quarterly, the formula becomes

$$A=P\left(1+\frac{r}{4}\right)^{4n}. \tag{3}$$

247. Present Worth.—The *present worth* of a sum of money due at some future time without interest is the principal which put at interest for the given time would amount to the given sum.

248. PROBLEM II. *To find the present worth, at compound interest, of a fixed sum due at a future date.*

In formula (1), if A denotes the given sum, r the current rate of interest, and n the given number of years, then P will evidently denote the present worth. Hence

$$P = \frac{A}{(1+r)^n} = A(1+r)^{-n}. \qquad (4)$$

To solve this by logarithms, it is necessary to take out the log of A, the colog of $(1+r)^n$, and the antilog of their sum.

Of course, if the interest is to be computed semi-annually or quarterly, P must be found from formula (2) or (3).

249. PROBLEM III. *To find the amount at a given time of a fixed sum invested at stated intervals at compound interest.*

Let P denote the fixed sum, and use A, r, and n as before. Then the amounts of the stated investments, on the supposition that they are made annually, will be as follow:

$$A_1 = P(1+r)^n,$$
$$A_2 = P(1+r)^{n-1},$$
$$A_3 = P(1+r)^{n-2},$$
$$A_4 = P(1+r)^{n-3},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$A_n = P(1+r)^{n-(n-1)} = P(1+r).$$

The sum of these amounts is

$$P(1+r) + P(1+r)^2 + P(1+r)^3 + \ldots . P(1+r)^n.$$

This is a geometrical progression, of which the first term is $P(1+r)$, the common ratio $(1+r)$, and the number of terms n. Hence

$$A = P(1+r)\frac{(1+r)^n - 1}{1+r-1} = P\frac{(1+r)^{n+1} - (1+r)}{r}. \qquad (5)$$

To solve this by means of logarithms, first find by logarithms the value of $(1+r)^{n+1}$, from this subtract $1+r$, find the logarithm of the result, of P, and the colog of r, and, finally, the antilog of the sum of the three.

250. Annuities.—An *annuity* is a fixed sum of money payable at equal intervals of time.

If the payment continue for a definite time, the annuity is called a *fixed annuity;* if only during a person's life, a *life annuity;* and if for all time, a *perpetuity.*

Annuities may pay annually, semi-annually, quarterly, or at any other stated times, but the principles of dealing with all the cases being the same, we shall consider only the case of annual annuities.

251. PROBLEM IV. *To find the present value of an annuity of a given amount payable at the end of each of n successive years.*

Let A denote the amount of each payment, P the present worth of the whole annuity, and P_1, P_2, etc., the present worth of the successive payments, beginning with the first. Then

$$P_1 = A(1+r)^{-1},$$

$$P_2 = A(1+r)^{-2},$$

$$\cdots\cdots\cdots$$

$$P_n = A(1+r)^{-n}.$$

Hence $$P = A\left\{\frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots \frac{1}{(1+r)^n}\right\}$$

$$= A\frac{1-\dfrac{1}{(1+r)^n}}{1+r-1} = \frac{A}{r}(1-(1+r)^{-n}). \quad (6)$$

In case of a perpetuity, n becomes ∞, and $\dfrac{1}{(1+r)^n}$ becomes 0. Therefore $P = \dfrac{A}{r}$.

That is, *the present worth of a perpetuity is the quotient obtained by dividing the amount of the annual payment by the interest of one dollar for one year.*

252. PROBLEM V. *To find the amount of an annuity to run for n years which can be purchased for a given sum of money, the rate of compound interest being known.*

In formula (6), P denotes the present value or the purchase-money, and A the amount of the annuity. From (6), we obtain

$$A = \frac{rP}{1 - \dfrac{1}{(1+r)^n}} = \frac{rP(1+r)^n}{(1+r)^n - 1}. \qquad (7)$$

Formula (7) is also the formula for finding by what fixed annual payment of A dollars an obligation of P dollars may be cancelled in a given number of years, r being the interest of one dollar for one year.

253. PROBLEM VI. *To find the present worth of an annuity to begin after m years and to continue for n years, allowing compound interest.*

By (6), the value of the annuity at the expiration of m years is

$$\frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right);$$

and by (4), the present worth of this sum due in m years is

$$P = \frac{\dfrac{A}{r}\left(1 - \dfrac{1}{(1+r)^n}\right)}{(1+r)^m} = \frac{A\left((1+r)^n - 1\right)}{r(1+r)^{m+n}}. \qquad (8)$$

EXERCISE CXXXII.

1. What will be the amount of 2000 dollars for 15 years at 5 per cent, the interest being compounded annually?

2. What will be the amount of 800 dollars for 9 years 3 months at 4 per cent, the interest being compounded quarterly?

3. What sum of money will amount to $1240.60 in 5 years 6 months at 6 per cent, the interest being compounded semi-annually?

4. In how many years will 968 dollars amount to $1269.40 at 5 per cent, the interest being compounded semi-annually?

5. What is the present worth of a note for 600 dollars due 9 years hence, allowing $4\frac{1}{2}$ per cent compound interest?

6. At what rate per annum will 2600 dollars give $416.40 in 3 years and 9 months, the interest being compounded quarterly?

7. In how many years will 500 dollars double itself at 5 per cent, the interest being compounded annually?

8. In how many years will a sum of money double itself at 4 per cent, the interest being compounded quarterly?

9. What is the present value of an annuity of 500 dollars to continue for 20 years, allowing 4 per cent compound interest?

10. What is the present value of a perpetuity of 300 dollars, allowing 5 per cent compound interest?

11. What is the present value of an annuity of 400 dollars to begin 8 years hence and to run for 15 years, allowing 4 per cent compound interest?

12. What fixed annual payment must be made to cancel an obligation of 3000 dollars in 8 years, allowing $3\frac{1}{2}$ per cent interest?

13. What annuity to continue 12 years can be purchased for 4000 dollars, allowing 5 per cent compound interest?

D. HARMONIC PROGRESSION.

254. Harmonic Progression.—Three quantities are said to be in harmonic progression when the first is to the third as the difference between the first and second is to the difference between the second and third. An harmonic progression is denoted by the abbreviation H.P.

a, b, and c are in H.P. when

$$a : b = a - b : b - c.$$

A series is said to be harmonic when every three consecutive terms are in H.P.

255. THEOREM I. *If three quantities are in harmonic progression, their reciprocals are in arithmetical progression.*

Let a, b, and c be three quantities in harmonic progression. Then

$$a : c = a - b : b - c.$$

Whence $$a(b - c) = c(a - b),$$

or $$ab - ac = ac - bc.$$

Dividing each term by abc, we have

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

Harmonical properties are interesting because of their importance in geometry and in the theory of sound. In algebra, the theorem just proved is the only one of any importance. There is no general formula for the sum of any number of terms in H.P. Questions in H.P. are solved usually by taking the reciprocals of their terms, and making use of the properties of the resulting A.P.

256. THEOREM II. *The harmonic mean of two quantities is equal to twice their product divided by their sum.*

If a, b, and c are in H.P., $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$ are in A.P.

$$\therefore \quad \frac{1}{a}+\frac{1}{c}=\frac{2}{b}.$$

$$\therefore \quad b=\frac{2ac}{a+c}.$$

257. THEOREM III. *The geometric mean of two quantities is also the geometric mean of the arithmetic and harmonic means of the quantities.*

Denote the arithmetic, geometric, and harmonic means of a and b by A, G, and H, respectively. Then

$$A=\frac{a+b}{2}.$$

$$G=\sqrt{ab}.$$

$$H=\frac{2ab}{a+b}.$$

$$\therefore \quad A\,.\,H=ab=G^2.$$

258. PROBLEM. *To insert n harmonic means between a and b.*

Insert n arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$, and the reciprocals of these will be the required harmonic means.

EXERCISE CXXXIII.

1. Insert two harmonic means between 3 and 12.

2. Insert two harmonic means between 2 and 1/5.

3. Find the fifth term of the H.P. 1/2, 1/4, 1/6.

4. Insert three harmonic means between 5 and 25.

5. If a, b, c, are in A.P., and b, c, d, are in H.P., prove that $a : b = c : d$.

6. Show that if a, b, c, d, be in H.P., then will

$$3(b - a)(d - c) = (c - b)(d - a).$$

7. Show that if a, b, c, be in A.P., b, c, d, in G.P., and c, d, e, in H.P., then will a, c, e, be in G.P.

CHAPTER XXVII.

BINOMIAL THEOREM.

259. THEOREM.—*When n is a positive integer,*

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\,.\,2}a^{n-2}x^2$$
$$+ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}x^3$$
$$+ \ldots \text{ to } n+1 \textit{ terms.}$$

1°. When $n = 1$, we have

$$(a+x)^1 = a + x = a^n + na^{n-1}x, \quad \text{since} \quad a^{1-1} = a^0 = 1.$$

By actual multiplication, when $n = 2$, we have

$$(a+x)^2 = a^2 + 2ax + x^2 = a^n + na^{n-1}x + \frac{n(n-1)}{1\,.\,2}a^{n-2}x^2,$$

since $a^{2-2} = a^0 = 1$.

When $n = 3$, we have

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3 = a^n + na^{n-1}x + \frac{n(n-1)}{1\,.\,2}\,.$$
$$a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}x^3.$$

When $n = 4$, we have

$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4 = a^n + na^{n-1}x$$
$$+ \frac{n(n-1)}{1\,.\,2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}x^3.$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}a^{n-4}x^4.$$

We thus see that the theorem holds true when $n = 1$, 2, 3, or 4.

2°. Now multiply each member of the expression

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 . 2}a^{n-2}x^2$$
$$+ \frac{n(n-1)(n-2)}{1 . 2 . 3}a^{n-3}x^3$$
$$+ \ldots \text{ to } (n + 1) \text{ terms},$$

which we have found to hold true when $n = 1$, 2, 3, and 4, by $a + x$, and we obtain

$$(a + x)^{n+1} = a^{n+1} + [a^n x + na^n x]$$
$$+ \left[na^{n-1}x^2 + \frac{n(n-1)}{1 . 2}a^{n-1}x^2\right]$$
$$+ \left[\frac{n(n-1)}{1 . 2}a^{n-2}x^3 + \frac{n(n-1)(n-2)}{1 . 2 . 3}a^{n-2}x^3\right]$$
$$+ \left[\frac{n(n-1)(n-2)}{1 . 2 . 3}a^{n-3}x^4 + \frac{n(n-1)(n-2)(n-3)}{1 . 2 . 3 . 4}a^{n-3}x^4\right]$$
$$+ \ldots \text{ to } n + 2 \text{ terms.}$$

Note that the second term of the last aggregate is obtained by multiplying the fifth term of the expression of $(a + x)^n$ by a.

Note also that each aggregate contains two terms in ax with identical exponents, and that, if we let $r + 1$ denote the number of the aggregate, the coefficient of these two terms of each aggregate after the first will be respectively

$$\frac{n(n-1)\ldots(n-(r-1))}{1 . 2 . \ldots . r} \text{ and } \frac{n(n-1)\ldots(n-r)}{1 . 2 . \ldots . r+1}.$$

$$\text{Now } \frac{n(n-1)\ldots(n-(r-1))}{1 . 2 . \ldots . r} + \frac{n(n-1)\ldots(n-r)}{1 . 2 . \ldots . r+1}$$
$$= \frac{n(n-1)\ldots(n-(r-1))}{1 . 2 . \ldots . r} . \left[1 + \frac{n-r}{r+1}\right]$$

$$= \frac{n(n-1)\ldots(n-(r-1))}{1.2\ldots.r} \times \frac{n+1}{r+1}$$

$$= \frac{(n+1)n(n-1)\ldots(n-(r-1))}{1.2\ldots.r(r+1)},$$

whatever r may be, and this is the general expression for the sum of the coefficients of the term in ax in each bracket after the first.

Therefore we have

$$(a+x)^{n+1} = a^{n+1} + (n+1)a^n x + \frac{(n+1)n}{1.2}a^{n-1}x^2$$

$$+ \frac{(n+1)n(n-1)}{1.2.3}a^{n-2}x^3 + \frac{(n+1)n(n-1)(n-2)}{1.2.3.4}a^{n-3}x^4$$

$$+ \ldots \text{ to } (n+2) \text{ terms.}$$

If we put $n+1=n'$, we will have

$$(a+x)^{n'} = a^{n'} + n'a^{n'-1}x + \frac{n'(n'-1)}{1.2}a^{n'-2}x^2$$

$$+ \frac{n'(n'-1)(n'-2)}{1.2.3}a^{n'-3}x^3 + \ldots \text{ to } (n'+1) \text{ terms,}$$

which agrees with the theorem.

We therefore conclude that the theorem will be true for the next higher value of n if it be true for any one value of n.

But by actual multiplication the theorem has been shown to hold true when $n=1$, 2, 3, and 4. It therefore must hold true when $n=5$, 6, 7, or any positive integer.

260. The Binomial Coefficients.—The quantities

$$n, \quad \frac{n(n-1)}{1.2}, \quad \frac{n(n-1)(n-2)}{1.2.3}, \text{ etc.,}$$

are known as the binomial coefficients.

Note that the factors in the numerators begin with n

and decrease by 1, and that their number is one less than the number of the term in which it occurs; also that the factors in the denominators begin with one and increase by unity, and that the number of factors in the denominator is the same as in the numerator.

e.g. The coefficient of the fifth term of the development of $(a+x)^n$ is $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$.

Note carefully that the binomial coefficient of the next term in the development of a binomial expression can be obtained by multiplying the coefficient of the last by the exponent of a in that term and dividing by the number of the term.

Thus the binomial coefficient of the third term is $\frac{n(n-1)}{1.2}$, and the exponent of a is $n-2$. The binomial coefficient of the fourth term is $\frac{n(n-1)(n-2)}{1.2.3}$. This is the coefficient of the third multiplied by $(n-2)$ and divided by 3.

261. Developments.—When a single algebraic expression is changed into the sum of a series of terms, it is said to be *developed,* and the series is called its *development.* A development may be true in form, yet may equal the function only for certain special values of x. No development can equal the function except for the values of x which make it convergent.

EXERCISE CXXXIV.

Find the binomial coefficients of the development of the following expressions:

1. $(a+x)^2$. 2. $(a+x)^3$. 3. $(a+x)^4$.
4. $(a+x)^5$. 5 $(a+x)^6$. 6. $(a+x)^7$.
7. $(a+x)^8$. 8. $(a+x)^9$. 9. $(a+x)^{10}$.

262. Coefficients.—Note in the above examples that after the middle of the development, the coefficients of the first half are repeated in the reverse order.

When n is odd, the number of terms in the development will be even. There will be no middle term, and the coefficients of the terms each side of the middle of the series will be the same. When n is even, the number of terms in the development will be odd, and there will be a middle term whose coefficient will be the largest of all.

263. Exponents.—Note also that the sum of the exponents of the two terms of the binomial in each term of the development is equal to n, and that the exponent of the second term of the binomial is always one less than the number of the term in which it occurs in the development. The exponent of the first term will be n, minus the exponent of the second term.

e.g. In the sixth term of the development of $(a+x)^9$ we have a^4x^5.

264. Signs.—When both terms of the binomial to be developed are positive, all the terms of the development are positive, since all powers of positive quantities are positive.

When the first term of the binomial is positive and the second term negative, every other term of the development beginning with the second is negative.

e.g. Write the product of the powers of the first and second terms of the binomial $(c-2x^2)^6$ in the fourth term of its development.

$$c^3(-2x^2)^3 = c^3 \times -8x^6 = -8c^3x^6.$$

EXERCISE CXXXV.

Write the product of the powers of the two terms of the following binomials in the given term of their development.

N.B.—When the terms of the binomial expression to be developed are complex, they should in all cases be thrown with their signs within parentheses, the powers to which these are to be raised should be indicated, and the binomial coefficient should be written before and then the indicated operation should be performed.

1. In the fifth term of $(a + 2x^3)^{17}$.
2. In the fourteenth term of $(3 - a)^{15}$.
3. In the fourth term of $(5a^3 - 7x^3)^7$.
4. In the eighth term of $(5a - x/5)^{14}$.
5. In the seventh term of $\left(2x - \frac{1}{2x}\right)^{12}$.
6. In the eleventh term of $\left(4x - \frac{1}{2\sqrt{x}}\right)^{15}$.

265. Practical Rules. — The work of developing a power of a binomial is facilitated by the following arrangement:

1°. In one line write all the powers of the first term beginning with the nth and ending with the 0th, or unity.

2°. Under these write the corresponding powers of the second term, beginning with the 0th, or unity, and ending with the nth.

3°. Under these, in a third line, write the binomial coefficients.

4°. Form the continued product of each column of three factors, and connect these products with the proper signs. The result will be the required development.

e.g. Develop $(2a - 3x^2)^5$.

Powers of $2a$,	$32a^5$	$+16a^4$	$+8a^3$	$+4a^2$	$+2a$	$+1.$
Powers of $-3x^2$,	1	$-3x^2$	$+9x^4$	$-27x^6$	$+81x^8$	$-243x^{10}.$
Binom. Coef.	1	$+5$	$+10$	$+10$	$+5$	$+1.$
$(2a - 3x^2)^5 =$	$32a^5$	$-240a^4x^2$	$+720a^3x^4$	$-1080a^2x^6$	$+810ax^8$	$-243x^{10}.$

Perhaps the easiest way to write out a binomial expression is first to throw the complex terms with their signs within parentheses, indicate the powers to which these are to be raised, and then find the binomial coefficients by successive applications of the rule already given for finding the coefficient of the next term to the one already obtained.

EXERCISE CXXXVI.

Develop the following expressions:

1. $(a+x)^7$. **2.** $(a-x)^8$. **3.** $(1+x)^9$.

4. $(x-3)^5$. **5.** $(3x+2y)^4$. **6.** $(2x-y)^5$.

7. $(1-3a^2)^6$. **8.** $(1-xy)^7$. **9.** $(3a-2/3)^6$.

10. $\left(\frac{2x}{3}+\frac{3}{2x}\right)^6$. **11.** $(c^{2/3}+d^{-3/4})^4$

12. $(m^{-1/2}-n^2)^6$. **13.** $(x^m-2y^{2n})^5$.

14. $(a^3+5\sqrt{x})^4$. **15.** $(\sqrt{a^3}+4\sqrt[3]{a})^4$.

16. $(x^{3/5}+3y^{-2/5})^5$. **17.** $(a^{1/2}b^{-2/3}+a^{-1/2}b^{2/3})^7$.

18. $(1-1/x)^{10}$.

266. The General Term.—The general term of the development of $(a+x)^n$ is usually designated the rth term, r standing for the number of the term.

In any term of the development of $(a+x)^n$:

1°. The exponent of x is one less than the number of the term.

2°. The exponent of a is n minus the exponent of x.

3°. The last factor of the numerator is one greater than the exponent of a.

4°. The last factor of the denominator is the same as the exponent of x.

Therefore, in the rth term,

The exponent of x will be $r-1$;

The exponent of a will be $n - (r - 1)$ or $n - r + 1$;
The last factor of the numerator will be $n - r + 2$;
The last factor of the denominator will be $r - 1$.
Hence the formula for the rth term is

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{1\,.\,2\,.\,3 \quad \ldots \quad (r-1)}a^{n-r+1}x^{r-1}.$$

e.g. The seventh term of $(2a^{1/2} - b^{-2})^{12}$.

In this case $n = 12$ and $r = 7$; hence the seventh term will be

$$\frac{12\,.\,11\,.\,10\,.\,9\,.\,8\,.\,7}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}(2a^{1/2})^6(-b^{-2})^6$$

$$= 924\,.\,(64a^3b^{-12}) = 59136a^3b^{-12}.$$

EXERCISE CXXXVII.

1. Find the fourth term of $(x - 5)^{13}$.
2. Find the tenth term of $(1 + 2x)^{12}$.
3. Find the twelfth term of $(2x - 1)^{13}$.
4. Find the fourth term of $(a/3) + 9b)^{10}$.
5. Find the fifth term of $(2a - b/3)^8$.
6. Find the seventh term of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.
7. Find the fifth term of $\left(\frac{x^{3/2}}{a^{1/2}} - \frac{y^{5/2}}{b^{3/2}}\right)^8$.
8. Find the value of $(x + \sqrt{2})^4 + (x - \sqrt{2})^4$.
9. Find the value of $(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6$.
10. Find the value of $[2 - \sqrt{(1 - x)}\,]^6 + [2 + \sqrt{(1-x)}]^6$.
11. Find the middle term of $(a/x + x/a)^{10}$.
12. Find the two middle terms of $\left(3a - \frac{a^3}{6}\right)^9$.
13. Find the term without x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

267. Binomial Theorem for any Rational Index.—We have seen that when n is a positive integer, the binomial function develops into a finite series, the number of whose terms is $n + 1$. This is because the factor $n - r + 1$ vanishes when $r = n + 1$.

Now as r is necessarily integral, $n - r + 1$ cannot vanish for any fractional or negative value of n. Hence when n is negative or fractional, a function when developed by the binomial theorem must produce an infinite series of terms.

It is shown in Higher Algebra that the development is true in form for all rational values of n. It must, however, be borne in mind that the series is in reality an expansion of the function only for those values of x which render the series convergent.

EXERCISE CXXXVIII.

Develop each of the following binomials to five terms:

1. $(a - x)^{1/4}$. **2.** $(a + x)^{3/2}$. **3.** $(1 - x)^{-4}$.

4. $(1 + x)^{-7}$ **5.** $(3 - 2x)^{1/4}$. **6.** $1/\sqrt[3]{1 - x}$.

7. $1/\sqrt[5]{1 + x}$. **8.** $1/(x^2 + 4y)$. **9.** $(a^4 - 2x^{-1/2})^{-1/4}$.

CHAPTER XXVIII.

PERMUTATIONS AND COMBINATIONS.

268. Permutation.—To *permute* a group of things is to arrange them in a different order, and the various different orders in which the things in a group may be arranged are called the *permutations* of the group.

Thus I permute the group formed by the three letters *abc* when I change their order into *acb*, and the six different orders in which the letters of this group may be written are called the permutations of this group. These permutations are

abc, *acb*, *bca*, *bac*, *cab*, *cba*.

269. Combination. — To combine a given number of things into groups each of which shall contain the same number of things is to select from the whole the requisite number of things and put them together without regard to the order in which they are placed, and the various groups that may be formed in this way out of the whole number are called the *combinations* of the things.

Thus the four letters *a*, *b*, *c*, *d*, may be combined two at a time, or by twos, in six different ways, namely,

ab, *ac*, *ad*, *bc*, *bd*, *cd*.

If the letters were taken three at a time, or by threes, it would be possible to make only four combinations, namely,

abc, *abd*, *acd*, *bcd*.

270. Symbols of Combination and Permutation. — If the whole number of things at our disposal be denoted by n, and the number to be put into each group be denoted by r, then the number of possible combinations will be denoted by the symbol ${}^nC_r$. This symbol is read, n things combined by r's.

Thus in the above example

$${}^4C_2 = 6,$$

and

$${}^4C_3 = 4.$$

When things are combined by 2's there are two possible permutations for each group. Thus we may write *ab*, or *ba*.

Of the four letters *a*, *b*, *c*, *d*, the possible combinations by 2's are

ab, *ac*, *ad*, *bc*, *bd*, *cd*.

Of each of these groups there are two possible permutations. Hence the possible permutations of the four letters by 2's are

ab, *ac*, *ad*, *bc*, *bd*, *cd*,
ba, *ca*, *da*, *cb*, *db*, *dc* = 12.

Of the same four letters the possible combinations by 3's are

abc, *abd*, *acd*, *bcd*.

Of each of these groups there are six possible permutations. Hence the possible permutations of the four letters by 3's are

abc, *abd*, *acd*, *bcd*,
acb, *adb*, *adc*, *bdc*,
bca, *bda*, *cda*, *cdb*,
bac, *bad*, *cad*, *cbd*,
cab, *dab*, *dac*, *dbc*,
cba, *dba*, *dca*, *dcb* = 24.

In any case, the number of permutations is equal to the product of the number of combinations and the number of permutations of each combination.

Using n and r as above, the number of permutations that are possible is denoted by the symbol ${}^nP_r$.

Thus, ${}^4P_2 = 12$ and ${}^4P_3 = 24$.

271. Number of Permutations.—The important fact to which attention was called a short time since may be symbolized thus:

$${}^nP_r = {}^nC_r \times {}^rP_r.$$

This is a special case of the following general principle: If one operation can be performed in m ways, and if after it has been performed in any one of these ways a second operation can be performed in n ways, the number of ways of performing the two operations will be $m \times n$.

The truth of this statement is evident. For there will be n ways of performing the second operation for each way of performing the first; that is, n ways of performing the two for each way of performing the first; and as there are m ways of performing the first, there must be $m \times n$ ways of performing the two.

e.g. There are ten steamers plying between Liverpool and Dublin. In how many ways can a man go from Liverpool to Dublin and return by a different steamer?

There are ten ways of making the first passage, and with each of these is a choice of nine ways of returning. Hence the number of possible ways of making the two journeys is $10 \times 9 = 90$.

This principle applies also to the case in which there are more than two operations each of which may be performed in a given number of ways.

e.g. Three travellers arrive at a town in which there are four hotels. In how many ways can they find accommodation, each at a different hotel?

The first traveller has a choice of four hotels, and after he has made his selection in any one way, the second has a choice of three. Hence the first two can make their choice in $4 \times 3 = 12$ ways. With any one of these selections, the third can select his hotel in two ways. Hence the possible number of ways is $4 \times 3 \times 2 = 24$.

272. PROBLEM I. *To find the number of permutations of n dissimilar things taken r at a time.*

This is equivalent to finding in how many different ways we may put one thing in each of r places when we have n different things at our disposal.

Evidently we may select any one of the n objects for the first place; hence we may fill that place in n different ways. After any object has been selected for the first place there remain $n - 1$ objects, any one of which may be selected for the second place. Hence the first two places may be filled in $n(n - 1)$ different ways. After any selection has been made for the first two places there remain $n - 2$ objects, any one of which may be selected for the third place. Hence the first three places can be filled in $n(n - 1)(n - 2)$ different ways. And so on.

Notice that a new factor is introduced for each place that is filled, so that the number of factors will be equal always to the number of places filled.

Notice also that the first factor is the number of objects at our disposal, and that each subsequent factor is diminished by unity, so that each factor is the number of things at our disposal diminished by a number which is one less than that of the corresponding place. Hence the rth factor will be $n - (r - 1) = n - r + 1$.

Hence the number of permutations of n things taken r at a time, or ${}^nP_r = n(n - 1)(n - 2) \ldots r$ factors,

or $$^nP_r = n(n - 1)(n - 2) \ldots (n - r + 1).$$

When r in the above formula for the number of permutations equals n, the last factor becomes 1, and the formula becomes

$$^{n}P_{n} = n(n-1)(n-2)\dots 3 \,.\, 2 \,.\, 1.$$

This product is called *factorial n.* It is usually denoted by the symbol $\underline{|n}$, or $n!$

e.g. 1°. Six persons enter a room in which there are six chairs. In how many ways may they be seated?

Here we have

$$^{6}P_{6} = \underline{|6} = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

e.g. 2°. Five persons enter a room where there are eight chairs. In how many ways may they be seated?

Here we have

$$^{8}P_{5} = 8 \times 7 \times 6 \times 5 \times 4 = 6720.$$

e.g. 3°. How many different numbers of six digits may be formed out of the nine digits 1, 2, 3, . . . 9?

Here we have

$$^{9}P_{6} = 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60480.$$

273. Problem II. *To find how many of the permutations* $^{n}P_{r}$ *contain a particular object.*

Denote the objects by the letters of the alphabet.

Find first how many permutations there are of all the letters but a when taken $r-1$ at a time. Then associate a with each of these in every possible way. The result of these two operations must be all the permutations of the n letters taken r at a time which contain the letter a.

The permutations of $n-1$ things taken $r-1$ at a time are

$$^{n-1}P_{r-1} = (n-1)(n-2)\dots(n-r+1).$$

In each of these groups a can have r positions, since it may occur first, or last, or in every intermediate position between the letters of each group.

Hence the number of permutations which contain the letter a is

$$r(n-1)(n-2)\dots(n-r+1).$$

In a similar way we may find that the number of permutations which contain two objects or letters is

$$r(r-1)(n-2)\dots(n-r+1).$$

For if the two letters a and b be left out and the remaining letters are arranged in groups of $r-2$ letters, the number of permutations would be

$$(n-2)(n-3)\dots(n-r+1).$$

Since each of these groups contains $r-2$ letters, b may be associated with each in $r-1$ different ways. Hence the number of permutations which contain b would be

$$(r-1)(n-2)(n-3)\dots(n-r+1).$$

As each of these groups contains $r-1$ letters, a may be associated with it in r different ways. Hence the number of permutations which contain a and b would be

$$r(r-1)(n-2)(n-3)\dots(n-r+1).$$

In a similar way, the number of permutations containing three objects or letters would be

$$r(r-1)(r-2)(n-3)\dots(n-r+1),$$

etc. etc,

e.g. How many numbers of four digits can be formed out of the six digits 1, 2, 3, 4, 5, 6? How many of these will contain 1? How many will contain 1 and 2? How many will contain 1, 2, and 3?

1°. ${}^6P_4 = 6 \times 5 \times 4 \times 3 = 360.$

2°. $r(n-1)(n-2)(n-3) = 4 \times 5 \times 4 \times 3 = 240.$

3°. $r(r-1)(n-2)(n-3) = 4 \times 3 \times 4 \times 3 = 144.$

4°. $r(r-1)(r-2)(n-3) = 4 \times 3 \times 2 \times 3 = 72.$

274. PROBLEM. *To find the number of permutations of n things all together, when u of the things are alike.*

Denote the required number of permutations by x. Now if the u things were all unlike they would give rise to ${}^uP_u$, or $u!$, permutations, each one of which might be combined with the x permutations, and thus give rise to ${}^nP_n$, or $n!$, permutations. Hence

$$x{}^uP_u = {}^nP_n,$$

or
$$x = \frac{n!}{u!}.$$

Similarly, if among the n objects there were u alike of one kind and v of another, then

$$x \, . \, {}^uP_u \, . \, {}^vP_v = {}^nP_n,$$

or
$$x = \frac{n!}{u! \; v!}, \text{ etc.}$$

e.g. How many permutations can be made from the letters in the word *Mississippi*?

Here there are 11 letters in all, and among them 4 s's, 4 i's, and 2 p's.

$$x = \frac{11!}{4! \; 4! \; 2!} = \frac{11 . 10 . 9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1}{4 . 3 . 2 . 1 . 4 . 3 . 2 . 1 . 2 . 1}$$
$$= 34650.$$

If the permutations were to contain no repeated letters, the number of different letters being 4, the permutations would be

$${}^4P_4 = 4 \, . \, 3 \, . \, 2 \, . \, 1 = 24.$$

EXERCISE CXXXIX.

Find the value of:

1. ${}^{13}P_7$. **2.** ${}^{15}P_8$. **3.** ${}^7P_7$.

4. How many permutations can be made of the letters in the word *number*?

5. How many permutations can be made of the letters in the word *quadruple*?

6. How many permutations can be made of the letters in the word *principle*?

7. In how many ways may 4 red, 3 blue, and 5 white cubes be arranged in a pile?

8. In how many ways can 7 cards each of a different prismatic color be arranged in piles of 4 cards each?

9. How many of these piles would contain red?

10. How many of them would contain red and green?

11. How many of them would contain red, green, and blue?

12. A pack consists of 8 white, 6 red, and 4 blue cards. In how many ways may they be arranged?

275. Problem. *To find the number of combinations of n things taken r at a time.*

As we have already seen,

$$ {}^nP = {}^nC_r \times {}^rP_r = {}^nC_r \times \underline{|r}. $$

$$ {}^nC_r = \frac{{}^nP_r}{\underline{|r}} = \frac{n(n-1)(n-2)\dots(n-r+1)}{\underline{|r}}. $$

e.g. How many different committees of 8 persons each can be formed out of a board of 16 men?

$$ \text{Here} \quad {}^{16}C_8 = \frac{16 \,.\, 15 \,.\, 14 \,.\, 13 \,.\, 12 \,.\, 11 \,.\, 10 \,.\, 9}{8 \,.\, 7 \,.\, 6 \,.\, 5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1} $$

$$ = 12870. $$

276. Problem. *To find the number of times any particular object, a, will be present in* ${}^nC_r$.

If we form ${}^{n-1}C_{r-1}$ combinations from all the objects except a taken $r-1$ together we can place a with each of these groups, and thus form all the combinations of the

n objects taken r together which contain a. Hence a occurs in ${}^{n-1}C_{r-1}$ of the combinations. Similarly, two particular objects will occur in ${}^{n-2}C_{r-2}$ of the combinations; etc.

e.g. Out of a guard of 14 men, how many different squads of 6 men can be drafted for duty each night?

In how many of these squads would any one particular man be?

In how many of these squads would any two given men be?

1°. ${}^{14}C_6 = 3003$.

2°. ${}^{13}C_5 = 1287$.

3°. ${}^{12}C_4 = 495$.

e.g. From 10 books in how many ways can a selection of 4 books be made, 1° when a specified book is included, 2° when a specified book is excluded?

1°. Since one book is to be included in each selection, we have only to choose 3 out of the remaining 9.

$$ {}^{9}C_3 = 84. $$

2°. Since one book is always to be excluded, we must select the 4 books out of the remaining 9.

$$ {}^{9}C_4 = 126. $$

EXERCISE CXL.

1. In a certain district 4 representatives are to be elected, and there are 8 candidates. In how many different ways may a ticket be made up, each ticket to contain four names?

2. Out of 9 red balls, 4 white balls, and 6 black balls, how many different combinations may be formed each consisting of 5 red balls, 1 white ball, and 3 black balls?

Out of the 9 red balls 126 combinations may be formed each containing 5 balls. Each of these may contain one of the 4 white balls, and there may be formed 20 combinations out of 6 black balls taken 2 at a time. As each of these may be combined with the 126 previous groups, hence the combinations will equal

$$126 \times 4 \times 20 = 10080.$$

3. How many combinations can be formed out of 5 red, 7 white, and 6 blue objects, each combination to consist of 3 red, 4 white, and 2 blue objects?

4. On the supposition that the colored objects of each set are all of different shape, how many permutations of these objects could be formed with 3 red, 4 white, and 2 blue in each resulting set?

5. Out of 12 doctors, 15 teachers, and 10 lawyers, how many different committees can be formed, each containing 4 doctors, 5 teachers, and 3 lawyers?

6. There are fifteen points in a plane no three of which are in a line. How many triangles can be formed by joining them in threes?

277. Meaning of the Binomial Coefficients.—

$$(a+x)^2 = (a+x)(a+x) = a^2 + 2ax + x^2;$$

$$(a+x)^3 = (a+x)(a+x)(a+x) = a^3 + 3a^2x + 3ax^2 + x^3;$$

$$\begin{aligned}(a+x)^4 &= (a+x)(a+x)(a+x)(a+x)\\ &= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4;\end{aligned}$$

$$\begin{aligned}(a+x)^n &= (a+x)(a+x)\ldots n \text{ factors}\\ &= a^n + na^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 \ldots \text{to } n+1 \text{ terms.}\end{aligned}$$

These products are formed by taking a letter from each of the n factors and combining them in every possible way.

We may take an a from each and combine these n a's into a product in every possible way. As the letters are all alike, there is only one way of combining them. Hence a^n is one term of the product.

The letter x can be taken once, and a the remaining $(n - 1)$ times, and the number of combinations of a^{n-1} and x will be the number of ways in which x may be taken out of the n factors, and this is the number of ways of taking n things 1 at a time, or ${}^nC_1 = n$. Hence the term $a^{n-1}x$ will occur ${}^nC_1$ times and we have

$$ {}^nC_1a^{n-1}x. $$

Again, the letter x can be taken twice, and a the remaining $(n - 2)$ times, and the number of ways in which 2 x's can be taken is the number of ways of taking n things 2 at a time, or ${}^nC_2 = \frac{n(n-1)}{1 . 2}$. Hence the term $a^{n-2}x^2$ will occur ${}^nC_2$ times, and we have

$$ {}^nC_2a^{n-2}x^2. $$

And, in general, x can be taken r times (r being a positive integer not greater than n), and a the remaining $(n - r)$ times, and the number of ways in which r x's can be taken is the number of ways of taking n things r at a time, or

$$ {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{1 . 2 . 3 \ldots . r}. $$

Hence we shall have ${}^nC_ra^{n-r}x^r$.

$$ \text{Hence } (a+x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 \\ + \ldots nC_ra^{n-r}x^r + \ldots \text{ to } [{}^nC_na^{n-n}x^n = x^n]. $$

We thus see that the binomial coefficients are simply the number of different ways in which n things can be taken 1, 2, 3, . . . up to n at a time.

They are $1, {}^nC_1, {}^nC_2, {}^nC_3, \ldots {}^nC_r \ldots$ up to ${}^nC_n$.

They are often written C_0, C_1, C_2, C_3, $\ldots C_r \ldots C_n$, C_0 being understood to be 1.

If we make both a and x equal to 1, the formula becomes

$$(1+1)^n = 1 + C_1 + C_2 + C_3 \ldots + C_r \ldots + Cn,$$

or $$2^n = 1 + C_1 + C_2 + C_3 \ldots + C_r \ldots + Cn.$$

That is, the sum of the binomial coefficients in any expression to $n+1$ terms is equal to $2^n - 1$.

Or the sum of all the possible ways of taking n things 1, 2, 3, up to n at a time is equal to $2^n - 1$.

CHAPTER XXIX.

DEPRESSION OF EQUATIONS.

278. General Equation of nth Degree in x.—The most general form of an integral equation of the nth degree in x is

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \dots A_{n-1}x + A_n = 0,$$

in which n is a positive integer.

If we divide this equation through by A_0, and put $\frac{A_1}{A_0} = a_1$, $\frac{A_2}{A_0} = a_2$, etc., we obtain

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_{n-1}x + a_n = 0, \qquad (1)$$

which we will consider as the general form of an integral equation of the nth degree in x.

The coefficients a_1, a_2, etc., may be integral, fractional, or surd, but we shall consider only the cases in which these coefficients are rational.

If none of the coefficients a_1, a_2, etc., are zero, the equation is said to be *complete;* and if one or more of them are zeros, *incomplete.*

Any value of x which causes the first member of (1) to vanish, or become zero, is called a root of the equation.

It is proved in Higher Algebra that every equation of the above form has at least one root, and we shall assume this to be true in the present chapter.

279. THEOREM I. *If a is a root of the equation*

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_{n-1}x + a_n = 0,$$

the first member of the equation is divisible by $x - a$.

The division of the first member by $x - a$ may be continued until the remainder does not contain x. Denote this remainder by R and the quotient obtained by Q. Then we have

$$(x - a) Q + R = 0,$$

as a form which the general equation may be made to assume.

But a is assumed to be a root of the equation. Hence if we put $x = a$, the first member must vanish.

$$\therefore \quad 0 \, . \, Q + R = 0,$$

or

$$R = 0.$$

Therefore $x - a$ is contained in the first member without a remainder.

280. Theorem II. *Conversely, if the first member of the equation*

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_{n-1} x + a_n = 0$$

is divisible by $x - a$, then a is a root of the equation.

In this case the equation may be made to take the form

$$(x - a) Q = 0,$$

the first member of which vanishes when $x = a$. Therefore a must be a root of the equation.

Cor. If the first member of the equation

$$A_0 x^n + A_1 x^{n-1} + A_2 x^{n-2} + \ldots A_{n-1} x + A_n = 0$$

be divisible by $ax + b$, then $-\frac{b}{a}$ is a root of the equation.

281. Theorem III. *An equation of the nth degree has n roots.*

We have assumed what may be proved in more advanced algebra that the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots a_{n-1} x + a_n = 0$$

has at least one root.

Denote this root by a. Then the first member is divisible by $x - a$, and the equation may be written

$$(x-a)(x^{n-1}+b_2x^{n-2}+\ldots b_{n-1}x+b_n)=0,$$

of which $x = a$ is a solution, and of which a farther solution may be obtained by putting

$$x^{n-1}+b_2x^{n-2}+\ldots b_{n-1}x+b_n=0.$$

This division lowers, or *depresses*, the degree of the equation by unity. The new equation is the same in form as (1), and therefore may be assumed to have at least one root.

Denote this root by b. Then the first member is divisible by $x - b$, and the equation may be written

$$(x-b)(x^{n-2}+c_3x^{n-3}+\ldots c_{n-1}x+c_n)=0,$$

of which $x = b$ is a solution, and of which a further solution may be obtained by putting

$$x^{n-2}+c_3x^{n-3}+\ldots c_{n-1}x+c_n=0.$$

The degree of this equation has been depressed two units from that of (1). It is still of the same general form as (1), and may be assumed to have at least one root.

Denote this root by c. As the first member is divisible by $x - c$, the equation may be written

$$(x-c)(x^{n-3}+d_4x^{n-4}+\ldots d_{n-1}x+d_n)=0,$$

and may be solved by putting

$$x-c=0,$$

and $$x^{n-3}+d_4x^{n-4}+\ldots d_{n-1}x+d_n=0.$$

The degree of our original equation has been depressed now by three units.

This process may be continued till the degree of the original equation has been depressed $n - 1$ units, and we reach an equation of the first degree of the form $x - k = 0$, of which k is the root.

As each division by a linear factor depresses the degree of the equation by unity, it must be divided by $n \dot{-} 1$ factors to depress it to the first degree. This implies $n - 1$ roots, which together with the root of the resulting linear equation make n roots.

Cor. 1. The equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \dots a_{n-1}x + a_n = 0$$

may be written

$$(x - a)(x - b)(x - c) \dots \text{to } n \text{ factors} = 0; \qquad (2)$$

and the equation

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \dots A_{n-1}x + A_n = 0$$

may be written

$$A_0(x - a)(x - b)(x - c) \dots \text{to } n \text{ factors} = 0. \qquad (3)$$

Cor. 2. The substitution of any other than one of the n values a, b, c, etc., for x in the first member of (2) or (3) would not cause it to vanish. Hence an equation of the nth degree has only n roots.

Of these n roots some may be rational, some may be surd, and some may be imaginary. Also some of the n roots may be equal.

Cor. 3. The solution of an equation of the nth degree consists merely in resolving it into its linear factors, and equating each of these factors to zero.

Cor. 4. The degree of an equation in x may be depressed by unity by dividing it through by x minus one of its roots.

Cor. 5. An equation in x may be tested for a suspected root by dividing it through by x minus the suspected root.

Cor. 6. When all the roots but two of an equation in x are known, the equation may be depressed to a quadratic equation, which may then be solved by the rule already given.

EXERCISE CXLI.

Form the equations which have the following roots:

1. 1, 2, and 3.
2. -2, -3, 4, and 5.
3. 1, -2, -3, and 0.
4. 4, -1, $-3/2$, and $1/3$.
5. -3, -3, $4/3$, and $4/3$.
6. 3, -4, $-1/4$, and $1/5$.

Prove that the numbers given are roots of the equation and find the other roots. In testing for suspected roots, use method of synthetic division:

	Equation.	Number.
7.	$x^3 - 37x + 84 = 0.$	$4.$
8.	$2x^3 + 5x^2 - 43x - 90 = 0.$	$-5.$
9.	$x^3 + 2x^2 - 11x + 6 = 0.$	$2.$
10.	$4x^4 - 4x^3 - 7x^2 - 4x + 4 = 0.$	$1/2$, 2.
11.	$9x^4 - 24x^3 - 2x^2 - 24x + 9 = 0.$	$1/3$, 3.
12.	$3x^3 - 14x^2 + 20x - 8 = 0.$	$2/3.$
13.	$x^4 - 15x^2 + 10x + 24 = 0.$	-1, 2.
14.	$x^5 - 4x^4 - 5x^3 + 20x^2 + 4x - 16 = 0.$	1, -1, 2.
15.	$x^5 - 74x^3 - 24x^2 + 937x - 840 = 0.$	1, 3, -5.

CHAPTER XXX.

UNDETERMINED COEFFICIENTS.

A. FUNCTIONS OF FINITE DIMENSIONS.

282. THEOREM I. *An integral expression of the nth degree in x cannot vanish for more than n values of x, except the coefficients of all the powers of x are zero.*

Let $$Ax^n + Bx^{n-1} + Cx^{n-2} + \dots$$

vanish for the n values of x, a, b, c, ... It must then be equivalent to $A(x-a)(x-b)(x-c)\dots$

If now we substitute for x any value k different from each of the n values a, b, c, ..., we have

$$A(k-a)(k-b)(k-c)\dots$$

Now as k is different from a, b, c, ..., the expression cannot vanish for the value $x = k$, except A itself is zero.

If A be zero, the original expression reduces to

$$Bx^{n-1} + Cx^{n-2} + \dots,$$

which is of the $(n-1)$th degree, and as before can vanish for only $n-1$ values of x, except $B = 0$. And so on.

Hence an expression of the nth degree in x *cannot vanish for more than n values of x, except the coefficients of all the powers of x are zero;* and when all these coefficients are zero, it is evident that the expression must vanish for all the powers of x.

283. THEOREM II. *If two integral expressions of the nth degree in x be equal to one another for more than n* *values of x, they will be equal for all values of x, and all the coefficients of the same powers of x in the two expressions must be equal.*

Let

$$Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots=A'x^n+B'x^{n-1}+C'x^{n-2}+\ldots$$

Then must $A=A', \quad B=B', \quad C=C'\ldots$

By transposition, we have

$$(A-A')x^n+(B-B')x^{n-1}+(C-C')x^{n-2}\ldots=0,$$

and this must be true for all values of x for which the two original expressions are equal, and therefore for more than n values of x. Hence by Theorem I,

$$A-A'=0, \quad B-B'=0, \quad C-C'=0, \ldots$$

or $$A=A', \qquad B=B', \qquad C=C', \ldots$$

When two integral expressions in x of finite dimensions are equal for *all values* of x, all the coefficients of the same power of x in the two expressions must be equal to each other. For in this case n is finite, and the possible values of x infinite, and therefore $> n$.

B. PARTIAL FRACTIONS.

284. Definition of Partial Fractions.—The sum of the two fractions $\frac{3}{1-x}$ and $\frac{2}{1+x}$ is $\frac{5+x}{1-x^2}$.

With reference to the last fraction, the parts which make it up by addition are called its *partial* fractions. It is often necessary to separate a fraction into its partials. In this separation it is understood that the denominators of the partials shall be of the first degree when practicable, but at any rate of a lower degree than that of the original fraction.

e.g. 1. Separate $\frac{2+8x}{1-x^2}$ into partial fractions.

Since the denominator $= (1 - x)(1 + x)$, assume

$$\frac{2 + 8x}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x},$$

in which A and B are coefficients to be determined.

Clearing of fractions, we have

$$\begin{aligned} 2 + 8x &= A(1 + x) + B(1 - x) \\ &= (A + B)x^0 + (A - B)x. \end{aligned}$$

And as this is to be true for all values of x, we may apply Theorem II, which gives

$$A + B = 2,$$

and $$A - B = 8.$$

$$\therefore \quad 2A = 10, \quad \text{and} \quad A = 5.$$

Also, $$2B = -6, \quad \text{and} \quad B = -3.$$

Hence the partials are

$$\frac{5}{1 - x} \quad \text{and} \quad -\frac{3}{1 + x}.$$

From the above example we may derive the following rule for separating a proper fraction into its partials:

Resolve the denominator, if possible, into real linear factors, and form fractions with undetermined numerators, and put their sum equal to the original fraction. Clear of fractions, and equate the coefficients of the like powers of x.

EXERCISE CXLII.

Separate the following fractions into partials with linear denominators:

1. $\dfrac{7x + 17}{x^2 + 5x + 6}$.

2. $\dfrac{34 - 2x}{x^2 + 2x - 8}$.

3. $\dfrac{25 - x}{x^2 - x - 12}$.

4. $\dfrac{13x - 26}{x^2 - 3x - 40}$.

5. $\dfrac{17x - 7}{2x^2 - 7x - 15}$. **6.** $\dfrac{10 - 15x}{6x^2 - 26x + 24}$.

e.g. 2. Separate $\dfrac{x^2 + 3x + 2}{6(x-1)(x-2)(x-3)}$ into partial fractions.

Assume

$$\frac{x^2 + 3x + 2}{6(x-1)(x-2)(x-3)} = \frac{A}{6(x-1)} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Then, clearing of fractions, we have

$$x^2 + 3x + 2$$
$$= A(x-2)(x-3) + 6B(x-1)(x-3) + 6C(x-1)(x-2)$$
$$= Ax^2 - 5Ax + 6A + 6Bx^2 - 24Bx + 18B + 6Cx^2 - 18Cx + 12C$$
$$\begin{array}{r|rr|rr|l} = A & x^2 - & 5A & x + & 6A & x^0 \\ +6B & & -24B & & +18B & \\ +6C & & -18C & & +12C & \end{array}$$

Therefore, equating coefficients, we have

$$A + 6B + 6C = 1,$$
$$5A + 24B + 18C = -3,$$

and
$$6A + 18B + 12C = 2.$$

Whence $A = 3$, $B = -2$, and $C = 5/3$.

$$\therefore \frac{x^2 + 3x + 2}{6(x-1)(x-2)(x-3)} = \frac{1}{2(x-1)} - \frac{2}{x-2} + \frac{5}{3(x-3)}.$$

There is, however, a shorter way of solving this example. Since in the expression

$$x^2 + 3x + 2 = A(x-2)(x-3)$$
$$+ 6B(x-1)(x-3) + 6C(x-1)(x-2)$$

x may have any value whatever, we may put $x = 1$.

Then we shall have

$$6 = 2A, \quad \text{and} \quad A = 3.$$

If we put $x = 2$, we shall have

$$12 = -6B, \quad \text{and} \quad B = -2.$$

If we put $x = 3$, we shall have

$$20 = 12C, \quad \text{and} \quad C = 5/3.$$

It is much shorter to use this method when by inspection we can find values of x which will cause all the terms except one of the right-hand member of the identity to vanish.

EXERCISE CXLIII.

Separate the following fractions into their partials:

1. $\dfrac{x^2 - 14x + 37}{(x-3)(x^2 - 9x + 20)}$.
2. $\dfrac{9x^2 - 36x - 69}{(2x+2)(x^2 - 9)}$.
3. $\dfrac{23x - 11x^2}{(2x-1)(9-x^2)}$.
4. $\dfrac{3x - 2}{(x-1)(x^2 - 5x + 6)}$.
5. $\dfrac{x}{(x+1)(x+3)(x+5)}$.
6. $\dfrac{x^2 + x + 1}{(x+1)(x^2 - 5x + 6)}$.

e.g. 3. Separate $\dfrac{7x^2 + 7x - 6}{(x+1)^2(x-2)}$ into its partials.

In forming this fraction by addition there may have been a fraction in the form of $\dfrac{A}{(x+1)^2}$, one in the form of $\dfrac{B}{x+1}$, and one in the form of $\dfrac{C}{x-2}$. Hence in our assumption we must make provision for all these.

Assume $$\frac{7x^2 + 7x - 6}{(x+1)^2(x-2)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-2}.$$

Clearing of fractions, we have

$$7x^2 + 7x - 6 = A(x - 2) + B(x + 1)(x - 2) + C(x + 1)^2.$$

Putting $x = -1$, we have

$$-6 = -3A, \quad \text{and} \quad A = 2.$$

Putting $x = 2$, we have

$$36 = 9C, \quad \text{and} \quad C = 4.$$

Equating coefficients of x^2, we have

$$B + C = 7.$$

$$\therefore \quad B = 7 - C = 3.$$

Hence $\dfrac{7x^2 + 7x - 6}{(x + 1)^2(x - 2)} = \dfrac{2}{(x + 1)^2} + \dfrac{3}{x + 1} + \dfrac{4}{x - 2}.$

e.g. 4. Separate $\dfrac{5x^2 + 1}{x^3 - 1}$ into partials.

The denominator $= (x - 1)(x^2 + x + 1)$, and the quadratic factor is not separable into real factors.

But a proper fraction which has a quadratic factor for its denominator may have a linear factor for its numerator. We must make provision for this by assuming that

$$\frac{5x^2 + 1}{x^3 - 1} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}.$$

$$5x^2 + 1 = A(x^2 + x + 1) + (Bx + C)(x - 1).$$

Putting $x = 1$, we have

$$6 = 3A, \quad \text{and} \quad A = 2.$$

Equating the coefficients of x^2, we have

$$A + B = 5.$$

$$\therefore \quad B = 5 - A = 3.$$

Equating the constant terms, we have

$$A + C = 1.$$

Whence $$C = 1 - 2 = -1.$$

Therefore $$\frac{5x^2 + 1}{x^3 - 1} = \frac{2}{x - 1} + \frac{3x - 1}{x^2 + x + 1}.$$

Observe that each of the separations into partial fractions given is characterized by this: that it introduces just as many undetermined coefficients as equations for them to satisfy. This is characteristic of any proper application of the method of undetermined coefficients in which the number of coefficients is finite.

EXERCISE CXLIV.

Separate the following fractions into partials:

1. $\dfrac{1}{x^3 + 1}$. **2.** $\dfrac{12x^2 - x + 10}{x^3 - 1}$.

3. $\dfrac{2x^3 + 2x^2 + 10}{x^4 + x^2 + 1}$. **4.** $\dfrac{x^2 - 3}{(x + 2)(x^2 + 1)}$.

5. $\dfrac{x^2 - x + 1}{(x^2 + 1)(x - 1)^2}$.

C. FUNCTIONS OF INFINITE DIMENSIONS.

285. THEOREM II. *If two integral functions of x of infinite dimensions, and arranged in ascending order, are equal to one another for all values of x which make the series convergent, the coefficients of the like powers of x in the two series will be equal.*

Let $A + bx + Cx^2 + \ldots = A' + B'x + C'x^2 + \ldots$

be true for all values of x which render both convergent.

Then will $A = A'$, $B = B'$, $C = C'$, etc.

For if the series are both convergent their difference will be convergent, and we shall have

$$A - A' + (B - B')x + (C - C')x^2 \ldots = 0$$

for all values of x for which the series is convergent.

But when x is sufficiently small, the series is convergent and $A - A'$ is greater than all that follows, and its sign must control that of the series; that is, the $A - A'$ will be $>$, $=$, or $<$ zero according as the series is $>$, $=$, or $<$ zero. But the whole series $= 0$.

$$\therefore \quad A - A' = 0, \quad \text{or} \quad A = A'.$$

By striking out A and A' as equal, we may in like manner prove $B = B'$; and then $C = C'$, etc. For since

$$(B - B')x + (C - C')x + \ldots = 0$$

for all values of x which make the original series convergent, and therefore for other values of x than zero, both members of the equation may be divided by x and the conclusion be drawn that

$$B - B' + (C - C')x + \ldots = 0$$

for values of x which make the original equation convergent.

D. EXPANSION OF FUNCTIONS.

A function may be developed into an infinite series in various ways; and whenever the series is convergent, the function is equal to its development, which is then called its *expansion*. It is important to bear in mind that when the series into which a finite function is developed becomes divergent for any value of x the function cannot equal its development.

A proper fraction may be developed into an infinite series in ascending powers of x by division.

The four following expansions by division are important:

1. $\dfrac{1}{1-x} \equiv 1 + x + x^2 + x^3 + x^4 + \ldots$

2. $\dfrac{1}{1+x} \equiv 1 - x + x^2 - x^3 + x^4 + \ldots$

$$3.\quad \frac{1}{(1-x)^2} \equiv 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$$

$$4.\quad \frac{1}{(1+x)^2} \equiv 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \dots$$

A function which is not a perfect power may be developed into an infinite series in ascending order by evolution.

$$\text{e.g.}\qquad \sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} \dots$$

If a function of x which has but one value for each value of x be expanded in ascending powers of x, the powers must all be integral.

For were the exponent of any term to become fractional, that term would be many-valued for each value of x, which contradicts the hypothesis.

The following example illustrates the expansion of a fraction by the method of undetermined coefficients.

Expand $\dfrac{1-x-x^2}{1+x+x^2}$ to five terms in ascending powers of x.

Assume

$$\frac{1-x-x^2}{1+x+x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + \dots$$

$$\begin{aligned} 1 - x - x^2 = {} & A(1 + x + x^2) + B(x + x^2 + x^3) \\ & + C(x^2 + x^3 + x^4) + D(x^3 + x^4 + x^5) + E(x^4 + x^5 + x^6) \\ & + F(x^5 + x^6 + x^7) + G(x^6 + x^7 + x^8) + \dots \end{aligned}$$

$$\begin{array}{r|r|r|r|r|r|l} = A + A & x + A & x^2 + B & x^3 + C & x^4 + D & x^5 + E & x^6 + \dots \\ + B & + B & + C & + D & + E & + F & \\ & + C & + D & + E & + F & + G & \end{array}$$

Whence $\qquad A = 1,$

$$\begin{array}{lllll} & A + B = -1, & \text{and} & B = -2, \\ & A + B + C = -1, & \text{and} & C = 0, \\ & B + C + D = 0, & \text{and} & D = 2, \\ & C + D + E = 0, & \text{and} & E = -2, \\ & D + E + F = 0, & \text{and} & F = 0, \\ \text{and} & E + F + G = 0, & \text{and} & G = 2. \end{array}$$

$$\frac{1 - x - x^2}{1 + x + x^2} = 1 - 2x + 2x^3 - 2x^4 + 2x^6 + \dots$$

In certain cases the operation of expanding fractions into series may be abridged.

1°. If the numerator and denominator of the fraction contain only even powers of x, we may assume a series containing only even powers, as $A + Bx^2 + Cx^4 + \dots$

2°. If the numerator of the fraction contains only odd powers of x and the denominator only even powers, we may assume a series containing only odd powers of x.

3°. If every term in the numerator contains x, but not every term in the denominator, we may assume a series beginning with the lowest power of x in the numerator.

4°. If the numerator does not contain x, we may find by actual division what power of x will occur in the first term of the expansion.

e.g. $\frac{1}{3x^2 - x^3}$ gives by division $1/3x^{-2}$ as the first term of the quotient. Hence we may assume

$$\frac{1}{3x^2 - x^3} = Ax^{-2} + Bx^{-1} + C + Dx + \dots$$

EXERCISE CXLV.

Expand each of the following fractions to five terms in ascending powers of x:

1. $\dfrac{1 - 2x + 3x^2}{1 + 3x - 4x^2}$.

2. $\dfrac{2 - 3x + 4x^2}{1 + 2x - 5x^2}$.

3. $\dfrac{3 - 4x^2}{1 + 5x^2}$.

4. $\dfrac{2 - 3x^2}{1 + 4x^2}$.

5. $\dfrac{3x}{4 - 3x^2}$.

6. $\dfrac{2x}{3 - 2x^2}$.

The following example illustrates the method of developing a radical by the method of undetermined coefficients.

To expand $\sqrt{1 + x}$.

Assume

$$\sqrt{1 + x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots$$

Then, squaring each member, we have

$$1 + x = A^2 + 2ABx + \left.\begin{array}{r}2AC\\ + B^2\\ \end{array}\right| x^2 + \left.\begin{array}{r}2AD\\ + 2BC\\ \end{array}\right| x^3 + \left.\begin{array}{r}2AE\\ + 2BD\\ C^2\end{array}\right| x^4 +$$

Whence

$$\begin{aligned} A^2 &= 1, & A &= 1,\\ 2AB &= 1, & B &= 1/2,\\ 2AC + B^2 &= 0, & C &= -1/8.\\ 2AD + 2BC &= 0, & D &= 1/16,\\ 2AE + 2BD + C^2 &= 0, & E &= -5/128. \end{aligned}$$

Therefore

$$\sqrt{(1 + x)} = 1 + 1/2x - 1/8x^2 + 1/16x^3 - 5/128x^4 + \dots$$

EXERCISE CXLVI.

1. Expand $\sqrt{(1+x+x^2)}$ to x^4.

2. Expand $\sqrt{\left(\frac{1-x}{1+x}\right)}$ to x^4.

3. Expand $\sqrt[3]{(1+x)}$ to x^4.

Ex. Let $y = 3x - 2x^2 + 3x^3 - 4x^4 + \dots$

Express x in ascending powers of y to y^4.

$$\begin{aligned} \text{Assume } x &= Ay + By^2 + Cy^3 + Dy^4 + \dots \\ &= A(3x - 2x^2 + 3x^3 - 4x^4 + \dots) \\ &+ B(9x^2 - 12x^3 + 22x^4 + \dots) \\ &+ C(27x^3 - 54x^4 + \dots) \\ &+ D(81x^4 + \dots). \end{aligned}$$

$$\therefore\quad x = 3Ax \begin{array}[t]{r|}-2A\\+9B\end{array}\,x^2 \begin{array}[t]{r|}+3A\\-12B\\+27C\end{array}\,x^3 \begin{array}[t]{r|}-4A\\+22B\\-54C\\+81D\end{array}\,x^4 + \dots$$

Whence
$$3A = 1$$
$$-2A + 9B = 0$$
$$3A - 12B + 27C = 0$$
$$-4A + 22B - 54C + 81D = 0.$$

Whence

$$A = 1/3,\ B = 2/27,\ C = -1/243,\ D = -14/2187.$$

Therefore $x = 1/3y + 2/27y^2 - 1/243y^3 - 14/2187y^4 + \dots$

EXERCISE CXLVII.

1. If $y = 2x + x^2 - 2x^3 - 3x^4 + \ldots$, find x in terms of y to y^4.

2. If $y = x + x^2 + x^3 + x^4 + \ldots$, find x in terms of y to y^4.

3. If $y = x - x^3 + x^5 - x^7 + \ldots$, find x in terms of y to y^4.

CHAPTER XXXI.

CONTINUED FRACTIONS.

286. Definition of a Continued Fraction.—An expression of the form

$$a \pm \cfrac{b}{c \pm \cfrac{d}{e \pm \cfrac{f}{g \pm \text{etc.}}}}$$

is called a *continued fraction.*

For convenience, continued fractions usually are written in the form

$$a \pm \frac{b}{c \pm}\,\frac{d}{e \pm}\,\frac{f}{g \pm}\ \text{etc.}$$

In this chapter we shall consider only the simpler form

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\ \text{etc.},$$

in which the numerators are each unity and a_1, a_2, a_3, etc., are positive integers.

The fractions a_1, $\frac{1}{a_2}$, $\frac{1}{a_3}$, etc., are called the first, second, third, etc., *elements* of the continued fraction.

287. The Convergents.—The fraction obtained by stopping at any element is called a *convergent* of the continued fraction. Thus a_1, $a_1 + \frac{1}{a_2}$, and $a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3}$ are the first, second, and third convergents of the continued fraction given above. These convergents may be reduced to the forms $\frac{a_1}{1}$, $\frac{a_1a_2+1}{a_2}$, and $\frac{(a_1a_2+1)a_3+a_1}{a_2a_3+1}$.

For, evidently, $a_1 = \frac{a_1}{1}$,

$$a_1 + \frac{1}{a_2} = \frac{a_1a_2}{a_2} + \frac{1}{a_2} = \frac{a_1a_2+1}{a_2},$$

and $$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}} = a_1 + \cfrac{1}{\cfrac{a_2a_3+1}{a_3}} = a_1 + \frac{a_3}{a_2a_3+1}$$

$$= \frac{a_1a_2a_3 + a_1 + a_3}{a_2a_3+1} = \frac{(a_1a_2+1)a_3+a_1}{a_2a_3+1}.$$

The rth convergent of a continued fraction will be denoted by $\frac{p_r}{q_r}$.

Each convergent may be reduced to an ordinary fraction, as above, by successive simplification of the complex fractions of which it is composed. In this simplification we begin always with the last complex denominator.

288. THEOREM I. *The numerator and denominator of any convergent beyond the second are formed by multiplying the numerator and denominator of the preceding convergent by the denominator of the new element considered and adding to the respective products the numerator and denominator of the last convergent but one.*

An examination of the first three convergents already obtained by actual reduction of the complex fractions to simpler ones will show that the numerator and denominator of the third convergent are formed in accordance with this theorem.

Denote the number of the convergent by n, and the nth convergent by $\frac{p_n}{q_n}$, the preceding convergent by $\frac{p_{n-1}}{q_{n-1}}$, the last but one by $\frac{p_{n-2}}{q_{n-2}}$.

Then in the case of the third convergent we have

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}. \tag{1}$$

Now each convergent differs from the one preceding it by having $a_n + \frac{1}{a_{n+1}}$ substituted in place of a_n. Thus the second convergent differs from the first simply in having $a_1 + \frac{1}{a_2}$ in place of a_1, the third differs from the second in having $a_2 + \frac{1}{a_3}$ in place of a_2, and the $(n+1)$st will differ from the n only in having $a_n + \frac{1}{a_{n+1}}$ in place of a_n.

Making this change in (1), we have

$$\frac{p_{n+1}}{q_{n+1}} = \frac{\left(a_n + \frac{1}{a_{n+1}} p_{n-1}\right) + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}} q_{n-1}\right) + q_{n-2}}$$

$$= \frac{a_{n+1}(a_n p_{n-1} + p_{n-2} +) p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2} +) q_{n-1}} = \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}},$$

which agrees with the theorem.

Hence the theorem which holds for the third convergent holds also for the fourth, the fifth, and each subsequent convergent.

Therefore the formula for the rth convergent is

$$\frac{p_r}{q_r} = \frac{a_r p_{r-1} + p_{r-2}}{a_r q_{r-1} + q_{r-2}}.$$

289. Partial and Complete Quotients.—The integers a_1, a_2, a_3, etc., may be called the *partial* quotients, a_n being the nth partial quotient. When the number of partial quotients is finite the continued fraction is said to be *terminating*. If the number of these quotients is unlimited the fraction is called an *infinite continued fraction*.

Since a_1, a_2, a_3, etc., are positive integers, a continued fraction of the form $a_1 + \frac{1}{a_2 +}\frac{1}{a_2} +$ etc. must be greater than unity; while a continued fraction of the form of $\frac{1}{a_1} + \frac{1}{a_2 +}\frac{1}{a_3 +} \dots$ must be less than unity.

The *complete* quotient at any stage is the quotient from that point on to the end. Thus a_n is the nth partial quotient, and $a_n + \frac{1}{a_{n+1} +}\frac{1}{a_{n+2} +} \dots$ is the corresponding complete quotient. The complete quotient at any stage may be denoted by K.

As we have seen, the nth convergent is

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}.$$

This value evidently may be converted into that of the whole continued fraction by substituting K in the place of a_n. Denote the value of the entire fraction by x. Then will

$$x = \frac{K p_{n-1} + p_{n-2}}{K q_{n-1} + q_{n-2}}.$$

290. THEOREM II. *The difference between two successive convergents is a fraction whose numerator is unity and whose denominator is the product of the denominators of the convergents, and this difference taken in regular order is alternately positive and negative.*

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} - \frac{p_{n-1}}{q_{n-1}}$$

$$= \frac{(a_np_{n-1} + p_{n-2})q_{n-1} - (a_nq_{n-1} + q_{n-2})p_{n-1}}{(a_nq_{n-1} + q_{n-2})q_{n-1}}.$$

$$\therefore \quad \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_{n-2}q_{n-1} - p_{n-1}q_{n-2}}{q_nq_{n-1}}$$

$$\therefore \quad p_nq_{n-1} - p_{n-1}q_n = -(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}).$$

So also in succession

$$p_{n-1}q_{n-2} - p_{n-2}q_{n-1} = -p_{n-2}q_{n-3} + p_{n-3}q_{n-2}.$$

. . . .

$$p_3q_2 - p_2q_3 = -p_2q_1 + p_1q_2.$$

But $p_2q_1 - p_1q_2 = (a_1a_2 + 1) - a_1a_2 = 1 = (-1)^2.$

Also, since the successive convergents, beginning with the first, are alternately less and greater than the fraction, the successive convergents are alternately greater and less than the preceding. Therefore the successive difference will be alternately positive and negative, so that the numerator of the fraction will be $(-1)^n$, in which n is the number of the convergent used as a subtrahend.

Hence $\quad p_nq_{n-1} - p_{n-1}q_n = (-1)^n. \quad$ (1)

Hence, also, $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}} = \dfrac{(-1)^n}{q_nq_{n-1}}. \quad$ (2)

Cor. 1. *All convergents are in their lowest terms.* For every common measure of p_n and q_n must also be a measure of $p_nq_{n-1} - p_{n-1}q_n$ and, from (1), of ± 1.

Hence p_n and q_n can have no common measure.

Cor. 2. In the continued fraction

$$\frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \ldots,$$

which is less than unity,

$p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}$ and $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_nq_{n-1}}$,

since the first convergent will be too large, the next too small, etc.

291. THEOREM III. *Each convergent is nearer in value to the continued fraction than any preceding convergent.*

Let x denote the continued fraction, and $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, and $\frac{p_{n+2}}{q_{n+2}}$ denote three consecutive convergents.

Then x differs from $\frac{p_{n+2}}{q_{n+2}}$ only in taking the complete $(n+2)$ quotient in place of a_{n+2}. Hence

$$x = \frac{Kp_{n+1} + p_n}{Kq_{n+1} + q_n}.$$

$$\therefore \quad x \sim \frac{p_n}{q_n} = \frac{K(p_{n+1}q_n \sim p_nq_{n+1})}{q_n(Kq_{n+1} + q_n)} = \frac{K}{q_n(Kq_{n+1} + q_n)},$$

and

$$\frac{p_{n+1}}{q_{n+1}} \sim x = \frac{p_{n+1}}{q_{n+1}} \sim \frac{Kp_{n+1} + p_n}{Kq_{n+1} + q_n}$$

$$= \frac{p_{n+1}q_n \sim p_nq_{n+1}}{q_{n+1}(Kq_{n+1} + q_n)} = \frac{1}{q_{n+1}(Kq_{n+1} + q_n)}.$$

Now $\qquad K > 1$ and $q_n < q_{n+1}$;

hence on both accounts

$$\frac{K}{q_nKq_{n+1} + q_n} > \frac{1}{q_{n+1}Kq_{n+1} + q_n}.$$

Combining the result of this article with that of article 290, it follows that

The convergents of an odd order continually increase, but are always less than the continued fraction;

The convergents of an even order continually decrease, but are always greater than the continued fraction.

292. THEOREM IV. *The value of x differs from $\frac{p_n}{q_n}$ by less than $\frac{1}{{q_n}^2}$ and by more than $\frac{1}{2q^2_{n+1}}$.*

Let $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, $\frac{p_{n+2}}{q_{n+2}}$ be three consecutive convergents, and let K denote the $(n+2)$th complete quotient.

Then
$$x = \frac{Kp_{n+1} + p_n}{Kq_{n+1} + q_n}.$$

$$\therefore \quad x - \frac{p_n}{q_n} = \frac{(Kp_{n+1} + p_n)q_n}{(Kq_{n+1} + q_n)q_n} - \frac{Kp_nq_{n+1} + p_nq_n}{q_n(Kq_{n+1} + q_n)}$$
$$= \frac{Kp_{n+1}q_n + p_nq_n - Kp_nq_{n+1} - p_nq_n}{q_n(Kq_{n+1} + q_n)}$$
$$= \frac{K(p_{n+1}q_n - p_nq_{n+1})}{q_n(Kq_{n+1} + q_n)} = \frac{K}{q_n(Kq_{n+1} + q_n)}$$
$$= \frac{1}{q_n\left(q_{n+1} + \frac{q_n}{K}\right)}.$$

Now K is greater than 1, therefore $\frac{p_n}{q_n}$ differs from x by less than $\frac{1}{q_nq_{n+1}}$ and by more than $\frac{1}{q_nq_{n+1} + {q_n}^2}$.

And since $q_n < q_{n+1}$, the difference between $\frac{p_n}{q_n}$ and x must be less than $\frac{1}{{q_n}^2}$ and greater than $\frac{1}{2q^2_{n+1}}$.

293. THEOREM V. *The last convergent preceding a large partial quotient is a close approximation to the value of the fraction.*

By the last theorem, the error in taking $\frac{p_n}{q_n}$ instead of the whole continued fraction is less than $\frac{1}{q_n q_{n+1}}$, or, since $q_{n+1} = a_{n+1}q_n + q_{n-1}$, less than $\frac{1}{q_n(a_{n+1}q_n + q_{n-1})}$, or less than $\frac{1}{a_{n+1}q_n^2}$. Hence the larger a_{n+1} is, the nearer does $\frac{p_n}{q_n}$ approximate to the continued fraction. Therefore when a_{n+1} is relatively large, the value of x differs but little from that of $\frac{p_n}{q_n}$.

294. Theorem VI. *Every fraction whose numerator and denominator are positive integers can be converted into a terminating continued fraction.*

Let $\frac{m}{n}$ be a fraction whose numerator and denominator are positive integers.

Divide m by n and let a_1 be the integral quotient and p the remainder. Then

$$\frac{m}{n} = a_1 + \frac{p}{n} = a_1 + \frac{1}{\frac{n}{p}}.$$

Divide n by p and let a_2 be the integral quotient and q be the remainder. Then

$$\frac{n}{p} = a_2 + \frac{q}{p} = a_2 + \frac{1}{\frac{p}{q}}.$$

Divide p by q and let a_3 be the integral quotient and r be the remainder, and so on.

Therefore $$\frac{m}{n} = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +} \ldots$$

If $m < n$, the first integral quotient will be zero.

Put $\dfrac{m}{n} = \dfrac{1}{\dfrac{n}{m}}$ and proceed as before.

The above process is the same as that of finding the greatest common measure of m and n, a_1, a_2, a_3 being the successive quotients. As m and n, being positive integers, are commensurable, the process must terminate after a finite number of divisions.

Cor. Evidently $\dfrac{m}{n}$ and $\dfrac{Km}{Kn}$ will give the same continued fraction.

e.g. 1. Reduce $\dfrac{251}{802}$ to a continued fraction.

Find the greatest common divisor of 251 and 802 by the usual method.

$$\begin{array}{r|r|l} 251 & 802 & 3 \\ 6 & 49 & 5 \\ & 1 & 8 \\ & & 6 \end{array}\left.\vphantom{\begin{array}{l}1\\1\\1\\1\end{array}}\right\}\text{quotients.}$$

$$\therefore\quad \frac{251}{802} = \frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{8+}\,\frac{1}{6}.$$

e.g. 2. Reduce 3.1416 to a continued fraction.

$$3.1416 = 3 + \frac{1416}{10000}.$$

$$\begin{array}{r|r|r} 1416 & 10000 & 7 \\ 8 & 88 & 16 \\ & & 11 \end{array} \qquad \therefore\quad \frac{1416}{10000} = \frac{1}{7+}\,\frac{1}{16+}\,\frac{1}{11},$$

and $$3.1416 = 3 + \frac{1}{7+}\,\frac{1}{16+}\,\frac{1}{11}.$$

e.g. 3. Show that $\frac{355}{113}$ is a close approximation to 3.14159, differing from it by less than .000004.

14159	100000	7
854	887	15
29	33	1
1	4	25
		1
		7
		4

$$3.14159 = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{25+}\,\frac{1}{1+}\,\frac{1}{7+}\,\frac{1}{4}.$$

The successive convergents are

$$\frac{3}{1},\quad \frac{22}{7},\quad \frac{333}{106},\quad \frac{355}{113},\ \cdots$$

The last convergent precedes the large quotient 25, and hence is a close approximation to x.

It differs from it by less than $\frac{1}{25 \times (113)^2}$, and therefore by less than $\frac{1}{25 \times (100)^2}$, or .000004.

EXERCISE CXLVIII.

Express the following as continued fractions:

1. $\frac{53}{59}$. **2** $\frac{72}{91}$. **3.** 3.61.

4. $\frac{112}{153}$. **5.** $\frac{749}{326}$. **6.** $\frac{144}{89}$.

7. $\frac{436}{345}$. **8.** $\frac{3015}{6961}$.

Calculate the successive convergents to the following continued fraction:

9. $2+\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{11+}\,\frac{1}{2}.$

10. $\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{6}.$

11. $3+\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9}.$

12. $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{4}.$

13. Find a series of fractions converging to .24226, the excess in days of the tropical year over 365 days.

14. A metre is 39.37079 inches; show by the theory of continued fractions that 32 metres are nearly equal to 35 yards.

15. A kilometre is very nearly equal to .62138 mile. Show that the fractions $\frac{5}{8}, \frac{18}{29}, \frac{23}{37}, \frac{64}{103}$ are successive approximations to the ratio of a kilometre to a mile.

16. Two scales of equal lengths are divided into 162 and 209 equal parts respectively. If their zero points are coincident, show that the thirty-first division of one nearly coincides with the fortieth of the other.

17. The modulus of the common system of logarithms is approximately equal to .43429. Express this decimal as a continued fraction, find its sixth convergent, and determine the limits to the error made in taking this convergent for the fraction itself.

18. The base of the Napierean system of logarithms is 2.7183 approximately. Express this decimal as a continued fraction, find its eighth convergent, and determine the limits to the error made in taking this convergent for the fraction itself.

295. Periodic Continued Fractions.—When the partial quotients of a continued fraction continually recur in the same order, the fraction is called a *periodic continued fraction.*

A periodic continued fraction is said to be simple or mixed according as the recurrence begins at the beginning or not. Thus,

$$a+\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}+\dots$$

is a simple periodic fraction.

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{b+}\,\frac{1}{b+}+\dots$$

is a mixed periodic fraction.

296. THEOREM VII. *A quadratic surd can be expressed as an infinite periodic continued fraction.*

e.g. Reduce $\sqrt{8}$ to a continued fraction.

The integer next below $\sqrt{8}$ is 2. Hence

$$\sqrt{8}=2+\sqrt{8}-2.$$

$\sqrt{8}-2$ expressed as an equivalent fraction with a rational numerator is

$$\frac{(\sqrt{8}-2)(\sqrt{8}+2)}{\sqrt{8}+2}=\frac{4}{\sqrt{8}+2}.$$

$$\therefore\quad \sqrt{8}=2+\frac{4}{\sqrt{8}+2}=2+\frac{1}{\frac{\sqrt{8}+2}{4}}.$$

The integer next below $\frac{\sqrt{8}+2}{4}$ is 1.

$$\therefore\quad \frac{\sqrt{8}+2}{4}=1+\frac{\sqrt{8}-2}{4}.$$

Hence

$$\sqrt{8}=2+\frac{1}{1+\frac{\sqrt{8}-2}{4}}=2+\frac{1}{1+}\,\frac{1}{\frac{\sqrt{8}-2}{4}}$$

$$= 2 + \frac{1}{1+}\,\frac{1}{\dfrac{(\sqrt{8}-2)(\sqrt{8}+2)}{4(\sqrt{8}+2)}}$$

$$= 2 + \frac{1}{1+}\,\frac{1}{\sqrt{8}+2}.$$

The integer next below $\sqrt{8}+2$ is 4. Hence

$$\sqrt{8}+2 = 4 + \sqrt{8} - 2 = 4 + \frac{(\sqrt{8}-2)(\sqrt{8}+2)}{\sqrt{8}+2}$$

$$= 4 + \frac{4}{\sqrt{8}+2} = 4 + \frac{1}{\dfrac{\sqrt{8}+2}{4}}.$$

At this point the steps begin to recur:

$$\sqrt{8} = 2 + \frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\ldots$$

Thus $\sqrt{8}$ is seen to be equivalent to a periodic fraction with one non-periodic element, which is half the last partial quotient of the recurring portion. This law holds good for every quadratic surd.

Note in the above example that the last partial quotient in the recurring portion is an integer + the given surd.

The following is a very compact and convenient form for working such examples:

$$\sqrt{8} = 2 + \sqrt{8} - 2 = 2 + \frac{4}{\sqrt{8}+2},$$

$$\frac{\sqrt{8}+2}{4} = 1 + \frac{\sqrt{8}-2}{4} = 1 + \frac{1}{\sqrt{8}+2},$$

$$\sqrt{8}+2 = 4 + \sqrt{8} - 2 = 4 + \frac{4}{\sqrt{8}+2},$$

$$\frac{\sqrt{8}+2}{4} = 1 + \frac{\sqrt{8}-2}{4} = 1 + \frac{1}{\sqrt{8}+2},$$

$$\sqrt{8}+2 = 4 + \sqrt{8} - 2 = 4 + \frac{4}{\sqrt{8}+2},$$

$$\therefore \quad \sqrt{8} = 2 + \frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4}, \text{ etc.}$$

296. THEOREM VIII. *An infinite periodic fraction may be expressed as a quadratic surd.*

Let the partial quotient be 1, 2, 3, 1, 2, 3, etc.

Then $$x = \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+x} = \frac{7+2x}{10+3x}.$$

$$\therefore \quad 10x + 3x^2 = 7 + 2x.$$

$$\therefore \quad 3x^2 + 8x - 7 = 0,$$

$$\therefore \quad x = 1/3(\sqrt{37} - 4).$$

EXERCISE CXLIX.

Express the following as periodic continued fractions:

1. $\sqrt{7}$. 2. $\sqrt{13}$. 3. $\sqrt{2}$.

4. $\sqrt{6}$. 5. $\sqrt{17}$. 6. $\sqrt{19}$.

Express the following continued fractions as quadratic surds:

7 $\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\ldots$ 8. $1 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\ldots$

9. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1+}\ldots$

ANSWERS.

EXERCISE I.

1. 105. **2.** 525. **3.** 2625.
4. 26460. **5.** 85050. **6.** 396900.
7. 91875. **8.** 1701000. **9.** 165375.
10. 1181250. **11.** $5a$ cts. **12.** $120a$ sq. in.
13. $20mn$. **14.** $25a^2c$.

EXERCISE II.

1. $67\frac{2}{3}$. **2.** $35\frac{1}{2}$. **3.** 1130.
4. Same. **5.** Same. **6.** Same.

EXERCISE III.

1. $2a^2b^3 + 10a^3b^2 + 12$. **2.** $a - 12b^2 + 3$.
3. $6x^2y + 5 - 5b^3$. **4.** $3a^2y + 9ay^2 - 7$.
5. $7a^3x - 5a^2x^3 + 6$.

EXERCISE IV.

1. $x = 12$. **2.** $y = 7$. **3.** $z = 7\frac{3}{7}$. **4.** $x = 15$.
5. $x = 284\frac{4}{19}$. **6.** $x = \frac{b + 4a}{3}$. **7.** $x = \frac{c - b}{9a}$.

EXERCISE V.

1. 27 and 36. **2.** 45 and 58. **3.** 30 and 120.
4. 17 and 85. **5.** 75, 150, and 225.
6. 72, 36, and 12. **7.** 525, 175, and 35.
8. Harness \$45 ; horse \$135 ; carriage \$270.
9. History \$1.38 ; arithmetic 69 cts. ; speller 23 cts.
10. Sister's age 10 ; boy's 13 ; brother's 18.

EXERCISE VI.

1. 70 and 105. 2. 27 and 45.
3. \$1.20, \$1.80, \$1.35, and \$0.54.
4. \$45000, \$30000, \$24000, and \$18000.
5. \$100.00, \$25.00, and \$200.00.

EXERCISE VII.

1. $3a - 4b - 6ab + 8ad + 6$.
2. $3m + 4n - 20cx + 25cy - 5c^2$.
3. $7 + 24c - 32b - 12x$. 4. $5x - ab - ac + 7a$.
5. $18m + 16a - 24b + 32c$. 6. $2x+6x+21$, or $8x+21$.
7. $5x - 3(a + 2b - 3c) + 9$; $5x + 3(-a - 2b + 3c) + 9$.
8. $7ab - 4c(2b - 4d - 6c) + 3$; $7ab + 4c(-2b + 4d + 6c) + 3$.
9. $27 - 2a^2(-3c + 5b - 6)$; $27 + 2a^2(3c - 5b + 6)$.
10. $10x - 5(-4x^2 - 5a^2x + 7)$; $10x + 5(4x^2 + 5a^2x - 7)$.

EXERCISE VIII.

1. 8 and 12. 2. 3 and 9.
3. Harness = \$60; horse = \$180; carriage = \$480.
4. \$2000 the first month, \$5920 the second month, and \$23720 the third month.
5. 20. 6. 36. 7. 18. 8. 132. 9. 99.
10. 25, 48, and 46.

EXERCISE IX.

1. 245 bushels in all, 98 bushels of rye, and 70 bushels of barley.

2. 231 in all, 154 baldwins, and 42 greenings.

3. First and second 30 miles, second and third 32 miles, and first and fourth 80 miles.

4. Louis had 320, and Howard 80.

5. First 77, second 81, and third 68.

6. Winning candidate 18156. Losing candidates 17344, 17624, and 17400, respectively.

7. M to N 21 miles, N to S 6 miles, and S to T 81 miles.

EXERCISE X.

1. 18. **2.** 6. **3.** 18. **4.** 6. **5.** -6.

6. -6. **7.** -18. **8.** -18. **9.** 18. **10.** -6.

11. 0. **12.** 0. **13.** 0. **14.** 0. **15.** -18.

16. 6. **17.** $2a$. **18.** $-2a$.

19. $12+(+6)$, $6+(+12)$, and $12-(-6)$;
$12+(-6)$, $-6+(+12)$, and $-6-(-12)$;
$a-(+a)$, $a+(-a)$; $-6+(-12)$,
$-12+(-6)$, $-6-(+12)$.

EXERCISE XI.

1. 6 B.C. **2.** n B.C. **3.** 40 A.D.

4. a A.D. **5.** 30 B.C. **6.** b B.C.

7. 50 A.D. **8.** c A.D. **9.** 20° below zero.

10. Has fallen 12°. **11.** Has risen 16°.

12. Has fallen 7°. **13.** Has fallen 8°.

14. Has risen a°. **15.** 17° warmer.

16. 8° colder. **17.** 12° warmer.

18. 3 miles west. **19.** 5 miles south.

20. 4 years younger. **21.** 6 years older.

22. 2 years younger.

23. The grocer owes Hermon 3 dollars.

24. 7 pounds less. **25.** 20000 dollars poorer.

EXERCISE XII.

1. $c-a-b$, $c-(a+b)$. **2.** $\frac{m}{a+b}$.

3. In 6 hours. First will have travelled 24 miles, and second 18 miles.

4. $\frac{a}{m+n}$ hours. First $\frac{ma}{m+n}$ miles, second $\frac{na}{m+n}$ miles.

5. 50 and 58.

EXERCISE XIII.

1. $25a$.

2. $37a^2x$.

3. $-36ab^2$.

4. $-56x$.

5. $-6x^2$.

6. $2ac^2x$.

7. $2y^2 - 2ac - 5$.

8. $4a^2x - ax^2 - 9ab - 8$.

9. $-9x + 6ab + c$.

10. $5x^2 - a^2b^3 - c + 7$.

11. $19/12x = 1\frac{7}{12}x$.

12. $-7/12y$.

13. $16(a + b)$.

14. $-a - (x + y)$.

15. $5(a + b) - 5(m + n)$.

16. $4a(b + x)$.

17. $c(a^2 - b^2)$.

18. $-2az - 4$.

19. 0.

20. $2a - b + 5c + 3d$.

21. $4x + 3y + 2 + 5g$.

22. $a^2 - xy$.

23. $a^2b^3 + x^2y$.

24. $3a + 10c + 5d - x$.

25. $x + b - c + d$.

26. First 7500, second 7000, third 6500, and fourth 6000.

27. $(4x - 50)$ dollars.

28. 8000 dollars.

29. $\dfrac{(a + b)x - mq}{5}$

EXERCISE XIV.

1. $(a + m)x + (b + n)y$.

2. $(mn + pq)x - 2by$.

3. $(3 + 6b + 7a)x - 6y + m + n$.

4. $8(a + b + 1)x + (b - 10)y$.

5. $x + 8$ and $2x + 8$.

6. Albert is 12 and Howard 24.

7. In 9 hours. 72 miles and 54 miles.

8. $\dfrac{a - b}{2}, \dfrac{a + b}{2}$.

9. $(a - m)x + (b - n)y + (c - p)z$.

10. $2(d - f)x + 3(e - d)y + 4(f + e)z$.

11. $1/12(8a + 9b)y - 2(1 - 3a)x$.

12. $(2a - 3b)x - (4a + b)y$.

13. Herbert is $1/2a$, and Horace a.

14. In $\dfrac{a}{b - c}$ hours, $\dfrac{ab}{b - c}$ miles, and $\dfrac{ac}{b - c}$ miles.

EXERCISE XV.

1. $-3x - y + 14z.$ **2.** $4a - b + 2c.$

3. $8a^4 - 2a^3 + 4a^2 - 15a + 14.$

4. $20a^2x^2 + 16a^2x.$ **5.** $4a^3 - 2.$

6. $4/3x^2 - 7/2x - 1/2.$ **7.** $a - b + c.$

8. $-2a - 9b - 8.$ **9.** $x - 8a^2x^2 + 12.$

10. $2a - 7b - 3.$ **11.** 9 and 18.

12. A has $72.50, and B $77.50.

13. $-x + y.$ **14.** $2x - 11.$

15. $5x + 4y + 7a - 11.$ **16.** $x^3 + x^2 + x + 1.$

17. $4(a + b).$ **18.** $2a(c - x).$

19. $2a^2(b-x)+4ab(a-b).$ **20.** $x - 8.$

21. $\frac{6a + m}{6}$ and $\frac{5m - 6a}{6}.$

EXERCISE XVI.

1. $-3ab - m - 2ax.$ **2.** $3x - 2a.$

3. $2b - 4c.$ **4.** $10x - 7y + 5z.$

5. $-9ax - 2by.$ **6.** 0. **7.** 0. **8.** $3m.$

EXERCISE XVII.

1. $x - (a + b).$ **2.** $x - (m + n).$

3. $x - (-a + 3x - 2y).$ **4.** $x - (3b - 2c - 5d).$

5. $x - (-x + 2a - 2b).$ **6.** $x - (-3 + a + b).$

7. $x-(-a+b-c-m+n).$ **8.** $x - (-x - a + b).$

9. $x - (-2x + 2m - 2n).$ **10.** $x - (-2x + 2ab - m).$

11. $x - (2m + 3a - 2b).$ **12.** $x-(2am+b+p-q-n).$

13. $x - (a + b + p - q + m - n).$

EXERCISE XVIII.

1. $m - p + q + a - b - c + d.$

2. $m + a - b + p + q - n + k.$

3. $15ax - 4by.$ **4.** 0.

5. $p + b + s + t + m + n.$

6. $8360, $16120, $23880, and $31640.

7. $\frac{a+b+c}{4}$, $\frac{a-3b+c}{4}$, and $\frac{a+b-c}{2}$.

8. $11ax$.

9. $-2ax-6by-cz$.

10. $-2x+2y$.

11. $-4bz$.

12. $-(m+6)y+2x+4ab-5$.

13. 6, 18, 36, 54, and 72.

EXERCISE XIX.

1. $21ab$.
2. $30a^3b$.
3. $-32a^2x^4y^2$.
4. $-a^5b^5xy^2$.
5. $3a^2b^2x^8z^2$.
6. $-42m^5n^3x^4y^3$.
7. a^4bcdm.
8. $-abcdx^4$.
9. $-a^3b^2cx^4$.
10. $30a^6b^3mx^2$.
11. $105a^3m^2xy^3$.
12. $4abmn$.
13. $168abkm^2x^3$.
14. $bgmny^3$.
15. $4ax^4y^6$.
16. $-30agx^4y^3z$.
17. $15a^2b^2nx^3yz$.
18. $-4abgxyz^4$.
19. $4bc^2gnx^2z^5$.
20. $-3ab^2c^3x^2y$.
21. $4abexy$.
22. $-24a^4x^2y^3$.
23. $-3a^4x^3y^3$.
24. $-m^6n^4x^2$.
25. $a^5bx^4y^2$.
26. $-apqx^4y^3$.
27. $3a^3bcd^2x^2$.
28. $9acm^2x^3y^2$.
29. $-2/5acm^3n^2x^2$.
30. $3a^3bcxy^2$.

EXERCISE XX.

1. $x = 20$.
2. $x = 40$.
3. \$19000, \$9000, \$12000, and \$7000.
4. 9, 10, 17, 19, and 26.

EXERCISE XXIII.

1. $-36x^2y^2z-48xy^2z^2$.
2. $a^3b^2c^3-a^2b^2c^4$.
3. $3x^2+3xy+3xz$.
4. $a^3bc-ab^3c+abc^3$.
5. $a^2b^2c-ab^2c^2+a^2bc^2$.
6. $14a^4b^3+28a^3b^4$.
7. $15x^3y^2-18x^2y^3+24x^3y^3$.
8. $56x^6y^4+40x^4y^6$.
9. $-5x^2y^3z^2+3x^2y^2z^3-8x^3y^2z^2$.
10. $-48x^5y^3z^5+96x^4y^2z^4$.
11. $91x^4y^5+105x^5y^4$
12. $-8x^2y^2z^2+10x^4y^2z^4$.
13. $-a^2b^2c^2+a^3b^2c^2+a^2b^3c^2$.
14. $a^3b^2c-a^2b^3c+a^2b^2c^2$.
15. $-3a^2+9/2ab-6ac$.
16. $-5/2x^2+5/3xy+10/3x$.
17. $1/4a^2x-1/16abx-3/8acx$.

18. $-2a^5x^3+7/2a^4x^4$. **19.** $5/2a^4x^2-5/3a^3x^3+a^2x^4$.
20. $21/2x^3y-x^2y^2$. **21.** $1/2x^5y^2-3x^3y^4$.
22. $-x^8y^3+16/49x^5y^6$.

EXERCISE XXIV.

1. x^2-1. **2.** $x^3y+x^2y^2+xy^3$.
3. $-3x^5+9x^4-3x^3+12x^2$. **4.** x^4+x^3-x-1.
5. $x^4+x^3+2x^2-x+3$. **6.** x^4-13x^2+36.
7. x^3-y^3. **8.** x^3+y^3 **9.** $x^4+x^2y^2+y^4$.
10. $x^5+5x^4+10x^3+10x^2+5x+1$. **11.** $x=11$.
12. $20000, $52000, and $48000. **13.** 5.
14. $x^5-5ax^4+10a^2x^3-13a^3x^2+13a^4x-6a^5$.
15. $x^5-4a^2x^3+3a^4x-a^5$.
16. $x^6+2x^5y-x^4y^2-4x^3y^3-x^2y^4+2xy^5+y^6$. **17.** x^4-a^4.
18. $x^3-(a-b+c)x^2-(bc-ca+ab)x+abc$.
19. $1-x+x^2-x^3+2x^4-x^5+x^6+x^8$.
20. a^6-b^6. **21.** $27x^3-64y^3$.
22. $125a^6x^3+27b^6y^6$. **23.** $64a^3z^6-125b^9x^3$.
24. $300, $550, and $350. **25.** 8.

EXERCISE XXV.

1. $9x^4+3x^3-2x^2+6x-4$.
2. $x^8+x^7-2x^6-2x^5-5x^4-x^3+5x^2+9$.
3. $2x^6-10x^5+5x^4-22x^3-5x^2+5x+1$.
4. $2x^6-7x^5+5x^4+3x^3-3x^2+4x-4$.
5. $1-6x^5+5x^6$. **6.** $1-7x^6+6x^7$.
7. $1+x-8x^2+19x^3-15x^4$.
8. $4-9x^2+12x^3-4x^4$.
9. $x^8+x^6-2x^5-x^3+x^2+x+1$.
10. $2x^9-5x^7+2x^5+6x^4-3x^2$.

EXERCISE XXVI.

1. $5xy$. **2.** $3a^2$. **3.** $9a^2$. **4.** $7x^2y$.
5. $-17x$. **6.** $-11x^3y$. **7.** $5z^2$. **8.** $9ac^2$.
9. $2x^2y^3$. **10.** $-3a^2b$. **11.** $4/5a^2y$.
12. $-9x^2y^2z^3$. **13.** $15(x+y)^5z^3$.

14. $-39(a-b)^5x^4$. **15.** $-30cd(a+b)^7x^4y^3$.
16. $56a^3b^4(c-d)^4xy^2$. **17.** $5(a+b)^2x$.
18. $-9ac(b-d)^2y$. **19.** $14d(b+c)x$.
20. $2a^2c^2$. **21.** $-21x^2y$.
22. $x = 60$. **23.** 48 and 132.

EXERCISE XXVII.

1. x^2+xy+y^2. **2.** a^2-ab+b.
3. $a^3-3a^2b+b^3$. **4.** $8x^3+36x^2y+27y^3$.
5. $5/6a^4-1/5a^3b-1/3a^2b^2$. **6.** $-2a^2b-4ab^2$.
7. $5x^2y-6xy^2+8x^2y^2$. **8.** $2a-3b+4c$.
9. $3x-2y-4$. **10.** $2/3a-1/6b-c$.

EXERCISE XXVIII.

1. $x-3$. **2.** $x+3$. **3.** $x-7$.
4. $x-2$. **5.** $2x-3$. **6.** $3x+8$.
7. $4x-3$. **8.** $5x+4$. **9.** $7x+5$.
10. x^2+xy+y^2. **11.** $x+y$.
12. $9a^2x^2+12abx+16b^2$. **13.** $2a^3x^2-3c^2b^3$.
14. $7x^2+5xy+2y^2$. **15.** x^3-2x^2+x+1.
16. x^3-3x^2+2x-1. **17.** x^2-xy+y^2.
18. x^3+x-y. **19.** x^2-2x+3.
20. x^2+5x+6. **21.** $7a^2-5ab+2b^2$.
22. 8. **23.** -8. **24.** 5. **25.** -5.
26. -18. **27.** 5. **28.** -10. **29.** 10.
30. -5. **31.** $7x-45$. **32.** 0. **33.** $-39x+27$.
34. $x+1/3$. **35.** $x^2-1/2x+3/4$.
36. $1-x+x^2-x^3+x^4-$ etc.
37. $1+2x+2x^2+2x^3+$ etc.
38. $2(x-y)^3-4(x-y)^2-(x-y)$.

EXERCISE XXIX.

1. $x=5\frac{1}{5}$. **2.** $x=-2$. **3.** $x=3$.
4. $x=20$. **5.** $x=11$. **6.** 31 doz.
7. 8 sheep.

EXERCISE XXXII.

1. a^2b^6.
2. x^9y^6.
3. $81x^8y^4z^4$.
4. $144c^{10}d^2x^4$.
5. $-125x^9y^6$.
6. $-32x^{15}y^{20}$.
7. $9a^6b^2$.
8. a^2c^6.
9. $25a^4b^4$.
10. $81a^4x^6$.
11. $49a^{10}b^8x^6$.
12. $4/9a^4x^{10}$.
13. $27a^9b^{12}$.
14. $-27a^9x^3$.
15. $-a^6b^9x^3$.
16. $-27/64x^{21}$.

EXERCISE XXXIII.

1. $a^2 + 6ab + 9b^2$.
2. $a^2 - 6ab + 9b^2$.
3. $x^2 - 10xy + 25y^2$.
4. $4x^2 + 12xy + 9y^2$.
5. $9x^2 - 6xy + y^2$.
6. $9x^2 + 30xy + 25y^2$.
7. $81x^2 - 36xy + 4y^2$.
8. $25a^2b^2 - 10abc + c^2$.
9. $p^2q^2 - 2pqr + r^2$.
10. $x^2 - 2abcx + a^2b^2c^2$.
11. $a^2x^2 + 4abxy + 4b^2y^2$.
12. $x^4 - 2x^2 + 1$.
13. $16 - 8x + x^2$.
14. $x^2 + 4/3ax + 4/9a^2$.
15. $x^2 - 2/3bx + 1/9b^2$.
16. $x^2 - 3ax + 9/4a^2$.
17. $x^2 + 2ax + a^2$.
18. $16 + 8x + x^2$.

EXERCISE XXXIV.

1. $1 + 4x^2 + 9x^4 + 4x + 6x^2 + 12x^3$.
2. $1 + 4x + 10x^2 + 20x^3 + 25x^4 + 24x^5 + 16x^6$.
3. $1 + 4x + 10x^2 + 20x^3 + 25x^4 + 34x^5 + 36x^6 + 30x^7 + 40x^8 + 25x^{10}$.
4. $a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2bc - 2ad + 2bd - 2cd$.
5. $9a^2 + 4b^2 + c^2 + d^2 + 12ab - 6ac - 4bc + 6ad. + 4bd - 2cd$.

EXERCISE XXXV.

1. $x^3 + 3ax^2 + 3a^2x + a^3$.
2. $x^3 - 3ax^2 + 3a^2x - a^3$.
3. $x^3 - 6x^2y + 12xy^2 - 8y^3$.
4. $8x^3 + 12x^2y + 6xy^2 + y^3$.
5. $27x^3 - 135x^2y + 225xy^2 - 125y^3$.
6. $a^3b^3 + 3a^2b^2c + 3abc^2 + c^3$.
7. $8a^3b^3 - 36a^2b^2c + 54abc^2 - 27c^3$.
8. $125a^3 - 75a^2bc + 15ab^2c^2 - b^3c^3$.

9. $x^6 + 12x^4y^2 + 48x^2y^4 + 64y^6$.
10. $64x^6 - 240x^4y^2 + 300x^2y^4 - 125y^6$.

EXERCISE XXXVI.

1. $x^3 - 2x^2 + x + 1$. **2.** $1/2x^2 - 1/3xy + 1/4y^2$.
3. 4 and 9. **4.** 5 and 8. **5.** 3 and 5. **6.** 42.

EXERCISE XXXVII.

1. $\pm a^4bc^6$. **2.** $\pm 8x^3y^9$. **3.** $3a^2bc^3$.
4. $-7a^4b^6$. **5.** x^2y^3z. **6.** $-x^2y^3$.

EXERCISE XXXVIII.

1. $a^2 + 2a - 1$. **2.** $x^2 - xy + y^2$.
3. $2a^3 - 3a^2x - ax^2$. **4.** $3x^3 - 4xy^2 - 2y^3$.
5. $2a^4 + 4a^2c^2 - 4c^4$. **6.** $2x^2 - 5x + 3$.
7. $4x^2 - 2bx + 2b^2$. **8.** $x^3 - 2x^2 + 3x - 4$.
9. $x^3 - 2x^2y + 2xy^2 - y^3$. **10.** $2 - 3a - a^2 + 2a^3$.
11. $5x^3 - 3x^2y - 4xy^2 + y^3$. **12.** $x^2 - 1/2xy - y^2$.
13. $x^2 - 2xy + y^2 - \frac{y^3}{x}$.

EXERCISE XXXIX.

1. 106929. **2.** 14356521. **3.** 714025.
4. 25060036. **5.** 387420489. **6.** 25836889.
7. .00092416. **8.** .00000784. **9.** 4816.36.
10. 1867.1041. **11.** 1435.6521. **12.** 64.128064.

EXERCISE XL.

1. 3789. **2.** 5006. **3.** 5083. **4.** 129.63.
5. 2.1319. **6.** .9486+. **7.** 2.4919+. **8.** .0923+.

EXERCISE XLI.

1. $a^3 + 3a^2 + 3a + 1$. **2.** $x^3 + 6x^2 + 12x + 8$.
3. $a^3x^3 - 3a^2x^2y^2 + 3axy^4 - y^6$.
4. $8m^3 - 12m^2 + 6m - 1$.
5. $64a^3 - 144a^2b + 108ab^2 - 27b^3$.

6. $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$.

7. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

8. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$.

9. $8a^6 - 36a^5 + 66a^4 - 63a^3 + 33a^2 - 9a + 1$.

10. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9$.

EXERCISE XLII.

1. $1 - x$. **2.** $1 + 2x$. **3.** $2x - 3y$.
4. $3xy - z^2$. **5.** $a + 8b$. **6.** $4x - 3x^2$.
7. $1 + x + x^2$. **8.** $1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} -$ etc.

EXERCISE XLIII.

1. 2460375. **2.** 11089567. **3.** 1191016.
4. 17173512. **5.** 109215352. **6.** 102503.232.
7. 820.025856. **8.** 20910.518875. **9.** 056623104.

EXERCISE XLIV.

1. 478. **2.** 3.84. **3.** 4.68. **4.** 9.36.
5. 27.55. **6.** 1.357+. **7.** .5848+. **8.** .2154+.
9. 1.587+. **10.** .7368+. **11.** 3.045+. **12.** 2.502+.
13. $9a^4x^2 + 6a^2b^3xy + 4b^6y^2$. **14.** $x = 2$.
15. 15 ft. by 12 ft. **16.** $2a^3 + 4c^2$.
17. $x = 1$. **18.** 48 ft. by 40 ft.
19. 90 of port and 150 of claret. **20.** 44.

EXERCISE XLV.

1. $14x^2 - 43x + 20$. **2.** $20x^2 + 62x + 48$.
3. $28 - 47x + 15x^2$. **4.** $30 - 20x - 80x^2$.
5. $x^2 + 16x + 63$. **6.** $x^2 - 8x + 15$.
7. $x^2 + 3x - 54$. **8.** $x^2 - 4x - 77$.
9. $x^2 - x - 30$. **10.** $x^2 + 3x - 28$.
11. $x^2 + 6x + 9$. **12.** $x^2 - 8x + 16$.
13. $x^2 - 64$. **14.** $x^2 - 36$.

15. $35x^2 + 39x - 108$. **16.** $72x^2 + 12x - 24$.
17. $24x^2 - 19x - 175$. **18.** $18x^2 - 6x - 180$.
19. $4ax^2-(5a+4b)x+5b$. **20.** $18ax^2+(24a+6c)x+8c$.
21. $5a^2x^2 + (ab - 5ac)x - bc$.
22. $(2a^2 + 2ab)x^2 - (ab + b^2 - 2ac)x - bc$.
23. $24 - 36x - 108x^2$. **24.** $63 - 44x - 32x^2$.

EXERCISE XLVI.

1. $20x^4 - 47x^2 + 21$. **2.** $21x^8 + 47x^4 + 20$.
3. $30x^6 - 16x^3 - 32$. **4.** $42x^{10} + 4x^5 - 6$.
5. $21z^4 - 20z^2 - 96$. **6.** $54y^{12} + 3y^6 - 77$.
7. $x + 12\sqrt{x} + 35$. **8.** $6x - 2\sqrt{x} - 48$.
9. $x - 49$. **10.** $9x + 24\sqrt{x} + 16$.
11. $x^2 - 5$. **12.** $m - 5$.
13. $5m^8 - 18m^4 + 16$. **14.** $3n^6 + 21n^3 - 180$.
15. $s^{10} + s^5 - 56$. **16.** $a^{14} - 2a^7 - 99$.
17. $x^8 - 49$. **18.** $m^6 - 36$.
19. $4x^4 - 16$. **20.** $25a^4x^6 - 9$.
21. $9x - 175$. **22.** $36x - 147$.
23. $x^2 + 4$. **24.** $4x^4 + 45$.

EXERCISE XLVII.

1. $(a+x)^2-3(a+x)-28 = x^2+(2a-3)x+a^2-3a-28$.
2. $(m+x)^2+m+x-72 = x^2-(2m-1)x+m^2+m-72$.
3. $(x-b)^2+4(x-b)-45 = x^2-(2b-4)x+b^2-4b-45$.
4. $(x-m)^2-5(x-m)-84=x^2-(2m+5)x+m^2+5m-84$.
5. $x^2 - m + 5$. **6.** $x^2 - 3 + a$.
7. $(x-4)^2 - (x-a)^2 = (2a-8)x - a^2 + 16$.
8. $x^3 + x^2 + x + 1$. **9.** $1 - 3x^2 + 2x^4$.
10. $x = 5\frac{11}{109}$. **11.** $5\frac{1}{7}$ days. **12.** 30 min.
13. $(x-5)^2 - (x+6)^2 = -22x - 11$.
14. $(x+7)^2 - (x-5)^2 = 24x + 24$.
15. $x - 23$. **16.** $x - 23$. **17.** $x^2 - x + 5$.
18. 11. **19.** 3. **20.** $220 - 16x$.
21. $7x + 148$. **22.** $x = -1\frac{2}{3}$. **23.** $4\frac{4}{5}$ hours.

EXERCISE XLVIII.

1. $x^3 + a^3$. **2.** $x^3 + 27$. **3.** $x^3 - 343$.
4. $x^3 - c^3$. **5.** $8x^6 - 27a^3$.
6. $x^2 + 4x + 16$. **7.** $4a^2x^4 - 14ax^2 + 49$.
8. $x^6 + 2x^5 + 3x^4 + 4x^3 + 3x^2 + 2x + 1$.
9. $1 - 9x^2 + 33x^4 - 63x^6 + 66x^8 - 36x^{10} + 8x^{12}$.
10. $a^6 + 1/27b^3$. **11.** $1/8a^6x^9 - 8/27b^9x^6$.
12. $1/125a^9x^{12} + 1/216b^{12}x^{15}$. **13.** $9a^4x^6 + a^3x^4 + 1/9a^2x^2$.
14. $1/16a^6x^{10} - 1/24a^3b^2x^{12} + 1/36b^4x^{14}$.

EXERCISE XLIX.

1. $x^2 + 8x + 16$. **2.** $m^2 - 10m + 25$.
3. $x^2 - 3x + 9/4$. **4.** $n^2 - 5n + 25/4$.
5. $x^2 + 7x + 49/4$. **6.** $y^2 - 9y + 81/4$.
7. $x^2 - 3/4x + 9/64$. **8.** $z^2 + 5/6z + 25/144$.
9. $x^2 + bx + b^2/4$. **10.** $x^2 - 5bx + 25b^2/4$.
11. $x^2 + x + 1/4$. **12.** $y^2 - y + 1/4$.

EXERCISE L.

1. $x^6 + 6x^3 + 9$. **2.** $m^4 - 12m^2 + 36$.
3. $x^4 - 5x^2 + 25/4$. **4.** $a^8 + 7a^4 + 49/4$.
5. $x^6 + bx^3 + b^2/4$. **6.** $z^4 - z^2 + 1/4$.
7. $x^{10} - 2/3x^5 + 1/9$. **8.** $n^6 - 3/4n^3 + 9/64$.
9. $(x+2)^2 + 6(x+2) + 9$. **10.** $(x-5)^2 - 3(x-5) + 9/4$.

EXERCISE LI.

1. $x^2 - 8x + 16 - 18$. **2.** $x^2 - 12x + 36 - 6$.
3. $x^2 + 7x + 49/4 - 52/4$. **4.** $x^2 - 7x + 49/4 - 233/20$.
5. $1/16x^4 + 1/2x^3 + 4x^2 + 32x + 256$.
6. 27 days. **7.** $3\frac{1}{9}$ days.
8. $y^2 - 9y + 81/4 - 69/4$. **9.** $z^2 + 11z + 121/4 - 149/4$.
10. $x^2 + bx + b^2/4 - \dfrac{b^2 - 4c}{4}$. **11.** $y^2 - by + b^2/4 - \dfrac{b^2 + 4c}{4}$.
12. $16/81x^4 - 40/27x^3 + 100/9x^2 - 250/3x + 625$.
13. 72 miles. **14.** $5\frac{5}{17}$ hours.

EXERCISE LII.

1. $x^2 + 3x + 9/4$. **2.** $x^2 - 5x + 25/4$.
3. $x^2 - 3x + 9/4$. **4.** $x^2 + 9x + 81/4$.
5. $(x + a)^2 - 5/3(x + a) + 25/36$.
6. $x = 2\frac{5}{7}$. **7.** $x^2 + \frac{b}{a}x + b^2/4a^2$.
8. $y^2 - n/my + n^2/4m^2$. **9.** $x^4 + 3/2x^2 + 9/16$.
10. $z^6 - 3z^3 + 9/4$.
11. $(z - 5)^4 + 3/7(z - 5)^2 + 9/196$. **12.** $x = 4\frac{1}{11}$.

EXERCISE LIII.

1. $2(x^2 + 3/2x + 9/16 + 39/16)$.
2. $3(x^2 - 6x + 9 - 13)$.
3. $4(x^2 - 3/2x + 9/16 + 19/16)$.
4. $5(x^2+5x+25/4-41/4)$. **5.** $6(x^2+7x+49/4-47/12)$.
6. $1.4142+$. **7.** $1.442+$.
8. $7(x^2-9x+81/4-53/4)$. **9.** $8(x^2-5x+25/4-31/4)$.
10. $9(x^2-9x+81/4-53/4)$. **11.** $10(x^2+7x+49/4-81/4)$.
12. $11(x^2 - 2/11x + 1/121 + 32/121)$.
13. $a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2}\right)$.
14. $m\left(z^2 - n/m\, z + \frac{n^2}{4m^2} - \frac{n^2 - 4mp}{4m^2}\right)$.

EXERCISE LIV.

1. $2a(3b + c)$. **2.** $2a^2b(x^2 - 4x + ab^2)$.
3. $5b^3c^2(bx + c^2y - 1)$. **4.** $7a(1 - a^2 + 2a^3)$.
5. $2x^3(3 + x + 2x^2)$. **6.** $15a^2(1 - 15a^2)$.
7. $5x^3(x^2 -- 2a^2 - 3a^3)$. **8.** $19a^3x^2(2x^3 + 3a)$.
9. $(3x^2 - x - 1)x$. **10.** $xy^2(2xy - 3x + 2y)$.

EXERCISE LV.

1. $(x + a)(x - a)$. **2.** $(x + 3)(x - 3)$.
3. $4(a + 4)(a - 4)$. **4.** $(3ax + 5b)(3ax - 5b)$.
5. $(9 + 4ax^2)(9 - 4ax^2)$.
6. $(7a^2x + 4a^3z^4)(7a^2x - 4a^3z^4)$.

7. $(x + 13)(x - 1)$.
8. $(y + 5)(y - 13)$.
9. $(a + 2)(a - 6)$.
10. $(b + 23)(b + 1)$.
11. 6 ft.
12. 86.
13. $3(2 + a)(2 - a)$.
14. $3a(4a + 6b)(4a - 6b)$.
15. $3a(3a^2 + 5x^2)(3a^2 - 5x^2)$.
16. $5x(5ax^3 + 3xy^2)(5ax^3 - 3xy^2)$.
17. $(x + 10)(x + 4)$.
18. $(x + 1)(x - 17)$.
19. $(x + 1)(x - 11)$.
20. $(x + 29)(x + 1)$.
21. $10\frac{10}{23}$ hours; $21\frac{9}{11}$ hours.
22. 67.

EXERCISE LVI.

1. $(x + 5)(x + 7)$.
2. $(x - 3)(x - 9)$.
3. $(x + 4)(x - 8)$.
4. $(x - 3)(x + 10)$.
5. $(x - 7)(x + 6)$.
6. $(x + 5)(x - 4)$.
7. $2(x - 8)(x + 3)$.
8. $(x + 5)(3x + 11)$.
9. $(2x - 1)(3x - 7)$.
10. $(4x + 1)(5x + 8)$.
11. $(5x - 3)(7x + 12)$.
12. $4(7x + 5)(2x - 5)$.
13. A can do it in $17\frac{1}{2}$ days, B in $14\frac{14}{19}$ days, and C in $13\frac{1}{3}$ days.
14. $2(2 - x)(3 + 4x)$.
15. $4(6 - 7x)(2 - 3x)$.
16. $(5 + 3x)(7 + 4x)$.
17. $(2x + 7)(3x - a)$.
18. $(ax - 5)(bx + 7)$.
19. $(ax + b)(cx - d)$.
20. $\big(x - (a - b)\big)\big(x + (a + b)\big)$.
21. $\big((a + b)x + 2\big)\big((a - b)x - 4\big)$.
22. $3(x + 6)(x - 3)$.
23. $7(x - 6)(x + 5)$.
24. $10(x - 2)(x + 7)$.
25. $5a^2(3x - 2)(5x + 3)$.
26. 93.

EXERCISE LVII.

1. 44.
2. -10.
3. $x^2 - x$.
4. $x^2 - x + 1$.
5. 28.
6. $c^3 + b^3 - b^3 - c^3 = 0$.
7. $y^5 - y^5 = 0$.
8. $y^6 - y^6 = 0$.
9. $y^n - y^n = 0$.
10. $-y^5 + y^5 = 0$.
11. $y^5 + y^5 = 2y^5$.
12. $y^4 + y^4 = 2y^4$.

13. $y^4 + y^4 = 2y^4$.
14. $-y^n + y^n = 0$.
15. $y^n + y^n = 2y^n$.
16. $y^n + y^n = 2y^n$.

EXERCISE LVIII.

1. It is.
2. It is.
3. It is.
4. It is not.
5. It is.
6. It is.
7. It is not.
8. It is not.
9. It is.
10. It is.
11. It is not.
12. It is not.
13. It is not.
14. It is not.
15. It is.
16. It is not.
17. It is.
18. It is.
19. It is not.
20. It is.
21. $x^6 + bx^5 + b^2x^4 + b^3x^3 + b^4x^2 + b^5x + b^6$.
22. $x^4 - bx^3 + b^2x^2 - b^3x + b^4$ with $-2b^5$ as remainder.
23. $x^5 + bx^4 + b^2x^3 + b^3x^2 + b^4x + b^5$ with $2b^6$ as remainder.
24. $x^7 - bx^6 + b^2x^5 - b^3x^4 + b^4x^3 - b^5x^2 + b^6x - b^7$ with $2b^8$ as remainder.
25. $x^4 + bx^3 + b^2x^2 + b^3x + b^4$ with $2b^5$ as remainder.
26. $x^6 - bx^5 + b^2x^4 - b^3x^3 + b^4x^2 - b^5x + b^6$.
27. $x^3 + bx^2 + b^2x + b^3$.
28. $x^5 - bx^4 + b^2x^3 - b^3x^2 + b^4x - b^5$.

EXERCISE LIX.

1. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.
2 $x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7$.
3. $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$.
4. $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$.
5. $x^2 + 3x + 9$.
6. $x^3 + 3x^2 + 9x + 27$.
7. $x^3 - 2x^2 + 4x - 8$.
8. $x^4 - 2x^3 + 4x^2 - 8x + 16$.
9. 0.
10. $b^3 - a^3$.
11. 0.
12. 0.
13. 0.
14. $-12b^2c^2$.
15. $2b^4 - 8b^3c - 8bc^3 + 2c^4$.
16. $(x + 2)(3x + 4)$.
17. $(x - 1)(x + 5)$.
18. $(x + 4)(x - 3)$.
19. $(x - 2)(3x - 2)$.
20. $(x - 1)(4x - 3)$.

EXERCISE LX.

1. $13\frac{1}{3}$ hours. **2.** 120 hours.
3. 50 days, $21\frac{3}{7}$ days. **4.** 36. **5.** 27 days.
6. A's $1650, B's $1320.

EXERCISE LXI.

1. $5x^3y$. **2.** $7x^2y^2z$. **3.** $18abcd$.
4. x^2y^2. **5.** $17a^4b^2c^3$. **6.** $14x^{m+1}y^{p+q-2}$.
7. $6x^{m+n}+18x^{2n}-15x^{n+p}y^q-8x^{m+2}-24x^{n+2}-20x^{p+2}y^q$
$+12x^{m+1}y^2+36x^{n+1}y^2-30x^{p+1}y^{q+2}$.
8. $3x^m+6x^n-4x^3$.

EXERCISE LXII.

1. $x+1$. **2.** $x+3$. **3.** $x-10$.
4. $x-2$. **5.** $x+a$. **6.** $x-y$.
7. $x-1$. **8.** $x+y$. **9.** $6(x+1)^2$.
10. x^2-y^2. **11.** $(x+a)(x-2a)$. **12.** $x-y$.

EXERCISE LXIII.

1. $x+1$. **2.** $x-3$. **3.** $x-2$.
4. $x-2$. **5.** $2x(x-3)$. **6.** $3a^2x(x-a)$.
7. $(x-a)^2$. **8.** $5x^2-1$. **9.** $(1-x)^2$.
10. x^2+4x+5. **11.** x^2+2x+3. **12.** x^2+2x+3.
13. x^2-3x+2. **14.** x^2+2x-3.
15. $18x^{m+3}-24x^{m+1}+30x^{n+4}+21x^{2m+2}-28x^{2m}+35x^{m+n+3}$.
16. $4x^{a+4}-12x^{3a+4}+20x^7-6x^{2a-2}+18x^{4a-2}-30x^{a+1}$.

EXERCISE LXIV.

1. 36 min. **2.** 8 o'clock.
3. $5\frac{5}{11}$ min. past 10. **4.** $5\frac{5}{11}$ minutes before 2.
5. $5\frac{5}{11}$ minutes after 4 and $38\,\frac{2}{1}$ minutes after 4.
6. 26. **7.** 69. **8.** $16\frac{2}{3}$ hours.
9. 42 hours. **10.** 16_3 years. **11.** 35 years.
12. 40 and 10.

EXERCISE LXV.

1. $126a^4b^3cd^5$. **2.** $60x^3y^3z^3$.
3. $x^2y - y^3$. **4.** $x^3 - 5x^2 - 9x + 45$.
5. $5x^3+35x^2-245x-1715$. **6.** $x^3 + 7x^2 - 28x - 160$.
7. $(x^2 - 1)(4x^2 - 1)$. **8.** $12(x - 2)(x^2 - 9)$.
9. $(x - 1)(x - 2)(x - 3)$. **10.** $3(x - 3a)^2(x^2 - 4a^2)$.

EXERCISE LXVI.

1. $12x^4 + 2ax^3 - 4a^2x^2 - 27a^3x - 18a^4$.
2. $(4a - b)(a - b)(3a^2 + b^2)$.
3. $(x^2 - 2x + 3)(6x^3 + x^2 - 44x + 21)$.
4. $(x^2 + 5x + 7)(7x^4 - 40x^3 + 75x^2 - 40x + 7)$.
5. $x(x + 1)(x + 2)(x - 2)(x + 3)$.
6. $x(x - 1)(x + 2)(x + 6)(x^2 - 2x + 4)$.
7. $2a(2a - b)(2a - 3b)(2a + 3b)$.
8. $6x(x + 1)(x - 3)(x - 4)$.
9. $(3x + 2)(8x^3 + 27)(8x^3 - 27)$.
10. $3(x - 3y)^2(x^2 - 4y^2)$. **11.** $x^{2m} - x^{2n}$.
12. $9a^{2n}x^{2m} - 16a^{2m}x^{2n}$. **13.** x^{-2m+1}. **14.** $2a^{m-n}x^{-2m+2p}$.

EXERCISE LXVII.

1. $5\frac{5}{11}$ minutes past 11 o'clock.
2. In 12 hours. 120 miles.
3. 12 miles an hour. 240 miles. **4.** $14\frac{2}{7}$ days.
5. $33\frac{1}{3}$ days. **6.** 24 days.
7. 120 days. **8.** $x^{2m} - x^my^n + y^{2n}$.
9. $x^{5m} + x^{4m}y^n + x^{3m}y^{2n} + x^{2m}y^{3n} + x^my^{4n} + y^{5n}$.

EXERCISE LXVIII.

. $\frac{a}{9} - \frac{4b}{9} + \frac{5c}{9}$. **2.** $\frac{2x}{a+b} - \frac{5y}{a+b} - \frac{8ac}{a+b} + \frac{9}{a+b}$.
3. $\frac{3a - 5b + 4}{2a + b}$. **4.** $\frac{x + a - c - 1}{4c}$.
5. $-\frac{7a+9b+c-d-18}{x^2 - 3}$. **6.** $-\frac{9x + 6a + 11b + 5c}{a^2 - b^2}$.

EXERCISE LXIX.

1. $\frac{2a}{x}$. **2.** $\frac{3ab}{4c}$. **3.** $\frac{1}{a}$. **4.** $\frac{x-a}{x}$.

5. $\frac{3x}{7}$. **6.** $\frac{a^3x^3}{b^2}$. **7.** $\frac{x+5}{x-7}$. **8.** $\frac{x+5}{x+3}$.

9. $\frac{2x-4}{x-3}$. **10.** $\frac{3x+1}{x+2}$. **11.** $\frac{x^2+4x+16}{x+7}$.

12. $\frac{x^2-3x+9}{x-3}$. **13.** $\frac{3x-y}{4x-y}$. **14.** $\frac{2x-3}{2x+3}$.

EXERCISE LXX.

1. $\frac{12a^3c^3}{9ac^2}$. **2.** $\frac{4a^2x^2-24a^2x^4}{20a^2x^3}$. **3.** $\frac{x^2-9x+18}{x^2+x-42}$.

4. $\frac{8x^2+22x+12}{12x^2+x-6}$. **5.** $\frac{20x^2-53x+35}{8x^2-34x+30}$.

6. $\frac{15a^4x^4}{5a^2x^3}$. **7.** $\frac{6b^2x^2-14ab^4x^5}{3-7ab^2x^3}$.

8. $\frac{21x^2-11x-40}{7x+8}$. **9.** $-\frac{15x^2-9x+42}{6-3x}$.

10. $-\frac{15x^2+13x-72}{9-5x}$. **11.** $\frac{15a^2x}{9a^3bx}$ and $\frac{7bx}{9a^3bx}$.

12. $\frac{x^2-9x+20}{x^2-16}$ and $\frac{x^2+10x+24}{x^2-16}$

Their sum $= \frac{2x^2+x+44}{x^2-16}$.

13. $\frac{x^2-x-30}{x^2-x-30}+\frac{x^2+12x+35}{x^2-x-30}-\frac{x^2-14x+48}{x^2-x-30}$

$$=\frac{x^2+25x-43}{x^2-x-30}.$$

14. $\frac{6a}{4a^2}-\frac{5x+6}{4a^2}=\frac{6a-5x-6}{4a^2}$.

15. $\frac{45x^3+20x^2-3x+18}{15x^2+14x-8}$. **16.** $-\frac{b^2-4ac}{4a^2}$.

17. $-\frac{25b^2-84ac}{36a^2}$. **18.** $\frac{(a+b)^2}{4ab}$.

19. $x^{3m} - x^{2m+n} + x^{m+2n} - x^{3n}$.

20. $x^{4m} + x^{3m+n} + x^{2m+2n} + x^{m+3n} + x^{4n}$.

EXERCISE LXXI.

1. 1. **2.** $\frac{a+x}{a}$. **3.** $1/x$. **4.** $\frac{a^2(a-y)}{x}$.

5. $\frac{x^2-13x+42}{x+4}$ **6.** $-\frac{15x^2-26x+8}{x+7}$.

7. $\frac{a^2b^2}{a^2-b^2}$. **8.** $\frac{ax}{a^2-x^2}$.

9. $\frac{a^4+b^4+a^2b^2}{a^2b^2}$. **10.** $\frac{2x-1}{2x-5}$.

EXERCISE LXXII.

1. 7/12. **2.** $\frac{ab}{2a-1}$.

3. $\frac{a-11}{a-2}$. **4.** $\frac{2x-1}{2x-3}$.

5. $\frac{x+1}{x+5}$. **6.** $\frac{6x^2-23x+20}{3x-2}$.

7. $\frac{14-17x-6x^2}{4x+6}$. **8.** $\frac{2x^2-21x+27}{x-5}$.

EXERCISE LXXIII.

1. x^2-x-12, $x^2-15x+56$, $6x+48$.

2. $9x^2-9x-28$, $10x^2-43x+28$, $5x^2+51x-44$.

3. $-40x^2+94x-48$, $-35x^2-19x-42$, $-3(x^2-6x+9)$.

4. x^2-64, x^3-64, $7x^2+28x-224$.

5. x^2+x-42, x^3+216, $5x^2-30x+180$.

6. $x^{6m}+x^{5m+n}+x^{4m+2n}+x^{3m+3n}+x^{2m+4n}+x^{m+5n}+x^{6n}$.

EXERCISE LXXIV.

1. A is 48 and B is 12.

2. $49\frac{1}{11}$ minutes after 3 o'clock. **3.** 17 and 28.

4. 35 dimes, 5 cents, and 10 dollars. **5.** 98 and 215.

6. A's age $= \frac{cm(m-1)}{m-n}$; B's age $= \frac{c(n-1)}{m-n}$.

7. $\frac{100(b-a)}{ac}$. 8. $\frac{11}{3}$.

9. $27\frac{3}{11}$ minutes after 5 o'clock.

10. $\frac{n-r}{q+1}$ and $\frac{nq+r}{q+1}$.

EXERCISE LXXV.

1. $-3\frac{1}{7}$. 2. 11. 3. 7. 4. -8.
5. 1. 6. $-1\frac{3}{4}$. 7. -10. 8. -1.
9. 7. 10. 3. 11. 1. 12. -2.
13. 1/4. 14. $-3\frac{3}{10}$. 15. $1\frac{6}{7}$.

EXERCISE LXXVI.

1. $5\frac{1}{5}$. 2. 6. 3. $9\frac{7}{17}$. 4. 1. 5. $1\frac{1}{2}$.

EXERCISE LXXVII.

1. 1/2. 2. $4\frac{2}{3}$. 3. $4\frac{1}{3}$.
4. $-4/7$. 5. $1\frac{13}{17}$. 6. 3.
7. $3\frac{1}{2}$. 8. $-4\frac{1}{2}$. 9. $2\frac{1}{25}$.
10. 0. 11. $\frac{ab-cd}{a+b-c-d}$. 12. $2\frac{7}{12}a$.

EXERCISE LXXVIII.

1. 55 minutes. 2. $37\frac{1}{2}$ min. and 25 min.
3. \$30000. 4. \$84000.
5. A 39 miles and B 27 miles. 6. 283. 7. 536.
8. Of the first $\frac{a(c-bd)}{a-b}$, and of the second $\frac{b(ad-c)}{a-b}$.
9. \$750 and \$500. 10. 192 miles.
11. \$15.36 and \$4.56. 12. Hound 72 and fox 108.
13. 300. 14. Man 84 cents, boy 42 cents.
15. 85 gallons of spirits and 35 of water.
16. 1500. 17. 28. 18. 3 shillings.

EXERCISE LXXIX.

1. $\sqrt{m^2}$. 2. $\sqrt{n^2}$. 3. $\sqrt{9a^2}$.
4. $\sqrt{25a^2b^2}$. 5. $\sqrt{49a^6}$. 6. $\sqrt{36x^4y^6}$.

7. $\sqrt{1/16a^4x^2}$ or $\dfrac{\sqrt{a^4x^2}}{\sqrt{16}}$.

8. $\sqrt{1/9a^6y^4}$ or $\dfrac{\sqrt{a^6y^4}}{\sqrt{9}}$.

9. $\sqrt{\dfrac{25a^6x^8}{9b^2}}$ or $\dfrac{\sqrt{25a^6x^8}}{\sqrt{9b^2}}$.

10. $\sqrt{a^2 + 2ab + b^2}$.

11. $\sqrt{x^2 - 2xy + y^2}$.

12. $\sqrt{9a^4 + 42a^2 + 49}$.

13. $\sqrt[3]{x^3}$.

14. $\sqrt[3]{27a^6x^3}$.

15. $\sqrt[3]{1/27a^9y^3}$ or $\dfrac{\sqrt[3]{a^9y^3}}{\sqrt[3]{27}}$.

16. $\sqrt[3]{x^3+15x^2+75x+125}$.

17. $\sqrt[3]{a^3 - 9a^2 + 27a - 27}$.

18. $\sqrt[3]{\dfrac{27a^9x^6}{64c^3}}$ or $\dfrac{\sqrt[3]{27a^9x^6}}{\sqrt[3]{64c^3}}$.

EXERCISE LXXX.

1. $2\sqrt{3}$.
2. $5\sqrt{3}$.
3. $6\sqrt{5}$.
4. $7\sqrt{15}$.
5. $16\sqrt{2}$.
6. $9\sqrt{7}$.
7. $3\sqrt[3]{5}$.
8. $4\sqrt[3]{7}$.
9. $8\sqrt[3]{11}$.
10. $4a\sqrt{3b}$.
11. $5ax^2\sqrt{5a}$.
12. $7a^2x^3\sqrt{3ax}$.
13. $2a(a + b)\sqrt{a}$.
14. $2x^2y(x - y)\sqrt{3xy}$.

EXERCISE LXXXI.

1. $\sqrt{99}$.
2. $\sqrt{208}$.
3. $\sqrt{252}$.
4. $\sqrt[3]{72}$.
5. $\sqrt[3]{320}$.
6. $\sqrt[3]{864}$.
7. $\sqrt{9a^3 - 9a^2b}$.
8. $\sqrt{3x^3 + 6x^2y + 3xy^2}$.
9. $\sqrt{45a^5b - 90a^4b^2 + 45a^3b^3}$.

EXERCISE LXXXII.

1. $1/2\sqrt{2}$.
2. $1/5\sqrt{5}$.
3. $1/3\sqrt{6}$.
4. $1/6\sqrt{15}$.
5. $\dfrac{1}{7x}\sqrt{21ax}$.
6. $\dfrac{1}{a-b}\sqrt{a^2-b^2}$.
7. $\dfrac{1}{x+6}\sqrt{x^2 + 10x + 24}$.
8. $\dfrac{1}{x+7}\sqrt{x^2 + 2x - 35}$.

9. $\frac{1}{2x+1}\sqrt{10x^2+x-2}$. **10.** $\frac{1}{3x-7}\sqrt{12x^2-46x+42}$.
11. $\frac{1}{3x+4}\sqrt{20+7x-6x^2}$. **12.** $\frac{1}{4x-3}\sqrt{12x^2+7x-12}$.

EXERCISE LXXXIII.

1. $18\sqrt{2}$. **2.** $37\sqrt{2}$. **3.** $\frac{43}{105}\sqrt{15}$.
4. $2/5\sqrt[3]{6}$. **5.** $25a^2x\sqrt{3x}$. **6.** $\frac{31}{2}\sqrt{3}$.
7. $18ab\sqrt[3]{2a^2b^2}$. **8.** $\frac{m(n+\sqrt{ns})}{n-s}$.

EXERCISE LXXXIV.

1. $4\sqrt{5}$. **2.** $-3a^2b\sqrt[3]{b}$. **3.** $2b\sqrt{b}$.
4. $2b\sqrt{2a}$. **5.** $\frac{31}{90}\sqrt[3]{6}$. **6.** $-19a\sqrt{ab}$.
7. $(13c-35cd)\sqrt{2c}$. **8.** $\left(c-x-\frac{1}{c-x}\right)\sqrt{c^2-x^2}$.

EXERCISE LXXXV.

1. $96\sqrt{3}$. **2.** $\frac{21}{24}\sqrt[3]{4}$. **3.** $24\sqrt{6}$.
4. $1/2\sqrt{6}$. **5.** $\frac{25ax}{2}\sqrt{b}$. **6.** $4a^2b^2$.
7. $6\sqrt{10}+7\sqrt{15}+8\sqrt{6}+24$.
8. $6+\sqrt{10}$. **9.** $6\sqrt{21}-46$. **10.** $2\sqrt{6}$.
11. $6a-6x+5\sqrt{ax}$. **12.** $3\sqrt{7}-47$.
13. $6\sqrt{5}+14$. **14.** $53-14\sqrt{5}$.
15. $32-10\sqrt{7}$. **16.** $x+7\sqrt{x}+12$.
17. $x+18\sqrt{x}+81$. **18.** $x+\sqrt{x}-30$.
19. $x-2\sqrt{3x}+3$. **20.** -2.

21. $15 + 4\sqrt{14}$. **22.** $\sqrt{x^2 - 3x - 40}$.
23. $2x + 2 + 2\sqrt{x^2 + 2x - 24}$. **24.** $\sqrt{x^2 - 49}$.
25. $2x + 2\sqrt{x^2 - 9}$. **26.** $15x^2\sqrt{a^2 - 13a + 42}$.
27. $24a^2x + 8a\sqrt{6bx} + 4b$. **28.** $35abx + 245ab$.
29. $25x - 58 - 24\sqrt{x^2 - x - 42}$. **30.** $63a^3\sqrt{x^2 - 16}$.
31. $34a^3 - 98a^2 + 30a^2\sqrt{a^2 - 2a - 15}$. **32.** $x - 29$.
33. $x + 2$. **34.** -13. **35.** $-7x - 26$.
36. $9a^4x^3 - 72a^4x^2 - 25x^5 - 175x^4$.

EXERCISE LXXXVI.

1. 113. **2.** -166. **3.** 172.
4. -6. **5.** $a - 4b$. **6.** $9c^2 - 4x$.
7. x. **8.** $2p - q$. **9.** $2x$.
10. $25x^2 + 75y^2 - 49a^2$. **11.** $-2ax$. **12.** $2x^2 + 6x$.

EXERCISE LXXXVII.

1. 44. **2.** 59.
3. $a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$. **4.** 64.

EXERCISE LXXXVIII.

1. $\frac{38}{5} + \frac{18}{5}\sqrt{6}$. **2.** $\frac{104 - 13\sqrt{42}}{26}$. **3.** $\frac{6\sqrt{7} - \sqrt{2}}{25}$.
4. $-6\sqrt{3} + 8\sqrt{2}$. **5.** $a - \sqrt{a^2 - b^2}$.
6. $\sqrt{a^2 + y^2} + y$. **7.** $\frac{3 - \sqrt{9 - a^4}}{a^2}$.
8. $\frac{7x + 3 + 8\sqrt{x^2 - 9}}{3x + 15}$. **9.** $\frac{x^2 + 9 - 4\sqrt{5 + x^2}}{1 + x^2}$. **10.** $\sqrt{3}$.
11. $\sqrt{3}$. **12.** $\sqrt[3]{3}$. **13.** $4\sqrt{5}$.
14. $a\sqrt{x}$. **15.** $\sqrt{x - 7}$. **16.** $\sqrt{x^2 + 2x + 4}$.
17. $\sqrt[3]{x + 3}$. **18.** $\sqrt[5]{x - 3}$. **19.** $\sqrt{x - 7}$.
20. $\sqrt{x - 9}$. **21.** $\sqrt{3x - 2}$. **22.** $\sqrt{5x - 7}$.
23. $\frac{11 - 3\sqrt{7}}{2}$. **24.** $\frac{3\sqrt{7} - 2\sqrt{3}}{3}$. **25.** $\frac{19 - 6\sqrt{2}}{17}$.

26. $2 + \sqrt{6}$. **27.** $\frac{\sqrt{xy}}{y}$. **28.** $\frac{\sqrt{5}}{5}$.
29. $\frac{1}{a - x}\sqrt{ax}$. **30.** $4 + \sqrt{15}$.

EXERCISE LXXXIX.

1. $\sqrt[6]{100}$, $\sqrt[6]{125}$, and $\sqrt[6]{11/2}$.
2. $\sqrt[12]{(a + b)^4}$, $\sqrt[12]{(a - b)^6}$, and $\sqrt[12]{(a^2 + x^2)^3}$.
3. $\sqrt[6]{10125}$. **4.** $1/2\sqrt[6]{\frac{27}{4}}$. **5.** $\sqrt[6]{a^5}$.
6. $2\sqrt[6]{\frac{a^3}{2b^2c}}$. **7.** $3/2\sqrt[6]{8/3}$.

EXERCISE XC.

1. 14. **2.** 8. **3.** 20. **4.** $2\frac{2}{5}$.
5. 13. **6.** 6/5. **7.** 144. **8.** 2.
9. $4\frac{21}{25}$. **10.** $1\frac{9}{16}$. **11.** 5. **12.** 12.
13. $(a - b)^2$. **14.** $\frac{(a - b)^2}{2a - b}$. **15.** 1/6.
16. $3a/4$. **17.** $(\sqrt{a} - 1)^2$. **18.** 2/5.
19. 23. **20.** $a - 1$. **21.** $42\frac{1}{4}$. **22.** $9a/10$.

EXERCISE XCI.

1. 24/17. **2.** $1\frac{31}{65}$. **3.** $3\frac{1}{5}$.
4. $12\frac{3}{20}$. **5.** $\frac{a(c - 1)^2}{4c}$. **6.** 8/45.
7. 25/168. **8.** $\frac{(a - b)^2}{2ab}$.

EXERCISE XCII.

1. 16. **2.** 1/64. **3** 1/5. **4.** 1/6.
5. 1/1000. **6.** 36. **7.** $a^2b^{7/3}$.
8. $a^{-8/15}b^{1/6}$. **9.** $a^{4/3}b^{12}$. **10.** $a^{-2}b^{-1/2}$.
11. $a^{1/5} + b^{1/2} + x^{4/3}$. **12.** $xy^{3/2} + a^{1/3}y^2$.
13. $a^{3/5}x^{4/5} + a^{5/3}y^{2/3}$. **14.** $x^{2/3}yz^{5/3} + a^{3/4}x^{1/2}$.

15. $\sqrt[3]{x^2} - \dfrac{1}{z^3}$. **16.** $\dfrac{1}{a^7\sqrt[5]{b}}$.

17. $\sqrt[4]{a^3} \times \dfrac{1}{b^2} - \dfrac{1}{\sqrt[4]{a^3}} \times b^2$.

18. $\dfrac{1}{a^5} \times \dfrac{1}{\sqrt[3]{b^2}} + \dfrac{3\sqrt[3]{a}}{\sqrt[4]{b^3}}$. **19.** $x^{4/5} - y^{4/5}$.

20. $1 - x^{3/5}$. **21.** $a^{3/4} + b^{3/4}$. **22.** $x^2 - 1$.
23. $x^3 + 2x^{3/2} + 3 + 2x^{-3/2} + x^{-3}$.
24. $\dfrac{1}{81}a^{4/3} - \dfrac{1}{54}ab^{1/3} + \dfrac{1}{72}a^{2/3}b^{2/3} - \dfrac{1}{96}a^{1/3}b + \dfrac{1}{256}b^{4/3}$.

25. $x^{5/2} + y^{5/2}$. **26.** $x^{\frac{4n}{5}} + x^{\frac{2n}{5}}y^{\frac{2n}{5}} + y^{\frac{4n}{5}}$.
27. $x^{8/3} - x^{4/3}y^{4/3} + y^{8/3}$.
28. $x^{8/5} - 2x^{6/5}y^{1/4} + 4x^{4/5}y^{1/2} - 8x^{2/5}y^{3/4} + 16y$.
29. $x^{2/3} - x^{-2/3}$.
30. $a^{4/10} + a^{3/10}x^{1/5} + a^{2/10}x^{2/5} + a^{1/10}x^{3/5} + x^{4/5}$.
31. $x + y$.

EXERCISE XCIII.

1. $x = 2, y = 3$. **2.** $x = 3, y = 5$. **3.** $x = 2, y = 1$.
4. $x = 4, y = -1$. **5.** $x = 1, y = 2$. **6.** $x = -3, y = 4$.
7. $x = 5, y = -6$. **8.** $x = -1, y = -2$.
9. $x = 3, y = -1$. **10.** $x = 7, y = 5$.
11. $x = 3, y = 8$. **12.** $x = 2, y = 3$.
13. $1 - \dfrac{x}{2} - \dfrac{x^2}{8} - \dfrac{x^3}{16}$. **14.** $x^2 - 2x + 1$.
15. $x = 15, y = 16$. **16.** $x = 3, y = 2$.
17. $x = 2, y = -1/2$. **18.** $x = 1, y = 7$.
19. $x = 5, y = 5$ **20.** $x = \dfrac{bc}{a+b}, y = \dfrac{ac}{a+b}$.
21. $x = \dfrac{b-c}{b-a}, y = \dfrac{a-c}{a-b}$. **22.** $x = 2b - a, y = 2a - b$.
23. $x = a, y = b$. **24.** $x = \dfrac{ac}{a^2+b^2}, y = \dfrac{bc}{a^2+b^2}$.

25. $x = \frac{7a + 8b}{9}$, $y = \frac{8a + 7b}{9}$.

EXERCISE XCIV.

1. 7 and 1. **2.** 8/15. **3.** 45. **4.** 54.
5. 58 years and 18 years. **6.** Each would do it in 50 days.
7. Tea 28 cents a pound, and sugar 3 cents.
8. 4 gals. from the first and 3 gals. from the second.
9. 2 gals. from the first and 10 gals. from the second.
10. Tea 30 cents a pound, and sugar 3½ cents.

EXERCISE XCV.

1. $x = 2$, $y = 3$, $z = 4$.
2. $x = 1$, $y = -2$, $z = 3$.
3. $x = 3$, $y = 5$, $z = -3$.
4. $x = 7$, $y = 3$, $z = -5$.
5. $x = 4$, $y = -3$, $z = 2$.
6. $x = 3/2$, $y = 2/3$, $z = 5/6$.
7. $x = 15$, $y = 18$, $z = 20$.
8. $x = 3$, $y = 6$, $z = 9$.
9. $x = 9$, $y = 18$, $z = 6$.
10. $x = 8$, $y = 6$, $z = 5$.
11. $3x^{\frac{5n}{6}} + 4x^{\frac{2n}{3}} - 13x^{\frac{7n}{12}} - 6x^{\frac{3n}{4}} + 10x^{\frac{n}{2}}$.
12. $x^{2n} + x^{\frac{3n+m}{2}} + x^{n+m} + x^{\frac{n+3m}{2}} + x^{2m}$.
13. $4x^{2/3} + 25x^{4/3} + 16x^2 - 12x - 24x^{5/3}$.

EXERCISE XCVI.

1. 9, 11, and 18. **2.** 37, 25, and 16.
3. \$24, \$32, and \$16. **4.** A, \$420; B, \$640; C, \$1040.
5. A in 40 days, B in 120 days, and C in 60 days.
6. A in 10 days, B in 15 days, and C in 12 days.
7. 234. **8.** 253. **9.** 428.
10. A, $\frac{2rst}{rs + st - rt}$; B, $\frac{2rst}{-rs + st + rt}$; C, $\frac{2rst}{rs - st + rt}$.

11. Rate of stream, 2 miles per hour; rate rowing in still water, 10 miles per hour.

12. Rate of the current, 3 miles per hour; rate of crew in still water, 12 miles per hour.

13. Rates 36 and 27 miles per hour respectively, and distance 756 miles.

14. Rates 25 and 30 miles per hour respectively, and distance 330 miles.

15. 15 persons, and 5 dollars a piece.

16. Number of persons $\frac{(a+b)mn}{bm-an}$; each received $\frac{ab(m+n)}{bm-an}$ dollars.

EXERCISE XCVII.

1. $(x+\sqrt{-5})(x-\sqrt{-5})$. **2.** $(x+\sqrt{7})(x-\sqrt{7})$.
3. $(x+4\sqrt{-1})(x-4\sqrt{-1})$. **4.** $3(x+\sqrt{3})(x-\sqrt{3})$.
5. $5(x+\sqrt{5})(x-\sqrt{5})$. **6.** $7(x+\sqrt{-2})(x-\sqrt{-2})$.
7. $2(x+1/2\sqrt{6})(x-1/2\sqrt{6})$.
8. $3(x+1/3\sqrt{-15})(x-1/3\sqrt{-15})$.
9. $5(x+1/5\sqrt{10})(x-1/5\sqrt{10})$.
10. $4(x+1/4\sqrt{-12})(x-1/4\sqrt{-12})$
$= 4(x+1/2\sqrt{-3})(x-1/2\sqrt{-3})$.
11. $3(x+2/3\sqrt{3})(x-2/3\sqrt{3})$.
12. $7(x+1/7\sqrt{-35})(x-1/7\sqrt{-35})$.

EXERCISE XCVIII.

1. $4\left(x+\frac{7+\sqrt{129}}{8}\right)\left(x+\frac{7-\sqrt{129}}{8}\right)$.

2. $2(x+1/2)(x+2)$.

3. $5\left(x+\frac{-6+\sqrt{176}}{10}\right)\left(x+\frac{-6-\sqrt{176}}{10}\right)$

$$= 5\left(x+\frac{-3+2\sqrt{11}}{5}\right)\left(x+\frac{-3-2\sqrt{11}}{5}\right).$$

4. $6\left(x+\frac{-4+\sqrt{88}}{12}\right)\left(x+\frac{-4-\sqrt{88}}{12}\right)$

$$= 6\left(x+\frac{-2+\sqrt{22}}{6}\right)\left(x+\frac{-2-\sqrt{22}}{6}\right).$$

5. $4\left(x+\frac{3+\sqrt{-87}}{8}\right)\left(x+\frac{3-\sqrt{-87}}{8}\right).$

6. $2(x+4)(x+1).$ 7. $66\frac{2}{3}$ at \$25, and $108\frac{1}{3}$ at \$40.

8. $7(x+1)(x+2/7).$ 9. $7(x+2+\sqrt{5})(x+2-\sqrt{5}).$

10. $3(x+3)(x-2/3).$

11. $4(x-3+\sqrt{+6})(x-3-\sqrt{+6}).$

12. $15(x-3/5)(x+2/3).$

13. $3\left(x-\frac{5+\sqrt{7}}{3}\right)\left(x-\frac{5-\sqrt{7}}{3}\right).$

14. $\frac{s-bm}{a-b}$ and $\frac{s-am}{b-a}$ acres. 15. $x = 4/9.$

16. $x = \frac{a(c-1)^2}{4c}.$

EXERCISE XCIX.

1. $-3, 6.$
2. $5, -9.$
3. $-5, -8.$
4. $5, 7.$
5. $-4, -4.$
6. $5, 5.$
7. $3, -3.$
8. $a, -a.$
9. $-a, -b.$
10. $-a, b.$
11. $-3/2, 1.$
12. $4/3, -3.$
13. $2/5, -4/3.$
14. $-8/7, -1.$
15. $3/2, 4.$
16. $5/3, 4.$

EXERCISE C.

1. $x^2-10x+21=0.$
2. $x^2+2x-24=0.$
3. $x^2+8x+7=0.$
4. $x^2-2x=0.$
5. $x^2+9x=0.$
6. $x^2-49=0.$
7. $x^2+16x+64=0.$
8. $x^2-22x+121=0.$
9. $4x^2-15x+9=0.$
10. $18x^2-18x+1=0.$
11. $16x^2-28x+11=0.$
12. $x^2-8x+22=0.$
13. $18x^2+27x+10=0.$
14. $2x^2-x-3=0.$
15. $5x^2-33x-14=0.$
16. $x^2-6x+4=0.$

17. $x^2 - 4x - 4 = 0.$ **18.** $x^2 - 10x + 22 = 0.$
19. $x^2 - 18x + 85 = 0.$ **20.** $25x^2 - 35x + 13 = 0.$
21. $24x^2 - 44x + 21 = 0.$ **22.** $-\frac{b^2 - 4ac}{4a^2}.$
23. $\frac{x^2 - 2x}{x + 2}.$ **24.** $\frac{x}{2 - x}.$

EXERCISE CI.

1. $7, -1/3.$ **2.** $2, -3.$ **3.** $2, 3.$
4. $4, 1/4.$ **5.** $-1, 2.$ **6.** $-3/4, -9/4.$
7. $5, -6\frac{7}{17}.$ **8.** $1, -7/32.$ **9.** $a, 1/a.$
10. $3, 13/11.$ **11.** $2, 1/2.$ **12.** $1/2, -3.$
13. $5, -1/6.$

EXERCISE CII.

2. $5/7, 3/4.$ **3.** $-a, b.$ **4.** $-3/4, -2.$
5. $2/3, -5/4.$ **6.** $\pm 6, \pm 9.$ **7.** $\pm 6, \pm 10.$
8. $\pm 2/3, \pm 3/4.$ **9.** $\pm\frac{2 + \sqrt{3}}{6}, \pm\frac{2 - \sqrt{3}}{6}.$

EXERCISE CIII.

1. 15 and 8, or − 23/2 and − 37/2.
2. 3, 4, and 5, or − 1, 0, and 1.
3. 20 and 8, or − 14 and − 26.
4. 5, 6, and 7, or − 1, 0, and 1. **5.** 4 and 2.
6. 1, 2, 3, 4; or 5, 6, 7, 8.
7. 3, 4, 5, 6, or − 4/3, − 1/3, 2/3, 5/3.
8. 20 barrels; 6 dollars a barrel. **9.** $80 or $20.
10. $60. **11.** 8 miles an hour. **12.** 7 miles an hour.

EXERCISE CIV.

1. $-2, -4.$ **2.** $20, -6.$ **3.** $5, -5/2.$
4. $1, 4\frac{2}{3}.$ **5.** $1, 2\frac{1}{7}.$ **6.** $3, 1/2.$
7. $4, -4\frac{1}{2}.$ **8.** $1, -3/4.$ **9.** $2, -2/9.$
10. $7, -1/3.$ **11.** $\frac{-6 \pm \sqrt{3}}{3}.$ **12.** $-a + b, -a - b.$

13. $-a, -3ab.$ **14.** $-\frac{b}{a}, ab.$ **15.** $a, b.$

16. $0, \frac{2ab}{a+b}.$ **17.** $2a-b, 3b-2a.$ **18.** $a, 1/a.$

19. $b, \frac{a^2}{b}.$ **20.** $\frac{a}{2}(-3 \pm \sqrt{3}).$ **21.** $\pm \sqrt{a^2+b^2}.$

22. $1/8(-25 \pm \sqrt{33}).$ **23.** $3/5, -2/3.$ **24.** $3, +1/6.$

EXERCISE CV.

1. 30 and 40 miles per hour. **2.** 40 and 45 miles per hour.

3. $2\frac{1}{7}$ hours. **4.** $2\frac{1}{10}$ hours.

5. $a^{-26/3}.$ **6.** $\frac{1}{a^4 \sqrt[4]{b^3}} + \frac{2\sqrt{a^3}}{\sqrt[3]{b^4}}.$

7. $\frac{1}{512}.$ **8.** $a\frac{(m-n)bd - cb + dx}{bd}.$

9. $\frac{c^4}{c^2 - b^2}.$ **10.** $(x - 2y)(7x - 11).$

EXERCISE CVI.

1. $5/2, 3/2.$ **2.** $\pm 2/3\sqrt{3}, \pm \sqrt{5}.$ **3.** $6, 5\frac{2}{3}.$

4. $\pm \sqrt{2}, \pm 1/2\sqrt{6}.$ **5.** $5, 6.$ **6.** $\pm 2/3\sqrt{3}, \pm 1/3\sqrt{21}.$

7. $\pm 1/2\sqrt{3}, \pm 1/3\sqrt{6}.$ **8.** $\pm 1/6\sqrt{6}, \pm 1/3\sqrt{2}.$

9. $3/2, -2.$ **10.** $3/5, -4/7.$

11. $\frac{1}{\sqrt[3]{3}}, -\frac{1}{\sqrt[3]{23}}.$ **12.** $\pm 1/6\sqrt{3}, \pm 1/3.$

13. $-1, -1/2.$ **14.** $\sqrt[3]{121}, 4.$

EXERCISE CVII.

1. 3 miles an hour, $3\frac{4}{7}$ hours.

2. 5 miles an hour, $5\frac{3}{4}$ hours.

3. $-\frac{3b^2 - 75a^2x^2}{25a^2}$ **4.** $-\frac{31\sqrt{5} + 85}{11}.$

5. 8 days. **6.** 16 days.

EXERCISE CVIII.

1. $x = 3, \quad y = \pm 5,$
$x = -3, \quad y = \pm 5.$

2. $x = 7/2, \quad y = \pm 5/2,$
$x = -7/2, \quad y = \pm 5/2.$

3. $x = 3\sqrt{2}, \quad y = \pm 2\sqrt{5},$
$x = -3\sqrt{2}, \quad y = \pm 2\sqrt{5}.$

4. $x = 3m - n, \quad y = \pm(m + 3n),$
$x = n - 3m, \quad y = \pm(m + 3n).$

5. $25, \ 9/16.$

6. $-243, \ \sqrt[3]{26^5}.$

7. $(a^2 + b^2)^2 a^{-2} b^{-2} - c^2.$

8. $(5x + 8b)(x - 2a).$

EXERCISE CIX.

1. $x = 2, \quad y = 3,$
$x = -7/5, \quad y = 49/5.$

2. $x = 4, \quad y = -6,$
$x = -6, \quad y = 4.$

3. $x = 5, \quad y = 3,$
$x = -3, \quad y = -5.$

4. $x = 5, \quad y = 9,$
$x = -1, \quad y = 3.$

5. $x = 6, \quad y = 8,$
$x = -8, \quad y = -6.$

6. $x = 5, \quad y = 3,$
$x = 3, \quad y = 5.$

7. $x = a + 1, \quad y = a,$
$x = -a, \quad y = -a - 1.$

8. $x = 4, \quad y = 6.$

9. $x = -1, \quad y = -1,$
$x = 1/2, \quad y = 2.$

10. $x = 4, \quad y = 12,$
$x = -36/7, \quad y = -12/7.$

11. $(\pm 2)^n, \ (-14/3)^{n/2}.$

12. $(3a - 2)(5x + 2b).$

EXERCISE CX.

1. $x = 7, \quad y = 6,$
$x = 6, \quad y = 7.$

2. $x = 8, \quad y = 3,$
$x = 3, \quad y = 8.$

3. $x = 5, \quad y = 2,$
$x = 2, \quad y = 5.$

4. $x = 3, \quad y = 7,$
$x = 7, \quad y = 3.$

5. $x = 1, \quad y = 5,$
$x = -5, \quad y = -1.$

6. $x = 2, \quad y = 8,$
$x = -8, \quad y = -2.$

7. $x = 2, \quad y = -9,$
$x = 9, \quad y = -2.$

8. $x = -6, \quad y = 12,$
$x = -12, \quad y = 6.$

9. $x = 7, \quad y = 4,$
$x = -4, \quad y = -7.$

10. $x = 5, \quad y = 3,$
$x = 3, \quad y = 5.$

11. $x = 6, \quad y = 4,$
$x = -4, \quad y = -6.$

12. $x = 5, \quad y = 8,$
$x = 8, \quad y = 5.$

13. $x = 6,\ y = 3,$
$x = 3,\ y = 6.$

14. $x = 9,\ y = 7,$
$x = -7,\ y - 9.$

15. $x = b + a,\ y = a - b,$
$x = b - a,\quad y = -a - b.$

16. $x = \pm(2a - b),\ y = \pm(a - 2b),$
$x = \pm(a - 2b),\ y = \pm(2a - b).$

EXERCISE CXI.

1. $x = \pm 4,\quad y = \pm 1,$
$x = \pm 14,\ y = \mp 4.$

2. $x = \pm 8,\ y = \mp 5,$
$x = \pm 3,\ y = \pm 5.$

3. $x = \pm 6,\ y = \pm 2.$

4. $x = \pm 9,\ y = \pm 4.$

5. $x = \pm 4,\qquad y = \pm 5,$
$x = \pm 3\sqrt{3},\ y = \pm\sqrt{3}.$

6. $x = \pm 2,\quad y = \pm 4,$
$x = \pm\sqrt{2},\ y = \pm 3\sqrt{2}.$

EXERCISE CXII.

1. 3 and 5.
2. 4 and 7.
3. 5 and 9.
4. 4 and 10.
5. 3 and 4.
6. 3 and 7.
7. 2 and 3.
8. 1 and 2.
9. Cows 30 dollars apiece and sheep 3 dollars apiece.
10. 13.
11. $\dfrac{a^2b^2}{a^2 - b^2}.$
12. $(4x - 3y^2)(3x^2 - 2y).$
13. $-\dfrac{25b^2 - 84ac}{36a^2}.$
14. 25.
15. $4\sqrt{2}.$
16. $\dfrac{\sqrt[3]{a^2}}{b} - \dfrac{b}{\sqrt[3]{a^2}}.$
17. $4 + \sqrt{2}.$

EXERCISE CXIII.

1. 2, 3.
2. 1, 10; 14, 2.
3. 4, 8; 13, 1.
4. 1, 11.
5. 7.
6. 9, 8, 3.
7. 5, 6, 7.
8. 4, 2, 7.
9. 3, 11, 1; 7, 4, 1; 2, 8, 2; 6, 1, 2; 1, 5, 3.
10. 1, 5, 2; 3, 1, 4; 2, 3, 3.
11. $x = 4 + 13p,\ y = 1 + 7p.$
12. $x = 11p - 2,\ y = 9p - 2.$
13. 8, 7.
14. 64, 44.
15. By using the 7-inch five times and the 13-inch once.

16. By using 6 four-pound weights and 3 seven-pound weights.

17. By using the fifty- and twenty-cent pieces respectively 1, 17; 3, 12; 5, 7; or 7, 2.

18. By using the half-dollars, quarter-dollars, and dimes respectively 1, 18, 1; 4, 10, 6; or 7, 2, 11.

19. 5 pigs, 10 sheep, and 15 calves. **20.** 92, 90.

21. 19/9, 2/5; 10/9, 7/5; or 1/9, 12/5.

EXERCISE CXIV.

3. $x > 2\frac{1}{8}$. **4.** $x > 2\frac{14}{15}$.
5. $x > 4\frac{2}{3}$. **6.** $x > 3.9$, $y > 4.9$.

EXERCISE CXVII.

1. 151 : 208. **2.** 6 : 11. **3.** 589 : 1008.
4. $x^2 - y^2 : x - y$. **5.** $x^3 - y^3 : x - y$.
6. $x^n - y^n : x - y$. **7.** 144 : 125. **8.** 15 : 8.
9. 0, 4, 16, ∞, -32. **10.** $-1\frac{5}{7}$. **11.** 18.
12. 385, 660. **13.** 11. **14.** 5 : 37.
15. 5 : 6 or $-3 : 5$. **16.** 9 : 7, or $-8 : 3$. **17.** 5.

EXERCISE CXVIII.

1. $\frac{ab}{c}$. **2.** $\frac{5b}{7}$. **3.** $\frac{1}{c}$. **4.** $\frac{9}{c}$.
5. $3\frac{1}{2}$. **6.** $-5\frac{5}{6}$. **7.** $\frac{nc}{m(b-a)}$. **8.** $\frac{4a}{3(c-b)}$.
9. 2. **10.** -4. **11.** $-2\frac{1}{3}$. **12.** 6.
13. $2\frac{1}{2}$. **14.** 1/2. **15.** $-3/14$.

EXERCISE CXIX.

1. $13\frac{1}{2}$. **2.** $2\frac{3}{5}$. **3.** 3.6. **4.** 16.
5. 1256.64. **6.** 523.5 cu. ft. **7.** 4752 cu. ft.
8. $2\frac{19}{28}$ cu. ft. **9.** 18 miles. **10.** 15/32.

EXERCISE CXX.

1. 2.9805. **2.** 1.7686. **3.** 0.3766. **4.** $\bar{2}.5119$.
5. 1.6990. **6.** $\bar{3}.4771$. **7.** 4.6021. **8.** 0.3010.

9. $\overline{6}.8451$. **10.** 4.4571. **11.** 1.2129. **12.** $\overline{3}.5538$.
13. 0.3923. **14.** 0.9034.

EXERCISE CXXI.

1. 862. **2.** .366. **3.** .0988.
4. 7665. **5.** 3.9645. **6.** .006823.
7. .2864. **8.** .09034. **9.** 6.42285.

EXERCISE CXXII.

1. 6.42221. **2.** 6.4024. **3.** 6.5383.
4. 10.3701. **5.** 11.1025. **6.** 12.0969.
7. 8.3010. **8.** 13.0969. **9.** 14.0458.

EXERCISE CXXIII.

1. 172. **2.** .677. **3.** — 127.205.
4. .000406. **5.** .0114289. **6.** — 1299.39.
7. — 2340.52. **8.** 118.916. **9.** 645300.
10 .000000636. **11.** 4.326. **12.** 1.71.
13. — 378.45. **14.** 7.12. **15.** .07852.
16. 2.3388. **17.** — .006535. **18.** .2475.
19. 23.2578. **20.** .8834. **21.** .15811. **22.** — .70214.

EXERCISE CXXIV.

1. 3.9073. **2.** 3.4022. **3.** 1.4999.
4. $\overline{2}.7871$. **5.** $\overline{2}.1683$. **6.** 18346.
7. 6. **8.** 5/2. **9.** — 1/3.

EXERCISE CXXV.

1. 1. **2.** ∞. **3.** a/b. **4.** b/a.
5. m/p. **6.** 0. **7.** — 10/7. **8.** — 9/4.
9. $2a$. **10.** $5z^4$. **11.** — 3/2. **12.** — 2.

EXERCISE CXXVI.

1. 64; 78; — 75; 8. **2.** 52; 83; —14; 55; —19⅔.
3. 11. **4.** 7. **5.** 2⅓. **6.** 1/6.
7. 3½. **8.** 0. **9.** 19th. **10.** 16th.
11. 9th. **12.** 10, 12, 14, . . . 52.
13. 8/9, 7/9, 6/9, . . . 1/9.

14. $4a - 5b, 3a - 4b, 2a - 3b \ldots - 5a + 4b$.
15. $d = 4, a = 2$. **16.** $d = -3, a = 21$. **17.** $-28\frac{4}{7}$.

EXERCISE CXXVII.

1. 624. **2.** 187. **3.** 255.
4. 810. **5.** 0. **6.** 357.
7. $1/2\,(n^3 + 3n^2)$. **8.** $n(a + b)^2 - n(n - 1)ab$.
9. 80. **10.** 1941. **11.** 1080.
12. 1160. **13.** $8 + 12 + 16 + \ldots + 76$. $S = 680$.
14. $12\frac{8}{41} + 14\frac{16}{41} + 16\frac{24}{41} + \ldots 97\frac{33}{41}$. $S = 2200$.
15. 8729. **16.** 41832.

EXERCISE CXXVIII.

1. 693. **2.** 3375. **3.** 13. **4.** 33. **5.** 10 days.
6 8 days. **7.** ± 5. **8.** $\pm 2\frac{1}{2}$. **9.** 9 days.
10. 50500 yards. **11.** $5195. **12.** ± 20, ± 30, ± 40.
13. ± 8, ± 12, ± 16, ± 20. **14.** ∓ 4, ± 2, ± 8, ± 14.

EXERCISE CXXIX.

1. 10, 50. **2.** ± 12, − 48, ± 192.
3. − 15, 45, − 135, 405.
4. ± .6, .12, ± .024, .0048, ± 00096
5. 1/3, 2/3, 4/3, 8/3, 16/3, 32/3.

EXERCISE CXXX.

1. 19680. **2.** −9840. **3.** 1281/512.
4. $191\frac{1}{4}$. **5.** −682. **6.** 53144/2187.
7. −463/192. **8.** 64/65. **9.** 27/58.
10. .999. **11.** 1/2. **12.** 4.
13. 6, 24, 96, 384, 1536. **14.** − 12, 36, − 108.
15. 24, 60, 150; or 27, 63, 147.

EXERCISE CXXXI.

1. 5/33. **2.** 5/27. **3.** 44/111. **4.** 3/7. **5.** 1/77.
6. 4/5. **7.** 52/165. **8.** 7/60. **9.** 143/740.

EXERCISE CXXXII.

1. \$4159.09.
2. \$1153.94.
3. \$897.00.
4. $5\frac{1}{2}$ yrs.
5. \$403.90.
6. .04.
7. 14 yrs. 2 mo. 12 da.
8. $17\frac{1}{2}$ yrs.
9. \$6785.71.
10. \$6000.
11. \$3246.42.
12. \$437.50.
13. 451.33.

EXERCISE CXXXIII.

1. 4 and 6.
2. 1/2 and 2/7.
3. 1/10.
4. $6\frac{1}{4}$, $8\frac{1}{3}$, $12\frac{1}{2}$.

EXERCISE CXXXIV.

1. 1.2.1.
2. 1.3.3.1.
3. 1.4.6.4.1.
4. 1.5.10.10.5.1.
5. 1.6.15.20.15.6.1.
6. 1.7.21.35.35.21.7.1.
7. 1.8.28.56.70.56.28.8.1.
8. 1.9.36.84.126.126.84.36.9.1.
9. 1.10.45.120.210.252.210.120.45.10.1.

EXERCISE CXXXV.

1. $a^{13} \times (2x^3)^4 = 16a^{13}x^{12}$.
2. $3^2 \times (-a)^{13} = -9a^{13}$.
3. $(5a^3)^4(-7x^3)^3 = -214375a^{12}x^9$.
4. $5^7a^7 \times -\frac{x^7}{5^7} = -a^7x^7$.
5. $(2x)^6\left(-\frac{1}{2x}\right)^6 = 1$.
6. $4^5x^5 \times \frac{1}{2^{10}x^5} = 1$.

EXERCISE CXXXVI.

1. $a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7$.
2. $a^8 - 8a^7x + 28a^6x^2 - 56a^5x^3 + 70a^4x^4 - 56a^3x^5 + 28a^2x^6 - 8ax^7 + x^8$.
3. $1 + 9x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$.
4. $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$.
5. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$.
6. $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$.
7. $1-18a^2 + 135a^4 - 540a^6 + 1215a^8 - 1458a^{10} + 729a^{12}$.

8. $1-7xy+21x^2y^2-35x^3y^3+35x^4y^4-21x^5y^5+7x^6y^6-x^7y^7.$

9. $729a^6-972a^5+540a^4-160a^3+\frac{80a^2}{3}-\frac{64a}{27}+\frac{64}{729}.$

10. $\frac{64x^6}{729}+\frac{32x^4}{27}+\frac{20x^2}{3}+20+\frac{135}{4x^2}+\frac{243}{8x^4}+\frac{729}{64x^6}.$

11. $c^{8/3}+4c^2d^{-3/4}+6c^{4/3}d^{-6/4}+4c^{2/3}d^{-9/4}+d^{-3}.$

12. $m^{-3}-6m^{-5/2}n^2+15m^{-2}n^4-20m^{-3/2}n^6+15m^{-1}n^8$
$-6m^{-1/2}n^{10}+n^{12}.$

13. $x^{5m}-10x^{4m}y^{2n}+40x^{3m}y^{4n}-80x^{2m}y^{6n}+80x^my^{8n}-32y^{10n}.$

14. $a^{12}+20a^9x^{1/2}+150a^6x+500a^3x^{3/2}+625x^2.$

15. $a^6+16a^{29/6}+96a^{11/3}+256a^{5/2}+256a^{4/3}.$

16. $x^3+15x^{12/5}y^{-2/5}+90x^{9/5}y^{-4/5}+270x^{6/5}y^{-6/5}+405x^{3/5}y^{-8/5}$
$+243y^{-2}.$

17. $a^{7/2}b^{-14/3}+7a^{5/2}b^{-10/3}+21a^{3/2}b^{-2}+35a^{1/2}b^{-2/3}+35a^{-1/2}b^{2/3}$
$+21a^{-3/2}b^2+7a^{-5/2}b^{10/3}+a^{-7/2}b^{14}/^{3}.$

18. $1-\frac{10}{x}+\frac{45}{x^2}-\frac{120}{x^3}+\frac{210}{x^4}-\frac{252}{x^5}+\frac{210}{x^6}-\frac{120}{x^7}+\frac{45}{x^8}$
$-\frac{10}{x^9}+\frac{1}{x^{10}}.$

EXERCISE CXXXVII.

1. $-35750x^{10}.$ **2.** $-112640x^9.$ **3** $-312x^2.$

4. $40a^7b^3.$ **5.** $\frac{1120}{81}a^4b^4.$ **6.** $\frac{10500}{x^3}.$

7. $\frac{70x^6y^{10}}{a^2b^6}.$ **8.** $2x^4+24x^2+8.$ **9.** $140\sqrt{2}.$

10. $2(365-363x+63x^2-x^3).$ **11.** $252.$

12. $\frac{189a^{17}}{8},-\frac{21a^{19}}{16}.$ **13.** $\frac{7}{18}.$

EXERCISE CXXXVIII.

1. $a^{1/4}-1/4a^{-3/4}x-3/32a^{-7/4}x^2-7/128a^{-11/4}x^3$
$-77/2048a^{-15/4}x^4.$

2. $a^{3/2}+3/2a^{1/2}x+3/8a^{-1/2}x^2-1/16a^{-3/2}x^3$
$+3/128a^{-5/2}x^4,$

3. $1 + 4x + 10x^2 + 20x^3 + 35x^4$.

4. $1 - 7x + 28x^2 - 84x^3 + 210x^4$.

5. $3^{1/4} - \frac{1}{2\sqrt[4]{27}}x - \frac{3}{8\sqrt[4]{3^7}}x^2 - \frac{7}{16\sqrt[4]{3^{11}}}x^3 - \frac{77}{128\sqrt[4]{3^{15}}}x^4$.

6. $1 + 1/3x + 2/9x^2 + 14/81x^3 + 35/243x^4$.

7. $1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4$.

8. $x^{-2} - 4x^{-4}y + 16x^{-6}y^2 - 64x^{-8}y^3 + 256x^{-10}y^4$.

9. $a^{-1} + 1/2a^{-5}x^{-1/2} + 5/8a^{-9}x^{-1} + 15/16a^{-13}x^{-3/2}$
$+ 195/128a^{-17}x^{-2}$

EXERCISE CXXXIX.

1. 8648640. 2. 259459200. 3. 5040.
4. 720. 5. 181440. 6. 90720.
7. 27720. 8. 840. 9. 480.
10. 240. 11. 96. 12. 9189180.

EXERCISE CXL.

1. 70. 2. 10080. 3. 5250.
4. 1512000. 5. 178378200. 6. 455.

EXERCISE CXLI.

1. $x^3 - 6x^2 + 11x - 6 = 0$.

2. $x^4 - 4x^3 - 19x^2 + 46x + 120 = 0$.

3. $x^4 + 4x^3 + x^2 - 6x = 0$.

4. $6x^4 - 11x^3 - 48x^2 - 19x + 12 = 0$.

5. $9x^4 + 30x^3 - 47x^2 - 120x + 144 = 0$.

6. $20x^4 + 21x^3 - 240x^2 - 13x + 12 = 0$.

7. $3, -7$. 8. $-2, \frac{9}{2}$.

9. $-2 \pm \sqrt{7}$. 10. $\frac{1}{4}(-3 \pm \sqrt{-7})$.

11. $\frac{1}{3}(-1 \pm \sqrt{-8})$. 12. $2, 2$.

13. $3, -4$. 14. $-2, 4$. 15. $-7, 8$.

EXERCISE CXLII.

1. $\frac{3}{x+2}$ and $\frac{4}{x+3}$. **2.** $\frac{5}{x-2}$ and $-\frac{7}{x+4}$.

3. $\frac{3}{x-4}$ and $-\frac{4}{x+3}$. **4.** $\frac{6}{x-8}$ and $\frac{7}{x+5}$.

5. $\frac{5}{2x+3}$ and $\frac{6}{x-5}$. **6.** $\frac{3}{3x-4}$ and $-\frac{7}{2x-6}$.

EXERCISE CXLIII.

1. $\frac{2}{x-3}+\frac{3}{x-4}-\frac{4}{x-5}$.

2. $\frac{3}{2x+2}-\frac{2}{x-3}+\frac{5}{x+3}$.

3. $\frac{1}{2x-1}+\frac{4}{3+x}-\frac{1}{3-x}$.

4. $\frac{1}{2(x-1)}-\frac{4}{x-2}+\frac{7}{2(x-3)}$.

5. $\frac{3}{4(x+3)}-\frac{5}{8(x+5)}-\frac{1}{8(x+1)}$.

6. $\frac{1}{12(x+1)}-\frac{7}{3(x-2)}+\frac{13}{4(x-3)}$.

EXERCISE CXLIV.

1. $\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$.

2. $\frac{7}{x-1}+\frac{5x-3}{x^2+x+1}$.

3. $\frac{5x+6}{x^2+x+1}-\frac{3x-4}{x^2-x+1}$.

4. $\frac{1}{5(x+2)}+\frac{4x-8}{5(x^2+1)}$.

5. $\frac{1}{2(x^2+1)}+\frac{1}{2(x-1)^2}$.

EXERCISE CXLV.

1. $1 - 5x + 22x^2 - 86x^3 + 346x^4.$

2. $2 - 7x + 28x^2 - 91x^3 + 322x^4.$

3. $3 - 19x^2 + 95x^4 - 475x^6 + 2375x^8.$

4. $2 - 11x^2 + 44x^4 - 176x^6 + 704x^8.$

5. $\frac{3}{4}x + \frac{9}{16}x^3 + \frac{27}{64}x^5 + \frac{81}{256}x^7 + \frac{243}{1024}x^9.$

6. $\frac{2}{3}x + \frac{4}{9}x^3 + \frac{8}{27}x^5 + \frac{16}{81}x^7 + \frac{32}{243}x^9.$

EXERCISE CXLVI.

1. $1 + \frac{1}{2}x + \frac{3}{8}x^2 - \frac{3}{16}x^3 + \frac{3}{128}x^4.$

2. $1 - x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{3}{8}x^4.$

3. $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4.$

EXERCISE CXLVII.

1. $x = \frac{1}{2}y - \frac{1}{8}y^2 + \frac{3}{16}y^3 - \frac{13}{128}y^4.$

2. $x = y - y^2 + y^3 - y^4.$

3. $x = y + y^3 + 2y^5 + 8y^7 + \dots$

EXERCISE CXLVIII.

1. $\frac{1}{1}+\frac{1}{8}+\frac{1}{1}+\frac{1}{5}.$ **2.** $\frac{1}{1}+\frac{1}{3}+\frac{1}{1}+\frac{1}{3}+\frac{1}{1}+\frac{1}{3}.$

3. $3+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{3}+\frac{1}{2}+\frac{1}{2}.$

4. $\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}.$

5. $2+\frac{1}{3}+\frac{1}{2}+\frac{1}{1}+\frac{1}{3}+\frac{1}{2}+\frac{1}{1}+\frac{1}{2}.$

6. $1+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{1}+\frac{1}{2}.$

7. $1+\frac{1}{3+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{3}$.

8. $\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\frac{1}{5+}\frac{1}{6+}\frac{1}{7}$.

9. 2/1, 13/6, 15/7, 28/13, 323/150, 674/313.

10. 1/2, 2/5, 7/17, 9/22, 25/61, 159/388.

11. 3/1, 10/3, 13/4, 36/11, 85/26, 121/37, 1174/359.

12. 1/2, 3/7, 4/9, 19/43.

13. 1/4, 7/29, 8/33, 39/161, 47/194.

17. $\frac{1}{2+}\frac{1}{3+}\frac{1}{3+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1}+\frac{1}{7+}\dots$; $\frac{76}{175}$; $\frac{1}{262325}$, $\frac{1}{231700}$.

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EXERCISE CXLIX.

1. $2+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\dots$

2. $3+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{6+}\dots$,

3. $1+\frac{1}{2+}\frac{1}{2}\dots$ **4.** $2+\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\frac{1}{4+}\dots$

5. $4+\frac{1}{8+}\frac{1}{8+}\dots$

6. $4+\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{8}$
$+\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{8+}\dots$

7. $\sqrt{2}-1$. **8.** $\sqrt{6}-1$. **9.** $1/5(2\sqrt{39}-9)$.

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