ELEMENTARY ALGEBRA

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PREFACE

THE subject-matter of this text follows the requirements of the College Entrance Examination Board both as to subjects treated as well as to those omitted, but especial emphasis is placed upon those principles which are the tools of more advanced work in mathematics. The philosophy *per se* of algebra and all algebraic puzzles are therefore omitted, to give place to a logical discussion, simply told, of the fundamental principles. The scheme of the whole text is to illustrate the meaning of a principle by carefully selected exercises; every principle is followed by such a group of examples as will exact a mastery of the principle involved before another topic is taken up. The examples are expressly prepared to illustrate various principles treated in the text. Nearly a thousand of these examples are taken from the most recent college entrance papers.

The attention of teachers is especially invited to the use of Graphical Methods throughout the book, the introduction of the Negative Number, the treatment of the Graphs of Equations, the introduction of Equations used in Physics, and the insertion of problems from Physics in Ratio and in Variation, and to the treatment of the Progressions and of Permutations and Combinations.

It is suggested that paragraphs, exercises, and examples marked by the * be omitted at first reading.

The author begs to acknowledge gratefully the valuable assistance of Professor Charles H. Ashton of the University of Kansas, of Miss Mary M. Wardwell of the Central High School, Buffalo, N.Y., and of Mr. Frank C. Robertson of the Pingry School, Elizabeth, N.J., not only for their careful reading of the proofs, but also for their criticisms of the text.

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TEACHERS MAY OBTAIN ANSWER-BOOKS,
FOR WHICH NO CHARGE IS MADE,
ON APPLICATION TO THE PUBLISHERS.

ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION AND DEFINITIONS

1. The science of number includes both Arithmetic and Algebra. Algebra may be defined as generalized Arithmetic.

2. In arithmetic every number represents a definite value. Thus, $4 = 1 + 1 + 1 + 1$. In algebra, *a set of symbols, usually letters of the alphabet, is used to represent numbers.* A letter can represent any number whatever, provided its value does not change during a particular range of operations.

SYMBOLS OF OPERATION

3. Addition is indicated by the sign $+$, read "plus."

Thus, $4 + 1$ means the sum of 4 and 1; $a + d$ means the sum of a and d.

4. Subtraction is indicated by the sign $-$, read "minus."

Thus, $3 - 2$ means that 2 is to be subtracted from 3; $b - c$ means that c is to be subtracted from b.

Multiplication is indicated by the sign $\times$, and by the sign $\cdot$, each read "times" or, "multiplied by"; and by the omission of sign.

Thus, $m \times n$, $m \cdot n$, and mn all mean the product of m and n, or of n and m.

The multiplication sign is never omitted in expressing the product of numbers in the form of digits.

Thus, 56 indicates $50+6$; $5 \cdot 6$ indicates 5×6.

Division is indicated by the signs $\div$, $/$, $:$, each read "divided by"; and by the fractional form.

Thus, $a \div b$, a/b, $a:b$, and $\frac{a}{b}$ all indicate the division of a by b.

Equality between two numbers is indicated by the sign $=$, read "is equal to."

Thus, $a=b$ indicates that a is equal to b.

EXERCISE I

If $a=1$, $b=2$, $c=3$, $d=4$, find the value of each of the following:

1. $\dfrac{a+b}{c}$.
2. $\dfrac{b+d}{c}$.
3. $\dfrac{c+d}{a}$.
4. $\dfrac{d-a}{c}$.
5. $\dfrac{c-a}{b}$.
6. $\dfrac{cd}{b}$.
7. $\dfrac{ad}{b}$.
8. $\dfrac{d}{a}-\dfrac{b}{a}$.
9. $\dfrac{ad}{b}+\dfrac{c}{a}$.
10. $\dfrac{ab+bc}{d}$.
11. $\dfrac{abcd}{a+c+d}$.
12. $\dfrac{ac+bc+a}{a+d}$.
13. $\dfrac{ab+bc+cd}{d+c-b}$.
14. $\dfrac{ac+ad+cb}{cd+1}$.
15. $\dfrac{ad+cd-bc}{a+b+c+d}$.

ALGEBRAIC EXPRESSIONS

4. An **algebraic expression** is a combination of number symbols connected by any of the symbols of operation.

Thus, a, $7 - a$, $6 + a \div 3 + b$ are algebraic expressions.

5. A **term** of an algebraic expression is a combination of number symbols not separated by the signs $+$ or $-$.

Thus, in the algebraic expression $6 + a \div 3 - b$, the terms are 6, $a \div 3$, and b. $\frac{a}{3}$

6. When two or more numbers multiplied together produce a certain product, each of these numbers is called a **factor** of the product.

Thus, a, b, and c, are factors of abc.

Each of the factors of a number or the product of any number of factors is called a **coefficient** of the rest of the term.

Thus, in $3\,a$, 3 is the coefficient of a; in ab, a is the coefficient of b; in $\frac{2}{3}\,abc$, $\frac{2}{3}$ is the coefficient of abc, $\frac{2}{3}\,a$ of bc, and $\frac{2}{3}\,ab$ of c.

The coefficient is generally understood to mean the number placed before the number symbols represented by the letters.

If the coefficient be 1, *it is always omitted.*

Thus, $$a = 1\,a.$$

7. The **exponent** of a number is the symbol in the form of an integer which represents how many factors equal to the number affected by the exponent are taken.

Thus, a^3 represents that a has been taken three times as a factor; or, $a^3 = a \cdot a \cdot a$.

The exponent affects only that number symbol which it follows, and at the upper right hand of which it is written.

Thus, $3\,a^2bc$ means that a alone has been taken twice as a factor; or $3\,a^2bc = 3 \cdot a \cdot a \cdot b \cdot c$.

If no number symbol be written as the exponent, it is always understood that 1 is that exponent.

Thus, in $3\,a^2bc$, 3, b, and c are to be understood as having the exponent 1 affecting each of these numbers; or $3\,a^2bc = 3^1a^2b^1c^1$.

Since the product of a number of equal factors can be called a **power** of that number, a^3 can be read "a with the exponent 3"; or "a third."

Thus, $a^4 = a \cdot a \cdot a \cdot a$ can be read "a with the exponent 4," "a fourth," or, "a to the fourth power."

The distinction between coefficient and exponent should be carefully noticed.

Thus, $3\,a = a + a + a$; and $a^3 = a \cdot a \cdot a$.

8. A **monomial** is an expression containing a single term.

Thus, $2\,a^2$, $3\,b$, and c^3 are monomials.

9. **Similar terms**, or **like terms**, are those which differ only in their numerical coefficients.

Thus, $3\,a^2b$, a^2b, and $7\,a^2b$ are similar, or like, terms.

10. A **polynomial** is an expression containing several terms.

Thus, $2\,a^2b + 3\,ab^2 + b^3$ is a polynomial.

A polynomial which contains two terms is called a **binomial**; and one which contains three terms is called a **trinomial.**

Thus, $a^2 + b^2$ is a binomial; and $a^2 - ab + b^2$ is a trinomial.

11. The **positive and negative terms** of an expression are those which are preceded by the plus and minus signs respectively.

Thus, the positive terms of $a^3 - 3a^2b + 3ab^2 - b^3$ are a^3 and $3ab^2$, and the negative terms are $3a^2b$ and b^3.

12. The **numerical value** of an expression is found by substituting for the letters their values in numbers, and performing the indicated operations.

Thus, the numerical value of $2a$, if $a = 4$, is 8.

EXERCISE II

If $a = 6$, $b = 4$, $c = 3$, $d = 2$, $e = 1$, find the value of each of the following expressions:

1. $2ab$.

2. $3cd$.

3. $4cde$.

4. a^2d.

5. c^3d.

6. $4a^2de$.

7. $2c^3d$.

8. $2b^2cd^3e$.

9. $6cd^4e^2$.

10. $7abcd^3e$.

11. $2a + 3c^2$.

12. $4a^2 - 3bc$.

13. $a^2 - 4c^2$.

14. $5ad - 2b^2e$.

15. $4a^2 - 2b^2d^2$.

16. $ab + bc + b^2$.

17. $2ac - c^2 + d^2$.

18. $a^2 + ab + b^2$.

19. $a^2 - 2ab + b^2$.

20. $a^3 + b^3$.

21. $a^4 + a^2b^2 + b^4$.

22. $a^4 - b^4$.

23. $b^3 + c^3$.

24. $b^3 - c^3$.

25. $c^2 + cd + d^2$.

26. $c^2 - cd + d^2$.

27. $2a^2 + b^2 - 5c^2$.

28. $b^2 - 4b + 4$.

29. $2a^2b^2cd^3e$.

30. $a^3 - a^2c + 3de^2$.

13. **Aggregation,** *the process of taking the result of several operations as a whole,* is indicated by the symbols ( ), { }, [ ], read respectively "parenthesis," "brace," "bracket."

Thus, $a(b+c)$, $a\{b+c\}$, $a[b+c]$ all mean that the sum of b and c is to be multiplied by a.

ORDER OF OPERATIONS

14. In any polynomial in which the various signs of operation occur, the plus and minus signs are used to separate terms.

The operations of multiplication and of division are to be performed before those of addition and subtraction.

Thus, $28 \div 4 - 2 \times 3$ contains two terms, a plus sign being understood as preceding 28; $+28 \div 4 - 2 \times 3 =$ first term $(28 \div 4) -$ the second term (2×3);

$$28 \div 4 - 2 \times 3 = (28 \div 4) - (2 \times 3) = 7 - 6 = 1.$$

Were this problem to be given orally in arithmetic, it might be understood: $28 \div 4 = 7$; $7 - 2 = 5$; $5 \times 3 = 15$.

The difference between the algebraic usage and the arithmetical oral statement is to be carefully noticed.

EXERCISE III

If $a=1$, $b=2$, $c=3$, $d=4$, find the value of the following expressions:

1. $a + d \div b$.
2. $2b^2 \times c - 2ab$.
3. $4a^2b^3 - c \times d$.
4. $5ac^2d \div d^2 + 3b^2$.
5. $(3a + 2d) \div 11a + b^2$.
6. $b(d-a)^2$.
7. $(3a - b)(3a + b)$.
8. $(b+a)^2 \div (d-a)$.
9. $3c^2d \div 9bc + b^2$.
10. $4a \times b^2 \div 2a^3d + b^2c^2$.

USE OF LITERAL NOTATION

15. The properties of numbers, whether expressed by integers or by letters, are identical.

The advantage, therefore, of representing numbers by letters lies in the fact that the letter, being a general number, often leads to a general conclusion, expressed as a formula. In arithmetic the principle is taught that interest = principal × time × rate per cent; or that $i = prt$, whatever may be the numerical values of the letters.

Moreover, literal notation is often advantageously used as a sort of shorthand. For example, four times a certain number equals the sum of 60 and three times that number. Expressing the problem in arithmetic,

4 times the number = 60 + 3 times the number.

Expressing the same problem in algebraic language, taking x to represent the number,

$$4x = 60 + 3x.$$

The advantage of the algebraic form of statement lies in the fact that it is merely a statement in shorthand, where x takes the place of the printed words "the number."

EXERCISE IV

1. Express in algebraic form the sum of twice a number, a, and three times that number; the product of five times a number and four times that number.

2. If 1 barrel of flour costs $5, how much will 2 barrels cost? 3 barrels? a barrels? b barrels?

3. If 20 barrels of flour cost $80, what will be the cost of 1 barrel? If a barrels cost $80, what will be the cost of 1 barrel?

4. If a man earns $5 a day, how much will he earn in 4 days? in b days? in c days?

5. The sum of two numbers is 20. If one of the numbers is 8, what is the other number? If one of the numbers is a, what is the other number?

6. If one part of 8 is 6, what is the other part?

7. If one part of a is 2, what is the other part?

8. If one part of 2 is a, what is the other part?

9. If one part of a is x, what is the other part?

10. If one part of x is b, what is the other part?

11. What is the product of two numbers, if one factor is a and the other b?

12. What is the divisor, if the dividend is 27 and the quotient 3? If the quotient is a?

13. The divisor of a certain number is a and the quotient b. What is the dividend?

14. How much is 8 increased by 3? 8 increased by a? a decreased by 4? m decreased by $2x$?

15. By how much does 12 exceed 8? 12 exceed a? a exceed 12? a exceed x?

16. What is the excess of 20 over 11? of 20 over x? of x over 20? of x over y?

17. What is the quotient of 20 divided by the excess of x over 200?

18. If x is the smaller part of 5, what is the larger part?

19. If 10 is the larger part of x, what is the smaller part?

20. How much does 8 lack of 13? of a?

21. How much does a lack of x? of 22?

22. How much does x lack of 13? of m?

23. If A is 30 years old now, how old will he be in 4 years? in x years?

24. If A is now a years old, what would half his age be? three times his age?

25. If A is 18 years old now, how old was he 4 years ago? a years ago?

26. If A is 25 years old now, what was three times his age a years ago?

27. What is the average age of two men, the age of the first being 30, and the second being a?

28. If 3 is the tens' digit of a number of two digits, and a the units' digit, what is the number?

29. If a is the greater part of a number, and the difference between the parts is 4, what is the other part?

30. If a is the smaller part of a number, and if the smaller part lacks 4 of the larger part, what is the larger part?

31. If $2a + 3$ represents a certain number, what represents a fourth of that number?

32. By how much does three times a exceed 22?

33. By how much is the third part of a below 9?

34. If A has x dollars, B twice as much as A, and C as much as A and B together, how much has B? how much has C?

POSITIVE AND NEGATIVE NUMBERS

16. Up to this time the restriction has always been made that the quantity to be subtracted, the **subtrahend**, must be less than the quantity, the **minuend**, from which the subtrahend is to be subtracted. Since 7 is greater than 4, it is possible to subtract 4 from 7. Expressed in arithmetical language, $7 - 4 = 3$. Since 4 is less than 7, it is not possible to subtract 7 from 4. But there is a mathematical necessity for making the process of subtraction always possible.

17. It is evident that a new sort of number must be employed if subtractions are always possible. Numbers hitherto employed can be represented as shown in Figure 1.

Fig. 1.

If a straight line of indefinite length is divided into units of length from zero, the natural numbers can be represented by successive repetition of this unit of length in a direction extending indefinitely towards the right. These numbers will be seen to increase by a unit, counting from left to right; and to decrease by a unit, counting from right to left. The addition of 2 and 3 can be illustrated by counting from zero, two units towards the right, and then by counting three more units from 2 towards the right. The subtraction of 2 from 3 can be illustrated by counting three units from zero towards the right, and then by counting two units from 3 in the *opposite direction* towards the left. If, however, the problem were to subtract a greater from a lesser number,— for example, to subtract 3 from 2,—the process is: count from zero two units towards the right; try to count three units from 2 towards the left; two units can be counted up to zero; the third unit will seem to be beyond zero to the left. It is evident that the counting cannot continue further unless there are new units which are different in character towards the left of zero.

18. An abstract number is used without application to things, as 3, 4, 6; a concrete number is used with application to things, as 3 men, 4 inches, 6 cubic feet. Concrete numbers, or quantities, are often opposite in character. The following are examples of opposite concrete quantities: $20 gain and $15 loss; 2 inches to the right and 4 inches to the left; 10 degrees above zero and 5 degrees below zero; 25 degrees north latitude and 4 degrees south latitude. If two concrete quantities of opposite kinds be combined, the effect of one is to decrease, destroy, or to reverse the state of the other. For example: $20 gain combined with $15 loss destroys the loss of $15 and leaves a gain of $5.

19. Differences that arise from subtracting quantities from lesser quantities are called **negative quantities.** Quantities that are not negative are called **positive quantities.** Positive quantities are represented thus: $+3$, $+5$; while negative quantities are represented thus: -3, -5. The former are read: "positive 3," "positive 5"; the latter are read: "minus (negative) 3," "minus (negative) 5." The signs $+$ and $-$ are also used to indicate the processes of addition and of subtraction. Therefore, for the present, positive numbers will be indicated thus: $(+3)$, $(+5)$; and minus (negative) numbers thus: (-3), (-5).

20. The series of positive and negative numbers can be represented as shown in Figure 2:

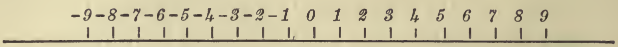

Fig. 2.

Numbers passing from zero in the positive direction increase indefinitely, and numbers passing from zero in the negative direction diminish indefinitely. Positive and negative numbers taken together are called **algebraic numbers.** The sign +, indicating a positive number, is sometimes omitted; the sign −, indicating negative numbers, is *never omitted.* When no sign is written before a number, the plus sign is always understood.

21. The **absolute** or **numerical value** of a number depends upon the number of units contained in the number, no reference being paid to its sign, or its quality of opposition, that is, its direction towards the right or towards the left. For example: $(+7)$ and (-7) are equal in absolute or numerical value.

22. A negative number may be considered as indicating a delayed or postponed subtraction. For example: (-1), since it is a difference obtained by subtracting a quantity one unit greater than a second quantity, indicates that $(+1)$ still remains to be subtracted. Since the addition of (-1) to a second number means the subtraction of $(+1)$ from the second number, by applying the same principle to any negative number, it is evident that *adding a negative number to a second number is equivalent to subtracting a positive number (of the same absolute value as the negative number) from the second number.*

23. 1. Add $(+3)$ and $(+5)$.

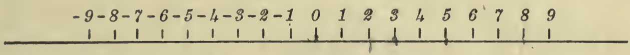

FIG. 2.

The sum of $(+3)$ and $(+5)$ is found by counting, from $(+3)$, five units in the positive direction; and is, therefore, $(+8)$.

2. Add (-3) and (-5).

The sum of (-3) and (-5) is found by counting, from (-3), five units in the negative direction; and is, therefore, (-8).

3. Add $(+5)$ and (-3).

The sum of $(+5)$ and (-3) is found by counting, from $(+5)$, three units in the negative direction; and is, therefore, $(+2)$.

4. Add (-5) and $(+3)$.

The sum of (-5) and $(+3)$ is found by counting, from (-5), three units in the positive direction; and is, therefore, (-2).

If a and b represent any two integers, positive or negative,

$$(+a)+(+b)=+a+b,$$
$$(-a)+(-b)=-a-b,$$
$$(+a)+(-b)=+a-b,$$
$$(-a)+(+b)=-a+b.$$

Zero *may be defined as the sum of that positive and that negative number which are equal in absolute value.*

RULE FOR ADDITION OF TWO NUMBERS

If both numbers are positive, the sum will be positive and equal to the sum of the absolute values of the numbers. If both numbers are negative, the sum will be negative and equal to the sum of the absolute values of the numbers. If one number is positive and the other negative, the absolute value of the sum will be the difference of the absolute values of the numbers, and will be positive or negative according as the number of greater absolute value is positive or negative.

24. Two operations are said to be **inverse** to each other when the effect of one is to undo the other.

Subtraction *is the inverse operation to addition;* and may be defined as the process of *finding from two given numbers a third number so that the sum of the first and the third is equal to the second.*

The process of subtraction depends upon the principle in § 22.

1. Subtract $(+3)$ from $(+5)$.

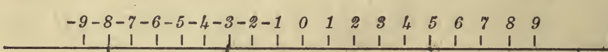

Fig. 2.

The result of subtracting $(+3)$ from $(+5)$ is found by counting, from $(+5)$, three units in the negative direction; and is, therefore, $(+2)$.

2. Subtract (-3) from (-5).

The result of subtracting (-3) from (-5) is found by counting, from (-5), three units in the positive direction; and is, therefore, (-2).

Three units are counted from (-5) *in the positive direction because the subtraction of a negative quantity is equivalent to the addition of its absolute value.*

3. Subtract (-3) from $(+5)$.

The result of subtracting (-3) from $(+5)$ is found by counting, from $(+5)$, three units in the positive direction; and is, therefore, $(+8)$.

4. Subtract $(+3)$ from (-5).

The result of subtracting $(+3)$ from (-5) is found by counting, from (-5), three units in the negative direction; and is, therefore, (-8).

If a and b represent any two integers, positive or negative,

$$(+a)-(+b)=+a-b,$$
$$(-a)-(-b)=-a+b,$$
$$(+a)-(-b)=+a+b,$$
$$(-a)-(+b)=-a-b.$$

Rule for Subtraction of Two Numbers: *Change the sign of the subtrahend and add the result to the minuend.*

EXERCISE V

Find the values of the following indicated operations:

1. $(+3)+(+5)$.
2. $(-3)-(-5)$.
3. $(+3)+(-5)$.
4. $(-5)+(+3)$.
5. $(+7)-(+4)$.
6. $(+6)-(+7)$.
7. $(+9)-(-12)$.
8. $(+12)-(-15)$.
9. $(+3)-(-7)$.
10. $(-1)-(+4)$.
11. $(+4)-(+5)$.
12. $(+6)-(+7)$.
13. $(+4)+(+5)$.
14. $(-6)-(-5)$.
15. $(+4)+(-4)$.
16. $(+4)-(+4)$.
17. $(-7)+(-4)$.
18. $(+8)-(+5)$.
19. $(+7)-(+8)$.
20. $(+12)-(-4)$.
21. $(+8)-(-9)$.

25. *The product of two algebraic numbers is a third number whose absolute value is the product of the absolute values of the two numbers; and is* (1) *positive if both numbers are each positive or negative, and negative* (2) *if one of the numbers is positive and the other negative.*

The operation of finding the product of two numbers is called **multiplication**. To find the product of a and b is to multiply a and b, or to multiply b and a. The product of a and b is indicated thus: $(a \times b)$, or (ab), or ab.

Since the arithmetical product of the absolute values of the factors is not determined by the order of the factors, by definition the product of a and b is the same as the product of b and a. If ab indicates the product of a and b, and ba indicates the product of b and a, $ab = ba$.

$$(+5) \times (+3) = (+15),$$
$$(-5) \times (-3) = (+15),$$
$$(+5) \times (-3) = (-15),$$
$$(-5) \times (+3) = (-15).$$

In general,
$$(+a) \times (+b) = (+ab),$$
$$(-a) \times (-b) = (+ab),$$
$$(+a) \times (-b) = (-ab),$$
$$(-a) \times (+b) = (-ab).$$

The Law of Signs in Multiplication: *Like signs give positive, and unlike signs give negative products.*

26. *The absolute value of the quotient of two numbers is the quotient of the absolute values of the numbers; and is* (1) *positive if both numbers are each positive or negative, and is* (2) *negative if one of the numbers is positive and the other negative.*

The operation of finding the quotient of two numbers is called **division**. Division is the operation inverse to multiplication.

Since, § 25,

$$(+5)\times(+3)=(+15),\ (+15)\div(+3)=(+5);$$
$$(-5)\times(-3)=(+15),\ (+15)\div(-3)=(-5);$$
$$(+5)\times(-3)=(-15),\ (-15)\div(-3)=(+5);$$
$$(-5)\times(+3)=(-15),\ (-15)\div(+3)=(-5).$$

In general,

$$(+ab)\div(+b)=(+a),$$
$$(-ab)\div(-b)=(+a),$$
$$(+ab)\div(-b)=(-a),$$
$$(-ab)\div(+b)=(-b).$$

The Law of Signs in Division is: *Like signs give positive, and unlike signs give negative quotients.*

EXERCISE VI

Find the values of the following indicated operations:

1. $(+3)(-2)$.
2. $(-4)(-5)$.
3. $(-8)(-3)$.
4. $(-9)(-4)$.
5. $(+6)(-4)$.
6. $(-7)(+3)$.
7. $(-5)(-6)$.
8. $(-8)(+3)$.
9. $(-9)(-5)$.
10. $(+6)(+7)$.
11. $(-9)\div(+3)$.
12. $(-8)\div(+4)$.
13. $(+10)\div(+5)$.
14. $(-10)\div(+2)$.
15. $(-12)\div(-4)$.
16. $(+12)\div(-12)$.
17. $(+15)\div(+3)$.
18. $(-16)\div(-8)$.

27. The sign $+$ may be used, § 3, to denote addition, and, § 19, to indicate positive numbers. In practice, however, the sign $+$ is omitted in indicating positive numbers. Thus, $(+4)$, 4, are identical. Henceforth, in this book, positive numbers will be represented by the absence of sign. Thus, 4 means positive 4, and $+4$ means the addition of positive 4.

The sign $-$ may be used, § 3, to denote subtraction, and, § 19, to indicate negative numbers. In conformity with general usage, negative numbers will be henceforth represented by numbers preceded by the sign $-$. Thus, (-5) and -5 are identical. The sign $-$, denoting a negative number, is never omitted.

EXERCISE VII

Simplify the following:

1. $(4) + (3)$.
2. $(4) - (3)$.
3. $(4) + (-3)$.
4. $(4) - (-3)$.
5. $(-4) + 3$.
6. $4 + 3$.
7. $4 - 3$.
8. $4 - (-3)$.
9. $4 + (-3)$.
10. $8 + (-2)$.
11. $(-2) + 8$.
12. $7 - (-5)$.
13. $7 + (5)$.
14. $7 + 5$.
15. $7 + (-5)$.
16. $(-5) + (-7)$.
17. $(4) \cdot (-3)$.
18. $(-5) \cdot (2)$.
19. $(-6) \cdot (-5)$.
20. $(-4) \cdot (4)$.
21. $(-9) \cdot 4$.
22. $(-12) \cdot (-3)$.
23. $8 \cdot 5$.
24. $-12 \cdot 3$.
25. $24 \div (-3)$.
26. $-36 \div (-6)$.
27. $-54 \div 18$.
28. $-39 \div (-13)$.
29. $-65 \div 5$.
30. $50 \div (-25)$.

CHAPTER II

ADDITION AND SUBTRACTION

28. The addition of two numbers, or quantities, whether positive or negative, has already been illustrated, and the rule given, in § 23.

The sum of three quantities is the sum of the first two quantities and the third quantity; similarly, the sum of four quantities is the sum of the first three and the fourth quantity.

Thus, $2a^2 + 3bc + c^2 + m = (2a^2 + 3bc) + c^2 + m$

$$= [(2a^2 + 3bc) + c^2] + m.$$

29. Addition is subject to two laws (whose truth is assumed), the first of which is the **Commutative Law**, — *the sum of two or more numbers is independent of the order in which the addition is performed.*

Thus, $4 + 5 = 5 + 4$; or, in general, $a + b = b + a$.

Addition is also subject to the **Associative Law**, — *the sum of three or more numbers is independent of the way in which successive terms are grouped in the process of addition.*

Thus, $4 + 5 + 2 = (4 + 5) + 2 = 9 + 2 = 11,$

and $4 + 5 + 2 = 4 + (5 + 2) = 4 + 7 = 11,$

in general, $a + b + c = (a + b) + c = a + (b + c).$

The Associative Law gives a short method for combining positive and negative terms.

Thus, the sum of the positive terms of $22 - 11 + 12 - 5 + 6 - 17$ is 40; the sum of the negative terms of $22 - 11 + 12 - 5 + 6 - 17$ is -33. Hence $22 - 11 + 12 - 5 + 6 - 17 = (22 + 12 + 6) + (-11 - 5 - 17) = 40 - 33 = 7$.

EXERCISE VIII

Find the sum of the following numbers:

1. $20 - 3 + 7 - 8$.
2. $16 - 22 + 12 - 5$.
3. $1 - 12 + 13 - 7$.
4. $8 - 9 - 10 - 11$.
5. $20 - 14 - 13 + 27$.
6. $30 - 14 - 16 + 5$.
7. $27 - 18 - 17 + 8$.
8. $6 - 22 + 33 + 12 - 6$.
9. $24 - 8 - 13 + 7 + 5 - 16$.
10. $6 + 8 - 21 + 17 - 8 - 5 - 13$.
11. $-6 + 5 - 19 + 13 - 20 + 4 + 7$.
12. $9 - 8 + 15 + 3 - 19 - 11 - 6$.
13. $12 - 8 - 7 - 14 + 15 - 13 + 20 + 5$.

ADDITION OF LIKE TERMS

30. Like terms can be combined into a single term. Just as in arithmetic, the sum of 4 bushels and 3 bushels is indicated by 4 bu. + 3 bu. = 7 bu., so, in algebra, $3a^2b + 5a^2b = 8a^2b$. Hence, **to add like terms**, *add their numerical coefficients, and prefix this sum as the numerical factor of the literal part.*

Thus, $\quad 3a^2b + 5a^2b + 2a^2b = 10a^2b,$

and $\quad -2a - 3a - 5a = -10a,$

and $\quad 2b^2 - 3b^2 + 10b^2 = 9b^2.$

EXERCISE IX

Find the sum of the like terms in the following:

1. $2a + 3a - 4a.$
2. $6m + 5m - 7m.$
3. $2c - 3c - 4c.$
4. $7a^2 - 4a^2 + 2a^2.$
5. $3ab - 2ab - 3ab + 12ab.$ 10 ab
6. $6x^2 - 4x^2 + 5x^2 - x^2.$
7. $11xy - 7xy + 4xy - 3xy.$
8. $12bc - 3bc + 6bc - 2bc.$
9. $-x^2y - 2x^2y - 3x^2y + 5x^2y.$
10. $10b^2c - 3b^2c - 5b^2c + 4b^2c.$
11. $2b^2c^2 - 3b^2c^2 - 7b^2c^2 - 6b^2c^2.$ $= -14b^2c^2$
12. $4ab - 5ab + 7ab - 11ab - 12ab.$
13. $5mn - 4mn - 6mn - 7mn - mn + 2mn.$
14. $6ab - 7ab - 2ab - ab + 12ab + 22ab.$
15. $x^2 - 11x^2 - 13x^2 + 7x^2 - 5x^2 - 4x^2 + 7x^2 - 9x^2.$
16. $mn + 2mn - 3mn - 7mn + 13mn - 14mn.$
17. $-ab + 7ab - 13ab + 12ab - 7ab - 15ab.$
18. $x^2 - 3x^2 - 4x^2 + 7x^2 - 9x^2 - 11x^2 - 4x^2 + 5x^2.$
19. $y^2 - 11y^2 - 13y^2 + 5y^2 - 4y^2 + 3y^2 - 22y^2.$
20. $a^2 - 3a^2 + 4a^2 - 6a^2 - 7a^2 - 32a^2 + 50a^2.$
21. $-ab + 4ab - 7ab + 5ab - 13ab + 17ab - 56ab.$
22. $a - 17a + 33a - 44a + 109a - 64a + 32a.$
23. $x^2y - 3x^2y + 5x^2y + 22x^2y - 17x^2y + 37x^2y.$
24. $-17b^2 - 33b^2 + 105b^2 + 62b^2 - 109b^2 - 56b^2.$
25. $6ab - 17ab + 33ab - 512ab + 203ab + 1002ab.$

ADDITION OF POLYNOMIALS

31. Let $A = b + c + d$, and let $E = m - n - p$. The addition of these two polynomials is indicated thus: $A + E = (b + c + d) + (m - n - p)$. (1) The parenthesis may be dropped, and the equivalent expression may be written: $A + E = b + c + d + m - n - p$. (2) Expression (1) indicates the sum of the numerical values of the polynomials $b + c + d$ and $m - n - p$; the numerical value of expression (2) is independent of the order of the terms, and may be considered as the sum of the numerical values of the first three, and the last three terms, which is exactly the result of expression (1). Hence expressions (1) and (2) are equivalent. Whence is the following **rule for the addition of polynomials**: *Write the polynomials in order, retaining the sign of each term.*

If the polynomials contain like terms, these terms should be united.

1. Add $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$.

The work will be simplified by arranging like terms under like terms before combining.

$$\begin{array}{r}
a^2 + 2ab + b^2 \\
a^2 - 2ab + b^2 \\
\hline
2a^2 \qquad\quad + 2b^2
\end{array}$$

If the sum of more than two polynomials is required, the process is similar.

2. Add $a^2 - 3ab$, $6ab - b^2$, $11a^2 + 3ab - 12b^2$.

$$\begin{array}{rrr}
a^2 & -\ 3ab & \\
 & 6ab & -\ \ b^2 \\
11a^2 & +\ 3ab & -\ 12b^2 \\
\hline
12a^2 & +\ 6ab & -\ 13b^2
\end{array}$$

32.* The process of finding the sum of several polynomials containing like terms, may be still further abridged by the method of **Detached Coefficients**; that is, by omitting the literal parts of several like terms.

Thus, in finding the sum of $2m + 3n - 6p$, $11m - 4n + 2p$, and $-7m + n - p$, omit all literal factors except in the first line and arrange the terms thus:

$$\begin{array}{rrr} 2m & +3n & -6p \\ 11 & -4 & +2 \\ -7 & +1 & -1 \\ \hline 6m & & -5p \end{array}$$

The advantage of this method is simply in the labor saved by omitting the literal factors.

CHECKS FOR OPERATIONS

33. It is often useful to test, or check, the results obtained in the processes of addition, subtraction, multiplication, and division with the results obtained in the same operations obtained from the numerical values.

1. $(3x^3 - 2x^2 + 5x + 1) + (3x^2 - 2x + 3) = 3x^3 + x^2 + 3x + 4.$

In each of the above expressions take $x = 1$, then

$$(3 - 2 + 5 + 1) + (3 - 2 + 3) = 3 + 1 + 3 + 4,$$

or, $$7 + 4 = 11,$$

$$11 = 11.$$

2. $(5a^2 - 6b^2 - 3a^2b + m) + (2b^2 + 3a^2b - 4m) = 5a^2 - 4b^2 - 3m.$

In each of the above expressions take $a = b = m = 1.$

$$(5 - 6 - 3 + 1) + (2 + 3 - 4) = 5 - 4 - 3,$$

or, $$-3 + 1 = -2,$$

$$-2 = -2.$$

EXERCISE X

Add the following polynomials:

1. $2a + 3b + 2c,\ a + b - c.$
2. $3b + 2c - d,\ 2b - 5c - d.$
3. $m + n - 7p,\ -m - n + 7p.$
4. $ab + bc + ab^2c,\ 2ab - 3bc - 4ab^2c.$
5. $2ab + a^2 + 3b^2,\ 2ab + 3a^2 - 4b^2.$
6. $4xy - x^2 + 4y^2,\ 6xy - 5x^2 - 7y^2.$
7. $3a^2 - 5ab + 7c^2,\ 2a^2 - 6ab + 4c^2.$
8. $x^2 - xy + y^2,\ 3x^2 - 6xy - 4y^2.$
9. $7m - 2n + p + 6q,\ 6m + 5n - 6p + 2q.$
10. $4x - 2y + 3z - 8,\ 9x - 5y - 8z + 6.$
11. $a^2 + b^2,\ a^2 - 3ab + b^2,\ 2ab - 2b^2.$
12. $m^2 + mn + p,\ 3m^2 - 2mn - p^2,\ 6m^2 - 3mn + 2p^2.$
13. $5m - 10n + np,\ m - 7n + np,\ 6m + 12n - 4np.$
14. $x^2 - xy + y^2,\ x^2 + 2xy + y^2,\ -x^2 - 4xy - 4y^2.$
15. $12a^2 - 11ab + 6b^2,\ -5a^2 + 2ab - 3b^2,\ 5a^2 + 8ab + 4b^2.$
16. $6m^2 - 3mn + 5n^2,\ 5m^2 + 8mn - 4n^2,$
    $-10m^2 + 5mn + 12n^2.$
17. $ab - ac + ad,\ ac - ab + ad,\ ad - ac + ab.$
18. $m^2 - n^2 + p^2,\ n^2 - m^2 - p^2,\ p^2 - m^2 - n^2.$
19. $a^2 - ab + b^2,\ b^2 - a^2 + ab,\ b^2 + ab - a^2.$
20. $2a - 3c^2 + 4d,\ b^2 - 3c^2 + 2d,\ b^2 - a - 2a^2.$
21. $5x^2 - 11xy + 12y^2,\ x^2y^2 - 3xy + y^2,\ x^2 - y^2.$
22. $12b^2 - 10bc + 15d,\ a^2 - 10b^2 + 11bc,\ d - 14b^2 - 11a^2$
23. $22x^2 - 3by + 4y^2,\ 15by - 4y^2 - 2x^2,\ 22by - y^2 + 9x^2$
24. $6a^2b - 7a^2c - 5c^2a + 8b^2a,\ 11c^3 + 8c^2a + 6a^2c - 9b^2a$
25. $9x^3y - x^4 - 12x^2y^2 - 14xy^3 + y^4,\ x^4 - 6x^3y + 10x^2y$
    $-2y^4.$

SUBTRACTION

34. The subtraction of two quantities has already been defined, and the rule given in § 24.

SUBTRACTION OF LIKE TERMS

35. Just as in arithmetic the process of subtracting 3 barrels from 4 barrels is indicated by 4 bbls. – 3 bbls. = 1 bbl., so, in algebra, the subtraction of $3a$ from $4a$ is indicated $4a - 3a = a$. But, § 19, $4a$ can be subtracted from $3a$, and is indicated $3a - 4a = -a$; that is, $-a$ must evidently be added to $4a$ to make $3a$.

Similarly, $2a - (-5a) = 7a$; $-5a - (6a) = -11a$.

In § 22 it was shown that adding a negative number is the same as subtracting that positive number whose absolute value is identical. Algebraic subtractions are usually changed into algebraic additions. These operations are equivalent in results, and the change of an algebraic subtraction of a negative number into an algebraic addition is to be interpreted as illustrated in § 24.

EXERCISE XI

Subtract the first from the second, and also the second from the first quantity of the following:

1. $2b, b$.

2. $-b, 2b$.

3. $-a, -2a$.

4. $-a, 2a$.

5. $a, 2a$.

6. $a, -2a$.

7. $3m, 4m$.

8. $7c, 4c$.

9. $x^2y, -3x^2y$.

10. $7a^2b, -8a^2b$.

11. $-a^2y, -3a^2y$.

12. $5x^2y^2, -13x^2y^2$.

SUBTRACTION OF POLYNOMIALS

36. Let $A=b+c-d+e$, and $F=m-n+p-q$. The subtraction of the second from the first polynomial is indicated thus:

$$A-F=(b+c-d+e)-(m-n+p-q). \qquad (1)$$

Or,

$$\begin{array}{r} A=(b+c-d+e) \\ F=(m-n+p-q) \\ \hline A-F=b+c-d+e-m+n-p+q. \end{array} \qquad (2)$$

The quantity $A-F$ must evidently be added to F to produce A; and the quantity $b+c-d+e-m+n-p+q$ must evidently be added to $m-n+p-q$ to make $b+c-d+e$. Expressions (1) and (2) are identical; hence, **to subtract a polynomial from a second polynomial**: *Write the first polynomial after the second, changing all the signs of the terms of the first polynomial; combine like terms.*

1. Subtract $2a^2-5ab-3b^2$ from $a^2-2ab+b^2$.

$$\begin{aligned}(a^2-2ab+b^2)&-(2a^2-5ab-3b^2)\\ &=a^2-2ab+b^2-2a^2+5ab+3b^2,\\ &=-a^2+3ab+4b^2.\end{aligned}$$

Or,

$$\begin{array}{r} a^2-2ab+b^2 \\ 2a^2-5ab-3b^2 \\ \hline -a^2+3ab+4b^2 \end{array}$$

The number $-a^2$ must evidently be added to $2a^2$ to make a^2; $3ab$ to $-5ab$ to make $-2ab$; $4b^2$ to $-3b^2$ to make b^2.

The work can be still further abridged by the method of Detached Coefficients.

$$\begin{array}{r} a^2-2ab+b^2 \\ 2\quad -5\quad -3 \\ \hline -a^2+3ab+4b^2 \end{array}$$

2. Subtract $m^2 - 2m + 1$ from $3m^2 - 7m + 1$.

$$\begin{array}{l}
3m^2 - 7m + 1 \\
\underline{1 \quad -2 \quad +1} \\
2m^2 - 5m
\end{array}$$

$$(3m^2 - 7m + 1) - (m^2 - 2m + 1) = 2m^2 - 5m.$$

In each of the above expressions take $m = 1$; then

$$(3 - 7 + 1) - (1 - 2 + 1) = 2 - 5,$$

or

$$(-3) - (0) \qquad = -3,$$

$$-3 = -3.$$

The results check, and the subtraction is therefore correct.

EXERCISE XII

Subtract the first from the second, and also the second from the first expression of the following:

1. $x + 5,\ x + 3.$
2. $x - 5,\ x - 3.$
3. $x + 5,\ x - 3.$
4. $x - 5,\ x + 3.$
5. $5 + x,\ 3 + x.$
6. $5 - x,\ 3 - x.$
7. $5 + x,\ 3 - x.$
8. $5 - x,\ 3 + x.$
9. $a,\ a + 1.$
10. $a,\ a - b.$
11. $4x,\ y + 5x.$
12. $a - b,\ 5.$
13. $7,\ 2a + b.$
14. $-x,\ -y - 3.$
15. $a - b,\ b + a.$
16. $3 - n,\ n + 1.$
17. $a - 8,\ b - 8.$
18. $4 - n,\ n + 4.$
19. $-a + 8b,\ -b - 7c.$
20. $a + b - c,\ 2a + b - c.$
21. $5a + 2b + 6,\ 7a - b - 8.$
22. $4a^2 - 7b^2 + 7,\ 4a^2 + 7b^2 - 1.$
23. $6m - 5n - p,\ -m - 8n - p - q.$
24. $-3k + m - 5n + 4p,\ 9k - m + 6n + 7p$

AGGREGATIONS

37. *An aggregation symbol preceded by the plus sign may be neglected*, because the expression within the aggregation symbol is to be added to the preceding number, which number is sometimes 0.

$$a + (b + c) = a + b + c;\ 0 + (a - b + c) = a - b + c.$$

An aggregation symbol preceded by the minus sign can be removed by changing the sign of every term contained within it; because the indicated process of subtraction is the addition of the several terms changed in sign but having the same absolute value by § 24.

Thus,

$$7a - (a - 2b - 3c) = 7a - a + 2b + 3c = 6a + 2b + 3c.$$

38. By § 37, *the terms of a polynomial can be enclosed by a symbol of aggregation which is preceded by the plus sign without change of sign; and can be enclosed by a symbol of aggregation preceded by the minus sign if the sign of every term be changed.*

$$x^2 - xy + y^2 = +(x^2 - xy + y^2);$$

$$-3x^2 + 4xy - y^2 = -(3x^2 - 4xy + y^2).$$

39. An aggregation enveloping several aggregations can be removed by the foregoing principles. Either the inner or the outer symbol may be removed first.

Thus, simplify $a - [a - \{2a - (3a - b)\}]$.

$$\begin{aligned} a - [a - \{2a - (3a - b)\}] &= a - [a - \{2a - 3a + b\}], \\ &= a - [a - 2a + 3a - b], \\ &= a - a + 2a - 3a + b, \\ &= -a + b; \end{aligned}$$

or, removing first the outer symbol,

$$\begin{aligned} a-[a-\{2a-(3a-b)\}] &= a-a+\{2a-(3a-b)\}, \\ &= a-a+2a-(3a-b), \\ &= a-a+2a-3a+b, \\ &= -a+b. \end{aligned}$$

EXERCISE XIII

Simplify the following expressions:

1. $(a-b)-[(c-d)-(e-f)]+(g-h)$.
2. $(a-b)+c-[(d+e)-f-(g-h)]$.
3. $a-[b-(c-d)]-[e+(f-g)-h]$.
4. $a-[b-(c+d)+e]-(f-g)+h$.
5. $[(a-b)+(c-d)]-[(e+f)+(g-h)]$.
6. $[(a+b)-(c+d)]+[(e-f)-(g+h)]$.
7. $[(a-b)-(c-d)]-[(e-f)-(g-h)]$.
8. $[(a+b)+(c-d)]+[(e-f)-(g-h)]$.
9. $(3x+5y)-[(7x-2y)-(8x-4y)]+(x-y)$.
10. $(7m-4)+3p-[(8g+3p-2)+5m-(3g-p)]$
11. $a-[b-(c-[d-(e-f)-g]+h)-k]$.
12. $a-[2a-(\{3a-7a\}-3c)]$.
13. $a-[-(-\{-3a-(2a-b)\})]$.
14. $m-[-n-\{-3n-(4m-6n)\}]$.
15. $a-[\{b-(c+d)\}+\{e+(f-g+h)-(k+l-m)\}$ $-(n-v)]$.
16. $a-(b+\{c+2x\}-\{y-z\})$.
17. $x-(2x-y-[3x-2y-(4x-3y)])$.

CHAPTER III

MULTIPLICATION AND DIVISION

MULTIPLICATION

40. In § 25 there was given a definition of the product of two algebraic numbers; the rule for finding the product; and a statement of the Law of Signs.

41. The product of three algebraic quantities is the product of the first two quantities multiplied by the third.

Thus, $$a \cdot b \cdot c = (ab) \cdot c.$$

The product of four algebraic quantities is the product of the first three quantities multiplied by the fourth; and so on.

Thus, $$a \cdot b \cdot c \cdot d = (ab) \cdot c \cdot d = (abc) \cdot d.$$

The absolute value of the product of three or more algebraic quantities is the product of their absolute values, and is positive when it contains an even number of negative factors, and negative when it contains an odd number of negative factors.

Thus, the product of $-a$, b, $-c$, and d is $abcd$; and the product of $-a$, $-b$, $-c$, and d is $-abcd$.

Since 0, § 23, $= a - a$, $a(a-a) = a^2 - a^2 = 0$; $a \cdot 0 = 0$.

42. By definition, § 7, $a^3 = aaa$, and $a^2 = aa$.

Therefore, $a^3 \times a^2 = aaa \times aa = aaaaa = a^5 = a^{2+3}$.

$$a^4 \times a^5 = aaaa \times aaaaa = aaaaaaaaa = a^9 = a^{4+5}.$$

In the above examples the exponents are positive whole numbers or integers. Restricting, for the present, exponents to positive integers, the product of any two powers of the same letter may be found thus:

$$a^m = aaa \text{ taken to } m \text{ factors},$$

$$a^n = aa \text{ taken to } n \text{ factors},$$

therefore,

$$\begin{aligned} a^m \cdot a^n &= (a \text{ taken to } m \text{ factors}) \times (a \text{ taken to } n \text{ factors}), \\ &= a \text{ taken to } (m+n) \text{ factors}, \\ &= a^{m+n}. \end{aligned}$$

In the same way, $a^x \cdot a^y \cdot a^z = a^{x+y+z}$

The principle just shown is called the **Index Law**, — *the exponent of the product of two powers of the same letter is the sum of the exponents of the factors.*

43. The process of multiplication is subject to three fundamental laws (whose truth is assumed), of which the first is the **Commutative Law**, — *the product of two or more quantities is independent of the order of the factors.*

Thus, $2 \cdot 3 = 3 \cdot 2$; and, in general, $a \cdot b = b \cdot a$.

44. Multiplication is also subject to the **Associative Law**, — *the product of three or more quantities is independent of the order in which the factors are grouped in finding the partial products.*

Thus, by § 41, $5 \cdot 4 \cdot 3 = (5 \cdot 4) \cdot 3 = 20 \cdot 3 = 60$,

and, by § 41, $5 \cdot 4 \cdot 3 = 5\,(4 \cdot 3) = 5 \cdot 12 = 60$,

and, in general, $a \cdot b \cdot c = (a \cdot b) \cdot c = a\,(b \cdot c)$.

MULTIPLICATION OF MONOMIALS

45. 1. Find the product of $2\,abx^2y$ and $5\,a^2bx^3$.

By the associative law,

$$(2\,abx^2y)\,(5\,a^2bx^3) = 2 \cdot a \cdot b \cdot x^2 \cdot y \cdot 5 \cdot a^2 \cdot b \cdot x^3,$$

by the commutative law, $= 2 \cdot 5 \cdot a \cdot a^2 \cdot b \cdot b \cdot x^2 \cdot x^3 \cdot y,$

by the associative law, $= 10\,a^3b^2x^5y.$

2. Find the product of $-3\,x^2$, $-5\,x^2y^2$, and $4\,xyz$.

By the associative law,

$$(-3\,x^2)\,(-5\,x^2y^2)\,(4\,xyz) = (-3) \cdot x^2\,(-5) \cdot x^2 \cdot y^2\,(4) \cdot x \cdot y \cdot z,$$

by the commutative law, $= (-3)\,(-5)\,(4) \cdot x^2 \cdot x^2 \cdot x \cdot y^2 \cdot y \cdot z,$

by the associative law and law of signs,

$$= 60\,x^5y^3z.$$

Hence, **the product of several monomials is found** *by annexing to the product of the numerical factors each literal factor, giving to it an exponent which is the sum of the exponents of this factor in the monomials.*

EXERCISE XIV

Perform the multiplications indicated:

1. $3\,xy \cdot -xy^2.$
2. $-a \cdot -a^2b \cdot -b^2.$
3. $ab \cdot 3\,ac \cdot -5\,bc.$
4. $2\,mn \cdot -3\,m^2 \cdot -4\,n^3.$
5. $2\,c \cdot 4\,xy \cdot 7\,ab.$
6. $2\,a^2 \cdot 4\,ax \cdot -11\,a^4x^3.$
7. $-m^2n^7 \cdot -m^3n \cdot -7\,mn^6.$
8. $4\,a^2b \cdot 6\,xy^2 \cdot -15\,a^4b^2x^3y^4.$
9. $a^2b \cdot -5\,b^3x \cdot -3\,a^2bx^4.$
10. $11\,ac \cdot -14\,bc^3 \cdot -13\,a^4c^6.$
11. $ab \cdot -ac \cdot -bc \cdot -cd \cdot -abcd.$
12. $2\,a^7 \cdot -3\,a^4 \cdot -3\,a^3 \cdot -3\,a^2 \cdot -2\,a.$

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

46. An **entire expression** is an expression no term of which contains a literal quantity in its denominator.

Thus, $\frac{3}{4}a^2 + 2\,ab + b^2$ is entire.

A **fractional expression** is an expression in which at least one term has a literal quantity in the denominator.

Thus, $\dfrac{3}{4\,a^2} + 2\,ab + b^2$ is fractional.

47. The **degree of a monomial** is found by taking the sum of the exponents of the literal factors.

Thus, $3\,a^3b^2c$ is of the sixth degree; and $13\,x$ is of the first degree.

The **degree of a polynomial** is found by taking the sum of the exponents in that term in which the sum is greatest.

Thus, $a^3 - 3\,ab^7 + d^3e^4$ is of the eighth degree because the sum of the exponents of ab^7 is eight.

A **homogeneous expression** is one in which the degree of the several terms is identical.

Thus, $a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$ is a homogeneous expression of the fourth degree.

48. The definition of the product of two numbers, § 25, applies to two expressions in the form $3(2+4)$.

By definition, § 25,

$$3(2+4) = 3 + 3 + \text{etc. to } (2+4) \text{ terms},$$

by associative law, § 29, $= (3 + 3 + \text{ etc. to } 2 \text{ terms}) + (3 + 3 + \text{etc. to } 4 \text{ terms}),$

by definition, § 25, $= 3\cdot 2 + 3\cdot 4,$

similarly, $a(b+c) = ab + ac.$

Note: The above law is assumed to hold for positive fractions and negative numbers.

By the commutative law, $a(b+c) = (b+c)a$,

by the commutative law, $ab + ac = ba + ca$,

therefore, $a(b+c) = (b+c)a = ab + ac = ba + ca.$

The statement of the foregoing principle is the third law of multiplication, the **Distributive Law**, — *the product of a (entire) polynomial by a monomial is found by multiplying each term of the polynomial by the monomial and adding the products thus obtained.*

1. Find the product of $2x^2 - 5xy - 2y^2$ by $3x$.

$$3x(2x^2 - 5xy - 2y^2) = (2x^2 - 5xy - 2y^2) \cdot 3x,$$
$$= 6x^3 - 15x^2y - 6xy^2.$$

The work may also be arranged thus:

$$\begin{array}{l} 2x^2 - 5xy - 2y^2 \\ 3x \\ \hline 6x^3 - 15x^2y - 6xy^2 \end{array}$$

EXERCISE XV

Perform the indicated multiplications:

1. $c(2a+b)$.
2. $p(3m-4n)$.
3. $3x(x-7y)$.
4. $5a(a-b)$.
5. $3n(4p-q)$.
6. $xy(x+y)$.
7. $a(a-2b+3c)$.
8. $kmn(4k-8m-7n)$.
9. $6(2a+5b-9c)$.
10. $(-1)(-5a+6b-c)$.
11. $(a-7b+c)(n)$.
12. $(11a-8b-5c)(-3y)$.
13. $(-5ab-3bc+4cd)(-6ad)$.

MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

49. The product of two polynomials is expressed thus, $(a+b)(c+d)$.

By definition, § 25,

$$(a+b)(c+d) = (c+d) + (c+d) + \text{etc. to } (a+b) \text{ terms},$$

by associative law, § 29,

$$= [(c+d) + (c+d) + \text{etc. to } a \text{ terms}] + [(c+d) + (c+d) + \text{etc. to } b \text{ terms}],$$

by definition, § 25, $\quad = (c+d)a + (c+d)b,$

by distributive and by commutative laws,

$$= ac + ad + bc + bd.$$

From the foregoing principle is derived the following **Rule for the Product of Any Polynomials:** *Multiply each term of the multiplicand by each term of the multiplier and add the successive products.*

1. Find the product of $2x^2 - 3xy + 4y^2$ and $x - y$.

Arrange the work thus:

$$\begin{array}{llll}
2x^2 & -3xy & +4y^2 & \\
x & -y & & \\
\hline
2x^3 & -3x^2y & +4xy^2 & \\
 & -2x^2y & +3xy^2 & -4y^3 \\
\hline
2x^3 & -5x^2y & +7xy^2 & -4y^3
\end{array}$$

The product of $2x^2 - 3xy + 4y^2$ and x is written in the third line, and the product of $2x^2 - 3xy + 4y^2$ and $-y$ in the fourth line. Like terms are arranged in columns so that they may be united.

The product of three or more polynomials is found by taking the product of the first two by the third, and so on.

50. A polynomial is said to be **arranged** with reference to a letter when the powers of that letter constantly increase or decrease. Any letter can be selected as the **letter of order.** If the exponents of the letter increase, the polynomial is said to be arranged in **ascending order.**

Thus, $x^3+3\,xy^2-3\,x^2y-y^3$, arranged with reference to x, in descending order, is $x^3-3\,x^2y+3\,xy^2-y^3$; and the same expression, arranged with reference to y, in descending order, is $-y^3+3\,xy^2-3\,x^2y+x^3$.

51.* The application of the method of Detached Coefficients will be facilitated if all of the terms of the expressions to be multiplied are arranged with reference to a single letter in the same order, *the coefficients of missing powers of the letter of arrangement being represented by zero.*

Multiply $a^3+a^2b+ab^2+b^3$ by a^2-b^2.

$$\begin{array}{l}
1+1+1+1 \\
\underline{1+0-1} \\
1+1+1+1 \\
\underline{-1-1-1-1} \\
1+1+0+0-1-1=a^5+a^4b-ab^4-b^5.
\end{array}$$

The result obtained may be checked by substituting $a=b=1$.

$$(1+1+1+1)\,(1-1)=1+1-1-1;\ \ 4\cdot 0=0.$$

Detached coefficients are most advantageously employed in finding the products of homogenous expressions.

The above example also illustrates the following principle: *The product of two homogeneous expressions is a homogeneous expression whose degree is the sum of the degrees of the multiplicand and multiplier.*

EXERCISE XVI

Perform the indicated multiplications:

1. $a+2$ by $a+3$.
2. $a-3$ by $a-4$.
3. $m+4$ by $m-3$.
4. $n-2$ by $n+5$.
5. $x-2$ by $-x+3$.
6. $-x-3$ by $x+4$.
7. $-x+3$ by $-x+6$.
8. $-x+2$ by $x-7$.
9. $a-4$ by $2a+1$.
10. $2a+5$ by $a-4$.
11. $2a-7$ by $3a+6$.
12. $3a+4$ by $-3a+5$.
13. x^2+xy+y^2 by $x-y$.
14. x^2-xy+y^2 by $x+y$.
15. $2a^2+ab+b^2$ by $2a-5$.
16. $3x^2-6xy+9$ by $-x+4y$.
17. a^2+ab+b^2 by a^2-ab+b^2.
18. $x^4+x^2y^2+y^4$ by $x^4-x^2y^2+y^4$.
19. $a^3+3a^2b+3ab^2+b^3$ by $a+b+c$.
20. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$ by $a-b$.
21. $3a^2-4ab+5b^2$ by $2a^2-3ab+2b^2$.
22. $5x^3-y^3+11x^2y-4xy^2$ by $2x^2+3y^2-xy$.
23. $10a^2b^2-13a^4-6a^3b+6ab^3+3b^4$ by x^2+y^2-xy.
24. $a^4-7a^2b-8ab^2-11a^3b$ by $a^2x-5ax^2+7a^3-x^3$.
25. $m^3-11ab^3-4xy^3-3n^3$ by $n^3-7-2ab^3+m^3$.
26. $a^2-ab+x-y$ by $a^2-ab-x+y$.
27. x^3-2x^2+x-3 by x^3-2x^2+x-3.
28. $2x^3-3x^2-x+2$ by $2x^3-3x^2+x-2$.
29. $5a^3-4a^2b+2ab^2-b^3$ by $2ab^2-b^3-5a^3+4a^2b$.
30. $x^3+y^3+z^3$ by $x^2+y^2+z^2-xy-xz-yz$.

DIVISION

52. In § 26 there was given a definition of division of two algebraic numbers, the rule for finding the quotient, and a statement of the law of signs.

If the indicated divisor be zero, since the product of a finite number and 0 is 0, it follows that the quotient cannot be found; that is, 0 cannot be used as a divisor.

If the dividend be zero, since $a \cdot 0 = 0$, $\frac{0}{a}$ may be defined as 0.

53. Since, by § 42, $a^m \cdot a^n = a^{m+n}$,

by § 26, $$\frac{a^{m+n}}{a^m} = a^n = a^{m+n-m},$$

and, by § 26, $$\frac{a^{m+n}}{a^n} = a^m = a^{m+n-n}.$$

This principle is called the **Index Law**,—*the exponent of the quotient of two powers of the same letter is the exponent of the dividend minus the exponent of the divisor.*

NOTE. m and n are, as in § 42, positive integers only; and m and n are restricted to such values that m is not less than n. A full discussion will be found in Chapter XVII.

DIVISION OF MONOMIALS

54. From §§ 26, 45, and 53, **the quotient of two monomials is found** *by annexing to the quotient of the numerical factors each literal factor whose exponent is its exponent in the dividend minus its exponent in the divisor.*

1. Divide $8x^3$ by $2x^2$.

$$\frac{8x^3}{2x^2} = 4x^{3-2} = 4x.$$

2. Divide $12\,b^3c^2m^3$ by $-3\,bcm^2$.

$$\frac{12\,b^3c^2m^3}{-3\,bcm^2} = -4\,b^{3-1}c^{2-1}m^{3-2} = -4\,b^2cm.$$

55. Since any quantity divided by itself produces 1, it is evident that $a^n \div a^n = 1$; and, by the Index Law, it is also evident that $\frac{a^n}{a^n} = a^{n-n} = a^0$. The quotients just derived must be equal, because the dividends and divisors are identical. Hence, *any finite quantity with the exponent zero may be defined as equal to* 1; *or*, $a^0 = 1$.

Divide $-30\,a^4b^2c$ by $-6\,a^4bc$.

$$\frac{-30\,a^4b^2c}{-6\,a^4bc} = 5\,a^{4-4}b^{2-1}c^{1-1} = 5\,a^0bc^0 = 5 \cdot 1 \cdot b \cdot 1 = 5\,b.$$

EXERCISE XVII

Perform the indicated divisions:

1. $\frac{2\,a^5}{a^2}$. $= 2a^3$
2. $\frac{3\,a^6}{a^6}$. $= 3$
3. $\frac{12\,a^4b}{-4\,b}$. $-3a^4$
4. $\frac{-25\,a^2b^3}{5\,a^2b^2}$. $-5b$
5. $\frac{-15\,x^3y^3}{-5\,x^2y}$. $+3xy^2$
6. $\frac{20\,a^3by}{5\,a^2y}$.
7. $\frac{x^3y^3z}{-y^2z}$.
8. $\frac{39\,a^3x^3}{13\,a^3x^3}$.
9. $\frac{-28\,a^4x^2}{-7\,a^2x}$.
10. $\frac{-64\,a^5b^2}{-16\,a^4b^2}$.
11. $\frac{-30\,a^4b^5y}{-6\,a^3by}$.
12. $\frac{-34\,x^3y^3z^7}{17\,x^2yz^4}$.
13. $\frac{91\,x^7y^8z^{10}}{13\,x^5y^4z^6}$.
14. $\frac{44\,m^8z^{10}}{-4\,m^8z^2}$.
15. $\frac{-33\,a^7b^9c^{11}d}{11\,a^5b^6c^8}$.
16. $\frac{60\,a^8b^7c^{16}}{-15\,a^3b^4c^{12}}$.
17. $\frac{84\,a^{19}b^{12}c^{23}}{-7\,a^{12}b^3c^{15}}$.
18. $\frac{63\,x^3y^7a^{12}}{7\,x^3y^4a^{12}}$.
19. $\frac{-78\,x^3y^7z^{17}}{-13\,x^2y^4z^{14}}$.
20. $\frac{42\,a^9b^7c^{45}}{-7\,a^9bc^{45}}$.
21. $\frac{-52\,a^9x^3y^{24}z^{54}}{-13\,a^6x^2y^{24}z^{44}}$.

DIVISION OF A POLYNOMIAL BY A MONOMIAL

56. It has been shown, § 48, that, by the distributive law, $ab + ac = a(b + c)$. By the definition in § 48, if the product $ab + ac$, and the factor a are given, the quotient will be $b + c$.

Whence is derived the following **Rule for the Division of a Polynomial by a Monomial:** *Divide each term of the polynomial by the monomial and add the quotients thus derived.*

Divide $a^3 - 2a^4b + 8a^5b^2$ by a^3.

$$\frac{a^3 - 2a^4b + 8a^5b^2}{a^3} = 1 - 2ab + 8a^2b^2.$$

EXERCISE XVIII

Perform the indicated divisions:

1. $\dfrac{x^2 + xy}{x}$. $x + y$

2. $\dfrac{x^4y^4 - x^2y^2}{xy}$.

3. $\dfrac{5m^2 + 10m^3}{-5m}$.

4. $\dfrac{a^2b^2 + a^3b^3 - 2a^4b^4}{ab^2}$.

5. $\dfrac{2x^4 - 6x^5 + 12x^7}{-2x^3}$.

6. $\dfrac{6a^4x^4 + 8a^2x^6 + 16x^8}{2x^4}$.

7. $\dfrac{-21x^3y - 91xy^7 + 56y^9}{-7y}$.

8. $\dfrac{16x^8y^8 - 48x^6y^{10} + 112x^4y^{12}}{-16x^4y^8}$.

9. $\dfrac{a^4b^4c^7 - 13a^3b^3c^9 - 21a^5b^3c^{12}}{a^3bc^4}$.

10. $\dfrac{51x^{20}y^{31} - 102x^{56}y^{41}}{17x^{13}y^{23}}$.

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

57. Since it is always true in exact division that the product of the divisor and quotient gives the dividend, and since $(x^2 - xy + y^2)(x - y) = x^3 - 2x^2y + 2xy^2 - y^3$, it is possible to take either $x^2 - xy + y^2$, or $x - y$, as the divisor, and the other expression as the quotient; while $x^3 - 2x^2y + 2xy^2 - y^3$ is the dividend. Take $x^2 - xy + y^2$ as the divisor. Then

$$(x^3 - 2x^2y + 2xy^2 - y^3) \div (x^2 - xy + y^2) = x - y.$$

The quotient $x - y$ is derived from the dividend and divisor by the following process:

Notice first that the dividend and divisor are both arranged in descending powers of x. The first term of the dividend is evidently the product of the first term of the divisor and the first term of the quotient, the first terms in each case being evidently the term of highest degree because of the order of arrangement. Therefore, x^3, the first term of the dividend, divided by x^2, the first term of the divisor, gives x, the first term of the quotient.

Now the first term of the quotient is a multiplier of each term of the divisor, as will be seen by referring to the case,

$$(x^2 - xy + y^2)(x - y) = x^3 - 2x^2y + 2xy^2 - y^3.$$

Therefore the partial products of *all* the terms of the divisor by the first term of the quotient form a part, at least, of the dividend. That is,

$$(x^2 - xy + y^2)x = x^3 - x^2y + xy^2$$

must be subtracted from the dividend since the dividend is the sum of the partial products found by multiplying all the terms of the divisor by all the terms of the quotient. The remainder, so derived, is

$$x^3 - 2x^2y + 2xy^2 - y^3 - (x^3 - x^2y + xy^2) = -x^2y + xy^2 - y^3$$

This remainder may be considered as a new dividend and is the product of the divisor and the remaining term (or terms) of the quotient.

As before, the first term of the remainder (new dividend) is the product of the first term of the divisor and the first term of the quotient. Hence $-x^2y$ divided by x^2 gives $-y$, the second term of the quotient. Since the second term of the quotient is a multiplier of each term of the divisor, the product of the whole of the divisor and the second term of the quotient is sought.

$$(x^2 - xy + y^2)(-y) = -x^2y + xy^2 - y^3.$$

Subtracting this product from the remainder, the new remainder will be 0; that is, the division is exact.

The above explanation may be expressed thus:

$$x^3 - 2x^2y + 2xy^2 - y^3 = (x^3 - x^2y + xy^2) + (-x^2y + xy^2 - y^3),$$

$$\frac{x^3 - 2x^2y + 2xy^2 - y^3}{x^2 - xy + y^2} = \frac{x^3 - x^2y + xy^2}{x^2 - xy + y^2} + \frac{-x^2y + xy^2 - y^3}{x^2 - xy + y^2},$$

$$= x - y.$$

It will be noticed that the dividend is separated into such terms that each may be exactly divided by the divisor.

The following arrangement is, therefore, more convenient:

$$\begin{array}{r|l}
\text{Dividend} = x^3 - 2x^2y + 2xy^2 - y^3 & x^2 - xy + y^2 = \text{Divisor} \\
\underline{x^3 - x^2y + xy^2\phantom{{} - y^3}} & \phantom{x^2 - {}} x - y = \text{Quotient} \\
- x^2y + xy^2 - y^3 & \\
\underline{- x^2y + xy^2 - y^3} &
\end{array}$$

If the quotient contains more than two terms, the process of division is the same.

Checking the division by substituting $x = y = 1$,

$$(x^3 - 2x^2y + 2xy^2 - y^3) \div (x^2 - xy + y^2) = x - y,$$

$$(1 - 2 + 2 - 1) \div (1 - 1 + 1) = 1 - 1,$$

$$0 \div 1 = 0.$$

58. From the foregoing principle is derived the following **Rule for the Division of a Polynomial by a Polynomial:**

1. *Arrange both polynomials in the descending or ascending order of some common letter.*

2. *Multiply each term of the divisor by the quotient obtained by dividing the first term of the dividend by the first term of the divisor.*

3. *Subtract the partial products so derived from the dividend.*

4. *With the remainder still arranged in the same order as before, continue the process until there is no remainder, or until the degree of the first term of the divisor is higher than that of the first term of the remainder.*

1. Divide $m^3 - 3m^2n + 3mn^2 - n^3$ by $m - n$.

$$\begin{array}{rrrr|l}
m^3 - 3m^2n & + 3mn^2 & - n^3 & & m - n \\
\underline{m^3 - m^2n} & & & & \overline{m^2 - 2mn + n^2} \\
- 2m^2n & + 3mn^2 & - n^3 & & \\
\underline{- 2m^2n} & \underline{+ 2mn^2} & & & \\
 & mn^2 & - n^3 & & \\
 & \underline{mn^2} & \underline{- n^3} & &
\end{array}$$

2. Divide $2a^4 - 5a^3b + 7a^2b^2 - 5ab^3 + 2b^4$ by $a^2 - ab + b^2$.

$$\begin{array}{rrrrr|l}
2a^4 - 5a^3b & + 7a^2b^2 & - 5ab^3 & + 2b^4 & & a^2 - ab + b^2 \\
\underline{2a^4 - 2a^3b} & \underline{+ 2a^2b^2} & & & & \overline{2a^2 - 3ab + 2b^2} \\
- 3a^3b & + 5a^2b^2 & - 5ab^3 & + 2b^4 & & \\
\underline{- 3a^3b} & \underline{+ 3a^2b^2} & \underline{- 3ab^3} & & & \\
 & 2a^2b^2 & - 2ab^3 & + 2b^4 & & \\
 & \underline{2a^2b^2} & \underline{- 2ab^3} & \underline{+ 2b^4} & &
\end{array}$$

3. Divide $15x^2 + 7x + 7x^3 + 15x^4 + 4$ by $1 + 3x^2 + 2x$. Arrange the dividend and divisor in the same order.

By the method of detached coefficients:

$$\begin{array}{r|l} 15+\ 7+15+7+4 & 3+2+1 \\ \cline{2-2} \underline{15+10+\ 5} & 5-1+4 \\ -\ 3+10+7+4 & \\ \underline{-\ 3-\ 2-1} & \\ 12+8+4 & \\ \underline{12+8+4} & \end{array}$$

The quotient $5-1+4$ must have x^2 in the first term, and an integer only in the last term; and is, $5x^2-x+4$.

4. Divide $x^6-x^5-2x^4+5x^3-4x^2+x+1$
by x^4-3x^2+2x+1.

$$\begin{array}{r|l} 1-1-2+5-4+1+1 & 1+0-3+2+1 \\ \cline{2-2} \underline{1+0-3+2+1} & 1-1+1 \\ -1+1+3-5+1+1 & x^2-x+1 \\ \underline{-1-0+3-2-1} & \\ 1+0-3+2+1 & \\ \underline{1+0-3+2+1} & \end{array}$$

The divisor x^4-3x^2+2x+1 contains no term in x^3; since 0 times x^3 equals 0, to make the method available, the x^3 appears with the coefficient 0.

In detaching coefficients, the coefficient of any missing power of the letter of arrangement is always written as 0.

EXERCISE XIX

Perform the indicated divisions:

1. a^2+5a+6 by $a+2$.
2. x^2-2x-3 by $x+1$.
3. x^2-16 by $x+4$.
4. $x^2-14x+49$ by $x-7$.

5. $4a^2 - 12a + 9$ by $2a - 3$.

6. $36x^2 - 60x + 25$ by $6x - 5$.

7. $4a^2 + 12ab + 9b^2$ by $2a + 3b$.

8. $9x^2 - 6xmn + m^2n^2$ by $3x - mn$.

9. $x^3 - 27$ by $x - 3$.

10. $64 + a^3$ by $4 + a$.

11. $6mx - 8am - 9x + 12a$ by $3x - 4a$.

12. $21ax - 35ay + 3bx - 5by$ by $3x - 5y$.

13. $20ac - 15ad - 12bc + 9bd$ by $5a - 3b$.

14. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$ by $a + b + c$.

15. $p^2 + q^2 + r^2 + 2pq - 2pr - 2qr$ by $p + q - r$.

16. $p^2 + 2pq + q^2 - r^2$ by $p + q + r$.

17. $12a^2 - 4b^2 - 5c^2 + 2ab + 4ac - 9bc$ by $2a - b - c$.

18. $1 - 18a^2 + 81a^4$ by $1 - 6a + 9a^2$.

19. $-2x^4 + 7x^3 + 82x^2 + 145x + 72$ by $9 + 8x - x^2$.

20. $216a^3 + 125$ by $36a^2 - 30a + 25$.

21. $1 - 32p^5$ by $1 + 2p + 4p^2 + 8p^3 + 16p^4$.

22. $128a^4b^3 - 160a^5b^2 + 2a^6b + 15a^7$ by $3a^2 - 8ab$.

23. $5ac + 7bc + 3a^2 - 7ab - 6b^2 - 2c^2$ by $a - 3b + 2c$.

24. $44y - 30 - 16y^2 + 3x + 9x^2$ by $4y + 3x - 5$.

25. $48x^2 - 192xy + 192y^2 - 27z^2$ by $4x - 8y + 3z$.

26. $4x^4 - 197x^2y^2 + 49y^4$ by $2x^2 + 15xy + 7y^2$.

27. $80bc + 18a - 64b^2 - 48b + 9a^2 + 30c - 25c^2$ by $3a - 8b + 5c$.

28. $a^2 - 6ac + 9c^2 - 4b^2 - 4bd - d^2$ by $a + 2b - 3c + d$.

29. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

30. $x^5 - y^5$ by $x - y$.

EXERCISE XX

Perform the indicated operations and check the results :

1. $a[a+(b-c)] \div (a+b-c)$.
2. $5x-(3x-4)-[7x(2-9x)]$.
3. $(a^2+ab+b^2)(a-b)-3(a^3-b^3)$.
4. $7a-2[3a-2b+\{(a+b)-(a-b)\}]$.
5. $[-c+(a+b)][-c-(a+b)]-(c^2-a^2)\div(c+a)$.
6. $(a-b)(a+b-c)+(b-c)(b+c-a)$
$+(c-a)(c+a-b)$.
7. $(a+b)(a-b)-\{(a+b-c)-(b-a-c)$
$+(b+c-a)\}\{a-b-c\}$.
8. $(x^2-y^2)\div(x+y)+3[(x-y)(x^2-2xy+y^2)]$
$\div[(x-y)(x-y)(x-y)]$.
9. $8a\div 4a+7-[6a\div 2]\cdot 3-6a^2\div 2a-7(1-a)$.
10. $(x^2-y^2-z^2+2yz)\div(x-y+z)$.
11. $3a-[b-a-4(2a+b-\{a-b\})]$.
12. $a-2[2a-b-(3a-2b-\{4a-3b\})]$.
13. $(x^2-2x+1)(x^3-3x^2+3x-1)$.
14. $(x^5-32)\div(x^4+2x^3+4x^2+8x+16)$.
15. $(15x^4+7x^3+15x^2+7x+4)\div(3x^2+2x+1)$.
16. $(x^5-y^5)\div(x^4+x^3y+x^2y^2+xy^3+y^4)$.
17. $(1+2p+4p^2+8p^3+16p^4)(1-2p)$.
18. $(216a^3+125)\div(36a^2-30a+25)$.
19. $(x^2+y^2-z^2)(x^2+y^2+z^2)-(x^2-y^2+z^2)(x^2-y^2-z^2)$.
20. $(2x^2-3xy+y^2-3)(3-y^2-3xy+2x^2)$.
21. $(77a^9b^3-55a^8b^2-35a^8b+25a^7)\div(5a-7a^2b)$.

CHAPTER IV

EQUATIONS AND PROBLEMS

59. An algebraic **equation** is a statement that the numerical values of two expressions are the same. The expression preceding the sign of equality is called the **first (or the left) member**, and the expression following the equal sign is called the **second (or the right) member.**

Thus, in the equations, $a+b=b+a$, and $x+1=3$, the first members are respectively $a+b$, and $x+1$; and the second members are respectively $b+a$, and 3.

60. If an equation is always true for any values of the letters involved, it is called an **equation of identity.** In the more advanced work the sign of higher equality in identities is written $\equiv$.

Thus, $a+b\equiv b+a$, for all values of a and b.

61. Equations other than identical are called **conditional equations,** since the equality does not hold for all values of the letters involved, but is conditional upon a certain value.

Thus, $x+1=3$, if $x=2$, but not if x has any other value.

Equations of identity are more briefly called identities; and equations of condition are more simply called equations.

62. The process of finding for the letters involved those values which make the members equal is called **solving the equation.** These values are called the **roots of the equation.** The equation is said to be **satisfied** if the numerical values of the members, found by substituting the values of the roots, are the same.

Thus, 2 is a root of the equation $x+1=3$, and the equation is satisfied by substituting 2 for x; $2+1=3$.

63. An **Axiom** may be defined as a self-evident truth.

The following principles are stated as algebraic axioms:

1. *If equal quantities are added to equal quantities, their sums will be equal.*

2. *If equal quantities are subtracted from equal quantities, the remainders will be equal.*

3. *If equal quantities are multiplied by the same quantity or by equal quantities, the products will be equal.*

4. *If equal quantities are divided by the same quantity or by equal quantities, the quotients will be equal.*

5. *Quantities equal to the same quantity, or equal quantities, are equal to each other.*

The axioms should be memorized in order.

64. In an algebraic problem the quantities whose values are given are called **known quantities**, and are usually represented by the first letters of the alphabet; those quantities whose values are not given but are to be determined are called the **unknown quantities**, and are usually represented by the last letters of the alphabet.

Thus, in $x+3=11$, x is the unknown quantity, and 3 and 11 are the known quantities; in $y+b=3a$, y is the unknown quantity, and b and $3a$ are the known quantities.

65. A **simple equation** is one which in its simplest form contains only the first power of the unknown quantity.

Thus, $3x + 7 = 28$, and $3x = 6a^2$ are simple equations.

The solution of a simple equation may depend upon any, or all, of the axioms.

1. Solve for x, the equation $2x + 3 = 11$.

Since	$2x + 3 = 11,$
by Ax. 1,	$2x + 3 - 3 = 11 - 3,$
by § 23,	$2x = 8,$
by Ax. 4,	$x = 4.$

The root may be tested and the equation satisfied by substituting the value of the root in the given equation.

Thus,	$2(4) + 3 = 11,$
	$11 = 11.$

66. The process of satisfying an equation is variously called **verification**, **testing**, and **checking**. The root should always be substituted in the given equation.

2. Solve for y, the equation $2y + 7 = y - 4$.

Since	$2y + 7 = y - 4,$
by Ax. 1,	$2y + 7 - 7 = y - 4 - 7,$
by § 23,	$2y = y - 4 - 7,$
combining,	$2y = y - 11,$
by Ax. 1,	$2y - y = y - y - 11,$
by § 23,	$2y - y = -11,$
combining,	$y = -11.$

VERIFICATION: $2(-11) + 7 = -11 - 4,$

$$-15 = -15.$$

67. By the use of Axiom 1, a term may be changed from the first to the second member, or *vice versa.*

If $x + 4 = 5,$

by Ax. 1, $x + 4 - 4 = 5 - 4,$

or, $x = 5 - 4.$

The $+ 4$ in the first member appears in the second member as -4.

Again, if $2x - 7 = x + 10,$

by Ax. 1, $2x - 7 + 7 = x + 10 + 7,$

or, $2x = x + 17,$

and, by Ax. 1, $2x - x = x - x + 17,$

or, $x = 17.$

VERIFICATION: $2(17) - 7 = 17 + 10,$

$$34 - 7 = 27,$$

$$27 = 27.$$

The process of changing a term from the first to the second member, or *vice versa*, is called **transposition**; *any term may be transposed if its sign be changed.*

Transposing in the equation $5x - 11 + 3 = 2x + 1 - x,$

$$5x - 2x + x = 11 - 3 + 1.$$

It is to be noticed that transposition is simply an application of Axiom 1.

68. From the foregoing principles is derived the following **Rule for the Solution of a Simple Equation**: *Transpose all the unknown terms to the first member and all the known terms to the second member; combine similar terms; and divide both members by the coefficient of the unknown quantity.*

1. Solve for x, the equation

$$6x - 22 = 4x - 30. \qquad (1)$$

Transposing in (1), $6x - 4x = 22 - 30,$ (2)

uniting in (2), $2x = -8,$ (3)

applying Ax. 4 in (3), $x = -4.$ (4)

Verification: $6(-4) - 22 = 4(-4) - 30,$

$$-24 - 22 = -16 - 30,$$

$$-46 = -46.$$

2. Solve for x, the equation

$$3(x - 7) + 3 = 4 - 2(6 + x). \qquad (1)$$

Simplifying in (1), $3x - 21 + 3 = 4 - 12 - 2x,$ (2)

transposing in (2), $3x + 2x = 21 - 3 + 4 - 12,$ (3)

uniting in (3), $5x = 10,$ (4)

applying Ax. 4 in (4), $x = 2.$

Verification: $3(2 - 7) + 3 = 4 - 2(6 + 2),$

$$-15 + 3 = 4 - 16,$$

$$-12 = -12.$$

3. Solve for x, the equation

$$(2x + 3)(3x + 1) = (6x + 1)(x + 5) - 22. \qquad (1)$$

Simplifying in (1),

$$6x^2 + 11x + 3 = 6x^2 + 31x + 5 - 22, \qquad (2)$$

transposing in (2),

$$6x^2 - 6x^2 + 11x - 31x = -3 + 5 - 22, \qquad (3)$$

uniting in (3), $-20x = -20,$ (4)

applying Ax. 4 in (4), $x = 1.$

Verification: $(2 + 3)(3 + 1) = (6 + 1)(1 + 5) - 22,$

$$20 = 42 - 22,$$

$$20 = 20.$$

EXERCISE XXI

Solve for x, the following equations and verify the results:

1. $7x - 5 + 2x = 13.$
2. $4 + 12x = x + 15.$
3. $34x = 6x + 5 + 51.$
4. $70 - 3x - 2x = 7x - 2.$
5. $x = 9x + 7 - 5x - 10.$
6. $x + 2x + 3x + 4x = 100.$
7. $-2x - 5 = 9x + 5x + 21 - 68 - 6.$
8. $0 = 5x + 7x - 9x - 11x + 107x - 74 + x - 26.$
9. $3x - (x - 7) = x + 15.$
10. $3x - (2x - 8) = 19.$
11. $3(x + 1) - 2x = 93.$
12. $81 - 4(x + 1) = x + 7.$
13. $103 - 3(x - 5) = 2x + 18.$
14. $13(2x - 1) = 5(5x + 4).$
15. $25 - 6(x - 6) = 20 - (2x - 13).$
16. $2(9 - x) + 5(2x + 3) = 81.$
17. $6(20 + 3x - 1) - 5(8x - 7) + 19 = 2(x - 72).$
18. $3 \cdot 5(x + 6) + 5 \cdot 7(1 + 2x) - 7 \cdot 9(x - 8) = 827.$
19. $(2x - 1)(3x + 1) = (6x - 12)(x + 3).$
20. $(5x + 7)(6x - 3) = (10x + 2)(3x + 2) - 9.$
21. $7(x - 1) - 3(1 - x) = -4(6 + x).$
22. $3(2x + 7) + 4(6 + x) = -4(x - 3) + 3(2x + 1) - 10.$
23. $6(2x - 4) - 3(2x - 1) = 7(3x + 2) - 8(4x - 2).$

69. If the equation contains fractions, it can be simplified by application of Axiom 3.

1. Solve for x, the equation

$$\frac{x}{6} - 4 = 10 - x. \qquad (1)$$

Applying Ax. 3 in (1), $\quad x - 24 = 60 - 6x, \qquad (2)$

transposing and uniting in (2), $\quad 7x = 84, \qquad (3)$

applying Ax. 4 in (3), $\quad x = 12. \qquad (4)$

VERIFICATION: $\quad \frac{12}{6} - 4 = 10 - 12,$

$$2 - 4 = 10 - 12,$$

$$-2 = -2.$$

2. Solve for x, the equation

$$\frac{8}{9} \cdot \frac{4 - x}{11} = \frac{x}{3} - \left[\frac{16x + 2}{33} - \frac{x + 2}{9}\right]. \qquad (1)$$

Simplifying in (1),

$$\frac{8(4 - x)}{99} = \frac{x}{3} - \frac{16x + 2}{33} + \frac{x + 2}{9}, \qquad (2)$$

applying Ax. 3 in (2),

$$8(4 - x) = 33x - 3(16x + 2) + 11(x + 2), \qquad (3)$$

simplifying in (3),

$$32 - 8x = 33x - 48x - 6 + 11x + 22, \qquad (4)$$

transposing and uniting in (4),

$$-4x = -16, \qquad (5)$$

applying Ax. 4 in (3), $\quad x = 4. \qquad (6)$

VERIFICATION: $\quad \frac{8}{9} \cdot \frac{4 - 4}{11} = \frac{4}{3} - \frac{64 + 2}{33} + \frac{6}{9},$

$$0 = \tfrac{4}{3} - 2 + \tfrac{2}{3}.$$

EXERCISE XXII

Solve for x, the following equations:

1. $8-\frac{x}{9}=\frac{x+4}{3}$.

2. $5+\frac{x}{2}=x-5$.

3. $\frac{x}{4}-5=x-23$.

4. $19-\left(7+\frac{x}{8}\right)=\frac{x}{2}+7$.

5. $5x-\frac{5}{2}=7x-\frac{7}{2}$.

6. $\frac{x}{8}+\frac{x}{6}-6=1$.

7. $\frac{7-3(x-5)}{4}=1$.

8. $\frac{8(2+5x)-5}{9}=\frac{9x+2}{2}$.

9. $\frac{x}{4}-\frac{x}{5}+\frac{x}{3}-\frac{x}{6}=13$.

10. $8x-\frac{x}{4}=\frac{x}{10}+153$.

11. $\frac{7x}{6}-\frac{1}{4}-\frac{17}{18}=\frac{11}{36}(3x+1)$.

12. $\frac{2(7x-1)}{7}=\frac{3(3x+5)}{14}$.

13. $\frac{7x+13}{16}-\frac{x+8}{13}=\frac{x+11}{8}$.

14. $30(x-2)+\frac{x}{3}=\frac{5x+1}{16}+30$.

15. $\frac{1}{2}(5x+1)-\frac{1}{3}(4x+5)=\frac{1}{4}(3x-1)-\frac{1}{20}(6x+4)$.

16. $\frac{3x+9}{72}+\frac{5x-33}{36}-\frac{48-x}{9}+\frac{x-17}{4}=\frac{3+x}{24}$.

17. $\frac{x}{8}-\frac{x-2}{5}+\frac{x-22}{10}+\frac{x-12}{20}=\frac{32-x}{40}$.

18. $\frac{3x-5}{16}-\frac{4(2x+4)}{9}-\left[\frac{9-x}{2}+\frac{x-7}{12}\right]=x-15$.

19. $\left(5+\frac{x}{2}\right)\left(5-\frac{x}{2}\right)+\frac{x^2}{4}=x+12$.

20. $(4x-\frac{1}{2})(5x+\frac{3}{4})=(2x+\frac{1}{2})(10x-\frac{3}{2})$.

70. The statement of a problem in algebraic language often leads to an equation. The problem is solved by finding the numerical value of the numbers which first appear as unknowns. Certain relations of the unknowns in definite numbers are given; from these relations the values of the unknowns are determined.

Little difficulty need be met in translating the statement of a problem into algebraic language if it be remembered that every algebraic expression represents some number.

EXERCISE XXIII

1. What is the value in cents of 2 two-dollar bills, 3 dollar bills, 4 quarters, and 5 nickels? of a two-dollar bills, b dollar bills, c quarters, and x nickels?

2. If x is the tens' digit and 4 the units' digit of a number of two digits, what is the number?

3. If 3 is the tens' digit and x the units' digit of a number of two digits, what is the number formed by reversing the order of the digits?

4. If in a number of three digits the tens' digit is x, and the hundreds' digit is twice the tens' digit, and the units' digit is four times the tens' digit, what is the number?

5. What is the cost of 20 articles bought at the rate of 3 for x cents?

6. If x represents a certain digit, what is the next higher digit? the next lower digit?

7. If x is a certain digit, what are the 2 next higher (consecutive) digits?

8. If x is an odd number, what are the next two even numbers? the next two odd numbers?

9. If x men contribute equally to a certain fund of $225, how much does each man contribute?

10. If a man spends a dollar a day more than on the preceding day, and on the tenth day spends x dollars, how much does he spend on the twenty-third day?

11. If the price of eggs is lowered 3 cents a dozen from the original price of a cents a dozen, how much does one egg now cost?

12. If the interest on a certain sum of money for a given time is computed at x per cent, what will be one per cent higher rate?

13. What is the value in cents of the same number, x, of dollars, cents, quarters, and dimes?

14. If in a certain number of two digits the units' digit is x, and the tens' digit is four times the units' digit, what is the sum of the digits?

15. If a newspaper increased x per cent over the preceding yearly circulation at the end of each year, and if the circulation at the end of the first year was 25,000, what was the circulation at the end of the second year?

16. If the rate of a stream is 2 miles per hour, what will be the rate down the river of a crew which rows 4 miles an hour in still water? up the river?

17. What is the perimeter of a rectangular field whose length is a feet and whose breadth is b feet?

18. What is the greater of two numbers if the greater is three times the excess of the less number, x, over 12?

71. After the conditions of a problem have been stated in algebraic language, the next step is to find two equal expressions. In the equation formed of these two equal expressions the roots are found by § 68.

1. The sum of a number and its double is 48. Find the number.

Let $x =$ the number,

then $2x =$ double the number,

and $x + 2x = 3x =$ the sum of the number and its double,

but $48 =$ the sum of the number and its double,

by Ax. 5, $3x = 48$,

by Ax. 4, $x = 16$.

VERIFICATION: $16 + 2(16) = 48$,

$$48 = 48.$$

2. Find that number which lacks as much of 18 as it exceeds 10.

Let $x =$ the number,

then $18 - x =$ the amount the number lacks of 18,

and $x - 10 =$ the amount the number exceeds 10,

but the amount the number lacks of 18 is the same amount that the number exceeds 10;

by Ax. 5, $18 - x = x - 10$,

or $-2x = -28$,

by Ax 4, $x = 14$.

VERIFICATION: $18 - 14 = 14 - 10$,

$$4 = 4.$$

3. A's age exceeds B's by 25 years. Five years ago A was six times as old as B. Find the age of each.

Let	$x = $ B's age,
then	$25 + x = $ A's age,
and	$x - 5 = $ B's age 5 years ago,
and	$25 + x - 5 = $ A's age 5 years ago,
and	$6(x - 5) = 6$ times B's age 5 years ago,
but	$20 + x = $ A's age 5 years ago,
by Ax. 5,	$6(x - 5) = 20 + x,$
simplifying,	$6x - 30 = 20 + x,$
uniting,	$5x = 50,$
by Ax. 4,	$x = 10.$

Verification: $6(10 - 5) = 20 + 10,$

$$30 = 30.$$

4. The units' digit of a number is double the tens' digit, and the sum of the digits is 12. Find the number.

Let	$x = $ tens' digit,
then	$2x = $ units' digit,
and	$x + 2x = $ sum of the digits,
but	$12 = $ sum of the digits,
by Ax. 5,	$x + 2x = 12,$
uniting,	$3x = 12,$
by Ax. 4,	$x = 4,$
by Ax. 3,	$2x = 8.$

Therefore the number $= 10(x) + 2x = 48.$

Verification: $4 + 8 = 12,$

$$12 = 12$$

5. The sum of the third part and twelfth part of a number is 25. Find the number.

Let $x =$ the number,

then $\frac{x}{3} =$ the third part of the number,

and $\frac{x}{12} =$ the twelfth part of the number,

and $\frac{x}{3} + \frac{x}{12} =$ the sum of the third and twelfth parts,

but $25 =$ the sum of the third and twelfth parts,

by Ax. 5, $\frac{x}{3} + \frac{x}{12} = 25,$

by Ax. 3, $4x + x = 300,$

uniting, $5x = 300,$

by Ax. 4, $x = 60.$

VERIFICATION: $\frac{60}{3} + \frac{60}{12} = 25,$

$20 + 5 = 25,$

$25 = 25.$

6. A man has the same number of half-dollars, quarters, dimes, and nickels. Find the number if he has all together \$3.60.

Let $x =$ the number of each coin,

then $50x =$ the value of the half-dollars in cents,

and $25x =$ the value of the quarters in cents,

and $10x =$ the value of the dimes in cents,

and $5x =$ the value of the nickels in cents,

and $90x =$ the values of all the coins in cents,

but $360 =$ the values of all the coins in cents, by Ax. 5, $90\,x = 360,$

by Ax. 4, $x = 4.$

VERIFICATION: $90(4) = 360,$

$360 = 360.$

EXERCISE XXIV

1. The sum of a number and three times that number is 48. What is the number?

2. The sum of 10 and twice a number equals four times that number. What is the number?

3. If 13 be subtracted from eight times a number, the remainder equals 35. What is the number?

4. If five times a certain number is subtracted from 27, the remainder is 7. Find the number.

5. Five times a number exceeds twice that number by 21. Find the number.

6. Find that number the sum of whose products by 3 and 4 respectively equals 119.

7. One number is twice another number and their difference is 14. Find the numbers.

8. The sum of 12 and three times a number equals the excess of 39 over six times the number. Find the number.

9. Twice a number lacks as much of 20 as three times the number exceeds 20. Find the number.

10. Twelve times a number exceeds 7 as much as ten times the number lacks of 15. Find the number.

11. The sum of 12 and four times a number exceeds by 2 nine times the number. Find the number.

12. The excess of four times a number over 24 equals the sum of 9 and the number. Find the number.

13. The greater part of 8 equals three times the smaller part. Find the parts.

14. Three times the smaller part of 15 exceeds by 5 twice the larger part. Find the parts.

15. The sum of two numbers is 47, and their difference is 3. Find the numbers.

16. The sum of two numbers is 26, and their difference is 6. Find the numbers.

17. The sum of two numbers is 120, and the greater exceeds the less by 21. Find the numbers.

18. The difference of two numbers is 26 and their sum is 52. Find the numbers.

19. The excess of 7 over the larger part of 5 equals twice the smaller part. Find the smaller part.

20. The sum of three consecutive numbers is 39. Find the numbers.

21. Find the ages of A and B if the sum of their ages is 62 years, A being 16 years older than B.

22. A has four times as much money as B, and both have $125. How much has each?

23. A, B, and C have together $28. A and B each has three times as much as C. How much has each?

24. A has twice as much money as B, and B has three times as much as C. All have together \$150. How much has each?

25. A has twice as many dollars as B, three times as many as C, and half as many as D. If they all have \$92, how much has each?

26. A, B, and C together have \$54. If A has twice as much as B, and C has as much as A and B together, how much has each?

27. A and B together have \$12; B and C, \$15; A and C, \$19. How much has each?

28. The same number each of dollars, dimes, and cents amount to \$8.88. Find the number of cents.

29. The sum of a certain number of quarters and four times that number of cents is \$5.80. Find the number of cents.

30. A has ten times as many cents as dimes and eight times as many dimes as dollars. If he has in all \$13, find the number of dimes.

31. A's age exceeds B's by 20 years. Ten years ago A was twice as old as B. Find the age of each.

32. A is now four times as old as B; 5 years ago he was seven times as old as B. Find the age of each.

33. A is now five times as old as B; in 12 years he will be three times as old as B. Find the age of each.

34. Six years ago a father was six times as old as his son, whose age now lacks 30 years of the father's age. Find the age of each.

35. If A is now 52 years old and B is now 12, find the number of years ago that A was five times as old as B.

36. The units' digit of a number of two digits is three times the tens' digit, and the sum of the digits is 12. Find the number.

37. The tens' digit of a number of two digits exceeds by 4 the units' digit, and the sum of the digits is 8. Find the number.

38. The tens' digit of a certain number of two digits is 3 times the units' digit. If 18 be subtracted from the number, the order of the digits will be reversed. Find the number.

39. The hundreds' digit of a number of three digits is twice the tens' digit and four times the units' digit. If 297 be subtracted from the number, the order of digits will be reversed. Find the number.

40. A fifth of a certain number exceeds the eighth of that number by 6. Find the number.

41. The excess of a certain number over 8 equals a third of that number. Find the number.

42. The quotient of a certain number divided by 9 exceeds the twelfth part of the number by 1. Find the number.

43. The twelfth part of a certain number is 8 less than the sixth part of that number. Find the number.

44. The eighth part of a certain number is 3 less than the fifth part of that number. Find the number.

45. The ninth part of a certain number exceeds by 1 the tenth part of that number. Find the number.

46. The third part of a certain number exceeds 5 by as much as the eighth part is less than 6. Find the number.

47. The fifth part of a certain number exceeds 7 by as much as the ninth part is less than 7. Find the number.

48. Two-thirds of a certain number exceeds one-sixth of that number by 15. Find the number.

49. Three-eighths of a certain number exceeds one-fourth of that number by 4. Find the number.

50. Two-thirds of a certain number exceeds four-sevenths of that number by 2. Find the number.

51. The sum of one-third and one-thirteenth parts of a certain number is 16. Find the number.

52. One and one-half times a certain number exceeds three-eighths of that number by 36. Find the number.

53. The sum of the ages of a father and son is 48 years. How many years ago was the son's age one-seventh of the father's age if the son's age is now 12 years?

54. A has four times as many cents as dimes and twice as many dimes as dollars. If he has in all $5.12, find the number of dollars.

55. Find that number of three digits in which the hundreds' digit is double the tens' digit, and in which the tens' digit is double the units' digit, if the sum of the digits is 14.

CHAPTER V

TYPE FORMS IN MULTIPLICATION

72. The products of certain expressions are so often required that it is convenient to have a shorthand method of writing the product without performing the multiplications as in § 49. These expressions and their products are called **type forms.**

CASE I

73. By multiplication, $(a + b)^2 = a^2 + 2\,ab + b^2$.

Here a and b represent the sum of any two quantities; the square of the sum is required.

The process may be represented thus:

(1st number + 2d number)^2 = (1st number)^2 + 2 (1st number)(2d number) + (2d number)^2.

RULE: *The square of the sum of two quantities is the square of the first quantity, plus twice the product of the first and second quantity, plus the square of the second quantity.*

EXERCISE XXV

Write the indicated squares by inspection:

1. $(m + n)^2$.
2. $(a + 2\,b)^2$.
3. $(c + d)^2$.
4. $(2\,c + d)^2$.
5. $(c + 2\,d)^2$.
6. $(2\,c + 3\,d)^2$.
7. $(c^2 + d)^2$.
8. $(a^2 + b)^2$.
9. $(a + b^2)^2$.
10. $(2\,c^2 + d)^2$.
11. $(4\,c + m)^2$.
12. $(2\,a + 12\,d^2)^2$.

CASE II

74. By multiplication, $(a-b)^2 = a^2 - 2\,ab + b^2$.

Here $a-b$ represents the difference of any two quantities. the square of the difference is required.

RULE: *The square of the difference of two quantities is the square of the first quantity, minus twice the product of the first and second quantity, plus the square of the second quantity.*

EXERCISE XXVI

Write the indicated squares by inspection:

1. $(m-n)^2$.
2. $(n-m)^2$.
3. $(c-d)^2$.
4. $(d-c)^2$.
5. $(7-5)^2$.
6. $(m-2\,d)^2$.
7. $(c-3\,d)^2$.
8. $(3\,d-c)^2$.
9. $(5\,m-n)^2$.
10. $(11\,m-1)^2$.
11. $(1-10\,d)^2$.
12. $(2\,m-3\,d^2)^2$.

CASE III

75. By multiplication, $(a+b)(a-b) = a^2 - b^2$.

Here the product of the sum and difference of the same two quantities is required.

RULE: *The product of the sum and difference of the same two quantities is the difference of the squares of the first and second quantities.*

EXERCISE XXVII

Write the indicated products by inspection:

1. $(a+c)(a-c)$. $a^2 - c^2$
2. $(m-n)(m+n)$. $m^2 - n^2$
3. $(d+e)(d-e)$. $d^2 - e^2$
4. $(2\,c+d)(2\,c-d)$.
5. $(a^2+b)(a^2-b)$.
6. $(2\,c-d^2)(2\,c+d^2)$.

76. It is sometimes possible to arrange the terms in both multiplicand and multiplier to take the form of Case III.

1. $(a+b+c)(a+b-c) = \{(a+b)+c\}\{(a+b)-c\},$
$$= (a+b)^2 - c^2,$$
$$= a^2 + 2\,ab + b^2 - c^2.$$

2. $(a-b+c)(a+b-c) = \{a-(b-c)\}\{a+(b-c)\},$
$$= a^2 - (b-c)^2,$$
$$= a^2 - b^2 + 2\,bc - c^2.$$

The rule of Case III applies to the product of terms so arranged.

EXERCISE XXVIII

Write the indicated products by inspection:

1. $(m+n+p)(m+n-p)$.
2. $(m-n+p)(m+n-p)$.
3. $(m-n-p)(m-n+p)$.
4. $(m-n-p)(m+n+p)$.
5. $(2\,a+b+c^2)(2\,a+b-c^2)$.
6. $(2\,a^2+3\,ab+b^2)(-2\,a^2+3\,ab+b^2)$.
7. $(c^2-2\,cd+d^2)(c^2+2\,cd-d^2)$.
8. $(c^2-ab+5\,d)(c^2+ab+5\,d)$.
9. $(s-sa-sb)(s+sa+sb)$.
10. $(s^2-s^2a+s^2b)(s^2+s^2a-s^2b)$.
11. $(-3\,x^2+4\,xy-5\,y^2)(3\,x^2+4\,xy-5\,y^2)$.
12. $(x^2y^2-3\,xy^3+4\,y^4)(x^2y^2+3\,xy^3-4\,y^4)$.
13. $(3\,x^2-2\,xy+y^2-4)(3\,x^2-2\,xy-y^2+4)$.
14. $(a^3-3\,ab+b^2-2\,b)(-a^3-b^2-2\,b-3\,ab)$.

CASE IV

77. By Case I,

$$(a+b+c)^2 = [(a+b)+c]^2,$$
$$= (a+b)^2 + 2(a+b)c + c^2,$$
$$= a^2 + 2\,ab + b^2 + 2\,ac + 2\,bc + c^2,$$
$$= a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc.$$

By Case II,

$$(a-b-c)^2 = \{(a-b)-c\}^2,$$
$$= (a-b)^2 - 2(a-b)c + c^2,$$
$$= a^2 - 2\,ab + b^2 - 2\,ac + 2\,bc + c^2,$$
$$= a^2 + b^2 + c^2 - 2\,ab - 2\,ac + 2\,bc.$$

RULE: *The square of any polynomial is the sum of the squares of the several terms, and twice the product of every term by every term that follows it, giving to every product the proper sign.*

EXERCISE XXIX

Write the indicated squares by inspection:

1. $(a+b-c)^2$.
2. $(-a+b+c)^2$.
3. $(-a-b+c)^2$.
4. $(a-b-c)^2$.
5. $(-a-b-c)^2$.
6. $(2\,a+b+c)^2$.
7. $(a+2\,b+7\,c)^2$.
8. $(1+2\,a+3\,a^2)^2$.
9. $(2\,a+b-3\,c)^2$.
10. $(2\,a^2+3\,bx+x^2)^2$.
11. $(1+2\,x+3\,x^2)^2$.
12. $(2\,a^2-3\,ab-5\,b^2)^2$.
13. $[(a+b)+c+2\,d]^2$.
14. $(2\,m^2-3\,n^2+4\,mn)^2$.
15. $[(2\,a-b)-c+3\,d]^2$.
16. $[6\,m-15\,mn+n(n-m^2)]^2$.
17. $(2\,a^3-a^2b+3\,ab^2-b^3)^2$.

CASE V

78. By multiplication,

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3,$$

and
$$(a-b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3.$$

RULE: *The cube of the sum of two quantities is the sum of the cubes of the quantities plus three times the product of the square of the first quantity and the second, plus three times the product of the first quantity and the square of the second.*

RULE: *The cube of the difference of two quantities is the difference of the cubes of the quantities minus three times the product of the square of the first quantity and the second, plus three times the product of the first quantity and the square of the second.*

The result of the two rules can be shown thus:

$$(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3,$$

where the sign $\pm$, read "plus or minus," means that in the cube of $a+b$ the signs are all plus; and that in the cube of $a-b$ the signs are alternately plus and minus.

Write by inspection $(a-2\,b)^3$.

$$(a-2\,b)^3 = (a)^3 - 3\,(a)^2(2\,b) + 3\,(a)\,(2\,b)^2 - (2\,b)^3,$$
$$= a^3 - 6\,a^2b + 12\,ab^2 - 8\,b^3.$$

EXERCISE XXX

Write the indicated cubes by inspection:

1. $(x+y)^3$.
2. $(x-y)^3$.
3. $(x^2+y^2)^3$.
4. $(2\,a+b)^3$.
5. $(2\,x^2+3\,y^2)^3$.
6. $(2\,x^3-3\,xyz)^3$.
7. $(m-5\,n)^3$.
8. $(2\,a^2b-7\,ab^2)^3$.
9. $(1-5\,x^2)^3$.

CASE VI

79. By multiplication the product of two binomials of the form $x + a$ and $x + b$ can be determined.

$$(x+2)(x+3) = x^2 + 5x + 6.$$
$$(x+2)(x-3) = x^2 - x - 6.$$
$$(x-2)(x+3) = x^2 + x - 6.$$
$$(x-2)(x-3) = x^2 - 5x + 6.$$

RULE: *The product of any two binomials whose first terms are identical is the product of the first terms of the binomials, the algebraic sum of the second terms as the coefficient of the common term, and the product of the second terms of the binomials.*

EXERCISE XXXI

Write the indicated products by inspection:

1. $(x+1)(x+2)$.
2. $(x+1)(x-2)$.
3. $(m+5)(m-4)$.
4. $(m-3)(m-4)$.
5. $(m-7)(m+3)$.
6. $(x-5n)(x+3n)$.
7. $(x^2-5)(x^2-5)$.
8. $(xy-3)(xy+4)$.
9. $(x^2-3)(x^2+4)$.
10. $(3-7xy)(3-xy)$.
11. $(a^2-4)(a^2+6)$.
12. $(ax+11)(ax+1)$.
13. $(a^2-21)(a^2+3)$.
14. $(xy-7)(xy+5)$.
15. $(16-5xy)(16-2xy)$.
16. $(5m^2n-3ny^2)(5m^2n-ny^2)$.
17. $[(a+b)+5][(a+b)-3]$.
18. $[1-(x+y)][1-4(x+y)]$.
19. $[(x-y)+2][(x-y)+7]$.

CASE VII

80. By multiplication the product of two binomials which contain the same letters can be determined.

$$(x+2\,y)(2\,x+3\,y)=2\,x^2+7\,xy+6\,y^2.$$
$$(x-2\,y)(2\,x+3\,y)=2\,x^2-\quad xy+6\,y^2.$$
$$(x+2\,y)(2\,x-3\,y)=2\,x^2+\quad xy-6\,y^2.$$
$$(x-2\,y)(2\,x-3\,y)=2\,x^2-7\,xy+6\,y^2.$$

RULE: *The product of two binomials which contain the same letters is the product of the first terms of the binomials, the algebraic sum of the cross products, and the product of the second terms of the binomials.*

Write by inspection $(3\,x+7\,y)(2\,x-4\,y)$.

$$\begin{aligned}(3\,x+7\,y)(2\,x-4\,y)&=6\,x^2+14\,xy-12\,xy-28\,y^2,\\&=6\,x^2+2\,xy-28\,y^2.\end{aligned}$$

The cross products, as $14\,xy$ and $-12\,xy$, are usually combined, without writing in full, into the middle term of the product.

EXERCISE XXXII

Write the indicated products by inspection:

1. $(2\,x-a)(3\,x+a)$.
2. $(2\,m+a)(m-2\,a)$.
3. $(2\,x+a)(3\,x-a)$.
4. $(2\,m-a)(m-2\,a)$.
5. $(2\,x-a)(3\,x-a)$.
6. $(2\,n-a)(n-2\,a)$.
7. $(x-5\,y)(2\,x-3\,y)$.
8. $(2\,x-m^2)(3\,x-m^2)$.
9. $(x+1)(3\,x-4)$.
10. $(5\,a-2\,b)(2\,b+5\,a)$
11. $(a-11\,c)(2\,a-c)$.
12. $(1-xy)(3-5\,xy)$.
13. $(6\,a^2-7\,x)(2\,a^2+x)$.
14. $(3\,x^2+4\,xy)(4\,x^2-7\,xy)$.

CASE VIII

81. By multiplication,

$$(a+b)(a^2-ab+b^2)=a^3+b^3,$$

and
$$(a-b)(a^2+ab+b^2)=a^3-b^3.$$

Here $a+b$ represents the sum of any two quantities and $a-b$ the difference of any two quantities.

RULE: *The product of the sum of two quantities and the sum of the squares of the quantities minus the product of the quantities is the sum of the cubes of the quantities.*

RULE: *The product of the difference of two quantities and the sum of the squares of the quantities plus the product of the quantities is the difference of the cubes of the quantities.*

EXERCISE XXXIII

Find the indicated products by inspection:

1. $(x-y)(x^2+xy+y^2)$.
3. $(2+a)(4-2a+a^2)$.

2. $(x+y)(x^2-xy+y^2)$.
4. $(a-2)(a^2+2a+4)$.

5. $(x^2+4)(x^4-4x^2+16)$.

6. $(5a-2b)(25a^2+10ab+4b^2)$.

7. $(2b-5a)(4b^2+10ab+25a^2)$.

8. $(7c-1)(49c^2+7c+1)$.

9. $(1+10b)(1-10b+100b^2)$.

10. $(8d-x^5)(64d^2+8dx^5+x^{10})$.

11. $[a+b+c][(a+b)^2-(a+b)c+c^2]$.

12. $[3(a-c)+4(b-d)]$
$[9(a-c)^2-12(a-c)(b-d)+16(b-d)^2]$.

13. $[2(x-y)+3][4(x-y)^2-6(x-y)+9]$.

REVIEW EXERCISE XXXIV

Write the indicated products by inspection:

1. $(2a+b)^2$.
2. $(x-11)(x-6)$.
3. $(1-2x-3y)^2$.
4. $(a+7)(a-7)$.
5. $(x-3)(x^2+3x+9)$.
6. $(1-4x)^3$.
7. $(2x+y+1)(2x+y-1)$.
8. $(3x^2-5y^2)^2$.
9. $(3x^2+4xy^2)(3x^2-4xy^2)$.
10. $(2m-3n)(3m-4n)$.
11. $(x+4)(x^2-4x+16)$.
12. $(2m^2-n^2-p^2)(2m^2-n^2+p^2)$.
13. $(2m^2-n^2-p^2)(2m^2-n^2-p^2)$.
14. $(2x^2-5y^2)(5x^2-2y^2)$.
15. $(14-a)(3-a)$.
16. $(1-4xyz)^3$.
17. $(2x^2y^2-z^2)^2$.
18. $(3x-a)(9x^2+3xa+a^2)$.
19. $(7a^2+3xy)(7a^2+3xy)$.
20. $(5xy-3ab)(5xy+3ab)$.
21. $(5xy+1)(25x^2y^2-5xy+1)$.
22. $(3m-2n+y)(3m+2n-y)$.
23. $(a^2+ax+x^2)(a^2-ax+x^2)$.
24. $(a+x+y)(a-x-y)$.
25. $(ax+m-n)(ax-d)$.
26. $(m-n-x+y)(m-n+x-y)$.
27. $(2ax-3by)(4a^2x^2+6abxy+9b^2y^2)$.
28. $(1+x+x^2)(1-x+x^2)$.

29. $(1+x+x^2)(1+x+x^2)$.

30. $(1+x+x^2)(1+x-x^2)$.

31. $[(m+2n)+(a+b)]^2$.

32. $(7m^2a^2-4d^2)^2$.

33. $(2a^2+7b)(2a^2-7b)$.

34. $[(2a+3b)+(3c+d)]^2$.

35. $\{2(a-b)-c\}^2$.

36. $(1-5x+2x^2)^2$.

37. $(2a^2-3ab+4)(-2a^2-3ab+4)$.

38. $(9x^2y^2-7xyz^2)^2$.

39. $(7x^2y^2z-4xyz^3)^3$.

40. $\{2(a-b)-3(c+d)\}^2$.

41. $(x^2-y^2+3)(x^2-y^2-11)$.

42. $\{3(a+b)+5m^2(a^2+b^2)\}^2$.

43. $(2m-4n)(3m+n)$.

44. $(a+b-c^2)(a+b+c^2)$.

45. $(5ny-cz)(7ny-4cz)$.

46. $(a^2-3b^2)(a^4+3a^2b^2+9b^4)$.

47. $\{(2a+b)+3c^2\}\{(2a+b)-3c^2\}$.

48. $[x-y+3a][x-y-2a]$.

49. $[2(a-b)+3c][2(a-b)-2c]$.

50. $(2c+2d+5a+5b)(2c+2d-5a-5b)$.

51. $[3(a+b)-2c][2(a+b)+5c]$.

52. $(x^2-xy-z^2)^3$.

53. $[a-3(x-y)][2a-4(x-y)]$.

54. $\{13(a+b)-5(a^2+b^2)\}\{13(a+b)+5(a^2+b^2)\}$.

55. $[7(m-n)-(a-b)][3(m-n)+4(a-b)]$.

CHAPTER VI

FACTORING

82. If a, b, and c are limited to integral expressions and if $a \cdot b = c$, then a and b are called **factors** of c, and c is called a **multiple** of a and of b.

An integral expression is **prime** when it has no factors except itself and 1.

The process of finding the prime factors of an integral expression is called **factoring.**

83. The factors of a monomial can be obtained by inspection.

Thus, the factors of $36\,a^3b^2$ are $2 \cdot 2 \cdot 3 \cdot 3 \cdot a \cdot a \cdot a \cdot b \cdot b$.

The factors of a polynomial are indicated by the form of the expression, which is often one of the type forms discussed in the previous chapter.

CASE I

84. When each of the terms of the expression contains a common monomial factor.

1. Factor $x^3 + 3\,x^2 + 5\,x$.

By inspection x is common to each term.

Therefore, $\quad x^3 + 3\,x^2 + 5\,x = x(x^2 + 3\,x + 5)$.

2. Factor $5\,a^2 + 5\,ab + 10\,a$.

By inspection $5\,a$ is a factor of each term.

Therefore, $\quad 5\,a^2 + 5\,ab + 10\,a = 5\,a(a + b + 2)$

EXERCISE XXXV

Factor:

1. $x^2 + ax.$ $x(x+a)$

2. $x^2 + 2\,bx + cx.$

3. $a^2 + 2\,ab + ac.$

4. $2\,ax - 2\,ay + 2\,a^2.$

5. $3\,a^3 - 3\,abc + 3\,ad.$

6. $-12\,ax^2 + 4\,axy - 8\,ay^2.$

7. $-2\,bm - 3\,bn^2 + 4\,bp.$

8. $-21\,ac + 99\,acd.$

9. $4\,a^2d - 4\,acd + 12\,c^2d.$

10. $2\,a^3x^2 + 2\,a^2x^4 - 6\,x^3y^2.$

11. $3\,x^3y - 3\,x^3y^2 + 9\,ax^2.$

12. $5\,mx - 15\,mnx - 5\,nx + 10\,px.$

13. $14\,a^3x^2y^4 + 7\,a^2bx^2y^2 + 49\,ab^2x^2y^3 - 21\,b^3x^2y^2.$

14. $91\,abd + 21\,b^2d - 7\,cd + 14\,d^2.$

15. $2\,a^4bc^2 - 2\,a^4b^3c^2 + 10\,a^2b^3c^2.$

16. $-a^2b^2c^4 + a^3bc^4 + ab^3c^4 - abc^6.$

17. $3\,a^2x^3 - 9\,acx^4 + 15\,d^2x^2 + 6\,x^3.$

18. $3\,a^6b - a^5b^2 + 2\,a^4b^3 - 4\,a^2b^5.$

19. $4\,m^2 + 10\,mn + 26\,mn^2 + 108\,m^3.$

20. $4\,c^7d^2 - 4\,c^6d^4 - 12\,c^5d^3 + 4\,c^5d^4.$

21. $5\,a^2b - 10\,ab^2 - 15\,b^3 - 20\,a^3.$

22. $2\,a^2x^2 - 3\,ax^3 - 5\,a^3x - 6\,a^2x^3.$

23. $6\,m^3 - 3\,m^2n + 12\,mn^2 - 27\,n^3.$

24. $x^4y^2 - 4\,x^3y + 6\,x^2y^2 - 4\,xy^3 + x^2y^4.$

25. $12\,a^5b^2c^3 - 24\,a^7bc^2 - 36\,ab^8c - 4\,ac^9.$

26. $14\,x^3y^4 - 91\,x^6y^5 - 56\,x^4y^2z^7 - 98\,x^3z^8.$

CASE II

85. When the expression is in the type form $a^2 \pm 2ab + b^2$.

Since, § 73, $(a+b)^2 = a^2 + 2ab + b^2$, the factors of $a^2 + 2ab + b^2$ are determined by inspection to be $(a+b)(a+b)$.

Similarly, § 74, the factors of $a^2 - 2ab + b^2$ are $(a-b)(a-b)$.

1. Factor $x^2 + 4x + 4$.

$$x^2 + 4x + 4 = (x+2)(x+2).$$

2. Factor $169a^2b^2 - 26ab + 1$.

$$169a^2b^2 - 26ab + 1 = (13ab - 1)(13ab - 1).$$

EXERCISE XXXVI

Factor:

1. $x^2 + 14x + 49$.
2. $x^2 - 14x + 49$.
3. $x^6 - 4x^3 + 4$.
4. $9x^2 - 24x + 16$.
5. $4x^2 - 20xy + 25y^2$.
6. $25a^2 + 70ab + 49b^2$.
7. $36a^4 - 84a^2c + 49c^2$.
8. $289x^2 + 136xy + 16y^2$.
9. $81x^2 - 234xy + 169y^2$.
10. $484x^4 - 44x^2y^2 + y^4$.
11. $256x^2y^2 - 96cxy + 9c^2$.

$(16xy) + 3c$

12. $49x^2y^2 + 14d^2xy + d^4$.
13. $144c^2 - 24c + 1$.
14. $1 + 28m^2 + 196m^4$.
15. $49x^4 - 28x^2 + 4$.
16. $81m^4 + 144m^3 + 64m^2$.
17. $25m^4 + 130m^2n + 169n^2$.
18. $(a+x)^2 - 2(a+x) + 1$.
19. $1 - 4(a+b) + 4(a+b)^2$.
20. $(a+b)^2 - 2(a+b)(c+d) + (c+d)^2$.
21. $(x-y)^2 - 2(x-y)(y-z) + (y-z)^2$.
22. $9(a-b)^2 - 6(a-b) + 1$.

CASE III

86. When the expression is in the type form $a^2 - b^2$.

Since, § 75, $(a+b)(a-b) = a^2 - b^2$, the factors of $a^2 - b^2$ are determined by inspection to be $(a+b)(a-b)$.

1. Factor $x^4 - 49$.

$$x^4 - 49 = (x^2 + 7)(x^2 - 7).$$

2. Factor $(a-b)^2 - c^2$.

$$(a-b)^2 - (c)^2 = \{(a-b)+c\}\{(a-b)-c\},$$
$$= (a-b+c)(a-b-c).$$

EXERCISE XXXVII

Factor:

1. $x^2 - 144$.
2. $16\,a^2 - 121\,b^2$.
3. $1 - 100\,c^2$.
4. $1 - 196\,y^2$.
5. $x^4 - 25\,y^2$.
6. $9\,a^2 - 4\,b^2$.
7. $64\,a^2b^2 - 1$.
8. $25\,x^2 - 4\,y^2$.
9. $49\,m^2 - 16\,x^4y^2$.
10. $121\,b^2x^2y^2 - 225\,z^2$.
11. $a^2 - (b-c)^2$.
12. $(a-b)^2 - m^2$.
13. $(a-b)^2 - (c+d)^2$.
14. $1 - (a-b)^2$.
15. $(a-b)^2 - 1$.
16. $25 - (a^2-b)^2$.
17. $(am+c^2)^2 - 36$.
18. $a^4 - (c^2-d^2)^2$.
19. $(a^2-bc)^2 - (a-d)^2$.
20. $(m-n^2)^2 - (m^2+n^2)^2$.
21. $(a+b+c)^2 - (c+d+e)^2$.
22. $(a+b-c-abc)^2 - 1$.
23. $(a^2+2\,bc-d^2)^2 - (2\,x+3\,y)^2$.
24. $(5\,c^2-7\,ad)^2 - 121\,b^2$.
25. $64\,a^2b^2 - (7\,d+11\,c)^2$.
26. $(a^2+m^2+n^2)^2 - (-b^3-c^3-d^3)^2$.

87. The terms of an expression may sometimes be arranged to show the type form $a^2 - b^2$.

1. Factor $a^2 - 2ab + b^2 - c^2 + 2cd - d^2$.

$$\begin{aligned} a^2 - 2ab + b^2 - c^2 + 2cd - d^2 \\ &= (a^2 - 2ab + b^2) - (c^2 - 2cd + d^2), \\ &= (a - b)^2 - (c - d)^2, \\ &= \{(a - b) + (c - d)\}\{(a - b) - (c - d)\}, \\ &= (a - b + c - d)(a - b - c + d). \end{aligned}$$

2. Factor $m^4 + n^4 - a^2 - b^2 + 2m^2n^2 + 2ab$.

$$\begin{aligned} m^4 + n^4 - a^2 - b^2 + 2m^2n^2 + 2ab \\ &= m^4 + 2m^2n^2 + n^4 - a^2 + 2ab - b^2, \\ &= (m^4 + 2m^2n^2 + n^4) - (a^2 - 2ab + b^2), \\ &= (m^2 + n^2)^2 - (a - b)^2, \\ &= \{(m^2 + n^2) + (a - b)\}\{(m^2 + n^2) - (a - b)\}, \\ &= (m^2 + n^2 + a - b)(m^2 + n^2 - a + b). \end{aligned}$$

3. Factor $4x^2 - 1 + 9y^2 - 16m^2 - 12xy + 8m$.

$$\begin{aligned} 4x^2 - 1 + 9y^2 - 16m^2 - 12xy + 8m \\ &= 4x^2 - 12xy + 9y^2 - 16m^2 + 8m - 1, \\ &= (4x^2 - 12xy + 9y^2) - (16m^2 - 8m + 1), \\ &= (2x - 3y)^2 - (4m - 1)^2, \\ &= \{(2x - 3y) + (4m - 1)\}\{(2x - 3y) - (4m - 1)\}, \\ &= (2x - 3y + 4m - 1)(2x - 3y - 4m + 1). \end{aligned}$$

It should be noticed that the terms containing cross products show the order in which the terms should be grouped.

EXERCISE XXXVIII

Factor:

1. $a^2 - 2ab + b^2 - 1.$
2. $1 - 4a + 4a^2 - x^2.$
3. $x^2 + y^2 + 2xy - 9b^2.$
4. $1 - x^2 + 2xy - y^2.$
5. $9 - m^2 + 2mn - n^2.$
6. $x^2 + 9 - 6x - 25a^2.$
7. $16x^2 + 8x - 100a^2 + 1$
8. $1 - 6a - 49b^2 + 9a^2.$
9. $9a^2 - 9b^2 + 42b - 49.$
10. $4x^2 - 16a^2b^2 - 20xy + 25y^2.$
11. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2.$
12. $c^2 - 2cd + d^2 - a^2 + 2ab - b^2.$
13. $c^2 + 2cd + d^2 - a^2 - 2ab - b^2.$
14. $4x^2 - 4xy + y^2 - a^2 + 2ab - b^2.$
15. $x^4 - 6x^2y + 9y^2 - 9a^2 + 30ad^2 - 25d^4.$
16. $4m^2 + 20mn + 25n^2 - 9c^2 - 12cd - 4d^2.$
17. $x^4 + 2x^2y^2 + y^4 - 49c^2 + 14c - 1.$
18. $1 + 4x + 4x^2 - x^4 - 4x^2y - 4y^2.$
19. $25 - 49c^4 - 10x^2y + 28c^2d + x^4y^2 - 4d^2.$
20. $1 - 4xy - c^4 - d^2e^2 + 4x^2y^2 - 2c^2de.$
21. $4y^2 - 1 - 25x^2 + 10x - 12yz + 9z^2.$
22. $1 - 9z^2 - x^4 + 16x^2y^2 - 6x^2z - 8xy.$
23. $-6x^2y^2 + 9y^4 - 9z^4 + x^4 - x^2y^2 + 6z^2xy.$
24. $4a^2 + 9b^2x^2 - 16a^4x^4 - 9b^2 + 24a^2bx^2 - 12abx.$
25. $49n^2x^2 - 169m^2x^4 - 16n^2y^2 + 25m^2y^2 + 104mnx^2y$
$- 70mnxy.$

88. An expression in the form $a^4 + a^2b^2 + b^4$ may be said to be in the disguised form of the difference of two perfect squares, and may be factored as before.

1. Factor $a^4 + a^2b^2 + b^4$.

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + a^2b^2 + b^4 + (a^2b^2 - a^2b^2), \\ &= a^4 + 2\,a^2b^2 + b^4 - a^2b^2, \\ &= (a^2 + b^2)^2 - (ab)^2, \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2). \end{aligned}$$

2. Factor $x^4 + 9\,x^2 + 81$.

$$\begin{aligned} x^4 + 9\,x^2 + 81 &= x^4 + 9\,x^2 + 81 + (9\,x^2 - 9\,x^2), \\ &= (x^4 + 18\,x^2 + 81) - (9\,x^2), \\ &= (x^2 + 9)^2 - (3\,x)^2, \\ &= \{(x^2 + 9) + 3\,x\}\{(x^2 + 9) - 3\,x\}, \\ &= (x^2 + 3\,x + 9)(x^2 - 3\,x + 9). \end{aligned}$$

3. Factor $a^4 - 13\,a^2 + 4$.

$$\begin{aligned} a^4 - 13\,a^2 + 4 &= a^4 - 4\,a^2 + 4 - 9\,a^2, \\ &= (a^2 - 2)^2 - (3\,a)^2, \\ &= \{(a^2 - 2) + 3\,a\}\{(a^2 - 2) - 3\,a\}, \\ &= (a^2 + 3\,a - 2)(a^2 - 3\,a - 2). \end{aligned}$$

EXERCISE XXXIX

Factor:

1. $a^4 + a^2b^2 + 25\,b^4$.
2. $1 + m^2 + m^4$.
3. $4\,x^4 - 61\,x^2y^2 + 9\,y^4$.
4. $x^4 + x^2y^2 + y^4$.
5. $a^4 + 9\,a^2 + 81$.
6. $x^4 + 11\,x^2y^2 + 36\,y^4$.
7. $a^4 + a^2 + 1$.
8. $9\,a^4 - 16\,a^2b^2 + 4\,b^4$.
9. $x^4 - 8\,x^2y^2 + 4\,y^4$.
10. $625\,a^4 + 25\,a^2 + 1$.

CASE IV

89. When the expression is in the type form

$$a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.$$

Since, § 77, $(a+b+c)^2=a^2+b^2+c^2+2\,ab+2\,ac+2\,bc$, the factors of $a^2+b^2+c^2+2\,ab+2\,ac+2\,bc$ are determined by inspection to be $(a+b+c)(a+b+c)$.

1. Factor $a^2+b^2+c^2-2\,ab+2\,ac-2\,bc$.

The cross products, $-2\,ab$ and $-2\,bc$, in both of which b occurs, show that b has the minus sign.

$$a^2+b^2+c^2-2\,ab+2\,ac-2\,bc=(a-b+c)(a-b+c).$$

EXERCISE XL

Factor:

1. $a^2+b^2+c^2+2\,ab-2\,ac-2\,bc$.
2. $a^2+b^2+c^2-2\,ab-2\,ac+2\,bc$.
3. $a^2+b^2+c^2-2\,ab+2\,ac-2\,bc$.
4. $x^2+4\,y^2+9\,z^2-4\,xy-6\,xz+12\,yz$.
5. $4\,a^2+9\,b^2+c^2-12\,ab-4\,ac+6\,bc$.
6. $25\,x^4+4\,x^2y^2+y^4-20\,x^3y+10\,x^2y^2-4\,xy^3$.
7. $4\,x^2+9\,x^2a^2+16\,x^2b^2-12\,x^2a+16\,x^2b-24\,x^2ab$.
8. $16\,b^2x^2+a^4+4\,b^2+16\,b^2x-8\,a^2bx-4\,a^2b$.
9. $4\,x^4+x^2y^2+9\,y^2-4\,x^3y+12\,x^2y-6\,xy^2$.
10. $25\,c^4+30\,abc^2-20\,ac^3+9\,a^2b^2-12\,a^2bc+4\,a^2c^2$.
11. $12\,ab^2c^3+4\,b^4c^2+a^4b^2-4\,a^2b^3c-6\,a^3bc^2+9\,c^4a^2$.
12. $60\,a^3b^3+1-10\,a^2b-12\,ab^2+36\,a^2b^4+25\,a^4b^2$.

CASE V

90. When the expression is in the type form

$$a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3.$$

Since, § 78, $(a \pm b)^3 = a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3$, the factors of $a^3 \pm 3\,a^2b + 3\,ab^2 \pm b^3$ are determined by inspection to be $(a \pm b)(a \pm b)(a \pm b)$.

1. Factor $x^3 - 3\,x^2y + 3\,xy^2 - y^3$.

$$x^3 - 3\,x^2y + 3\,xy^2 - y^3 = (x - y)(x - y)(x - y).$$ $(x-y)^3$

2. Factor $1 - 6\,x + 12\,x^2 - 8\,x^3$.

$$1 - 6\,x + 12\,x^2 - 8\,x^3 = 1 - 3(1)^2(2\,x) + 3(1)(2\,x)^2 - (2\,x)^3,$$
$$= (1 - 2\,x)(1 - 2\,x)(1 - 2\,x).$$

EXERCISE XLI

Factor:

1. $m^3 - 3\,m^2n + 3\,mn^2 - n^3$.
2. $1 - 3\,y + 3\,y^2 - y^3$.
3. $a^3 + 3\,a^2 + 3\,a + 1$.
4. $8\,a^3 - 12\,a^2 + 6\,a - 1$.
5. $27\,x^6 - 54\,x^4b + 36\,x^2b^2 - 8\,b^3$.
6. $125\,a^3 + 75\,a^2 + 15\,a + 1$.
7. $8\,b^3 + 36\,b^2c + 54\,bc^2 + 27\,c^3$.
8. $1 - 21\,x + 147\,x^2 - 343\,x^3$.
9. $216\,a^3 + 108\,a^2x + 18\,ax^2 + x^3$.
10. $8\,a^3 - 36\,a^2b + 54\,ab^2 - 27\,b^3$.
11. $27\,a^3x^3 + 108\,a^2bx^2y + 144\,ab^2xy^2 + 64\,b^3y^3$.
12. $125\,x^3y^3 - 300\,abx^2y^2 + 240\,a^2b^2xy - 64\,a^3b^3$.
13. $1331\,a^6 - 1089\,a^4b + 297\,a^2b^2 - 27\,b^3$.

CASE VI

91. When the expression is in the type form

$$x^2 + (a + b)x + ab.$$

Since, § 79, $(x + a)(x + b) = x^2 + (a + b)x + ab$, the factors of $x^2 + (a + b)x + ab$ are determined by inspection to be $(x + a)(x + b)$.

1. Factor $x^2 + 8x + 15$.

$$x^2 + 8x + 15 = (x + 3)(x + 5).$$

2. Factor $x^2 - 2x - 15$.

$$x^2 - 2x - 15 = (x + 3)(x - 5).$$

It is to be noticed that the algebraic sum of the factors of the last term is the coefficient of the middle term in the trinomial, and it is therefore necessary to find two numbers such that their product is the last term and their sum the coefficient of the middle term.

EXERCISE XLII

Factor:

1. $x^2 + 5x + 6$.	**5.** $x^2 - 3x - 10$.	**9.** $x^2 - 8x + 7$.
2. $x^2 - 2x - 3$.	**6.** $x^2 - 3x - 28$.	**10.** $x^2 - 5x - 36$.
3. $x^2 - 4x - 5$.	**7.** $x^2 + 5x - 24$.	**11.** $a^2 + 2a - 35$.
4. $x^2 - 5x + 4$.	**8.** $x^2 - x - 20$.	**12.** $a^2 - 7a - 18$.

13. $a^4 - 8a^2 - 33$.	**19.** $x^2 + bx + cx + bc$.
14. $a^4 + 12a^2 + 11$.	**20.** $x^2 - bx + cx - bc$.
15. $a^2 - 11a - 42$.	**21.** $x^2 - bx - cx + bc$.
16. $a^4b^4 - 9a^2b^2 - 136$.	**22.** $x^2 + bx - cx - bc$.
17. $a^2 - 26a + 133$.	**23.** $x^2 + (a - d)x - ad$.
18. $a^4x^4 - 19a^2x^2 - 92$.	**24.** $x^2 + (a + d)x + ad$.

CASE VII

92. When the expression is in the type form

$$acx^2 + x(bc + ad) + bd.$$

Since, § 80, $(ax + b)(cx + d) = acx^2 + x(bc + ad) + bd$, the factors of $acx^2 + x(bc + ad) + bd$ are determined by inspection to be $(ax + b)(cx + d)$.

1. Factor $2x^2 + 5xy - 3y^2$.

$$2x^2 + 5xy - 3y^2 = (2x - y)(x + 3y).$$

Since the first term of the trinomial is the product of the first terms of the binomials, the first terms of the binomials must be $2x$ and x; since the last term of the trinomial is the product of the last terms of the binomials, the last terms of the binomials must be $3y$ and y. The sign of the last term of the trinomial is minus; hence the last terms of the binomials must have opposite signs. By trial the factors are now found as given above.

If the trinomial contains no common monomial factor, the binomial contains no common monomial factor.

The middle term is found by multiplying the first term of the first binomial by the second term of the second binomial, and by multiplying the second term of the first binomial by the first term of the second binomial, and taking the algebraic sum of these products for the middle term. The process is represented:

$$2x^2 + 5xy - 3y^2 = (2x - y)(x + 3y).$$

Writing the possible factors of $2x^2$,

$$2x^2 + 5xy - 3y^2 \qquad (2x \qquad)(x \qquad),$$

and in the parentheses writing also the possible factors of $-3y^2$, the factors of $2x^2 + 5xy - 3y^2$ are either $(2x + 3y)(x - y),$ (1)

or $(2x - 3y)(x + y),$ (2)

or $(2x + y)(x - 3y),$ (3)

or $(2x - y)(x + 3y).$ (4)

Each of the possibilities (1), (2), (3), (4) must be tried by actual multiplication until the proper factors are discovered.

2. Factor $2x^2 + 5xy - 12y^2$.

The possible factors of $2x^2$ are $2x$ and x; the possible factors of $12y^2$ are $12y$ and y, y and $12y$, $6y$ and $2y$, $2y$ and $6y$, $4y$ and $3y$, $3y$ and $4y$. That is, $2x$ and x must be tried with each of the six possible factors of $12y^2$.

$(2x \quad 12y)(x \quad y),$ (1)

$(2x \quad y)(x \quad 12y),$ (2)

$(2x \quad 6y)(x \quad 2y),$ (3)

$(2x \quad 2y)(x \quad 6y),$ (4)

$(2x \quad 4y)(x \quad 3y),$ (5)

$(2x \quad 3y)(x \quad 4y).$ (6)

Possibilities (1), (3), (4), and (5) are immediately eliminated because the binomials contain a factor which is not a factor of the trinomial. By trial,

$$2x^2 + 5xy - 12y^2 = (2x - 3y)(x + 4y).$$

3. Factor $2x^2 - x(a + 2b) + ab$.

The possible factors are:

$(2x \quad a)(x \quad b),$ (1)

$(2x \quad b)(x \quad a),$ (2)

$(2x \quad ab)(x \quad 1),$ (3)

$(2x \quad 1)(x \quad ab),$ (4)

$$2x^2 - x(a + 2b) + ab = (2x - a)(x - b).$$

$6x +$

EXERCISE XLIII

Factor:

1. $6x^2 + 7x + 2$.
2. $4x^2 + 8x + 3$.
3. $2x^2 - 3x + 1$.
4. $3x^2 - 8x + 4$.
5. $6x^2 - x - 1$.
6. $2x^2 + 3x - 2$.
7. $3x^2 - 8x + 5$.
8. $2x^2 + x - 21$.
9. $3x^2 + 11x - 20$.
10. $8x^2 + 10x - 12$.
11. $6x^2 - 7x - 20$.
12. $3x^2 - 29x + 40$.
13. $16x^2 - 82x - 33$.
14. $26x^2 - 141x - 11$.
15. $34x^2 + 131x - 99$.
16. $30x^2 - 9x - 3$.
17. $28x^2 + 23x - 15$.
18. $18x^2 - x - 39$.
19. $10x^2 - 59x + 85$.
20. $95x^2 - 138x + 7$.
21. $16x^2 - 211x + 39$.
22. $24x^2 - 54x - 105$.
23. $84x^2 + 148x - 112$.
24. $15x^2 - 101x - 28$.
25. $6x^2 - 61x - 55$.
26. $60x^2 - 147x - 156$.
27. $6x^2 - 77x + 92$.
28. $80x^2 - 28x - 52$.
29. $10x^2 + 9x - 91$.
30. $84x^2 - 158x + 70$.
31. $abx^2 - x(b + ac) + c$.
32. $ax^2 + x(1 + ab) + b$.
33. $abx^2 + x(ad - bc) - cd$.
34. $2a^2x^2 + x(2ac - ab) - bc$.
35. $abx^2 + x(ac - b^2) - bc$.
36. $abx^2 - x(a^2 + bc) + ac$.
37. $2abx^2 - x(bc + 4a^2) + 2ac$.
38. $mnx^2 - x(an - 3bm) - 3ab$.
39. $2ax^2 - x(3ad + 4c) + 6cd$.
40. $7ax^2 - x(a - 14d) - 2d$.
41. $8mnx^2 - x(12an - 10bm) - 15ab$.

CASE VIII

93. When the expression is in the type form $a^3 \pm b^3$.

By § 81, $a^3 + b^3 = (a^2 - ab + b^2)(a + b)$.

$$a^3 - b^3 = (a^2 + ab + b^2)(a - b).$$

Hence the factors of $a^3 + b^3$ are determined by inspection to be $(a + b)(a^2 - ab + b^2)$; and the factors of $a^3 - b^3$ are $(a - b)(a^2 + ab + b^2)$.

1. Factor $x^3 + y^3$.

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

2. Factor $x^3 + 27$.

$$\begin{aligned} x^3 + 27 &= (x)^3 + (3)^3, \\ &= (x + 3)[(x)^2 - (x)(3) + (3)^2], \\ &= (x + 3)(x^2 - 3x + 9). \end{aligned}$$

3. Factor $125x^3 - 64z^3$.

$$\begin{aligned} 125x^3 - 64z^3 &= (5x)^3 - (4z)^3, \\ &= (5x - 4z)[(5x)^2 + (5x)(4z) + (4z)^2], \\ &= (5x - 4z)(25x^2 + 20xz + 16z^2). \end{aligned}$$

4. Factor $x^3 + y^6$.

$$\begin{aligned} x^3 + y^6 &= (x)^3 + (y^2)^3, \\ &= [x + y^2][(x)^2 - (x)(y^2) + (y^2)^2], \\ &= (x + y^2)(x^2 - xy^2 + y^4). \end{aligned}$$

5. Factor $64 - (a - b)^3$.

$$\begin{aligned} 64 - (a - b)^3 &= (4)^3 - (a - b)^3, \\ &= [4 - (a - b)][(4)^2 + 4(a - b) + (a - b)^2], \\ &= (4 - a + b)(16 + 4a - 4b + a^2 - 2ab + b^2). \end{aligned}$$

EXERCISE XLIV

Factor:

1. $x^3 - y^3$.
2. $x^3 + y^3$.
3. $8 + a^3$.
4. $a^3 - 8$.
5. $x^6 + 64$.
6. $125\,a^3 - 8\,b^3$.
7. $8\,b^3 - 125\,a^3$.
8. $343\,c^3 - 1$.
9. $1 + 1000\,b^3$.
10. $512\,d^3 - x^{15}$.
11. $8\,x^3 + y^3$.
12. $y^3 - 27\,z^3$.
13. $1 - 125\,a^3$.
14. $216\,x^3y^3 + 27\,z^9$.
15. $a^3b^6 - c^{12}$.
16. $64\,c^3 - b^3$.
17. $1 - 343\,b^6$.
18. $729\,m^3 - 1$.
19. $1000\,a^3 + 1$.
20. $m^3 - (a-b)^3$.
21. $(a+b)^3 + c^3$.
22. $(m-n)^3 + (a+b)^3$.
23. $8(a^2 - b^2)^3 - 1$.
24. $27(a-c)^3 + 64(b-d)^3$.
25. $343(c^2 + d^2)^3 - 1$.
26. $(6\,c^3 - a^3)^3 - (a^2 + b^2)^3$.
27. $(s+a)^3 - (s-c)^3$.
28. $1728(x+y)^3 - 343(x-y)^3$.
29. $729(a-b)^3 + 125(a+b)^3$.
30. $512(a+b+c)^3 - 1331(a-b-c)^3$.
31. $x^3(a-b+c)^3 + a^3(x-y+z)^3$.
32. $x^6(a+b+c)^3 - b^3(x+y+z)^3$.
33. $a^6b^3(x^2+y-z)^3 + a^3b^9(x-y-z^2)^3$.
34. $a^3(x+y-z)^3 - b^3(y+z-x)^3$.
35. $343\,a^6(s+a-b)^3 + 1000\,b^6(a+b+s)^3$.

CASE IX

94. When the terms of the expression can be arranged to show a common binomial factor.

1. Factor $ax + ay + bx + by$.

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by), \\ &= a(x+y) + b(x+y), \\ &= (a+b)(x+y). \end{aligned}$$

After the terms have been arranged to show the common binomial factor, the process is really one of division, thus:

$$\begin{array}{r|l} a(x+y) + b(x+y) & (x+y) \\ \cline{2-2} a(x+y) \phantom{{}+ b(x+y)} & a+b \\ \hline b(x+y) & \\ b(x+y) & \end{array}$$

Case IX may then be considered as an extension of Case I; in Case I, a common monomial factor is removed; in Case IX, a common binomial factor is removed.

2. Factor $2\,x^3 - 3\,ax + 4\,bx^2 - 6\,ab$.

$$\begin{aligned} 2\,x^3 - 3\,ax + 4\,bx^2 - 6\,ab &= (2\,x^3 - 3\,ax) + (4\,bx^2 - 6\,ab), \\ &= x(2\,x^2 - 3\,a) + 2\,b(2\,x^2 - 3\,a), \\ &= (x + 2\,b)(2\,x^2 - 3\,a). \end{aligned}$$

3. Factor $2\,x^3 - 4\,x^2y - 3\,x + 6\,y$.

$$\begin{aligned} 2\,x^3 - 4\,x^2y - 3\,x + 6\,y &= (2\,x^3 - 4\,x^2y) - (3\,x - 6\,y), \\ &= 2\,x^2(x - 2\,y) - 3(x - 2\,y), \\ &= (2\,x^2 - 3)(x - 2\,y). \end{aligned}$$

4. Factor $12\,a^2b^2 - 42\,b^3c + 16\,a^3 - 56\,abc$.

$$\begin{aligned} &12\,a^2b^2 - 42\,b^3c + 16\,a^3 - 56\,abc \\ &\qquad = 2\,[3\,b^2(2\,a^2 - 7\,bc) + 4\,a(2\,a^2 - 7\,bc)], \\ &\qquad = 2(3\,b^2 + 4\,a)(2\,a^2 - 7\,bc). \end{aligned}$$

EXERCISE XLV

Factor:

1. $mx + am + nx + an.$
2. $mx - am - nx + an.$
3. $2\,bx^2 - abx + 4\,cx - 2\,ac.$
4. $m^2n - 3\,abn - 2\,m^2p + 6\,abp.$
5. $x^2 + ax + bx + ab.$
6. $x^2 - ax - bx + ab.$
7. $6\,bx - 15\,ab - 4\,dx + 10\,ad.$
8. $dmnx - ac^2dx + mnrs - ac^2rs.$
9. $-2\,an + 3\,ap + 2\,bn - 3\,bp.$
10. $6\,ce - 9\,de + 4\,ac - 6\,ad.$
11. $a^4b^2 - 2\,a^2b^4 + a^3b - 2\,ab^3.$
12. $2\,bcm - 4\,ab^2c + 7\,amn - 14\,a^2bn.$
13. $2\,m^2n - 2\,a^2b - 3\,cm^2n + 3\,a^2bc.$
14. $-a^2bx^2 - 4\,bc - 2\,a^2x^2y - 8\,cy.$
15. $8\,as^2 + 10\,art + 12\,bs^2 + 15\,brt.$
16. $-mnx - 2\,mn + p^2x + 2\,p^2.$
17. $14\,a^2ce^2f + 35\,b^2de^2f + 6\,a^2cxy + 15\,b^2dxy.$
18. $10\,ac + bc - 110\,ad - 11\,bd.$
19. $rs + a^2n - 3\,d^2rs - 3\,a^2d^2n.$
20. $8\,acxy - 14\,a^2xz + 21\,acdz - 12\,c^2dy.$
21. $6\,a^3 - 8\,a^2b^2 - 15\,abc + 20\,b^3c.$
22. $6\,x^3 - 33\,acx - 8\,cx^2 + 44\,ac^2.$

95. A **theorem** is a statement of a general truth which requires demonstration.

THE FACTOR THEOREM

96. *If any expression containing x reduces to 0 when a is substituted for x, then $x-a$ is a factor of that expression.*

Let E represent the expression. Divide E by $x-a$ until the remainder does not contain any power of x. Let R be the remainder, and Q be the quotient. Then

$$E = Q(x-a)+R. \tag{1}$$

Equation (1) is always true whatever may be the value of x. Take $x=a$, and substitute in (1);

$$0 = Q(a-a)+R, \tag{2}$$

$$0 = Q(0)+R, \tag{3}$$

$$0 = R. \tag{4}$$

In (2), E becomes 0, because the expression is taken as one containing x, which becomes 0 when a is substituted for x; also in (2), $a-a=0$, and $Q \cdot 0 = 0$. In (4), R becomes 0; or, in other words, there is no remainder. Consequently $x-a$ is an exact divisor or factor of E.

Since $x+a = x-(-a)$, the theorem holds true if (a) be replaced by $(-a)$ in the statement, thus: if any expression containing x reduces to 0 when $(-a)$ is substituted for x, then $x-(-a)$, or $x+a$, is a factor of that expression.

The Factor Theorem has a wide application, and may be applied as a check to most of the preceding cases, and to many forms which are not included in those already given.

1. Factor $x^3 + 1$.

Take $x = 1$, and substitute in $x^3 + 1$: $1^3 + 1 = 1 + 1 = 2$. Here the expression $x^3 + 1$ does not become 0, or vanish, and so $x - 1$ is not a factor. Take $x = -1$, and substitute in $x^3 + 1$: $(-1)^3 + 1 = -1 + 1 = 0$. The expression vanishes, and $x + 1$ is, therefore, a factor of $x^3 + 1$. By division the other factor or factors may be established.

2. Factor $x^5 + y^5$.

Take $x = y$, and substitute: $(y)^5 + y^5 = 2y^5$; $x - y$ is not a factor. Substitute $x = -y$: $(-y)^5 + y^5 = 0$; $x + y$ is a factor.

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

The second factor, $x^4 - x^3y + x^2y^2 - xy^3 + y^4$, is found by division.

3. Factor $x^5 + 32y^5$.

By substitution, $x + y$ and $x - y$ are shown not to be factors. Try $x = 2y$: $(2y)^5 + 32y^5 = 64y^5$; $x - 2y$ is not a factor. Try $x = -2y$: $(-2y)^5 + 32y^5 = 0$; $x + 2y$ is a factor. By division $x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4$ is the other factor.

$$x^5 + 32y^5 = (x + 2y)(x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4).$$

4. Factor $x^3 + x^2 - 7x + 2$.

The substitution of $x = 1$, $x = -1$ both give remainders; now try $x = 2$: $(2)^3 + (2)^2 - 7(2) + 2 = 8 + 4 - 14 + 2 = 0$; $x - 2$ is a factor.

By division, $x^2 + 3x - 1$ is the other factor.

$$x^3 + x^2 - 7x + 2 = (x - 2)(x^2 + 3x - 1).$$

The factor obtained by division must be carefully inspected to determine if it is prime.

EXERCISE XLVI

By use of the Factor Theorem, separate in factors:

1. $x^3 + y^3$.
2. $x^5 + y^5$.
3. $a^4 - 1$.
4. $x^3 + 8$.
5. $1 + x^5$.
6. $x^3 - 27$.
7. $64 - x^3$.
8. $32 - x^5$.
9. $a^2 - 16a + 64$.
10. $x^2 - 25$.
11. $x^2 - 57x + 56$.
12. $343 - x^3$.
13. $x^4 - 81$.
14. $x^5 + 243$.
15. $32a^5 + b^5$.
16. $a^7 + b^7$.
17. $a^7 - b^7$.
18. $1 - 8x^3$.
19. $27x^3 - y^3$.
20. $32x^5 + y^5$.
21. $x^6 - y^6$.
22. $x^6 + y^6$.
23. $x^3 + y^6$.
24. $x^5 + y^{10}$.
25. $x^3 - x^2 - x + 1$.
26. $x^3 - x^2 - 3x - 1$.
27. $x^3 - 2x^2 - 5x - 2$.
28. $x^3 - 3x - 2$.
29. $x^3 - x^2 - x - 2$.
30. $x^3 + x^2 - x + 2$.
31. $x^3 - 5x + 2$.
32. $x^3 - x^2 - 2x - 12$.
33. $x^3 + 2x^2 - 3x + 20$.
34. $x^3 + x^2 - 13x - 21$.
35. $x^3 - 8x^2 + 12x + 9$.
36. $x^3 + 13x^2 + 43x + 6$.
37. $x^4 + 3x^3 - x^2 - 2x + 1$.
38. $a^4 - a^3 + 5a^2 + 14a - 16$.
39. $2x^4 - 5x^3 + 13x^2 - 9x - 1$.
40. $3x^4 + 8x^3 + 8x^2 + 7x - 2$.
41. $2a^4 - 7a^3 + 8a^2 - 6a + 4$.
42. $a^4 + 6a^3 + 11a^2 - a - 21$.
43. $a^4 - 8a^3 + 17a^2 - 14a + 8$.
44. $2x^4 - 13x^3 + 16x^2 - 6x + 5$.
45. $x^5 - 4x^4 + 10x^3 - 5x^2 - 4x + 2$.
46. $3x^5 + 3x^4 - 5x^3 - 6x - 4$.

HINTS ON FACTORING

97. It is impossible to give any definite method of attack in factoring. A monomial factor should at once be removed. Every factor should be carefully inspected for further factors. It may happen that an expression can be factored by different methods. If the expression can be factored as the difference of two squares, it is generally preferable to do so.

1. Factor $x^6 - y^6$.

By Case III,

$$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3),$$

by Case VIII, $= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$.

By Factor Theorem,

$$x^6 - y^6 = (x - y)(x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5),$$

$$= (x - y)[x^4(x + y) + x^2y^2(x + y) + y^4(x + y)],$$

by Case IX, $= (x - y)(x + y)(x^4 + x^2y^2 + y^4)$,

by § 88, $= (x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$.

2. Factor $x^6 + y^6$.

By Case VIII, $x^6 + y^6 = (x^2)^3 + (y^2)^3 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

3. Factor $x^{12} + 64$.

By Case VIII, $x^{12} + 64 = (x^4)^3 + (4)^3 = (x^4 + 4)(x^8 - 4x^4 + 16)$.

4. Factor $x^9 - y^9$.

By Case VIII, $x^9 - y^9 = (x^3)^3 - (y^3)^3 = (x^3 - y^3)(x^6 + x^3y^3 + y^6)$,

$$= (x - y)(x^2 + xy + y^2)(x^6 + x^3y^3 + y^6).$$

5. Factor $x^{10} + y^{10}$.

By Factor Theorem,

$$x^{10} + y^{10} = (x^2)^5 + (y^2)^5 = (x^2 + y^2)(x^8 - x^6y^2 + x^4y^4 - x^2y^6 + y^8).$$

6. Factor $3(a-b)^3-a+b$.

$$3(a-b)^3-a+b=3(a-b)^3-(a-b),$$

by Case IX, $\qquad = (a-b)[3(a-b)^2-1],$

$$=(a-b)(3a^2-6ab+3b^2-1).$$

7. Factor $a^3+b^3+c^3-3abc$.

$$a^3+b^3+c^3-3abc=(a^3+b^3)+(c^3-3abc). \qquad (1)$$

Now

$$a^3+b^3=(a+b)^3-3a^2b-3ab^2=(a+b)^3-3ab(a+b). \qquad (2)$$

Substitute $a^3+b^3=(a+b)^3-3ab(a+b)$ in (1),

$$a^3+b^3+c^3-3abc=(a+b)^3-3ab(a+b)+c^3-3abc, \qquad (3)$$

$$=[(a+b)^3+c^3]-3ab(a+b)-3abc, \qquad (4)$$

by Case VIII, $\qquad =(a+b+c)[(a+b)^2-c(a+b)+c^2]$

$$-3ab[a+b+c], \qquad (5)$$

by Case IX, $\qquad =(a+b+c)[(a+b)^2-c(a+b)+c^2-3ab], \qquad (6)$

$$=(a+b+c)(a^2+2ab+b^2-ac-bc+c^2$$

$$-3ab), \qquad (7)$$

$$=(a+b+c)(a^2+b^2+c^2-ab-ac-bc). \qquad (8)$$

REVIEW EXERCISE XLVII

Factor:

1. $x^2-22x+121$. **4.** $343x^3-1$.

2. $4x^2-49a^4$. **5.** x^4-4x^2-60.

3. $x^2+28xy+196y^2$. **6.** $9a^2b^2-y^2$.

7. $1+a^2+b^2-2a-2b+2ab$.

8. $1-3a+3a^2-a^3$.

9. $3x^2+10xy+3y^2$. **11.** $81a^4-16b^4$.

10. $m^4+m^2n^2+n^4$. **12.** $a^2-b^2+2bc-c^2$.

13. $(m-n)^2+2x(m-n)+x^2$.

14. $a^4x^4+a^2x^2+1$.

15. $1-x^2-y^2-2xy$.

16. $x^2-6ax-9b^2-18ab$.

17. $x^3+x^2y+xy^2+y^3$.

18. $(2a+b)^2-(2b+a)^2$.

19. y^2-y-6.

20. $6x^2+13x-5$.

21. $(27y^3)^2-2(27y^3)(8b^3)+(8b^3)^2$.

22. $1-a^2x^2-b^2y^2+2abxy$.

23. $a^2+b^2+2ab-4a^2b^2$.

24. $1-18x-63x^2$.

25. $4x^2-5x+1$.

26. $4a^2b^2-(a^2+b^2-c^2)^2$.

27. $nx-x+y-ny$.

28. x^4-16y^4.

29. $a^4b^2-6a^3b+9a^2$.

30. $28x^4y+64x^3y-60x^2y$.

31. $x^4+y^4-18x^2y^2$.

32. $3x^3+2x^2-2x-1$.

33. $(7x+4y)^2-(2x-y)^2$.

34. $250(a-b)^3+2$.

35. $a^2b^2-a^2-b^2+1$.

36. $x^2y^2z^2-x^2z-y^2z+1$.

37. $3x^2-6x+9$.

38. $2x^3+5x^2-12x$.

39. $27a^3+125b^3$.

40. $x^2y+3xy^2-3x^3-y^3$.

41. $56+x-x^2$.

42. $5x^3y^2+5x^2yz-60xz^2$.

43. x^3+x^2+x+1.

44. $x^2-18x+32$.

45. $x^2-2ax-2bx+4ab$.

46. $x^4y^2+2x^2y^2z^2+y^2z^4$.

47. $a^2x^2-2acxz-b^2y^2+c^2z^2$.

48. a^6-b^6.

49. a^3+b^3+a+b.

50. $a^3-2a^2b+2ab^2-b^3$.

51. $(5x-2)^2-(x-4)^2$.

52. $1+bx-(a^2+ab)x^2$.

53. $a^2x^2-a^2y^2-b^2x^2+b^2y^2$.

54. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.

55. $a^3-5a^2b+4b^3$.

56. $64a^6-1$.

57. $y^2 - c^2 + 2cx - x^2.$

58. $bc(b - c) + ca(c - a) + ab(a - b).$

59. $x^3 - 2x^2y + x^2 - 4x + 8y - 4.$

60. $a^3 - a^2b + ab^2 - b^3.$

61. $a^2(b - c) + b^2(c - a) + c^2(a - b).$

62. $x^{12} - y^{12}.$

63. $x^{12} + y^{12}.$

64. $110 - x - x^2.$

65. $acx^2 - bcx + adx - bd.$

66. $a^4b + 8ac^3bm^6.$

67. $4c^3x^2 + 4c^2xy + cy^2.$

68. $9x^4 - 40x^2y^2 + 16y^4.$

69. $x^5 + y^5.$

70. $a^4 - a^3y + az^3 - yz^3.$

71. $x^4 - (x - 6)^2.$

72. $m^6 - 64n^6.$

73. $a^9 - x^6.$

74. $8(x + y)^3 - (2x - y)^3.$

75. $x^3 - 7x^2 + 14x - 8.$

76. $3x^2 + x(3a + b) + ab.$

77. $x^4 - 5x^2y^2 + 4y^4.$

78. $(x^3 + x - 1)^2 - (x^3 - x - 1)^2.$

79. $x^2 + 3x^3 - x^4 - 3x.$

80. $a^{10} + b^{10}.$

81. $x^7 + y^7.$

82. $x^3 - 3x - 2.$

83. $(a - 2b)^2 - 9 - 3(a - 2b + 3).$

84. $a^2 - b^2 + x^2 - y^2 + 2(ax - by).$

85. $mx^3 + anx^2 - b^2nx - ab^2m + nx^3 + amx^2 - b^2mx - ab^2n.$

86. $x^4 + 2x^3 + 2x^2 + 2x + 1.$

87. $2x^4 + x^3 + 4x^2 + 6x + 2.$

88. $m^4 - m^3z - m^2z - mz - z - 1.$

89. $2x^2 + 3xy + y^2 - 2x - y.$

90. $9(a^2 - ac)^2 - 6ac^2(a - c) + c^4.$

91. $(a-b)(2x^2-2xy)+(b-a)(2xy-2y^2)$.

92. $(x-1)(x-2)(x-3)+(x-1)(x-2)-(x-1)$.

93. $x^3-2x-21$.

94. $x^2-2xy-3y^2-x^3+2x^2y+3xy^2$.

95. $(x+y)^3+x+y$.

96. $(m+n-3)^2-3(m+n-3)-4$.

97. $x^6-y^6-(x-y)^2$.

98. $(a^2-b^2)+xy(3b+a)-2x^2y^2$.

99. $m^3-3m^2n+3mn^2-n^3-m+n$.

100. $x^3+y^3+z^3-3xyz$.

101. $(x-y)^3+1-3(x-y+1)$.

102. $x^4+x^3+x^2y^2+x^2+y^2+xy^2$.

103. $(x+y)^4-x^4-y^4$.

104. $(a^3-b^3)-(a^2-b^2)-(a-b)^2$.

105. $a^4+b^4+c^4-2a^2b^2-2a^2c^2+2b^2c^2$.

106. $4b^4c^4-b^4-2b^2c^2-c^4$.

107. $x^4+x^3+x^2y^2-y^3+y^4$.

108. $3x^3+x^2(2a-9)+x(3-6a)+2a$.

109. $x^4-x^3-x^2+3x-2$.

110. $(a+b)^3-c^2(a+b)-c(a+b)^2+c^3$.

111. $x^4-2x^3-2x^2-2x-3$.

112. $1+b^3+c^3-3bc$.

113. $a^4+2a^2+1-5(a^2+1)+6$.

114. x^4-2x^2-5x+2.

115. $(x-y)^7-x+y$.

CHAPTER VII

HIGHEST COMMON FACTORS. LOWEST COMMON MULTIPLES

THE HIGHEST COMMON FACTOR

98. A **common factor** of two or more algebraic expressions is an exact divisor of each of the expressions. Two expressions are said to be **prime** to each other when they have no common factor other than 1. The **highest common factor** of two or more algebraic expressions is the product of all the common prime factors. Thus, a^2 and 2 are common factors of $2a^3x$ and $6a^4b$, and the highest common factor is $2a^3$.

99. The highest common factor — abbreviated H. C. F. — of several monomials is readily found by inspection. Thus, find the H. C. F. of $6x^3y^2$, $12x^2y^3$, $40x^5y^2$.

$$6x^3y^2 = 2 \cdot 3 \cdot x^3 \cdot y^2,$$

$$12x^2y^3 = 2 \cdot 2 \cdot 3 \cdot x^2 \cdot y^3,$$

$$40x^5y^2 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot x^5 \cdot y^2,$$

$$\text{H.C.F.} = 2 \cdot x^2 \cdot y^2 = 2x^2y^2.$$

Note. The H. C. F. of two algebraic expressions has reference to the degree of the factor; the greatest common divisor of two arithmetic quantities has reference to value. The H. C. F. of a and a^3 is a; if a is any common fraction, and equal, say, to $\frac{1}{3}$, the greatest common divisor of $a = \frac{1}{3}$ and $a^3 = \frac{1}{27}$ is $\frac{1}{27}$. The terms H. C. F. and G. C. D. are not, therefore, interchangeable.

THE H.C.F. BY FACTORING

100. 1. Find the H. C. F. of $a^2 + ab$, $a^3 + b^3$.

$$a^2 + ab = a(a+b),$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2),$$
$$\text{H.C.F.} = a + b.$$

2. Find the H. C. F. of $a^2 - b^2$, $a^3 - b^3$, $a^4 - b^4$.

$$a^2 - b^2 = (a+b)(a-b),$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$
$$a^4 - b^4 = (a^2 + b^2)(a+b)(a-b),$$
$$\text{H.C.F.} = a - b.$$

3. Find the H. C. F. of $m^3 - 27$, $m^2 - 6m + 9$, $m^2 + m - 12$.

$$m^3 - 27 = (m-3)(m^2 + 3m + 9),$$
$$m^2 - 6m + 9 = (m-3)(m-3),$$
$$m^2 + m - 12 = (m+4)(m-3),$$
$$\text{H.C.F.} = (m-3).$$

4. Find the H. C. F. of $4x^4 - 7x^2y^2 + 3y^4$, $x^3 - x^2y - xy^2 + y^3$, $x^4 - 2x^2y^2 + y^4$.

$$4x^4 - 7x^2y^2 + 3y^4 = (4x^2 - 3y^2)(x^2 - y^2),$$
$$= (4x^2 - 3y^2)(x+y)(x-y),$$
$$x^3 - x^2y - xy^2 + y^3 = x^2(x-y) - y^2(x-y),$$
$$= (x^2 - y^2)(x-y) = (x+y)(x-y)(x-y),$$
$$x^4 - 2x^2y^2 + y^4 = (x^2 - y^2)^2 = (x+y)(x-y)(x+y)(x-y),$$
$$\text{H.C.F.} = (x+y)(x-y).$$

The H. C. F. of several algebraic expressions is found *by taking the product of the common prime factors the least number of times they occur in any of the given expressions.*

EXERCISE XLVIII

Find, by factoring, the H. C. F. of the following expressions:

1. $ab + a$, $b^2 + b$.
2. $15x - 9$, $6 - 10x$.
3. $ax^2 - 2axy$, $2ax^2 - axy$.
4. $am - an + bm - bn$, $am - an$.
5. $(x+1)^2$, $x^2 + x$.
6. $a^2 - 4a + 4$, $3ab - 6b$.
7. $a^2 - 6a + 9$, $ab - 3a - 3b + 9$.
8. $x^2 - 1$, $x^2 - x$.
9. $x^4 - y^4$, $x^2 + y^2$.
10. $a^3 + 1$, $a^2 - 1$.
11. $25a^2 - 9(b-1)^2$, $6b - 10a - 6$.
12. $x^2 + 3x + 2$, $x^2 + x - 2$.
13. $x^2 - 9x + 20$, $x^2 - 16$.
14. $4a^2 - 5ab - 6b^2$, $8a^2 + 2ab - 3b^2$.
15. $x^4 + x^2y^2 + y^4$, $x^3 + y^3$.
16. $x^2 - 8x + 12$, $x^2 - 7x + 6$, $x^3 - 216$.
17. $3(x-1)^3$, $x^4 - 1$, $x^3 + x^2 - 2$.
18. $x^3 - y^3$, $x^4 + x^2y^2 + y^4$, $x^3 + 4x^2y + 4xy^2 + 3y^3$.
19. $2x^2 + 17x + 21$, $8x^3 + 27$, $2x^2 + 5x + 3$.
20. $m^2 - 4m + 3$, $m^2 - 6m + 9$, $m^3 - 9m^2 + 27m - 27$.
21. $a^4 - 16$, $a^3 + 2a^2 + 4a + 8$, $3a^3 - 2a^2 + 12a - 8$.
22. $m^3 - n^6$, $m^3 - 3m^2n^2 + 3mn^4 - n^6$, $m^4 - mn^6 + m^3n - n^7$.
23. $x^3 - 27$, $x^4 + 9x^2 + 81$, $x^3 + 2x^2 + 6x - 9$.
24. $3x^3 + 2x^2 - 2x - 1$, $x^4 - 1$, $ax + a - bx - b$.
25. $(a-b)^3 + 1$, $(a-b)^2 - 1$, $(a-b)^2 - 2(a-b) - 3$.

THE H. C. F. PARTLY BY FACTORING

101. If difficulty be met in factoring one of the expressions, the factors of this expression may often be found by trial of the factors of the other expressions.

Find the H. C. F. of

$a^{12}-b^{12},\ 2a^6-3a^5b-a^4b^2+3a^3b^3+a^2b^4-3ab^5+b^6.$

$$a^{12}-b^{12}=(a^6+b^6)(a^6-b^6)=(a^2+b^2)(a^4-a^2b^2+b^4)(a^3+b^3)(a^3-b^3),$$
$$=(a^2+b^2)(a^4-a^2b^2+b^4)(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2).$$

By division, a^2+b^2 is not a factor of $2a^6-3a^5b-a^4b^2+3a^3b^3+a^2b^4-3ab^5+b^6$; and $a^4-a^2b^2+b^4$ is a factor, producing the quotient $2a^2-3ab+b^2$.

$$2a^6-3a^5b-a^4b^2+3a^3b^3+a^2b^4-3ab^5+b^6$$
$$=(a^4-a^2b^2+b^4)(2a^2-3ab+b^2),$$
$$=(a^4-a^2b^2+b^4)(2a-b)(a-b),$$
$$\text{H. C. F.}=(a^4-a^2b^2+b^4)(a-b).$$

EXERCISE XLIX

Find the H. C. F. of the following expressions:

1. $x^2+5x+4,\ x^3-x^2-3x-1.$
2. $x^3-2x^2+1,\ x^4-1.$
3. $2x^2+24x+70,\ x^4+7x^3-x^2-6x+7.$
4. $a^4-2a^3+4a^2+6a-21,\ 3a^4-11a^2+6.$
5. $1+2x^2+2x^4+2x^6+x^8,\ 1+x^6.$
6. $x^3-2x^2y+2xy^2-y^3,\ ax+bx-ay-by.$
7. $x^2+y^2-2xy+z(-2-x)-y(2-z)+2x,\ x^2-4+y^2-2xy.$
8. $a^4+a^2b^2+b^4,\ a^2(2m-3n)+m(2b^2-2ab)+3bn(a-b).$

THE H. C. F. BY DIVISION

102.* If the expressions are such that they are not readily factorable, the H. C. F. can be found by a process which depends upon the following principles:

1. *A factor of an expression is also a factor of any multiple of that expression.*

Let a be contained b times in R. Then $R = ab$. Let mR be any multiple of R. Then $mR = mab$; that is, a is a factor of mR.

2. *A factor of two expressions is a factor of the sum, or of the difference, of any two multiples of these expressions.*

Let a be contained b times in R and c times in S. Then $R = ab$ and $S = ac$; or, applying the preceding principle, $mR = mab$ and $nS = nac$. Adding or subtracting the two last equations,

$$mR \pm nS = mab \pm nac = a(mb \pm nc);$$

that is, a is a factor of $mR \pm nS$.

103.* Let A and B be any two expressions, arranged in descending order of the same letter. Let A be contained m times in B, with a remainder of C; let C be contained n times in A, with a remainder of D; let D be contained exactly p times in C.

$$\begin{array}{l}
A)B(m \\
\quad \underline{mA} \\
\qquad C)A(n \\
\qquad \quad \underline{nC} \\
\qquad \qquad D)C(p \\
\qquad \qquad \quad \underline{pD}
\end{array}$$

Since D is contained p times in C, $pD = C$; since the dividend equals the product of the quotient and divisor, plus the remainder, and since C is contained n times in A, with a remainder D, $A = nC + D$; since A is contained m times in B, with a remainder C, $B = mA + C$. That is,

$$C = pD, \tag{1}$$

$$A = nC + D, \tag{2}$$

$$B = mA + C. \tag{3}$$

D may be shown to be a factor of each of the equations (1), (2), and (3).

D has already been shown to be a factor of C, since it is contained p times in C.

Substitute the value of C from (1) in (2),

$$A = npD + D \tag{4}$$

$$= D(np + 1) \tag{5}$$

Substitute the value of A from (5) in (3); and the value of C from (1) in (3) also,

$$B = mD(np + 1) + pD \tag{6}$$

$$= D(mnp + m + p). \tag{7}$$

Hence, *D is a common factor of A, B, and C.*

Moreover, *D is the highest common factor of A and B.*

From (3), $$B - mA = C, \tag{8}$$

from (2), $$A - nC = D. \tag{9}$$

By § 102, 2, a factor of A and B is a factor of $B - mA$, or of C; and a factor of A and C is a factor of $A - nC$, or of D. That is, a factor of A and B is also a factor of D. Since there can be no factor of D of higher degree than D itself, D is the highest common factor of A and B.

104.* From § 103 is derived the statement of the **Rule for finding the H. C. F. by division**: *arrange the expressions in the descending powers of the common letter; remove a monomial factor, if any, from either expression, and if the monomial factors so removed have a common factor write such a factor as a factor of the H. C. F. subsequently found; divide the expression of higher degree by the remaining expression until the remainder is of less degree than the divisor; continue the division with the remainder as a divisor, and the former divisor as a dividend, as before; the last divisor will be the H. C. F. if there is no common monomial factor; but if there is a common monomial factor, the H. C. F. is found by multiplying the last divisor by that factor.*

105.* The H. C. F. of two expressions remains unchanged if either of the expressions be multiplied or divided by a quantity which is not common to both expressions, since, by definition, the H. C. F. is the product of all the common prime factors. Thus, at any stage in finding the H. C. F. by division, a factor not common to both expressions may be removed by division; or, if at any stage the expressions are such that the first terms are not exactly divisible, they can be made so by multiplying either of the expressions by a quantity which will make them divisible — thus avoiding the use of fractions — without altering the value of the H. C. F.

1. Find the H. C. F. of $x^5+2x^4-2x^3+4x^2+3x$ and $x^6+2x^5-x^4+8x^3+5x^2-3x$.

$$x^5+2x^4-2x^3+4x^2+3x=x(x^4+2x^3-2x^2+4x+3),$$

$$x^6+2x^5-x^4+8x^3+5x^2-3x=x(x^5+2x^4-x^3+8x^2+5x-3).$$

The factor x is common to both expressions; therefore x is a part of the H. C. F.

$$\begin{array}{l|r|l} x^4+2x^3-2x^2+4x+3 & x^5+2x^4-x^3+8x^2+5x-3 & x \\ & \underline{x^5+2x^4-2x^3+4x^2+3x} & \\ & x^3+4x^2+2x-3 & \end{array}$$

The remainder is now of lower degree than the divisor.

$$\begin{array}{l|r|l} x^3+4x^2+2x-3 & x^4+2x^3-2x^2+4x+3 & x-2 \\ & \underline{x^4+4x^3+2x^2-3x} & \\ & -2x^3-4x^2+7x+3 & \\ & \underline{-2x^3-8x^2-4x+6} & \\ & 4x^2+11x-3 & \end{array}$$

The remainder is now of lower degree than the divisor.

$$\begin{array}{l|r|l} 4x^2+11x-3 & x^3+4x^2+2x-3 & \\ & \underline{4} & \\ & 4x^3+16x^2+8x-12 & x \\ & \underline{4x^3+11x^2-3x} & \\ & 5x^2+11x-12 & \\ & \underline{4} & \\ & 20x^2+44x-48 & 5 \\ & \underline{20x^2+55x-15} & \\ & -11 \,\big|\, \underline{-11x-33} & \\ & x+3 & \end{array}$$

To avoid fractions, multiply the expression x^3+4x^2+2x-3 by 4. This will not alter the H. C. F., because 4 is not a factor of $4x^2+11x-3$. Multiply the remainder, $5x^2+11x-12$, by 4 to make the expressions exactly divisible. Divide the remainder $-11x-33$ by -11. This will not alter the H. C F., since -11 is not a factor of $4x^2+11x-3$.

$$\begin{array}{r} x+3)\,4x^2+11x-3\,(4x-1 \\ \underline{4x^2+12x} \\ -x-3 \\ \underline{-x-3} \end{array}$$

Therefore, the H. C. F. $= x(x+3)$.

2. Find the H. C. F. of $4x^4 - 2x^3 - 12x^2 - 16x - 10$ and $3x^5 + 3x^4 + 9x^2 + 9x + 12$.

$$4x^4 - 2x^3 - 12x^2 - 16x - 10 = 2(2x^4 - x^3 - 6x^2 - 8x - 5),$$

$$3x^5 + 3x^4 + 9x^2 + 9x + 12 = 3(x^5 + x^4 + 3x^2 + 3x + 4).$$

The monomials removed contain no common factor.

$$\begin{array}{r|l|l}
2x^4 - x^3 - 6x^2 - 8x - 5 & x^5 + x^4 + 3x^2 + 3x + 4 & \\
& \underline{2} & \\
& 2x^5 + 2x^4 + 6x^2 + 6x + 8 & x \\
& \underline{2x^5 - x^4 - 6x^3 - 8x^2 - 5x} & \\
& \quad 3x^4 + 6x^3 + 14x^2 + 11x + 8 & \\
& \quad \underline{2} & \\
& \quad 6x^4 + 12x^3 + 28x^2 + 22x + 16 & 3 \\
& \quad \underline{6x^4 - 3x^3 - 18x^2 - 24x - 15} & \\
& \qquad 15x^3 + 46x^2 + 46x + 31 &
\end{array}$$

$$\begin{array}{r|l|l}
15x^3 + 46x^2 + 46x + 31 & 2x^4 - x^3 - 6x^2 - 8x - 5 & \\
& \underline{15} & \\
& 30x^4 - 15x^3 - 90x^2 - 120x - 75 & 2x \\
& \underline{30x^4 + 92x^3 + 92x^2 + 62x} & \\
& \quad -107x^3 - 182x^2 - 182x - 75 & \\
& \quad \underline{-15} & \\
& \quad 1605x^3 + 2730x^2 + 2730x + 1125 & 107 \\
& \quad \underline{1605x^3 + 4922x^2 + 4922x + 3317} & \\
& \quad -2192 \,|\, \underline{-2192x^2 - 2192x - 2192} & \\
& \qquad\qquad x^2 + x + 1 &
\end{array}$$

$$\begin{array}{r|l|l}
x^2 + x + 1 & 15x^3 + 46x^2 + 46x + 31 & 15x + 31 \\
& \underline{15x^3 + 15x^2 + 15x} & \\
& \quad 31x^2 + 31x + 31 & \\
& \quad \underline{31x^2 + 31x + 31} &
\end{array}$$

Therefore, the H. C. F. $= x^2 + x + 1$.

3. Find the H. C. F. of $4x^6-2x^5-2x^4+8x^3-2x^2-6x$ and $4x^6-4x^5+x^4-x^2-6x-9$.

$$4x^6-2x^5-2x^4+8x^3-2x^2-6x=2x(2x^5-x^4-x^3+4x^2-x-3).$$

$$\begin{array}{r|r|l}
2x^5-x^4-x^3+4x^2-x-3 & 4x^6-4x^5+x^4-x^2-6x-9 & 2x-1 \\
& \underline{4x^6-2x^5-2x^4+8x^3-2x^2-6x} & \\
& -2x^5+3x^4-8x^3+x^2-9 & \\
& \underline{-2x^5+x^4+x^3-4x^2+x+3} & \\
& 2x^4-9x^3+5x^2-x-12 &
\end{array}$$

$$\begin{array}{r|r|l}
2x^4-9x^3+5x^2-x-12 & 2x^5-x^4-x^3+4x^2-x-3 & x+4 \\
& \underline{2x^5-9x^4+5x^3-x^2-12x} & \\
& 8x^4-6x^3+5x^2+11x-3 & \\
& \underline{8x^4-36x^3+20x^2-4x-48} & \\
& 15\,\underline{|\,30x^3-15x^2+15x+45} & \\
& 2x^3-x^2+x+3 &
\end{array}$$

$$\begin{array}{r|r|l}
2x^3-x^2+x+3 & 2x^4-9x^3+5x^2-x-12 & x-4 \\
& \underline{2x^4-x^3+x^2+3x} & \\
& -8x^3+4x^2-4x-12 & \\
& \underline{-8x^3+4x^2-4x-12} &
\end{array}$$

Therefore, the H. C. F. $=2x^3-x^2+x+3$.

A more compact arrangement of the above example is the following:

$$\begin{array}{r|r|l}
2x^5-x^4-x^3+4x^2-x-3 & 4x^6-4x^5+x^4-x^2-6x-9 & 2x-1 \\
\underline{2x^5-9x^4+5x^3-x^2-12x} & \underline{4x^6-2x^5-2x^4+8x^3-2x^2-6x} & \\
8x^4-6x^3+5x^2+11x-3 & -2x^5+3x^4-8x^3+x^2-9 & \\
\underline{8x^4-36x^3+20x^2-4x-48} & \underline{-2x^5+x^4+x^3-4x^2+x+3} & \\
15\,\underline{|\,30x^3-15x^2+15x+45} & 2x^4-9x^3+5x^2-x-12 & x+4 \\
2x^3-x^2+x+3 & \underline{2x^4-x^3+x^2+3x} & \\
& -8x^3+4x^2-4x-12 & x-4 \\
& \underline{-8x^3+4x^2-4x-12} &
\end{array}$$

The H. C. F. of three or more expressions is found by division by first finding the H. C. F. of the first two expressions; and then finding the H. C. F. of that result and the next expression.

EXERCISE L*

Find the H. C. F. of the following expressions:

1. $4x^2+3x-10$, $4x^3+7x^2-3x-15$.
2. x^3+2x^2+2x+1, x^3-2x^2-2x-3.
3. $4x^3-6x^2-4x+6$, $12x^3-2x^2-20x-6$.
4. $6x^3+7x^2-5x$, $15x^4+31x^3+10x^2$.
5. $2x^4+x^3-9x^2+8x-2$, $2x^4-7x^3+11x^2-8x+2$.
6. $4x^4+3x^2+4x-3$, $2x^4-3x^3+2x^2-2x-3$.
7. $3x^5+2x^4+x^2$, $3x^4+2x^3-3x^2+2x-1$.
8. $2x^3-5x^2-22x-15$, $6x^4-21x^3-41x^2-14x-30$.
9. $4x^5+14x^4+20x^3+70x^2$,
   $8x^7+28x^6-8x^5-12x^4+56x^3$.
10. $2x^5-11x^2-9$, $4x^5+11x^4+81$.
11. x^4+2x^2+9, $x^4-4x^3+10x^2-12x+9$.
12. $6x^4-5x^3-10x^2+3x-10$, $4x^3-4x^2-9x+5$.
13. $6x^4-13x^3+3x^2+2x$, $6x^4-9x^3+15x^2-27x-9$.
14. $3x^4-x^3-2x^2+2x-8$, $6x^3+13x^2+3x+20$.
15. $9x^5-7x^3+8x^2+2x-4$, $6x^4-7x^3-10x^2+5x+2$.
16. $6x^5-2x^4-11x^3+5x^2-10x$,
    $9x^5+3x^4-11x^3+9x^2-10x$.
17. $x^6+3x^5+3x^4+9x^3-4x^2-12x$, $x^6+3x^5-x^3-3x^2$.
18. $2x^4+x^3-8x^2-x+6$, $4x^4+12x^3-x^2-27x-18$.
19. $6x^5-9x^4+11x^3+6x^2-10x$,
    $4x^5+10x^4+10x^3+4x^2+60x$.
20. $4x^5+x^3-17x^2+9x-9$,
    $2x^5+3x^4+7x^3-6x^2-9x-27$.
21. $4x^4-6x^3+9x^2-5x+3$, $8x^3+8x^2+9$.

THE LOWEST COMMON MULTIPLE

106. A **multiple** of an algebraic expression is an expression which contains all the prime factors of the first expression and is therefore exactly divisible by it. A **common multiple** of two or more algebraic expressions is an expression which contains all the prime factors of each expression. The **lowest common multiple** of two or more expressions is that expression which contains, only, all the prime factors of each of the given expressions.

Thus, $2a^3b$ is a multiple of $2ab$; $6a^3x^2$ is a common multiple of $2a$ and $3x^2$; $6ax^2$ is the lowest common multiple of $2a$ and $3x^2$.

107. The lowest common multiple—abbreviated L.C.M.—of several monomials is readily found by inspection.

1. Find the L. C. M. of $4a^2b$, $6a^3b$, $12b^3$.

$$4a^2b = 2 \cdot 2 \cdot a^2 \cdot b,$$

$$6a^3b = 2 \cdot 3 \cdot a^3 \cdot b,$$

$$12b^3 = 2 \cdot 2 \cdot 3 \cdot b^3,$$

$$\text{L. C. M.} = 2 \cdot 2 \cdot 3 \cdot a^3 \cdot b^3 = 12a^3b^3.$$

2. Find the L. C. M. of $8x^2y$, $10x^4y^5$, $15x^3y^3$.

$$8x^2y = 2 \cdot 2 \cdot 2 \cdot x^2 \cdot y,$$

$$10x^4y^5 = 2 \cdot 5 \cdot x^4 \cdot y^5,$$

$$15x^3y^3 = 3 \cdot 5 \cdot x^3 \cdot y^3,$$

$$\text{L. C. M.} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot x^4 \cdot y^5 = 120x^4y^5.$$

NOTE. As in the case of the H. C. F. there are two forms of the L. C. M., one being the negative of the other.

THE L. C. M. BY FACTORING

108. 1. Find the L. C. M. of $3(a^2-b^2)$, $4a^2+8ab+4b^2$, $a^2-2ab+b^2$.

$$3(a^2-b^2)=3(a+b)(a-b),$$
$$4a^2+8ab+4b^2=2\cdot 2(a+b)(a+b),$$
$$a^2-2ab+b^2=(a-b)(a-b),$$
$$\text{L. C. M.}=2\cdot 2\cdot 3\cdot (a+b)(a+b)(a-b)(a-b).$$
$$=12(a+b)^2(a-b)^2,$$
$$=12(a^2-b^2)^2.$$

2. Find the L. C. M. of $x^2-8x+15$, $x^2-3x-10$, x^2-x-6, x^3-6x^2-x+30.

$$x^2-8x+15=(x-3)(x-5),$$
$$x^2-3x-10=(x+2)(x-5),$$
$$x^2-x-6=(x+2)(x-3),$$
$$x^3-6x^2-x+30=(x+2)(x-3)(x-5),$$
$$\text{L. C. M.}=(x+2)(x-3)(x-5).$$

3. Find the L. C. M. of $x^4+x^2y^2+y^4$, x^3+y^3, x^3-y^3.

$$x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2),$$
$$x^3+y^3=(x+y)(x^2-xy+y^2),$$
$$x^3-y^3=(x-y)(x^2+xy+y^2),$$
$$\text{L. C. M.}=(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2).$$

Rule: *Separate each expression into its prime factors, and write the product of all the different prime factors, giving to each prime factor the highest exponent which it has in any of the given expressions.*

EXERCISE LI

Find, by factoring, the L. C. M. of the following expressions:

1. $2x^2y^3$, $3x^3y^2$, $5x^4y$, $7xy^7$.
2. $4x^3y$, $5x^2y^3$, $6xy^4$, $15y^4$.
3. $4a^5b$, $6a^4b^2$, $18a^3b^3$, $36a^2b^4$.
4. $5a^3b^2c^2$, $7a^2b^7c^4$, $91a^4b^3c^5$, $65a^7b^4c^6$.
5. $x^2 - xy$, $7x^2 - 7y^2$.
6. $(a + b)$, $(a^2 - b^2)$, $a^2 + 2ab + b^2$.
7. $x^2 - y^2$, $x^3 + y^3$, $x^3 - y^3$.
8. $a^4 - b^4$, $a^4 + 2a^2b^2 + b^4$, $a^4 + b^4$.
9. $(m - n)^3$, $m^3 - n^3$, $m^4 - n^4$.
10. $x^5 - y^5$, $x^3 + y^3$, $x^6 - y^6$.
11. $m^2 - 2m - 3$, $m^3 - 27$.
12. $x^2 - 2xy + y^2 - 1$, $(x - y)^3 - 1$.
13. $x^3 + 64$, $x^2 + x - 12$.
14. $x^{12} - y^{12}$, $x^6 + y^6$, $x^4 + y^4$.
15. $4x^2 + 4x + 4$, $6x^3 - 6$.
16. $3(a - b)^3$, $27a^3 - 27b^3$.
17. $3x^2 + 14x + 8$, $27x^3 + 8$.
18. $(a - b)^2 - c^2$, $(a - b + c)^2$.
19. $x^3 - 5x + 2$, $x^4 - 16$.
20. $x^3 + x - 2$, $(x^2 + 2)^2 - x^2$.
21. $(x - y)^2 - z^2$, $(x - z)^2 - y^2$, $(y - z)^2 - x^2$.
22. $6x^3(x^3 - y^3)$, $3xy(x - y)^3$, $2y^3(x^6 - y^6)$.

THE L. C. M. OF EXPRESSIONS NOT READILY FACTORABLE

109.* If the expressions are not readily factorable, the H. C. F. can be found by § 104; and the expressions then split into factors by dividing each of them by the H. C. F.

1. Find the L. C. M. of $4x^4 + 12x^3 + 2x^2 - 8x + 2$ and $6x^4 + 3x^3 - 39x^2 - 21x + 15$.

The H. C. F. is found by division to be $x^2 + 2x - 1$. The factors of each expression are now found by dividing each expression by the H. C. F.

$$4x^4 + 12x^3 + 2x^2 - 8x + 2 = 2(x^2 + 2x - 1)(2x^2 + 2x - 1),$$
$$6x^4 + 3x^3 - 39x^2 - 21x + 15 = 3(x^2 + 2x - 1)(2x^2 - 3x - 5),$$
$$= 3(x^2 + 2x - 1)(x + 1)(2x - 5),$$
$$\text{L. C. M.} = 6(x^2 + 2x - 1)(x + 1)(2x - 5)(2x^2 + 2x - 1).$$

EXERCISE LII*

Find the H. C. F. and the L. C. M. of the following expressions:

1. $6x^3 + 7x^2 - 5x$, $15x^4 + 31x^3 + 10x^2$.
2. $12x^2 - 29x + 14$, $18x^2 + 3x - 10$.
3. $9x^4 - x^2 + 10x - 25$, $6x^4 - 2x^3 + 7x^2 + x - 5$.
4. $6x^4 - x^3 - 14x^2 - x + 6$, $2x^4 - 3x^3 + 2x^2 + x - 6$.
5. $12x^3 - 2x^2 - 20x - 6$, $4x^3 - 6x^2 - 4x + 6$.
6. $4x^3 - 12x - 8$, $x^4 - 6x^2 - 8x - 3$.
7. $x^3 + 4x^2 + 4x + 3$, $x^3 + 3x^2 + 4x + 12$.
8. $2x^2 + x + 4$, $3x^3 + 11x - 2x^4 + 12 + 4x^2$.
9. $x^3 - 1$, $x^3 + x^2 - 3x + 1$, $x^4 + 3x^3 + x - 1$.
10. $x^2 + x - 6$, $x^3 + 2x^2 - 10x - 21$, $x^3 - 3x^2 - 5x + 14$.
11. $x^3 + 5x^2 + 6x + 8$, $x^4 + 2x^3 + x^2 - 4$, $x^3 + 5x^2 + 2x - 8$.

CHAPTER VIII

FRACTIONS

110. The indicated quotient of two expressions a and b, written in the form $\frac{a}{b}$, is called an **algebraic fraction.** Since $a \div b$ also expresses the quotient of a divided by b, the forms $\frac{a}{b}$ and $a \div b$ are equivalent. Hence, by § 26, $\frac{a}{b} \times b = a$.

As in arithmetic, the dividend is called the **numerator,** and the divisor, the **denominator,** of the fraction. The numerator and denominator are called the **terms** of the fraction.

111. *If the terms of a fraction are both divided, or both multiplied, by the same quantity, the value of the fraction remains unchanged.*

Let $\frac{a}{b}$ be any fraction, and let the quotient be q. By definition,
$$\frac{a}{b} = q,$$
by Ax. 3,
$$a = bq,$$
by Ax. 3,
$$am = bmq,$$
by Ax. 4,
$$\frac{am}{bm} = q = \frac{a}{b},$$
or,
$$\frac{a}{b} = \frac{am}{bm}.$$

REDUCTION OF FRACTIONS TO LOWEST TERMS

112. An algebraic fraction is said to be in its **lowest terms** when the terms of the fraction contain no common factor.

Rule for Reduction of a Fraction to Lowest Terms: *divide both numerator and denominator by their H. C. F.*

1. Reduce $\dfrac{3a^3x^5y^2}{36a^4x^4y^3}$ to lowest terms.

$$\frac{3a^3x^5y^2}{36a^4x^4y^3} = \frac{(3a^3x^4y^2)(x)}{(3a^3x^4y^2)(12ay)} = \frac{x}{12ay}.$$

2. Reduce $\dfrac{x^4 - y^4}{x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4}$ to lowest terms.

$$\frac{x^4 - y^4}{x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4} = \frac{(x^2+y^2)(x+y)(x-y)}{(x^2+y^2)(x-y)(x-y)} = \frac{x+y}{x-y}.$$

3.* Reduce $\dfrac{x^4 - 3x^3 - 4x^2 + 7x + 5}{x^4 - x^3 - 6x^2 - 7x - 5}$ to lowest terms.

By § 104, the H.C.F. is discovered.

$$\frac{x^4 - 3x^3 - 4x^2 + 7x + 5}{x^4 - x^3 - 6x^2 - 7x - 5} = \frac{(x^2 - 2x - 5)(x^2 - x - 1)}{(x^2 - 2x - 5)(x^2 + x + 1)},$$

$$= \frac{x^2 - x - 1}{x^2 + x + 1}.$$

The process of dividing both numerator and denominator of a fraction by the same quantity is often called **cancellation.** Cancellation can exist, therefore, only between the factors, and never between the terms.

Thus, $\dfrac{2x + a}{3x + a}$ is not equal to $\dfrac{2x}{3x} = \dfrac{2}{3}$, since no factor is common between numerator and denominator.

EXERCISE LIII

Reduce the following fractions to lowest terms:

1. $\dfrac{a^3b^2c}{a^2bc^2}$.

2. $\dfrac{7\,a^4bc^3}{2\,a^2b^5c}$.

3. $\dfrac{12\,a^3b^4c}{20\,a^4bc^3}$.

4. $\dfrac{15\,x^2y}{35\,xy^2}$.

5. $\dfrac{18\,x^5y^3z^4}{72\,x^3y^5z^2}$.

6. $\dfrac{8\,a+8\,b}{9\,a+9\,b}$.

7. $\dfrac{mx-my}{mp+mq}$.

8. $\dfrac{a^2+ab}{ab+b^2}$.

9. $\dfrac{ax+a}{b+bx}$.

10. $\dfrac{b^2+b}{1+b}$.

11. $\dfrac{x^2-1}{3(x-1)}$.

12. $\dfrac{1-y^2}{4(y+1)}$.

13. $\dfrac{2\,a-2\,b}{a^2-2\,ab+b^2}$.

14. $\dfrac{(a-b)c-(a-b)d}{(a+b)c-(a+b)d}$.

15. $\dfrac{ac-ad+bc-bd}{ac+ad+bc+bd}$.

16. $\dfrac{x^3+x^2-x-1}{3(x^2-1)}$.

17. $\dfrac{a^2-2\,ab-3\,b^2}{a^2-4\,ab+3\,b^2}$.

18. $\dfrac{4\,a^2-5\,ab-6\,b^2}{8\,a^2+2\,ab-3\,b^2}$.

19. $\dfrac{a^4+a^2+1}{a^3-1}$.

20. $\dfrac{a^2-b^2-c^2+2\,bc}{a^2+b^2-c^2+2\,ab}$.

21. $\dfrac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$.

22. $\dfrac{x^3+x^2-13\,x-4}{x^3+2\,x^2-16\,x-5}$.

By means of § 104, reduce the following fractions to lowest terms:

23.* $\dfrac{x^3-x^2-11\,x+15}{x^4-6\,x^2+16\,x-15}$.

24.* $\dfrac{x^4-6\,x^3+12\,x^2-9\,x-10}{2\,x^4-5\,x^3-5\,x^2-8\,x-4}$.

25.* $\dfrac{m^5-5\,m^4+5\,m^3+4\,m^2-5\,m+6}{m^5-5\,m^3-m^2-4}$.

THE LAWS OF SIGNS IN FRACTIONS

113. Since a fraction is an indicated quotient, the laws o signs are derived from the laws of signs in division, § 26

Therefore, $$-\frac{a}{b}=+\frac{-a}{b}=+\frac{a}{-b}.$$

By § 106, $$\frac{a}{b}=\frac{a(-1)}{b(-1)}=\frac{-a}{-b}.$$

Hence, $$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b}.$$

From the foregoing laws is derived the **Rule for Change of Signs in Fractions:** *The value of the fraction is not changed if* (1) *the signs of the numerator and denominator are changed simultaneously, or if* (2) *the sign before the fraction and the sign of either the numerator or the denominator are changed simultaneously.*

114. If the numerator and denominator are expressed in factors, since by the laws of signs in multiplication, § 25, the product of an even number of positive or negative factors is positive, and the product of an odd number of negative factors is negative, *the value of the fraction is not changed if* (1) *the signs of an even number of factors in the numerator, or in the denominator, or in both of them, are changed; and if* (2) *the sign before the fraction and the signs of an odd number of factors in the numerator, or in the denominator, or in both of them, are changed.*

1. $$\frac{(b-a)(b-a)}{(a-b)(a-b)(a-b)}=\frac{(a-b)(a-b)}{(a-b)(a-b)(a-b)}=\frac{1}{a-b}.$$

2. $$\frac{(b-a)(c-d)(m-n)}{(a-b)(d-c)(n-m)}=-\frac{(a-b)(c-d)(m-n)}{(a-b)(c-d)(m-n)}=-1.$$

The numerator and denominator, or either of them, may consist of several terms. A change of sign of the numerator or denominator means a change of the sign of every term of the numerator or denominator.

Thus, $\dfrac{-x^2+2xy-y^2}{-x^2+y^2}=\dfrac{x^2-2xy+y^2}{x^2-y^2}$,

$$=-\frac{x^2-2xy+y^2}{-x^2+y^2}=-\frac{-x^2+2xy-y^2}{x^2-y^2}.$$

EXERCISE LIV

Reduce the following fractions to lowest terms:

1. $\dfrac{x^2-a^2}{a^2-ax}$.

2. $\dfrac{ab^2-ac^2}{(c-b)(c-d)}$.

3. $\dfrac{m^2-n^2}{n^3-m^3}$.

4. $\dfrac{x^2-(a-b)^2}{(b-a)^2-x^2}$.

5. $\dfrac{a^2-ac-ab+bc}{bc-ab+ac-a^2}$.

6. $\dfrac{12-x-x^2}{6x^2+2x-60}$.

7. $\dfrac{x^3-4x^2+x+6}{x^3-6x^2+11x-6}$.

8. $\dfrac{(a+b)^2-c^2}{(c-a)^2-b^2}$.

9. $\dfrac{(a-b)(b-c)(c-d)}{(a+b)(b+c)(d-c)}$.

10. $\dfrac{x^3-4x^2-4x+16}{16+4x-4x^2-x^3}$.

11. $\dfrac{(x-a)(x-b)(x-c)(c-x)}{(x+a)(b-x)(c-x)(c-x)}$.

12. $\dfrac{(a-b)(b-c)(c-d)(d-a)}{(a+b)(c-b)(d-c)(d+a)}$.

13. $\dfrac{-m^4+2m^3n-2mn^3+n^4}{(n-m)(n-m)(n-m)(m-n)}$.

14. $\dfrac{(a^2-b)(b^2-c)(c^2-d)(d^2-a)}{(b-a^2)(c-b^2)(d-c^2)(d^2+a)}$.

115. An **integral expression** is one that does not contain any literal quantity in the denominator of any term.

Thus, $2a^2 + 3ab^3 + \frac{a}{2}$ is an integral expression.

A **fractional expression** is an expression which contains a literal quantity in the denominator of one or more of its terms.

Thus, $x^2 + xy + \frac{y^2}{a^3}$ is a fractional expression.

A **mixed expression** consists of an integral expression and a fraction.

Thus, $a + \frac{a}{b}$ and $x^2 + xy + y^2 - \frac{x^3 - 1}{x^4 + 2}$ are mixed expressions.

If the numerator of a fraction is of higher degree than the denominator, the fraction is called an **improper fraction**; if the numerator is of lower degree than the denominator the fraction is called a **proper fraction.**

Thus, $\frac{x^2 + 1}{x + 1}$ is an improper fraction; and $\frac{x + 1}{x^2 + 1}$ is a proper fraction.

REDUCTION OF IMPROPER FRACTIONS TO INTEGRAL OR MIXED EXPRESSIONS

116. If the denominator is a factor of the numerator the quotient is an integral expression; if the denominator is not a factor of the numerator, the quotient is a mixed expression.

Thus, $\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$ is an integral expression,

and $\frac{a^3 + b^3}{a - b} = a^2 + ab + b^2 + \frac{2b^3}{a - b}$ is a mixed expression.

Rule for Reduction of an Improper Fraction to Integral or Mixed Expression: *Divide the numerator by the denominator.*

Thus, $$\frac{x^3+x^2+1}{x}=x^2+x+\frac{1}{x}.$$

1. Reduce $\dfrac{4x^4-4x^3+3x^2+x+1}{2x^2+x-1}$ to an integral or mixed expression.

$$\begin{array}{r|r|l}
2x^2+x-1 & 4x^4-4x^3+3x^2+x+1 & 2x^2-3x+4 \\
 & \underline{4x^4+2x^3-2x^2} & \\
 & -6x^3+5x^2+x+1 & \\
 & \underline{-6x^3-3x^2+3x} & \\
 & 8x^2-2x+1 & \\
 & \underline{8x^2+4x-4} & \\
 & -6x+5 &
\end{array}$$

$$\frac{4x^4-4x^3+3x^2+x+1}{2x^2+x-1}=2x^2-3x+4+\frac{-6x+5}{2x^2+x-1}.$$

Or, by § 113, $$=2x^2-3x+4-\frac{6x-5}{2x^2+x-1}.$$

EXERCISE LV

Reduce the following improper fractions to integral or mixed expressions:

1. $\dfrac{x^2-xy+y^2}{x}$.

2. $\dfrac{2x^2+4x+1}{2x}$.

3. $\dfrac{x^2+2xy-y^2}{x+y}$.

4. $\dfrac{3m^2+4m+5}{m+1}$.

5. $\dfrac{x^3-y^3}{x+y}$.

6. $\dfrac{3x^3+2x+1}{x^2+2}$.

7. $\dfrac{x^4-2x^3+2x^2+x-1}{x^2+x-1}$.

8. $\dfrac{a^5-b^5}{a^3-3a^2+2a-1}$.

REDUCTION OF FRACTIONS TO EQUIVALENT FRACTIONS HAVING THE LEAST COMMON DENOMINATOR

117. As in arithmetic, the least common denominator — abbreviated L. C. D. — of a number of fractions is the L. C. M. of the denominators.

1. Reduce $\frac{2m}{3a^2}, \frac{6m^2}{5a}, \frac{12mn}{10a^3}$ to equivalent fractions having the least common denominator.

The L. C. M. of the respective denominators is, § 107, $30a^3$.

Take $30a^3$ as the L. C. D.; and divide $30a^3$ by the respective denominators, $3a^2$, $5a$, and $10a^3$, thus obtaining the respective quotients, $10a$, $6a^2$, 3.

$$\frac{2m(10a)}{3a^2(10a)}=\frac{20am}{30a^3};\quad \frac{6m^2(6a^2)}{5a(6a^2)}=\frac{36a^2m^2}{30a^3};\quad \frac{12mn(3)}{10a^3(3)}=\frac{36mn}{30a^3}.$$

2. Reduce $\frac{1}{a^2-ab-ac+bc}$ and $\frac{1}{bc+ac-ab-c^2}$ to equivalent fractions having the L. C. D.

Factor each denominator, and simplify by § 114, if possible:

$$\frac{1}{a^2-ab-ac+bc}=\frac{1}{(a-b)(a-c)},$$

$$\frac{1}{bc+ac-ab-c^2}=\frac{1}{(b-c)(c-a)}=\frac{1}{(c-b)(a-c)}.$$

The L. C. M. of the denominators is $(a-b)(a-c)(c-b)$. Divide the L. C. D. by the factors of the respective denominators, thus obtaining the respective quotients, $c-b$ and $a-b$.

$$\frac{1(c-b)}{(a-b)(a-c)(c-b)}=\frac{c-b}{(a-b)(a-c)(c-b)},$$

$$\frac{1(a-b)}{(c-b)(a-c)(a-b)}=\frac{a-b}{(a-b)(a-c)(c-b)}.$$

Rule for the Reduction of Fractions to Equivalent Fractions having the L. C. D.: *Simplify each fraction, and express the denominator as the product of prime factors. Take the L. C. M. of the denominators as the L. C. D. Multiply both terms of each fraction by the quotient found by dividing the L. C. D. by each denominator.*

EXERCISE LVI

Reduce the following fractions to equivalent fractions having the L. C. D.:

1. $\dfrac{3}{10a}, \dfrac{5}{4a^2}, \dfrac{4}{a^3}.$

2. $\dfrac{2x^3}{3y}, \dfrac{5x}{2y^2}, \dfrac{6}{12y^3}.$

3. $\dfrac{3x-y}{7a^3}, \dfrac{y-3x}{4am}.$

4. $\dfrac{a-b}{5xz}, \dfrac{a+b}{12x^2}.$

5. $\dfrac{a}{a+b}, \dfrac{a}{a-b}.$

6. $\dfrac{1}{x+y}, \dfrac{x+y}{x-y}.$

7. $\dfrac{a-c}{c-d}, \dfrac{a+c}{c+d}.$

8. $\dfrac{x^2+y}{x^2-y^2}, \dfrac{x+y^2}{x+y}.$

9. $\dfrac{x^2}{x^3-y^3}, \dfrac{y^2}{x^6-y^6}.$

10. $\dfrac{3x-4y}{x^2-y^2}, \dfrac{5y-3x}{y^2-x^2}.$

11. $\dfrac{1}{(a-b)(m-n)}, \dfrac{1}{(b-a)(m+n)}.$

12. $\dfrac{a^2}{(a-b)(b-c)}, \dfrac{b^2}{(b-a)(c-b)}.$

13. $\dfrac{a^2-ac}{(a-b)(b+c)(a-c)}, \dfrac{b^2-bc}{(a-b)(b-c)(c-a)}$

14. $\dfrac{2m-3}{2m^2-5m+3}, \dfrac{3m+7}{3m^3-2m^2-18m+7}.$

ADDITION AND SUBTRACTION OF FRACTIONS

118. By § 56, $\frac{x+y+z}{a}=\frac{x}{a}+\frac{y}{a}+\frac{z}{a}$.

Therefore, *the sum of a number of fractions having a common denominator is a fraction whose numerator is the algebraic sum of the numerators and whose denominator is the common denominator of the fractions.*

By the law of signs in fractions, § 113, a fraction in the form $-\frac{a}{b}$ may be changed to the equivalent form, $+\frac{-a}{b}$.

Find the algebraic sum of $\frac{2x}{3a}+\frac{5x^2}{3a}-\frac{b}{3a}$.

By § 113, $-\frac{b}{3a}=+\frac{-b}{3a}$.

The algebraic sum of the numerators $= 2x+5x^2-b$.

The common denominator $= 3a$.

$$\frac{2x}{3a}+\frac{5x^2}{3a}+\frac{-b}{3a}=\frac{2x+5x^2-b}{3a}.$$

119. If the denominators are not common, the L. C. D. may be found by § 117 and the fractions added as before.

1. Find the algebraic sum of $\frac{2x}{3a}+\frac{3a}{2x}-\frac{1}{a^2x}$.

The L. C. D. $= 6a^2x$.

$$\frac{2x}{3a}=\frac{2x(2ax)}{3a(2ax)}=\frac{4ax^2}{6a^2x},$$

$$\frac{3a}{2x}=\frac{3a(3a^2)}{2x(3a^2)}=\frac{9a^3}{6a^2x},$$

$$-\frac{1}{a^2x}=\frac{-1(6)}{a^2x(6)}=\frac{-6}{6a^2x}.$$

$$\frac{2x}{3a}+\frac{3a}{2x}+\frac{-1}{a^2x}=\frac{4ax^2+9a^3-6}{6a^2x}.$$

Rule for Addition (or Subtraction) of Fractions: *Reduce the fractions, in their lowest terms, to equivalent fractions having the least common denominator; the sum of the fractions is a fraction whose numerator is the algebraic sum of the numerators and whose denominator is the least common denominator of the fractions.*

120. It should be carefully noticed that the sign of division in fractions is a sign of aggregation.

Thus, $\dfrac{a^2+ab+b^2}{a^2-2\,ab+b^2}$ means that the whole of the numerator, a^2+ab+b^2, is to be divided by the whole of the denominator, $a^2-2\,ab+b^2$. If the fraction $\dfrac{a^2+ab+b^2}{a^2-2\,ab+b^2}$ be preceded by the minus sign, the whole process is indicated:

$$-(a^2+ab+b^2)\div(a^2-2\,ab+b^2).$$

That is, *the minus sign before the fraction is to be interpreted as affecting the whole of the numerator.*

Thus,

$$\frac{a+b}{a^2+ab+b^2}-\frac{a-b}{a^2+ab+b^2}=\frac{(a+b)-(a-b)}{a^2+ab+b^2}=\frac{2\,b}{a^2+ab+b^2},$$

or, by § 114,

$$\frac{a+b}{a^2+ab+b^2}-\frac{a-b}{a^2+ab+b^2}=\frac{a+b}{a^2+ab+b^2}+\frac{-a+b}{a^2+ab+b^2}=\frac{2\,b}{a^2+ab+b^2}.$$

1. Find the algebraic sum of $\dfrac{a+b}{3\,a}-\dfrac{a-b}{4\,a}$.

The L. C. D. $=12\,a$.

$$\frac{a+b}{3\,a}=\frac{4\,(a+b)}{12\,a};\quad -\frac{a-b}{4\,a}=-\frac{3\,(a-b)}{12\,a}.$$

$$\frac{a+b}{3\,a}-\frac{a-b}{4\,a}=\frac{4\,(a+b)-3\,(a-b)}{12\,a}=\frac{4\,a+4\,b-3\,a+3\,b}{12\,a}=\frac{a+7\,b}{12\,a}.$$

EXERCISE LVII

Find the algebraic sum of the following fractions:

1. $\frac{3}{m}-\frac{x}{m}$.

2. $\frac{3a}{7}-\frac{a}{7}$.

3. $\frac{m}{x^2}+\frac{n}{x}$.

4. $\frac{a}{3x}+\frac{b}{4x}$.

5. $\frac{1}{6a}+\frac{1}{9b}$.

6. $\frac{a}{x^2}-\frac{b}{xy}$.

7. $7+\frac{1}{a}$.

8. $\frac{a-x}{x}+1$.

9. $1-\frac{b}{a+b}$.

10. $\frac{5a-17b}{16}+\frac{7b-2a}{12}-\frac{a-4b}{8}$.

11. $\frac{2x-5y-3}{15}-\frac{3x-8y+45}{25}+2$.

12. $\frac{a+4b}{10}-\frac{2(a-3b)}{15}-\frac{11(a+b)}{20}$.

13. $\frac{3(x-y)}{4}-\frac{5(2x-3y)}{6}+\frac{7(x-2y)}{8}$.

14. $\frac{5a+16b-14}{33(a+1)}+\frac{a-6b+7}{36(a+1)}-\frac{3a+14b-15}{44(a+1)}$.

15. $\frac{4a+7c}{6bc}-\frac{3}{c}+\frac{4b-3a}{2ab}+\frac{3a-2c}{ac}$.

16. $\frac{5x+3y}{3x^2}+\frac{y+2z}{4yz}-\frac{3x+4y}{6xy}-\frac{x+3z}{4xz}-\frac{1}{4x}$.

17. $\frac{a^2+4bc}{8ac}-\frac{b^2-3ac}{6ab}-\frac{ab+4c^2}{4bc}-\frac{b}{3a}+\frac{c}{2b}$.

18. $\frac{2}{5a}+\frac{2(6a-b)}{15ab}-\frac{1}{3b}-\frac{a+b}{10a^2}-\frac{3a^2-b^2}{10a^2b}$.

19. $\frac{a-2b}{4ab}+\frac{3a-4b}{6a^2}-\frac{1}{2b}-\frac{6a-5b}{20b^2}+\frac{a^2+2b^2}{3a^2b}$.

20. $\dfrac{4(ab+xy)}{15\,bx}+\dfrac{ay-8\,b^2}{6\,ab}-\dfrac{a^2-5\,bx}{6\,ax}+\dfrac{b}{2\,a}-\dfrac{y}{3\,b}-\dfrac{a}{10\,x}.$

21. $\dfrac{(1+x)^2}{x^2}-\dfrac{(1+y)^2}{y^2}+\dfrac{2\,x-2\,y+1}{xy}+\dfrac{2\,x+y}{xy^2}.$

22. $\dfrac{2}{3}+\dfrac{1}{a-1}.$

23. $\dfrac{a}{b}-\dfrac{a+m}{b+m}.$

24. $\dfrac{x}{y}-\dfrac{y}{x}-\dfrac{x-y}{x+y}.$

25. $\dfrac{a}{1-x}-\dfrac{a}{1+x}.$

26. $\dfrac{1}{a-b}-\dfrac{1}{a+b}.$

27. $\dfrac{a}{a+b}-\dfrac{b}{a-b}.$

28. $\dfrac{3}{x-11}-\dfrac{2}{x-7}.$

29. $\dfrac{3(x+2)}{x+5}-\dfrac{5(x-2)}{2\,x+1}.$

30. $\dfrac{x-13}{10(x-3)}-\dfrac{x-18}{14(x-2)}.$

31. $\dfrac{5}{3\,x-3}-\dfrac{3}{2\,x-2}.$

32. $\dfrac{2\,x-13}{10\,x+10}-\dfrac{3\,x-16}{15\,x+45}.$

33. $\dfrac{3}{2\,x-4}-\dfrac{5}{6\,x-12}-\dfrac{2}{3\,x+6}.$

34. $\dfrac{1}{y}+\dfrac{x-y}{x^2+xy}-\dfrac{2}{x+y}.$

35. $\dfrac{x-y}{xy}-\dfrac{x-a}{ax+bx}+\dfrac{y-b}{ay+by}.$

36. $\dfrac{5}{4(a+b)}+\dfrac{3}{a-b}-\dfrac{13\,a+7\,b}{4(a^2-b^2)}.$

37. $\dfrac{5}{x-2}+\dfrac{7}{x-3}-\dfrac{x-4}{x^2-5\,x+6}.$

38. $\dfrac{21\,x+13}{12\,x+24}-\dfrac{5\,x}{3\,x-3}+\dfrac{16\,x-3}{4\,x^2+4\,x-8}.$

121. If some of the factors of the denominators are alike except that their terms are not arranged in the same order, they may be made to take the same order by § 113.

1. Find the algebraic sum of

$$\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(c-a)}+\frac{1}{(a-c)(a-b)}.$$

$$\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(c-a)}+\frac{1}{(a-c)(a-b)}$$

$$=\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(c-a)}-\frac{1}{(c-a)(a-b)},$$

$$=\frac{c-a-(a-b)-(b-c)}{(a-b)(b-c)(c-a)}=\frac{2(c-a)}{(a-b)(b-c)(c-a)},$$

$$=\frac{2}{(a-b)(b-c)}.$$

2. Find the algebraic sum of

$$\frac{1}{x+1}+\frac{1}{1-x}-\frac{2}{-1-x^2}+\frac{4}{1+x^4}.$$

$$\frac{1}{x+1}-\frac{1}{x-1}-\frac{2}{-1-x^2}+\frac{4}{1+x^4}$$

$$=\frac{x-1-x-1}{(x+1)(x-1)}+\frac{2}{x^2+1}+\frac{4}{x^4+1}=\frac{-2}{x^2-1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}$$

$$=\frac{-2x^2-2+2x^2-2}{x^4-1}+\frac{4}{x^4+1}=\frac{-4}{x^4-1}+\frac{4}{x^4+1},$$

$$=\frac{-4x^4-4+4x^4-4}{x^8-1}=\frac{-8}{x^8-1}=\frac{8}{1-x^8}.$$

EXERCISE LVIII

Find the algebraic sum of the following fractions:

1. $\dfrac{1}{x^2-1}-\dfrac{1}{x+1}-\dfrac{1}{1-x}.$

2. $\dfrac{x^2}{x^2-y^2}-\dfrac{1}{y-x}+\dfrac{x^2}{y^2-x^2}.$

3. $\dfrac{a}{a-b}-\dfrac{2\,a}{a+b}+\dfrac{3\,ab}{b^2-a^2}.$

4. $\dfrac{3}{x-3\,y}-\dfrac{2}{3\,y+x}+\dfrac{15\,y}{9\,y^2-x^2}.$

5. $\dfrac{1}{(x-1)(x-2)}-\dfrac{1}{(2-x)(3-x)}-\dfrac{1}{(1-x)(x-3)}.$

6. $\dfrac{1}{(a-b)(b-c)}-\dfrac{1}{(b-a)(a-c)}+\dfrac{1}{(c-b)(c-a)}.$

7. $\dfrac{c+a}{(a-b)(b-c)}+\dfrac{b+c}{(a-b)(a-c)}+\dfrac{a+b}{(c-b)(c-a)}.$

8. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}+\dfrac{c^3}{(c-a)(c-b)}.$

9. $\dfrac{(a-m)(b-m)}{(a-c)(b-c)}+\dfrac{(b-m)(c-m)}{(b-a)(c-a)}+\dfrac{(c-m)(a-m)}{(c-b)(a-b)}.$

10. $\dfrac{a}{(a-b)(a-c)(a-d)}+\dfrac{b}{(b-a)(b-c)(b-d)}$
$\qquad+\dfrac{c}{(c-a)(c-b)(c-d)}+\dfrac{d}{(d-a)(d-b)(d-c)}$

11. $\dfrac{6}{a}-\dfrac{4}{b+a}+\dfrac{4}{b-a}+\dfrac{1}{a-2\,b}-\dfrac{1}{-2\,b-a}.$

12. $\dfrac{1}{a+2\,x}+\dfrac{1}{2\,x-a}+\dfrac{4\,x}{a^2+4\,x^2}-\dfrac{32\,x^3}{a^4+16\,x^4}.$

REDUCTION OF MIXED EXPRESSIONS TO FRACTIONS

122. Since $a \div 1 = a$, any integral expression may be written in the form of a fraction having the integral expression for a numerator and 1 for a denominator.

Thus, $$a + b = \frac{a}{1} + \frac{b}{1} = \frac{a+b}{1}.$$

Hence, the **Rule for Reduction of Mixed Expressions to Fractions**: *Write the integral part of the expression as a fraction having the integral expression for a numerator and 1 for a denominator, and add the fractions.*

1. Reduce $a - b - \dfrac{a^2+b^2}{a+b}$ to a fraction.

$$a - b - \frac{a^2+b^2}{a+b} = \frac{a-b}{1} - \frac{a^2+b^2}{a+b} = \frac{-2b^2}{a+b}.$$

EXERCISE LIX

Reduce the following mixed expressions to fractions:

1. $a + b - \dfrac{a^2+b^2}{a+b}$.

2. $m + 3 - \dfrac{m^2+9}{m-3}$.

3. $x^2 + xy + y^2 - \dfrac{x^3}{x-y}$.

4. $x^2 - xy + y^2 - \dfrac{y^3}{x+y}$.

5. $a^2 + ab + b^2 - \dfrac{a^2b^2}{a^2 - ab + b^2}$.

6. $\dfrac{m-2}{3} + 2 - m$.

7. $\dfrac{a^2+b^2}{a-b} - a + b$.

8. $x + y - \dfrac{x-y}{x+y}$.

9. $2a - 3b - \dfrac{6(a^2-b^2)}{3a-2b}$.

10. $m^2 - n^2 - \dfrac{n^4 - m^4}{m^2+n^2}$.

MULTIPLICATION OF FRACTIONS

123. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions; and let the product of these fractions be P.

Then, $$\frac{a}{b} \cdot \frac{c}{d} = P, \tag{1}$$

multiplying (1) by b, $$\frac{ab}{b} \cdot \frac{c}{d} = bP, \tag{2}$$

multiplying (2) by d, $$\frac{ab}{b} \cdot \frac{cd}{d} = bdP, \tag{3}$$

simplifying (3), $$ac = bdP, \tag{4}$$

dividing (4) by bd, $$\frac{ac}{bd} = P, \tag{5}$$

applying Ax. 5 to (1) and (5),

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}. \tag{6}$$

The product of three fractions can be found by multiplying the product of the first two by the third, and so on.

Rule for Product of Several Fractions: *The product of any number of fractions is a fraction whose numerator is the product of the numerators, and whose denominator is the product of the denominators of the given fractions.*

1. Find the product of $\frac{2x^2}{3y^2} \times \frac{5xy}{8} \times \frac{y^2}{10}$.

$$\frac{2x^2}{3y^2} \cdot \frac{5xy}{8} \cdot \frac{y^2}{10} = \frac{10x^3y^3}{240y^2} = \frac{x^3y}{24}.$$

In multiplying several fractions the process may be simplified by cancelling the common factors before finding the product.

2. Find the product of $\dfrac{x^2+y^2}{x^3-y^3} \cdot \dfrac{x^2+xy+y^2}{x^2-y^2} \cdot \dfrac{(x-y)^5}{x^4-y^4}$.

$$\frac{x^2+y^2}{x^3-y^3} \cdot \frac{x^2+xy+y^2}{x^2-y^2} \cdot \frac{(x-y)^5}{x^4-y^4}$$

$$= \frac{\cancel{(x^2+y^2)}\cancel{(x^2+xy+y^2)}\overset{(x-y)^2}{\cancel{(x-y)^5}}}{\cancel{(x-y)}\cancel{(x^2+xy+y^2)}(x+y)\cancel{(x-y)}\cancel{(x^2+y^2)}(x+y)\cancel{(x-y)}}$$

$$= \frac{(x-y)^2}{(x+y)^2} = \left(\frac{x-y}{x+y}\right)^2.$$

EXERCISE LX

Find the product of:

1. $\dfrac{2a^2b}{3x^3} \cdot \dfrac{6x^4}{a^2b^2}$.

2. $\dfrac{5x^2}{9y^2} \cdot \dfrac{21y^3}{20x^3z}$.

3. $\dfrac{4m^7n^5p}{7x^3y^2z} \cdot \dfrac{28x^4yz^3}{m^6n^6p}$.

4. $\dfrac{7a^2m^5c^4}{13x^3yz^7} \cdot \dfrac{26x^4y^3z^8}{21a^2m^4c^3}$.

5. $\dfrac{44a^2bx^3}{65am^3} \cdot \dfrac{26m^2x}{33abx^2}$.

6. $\dfrac{27a^5b^6c^2}{32a^4b^4} \cdot \dfrac{132b^3c^4d^6}{81a^2c^5d^3}$.

7. $\dfrac{a^2-b^2}{a^3} \cdot \dfrac{a^4}{a+b}$.

8. $\dfrac{x^2-2xy+y^2}{x^2+y^2} \cdot \dfrac{x^4-y^4}{x+y}$.

9. $\dfrac{3a^3}{a^3+b^3} \cdot \dfrac{a^2-ab+b^2}{12a^4}$.

10. $\dfrac{2x+14}{a-3} \cdot \dfrac{a^2-9}{3x+21}$.

11. $\dfrac{(x-y)^4}{x^2+y^2} \cdot \dfrac{x^4-y^4}{(x-y)^5}$.

12. $\dfrac{a^3+27}{a^2-16} \cdot \dfrac{a+4}{a^2-3a+9}$.

13. $\dfrac{a^3-64}{a^3+64} \cdot \dfrac{a^2-16}{a^2+16}$.

14. $\dfrac{ab+b^2}{a^3-b^3} \cdot \dfrac{a^2-b^2}{a^3+b^3}$.

15. $\dfrac{x^3-1}{x^2+x}\cdot\dfrac{x^2-1}{x^4+x^2+1}.$

16. $\dfrac{m^2-7m+12}{m^2-3m+2}\cdot\dfrac{m^2-6m+8}{m^2-8m+16}.$

17. $\dfrac{2a^3+a^2b-2ab^2-b^3}{a^3+2a^2b+ab^2+2b^3}\cdot\dfrac{a+2b}{2a+b}.$

18. $\dfrac{m^3+2m^2+2m+1}{m^3+1}\cdot\dfrac{m^2+2m+1}{m^3-1}.$

19. $\dfrac{x^4-y^4}{x^5+x^4y+xy^4+y^5}\cdot\dfrac{x^4+y^4}{x^2+y^2}.$

20. $\dfrac{x^2-x-6}{x^2y+xy-2y}\cdot\dfrac{x^3+3x^2-4x}{x^2+x-12}.$

21. $\dfrac{2x^2-5x+3}{3x^2-8x+4}\cdot\dfrac{3x^2-11x+6}{2x^2+5x-12}\cdot\dfrac{8-2x-x^2}{x^2-4x+3}.$

22. $\left(\dfrac{9a^2}{b^2}-1\right)\left(\dfrac{3a}{3a-b}-1\right)\left(1-\dfrac{3a}{3a+b}\right).$

23. $\dfrac{(a^3-b^3)}{a^2-b^2}\cdot\dfrac{(a-b)^3}{a^2(a+b)}\cdot\dfrac{a^3+b^3}{a^4+a^2b^2+b^4}.$

24. $\dfrac{2x^2+2ax-8x-8a}{x^2-ax+2x-2a}\cdot\dfrac{2x^2-2ax-3x+3a}{x^2+ax-4x-4a}\cdot\dfrac{x+2}{2x-3}.$

25. $\dfrac{x^3-2x-1}{x^3+2x^2+2x+1}\cdot\dfrac{x^4-x^3+2x^2-x+1}{x^3-1}\cdot\dfrac{x+1}{x^2+1}.$

26. $\dfrac{x^3-2x^2+1}{x^4+x^2+1}\cdot\dfrac{x^2-1}{x^2+1}\cdot\dfrac{(x^2+x+1)^2}{x^3-1}.$

27. $\dfrac{m^2-a^2+2a-1}{m^2+2am+a^2-1}\cdot\dfrac{(m+a+1)^2}{m^2-am+a-1}\cdot\dfrac{m-1}{m-a+1}$

$$\cdot\dfrac{m^2+3m-4}{m^2+am+5m+4a+4}.$$

DIVISION OF FRACTIONS

124. The **reciprocal** of a fraction is a fraction formed by interchanging the terms of the given fraction.

Thus, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$; the reciprocal of a is $\frac{1}{a}$.

The product of a fraction and its reciprocal is **1**.

Thus, $$\frac{m}{n}\cdot\frac{n}{m}=\frac{mn}{nm}=1.$$

125. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions; and let

$$\frac{a}{b}\div\frac{c}{d}=Q. \qquad (1)$$

Then, by § 26, $$\frac{a}{b}=Q\cdot\frac{c}{d}, \qquad (2)$$

multiplying (2) by $\frac{d}{c}$, $$\frac{a}{b}\cdot\frac{d}{c}=Q\cdot\frac{c}{d}\cdot\frac{d}{c}=Q, \qquad (3)$$

applying Ax. 5 to (1) and (3), $$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}. \qquad (4)$$

Rule for Quotient of Two Fractions: *Multiply the dividend by the reciprocal of the divisor.*

1. Divide $\frac{12(x-1)^2}{35(a^2-b^2)}$ by $\frac{15(1-x)}{28(a-b)^2(a+b)}$.

$$\frac{12(x-1)^2}{35(a^2-b^2)}\div\frac{15(1-x)}{28(a-b)^2(a+b)},$$

$$=\frac{\overset{4}{\cancel{12}}\cancel{(1-x)}(1-x)}{\underset{5}{\cancel{35}}\cancel{(a+b)}\cancel{(a-b)}}\cdot\frac{\overset{4}{\cancel{28}}(a-b)\cancel{(a-b)}\cancel{(a+b)}}{\underset{5}{\cancel{15}}\cancel{(1-x)}},$$

$$=\frac{16(1-x)(a-b)}{25}.$$

EXERCISE LXI

Find the quotient of:

1. $\dfrac{3\,ax}{8} \div \dfrac{5\,a}{12\,xy}$.

2. $\dfrac{55\,a}{24\,x} \div \dfrac{15\,a^2x^2}{4}$.

3. $\dfrac{36\,x^3y^2}{27\,a^2b^3} \div \dfrac{24\,x^2y^3}{81\,a^3b}$.

4. $\dfrac{40\,a^7b^5c^6}{22\,m^3x^4z^5} \div \dfrac{35\,a^6b^6c^6}{88\,m^6xz^7}$.

5. $\dfrac{9\,y(a-4\,b)}{22\,a^2x(a+b)} \div \dfrac{6\,y^2(4\,b-a)}{55\,ax^2(a-b)}$.

6. $\dfrac{b-a}{b} \div (a-b)$.

7. $\left(1+\dfrac{a}{b}\right) \div \left(1+\dfrac{b}{a}\right)$.

8. $(\frac{1}{3}\,a + \frac{1}{2}\,b) \div (\frac{1}{4}\,a + \frac{1}{3}\,b)$.

9. $\left[\dfrac{a^3+b^3}{a^4+b^4} \div \dfrac{a^2+ab+b^2}{a^8-b^8}\right] \div \dfrac{a^4-b^4}{a+b}$.

10. $\left[\dfrac{a^3-b^3}{a^2+b^2} \div \dfrac{(a^2-b^2)^2}{a^4-b^4}\right] \div \dfrac{a^4+b^4}{a^8-b^8}$.

11. $\left[\left(x-\dfrac{y^2}{x}\right) \div (x-y)\right]\dfrac{x^2}{x+y}$.

12. $\left[\dfrac{(a+b)^2-c^2}{(b+c)^2-a^2} \div \dfrac{a^2-(b-c)^2}{c^2-(b-a)^2}\right]\dfrac{(a+b+c)^2}{(a-b-c)^2}$.

13. $\left[\dfrac{x^3-2\,x^2-2\,x+1}{x^2-2\,x+1} \div \dfrac{x^2+x+1}{x^2-1}\right]\dfrac{x^3-1}{(x+1)^3}$.

14. $\left[\dfrac{x^2-7\,x+6}{x^2-3\,x+2} \div \dfrac{x^2-9\,x+20}{x^2-7\,x+12}\right]\dfrac{x^2-7\,x+10}{x^2-9\,x+18}$.

15. $\left[\dfrac{(x+y)^2-z^2}{(x+y+z)^2} \cdot \dfrac{(x+z)^2-y^2}{(x-y+z)^2}\right] \div \dfrac{y^2-(x-z)^2}{x^2-(y+z)^2}$.

16. $\left[\dfrac{4\,x^2-5\,x+1}{8\,x^2-2\,x-1} \div \dfrac{3\,x^2-4\,x+1}{3\,x-1-2\,x^2}\right] \div \dfrac{1-3\,x+2\,x^2}{5\,x-1-6\,x^2}$.

17. $\left[\dfrac{2\,m^2-m-6}{m^2-5\,m+6} \cdot \dfrac{2\,m^2-m-3}{6\,m^2-11\,m+3}\right] \div \dfrac{2\,m^2-m-3}{3\,m^2-10\,m+3}$.

COMPLEX FRACTIONS

126. A **complex fraction** is a fraction having one or both of its terms in the form of a fractional expression.

Thus, $\dfrac{a+\dfrac{1}{a}}{x}$, $\dfrac{a}{1+\dfrac{1}{x}}$, and $\dfrac{1+\dfrac{1}{a}}{1+\dfrac{1}{x}}$ are complex fractions.

The process indicated is merely one of division, — after the numerator and denominator have been simplified.

Hence, the **Rule to Simplify Complex Fractions**: *Divide the numerator by the denominator.*

1. Simplify $\dfrac{1-\dfrac{a}{b}}{\dfrac{a-b}{b}}$.

$$\frac{1-\dfrac{a}{b}}{\dfrac{a-b}{b}}=\left(1-\frac{a}{b}\right)\div\left(\frac{a-b}{b}\right),$$

$$=\frac{b-a}{b}\div\frac{a-b}{b},$$

$$=\frac{\cancel{b-a}}{\cancel{b}}\cdot\frac{\cancel{b}}{\cancel{a-b}}=-1.$$

If the L. C. D. of the denominators of the several fractions is easily found by inspection, it is sometimes preferable to simplify the complex fraction by multiplying both numerator and denominator by that L. C. D.

$$\frac{1-\dfrac{a}{b}}{\dfrac{a-b}{b}}=\frac{b\left(1-\dfrac{a}{b}\right)}{b\left(\dfrac{a-b}{b}\right)}=\frac{b-a}{a-b}=-1.$$

127. **A continued fraction**, that is, a complex fraction in the form $\dfrac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}$, is simplified by beginning at the last fractional expression and working up.

1. Simplify $\dfrac{1}{4-\dfrac{4}{2-\dfrac{1}{1-\dfrac{1}{2x}}}}$.

$$\frac{1}{4-\dfrac{4}{2-\dfrac{1}{1-\dfrac{1}{2x}}}}=\frac{1}{4-\dfrac{4}{2-\dfrac{1}{\dfrac{2x-1}{2x}}}},$$

$$=\frac{1}{4-\dfrac{4}{2-\dfrac{2x}{2x-1}}}=\frac{1}{4-\dfrac{4}{\dfrac{2(x-1)}{2x-1}}},$$

$$=\frac{1}{4-\dfrac{2(2x-1)}{x-1}}=\frac{1}{\dfrac{-2}{x-1}},$$

$$=\frac{x-1}{-2}=\frac{1-x}{2}.$$

NOTE. In § 52 it was stated that 0 can never be taken as a divisor; hence, if $b=0$, the form $\frac{a}{b}$ may be considered impossible. The definitions of fractions hitherto given must be understood to exclude 0 as a denominator.

EXERCISE LXII

Simplify the following fractions:

1. $\dfrac{1+\dfrac{1}{x}}{\dfrac{1}{y}+1}.$

2. $\dfrac{\dfrac{m}{n}-\dfrac{x}{y}}{\dfrac{x}{n}+\dfrac{m}{y}}.$

3. $\dfrac{a-\dfrac{b}{c}}{a+\dfrac{b}{c}}.$

4. $\dfrac{1+\dfrac{a}{b}}{a+\dfrac{b}{a}}.$

5. $\dfrac{\dfrac{a}{b}+\dfrac{a+b}{a-b}}{\dfrac{b}{a}+\dfrac{a-b}{a+b}}.$

6. $\dfrac{1+\dfrac{x}{y}}{1+\dfrac{(x-y)^2}{4xy}}.$

7. $\dfrac{1-\dfrac{x-3y}{x+y}}{\dfrac{3x+y}{x-y}-3}.$

8. $\dfrac{\dfrac{x-2}{x-3}-\dfrac{x+3}{x+4}}{\dfrac{1}{x+1}+\dfrac{7}{x-3}}.$

9. $\dfrac{1}{11-\dfrac{4}{3-\dfrac{10x}{x-y}}}.$

10. $\dfrac{\dfrac{x}{a^2}+\dfrac{a^2b^4}{x^3}-\dfrac{23b^2}{x}}{\dfrac{1}{a}+\dfrac{5b}{x}+\dfrac{ab^2}{x^2}}.$

11. $\dfrac{\dfrac{a^2}{a-b}-\dfrac{b^2}{a+b}-a}{\dfrac{b}{(a-b)^2}+\dfrac{b}{(a+b)^2}}.$

12. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{x}}}}.$

13. $\dfrac{2}{1-\dfrac{a}{a-\dfrac{1}{1-\dfrac{a}{2}}}}.$

14. $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{a}}}}.$

15. $\dfrac{2}{3-\dfrac{2}{3+\dfrac{1}{1-\dfrac{1}{x}}}}.$

REVIEW EXERCISE LXIII

1. Simplify the following fractions:

1. $\dfrac{5x^2+4x-1}{20x^2+21x-5}$.

2. $\dfrac{12x^2+24x-15}{4x^2+12x+5}$.

3. $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}} \cdot \dfrac{x^2-y^2}{x^3+y^3}$.

4. $\dfrac{3x^3+6x^2-3x-6}{x^3+3x^2+2x}$.

5. $1-\dfrac{\dfrac{a^2+3a+2}{a^2+2a+1}}{\dfrac{a^2+7a+12}{a^2+5a+4}}$.

6. $\dfrac{x^3-a^3}{x^3+a^3}+\dfrac{(x-a)^2}{x^2-a^2}$.

7. $\dfrac{x^3-8y^3}{x^2-y^2} \cdot \dfrac{x+y}{x^2+2xy+y^2}$.

8. $\dfrac{xy^2+y^3}{x^2+xy+y^2} \div \dfrac{x^2y^2-x^4}{y^3-x^3}$.

9. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{a}{b}-1} \cdot \dfrac{a^2-b^2}{a^3+b^3}$.

10. $\dfrac{a^4-x^4}{a^2-2ax+x^2} \div \dfrac{a^2x+x^3}{a^3-x^3}$.

11. $\dfrac{x^2-9x+20}{x^2-6x} \cdot \dfrac{x^2-13x+42}{x^2-5x}$.

12. $\left(a+\dfrac{b-a}{1+ab}\right)\dfrac{a}{b} \div \left(1-a\dfrac{b-a}{1+ab}\right)$.

13. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right) \div \left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$.

14. $\left(\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right) \div \left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$.

15. $\dfrac{xy}{x^2+y^2}\left(\dfrac{x+y}{x-y}+\dfrac{x^3+y^3}{x^3-y^3}\right) \div \left(\dfrac{x+y}{x-y}-\dfrac{x^3+y^3}{x^3-y^3}\right)$.

16. $\dfrac{3}{x+1}-\dfrac{2x-1}{x^2+\dfrac{x}{2}+\dfrac{1}{2}}.$

17. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}.$

18. $\dfrac{2m-3+\dfrac{1}{m}}{\dfrac{2m-1}{m}}.$

19. $\dfrac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}}\cdot\dfrac{a^2-b^2}{a^2+b^2}.$

20. $\dfrac{a^4-x^4}{a^2-2ax+x^2}\div\dfrac{a^2-x^2}{(a-x)^3}.$

21. $\dfrac{2x^2+5x+2}{x^2-4}\div\dfrac{2x^2+9x+4}{x+4}.$

22. $\dfrac{\dfrac{1}{x-y}-\dfrac{x}{x^2-y^2}}{\dfrac{x}{xy+y^2}-\dfrac{y}{x^2+xy}}\cdot\dfrac{x^2-2xy+y^2}{xy}.$

23. $\left(\dfrac{x^4-y^4}{x^2-y^2}\div\dfrac{x+y}{x^2-xy}\right)\div\left(\dfrac{x^2+y^2}{x-y}\div\dfrac{x+y}{xy-y^2}\right).$

24. $\left(1-\dfrac{ab}{a^2-ab+b^2}\right)\left(1-\dfrac{ab}{a^2+2ab+b^2}\right)\div\dfrac{a^3-b^3}{a^3+b^3}.$

25. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ca}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}.$

26. $\dfrac{x^2-yz}{(x+y)(x+z)}+\dfrac{y^2-zx}{(y+z)(y+x)}+\dfrac{z^2-xy}{(z+x)(z+y)}.$

27. $\dfrac{1}{x^2-\dfrac{x^3+1}{x+\dfrac{1}{x-1}}}.$

28. $\dfrac{a}{1-\dfrac{a}{1-\dfrac{1}{a-1}}}.$

29. $$\frac{\dfrac{a-b}{1+ab}+\dfrac{b-c}{1+bc}}{1-\dfrac{(a-b)(b-c)}{(1+ab)(1+bc)}}.$$

30. $$\frac{\left(a^2+\dfrac{b^4}{a^2+b^2}\right)(a^2-b^2)}{1-\dfrac{b}{a+b}-\dfrac{1}{2}\left(1-\dfrac{a-b}{a+b}\right)}.$$

31. $$(a^2+b^2)\left\{\frac{\dfrac{b^4}{b^2-a^2}-a^2}{\dfrac{a}{a+b}+\dfrac{b}{a-b}}\right\}.$$

32. $$\frac{\dfrac{1-x^2}{1+y}\left(\dfrac{x}{1+x}-1\right)}{1-\left(\dfrac{1}{1-y}-\dfrac{x^2+y^2-x+y}{1-y^2}\right)}.$$

33. $$\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\div\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right).$$

34. $$\frac{\dfrac{1}{a-x}-\dfrac{1}{a-y}+\dfrac{x}{(a-x)^2}-\dfrac{y}{(a-y)^2}}{\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-x)(a-y)^2}}.$$

35. $$\frac{y-z}{y+z}+\frac{z-x}{z+x}+\frac{x-y}{x+y}+\frac{(y-z)(z-x)(x-y)}{(y+z)(z+x)(x+y)}.$$

36. $$\frac{a^2-bc}{(a+b)(a+c)}+\frac{b^2-ca}{(b+c)(b+a)}+\frac{c^2-ab}{(c+a)(c+b)}.$$

37. If $\dfrac{x}{y+z}=a$, $\dfrac{y}{x+z}=b$, $\dfrac{z}{x+y}=c$, find the value of

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}.$$

CHAPTER IX

SIMPLE EQUATIONS

128. Some forms of equations have already been defined and discussed in Chapter IV. As before, § 64, the last letters of the alphabet are used to represent unknown quantities, and the first letters are used to represent known quantities.

129. An **integral equation** is one which does not contain the unknown quantity in any denominator. A **numerical equation** is one which contains the unknown quantities and numerical quantities only. A **literal equation** is one which contains other literal quantities than the unknown quantity.

Thus, $2x + 3 = 11$ is both integral and numerical; $a + x = b$ is both integral and literal; $\frac{2}{x} + 3 = \frac{4}{x}$ is both fractional and numerical; $\frac{a}{x} + b = c$ is both fractional and literal.

130. The **degree** of an equation in one unknown quantity depends upon the highest degree which that unknown quantity may have in any term. If the equation, in its simplest integral form, contains the first degree of the unknown number as the highest degree, the equation is said to be of the **first degree**, or a **simple**, or **linear equation**.

Thus, $ax + b = c$ is a simple equation.

NUMERICAL FRACTIONAL EQUATIONS

131. Two equations are said to be **equivalent** when the roots of the equations are identical. The general method for the solution of simple equations consists, as in § 68, in the transformation of the original equation into a series of equivalent equations, until such a simple form is obtained that it contains as a left member only the unknown quantity, and as a right member only the known quantity.

1. Solve $7\left(x-\frac{1}{9}\right)+\frac{1}{12}=\frac{2x-3}{6}.$ (1)

Simplifying in (1), $7x-\frac{7}{9}+\frac{1}{12}=\frac{2x-3}{6}.$ (2)

Multiplying (2) by 36, the L. C. M. of the denominators,

$$252x-7(4)+3=6(2x-3), \quad (3)$$

Simplifying in (3), $252x-28+3=12x-18,$ (4)

Transposing in (4), $252x-12x=28-3-18,$ (5)

Uniting in (5), $240x=7,$ (6)

Dividing (6) by 240, $x=\frac{7}{240}.$ (7)

2. Solve $\frac{8}{x}+\frac{5+x}{7x}-\frac{3}{2x}=\frac{15}{4}.$ (1)

Multiplying (1) by $28x$,

$$28(8)+4(5+x)-14(3)=7x(15), \quad (2)$$

Simplifying in (2),

$$224+20+4x-42=105x, \quad (3)$$

Transposing in (3), $4x-105x=-224-20+42,$ (4)

Uniting in (4), $-101x=-202,$ (5)

Dividing (5) by -101, $x=2.$ (6)

To solve a simple equation in the fractional form and containing one unknown quantity :

Multiply every term of each member of the equation by the L. C. M of the denominators ; transpose the unknown terms to the left member, and the known terms to the right member ; unite similar terms. Divide every term of each member of the equation by the coefficient of the unknown quantity.

Multiplying by the L. C. M. of the denominators is called **clearing the equation of fractions.**

EXERCISE LXIV

Solve the following equations :

1. $\frac{4}{x}=\frac{5}{x}-1.$

2. $\frac{1}{x}+\frac{2}{x}+\frac{3}{x}=1.$

3. $\frac{42}{x}-\frac{1}{x}+\frac{1}{3x}=\frac{40}{3}.$

4. $\frac{4}{5x}-\frac{7}{10x}=\frac{1}{10}.$

5. $\frac{56}{9x}+\frac{12}{x}-\frac{7}{3x}+\frac{1}{9x}=4.$

6. $\frac{1}{9x}+\frac{1}{12x}+\frac{1}{8x}+\frac{1}{24x}-\frac{13}{72}=0.$

7. $4+\frac{3}{x}-\frac{2x-5}{11x}-\frac{1}{3}+\frac{7}{x}=0.$

8. $\frac{5}{x}+\frac{97}{7}-\frac{5(11-3x)}{6x}=\frac{7-9x}{2x}-\frac{5}{7x}.$

9. $\frac{3x+1}{4x}-\frac{5x-1}{8x}+7-\frac{9x+5}{5x}=\frac{21}{5}+\frac{5}{x}.$

10. $\frac{7}{8x}-\frac{11}{12x}+\frac{13}{16x}=\frac{4}{3}-\frac{1}{6}+\frac{3}{8}.$

132. The method of procedure in case the denominators contain several terms is the same as in § 131.

1\. Solve $\frac{7}{8x+2}-\frac{11}{20x+5}=13.$ (1)

Factoring the denominators in (1),

$$\frac{7}{2(4x+1)}-\frac{11}{5(4x+1)}=13, \quad (2)$$

multiplying (2) by $10(4x+1)$,

$$5(7)-2(11)=13(10)(4x+1), \quad (3)$$

simplifying in (3), $35-22=520x+130,$ (4)

transposing in (4), $-520x=-35+22+130,$ (5)

uniting in (5), $-520x=117,$ (6)

dividing (6) by -520, $x=-\frac{117}{520}=-\frac{9}{40}.$ (7)

2\. Solve $\frac{5x+1}{x-1}-\frac{x-9}{x+1}=4.$ (1)

Multiplying (1) by $(x-1)(x+1)$,

$$(5x+1)(x+1)-(x-9)(x-1)=4(x-1)(x+1), \quad (2)$$

simplifying in (2),

$$5x^2+6x+1-x^2+10x-9=4x^2-4, \quad (3)$$

transposing in (3),

$$5x^2-x^2-4x^2+6x+10x=-1+9-4, \quad (4)$$

uniting in (4), $16x=4,$ (5)

dividing (5) by 16, $x=\frac{1}{4}.$ (6)

Although (3) contains x^2, yet the equation can be solved as a simple equation because the simplified form, (5), contains only the first power of the unknown quantity.

NOTE. Each term of a fractional equation which is in the fractional form should be reduced to its lowest terms.

EXERCISE LXV

Solve the following equations:

1. $\dfrac{1}{x+2}+\dfrac{7}{3(x+2)}=\dfrac{2}{3}.$

2. $\dfrac{5}{x+1}-\dfrac{3}{2x+2}=\dfrac{5}{2}.$

3. $\dfrac{9}{2x+2}-\dfrac{7}{3x+3}=\dfrac{13}{12}.$

4. $\dfrac{1}{2-3x}-\dfrac{1}{4-6x}=\dfrac{1}{6}.$

5. $\dfrac{7}{x+1}-\dfrac{11}{2x+2}=\dfrac{5}{2}.$

6. $\dfrac{4}{2x+2}+\dfrac{31}{3x+3}=\dfrac{1}{6}.$

7. $\dfrac{13}{x-2}+\dfrac{7}{5x-10}=4.$

8. $\dfrac{15}{3-2x}-\dfrac{2}{6-4x}=14.$

9. $\dfrac{1}{3(x-7)}+\dfrac{1}{6}=\dfrac{1}{2x-14}.$

10. $\dfrac{120}{144-x^2}=\dfrac{x-10}{12+x}+\dfrac{x+10}{12-x}.$

11. $\dfrac{5x}{x-3}+\dfrac{7+4x}{4x-7}=1-\dfrac{6-5x}{x-3}.$

12. $\dfrac{3x^2+4x+5}{6x^2+7x+8}=\dfrac{1}{2}.$

13. $\dfrac{x-9}{x-12}+\dfrac{x-4}{x-7}=2.$

14. $\dfrac{5x^2+7x+4}{15x^2+x-6}=\dfrac{3x^2+6x+7}{9x^2+6x+3}.$

15. $\dfrac{7x-2}{5x+3}-\dfrac{2x+5}{3x+9}=\dfrac{11x+3}{15x+9}.$

16. $\dfrac{5x+2}{4x+3}-\dfrac{3x+1}{6x+2}=\dfrac{3x+2}{4x-6}.$

17. $\dfrac{1}{2x+1}-\dfrac{11}{12}=\dfrac{x}{2x+1}-\dfrac{4}{3(2x+1)}.$

18. $\dfrac{5x+4}{x-1}-\dfrac{3(x-7)}{5(x-1)}=6-\dfrac{9}{5(x-1)}.$

19. $\dfrac{x+3}{4}+\dfrac{2x-1}{3x-12}-\dfrac{7x+5}{8x-32}=\dfrac{x^2-25}{4x-16}.$

20. $\dfrac{13x+10}{28x-32}+\dfrac{2(10x+1)}{49x-56}-\dfrac{7-11x}{35x-40}=3.$

21. $\dfrac{\frac{1}{2}+x}{\frac{1}{3}-x}+5+\dfrac{2x^2}{\frac{1}{9}-x^2}=3-\dfrac{\frac{1}{4}+x}{\frac{1}{3}+x}.$

133. If the equation contains several terms in one denominator and several simple denominators, the process of solution is much simplified by first multiplying every term of each member of the equation by the L. C. M. of the simple denominators and then simplifying the resulting equivalent equation.

1. Solve $\frac{5x}{x+4} - \frac{16x+59}{15} = \frac{3x+2}{5} - \frac{5x+1}{3}$. (1)

Multiplying every term of each member of (1) by 15,

$$\frac{75x}{x+4} - (16x+59) = 3(3x+2) - 5(5x+1), \tag{2}$$

simplifying in (2),

$$\frac{75x}{x+4} - 16x - 59 = 9x + 6 - 25x - 5, \tag{3}$$

transposing integral terms in (3),

$$\frac{75x}{x+4} = 16x + 59 + 9x + 6 - 25x - 5, \tag{4}$$

uniting integral terms in (4), $\frac{75x}{x+4} = 60$ (5)

dividing each member of (5) by 15, $\frac{5x}{x+4} = 4,$ (6)

multiplying each member of (6) by $x+4$, $5x = 4x + 16,$ (7)

transposing and uniting in (7), $x = 16.$ (8)

Some equations which appear to be higher than first degree equations may be solved, by various devices, as first degree equations.

2. Solve $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}$. (1)

Uniting the members in (1) and simplifying,

$$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-4)(x-5)}, \qquad (2)$$

multiplying each member of (2) by the L. C. M.,

$$(x-4)(x-5)=(x-2)(x-3), \qquad (3)$$

simplifying in (3), $x^2-9x+20=x^2-5x+6$ (4)

transposing and uniting in (4), $-4x=-14$, (5)

dividing each member of (5) by -4, $x=\frac{7}{2}$. (6)

3. Solve $\dfrac{2x+3}{x-1}+\dfrac{3x+4}{x-2}=\dfrac{5x+30}{x+3}$. (1)

Reduce each fraction in (1) to a mixed expression,

$$2+\frac{5}{x-1}+3+\frac{10}{x-2}=5+\frac{15}{x+3}, \qquad (2)$$

uniting integral terms in (2),

$$\frac{5}{x-1}+\frac{10}{x-2}=\frac{15}{x+3}, \qquad (3)$$

dividing every term of each member of (3) by 5,

$$\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x+3}, \qquad (4)$$

multiplying every term of each member of (4) by the L. C. M.,

$$(x-2)(x+3)+2(x-1)(x+3)=3(x-1)(x-2), \qquad (5)$$

simplifying in (5),

$$x^2+x-6+2x^2+4x-6=3x^2-9x+6 \qquad (6)$$

transposing and uniting in (6), $14x=18$, (7)

dividing each member of (7) by 14, $x=\frac{9}{7}$. (8)

EXERCISE LXVI

Solve the following equations:

1. $\dfrac{2x+7}{4}-\dfrac{3x+8}{5x+3}=\dfrac{4x+3}{8}$.

2. $\dfrac{6x+1}{15}+\dfrac{11x-1}{4x+3}=\dfrac{2x+11}{5}$.

3. $\dfrac{5x+1}{12}-\dfrac{3x+2}{5x-8}=\dfrac{15x-39}{36}$.

4. $\dfrac{2x+3}{10}+\dfrac{2x-1}{4x+2}=\dfrac{x+3}{5}$.

5. $\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}$.

6. $\dfrac{x+1}{x+2}-\dfrac{x+2}{x+3}=\dfrac{x+3}{x+4}-\dfrac{x+4}{x+5}$.

7. $\dfrac{21x+13}{3x+1}-\dfrac{8x+13}{4x+1}=5$.

8. $\dfrac{9x+2}{3x-1}+\dfrac{5x+2}{x+1}=8$.

9. $\dfrac{2x+3}{x-4}+\dfrac{4x+5}{x-6}=6$.

10. $\dfrac{x+3}{x+4}-\dfrac{x+4}{x+5}=\dfrac{x+5}{x+6}-\dfrac{x+6}{x+7}$.

11. $\dfrac{4x+5}{9}-\dfrac{14x+3}{35x+1}=\dfrac{16x+3}{36}$.

12. $\dfrac{x-10}{7}+\dfrac{13x-2}{10x+7}-\dfrac{47x-1}{35}+\dfrac{6x+7}{5}=1$.

13. $\dfrac{3x+8}{x+1}+\dfrac{5x+8}{x+2}+\dfrac{10x+27}{x+3}=18$.

LITERAL FRACTIONAL EQUATIONS

134. Literal fractional equations are solved by the Rule given in § 131.

1. Solve $a+\frac{b}{x}=\frac{a}{x}-b.$ (1)

Multiplying each term of (1) by x, $ax+b=a-bx,$ (2)

transposing in (2), $ax+bx=a-b,$ (3)

factoring in (3), $x(a+b)=a-b,$ (4)

dividing each member of (4) by $a+b$, $x=\frac{a-b}{a+b}.$ (5)

2. Solve $\frac{ax-b}{c}+d=x+a.$ (1)

Multiplying (1) by c, $ax-b+cd=cx+ac,$ (2)

transposing in (2), $ax-cx=b-cd+ac,$ (3)

factoring in (3), $x(a-c)=b-cd+ac,$ (4)

dividing (4) by $a-c$, $x=\frac{b-cd+ac}{a-c}.$ (5)

3. Solve $\frac{bx}{a^2-b^2}=\frac{x+3b}{a+b}-1.$ (1)

Multiplying (1) by a^2-b^2,

$bx=(x+3b)(a-b)-(a^2-b^2),$ (2)

simplifying in (2), $bx=ax-bx+3ab-3b^2-a^2+b^2,$ (3)

transposing in (3), $bx-ax+bx=3ab-3b^2-a^2+b^2,$ (4)

uniting in (4), $2bx-ax=-a^2+3ab-2b^2,$ (5)

factoring in (5), $x(2b-a)=-(a-b)(a-2b),$ (6)

dividing (6) by $2b-a$, $x=-(a-b)(-1),$ (7)

simplifying in (7), $x=a-b.$ (8)

EXERCISE LXVII

Solve the following equations:

1. $\dfrac{ax}{b}=1.$

2. $\dfrac{a}{bx}=\dfrac{b}{a}.$

3. $\dfrac{a}{x}-1=\dfrac{a-1}{x}.$

4. $\dfrac{x}{m}+\dfrac{x}{n}=a.$

5. $\dfrac{m}{ax}+\dfrac{n}{bx}=c.$

6. $\dfrac{x}{m}+\dfrac{x}{n}+\dfrac{x}{p}=a.$

7. $\dfrac{ax}{m}+\dfrac{bx}{n}+\dfrac{cx}{p}=d.$

8. $\dfrac{a}{mx}+\dfrac{b}{nx}+\dfrac{c}{xp}=d.$

9. $\dfrac{a}{bcx}+\dfrac{b}{acx}+\dfrac{c}{abx}=a^2+b^2+c^2.$

10. $\dfrac{ab}{cx}+\dfrac{ac}{bx}+\dfrac{bc}{ax}=\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}.$

11. $\dfrac{x+a}{b}+\dfrac{7}{3}-\dfrac{2(x-b)}{3a}=4.$

12. $\dfrac{3(x-4b)}{a}-\dfrac{5(x-3a)}{4b}=4.$

13. $\dfrac{ax-1}{b}+\dfrac{bx-1}{a}=\dfrac{a^3+b^3}{a^2b^2}.$

14. $\dfrac{cx+ab}{a}+b=\dfrac{ax+b^2}{b}+\dfrac{a^2}{c}.$

15. $\dfrac{b}{ax}-\dfrac{b^2}{a^2}-\dfrac{a}{2bx}=\dfrac{1}{2}-\dfrac{a^2}{2b^2}.$

16. $\dfrac{b+c}{bcx}+\dfrac{a+c}{acx}-\dfrac{a+b}{abx}=\dfrac{2}{c}.$

17. $\dfrac{ax}{a-b}=1-x.$

18. $\dfrac{x}{a+b}=\dfrac{b+x}{a}.$

19. $\dfrac{a+x}{b+c}-\dfrac{x}{a}=1.$

20. $\dfrac{ab-c^2x}{a-b}+\dfrac{a}{3c}=\dfrac{cx}{3b}.$

21. $\dfrac{x}{b+cx}=\dfrac{a}{c(1+a)}.$

22. $\dfrac{ax}{a-b}-\dfrac{bx}{a+b}=a^2+b^2.$

23. $\dfrac{x-2a+3b}{x+2a-3b}=\dfrac{5b}{4a-b}.$

24. $\dfrac{x-a}{a-b}+\dfrac{x+3b}{a+b}=3.$

25. $\dfrac{ax+bc}{b+c}-\dfrac{bx-ac}{a^2+b^2}=\dfrac{c}{a}.$

26. $\dfrac{ax^2+bx+1}{bx^2+ax+1}=\dfrac{a}{b}.$

27. $a\left(\dfrac{x}{c}-b\right)+\dfrac{x}{a+b}-a^2=\dfrac{x-2bc}{a-b}.$

28. $\dfrac{2bx}{ab+cd}+\dfrac{cd-x}{a}-\dfrac{a(x-2ab)}{ab-cd}=a+b.$

29. $\dfrac{x-ab}{ac}+\dfrac{x-ac}{bc}+\dfrac{x-bc}{ab}-3=\dfrac{bc+a^2}{ac}+\dfrac{c}{b}.$

30. $\dfrac{(b-c)x}{bx-c^2}+\dfrac{cx+b^2}{c(x+b)}=\dfrac{2b-c}{b}.$

31. $\dfrac{ax+bc}{bx+ac}-\dfrac{bx+a(c-1)}{ax-bc}=\dfrac{a-b}{a}+\dfrac{(a-b)(x-c)+b}{bx+ac}.$

32. $\dfrac{a}{2x-ab}+\dfrac{x-ac}{b-c}-\dfrac{1}{c}=\dfrac{bx-ac^2}{b(b-c)}-\dfrac{ac}{b}.$

33. $\dfrac{3a-x}{b}-\dfrac{2(x-c)}{3a-2b}+\dfrac{x+2b}{3a+c}=1-\dfrac{c}{b}.$

34. $\dfrac{ax+b}{bx+c}-\dfrac{ax-c}{bx+2c}=\dfrac{b^2+c(a+b)}{b(bx+3c)}.$

35. $\dfrac{abx+1}{a+b}+\dfrac{acx-1}{a-c}+\dfrac{bcx-1}{b-c}=\dfrac{ab+ac+bc}{abc}.$

36. $\dfrac{ax+b}{x+1}-\dfrac{b(ax-b)}{bx-a}=\dfrac{2b(b-a)(2b+a)}{2b(bx-a)+b^2-a^2}.$

37. $\dfrac{a(x-2c)}{a-b-c}-\dfrac{b(x-2a)}{c-a-b}+\dfrac{c(x+2b)}{a+b+c}=x.$

38. $\dfrac{2a-x}{a-b-c}+\dfrac{2b-x}{b-a-c}-\dfrac{2c-x}{c-a-b}=\dfrac{x}{a+b+c}.$

135. A quantity q is said to be expressed **in terms of** the quantities m and n when q is the left member of an equation which contains m and n and numerical quantities only in the right member.

Thus, if $a=b+c$, a is expressed in terms of b and c.

EXERCISE LXVIII

In the following equations express each literal quantity in terms of the other literal quantities :

1. $\dfrac{a}{b}=\dfrac{c}{d}$.

2. $\dfrac{x}{r}=\dfrac{b}{a}$.

3. $V=\dfrac{S}{T}$.

4. $\dfrac{1}{A}=\dfrac{1}{ab}$.

5. $\dfrac{1}{F}=\dfrac{2}{a+b}$.

6. $\dfrac{1}{D_o}+\dfrac{1}{D_i}=\dfrac{1}{F}$.

7. $s=\dfrac{(v_1-v_0)t}{2}$.

8. $\dfrac{V_1P_1}{T_1}=\dfrac{V_2P_2}{T_2}$.

9. $tv-tw=t-1$.

10. $C=\frac{5}{9}(F-32)$.

11. $Ma-Rb=a-b$.

12. $C=\dfrac{E}{R}$.

13. $d=\dfrac{W}{abc}$.

14. $ax=bw$.

15. $V=abc$.

16. $\dfrac{V_1}{V_2}=\dfrac{P_2}{P_1}$.

17. $G=\dfrac{AW}{L}$.

18. $abc=4Rs$.

19. $a=\dfrac{v_1-v_0}{t}$.

20. $T=\dfrac{a}{2}(b+b')$.

21. $d=\dfrac{a}{2}(2t+1)$.

22. $\dfrac{mp-rs}{w}=\dfrac{s-rp}{n}$.

INDETERMINATE EQUATIONS

136. If a single simple equation contains *two* unknown quantities, there is an infinite number of solutions: hence, such an equation is called **indeterminate.**

Thus, $x+y=5$ is a simple indeterminate equation. If in $x+y=5$, $x=0$, then $y=5$; if $x=\frac{1}{8}$, then $y=4\frac{7}{8}$; if $x=2$, then $y=3$; and so on.

137. The solutions of indeterminate equations are often restricted to those in which the roots are both positive and integral. Such solutions can often be found by inspection.

1. Find the positive integral solutions of $2x+5y=14$.

If $y=0$, $x=7$; if $y=1$, $x=4\frac{1}{2}$; if $y=2$, $x=2$; if $y=3$, $x=-\frac{1}{2}$.

Whence the positive integral solutions are: $y=0$, $x=7$; $y=2$, $x=2$.

2. Find the positive integral solutions of $2x+3y=19$.

If $y=0$, $x=9\frac{1}{2}$; if $y=1$, $x=8$; if $y=2$, $x=6\frac{1}{2}$; if $y=3$, $x=5$; if $y=4$, $x=3\frac{1}{2}$; if $y=5$, $x=2$; if $y=6$, $x=\frac{1}{2}$; if $y=7$, $x=-1$.

Whence the positive integral solutions are:

$$y=1,\ x=8;\ y=3,\ x=5;\ y=5,\ x=2.$$

EXERCISE LXIX

Find the positive integral solutions of the following:

1. $7x+5y=38$.
2. $6x+11y=125$.
3. $x+20y=53$.
4. $2x+17y=70$.
5. $32x+3y=1624$.
6. $11x=576-13y$.

7. $14x - 9y = 1$.

8. $8x - 15y = 33$.

9. $x - 10y = 6$

10. $11x = 7y + 114$.

11. $9x = 11y$.

12. $\frac{3x}{4} = 7y - 29$.

13. $\frac{3x}{2} + \frac{5y}{3} = 36$.

14. $\frac{3x-1}{5} + \frac{4y+2}{10} = 7$.

15. $\frac{3x+5}{4} = \frac{7x+y-6}{8}$.

16. $\frac{3(x+4y-50)}{19x+y-200} = 1$.

Find the least positive solutions of the following:

17. $\frac{2x}{4} + \frac{5y}{3} = 11$.

18. $\frac{3x+7}{4} = \frac{4x-y+1}{2}$.

19. $\frac{4x}{3} + \frac{3y}{8} = \frac{2x+y+2}{2}$.

20. $\frac{15x}{9} - \frac{8y}{4} = -\frac{y-2x}{2}$.

21. $\frac{2x-y+8}{2y-x+5} = 1$.

22. $\frac{3(2x-y+2)}{4x-3y+10} = 5$.

23. In how many ways can $110 be made up of ten-dollar bills and two-dollar bills?

24. If A spends 76 cents in buying pencils at 3 cents each, and penholders at 2 cents each, how many of each does he buy?

25. How many golf balls at 50 cents each, and how many baseballs at $1.25 each, can be bought for $9?

26. How many baseballs at $1.25 each, and how many baseball bats at 75 cents each can be bought for $21?

27. In how many ways can railroad stocks at $105 and $95 respectively per share be bought for $5900?

REVIEW EXERCISE LXX

Solve the following equations:

1. $7x - \dfrac{11x-3}{4} = 3x + 7.$

2. $\dfrac{9x+7}{2} - \left(x - \dfrac{x-2}{7}\right) = 36.$

3. $\dfrac{x+3}{2} - \dfrac{x-2}{3} = \dfrac{3x-5}{12} + \dfrac{1}{4}.$

4. $\dfrac{5x-1}{8} - \dfrac{3x-2}{7} = \dfrac{5-x}{4}.$

5. $3x - \dfrac{x-4}{4} - 4 = \dfrac{5x+14}{3}.$

6. $\dfrac{2x+1}{3} - \dfrac{4x+5}{4} = \dfrac{2x+5}{8} - \dfrac{x+8}{6}.$

7. $\dfrac{x-a}{2x-b} - \dfrac{3x-c}{6x-d} = 0.$

8. $\dfrac{x-2a}{x+3a} - \dfrac{13a^2-2x^2}{x^2-9a^2} = 3.$

9. $\dfrac{x+(a-b)x}{a-b} = \dfrac{cx-d}{c}.$

10. $\dfrac{ab+x}{b^2} - \dfrac{b^2-x}{a^2b} = \dfrac{x-b}{a^2} - \dfrac{ab-x}{b^2}.$

11. $\dfrac{ax}{a-b} + 4b = \dfrac{cx}{3a+b}.$

12. $\dfrac{3}{1-3x} + \dfrac{5}{1-5x} = \dfrac{4}{1-2x}.$

13. $\dfrac{x+2a}{x-2b} = \dfrac{(x+a)^2}{(x-b)^2}.$

14. $\dfrac{3x-2}{2x-1}-\dfrac{2x+1}{3x+2}=\dfrac{5}{6}$.

15. $ax-\dfrac{bx+1}{x}=\dfrac{a(x^2-1)}{x}$.

16. $\dfrac{x^2+a^2}{4x^2-a^2}-\dfrac{x}{2x+a}=-\dfrac{1}{4}$.

17. $\dfrac{1}{2(3x+7)}-\dfrac{2}{3x^2+22x+35}+\dfrac{1}{2x+10}=0$.

18. $\dfrac{x+1}{3x+1}+\dfrac{2x}{5-6x}=\dfrac{5}{5+9x-18x^2}$.

19. $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{b^2-a^2}{b^2-bx}$.

20. $\dfrac{a-x}{bc}+\dfrac{b-x}{ca}+\dfrac{c-x}{ab}=0$.

21. $\dfrac{ax-b}{ax+b}-\dfrac{bx-a}{bx+a}=\dfrac{a-b}{(ax+b)(bx+a)}$.

22. $\dfrac{a^2+4a}{x^2+x-a^2+a}-\dfrac{a}{x+a}=\dfrac{1}{x-a+1}$.

23. $\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}$.

24. $\dfrac{8x+1}{18}-\dfrac{7x-6}{5x-4}=\dfrac{4x+2}{9}-\dfrac{7}{6}$.

25. $\dfrac{b(x-b)}{2a+b+2c}+\dfrac{a(x-a)}{a+2b+2c}+\dfrac{c(x-c)}{2a+2b+c}=\dfrac{x}{2}$.

CHAPTER X

GRAPHS

138. In Chapter I it was shown that numbers can be represented by distances along a line from a given fixed starting-point. In the present chapter it will be shown how the relation between two algebraic quantities, which are connected in any way, can be represented by drawings to scale. As a first step in this direction, it is necessary to establish certain conventions, by the aid of which the position of any point in a single flat surface, or plane, can be fixed by two algebraic quantities.

139. Constructing a pair of perpendicular lines, called **axes**, $X'X$ and $Y'Y$, as shown in Fig. 3, a point can be located by saying that it is m units above or below $X'X$, and n units to the right or left of $Y'Y$.

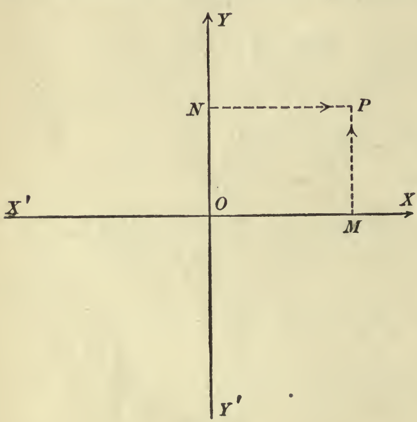

FIG. 3.

If, instead of using the words "above" and "below," "right" or "left," it is understood that *all distances measured upward or to the right are positive, and those measured downward or to the left are negative, two numbers with the proper signs attached will represent* *the distances of the point from the two lines, and these two numbers taken together will locate absolutely the position of any point in the same plane with the lines.*

140. The distance of a point to the right or left of $Y'Y$ is given first; and the distance of this point above or below $X'X$ is given second. These two distances are called the **coördinates** of the point. The coördinates are written in parenthesis; thus, $P = (3, 4)$ means that the point P is 3 units to the right of the vertical line $Y'Y$, and 4 units above the horizontal line $X'X$.

141. Any point whose coördinates are known can be definitely located.

Thus the locations of the points, $A = (3, 4)$, $B = (-2, 6)$, $C = (-5, -7)$, $D = (6, -3)$, are shown in Fig. 4.

If either of the coördinates is 0, the point will lie on one of the axes.

Thus, the location of $E = (0, -5)$ and of $F = (2, 0)$ is shown.

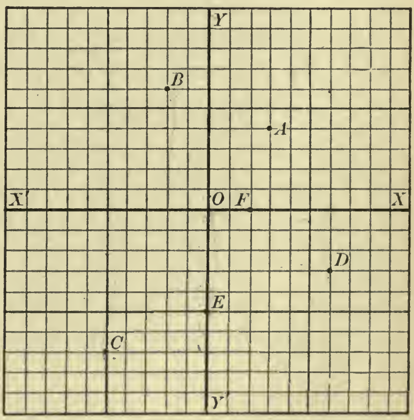

FIG. 4.

EXERCISE LXXI

Locate the following points whose coördinates are:

1. $(2, 4)$.
2. $(-3, 4)$.
3. $(-3, -4)$.
4. $(-4, -8)$.
5. $(0, -9)$.
6. $(0, 0)$.
7. $(-4, -3\frac{1}{2})$.
8. $(-3, 2\frac{1}{2})$.
9. $(-7\frac{1}{2}, 9)$.

GRAPHS OF SIMPLE EQUATIONS IN ONE UNKNOWN QUANTITY

142. If a single equation in x and y is given, it is evident that the coördinates of points taken at random will not satisfy it, since, if a value is assigned to one of the coördinates, the other will be determined by such an equation. There are then only certain points whose coördinates satisfy the given equation, and it will be discovered that these points lie consecutively, and hence form a curve (or straight line). Such a curve, which contains all the points which satisfy a given equation, is called the **graph** of that equation.

143. In case the equation contains only one unknown quantity, as $x = 5$, the graph is very easily determined, since the equation says that every point which satisfies it must have its x-coördinate equal to 5, but places no restriction upon the y-coördinate. All such points lie in MN, Fig. 5, 5 units to the right of the axis $Y'Y$, and MN is, therefore, the graph of the equation, $x = 5$. Similarly, the graph of any simple equation in one unknown can be shown to be a line parallel to one of the axes.

EXERCISE LXXII

Construct the graphs of the equations:

1. $x + 5 = 6.$
2. $y + 4 = 9.$
3. $9 + 5x = 16 + 4x.$
4. $8y = 5 + 10y - 11.$
5. $5x - (3x - 7) = 17.$
6. $\frac{y}{6} + 1 = 2.$
7. $\frac{x}{4} + \frac{x}{5} = 9.$
8. $\frac{3x + 5}{4} - \frac{x}{2} = \frac{13 - 4x}{2}.$

GRAPHS OF SIMPLE EQUATIONS IN TWO UNKNOWNS

144. As the simplest type of equations in two unknowns, consider those in which the known quantity is wanting. Any such equation may be put into the form, $y = ax$, where a can have any value — positive, negative, or fractional. All points whose coördinates satisfy this equation must have their y-coördinate a times their x-coördinate, and hence must lie on a straight line through the origin, as KL in Fig. 5. To determine the graph of any such equation, plot any one point which satisfies it, and draw a line of indefinite length through this point and the origin.

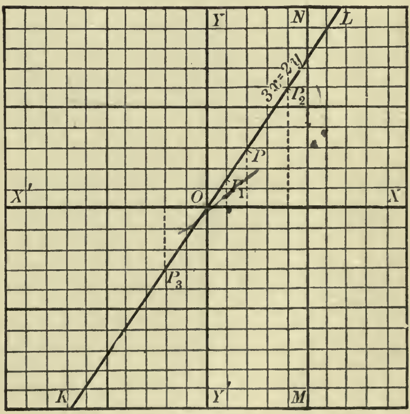

Fig. 5.

For example, the equation $3x = 2y$ is satisfied by (2, 3). Hence a line through $P = (2, 3)$ and (0, 0) is its graph. If the pupil has not had enough geometry to be sure that all points whose coördinates satisfy the equation must lie on the line, let him plot a number of such points, as $(1, \frac{3}{2})$, $(3, \frac{9}{2})$, (4, 6), etc., and convince himself that they all do lie on the line.

EXERCISE LXXIII

Construct the graphs of the following equations:

1. $y = 3x$.	**3.** $x + y = 0$.	**5.** $3x + 8y = 0$.
2. $5y = 7x$.	**4.** $2x - 5y = 0$.	**6.** $x - 5y = 0$.

145. Any simple equation in x and y, which contains a known quantity, can be reduced to the form $y = ax + b$. If the graph of the equation $y = ax$ is plotted, and from every point on this line lines parallel to $Y'Y$ and equal in length to b are drawn, the extremities of these lines will evidently be the points whose coördinates satisfy the equation $y = ax + b$. These points are also on a straight line. It will be noticed that the graph of every equation of the first degree in x and y is a straight line. To find the graph, it is only necessary to determine two points and draw a line through them. These two points are usually taken on the axes.

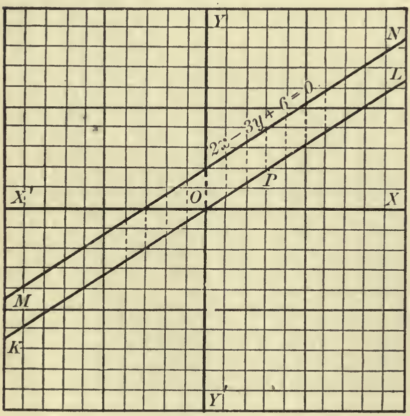

Fig. 6.

For example, the equation $2x - 3y + 6 = 0$ is satisfied by $(-3, 0)$ and $(0, 2)$; its graph has the position of MN in Fig. 6. The pupil should assure himself by trial that this line contains every point which satisfies the given equation; for example, the points $(1, 2\frac{2}{3})$, $(2, 3\frac{1}{3})$, $(4, 4\frac{2}{3})$, $(-1, 1\frac{1}{3})$, etc.

EXERCISE LXXIV

Construct the graphs of the following equations:

1. $x + y = 3$.

2. $x + 5y = 16$.

3. $4x + y = 10$.

4. $3x + 2y = 13$.

5. $3x + 4y = 21$.

6. $4x + 5y = 25$.

7. $x + 6y = 20$.

8. $3x + 2y = 24$.

CHAPTER XI

SIMULTANEOUS SIMPLE EQUATIONS

146. Two or more **simultaneous equations** are those which can be satisfied by the same values of the unknowns.

Thus, $2x + 3y = 8$, and $3x + 2y = 7$, are simultaneous simple equations, since each equation is satisfied if $x = 1$, and $y = 2$. Similarly, $x + y + z = 6$, $2x - y + z = 3$, and $3x + 2y - 4z = -5$, are simultaneous simple equations, since each equation is satisfied if $x = 1$, $y = 2$, and $z = 3$.

147. Two or more equations are **inconsistent** when they cannot be satisfied by the same values of the unknowns.

Thus, $x + y = 5$, and $x + y = 4$, are inconsistent equations, since the unknowns cannot have the same values in both equations.

148. Two or more equations are **dependent** when each equation can be derived from the others.

Thus, $x + y = 4$, and $2x + 2y = 8$, are dependent equations, since when the second equation is divided by 2 it gives $x + y = 4$, identical with the first equation.

Dependent equations, though simultaneous, are reducible to a single indeterminate equation.

149. Two or more equations are **independent** when none of them can be derived from the others.

Thus, $2x + y = 5$, and $x + 3y = 10$, are independent since neither can be derived from the other.

150. A system of equations is a group of two or more equations.

A solution of a system of equations is a set of numbers which satisfy each of the equations in that system. The process of finding the solution of a system of equations is called **solving the equations.**

GRAPHS OF SIMULTANEOUS SIMPLE EQUATIONS

151. Graphs of simultaneous simple equations in two unknowns can be constructed by the method of §§ 144 and 145.

Consider the simultaneous simple equations:

$$\begin{cases} x + 2y = 4, & (1) \\ x + y = 5. & (2) \end{cases}$$

In (1), $B = (0, 2)$, $A = (4, 0)$; in (2), $D = (0, 5)$, $C = (5, 0)$.

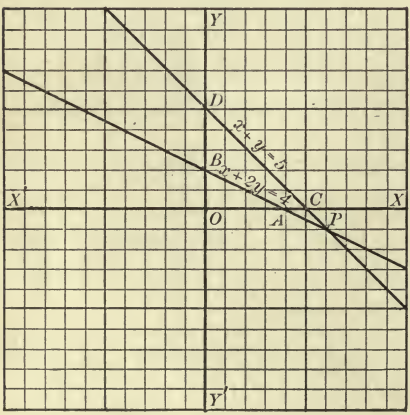

FIG. 7.

In Fig. 7, the location of the points A and B gives the line AB; and the location of the points C and D gives the line CD. The lines AB and CD cross, or intersect, at P; and since P is on both lines, its coördinates must satisfy both equations. Hence its coördinates are the values of x and y which would be determined by solving the two equations simultaneously. These are found by measurement to be $x = 6$ and $y = -1$. Two lines which intersect represent simultaneous equations which have a single solution.

GRAPHS OF TWO INCONSISTENT SIMPLE EQUATIONS

152. Inconsistent equations, § 147, may be shown to have no common solution by constructing their graphs.

Thus, find a solution, if possible, of

$$\begin{cases} 2x + y = 4, & (1) \\ 2x + y = 8. & (2) \end{cases}$$

In (1), if $B = (0, 4)$, $A = (2, 0)$; in (2), $D = (0, 8)$, $C = (4, 0)$.

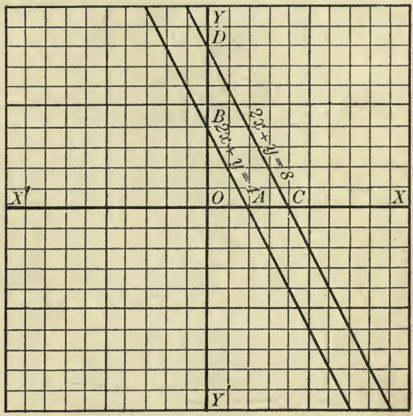

FIG. 8.

In Fig. 8 the graphs of (1) and (2) are such that they never meet; that is, AB and CD are parallel lines. Hence there is evidently no common solution of (1) and (2).

GRAPHS OF TWO DEPENDENT EQUATIONS

153. Two dependent equations, § 148, may be shown to be reducible to a single indeterminate simple equation by constructing their graphs.

Thus, find a solution, if possible, of

$$\begin{cases} 2x + 3y = 8, & (1) \\ \dfrac{x}{3} + \dfrac{y}{2} = \dfrac{4}{3}. & (2) \end{cases}$$

In (1), $B = (0, \frac{8}{3})$, $A = (4, 0)$; in (2), $D = (0, \frac{8}{3})$, $C = (4, 0)$.

Since B and D and A and C have respectively the same coördinates, the graph is a single line; and the given equations are therefore reducible to an indeterminate simple equation whose graph has been shown, § 145, to be a line crossing the axes.

EXERCISE LXXV

Determine the nature of the following systems of equations by the graphical method:

1. $\begin{cases} x+y=8, \\ x-3y=0. \end{cases}$

2. $\begin{cases} -x+y=1, \\ 2x+y=10. \end{cases}$

3. $\begin{cases} x+y=14, \\ x-y=2. \end{cases}$

4. $\begin{cases} 2x+3y=2, \\ -3x-7y=2. \end{cases}$

5. $\begin{cases} 4x+3y=10, \\ 2x+y=6. \end{cases}$

6. $\begin{cases} 2x+3y=12, \\ 4x+5y=20. \end{cases}$

7. $\begin{cases} x+y=5, \\ 2x+y=6. \end{cases}$

8. $\begin{cases} 3x+2y=7, \\ 2x+3y=8. \end{cases}$

9. $\begin{cases} 2x-3y=2, \\ x-5y=-5. \end{cases}$

10. $\begin{cases} 2x+3y=5, \\ 3x+2y=5. \end{cases}$

11. $\begin{cases} x-2y=4, \\ 2x-4y=8. \end{cases}$

12. $\begin{cases} 5x-3y=-2, \\ 4x+2y=-6. \end{cases}$

13. $\begin{cases} 3x+2y=6, \\ 9x+6y=18. \end{cases}$

14. $\begin{cases} 3x+4y=9, \\ 3x+4y=12. \end{cases}$

15. $\begin{cases} 4x+2y=8, \\ 10x+5y=20. \end{cases}$

16. $\begin{cases} 2x-3y=24, \\ 2x-3y=6. \end{cases}$

17. $\begin{cases} 2x+5y=10, \\ 4x-3y=12. \end{cases}$

18. $\begin{cases} 5x-6y=3, \\ 10x-12y=6. \end{cases}$

19. $\begin{cases} 2x-3y=0, \\ 3x-4y=0. \end{cases}$

20. $\begin{cases} 4x-5y=1, \\ 5x-4y=9. \end{cases}$

21. $\begin{cases} 5x+3y=5, \\ 9x+4y=9. \end{cases}$

22. $\begin{cases} 6x-5y=-7, \\ -2x+y=3. \end{cases}$

23. $\begin{cases} 3x+4y=2, \\ 4x+3y=6. \end{cases}$

24. $\begin{cases} 3x-5y=12, \\ 6x-10y=24. \end{cases}$

154. Elimination of one of two or more unknowns in a system of simultaneous equations is the process of combining the equations in such a way as to obtain fewer equations containing less unknown quantities. The quantity which has been caused to disappear is said to be **eliminated.**

TWO UNKNOWN QUANTITIES

I. ELIMINATION BY ADDITION OR SUBTRACTION

155. 1. Solve $\begin{cases} 3x + 2y = 12, & (1) \\ -2x + 3y = 5. & (2) \end{cases}$

Multiplying (1) by 2, $\quad 6x + 4y = 24, \quad (3)$

multiplying (2) by 3, $\quad \underline{-6x + 9y = 15,} \quad (4)$

adding (3) and (4), $\quad 13y = 39, \quad (5)$

dividing (5) by 13, $\quad y = 3. \quad (6)$

Substituting y from (6) in (1), $\quad 3x + 6 = 12, \quad (7)$

transposing in (7), $\quad 3x = 6, \quad (8)$

dividing (8) by 3, $\quad x = 2. \quad (9)$

VERIFICATION: $\quad 6 + 6 = 12; \; -4 + 9 = 5.$

The above equations can be solved by this method by multiplying the first equation by 3 and the second equation by 2, and subtracting the equivalent equations thus derived.

It is to be noticed that the equations are checked by substituting the values of the unknowns in the original equations.

2. Solve $\begin{cases} 11x + 2y = 23, & (1) \\ 9x + y = 8. & (2) \end{cases}$

Multiplying (2) by 2, $18x + 2y = 16,$ (3)

rewriting (1), $\underline{11x + 2y = 23,}$ (4)

subtracting (4) from (3), $7x = -7,$ (5)

dividing (5) by 7, $x = -1.$ (6)

Substituting x from (6) in (1), $-11 + 2y = 23,$ (7)

transposing in (7), $2y = 34,$ (8)

dividing (8) by 2, $y = 17.$ (9)

VERIFICATION: $-11 + 34 = 23$; $-9 + 17 = 8.$

The above equations can be solved by this method by multiplying the first equation by 9 and the second equation by 11, and subtracting the equivalent equations thus derived.

That unknown is preferably chosen for elimination whose coefficients are such that they can be made equal by the smaller multipliers.

Rule for Elimination by Addition or Subtraction: *Make equal the coefficients of one of the unknowns in each equation by multiplying one or both of the equations by the necessary numbers. Add or subtract the resulting equations according as the equal coefficients have unlike or like signs. Find the other unknown number by substituting the value of the unknown already found in that one of the given equations which has the least coefficients. Verify the solution by substitution in each of the given equations.*

EXERCISE LXXVI

Solve the following systems of equations:

1. $\begin{cases} x - 2y = 6, \\ x + 2y = 34. \end{cases}$

2. $\begin{cases} 2x + y = 7, \\ -2x + 3y = 13. \end{cases}$

3. $\begin{cases} 7x - 3y = 15, \\ 5x + 6y = 27. \end{cases}$

4. $\begin{cases} 8x + 17y = 42, \\ 2x + 19y = 40. \end{cases}$

5. $\begin{cases} 4x + 5y = 40, \\ 6x - 7y = 2. \end{cases}$

6. $\begin{cases} 17x - 18y = 15, \\ 5x + 12y = 39. \end{cases}$

7. $\begin{cases} 28x + y = 33, \\ -21x + 11y = 34. \end{cases}$

8. $\begin{cases} 33x - (y + 9) = 23, \\ 44x + 3(y + 1) = 50. \end{cases}$

9. $\begin{cases} 3x - 7y = 1, \\ 5x + 3y = 2. \end{cases}$

10. $\begin{cases} 9x - 6y = 2, \\ 45x + 8 = 72y. \end{cases}$

11. $\begin{cases} 19x - 16y = 91, \\ 27x - 20y = 130. \end{cases}$

12. $\begin{cases} 8x - 9y = 34, \\ 9x - 8y = 17. \end{cases}$

13. $\begin{cases} 6x + 5y = 68, \\ 4x - 13y = 78. \end{cases}$

14. $\begin{cases} 18x + 5y = 38, \\ 12x - y = -5. \end{cases}$

15. $\begin{cases} 8x + 9y = 26, \\ 32x - 3y = 26. \end{cases}$

16. $\begin{cases} 33x + 54y = -24, \\ 44x - 80y = 44. \end{cases}$

17. $\begin{cases} 21x - 23y = 2, \\ 7x - 19y = 12. \end{cases}$

18. $\begin{cases} 15x + 28y = 157, \\ 20x + 21y = 144. \end{cases}$

19. $\begin{cases} 65x + 68y = -3, \\ 39x - 119y = 158. \end{cases}$

20. $\begin{cases} 63x - 46y = 29, \\ 42x - 69y = 96. \end{cases}$

21. $\begin{cases} 27x - 5y = -37, \\ 81x - 7y = -151. \end{cases}$

22. $\begin{cases} 13x - 15y = 11, \\ 12x - 7y = 17. \end{cases}$

23. $\begin{cases} 11x + 13y = -9, \\ 15x - 14y = -44. \end{cases}$

24. $\begin{cases} 19x - 23y = -11, \\ 22x + 25y = -16. \end{cases}$

II. Elimination by Substitution

156. Solve $\begin{cases} 2x + 3y = 13, & (1) \\ -4x + 21y = 55. & (2) \end{cases}$

Transposing in (1), $\quad 2x = 13 - 3y, \quad (3)$

dividing (3) by 2, $\quad x = \dfrac{13 - 3y}{2}, \quad (4)$

substituting x from (4) in (2),

$$-4\left(\frac{13 - 3y}{2}\right) + 21y = 55, \quad (5)$$

simplifying in (5),

$$\frac{-52 + 12y}{2} + 21y = 55, \quad (6)$$

multiplying (6) by 2,

$$-52 + 12y + 42y = 110, \quad (7)$$

transposing and uniting in (7), $\quad 54y = 162, \quad (8)$

dividing (8) by 54, $\quad y = 3. \quad (9)$

Substituting y from (9) in (1), $2x + 9 = 13, \quad (10)$

transposing and uniting in (10), $\quad 2x = 4, \quad (11)$

dividing (11) by 2, $\quad x = 2. \quad (12)$

Verification: $4 + 9 = 13$; $-8 + 63 = 55$.

It is to be noticed that the above equations may also be solved by the Addition and Subtraction method.

Rule for Elimination by Substitution: *In one of the equations find the value of one unknown quantity in terms of the other. Substitute the value thus obtained in the other equation. Reduce this equation. Verify the solution in each of the given equations.*

EXERCISE LXXVII

Solve the following systems of equations by substitution:

1. $\begin{cases} x - y = 3, \\ 2x - y = 0. \end{cases}$

2. $\begin{cases} -2x + y = -3, \\ -3x + 4y = 8. \end{cases}$

3. $\begin{cases} -2x + y = 3, \\ 3x + y = 13. \end{cases}$

4. $\begin{cases} 2x + 3y = 46, \\ x + y = 18. \end{cases}$

5. $\begin{cases} 4x + y = 23, \\ 3x - 2y = 9. \end{cases}$

6. $\begin{cases} 3x + 5y = 94, \\ 2x - y = 15. \end{cases}$

7. $\begin{cases} 4x + 3y = 81, \\ -x + 2y = 21. \end{cases}$

8. $\begin{cases} 4x + 2y = 38, \\ 3x - 3y = 6. \end{cases}$

9. $\begin{cases} 2x + y = 20, \\ 4x + 3y = 70. \end{cases}$

10. $\begin{cases} x - 9y = 0, \\ 4x - y = 70. \end{cases}$

11. $\begin{cases} 4x - 5y = 3, \\ 8x + 2y = 66. \end{cases}$

12. $\begin{cases} 2x - y = 10, \\ 3y + 17x = 177. \end{cases}$

13. $\begin{cases} 3x - 2y = 91, \\ 7x + 3y = 82. \end{cases}$

14. $\begin{cases} 3x - 5y = 2, \\ 4x + 7y = -93. \end{cases}$

15. $\begin{cases} 4x + 3y = 4, \\ -7x + 5y = 75. \end{cases}$

16. $\begin{cases} 8x - 5y = 6, \\ 7x + 10y = 149. \end{cases}$

17. $\begin{cases} 3x + 12y = 57, \\ 2x + y = 10. \end{cases}$

18. $\begin{cases} 7x + 4y = 95, \\ x - 2y = -7. \end{cases}$

19. $\begin{cases} 27x + 14y = 41, \\ 36x + 51y = 87. \end{cases}$

20. $\begin{cases} 100x - 143y = 757, \\ 11x - 91y = 8. \end{cases}$

21. $\begin{cases} 55x + 31y = 171, \\ 27x - 11y = 18.4. \end{cases}$

22. $\begin{cases} 109x + 110y = 86, \\ 107x + 146y = 98. \end{cases}$

23. $\begin{cases} 33x + 25y = 4, \\ 21x + 85y = 6. \end{cases}$

24. $\begin{cases} 39x - 98y = 3, \\ 51x + 182y = 63. \end{cases}$

III. Elimination by Comparison

157. Solve $\begin{cases} 2x + 3y = 16, & (1) \\ x + y = 18. & (2) \end{cases}$

Transposing in (1), $2x = 16 - 3y,$ (3)

dividing (3) by 2, $x = \dfrac{16 - 3y}{2},$ (4)

transposing in (2), $x = 18 - y,$ (5)

comparing x in (5) and (4), $18 - y = \dfrac{16 - 3y}{2},$ (6)

multiplying (6) by 2, $36 - 2y = 16 - 3y,$ (7)

transposing and uniting in (7), $y = -20,$ (8)

substituting y in (5), $x = 38.$ (9)

Verification: $76 - 60 = 16$; $38 - 20 = 18$.

Rule for Elimination by Comparison: *In each equation find the value of one unknown in terms of the other. Place these values equal, and solve the resulting equation. Verify the solution in each of the given equations.*

EXERCISE LXXVIII

Solve the following systems of equations by comparison, and check the results on the graph:

1. $\begin{cases} 5x + y = 7, \\ -5x + y = -3. \end{cases}$

2. $\begin{cases} x + y = 0, \\ 2x + 3y = 1. \end{cases}$

3. $\begin{cases} 2x + 3y = 10, \\ 3x + 2y = 10. \end{cases}$

4. $\begin{cases} x + 4y = 7, \\ 2x + 3y = 9. \end{cases}$

5. $\begin{cases} x + 3y = 3, \\ 3x + 4y = -1. \end{cases}$

6. $\begin{cases} 3x + 2y = 0, \\ 2x - y = -7. \end{cases}$

158. If either, or both, of the equations in a system of equations contain aggregations or fractions, it is, in general, best to simplify the equations before elimination.

Solve $\begin{cases} 4(x-3y)=8, & (1) \\ \dfrac{x+y}{x-2y}=3. & (2) \end{cases}$

Simplifying in (1), $4x-12y=8,$ (3)

multiplying (2) by $x-2y$, $x+y=3x-6y,$ (4)

transposing and uniting in (4), $-2x+7y=0,$ (5)

multiplying (5) by 2, $-4x+14y=0,$ (6)

rewriting (3), $4x-12y=8,$ (7)

adding (6) and (7), $2y=8,$ (8)

dividing (8) by 2, $y=4.$ (9)

Substituting y from (9) in (3), $4x-48=8,$ (10)

transposing and uniting in (10), $4x=56,$ (11)

dividing (11) by 4, $x=14.$ (12)

Verification: $4(14-12)=8$; $\dfrac{14+4}{14-8}=3.$

EXERCISE LXXIX

Solve the following systems of equations, selecting the best method:

1. $\begin{cases} 4(3x-5)-2(y-x)=2, \\ 2(5x-y)-3y=5. \end{cases}$

2. $\begin{cases} \frac{4}{5}(x-y)-\frac{1}{10}x-\frac{1}{20}y=14, \\ \frac{5}{6}(x-14)-\frac{7}{12}(y+12)=-2. \end{cases}$

3. $\begin{cases} \dfrac{7}{2x-y} = \dfrac{1}{x-y}, \\ 3x + y = 23. \end{cases}$

4. $\begin{cases} 9 - \frac{1}{2}(x - \frac{1}{2}y) = 14, \\ 5 + \frac{5}{4}(x + y) = 9. \end{cases}$

5. $\begin{cases} 5x - (3y - \frac{1}{2}) = \frac{3}{4}, \\ 4 + x - 2(y - \frac{1}{3}) = \frac{16}{25}. \end{cases}$

6. $\begin{cases} \dfrac{2}{10+x} + \dfrac{1}{10+y} = 0, \\ \dfrac{17}{2x-5y} = \dfrac{3}{x+5}. \end{cases}$

7. $\begin{cases} \dfrac{x+y}{x-y} = 6, \\ \dfrac{x+7}{x+7y} = \dfrac{1}{3}. \end{cases}$

8. $\begin{cases} \dfrac{6x-y}{5(x+2y)} = \dfrac{1}{29}, \\ \dfrac{5x+y}{y+1} = 2. \end{cases}$

9. $\begin{cases} \dfrac{x+y+4}{x-y+6} = 6, \\ \dfrac{2x-y+7}{x-2y+7} = -2. \end{cases}$

10. $\begin{cases} \dfrac{8x-3y}{5x-2y+3} = 4, \\ \dfrac{4x+2y+11}{6x-7y+6} = -\dfrac{4}{3}. \end{cases}$

11. $\begin{cases} \dfrac{5x-4}{3} - x = \dfrac{4y+1}{11} + 3, \\ \dfrac{3y-7x}{4} + 13 = \dfrac{x-y}{8} - \dfrac{y}{3}. \end{cases}$

12. $\begin{cases} \dfrac{x+1}{4} + \dfrac{y+2}{10} = \dfrac{2(y-x)}{5}, \\ \dfrac{x-1}{4} - \dfrac{y-2}{12} = \dfrac{3y-8x}{18}. \end{cases}$

13. $\begin{cases} \dfrac{4x+y-4}{3} + \dfrac{6x+2y-7}{9} = 0, \\ \dfrac{2x-y+1}{8} - \dfrac{10x-4y}{3} = 1. \end{cases}$

14. $$\begin{cases} \dfrac{3x+y+5}{9}+\dfrac{2x-y+5}{6}=\dfrac{x+2y+1}{2}, \\[2ex] \dfrac{2x-y+7}{8}-\dfrac{4x-3y-1}{4}=\dfrac{5x+8y+3}{16}. \end{cases}$$

15. $$\begin{cases} (3x+8)(4y-3)=(2x+9)(6y-5), \\ (2x-1)(12y-1)=(3x+8)(8y-7). \end{cases}$$

16. $$\begin{cases} \dfrac{7y}{10}-\dfrac{5y+22}{7}=\dfrac{x}{5}+\dfrac{55-8y}{6}, \\[2ex] \dfrac{3+x}{7}+\dfrac{5x-3y}{11}=\dfrac{1}{11}x+2y-19. \end{cases}$$

17. $$\begin{cases} \dfrac{2x+7y+5}{7}+3y=29x-\dfrac{4x+11y+5}{8}, \\[2ex] 3x-\dfrac{5x+3y}{11}=\dfrac{11x-14y+241}{154}. \end{cases}$$

18. $$\begin{cases} 3x+\dfrac{4y+5x}{6}+\dfrac{9x+8y-12}{30}=\dfrac{1}{12}+\dfrac{11x+6y+1}{6} \\[2ex] \dfrac{x+3y}{7}+\dfrac{7}{12}=\dfrac{4y+5}{8}. \end{cases}$$

19. $$\begin{cases} x-\dfrac{2y-x}{22}=20-\dfrac{49-2x}{9}, \\[2ex] x+\dfrac{x+7}{9}=25-\dfrac{73-3y}{5}. \end{cases}$$

20. $$\begin{cases} \dfrac{x-4}{3}+\dfrac{x-3}{4}=\dfrac{8+y}{6}, \\[2ex] \dfrac{2y+7}{8}-\dfrac{3x-y}{7}=\dfrac{3y-2x+4}{8}. \end{cases}$$

21. $$\begin{cases} \dfrac{3y-2}{4}-\dfrac{x-5}{2}=5-\dfrac{2x+3y-1}{8}, \\[2ex] \dfrac{5x+6y-3}{11}-\dfrac{2x+9y-2}{7}=-2. \end{cases}$$

159. It is often convenient in simultaneous equations containing fractions to eliminate one of the fractions.

1. Solve $\begin{cases} \dfrac{x}{2} - \dfrac{y}{4} = 1, & (1) \\ \dfrac{x}{4} - \dfrac{5y}{12} = -3. & (2) \end{cases}$

$$\frac{x}{4} - \frac{5y}{12} = -3.$$

Multiply (1) by $\frac{1}{2}$, $\quad \dfrac{x}{4} - \dfrac{y}{8} = \dfrac{1}{2},$ (3)

subtracting (3) from (2),

$$-\frac{5y}{12} + \frac{y}{8} = -3 - \frac{1}{2}, \qquad (4)$$

multiplying (4) by 24,

$$-10y + 3y = -72 - 12, \qquad (5)$$

uniting in (5), $\quad -7y = -84,$ (6)

dividing (6) by -7, $\quad y = 12.$ (7)

Substituting y from (7) in (2),

$$\frac{x}{4} - 5 = -3, \qquad (8)$$

transposing and uniting in (8), $\dfrac{x}{4} = 2,$ (9)

multiplying (9) by 4, $\quad x = 8.$ (10)

VERIFICATION: $\dfrac{8}{2} - \dfrac{12}{4} = 1$; $\dfrac{8}{4} - \dfrac{60}{12} = -3.$

This method is especially valuable in solving, by the foregoing methods, equations which contain the unknowns in the denominators.

2. Solve $\begin{cases} \dfrac{3}{x} + \dfrac{4}{y} = 5, & (1) \\ \dfrac{16}{x} - \dfrac{28}{y} = 2. & (2) \end{cases}$

$$\frac{16}{x} - \frac{28}{y} = 2.$$

Multiplying (1) by 7, $\dfrac{21}{x} + \dfrac{28}{y} = 35,$ (3)

adding (2) and (3), $\dfrac{37}{x} = 37,$ (4)

multiplying (4) by x, $37 = 37x,$ (5)

dividing (5) by 37, $x = 1.$ (6)

Substituting x from (6) in (1),

$$3 + \frac{4}{y} = 5, \qquad (7)$$

transposing and uniting in (7),

$$\frac{4}{y} = 2, \qquad (8)$$

multiplying (8) by y, $4 = 2y,$ (9)

dividing (9) by 2, $y = 2.$ (10)

VERIFICATION: $\dfrac{3}{1} + \dfrac{4}{2} = 5$; $\dfrac{16}{1} - \dfrac{28}{2} = 2$

Although equations (1) and (2) can be solved by first multiplying each equation by xy, and then multiplying the resulting equations by 2 and 5 respectively and next subtracting these last equivalent equations, this method is not recommended. If equations are solved by the latter method, it may happen that roots are introduced which do not verify.

EXERCISE LXXX

Solve the following systems of equations by eliminating the fractions:

1. $\begin{cases} \dfrac{x}{6}+\dfrac{y}{5}=7, \\ \dfrac{x}{3}-\dfrac{y}{15}=7. \end{cases}$

2. $\begin{cases} \dfrac{x}{3}+\dfrac{y}{4}=11, \\ \dfrac{x}{7}+\dfrac{y}{8}=5. \end{cases}$

3. $\begin{cases} \dfrac{x}{6}-\dfrac{y}{4}=6, \\ \dfrac{x}{7}-\dfrac{y}{2}=4. \end{cases}$

4. $\begin{cases} \dfrac{x}{5}+\dfrac{7y}{8}=17, \\ \dfrac{x}{3}-\dfrac{3y}{4}=-7. \end{cases}$

5. $\begin{cases} \dfrac{x}{9}+\dfrac{y}{7}=\dfrac{63}{10}, \\ \dfrac{x}{3}+\dfrac{53y}{56}=\dfrac{392}{10}. \end{cases}$

6. $\begin{cases} \dfrac{9x}{10}-\dfrac{4y}{7}=16, \\ \dfrac{5x}{8}-\dfrac{9y}{35}=16. \end{cases}$

7. $\begin{cases} \dfrac{8}{x}+\dfrac{9}{y}=5, \\ \dfrac{40}{x}-\dfrac{21}{y}=3. \end{cases}$

8. $\begin{cases} \dfrac{17}{4x}+\dfrac{19}{6y}=18, \\ \dfrac{11}{3x}+\dfrac{14}{9y}=12. \end{cases}$

9. $\begin{cases} \dfrac{3}{2x}+\dfrac{5}{y}=\dfrac{11}{4}, \\ \dfrac{5}{3x}+\dfrac{13}{9y}=1. \end{cases}$

10. $\begin{cases} \dfrac{9}{7x}-\dfrac{4}{5y}=7, \\ \dfrac{11}{6x}-\dfrac{13}{5y}=\dfrac{19}{3}. \end{cases}$

11. $\begin{cases} 3x+\dfrac{8}{5y}=\dfrac{44}{3}, \\ \dfrac{x}{4}-\dfrac{1}{3y}=\dfrac{4}{9}. \end{cases}$

12. $\begin{cases} \dfrac{1}{x}+\dfrac{7}{y}=9, \\ \dfrac{x}{y}=\dfrac{8}{7}. \end{cases}$

160. Literal simultaneous equations are solved in the same way as are numerical equations. Especial care should be taken to express the values of the unknowns in terms of the knowns; and to that end the known terms should always be transposed to the right member of the equation.

1. Solve $\begin{cases} x + y = 2a, & (1) \\ y = 5a - 4x. & (2) \end{cases}$

Transposing in (2), $4x + y = 5a,$ (3)
rewriting (1), $x + y = 2a,$ (4)
subtracting (4) from (3), $3x \quad = 3a,$ (5)
dividing (5) by 3, $x = a.$ (6)

Substituting x from (6) in (1), $a + y = 2a,$ (7)
transposing and uniting in (7), $y = a.$ (8)

VERIFICATION: $a + a = 2a$; $a = 5a - 4a$.

2. Solve $\begin{cases} ax + by = a^3, & (1) \\ bx + ay = b^3. & (2) \end{cases}$

Multiplying (1) by b, $abx + b^2y = a^3b,$ (3)
multiplying (2) by a, $abx + a^2y = ab^3,$ (4)
subtracting (4) from (3), $b^2y - a^2y = a^3b - ab^3,$ (5)
factoring in (5), $y(b^2 - a^2) = ab(a^2 - b^2),$ (6)
dividing (6) by $b^2 - a^2$, $y = -ab.$ (7)

Substituting y in (1), $ax - ab^2 = a^3,$ (8)
dividing (8) by a, $x - b^2 = a^2,$ (9)
transposing in (9), $x = a^2 + b^2$ (10)

VERIFICATION:

$$a(a^2 + b^2) + b(-ab) = a^3;\ b(a^2 + b^2) + a(-ab) = b^3.$$

3. Solve $\begin{cases} \dfrac{b}{ax} + \dfrac{c}{by} = b^2 + c^2, & (1) \\ \dfrac{a}{bx} - \dfrac{b}{cy} = a^2 - b^2. & (2) \end{cases}$

Multiplying (1) by $\dfrac{a}{b}$, $\qquad \dfrac{1}{x} + \dfrac{ac}{b^2y} = ab + \dfrac{ac^2}{b}$, (3)

multiplying (2) by $\dfrac{b}{a}$, $\qquad \dfrac{1}{x} - \dfrac{b^2}{acy} = ab - \dfrac{b^3}{a}$. (4)

subtracting (4) from (3), $\qquad \dfrac{ac}{b^2y} + \dfrac{b^2}{acy} = \dfrac{ac^2}{b} + \dfrac{b^3}{a}$, (5)

multiplying (5) by ab^2cy, $\qquad a^2c^2 + b^4 = a^2bc^3y + b^5cy$, (6)

factoring the right member in (6), $a^2c^2 + b^4 = bcy(a^2c^2 + b^4)$, (7)

dividing (7) by $(a^2c^2 + b^4)$, $\qquad 1 = bcy$, (8)

dividing (8) by bc, $\qquad y = \dfrac{1}{bc}$. (9)

Substituting y from (9) in (1), $\dfrac{b}{ax} + \dfrac{c}{\frac{b}{bc}} = b^2 + c^2$, (10)

simplifying in (10), $\qquad \dfrac{b}{ax} + c^2 = b^2 + c^2$, (11)

transposing and uniting in (11), $\qquad \dfrac{b}{ax} = b^2$, (12)

dividing (12) by b, $\qquad \dfrac{1}{ax} = b$, (13)

multiplying (13) by ax, $\qquad 1 = abx$, (14)

dividing (14) by ab, $\qquad x = \dfrac{1}{ab}$. (15)

Verification: $\dfrac{b}{\frac{1}{b}} + \dfrac{c}{\frac{1}{c}} = b^2 + c^2$; $\dfrac{a}{\frac{1}{a}} - \dfrac{b}{\frac{1}{b}} = a^2 - b^2$.

EXERCISE LXXXI

Solve the following systems of equations and verify the results:

1. $\begin{cases} x + ay = a^2, \\ x - by = b^2. \end{cases}$

2. $\begin{cases} ax + y = b, \\ x + by = b^2. \end{cases}$

3. $\begin{cases} ax + by = c, \\ \dfrac{x}{y} = \dfrac{m}{n}. \end{cases}$

4. $\begin{cases} a(x+y) + b(x-y) = c, \\ \dfrac{x}{y} = \dfrac{m}{n}. \end{cases}$

5. $\begin{cases} ax + by = c, \\ a_1x + b_1 = c_1y. \end{cases}$

6. $\begin{cases} ax + by = 2, \\ ab(x + y) = a + b. \end{cases}$

7. $\begin{cases} ax = b(y - 2), \\ y - x = \dfrac{a^2 + b^2}{ab}. \end{cases}$

8. $\begin{cases} ay = c(x + 1) - a, \\ y = x + \dfrac{(a+b+c)(c-a)}{ac}. \end{cases}$

9. $\begin{cases} a^2x - b^2y = a + b, \\ bx - ay = -1. \end{cases}$

10. $\begin{cases} ax - (a-b)y = (a-b)^2, \\ bx - y = b(a-b-1). \end{cases}$

11. $\begin{cases} a(x+c) + b(y-c) = a^2 - b^2, \\ y - x = 2c. \end{cases}$

12. $\begin{cases} x - (a + b)y = \dfrac{b^2 - a^2}{b}, \\ (b - a)x + aby = b^2. \end{cases}$

13. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{2a}{a^2 - b^2}, \\ (a+b)x + (a-b)y = a + b. \end{cases}$

14. $\begin{cases} \dfrac{ax}{b} + \dfrac{by}{a} = a + b, \\ \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a^4 + b^4}{a^2b^2}. \end{cases}$

15. $\begin{cases} \dfrac{x + y}{a + b} = ab, \\ \dfrac{bx + a^2}{ay} = 1 + \dfrac{1}{b^2}. \end{cases}$

16. $\begin{cases} \dfrac{x - a}{y - b} = \dfrac{b + c}{a - c}, \\ \dfrac{x - b}{y - a} = \dfrac{a + c}{b - c}. \end{cases}$

17. $\begin{cases} \dfrac{x}{b + c} - \dfrac{y}{a + c} = a - c, \\ \dfrac{x}{a + b} - \dfrac{y}{b + c} = b - a. \end{cases}$

THREE OR MORE UNKNOWN QUANTITIES

161. Three simultaneous equations containing three unknowns are solved by the elimination of one of the unknowns between a pair of the given equations, and by the further elimination of the same unknown between a different pair of the given equations; the resulting equations are then solved as in §§ 155–7.

Elimination is performed by the addition and subtraction method. That quantity is generally chosen for elimination whose coefficients are smallest. It is evident that of three given equations the first may be combined with the second, the first with the third, and the second with the third.

1. Solve $\begin{cases} x+y+z=14, & (1) \\ 4x+2y+z=43, & (2) \\ 9x+3y+z=88. & (3) \end{cases}$

Subtracting (2) from (1), $-3x-y=-29,$ (4)

subtracting (3) from (2), $-5x-y=-45,$ (5)

subtracting (5) from (4), $2x=16,$ (6)

dividing (6) by 2, $x=8.$ (7)

Substituting x from (7) in (5),

$$-40-y=-45, \quad (8)$$

transposing and uniting in (8), $y=5.$ (9)

Substituting x from (7) and y from (9) in (1),

$$8+5+z=14, \quad (10)$$

transposing and uniting in (10), $z=1.$ (11)

Verification:

$$8+5+1=14;\ 32+10+1=43;\ 72+15+1=88.$$

162. Four or more simultaneous equations containing four or more unknowns are solved by the elimination of one of the unknowns between three or more pairs of the given equations, in the resulting equations another unknown is eliminated between two or more pairs of the resulting equations, and the process is continued until three resulting equations are obtained. These latter equations are solved by the method shown in § 161.

Care must be taken to keep the same number of equations as unknowns; otherwise, dependent equations will be obtained.

1. Solve
$$\begin{cases} x+y+z+w=\frac{15}{8}, & (1)\\ 2x+y+3z+2w=\frac{7}{2}, & (2)\\ 3x+2y+2z+8w=\frac{11}{2}, & (3)\\ 4x+3y+4z+6w=\frac{29}{4}. & (4)\end{cases}$$

Eliminate y

Subtracting (2) from (1), $-x-2z-w=-\frac{13}{8}$, (5)

subtracting (3) from (1) × 2, $-x \quad -6w=-\frac{7}{4}$, (6)

subtracting (4) from (1) × 3, $-x-z-3w=-\frac{13}{8}$. (7)

Eliminate z

Subtracting (7) × 2 from (5), $x+5w=\frac{13}{8}$, (8)

rewriting (6), $-x-6w=-\frac{7}{4}$. (6)

Eliminate x

Adding (8) and (6), $-w=-\frac{1}{8}$, (9)

dividing (9) by -1, $w=\frac{1}{8}$. (10)

Whence, by substitution,

$$x=1,\ y=\tfrac{1}{2},\ z=\tfrac{1}{4},\ w=\tfrac{1}{8}.$$

EXERCISE LXXXII

Solve the following systems of equations:

1. $\begin{cases} 4x+5y+9z=13, \\ 5x+y+2z=-5, \\ 7x-5y-8z=-31. \end{cases}$

2. $\begin{cases} x+5y-2z=5, \\ 3x+8y+4z=31, \\ 7x+25y-4z=45. \end{cases}$

3. $\begin{cases} 2x-9y+10z=55, \\ 11x-3y-5z=7, \\ 13x-4y-6z=1. \end{cases}$

4. $\begin{cases} 5x+3y-2z=\frac{9}{8}, \\ 4x-5y+\dfrac{2z}{3}=-1, \\ 8x-\frac{1}{2}y-z=\frac{7}{12}. \end{cases}$

5. $\begin{cases} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{6}z=14, \\ \frac{1}{3}x+\frac{1}{5}y-\frac{1}{9}z=5, \\ \frac{1}{4}x+y-\frac{1}{3}z=12. \end{cases}$

6. $\begin{cases} \frac{1}{5}x+3y-\frac{5}{8}z=16, \\ 2x-\frac{1}{2}y+\frac{5}{6}z=25, \\ \frac{1}{3}x-\frac{4}{5}y+z=17\frac{2}{3}. \end{cases}$

7. $\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}=2, \\ \dfrac{1}{y}-\dfrac{1}{z}=3, \\ \dfrac{1}{z}+\dfrac{1}{x}=9. \end{cases}$

8. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=7, \\ \dfrac{1}{y}+\dfrac{2}{z}=14, \\ \dfrac{2}{x}+\dfrac{3}{z}=21. \end{cases}$

9. $\begin{cases} \dfrac{1}{x}+\dfrac{3}{y}+\dfrac{4}{z}=8, \\ \dfrac{4}{x}+\dfrac{5}{y}-\dfrac{2}{z}=16, \\ \dfrac{7}{x}-\dfrac{2}{y}+\dfrac{4}{z}=21. \end{cases}$

10. $\begin{cases} \dfrac{3}{x}+\dfrac{4}{y}-\dfrac{8}{z}=15, \\ \dfrac{5}{x}-\dfrac{1}{2y}+\dfrac{2}{z}=\frac{1}{6}, \\ \dfrac{9}{4x}+\dfrac{3}{y}+\dfrac{1}{z}=13. \end{cases}$

11. $\begin{cases} x+y=2a, \\ ay+z=a^2, \\ bx-z=b^2. \end{cases}$

12. $\begin{cases} ax+y=1, \\ bx+z=b, \\ cz+x=bc. \end{cases}$

13. $\begin{cases} bx+ay=2ab, \\ cy+bz=2bc, \\ cx+az=2ac. \end{cases}$

14. $\begin{cases} x + ay = a(a+b), \\ a^2z - bx = a^3, \\ y = z - a. \end{cases}$

15. $\begin{cases} (b+c)x + by = c, \\ (a+c)y + cz = a, \\ (a+b)z + ax = b. \end{cases}$

16. $\begin{cases} ax + y + z = abc + a(b+c), \\ x + by + z = abc + b(a+c), \\ x + y + cz = abc + c(a+b). \end{cases}$

17. $\begin{cases} x + y + z + u = 55, \\ x + 2y - z - u = 1, \\ 2x + 3y + 2z - u = 68, \\ 3x - 2y + 2z + u = 54. \end{cases}$

18. $\begin{cases} x + 2y + z - u = 10, \\ x - y + 2z + u = 23, \\ x + 3y + 4z - 2u = 39, \\ x - 5y - 4z - 3u = 41. \end{cases}$

19. $\begin{cases} x + y + z + u = 10, \\ x + 3y + 5z + 7u = 30, \\ x + 6y + 15z + 28u = 80, \\ x + 10y + 35z + 84u = 188. \end{cases}$

20. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{5}z - \frac{1}{6}u = 65, \\ \frac{1}{3}x - \frac{1}{4}y + \frac{1}{8}z - \frac{1}{2}u = 8, \\ \frac{1}{4}x - \frac{1}{6}y + \frac{1}{3}z + \frac{1}{5}u = 53, \\ \frac{2}{5}x - \frac{3}{8}y + \frac{3}{4}z - \frac{2}{3}u = 76. \end{cases}$

21. $\begin{cases} x + 5y = 23, \\ y + 4z = -1, \\ z + 3u = 20, \\ u + 2v = 3, \\ v + x = 6. \end{cases}$

22. $\begin{cases} 2x + 3y = 57, \\ 5x - 4z = 20, \\ 3z + 2u = 48, \\ 4y + 3v = 68, \\ 7u - 6v = 15. \end{cases}$

REVIEW EXERCISE LXXXIII

Solve the following simultaneous equations:

1. $\begin{cases} ax + by = 1, \\ bx - ay = 1. \end{cases}$

2. $\begin{cases} ax = by, \\ bx + ay = c. \end{cases}$

3. $\begin{cases} \dfrac{3}{x} + y = -\dfrac{15}{4}, \\ \dfrac{2}{x} - \dfrac{y}{2} = 1. \end{cases}$

4. $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = 4, \\ \dfrac{1}{x} + \dfrac{7}{y} = 6. \end{cases}$

5. $\begin{cases} \dfrac{2x-3}{5} - \dfrac{y-2}{10} + x = 7, \\ \dfrac{2x-y}{3} + \dfrac{2y-x}{4} = -\dfrac{49}{12}. \end{cases}$

6. $\begin{cases} 3x - \dfrac{2y-5}{x-2} = -\dfrac{4-9x}{3}, \\ 7x - 3y = 10. \end{cases}$

7. $\begin{cases} x + y = 2a, \\ (a-b)x = (a+b)y. \end{cases}$

8. $\begin{cases} \dfrac{1}{ax} + \dfrac{1}{by} = c, \\ \dfrac{1}{bx} - \dfrac{1}{ay} = d. \end{cases}$

9. $\begin{cases} 3x + 2y - 4z = 15, \\ 5x - 3y + 2z = 28, \\ 3y + 4z - x = 24. \end{cases}$

10. $\begin{cases} \dfrac{x}{a+b} + \dfrac{y}{a-b} = 2a, \\ \dfrac{x-y}{4ab} = 1. \end{cases}$

11. $\begin{cases} 5(x-2y) - (x-y) = -24, \\ 11(2x+3y) + (2x-y) = 200. \end{cases}$

12. $\begin{cases} qx - rb = p(a-y), \\ q\dfrac{x}{a} + r = p\left(1 + \dfrac{y}{b}\right). \end{cases}$

13. $\begin{cases} \left(\dfrac{1}{a} + \dfrac{1}{b}\right)x + y\left(\dfrac{1}{a} - \dfrac{1}{b}\right) = 4, \\ \dfrac{x}{a+b} + \dfrac{y}{a-b} = 2. \end{cases}$

14. $\begin{cases} (a-b)x-(a+b)y=2a^2-2b^2, \\ (a+b)x-(a-b)y=4ab. \end{cases}$

15. $\begin{cases} ax-by=a^2-b^2-2ab, \\ bx+ay=2ab+a^2-b^2. \end{cases}$

16. $\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{b+c}=b-a, \\ \dfrac{y}{c-a}+\dfrac{z}{c+a}=c+a, \\ \dfrac{x}{b-c}-\dfrac{z}{a-b}=b-c. \end{cases}$

17. $\begin{cases} (a+b)(x+y)-(a-b)(x-y)=a^2, \\ (a-b)(x+y)+(a+b)(x-y)=b^2. \end{cases}$

18. $\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{a-b}=2a, \\ \dfrac{x-y}{2ab}=\dfrac{x+y}{a^2+b^2}. \end{cases}$

19. $\begin{cases} x+y-z=7, \\ y+z-u=9, \\ z+u-x=19, \\ u+x-y=13. \end{cases}$

20. $\begin{cases} x+y+z=0, \\ (c+b)x+(a+c)y+(b+a)z=0, \\ (c-b)x+(a-c)y+(b-a)z=2(ab+ac+bc) \\ \quad -2(a^2+b^2+c^2). \end{cases}$

CHAPTER XII

PROBLEMS INVOLVING SIMPLE EQUATIONS

EXAMPLES

163. 1. The sum of two numbers is 27, and if the greater be divided by the less, the quotient is 1 and the remainder is 5. Find the numbers.

Let $x =$ the greater number, and $y =$ the less number.

By the first condition, $x + y = 27$, (1)

by the second condition, $\frac{x-5}{y} = 1$. (2)

Solving (1) and (2), $x = 16$ and $y = 11$.

VERIFICATION: $16 + 11 = 27$; $\frac{16-5}{11} = 1$.

It should be noticed that in this, as in many of the following problems, one, two, or more unknowns may be employed to find the solution.

Let $x =$ the greater number, and $27 - x =$ the less number.

By the second condition, $\frac{x-5}{27-x} = 1$. (1)

Solving (1), $x = 16$ and $27 - x = 11$.

In general, if an equation can be solved with a single unknown, this method is preferable.

2. The width of a rectangular room is $\frac{5}{6}$ of its length. If the width were 5 feet more, the room would be square. Find the dimensions.

Let x = number of feet in the length, and $\frac{5x}{6}$ = number of feet in the width.

By the conditions, $\frac{5x}{6} + 5 = x$. (1)

Solving (1), $x = 30$; hence $\frac{5x}{6} = 25$.

VERIFICATION: $25 + 5 = 30$.

3. A's age is $\frac{1}{5}$ of B's age, but 5 years ago A was $\frac{1}{9}$ as old as B. Find their present ages.

Let x = the number of years in A's age, and $5x$ = the number of years in B's age.

By the conditions, $9(x-5) = 5x - 5$. (1)

Solving (1), $x = 10$; hence $5x = 50$.

4. A can row 4 miles an hour down a stream, and 2 miles an hour against the stream. Find A's rate in still water, and the rate of the stream.

Let x = A's rate in still water, in miles per hour; and y = rate of stream, in miles per hour.

By the first condition, $x + y = 4$; (1)

by the second condition, $x - y = 2$. (2)

Solving (1) and (2), $x = 3$, $y = 1$.

5. At what time between 2 and 3 will the hands of a clock be (*a*) together? (*b*) exactly opposite?

In the same period of time the minute hand moves twelve times as fast as the hour hand. Thus, the minute and hour hand cover in an hour respectively 60 and 5 minute-spaces; and in 12 minutes respectively 12 and 1 minute-spaces.

Let $x=$ number of minute-spaces passed over by the minute hand in given time, and $\dfrac{x}{12}=$ number of minute-spaces passed over by the hour hand in given time.

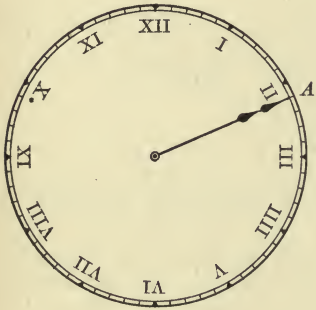

FIG. 9.

(*a*) Since the minute hand starts at XII and moves to A, where it meets the hour hand which starts from II, which is 10 minute-spaces from XII, and *in the same time* moves to A, by the conditions,

$$x=10+\frac{x}{12}. \qquad (1)$$

Solving (1), $x=10\tfrac{10}{11}$.

(*b*) Since the minute hand starts at XII and moves to B, where it is exactly opposite the hour hand, which starts from II, 10 minute-spaces from XII, and in the same time moves to A, by the conditions,

$$x=10+\frac{x}{12}+30. \qquad (1)$$

Solving (1), $x=43\tfrac{7}{11}$.

FIG. 10.

6. The sum of the two digits of a number is 6, and if 36 be added to the number the order of the digits is reversed. Find the number.

Since in arithmetic, position indicates the value of the digits in a number, $(56=10\cdot 5+6)$, let

$x =$ the digit in the tens' place,

and $y =$ digit in the units' place,

and $10x + y =$ the number.

By the first conditions, $x + y = 6,$ (1)

by the second condition, $10x + y + 36 = 10y + x.$ (2)

Solving (1) and (2), $x = 1$, $y = 5$; hence the number is 15.

7. A can do a piece of work in 5 days, and with the help of B can do it in 3 days. How long would it take B alone to do the work?

Let $x =$ the number of days it takes B alone to do the work,

then $\frac{1}{x} =$ part that B can do in 1 day,

and $\frac{1}{5} =$ part that A can do in 1 day,

and $\frac{1}{3} =$ part that A and B can do in 1 day.

By the conditions, $\frac{1}{5} + \frac{1}{x} = \frac{1}{3}.$ (1)

Solving (1), $x = 7\frac{1}{2}.$

8. A train runs 84 miles in the same time that a second train runs 96 miles. If the rate of the first train is 3 miles per hour less than that of the second train, find the rate of each.

Let $x =$ rate of the first train, and $x + 3 =$ rate of the second train.

By the conditions, $\frac{84}{x} = \frac{96}{x + 3}.$ (1)

Solving (1), $x = 21$; hence $x + 3 = 24.$

9. A number of 4 % bonds were sold at 90, and the proceeds invested in $3\frac{1}{2}$ % bonds at 75, the par value of each bond being \$100. If the gain in income is \$4, find the number of 4 % bonds.

Let $x =$ the number of 4 % bonds,

then $4x =$ the income in dollars of the 4 % bonds,

and $90x =$ the value in dollars of the 4 % bonds,

then $\frac{90x}{75} =$ the number of $3\frac{1}{2}$ % bonds,

and $3\frac{1}{2}\left(\frac{90x}{75}\right) =$ the income in dollars from the $3\frac{1}{2}$ % bonds.

By the conditions, $$3\tfrac{1}{2}\left(\frac{90x}{75}\right) - 4x = 4. \tag{1}$$

Solving (1), $$x = 20.$$

EXERCISE LXXXIV

1. The sum of half a number and its third part is 135. Find the number.

2. The difference between the third and seventh parts of a number is 40. Find the number.

3. The excess of the sum of the fourth and twelfth parts over the ninth part of a number is 8. Find the number.

4. The excess of the sum of the fifth and seventh parts over the difference of the half and the third parts of a number is 259. Find the number.

5. Find that number which is $1\frac{1}{5}$ times the excess of the number over 2.

6. The sum of two numbers is 32, and their difference is 8. Find the numbers.

7. The difference of two numbers is 13, and if 144 be subtracted from 8 times the first, the remainder is 56. Find the numbers.

8. The fourth part of the larger of two consecutive numbers exceeds the fifth part of the smaller by 1. Find the numbers.

9. The sum of two numbers is 18, and if the greater number be divided by the less, the quotient is 2. Find the numbers.

10. Find the two numbers such that their difference is 20, and the quotient of the greater divided by the less is 3.

11. The sum of two numbers is 26, and if the greater number be divided by the less, the quotient is 1 and the remainder is 4. Find the numbers.

12. The difference of two numbers is 9, and if the greater be divided by the less, the quotient is 2 and the remainder is 2. Find the numbers.

13. The difference of two numbers is 18, and if the less be divided by the greater, the quotient is $\frac{1}{4}$. Find the numbers.

14. The sum of two numbers is 22, and if the less be divided by the greater diminished by 7, the quotient is $\frac{1}{2}$. Find the numbers.

15. The sum of two numbers is 200, and their difference is $\frac{2}{3}$ of the less number. Find the numbers.

16. The sum of two numbers is 59, and if the greater be divided by the less, the quotient and the remainder is 4. Find the numbers.

17. The difference of two numbers is 16, and if the greater be divided by the less, the quotient is 2 and the remainder is 4. Find the numbers.

18. If 59 be added to half of a certain number, the sum obtained is $7\frac{3}{16}$ times a seventh of the number. Find the number.

19. A number is 10 times a second number. The quotient of the first number divided by 22 exceeds by $\frac{8}{11}$ the quotient of the second number divided by 3. Find the numbers.

20. If a certain number be added to the terms of $\frac{3}{5}$, it becomes $\frac{5}{7}$. Find the number.

21. Find the fraction such that if 1 be added to the numerator it becomes $\frac{1}{3}$; but if 1 be subtracted from the denominator it becomes $\frac{1}{4}$.

22. Find the fraction such that if 3 be added to the numerator it becomes $\frac{2}{3}$; but if 1 be subtracted from the denominator it becomes $\frac{1}{2}$.

23. Find the fraction such that if 4 be subtracted from its terms it becomes $\frac{1}{3}$; but if 5 be added to its terms it becomes $\frac{5}{6}$.

24. The sum of two fractions whose numerators are respectively 7 and 9 is $\frac{158}{99}$; but if the numerators be interchanged, the sum of the fractions is $\frac{18}{11}$. Find the fractions.

25. A certain fraction becomes $\frac{7}{12}$ if 1 be subtracted from the numerator, and becomes $\frac{1}{2}$ if 4 be added to the denominator. Find the fraction.

26. If 3 be added to the numerator and 1 be added to the denominator of a certain fraction, it becomes $\frac{3}{4}$; but if 1 be subtracted from the numerator and 3 be subtracted from the denominator, it becomes $\frac{1}{2}$. Find the fraction.

27. The sum of two fractions whose numerators are each 1 is $\frac{15}{56}$. The first fraction exceeds the second by $\frac{1}{56}$. Find the fractions.

28. The width of a rectangular room is $\frac{3}{4}$ of its length. If the width were 3 feet more, the room would be square Find the dimensions of the room.

29. The dimensions of a rectangle are respectively 12 feet more and 8 feet less than the side of an equivalent square. Find the dimensions of the rectangle.

30. The length of a rectangular floor exceeds the width by 6 feet. If the width be increased by 3 feet and the length by 2 feet, the area is increased by 134 square feet. Find the area.

31. A square contains the same area as a rectangle whose dimensions are respectively the half and the double of the side of the square. If the width of the rectangle be increased by 3 feet and its length be diminished by 5 feet, the area is increased 34 square feet. Find the side of the square.

32. Seven men and 5 boys earn $11.25 per day, and at the same wages 12 boys and 4 men earn $11 per day. What are the wages per day of a man?

33. A sum of money is divided equally among a certain number of men. If there were 4 more men, each would receive $1 less; if 5 less men, each would receive $2 more. Find the number of men.

34. A could have bought 5 more oranges, each at half a cent less, for the same amount of money that he could have bought 3 less oranges, each at half a cent more. Find the cost of the oranges.

35. A's age is $\frac{1}{4}$ of B's. Five years ago A was $\frac{1}{7}$ as old as B. Find their present ages.

36. A's age is five times B's. In 12 years B's age will be $\frac{1}{3}$ of A's. Find their present ages.

37. A is 50 years old, and B is 25. In how many years will B be $\frac{7}{12}$ as old as A?

38. A's age is twice that of his son, but 10 years ago it was three times as great. Find the present age of each.

39. If A was four times as old as B 7 years ago, and if A will be twice as old as B in 7 years, what is the present age of each?

40. If A is $\frac{12}{5}$ as old as B, and if he was eight times as old as B 20 years ago, find the present age of each.

41. A's age exceeds B's by 21 years. In 8 years A will be $1\frac{7}{8}$ times as old as B. Find the present age of each.

42. A's age exceeds B's by 12 years. Twelve years ago A's age was $\frac{4}{3}$ of B's age. Find the present age of each.

43. Find three numbers such that the sums of the numbers in pairs of two are 6, 8, and 12.

44. A has \$15 more than B; B has \$5 less than C; A and B and C together have \$65. How much has each?

45. A and B and C have \$54. A has six times as much as B; B and C together have as much as A. How much has each?

46. A and B have only $\frac{2}{3}$ as much money as C; B and C together have six times as much as A; B has \$680 less than A and C together. How much has each?

47. A can row 6 miles an hour down a stream, and 2 miles an hour against the stream. Find A's rate in still water, and the rate of the current.

48. A crew can row 20 miles in 2 hours down a stream, and 12 miles in 3 hours against the stream. Find the rate of the current, and the rate per hour of the crew in still water.

49. A man can row $3\frac{1}{2}$ miles down a river in 56 minutes. If the river has a current of 2 miles per hour, find the rate of the man in still water.

50. At what time between 3 and 4 will the hands of a clock be together? between 7 and 8? between 9 and 10?

51. At what time between 5 and 6 will the hands of a clock first be at right angles? between 6 and 7? between 10 and 11?

52. At what time between 12 and 1 will the hands of a clock be exactly opposite? between 4 and 5? between 11 and 12?

53. At what time between 8 and 9 is the hour hand of a clock 20 minute-spaces ahead of the minute hand?

54. At what time between 4 and 5 is the minute hand of a clock exactly 5 minutes ahead of the hour hand?

55. The sum of the two digits of a number is 9, and if 9 be subtracted from the number the digits will be reversed. Find the number.

56. The tens' digit exceeds the units' digit of a number of two digits by 1, and if 9 be subtracted from the number, the digits will be reversed. Find the number.

57. The sum of the digits of a number of three digits is 17; the hundreds' digit is twice the units' digit; if 396 be subtracted from the number, the order of the digits will be reversed. Find the number.

58. The sum of the digits of a number of three digits is 5; the hundreds' digit is $\frac{1}{4}$ of the units' digit; if the number be divided by the sum of the digits, the quotient so derived is $83\frac{1}{5}$ less than the number. Find the number.

59. A number is expressed by three digits whose sum is 18. If the digits in the hundreds' and units' places be interchanged, the number will be diminished by 792. The digit in the tens' place is $\frac{4}{5}$ of the sum of the other two digits. Find the number.

60. A can do a piece of work in 3 days, and B can do it in 5 days. In how many days can A and B, working together, do the work?

61. A can do a piece of work in 3 days, B in 7 days, and C in 5 days. How many days will it take all together to do the work?

62. A can dig a ditch in $7\frac{1}{2}$ days, B in $5\frac{1}{3}$ days, and C in $6\frac{1}{5}$ days. How many days will it take all together to do the work?

63. A and B together can plough a field in 15 days, while A and C together can plough it in 18 days, and C in 30 days. In how many days can B and C together plough the field?

64. A and B can build a walk in 6 days, B and C in $7\frac{1}{2}$ days, and A and C in 10 days. How many days will t take A, B, and C together to build the walk?

65. A and B can do $\frac{1}{4}$ of a piece of work in 2 days; B an do $\frac{1}{3}$ of it in 6 days. How long will it take A alone o do $\frac{1}{3}$ of the work?

66. Two pipes, A and B, can fill a cistern in 70 minutes, A and C in 84 minutes, and B and C in 140 minutes. How long will it take for each alone to fill it?

67. One tap will empty a vessel in 80 minutes, a second n 200 minutes, and a third in 5 hours. How long would t take to empty the vessel if all the taps were open?

68. A and B can do a piece of work in m days, B and C n n days, and C and A in p days. How many days will t take A, B, and C, all working together, to do the work?

69. A cistern can be filled by two pipes in 5 and 7 hours espectively, and can be emptied by a third pipe in a hours. n what time can the cistern be filled if the first two are unning into, and the third is emptying the cistern?

70. A train runs 100 miles in the same time that a econd train, whose rate is $3\frac{1}{3}$ miles an hour less, runs 0 miles. Find the rate of each train.

71. Two trains leave A at the same time, and run in pposite directions. The first train runs at a rate, in iles per hour, $\frac{1}{7}$ faster than the second. How many ours will each train have run when they are 425 miles part, if the distance covered by the first train in 10 hours xceeds that covered by the second train in 8 hours by 120 iles?

72. A and B are 240 miles apart. If at the same time a train leaves A and B, and runs for the other place, how far from A will they meet if the train from A runs at the rate of 45 miles an hour, and the other $\frac{7}{9}$ as fast?

73. A leaves the place X at 8 A.M., and 2 hours later B leaves Y, 100 miles from X, and meets A at noon. If A had left at 8.30 A.M., and B at 9 A.M., they would also have met at noon. Find the rate of A, and of B.

74. A is 100 units east from B. If A and B move toward each other, they will meet in 4 minutes; but if both move west, A overtakes B in 20 minutes. Find their rates of speed.

75. A left a certain town and travels at the rate of a miles an hour, and in n hours was followed by B at the rate of b miles an hour. In how many hours did B overtake A?

76. A leaves New York and travels at the rate of 1 miles in 5 hours; 8 hours after, B leaves New York, and travels after A at the rate of 13 miles in 3 hours. How far must B travel to overtake A?

77. A and B run a mile. First, A gives B a start of 44 yards and beats him 51 seconds; in the second heat A gives B an allowance of 1 minute 15 seconds, and is beaten by 88 yards. Find the time it takes B to run a mile.

78. A fox is pursued by a hound. The fox takes leaps while the hound is taking $3\frac{1}{2}$. Four of the hound's leaps are equivalent to 7 of the fox. The fox has 45 of her own leaps the start. How many leaps will each make before the fox is caught?

79. Find the principal upon which the simple interest for 3 years and 3 months at $3\frac{3}{5}\%$ is \$93.60.

80. Find the time required for \$2275 to amount to \$2378.74 at $3\frac{4}{5}\%$.

81. Find the rate per cent at which \$20,000 doubles itself in 27 years, 9 months, and 10 days.

82. A sum of money at simple interest in 5 years amounted to \$2400, and in 7 years to \$2560. Find the principal.

83. A has twice as many 4% bonds as 5% bonds, whose par values are each \$1000. The bonds produce an annual income of \$1950. Find the number of 4% and of 5% bonds.

84. A has \$20,000 invested between real estate and stocks, the par value of each share being \$100. On the real estate he nets, at $5\frac{1}{2}\%$, \$440; on the stocks, at $3\frac{3}{5}\%$, he nets \$8 less than on the real estate. Find the amount in stocks.

85. The sum of A's income for 3 years at simple interest on \$12,500, and on \$15,000 for $4\frac{1}{2}$ years at simple interest, is \$4020. If the rates of interest were interchanged he would receive, in the same time, \$3975. Find the different rates.

86. The sum of the capitals of A, B, and C is \$120,000. A's capital is invested at $3\frac{4}{5}\%$, B's at 4%, and C's at $3\frac{3}{5}\%$, and the sum of their incomes is \$4530. If the rates at which A's and B's capitals are invested are interchanged, the income of all is \$30 less. Find their capitals.

87. A mass of gold and silver which weighs 10 pounds loses, when weighed in water, $\frac{1}{16}$ of itself. If gold loses $\frac{1}{19}$, and silver $\frac{1}{10}$ of its weight, when weighed in water, how many pounds of gold and silver are there in the mass?

88. A mass of tin and copper, which weighs in air 687 pounds, weighs in water $603\frac{2}{3}$ pounds. If one pound of tin loses $\frac{100}{729}$ of a pound, and one pound of copper loses $\frac{25}{222}$ of a pound, when weighed in water, how many pounds of tin and copper are there in the mass?

89. If a number of soldiers be formed in a solid square, 24 men fail to get places; but if another solid square be formed, with one more man on a side, there are 29 places unfilled. Find the number of soldiers.

90. How many ounces of 14 carat gold must be mixed with 40 ounces of 15 carat gold to make a mixture of $14\frac{1}{3}$ carat gold?

91. Five pounds of gold 840 points pure are melted with 7 pounds of another sort, and produce a mass 700 points pure. How many points pure is the second sort?

92. How many quarts of water must be mixed with 250 quarts of alcohol 80% pure to make a mixture 75% pure?

93. A piece of work can be done by 20 workmen in 11 days, and by 30 master workmen in 7 days. In how many days can the work be done by 22 workmen and 21 master workmen?

94. At a gathering of 14 men and 23 women the ratio of unmarried men to unmarried women is 2 to 5. Find the number of married couples present.

CHAPTER XIII

INEQUALITIES

164. The signs $>$ and $<$ express **inequality**: $a > b$ is read "a is greater than b"; $a < b$ is read "a is less than b." Two quantities, a and b, can be compared in three different ways: (1) $a = b$, (2) $a > b$, (3) $a < b$. When $a > b$, $a - b$ is positive; when $a < b$, $a - b$ is negative. In general, **a quantity is said to be greater than a second quantity** when the first quantity less the second quantity is positive; and **a quantity is said to be less than a second quantity** when the first quantity minus the second quantity is negative. Since, by § 20, all positive quantities are greater than zero, if $a > b$, then $a - b > 0$; and, since all negative quantities are less than zero, if $a < b$, $a - b < 0$.

165. An **inequality** is a statement that one of two expressions is not equal to (that is, is greater, or less than) the other. The **first member** of an inequality is the expression to the left of the sign of inequality; and the **second member** is the expression to the right of that sign.

Thus, $a^2 + b^2$ is the first, and $2\,ab$ the second, member of the inequality, $a^2 + b^2 > 2\,ab$.

A **term** of an inequality is any term of either the first or second member. Two inequalities **subsist in the same sense** when they have the same sign of inequality.

Thus, $a > b$ and $c > d$ are inequalities subsisting in the same sense.

Inequalities **subsist in the opposite sense** when they have opposite signs of inequality.

Thus, $a > b$, $c < d$, are inequalities which subsist in the opposite sense.

166. The general principles upon which inequalities rest are:

I. *If equals be added to unequals, the sums are unequals subsisting in the same sense.*

If $$a > b, \tag{1}$$
then $$a - b > 0. \tag{2}$$
Now, $$(a + c) - (b + c) = a - b, \tag{3}$$
or, substituting (3) in (2),
$$(a + c) - (b + c) > 0, \tag{4}$$
or, rewriting (4), $$a + c > b + c. \tag{5}$$

II. *If equals be subtracted from unequals, the remainders are unequals subsisting in the same sense.*

If $$a > b, \tag{1}$$
then $$a - b > 0. \tag{2}$$
Now, $$(a - c) - (b - c) = a - b, \tag{3}$$
or, substituting (3) in (2),
$$(a - c) - (b - c) > 0, \tag{4}$$
or, rewriting (4), $$a - c > b - c. \tag{5}$$

Application of I and II: *Any quantity in an inequality may be transposed from member to member if the sign of that quantity be changed.*

If $$a - c > b, \tag{1}$$
by I, $$a > b + c. \tag{2}$$

If $$a + b > c, \tag{1}$$
by II, $$a > c - b. \tag{2}$$

If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.

If $$a - b > c - d, \tag{1}$$
transposing all the terms in (1),
$$d - c > b - a, \tag{2}$$
or, rewriting (2), $$b - a < d - c. \tag{3}$$

III. *If unequals be subtracted from equals, the remainders subsist in the opposite sense.*

If $$b > c, \tag{1}$$
rewriting (1), $$b - c > 0, \tag{2}$$
changing all signs in (2), $$c - b < 0. \tag{3}$$
Now, $$(a - b) + (-a + c) = -b + c, \tag{4}$$
substituting (4) in (3),
$$(a - b) + (-a + c) < 0, \tag{5}$$
rewriting (5), $$a - b < a - c. \tag{6}$$

IV. *If unequals be multiplied by positive equals, the products subsist in the same sense.*

If $$a > b, \tag{1}$$
then $$a - b > 0. \tag{2}$$

Let m be any positive quantity. Then $m(a - b)$ must be a positive quantity, since the product of two positive quantities must be positive.

Therefore, $$m(a - b) > 0, \tag{3}$$
or, rewriting (3), $$ma - mb > 0, \tag{4}$$
or, $$ma > mb. \tag{5}$$

Since the process of division is multiplication by the reciprocal of the divisor, it follows from IV that if unequals be divided by positive equals the quotients subsist in the same sense.

Application of IV : *To clear an inequality of fractions multiply each term by the L. C. D. taken as a positive quantity.*

Thus, if $$\frac{x}{-4}+\frac{x}{3}>\frac{x}{24}, \qquad (1)$$

multiplying (1) by 24, $$-6x+8x>x. \qquad (2)$$

V. *If unequals be multiplied by negative equals, the products subsist in the opposite sense.*

If $$a>b, \qquad (1)$$

then $$a-b>0. \qquad (2)$$

Let $-n$ be any negative number. Then $-n(a-b)$ must be a negative quantity, since the product of a negative and a positive quantity is a negative quantity.

Therefore, $$-n(a-b)<0, \qquad (3)$$

or, rewriting (3), $$-na+nb<0, \qquad (4)$$

or, $$-na<-nb. \qquad (5)$$

Since the process of division is multiplication by the reciprocal of the divisor, it follows from V that if unequals be divided by negative equals the quotients subsist in the opposite sense.

Henceforth, *in this chapter, literal quantities are used to represent only positive and unequal quantities.* This fact must be kept in mind, for otherwise the proofs will not hold.

167. A **conditional inequality** is true only for some value or values of the letters involved. An **absolute inequality** is true for all values of the letters involved.

Thus, $2x-3>x+2$ is a conditional, and $a^2+b^2>2ab$ is an absolute, inequality.

A. Prove that $\boldsymbol{a^2+b^2>2ab}$.

Either (1), $a-b>0$, or (2), $a-b<0$.

1. If $a-b>0$, (1)

multiplying (1) by itself, $a^2-2ab+b^2>0$, (2)

transposing in (2), $a^2+b^2>2ab$. (3)

2. If $a-b<0$, (1)

multiplying (1) by itself, $a^2-2ab+b^2>0$. (2)

(1) is negative: multiplying a negative number by itself is, by V, an inequality subsisting in the opposite sense.

Transposing in (2), $a^2+b^2>2ab$.

B. Prove that $\boldsymbol{a^3+b^3>ab(a+b)}$.

Now, $a^2-2ab+b^2>0$, (*A*)

transposing $-ab$ in (*A*), $a^2-ab+b^2>ab$, (1)

multiplying (1) by $a+b$,

$$(a+b)(a^2-ab+b^2)>ab(a+b), \quad (2)$$

$$a^3+b^3>ab(a+b). \quad (3)$$

C. Prove that $\boldsymbol{a^2+b^2+c^2>ab+bc+ca}$.

Now, by *A*, $a^2+b^2>2ab$, (1)

and, by *A*, $b^2+c^2>2bc$, (2)

and, by *A*, $c^2+a^2>2ca$, (3)

adding (1), (2), and (3), $2(a^2+b^2+c^2)>2(ab+bc+ca)$, (4)

dividing (4) by 2, $a^2+b^2+c^2>ab+bc+ca$. (5)

D. Prove that $\boldsymbol{a^3 + b^3 + c^3 > 3\,abc}$.

Now, by *C*, $$a^2 + b^2 + c^2 > ab + bc + ca, \tag{1}$$
multiplying (1) by a, $$a^3 + ab^2 + ac^2 > a^2b + abc + a^2c, \tag{2}$$
multiplying (1) by b, $$a^2b + b^3 + bc^2 > ab^2 + b^2c + abc, \tag{3}$$
multiplying (1) by c, $$a^2c + b^2c + c^3 > abc + bc^2 + c^2a, \tag{4}$$
adding (2), (3), and (4), and uniting,
$$a^3 + b^3 + c^3 > 3\,abc. \tag{5}$$

The type forms, *A*, *B*, *C*, and *D*, should be remembered.

168. The solutions of various problems in conditional inequalities are illustrated in the following problems.

1. In the conditional inequality, $3\,x + \frac{4}{3} > x + 8$, find one limit of x.

Let $$3\,x + \tfrac{4}{3} > x + 8. \tag{1}$$
Multiplying (1) by 3, $$9\,x + 4 > 3\,x + 24, \tag{2}$$
transposing and uniting in (2), $$6\,x > 20, \tag{3}$$
dividing (3) by 6, $$x > 3\tfrac{1}{3}. \tag{4}$$

2. In the conditional inequalities, (1) $x + 7 > \dfrac{2\,x}{3} + 9$, (2) $\dfrac{2\,x}{5} < \dfrac{x}{4} + 2$, find the integral values of x.

Multiplying (1) by 3, $$3\,x + 21 > 2\,x + 27, \tag{3}$$
transposing and uniting in (3), $$x > 6, \tag{4}$$
multiplying (2) by 20, $$8\,x < 5\,x + 40, \tag{5}$$
transposing and uniting in (5), $$3\,x < 40, \tag{6}$$
dividing (6) by 3, $$x < 13\tfrac{1}{3}. \tag{7}$$

From (4) and (7), x lies between the limits 6 and $13\frac{1}{3}$; and may therefore take the integral values, 7, 8, 9, 10, 11, 12, 13.

EXERCISE LXXXV

1. Between what limits must x lie, to satisfy the inequalities $2x-3>20$ and $3x-7<2x+6$?

2. Given $2x-3<x+5$, and $11+2x<3x+5$, find the limits of x.

3. Given $\frac{2x+3}{2}+3x<14$, and $\frac{4x-2}{3}+2x>9$, find the limits of x.

4. Given $3x-5>2x+1$, and $3x+15>4x+5$, find the limits of x.

Prove the following inequalities, the letters being positive and the sign $\neq$ being read, "not equal to":

5. $\frac{a}{b}+\frac{b}{a}>2$, if $a \neq b$.

6. $a^2>2ab-b^2$, if $a \neq b$.

7. $m^2+n^2+p^2>mn+mp+np$, if $m \neq n$, $n \neq p$, $m \neq p$.

8. $a^3b^3+b^3c^3+c^3a^3>3a^2b^2c^2$, if $a \neq b$, $a \neq c$, $b \neq c$.

9. $an+bm<1$, if $a^2+b^2=1$ and if $m^2+n^2=1$ and if $a \neq n$, and $b \neq m$.

10. $ax+by<15$, if $a^2+b^2=25$ and if $x^2+y^2=5$.

11. $2a^3+b^3>a(a^2+ab+b^2)$, if $a \neq b$.

12. $a^3-b^3>3ab(a-b)$, if $a>b$.

13. $(a+b)^3+(c+d)^3>(a+b+c+d)(a+b)(c+d)$, if $(a+b) \neq (c+d)$.

14. $a^2+4b^2+c^2>2ab+2bc+ac$, if $a \neq 2b$, $a \neq c$, $2b \neq c$.

15. $\frac{m}{p}+\frac{n}{m}+\frac{p}{n}>\frac{n}{p}+\frac{m}{n}+\frac{p}{m}$, if $m>n$, $n>p$, and $m>p$.

CHAPTER XIV

INVOLUTION AND EVOLUTION

INVOLUTION

169. The operation of raising an expression to any given power is called **involution**. An expression is said to be **expanded** when the indicated multiplications have been performed.

Thus, $(a)^2$ and $(a+b)^2$ have been expanded when the respective products have been found to be a^2 and $a^2+2\,ab+b^2$.

MONOMIALS

170. Involution of monomials is subject to the following Index Laws, in the proofs of which $a \neq 0$, and m and n are restricted to positive integers.

I. $$(a^m)^n = a^{mn}.$$

By definition, $(a^m)^n = [(a \text{ to } m \text{ factors}) \text{ to } n \text{ factors}]$,

by associative law, $= a$ to mn factors,

by definition, $= a^{mn}$.

The exponent of the power of any given monomial is found by multiplying the exponent of the given monomial by the index of the required power.

II. $$(ab)^m = a^m b^m.$$

By commutative and associative laws,

$$(ab)^m = (a \text{ to } m \text{ factors})(b \text{ to } m \text{ factors}),$$

by definition, $= a^m b^m$.

Similarly, $(abc)^m = a^m b^m c^m$.

The mth power of the product of two quantities is equal to the product of their mth powers.

III. $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$$

By commutative and associative laws,

$$\left(\frac{a}{b}\right)^m = (a \text{ to } m \text{ factors}) \div (b \text{ to } m \text{ factors}),$$

by definition, $$= a^m \div b^m = \frac{a^m}{b^m}.$$

The mth power of the quotient of two quantities is the quotient of their mth powers.

171. Involution is also subject to the **Law of Signs.**

$$(-a)(-a) = (-a)^2 = a^2,$$

$$(a)(a) = (a)^2 = a^2,$$

$$(-a)(-a)(-a) = (-a)^3 = -a^3, \text{ etc.}$$

All even powers of a negative monomial are positive, while all odd powers of a negative monomial are negative; all powers of a positive monomial are positive.

EXERCISE LXXXVI

Expand the following expressions:

1. $(a^5)^3$.

2. $(a^4)^3$.

3. $(-a^4)^3$.

4. $(2a)^3$.

5. $-(5m^2)^3$.

6. $-(-4c^2d)^5$.

7. $(2x^2yz^5)^3$.

8. $(-2x^2yz^7)^7$.

9. $-(-4x^5z)^6$.

10. $\left(\frac{3x^2y}{7z^2}\right)^2$.

11. $\frac{(-11ab^3)^4}{(6a^3b)^5}$.

12. $-\left(\frac{-5x^7z}{2ac}\right)^5$.

13. $\left(\frac{-10a^5b}{-3xy^4}\right)^7$.

BINOMIALS

172. The expansion of binomials may be shortened by employment of the Binomial Theorem, a proof of which is given in Chapter XXIV. The use of this theorem is evident from the following type forms, which are derived by multiplication:

$$(a+b)^2 = a^2 + 2\,ab + b^2, \tag{1}$$

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3, \tag{2}$$

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4, \tag{3}$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5, \tag{4}$$

$$(a+b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6. \tag{5}$$

Similarly, it may be shown that the expansion of the binomial $(a-b)$ gives, if the exponents are those of the left members respectively, the results in (1) to (5), except that the signs of the terms are alternately plus and minus, the first term being plus.

173. Examination of the expanded forms shows, if $\boldsymbol{n}$ be the exponent indicating the power, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are respectively the first and second terms of the binomial, that

1. The number of terms in the expansion is $n+1$.

2. Every term, except the last, in the expansion contains a; and every term, except the first, contains b.

3. The exponent of a in the first term is n, and decreases by 1 in each succeeding term; the exponent of b in the second term is 1, and increases by 1 in each succeeding term.

4. The first coefficient is 1, the second n; the third, and any subsequent coefficient, is derived from the preceding

term by multiplying the coefficient by the exponent of a and dividing this product by the exponent of b increased by 1.

Any binomial may be expanded by this method if in the right member a equals the first term and b equals the second term.

1. Expand $(a^2 - 2b)^4$.

By type form (3), § 172,

$$(a^2 - 2b)^4 = (a^2)^4 - 4(a^2)^3(2b) + 6(a^2)^2(2b)^2 - 4(a^2)(2b)^3 + (2b)^4$$
$$= a^8 - 8a^6b + 24a^4b^2 - 32a^2b^3 + 16b^4.$$

In a similar way, a polynomial, in the form of a binomial, may be expanded.

2. Expand $(x - 2y + 3z)^3$.

By type form (2), § 172,

$$[(x-2y)+3z]^3 = (x-2y)^3 + 3(x-2y)^2(3z) + 3(x-2y)(3z)^2 + (3z)^3$$
$$= x^3 - 6x^2y + 12xy^2 - 8y^3 + 9x^2z - 36xyz + 36y^2z$$
$$+ 27xz^2 - 54yz^2 + 27z^3.$$

EXERCISE LXXXVII

Expand the following expressions:

1. $(p+q)^4$.
2. $(x+y)^4$.
3. $(1+a)^4$.
4. $(p+q)^5$.
5. $(x-y)^5$.
6. $(b+1)^5$.
7. $(x+y)^6$.
8. $(2a+1)^5$.
9. $(x+2y)^4$.
10. $(x^2-y^2)^4$.
11. $(1-q)^6$.
12. $(2x-3y)^5$.
13. $(3x-2y^2)^4$.
14. $(3mn-4p)^4$.
15. $(2x^2-5y)^5$.
16. $(2a^2-3b^2)^6$.
17. $(a-b+c)^3$.
18. $(a-b-2c)^3$.
19. $(2x-y+3z)^3$.
20. $(a-b-c)^4$.
21. $(2x^2-3xy+y^2)^4$.

EVOLUTION

174. The operation of extracting a root of an expression is called **evolution**, and is indicated by the **radical sign,** $\sqrt{\ }$. The quantities whose roots are to be extracted, called **radicands**, are written after the radical sign. The particular root to be extracted is indicated by a small number, called the **index of the root**, written above the radical sign. The index 2 is generally omitted. If the index of the root is an even number, the root is called an **even root**; if an odd number, the root is called an **odd root.**

Thus, $\sqrt{4}$, $\sqrt[4]{81}$, $\sqrt[2n]{a^{2n}}$, are even roots; $\sqrt[3]{8}$, $\sqrt[5]{-243}$, $\sqrt[2n+1]{a^{2n+1}}$, are odd roots.

175. If a quantity can be expressed as the product of two equal factors, one of these factors is called the **square root** of the quantity; one of the three equal factors of a quantity is called the **cube root**; and, in general, one of the n equal factors is called the ***n*th root.**

Since involution and evolution are inverse processes,

$$\sqrt{a^2} = a = (\sqrt{a})^2; \ \sqrt[3]{a^3} = a = (\sqrt[3]{a})^3; \ \sqrt[n]{a^n} = a = (\sqrt[n]{a})^n.$$

176. *The one positive root of a positive number is called its* **principal root**; *the one negative root of a negative number is called its* **principal odd root.**

The radical sign will be used to indicate the principal roots only.

Thus, $\sqrt{4}$ means the positive square root of 4; that is, $\sqrt{4} = +2$; similarly, $\sqrt{25} = +5$; $\sqrt[3]{-27} = -3$; $\sqrt[5]{-243} = -3$; $\sqrt[n]{a^n} = a$.

Note. Only expressions whose exponents are multiples of the indices of the roots will be discussed in this chapter.

MONOMIALS

177. The Index Laws for the evolution of monomials are the inverse forms of the Index Laws for involution.

I. $\sqrt[n]{a^{mn}} = a^m.$

By I, § 170, $\quad (a^m)^n = a^{mn},$

by definition, $\quad \sqrt[n]{a^{mn}} = a^m.$

II. $\sqrt[n]{a^n b^n c^n} = abc.$

By II, § 170, $\quad (abc)^n = a^n b^n c^n,$

by definition, $\quad \sqrt[n]{a^n b^n c^n} = abc.$

From I and II is derived the **Rule for the Root of a Monomial in the form of a Product**: *Divide the exponent of each factor by the index of the required root.*

III. $\sqrt[n]{\dfrac{a^n}{b^n}} = \dfrac{a}{b}.$

By III, § 170, $\quad \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n},$

by definition, $\quad \sqrt[n]{\dfrac{a^n}{b^n}} = \dfrac{a}{b}.$

From III is derived the **Rule for the Root of a Monomial in the form of a Quotient**: *Divide the exponent of each factor in the terms of the fraction by the index of the required root.*

1. Simplify $\sqrt[3]{\dfrac{64\,x^3y^6}{343\,a^{12}b^9}}.$

$$\sqrt[3]{\frac{64\,x^3y^6}{343\,a^{12}b^9}} = \sqrt[3]{\frac{2^6\,x^3y^6}{7^3\,a^{12}b^9}} = \frac{2^2\,xy^2}{7\,a^4b^3} = \frac{4\,xy^2}{7\,a^4b^3}.$$

EXERCISE LXXXVIII

Simplify the following expressions:

1. $\sqrt{a^2b^2}$.
2. $\sqrt{4 \cdot 9}$.
3. $\sqrt{m^4n^4}$.
4. $\sqrt[3]{a^3b^3c^3}$.
5. $\sqrt[3]{8 \cdot 27}$.
6. $\sqrt[4]{a^8b^4x^{12}}$.
7. $\sqrt[5]{p^5q^5}$.
8. $\sqrt[6]{x^{12}y^{18}}$.
9. $\sqrt{9\,a^2b^4c^6}$.
10. $\sqrt{64\,a^8b^2c^6}$.
11. $\sqrt[3]{27\,a^6b^3}$.
12. $\sqrt[4]{16\,a^4b^8}$.
13. $\sqrt{(a+b)^2}$.
14. $\sqrt{a^2-2\,ab+b^2}$.
15. $\sqrt[3]{(a+b)^3}$.
16. $\sqrt{a^2(x+y)^2}$.
17. $\sqrt[3]{a^3(x-y)^3}$.
18. $\sqrt[3]{-8\,x^6y^{12}}$.
19. $\sqrt{\dfrac{9\,a^4b^6}{16\,x^8}}$.
20. $\sqrt[3]{-\dfrac{64\,a^9b^6}{343\,x^3y^{12}}}$.
21. $\sqrt[5]{-\dfrac{32\,x^{10}y^{25}}{243\,m^{10}n^{15}}}$.

SQUARE ROOT OF POLYNOMIALS

178. Since, § 173, a polynomial may be squared as a binomial, $(t+u)^2$ may be taken as the type form of the square of a polynomial. Examination of the way that the square root of $t^2+u(2\,t+u)$, which is called **the square root formula**, is obtained, will disclose a method by which the square root of any polynomial may be obtained.

Since $(t+u)^2 = t^2 + 2\,tu + u^2$, $\sqrt{t^2+2\,tu+u^2} = t+u$.

(A) *The first term of the root is the square root of the first term of the formula.*

(B) *The second term of the root is obtained by dividing the second term of the formula by twice the part of the root already found.*

The formula may be applied to any polynomial, if t represents the part of the root already found and if u represents the next term of the root.

1. Extract the square root of $4x^2 + 4xy + y^2$.

Let $\qquad t^2 + 2tu + u^2 = 4x^2 + 4xy + y^2, \qquad (1)$

by (A), $\qquad t = 2x, \qquad (2)$

squaring (2), $\qquad t^2 = 4x^2, \qquad (3)$

subtracting (3) from (1), $u(2t + u) = 4xy + y^2, \qquad (4)$

by (B), $\qquad u = y. \qquad (5)$

Substituting $t = 2x$, and $u = y$, in (4),

$$u(2t + u) = y(4x + y) = 4xy + y^2. \qquad (6)$$

Since

$$\sqrt{4x^2 + 4xy + y^2} = \sqrt{t^2 + 2tu + u^2} = t + u, \qquad (7)$$

and since $t = 2x$, and $u = y$, $\sqrt{4x^2 + 4xy + y^2} = 2x + y. \qquad (8)$

The work may be more compactly written:

$$\begin{array}{r|l}
 & 4x^2 + 4xy + y^2 \,\underline{|2x + y} \\
t = 2x & 4x^2 \\
\hline
2t = 4x & \qquad 4xy + y^2 \\
u = y & \\
2t + u = 4x + y & \\
\underline{u(2t + u) = y(4x + y)} & \qquad \underline{4xy + y^2}
\end{array}$$

The terms of the polynomial should be arranged either in ascending or in descending order of some one of its letters; otherwise the formula method is not available.

If the polynomial contains more than three terms, it should be carefully noticed that the part of the root already found in every case is represented by t.

Since $(a + b - c)^2 = [a + (b - c)]^2 = [(a + b) - c]^2$,

and since $\qquad (t + u)^2 = (a + b - c)^2$,

t is represented successively by a and $a + b$.

2. Extract the square root of $a^2+4\,c^2+b^2-2\,ab+4\,bc-4\,ac$.

$$
\begin{array}{r|l}
 & \underline{a-b-2\,c} \\
 & a^2-2\,ab-4\,ac+b^2+4\,bc+4\,c^2 \\
t=a & a^2 \\
\hline
2\,t=2\,a & \quad -2\,ab-4\,ac+b^2+4\,bc+4\,c^2 \\
u=-b & \\
2\,t+u=2\,a-b & \\
u(2t+u)=-b(2\,a-b) & \quad -2\,ab \qquad\quad +b^2 \\
\hline
2\,t=2\,a-2\,b & \qquad\qquad -4\,ac \qquad\quad +4\,bc+4\,c^2 \\
u=-2\,c & \\
2\,t+u=2\,a-2\,b-2\,c & \\
u(2t+u)=-2\,c(2\,a-2\,b-2\,c) & \qquad\qquad -4\,ac \qquad\quad +4\,bc+4\,c^2 \\
\hline
\end{array}
$$

In the above example, after the second term of the root has been found, the first two terms are together equal to t. Since $t=(a-b)$, and t has been squared and subtracted, the remainder again corresponds to the expression $u(2\,t+u)$.

EXERCISE LXXXIX

Extract the square roots of the following expressions:

1. $25\,a^2-70\,ac+49\,c^2$.
2. $a^2+2\,ab+b^2+2\,ac+2\,bc+c^2$.
3. $b^2+2\,bc+c^2-2\,ab-2\,ac+a^2$.
4. $4\,a^2+12\,ab+9\,b^2+16\,ac+24\,bc+16\,c^2$.
5. $49\,a^2+4\,b^2-28\,ab+42\,ac+9\,c^2-12\,bc$.
6. $9\,a^4+30\,a^3b+49\,a^2b^2+40\,ab^3+16\,b^4$.
7. $89\,a^2b^2-70\,ab^3+16\,a^4-56\,a^3b+25\,b^4$.
8. $4\,a^6-12\,a^4b-28\,a^3b^3+9\,a^2b^2+42\,ab^4+49\,b^6$.
9. $49\,m^2+4\,n^2+16\,p^2+28\,mn+16\,np+56\,mp$.
10. $a^4b^2c^2+a^2b^4c^2-2\,a^3b^3c^2-2\,a^3b^2c^3+a^2b^2c^4+2\,a^2b^3c^3$.

179. The extraction of the square root of an expression containing fractions is often made easier by arranging the terms in descending order of some one of its letters.

Extract the square root of $a^2+\frac{1}{a^2}-\frac{2}{a}+2-\frac{2}{a^3}+\frac{1}{a^4}$.

$$
\begin{array}{r|l|l}
 & a^2+2-\frac{2}{a}+\frac{1}{a^2}-\frac{2}{a^3}+\frac{1}{a^4} & a+\frac{1}{a}-\frac{1}{a^2} \\
t=a & a^2 & \\
\hline
2t=2a & & \\
u=\frac{1}{a} & & \\
2t+u=2a+\frac{1}{a} & & \\
u(2t+u)=\frac{1}{a}\left(2a+\frac{1}{a}\right) & 2 \quad +\frac{1}{a^2} & \\
\hline
2t=2a+\frac{2}{a} & -\frac{2}{a} \quad -\frac{2}{a^3}+\frac{1}{a^4} & \\
u=-\frac{1}{a^2} & & \\
2t+u=2a+\frac{2}{a}-\frac{1}{a^2} & & \\
u(2t+u)=-\frac{1}{a^2}\left(2a+\frac{2}{a}-\frac{1}{a^2}\right) & -\frac{2}{a} \quad -\frac{2}{a^3}+\frac{1}{a^4} & \\
\hline
\end{array}
$$

180. Under certain conditions the formula method may be applied to polynomials not in the form of perfect squares. These conditions are discussed in the following example. The square roots of such expressions are called **approximate square roots.** If the polynomial has a true square root, the square of the root equals the given polynomial; if the polynomial has an approximate square root only, the square of the root plus the remainder equals the given polynomial.

The symbol $\cdots$ is called the *symbol of continuation*, and is read, "and so on."

Thus, $x+x^2+x^3\cdots$ is read, "$x+x^2+x^3$ and so on."

1. Extract the approximate square root of $1+x$.

$$\begin{array}{r|l}
 & 1+x \;\Big|\; 1+\frac{x}{2}-\frac{x^2}{8}\cdots \\
t=1 & 1 \\
\hline
2t=2 & \quad x \\
u=\frac{x}{2} & \\
2t+u=2+\frac{x}{2} & \\
u(2t+u)=\frac{x}{2}\left(2+\frac{x}{2}\right) & \quad x+\frac{x^2}{4} \\
\hline
2t=2+x & \quad -\frac{x^2}{4} \\
u=-\frac{x^2}{8} & \\
2t+u=2+x-\frac{x^2}{8} & \\
u(2t+u)=-\frac{x^2}{8}\left(2+x-\frac{x^2}{8}\right) & \quad -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
\end{array}$$

$$\sqrt{1+x}=\sqrt{\left[\left(1+x+\frac{x^2}{4}-\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}\right)+\frac{x^3}{8}-\frac{x^4}{64}\right]}.$$

The square root is extracted by the formula.

In the above example the root is not approximate if the remainder, $\frac{x^3}{8}-\frac{x^4}{64}$, is larger algebraically and numerically than $1+x$. In general, approximate square roots are to be interpreted as approximate square roots of such expressions only as produce remainders less than the given expressions.

EXERCISE XC

Extract the square roots of the following expressions:

1. $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$.

2. $\frac{a^8}{9\,b^{10}} + \frac{4\,a^4}{3\,b^6} - \frac{24\,b^2}{a^4} + \frac{36\,b^6}{a^8}$.

3. $a^2 + 2 - \frac{2}{a} + \frac{1}{a^2} - \frac{2}{a^3} + \frac{1}{a^4}$.

4. $\frac{a^2}{b^2} - \frac{2\,a}{c} + \frac{b^2}{c^2} + \frac{2\,c}{b} - \frac{2\,b}{a} + \frac{c^2}{a^2}$.

5. $\frac{9\,a^4}{b^6} + \frac{8\,b^2}{3} + \frac{4\,b^6}{9\,a^2} - \frac{6\,a^3}{b^4} + \frac{13\,a^2}{b^2} + \frac{8\,b^4}{3\,a}$.

6. $\frac{a^2}{b^2} + \frac{2\,ac}{bd} + \frac{c^2}{d^2} - \frac{3\,am}{2\,bn} - \frac{3\,cm}{2\,dn} + \frac{9\,m^2}{16\,n^2}$.

7. $x^4 + 4\,x^3 + 10\,x^2 + 13\,x + 13 + \frac{7}{x} + \frac{25}{4\,x^2} + \frac{1}{x^3} + \frac{1}{x^4}$.

8. $4 + \frac{121\,x^6}{64} + \frac{57\,x^3}{4} + \frac{77\,x^5}{16} + \frac{159\,x^4}{16} + \frac{53\,x^2}{4} + 10\,x$.

9. $a^4 + 2\,a^3 + a^2 + 2\,a + 4 + \frac{2}{a} + \frac{1}{a^2} + \frac{2}{a^3} + \frac{1}{a^4}$.

10. $x^8 + 2\,x^6 + x^4 + 2\,x^2 + 4 + \frac{2}{x^2} + \frac{1}{x^4} + \frac{2}{x^6} + \frac{1}{x^8}$.

11. $x^4 - 2\,x^3 + 3\,x^2 - 4\,x + 5 - \frac{4}{x} + \frac{3}{x^2} - \frac{2}{x^3} + \frac{1}{x^4}$.

Express to four terms the approximate square roots of the following expressions:

12. $1 - x$.

13. $a^2 + b$.

14. $1 - x + x^2$.

15. $x^4 + x^2 + 1$.

16. $x^6 + x$.

17. $x^2 + 3\,x + 2$.

ARITHMETICAL SQUARE ROOTS

181. Square roots of arithmetical numbers may be extracted by the formula method.

Since $\sqrt{1}=1$, $\sqrt{100}=10$, $\sqrt{10{,}000}=100$, $\sqrt{1{,}000{,}000}=1000$, etc., the square root of a number >1 and <100 has one digit, the square root of a number >100 and $<10{,}000$ has two digits, the square root of a number $>10{,}000$ and $<1{,}000{,}000$ has three digits; and so on. If, therefore, the number be separated into periods of two digits each, running from right to left, the number of periods will equal the number of digits in the root.

Thus $\sqrt{1764}$ has two digits, $\sqrt{811{,}801}$ has three digits.

182. Every integral number may be considered as made up of tens and units. Hence $(t+u)^2$, where t represents the part of the root already found and u represents the next term of the root, will correspond to any integral number in the form of a perfect square.

$$42 = 40 + 2 = t + u, \tag{1}$$

squaring (1), $$(42)^2 = (40+2)^2 = (t+u)^2, \tag{2}$$

simplifying (2), $$1764 = 1600 + 160 + 4 = t^2 + 2\,tu + u^2, \tag{3}$$

indicating square roots in (3),

$$\sqrt{1764} = \sqrt{1600+160+4} = \sqrt{t^2+2\,tu+u^2}. \tag{4}$$

1. Extract the square of $1764 = 1600 + 160 + 4$.

$$\begin{array}{r|l}
t = 40 & 1600 + 160 + 4 \;\underline{|\; 40 + 2 = 42} \\
t^2 = 1600 & 1600 \\
\hline
2\,t = 80 & \qquad\quad 160 + 4 \\
u = \;\;2 & \\
2\,t + u = 80 + 2 & \\
u(2\,t+u) = 2(80+2) & \qquad\quad 160 + 4 \\
\hline
\end{array}$$

The work necessary in writing a number in the form $t^2 + 2tu + u^2$ is tedious, and may be abridged; the preceding written in the abridged form is:

$$
\begin{array}{r|r|l}
t = 40 & 17\ 64 & \underline{40 + 2 = 42} \\
t^2 = 1600 & 16\ 00 & \\
\hline
2t = 80 & 1\ 64 & \\
u = 2 & & \\
2t + u = 82 & & \\
u(2t + u) = 2(82) & 1\ 64 & \\
\hline
\end{array}
$$

In the above example, if $t =$ value of the digit in the tens' place, and $u =$ value of the digit in the units' place, t is the greatest multiple of 10 whose square is < 1764; that is, $t = 40$. Subtracting $t^2 = 1600$, the remainder is 164. Dividing 164 by $2t = 80$, the quotient is 2, which is u. Hence $u(2t + u) = 2(80 + 2) = 164$ is to be subtracted from the remainder, 164. The remainder being 0, the square root is $40 + 2 = 42$. In the above example the work may be further abridged by omitting the two zeros in the square of 40.

2. Extract the square root of 4,414,201.

$$
\begin{array}{r|r|l}
t = 2 & 4\ 41\ 42\ 01 & \underline{2101} \\
t^2 = 4 & 4 & \\
\hline
2t = 40 & 41 & \\
u = 1 & & \\
2t + u = 41 & & \\
u(2t + u) = 1(41) & 41 & \\
\hline
2t = 420 & 42 & \\
u = 0 & & \\
2t + u = 420 & & \\
u(2t + u) = 0(420) & 0 & \\
\hline
2t = 4200 & 42\ 01 & \\
u = 1 & & \\
2t + u = 4201 & & \\
u(2t + u) = 1(4201) & 42\ 01 & \\
\hline
\end{array}
$$

183. Since $\sqrt{0.01} = 0.1$, $\sqrt{0.0001} = 0.01$, $\sqrt{0.000001} = 0.001$, etc., the square root of a decimal in the form of a perfect square has half as many decimal places as the number itself. A decimal is therefore separated into periods of two digits each, running from left to right.

After pointing off the decimal, the square root is extracted as if the decimal were an integer.

1. Extract the square root of 0.01301881.

$t = 1$	0.01 30 18 81	0.1141
$t^2 = 1$	1	
$2t = 20$	30	
$u = 1$		
$2t + u = 21$		
$u(2t + u) = 1(21)$	21	
$2t = 220$	9 18	
$u = 4$		
$2t + u = 224$		
$u(2t + u) = 4(224)$	8 96	
$2t = 2280$	22 81	
$u = 1$		
$2t + u = 2281$		
$u(2t + u) = 1(2281)$	22 81	

Since there are eight decimal places in the number there are four decimal places in the root.

184. The approximate square roots of numbers, whether integral or decimal, or both, not in the form of perfect squares, may be found by annexing zeros to fill out the periods of two digits each until the number of periods equals the number of root digits required.

1. Extract the square root of 7.1 to three decimals.

$$\begin{array}{r|rl}
t = 2 & 7.10\ 00\ 00 & \underline{|\ 2.469\cdots} \\
t^2 = 4 & 4 & \\
\hline
2\,t = 40 & 3\ 10 & \\
u = \ \ 6 & & \\
2\,t + u = 46 & & \\
u(2\,t+u) = 6(46) & 2\ 76 & \\
\hline
2\,t = 480 & 34\ 00 & \\
u = \ \ \ 6 & & \\
2\,t + u = 486 & & \\
u(2\,t+u) = 6(486) & 29\ 16 & \\
\hline
2\,t = 4920 & 4\ 84\ 00 & \\
u = \ \ \ \ 9 & & \\
2\,t + u = 4929 & & \\
u(2\,t+u) = 9(4929) & 4\ 43\ 61 & \\
\hline
\end{array}$$

EXERCISE XCI

Extract the square root of the following numbers:

1. 361.
2. 1681.
3. 7396.
4. 71,824.
5. 15,129.
6. 136,161.
7. 3,404,025.
8. 1,225,449.
9. 3,466,383,376.
10. 0.0081.
11. 0.1369.
12. 0.134689.
13. 0.094864.
14. 8475.0436.
15. 2499.700009.

Extract the approximate square root to four decimals of the following numbers:

16. 2.
17. 3.
18. 5.
19. 6.
20. 7.
21. 10.
22. 0.831.
23. 10.4.
24. 32.701.

CHAPTER XV

RADICALS

185. The quantity $\sqrt[n]{a}$ has already been defined, § 17
as the quantity whose nth power is a, or $(\sqrt[n]{a})^n = a$. I
a is an exact nth power, the existence of such a quantity i
at once evident, as $\sqrt[3]{8} = 2$. But if a is not an exact nt
power, it becomes necessary to prove the existence of $\sqrt[n]{a}$
Such a proof is beyond the province of this book; an
a simple numerical example must suffice. It is not pos
sible to obtain exactly the value of $\sqrt{2}$, since there is n
number, integral or fractional, whose square is exactly 2
But,

$$(1.4)^2 < 2 < (1.5)^2, \qquad (A$$

$$(1.41)^2 < 2 < (1.42)^2, \qquad (B$$

$$(1.414)^2 < 2 < (1.415)^2. \qquad (C$$

In (A), since 2 lies between $(1.4)^2$ and $(1.5)^2$, $\sqrt{}$
differs from 1.4 and 1.5 by less than they differ from eac
other: that is, since 1.4 and 1.5 differ from each othe
by 0.1, $\sqrt{2}$ differs from either by less than 0.1; similarl
in (B), $\sqrt{2}$ differs from 1.41 and 1.42 by less than 0.01
and in (C), $\sqrt{2}$ differs from 1.414 and 1.415 by less tha
0.001. Continuing the process shown in (A), (B), an
(C), a number may be found which will represent a
close an approximation of $\sqrt{2}$ as is required.

FIG. 11.

The value of $\sqrt{2}$ may be represented graphically. On the line OX, Fig. 11, let equal distances be laid off from O toward the right, and OA represent the number 1, OB the number 2, etc. Then 1.4 will be represented by OC, 1.5 by OD. The numbers 1.4, 1.41, 1.414 will be seen to be represented by lines whose terminal points move toward the right, while the numbers 1.5, 1.42, 1.415 will be represented by lines whose terminal points move toward the left. The terminal points representing these two sets of numbers will approach each other, but no terminal point in either set can cross into the region of the other. Yet the numbers show that the terminal points may be made as near to each other as may be required. There will be some point P which will be the limiting position of both sets of terminal points; and the line OP will represent $\sqrt{2}$.

186. An indicated root of a quantity is called a **radical.**

Thus, $\sqrt{a}$, $\sqrt[3]{27}$, are radicals.

An expression which is composed of radicals is called a **radical expression.**

Thus, $\sqrt{x}+\sqrt[3]{27}$, $\sqrt{a}-\sqrt{b}$, are radical expressions.

All integers and fractions are called **rational quantities.**

All other numbers are called **irrational quantities.** The simplest class of irrational quantities consists of indicated roots which cannot be extracted.

Thus, 2, and $\frac{5}{8}$, are rational; $\sqrt{2}$, $\sqrt{1+\sqrt{2}}$, are irrational.

An expression which contains rational quantities only is called a **rational expression.**

Thus, $a+\frac{2}{3}$ is a rational expression.

An expression which contains an irrational quantity is called an **irrational expression.**

Thus, $a+\sqrt{2}$ is an irrational expression.

187. A radical whose radicand is rational and whose root is irrational is called a **surd.**

Thus, $\sqrt{a}$ and $\sqrt[3]{4}$ are surds; while $\sqrt{1+\sqrt{3}}$, being the indicated root of a quantity not rational, is not a surd.

The **order of a surd** depends upon the index of the root. A **quadratic surd**, or a surd of the second order, has 2 for the index of the root; a **cubic surd**, or a surd of the third order, has 3 for the index of the root; a **biquadratic surd**, or a surd of the fourth order, has 4 for the index of the root, etc.

Thus, $\sqrt{a}$, $\sqrt[3]{b}$, $\sqrt[4]{c}$, are respectively quadratic, cubic, and biquadratic surds.

188. A rational factor of a surd is called the **coefficient** of the surd.

Thus, $\frac{3}{4}$ is the coefficient of $\frac{3}{4}\sqrt{ax}$.

Surds which have 1 as a coefficient, expressed or implied, are called **entire surds.**

Thus, $\sqrt{ay}$ and $\sqrt[3]{4}$ are entire surds.

Surds which have other coefficients than 1 are called **mixed surds.**

Thus, $2\sqrt{x}$ and $3\sqrt{a-b}$ are mixed surds.

A surd is called a **monomial surd** if it consists of a single surd.

Thus, $\sqrt[3]{x^2}$ and $5\sqrt{3}$ are monomial surds.

The sum of a rational, and a surd quantity, or the sum of two monomial surds, is called a **binomial surd.**

189. The difference between algebraic and arithmetical irrational quantities should be noticed. Such a quantity as $\sqrt{2}$ is an arithmetical irrational quantity; similarly, quantities such as $\sqrt{a}$ are considered algebraic irrational quantities, although if $a = 4$, $\sqrt{a}$ is an arithmetical rational quantity.

In this, as in the preceding chapter, the principal roots only are discussed, and the quantity under the radical sign is restricted to positive values.

Thus, $\sqrt{4} \neq \pm 2$, but $\sqrt{4} = 2$. This fact must be kept in mind, for otherwise some of the proofs of the principles will not hold.

PRINCIPLES OF RADICALS

190. I. *The product of the nth roots of any number of quantities is equal to the nth root of their products.*

By II, § 170, $(\sqrt[n]{a}\sqrt[n]{b}\sqrt[n]{c})^n = abc,$

by definition, $\sqrt[n]{a}\sqrt[n]{b}\sqrt[n]{c} = \sqrt[n]{abc}.$

If the radicand contains a factor whose exponent is a multiple of the index of the root, the surd may be simplified by I. Since $\sqrt[n]{a^n} = a$,

by I, $\sqrt[n]{a^n b} = \sqrt[n]{a^n}\sqrt[n]{b} = a\sqrt[n]{b}.$

1. Simplify $\sqrt[3]{16}$.

$$\sqrt[3]{16} = \sqrt[3]{2^4} = \sqrt[3]{2^3}\sqrt[3]{2} = 2\sqrt[3]{2}.$$

2. Simplify $\sqrt[4]{32\,x^5y}$.

$$\sqrt[4]{32\,x^5y} = \sqrt[4]{2^5\,x^5y} = \sqrt[4]{2^4\,x^4}\sqrt[4]{2\,xy} = 2\,x\sqrt[4]{2\,xy}.$$

3. Simplify $\sqrt{25\,a - 25\,b}$

$$\sqrt{25\,a - 25\,b} = \sqrt{25\,(a-b)} = \sqrt{5^2}\sqrt{a-b} = 5\sqrt{a-b}.$$

EXERCISE XCII

Simplify the following surds:

1. $\sqrt{8}$.
2. $\sqrt{18}$.
3. $\sqrt{75}$.
4. $\sqrt{27}$.
5. $\sqrt{48}$.
6. $\sqrt{150}$.
7. $\sqrt[3]{16}$.
8. $\sqrt[3]{54}$.
9. $\sqrt[3]{-54}$.
10. $\sqrt[4]{80}$.
11. $\sqrt[5]{96}$.
12. $\sqrt[6]{192}$.
13. $\sqrt{a^2x}$.
14. $\sqrt{25\,ax^2}$.
15. $\sqrt{5\,m^2n^2}$.
16. $\sqrt[3]{4\,x^3y^3}$.
17. $\sqrt[3]{27\,xy^3}$.
18. $\sqrt[4]{x^7}$.
19. $\sqrt[5]{y^9}$.
20. $\sqrt{16\,x-16\,y}$.
21. $\sqrt[4]{x^7(a-b)}$.
22. $2\sqrt[3]{27\,a-54\,b}$.
23. $3\sqrt[3]{(a-b)^4}$.
24. $5\sqrt[4]{x^3(a-b)^5}$.
25. $a-3\sqrt{4\,x^2-8\,y}$.
26. $(a-2)\sqrt{a^2-a^4}$.

191. The coefficient of a radical may be introduced under the radical by raising the coefficient to the power indicated by the index of the root. Since $a=\sqrt[n]{a^n}$,

$$a\sqrt[n]{b}=\sqrt[n]{a^n}\,\sqrt[n]{b}=\sqrt[n]{a^nb}.$$

Reduce the mixed surds $2\,a\sqrt{2\,b}$, and $-3\,ax\sqrt[3]{a^2x}$, to entire surds.

$$2\,a\sqrt{2\,b}=\sqrt{(2\,a)^2}\sqrt{2\,b}=\sqrt{8\,a^2b}.$$

$$-3\,ax\sqrt[3]{a^2x}=\sqrt[3]{(-3\,ax)^3}\sqrt[3]{a^2x}=\sqrt[3]{-27\,a^5x^4}.$$

EXERCISE XCIII

Reduce the following mixed surds to entire surds:

1. $2\sqrt{3}$.
2. $3\sqrt{2}$.
3. $4\sqrt[3]{2}$.
4. $2\sqrt[3]{4}$.
5. $2\sqrt[4]{5}$.
6. $3\sqrt[5]{2}$.
7. $-5\sqrt[3]{4}$.
8. $-2\,a\sqrt[3]{x}$.
9. $-ab\sqrt[5]{bx}$.
10. $a-b\sqrt{a-b}$.
11. $(a-b)\sqrt{a-b}$.
12. $-(a-x)\sqrt[3]{x-y}$.

192. When the radicand is in the fractional form it may be made integral, by I, § 190. Since $\sqrt[n]{\left(\frac{1}{a}\right)^n} = \frac{1}{a}$,

$$\sqrt[n]{\frac{b}{a^n}} = \sqrt[n]{\frac{1}{a^n}}\sqrt[n]{b} = \frac{1}{a}\sqrt[n]{b}.$$

NOTE. $(1)^n = 1$, hence $\sqrt[n]{1^n} = 1$. It is usual to omit both the exponent of the power, and the index of the root, of 1.

When the denominator of a radicand in the fractional form does not contain a quantity whose exponent is a multiple of the index of the root, the denominator may be put into such a form that the indicated root may be extracted by multiplying both numerator and denominator by the same quantity.

Thus,

$$\sqrt{\frac{a}{b}} = \sqrt{\frac{a}{b}\left(\frac{b}{b}\right)} = \sqrt{\frac{ab}{b^2}} = \sqrt{\frac{1}{b^2}}\sqrt{ab} = \frac{1}{b}\sqrt{ab}.$$

1. Simplify $\sqrt{\frac{2}{3}}$.

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{2}{3}\left(\frac{3}{3}\right)} = \sqrt{\frac{2\cdot 3}{3^2}} = \sqrt{\frac{1}{3^2}}\sqrt{6} = \frac{1}{3}\sqrt{6}.$$

2. Simplify $\sqrt[3]{\frac{2\,xy}{9\,ab}}$.

$$\sqrt[3]{\frac{2\,xy}{9\,ab}} = \sqrt[3]{\frac{2\,xy}{3^2ab}} = \sqrt[3]{\frac{2\,xy}{3^2ab}\left(\frac{3\,a^2b^2}{3\,a^2b^2}\right)}$$

$$= \sqrt[3]{\frac{1}{3^3a^3b^3}}\sqrt[3]{6\,a^2b^2xy} = \frac{1}{3\,ab}\sqrt[3]{6\,a^2b^2xy}.$$

EXERCISE XCIV

Simplify the following surds:

1. $\sqrt{\frac{1}{2}}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{1}{5}}$.
4. $\sqrt[3]{\frac{1}{2}}$.
5. $\sqrt[3]{\frac{1}{4}}$.
6. $\sqrt[4]{\frac{1}{6}}$.
7. $\sqrt[5]{\frac{1}{2}}$.
8. $\sqrt{\frac{a}{x}}$.
9. $\sqrt[3]{\frac{a}{x}}$.
10. $\sqrt[3]{\frac{3a}{4b}}$.
11. $\sqrt[5]{\frac{m}{n}}$.
12. $\sqrt{\frac{2a}{3b}}$.
13. $\sqrt[3]{\frac{3ax}{2bc}}$.
14. $\sqrt[3]{\frac{5a^2}{16}}$.
15. $\sqrt[3]{\frac{2a}{125c^2}}$.
16. $\sqrt[4]{\frac{b}{x^2}}$.
17. $\sqrt[4]{\frac{5a}{7b^2}}$.
18. $\sqrt[3]{\frac{3a}{7c^4}}$.
19. $\sqrt{\frac{x^2y}{12a^3b}}$.
20. $\sqrt[3]{\frac{5a}{16x^4y}}$.
21. $\sqrt[4]{\frac{7m}{162x^3y^2}}$.
22. $\sqrt[5]{\frac{3a^3}{2048x^7y}}$.
23. $\sqrt[3]{\frac{21abc}{250x^3y}}$.
24. $\sqrt{\frac{a^5c}{242x^7z}}$.
25. $\sqrt{\frac{6a^4b^5}{343x^5y^2}}$.
26. $\sqrt[3]{\frac{32a^5b^7}{243mx}}$.
27. $\sqrt[4]{\frac{10a^6c^2}{81x^7y^2}}$.
28. $\sqrt{\frac{8(a-b)}{27(x-y}}$
29. $\sqrt[3]{\frac{15x^7y^8}{16(a-b}}$
30. $\sqrt[5]{\frac{8a^7b^9z^{12}}{27a^{12}b^8c}}$
31. $\sqrt{\frac{ab}{a-b}}$.
32. $\sqrt{\frac{x+y}{x-y}}$.
33. $\sqrt[3]{\frac{2}{3}}$.
34. $\sqrt[3]{\frac{27}{10}}$.
35. $8\sqrt[3]{\frac{3}{4}}$.
36. $20\sqrt[3]{\frac{81}{100}}$.
37. $45\sqrt[3]{\frac{2}{75}}$.
38. $10a\sqrt[3]{\frac{4b}{5a}}$.

193. II. *The quotient of the nth roots of two quantities is equal to the nth root of their quotient, or* $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}.$

By III, § 170, $$\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n=\frac{a}{b},$$

by definition, $$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}.$$

1. Express $\sqrt{12}\div\sqrt{5}$ as a single radical.

$$\sqrt{12}\div\sqrt{5}=\frac{\sqrt{12}}{\sqrt{5}}=\sqrt{\frac{12}{5}}=\sqrt{\frac{2^2\cdot 3}{5}\left(\frac{5}{5}\right)},$$

$$=\sqrt{\frac{2^2}{5^2}}\sqrt{3\cdot 5}=\frac{2}{5}\sqrt{15}.$$

2. Express $\sqrt{2\,a}\div\sqrt{3\,b}$ as a single radical.

$$\sqrt{2\,a}\div\sqrt{3\,b}=\sqrt{\frac{2\,a}{3\,b}}=\sqrt{\frac{2\,a}{3\,b}\left(\frac{3\,b}{3\,b}\right)},$$

$$=\sqrt{\frac{1}{3^2b^2}}\sqrt{2\cdot 3\,ab}=\frac{1}{3\,b}\sqrt{6\,ab}.$$

EXERCISE XCV

In the following radicals, simplify the quotients expressed as single radicals:

1. $\sqrt{6}\div\sqrt{5}.$
2. $\sqrt[3]{4}\div\sqrt[3]{3}.$
3. $\sqrt[4]{a}\div\sqrt[4]{b}.$
4. $\sqrt[5]{x}\div\sqrt[5]{y}.$
5. $\sqrt{3\,ab}\div\sqrt{7\,xy}.$
6. $\sqrt[5]{x^2y}\div\sqrt[5]{16\,m^4n}.$
7. $\sqrt[3]{49\,x^2y^2}\div\sqrt[3]{13\,bc}.$
8. $\sqrt[3]{(a-b)^2}\div\sqrt[3]{2\,a}.$
9. $\sqrt[4]{(m-n)^3}\div\sqrt[4]{8\,mn}$
10. $\sqrt{32\,xy}\div\sqrt{27\,ab^5}.$

194. III. *The nth power of the mth root of any quantity is equal to the mth root of the nth power, or* $(\sqrt[m]{a})^n = \sqrt[m]{a^n}$.

By definition, $(\sqrt[m]{a})^n = \sqrt[m]{a}$ to n factors,

by I, § 190, $= \sqrt[m]{a^n}$.

Simplify $(\sqrt[3]{2\,a^5c})^4$.

$$(\sqrt[3]{2\,a^5c})^4 = \sqrt[3]{(2\,a^5c)^4} = \sqrt[3]{2^4a^{20}c^4} = \sqrt[3]{2^3a^{18}c^3}\,\sqrt[3]{2\,a^2c} = 2\,a^6c\,\sqrt[3]{2\,a^2c}.$$

EXERCISE XCVI

Simplify the following radicals:

1. $(\sqrt{2})^3$. **4.** $(\sqrt{2\,a})^3$. **7.** $(\sqrt[3]{-6\,x^2y})^3$.

2. $(\sqrt[3]{3})^2$. **5.** $(\sqrt[4]{3\,a})^3$. **8.** $(\sqrt[7]{7})^3$.

3. $(\sqrt[4]{2})^3$. **6.** $(\sqrt[5]{2\,a^2bx})^2$. **9.** $(\sqrt[3]{a+b})^2$.

10. $(\sqrt[5]{-4\,ab^2})^3$. **12.** $(\sqrt{x^2+2\,xy+y^2})^3$.

11. $(-\sqrt[7]{-2\,a})^3$. **13.** $(\sqrt[3]{-a-b})^2$.

195. IV. *The mth root of the nth root of any quantity is equal to the mnth root of the quantity, or* $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

Let $$x = \sqrt[m]{\sqrt[n]{a}}, \quad (1)$$

raising both members of (1) to mth power,

$$x^m = \sqrt[n]{a}, \quad (2)$$

raising both members of (2) to nth power,

$$x^{mn} = a, \quad (3)$$

extracting mnth roots in (3),

$$x = \sqrt[mn]{a}, \quad (4)$$

from (4), $$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}. \quad (5)$$

A surd is said to be in its **simplest form** when neither of the reductions explained under I, §§ 190, 192, and IV, § 195, may be applied; that is, when the radicand is integral and contains no factor whose exponent is a multiple of the index of the root, and when the index of the root is as small as possible. **Similar surds** are those which, when reduced to their simplest form, differ in their coefficients only. Surds which are not similar are called **dissimilar surds.**

Thus, $\frac{1}{3}\sqrt{a}$ and $b\sqrt{a}$ are similar surds, and $\frac{1}{3}\sqrt{a}$ and $b\sqrt[3]{a}$ are dissimilar surds.

Reduce $\sqrt[6]{27}$ and $\sqrt[15]{243\,a^5}$ to their simplest forms.

$$\sqrt[6]{27} = \sqrt[6]{3^3} = \sqrt{\sqrt[3]{3^3}} = \sqrt{3}.$$

$$\sqrt[15]{243\,a^5} = \sqrt[15]{3^5\,a^5} = \sqrt[3]{\sqrt[5]{3^5\,a^5}} = \sqrt[3]{3\,a}.$$

EXERCISE XCVII

Reduce the following radicals to their simplest forms:

1. $\sqrt[6]{4}$. **9.** $\sqrt[15]{a^5}$. **17.** $\sqrt[3]{\sqrt{a}}$.

2. $\sqrt[6]{25}$. **10.** $\sqrt[15]{-a^3}$. **18.** $\sqrt[5]{\sqrt{a}}$.

3. $\sqrt[12]{16}$. **11.** $\sqrt[6]{-1000}$. **19.** $\sqrt[5]{\sqrt[3]{a^2}}$.

4. $\sqrt[15]{64}$. **12.** $\sqrt[15]{-32}$. **20.** $\sqrt[3]{\sqrt[4]{ab}}$.

5. $\sqrt[10]{100}$. **13.** $\sqrt[24]{64}$. **21.** $\sqrt[3]{\sqrt[12]{a^{13}}}$.

6. $\sqrt[10]{243}$. **14.** $\sqrt[35]{a^{15}}$. **22.** $\sqrt[9]{\sqrt[5]{x^{11}}}$.

7. $\sqrt[6]{a^2}$. **15.** $\sqrt[24]{a^{12}}$. **23.** $\sqrt[3]{\sqrt[15]{x^{19}}}$.

8. $\sqrt[12]{a^3}$. **16.** $\sqrt{\sqrt[3]{4\,a^2}}$. **24.** $\sqrt[3]{a\sqrt{a}}$.

25. $\sqrt[4]{a^2\sqrt[3]{2\,a}}$. **26.** $\sqrt[5]{x^2y\sqrt[4]{x^3}}$.

196. V. *The nth root of any quantity is equal to the mnth root of the mth power of the quantity, or* $\sqrt[n]{a}=\sqrt[mn]{a^m}$.

Since $\sqrt[n]{a}=\sqrt[m]{(\sqrt[n]{a})^m}$,

by III, § 194, $=\sqrt[m]{\sqrt[n]{a^m}}$,

by IV, § 195, $=\sqrt[mn]{a^m}$.

By V, surds of different orders may be reduced to equivalent surds of the same order — that order being the least common multiple of the original orders.

Reduce $\sqrt{2}$ and $\sqrt[3]{5}$ to equivalent surds of the same order.

$$\sqrt{2}=\sqrt[6]{(2)^3}=\sqrt[6]{8}.$$

$$\sqrt[3]{5}=\sqrt[6]{(5)^2}=\sqrt[6]{25}.$$

Since $\sqrt[6]{25}>\sqrt[6]{8}$, $\sqrt[3]{5}>\sqrt{2}$.

EXERCISE XCVIII

Reduce the following surds to equivalent surds of the same order:

1. $\sqrt{a}$, $\sqrt[4]{a^3}$.

2. $\sqrt{a}$, $\sqrt[3]{a}$.

3. $\sqrt{2x}$, $\sqrt[3]{4x^2}$.

4. $\sqrt[3]{a^2b}$, $\sqrt[4]{ab^2}$.

5. $\sqrt{3a^2}$, $\sqrt[5]{2xy}$.

6. $\sqrt[3]{7x^2y}$, $\sqrt[4]{3ab^2}$.

7. $4\sqrt{2x^2y}$, $\sqrt[6]{2xy^2}$.

8. $\sqrt[5]{8x^3}$, $\sqrt[4]{27y^3}$.

9. $\sqrt[6]{5a^4b^3}$, $\sqrt[5]{3a^4b^3}$.

10. $\sqrt{11axy}$, $\sqrt[3]{4a^2xy^2}$.

Arrange the following surds in order of magnitude:

11. $\sqrt[3]{4}$, $\sqrt[4]{8}$.

12. $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[4]{5}$.

13. $\sqrt[6]{9}$, $\sqrt[12]{10}$, $\sqrt[24]{100}$.

14. $\sqrt{12}$, $\sqrt[3]{48}$, $\sqrt[6]{81}$.

15. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{4}}$, $\sqrt[4]{\frac{1}{8}}$.

16. $\sqrt{\frac{3}{4}}$, $\sqrt[3]{\frac{4}{5}}$, $\sqrt[4]{\frac{5}{6}}$.

ADDITION AND SUBTRACTION OF RADICALS

197. Similar radicals may be added or subtracted by writing the algebraic sum of their coefficients as the coefficient of the common radical, taken as the unit of addition. Each radical should be reduced to its simplest form before the process of addition or subtraction is begun. Addition or subtraction of dissimilar radicals may only be indicated by connecting them with the proper signs.

1. Simplify $\sqrt{a^3} - 2a\sqrt{a^5} + 3\sqrt{a}$.

$$\begin{aligned} \sqrt{a^3} &= \sqrt{a^2}\sqrt{a} = & a\sqrt{a},\\ -2a\sqrt{a^5} &= -2a\sqrt{a^4}\sqrt{a} = & -2a^3\sqrt{a},\\ 3\sqrt{a} &= & 3\sqrt{a},\\ \hline \sqrt{a^3} - 2a\sqrt{5} + 3\sqrt{a} &= (a - 2a^3 + 3)\sqrt{a}. \end{aligned}$$

2. Simplify $2\sqrt{\frac{1}{2}} - 4\sqrt{\frac{1}{8}} + 8\sqrt{2}$.

$$\begin{aligned} 2\sqrt{\frac{1}{2}} &= 2\sqrt{\frac{1}{2^2}}\sqrt{2} = & \sqrt{2},\\ -4\sqrt{\frac{1}{8}} &= -4\sqrt{\frac{1}{2^4}}\sqrt{2} = & -\sqrt{2},\\ 8\sqrt{2} &= & 8\sqrt{2},\\ \hline 2\sqrt{\tfrac{1}{2}} - 4\sqrt{\tfrac{1}{8}} + 8\sqrt{2} &= 8\sqrt{2}. \end{aligned}$$

EXERCISE XCIX

Simplify the following radical expressions:

1. $2\sqrt{x} - 4\sqrt{x} + \sqrt{x}$.
2. $3\sqrt{a} - 4\sqrt{a} + 2\sqrt{a}$.
3. $\sqrt{32} + \sqrt{8} - \sqrt{128}$.
4. $\sqrt[3]{16} + \sqrt[3]{54} - \sqrt[3]{128}$.
5. $2\sqrt[4]{x} + 5\sqrt[4]{x^5} - x\sqrt[4]{16x}$.

6. $4a\sqrt[5]{a^2b} - \sqrt[5]{32a^7b} - a\sqrt[5]{243a^2b}$.

7. $\sqrt{xy} + \frac{1}{4}\sqrt{xy} + \frac{9}{4}\sqrt{xy}$.

8. $a\sqrt{x} - \sqrt{a^3x} + 7a\sqrt{a^4x}$.

9. $\sqrt{16a} - \sqrt{25a} + \sqrt{49a}$.

10. $\sqrt[3]{27x^2y} + \sqrt[3]{8x^2y} - \sqrt[3]{125x^2y}$.

11. $\sqrt{16a} + \sqrt{81a} + \sqrt{144a^2b^2}$.

12. $\sqrt{a^2bc} + \sqrt{ab^2c} + \sqrt{abc^2}$.

13. $\sqrt{a^3bc} + \sqrt{ab^3c} + \sqrt{abc^3}$.

14. $\sqrt[3]{ax^3} + \sqrt[4]{bx^4} + \sqrt[5]{cx^5}$.

15. $\sqrt{a^3bc} + 2\sqrt{ab^3c} - 5\sqrt{abc^2}$.

16. $\sqrt{9a^5b^3c} + \sqrt{25a^3b^5c} + \sqrt{36a^3b^3c^3}$.

17. $\sqrt{9x} - \sqrt{16x} + 3\sqrt{25x} - \sqrt{100x}$.

18. $\sqrt{ax^2} + 3\sqrt{16ax^2} - \sqrt{64ax^2} - 2\sqrt{9ax^2}$.

19. $1 - \sqrt{3} + 4\sqrt{5} + 2\sqrt{9} + \sqrt{108} - \sqrt{80}$.

20. $\sqrt{18x} + \sqrt{147x} - 2\sqrt{32x} - \sqrt{192x} + \sqrt{72x}$.

21. $3\sqrt{x(a+b)^2} - \sqrt{4a^2x} - \sqrt{25b^2x} - \sqrt{(a+b)^2x}$.

22. $\sqrt{4x-4y} + \sqrt{16x-16y} + 5\sqrt{49x-49y}$
$- 8\sqrt{25x-25y}$.

23. $b\sqrt{a^5b} - (a+b)\sqrt{a^3b^3} + a\sqrt{ab^5} + b\sqrt{a^7b^7}$.

24. $\sqrt[3]{a^3b} + \sqrt[3]{bc^3} + \sqrt[3]{b(a-c)^3} + \sqrt[3]{a^3b^4c^3}$.

25. $\sqrt[3]{54} - \sqrt[3]{128} + \sqrt[3]{250}$.

26. $\sqrt{\frac{1}{2}} + \frac{3}{2}\sqrt{2} - \sqrt{\frac{2}{9}}$.

27. $\frac{2}{3}\sqrt{3} - \sqrt{\frac{3}{25}} + \frac{8}{5}\sqrt{\frac{1}{3}}$.

28. $3\sqrt[3]{\frac{1}{9}} - 2\sqrt[3]{81} + 6\sqrt[3]{3}$.

29. $2a\sqrt[4]{a} - \sqrt[4]{\frac{1}{a^3}} - 8\sqrt[4]{a^5}$.

30. $\sqrt{\frac{1}{32}} - \frac{5}{2}\sqrt{8} + 4\sqrt{\frac{1}{8}}$.

31. $\sqrt{\frac{x-a}{x+a}} + \sqrt{x^2 - a^2} + \sqrt{(x^2-a^2)^3}$.

MULTIPLICATION OF RADICALS

198. The product of any two monomial radicals may be found by applying V, § 196, and I, § 190.

1. Multiply $2\sqrt{5}$ by $3\sqrt[3]{4}$.

$$2\sqrt{5}=2\sqrt[6]{5^3}=2\sqrt[6]{125},$$

$$3\sqrt[3]{4}=3\sqrt[6]{2^4}=3\sqrt[6]{16}.$$

$$2\sqrt{5}\cdot 3\sqrt[3]{4}=6\sqrt[6]{2000}.$$

2. Multiply $3\sqrt[3]{ax^2}$ by $4\sqrt[4]{bx^3}$.

$$3\sqrt[3]{ax^2}=3\sqrt[12]{(ax^2)^4}=3\sqrt[12]{a^4x^8},$$

$$4\sqrt[4]{bx^3}=4\sqrt[12]{(bx^3)^3}=4\sqrt[12]{b^3x^9},$$

$$3\sqrt[3]{ax^2}\cdot 4\sqrt[4]{bx^3}=12\sqrt[12]{a^4b^3x^{17}}=12\sqrt[12]{x^{12}}\sqrt[12]{a^4b^3x^5}=12\,x\sqrt[12]{a^4b^3x^5}.$$

EXERCISE C

Find the products of the following radicals:

1. $\sqrt{a}\cdot\sqrt[4]{b}.$

2. $\sqrt{x}\cdot\sqrt[6]{y}.$

3. $\sqrt[3]{b}\cdot\sqrt[12]{c}.$

4. $\sqrt[3]{5}\cdot\sqrt[6]{2}.$

5. $\sqrt[3]{a}\cdot\sqrt[4]{b}.$

6. $\sqrt{a}\cdot\sqrt[3]{x}.$

7. $\sqrt[4]{a}\cdot\sqrt[6]{b}.$

8. $\sqrt[6]{x}\cdot\sqrt[12]{y}.$

9. $\sqrt[5]{x^6}\cdot\sqrt[20]{x}.$

10. $\sqrt[4]{a^7}\cdot\sqrt[10]{a^3}.$

11. $\sqrt{2}\cdot\sqrt[3]{3}.$

12. $\sqrt[3]{12}\cdot\sqrt[4]{2}.$

13. $\sqrt[4]{x}\cdot\sqrt[6]{x^8}.$

14. $\sqrt[5]{x^7}\cdot\sqrt[10]{y}.$

15. $\sqrt{2\,a^9}\cdot\sqrt[3]{3\,x}.$

16. $\sqrt[4]{a^7x^5}\cdot\sqrt[12]{2\,x}.$

17. $\sqrt[4]{\frac{4}{5}}\cdot\sqrt[6]{\frac{5}{2}}.$

18. $\sqrt[4]{\frac{25}{7}}\cdot\sqrt[3]{\frac{14}{5}}.$

19. $\sqrt{\frac{a}{b}}\cdot\sqrt[4]{\frac{b^3}{a}}.$

20. $\sqrt[3]{\frac{a^5}{b^2}}\cdot\sqrt[9]{\frac{b^8}{a^4}}.$

21. $\sqrt[4]{72}\cdot\sqrt[6]{108}.$

22. $\sqrt[4]{5\,a}\cdot\sqrt[5]{2\,a^6}.$

23. $\sqrt[3]{5\,b^2x}\cdot\sqrt[4]{10\,x}.$

24. $\sqrt[3]{1024}\cdot\sqrt[4]{\frac{1}{16}}.$

25. $\sqrt[4]{1024\,a^3}\cdot\sqrt[5]{2}.$

MULTIPLICATION OF POLYNOMIALS INVOLVING RADICALS

199. The product of two or more polynomials involving radicals is found in the same way that the product of two rational expressions is found; the terms of the polynomials being expressed in their simplest form before the process of multiplication is attempted.

Multiply $\sqrt{60}+\sqrt{24}$ by $\sqrt{27}-\sqrt{8}$.

$$\sqrt{60}+\sqrt{24}=2\sqrt{15}+2\sqrt{6},$$
$$\sqrt{27}-\sqrt{8}\ =3\sqrt{3}\ -2\sqrt{2},$$

$$\begin{array}{l} 2\sqrt{15}+2\sqrt{6} \\ \ \ 3\sqrt{3}-2\sqrt{2} \\ \hline 6\sqrt{45}+6\sqrt{18}-4\sqrt{30}-4\sqrt{12}= \\ 18\sqrt{5}+18\sqrt{2}-4\sqrt{30}-8\sqrt{3}. \end{array}$$

EXERCISE CI

Multiply the following expressions involving radicals:

1. $2a\sqrt{x}-6\sqrt{a^3xy}$ by $2\sqrt{a^3x}+5a\sqrt{y}$.
2. $4\sqrt{8}-\sqrt{32}+2\sqrt{50}$ by $\sqrt{2}$.
3. $\sqrt{75}-\sqrt{150}+2\sqrt{243}$ by $\sqrt{3}$.
4. $\sqrt{\frac{1}{5}}+\sqrt{\frac{1}{6}}+\sqrt{\frac{1}{8}}$ by $\sqrt{5}+\sqrt{6}+\sqrt{8}$.
5. $2\sqrt{27}-3\sqrt{75}+4\sqrt{12}$ by $2\sqrt{3}+5\sqrt{48}+4\sqrt{5}$.
6. $3\sqrt{12}+\sqrt{32}+2\sqrt{80}$ by $4\sqrt{45}-2\sqrt{48}-2\sqrt{18}$.
7. $3\sqrt{27}-5\sqrt{20}-2\sqrt{343}$ by $2\sqrt{48}-3\sqrt{45}-3\sqrt{112}$
8. $3\sqrt{18}+4\sqrt{54}-4\sqrt{75}$ by $4\sqrt{48}+2\sqrt{162}-3\sqrt{150}$
9. $2\sqrt[3]{16}+4\sqrt[3]{24}-3\sqrt[3]{108}$ by $2\sqrt[3]{54}-3\sqrt[3]{81}+4\sqrt[3]{32}$,
10. $4\sqrt[3]{24}+3\sqrt[3]{256}-5\sqrt[3]{135}$ by $3\sqrt[3]{81}-\sqrt[3]{32}+2\sqrt[3]{625}$

DIVISION OF RADICALS

200. The quotient of any two monomial radicals may be found by applying V, § 196, and II, § 193.

Divide $\sqrt[3]{12}$ by $\sqrt{2}$; and $3\sqrt{2x}$ by $2\sqrt[4]{x^3}$.

$$\frac{\sqrt[3]{12}}{\sqrt{2}}=\frac{\sqrt[6]{(2^2\cdot 3)^2}}{\sqrt[6]{2^3}}=\sqrt[6]{\frac{2^4\cdot 3^2}{2^3}}=\sqrt[6]{2\cdot 3^2}=\sqrt[6]{18}.$$

$$\frac{3\sqrt{2x}}{2\sqrt[4]{x^3}}=\frac{3\sqrt[4]{(2x)^2}}{2\sqrt[4]{x^3}}=\frac{3}{2}\sqrt[4]{\frac{2^2x^2}{x^3}}=\frac{3}{2}\sqrt[4]{\frac{2^2x^2}{x^3}\left(\frac{x}{x}\right)}=\frac{3}{2}\sqrt[4]{\frac{1}{x^4}}\sqrt[4]{4x^3}=\frac{3}{2x}\sqrt[4]{4x^3}.$$

EXERCISE CII

Find the quotient of the following radicals:

1. $\sqrt{a}\div\sqrt[4]{b}$.
2. $\sqrt[3]{x^2}\div\sqrt[6]{x}$.
3. $\sqrt[3]{a}\div\sqrt[4]{b}$.
4. $\sqrt{a}\div\sqrt[3]{a}$.
5. $\sqrt[4]{a}\div\sqrt[6]{a}$.
6. $\sqrt[4]{a^3}\div\sqrt[5]{a^4}$.
7. $\sqrt[3]{x^2}\div\sqrt[6]{x^5}$.
8. $\sqrt{30}\div\sqrt[3]{45}$.
9. $a\sqrt{a}\div\sqrt[3]{a^2}$.
10. $\sqrt[6]{\frac{a^5}{bc}}\div\sqrt[9]{\frac{a^7}{b^2c}}$.
11. $3\sqrt{5}\div\sqrt[4]{15}$.
12. $12\sqrt{27}\div\sqrt[3]{9}$.
13. $4\sqrt[3]{49}\div 3\sqrt{7}$.
14. $4\sqrt[3]{100}\div\sqrt{10}$.
15. $3\sqrt[6]{90}\div 2\sqrt[3]{18}$.
16. $3\sqrt{343}\div\sqrt[3]{49}$.
17. $\sqrt[16]{\frac{x^3y^9}{z^7}}\div\sqrt[12]{\frac{x^5y^{11}}{z^{10}}}$.
18. $2\sqrt[3]{686}\div 5\sqrt{56}$.
19. $10\sqrt{216}\div 2\sqrt[3]{36}$.
20. $\sqrt[4]{1024}\div 2\sqrt[3]{192}$.
21. $\sqrt[3]{\frac{(a-b)^2}{(x-y)^2}}\div\sqrt[4]{\frac{(a-b)^3}{(x-y)^7}}$.
22. $\sqrt[4]{\frac{x^5}{(a-b)^6}}\div\sqrt[5]{\frac{x^3}{(a-b)^8}}$.

DIVISION OF POLYNOMIALS INVOLVING RADICALS

201. The quotient of a polynomial involving radicals by a monomial radical may be found in the same way as in rational expressions.

The division of two polynomials involving radicals is best effected by a process called division by means of **rationalization**; by this process the radical denominator is transformed into a rational quantity. The least factor, the product of which and the given radical produces a rational quantity, is called the **rationalizing factor.**

Thus, the rationalizing factors of $\sqrt{2}$, $\sqrt[3]{ax}$, $\sqrt[4]{a^3}$ are respectively $\sqrt{2}$, $\sqrt[3]{a^2x^2}$, $\sqrt[4]{a}$.

Two binomial quadratic surds in the forms $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are called **conjugates** of each other. Since $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=x-y$, the rationalizing factor of a quadratic binomial surd is its conjugate. It is necessary to multiply both the terms of the fraction expressing the quotient, by the rationalizing factor.

1. Rationalize $\dfrac{2}{\sqrt{3}}$.

$$\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2}{3}\sqrt{3}.$$

2. Rationalize $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}\cdot\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{a+2\sqrt{ab}+b}{a-b}.$$

EXERCISE CIII

Rationalize the denominators of the following fractions:

1. $\dfrac{5}{\sqrt{5}}$.

2. $\dfrac{12}{5\sqrt{3}}$.

3. $\dfrac{48}{5\sqrt{32}}$.

4. $\dfrac{a+b}{\sqrt{a+b}}$.

5. $\dfrac{a^2-b^2}{\sqrt{a-b}}$.

6. $\dfrac{a}{\sqrt{\dfrac{a}{b}}}$.

7. $\dfrac{3\sqrt{7}}{7\sqrt{3}}$.

8. $\dfrac{6\sqrt{5}}{5\sqrt{3}}$.

9. $\dfrac{a}{b\sqrt{ab}}$.

10. $\dfrac{4\sqrt{18}}{5\sqrt{12}}$.

11. $\dfrac{1}{3+\sqrt{2}}$.

12. $\dfrac{1}{a+\sqrt{b}}$.

13. $\dfrac{a\sqrt{b}}{\sqrt{b}+c}$.

14. $\dfrac{a}{\sqrt{b}+\sqrt{c}}$.

15. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.

16. $\dfrac{10+\sqrt{21}}{\sqrt{7}+\sqrt{3}}$.

17. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

18. $\dfrac{\sqrt{8}-\sqrt{6}}{\sqrt{8}+\sqrt{6}}$.

19. $\dfrac{a^2-a\sqrt{b}+b}{a-\sqrt{b}}$.

20. $\dfrac{a-b}{\sqrt{a+2b}-\sqrt{3b}}$.

21. $\dfrac{xy}{\sqrt{1-x}+\sqrt{1-y}}$.

22. $\dfrac{a+b}{\sqrt[3]{a}+\sqrt[3]{b}}$.

23. $\dfrac{\sqrt{a}+\sqrt{b}}{a+\sqrt{ab}+b}$.

24. $\dfrac{10+4\sqrt{5}}{1+\sqrt{2}+\sqrt{5}}$.

25. $\dfrac{4}{1+\sqrt{2}+\sqrt{3}}$.

26. $\dfrac{12}{\sqrt{2}-\sqrt{3}+\sqrt{5}}$.

27. $\dfrac{14\sqrt{5}}{2\sqrt{3}+\sqrt{5}+\sqrt{7}}$.

SQUARE ROOT OF A BINOMIAL QUADRATIC SURD

202. The square root of a binomial quadratic surd, in the form of a perfect square, may often be extracted by inspection if it be remembered that the binomial quadratic surd is a disguised form of $a^2 + 2ab + b^2$. Since $(\sqrt{a}+\sqrt{b})^2 = a + 2\sqrt{ab} + b = (a+b) + 2\sqrt{ab}$, the square root of $(a+b)+2\sqrt{ab}$ is found by obtaining two quantities whose sum is $a+b$ and whose product is ab. Since $(\sqrt{2}+\sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$, the square root of $5+2\sqrt{6}$ is $\sqrt{2}+\sqrt{3}$.

The binomial quadratic surd may be still further disguised by the introduction of the coefficient of $\sqrt{ab}$ under the radical, thus: $(a+b)+\sqrt{4ab}$. Before finding the square root it is necessary that the term corresponding to $2\sqrt{ab}$ shall be written with the coefficient 2.

1. Extract the square root of $8+\sqrt{60}$.

$$\sqrt{8+\sqrt{60}} = \sqrt{8+2\sqrt{15}} = \sqrt{3}+\sqrt{5}.$$

2. Extract the square root of $\sqrt{24}+\sqrt{25}$.

$$\sqrt{\sqrt{24}+\sqrt{25}} = \sqrt{2\sqrt{6}+5} = \sqrt{3}+\sqrt{2}.$$

EXERCISE CIV

Extract the square root of the following binomial quadratic surds:

1. $3+2\sqrt{2}$.
2. $4+2\sqrt{3}$.
3. $7+2\sqrt{10}$.
4. $8-2\sqrt{15}$.
5. $6+2\sqrt{5}$.
6. $9-2\sqrt{14}$.
7. $11-2\sqrt{24}$.
8. $11-2\sqrt{28}$.
9. $\sqrt{121}-2\sqrt{10}$.
10. $\sqrt{81}-\sqrt{80}$.
11. $\sqrt{121}-\sqrt{120}$.
12. $\sqrt{64}-\sqrt{28}$.
13. $\sqrt{256}-\sqrt{156}$.
14. $\frac{15}{4}-\frac{1}{2}\sqrt{14}$.
15. $2-\sqrt{3}$.

RADICAL EQUATIONS

203. An equation which involves the indicated root of he unknown is called a **radical** or **irrational equation.**

Thus, $\sqrt{3+x}=2$ is a radical equation.

A radical equation which involves square roots only can ften be solved as a simple equation by isolating one or ore of the radicals and rationalizing the resulting equaion by squaring. But since two equations with different igns may give the same result when squared, the solution btained by solving the squared equation does not necesarily satisfy the given equation. *It is necessary to test the olution in every case by substituting in the given equation.*

If the equation contains a single radical, it is simpler to solate the radical and then square the resulting equation; the equation contains two or more radicals, the more volved radical is isolated. The squared equation should hen be simplified, especial care being taken to reduce he resulting equivalent equation to the simplest integral orm.

1. Solve the equation: $\sqrt{x+6}+\sqrt{x-2}=4.$ (1)

Transposing in (1), $\sqrt{x+6}=4-\sqrt{x-2},$ (2)

quaring (2), $x+6=16-8\sqrt{x-2}+x-2,$ (3)

ransposing and uniting in (3),

$$8\sqrt{x-2}=8, \quad (4)$$

ividing (4) by 8, $\sqrt{x-2}=1,$ (5)

quaring (5), $x-2=1,$ (6)

ransposing and uniting in (6), $x=3.$ (7)

VERIFICATION: $\sqrt{9}+\sqrt{1}=4$; therefore 3 is a root of (1).

2. Solve the equation: $\sqrt{x+6}-\sqrt{x-2}=4.$ (1)

Transposing in (1), $\sqrt{x+6}=4+\sqrt{x-2},$ (2)

squaring (2), $x+6=16+8\sqrt{x-2}+x-2,$ (3)

transposing and uniting in (3),

$8\sqrt{x-2}=-8,$ (4)

dividing (4) by 8, $\sqrt{x-2}=-1,$ (5)

squaring (5), $x-2=1,$ (6)

transposing and uniting in (6), $x=3.$ (7)

Substituting in (1), $\sqrt{9}-\sqrt{1}\neq 4$; therefore 3 is *not* a root of (1).

3. Solve the equation: $(a-b)\sqrt{\dfrac{x}{a-b}}+b=a.$ (1)

Transposing in (1), $(a-b)\sqrt{\dfrac{x}{a-b}}=a-b,$ (2)

dividing (2) by $a-b$, $\sqrt{\dfrac{x}{a-b}}=1,$ (3)

squaring (3), $\dfrac{x}{a-b}=1,$ (4)

multiplying (4) by $a-b$, $x=a-b.$ (5)

VERIFICATION: $(a-b)\sqrt{\dfrac{a-b}{a-b}}+b=a$; $a-b+b=a.$

4. Solve the equation: $\sqrt{2+x}+\sqrt{x-3}=\sqrt{4x-3}.$ (1)

Squaring (1), $2+x+2\sqrt{x^2-x-6}+x-3=4x-3,$ (2)

transposing and uniting in (2), $2\sqrt{x^2-x-6}=2x-2,$ (3)

dividing (3) by 2, $\sqrt{x^2-x-6}=x-1,$ (4)

squaring (4), $x^2-x-6=x^2-2x+1,$ (5)

transposing and uniting in (5), $x=7.$ (6)

VERIFICATION: $\sqrt{9}+\sqrt{4}=\sqrt{25}.$

EXERCISE CV

Solve the following radical equations:

1. $\sqrt{x+5}=3.$
2. $6\sqrt{x+4}=11.$
3. $7=3\sqrt{x}-4.$
4. $\sqrt{7x+2}=4.$
5. $\sqrt{5+x}=5-\sqrt{x}.$
6. $\sqrt{15+x}=3\sqrt{5}-\sqrt{x}.$
7. $\sqrt{2x+11}+\sqrt{2x-5}=8.$
8. $\sqrt{27x+1}=2-3\sqrt{3x}.$
9. $\sqrt{4+x\sqrt{24+x^2}}=x+2.$
10. $\dfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}}=b.$
11. $\sqrt{x}+\sqrt{a+x}=\dfrac{b}{\sqrt{a+x}}.$
12. $\sqrt{x}+\sqrt{3+x}=\dfrac{24}{\sqrt{3+x}}.$
13. $\sqrt{x+4ab}=2a+\sqrt{x}.$
14. $\sqrt{x+a}=a-\sqrt{x-a}.$
15. $b-a\sqrt{x}=\sqrt{a^2x}.$
16. $\dfrac{\sqrt{x+1}}{\sqrt{x+3}}=\dfrac{\sqrt{x+3}}{\sqrt{x+6}}.$
17. $x=a-\sqrt{a^2-x\sqrt{x^2+8a^2}}.$
18. $\sqrt{5+2x}=\sqrt{2(8+9x)}-\sqrt{1+8x}.$
19. $3\sqrt{1+2x}-\sqrt{8x-15}=\sqrt{2(x+6)}.$
20. $\sqrt{9x-14}+3\sqrt{x+2}=2\sqrt{9x-2}.$

REVIEW EXERCISE CVI

Simplify the following expressions:

1. $\sqrt[4]{\frac{4}{7}}$.

2. $\sqrt{\dfrac{x}{a^3}}$.

3. $\sqrt[8]{\dfrac{1}{2a}}$.

4. $\sqrt[3]{\dfrac{a}{3x}}$.

5. $(\sqrt[4]{5})^3$.

6. $\sqrt[3]{40x^3}$.

7. $\sqrt[4]{48a^5}$.

8. $\sqrt[3]{\sqrt[4]{27}}$.

9. $\sqrt[4]{49x^2}$.

10. $\sqrt[5]{64x^7}$.

11. $\sqrt[3]{\sqrt[12]{a^{18}}}$.

12. $\sqrt{3}\cdot\sqrt[3]{9}$.

13. $\sqrt[4]{a^5}\cdot\sqrt{a^2}$.

14. $\sqrt[3]{2}\div\sqrt{14}$.

15. $\sqrt[3]{\frac{1}{7}}-3\sqrt[3]{49}$.

16. $\sqrt[3]{\frac{8}{3}}$.

17. $\sqrt[6]{\dfrac{x}{a^2}}$.

18. $\sqrt[5]{\dfrac{a^9}{x^{10}}}$.

19. $\sqrt[4]{\dfrac{a^3}{3bx}}$.

20. $(\sqrt[8]{a^5})^4$.

21. $\sqrt[5]{-\frac{1}{64}}$.

22. $\sqrt[5]{\sqrt[4]{2x^7}}$.

23. $\sqrt[3]{\sqrt[6]{1024}}$.

24. $\dfrac{x}{\sqrt{2}-\sqrt{x}}$.

25. $\dfrac{\sqrt{3}}{\sqrt{5}-\sqrt{2}}$.

26. $\sqrt[4]{20}\div\sqrt{8}$.

27. $\sqrt[6]{a}\cdot\sqrt[5]{9x}$.

28. $\sqrt[4]{\frac{1}{2}}+\sqrt[4]{168}$.

29. $(\sqrt[5]{ax^7})^3$.

30. $\sqrt[9]{\sqrt[4]{x^6}}$.

31. $\sqrt[6]{\dfrac{2a}{49b}}$.

32. $\sqrt{\dfrac{49x^3}{25a}}$.

33. $\sqrt{\dfrac{a+x}{x-a}}$.

34. $(\sqrt[3]{2a})^2$.

35. $\sqrt[6]{3}\cdot\sqrt[3]{2}$.

36. $\sqrt[4]{\sqrt[3]{36x^5}}$.

37. $\sqrt{a^4}\div\sqrt[3]{a}$.

38. $\sqrt[6]{a^5}\div\sqrt[5]{a^4}$.

39. $\sqrt[3]{21}\div\sqrt[4]{12}$.

40. $\sqrt[6]{2a^9}\div\sqrt[4]{a^7}$.

41. $(\sqrt[3]{-a+b})^4$.

42. $\sqrt{21-4\sqrt{17}}$

43. Solve for x: $\dfrac{5\sqrt{x}+4}{5\sqrt{x}-4}=\dfrac{3}{2}$.

44. $\sqrt{18+12\sqrt{2}}+(2-\sqrt{3})(2+\sqrt{3})$.

CHAPTER XVI

IMAGINARIES

204. An indicated even root of a negative number is called a **pure imaginary quantity.** Quantities which are not imaginary are called **real.** Since $(+2)^2 = (-2)^2 = 4$, $\sqrt{-4} \neq \pm 2$, or *the even root of a negative number is defined as impossible;* that is, $\sqrt{-4}$ can be expressed neither as a positive nor as a negative quantity. Therefore, pure imaginary quantities must be excluded from the number system, which up to this point includes rational and irrational quantities; or the number system must be enlarged to include pure imaginary quantities. Because such quantities are frequently met in the solutions of quadratic equations, Chapter XVIII, and elsewhere, they are included within the number system.

205. It will be assumed that the fundamental laws, the commutative, associative, etc., govern operations of expressions involving imaginary quantities. The proof of these laws, as applied to such expressions, is beyond the province of this book.

206. The **general form** of a pure imaginary is $\sqrt{-a}$. Since $\sqrt{ab} = \sqrt{a}\sqrt{b}$, similarly, $\sqrt{-a} = \sqrt{a}\sqrt{-1}$, the latter being called the **typical form** of a pure imaginary.

The sum or difference of a real and a pure imaginary quantity is called a **complex quantity.**

Thus, $a + \sqrt{-b}$ is a complex quantity.

In all operations with complex quantities the pure imaginary should first be reduced to the typical form.

Thus, $2\sqrt{-4} = 2\sqrt{4}\sqrt{-1} = 4\sqrt{-1}$.

207. By § 175, $(\sqrt{-b})^2 = -b$; or $(\sqrt{-1})^2 = -1$.

The form $\sqrt{-1}$ is generally written in the shorter form i.

The following list gives some of the integral powers of i:

$$i = i, \qquad i^5 = i^4 \cdot i = i,$$

$$i^2 = -1, \qquad i^6 = i^4 \cdot i^2 = -1,$$

$$i^3 = i^2 \cdot i = -i, \qquad i^7 = i^6 \cdot i = -i,$$

$$i^4 = i^2 \cdot i^2 = +1, \qquad i^8 = i^4 \cdot i^4 = +1, \text{ etc.}$$

That is, in general, $i^n = i^{4+n}$.

208. Imaginary and complex quantities can be employed in the various operations.

1. Reduce $\sqrt{-16}$, $\sqrt{-x^6}$, $\sqrt{-4x^5y}$ to the typical form.

$$\sqrt{-16} = \sqrt{16}\, i = 4i,$$

$$\sqrt{-x^6} = \sqrt{x^6}\, i = x^3 i,$$

$$\sqrt{-4x^5y} = \sqrt{2^2x^4}\sqrt{xy}\, i = 2x^2\sqrt{xy}\, i.$$

2. Add $\sqrt{-16}$, $\sqrt{-25}$, and $\sqrt{-36}$.

$$\sqrt{-16} = \sqrt{16}\, i = 4i,$$

$$\sqrt{-25} = \sqrt{25}\, i = 5i,$$

$$\sqrt{-36} = \sqrt{36}\, i = 6i,$$

$$\sqrt{-16} + \sqrt{-25} + \sqrt{-36} = (4 + 5 + 6)i = 15i.$$

3. Multiply $\sqrt{-4}+\sqrt{2}$ by $2\sqrt{-4}+3\sqrt{2}$.

$$\begin{array}{l} 2i+\sqrt{2} \\ \underline{4i+3\sqrt{2}} \\ 8i^2+4\sqrt{2}\,i \\ \underline{\qquad +6\sqrt{2}\,i+6} \\ 8i^2+10\sqrt{2}\,i+6 \end{array}$$

$$=-8+10\sqrt{2}\,i+6=10\sqrt{2}\,i-2.$$

4. Divide 1 by $\sqrt{-2}+\sqrt{-3}$.

$$\frac{1}{\sqrt{2}\,i+\sqrt{3}\,i}\cdot\frac{\sqrt{2}\,i-\sqrt{3}\,i}{\sqrt{2}\,i-\sqrt{3}\,i}=\frac{\sqrt{2}\,i-\sqrt{3}\,i}{2\,i^2-3\,i^2}=\frac{\sqrt{2}\,i-\sqrt{3}\,i}{-2+3},$$

$$=(\sqrt{2}-\sqrt{3})i.$$

5. Expand $(\sqrt{-3}+\sqrt{-5})^2$.

$$(\sqrt{3}\,i+\sqrt{5}\,i)^2=[i(\sqrt{3}+\sqrt{5})]^2=i^2(\sqrt{3}+\sqrt{5})^2$$

$$=-1(3+2\sqrt{15}+5)=-8-2\sqrt{15}.$$

6. Extract the square root of $1+2\sqrt{-6}$.

$$\sqrt{1+2\sqrt{6}\,i}=\sqrt{2}\,i+\sqrt{3}.$$

EXERCISE CVII

Reduce the following pure imaginary quantities to the typical forms:

1. $\sqrt{-25}$. **4.** $\sqrt{-a^2}$. **7.** $\sqrt{-225\,y^4}$.

2. $\sqrt{-36}$. **5.** $\sqrt{-x^2y^2}$. **8.** $\sqrt{-484\,x^8}$.

3. $\sqrt{-100}$. **6.** $\sqrt{-4\,x^6y^3}$. **9.** $\sqrt{-625\,x^8y^2}$.

Simplify the following expressions:

10 $\sqrt{-16}+\sqrt{-25}+\sqrt{-64}-5\sqrt{-100}$.

11. $\sqrt{-49}+\sqrt{-121}-\sqrt{-169}-\sqrt{-196}$.

12. $\sqrt{-x^2}-\sqrt{-4x^2}-\sqrt{-9x^2}+\sqrt{-25x^2}$.

13. $3+\sqrt{-4}+5\sqrt{-16}+16+7\sqrt{-225}$.

14. $a+b\sqrt{-x^2}+2a-b\sqrt{-x^2}+3a-4b\sqrt{-x^2}$.

15. $\sqrt{-4}\cdot\sqrt{-9}$.

16. $\sqrt{-25}\cdot\sqrt{-36}$.

17. $\sqrt{-9}\cdot\sqrt{-16}$.

18. $\sqrt{-\frac{1}{4}}\cdot\sqrt{-\frac{1}{9}}$.

19. $-\sqrt{-x^2}\cdot\sqrt{-x^4}$.

20. $\sqrt{-a^3b^2}\cdot\sqrt{-ab^4}$.

21. $(\sqrt{-a}+\sqrt{-b})(\sqrt{-a}-\sqrt{-b})$.

22. $(\sqrt{-x}+\sqrt{-y})(2\sqrt{-x}+3\sqrt{-y})$.

23. $\sqrt{-a^3b}\cdot\sqrt{-a^2b^2}\cdot\sqrt{-ab^3}$.

24. $\sqrt{-2}\cdot\sqrt{-3}\cdot\sqrt{-6}$.

25. $(1+\sqrt{-4})^2$.

26. $(\sqrt{-x}+\sqrt{-y})^2$.

27. $(2\sqrt{-9}+3\sqrt{-4})^2$.

28. $(3\sqrt{-9}-4\sqrt{-4})^2$.

29. $(1-\sqrt{-4})^3$.

30. $(\sqrt{-9}+1)^3$.

31. $(1+\sqrt{-16})^4$.

32. $(1+\sqrt{-1})^5$.

33. $(\sqrt{-\frac{1}{4}}+\sqrt{-\frac{1}{9}})^3$.

34. $(\sqrt{-\frac{1}{16}}-\sqrt{-\frac{1}{9}})^3$.

35. $(\sqrt{-4}+\sqrt{-9}+\sqrt{-16})^2$.

36. $(\sqrt{-x^2}-\sqrt{-y^2}-\sqrt{-z^2})^2$.

37. $\dfrac{\sqrt{-4}}{\sqrt{-9}}$.

38. $\dfrac{1}{\sqrt{-16}}$.

39. $\dfrac{3}{\sqrt{-9}}$.

40. $\dfrac{9}{\sqrt{-9}}$.

41. $\dfrac{\sqrt{a}}{\sqrt{-a}}$.

42. $\dfrac{\sqrt{x}}{\sqrt{-x^3}}$.

43. $\dfrac{\sqrt{-49}}{-\sqrt{-25}}$.

44. $\dfrac{1}{1+\sqrt{-2}}$.

45. $\dfrac{2}{2+\sqrt{-9}}$.

46. $\dfrac{a+\sqrt{-b}}{a-\sqrt{-b}}$.

47. $\dfrac{a-\sqrt{-b}}{a+\sqrt{-b}}$.

48. $\dfrac{1}{\sqrt{2}+\sqrt{-2}}$.

49. $\dfrac{5+\sqrt{-3}}{1+\sqrt{-1}}.$

50. $\dfrac{4-3\sqrt{-5}}{3+2\sqrt{-5}}.$

51. $\dfrac{3-5\sqrt{-3}}{5+3\sqrt{-3}}.$

52. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{-5}}.$

53. $\dfrac{\sqrt{-3}}{\sqrt{3}+\sqrt{7}+\sqrt{-10}}.$

54. $\dfrac{\sqrt{2}+\sqrt{-3}}{1+\sqrt{2}+\sqrt{-3}}.$

55. $(2\sqrt{3}-\sqrt{-5})(4\sqrt{3}-2\sqrt{-5}).$

56. $(x-5+2\sqrt{-1})(x-5-2\sqrt{-1}).$

57. $(\sqrt{x}-2+\sqrt{-3})(\sqrt{x}+2-\sqrt{-3}).$

58. $(x-2\sqrt{5}+3\sqrt{-5})(x-2\sqrt{5}-3\sqrt{-5}).$

59. $(2-\sqrt{-3}-3\sqrt{-2})(4\sqrt{-3}+6\sqrt{-2}).$

60. $(x-\frac{1}{2}+\frac{1}{2}\sqrt{-3})(x-\frac{1}{2}-\frac{1}{2}\sqrt{-3}).$

61. $(x-2-\sqrt{3})(x-2+\sqrt{3})(x-3+\sqrt{-1})$
$(x-3-\sqrt{-1}).$

62. $(x-1-\sqrt{-2})(x-1+\sqrt{-2})(x-2+\sqrt{-3})$
$(x-2-\sqrt{-3}).$

63. $\sqrt{1+\sqrt{-1}}\cdot\sqrt{1-\sqrt{-1}}\cdot\sqrt{3-\sqrt{-2}}\cdot\sqrt{3+\sqrt{-2}}.$

64. $-\dfrac{5}{4-\sqrt{-4}}+\dfrac{3}{1+\sqrt{-1}}+\dfrac{4}{1-\sqrt{-1}}.$

65. $(\sqrt{-1})^2+(\sqrt{-1})^3+(\sqrt{-1})^4+(\sqrt{-1})^5$
$+(\sqrt{-1})^6+(\sqrt{-1})^7+(\sqrt{-1})^8.$

CHAPTER XVII

THEORY OF EXPONENTS

THE EXPONENT IN THE FORM OF A POSITIVE FRACTION

209. In § 177 it was shown, if m and n are integers and n is a multiple of m, that $\sqrt[m]{a^n} = a^{\frac{n}{m}}$. If, however, n is not an exact multiple of m, there can be no meaning attached to $a^{\frac{n}{m}}$ according to the previous definition, § 7, of an exponent. Thus, it is impossible to speak of $a^{\frac{3}{4}}$ as meaning a taken three-fourths of a time as a factor. The definition of an exponent is therefore extended to include the exponent $\dfrac{n}{m}$, it being understood that $a^{\frac{n}{m}}$ (where n and m are positive integers and a is a positive real quantity) is simply an alternative way of writing $\sqrt[m]{a^n}$, or the principal value of the mth root of the nth power of a.

This extension of the definition of an exponent is valid only in case exponents in the form of a positive fraction conform to the laws of exponents which have been shown to hold for positive integers. That is, exponents in the form of a positive fraction must be shown to obey the laws,

$$a^x a^y = a^{x+y}, \qquad \text{I}$$

$$a^x \div a^y = a^{x-y}, \qquad \text{II}$$

$$(a^x)^y = a^{xy}, \qquad \text{III}$$

$$(ab)^x = a^x b^x. \qquad \text{IV}$$

210. I. $a^{\frac{n}{m}} a^{\frac{r}{s}} = a^{\frac{n}{m}+\frac{r}{s}}.$

By definition, $a^{\frac{n}{m}} a^{\frac{r}{s}} = \sqrt[m]{a^n}\sqrt[s]{a^r},$

by V, § 196, $= \sqrt[ms]{a^{ns}}\sqrt[ms]{a^{mr}},$

by I, § 190, $= \sqrt[ms]{a^{ns+mr}},$

by definition, $= a^{\frac{ns+mr}{ms}},$

or, $= a^{\frac{n}{m}+\frac{r}{s}}.$

II. $a^{\frac{n}{m}} \div a^{\frac{r}{s}} = a^{\frac{n}{m}-\frac{r}{s}}.$

By definition, $a^{\frac{n}{m}} \div a^{\frac{r}{s}} = \sqrt[m]{a^n} \div \sqrt[s]{a^r},$

by V, § 196, $= \sqrt[ms]{a^{ns}} \div \sqrt[ms]{a^{mr}},$

by II, § 193, $= \sqrt[ms]{a^{ns-mr}},$

by definition, $= a^{\frac{ns-mr}{ms}},$

or, $= a^{\frac{n}{m}-\frac{r}{s}}.$

III. $(a^{\frac{n}{m}})^{\frac{r}{s}} = a^{\frac{nr}{ms}}.$

By definition, $a^{\frac{n}{m}} = \sqrt[m]{a^n},$

by conditions, $(a^{\frac{n}{m}})^{\frac{r}{s}} = (\sqrt[m]{a^n})^{\frac{r}{s}},$

by definition, $= \sqrt[s]{(\sqrt[m]{a^n})^r},$

by III, § 194, $= \sqrt[s]{\sqrt[m]{a^{nr}}},$

by IV, § 195, $= \sqrt[ms]{a^{nr}},$

by definition, $= a^{\frac{nr}{ms}}.$

IV. $(ab)^{\frac{n}{m}} = a^{\frac{n}{m}} b^{\frac{n}{m}}$.

By definition, $(ab)^{\frac{n}{m}} = \sqrt[m]{(ab)^n}$,

by III, § 194, $= \sqrt[m]{a^n b^n}$,

by I, § 190, $= \sqrt[m]{a^n}\sqrt[m]{b^n}$,

by definition, $= a^{\frac{n}{m}} b^{\frac{n}{m}}$.

It may also be shown that $a^{\frac{n}{m}} = a^{\frac{kn}{km}}$.

By definition, $a^{\frac{n}{m}} = \sqrt[m]{a^n}$,

by V, § 196, $= \sqrt[km]{a^{kn}}$,

by definition, $= a^{\frac{kn}{km}}$.

A NEGATIVE INTEGER AS EXPONENT

211. If m and n are integers and a is a positive real quantity, the quotient of $a^m \div a^n$ is a^{m-n}. If $m > n$, there is no difficulty: but if $m < n$, a quotient with a negative integer as exponent is obtained; as, $-p$, where $p = n - m$, or $-p = m - n$. Such a quantity as a^{-p} has no meaning according to the original definition of an exponent. But it is convenient to extend still further that definition, and to speak of $-p$ as an exponent.

If $p = n - m$, $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n} = a^{-p}$; if the numerator and denominator of the fraction $\dfrac{a^m}{a^n}$ be divided by a^m, the quotient is evidently $\dfrac{1}{a^{n-m}} = \dfrac{1}{a^p}$. Hence $a^{-p} = \dfrac{1}{a^p}$; *a quantity with a negative integral exponent is equal to the reciprocal of the same quantity with an equal positive exponent.*

212. It is necessary to show that negative integral exponents conform to the laws of exponents for positive integers.

I. $$a^{-p}a^{-q} = a^{-p-q}.$$

$$a^{-p}a^{-q} = \frac{1}{a^p} \cdot \frac{1}{a^q} = \frac{1}{a^{p+q}} = a^{-p-q}.$$

II. $$a^{-p} \div a^{-q} = a^{-p+q}.$$

$$a^{-p} \div a^{-q} = \frac{1}{a^p} \div \frac{1}{a^q} = \frac{a^q}{a^p} = a^{-p+q}.$$

III. $$(a^{-p})^{-q} = a^{pq}.$$

$$(a^{-p})^{-q} = \frac{1}{(a^{-p})^q} = \frac{1}{\left(\frac{1}{a^p}\right)^q} = \frac{1}{\frac{1}{a^{pq}}} = a^{pq}.$$

IV. $(ab)^{-p} = a^{-p}b^{-p}$.

$$(ab)^{-p} = \frac{1}{(ab)^p} = \frac{1}{a^pb^p} = \frac{1}{a^p} \cdot \frac{1}{b^p} = a^{-p}b^{-p}.$$

THE EXPONENT IN THE FORM OF A NEGATIVE FRACTION

213. If m is a positive integer and n is a negative integer which is not an exact multiple of m, a being a real quantity, $a^{-\frac{n}{m}}$ may be defined as the alternative form of $\sqrt[m]{a^{-n}}$.

Exponents in the form of a negative fraction must be shown to conform to the index laws for exponents in the form of positive fractions. Such exponents are made positive by application of § 211; and hence, by § 210, obey the index laws.

214. It now remains to prove in general that

$$\sqrt[m]{a^n} = a^{\frac{n}{m}}. \qquad \text{V}$$

A negative integral exponent and a negative fractional exponent can be made positive, respectively, by §§ 211 and 213. It is therefore necessary to prove V only for positive fractional exponents.

V. $$\sqrt[m]{a^{\frac{r}{s}}} = a^{\frac{r}{ms}}.$$

By definition, $$\sqrt[m]{a^{\frac{r}{s}}} = \left(a^{\frac{r}{s}}\right)^{\frac{1}{m}},$$

by III, $$= a^{\frac{r}{ms}}.$$

The index laws hold when one exponent is integral and the other is fractional, since the integer may be written as a fraction whose denominator is 1.

ZERO AS EXPONENT

215. The case of a zero exponent naturally arises when any quantity is divided by itself:

$$\frac{a^n}{a^n} = a^{n-n} = a^0; \text{ but } \frac{a^n}{a^n} = 1. \quad \text{Hence } \boldsymbol{a^0 = 1}.$$

It is seen that *the value which must be attached to any finite quantity with the zero exponent is unity.*

Zero as an exponent may be shown to conform to the ordinary laws of exponents.

NOTE. It has been shown in the preceding articles that any rational quantity can be used as an exponent. Examples in which irrational numbers are used as exponents are given in Chapter XXV.

EXAMPLES

1. $a^{\frac{3}{4}} = \sqrt[4]{a^3}$.

2. $a^{\frac{2}{3}} \cdot a^{-\frac{3}{4}} \cdot a^{-1} \cdot a^{\frac{3}{2}} = a^{\frac{8-9-12+18}{12}} = a^{\frac{5}{12}}$.

3. $25^{\frac{1}{2}} \cdot 8^{-\frac{2}{3}} = (5^2)^{\frac{1}{2}} \cdot (2^3)^{-\frac{2}{3}} = 5 \cdot 2^{-2} = 5 \cdot \frac{1}{2^2} = 5 \cdot \frac{1}{4} = \frac{5}{4}$.

4. $\left(\frac{16\, a^0 b^{-2}}{c^{-6}}\right)^{\frac{1}{4}} = \left(\frac{2^4 \cdot 1 \cdot b^{-2}}{c^{-6}}\right)^{\frac{1}{4}} = \left(\frac{2^4 c^6}{b^2}\right)^{\frac{1}{4}} = \frac{2\, c^{\frac{3}{2}}}{b^{\frac{1}{2}}}$.

5. $\left(\frac{3^{-3} a^{-\frac{3}{2}} b^6}{64}\right)^{-\frac{1}{3}} = \left(\frac{3^{-3} a^{-\frac{3}{2}} b^6}{2^6}\right)^{-\frac{1}{3}} = \frac{3\, a^{\frac{1}{2}} b^{-2}}{2^{-2}} = \frac{3 \cdot 2^2 a^{\frac{1}{2}}}{b^2} = \frac{12\, a^{\frac{1}{2}}}{b^2}$.

EXERCISE CVIII

Change each of the following expressions into radicals:

1. $a^{\frac{1}{2}}$.
2. $a^{\frac{3}{4}}$.
3. $x^{0.3}$.
4. $9^{\frac{3}{2}}$.
5. $a^{\frac{1}{n}}$.
6. $a^{-\frac{4}{3}}$.
7. $36^{-\frac{1}{2}}$.
8. $(\frac{4}{25})^{-\frac{3}{2}}$.
9. $3\, a^{\frac{1}{3}} + b^{\frac{1}{3}}$.
10. $5\, b^{-\frac{7}{8}} - 2\, c^{\frac{1}{4}}$.
11. $(x+y)^{\frac{1}{4}}$.
12. $3(a + 2\, b)^{-\frac{2}{5}}$.

Change each of the following radicals into expressions containing exponents in the form of fractions:

13. $\sqrt[3]{a^2}$.
14. $\sqrt[n]{a}$.
15. $\sqrt[m]{2\, x^p}$.
16. $7\, a\sqrt[4]{b^3}$.
17. $\sqrt[3]{a+b}$.
18. $\sqrt{3\, a^{-1}}$.
19. $\sqrt[3]{4\, x^{-4}}$.
20. $\sqrt[n]{9\, a^{-x}} + \sqrt[m]{b^{-n}}$.
21. $\sqrt[5n]{4\, x^{-10}}$.
22. $\sqrt[4]{a^{p-q}}$.
23. $\sqrt[4]{x-y}$.
24. $\sqrt[3]{(a+b)^{-2}}$.

Free each of the following expressions from negative exponents:

25. a^{-2}.

26. $3\,a^{-\frac{1}{2}}$.

27. $9\,c^{-6}$.

28. $2\,a^{-5}b^{2}$.

29. $\dfrac{5\,a^{0}b^{-3}}{2\,c^{-5}}$.

30. $\dfrac{16^{-2}\,x^{-\frac{5}{2}}y^{2}}{4\,a^{-\frac{3}{2}}b^{-\frac{2}{3}}}$.

31. $\dfrac{2\,a^{-1}b^{-3}c^{-4}}{5\,x^{-2}y^{-4}z^{-6}}$.

32. $\dfrac{a^{0}c^{-5}d^{-\frac{5}{3}}}{9\,x^{-\frac{3}{4}}y^{-\frac{1}{m}}}$.

33. $\dfrac{25^{-2}x(a-b)^{-3}}{4\,a^{-2}(x-y)^{-4}}$.

Find the value of each of the following expressions:

34. $9^{\frac{3}{2}}$.

35. $64^{\frac{5}{6}}$.

36. $(100^{2})^{\frac{1}{2}}$.

37. $81^{\frac{3}{4}}$.

38. $3(49)^{\frac{1}{2}}$.

39. $2(81)^{\frac{3}{4}}$.

40. $2 \cdot 16^{-\frac{1}{4}}$.

41. $216^{-\frac{2}{3}}$.

42. $243^{-\frac{3}{5}}$.

43. $256^{-\frac{3}{4}}$.

44. $64^{-\frac{2}{3}}$.

45. $(27+5)^{-\frac{4}{5}}$.

Find the product of each of the following expressions:

46. $a^{8} \cdot a^{-2}$.

47. $x^{-7} \cdot x^{-10}$.

48. $3\,x^{-11} \cdot 4\,x^{0}$.

49. $a \cdot a^{-1}$.

50. $5\,b^{m} \cdot 4\,b^{-m}$.

51. $a^{m-3} \cdot a^{3-2m}$.

52. $(3\,a)^{-5} \cdot (3\,a)^{5}$.

53. $5\,x \cdot 5\,x^{-1}$.

54. $(a+b)^{-3} \cdot (a+b)^{5}$.

55. $(-a)^{-m} \cdot (-a)^{4+m}$.

56. $(-ax)^{-m-n} \cdot (-ax)^{m-n}$.

57. $4^{n+1} \cdot 2^{2-m}$.

Find the quotient of each of the following expressions:

58. $b^{-6} \div b^{-8}$.

59. $x^{-5} \div x$.

60. $a^{-7} \div a^{0}$.

61. $a^{-m+1} \div a^{-m-1}$.

62. $4\,x \div 3\,x^{1-n}$.

63. $a^{-2m-5n} \div a^{-m-6n}$.

64. $a^{-4m}b^{4-m} \div a^{m-1}b^{4}$.

65. $-a \div 3\,a^{-4}$.

66. $7\,a^{-2x}b^{3y} \div c^{m}d^{-n}$.

67. $8\,a^{-x}b^{-y^{2}} \div a^{3x}b^{5y^{2}}$.

216. The index laws which apply to monomials apply to the terms of a polynomial.

1. Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}}$ by $a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}}$.

$$\begin{array}{lllll}
a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} & & & & \\
a^{\frac{1}{3}} + 1 \;\; - a^{-\frac{1}{3}} & & & & \\
\hline
a - a^{\frac{2}{3}} + a^{\frac{1}{3}} - 1 & & & & \\
\qquad\quad a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} & & & & \\
\qquad\qquad\;\; - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}} & & & & \\
\hline
a \qquad\; - a^{\frac{1}{3}} + 1 - 2a^{-\frac{1}{3}} + a^{-\frac{2}{3}} = a - a^{\frac{1}{3}} + 1 - \dfrac{2}{a^{\frac{1}{3}}} + \dfrac{1}{a^{\frac{2}{3}}} & & & &
\end{array}$$

In this chapter, unless the contrary is stated, results are to be written with exponents in the form of positive integers or positive fractions.

2. Divide $x^2\sqrt[3]{y^{-4}} - 2 + x^{-1}\sqrt[3]{y^2}$ by $\sqrt{x}\sqrt[3]{y^{-1}} - \sqrt{x^{-1}}\sqrt[3]{y}$.

$$\begin{array}{l|l}
x^2y^{-\frac{4}{3}} - 2 + x^{-1}y^{\frac{2}{3}} & x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}} \\
\cline{2-2}
x^2y^{-\frac{4}{3}} - xy^{-\frac{2}{3}} & x^{\frac{3}{2}}y^{-1} + x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}} = \\
\cline{1-1}
\quad xy^{-\frac{2}{3}} - 2 + x^{-1}y^{\frac{2}{3}} & \quad \dfrac{x^{\frac{3}{2}}}{y} + \dfrac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} - \dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{2}}} \\
\quad xy^{-\frac{2}{3}} - 1 & \\
\cline{1-1}
\qquad\quad -1 + x^{-1}y^{\frac{2}{3}} & \\
\qquad\quad -1 + x^{-1}y^{\frac{2}{3}} &
\end{array}$$

3. Extract the square root of $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

$$\begin{array}{l|r|l}
 & 9x \quad -12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1} & 3x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}} = \\
 & 9x \qquad\qquad\qquad\qquad\qquad\qquad & 3x^{\frac{1}{2}} - 2 + \dfrac{1}{x^{\frac{1}{2}}} \\
\hline
6x^{\frac{1}{2}} - 2 & -12x^{\frac{1}{2}} + 4 \qquad\qquad\qquad & \\
\hline
 & 6 - 4x^{-\frac{1}{2}} + x^{-1} & \\
6x^{\frac{1}{2}} - 4 + x^{-\frac{1}{2}} & 6 - 4x^{-\frac{1}{2}} + x^{-1} & \\
\hline
\end{array}$$

EXERCISE CIX

Find the products of the following expressions:

1. $(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} - y^{\frac{1}{4}})$.

2. $(x^{\frac{3}{4}} + 3x)(x^{\frac{3}{4}} - 3x)$.

3. $(a^{m-1}b + a^m b^{-2} - a^{n-2}b^{-1})ab$.

4. $(x + x^{-1} - 1)(x - x^{-1} + 1)$.

5. $(x^{\frac{1}{3}} - y^{\frac{2}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}})$.

6. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}})$.

7. $(2a^{-2} + 3a^{-1} - 5)(2a^{-2} - 3a^{-1} + 5)$.

8. $(5x^{p-3}y^{r+3} - 2x^{p-1}y^{r+1} - x^{p-2}y^{r+2})$
$(3x^{p+4}y^{r-1} + 4x^{p+5}y^{r-2} - x^{p+3}y^r)$.

Find the quotients of the following expressions:

9. $(x^{2n} - y^{2n}) \div (x^n + y^n)$.

10. $(a^{-3n} - b^{6n}) \div (a^{-n} - b^{2n})$.

11. $(x^{\frac{2}{3}} + x^{\frac{1}{3}} - 6) \div (x^{\frac{1}{3}} - 2)$.

12. $(x^2 + x^{-2} - 2) \div (x - x^{-1})$.

13. $(x^{\frac{3}{4}} - y^{\frac{3}{4}}) \div (x^{\frac{1}{4}} - y^{\frac{1}{4}})$.

14. $(p^{\frac{1}{2}} - q) \div (p^{\frac{1}{10}} - q^{\frac{1}{5}})$.

15. $(a^{\frac{3}{4}} - b^{\frac{3}{4}}) \div (a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}})$.

16. $(x^3 - y^2) \div (x^{\frac{1}{2}} + y^{\frac{1}{3}})$.

17. $(2a^5b^{-3} - 5a^4b^{-2} + 7a^3b^{-1} - 5a^2 + 2ab)$
$\div (a^3b^{-3} - a^2b^{-2} + ab^{-1})$.

18. $(\sqrt[3]{x^4} - 4xy + 4y\sqrt[3]{x^2} + 4y^2) \div (\sqrt[3]{x^2} + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + 2y)$.

Extract the square root of the following expressions:

19. $x^{-8} - 6x^{-5} + 11x^{-2} - 6x + x^4$.

20. $4x^n + 9x^{-n} + 28 - 24x^{-\frac{n}{2}} - 16x^{\frac{n}{2}}$.

21. $1 + 4x^{-\frac{1}{3}} - 2x^{-\frac{2}{3}} - 4x^{-1} + 25x^{-\frac{4}{3}} - 24x^{-\frac{5}{3}} + 16x^{-2}$.

217. By the principles of the preceding articles many expressions may be simplified.

1. $$ab \cdot \frac{a^{-1}-b^{-1}}{(a-b)^{-1}} = ab \cdot (a-b)\left(\frac{1}{a}-\frac{1}{b}\right) = ab\,(a-b)\left(\frac{b-a}{ab}\right)$$
$$= -(a-b)^2.$$

2. $$(a^{n^2-1})^{\frac{n}{n+1}} + \frac{\sqrt[n]{a^{2n}}}{a} = a^{n(n-1)} + \frac{a^2}{a} = a^{n^2-n} + a = \frac{a^{n^2}}{a^n} + a.$$

3. $$\frac{2^n \cdot (2^{n-1})^n}{2^{n+1} \cdot 2^{n-1}} = \frac{2^n \cdot 2^{n^2-n}}{2^{2n}} = \frac{2^{n^2}}{2^{2n}}.$$

EXERCISE CX

Simplify each of the following expressions, giving each result in a form free from radicals and from negative exponents:

1. $\left(\dfrac{4\,a^{-\frac{1}{4}}x^2}{9\,b^{-3}y^{\frac{1}{2}}}\right)^2.$

2. $\dfrac{\sqrt{a^{-\frac{5}{3}}b^3c^{-\frac{2}{3}}}}{\sqrt[3]{a^{\frac{1}{2}}b^4c^{-1}}}.$

3. $\left(\dfrac{8\,a^2}{27\,a^{-3}y^{\frac{1}{3}}}\right)^{-\frac{1}{3}}.$

4. $\dfrac{\sqrt{xa^{-1}}}{b^{\frac{1}{2}}y^{-\frac{3}{4}}} \div \dfrac{xa^{-\frac{3}{2}}}{b^2y^{-\frac{1}{2}}}.$

5. $\left[\dfrac{(-a)^{-\frac{5}{2}} \cdot a^{\frac{7}{2}}}{(-b)^2}\right]^2.$

6. $(\sqrt[b]{x^b} \div \sqrt[a]{x})^{\frac{1}{1-a}}.$

7. $(64\sqrt[3]{36\,a^4b^{-2}})^{\frac{1}{2}}.$

8. $\dfrac{a^{2n+1}b^{n-1}}{a^{n+2}b^{1-n}}.$

9. $\dfrac{4^{\frac{3}{4}} \cdot 3^{\frac{1}{2}} \cdot 2^{\frac{1}{3}}}{2^{\frac{3}{4}} \cdot 9^{\frac{1}{4}} \cdot 4^{\frac{1}{2}}}.$

10. $\dfrac{2\,x^2y^{-\frac{1}{2}}}{3\,a^2b^{-4}} \cdot \dfrac{6\,a^{-5}b^{\frac{1}{2}}}{7\,x^3y^{\frac{3}{4}}}.$

11. $(81\,a^4x^{-2}y^{\frac{2}{3}}z^{-\frac{1}{6}})^{\frac{1}{2}}.$

12. $a^{\frac{1}{2}}y^{\frac{1}{3}} \cdot \left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^2 \div \dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}.$

13. $\dfrac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{x^{-\frac{2}{3}}} \div \dfrac{y^{-1}z^2}{x^{\frac{1}{2}}} \cdot \dfrac{z^3}{\sqrt{xy^{-\frac{3}{2}}}}.$

14. $(x^{q-r})^p \cdot (x^{r-p})^q \cdot (x^{p-q})^r.$

15. $\dfrac{(a^{p-q})^{p+q}\cdot(a^q)^{q+r}}{(a^p)^{p-q}}$. **17.** $\left(\dfrac{a^{-2}b}{a^3b^{-4}}\right)^{-3}\div\left(\dfrac{ab^{-1}}{a^{-3}b^2}\right)^5$.

16. $\dfrac{\sqrt{a^{-4}b^6}\cdot\sqrt{a}\cdot\sqrt{b^{-3}}}{a^2b^{-3}}$. **18.** $\sqrt{\left[\sqrt[3]{x^2}\cdot\sqrt{\left(\dfrac{\sqrt{x}}{\sqrt[3]{x}}\right)^5}\,\right]^3}$.

19. $\left\{\dfrac{a^{p-q}}{\sqrt[q]{a^{q^2-pq}}}\cdot x^{2(p-q)}\right\}^n$.

20. $\dfrac{2^{n+1}\cdot 2^{2n}}{(2^n)^{n+1}}\div\dfrac{4^{n+1}}{(2^{n-1})^{n+1}}$.

21. $\dfrac{\{(a^3)^{\frac{1}{13}}\cdot(a^5)^{\frac{1}{7}}\}^{91}}{\{(b^{\frac{1}{5}})^7\cdot(b^{\frac{1}{3}})^{13}\}^{15}}\div\left(\dfrac{a^5}{b^5}\right)^{13}$.

22. $\dfrac{10\sqrt[3]{a^2}}{3\sqrt[4]{b^5}}\cdot\dfrac{9\,b^{-3}}{\sqrt{5}}\div\sqrt{\dfrac{5\,a\sqrt[3]{a^2}}{4\sqrt[5]{a^2b^9}}}$.

23. $x^3-1+\dfrac{\frac{1}{3}}{x^{\frac{1}{3}}-1}+\dfrac{1}{x^{\frac{1}{3}}+\dfrac{1}{x^{\frac{1}{3}}+1}}$.

24. $\dfrac{\{(a^m)^{\frac{1}{r}}\cdot(a^q)^{\frac{1}{n}}\}^{nr}}{\{\sqrt[q]{b^n}\cdot(\sqrt[m]{b})^r\}^{mq}}\div\left\{\left(\dfrac{a}{b}\right)^q\right\}^r$.

25. $\dfrac{x}{x^{\frac{1}{3}}-1}-\dfrac{x^{\frac{2}{3}}}{x^{\frac{1}{3}}+1}-\dfrac{1}{x^{\frac{1}{3}}-1}+\dfrac{1}{x^{\frac{1}{3}}+1}$.

26. $\dfrac{3\,a}{2\,b^2}\cdot\sqrt{\left(\dfrac{b}{\sqrt[6]{16\,a}}\right)^3}\div\left\{\dfrac{1}{12\,b\sqrt{a}}\left(\dfrac{a}{\sqrt[3]{b}}\right)^2\right\}$.

27. $\dfrac{[(x^2)^{\frac{1}{7}}\cdot(x^3)^{-\frac{1}{2}}]^{-14}\cdot[(y^{\frac{1}{3}})^{-2}\cdot(y^{\frac{1}{2}})^7]^{-6}}{(x^3y^{-3})^7}$.

REVIEW EXERCISE CXI

Simplify the following expressions :

1. $\sqrt[4]{49}$.
2. $\sqrt[5]{a^{10}}$.
3. $a^0 b^{-m}$.
4. $\sqrt[6]{144}$.
5. $\sqrt[8]{256}$.
6. $\sqrt[3]{8\,x^6}$.
7. $\sqrt[3]{a^6 x^{12}}$.
8. $\sqrt[3]{a^2 b^6 c^9}$.
9. $\sqrt{\dfrac{a+b}{a-b}}$.
10. $(3\sqrt[4]{\frac{1}{9}})^2$.
11. $\frac{1}{3}x^{\frac{2}{3}} \cdot 6\,x^{\frac{3}{2}}$.
12. $\sqrt{18} \cdot \sqrt{8}$.
13. $\sqrt{32\,a^{2m-1}b}$.
14. $\sqrt[3]{3\,a} \cdot \sqrt{2\,b}$.
15. $a^{\frac{3}{2}}$.
16. $\sqrt[10]{4}$.
17. $81^{\frac{3}{2}}$.
18. $\dfrac{a^a}{a^{-2}}$.
19. $\sqrt[3]{a^2}$.
20. $\dfrac{4\sqrt{6}}{3\sqrt{2}}$.
21. $(x^{\frac{m}{n}})^{\frac{p}{q}}$.
22. $(a+b)^0$.
23. $(2\sqrt{a^3})^2$.
24. $\sqrt{2} \cdot \sqrt[3]{3}$.
25. $\sqrt[4]{36\,a^2 b^2}$.
26. $\sqrt{4\,a^2 b^4 c^8}$.
27. $(a^{2n+n^2})^{\frac{1}{3n}}$.
28. $(2\,ax^2 y^{\frac{3}{2}})^2$.
29. $\sqrt{\dfrac{a^0 b^4}{c^6 d^2}}$.
30. $\dfrac{7\sqrt{48}}{3\sqrt{27}}$.
31. $\left(\dfrac{a^5}{a^{-3}}\right)^{\frac{1}{2}}$.
32. $\dfrac{a+x}{\sqrt{a+x}}$.
33. $\dfrac{\sqrt{50\,a^6 b^4}}{\sqrt{2\,a^2 b^2}}$.
34. $\sqrt{a^m} \cdot \sqrt[3]{b^n}$.
35. $\sqrt[3]{\dfrac{a}{bc}} \cdot \sqrt{\dfrac{a}{bd}}$.
36. $\sqrt{12} \cdot \sqrt[3]{72}$.
37. $\sqrt{108} - \sqrt{72}$.
38. $\sqrt[3]{54} + \sqrt[3]{128}$.
39. $\sqrt[3]{ab^{-1}c^{-2}} \cdot (a^{-1}b^{-2}c^{-4})^{-\frac{1}{6}}$.
40. $(2-\sqrt{-5})(3+4\sqrt{-5})$.
41. $x^{3p+q} \cdot x^{p-4r} \cdot (x^2)^{q-2r} \div x^{4p-8r}$.
42. $8^{-\frac{5}{3}} \cdot x^{\frac{1}{3}} \cdot \sqrt[3]{x\sqrt{9\,x^{-3}}} \div (64\,x^{-\frac{2}{3}})^{-\frac{1}{2}}$.
43. $\left\{x - \dfrac{1}{x} - 3\left(x^{\frac{1}{3}} - \dfrac{1}{x^{\frac{1}{3}}}\right)\right\} \div (x^{\frac{1}{3}} - x^{-\frac{1}{3}})$.

44. $\left(\dfrac{a^{-\frac{1}{2}}}{4c^2}\right)^{-\frac{1}{2}}$.

45. $8^{-\frac{2}{3}}+16^{\frac{3}{4}}$.

46. $\left(\dfrac{64a^3}{27b^3}\right)^{-\frac{1}{3}}$.

47. $\sqrt{a^m}\cdot\sqrt{a^{2m}}$.

48. $2\sqrt[3]{6}\div 6\sqrt{2}$.

49. $\frac{2}{5}\sqrt{\frac{4}{5}}\cdot\frac{1}{3}\sqrt{\frac{1}{20}}$.

50. $(x^6)^{\frac{1}{3}}+4\sqrt[4]{x^8}$.

51. $\sqrt[3]{2}\cdot\sqrt[6]{\frac{1}{3}}\cdot\sqrt[3]{3}$.

52. $\sqrt[3]{2}\cdot\sqrt[6]{\frac{1}{3}}\cdot\sqrt[8]{2}$.

53. $\dfrac{3\sqrt{11}}{2\sqrt{98}}\cdot\dfrac{7\sqrt{22}}{5}$.

54. $(3\sqrt[n]{a^{-m}b^p})^{-mp}$.

55. $\sqrt{a+b}\cdot\sqrt{a-b}$.

56. $3\sqrt[6]{4a^2}+2\sqrt[3]{2a}$.

57. $\sqrt{24}+\sqrt{54}-\sqrt{6}$.

58. $2\sqrt{3}-\sqrt{12}+\sqrt[4]{9}$.

59. $\sqrt{20}+\sqrt{45}-\sqrt{\frac{4}{5}}$.

60. $\left(\dfrac{a^2b}{a^3b^{-4}}\right)^3\div\left(\dfrac{ab^{-1}}{a^{-3}b^2}\right)^5$.

61. $\sqrt{75}+\sqrt{48}-\sqrt{243}$.

62. $(2+3\sqrt{5})(3\sqrt{5}-2)$.

63. $2abc\sqrt{20}+3a\sqrt{5b^2c^2}$.

64. $\dfrac{2}{\sqrt{5}+\sqrt{2}}$.

65. $\dfrac{\sqrt{7}+\sqrt{2}}{9+2\sqrt{14}}$.

66. $\sqrt{19-8\sqrt{3}}$.

67. $\sqrt{27-12\sqrt{5}}$.

68. $\sqrt[m]{a^2}\cdot b^3\cdot\sqrt{c^4}$.

69. $\dfrac{\sqrt{-3}+3}{\sqrt{-4}-2\sqrt{3}}$.

70. $\dfrac{2\sqrt[3]{4}+5\sqrt[3]{32}}{\sqrt[3]{108}}$.

71. $\dfrac{a^{2m-3n}\cdot a^{-5m-n}}{a^{3m-4n}}$.

72. $\dfrac{a^{\frac{3}{5}}c^{-1}\sqrt{b}}{\sqrt[4]{b^3}}\cdot a^{-3}\sqrt[5]{ab^5}$.

73. $\sqrt{a^2-x^2}+\dfrac{x^2}{\sqrt{a^2-x^2}}$.

74. $2\sqrt{3}+3\sqrt{\frac{4}{3}}-5\sqrt{\frac{16}{3}}$.

75. $(6x^2-6)\left(\dfrac{3x-3}{2x+2}\right)^{\frac{1}{2}}$.

76. $4\sqrt{\frac{3}{4}}-\frac{2}{7}\sqrt{\frac{3}{16}}-2\sqrt{27}$.

77. $\dfrac{3\sqrt{1-x^2}-(1-x^2)^{-\frac{1}{2}}}{1-x^2}$.

78. $2\sqrt{24}\cdot 3\sqrt[4]{18}\cdot 4\sqrt[6]{24}$.

79. $\sqrt{a^{4n}b^{11}c^{2p}}\cdot\sqrt{a^nbc^{2-p}}$.

80. $\dfrac{x^4-2+x^{-4}}{x^2+2+x^{-2}}$.

81. $\dfrac{2\sqrt{3}-3}{1+\sqrt{3}-\sqrt{5}}$.

82. $\dfrac{\sqrt{-2}-3\sqrt{-6}}{\sqrt{-2}+2\sqrt{-5}}$.

83. $\dfrac{x+y^{-1}}{x^{\frac{1}{3}}+y^{-\frac{1}{3}}}-\dfrac{x-y^{-1}}{x^{\frac{1}{3}}-y^{-\frac{1}{3}}}$.

84. $\dfrac{\sqrt{3}}{2-\sqrt{3}}-\dfrac{2-\sqrt{-2}}{2+\sqrt{-2}}$.

85. $\dfrac{2\,x^2}{(1-x^2)^{\frac{3}{6}}}+\dfrac{1}{(1-x^2)^{\frac{1}{2}}}$.

86. $\sqrt{22+10\sqrt{-3}}$.

87. $(\sqrt[b]{x^b}\div\sqrt[a]{x^a})^{\frac{1}{1-a}}$.

88. $a^{\frac{1}{2}}y^{\frac{1}{3}}\cdot\left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^4\div\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}}$.

89. $(a^3x^{-3}\div\sqrt[3]{a^2x^{-2}})^{\frac{2}{5}}$.

90. $\left(\dfrac{x^m}{y^n}\right)^{m-n}\div\left(\dfrac{y^{m+n}}{x^m}\right)^n$.

91. $\dfrac{1}{\sqrt{a}+\sqrt{b}}-\dfrac{1}{\sqrt{a}-\sqrt{b}}$.

92. $7\sqrt[3]{54}+\sqrt[6]{256}+\sqrt[3]{432}$.

93. $(81\,x-16)\div(3\,x^{\frac{1}{4}}+2)$.

94. $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$.

95. $\dfrac{a^{\frac{5}{3}}b^{\frac{1}{3}}-(ab)^{\frac{3}{2}}}{1-3\,a^{-\frac{1}{6}}b^{\frac{7}{6}}+3\,a^{-\frac{1}{3}}b^{\frac{7}{3}}-a^{-\frac{1}{2}}b^{\frac{7}{2}}}$.

96. $(9\,a^5-21\,a^3\sqrt{x}-\frac{25}{2}\,ax+12\,a^{-1}x^{\frac{3}{2}})\div(\frac{3}{2}\,a^3-4\,ax^{\frac{1}{2}})$.

97. Arrange in order of magnitude: $\sqrt{\frac{4}{3}}$, $\sqrt[3]{\frac{3}{2}}$, $\sqrt[4]{\frac{7}{4}}$.

98. Arrange in order of magnitude: $(\frac{1}{2})^{\frac{1}{2}}$ and $(\frac{2}{3})^{\frac{2}{3}}$.

99. Extract the square root of $x^4+4\,xy^2+y^{-2}+4\,x^{\frac{5}{2}}y-2\,x^2y^{-1}-4\,x^{\frac{1}{2}}$.

100. $(a^{-4}-5\,b^4+4\,ab^5)\div(a^{-3}+2\,a^{-2}b+3\,a^{-1}b^2+4\,b^3)$.

CHAPTER XVIII

QUADRATIC EQUATIONS

218. An equation which contains, in its simplified form, the second power of the unknown quantity as the highest power of that unknown quantity is called a **quadratic equation.**

Thus, $\frac{x^3}{4}+\frac{x^2}{6}=\frac{1+3x^3}{12}+\frac{x}{2}$, which, when simplified, becomes $2x^2-6x-1=0$, is a quadratic equation.

219. Every quadratic equation may be reduced by the fundamental laws of algebra to the **general form,**

$$\boldsymbol{ax^2+bx+c=0},$$

wherein a, b, and c are known quantities, and wherein $a \neq 0$. If $a=0$, the general form becomes $bx+c=0$, which is a simple equation.

If, in the general form, $b=0$, the resulting equation, $ax^2+c=0$, is called an **incomplete pure quadratic** equation.

If, in the general form, $c=0$, the resulting equation, $ax^2+bx=0$, is called an **incomplete quadratic** equation.

If, in the general form, neither $b=0$, nor $c=0$, the resulting equation, $ax^2+bx+c=0$, is called a **complete** (or **affected**) **quadratic** equation.

The known numbers, a, b, and c, are called the **coefficients** of the equation; and c is further called the **absolute** (or **constant**) **term.**

Thus, $x^2=4$, or $x^2-4=0$, is an incomplete pure quadratic equation in which $a=1$, $b=0$, $c=-4$; $3x^2+4x=0$ is an incomplete quadratic equation in which the coefficients are $a=3$, $b=4$, $c=0$; $4x^2+4x+3=0$ is a complete, or affected, quadratic equation in which $a=4$, $b=4$, $c=3$.

PURE QUADRATIC EQUATIONS

220. 1. Solve: $x^2=a^2$. (1)

Extracting the square roots in (1), $\pm x = \pm a$. (2)

The complete form of (2) is
$$\begin{cases} +x=+a, & (3)\\ -x=-a, & (4)\\ -x=+a, & (5)\\ x=-a. & (6)\end{cases}$$

A value of $-x$ is not required; therefore,

multiplying (4) by -1, $x = a$, (7)

multiplying (5) by -1, $x=-a$. (8)

It is evident that (3) and (7) are indentical; and that (6) and (8) are identical. Hence, *if the double sign be used only in the right member, the roots are not altered in value.* Thus,

Extracting the square roots in (1), $x=\pm a$.

VERIFICATION: $a^2=a^2$.

2. Solve: $\frac{x^2}{4}-20=\frac{x^2}{5}$. (1)

Clearing of fractions in (1), $5x^2-400=4x^2$, (2)

transposing and uniting in (2), $x^2=400$, (3)

extracting square roots in (3), $x=\pm 20$.

VERIFICATION: $\frac{400}{4}-20=\frac{400}{5}$.

NOTE. If x^2 is negative, the signs of all terms must be changed, since the square root of a negative number cannot be obtained.

EXERCISE CXII

Solve the following equations:

1. $x^2 = 169.$

2. $x^2 - a^2 = 0.$

3. $x^2 - 81 = 0.$

4. $3x^2 = 48.$

5. $25x^2 - b^2 = 0.$

6. $a^2x^2 = b^2x^2.$

7. $11x^2 = 36 + 2x^2.$

8. $x^2 = a^2 + 2ab + b^2.$

9. $ax^2 - ab = 2ax^2.$

10. $(7x)^2 = 296 - (5x)^2.$

11. $\dfrac{x+5}{x+13} = \dfrac{2x+7}{3x+18}.$

12. $\dfrac{2}{3} + \dfrac{x-\frac{3}{4}}{x^2} = -19\tfrac{1}{3} + \dfrac{x+\frac{1}{2}}{x^2}.$

13. $\dfrac{x+\frac{1}{2}}{x-\frac{1}{2}} + \dfrac{x-\frac{1}{2}}{x+\frac{1}{2}} = \dfrac{2}{x^2-\frac{1}{4}}.$

14. $\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a} = b.$

15. $\dfrac{ax+b}{cx+d} = \dfrac{cx-d}{ax-b}.$

16. $\dfrac{a(x-b)}{bx} = \dfrac{a}{b} - \dfrac{bx}{a}.$

17. $\dfrac{x-a^2}{x+a^2} + \dfrac{x+a^2}{x-a^2} = 3a^4 + 1.$

18. $\dfrac{12}{5}\left(x - 1 + \dfrac{2}{x+1}\right) = \dfrac{5}{x+1}.$

SOLUTION OF QUADRATIC EQUATIONS BY FACTORING

221. If the product of two quantities be zero, either of the two quantities may be taken as equal to zero. When the left member of a quadratic equation, reduced to the general form, can be factored, either factor may therefore be taken equal to zero, or **equated to zero**. The roots of the factors are therefore the roots of the equation.

The Factor Method holds for all forms of quadratic equations, both complete and incomplete.

1. Solve by factoring : $9x^2 = 36.$ (1)

Dividing (1) by 9, $x^2 = 4,$ (2)

transposing in (2), $x^2 - 4 = 0,$ (3)

factoring in (3), $(x+2)(x-2) = 0,$ (4)

equating each factor in (4) to zero, $\begin{cases} x+2=0, \\ x-2=0, \end{cases}$ (5)

transposing in (5), $x = -2,$ or $x = 2.$ (6)

VERIFICATION: $9(-2)^2 = 36;\ 9(2)^2 = 36.$

2. Solve by factoring : $ax^2 + bx = 0.$ (1)

Factoring in (1), $x(ax+b) = 0,$ (2)

equating each factor in (2) to zero, $\begin{cases} x=0, \\ ax+b=0, \end{cases}$ (3)

transposing in (3), $x = 0,\ ax = -b,$ (4)

dividing $ax = -b$ by a, $x = -\frac{b}{a}.$ (5)

VERIFICATION:

$$a(0) + b(0) = 0;\ a\left(-\frac{b}{a}\right)^2 + b\left(-\frac{b}{a}\right) = \frac{b^2}{a} - \frac{b^2}{a} = 0.$$

3. Solve by factoring : $x^2 - 4x - 21 = 0.$ (1)

Factoring in (1), $(x-7)(x+3) = 0,$ (2)

equating each factor in (2) to zero, $\begin{cases} x-7=0, \\ x+3=0, \end{cases}$ (3)

transposing in (3), $x = 7,$ or $x = -3.$ (4)

VERIFICATION:

$$\begin{cases} (7)^2 - 4(7) - 21 = 49 - 28 - 21 = 0. \\ (-3)^2 - 4(-3) - 21 = 9 + 12 - 21 = 0. \end{cases}$$

EXERCISE CXIII

Solve the following equations by factoring:

1. $x^2 + 7x + 12 = 0.$
2. $x^2 + x - 30 = 0.$
3. $x^2 - x - 12 = 0.$
4. $x^2 + 9x + 20 = 0.$
5. $x^2 + 2x - 224 = 0.$
6. $x^2 - 7x - 260 = 0.$
7. $2x^2 + 9x - 5 = 0.$
8. $3x^2 - 25x + 28 = 0.$
9. $15x^2 + 23x - 28 = 0.$
10. $-63x^2 + 16x - 1 = 0.$
11. $x^2 - \frac{9x}{20} + \frac{1}{20} = 0.$
12. $40x^2 - x - \frac{1}{20} = 0.$
13. $x^2 - (a + b)x + ab = 0.$
14. $x^2 - x(2p + 5q) + 10pq = 0.$

NUMERICAL COMPLETE QUADRATIC EQUATIONS

222. If the coefficients of the equation $ax^2 + bx + c = 0$ are numerical, the equation is called a numerical complete quadratic equation.

Thus, $5x^2 + 7x - 3 = 0$ is a numerical complete quadratic equation.

223. Solution by completing the Square. By §§ 73 and 74, $(x \pm n)^2 = x^2 \pm 2nx + n^2$. The third term is evidently the square of half the coefficient of x. If the left member of a complete quadratic equation contains the unknowns only, and the right member the absolute term, the equation may be made to assume the form $x^2 \pm 2nx$ by dividing the equation by the coefficient of x^2. The left member may be put into the form of the square of a binomial by adding the square of half the coefficient of x, a process which is called **completing the square**. This process is best understood by examples.

1. Solve the equation: $x^2 - 6x = 16$. (1)

The left member is already in the form $x^2 - 2nx$; that is, the coefficient of x^2 is unity. Half the coefficient of x is -3; $(-3)^2 = 9$. Therefore, adding 9 to *both* members of (1), so as not to destroy the equality, or,

completing the square in (1), $x^2 - 6x + 9 = 25$, (2)

extracting the square roots in (2), $x - 3 = \pm 5$, (3)

transposing and uniting in (3), or,

$$\left.\begin{aligned} x &= 3 + 5, \\ x &= 3 - 5, \end{aligned}\right\} \quad (4)$$

combining in (4), $x = 8$, or $x = -2$. (5)

VERIFICATION:

$$(8)^2 - 6(8) = 16; \ (-2)^2 - 6(-2) = 16.$$

2. Solve the equation: $x^2 - 14x - 11 = 0$. (1)

Transposing in (1), $x^2 - 14x = 11$, (2)

completing the square in (2), $x^2 - 14x + 49 = 60$, (3)

extracting the square roots in (3), $x - 7 = \pm 2\sqrt{15}$, (4)

transposing and uniting in (4), or,

$$\left.\begin{aligned} x &= 7 + 2\sqrt{15}, \\ x &= 7 - 2\sqrt{15}. \end{aligned}\right\} \quad (5)$$

VERIFICATION:

$$\left\{\begin{aligned} &(7 + 2\sqrt{15})^2 - 14(7 + 2\sqrt{15}) - 11 = \\ &\quad 49 + 28\sqrt{15} + 60 - 98 - 28\sqrt{15} - 11 = 109 - 109 = 0. \\ &(7 - 2\sqrt{15})^2 - 14(7 - 2\sqrt{15}) - 11 = \\ &\quad 49 - 28\sqrt{15} + 60 - 98 + 28\sqrt{15} - 11 = 109 - 109 = 0. \end{aligned}\right.$$

3. Solve the equation: $x^2 - 3x = 4.$ (1)

Completing the square in (1), $x^2 - 3x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4},$ (2)

extracting the square roots in (2), $x - \frac{3}{2} = \pm \frac{5}{2},$ (3)

transposing and uniting in (3), $x = 4$, or $x = -1.$ (4)

VERIFICATION: $(4)^2 - 3(4) = 4$; $(-1)^2 - 3(-1) = 4.$

4. Solve the equation: $\dfrac{1}{2x+1} + \dfrac{1}{3-x} = \dfrac{6}{5}.$ (1)

Clearing of fractions in (1),

$$5(3 - x) + 5(2x + 1) = 6(2x + 1)(3 - x), \quad (2)$$

simplifying in (2),

$$15 - 5x + 10x + 5 = -12x^2 + 30x + 18, \quad (3)$$

transposing and uniting in (3),

$$12x^2 - 25x = -2, \quad (4)$$

dividing (4) by 12, $x^2 - \dfrac{25x}{12} = -\dfrac{2}{12},$ (5)

completing the square in (5),

$$x^2 - \frac{25x}{12} + \left(\frac{25}{24}\right)^2 = \left(\frac{25}{24}\right)^2 - \frac{2}{12} = \frac{529}{576}, \quad (6)$$

extracting square roots in (6),

$$x - \tfrac{25}{24} = \pm \tfrac{23}{24}, \quad (7)$$

transposing and uniting in (7), $x = 2$, or $x = \frac{1}{12}.$ (8)

VERIFICATION:

$$\frac{1}{5} + 1 = \frac{6}{5};\ \frac{1}{\frac{1}{6} + 1} + \frac{1}{3 - \frac{1}{12}} = \frac{6}{7} + \frac{12}{35} = \frac{42}{35} = \frac{6}{5}.$$

Rule for solving Numerical Complete Quadratic Equations: *After clearing the equation of fractions (if any exist), trans-*

pose the unknowns to the left member and the absolute term to the right member; divide the equation by the coefficient of x^2*; complete the square by adding to each member the square of half the coefficient of* x*; extract the square root of each member; solve the simple equations thus derived.*

EXERCISE CXIV

Solve the following equations by completing the square:

1. $x^2 + 6x = 55.$

2. $x^2 + 12x = 13.$

3. $x^2 + x - 2 = 0.$

4. $x^2 + x - 6 = 0.$

5. $8x^2 + 6x + 1 = 0.$

6. $x^2 + \frac{2x}{3} - \frac{8}{9} = 0.$

7. $x^2 + 2x + 40 = 0.$

8. $x^2 - 3x + 1 = 0.$

9. $x^2 + 5x - 7 = 0.$

10. $3x^2 + 5x = 2.$

11. $3x^2 - 7x = 16.$

12. $2x^2 - 5x + 3 = 0.$

13. $x(x+1) = 12.$

14. $x + 3 = \frac{10}{x}.$

15. $x + \frac{1}{x} = 2.$

16. $\frac{2}{3x} + 4 = \frac{5x}{6x+7}.$

17. $2x + 14 + \frac{7}{x} = 0.$

18. $x^2 - \frac{23x}{4} - 18 = 0.$

19. $\frac{1}{x^2} + \frac{5}{x} - 6 = 0.$

20. $11\frac{3}{4}x - 3\frac{1}{2}x^2 = -41\frac{1}{4}.$

21. $\frac{2}{x-1} - \frac{1}{x+3} = \frac{3}{8}.$

22. $\frac{5}{x+2} + \frac{2}{x} = \frac{14}{x+4}.$

23. $\frac{5}{4x^2-1} - \frac{3}{2x+1} = \frac{2}{3}.$

24. $\frac{x+3}{x+2} + \frac{x-3}{x-2} = \frac{2x-3}{x-1}.$

25. $\frac{x^2+6}{x^2-4} + \frac{3}{2-x} = 5 - \frac{7}{x+2}.$

26. $9\frac{1}{3}x^2 - 90\frac{1}{3}x = -195.$

27. $\frac{17}{x} - \frac{32-11x}{3x^2} = 7\frac{2}{3}.$

LITERAL COMPLETE QUADRATIC EQUATIONS

224. If the coefficients of the equation $ax^2 + bx + c = 0$ are literal, the equation is called a literal complete quadratic equation.

Thus, $2\,ax^2 + mx + 6\,n = 0$ is a literal complete quadratic equation.

The solution is found in the same manner as in the preceding paragraph.

1. Solve the equation: $x^2 - bx - cx = (a + b)(a - c)$. (1)

Factoring in (1) to show coefficient of x,

$$x^2 - x(b + c) = (a + b)(a - c), \tag{2}$$

completing the square in (2),

$$x^2 - x(b + c) + \left(\frac{b + c}{2}\right)^2 = \left(\frac{b + c}{2}\right)^2 + (a + b)(a - c), \tag{3}$$

Simplifying the right member in (3),

$$x^2 - x(b + c) + \left(\frac{b + c}{2}\right)^2 = \frac{4\,a^2 + 4\,ab - 4\,ac + b^2 - 2\,bc + c^2}{4}, \tag{4}$$

extracting square roots in (4),

$$x - \frac{b + c}{2} = \pm\frac{2\,a + b - c}{2}, \tag{5}$$

transposing and uniting in (5),

$$x = a + b, \text{ or } x = c - a. \tag{6}$$

Verification:
$$\begin{cases} (a+b)^2 - (a+b)(b+c) = (a+b)(a-c), \\ \qquad (a+b)(a-c) = (a+b)(a-c). \\ (c-a)^2 - (c-a)(b+c) = (a+b)(a-c), \\ \qquad (c-a)(-a-b) = (a+b)(a-c). \end{cases}$$

2. Solve the equation: $ax^2 + bc - bx = acx.$ (1)

Transposing in (1), $ax^2 - bx - acx = -bc,$ (2)

factoring in (2), $ax^2 - x(b + ac) = -bc,$ (3)

dividing (3) by a, $x^2 - x\left(\frac{b + ac}{a}\right) = -\frac{bc}{a},$ (4)

completing the square in (4),

$$x^2 - x\left(\frac{b+ac}{a}\right) + \left(\frac{b+ac}{2a}\right)^2 = \left(\frac{b+ac}{2a}\right)^2 - \frac{bc}{a}, \tag{5}$$

simplifying the right member in (5),

$$x^2 - x\left(\frac{b+ac}{a}\right) + \left(\frac{b+ac}{2a}\right)^2 = \frac{b^2 - 2abc + a^2c^2}{4a^2}, \tag{6}$$

extracting square roots in (6),

$$x - \frac{b+ac}{2a} = \pm\frac{b-ac}{2a}, \tag{7}$$

transposing and uniting in (7), $x = \frac{b}{a}$, or $x = c.$ (8)

VERIFICATION:

$$\begin{cases} a\left(\dfrac{b^2}{a^2}\right) + bc - b\left(\dfrac{b}{a}\right) = ac\left(\dfrac{b}{a}\right), \ \dfrac{b^2}{a} + bc - \dfrac{b^2}{a} = bc. \\ ac^2 + bc - bc = ac^2. \end{cases}$$

The left member should always be factored to show the coefficients of x^2 and of x.

EXERCISE CXV

Solve the following equations by completing the square:

1. $x^2 + 4bx = -4b^2.$
2. $x^2 - 5ax + 6a^2 = 0.$
3. $x^2 + ax - 2a^2 = 0.$
4. $9x^2 - 6pq = 2pq - 3qx.$
5. $bx^2 + ac = (a + bc)x.$
6. $x^2 + (a + b)x + ab = 0.$

7. $x^2 + ax = a^2.$

8. $x + \frac{1}{x} = a + \frac{1}{a}.$

9. $lx^2 + mx + n = 0.$

10. $x^2 - 2\,ax + b = 0.$

11. $\frac{x-a}{x+a} - \frac{x+a}{x-a} = 5.$

12. $ax^2 - 2\,bx + c = 0.$

13. $x^2 - 4\,ax + a^2 = 0.$

14. $\frac{x}{a^2b\,(x+c)} = \frac{x+c}{d^2bc}.$

15. $ax^2 + a = (a^2+1)\,x.$

16. $x^2 - 2\,ax + a^2 - b^2 = 0.$

17. $\frac{1}{a-x} - \frac{1}{a+x} = \frac{3+x^2}{a^2-x^2}.$

18. $mqx^2 - mnx + pqx = np.$

19. $\frac{(4\,a^2-b^2)(x^2+1)}{4\,a^2+b^2} = 2\,x.$

20. $\frac{a}{x-b} + \frac{b}{x-a} = 2.$

21. $x\,(1-x) = ax^2 + b.$

22. $1 - 7\,x^2 = 2\,ax - bx^2.$

23. $\frac{(x-a)^2}{a(x+a)} = \frac{b - \frac{2\,c}{x+a}}{b - \frac{c}{x-a}}.$

24. $\frac{x}{a} + \frac{1}{a} \cdot \frac{1+a}{x} = x + \frac{1}{a^2}.$

25. $a^2x - 2\,b^2 = ab \cdot \frac{x^2+1}{x+1}.$

26. $\frac{a}{b+x} + \frac{b}{a+x} = \frac{2\,a}{2\,a-b}.$

27. $\frac{x-a}{2\,b} - \frac{x-2\,b}{x-b} = \frac{b}{a+b}.$

28. $a^2 \cdot \frac{2\,x-1}{x+2} = b^2 \cdot \frac{x+2}{2\,x-1}.$

29. $\frac{a+x}{a-x} - \frac{a-x}{a+x} = \frac{4\,b\,(a+b)}{a(a+2\,b)}.$

30. $\frac{3\,x}{a-b+2\,x} = \frac{5\,x-5\,a+7\,b}{a+b-x}.$

31. $cx^2 - (a+b+c)\,x + (a+b) = 0.$

32. $mnx^2 - (m+n)(mn+1)x + (m+n)^2 = 0.$

33. $2\,x^2(a^2-b^2) - (3\,a^2+b^2)(x-1) = (3\,b^2+a^2)(x+1)$

34. $\dfrac{x-2a}{b+x}+\dfrac{x-b}{x-a}=\dfrac{b+4a}{2(a+b)}.$

35. $x^2(a+b)^2-x(a^2-b^2)=ab.$

36. $\dfrac{2(a+b)}{x-b}+\dfrac{2b}{x-a}=\dfrac{3(a-b)}{x-3b}.$

37. $\dfrac{x-a}{x+3b}+\dfrac{x}{x-b}=\dfrac{2(x+a)}{x+3a-4b}.$

38. $\dfrac{a(1-x^2)}{bx}+\dfrac{(2a-b)x}{a}=\dfrac{4a}{a+b}.$

39. $\dfrac{a}{b}(x+1)+\dfrac{b}{a}(x-1)=\dfrac{2x^2-1}{x}.$

40. $(a-x)\left(1-\dfrac{3a+3x}{c-\frac{1}{2}}\right)-2=\dfrac{1-2a}{1-2c}(c-3)-\dfrac{1+a}{1-\frac{1}{2}c}.$

41. $\dfrac{x+b}{2a}+\dfrac{2a}{x-b}=1-\dfrac{2a}{b}\left(1-\dfrac{2a-b}{x-b}\right).$

42. $(a^2+b^2)(4x^2+1)+2ab(4x^2-1)=4x(a^2-b^2).$

43. $ax-\dfrac{4}{ax+b}[b^2(1+x)x-a^2(1-x)]=b.$

44. $(a+1)\dfrac{x-2}{x-1}-\dfrac{a}{a+1}\left(2+\dfrac{1}{x}-\dfrac{(a+1)^2}{a(x-1)}\right)=0.$

45. $\dfrac{2a(a+b)-b^2x}{bx-2a}=\dfrac{2}{\dfrac{b}{a^2}\left(\dfrac{1}{x}-\dfrac{b}{2a}\right)}-b\left[1-\dfrac{b}{a-\dfrac{b}{2x}}\right].$

46. $\dfrac{2}{\dfrac{1}{b}\left(\dfrac{x}{2a}-2\right)}-\dfrac{b}{a}\left[\dfrac{4a}{\dfrac{4b}{x}-\dfrac{b}{a}}-\dfrac{1}{\dfrac{1}{x}-\dfrac{1}{b}}\right]=0.$

SOLUTION OF QUADRATIC EQUATIONS BY A FORMULA

225. Every quadratic equation may be reduced to the general form, $ax^2 + bx + c = 0$.

Solve the equation: $ax^2 + bx + c = 0$. (1)

Transposing in (1), $ax^2 + bx = -c$, (2)

dividing (2) by a, $x^2 + x\left(\frac{b}{a}\right) = -\frac{c}{a}$, (3)

completing the square in (3),

$$x^2 + x\left(\frac{b}{a}\right) + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}, \tag{4}$$

simplifying the right member in (4),

$$x^2 + x\left(\frac{b}{a}\right) + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}, \tag{5}$$

extracting square roots in (5),

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}, \tag{6}$$

transposing and uniting in (6), $\boldsymbol{x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}}$,

or, $\boldsymbol{x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$. (7)

The values of x in (7) are general values. The values of the roots in any particular equation are found by substituting in the formulas in (7) the values of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in any particular equation.

1. Solve by the formula: $2x^2 - 5x = 3$. (1)

Putting (1) in the general form, $2x^2 - 5x - 3 = 0$. (2)

In (2), $a = 2,\ b = -5,\ c = -3$. (3)

Substitute for a, b, and c their values from (3) in the formulas in (7),

$$\left.\begin{aligned} x &= \frac{5+\sqrt{(-5)^2-4(2)(-3)}}{2(2)} = 3, \\ \text{or,}\quad x &= \frac{5-\sqrt{(-5)^2-4(2)(-3)}}{2(2)} = -\tfrac{1}{2}. \end{aligned}\right\} \qquad (4)$$

VERIFICATION: $\begin{cases} 2(3)^2 - 5(3) - 3 = 18 - 15 - 3 = 0, \\ 2(-\frac{1}{2})^2 - 5(-\frac{1}{2}) - 3 = \frac{1}{2} + \frac{5}{2} - 3 = 0. \end{cases}$

The formulas are written in the more compact form,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

EXERCISE CXVI

Solve the following equations by the formula:

1. $x^2 - 10x + 25 = 0.$
2. $6x^2 + 13x - 8 = 0.$
3. $5x^2 + 11x + 83 = 0.$
4. $x(x-2) = 5(x-6)^2.$
5. $x^2 + \dfrac{3x}{7} = 10\frac{3}{4}.$
6. $x(x+1) = \frac{103}{36}.$
7. $x^2 - 2ax + b = 0.$
8. $ax^2 - 2bx + c = 0.$
9. $ax^2 - (a+b)x + b = 0.$
10. $\dfrac{2x(2x-5)}{2x-1} - \dfrac{2}{2x-1} = 3.$
11. $(a+b)bx^2 + a^2 = a(a+2b)x.$
12. $(x-2)^2 + (x+5)^2 = (x+6)^2.$
13. $(b^2-1)x^2 - 2(ab-1)x + a^2 = 1.$
14. $(a-1)x^2 + (a+1)x + \dfrac{a}{a-1} = 0.$

IRRATIONAL QUADRATIC EQUATIONS

226. Quadratic equations which contain indicated roots of the unknown quantities are called **irrational**, or **radical quadratic equations.**

Thus, $\sqrt{x+3}+x=9$, is an irrational quadratic equation.

Roots obtained in solving quadratic equations involving radicals must be substituted in the original equation for the purpose of verification.

1. Solve the equation : $\sqrt{x+7}=x+1$. (1)

Squaring (1), $x+7=x^2+2x+1$, (2)

transposing and uniting in (2), $x^2+x-6=0$, (3)

solving (3), $x=2$, or $x=-3$. (4)

VERIFICATION : $\begin{cases} \sqrt{2+7}=2+1; \text{ 2 is a root of (1).} \\ \sqrt{-3+7}\neq -3+1; -3 \text{ is not a root of (1).} \end{cases}$

2. Solve the equation : $\sqrt{2x+21}-\sqrt{x+7}=2$. (1)

Squaring (1),

$2x+21-2\sqrt{2x^2+35x+147}+x+7=4$, (2)

transposing and uniting in (2),

$$-2\sqrt{2x^2+35x+147}=-24-3x, \quad (3)$$

squaring (3), $8x^2+140x+588=576+144x+9x^2$, (4)

transposing and uniting in (4),

$$x^2+4x-12=0, \quad (5)$$

solving (5), $x=2$, or $x=-6$. (6)

VERIFICATION : $\begin{cases} \sqrt{4+21}-\sqrt{2+7}=2; \text{ 2 is a root of (1).} \\ \sqrt{-12+21}-\sqrt{-6+7}=2; -6 \text{ is a root of (1)} \end{cases}$

EXERCISE CXVII

Solve the following equations and verify the roots:

1. $\frac{1}{2}\sqrt{x}+1=3x.$

2. $2x-\sqrt{x+3}=-5.$

3. $\sqrt{x+5}+x=7.$

4. $4x-\sqrt{x+3}=x-5.$

5. $9x-\sqrt{9x+1}=2x-1.$

6. $\dfrac{5+\sqrt{x}}{3\sqrt{x}+1}+\sqrt{x}=3.$

7. $\dfrac{\sqrt{x+1}-2}{2\sqrt{x}-1}=2\sqrt{x}+1.$

8. $\sqrt{\dfrac{9x-2}{x-1}}-3=\dfrac{2}{x}.$

9. $\sqrt{x+1+\sqrt{5(x+2)}}=3.$

10. $\sqrt{2x-7}+\sqrt{7x+8}=11.$

11. $\sqrt{3x+4}+\sqrt{5(x+1)}=9.$

12. $\sqrt{a+x}+\sqrt{b-x}=\sqrt{a+b}.$

13. $2\sqrt{3x+1}-3\sqrt{x+3}+2=0.$

14. $3\sqrt{3x-4}+4x=10(x-1).$

15. $\dfrac{\sqrt{5x^2-2x+1}}{2\sqrt{x}}=\sqrt{4x-3}.$

16. $3\sqrt{x+8}-\sqrt{x-8}=2\sqrt{2x+2}.$

17. $3\sqrt{3x+1}-2\sqrt{x+3}=\sqrt{2(x+1)}.$

18. $\sqrt{4x+1}+3\sqrt{9x-2}=5\sqrt{5x-1}.$

19. $\dfrac{x}{\sqrt{x}+\sqrt{a-x}}+\dfrac{x}{\sqrt{x}-\sqrt{a-x}}=\dfrac{8a}{3\sqrt{x}}.$

SOLUTIONS OF EQUATIONS IN THE QUADRATIC FORM

227. An equation which contains only two different powers of the unknown quantity one of which is double the other is said to be in the **quadratic form.** The general type of equations in the quadratic form is $ax^{2n}+bx^n+c=0$. Equations in the quadratic form may be solved like quadratics.

1. Solve the equation: $x^4-2x^2+1=0$. (1)

Writing (1) in the quadratic form,

$$(x^2)^2-2(x^2)+1=0, \tag{2}$$

factoring (2), $$(x^2-1)(x^2-1)=0, \tag{3}$$

equating the factors in (3) to zero,

$$\begin{cases} x^2-1=0, \\ x^2-1=0, \end{cases} \tag{4}$$

transposing in (4), $$x^2=1,\ x^2=1, \tag{5}$$

extracting square roots in (5),

$$x=\pm 1, \text{ or } x=\pm 1. \tag{6}$$

Verification: $1-2+1=0$.

2. Solve the equation: $x^{\frac{3}{2}}-9x^{\frac{3}{4}}+8=0$. (1)

Writing (1) in the quadratic form,

$$(x^{\frac{3}{4}})^2-9(x^{\frac{3}{4}})+8=0, \tag{2}$$

factoring in (2), $$(x^{\frac{3}{4}}-8)(x^{\frac{3}{4}}-1)=0, \tag{3}$$

equating the factors in (3) to zero,

$$\begin{cases} x^{\frac{3}{4}}-8=0, \\ x^{\frac{3}{4}}-1=0, \end{cases} \tag{4}$$

transposing in (4), $$x^{\frac{3}{4}}=8,\ x^{\frac{3}{4}}=1, \tag{5}$$

raising each equation in (5) to $\frac{4}{3}$ power,

$$(x^{\frac{3}{4}})^{\frac{4}{3}} = 8^{\frac{4}{3}},\ (x^{\frac{3}{4}})^{\frac{4}{3}} = (1)^{\frac{4}{3}}, \tag{6}$$

simplifying in (6), $\qquad x = 16,$ or $x = 1.$ $\qquad$ (7)

VERIFICATION: $\begin{cases} (16)^{\frac{3}{2}} - 9(16)^{\frac{3}{4}} + 8 = 64 - 72 + 8 = 0. \\ 1 - 9 + 8 = 0. \end{cases}$

3. Solve the equation:

$$x^2 - 7x - \sqrt{x^2 - 7x + 18} = 12. \tag{1}$$

Adding 18 to each member in (1),

$$(x^2 - 7x + 18) - \sqrt{x^2 - 7x + 18} = 30, \tag{2}$$

writing (2) in the quadratic form,

$$(\sqrt{x^2 - 7x + 18})^2 - \sqrt{x^2 - 7x + 18} = 30, \tag{3}$$

transposing in (3),

$$(\sqrt{x^2 - 7x + 18})^2 - \sqrt{x^2 - 7x + 18} - 30 = 0, \tag{4}$$

factoring in (4),

$$(\sqrt{x^2 - 7x + 18} - 6)(\sqrt{x^2 - 7x + 18} + 5) = 0, \tag{5}$$

equating the factors in (5) to zero,

$$\begin{cases} \sqrt{x^2 - 7x + 18} - 6 = 0, \\ \sqrt{x^2 - 7x + 18} + 5 = 0, \end{cases} \tag{6}$$

transposing in (6),

$$\sqrt{x^2 - 7x + 18} = 6,\ \sqrt{x^2 - 7x + 18} = -5. \tag{7}$$

Solving $\sqrt{x^2 - 7x + 18} = 6$, $x = 9$, or -2.

$\sqrt{x^2 - 7x + 18} = -5$ is impossible since the radical cannot equal a negative quantity.

VERIFICATION: on substitution in (1), both 9 and -2 are roots.

EXERCISE CXVIII

Solve the following equations:

1. $x - \sqrt{x} = 2.$

2. $x^4 - 5x^2 - 126 = 0.$

3. $x^4 - 30x^2 + 125 = 0.$

4. $x^{\frac{4}{3}} + 5x^{\frac{2}{3}} = 36.$

5. $x^{\frac{2}{3}} - 6x^{\frac{1}{3}} = 16.$

6. $2x^{-\frac{2}{3}} - x^{-\frac{1}{3}} - 6 = 0.$

7. $x^{-3} - x^{-\frac{3}{2}} = -\frac{7}{64}.$

8. $\left(x + \frac{8}{x}\right)^2 - x - 72 = \frac{8}{x}.$

9. $\frac{55}{(x+7)^2} - \frac{4}{(x+7)} = \frac{1}{11}.$

10. $\sqrt{x^2 - a^2} = 2 - \frac{1}{\sqrt{x^2 - a^2}}.$

11. $\sqrt{x^2 + 12} + \sqrt[4]{x^2 + 12} = 6.$

12. $x + \sqrt{x^2 - ax + b^2} = \frac{x^2}{a} + b.$

13. $\frac{120}{x^2 + 8x + 16} + \frac{11}{x + 4} = 17.$

14. $\sqrt[4]{x - 1} + 2\sqrt{x - 1} - 1 = 0.$

15. $x^2 - 2\sqrt{x^2 + 4x - 5} = 13 - 4x.$

16. $49x^2 + 42x + 9 = 1 - (7x + 3).$

17. $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2.$

18. $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3.$

19. $\frac{3}{(x^2 - 5x + 7)^2} - \frac{2}{x^2 - 5x + 7} + \frac{1}{3} = 0.$

20. $4x^2 + 22x - 3\sqrt{2x^2 + 11x + 13} = 78.$

21. $\sqrt[6]{x^2 + 5x + 28} + \sqrt[3]{x^2 + 5x + 28} - 6 = 0.$

SOLUTIONS OF CERTAIN HIGHER EQUATIONS BY QUADRATIC METHODS

228. An equation which in its simplest form contains higher than the second power of the unknown quantity is, in general, beyond the province of this book. Some forms of higher equations have been solved in the preceding paragraph.

1. Solve the equation : $4x^3 + 8x^2 - 140x = 0.$ (1)

Factoring (1), $4x(x^2 + 2x - 35) = 0,$ (2)

equating the factors in (2) to zero,

$$\begin{cases} 4x = 0, \\ x^2 + 2x - 35 = 0, \end{cases} \qquad (3)$$

solving the equation in (3) separately,

$$\begin{array}{r|l} 4x = 0, & x^2 + 2x - 35 = 0, \\ x = 0. & (x+7)(x-5) = 0, \\ & \quad x = 5, \text{ or } x = -7. \end{array}$$

VERIFICATION : $\begin{cases} 0 = 0. \\ 4(5)^3 + 8(5)^2 - 140(5) = 0. \\ 4(-7)^3 + 8(-7)^2 - 140(-7) = 0 \end{cases}$

2. Solve the equation : $x^3 = 21x - 20.$ (1)

Transposing in (1), $x^3 - 21x + 20 = 0.$ (2)

By § 96, $x - 1$ is a factor of $x^3 - 21x + 20$.

Factoring (2), $(x-1)(x^2 + x - 20) = 0,$ (3)

writing (3) in prime factors,

$$(x-1)(x-4)(x+5) = 0, \qquad (4)$$

from (4), $x = 1$, or $x = 4$, or $x = -5.$ (5)

VERIFICATION: $1 = 21 - 20$; $64 = 84 - 20$; $-125 = -105 - 20.$

3. Solve the equation: $x^4+4x^3+2x^2-4x-3=0.$ (1)

The left member in (1) is found by trial to be the square of the trinomial x^2+2x-1 if 4 be added to that member.

Adding 4 to each member in (1),

$$x^4+4x^3+2x^2-4x+1=4, \tag{2}$$

extracting the square roots in (2), $x^2+2x-1=\pm 2,$ (3)

whence, solving (3), $x=1$, or $x=-1$, or $x=-3$. (4)

VERIFICATION: $\begin{cases} 1+4+2-4-3=0. \\ 1-4+2+4-3=0. \\ 81-108+18+12-3=0. \end{cases}$

NOTE. It should be noticed that the methods of solution shown above apply only to particular forms of higher equations and are in no sense general solutions of such equations.

EXERCISE CXIX

Solve the following equations:

1. $x(x+2)(x^2-4)=0.$

2. $x(x+a)(x^2-b^2)=0.$

3. $ax(bx-2)(x^2-9)=0.$

4. $x^3+x^2+x+1=0.$

5. $3x^3+4x^2-6x=7.$

6. $x^3-6x^2-45x+50=0.$

7. $x(x^2-4)+(x-2)=0.$

8. $3x^3-8x^2+3x+2=0.$

9. $(x-1)(x+2)(x^2-6x+9)=0.$

10. $x^4-2x^3-x+2=0.$

11. $x^4-10x^3+13x^2+60x-64=0.$

12. $4x^4-12x^3+17x^2-12x-5=0.$

13. $7x^4-5x^3-157x^2+5(x+30)=0.$

14. $x^5-3x^4-15x^3+35x^2+54x-72=0.$

CHARACTER OF THE ROOTS

229. The roots of the general form $ax^2 + bx + c = 0$ have been found, § 225, to be: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Upon the nature of $\sqrt{b^2 - 4ac}$ will depend the character of the roots. The quantity, $b^2 - 4ac$, is called the **discriminant.**

(1) **If $b^2 - 4ac$ is positive**, that is, if $b^2 - 4ac > 0$, the roots are real and unequal, and either (*a*) rational or (*b*) irrational. If the discriminant is (*a*) a perfect square, the roots are real and rational; if (*b*) not a perfect square, the roots are real and irrational.

Thus, in the equation $6x^2 + 5x - 21 = 0$, since $a = 6$, $b = 5$, $c = -21$, the discriminant is $529 = 23^2$. Therefore, the roots are real, rational, and unequal.

In the equation $2x^2 + 5x - 4 = 0$, since $a = 2$, $b = 5$, $c = -4$, the discriminant is 57. Therefore, the roots are real, irrational, and unequal.

(2) **If $b^2 - 4ac$ is zero**, that is, if $b^2 = 4ac$, the roots are real, rational, and equal.

Thus, in the equation $4x^2 - 12x + 9 = 0$, since $a = 4$, $b = -12$, $c = 9$, the discriminant is 0. Therefore, the roots are real, rational, and equal.

(3) **If $b^2 - 4ac$ is negative**, that is, if $b^2 - 4ac < 0$, the roots are imaginary and unequal.

Thus, in the equation $x^2 - 2x + 4 = 0$, since $a = 1$, $b = -2$, and $c = 4$, the discriminant is -12. Therefore, the roots are imaginary and unequal.

The character of the roots of any given equation may therefore be found by evaluating the discriminant.

The following summary will be found useful :

(1) **If $b^2 - 4ac > 0$, the roots are real and unequal.**

(2) **If $b^2 = 4ac$, the roots are real and equal.**

(3) **If $b^2 - 4ac < 0$, the roots are imaginary and unequal.**

1. Determine, without solving, the character of the roots of $2x^2 - 7x + 5 = 0$.

$$a = 2, b = -7, c = 5.$$

$$b^2 - 4ac = 49 - 4(2)(5) = 9 = 3^2.$$

Roots are real, rational, and unequal.

2. Determine, without solving, the character of the roots of $9x^2 - 12x + 4 = 0$.

$$a = 9, b = -12, c = 4.$$

$$b^2 - 4ac = 144 - 4(9)(4) = 0.$$

Roots are real and equal.

3. Determine, without solving, the character of the roots of $4x^2 - 4x + 5 = 0$.

$$a = 4, b = -4, c = 5.$$

$$b^2 - 4ac = 16 - 4(4)(5) = -64.$$

Roots are imaginary and unequal.

4. For what value of m are the roots equal in the equation $3x^2 + 4x + m = 0$?

$$a = 3, \ b = 4, \ c = m.$$

If the roots are equal, $\quad b^2 - 4ac = 0,$

$$16 - 4(3)m = 0,$$

$$16 - 12m = 0,$$

$$12m = 16,$$

$$m = \tfrac{4}{3}.$$

EXERCISE CXX

Determine by the use of the discriminant the character of the roots in the following equations:

1. $x^2 - 4x + 4 = 0$.
2. $x^2 - 5x + 6 = 0$.
3. $x^2 - 2x - 1 = 0$.
4. $2x^2 - 3x + 5 = 0$.
5. $5x^2 - 2x + 1 = 0$.
6. $x^2 - 3x + 1 = 0$.
7. $x^2 - 4x + 7 = 0$.
8. $x^2 - 7x + 12 = 0$.
9. $3x^2 - 4x + 1 = 0$.
10. $2x^2 - 13x + 5 = 0$.
11. $3x^2 - 4x + 12 = 0$.
12. $2x^2 - 5x - 5 = 0$.
13. $3x^2 - 5x = 2$.
14. $x^2 - 2ax = (b + a)(b - a)$.

Determine the value of m for which the roots are equal in the following equations:

15. $2x^2 + 4x + m = 0$.
16. $mx^2 + 6x + 3 = 0$.
17. $3x^2 + 4x - m = 0$.
18. $16x^2 + 8mx + 1 = 0$.
19. $4x^2 - 12x + m = 0$.
20. $mx^2 - (8 + m)x + 9 = 0$.

RELATION BETWEEN ROOTS AND COEFFICIENTS

230. It is convenient to derive the formula for the general equation $ax^2 + bx + c = 0$, where the coefficient of x^2 is unity. Dividing the general equation by a, $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$; in the last equation, putting $p = \frac{b}{a}$, and $q = \frac{c}{a}$, the equation is $\boldsymbol{x^2 + px + q = 0}$.

The roots of $x^2 + px + q = 0$ are found to be

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Let $$\alpha = \frac{-p + \sqrt{p^2 - 4q}}{2}, \quad (1)$$

and $$\beta = \frac{-p - \sqrt{p^2 - 4q}}{2}, \quad (2)$$

adding (1) and (2), $\alpha + \beta = -p$, (3)

multiplying (1) and (2), $\alpha\beta = q$. (4)

Hence, in the equation $x^2 + px + q = 0$:

(1) *The sum of the roots equals the coefficient of x with its sign changed.*

(2) *The product of the roots equals the absolute term.*

231. Since the equation $x^2 + px + q = 0$ is the general form of complete quadratics, the sum and the product of the roots of any complete quadratic may be found by inspection.

1. Find by inspection the sum and product of the roots of $2x^2 + 3x + 1 = 0$. (1)

Dividing (1) by 2, $x^2 + \frac{3}{2}x + \frac{1}{2} = 0$, (2)

if α and β are the roots, by the rule, $\alpha + \beta = -\frac{3}{2}$, (3)

$\alpha\beta = \frac{1}{2}$. (4)

The equation $x^2 + px + q = 0$, wherein α and β (read respectively, "alpha" and "beta") are the roots, may be written, $\mathbf{x^2 - (\alpha + \beta)x + \alpha\beta = 0}$.

2. Form an equation whose roots are -2 and 3.

Take $\alpha = -2$, and $\beta = 3$.

Then, $\alpha + \beta = -2 + 3 = 1$; $\alpha\beta = (-2)(3) = -6$.

Substituting for $\alpha + \beta$ and $\alpha\beta$ these values in $x^2 - (\alpha + \beta)x + \alpha\beta = 0$,

$$x^2 - (1)x - 6 = 0, \text{ or, } x^2 - x - 6 = 0.$$

3. Form an equation whose roots shall be the squares of the roots of the equation $x^2+px+q=0$. (1)

Let α and β be the roots of (1).

By the conditions the required equation is,

$$x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0. \qquad (2)$$

Now $\alpha\beta=q$; hence, $\alpha^2\beta^2=q^2$.

Again, $\alpha+\beta=-p$; hence, $\alpha^2+2\,\alpha\beta+\beta^2=p^2$,

$$\alpha^2+\beta^2=p^2-2\,\alpha\beta,$$

$$\alpha^2+\beta^2=p^2-2\,q.$$

Substituting $\alpha^2+\beta^2=p^2-2\,q$ and $\alpha^2\beta^2=q^2$, in (2),

$$x^2-(p^2-2\,q)x+q^2=0.$$

4. Form an equation whose roots are reciprocals of the roots of the equation $x^2+px+q=0$. (1)

Let α and β be the roots of (1).

By the conditions, the required equation is,

$$x^2-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x+\frac{1}{\alpha\beta}=0, \text{ or, } x^2-\left(\frac{\alpha+\beta}{\alpha\beta}\right)x+\frac{1}{\alpha\beta}=0. \qquad (2)$$

Now, $\alpha+\beta=-p$; and $\alpha\beta=q$.

Substituting $\alpha+\beta=-p$, and $\alpha\beta=q$, in (2),

$$x^2-\left(\frac{-p}{q}\right)x+\frac{1}{q}=0, \qquad (3)$$

$$x^2+\frac{p}{q}x+\frac{1}{q}=0,$$

$$qx^2+px+1=0.$$

The results obtained in the preceding examples may be verified by solving the equation.

EXERCISE CXXI

Form that equation whose roots are respectively:

1. 2 and 3.
2. 5 and 2.
3. 6 and -2.
4. -3 and -5.
5. $\frac{1}{2}$ and $\frac{1}{3}$.
6. 4 and $-\frac{3}{5}$.
7. -5 and $\frac{2}{7}$.
8. -4 and $\frac{1}{4}$.
9. $2+\sqrt{3}$ and $2-\sqrt{3}$.
10. $\dfrac{1+\sqrt{3}}{2}$ and $\dfrac{1-\sqrt{3}}{2}$.
11. $-\dfrac{1-2\sqrt{5}}{3}$ and $-\dfrac{1+2\sqrt{5}}{3}$.
12. $1+2\sqrt{-3}$ and $1-2\sqrt{-3}$.
13. $2+\sqrt{-2}$ and $2-\sqrt{-2}$.
14. $a+\sqrt{b}$ and $a-\sqrt{b}$.
15. $a+\sqrt{-b}$ and $a-\sqrt{-b}$.
16. $-c+\sqrt{-d}$ and $-c-\sqrt{-d}$.

17. Form a quadratic equation whose second member shall be 0, whose absolute term in the first member shall be -4, and one of whose roots shall be $-\frac{1}{3}$.

18. One root of the equation $4x^2-16x+4=0$ is $2+\sqrt{3}$: find the second root.

19. Find, without solving, the sum and product of the roots of the equation $3x^2-7x-5=0$.

20. Form an equation whose roots shall be the reciprocals of the roots of the equation $2x^2-x+1=0$.

21. Form an equation whose roots shall have the same absolute value as, but signs opposite to, the roots of $x^2+px+q=0$.

GRAPHS OF EQUATIONS OF THE SECOND DEGREE

232. By the method employed in § 145 it is possible to construct the graph of any equation of the second degree in two unknowns.

Consider the equation $y = ax^2 + bx + c$,

the right hand member of which is evidently a part of the general form of complete quadratic equation in one unknown. The graphs of certain numerical forms of $y = ax^2 + bx + c$ for various characters of the roots are interesting.

233. (1) When $b^2 > 4ac$, and when $\sqrt{b^2 - 4ac}$ is rational.

1. Plot the graph of $y = x^2 - 4x + 3$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	15	8	3	0	-1	0	3	8	15

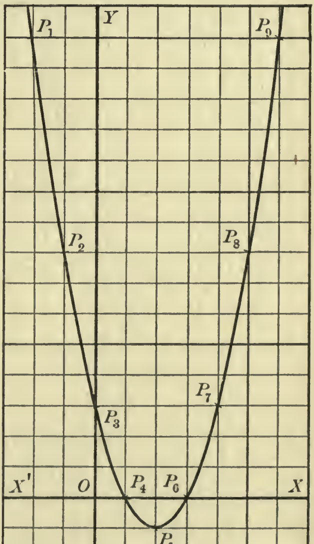

FIG. 12.

In the table are found the coördinates of the various points. Locating convenient points, and drawing a smooth curve through these points, the curve $P_1P_2P_3P_4P_5P_6P_7P_8P_9$, Fig. 12, is the graph. The graph is seen to cut the X-axis at the points P_4 and P_6, whose coördinates are respectively (1, 0) and (3, 0). But, since the y-coördinates of the points where the graph crosses the X-axis are zero, the x-coördinates of these points are the solutions of the equation, $x^2 - 4x + 3 = 0$.

In like manner, if the graph of any equation in the form $y=ax^2+bx+c$ is plotted, the x-coördinates of the points where the graph crosses the X-axis, will evidently be the solutions of $ax^2+bx+c=0$. The nature and approximate values of the solutions can therefore be determined from the graph.

234. (2) When $b^2>4\,ac$, and when $\sqrt{b^2-4\,ac}$ is irrational.

Plot the graph of $y=x^2-4\,x+2$.

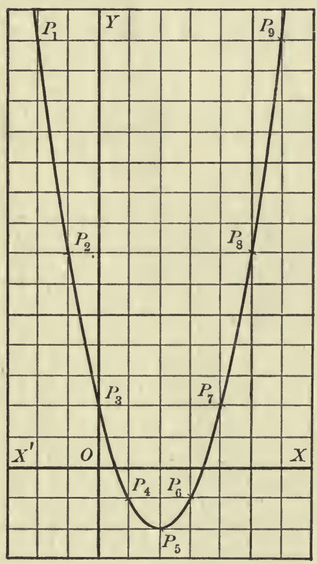

FIG. 13.

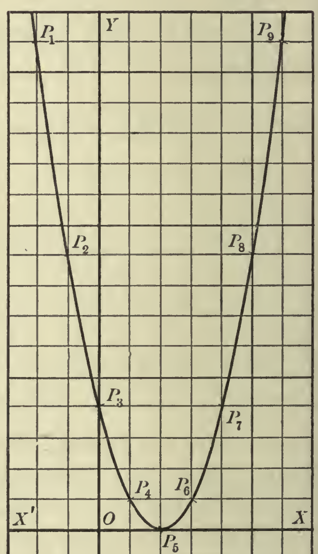

FIG. 14.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	14	7	2	-1	-2	-1	2	7	14

The graph is constructed as shown in Fig. 13, and is seen to cut the X-axis at points whose x-coördinates are between 0 and 1, and between 3 and 4. By the usual method of solving the equation $x^2 - 4x + 2 = 0$, the roots are found to be $2 \pm \sqrt{2}$, or 0.26795+ and 3.73205+. These must therefore be the exact values of x where the graph crosses the X-axis.

235. (3) When $b^2 = 4ac$.

1. Plot the graph of $y = x^2 - 4x + 4$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	16	9	4	1	0	1	4	9	16

Here the equation,

$$x^2 - 4x + 4 = 0,$$

has equal roots, $x = 2$, and the graph, Fig. 14, touches the X-axis at the single point P_5, whose coördinates are (2, 0).

236. (4) When $b^2 < 4ac$.

Plot the graph of

$$y = x^2 - 4x + 5.$$

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
$x=$	-2	-1	0	1	2	3	4	5	6
$y=$	17	10	5	2	1	2	5	10	17

If $y=0$, and the resulting equation, $x^2 - 4x + 5 = 0$, is solved, the roots are found to be $x = 2 \pm \sqrt{-1}$. Since these values are imaginary, they cannot represent any real distance. Hence the graph, Fig. 15, does not cut the X-axis.

The graphs of the equations which have been plotted have the same general shape, which will be found to be the same for all equations of the form $y = ax^2 + bx + c$. This curve is called the **parabola**.

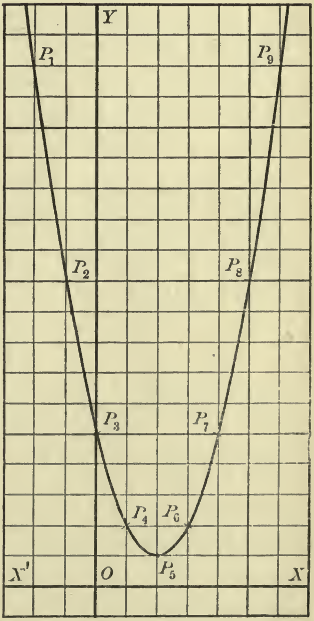

FIG. 15.

GRAPHS OF EQUATIONS CONTAINING y^2

237. 1. Plot the graph of $x^2+y^2=36$.

Solving $x^2+y^2=36$, $y=\pm\sqrt{36-x^2}$. The nature of $\sqrt{36-x^2}$ is such that if x takes any values less than -6 or greater than 6, y becomes imaginary. It is necessary to construct a table only for values of x between -6 and $+6$.

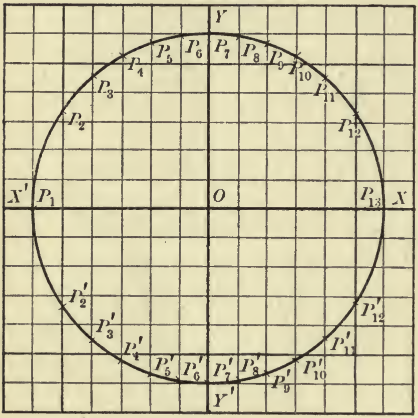

Fig. 16.

$x=$	± 6	± 5	± 4	± 3	± 2	± 1	0
$y=$	0	$\pm\sqrt{11}$	$\pm 2\sqrt{5}$	$\pm 3\sqrt{3}$	$\pm 4\sqrt{2}$	$\pm\sqrt{35}$	6

The graph is constructed as shown in Fig. 16, using approximations of the double values of the surd values of y. Points may be located closer together by taking fractional values of x, as $1\frac{1}{2}$, $1\frac{3}{4}$, etc. The graph is seen to be a circle.

EXERCISE CXXII

Plot the graphs of the following equations:

1. $y=x^2-7x+10$.
2. $y=x^2-3x+5$.
3. $y=x^2-2x+1$.
4. $y=x^2-7x+4$.
5. $y=x^2-5x+6$.
6. $y=x^2-6x+9$.
7. $x^2+y^2=25$.
8. $x^2+y^2=16$.
9. $y^2=4x$.
10. $x^2+y^2-x-8=0$.

CHAPTER XIX

SIMULTANEOUS EQUATIONS SOLVABLE BY QUADRATICS
TWO UNKNOWN QUANTITIES

238. A system of two simultaneous quadratic equations involving two unknown quantities cannot, in general, be solved by quadratics.

Solve the equations : $\begin{cases} x^2 - y^2 = 3, & (1) \\ x^2 + 2x + y = 8. & (2) \end{cases}$

Substituting in (2) the value of y in (1) and simplifying,

$$x^4 + 4x^3 - 13x^2 - 32x + 67 = 0. \quad (3)$$

Equation (3) cannot be solved by the method of quadratics ; and, in general, the solution of a pair of quadratic equations, chosen at random, will involve the solution of an equation of the fourth degree.

There are, however, certain forms of simultaneous equations which can be solved by means of quadratics.

SIMULTANEOUS EQUATIONS SOLVABLE BY QUADRATICS

239. In § 151 it was shown that the coördinates of the point of intersection of two lines were the values of x and y in the solution of the two equations which the lines represent, since the coördinates of this point must satisfy both equations. For the same reason, if the graphs of two quadratic equations or a simple and a quadratic equation are plotted, the coördinates of the points of intersection of these graphs must be the solutions of the pair of equations.

240. 1. Plot the graphs of the system : $\begin{cases} x+y=3, & (1) \\ x^2+y=5. & (2) \end{cases}$

By § 145, construct the graph, AB, of $x+y=3$.

By § 237, construct the graph of $x^2+y=5$.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$x=$	0	± 1	$\pm\sqrt{2}$	$\pm\sqrt{3}$	± 2	$\pm\sqrt{5}$	$\pm\sqrt{6}$	$\pm\sqrt{7}$
$y=$	5	4	3	2	1	0	-1	-2

Locate suitable points and draw the smooth curve $P_8P_1P'_8$.

The intersections of the graphs AB and the smooth curve $P'_8P_1P_8$, Fig. 17, will be points whose coördinates are solutions of the given system.

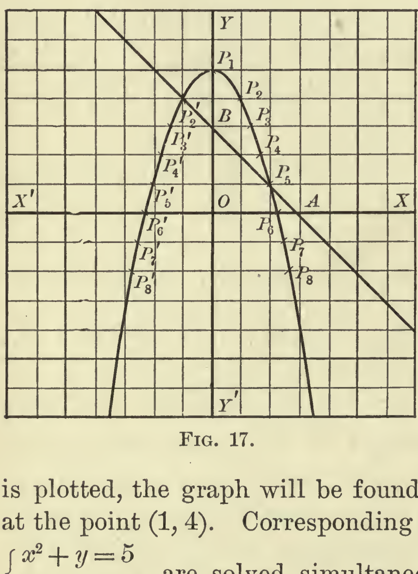

Fig. 17.

If, in place of $x+y=3$, the graph of $x+y=6$ is plotted, the graph will be found not to cut the parabola which is the graph of $x^2+y=5$. Corresponding to this non-intersection of the two graphs are found imaginary values for x and y when the equations $\left\{\begin{matrix} x^2+y=5 \\ x+y=6 \end{matrix}\right\}$ are solved simultaneously.

If the graph of $2x+y=6$ is plotted, the graph will be found just to touch the parabola at the point (1, 4). Corresponding to this fact, if the equations $\begin{cases} x^2+y=5 \\ 2x+y=6 \end{cases}$ are solved simultaneously, they have the single solution, $x=1$, $y=4$.

241. 1. Plot the graphs of the system: $\begin{cases} x^2+2y^2=64, & (1) \\ x^2-y^2=16. & (2) \end{cases}$

By the same method used in the preceding paragraph the graphs of the two equations are plotted as shown in Fig. 18.

They intersect in the four points, P_1, P_2, P_3, P_4, whose coördinates are found by measurement to agree with the solutions of the two equations,

$(4\sqrt{2}, 4)$, $(4\sqrt{2}, -4)$, $(-4\sqrt{2}, 4)$, $(-4\sqrt{2}, -4)$.

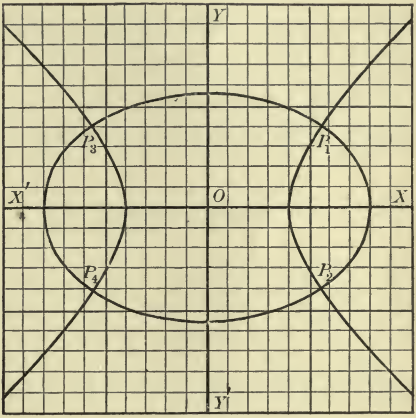

FIG. 18.

The graph of (1) is called an **ellipse**; of (2), an **hyperbola.**

EXERCISE CXXIII

Plot the graphs of the following systems and determine by measurement the coördinates of their points of intersection.

1. $\begin{cases} x+y=2, \\ xy=1. \end{cases}$

2. $\begin{cases} x+y=4, \\ x^2+y^2=10. \end{cases}$

3. $\begin{cases} x^2+y^2=5, \\ xy=2. \end{cases}$

4. $\begin{cases} x^2-y^2=16, \\ x+y=8. \end{cases}$

5. $\begin{cases} x^2-y^2=5, \\ xy=6. \end{cases}$

6. $\begin{cases} x^2-y^2=24, \\ 3x^2-20y^2=55. \end{cases}$

7. $\begin{cases} 4x^2-xy=6, \\ 3xy-y^2=5. \end{cases}$

8. $\begin{cases} x^2+y^2=17, \\ x^2-y^2=15. \end{cases}$

CASE I

242. A simple equation and a quadratic equation.

A system of simultaneous equations in which one equation is simple and the other quadratic can always be solved by substituting in the quadratic equation the value of one of the unknowns obtained from the simple equation.

1. Solve the equations : $\begin{cases} x + y = 7, & (1) \\ x^2 + 2y^2 = 34. & (2) \end{cases}$

Substituting in (2), $x = 7 - y$ from (1),

$$(7 - y)^2 + 2y^2 = 34, \qquad (3)$$

simplifying in (3), $3y^2 - 14y + 15 = 0,$ (4)

factoring in (4), $(y - 3)(3y - 5) = 0,$ (5)

from (5), $y = 3$, or $y = \frac{5}{3}$. (6)

Substituting values of y from (6) in (1),

$$x = 4, \text{ or } x = \frac{16}{3}. \qquad (7)$$

The given equations check if $x = 4$ and $y = 3$ be substituted; and the given equations also check if $x = \frac{16}{3}$ and $y = \frac{5}{3}$ be substituted. Such values of the unknowns which, taken together, satisfy the given equations are called *dependent values.*

Dependent values should always be found by substituting the value of the unknown first found in the ***simple*** *equation, and* ***never in the quadratic equation.***

It is to be noticed that the given equations are not verified by values which are not dependent.

243. The use of the double signs, $\pm$, read "plus or minus," and $\mp$, read "minus or plus," taken together are to be interpreted in the order in which the signs are read.

Thus, $\begin{cases} x = \pm 1, \\ y = \pm 2, \end{cases}$ is equivalent to $\begin{cases} x = +1, \\ y = +2, \end{cases}$ and $\begin{cases} x = -1, \\ y = -2. \end{cases}$

Similarly $\begin{cases} x = \mp 1, \\ y = \pm 2, \end{cases}$ is equivalent to $\begin{cases} x = -1, \\ y = +2, \end{cases}$ and $\begin{cases} x = +1, \\ y = -2. \end{cases}$

EXERCISE CXXIV

Solve the following systems of equations:

1. $\begin{cases} x - y = 3, \\ 3x^2 - 11y^2 = 1. \end{cases}$

2. $\begin{cases} xy - 5x = 1, \\ 7x - y = 1. \end{cases}$

3. $\begin{cases} x + 2y = 3, \\ 2x^2 + xy = 3. \end{cases}$

4. $\begin{cases} x - y = 4, \\ \dfrac{3x - 2}{y + 5} + \dfrac{y}{x} = 2. \end{cases}$

5. $\begin{cases} 2x - 3y = 2, \\ 3x^2 - 2y^2 = 115. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 50, \\ 9x + 7y = 80. \end{cases}$

7. $\begin{cases} x + 2y = 3, \\ 2x^2 + y^2 = \frac{18}{35}. \end{cases}$

8. $\begin{cases} x - \dfrac{x - y}{2} = 4, \\ y - \dfrac{x + 3y}{x + 2} = 1. \end{cases}$

9. $\begin{cases} 4x + 3y = 1, \\ \dfrac{5y}{x} + \dfrac{2y + 3}{x + y} = 0. \end{cases}$

10. $\begin{cases} \dfrac{1}{x(b - a)} - \dfrac{3}{y(a + b)} + \dfrac{1}{a^2 - b^2} = 0, \\ \dfrac{a}{y + 4b} = \dfrac{2b}{x - y}. \end{cases}$

CASE II

244 When one of two simultaneous quadratic equations is homogeneous.

A quadratic equation is said to be **homogeneous** when all of the terms involved are of the second degree in the unknown quantities.

Thus, $x^2 - 3xy + 2y^2 = 0$ is a homogeneous quadratic equation.

1. Solve the system: $\begin{cases} x^2 - xy - 2y^2 = 0, & (1) \\ x^2 + y = 5. & (2) \end{cases}$

Dividing (1) by y^2, $\left(\frac{x}{y}\right)^2 - \left(\frac{x}{y}\right) - 2 = 0,$ (3)

factoring (3), $\left(\frac{x}{y} - 2\right)\left(\frac{x}{y} + 1\right) = 0,$ (4)

from (4), $x = 2y$, or $x = -y$, (5)

substituting $x = 2y$ in (2), $4y^2 + y = 5,$ (6)

solving (6), $y = 1$, or $y = -\frac{5}{4}$, (7)

substituting values of y from (7) in (5), $x = 2$, or $x = -\frac{5}{2}$, (8)

substituting $x = -y$ in (2), $y^2 + y = 5,$ (9)

solving (9), $y = \dfrac{-1 \pm \sqrt{21}}{2},$ (10)

substituting $y = \dfrac{-1 \pm \sqrt{21}}{2}$ in (5), $x = \dfrac{1 \mp \sqrt{21}}{2}.$ (11)

The solutions are:

$$\begin{cases} x = 2, \\ y = 1, \end{cases} \quad \begin{cases} x = -\frac{5}{2}, \\ y = -\frac{5}{4}, \end{cases} \quad \begin{cases} x = \dfrac{1 - \sqrt{21}}{2}, \\ y = \dfrac{-1 + \sqrt{21}}{2}, \end{cases} \quad \begin{cases} x = \dfrac{1 + \sqrt{21}}{2}, \\ y = \dfrac{-1 - \sqrt{21}}{2}. \end{cases}$$

EXERCISE CXXV

Solve the following systems of equations:

1. $\begin{cases} x^2 + xy - 6y^2 = 0, \\ 2x^2 + 3y^2 = 11. \end{cases}$

2. $\begin{cases} 2x^2 - 3xy + y^2 = 0, \\ y^2 - x + 2y = 6. \end{cases}$

3. $\begin{cases} x^2 + xy = 0, \\ 2x^2 - 3x - y = 4. \end{cases}$

4. $\begin{cases} 2x^2 + xy - 10y^2 = 0, \\ x^2 + 3xy + y = -7. \end{cases}$

5. $\begin{cases} x^2 - 3xy + 2y^2 = 0, \\ xy - x + y = 4. \end{cases}$

6. $\begin{cases} x^2 - 9y^2 = 0, \\ 3x^2 + x - y = 29. \end{cases}$

7. $\begin{cases} x^2 - xy = 20y^2, \\ x^2 - x + y = 54. \end{cases}$

8. $\begin{cases} x^2 + y^2 = 2xy, \\ 2x^2 - xy + y = 30. \end{cases}$

9. $\begin{cases} x^2 - xy - 2y^2 = 0, \\ 2x^2 - 3x + y = 3. \end{cases}$

10. $\begin{cases} 5x^2 + 11xy + 2y^2 = 0, \\ x^2 - xy + y = 5. \end{cases}$

11. $\begin{cases} 3x^2 + 2xy - y^2 = 0, \\ x - 2y + 3y^2 = 32. \end{cases}$

12. $\begin{cases} 15x^2 - 34xy + 15y^2 = 0, \\ x + y - 2y^2 = -10. \end{cases}$

13. $\begin{cases} 8x^2 + 2xy - 3y^2 = 0, \\ x^2 + x + y^2 = 22. \end{cases}$

14. $\begin{cases} 3x^2 + 8xy + 5y^2 = 0, \\ 3x^2 + 4xy + y = -30. \end{cases}$

15. $\begin{cases} 2x^2 + 9xy = 35y^2, \\ 2x(x + y) - 11y = 236. \end{cases}$

16. $\begin{cases} 6x^2 = 11xy + 35y^2, \\ x^2 - 17xy - 180y = -260. \end{cases}$

17. $\begin{cases} 9x^2 - 39xy + 22y^2 = 0, \\ 3x^2 - 7x + y = 289. \end{cases}$

18. $\begin{cases} 10x^2 + 23xy + 12y^2 = 0, \\ 9x^2 + 7xy + 6y = 132. \end{cases}$

CASE III

245. When each of two simultaneous quadratic equations is homogeneous only in the unknowns involved.

A system of two simultaneous quadratic equations which are homogeneous except in the absolute terms may be solved as in Case II, by combining such multiples of the two equations as will make equal the absolute terms.

1. Solve the system: $\begin{cases} x^2 + xy = 12, & (1) \\ xy - 2y^2 = 1. & (2) \end{cases}$

Multiplying (2) by 12, $\quad 12xy - 24y^2 = 12, \quad (3)$

subtracting (3) from (1), $\quad x^2 - 11xy + 24y^2 = 0. \quad (4)$

Equation (4) may be solved as in Case II; or it may be solved by factoring.

Factoring (4), $\quad (x - 3y)(x - 8y) = 0, \quad (5)$

from (5), $\quad x = 3y,$ or $x = 8y, \quad (6)$

substituting $x = 3y$ and $x = 8y$ in (1), and solving the resulting equations,
$$y = \pm 1,\ y = \pm \tfrac{1}{6}\sqrt{6},$$

by substitution in (6), $\quad x = \pm 3,\ x = \pm \tfrac{4}{3}\sqrt{6}.$

The solutions are: $\begin{cases} x=3, \\ y=1, \end{cases} \begin{cases} x=-3, \\ y=-1, \end{cases} \begin{cases} x=\tfrac{4}{3}\sqrt{6}, \\ y=\tfrac{1}{6}\sqrt{6}, \end{cases} \begin{cases} x=-\tfrac{4}{3}\sqrt{6}, \\ y=-\tfrac{1}{6}\sqrt{6}. \end{cases}$

246. An alternative method for solving equations of the class of Case III is called the ***vx* method.**

1. Solve the system: $\begin{cases} x^2 + xy + 4y^2 = 6, & (1) \\ 3x^2 + 8y^2 = 14. & (2) \end{cases}$

Let $y = vx$, and substitute in (1) and in (2),

$$x^2 + x^2v + 4x^2v^2 = 6, \quad (3) \qquad 3x^2 + 8x^2v^2 = 14, \quad (4)$$

factoring, $\quad x^2(1 + v + 4v^2) = 6, \quad (5) \qquad x^2(3 + 8v^2) = 14, \quad (6)$

$$x^2 = \frac{6}{1+v+4v^2}, \quad (7) \qquad\qquad x^2 = \frac{14}{3+8v^2}, \quad (8)$$

equating x^2 in (7) and (8), $\dfrac{6}{1+v+4v^2} = \dfrac{14}{3+8v^2}$, (9)

clearing and simplifying in (9), $4v^2 + 7v - 2 = 0$, (10)

from (10), $v = \frac{1}{4}$, or $v = -2$, (11)

substituting values of v from (11) in (7),

$$\begin{cases} x^2 = \dfrac{6}{1+\frac{1}{4}+\frac{1}{4}} = 4, & (12) \\[2ex] x^2 = \dfrac{6}{1-2+16} = \dfrac{6}{15}, & (13) \end{cases}$$

extracting square roots in (12) and in (13),

$$\begin{cases} x = \pm 2, & (14) \\ x = \pm \frac{1}{5}\sqrt{10}. & (15) \end{cases}$$

When $v = \frac{1}{4}$, $x = \pm 2$; substituting $v = \frac{1}{4}$ in $y = vx$,

$$y = \tfrac{1}{4}(\pm 2) = \pm \tfrac{1}{2}. \qquad (16)$$

When $v = -2$, $x = \pm \frac{1}{5}\sqrt{10}$; substituting $v = -2$ in $y = vx$,

$$y = \mp \tfrac{2}{5}\sqrt{10}. \qquad (17)$$

The solutions are:

$$\begin{cases} x = 2, \\ y = \frac{1}{2}, \end{cases} \quad \begin{cases} x = -2, \\ y = -\frac{1}{2}, \end{cases} \quad \begin{cases} x = \frac{1}{5}\sqrt{10}, \\ y = -\frac{2}{5}\sqrt{10}, \end{cases} \quad \begin{cases} x = -\frac{1}{5}\sqrt{10}, \\ y = \frac{2}{5}\sqrt{10}. \end{cases}$$

The values of x must always be substituted in $y = vx$.

Since equations of the type of Case III may be reduced to a quadratic equation homogeneous in all its terms, and since such an equation may always be expressed as a quadratic in $\frac{x}{y}$, for $\frac{x}{y}$ any quantity v may be substituted. If $\frac{x}{y} = v$, $x = vy$; if $\frac{y}{x} = v$, $y = vx$.

EXERCISE CXXVI

Solve the following systems of equations:

1. $\begin{cases} x^2 + 3\,xy = 27, \\ xy + 2\,y^2 = 14. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 20, \\ x^2 - xy = 8. \end{cases}$

3. $\begin{cases} xy + 4 = 0, \\ 9\,x^2 - y^2 = 7. \end{cases}$

4. $\begin{cases} x^2 - xy = 15, \\ x^2 - y^2 = 21. \end{cases}$

5. $\begin{cases} 2\,x^2 + xy = 52, \\ 2\,y^2 - xy = 30. \end{cases}$

6. $\begin{cases} xy = 4 - y^2, \\ 2\,x^2 - y^2 = 17. \end{cases}$

7. $\begin{cases} 2\,x^2 - 3\,y^2 = 60, \\ 3\,x^2 - 4\,xy + y^2 = 64. \end{cases}$

8. $\begin{cases} 6\,x^2 - 5\,xy + 2\,y^2 = 12, \\ 3\,x^2 + 2\,xy - 3\,y^2 = -3. \end{cases}$

9. $\begin{cases} 2\,x^2 - 2\,xy - y^2 = 3, \\ x^2 + 3\,xy + y^2 = 11. \end{cases}$

10. $\begin{cases} 3\,x^2 - 7\,xy + 4\,y^2 = -1, \\ 2\,x^2 + xy - 3\,y^2 = 22. \end{cases}$

CASE IV

247. When two simultaneous quadratic equations are each symmetric with respect to the unknowns involved.

An equation is said to be **symmetric** with respect to the unknowns involved when the interchange of the unknowns does not change the form of that equation.

Thus, $x^2 + xy + y^2 = 7$, and $xy + x + y = 5$, are symmetric quadratic equations.

A solution of a system of such equations may always be found by substituting $x = u + v$, and $y = u - v$, in the given equations.

Solve the system : $\begin{cases} x^2 + xy + y^2 = 7, & (1) \\ xy + x + y = 5. & (2) \end{cases}$

Let $x = u + v$, and let $y = u - v$.

Substituting $x = u + v$, and $y = u - v$, in (1) and in (2),

$$(u+v)^2 + (u+v)(u-v) + (u-v)^2 = 7, \qquad (3)$$

$$(u+v)(u-v) + (u+v) + (u-v) = 5, \qquad (4)$$

simplifying in (3) and in (4), $3u^2 + v^2 = 7, \qquad (5)$

$$u^2 + 2u - v^2 = 5, \qquad (6)$$

transposing in (5) and in (6), $v^2 = 7 - 3u^2, \qquad (7)$

$$v^2 = u^2 + 2u - 5, \qquad (8)$$

equating v^2 in (7) and in (8), $7 - 3u^2 = u^2 + 2u - 5, \qquad (9)$

solving (9), $u = \frac{3}{2}$, or $u = -2. \qquad (10)$

Substituting $u = \frac{3}{2}$, in (5), $v = \pm\frac{1}{2}$,

$$x = u + v = \tfrac{3}{2} \pm \tfrac{1}{2} = 2, \text{ or } 1,$$

and, $y = u - v = \frac{3}{2} \mp \frac{1}{2} = 1$, or 2.

Substituting $u = -2$, in (5), $v = \pm\sqrt{-5}$,

$$x = u + v = -2 \pm \sqrt{-5},$$

and, $y = u - v = -2 \mp \sqrt{-5}$.

The solutions are:

$$\begin{cases} x = 2, \\ y = 1, \end{cases} \quad \begin{cases} x = 1, \\ y = 2, \end{cases} \quad \begin{cases} x = -2 + \sqrt{-5}, \\ y = -2 - \sqrt{-5}, \end{cases} \quad \begin{cases} x = -2 - \sqrt{-5}, \\ y = -2 + \sqrt{-5}. \end{cases}$$

Two simultaneous quadratic equations which are symmetric, except in respect to signs, can often be solved by Case IV.

The proof that equations of the type of Case IV can be solved by substituting $x = u + v$, and $y = u - v$, is beyond the province of this book.

EXERCISE CXXVII

Solving the following systems of equations:

1. $\begin{cases} x + xy + y = 29, \\ x^2 + xy + y^2 = 61. \end{cases}$

2. $\begin{cases} 3(x^2 + y^2) - 5xy = 15, \\ 3(x + y) = 4xy. \end{cases}$

3. $\begin{cases} x^2 + y^2 - x - y = 22, \\ x + y + xy = -1. \end{cases}$

4. $\begin{cases} xy + x(x + 1) + y(y + 1) = 24, \\ xy = 6. \end{cases}$

5. $\begin{cases} xy - 2x^2 - 2y^2 = -20, \\ 4xy + x + y = 29. \end{cases}$

6. $\begin{cases} 3x^2 + 3y^2 = 8(x + y) - 1, \\ xy - x - y = 1. \end{cases}$

7. $\begin{cases} x^2 + y^2 + xy + x + y = 17, \\ x^2 + y^2 - 3xy + 2x + 2y = 9. \end{cases}$

8. $\begin{cases} 2x + 2y + xy = 16, \\ 3x(1 + x) + 3y(1 + y) = 54. \end{cases}$

9. $\begin{cases} x^2 + y^2 + x + y = 62, \\ 5xy + 4(x^2 + y^2) = 328. \end{cases}$

10. $\begin{cases} x^2 + 2xy + y^2 + 5x + 5y = 84, \\ x^2 + y^2 + x + y = 32. \end{cases}$

11. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{1}{y^2} = 7, \\ x(x - y) + y(x + y) = \frac{3}{4} + xy. \end{cases}$

12. $\begin{cases} x^2 + y^2 + x + y = a^2, \\ xy + x + y = \dfrac{3a}{2}. \end{cases}$

SPECIAL DEVICES

248. Special devices may be employed in finding solutions by shorter methods for some of the systems in the preceding cases, as well as for certain other systems whose equations are often of higher degree than the second.

1. Solve the system: $\begin{cases} x+y=3, & (1) \\ x^2+y^2=29. & (2) \end{cases}$

Squaring (1) and subtracting from (2),

$$-2xy=20, \qquad (3)$$

adding (3) and (2), $x^2-2xy+y^2=49,$ (4)

extracting square roots in (4), $x-y=\pm 7,$ (5)

adding (5) and (1), $x=5, \text{ or } -2,$ (6)

subtracting (1) from (5), $y=-2, \text{ or } 5.$ (7)

The solutions are: $\begin{cases} x=5, \\ y=-2, \end{cases} \quad \begin{cases} x=-2, \\ y=5. \end{cases}$

2. Solve the system: $\begin{cases} x^3+y^3=1001, & (1) \\ x+y=11. & (2) \end{cases}$

Dividing (1) by (2), $x^2-xy+y^2=91,$ (3)

squaring (2) and subtracting from (3),

$$-3xy=-30, \qquad (4)$$

dividing (4) by -3 and subtracting from (3),

$$x^2-2xy+y^2=81, \qquad (5)$$

extracting square roots in (5), $x-y=\pm 9,$ (6)

combining (2) and (6), $x=10 \text{ or } 1, \; y=1 \text{ or } 10.$ (7)

The solutions are: $\begin{cases} x=10, \\ y=1, \end{cases} \quad \begin{cases} x=1, \\ y=10. \end{cases}$

EXERCISE CXXVIII

Solve the following systems of equations:

1. $\begin{cases} x+y=6, \\ xy=5. \end{cases}$

2. $\begin{cases} x+y=20, \\ xy=51. \end{cases}$

3. $\begin{cases} x^2+y^2=170, \\ xy=13. \end{cases}$

4. $\begin{cases} x^2+y^2=34, \\ xy=15. \end{cases}$

5. $\begin{cases} x^2+y^2=25, \\ 2\,xy=24. \end{cases}$

6. $\begin{cases} x+y=12, \\ x^2+y^2=74. \end{cases}$

7. $\begin{cases} x-y=2, \\ x^2-y^2=20. \end{cases}$

8. $\begin{cases} x^2+y^2=34, \\ x+y=8. \end{cases}$

9. $\begin{cases} x^2+y^2=74, \\ x-y=2. \end{cases}$

10. $\begin{cases} xy=a, \\ x^2+y^2=b. \end{cases}$

11. $\begin{cases} x+y=a, \\ x^2-y^2=b. \end{cases}$

12. $\begin{cases} y^2+xy=15, \\ x^2+xy=10. \end{cases}$

13. $\begin{cases} x^2+y^2=436, \\ x-y=14. \end{cases}$

14. $\begin{cases} x^2+xy=15, \\ xy+y^2=3. \end{cases}$

15. $\begin{cases} x^2-xy+y^2=13, \\ x+y=-2. \end{cases}$

16. $\begin{cases} x^2+3\,xy=28, \\ xy+4\,y^2=8. \end{cases}$

17. $\begin{cases} x+xy+y=29, \\ x^2+xy+y^2=61. \end{cases}$

18. $\begin{cases} x^2+xy+y^2=19, \\ x^2-xy+y^2=7. \end{cases}$

19. $\begin{cases} 2\,x^2+5\,xy=38, \\ 2\,y^2-xy=12. \end{cases}$

20. $\begin{cases} x^2+5\,xy+y^2=43, \\ x^2+5\,xy-y^2=25. \end{cases}$

21. $\begin{cases} x^3-y^3=98, \\ x-y=2. \end{cases}$

22. $\begin{cases} x^3+y^3=35, \\ x+y=5. \end{cases}$

23. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=7, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=25. \end{cases}$

24. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=b. \end{cases}$

25. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=5, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=13. \end{cases}$

26. $\begin{cases} x^2-xy=153, \\ x+y=1. \end{cases}$

27. $\begin{cases} x+y=3, \\ x^4+y^4=17. \end{cases}$

28. $\begin{cases} x^2+xy=10, \\ xy-y^2=-3. \end{cases}$

29. $\begin{cases} x+y=-3, \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{6}. \end{cases}$

30. $\begin{cases} 5\,xy=84-x^2y^2, \\ x-y=6. \end{cases}$

31. $\begin{cases} x+y+3=0, \\ x^2+2\,y^2=8. \end{cases}$

32. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=2, \\ xy+\dfrac{1}{x}+\dfrac{1}{y}=8. \end{cases}$

33. $\begin{cases} \dfrac{x^2+y^2}{x^2-y^2}=\dfrac{29}{21}, \\ x+y=7. \end{cases}$

34. $\begin{cases} x+\dfrac{1}{y}=3, \\ y+\dfrac{1}{x}=\dfrac{12}{5}. \end{cases}$

35. $\begin{cases} x+\dfrac{1}{y}=1, \\ y+\dfrac{1}{x}=4. \end{cases}$

36. $\begin{cases} xy-\dfrac{x}{y}=2, \\ xy-\dfrac{y}{x}=\dfrac{1}{2}. \end{cases}$

37. $\begin{cases} ay^2+bxy=b, \\ bx^2+axy=a. \end{cases}$

38. $\begin{cases} \dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=10, \\ \dfrac{ab}{xy}=3. \end{cases}$

39. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=1, \\ \dfrac{a}{x}+\dfrac{b}{y}=4. \end{cases}$

40. $\begin{cases} \sqrt{x+y}+\sqrt{x-y}=4, \\ x^2-y^2=9. \end{cases}$

41. $\begin{cases} x+y+\sqrt{x+y}=12, \\ xy=20. \end{cases}$

42. $\begin{cases} x^2y-xy^2=12, \\ x^3-y^3=63. \end{cases}$

THREE OR MORE UNKNOWN QUANTITIES

249.* Three simultaneous quadratic equations involving three unknown quantities cannot in general be solved by quadratic equations. The solutions of certain forms are illustrated in the following examples.

1. Solve the system : $\begin{cases} x^2+2y^2-z^2=5, & (1) \\ 2x+y+z=6, & (2) \\ x+4y-z=5. & (3) \end{cases}$

Eliminating z in (2) and (3), $x=\dfrac{11-5y}{3}$, (4)

eliminating x in (2) and (3), $z=\dfrac{7y-4}{3}$, (5)

substituting x and z from (4) and (5) in (1),

$$\left(\frac{11-5y}{3}\right)^2+2y^2-\left(\frac{7y-4}{3}\right)^2=5, \quad (6)$$

simplifying and solving (6), $y=1$, or $y=-10$. (7)

Substituting values of y in (4) and (5), $\begin{cases} x=2, \\ y=1, \\ z=1, \end{cases}$ $\begin{cases} x=\frac{61}{3}, \\ y=-10, \\ z=-\frac{74}{3}. \end{cases}$

2. Solve the system : $\begin{cases} x(y+z)=-4, & (1) \\ y(x+z)=-10, & (2) \\ z(x+y)=-54. & (3) \end{cases}$

Dividing the sum of (1), (2), and (3) by 2,

$$xy+yz+xz=-34, \quad (4)$$

subtracting (1) from (4), $y=-\dfrac{30}{z}$, (5)

subtracting (2) from (4), $x=-\dfrac{24}{z}$, (6) substituting y and x from (5) and (6) in (4),

$$\frac{720}{z^2} - 30 - 24 = -34, \tag{7}$$

solving (7), $z = \pm 6,$ (8)

substituting z from (8) in (6), $x = \mp 4,$ (9)

substituting z from (8) in (5), $y = \mp 5.$ (10)

EXERCISE CXXIX*

Solve the following systems of equations:

1. $\begin{cases} xy = -42, \\ xz = 48, \\ yz = -56. \end{cases}$

2. $\begin{cases} x(y+z) = 8, \\ y(z+x) = 18, \\ z(x+y) = 20. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 13, \\ y^2 + z^2 = 25, \\ z^2 + x^2 = 20. \end{cases}$

4. $\begin{cases} xy + xz + yz = 3, \\ x + 2y + 3z = 6, \\ 3x + 2y + z = 6. \end{cases}$

5. $\begin{cases} x + y = 15, \\ u + z = 3, \\ x + u^2 = 8, \\ y + z^2 = 12. \end{cases}$

6. $\begin{cases} x^2 + xy + y^2 = 19, \\ y^2 + yz + z^2 = 37, \\ z^2 + xz + x^2 = 28. \end{cases}$

7. $\begin{cases} x^2 + xy + z = 2, \\ x + 2y + z = 3, \\ x - y + z = 0. \end{cases}$

8. $\begin{cases} x^2 + y^2 + z^2 = 21, \\ xy + xz + yz = 14, \\ x + y - z = -1. \end{cases}$

9. $\begin{cases} x + y + z = 4, \\ xy + xz + yz = -4, \\ x - y + z = 8. \end{cases}$

10. $\begin{cases} xy + zu = 14, \\ xz + yu = 11, \\ xu + yz = 10, \\ x + y + z + u = 10. \end{cases}$

REVIEW EXERCISE CXXX

Solve the following equations:

1. $\dfrac{x^2-1}{3x^2-5}=\dfrac{5x-1}{15x-9}.$

2. $\sqrt{4x-3}-\sqrt{x+1}=1.$

3. $\dfrac{5}{x+2}-\dfrac{2x-3}{2(x-2)}=-\dfrac{3}{6}.$

4. $x^3-8x^{-3}=7.$

5. $a^2x-2b^2=ab\dfrac{x^2+1}{x+1}.$

6. $\dfrac{x^2+1}{x}=\dfrac{a+b}{c}+\dfrac{c}{a+b}.$

7. $3x-\dfrac{2x+4}{3x-5}=4-\dfrac{2x-3}{2}.$

8. $\dfrac{a-c}{x-a}-\dfrac{x-a}{a-c}=\dfrac{3b(x-c)}{(a-c)(x-a)}.$

9. $\sqrt{x+3}+\sqrt{x+6}-\sqrt{x+11}=0.$

10. $x^2+8x+6\sqrt{x^2+8x-8}-3=0.$

Solve the following systems of equations:

11. $\begin{cases}\dfrac{4}{x}+\dfrac{7}{y}=\dfrac{8a}{xy},\\ x^2+y^2=a^2.\end{cases}$

12. $\begin{cases}x^2-xy=5,\\ x=\sqrt{y^2+9}.\end{cases}$

13. $\begin{cases}x^2+xy+y=1,\\ 3x+2y-5=0.\end{cases}$

14.* $\begin{cases}xy=12,\\ zx=15,\\ yz=20.\end{cases}$

15. $\begin{cases}x^2-y^2=1,\\ x^3-y^3=3.\end{cases}$

16. $\begin{cases}x^2+5xy+3y^2=3,\\ 3x^2+7xy+4y^2=5.\end{cases}$

17. $\begin{cases}xy=-1,\\ 4x^2+(2y-x)(2y+x)=7.\end{cases}$

18. $$\begin{cases} \dfrac{ay}{x+a} - \dfrac{bx}{y-b} = \dfrac{a-b}{2}, \\ \dfrac{x}{a} - \dfrac{y}{b} = 2. \end{cases}$$

19. $$\begin{cases} y + \sqrt{x^2-1} = 2, \\ \sqrt{x+1} - \sqrt{x-1} = \sqrt{y}. \end{cases}$$

20. $$\begin{cases} (x+y)^2 + 3x + 3y + 2 = 0, \\ 2xy + 4x - y - 2 = 0. \end{cases}$$

21. Construct the equation whose roots are $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$.

22. What must be the value of c if the roots shall be equal in the equation, $3x^2 - 2x + c = 0$?

23. Determine the values of k if the roots of the equation, $kx^2 + 2kx - 3x + 2 = 0$, are real and equal, and verify the results.

24. Determine without solving the nature of the roots of $2x^2 - 3x + 5 = 0$; $5x^2 - 6x + 1 = 0$.

25. Find the values of k in order that the equation, $(x^2 - 3x + 2) + k(x^2 - x) = 0$, may have equal roots.

26. The two distinct equations, $x^2 + 2px + q = 0$, $x^2 + 2qx + p = 0$, are such that the roots of the first have the same difference as the roots of the second. Prove that either $p + q = -1$, or $p = q$.

CHAPTER XX

PROBLEMS INVOLVING QUADRATIC EQUATIONS

250. Since the two roots of a quadratic equation can be rational, irrational, or imaginary, problems solved by means of such equations can have apparently such solutions. But because it is impossible to translate all the restrictions expressed or implied in the problem, into the equations formed from the conditions of the problem, solutions must always be verified by substitution in the problem itself.

EXAMPLES

1. One of the two factors of 108 exceeds the other by 3. Find the factors.

Let $x =$ the first factor, and $x + 3 =$ the second factor.

By the conditions, $$x(x+3) = 108. \quad (1)$$

Solving (1), $x = 9$ or -12; whence $x + 3 = 12$, or -9.

Hence the factors of 108 are 9 and 12; or -12 and -9.

Each of the above solutions satisfies (1) and the problem; but if restrictions were imposed that both factors should be positive, the second pair would be rejected; and if it were necessary that factors should be negative, the first pair would be rejected.

2. A company of 76 men and boys are seated in chairs arranged in such a way that the number of chairs in each row is 3 more than twice the number of boys; and that

the number of rows is 4 less than the number of boys. Find the number of boys.

Let $x =$ the number of boys.

By the conditions, $(2x + 3)(x - 4) = 76.$ (1)

Solving (1), $x = 8$, or $-\frac{5}{2}$.

The restriction implied in the problem is that the solution shall be in positive integers, since it is absurd to speak of $\frac{5}{2}$ of a boy. Hence the root $-\frac{5}{2}$ must be rejected as a solution of the problem.

In the following problems if possible use a single unknown, rather than several unknowns.

EXERCISE CXXXI

1. The product of a number and its half is 18. Find the number.

2. The product of the third and seventh parts of a number is 21. Find the number.

3. What number is $2\frac{1}{4}$ times its reciprocal?

4. Find a number the sum of which and its reciprocal is 2.

5. Find a number the sum of which and 12 times its reciprocal is 8.

6. The sum of the squares of two consecutive integers is 145. Find the numbers.

7. One of two factors of a number exceeds the other by 2. If the product of the factors is 80, find the numbers.

8. The product of two factors of a number is $18\frac{3}{4}$. Find these factors if one factor exceeds the other by 5.

9. The sum of two numbers is 9, and their product is 18. Find the numbers.

10. The sum of two numbers is 7, and the sum of their squares is 29. Find the numbers.

11. The difference of two numbers is 7, and their product is 120. Find the numbers.

12. The difference of two numbers is 4, and the difference of their squares is 72. Find the numbers.

13. The sum of two numbers is 8, and the sum of their cubes is 152. Find the numbers.

14. Find two numbers such that the sum of the numbers and the difference of their squares is 11.

15. Find two numbers such that their sum is 15, and their product is 36.

16. If the length and breadth of a rectangle are each increased by 4 feet, the area is increased by 100 square feet; but if the length and breadth are each diminished by 1 foot, the area is 88 square feet. Find the dimensions.

17. A rectangle whose area is 160 square inches is surrounded by a border 2 inches wide. The border contains 120 square inches. Find the dimensions of the rectangle.

18. The diagonal of a rectangle is 50 feet, and the perimeter is 140 feet. Find the area.

19. Find the length of a rectangle whose area is 1161 square feet, if the sum of its length and breadth is 70 feet.

20. A number of men each subscribed a certain amount to take up a deficit of \$100; but 5 men failed to pay and thus increased the share of the others by \$1 each. Find the share of each.

21. It took as many days to do a piece of work as there were men; but if there had been 4 more men, these men could have done the work in 9 days. Find the number of men.

22. Divide 10 into two such parts that their product shall be 12 times their difference.

23. A number exceeds a second number by 4. Find these numbers if the sum of their reciprocals is $\frac{4}{15}$.

24. In a number of two digits the units' digit exceeds the tens' digit by 4, and the product of the number and the tens' digit is 192. Find the number.

25. A can do a piece of work in 3 more days than B; and both can do the work in $5\frac{1}{7}$ days. How long will it take each alone?

26. Divide 10 into two such parts that the quotient of 10 and the greater part equals the quotient of the greater and less part.

27. The quotient of a number of two digits, divided by the sum of the digits, is 6; and if the sum of the squares of the digits be subtracted from the number, the remainder is 13. Find the number.

28. A sold goods for \$56, and gained as many per cent as the goods cost. How much did the goods cost?

29. A number exceeds a second number by 5; the difference of their cubes is 665. Find the numbers.

30. Separate 250 into two such numbers that the sum of their square roots shall be 22.

31. If A had sold 7 books less for \$42, he would have received \$1 a book more. Find the price of each book.

32. A sold a number of yards of cloth for \$40. Had the price of a yard been 50 cents less he could have sold 4 more yards for the same money. Find the price per yard.

33. A bought two pieces of cloth, which together measured 36 yards. Each piece cost as many dollars per yard as there were yards in the piece, and the cost of the first was 4 times the cost of the second piece. Find the number of yards in each piece.

34. A can row in still water $1\frac{1}{2}$ miles an hour faster than the current. It takes him 8 hours to make a round trip of 18 miles. Find the rate of the current.

35. A tap A can fill a cistern in 9 minutes less than a second tap B can empty it. If A and B are running, it takes 3 hours to fill the cistern. How long will it take B alone to empty it?

36. In a number of two digits the tens' digit is double the units' digit; and if the number be multiplied by the sum of the digits, the product is 567. Find the number.

37. Find two numbers whose difference multiplied by the greater produces 35, and whose sum multiplied by the less produces 18.

38. What is the price of eggs when 10 more for \$1 lowers the price 4 cents per dozen?

39. A sum of money at simple interest for 1 year amounted to \$20,800; if the rate were 1% less, the amount would be \$200 less. Find the principal and the rate per cent.

40. A party of friends went on a pleasure excursion, the expense of which they share equally. If the number of the party had been decreased by 7, and if the total expenses had been \$150, the assessment for each person would have been \$1 more than it was; but if the number of the party had been increased by 8, and if the total expense had been \$160, the assessment for each person would have been \$1 less than it was. Find the number of the party, and the assessment for each person.

41. A and B had a money box containing \$210, from which each drew a certain sum daily — this sum being fixed for each, but different for the two. After 6 weeks, the box was empty. Find the sum which each drew daily from the box, knowing that A alone would have emptied it 5 weeks earlier than B alone.

42. On a certain road the telegraph poles are placed at equal intervals, and their number per mile is such that if that number were less by 1, each interval between two poles would be increased by $2\frac{14}{15}$ yards. Find the number of poles, and the number of intervals in a mile.

43. A broker sells certain railroad shares for \$3240. A few days later, the price having fallen \$9 per share, he buys, for the same sum, 5 more shares than he had sold. Find the price and the number of shares transferred on each day.

CHAPTER XXI

RATIO, PROPORTION, VARIATION

RATIO

251. The **ratio of one number to another number** is the quotient obtained by dividing the first by the second number. The quotient shows how the numbers compare.

Thus, the ratio of 5 to 7 is indicated: $5 \div 7$, $\frac{5}{7}$, $5:7$.

The **ratio of one quantity to another quantity of the same kind** is the ratio of the numerical values of the quantities.

Thus, the ratio of a dollars to b dollars is $\dfrac{a}{b}$.

The **terms of a ratio** are the terms of the fraction indicating the ratio; the numerator is called the **antecedent**, and the denominator the **consequent** of the ratio.

Thus, a and b are the terms, a is the antecedent, and b the consequent of the ratio $\dfrac{a}{b}$.

There is *no ratio of one quantity to another of a different kind*, since it is impossible to compare such quantities.

Thus, no ratio exists between a inches and b pounds.

252. If the ratio of two quantities can be expressed as a rational number, they are said to be **commensurable**; if the ratio of two quantities is an irrational number, they are said to be **incommensurable**.

Thus, when $\frac{a}{b} = \frac{3}{4}$, a and b are commensurable; when $\frac{a}{b} = \sqrt{2}$, a and b are incommensurable.

The ratio of two commensurable quantities is called a **commensurable ratio**; the ratio of two incommensurable quantities is called an **incommensurable ratio.**

Thus, when $\frac{a}{b} = 5$ and when $\frac{a}{b} = \sqrt{3}$, 5 and $\sqrt{3}$ are respectively commensurable and incommensurable ratios.

253.* *An incommensurable ratio can always be expressed as a commensurable ratio whose value differs from the incommensurable ratio by less than any assigned quantity, however small.*

If a is a diagonal of a square of which b is a side, $\frac{a}{b} = \sqrt{2}$. In § 185 it was shown that $\sqrt{2}$ may be determined to any required degree of accuracy.

In general, let a and b be any two incommensurable quantities. Let p be contained in b integrally (say) m times, and let p be contained in a more than (say) n times, and less than $n + 1$ times. That is, let,

$$mp = b, \tag{1}$$

$$np < a, \tag{2}$$

$$a < (n + 1)p. \tag{3}$$

Dividing (2) by (1), $\frac{n}{m} < \frac{a}{b}$, or $\frac{a}{b} > \frac{n}{m}$, (4)

Dividing (3) by (1), $\frac{a}{b} < \frac{n}{m} + \frac{1}{m}$. (5)

Since from (4), $\frac{a}{b} > \frac{n}{m}$, and from (5), $\frac{a}{b} < \frac{n}{m} + \frac{1}{m}$, $\frac{a}{b}$ differs from $\frac{n}{m}$ by less than $\frac{1}{m}$; or, $\frac{a}{b} - \frac{n}{m} < \frac{1}{m}$ (6)

Since it is always true that $mp = b$, by taking p smaller and smaller, m will increase: hence $\frac{1}{m}$ will decrease and may be made less than any assigned quantity. Therefore $\frac{a}{b}$ can be made to differ from the commensurable ratio $\frac{n}{m}$ by less than any assigned quantity, however small.

NOTE. If p is very small, $\frac{a}{b}$ is nearly equal to $\frac{n}{m}$; but $\frac{a}{b} \neq \frac{n}{m}$.

254. The reciprocal of a given ratio is called an **inverse ratio.**

Thus, $\frac{3}{7}$ is the inverse ratio of $\frac{7}{3}$.

A **ratio of equality** is one in which the antecedent and consequent are equal; a ratio of **greater inequality** is one in which the antecedent is greater than the consequent; a ratio of **less inequality** is one in which the antecedent is less than the consequent.

Thus, $\frac{3}{3}$, $\frac{7}{4}$, $\frac{5}{6}$, are respectively ratios of equality, greater inequality, and less inequality.

The ratio found by squaring the terms of a given ratio is called a **duplicate ratio**; the ratio found by cubing the terms of a given ratio is called a **triplicate ratio**.

Thus, $\frac{a^2}{b^2}$ and $\frac{a^3}{b^3}$ are respectively the duplicate and the triplicate ratios of $\frac{a}{b}$.

EXERCISE CXXXII

1. Express the ratio of 5 to 7; $4\frac{1}{2}$ to 12; 6 to 1; $3\frac{1}{11}$ to $7\frac{1}{9}$.

2. Express the ratio of a cents to b cents; m inches to n inches; c dollars to a dollars; m^2 feet to n^2 inches.

3. Determine which of the following ratios are commensurable:

$$\frac{2}{3},\ \frac{m}{n},\ \frac{6\sqrt{2}}{\sqrt{2}},\ \frac{6\sqrt{2}}{\sqrt{3}},\ \frac{5\frac{1}{7}}{2\frac{1}{11}},\ \frac{a\sqrt{b}}{c\sqrt{b}}.$$

4. Determine which of the following ratios are incommensurable:

$$\frac{m}{n},\ \frac{12\sqrt{5}}{\sqrt{5}},\ \frac{1}{\sqrt{3}},\ \frac{11}{\sqrt{7}},\ \frac{16}{\sqrt{16}},\ \frac{\sqrt{9}}{\sqrt{4}},\ \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

5. Find both the duplicate and triplicate ratios of:

$$\frac{c}{d},\ \frac{\sqrt{3}}{2},\ \frac{6}{7},\ \frac{\sqrt{3}}{\sqrt{5}},\ \frac{\sqrt[3]{3}}{\sqrt[3]{2}},\ \frac{\sqrt{2}}{\sqrt[3]{2}},\ \frac{\sqrt{m}}{\sqrt[3]{n}}.$$

6. Determine which of the following ratios are those of greater inequality and which are those of less inequality:

$$\frac{2}{4},\ \frac{6}{7},\ \frac{9}{8},\ \frac{a}{b},\ \frac{\sqrt{c}}{d},\ \frac{\sqrt{3}}{2},\ \frac{\sqrt{5}+4}{\sqrt{30}},\ \frac{4}{2+\sqrt{5}}.$$

7. Prove that a ratio of greater inequality is diminished if the same positive quantity is added to both terms.

8. Prove that a ratio of greater inequality is increased if the same positive quantity is subtracted from both terms.

9. Prove that a ratio of less inequality is increased if the same positive quantity is added to both terms.

PROPORTION

255. A **proportion** is an equation whose members are ratios. A proportion may be expressed thus: $\frac{a}{b}=\frac{c}{d}$, $a : b = c : d$, $a : b : : c : d$.

The terms of the equal ratios forming a proportion are called the **terms of the proportion.** The antecedents and consequents of the ratios are called the **antecedents** and **consequents** of the proportion. The first and fourth terms of a proportion are called the **extremes** and the second and third terms are called the **means.** The terms of a proportion are said to be **proportional.** The fourth term of a proportion is called a **fourth proportional.** When the second and third terms of a proportion are identical, it is called a **mean proportional**, and the consequent of the second ratio is called a **third proportional.**

Thus, in the proportion, $\frac{a}{b}=\frac{c}{x}$, a, b, c, and x are proportional, x is a fourth proportional; in the proportion, $\frac{a}{b}=\frac{b}{x}$, x is a third proportional, and b is a mean proportional.

A **continued proportion** is a series of equal ratios in which the consequent of each ratio is the antecedent of the next ratio.

Thus, $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$ is a continued proportion.

256.* *If two incommensurable ratios, $\frac{a}{b}$ and $\frac{c}{d}$, are so related to the commensurable ratio $\frac{n}{m}$, that $\frac{n}{m}<\frac{a}{b}<\frac{n}{m}+\frac{1}{m}$, when $\frac{n}{m}<\frac{c}{d}<\frac{n}{m}+\frac{1}{m}$, however much n and m are increased, then $\frac{a}{b}=\frac{c}{d}$.*

If $\frac{a}{b} \neq \frac{c}{d}$, since both $\frac{a}{b}$ and $\frac{c}{d}$ lie between $\frac{n}{m}$ and $\frac{n}{m}+\frac{1}{m}$, their difference must be some quantity less than $\frac{1}{m}$. But, since m can be made to increase, $\frac{1}{m}$ can be made less than any assigned quantity: hence $\frac{a}{b}-\frac{c}{d}$ can be made as small as is required; a fact which is true only when $\frac{a}{b}=\frac{c}{d}$.

Two incommensurable ratios are therefore equal under the conditions named above, and hence may form a proportion.

PRINCIPLES OF PROPORTION

257. I. *In any proportion the product of the means equals the product of the extremes.*

If $$\frac{a}{b}=\frac{c}{d}, \quad (1)$$

multiplying (1) by bd, $$ad=bc. \quad (2)$$

II. *If two products are each composed of two factors, these factors form a proportion in which the factors of either product can be made the means, and the other two factors the extremes.*

If $$ad=bc, \quad (1)$$

dividing (1) by bd, $$\frac{a}{b}=\frac{c}{d}. \quad (2)$$

Similarly, $$\frac{b}{d}=\frac{a}{c}, \quad \frac{a}{c}=\frac{b}{d}, \text{ etc.}$$

III. *The products of corresponding terms of two or more proportions are in proportion.*

If $$\frac{a}{b}=\frac{c}{d}, \qquad (1)$$

and if $$\frac{m}{n}=\frac{r}{s}, \qquad (2)$$

by Axiom 3, $$\frac{a}{b}\cdot\frac{m}{n}=\frac{c}{d}\cdot\frac{r}{s}, \qquad (3)$$

or, rewriting (3), $$\frac{am}{bn}=\frac{cr}{ds}. \qquad (4)$$

IV. *The quotients of the corresponding terms of two proportions are in proportion.*

If $$\frac{a}{b}=\frac{c}{d}, \qquad (1)$$

and if $$\frac{m}{n}=\frac{r}{s}, \qquad (2)$$

by Axiom 4, $$\frac{a}{b}\div\frac{m}{n}=\frac{c}{d}\div\frac{r}{s}, \qquad (3)$$

or, simplifying in (3), $$\frac{an}{bm}=\frac{cs}{dr}. \qquad (4)$$

V. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* **inversion** *; that is,* $\frac{b}{a}=\frac{d}{c}$.

If $$\frac{a}{b}=\frac{c}{d}, \qquad (1)$$

by I, $$ad=bc, \qquad (2)$$

dividing (2) by ac, $$\frac{d}{c}=\frac{b}{a}, \text{ or } \frac{b}{a}=\frac{d}{c}. \qquad (3)$$

VI. *If four quantities of the same kind, a, b, c, d, are in proportion, they are in proportion by* **alternation**; *that is,*

$$\frac{a}{c}=\frac{b}{d}.$$

If $$\frac{a}{b}=\frac{c}{d}, \qquad (1)$$

by I, $$ad=bc, \qquad (2)$$

dividing (2) by cd, $$\frac{a}{c}=\frac{b}{d}. \qquad (3)$$

NOTE. $\frac{5 \text{ inches}}{3 \text{ inches}}=\frac{10 \text{ pounds}}{6 \text{ pounds}}$ cannot be written by alternation, since $\frac{5 \text{ inches}}{10 \text{ pounds}}$ is impossible.

VII. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* **composition**; *that is,* $\frac{a+b}{b}=\frac{c+d}{d}$, *or* $\frac{a+b}{a}=\frac{c+d}{c}$.

If $$\frac{a}{b}=\frac{c}{d}, \qquad (1)$$

adding 1 to each member of (1),

$$\frac{a}{b}+1=\frac{c}{d}+1, \qquad (2)$$

or, rewriting (2), $$\frac{a+b}{b}=\frac{c+d}{d}. \qquad (3)$$

Similarly, (1), written first by V, and then by composition, is $\frac{a+b}{a}=\frac{c+d}{c}$.

VIII. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* **division**; *that is,* $\frac{a-b}{b}=\frac{c-d}{d}$, *or* $\frac{a-b}{a}=\frac{c-d}{c}$.

If $$\frac{a}{b}=\frac{c}{d}, \tag{1}$$

subtracting 1 from each member of (1),

$$\frac{a}{b}-1=\frac{c}{d}-1, \tag{2}$$

or, rewriting (2), $$\frac{a-b}{b}=\frac{c-d}{d}, \tag{3}$$

writing (1) by V, $$\frac{b}{a}=\frac{d}{c}, \tag{4}$$

subtracting 1 from each member of (4),

$$\frac{b}{a}-1=\frac{d}{c}-1, \tag{5}$$

or, rewriting (5), $$\frac{b-a}{a}=\frac{d-c}{c}, \tag{6}$$

multiplying (6) by -1,

$$\frac{a-b}{a}=\frac{c-d}{c}. \tag{7}$$

IX. *If four quantities, a, b, c, d, are in proportion, they are in proportion by* **composition and division**; *that is,* $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

If $$\frac{a}{b}=\frac{c}{d}, \tag{1}$$

writing (1) by VII, $\frac{a+b}{b}=\frac{c+d}{d},$ (2)

writing (1) by VIII, $\frac{a-b}{b}=\frac{c-d}{d},$ (3)

by IV, $\frac{a+b}{a-b}=\frac{c+d}{c-d}.$ (4)

X. *Like powers, or like roots of four quantities, a, b, c, d, which are in proportion, are in proportion; or* $\frac{a^n}{b^n}=\frac{c^n}{d^r}.$

If $\frac{a}{b}=\frac{c}{d},$ (1)

raising each member of (1) to the nth power, whether n is integral or fractional,

$$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$

XI. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its own consequent.*

If $\frac{a}{b}=\frac{c}{d}=\frac{m}{n}=\frac{x}{y},$ (1)

let $\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{m}{n}=r,\ \frac{x}{y}=r,$ (2)

clearing of fractions in (2), $a=br,\ c=dr,\ m=nr,\ x=yr,$ (3)

by Ax. 1, $a+c+m+x=(b+d+n+y)r,$ (4)

dividing each member of (4) by $(b+d+n+y)$,

$$\frac{a+c+m+x}{b+d+n+y}=r=\frac{a}{b}=\frac{c}{d}=\frac{m}{n}=\frac{x}{y}. \quad (5)$$

EXAMPLES

1. Solve for x, $\dfrac{\sqrt{2x+3}-\sqrt{2x-5}}{\sqrt{2x+3}+\sqrt{2x-5}}=\dfrac{1}{2}.$ (1)

By IX, $\dfrac{2\sqrt{2x+3}}{-2\sqrt{2x-5}}=\dfrac{3}{-1},$ (2)

simplifying (2), $\sqrt{2x+3}=3\sqrt{2x-5},$ (3)

solving (3), $x=3.$ (4)

2. If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{pa^2-qc^2}{pb^2-qd^2}=\dfrac{a^2}{b^2}.$ (1)

By X, $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2},$ (2)

since $\dfrac{p}{p}=\dfrac{q}{q}=1$, $\dfrac{p}{p}=\dfrac{q}{q},$ (3)

by III, $\dfrac{pa^2}{pb^2}=\dfrac{qc^2}{qd^2},$ (4)

by VI, $\dfrac{pa^2}{qc^2}=\dfrac{pb^2}{qd^2},$ (5)

by VIII, $\dfrac{pa^2-qc^2}{qc^2}=\dfrac{pb^2-qd^2}{qd^2},$ (6)

by VI, $\dfrac{pa^2-qc^2}{pb^2-qd^2}=\dfrac{qc^2}{qd^2}=\dfrac{c^2}{d^2}=\dfrac{a^2}{b^2}.$ (7)

An alternative method for this example is:

by I, $a^2b^2p-b^2c^2q=a^2b^2p-a^2d^2q,$ (2)

simplifying (2), $b^2c^2=a^2d^2,$ (3)

by X, $bc=ad,$ (4)

by II, $\dfrac{a}{b}=\dfrac{c}{d}.$

The first method is preferable.

EXERCISE CXXXIII

1. Find a fourth proportional to 462, 77, and 90.

2. Find a third proportional to 35 and 91.

3. Find a mean proportional to $2+\sqrt{3}$ and $2-\sqrt{3}$.

4. Solve for x: $\dfrac{4x^2-5x+11}{3x-7}=\dfrac{4x^2-5x-7}{3x-16}.$

5. Solve for x: $\dfrac{5x^2-4x+10}{3x^2-x+1}=\dfrac{5x^2-4x-2}{3x^2-x-5}.$

6. Solve for a: $\dfrac{\sqrt{a+5}+\sqrt{5-a}}{\sqrt{a+5}-\sqrt{5-a}}=2.$

7. Solve for x:

$$\frac{\sqrt{a-bx}+\sqrt{c-mx}}{\sqrt{a-bx}+\sqrt{nx-d}}=\frac{\sqrt{a-bx}-\sqrt{c-mx}}{\sqrt{a-bx}-\sqrt{nx-d}}.$$

If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that:

8. $\dfrac{a^2}{c^2}=\dfrac{a^2-b^2}{c^2-d^2}.$

9. $\dfrac{a^2+c^2}{b^2+d^2}=\dfrac{ac}{bd}.$

10. $\dfrac{ab+cd}{ab-cd}=\dfrac{a^2+c^2}{a^2-c^2}.$

11. $\dfrac{a^2+ab}{c^2+cd}=\dfrac{b^2-2ab}{d^2-2cd}.$

12. $\dfrac{a+b}{c+d}=\dfrac{a-b}{c-d}=\dfrac{a}{c}=\dfrac{b}{d}.$

13. $\dfrac{a+b+c+d}{a+b-c-d}=\dfrac{a-b+c-d}{a-b-c+d}.$

14. $\dfrac{\sqrt[3]{a^3+c^3}}{\sqrt[3]{b^3+d^3}}=\dfrac{c}{d}.$

15. $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}=\dfrac{c^2+cd+d^2}{c^2-cd+d^2}.$

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove that :

16. $\frac{a+c+e}{b+d+f}=\frac{a}{b}$.

17. $\frac{ha+kc+le}{hb+kd+lf}=\frac{a}{b}$.

18. $\frac{a^3+c^3+e^3}{b^3+d^3+f^3}=\frac{ace}{bdf}$.

19. $\frac{ma^2+nc^2+pe^2}{mb^2+nd^2+pf^2}=\frac{ac}{bd}$.

20. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a+b}{b+c}=\frac{b+c}{c+d}$.

21. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$, prove that $\frac{\sqrt{px^2+qy^2+rz^2}}{\sqrt{pa^2+qb^2+rc^2}}=k$.

22. If $\frac{a}{b}=\frac{c}{d}$, if x is a third proportional to a and b, and if y is a third proportional to c and d, prove that $\frac{x}{y}=\frac{b}{d}$.

23. What is the ratio of the mean proportional between a and b, to the mean proportional between c and d?

24. Two numbers are as 3 : 4, and if 7 be subtracted from each, the remainders are as 2 : 3. Find the numbers.

25. What two numbers whose difference is d are to each other as $a : b$?

26. Two numbers x and y (the first being negative) are in the ratio 8 to -9 : if 16 be subtracted from each one, the resulting numbers are in the ratio -9 to 8 ; find the numbers.

27. If $\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$, prove that $x+y+z=0$.

28. If $\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}=1$, prove that

$$(a+b+c)(x+y+z)=a^3+b^3+c^3-3\,abc.$$

VARIATION

258. A quantity whose value is dependent upon the value of another quantity is called a **function** of that quantity.

Thus, if $y = 2x^3$, y is called a function of x.

A function of x is indicated in any of the following ways: $F(x)$, $f(x)$, $\phi(x)$, etc.

When the value of a quantity is always the same in a particular investigation, the quantity is called a **constant.**

Thus, x is a constant whose value is 2, in $2x + 5 = 7 + x$.

When the value of a quantity changes in a particular investigation, the quantity is called a **variable.**

Thus, in the expression $x^3 + 1$, x is a variable, since it may take any value.

The theory of the dependence of a quantity upon another quantity is called **variation,** or **functionality.** Only the simplest forms of variation are discussed in this chapter.

The symbol $\propto$, called the **symbol of variation,** is used to indicate variation.

Thus, $x \propto y$ is read "x varies as y."

KINDS OF VARIATION

259. 1. If the ratio of two variables is constant, the variables are said to be in **direct variation.**

Thus, when m is a constant, if $\frac{x}{y} = m$, x varies directly as y; or $x \propto y$.

The height of a column of mercury in a thermometer is known to vary as the temperature. If H and H' represent the different heights of the mercury when the temperatures are respectively T and T', $H \propto T$; or, $H : H' = T : T'$.

2. If the ratio of a variable to the reciprocal of a second variable is constant, the variables are said to be in **inverse variation.**

Thus, when m is a constant, if $x : \frac{1}{y} = m$, x varies inversely as y, or $x \propto \frac{1}{y}$.

The volume of a gas is known to vary inversely as the pressure. If V and V' represent the volumes of a gas under the respective pressures P and P', $V \propto \frac{1}{P}$; or, $V : V' = \frac{1}{P} : \frac{1}{P'}$ which may be more conveniently written $V : V' = P' : P$.

3. If the ratio of a variable to the product of two other variables is a constant, the first variable is said to be in **joint variation** with the other two variables.

Thus, when m is a constant, if $x : yz = m$, x varies jointly as y and z; or, $x \propto yz$.

The distance travelled depends upon the rate and the time. If D and D' represent the distances travelled when the rates and times are respectively R and R', T and T', $D \propto RT$; or, $D : D' = RT : R'T'$.

4. If the ratio of a variable to a second variable multiplied by the reciprocal of a third variable is a constant, the first variable is said to be in **direct and inverse variation** with the second and third variables.

Thus, when m is a constant, if $x : \left(y \cdot \frac{1}{z}\right) = m$, x is in direct and inverse variation with y and z; or, $x \propto \frac{y}{z}$.

The base of a rectangle is known to vary as the area divided by the altitude. If B and B' represent the bases when the areas and altitudes are respectively S and S', A and A', $B \propto S \cdot \frac{1}{A}$; or, $B : B' = \frac{S}{A} : \frac{S'}{A'}$.

PRINCIPLES OF VARIATION

260. I. *If $x \propto y$, and $y \propto z$, then $x \propto z$.*

When m and n are constants, let

$$\frac{x}{y} = m, \text{ or } x = ym, \tag{1}$$

and let
$$\frac{y}{z} = n, \text{ or } y = zn, \tag{2}$$

multiplying (1) and (2),
$$xy = yzmn, \tag{3}$$

dividing (3) by yz,
$$\frac{x}{z} = mn. \tag{4}$$

In (4), since m and n are constants, mn is also a constant: hence $x \propto z$.

II. *If $x \propto y$, and $x' \propto y'$, then $xx' \propto yy'$.*

When m and n are constants, let

$$\frac{x}{y} = m, \tag{1}$$

and let
$$\frac{x'}{y'} = n, \tag{2}$$

multiplying (1) and (2),
$$\frac{xx'}{yy'} = mn. \tag{3}$$

Hence $xx' \propto yy'$.

Similarly, if $x \propto y$, $x^n \propto y^n$.

III. *If* $x \propto y$, *then* $kx \propto ky$.

Let k be either a constant or a variable; and let m be a constant. Let

$$\frac{x}{y} = m, \qquad (1)$$

multiplying $\frac{x}{y}$ in (1) by $\frac{k}{k}$, $\quad \frac{kx}{ky} = m. \qquad (2)$

Hence $\qquad kx \propto ky.$

IV. *If* $x \propto yz$, *then* $y \propto \frac{x}{z}$, *and* $z \propto \frac{x}{y}$.

When m is a constant, let

$$m = \frac{x}{yz}, \text{ or } myz = x, \qquad (1)$$

dividing (1) by mz, $\qquad y = \frac{x}{mz}, \qquad (2)$

dividing (2) by $\frac{x}{z}$, $\qquad \frac{y}{\frac{x}{z}} = \frac{1}{m}. \qquad (3)$

Hence $\qquad y \propto \frac{x}{z}$. Similarly, $z \propto \frac{x}{y}$.

V. *If* $x \propto y$ *when* z *is constant, and if* $x \propto z$ *when* y *is constant, then* $x \propto yz$ *when both* x *and* y *are variables.*

Let x, y, z; x', y', z; x'', y', z', be three sets of corresponding values of x, y, and z.

If z is constant, $\qquad \frac{x}{x'} = \frac{y}{y'}, \qquad (1)$

if y is constant, $\qquad \frac{x'}{x''} = \frac{z}{z'}, \qquad (2)$

multiplying (1) and (2), $\qquad \frac{x}{x''} = \frac{yz}{y'z'}, \qquad (3)$

or, rewriting (3), $$\frac{x}{yz} = \frac{x''}{y'z'}. \qquad (4)$$

Hence $$x \propto yz.$$

EXAMPLES

1. If $x \propto y$, and if $x = 3$ when $y = 2$, find x when $y = 6$.

Let $$\frac{x}{y} = m, \qquad (1)$$

substituting in (1) $x = 3$, $y = 2$, $$\tfrac{3}{2} = m, \qquad (2)$$

substituting in (1) $y = 6$, $$\frac{x}{6} = m = \frac{3}{2}, \qquad (3)$$

solving (3), $$x = 9. \qquad (4)$$

2. If y varies inversely as the square of x, and if $y = 8$ when $x = 3$, find x when $y = 2$.

Let $$\frac{y}{\frac{1}{x^2}} = m, \qquad yx^2 = m, \qquad (1)$$

substituting in (1) $y = 8$ and $x = 3$, $$72 = m, \qquad (2)$$

substituting in (1) $y = 2$ and $m = 72$, $$2x^2 = 72, \qquad (3)$$

solving (3), $$x = \pm 6. \qquad (4)$$

3. If $x \propto \frac{y}{z}$, and if $x = 4$ when $y = 6$ and $z = 3$, what is the value of x when $y = 6$ and $z = 9$?

Let $$\frac{x}{\frac{y}{z}} = m, \qquad \frac{xz}{y} = m, \qquad (1)$$

substituting in (1), $x = 4$, $y = 6$, $z = 3$, $$m = 2, \qquad (2)$$

substituting in (1), $y = 6$, $z = 9$, $m = 2$, $$\frac{9x}{6} = 2, \qquad (3)$$

solving (3), $$x = \tfrac{4}{3}. \qquad (4)$$

4. The volume of a sphere varies as the cube of the radius, and the volume of a sphere is $1437\frac{1}{3}$ when the radius is 7. Find the volume of a sphere whose radius is 14.

Let V represent the volume and R the radius of the sphere.

Then $$\frac{V}{R^3}=m, \text{ or } V=mR^3, \tag{1}$$

substituting in (1) $V=1437\frac{1}{3}$, and $R=7$, $$m=\frac{4312}{3\cdot 7^3}, \tag{2}$$

hence $$\text{volume}=\frac{4312}{3\cdot 7^3}\cdot 14^3=\frac{4312}{3}\cdot 8=11498\tfrac{2}{3}. \tag{3}$$

EXERCISE CXXXIV

1. If $x \propto y$, and if $x=5$ when $y=4$, find x when $y=9$.

2. If $x \propto \frac{1}{y^2}$, and if $x=4$ when $y=3$, find y when $x=2$.

3. If $x \propto yz$, and if $x=2$ when $y=3$ and $z=4$, find x when $y=2$ and $z=6$.

4. If $x \propto \frac{y}{z}$, and if $x=16$ when $y=3$ and $z=8$, find z when $x=12$ and $y=2$.

5. If $x \propto \frac{1}{y}+\frac{1}{z}$, and if $x=4$ when $y=3$ and $z=5$, find y when $x=3$ and $z=2$.

6. If x varies directly as y and inversely as z, and is equal to 4 when $y=2$ and $z=3$, what is the value of x when $y=35$ and $z=15$?

7. If $y=u-v$, if u varies as x, and v as x^2, and if $y=2$ when $x=1$, and $y=3$ when $x=2$, find the value of y in terms of x.

8. If a^2-b^2 varies as c^2, and if $c=2$ when $a=5$ and $b=3$, find the equation between a, b, and c.

9. If $x \propto y$, and $z \propto y$, prove that $x - z \propto y$.

10. If $x \propto y$, prove that $x^2 + y^2 \propto xy$.

11. If $x + y \propto x - y$, prove that $x^2 + y^2 \propto xy$.

12. If $x \propto y$, and $x \propto z$, and $x \propto w$, when z and w, y and w, y and z, are constants, prove that $x \propto yzw$.

13. The area of a circle varies as the square of the radius; show that the area of a circle of 5 feet radius is equal to the sum of a circle of 3 feet radius and another of 4 feet radius.

14. Knowing that the volume, V, of a gas varies directly as the temperature, T, when $T = 273° +$ the number of degrees in temperature (in the Centigrade System): if the volume of a certain gas is 400 c.c. when the temperature is 27° C., find the volume of the gas at 127° C.

15. Find, under the law given in the preceding example, the volume of a gas at 0° C., if the volume is 250 c.c. at 18° C.

16. Knowing that the volume, V, of a gas varies inversely as the pressure, P, upon it: if the volume of a gas is 100 c.c. when the pressure is 76 cm., find the volume when the pressure is 38 cm.

17. Under the conditions given in the preceding problem, if the volume of a gas is 600 c.c. when the pressure is 60 cm., find the pressure if the volume is 150 c.c.

18. Knowing that the intensity of illumination, I, varies inversely as the square of the distance, D: if a candle throws a certain amount of light on a screen 2 feet distant, what will be its relative illuminating power at a distance of 7 feet?

19. Under the conditions given in the preceding problem, if a candle and a gas flame are 12 feet apart, and if the gas flame is equivalent to 4 candles, where must a screen be placed on a line joining the candle and gas flame so that the screen may be equally illumined by each of them?

20. Knowing that $\frac{V_1P_1}{T_1} = \frac{V_2P_2}{T_2}$ where V_1, V_2, etc., are as given in Problems 14 and 16: if a mass of air at 0° C. has a volume of 600 c.c. at a pressure of 76 cm., find the volume when the temperature is 91° C. and the pressure is 190 cm.

21. Under the conditions given in the preceding problem, if the volume of a certain mass of air at 27° C., and under a pressure of 225 cm. is 2000 c.c., find its volume at 127° C., under a pressure of 75 cm.

22. Knowing that the amount of bending, B, of a rod varies jointly as the load, L, and the cube of the length, L', and inversely and jointly as the width, W, and the cube of the thickness, T, that is, $B \propto \frac{LL'^3}{WT^3}$: if a rod 8 feet long, 4 inches wide, 1 inch thick, is bent 0.2 inch by a weight of 50 pounds, how much would a weight of 5 pounds bend a rod of like material, 24 feet long, 8 inches wide, and 2 inches thick?

23. Under the conditions given in the preceding problem, if a beam 16 feet long, 8 inches wide, 4 inches thick, is bent $\frac{1}{2}$ inch by a weight of 1000 pounds, how much would a beam 10 feet long, 6 inches wide, 8 inches thick, be bent by the same weight?

CHAPTER XXII

PROGRESSIONS

ARITHMETICAL PROGRESSION

261. A succession of terms, each of which is obtained from the preceding term by the addition of the same positive or negative quantity (the **common difference**), is called an **arithmetical progression.**

Thus, 2, 5, 8, 11, etc., and -1, -2, -3, etc., are arithmetical progressions.

The first term is usually represented by a, and the common difference by d; hence the progression is a, $a+d$, $a+2d$, $a+3d$, etc. The number of terms in a progression is represented by n; and the nth term by l.

Since each term is formed from the preceding term by the addition of d, the coefficient of d, in any term, is one less than the number of the term in the progression. Thus, the third term is $a+2d$; hence

$$l = a + d(n-1). \qquad \text{I.}$$

1. Find the 10th term of the progression 2, 5, 8, etc.

By the conditions, $a=2$, $d=3$, $n=10$,

by I, 10th term $= 2 + 3(10-1) = 29.$

2. Find the 10th term of the progression in which the 3d term is 11, and the 7th term is 27.

By the conditions, $a + 2d = 11,$ (1)

and, $a + 6d = 27,$ (2)

subtracting (1) from (2), $4d = 16,$ (3)

or, $d = 4,$ (4)

substituting $d = 4$, in (1), $a = 3,$ (5)

by I, $\text{10th term} = a + 9d = 39.$ (6)

EXERCISE CXXXV

Find the last term of each of the following progressions:

1. 2, 5, 8, ⋯ to 10 terms. **2.** 8, 5, 2, ⋯ to 10 terms.

3. 100, 95, 90, ⋯ to 15 terms.

4. $b, b - c, b - 2c, \cdots$ to 13 terms.

Find the nth term of the following progressions in which:

5. $a = 3\frac{1}{4}$, $d = 2\frac{5}{6}$, $n = 10$.

6. $a = 76\frac{3}{5}$, $d = -4\frac{2}{7}$, $n = 8$.

7. $a = b + c$, $d = b - c$, $n = p$.

8. $a = x - y$, $d = -y$, $n = x^2 - y^2$.

Find the indicated terms in the following progressions:

9. 7th term; the 3d being 10, and the 10th, -5.

10. 6th term; the 4th being 0, and the 9th, 15.

11. 1st term; the 7th being -48, and the 13th, -108.

12. 10th term; the 5th being 28, and the 9th, 52.

13. 15th term; the 31st being -40, and the sum of the 3d and 11th, 4.

262. When three quantities are in arithmetical progression, the middle term is called the **arithmetical mean** between the other two.

If a, b, and c are in arithmetical progression, the arithmetical mean b can be found in terms of the other two. Since $b - a = c - b$, $b = \frac{1}{2}(a + c)$.

Hence, *the arithmetical mean between two quantities is one-half the sum of the quantities.*

In an arithmetical progression containing any number of terms, all the terms between the first and last are called **arithmetical means** between those terms.

Insert 6 arithmetical means between 8 and 29.

The progression evidently contains 8 terms; $a = 8$, $n = 8$, $l = 29$.

By I, $$29 = 8 + d(8 - 1), \tag{1}$$

solving (1), $$d = 3. \tag{2}$$

Hence the progression is, 8, [11, 14, 17, 20, 23, 26,] 29.

EXERCISE CXXXVI

1. Insert 7 arithmetical means between 69 and 95.

2. Insert 13 arithmetical means between 13 and 209.

3. Insert 98 arithmetical means between 6 and -489.

4. Insert 99 arithmetical means between -5780 and 0.

5. Insert 4 arithmetical means between k and $\frac{2k - 3m}{2}$.

6. Insert 10 arithmetical means between $m + \sqrt{3}$ and $m + \sqrt{3} + 729$.

7. Insert r arithmetical means between 1 and $3r - 2$.

263. If S denotes the sum of n terms of an arithmetical progression,

$$S = a + (a + d) + (a + 2d) + \cdots + (l - d) + l, \quad (1)$$

or, $$S = l + (l - d) + (l - 2d) + \cdots + (a + d) + a, \quad (2)$$

adding (1) and (2),

$$2S = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l), \quad (3)$$

or, $$2S = n(a + l), \quad (4)$$

whence, $$S = \frac{n}{2}\left(a + l\right). \quad \text{II.}$$

Since, by I, $l = a + d(n - 1)$, substituting I in II,

$$S = \frac{n}{2}\left\{2a + d(n - 1)\right\}. \quad \text{III.}$$

Equations I, II, and III are called the **formulas** of arithmetical progression.

1. Find the sum of 6 terms of the progression, 5, 3, 1, -1, etc.

By the conditions, $a = 5$, $d = -2$, $n = 6$.

Substituting a, d, and n in III, $S = \frac{6}{2}\{10 - 2(5)\} = 0$.

2. How many terms of the progression, 4, 7, 10, $\cdots$ must be taken in order that the sum may be 69?

By the conditions, $a = 4$, $d = 3$, $S = 69$.

Substituting a, d, and S in III, $$69 = \frac{n}{2}\left\{8 + 3(n - 1)\right\}, \quad (1)$$

reducing (1), $$3n^2 + 5n - 138 = 0, \quad (2)$$

solving (2), $$n = 6, \text{ or } -\tfrac{23}{3}. \quad (3)$$

Since n must always be a positive integer, $n = 6$ is the only solution.

Problems of the class stated above will evidently always involve the solution of a quadratic equation, and it is therefore possible to obtain one, two, or no correct solutions according as one, two, or no solutions of the quadratic equation are positive integers.

3. In an arithmetical progression whose first term is 3, the sum of 7 terms is 105. Find the common difference.

By the conditions, $a = 3$, $n = 7$, $S = 105$.

Substituting a, n, and S in III, $105 = \frac{7}{2}(6 + 6d)$, (1)

solving (1), $d = 4$.

EXERCISE CXXXVII

Find the sum in each of the following progressions:

1. 1, 2, 3, 4, ⋯ to 10 terms.

2. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, ⋯ to 12 terms.

3. 7, 17, 27, ⋯ to 8 terms.

4. $2, 2\frac{2}{3}, 3\frac{1}{3}$, ⋯ to m terms.

5. $6\frac{2}{5}, 9\frac{21}{40}, 12\frac{13}{20}$, ⋯ to 13 terms.

6. 100, 90, 80, ⋯ to 21 terms.

7. 178, 171, 164, ⋯ to 11 terms.

8. $1, 1 + \sqrt{2}, 1 + 2\sqrt{2}$, ⋯ to r terms.

Find the number of terms in each of the following progressions, so that the given sum may be obtained:

9. $S = 45$; 15, 12, 9, ⋯.

10. $S = -1545$; 50, 43, 36, ⋯.

11. $S = 1200$; 31, 38, 45, ⋯.

12. $S = 52\frac{1}{2}$; $\frac{3}{4}, \frac{7}{8}, 1$, ⋯.

13. $S = 30(1 + \frac{7}{6}\sqrt{2})$; $3 - \sqrt{2}, 3, 3 + \sqrt{2}$, ⋯.

In the following arithmetical progressions:

14. Find d, and l, if $a = 3$ and the sum of the first 13 terms is 351.

15. Find d, if the 12th term is 38 and the sum of the first 13 terms is 351.

16. Find d, and l, if $a = 222$, $n = 223$, and $S = 0$.

17. Find a, and l, if $d = 6$, $n = 10$, and $S = 310$.

18. Find n, and d, if $a = 4$, $l = -22$, and $S = -99$.

19. Find n, and d, if $a = \frac{1}{3}$, $l = 15\frac{1}{3}$, and $S = 47$.

20. Find n, and a, if $d = x - 1$, $l = x^3 + x^2 + 3x - 1$, and $S = 3x^3 + 3x^2 + 6x$.

21. The sum of the first 6 terms is 261, and the sum of the first 9 terms is 297. Find the first 9 terms.

22. The sum of the first 3 terms is 14, and the sum of the squares of these terms is 78. Find the terms.

23. The sum of the first half of the terms is 28, the sum of the second half is 222, the sum of the first and last terms is 50. Find the number of terms.

24. The sum of the last four terms is 20, the product of the second and fifth is 16. If the progression contains five terms, find the progression.

25. In a progression of eighteen terms the product of the two middle terms is 90, and the product of the first and eighteenth terms is 18. Find the first and last terms.

GEOMETRICAL PROGRESSION

264. A succession of terms, each of which is obtained from the preceding term by multiplying it by the same positive or negative quantity (the **common ratio**), is called a **geometrical progression.**

Thus, 2, 4, 8, 16, etc., and 1, -3, 9, -27, etc., are geometrical progressions.

The first term is usually represented by a, and the common ratio by r; hence the progression is, $a + ar + ar^2 + ar^3$, etc. The number of terms in a progression is represented by n, and the nth term by l.

Since each term is formed from the preceding term by multiplying it by r, the exponent of r in any term is one less than the number of the term. Thus, the third term is ar^2; and the nth term or

$$l = ar^{n-1}. \qquad \text{I.}$$

1. Find the 7th term of the progression $1, -3, 9, \cdots$.

By the conditions, $a = 1$, $r = -3$, $n = 7$,

by I, 7th term $= 1(-3)^6 = 729$.

2. If the 4th term of a geometrical progression is 1, and the 7th term is $\frac{1}{8}$, find the 1st term.

By the conditions, $ar^3 = 1$, (1)

and, $ar^6 = \frac{1}{8}$, (2)

dividing (2) by (1), $r^3 = \frac{1}{8}$, (3)

from (3), $r = \frac{1}{2}$, (4)

substituting $r = \frac{1}{2}$ in (1), $a = 8$. (5)

EXERCISE CXXXVIII

Find the last term in each of the following geometrical progressions:

1. $2, 6, 18, \cdots$ to 7 terms.
2. $3, -6, 12, \cdots$ to 6 terms.
3. $4, 8, 16, \cdots$ to 7 terms.
4. $27, 9, 3, \cdots$ to 8 terms.
5. $6, 3, \frac{3}{2}, \cdots$ to 10 terms.
6. $1, -\frac{5}{3}, \frac{25}{9}, \cdots$ to 11 terms.

In the following geometrical progressions:

7. Find the 7th term, the 2d term being 75, and the 5th, $-\frac{3}{5}$.

8. Find the 2d term, the 4th term being -5, and the 7th, 625.

9. Find the 15th term, the 5th term being $\frac{27}{8}$, and the 10th, $\dfrac{3^8}{2^8}$.

10. Find the 50th term, the 19th being 1200, and the 29th, 1200.

11. Find the 11th term, the 2d term being $b^2 - c^2$, and the 5th, $(b + c)(b - c)^4$.

12. Find the 10th term, the 3d term being b^2, and the 7th, $\dfrac{b^6}{a^4}$.

13. Find the 7th term, the 2d term being 1, and the 4th, $17 - 12\sqrt{2}$.

14. Find the 8th term, the 4th term being $49 - 20\sqrt{6}$, and the 6th, $485 - 198\sqrt{6}$.

15. Find the 7th term, the 3d term being -2, and the 8th, $-2i$.

265. When three quantities are in geometrical progression, the middle term is called the **geometrical mean** between the other two.

If a, b, and c are in geometrical progression, the geometrical mean, which is a mean proportional, can be found in terms of the other two. Since $\frac{b}{a} = \frac{c}{b}$,

$$b^2 = ac, \qquad (1)$$

extracting square roots in (1), $b = \sqrt{ac}$. (2)

Hence, *the geometrical mean between two quantities is the square root of the product of those quantities.*

In a geometrical progression containing any number of terms, all the terms between the first and last are called **geometrical means** between those terms.

Insert 3 geometrical means between 6 and 486.

The progression evidently contains 5 terms; $a = 6$, $n = 5$, $l = 486$.

By I, $486 = 6\,r^4$, (1)

solving (1), $r = 3$. (2)

Hence the progression is 6, [18, 54, 162,] 486.

EXERCISE CXXXIX

1. Insert 2 geometric means between 1 and 64.
2. Insert 6 geometric means between $\frac{a^4}{b^4}$ and $-\frac{b^3}{a^3}$.
3. Insert 11 geometric means between 1 and 2.
4. Insert 5 geometric means between 1875 and 3.
5. Insert 5 geometric means between 36 and $-\frac{2}{243}$.

266. If S denotes the sum of n terms of a geometrical progression,

$$S = a + ar + ar^2 + ar^3 + \cdots ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiplying (1) by r,

$$rS = ar + ar^2 + ar^3 + ar^4 + \cdots ar^{n-1} + ar^n, \qquad (2)$$

subtracting (1) from (2),

$$S(r-1) = ar^n - a, \qquad (3)$$

from (3),
$$\boldsymbol{S = \frac{ar^n - a}{r-1} = \frac{a(r^n - 1)}{r-1}}. \qquad \text{II.}$$

Since $l = ar^{n-1}$, $rl = ar^n$, substituting rl for ar^n in II,

$$\boldsymbol{S = \frac{rl - a}{r-1}}. \qquad \text{III.}$$

1. Find the sum of the progression, 2, 6, 18, $\cdots$ to 6 terms.

By the conditions, $a = 2$, $r = 3$, $n = 6$.

By II, $S = \dfrac{2(3)^6 - 2}{3-1} = 728.$

2. The 3d term of a geometrical progression is 27, the 5th is 81. Find the sum of the first 5 terms.

By the conditions, $ar^2 = 27$, (1)

and, $ar^4 = 81$, (2)

dividing (2) by (1), $r^2 = 3$, (3)

from (3), $r = \sqrt{3}$, (4)

substituting in (1), $a = 9$, (5)

substituting in II, $S = \dfrac{9(\sqrt{3})^5 - 9}{\sqrt{3} - 1} = 117 + 36\sqrt{3}.$ (6)

EXERCISE CXL

In the following geometrical progressions:

1. Find the sum of 3, -6, 12, $\cdots$ to 6 terms.

2. Find the sum of 6, $\frac{3}{2}$, $\frac{3}{8}$, $\cdots$ to 10 terms.

3. Find the sum of $\frac{2}{3}$, $\frac{1}{2}$, $\frac{3}{8}$, $\cdots$ to 10 terms.

4. Find the sum of $\sqrt{2}$, 2, $2\sqrt{2}$, $\cdots$ to 8 terms.

5. Find the sum of $\sqrt{2}+1$, 1, $\sqrt{2}-1$, $\cdots$ to 5 terms.

6. Find the sum of the first 7 terms, if the 2d term is 4, and the 5th, 256.

7. Find the sum of the first 5 terms, if the 3d term is 27, and the 5th, 48.

8. If $a=6$, and $r=-2$, find n, if the sum of n terms is -30

9. The sum of the first 5 terms is 242, and the common ratio is 3. Find the 5th term.

10. The sum of the first 4 terms is $9\frac{1}{3}$, and the common ratio is $\frac{1}{3}$. Find the 1st term.

11. Find the sum of the first 6 terms, if the 6th term is $-\frac{40}{243}$ and the common ratio is $-\frac{2}{3}$.

12. Find the common ratio, and the sum of the first 5 terms, if the 1st term is $\frac{2}{3}$ and the 6th term is 864.

13. Find the sum of the first 10 terms of a geometric progression in which the 1st term is 243 and the common ratio is $-\dfrac{1}{\sqrt{3}}$.

14. If the 4th term is $\frac{1}{16}$, and the 7th term is $\frac{1}{128}$, how many terms, beginning with the 1st, must be taken so that their sum is $\frac{4095}{4096}$?

267. As in § 258, if a quantity retains the same value throughout a particular investigation, it is called a **constant.** If a quantity changes in value during a particular investigation, it is called a **variable.**

When the value of a variable can be made to approach the value of a constant in such a way that the difference of the variable and the constant can be made less than any assigned quantity, however small, the constant is called the **limit** of the variable.

SUM OF AN INFINITE GEOMETRICAL PROGRESSION

268. If $r > 1$, each term of a geometrical progression is larger than the preceding term, and the sum of n terms must increase indefinitely as n increases. If $r = 1$, the terms are all equal, and the sum of n terms must again increase indefinitely as n increases. If $r < 1$, and $r > -1$, each term is less than the preceding term; and it will be seen that the sum of n terms always remains less than some definite, finite quantity; from which, however, by increasing n, it can be made to differ by less than any assigned quantity, however small.

As an illustration, consider the geometrical progression, $1 + \frac{1}{2} + \frac{1}{4} \cdots$. Applying III, $S = \dfrac{1 - \frac{1}{2}l}{1 - \frac{1}{2}} = 2 - l$. Hence, in this progression, the sum of any number of terms differs from 2 by just the last term. But, by increasing n the last term can be made as small as may be required. Evidently the sum of n terms can never be as large as 2, but it can be made to differ from 2 by a quantity less than any assigned value. Hence 2 is the limit of the sum of n terms, as n increases indefinitely.

269. When $r < 1$, it is convenient to write II in the form,

$$S = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Here r^n can be made as small as is required by increasing n. The second fraction, $\frac{ar^n}{1-r}$, can, therefore, be made as small as is required by increasing the number of terms; and S can be made to differ from $\frac{a}{1-r}$ by less than any assigned quantity. $\frac{a}{1-r}$ is, therefore, the limit approached by S as n increases indefinitely. It is usually called the *sum of the infinite geometrical progression*, but this must always be understood to mean *the limit of the sum of the progression as n increases indefinitely.*

If S represents the limit of that sum,

$$\mathbf{S = \frac{a}{1 - r}}. \qquad \text{IV.}$$

1. Find the sum of an infinite number of terms in the progression, $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{18}$, etc.

By IV,
$$S = \frac{\frac{1}{2}}{1 - \frac{1}{3}} = \frac{3}{4}.$$

2. Find the value of $0.4545 \cdots$.

The decimal 0.4545 is evidently the geometric progression

$$\tfrac{45}{100} + \tfrac{45}{10000} + \tfrac{45}{1000000} + \cdots,$$

in which
$$a = \tfrac{45}{100},\ r = \tfrac{1}{100}.$$

By IV,
$$S = \frac{\frac{45}{100}}{1 - \frac{1}{100}} = \frac{5}{11}.$$

3. Find the value of $0.4555 \cdots$.

The decimal 0.4555 is evidently $\frac{4}{10}$ + the progression

$$\tfrac{5}{100} + \tfrac{5}{1000} + \tfrac{5}{10000} + \cdots,$$

in which $a = \frac{5}{100}$, $r = \frac{1}{10}$.

By IV, $$S = \frac{\frac{5}{100}}{1 - \frac{1}{10}} = \frac{1}{18}. \qquad (1)$$

Hence $$0.4555 \cdots = \tfrac{4}{10} + \tfrac{1}{18} = \tfrac{41}{90}. \qquad (2)$$

EXERCISE CXLI

In the following infinite geometrical progressions:

1. Sum to infinity, $2, -\frac{4}{3}, \frac{8}{9}, \cdots$.
2. Sum to infinity, $5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots$.
3. Sum to infinity, $3\frac{3}{8}, -2\frac{1}{4}, 1\frac{1}{2}, \cdots$.
4. Sum to infinity, $4, -\frac{4}{3}, \frac{4}{9}, \cdots$.
5. Find the value of $0.2544 \cdots$.
6. Find the value of $0.86464 \cdots$.
7. Find the value of $0.5124545 \cdots$.
8. Find the value of $0.2162525 \cdots$.
9. Find the value of $0.1248248 \cdots$.
10. Find the value of $0.18301830 \cdots$.

11. Find the sum to infinity, if the 4th term is 56 and the 7th is $-10\frac{2}{3}$.

12. Find the 1st term, if the sum to infinity is $-1\frac{4}{5}$ and the 2d term is 2.

13. Find the 4th term, if the 1st term is 100 and the sum to infinity is $111\frac{1}{9}$.

270. A succession of quantities, whose successive terms are arranged in accordance with some law, is called a **series.**

Thus, arithmetical and geometrical progressions are series.

If a series of quantity be given, it must be tested to determine the nature of the series.

The abbreviations A. P. and G. P. indicate respectively arithmetical and geometrical progression.

REVIEW EXERCISE CXLII

1. Show that $2a^2(a+3b)$, $(a+b)^3$, and $2b^2(b+3a)$, are in A. P.

2. How many terms of the series 1, 3, 5, 7, $\cdots$ amount to 1,234,321?

3. The arithmetic mean between two quantities is $\frac{17}{4}$, and the geometric mean is 2. Find the quantities.

4. Find the sum of the terms in the series 1, $1+b$, $1+2b$, $1+3b$, $\cdots$ $1+nb$, when $b=2$, $n=11$.

5. Sum the series -3, 6, first as G. P., then as A. P., each to 5 terms.

6. If the arithmetic mean between a and b be double the geometric mean, find $\frac{a}{b}$.

7. How many terms of the series 42, 39, 36, $\cdots$ make 315?

8. Find the sum of 16 terms of the series $27+22\frac{1}{2}+18+13\frac{1}{2}+\cdots$.

9. Find the sum of k terms of the series $1, \frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \cdots$.

10. If a, b, c, and d are four quantities in G. P., show that $b+c$ is the geometric mean between $a+b$ and $c+d$.

11. Find the sum of all integral numbers between 1 and 207, which are divisible by 5.

12. Find the sum of all odd integral numbers between 74 and 692.

13. How many positive integral numbers of three digits are there which are divisible by 9? Find their sum.

14. Find four numbers in A. P., such that the sum of their squares shall be 120, and that the product of the first and last terms shall be less than the product of the other two by 8.

15. Find a G. P. in which the sum of the first two terms is 2, and the sum to infinity is 4.

16. The 1st term of an A. P. is 2, and $d=\frac{4}{3}$. How many terms must be taken that their sum amounts to 192?

17. Find the G. P. whose sum to infinity is 4, and whose second term is $\frac{3}{4}$.

18. The sum of three numbers in A. P. is -3, and their product is 8. Find the numbers.

19. Prove that in an A. P. of a limited number of terms, the sum of two terms, equally distant from the end terms, is equal to a constant.

20. Prove that if each term of an A. P. be multiplied by the same quantity, the resulting series will be in A. P.

21. Prove that in a G. P. of a limited number of terms, the product of two terms, equally distant from the end terms, is constant.

22. A body slides down an inclined plane 1290 feet long in 15 seconds. If it slides 9 feet the first second, and thereafter gains in distance traversed a fixed amount each second, find this gain.

23. A man deposits money in a bank every week-day for two weeks. The first day he deposits \$1.50, and on each succeeding day deposits three times as much as on the day previous. Find the amount to his credit at the end of the two weeks.

24. In starting an engine it was observed that the fly-wheel made $\frac{3}{4}$ of a revolution the first second, $3\frac{3}{4}$ revolutions the second second, and $18\frac{3}{4}$ revolutions the third second. If it continued to gain speed at this rate, how many revolutions would it make in the eighth second? If the wheel has a diameter of seven feet, how far would a point in its rim travel in nine seconds?

25. During a freshet the overflow pipe of a reservoir discharged in a certain number of hours 1,562,496 gallons. If it discharged during the first hour 16 gallons and it continued to discharge on each succeeding hour five times as much as on the hour previous, find the number of hours the overflow continued to increase and the amount discharged the last hour.

CHAPTER XXIII

PERMUTATIONS AND COMBINATIONS

271. The various orders in which a number of things can be arranged are called their **permutations.**

Thus, a and b can be arranged ab, ba; while a, b, and c, can be arranged abc, acb, bac, bca, cab, cba.

272. The various groups that can be selected out of a number of things, without reference to their order, are called their **combinations.**

Thus, the groups of two things that can be selected from a, b, and c, are ab, ac, and bc.

Unless the contrary is expressly stated, the things whose permutations or combinations are required will be understood as different things.

Thus, the number of permutations of three different things, when taken two at a time, may be required.

273. *If a single operation can be done in m different ways, and when this operation has been done, if a second operation can be done in n different ways, the two operations can be done together in mn different ways.*

With the first way of performing the first operation there may be associated any one of the n ways of performing the second operation; with the second way of performing the first operation there may be associated any one of the n ways of performing the second operation, etc. That is, with each one of the m different ways of performing the first operation there may be associated n ways of performing the second operation. Therefore there are mn different ways of performing the two operations.

Thus, the offices of president and vice-president can be filled from five candidates in 20 ways; since any one of the five can be selected for president, the office of president can be filled in five different ways; when the office of president has been filled, any one of the remaining four candidates can be selected for vice-president. Any one way of the five ways of filling the office of president can be associated with any one way of the four ways of filling the office of vice-president. Therefore the two offices may be filled in $5 \cdot 4 = 20$ different ways.

Similarly, the above principle applies to more than two operations, each one of which can be performed in a definite number of ways.

Thus, if a man has 5 coats, 3 waistcoats, and 6 pairs of trousers, he can dress himself in $5 \cdot 3 \cdot 6 = 90$ different ways.

PERMUTATIONS

274. *The number of permutations of n different things taken r at a time is $n(n-1)(n-2) \cdots (n-r+1)$.*

The problem of computing the number of permutations of n different things taken r at a time is equivalent to the problem of filling r different places with n different things.

The first place can evidently be filled with any one of the n different things. After the first place has been filled there remain $n-1$ different things, any one of which can be put into the second place; that is, the second place can be filled in $n-1$ different ways for each way that the first can be filled. Hence the first two places can be filled in $n(n-1)$ different ways.

After filling the second place, there remain $n-2$ different things, any one of which can be put into the third place; that is, the third place can be filled in $n-2$ different ways. Hence the first three places can be filled in $n(n-1)(n-2)$ different ways, etc.

Place	1st	2d	3d	4th	...	rth
Number of ways . .	n	$n-1$	$n-2$	$n-3$	...	$n-(r-1)$

Continuing the process, it is evident that the number of ways in which each place can be filled is found by subtracting from n that number which is one less than the number of the place. Hence the rth place can be filled in $n-(r-1)=n-r+1$ different ways. Therefore the r different places can be filled by n different things in $n(n-1)(n-2)\cdots(n-r+1)$ different ways.

The symbol for the number of permutations of n different things taken r at a time is written ${}_nP_r$. Hence

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+2)(n-r+1). \qquad \text{I.} $$

275. The number of permutations of n different things taken n at a time can evidently be found by substituting n for r in I,

$${}_nP_n = n(n-1)(n-2) \cdots (2)(1). \qquad \text{II.}$$

The product of the factors of ${}_nP_n$, that is, the product of the first n integral numbers, is called **factorial n**, and is written $\underline{|n}$ or $n!$ Formula II may therefore be written

$${}_nP_n = n! \qquad \text{II.}$$

EXAMPLES

1. In how many ways can 8 different letters be inserted in 3 different letter-boxes, one and only one being placed in each box?

The first letter-box can be filled in 8 different ways; the second in 7 different ways; the third in 6 different ways; and the three in $8 \cdot 7 \cdot 6 = 336$ different ways. That is,

by I, $${}_8P_3 = 8(8-1) \cdots (8-3+1),$$
$$= 8 \cdot 7 \cdot 6 = 336.$$

2. In how many ways can the letters of the word *Pingry* be arranged?

Since there are 6 different letters, the 6 different letters may be arranged in the 6 different places occupied by the letters in $6!$ different ways; or,

by II, $${}_6P_6 = 6! = 720.$$

3. In how many different ways can 5 people be seated at a round table?

The order of arrangement cannot be that of position on a straight line, but on a closed curve. If one of the 5 be seated, so as to give a starting-point from which to reckon the order, the remaining 4 can be seated in the remaining 4 places in $4!$ different ways; or,

by II, $${}_4P_4 = 4! = 24.$$

276. *The number of combinations of* n *different things taken* r *at a time is* $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$.

The symbol for the number of combinations of n different things taken r at a time is written ${}_nC_r$.

Each one of the combinations of ${}_nC_r$ is a selection of r different things which can be arranged, by II, in $r!$ different ways; hence the number of combinations of n different things taken r at a time, or ${}_nC_r$, when multiplied by $r!$ equals the number of permutations of ${}_nP_r$; that is,

$$ {}_nC_r \cdot r! = n(n-1)\cdots(n-r+1), $$

or

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \qquad \text{III.} $$

The combinations of a, b, and c, taken two at a time, are ab, ac, and bc. Each combination can be arranged in two different ways. Hence ${}_3P_2 = {}_3C_2 \cdot 2$; or

$$ {}_3C_2 = \frac{{}_3P_2}{2} = \frac{3 \cdot 2}{2} = 3. $$

277. Formula III is employed in obtaining arithmetical results, but the better form of III for algebraic use is

$$ {}_nC_r = \frac{n!}{r!(n-r)!}. \qquad \text{IV.} $$

Since by III,

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}, $$

$$ = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \cdot \frac{(n-r)!}{(n-r)!} $$

$$=\frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{r!(n-r)!}$$

$$=\frac{n!}{r!(n-r)!}.$$

$${}_nC_{n-r}=\frac{n!}{(n-r)![n-(n-r)]!}=\frac{n!}{r!(n-r)!}.$$

By ${}_{11}P_4$.

${}_2P_6$.

There [illegible] C_{n-r}.

5.

how ma[illegible]

EXAMPLES

1. How [illegible] committees of 4 men can be formed from 10 men?

Four men are to be selec[illegible]om 10 men; hence, by III,

$${}_{10}C_4=\frac{10\cdot 9\cdot 8\cdot [illegible]}{1\cdot 2\cdot 3\cdot 4}=[illegible]$$

2. From 11 men find how many committees of 4 men can be selected, when one man is always included on the committee.

Since one man is always included on the committee, the problem is to select 3 men from the remaining 10; hence, by III,

$${}_{10}C_3=\frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3}=120.$$

3. From 9 men find how many committees of 3 men can be selected, when one man is always excluded from the committee.

Since one man is always excluded from the committee, the problem is to select 3 men from the remaining 8 men; hence, by III,

$${}_8C_3=\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}=56.$$

4. How ma[illegible]aseball nines can be made up out [illegible] 12 players?

Nine men are selected from 12 men; hence, by III and IV,

$_{12}$[illegible] $12 \cdot 11$ [illegible] 20. [illegible]

5. A club consists of 8 senior[illegible] many different committees of five [illegible] the club, each committee to consist [illegible] 2 juniors?

By III, the seniors can be chosen in $_8C_3 = \frac{[illegible]}{1 \cdot 2 \cdot 3} = 56$ ways.

By III, the [illegible] can be chosen in $_6C_2 = \frac{6 \cdot 5}{1 \cdot 2} = 15$ ways.

By § 273, the entire committee can be chosen in $56 \cdot 15 = 840$ different ways.

6. If letters in any order form a word, how many words can be formed from 8 consonants and 5 vowels, each word consisting of 4 consonants and 3 vowels?

By III, the selections of consonants and vowels are respectively $_8C_4$ and $_5C_3$.

$$_8C_4 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70,$$

$$_5C_3 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10.$$

By § 273, the total number of selections of consonants and vowels is $70 \cdot 10 = 700$. Since each of the 700 combinations consists of 7 different letters, each combination can be permuted in $7! = 5040$ different ways. There are $700 \cdot 5040 = 3{,}528{,}000$ different words.

EXERCISE CXLIII

Find the values of:

1. ${}_{10}P_3$.	**5.** ${}_{7}P_7$.	**9.** ${}_{8}C_5$.
2. ${}_{11}P_4$.	**6.** ${}_{n}P_a$.	**10.** ${}_{12}C_8$.
3. ${}_{12}P_6$.	**7.** ${}_{6}C_2$.	**11.** ${}_{12}C_{10}$.
4. ${}_{7}P_5$.	**8.** ${}_{7}C_3$.	**12.** ${}_{15}C_{14}$.

13. In how many ways can 10 people sit in 4 chairs?

14. In how many ways can the first 4 letters of the alphabet be arranged?

15. How many numbers of 3 digits each, no digit being repeated, can be formed from the digits 1 to 9 inclusive?

16. In how many different ways can 2310 be written as the product of its prime factors?

17. A man has n different books, which he can place in 5040 different arrangements. Find the number of books.

18. How many combinations can be made of 10 different things taken in sets of 7?

19. On how many nights can a different guard of 5 men be selected from a body of 20? On how many of these would any one man serve?

20. There are 20 things of one kind, and 10 of another. How many different sets can be made each containing 3 of the first kind and 2 of the second?

21. In an examination paper of 10 questions any 3 can be omitted. Find the number of selections.

22. In how many ways can 5 people form a ring? In how many ways a line?

23. How many different committees of 3 Republicans and 3 Democrats can be formed from 10 Republicans and 7 Democrats?

24. How many even numbers of 4 digits each, no digit being repeated, can be formed from the digits 1 to 9 inclusive?

25. In a boat's crew of 8 men one man can row only on stroke side. How many ways can the crew be seated?

26. In how many different ways can a ball nine be arranged, the pitcher and catcher being always the same, but the others playing in any position?

27. How many different sums of money can be formed with a cent piece, a nickel, a dime, a quarter, and a half-dollar?

28. How many different quantities of anything ponderable can be weighed with n different weights?

29. How many changes can be rung with 3 bells out of 6 different bells? How many with the whole peal?

30. From 100 men how many juries of 12 men each can be selected if 25 men are excused and if A is always included?

31. If letters in any order form a word, how many words can be formed from 7 consonants and 5 vowels, each word containing 3 consonants and 3 vowels, and ending in a consonant?

32. If each of n straight lines intersects all the others, not more than 2 lines intersecting in the same point, how many points of intersection will there be?

278.* *The number of permutations of* **n** *different things, taken* **r** *at a time,* ***when each of the n things can be repeated, is*** n^r.

After the first place has been filled by one of the n things, the second place can be filled by any one of the n things; and the first two places can be filled in n^2 ways, etc.

Continuing the process, the first three places can be filled in n^3 ways. The exponent of n is evidently the same as the number of places filled. Hence the first r places can be filled in n^r different ways. If x be the number of permutations of n different things, taken r at a time, when each of the n things can be repeated,

$$x = n^r. \qquad \text{V.}$$

279.* *The number of permutations of* **n** *things, taken* **n** *at a time, when p, q, and r ⋯ of the n things are respectively a, b, and c, ⋯ is* $\dfrac{n!}{p!\,q!\,r!\cdots}$.

The proof will be best understood by taking a specific example: find the number of permutations of $a^3b^2c = a \cdot a \cdot a \cdot b \cdot b \cdot c$.

Place a distinguishing sign of each of the three letters a, and also upon the two letters b, thus: a_1, a_2, a_3, b_1, b_2. Then $a_1, a_2, a_3, b_1, b_2, c$, are 6 different things which can, by II, be arranged in $6!$ different ways.

Let x be the total number of permutations of a^3b^2c, in which 3 of the letters are a, 2 are b, and 1 is c. Since, by II, the 3 letters a, considered as a_1, a_2, a_3, can be arranged in $3!$ ways, and the 2 letters b can be arranged in $2!$ ways, the total number of permutations of the letters a^3b^2c, considered as different letters, is $x \cdot 3!\,2!$, or $6! = x \cdot 3!\,2!$ Hence $x = \dfrac{6!}{3!\,2!} = 60$.

In general, let x represent the number of permutations of n things, taken n at a time, when p, q, r, $\cdots$ of the n things are respectively a, b, c, $\cdots$. If in any one of the x permutations the p things a were different from each other and all the others, there will be $p\,!$ different permutations instead of a single permutation. Hence, if all the letters a were changed into p different letters, there would be in all $x \cdot p\,!$ permutations. Similarly, if in any one of the $x \cdot p\,!$ permutations, if the q letters b were different from each other and all the others, there would be $x \cdot p!\,q!$ permutations. Continuing the process of changing the letters until they are all different, the total number of permutations will be $x \cdot p\,!\,q\,!\,r\,!\cdots$. Since $n\,!$ also is the total number of permutations of n different letters, taken n at a time, $n\,! = x \cdot p\,!\,q\,!\,r\,!\cdots$, or

$$x = \frac{n\,!}{p\,!\,q\,!\,r\,!\cdots}. \qquad \text{VI.}$$

EXAMPLES

1. Find the number of ways in which a number of 3 digits can be formed of the 9 significant digits, repetitions being allowed.

Each place can be filled in 9 different ways. Hence, by V,

$$x = 9^3 = 729.$$

2. Find the number of arrangements of the letters in the word *Cincinnati*.

Of the 10 letters in the word *Cincinnati*, c is repeated twice, i is repeated three times, and n is repeated three times. Hence, by VI,

$$x = \frac{10\,!}{2\,!\,3\,!\,3\,!} = 50{,}400.$$

EXERCISE CXLIV*

1. In how many ways can the following products be written as a different succession of factors: (1), $abcdef$; (2), a^3bc; (3), $a^2b^3c^3$; (4), $a^5b^4c^3$?

2. How many different arrangements can be made of the letters in the following words: (1), *permutation*; (2), *parallel*; (3), *combination*; (4), *Massachusetts*; (5), *incommensurable*?

3. How many words, of 3 letters each, can be formed from a, b, c, e, i, o, u, if repetitions are allowed, and if any order of letters form a word?

4. How many numbers of 3 digits each, repetitions being allowed, can be formed from the first 5 digits?

5. How many odd numbers of 5 digits each, repetitions being allowed, can be formed from 0, 1, 2, $\cdots$ 9?

6. How many even numbers of 4 digits each, repetitions being allowed, can be formed from the digits 0, 1, $\cdots$ 9?

7. In how many ways can groups of 4 letters each, repetitions being allowed, be formed from m, n, r, s, u, v, w?

8. In how many ways can groups of 3 letters each be formed from the word *Illinois*?

9. In how many ways can groups of 3 books each be selected from 10 books, 3 of which are the same text in algebra, and 2 of which are the same text in geometry?

10. How many different signals can be formed from 12 flags, 2 being red, 3 green, the rest yellow, if all the flags, placed in line, must be used to make a signal?

CHAPTER XXIV

BINOMIAL THEOREM

280. The type forms given in § 172 when $n = 2, 3, 4, 5$, or 6 may be combined into the general form

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2$$
$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \cdots nab^{n-1} + b^n. \qquad \text{I.}$$

A proof — called the **Binomial Theorem** — that the laws governing the expansion of $(a+b)^n$, when n is any positive integer, give the type form of I will now be given.

1. That I is true when $n = 2, 3, 4, 5$, or 6, may be seen by substituting in I, for example, $n = 3$.

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

If $n = 6$,

$$(a+b)^6 = a^6 + 6\,a^5b + \frac{6 \cdot 5}{1 \cdot 2}a^4b^2 + \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}a^3b^3$$
$$+ \frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4}a^2b^4 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}ab^5 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}b^6.$$

2. If I is true, when $n = k$, k being any positive integer,

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1 \cdot 2}a^{k-2}b^2$$
$$+ \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}a^{k-3}b^3 + \cdots kab^{k-1} + b^k. \qquad (2)$$

3. Multiplying both members of (2) by $a+b$,

$$(a+b)^{k+1} = a^{k+1} + ka^kb + \frac{k(k-1)}{1 \cdot 2}a^{k-1}b^2$$

$$+ \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}a^{k-2}b^3 + \cdots + ab^k$$

$$+ a^kb + ka^{k-1}b^2 + \frac{k(k-1)}{1 \cdot 2}a^{k-2}b^3 + \cdots + kab^k + b^{k+1}$$

$$= a^{k+1} + (k+1)a^kb + \frac{(k+1)k}{1 \cdot 2}a^{k-1}b^2$$

$$+ \frac{(k+1)(k)(k-1)}{1 \cdot 2 \cdot 3}a^{k-2}b^3 + \cdots + (k+1)ab^k + b^{k+1}. \quad (3)$$

The right member of (3) has the same form as the right member of (2), $(k+1)$ taking the place of k. Hence if the theorem is true for any particular power, it is true for the next higher power.

4. The theorem was shown in 1 to be true for the 6th power; hence it is true for the 7th power: being now true for the 7th power, it is true for the 8th power, and so on for *any* power.

5. The theorem is true for $(a-b)^n$, since $(a-b)^n = [a+(-b)]^n$, the signs of the successive terms being alternately plus and minus, the first term being plus.

281. Any required term can be written without completing the expansion by observing the laws for the formation of particular terms. Thus, the fourth term of $(a+b)^n$ is known to be $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3$; the third term of $(a+b)^{k+1}$ is known to be $\frac{(k+1)k}{1 \cdot 2}a^{k-1}b^2$, etc.

Similarly the rth term of $(a+b)^n$ is,

$$\frac{n(n-1)(n-2)\cdots(n-r+3)(n-r+2)}{1\cdot 2\cdot 3\cdots(r-2)(r-1)}a^{n-r+1}b^{r-1};$$

and the $(r+1)$st term of $(a+b)^n$ is,

$$\frac{n(n-1)(n-2)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)(r)}a^{n-r}b^{r}.$$

282. The number of terms in the expansion of $(a+b)^n$, when n is a positive integer, is limited. Thus, by I,

$$(a+b)^4 = a^4 + 4\,a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3$$

$$+\frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4 + \frac{4\cdot 3\cdot 2\cdot 1\cdot 0}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^{-1}b^5 + \cdots.$$

Since the coefficients of all terms following the fifth contain a zero factor, all such terms disappear. In general, if n is a positive integer, the expansion of $(a+b)^n$ ends with the $(n+1)$st term.

The coefficients of terms equally distant from the end terms are equal. It is evident that

$$(a+b)^n = (b+a)^n.$$

$$(b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{1\cdot 2}b^{n-2}a^2 + \cdots nba^{n-1} + a^n. \quad (4)$$

$(b+a)^n$ is merely the expansion of $(a+b)^n$ written in descending powers of b. The last term of I is the same as the first term of (4); the second term of I is the second from the last of (4), etc.

Hence in the expansion of a binomial, *terms after the middle term (or terms) take their coefficients in reverse order.*

EXAMPLES

1. Expand $(3a-1)^5$.

By I,

$$(3a-1)^5 = (3a)^5 + 5(3a)^4(-1) + \frac{5 \cdot 4}{1 \cdot 2}(3a)^3(-1)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}(3a)^2(-1)^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}(3a)(-1)^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(-1)^5$$

$$= 243a^5 - 405a^4 + 270a^3 - 90a^2 + 15a - 1.$$

2. Find the first 4 terms and the last 4 terms of $(x-y)^{31}$.

By I, and § 282,

$$(x-y)^{31} = x^{31} + 31(x)^{30}(-y) + \frac{31 \cdot 30}{1 \cdot 2}(x)^{29}(-y)^2 + \frac{31 \cdot 30 \cdot 29}{1 \cdot 2 \cdot 3}(x)^{28}(-y)^3 + \cdots + \frac{31 \cdot 30 \cdot 29}{1 \cdot 2 \cdot 3}(x)^3(-y)^{28} + \frac{31 \cdot 30}{1 \cdot 2}(x)^2(-y)^{29} + 31(x)(-y)^{30} + (-y)^{31}$$

$$= x^{31} - 31x^{30}y + 465x^{29}y^2 - 4495x^{28}y^3 \cdots + 4495x^3y^{28} - 465x^2y^{29} + 31xy^{30} - y^{31}.$$

3. Find the 6th term of $\left(1 - \frac{\sqrt{b}}{2}\right)^{11}$.

By § 281, the 6th term of

$$\left(1 - \frac{\sqrt{b}}{2}\right)^{11} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(1)^6\left(-\frac{\sqrt{b}}{2}\right)^5$$

$$= -\frac{3 \cdot 7 \cdot 11\, b^2\sqrt{b}}{2^4}$$

$$= -\frac{231\, b^2\sqrt{b}}{16}.$$

EXERCISE CXLV

Expand the following binomials:

1. $(a-b)^5$.

2. $(a-b^2)^5$.

3. $(a^2+b^2)^4$.

4. $(1+x^2)^7$.

5. $(ab-1)^{11}$.

6. $(x^{-1}+y^{-2})^5$.

7. $(a^3-b^3)^7$.

8. $(2x-y)^5$.

9. $(a-2x)^6$.

10. $(3x-2y)^5$.

11. $\left(\frac{1}{x}-2xy\right)^6$.

12. $\left(\frac{2}{a}+\frac{b}{3x}\right)^5$.

13. $\left(\frac{6a}{7}-1\right)^4$.

14. $(2\sqrt{a}-1)^5$.

15. $\left(3a^{\frac{1}{2}}+\frac{1}{b}\right)^6$.

16. $\left(\frac{2a^{-1}}{\sqrt{b}}+\frac{a\sqrt{ab}}{2}\right)^6$.

17. $\left(\frac{2a\sqrt{b}}{a^{-\frac{3}{4}}}-\frac{1}{\sqrt[3]{b}}\right)^5$.

18. $\left(\frac{2a}{x^2}-\frac{x\sqrt{a}}{2}\right)^6$.

19. $\left(\frac{2a^{-1}}{\sqrt{b}}+\frac{a\sqrt{ab}}{2}\right)^7$.

20. $\left(\frac{2a\sqrt{b^{-3}}}{\sqrt[3]{xy}}-\frac{\sqrt{x^3y^3}}{b}\right)^5$.

21. $\left(\frac{2\sqrt{-a}}{b}-\frac{b}{3\sqrt{-a}}\right)^6$.

Express in simplest form the indicated terms of the following binomials:

22. 4th term of $(x-y)^{27}$.

23. 2d term of $(x-y)^{51}$.

24. 11th term of $(a-b)^{12}$.

25. 5th term of $\left(a^3b-\frac{3b^{-2}}{\sqrt{a^5}}\right)^{31}$.

26. 6th term of $\left(\frac{6a^2}{7b\sqrt{b}}-\frac{b}{\sqrt{3a}}\right)^7$.

27. 8th term of $\left(\frac{\sqrt[3]{a}}{b}-\frac{b^2}{2a}\right)^{29}$.

28. 10th term of $\left(\frac{9a}{\sqrt{b}}-\frac{2b}{\sqrt{a}}\right)^{27}$.

29. 6th term of $\left(\frac{2\sqrt{a}}{3} - \frac{6\sqrt[3]{b^2}}{a}\right)^{21}$.

30. 8th and 11th terms of $\left(\frac{a\sqrt{a}}{\sqrt[9]{b^2}} - 6\sqrt{b^3}\right)^{17}$.

31. 4th and 17th terms of $\left(\frac{3\sqrt{a}}{\sqrt[4]{2}\sqrt[11]{b^8}} - \frac{\sqrt{2}}{3a^4\sqrt[10]{b}}\right)^{27}$.

32. $(r+1)$st term of $(2a-b)^m$.

33. $(n-2)$d term of $\left(a - \frac{2}{\sqrt{b}}\right)^{k+1}$.

34. Find the first 4 and the last 4 terms of $(\sqrt{a} - 2\sqrt[3]{b})^{20}$.

35. Find the first 6 and the last 3 terms of $\left(1 - \frac{2\sqrt{a}}{\sqrt[3]{b}}\right)^{32}$.

36. Find the terms that do not contain radicals in $\left(\sqrt{2a} - \sqrt{\frac{b}{a}}\right)^4$.

37. Find the coefficient of x^{20} in $(x+2x^2)^{16}$.

38. Find the coefficient of a^2 in $\left(a + \frac{4}{a}\right)^{12}$.

39. Find the coefficient of a^{46} in $\left(a^3 - \frac{2b}{\sqrt{a}}\right)^{20}$.

40. Find the term independent of b in $\left(\sqrt{\frac{a}{b}} - b^2\right)^{15}$.

41. Find the term independent of x in $\left(\sqrt{\frac{a}{x^3}} - x\right)^{20}$.

42. Find the term independent of a in $\left(\frac{2a}{\sqrt{x}} - \frac{x}{\sqrt{a}}\right)^{36}$.

CHAPTER XXV

LOGARITHMS

283. The **logarithm** of any number is the exponent indicating the power to which a certain fixed number, called the **base**, must be raised in order to produce the given number.

EXAMPLES

1. Find the logarithm of 25 if the base is 5.

Since $25=(5)^2$, the logarithm of 25 is 2.

2. Find the logarithm of 243 if the base is 9.

Since $243=(3)^5=(3^2)^{\frac{5}{2}}=(9)^{\frac{5}{2}}$, the logarithm of 243 is $\frac{5}{2}=2.5$.

3. Find the logarithm of 16 if the base is 8.

Since $16=(2)^4=(2^3)^{\frac{4}{3}}=(8)^{\frac{4}{3}}$, the logarithm of 16 is $\frac{4}{3}=1.3333\cdots$.

4. Find the logarithm of $\frac{1}{27}$ if the base is 3.

Since $\dfrac{1}{27}=\dfrac{1}{(3)^3}=(3)^{-3}$, the logarithm of $\frac{1}{27}$ is -3.

EXERCISE CXLVI

Find the logarithms of the following numbers:

1. $8, 32, 2\sqrt{2}, \frac{1}{8}, \frac{1}{128}$, the base being 4.
2. $3, 27, 81\sqrt{3}, \frac{1}{9}, \frac{1}{81}$, the base being 9.
3. $2, \frac{1}{8}, 2^{32}, \dfrac{1}{2\sqrt{2}}$, the base being 16.

284. In the **common (or Briggs) System**, the number **10** is always taken as the base. It may be shown that

$$10^0 = 1, \qquad 10^0 = 1,$$

$$10^1 = 10, \qquad 10^{-1} = \frac{1}{10} = 0.1,$$

$$10^2 = 100, \qquad 10^{-2} = \frac{1}{10^2} = 0.01,$$

$$10^3 = 1000, \qquad 10^{-3} = \frac{1}{10^3} = 0.001,$$

$$10^4 = 10000. \qquad 10^{-4} = \frac{1}{10^4} = 0.0001.$$

285. Log $1 = 0$ is a short way of writing that, in the system in which the base is 10, the exponent of the power of 10, which produces 1, is 0. Hence,

$$\log 1 = 0, \qquad \log 1 = 0,$$

$$\log 10 = 1, \qquad \log 0.1 = -1,$$

$$\log 100 = 2, \qquad \log 0.01 = -2,$$

$$\log 1000 = 3, \qquad \log 0.001 = -3,$$

$$\log 10000 = 4. \qquad \log 0.0001 = -4.$$

286. It is evident that a number between 1 and 10 has a logarithm between 0 and 1; a number between 10 and 100 has a logarithm between 1 and 2; a number between 100 and 1000 has a logarithm between 2 and 3; a number between 1 and 0.1 has a logarithm between 0 and -1; a number between 0.1 and 0.01 has a logarithm between -1 and -2; a number between 0.01 and 0.001 has a logarithm between -2 and -3, etc. In general, the logarithm of a number greater than 1 is positive, and the logarithm of a number less than 1 is negative.

287. The logarithm of a number, not an exact power of 10, consists of two parts, — the **characteristic**, which is the integral part, and the **mantissa**, which is a fractional part expressed as a decimal.

The characteristic of the logarithm of any number greater than 1 is always positive, and depends upon the number of significant digits in the number to the left of the decimal point. From the table in the preceding paragraph, it may be seen that any number containing two digits to the left of the decimal point has a characteristic of 1; that any number containing three digits to the left of the decimal point has a characteristic of 2, etc. Hence:

The characteristic of the logarithm of any number greater than 1 *is always one less than the number of digits preceding the decimal point.*

The characteristic of the logarithm of any number less than 1 is always negative, and depends upon the number of zeros between the decimal point and the first significant digit. From the table in the preceding paragraph, it may be seen that any number less than 1 and containing no zeros between the decimal point and the first significant digit is -1; that any number containing one zero between the decimal point and the first significant digit is -2, etc. The characteristic of the logarithm of a number less than 1 is rarely written in a negative form, but thus:

-1 is written $9(+\text{ decimal}) - 10$,

-2 is written $8(+\text{ decimal}) - 10$,

-3 is written $7(+\text{ decimal}) - 10$.

The logarithm of a number less than 1 will have a characteristic which is the difference between 9 and the number of zeros between the decimal point and the first significant digit, minus 10. Hence:

The characteristic of the logarithm of any number less than 1 *is negative, and is the difference between* 9 *and the number of zeros between the decimal point and the first significant digit, writing* -10 *after the mantissa.*

288. The mantissa of the logarithm of any number is given in the table on pages 394 and 395.

PRINCIPLES OF LOGARITHMS

289. I. *The logarithm of the product of two or more factors is the sum of the logarithms of the factors.*

Let $\qquad 10^a = x, \quad \text{or} \quad \log x = a, \qquad (1)$

and let $\qquad 10^b = y, \quad \text{or} \quad \log y = b, \qquad (2)$

multiplying (1) and (2),

$$10^{a+b} = xy, \text{ or } \log xy = a + b = \log x + \log y. \qquad (3)$$

Similarly, I can be proved for the product of three or more factors.

II. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

Let $\qquad 10^a = x, \quad \text{or} \quad \log x = a, \qquad (1)$

and let $\qquad 10^b = y, \quad \text{or} \quad \log y = b, \qquad (2)$

dividing (1) by (2),

$$10^{a-b} = \frac{x}{y}, \text{ or } \log \frac{x}{y} = a - b = \log x - \log y. \qquad (3)$$

III. *The logarithm of the power of a number is the product of the logarithm of the number by the exponent of the power.*

Let $\quad 10^a = x, \quad$ or $\log x = a, \qquad (1)$

raising both members of (1) to the bth power,

$$10^{ab} = x^b, \text{ or } \log x^b = ab = b \log x. \qquad (2)$$

IV. *The logarithm of the root of a number is the quotient obtained by dividing the logarithm of the number by the index of the root.*

Let $\quad 10^a = x, \quad$ or $\log x = a, \qquad (1)$

extracting the bth root of both members of (1),

$$10^{\frac{a}{b}} = x^{\frac{1}{b}}, \text{ or } \log x^{\frac{1}{b}} = \frac{a}{b} = \frac{\log x}{b} = \frac{1}{b} \log x. \qquad (2)$$

NOTE. The above principles hold for any number whatever.

290. *The mantissa of the logarithms of all numbers which have the same sequence of digits is the same.*

Let $\quad \log 214.5 = 2.3314,$

then $\quad \log 2145 = \log(214.5 \times 10) = \log 214.5 + \log 10$

$$= 2.3314 + 1 = 3.3314.$$

Let $\quad \log 214.5 = 2.3314,$

then $\quad \log 0.002145 = \log(214.5 \div 100{,}000)$

$$= \log 214.5 - \log 100{,}000$$

$$= 2.3314 - 5 = 7.3314 - 10.$$

From the above examples, it is evident that changing the position of the decimal point is merely multiplying or dividing the given number by a power of 10.

USE OF THE TABLE

291. To find the logarithm of a number consisting of three digits:

On pages 394–395 *find in the column under N the first two digits of the given number. The mantissa required will be found at the intersection of the horizontal line containing the first two digits and the vertical column headed by the third digit. Prefix the proper characteristic.*

$$\log 21.7 = 1.3365,$$

$$\log 0.429 = 9.6325 - 10,$$

$$\log 970 = 2.9868,$$

$$\log 0.0211 = 8.3243 - 10.$$

Numbers containing less than three digits are similarly found.

$$\log 0.27 = 9.4314 - 10,$$

$$\log 5 = 0.6990,$$

$$\log 0.0029 = 7.4624 - 10.$$

292. To find the logarithm of a number consisting of more than three digits:

1. Find the logarithm of 92.04.

Mantissa of the log of the sequence $920 = 9638$,

mantissa of the log of the sequence $921 = 9643$.

An increase of one unit in the sequence gives an increase of 0.0005 in the mantissa; an increase of 0.4 of a unit in the sequence gives an increase of $0.4 \times 0.0005 = 0.0002$ in the mantissa. Therefore,

mantissa of the log of the sequence $9204 = 9640$,

prefixing required characteristic, $\log 92.04 = 1.9640.$

2. Find the logarithm of 0.01238.

Mantissa of the log of the sequence $123 = 0899$,

mantissa of the log of the sequence $124 = 0934$.

An increase of one unit in the sequence gives an increase of 0.0035 in the mantissa; an increase of 0.8 of a unit in the sequence gives an increase of $0.8 \times 0.0035 = 0.0028$ in the mantissa. Therefore

mantissa of the log of the sequence $1238 = 0927$,

prefixing required characteristic, $\log 0.01238 = 8.0927 - 10$.

293. The process of making the proper correction in the logarithms of numbers of more than three digits is called **Interpolation**, and is based upon the hypothesis that adjacent mantissas increase proportionally with the corresponding numbers. Corrections made in this manner are not strictly accurate; and even the mantissas given are only approximate, but are correct to 0.00005. If the correction in the fifth decimal place be 5 or more, the fourth decimal place is increased by 1.

In the table on pages 394–395 *find the mantissa of the first three significant digits, disregarding the position of the decimal point; subtract the mantissa thus found from the mantissa of the next higher number of three significant digits; multiply the difference thus found by the decimal represented by the remaining digits of the given number; add the product (to the fourth decimal) to the mantissa of the first three digits. Prefix the proper characteristic.*

294. To find the number corresponding to a given logarithm.

1. Find the number whose logarithm is $7.5521-10$.

From the table, 5514 is the mantissa of the sequence 356, and 5527 is the mantissa of the sequence 357; that is, a difference of 0.0013 in the mantissa gives a difference of one unit in the sequence; hence the mantissa 5521, being 0.0007 more than the mantissa 5514, gives a difference of $\frac{7}{13}$ of one unit $(=0.5)$ in the sequence. Therefore, applying § 287,

$$\log 0.003565 = 7.5521 - 10.$$

The number corresponding to a given logarithm is called the ***antilogarithm***.

EXERCISE CXLVII

Find the logarithms of the following numbers:

1. 254.	**7.** 362.	**13.** 8.437.
2. 465.	**8.** 5685.	**14.** 0.003.
3. 200.	**9.** 6297.	**15.** 0.000569.
4. 908.	**10.** 1004.	**16.** 0.009186.
5. 2.	**11.** 0.8562.	**17.** 0.01089.
6. 20.	**12.** 0.003547.	**18.** 0.9989.

Find the antilogarithms of:

19. 0.3927.	**25.** 0.9321.	**31.** 0.0250.
20. 1.6395.	**26.** 1.6872.	**32.** $9.5299-10$.
21. 3.7235.	**27.** 3.5589.	**33.** $8.7467-10$.
22. $9.8420-10$.	**28.** 5.6372.	**34.** 2.8837.
23. $7.9069-10$.	**29.** 4.3204.	**35.** $8.9432-10$.
24. $6.9903-10$.	**30.** 2.3974.	**36.** $7.0161-10$.

USE OF LOGARITHMS WHICH HAVE NEGATIVE CHARACTERISTICS

295. *In finding the antilogarithm of a negative logarithm,* -10 *should always appear at the end of the logarithm.*

EXAMPLES

1. Add the following logarithms:

$$\begin{array}{r} 9.6253 - 10 \\ 8.5145 - 10 \\ \hline 18.1398 - 20 \end{array} = 8.1398 - 10.$$

2. Subtract the logarithm 3.1461 from the logarithm 2.1430.

$$\begin{array}{rl} 2.1430 = & 12.1430 - 10 \\ 3.1461 = & 3.1461 \\ \hline & 8.9969 - 10. \end{array}$$

3. Subtract the logarithm $9.3141 - 10$ from the logarithm $8.6537 - 10$.

$$\begin{array}{rr} 8.6537 - 10 = & 18.6537 - 20 \\ 9.3141 - 10 = & 9.3141 - 10 \\ \hline & 9.3396 - 10. \end{array}$$

4. Multiply the logarithm $8.1461 - 10$ by 2.

$$\begin{array}{r} 8.1461 - 10 \\ 2 \\ \hline 16.2922 - 20 \end{array} = 6.2922 - 10.$$

5. Divide the logarithm $7.9101 - 10$ by 3.

$$7.9101 - 10 = 27.9101 - 30$$

$$\begin{array}{r} 3\underline{)27.9101 - 30} \\ 9.3034 - 10. \end{array}$$

In multiplying a logarithm by a fraction, multiply the logarithm by the numerator and divide this product by the denominator, in the order stated, taking care to simplify at each step.

6. Multiply the logarithm $8.3196-10$ by $\frac{2}{3}$.

$$\begin{array}{r} 8.3196-10 \\ 2 \\ \hline 16.6392-20 \end{array} = 26.6392-30$$

$$3\overline{)26.6392-30}$$
$$8.8797-10.$$

EXERCISE CXLVIII

Perform the indicated operations in the following logarithms:

1. $(9.7305-10)+(9.3457-10)$.
2. $(8.5478-10)+(9.8438-10)$.
3. $(0.6544)+(9.7253-10)$.
4. $(0.8733)-(2.7459)$.
5. $(9.3476)-(9.5244)$.
6. $(8.2386-10)\times 5$.
7. $(8.8300-10)\div 3$.
8. $(9.1436-10)\times 4$.
9. $(6.8433-10)\times \frac{2}{3}$.
10. $(9.8010-10)\div \frac{4}{5}$.
11. $(7.1431-10)\times \frac{4}{3}+(8.7153-10)$.
12. $(2.5157)\times \frac{1}{3}-(9.9918-10)$.
13. $(6.5000)-(8.5431)\times \frac{2}{5}$.
14. $(7.2511-10)+(8.2190)\times \frac{5}{3}$.
15. $(9.0909)\times 5-(8.1650)\times \frac{7}{4}$.
16. $(2.0001)\times \frac{3}{5}-(8.0999)\times \frac{1}{6}$.

COMPUTATIONS BY LOGARITHMS

296. 1. Find the value of $\dfrac{192.7 \times 6.54 \times 0.4683}{1624 \times 0.0329 \times 1.028}$.

$$\begin{aligned} \log 192.7 &= 2.2849 \\ \log 6.54 &= 0.8156 \\ \log 0.4683 &= \underline{9.6705-10} \\ \log \text{numerator} &= 2.7710 \\ \log \text{denominator} &= \underline{1.7398} \\ \log \text{fraction} &= 1.0312 \\ \text{fraction} &= 10.75 \end{aligned} \qquad \begin{aligned} \log 1624 &= 3.2106 \\ \log 0.0329 &= 8.5172-10 \\ \log 1.028 &= \underline{0.0120} \\ \log \text{denominator} &= 1.7398 \end{aligned}$$

2. Find the value of $\sqrt{32.5 \times 68.7 \times 32.74}$.

$$\begin{aligned} \log 32.5 &= 1.5119 \\ \log 68.7 &= 1.8370 \\ \log 32.74 &= \underline{1.5151} \\ \log \text{product} &= 4.8640 \\ \tfrac{1}{2} \log \text{product} &= 2.4320 \\ \text{product} &= 270.4. \end{aligned}$$

3. Find the value of $(5.235)^3$.

$$\begin{aligned} \log 5.235 &= 0.7189 \\ 3 \log 5.235 &= 2.1567 \\ (5.235)^3 &= 143.5. \end{aligned}$$

4. Find the value of $0.763 \times 62.8 + 8632 \div 3.265$.

$$\begin{aligned} \log 0.763 &= 9.8825-10 \\ \log 62.8 &= \underline{1.7980} \\ \log \text{product} &= 1.6805 \\ \text{product} &= 47.92 \\ \text{quotient} &= \underline{2644.} \\ \text{sum} &= 2691.92. \end{aligned} \qquad \begin{aligned} \log 8632 &= 3.9361 \\ \log 3.265 &= \underline{0.5139} \\ \log \text{quotient} &= 3.4222 \\ \text{quotient} &= 2644. \end{aligned}$$

NOTE. The last two digits are not accurate since a four-place table is used.

5. Find the value of $-\sqrt{8} \times \sqrt[3]{\frac{1}{7}}$.

$$\begin{aligned} \log 8 &= 0.9031 \\ \tfrac{1}{2}\log 8 &= 0.4516 \\ \tfrac{1}{3}\log \tfrac{1}{7} &= 9.7183 - 10 \\ \log \text{product} &= 0.1699 \\ \text{product} &= -0.1479. \end{aligned}$$

$$\begin{aligned} \log 1 &= 10.0000 - 10 \\ \log 7 &= 0.8451 \\ \log \tfrac{1}{7} &= 29.1549 - 30 \\ \tfrac{1}{3}\log \tfrac{1}{7} &= 9.7183 - 10 \end{aligned}$$

Note that the product is negative in accordance with the law of signs.

6. Solve the equation $3^x = 4$, by the use of logarithms.

$$\log 3^x = \log 4,$$

$$x \log 3 = \log 4,$$

$$x = \frac{\log 4}{\log 3} = \frac{0.6021}{0.4771} = 1.26 \cdots$$

Notice that the above example is a case of an irrational number employed as exponent.

EXERCISE CXLIX

Compute by the use of logarithms:

1. 21.4×9.87.
2. 6.92×53.4.
3. 0.908×201.
4. 65.31×0.319.
5. 0.8642×589.7.
6. 0.9034×0.00154.
7. $698 \div 20$.
8. $0.583 \div 2982$.
9. $0.9085 \div 9.805$.
10. $0.9651 \div 0.8939$.
11. $251.2 \div 0.785$.
12. $0.09891 \div 0.001234$.
13. $200.9 \div 10.01$.
14. $8957 \div 0.9081$.
15. $0.7154 \div 9.003$.
16. $0.2167 \div 0.0375$.
17. $0.04678 \div 892$.
18. $0.0001 \div 894.5$.
19. $8.9 \times 0.32 \times 0.065$.
20. $0.8 \times 3 \times 500$.

21. $0.3 \times 0.09 \times 0.1986$.

22. $6.98 \times 0.6851 \times 0.32$.

23. $0.91 \times 0.81 \times 0.09$.

24. $0.0061 \times 3159 \div 0.005468$.

25. $\dfrac{6.83 \times 0.7816 \times 0.9181}{9.2184 \times 0.07436}$.

26. $\dfrac{215.4 \times 89.72 \times 0.896}{0.6671 \times 19.2 \times 88.32}$.

27. $\dfrac{2.754 \times 0.9803 \times 2001}{3721 \times 0.1596 \times 0.31}$.

28. $\dfrac{6456 \times 0.6456 \times 0.06456}{27 \times 270 \times 2700}$.

29. $\dfrac{0.4692 \times 9231 \times 64.82}{0.1492 \times 0.8351 \times 6987}$.

30. $\dfrac{0.5533 \times 419.2 \times 0.3265}{60.90 \times 5.432 \times 0.8406}$.

31. $\dfrac{6384 \times 0.0987 \times 0.012}{2007 \times 0.3388 \times 0.871}$.

32. $\dfrac{0.7188 \times 0.8159 \times 0.0001}{0.01897 \times 0.8963 \times 0.3031}$.

33. $(6.608)^2$.

34. $(2.755)^2$.

35. $(1.01)^{25}$.

36. $(99.81)^3$.

37. $(49.73)^4$.

38. $(0.9801)^5$.

39. $\sqrt{64.91}$.

40. $\sqrt{9.181}$.

41. $\sqrt[3]{0.0182}$.

42. $\sqrt[3]{6503}$.

43. $\sqrt[3]{52.4}$.

44. $\sqrt[6]{0.1257}$.

45. $\sqrt[3]{0.6608}$.

46. $\sqrt[3]{0.2755}$.

47. $\sqrt[3]{0.1622}$.

48. $\sqrt[4]{851.2}$.

49. $\sqrt[3]{\frac{5}{6}}$.

50. $\sqrt[7]{\frac{1}{8}}$.

51. $\sqrt{\dfrac{23 \times 75}{13 \times 0.85}}$.

52. $\sqrt{\dfrac{0.525 \times 0.054}{351 \times 0.062}}$.

53. $\sqrt{\dfrac{0.768 \times 0.0345}{2512 \times 0.071}}$.

54. $\sqrt{\dfrac{2.016 \times 0.06932}{0.1126 \times 987}}$.

55. $\sqrt[3]{\dfrac{0.0435 \times 3986}{4534 \times 0.087}}$.

56. $\sqrt[3]{\dfrac{0.152 \times 0.025}{25 \times 0.035}}$.

57. $\sqrt[3]{\dfrac{0.3756 \times 0.265}{0.227 \times 863}}$.

58. $\dfrac{(0.03472)^{\frac{1}{2}} \times \sqrt[3]{4011}}{(1.21)^2}$.

59. $\dfrac{5076\sqrt{0.007109}}{9834\sqrt[3]{0.045}}$.

60. $\dfrac{(0.3143)^{\frac{2}{3}}}{1.63 - \sqrt{0.163}}$.

61. $(\frac{5}{9})^3 \sqrt[3]{36}$.

62. $\sqrt[7]{0.38\sqrt{3}}$.

63. $(\frac{14}{19})^6 \sqrt[3]{8.21}$.

64. $\sqrt[3]{8 + \sqrt[5]{7}}$.

65. $\sqrt[9]{384 + \sqrt[4]{31}}$.

66. $\sqrt[5]{937 - \sqrt[4]{21}}$.

67. $\frac{7}{818} \sqrt[6]{\frac{19}{37}}$.

68. $\sqrt[6]{34 \sqrt[5]{0.93}}$.

69. $(\frac{13}{29})^4 \sqrt[3]{0.8357}$

70. $\sqrt[16]{\dfrac{43 + 5\sqrt[3]{278}}{\sqrt[5]{17}}}$.

71. $5\sqrt[7]{\frac{18}{47}} \cdot \sqrt[6]{0.674}$.

72. $\sqrt[6]{2.7 + 3\sqrt[5]{0.15}}$.

73. $\sqrt[5]{9.921 - 3\sqrt[5]{502}}$.

74. $\sqrt[5]{0.783 - 6\sqrt{0.0431}}$.

75. $\sqrt[11]{\dfrac{3.19\sqrt[6]{\frac{2}{3}}}{8097\sqrt{0.35}}}$.

76. $\sqrt[10]{\dfrac{6.923 - \frac{2}{3}\sqrt[4]{9999}}{\sqrt[3]{0.1807}}}$.

77. $\sqrt[7]{\dfrac{52.38 - 17\sqrt[3]{0.138}}{5\sqrt[4]{6.946}}}$.

78. $[(1.048)^3 - \sqrt[6]{0.347}]^4$.

79. Solve for x: $3^x = 13$.

80. Solve for x: $12^x = 25$.

81. Solve for x: $6^x = 54.83$.

82. Solve for x: $3^x = 1.923$.

83. Solve for x: $5^x = 0.1987$.

84. Solve for x: $2x^3 = 1.6254$.

85. Solve for x: $(2 + 0.3)^{\frac{x}{2}} = 10$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996