THE MODERN MATHEMATICAL SERIES

LUCIEN AUGUSTUS WAIT . . . GENERAL EDITOR

(SENIOR PROFESSOR OF MATHEMATICS IN CORNELL UNIVERSITY)

THE MODERN MATHEMATICAL SERIES.

LUCIEN AUGUSTUS WAIT,
(Senior Professor of Mathematics in Cornell University,)
GENERAL EDITOR.

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ELEMENTARY ALGEBRA

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NEW YORK ·:· CINCINNATI ·:· CHICAGO
AMERICAN BOOK COMPANY

TANNER'S ELEM. ALG.

W. P. I

Gift
Pres. Wheeler

PREFACE

In writing this book one of the chief aims of the author has been to make the transition from arithmetic to algebra as easy and natural as possible, and at the same time to arouse and sustain the student's interest in the new field of work.

Accordingly the first few pages are devoted to a restatement and slight extension of the meaning of the ordinary arithmetical operations. Then the literal notation is introduced, and the innovation immediately justified by showing that, among other advantages, it enables the student to solve with ease a class of problems which, by unaided arithmetical analysis, had previously been very difficult for him.

In Chapter II negative numbers are introduced, but only after it has been shown, by concrete examples, that these numbers are essential to man's needs, and that they arise naturally from positive numbers. Moreover, to make this extension of the number system seem less startling, it is pointed out that an altogether similar extension has already been made in arithmetic by the introduction of fractions.

And so on throughout the book, wherever an essentially new step is to be taken, its naturalness and advantages are presented with it, and it is thereafter freely employed until it becomes a useful tool in the student's hands.

Moreover, in order to avoid every unnecessary discouragement to the student, the proofs of the various principles involved in his work are deferred, not only until after he has correctly apprehended and freely employed those principles, but also until after he has been convinced of the necessity of a proof; compare §§ 49, 62 (note), 95, 146 (footnote), 176, etc.

Another important object of this book is to teach the student to think clearly. "There is considerable danger of the true educational value of arithmetic and algebra being seriously impaired by reason of a tendency to sacrifice clear understanding to mere mechanical skill." * The mere manipulation of algebraic

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* From the report of a Committee of the London Mathématical Society appointed to consider the subject of the teaching of elementary mathematics.

symbols, however cleverly performed, is of no advantage whatever in after life to the vast majority of those who study algebra in the schools; but the training in correct reasoning and in an appreciation of the validity of conclusions that may be drawn from given data, which algebra when rightly taught affords, is of vast importance to every one.

Accordingly, although the early part of each new topic has been presented as concretely and simply as possible, and although the student has been led, often without conclusive proofs, to infer correctly the principles involved and to perform the various operations freely, his attention has always been called to the fact that results obtained in this way must be regarded as tentative until after the proofs have been given; and the discussion of no topic has been finally closed without a rigorous demonstration of all the principles involved therein.

New topics have always been brought in where they were needed, and this has made it necessary in some cases to defer the final proofs considerably (cf. Chapters VI, XVIII, and the Appendices); this arrangement has the further advantage, however, of making it possible, if the teacher prefers, to omit the harder proofs altogether on a first reading, without breaking the continuity of the subject.

While this book is designed to meet the most exacting entrance examination requirements in Elementary Algebra of any college or university in this country, and especially the excellent revised requirements of the College Entrance Board, yet the arrangement of the book will be found to be peculiarly suited to a briefer course where that should be desired.

The author takes pleasure in acknowledging his indebtedness to his colleagues in Cornell University for valuable suggestions, especially to Professors Wait and McMahon, who have read both the manuscript and the proof-sheets; to Miss Lelia J. Harvie, formerly of the Virginia State Normal School, who assisted in preparing and grading the exercises in a large part of the book; to Dr. William J. Milne of the State Normal College, Albany, N.Y., for his kind permission to make free use of the exercises in his books; to Professor H. W. Kuhn of the Ohio State University, and to several colleagues in the secondary schools, whose advice has been helpful.

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NOTICE

It is not expected that pupils will be asked to solve *all* of the very large number of exercises and problems, but rather that the teacher will make such selections as will best suit the needs of his or her classes.

If the teacher desires a briefer course than that provided in the book, or prefers to omit the proofs on a first reading, the following articles, together with their attached exercises, may be omitted without breaking the continuity of the work:

Articles	50–54	Pages	74–83	Omit exercises 1–14, pp. 84–86
“	77–79	“	116–122	
“	95	“	143–144	Take exercises 5–15, p. 145
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“	176	“	298–299	
“	183–185	“	314–317.	

The teacher will also find it easy to abbreviate somewhat the work of Chapters XIV and XV.

If the above omissions are made, it will be necessary to pass over a few isolated exercises and notes such as Ex. 3, p. 184, and note 1, p. 301, and also to change slightly the headings to some sets of exercises such as those on p. 145.

ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION

1. Algebra may be regarded as, in a certain sense, a continuation and extension of arithmetic; it may be best, therefore, to recall briefly the subject matter and some of the processes of arithmetic before taking up the study of algebra.

It will presently appear (§ 6) that algebra abbreviates and greatly simplifies the solution of certain kinds of problems. It will also be shown that the meaning hitherto attached to number, as well as its mode of representation, is greatly extended in algebra; and that the "equation," which plays a very minor part in arithmetic, is of great importance in algebraic investigations.

2. Number. The first numbers that present themselves are those which arise from counting and from measuring *things;** they are usually called **whole numbers**, and also **integers**, but may quite appropriately be called the **natural numbers**. These numbers are always definite, and are represented by one or more of the Arabic characters 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Out of combinations of these natural numbers have grown other kinds of numbers, such as fractions, which have already been studied in arithmetic, and still other kinds which will be presented in later chapters of this book.

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* Numbers themselves are not found ready made in nature; there are, however, everywhere *things*, and the counting or the measuring of these gives rise to *numbers*. Since much of the intercourse of life is concerned with the things about us, and with their relations to one another, and since these relations are expressed by means of numbers, it is for this reason alone — to say nothing of other excellent reasons — of fundamental importance that numbers and their combinations be carefully studied. It will be found advantageous, and will add clearness of view, if in our reasoning about *numbers* we frequently go back to the *things* themselves from which these numbers may have arisen.

3. Arithmetical processes. (i) *Addition.* Fundamentally, addition of natural numbers is merely counting.

E.g., to add 4 to 7, means to find that number which is four greater than seven; we begin therefore with 7 and count four, forward, which gives 11. Similarly in general.

The sign of addition is an upright cross (+), which is read **plus** (meaning *more*); when written between two numbers, it means that the second is to be added to the first.

E.g., $7+4$ is read "seven plus four," and means that 4 is to be added to 7.

The result of adding two or more numbers is called their **sum**; the numbers to be added are called the **summands.**

It is evident that addition, in the case of natural numbers, is always a possible arithmetical operation; that this is not true of subtraction will be seen in (ii) below.

Two short parallel horizontal lines (=) are used to express that one of two numbers is equal to, *i.e.*, is the same as, the other; *e.g.*, $7+4=11$. This expression is called an **equation,** and is read "seven plus four equals eleven."

(ii) *Subtraction.* Subtraction is the inverse* of addition; with natural numbers it is a counting *off.*

E.g., to subtract 3 from 15, we begin with 15 and count off (or backward) 3 units, thus: 14, 13, 12; and 12 is the result of the subtraction.

In other words, *to subtract the first of two numbers from the second is to find a third number such that this third number plus the first number equals the second number.*

The sign of subtraction is a short horizontal line (–), which is read **minus** (meaning *less*); when written between two numbers, this sign means that the second number is to be subtracted from the first.

E.g., $7-4$ is read "seven minus four," and means that 4 is to be subtracted from 7.

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* An inverse operation may be defined as one whose effect is neutralized by the corresponding *direct* operation. Addition and multiplication are direct operations; their inverses are subtraction and division.

The *result* of subtracting one number from another is called their **difference**, and also the **remainder**; the number which is subtracted is called the **subtrahend**, and the one from which the subtraction is made is called the **minuend**.

In the above example, 7 is the minuend, 4 the subtrahend, and 3 the remainder, all of which is expressed by the equation $7 - 4 = 3$, which is read "seven minus four equals three."

From the above definition it follows that subtraction is a possible arithmetical operation only when the minuend is at least as great as the subtrahend.

(iii) *Multiplication* is usually defined as the process (or operation) of taking one of two numbers, called the **multiplicand**, as many times as there are units in the other, which is called the **multiplier**. In this sense multiplication is, fundamentally, the same as addition.

E.g., 8 multiplied by 5 means that 8 is to be used 5 times as a summand; *i.e.*, the product of 8 multiplied by 5 is $8 + 8 + 8 + 8 + 8$.

The sign of multiplication is an oblique cross (×), which is read **multiplied by**; when written between two numbers, it means that the first is to be multiplied by the second. The result of multiplying one number by another is called their **product**.

NOTE. The definition of multiplication just given applies only when the multiplier is an integer. Under it, multiplication by a fraction or by a mixed number has, strictly speaking, no meaning. For example, let it be required to multiply 8 by $5\frac{2}{3}$; to do this under the definition just given, it is necessary to take 8 as many times as there are *units* in $5\frac{2}{3}$, but manifestly, while 8 may be taken additively five times, it can not be taken *two thirds of a time*,* and the proposed problem, therefore, does not admit of solution under this definition.

A far more *useful* definition of multiplication than that given above, and one that will serve all future needs, may be stated thus:

The product of two numbers is the result obtained by performing upon the first of these numbers (the multiplicand) the same operation that must be performed upon the unit to obtain the second (the multiplier).

This definition not only includes the former one, but it also gives an intelligible meaning to multiplication when the multiplier is a fraction or a mixed number.

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* This is as meaningless as "to fire a gun two-thirds of a time."

E.g., consider again the question of multiplying 8 by $5\frac{2}{3}$; the multiplier, $5\frac{2}{3}$, is obtained from the unit by taking the unit five times, and $\frac{1}{3}$ of the unit twice, as summands;

i.e.,
$$5\tfrac{2}{3} = 1 + 1 + 1 + 1 + 1 + \tfrac{1}{3} + \tfrac{1}{3},$$
and, therefore, by this new definition of multiplication,
$$\begin{aligned} 8 \times 5\tfrac{2}{3} &= 8 + 8 + 8 + 8 + 8 + \tfrac{8}{3} + \tfrac{8}{3} \\ &= 40 + \tfrac{16}{3} = 45\tfrac{1}{3}. \end{aligned}$$

(iv) *Division.* In algebra as in arithmetic, to **divide** one of two given numbers by another is to find a number which, being multiplied by the second of the given numbers, will produce the first; the symbol of division is ÷, and is read **divided by.**

E.g., $15 \div 5 = 3$, *because* $3 \times 5 = 15$; the first of these equations is read "fifteen divided by five equals three."

The operation of dividing one number by another is called **division**, the first of the given numbers is called the **dividend**, the second is the **divisor**, and the result, *i.e.*, the number sought, is the **quotient.**

E.g., in $15 \div 5 = 3$ the dividend, divisor, and quotient are 15, 5, and 3, respectively.

NOTE 1. Observe that, under the above definition, the test of the correctness of a quotient is
$$\text{quotient} \times \text{divisor} = \text{dividend}.$$
Division is therefore the inverse of multiplication (cf. footnote, p. 2).

NOTE 2. Observe also that while the sum, the difference, and also the product of any two integers is an integer, their quotient may or may not be an integer; for instance, $6 \div 3$ is an integer, but $7 \div 3$ and $5 \div 9$ are called *fractions* [cf. § 7 (v)].

4. Symbols of continuation and deduction. The **symbol of continuation** is $\cdots$; it is read "and so on," or "and so on to," and is used to denote that a given succession of numbers is to continue, either without end or up to a given number.

E.g., $1, 2, 3, \cdots$ is read "one, two, three, *and so on*"; while $1, 2, 3, \cdots 27$ is read "one, two, three, *and so on to* twenty-seven."

The **symbols of deduction** are $\because$ and $\therefore$, and are read "since" and "therefore," respectively.

E.g., $\because 3 \times 5 = 15, \therefore 15 \div 5 = 3$; this expression is read "*since* three multiplied by five equals fifteen, *therefore* fifteen divided by five equals three."

The symbols explained in this section are, like all other signs and symbols, merely abbreviations for longer expressions.

EXERCISES

Read the following expressions, and give the names of their parts:

1. $3 + 7 = 10.$ **3.** $15 \div 3 = 5.$

2. $13 - 8 = 5.$ **4.** $4 \times 6 = 24.$

5. State the definitions of the operations indicated in exercises 1–4. Show that your definition of multiplication applies also to cases in which the multiplier is a fraction or a mixed number.

6. Which of the operations in exercises 1–4 are direct, and what are their respective inverse operations? Explain your answer.

7. How is the correctness of an inverse operation to be tested? Illustrate your answer by testing the correctness of $15 \div 3 = 5$.

Read the following expressions:

8. $\because 5 \times 3 = 15, \therefore 15 \div 3 = 5.$ **9.** $\because 5 + 8 = 13, \therefore 13 - 8 = 5.$

10. The numbers 1, 3, 5, ⋯ are called odd numbers. The sum of the numbers 1, 3, 5, ⋯ 13 is 49.

5. Literal notation. The Arabic characters of arithmetic, viz., 0, 1, 2, 3, ⋯ 9, and also the signs $+$, $-$, $\times$, $\div$, and $=$, are all retained in algebra, and each with its precise arithmetical meaning; but algebra also frequently employs some of the letters of the alphabet to stand for, or represent, numbers.*

E.g., in a certain problem it may be agreed (possibly merely for brevity) to let n stand for a particular number, say 786; in that case $\frac{n}{2}$ (*i.e.*, one half of n) would, in the same problem, stand for 393, while $3n$ (*i.e.*, $n + n + n$) would stand for 2358, etc. In another problem, however, n may be employed to represent any other desired number.

One advantage of representing numbers by letters is explained in § 6 below; others will appear later. For the present it is perhaps sufficient to say that, just as in arithmetic we speak of 4 books, 7 bicycles, 85 pounds, 3 men, etc., so in algebra we shall frequently, in addition to these expressions, use such expressions as a books, n bicycles, x pounds, y men, etc.

When it is necessary to distinguish between numbers which are represented by the Arabic characters 0, 1, 2, ⋯, and numbers which are represented by letters, the latter will be called **literal numbers.**

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* This way of representing numbers is, however, not entirely new to the student because, even in arithmetic, in problems concerning "interest," the *principal, amount, rate, interest,* and *time* are often represented by the letters p, a, r, i, and t, respectively.

The *properties* of numbers are, of course, precisely the same whether these numbers are represented by the Arabic characters, by letters, by words, or in any other way.

E.g., just as 3 books + 8 books = 11 books, so m books + n books = $(m+n)$ books; and if k stands for 20, then $3k + \frac{k}{4} - 2k = 25$.

Again, just as $7 - 3$ means that 3 units are to be subtracted from 7 units, so $a - b$ means that b units are to be subtracted from a units.

EXERCISES

1. If s represents 16, what number is represented by $2s$? by $\frac{1}{4}$ of s, *i.e.*, by $\frac{s}{4}$? by $2s + \frac{s}{4}$?*

2. If a, b, and c represent, respectively, 2, 5, and 8, what is the value of $3a - b$? of $a + b + c$? of $\frac{2b - c}{a}$?

3. If x represents the number of panes of glass in a window, how may the number of panes of glass in 3 such windows be represented?

4. If a suit of clothes costs 8 times as much as a hat, and if d stands for the number of dollars which the hat costs, what will represent the cost of the suit? How may the combined cost of the suit and hat be represented?

5. Since $\frac{1}{3}$ of any number is the same as $\frac{4}{12}$ of that number, and $\frac{1}{4}$ of a number is the same as $\frac{3}{12}$ of that number, what is the remainder when $\frac{1}{4}$ of n is subtracted from $\frac{1}{3}$ of n, where n represents any number whatever? *i.e.*, $\frac{n}{3} - \frac{n}{4} = ?$

6. Just as 37 may be represented by $10 \times 3 + 7$, so $10t + u$ represents a number whose tens' digit is t and whose units' digit is u. If the units', tens', and hundreds' digits of a number are represented by x, y, and z, respectively, how may the number itself be represented?

7. If x represents the number of years in a man's present age, how may his age 5 years ago be represented? What will represent his age 12 years hence?

8. If x represents any integer, how may the next higher integer be represented? The next above that? If n represents any integer, does $2n$ represent an even or an odd number? How may the next higher even number be represented? Show that $2n - 3$, $2n - 1$, $2n + 1$, $2n + 3$, $\cdots$, represent consecutive odd numbers.

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* In these exercises, and throughout the first five chapters of this book, a knowledge of the ordinary arithmetical processes is assumed; the fundamental principles involved will be studied in Chapter VI.

9. A thermometer reads 80° at noon and falls $y°$ during the next 6 hours. What is its reading at 6 o'clock?

10. What number multiplied by 8 gives the product 40? If $8x = 40$, what is the value of x? If $3y + 5y - 2y = 54$, what is the value of y?

6. One advantage of literal notation. The use of letters to represent numbers greatly simplifies the solution of certain kinds of arithmetical problems. This is illustrated in the examples that follow.

Prob. 1. A gentleman paid $45 for a suit of clothes and a hat. If the clothes cost 8 times as much as the hat, what was the cost of each?

ARITHMETICAL SOLUTION

The hat cost "some number of dollars," and since the clothes cost 8 times as much as the hat, therefore the clothes cost 8 times "that number of dollars," and therefore the two together cost 9 times "that number of dollars"; hence 9 times "that number of dollars" is $45, therefore "that number of dollars" is $5, and 8 times "that number of dollars" is $40; *i.e.*, the hat cost $5, and the clothes cost $40.

This solution may be put into the following more systematic form, still retaining its arithmetical character.

Some number of dollars = the cost of the hat;
then 8 times that number of dollars = the cost of the clothes,
$\therefore$ 9 times that number of dollars = the cost of both,
i.e., 9 times that number of dollars = $45,
$\therefore$ that number of dollars = $5, the cost of the hat,
and 8 times that number of dollars = $40, the cost of the clothes.

ALGEBRAIC SOLUTION

The solution just given becomes very much simplified by letting a single letter, say x, stand for "some number" and "that number" which occur so often above; thus:

Let $x =$ the number of dollars* the hat cost.
Then $8x =$ the number of dollars the clothes cost,
and $x + 8x =$ the number of dollars both cost,
i.e., $9x = 45$,
$\therefore$ $x = 5$, and $8x = 40$;
i.e., the hat cost $5, and the clothes cost $40.

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* The letter x here stands for a *number*, not for the *cost of the hat;* the equations are *numerical.*

Prob. 2. Three men, A, B, and C, form a business partnership with a capital of $30,000; if A furnishes twice as much of this capital as B, and C furnishes as much as A and B together, how much does each furnish?

Solution

Let $x =$ the number of dollars furnished by B.

Then $2x =$ the number of dollars furnished by A,

and $3x =$ the number of dollars furnished by C;

and the *algebraic statement* of the conditions of the problem becomes

$$x + 2x + 3x = 30{,}000,$$

i.e., $6x = 30{,}000$,

whence $x = 5000$, $2x = 10{,}000$, and $3x = 15{,}000$;

i.e., A furnishes $10,000, B $5000, and C $15,000 of the capital.

Prob. 3. Of three numbers the second is 5 times, and the third 2 times, the first, and the sum of these numbers exceeds the third number by 42; what are the numbers?

Solution

Let $x =$ the first of the three numbers.

Then $5x =$ the second of the three numbers,

and $2x =$ the third of the three numbers;

and the *algebraic statement* of the conditions of the problem becomes

$$x + 5x + 2x = 2x + 42,$$

i.e., $8x = 2x + 42$,

hence $6x = 42$, [Subtract $2x$ from each member

therefore $x = 7$, $5x = 35$, and $2x = 14$;

and the required numbers are, respectively, 7, 35, and 14.

Note. Observe that the *plan* of each of the foregoing solutions is to let some letter, say x, stand for one of the *unknown* numbers (preferably the smallest), then to express the other unknown numbers in terms of x, and finally to translate into *algebraic language* the conditions which are *verbally* stated in the problem; this last statement is an equation, and from it the required numbers are easily found.

Observe also that while the above problems *can* be solved by arithmetical analysis, the algebraic solution is much simpler.

PROBLEMS

4. In a room containing 45 pupils there are twice as many boys as girls. How many boys are there in the room?

5. If a horse and saddle together cost $90, and the horse cost 5 times as much as the saddle, how much did each cost?

6. In a business enterprise, the combined capital of A, B, and C is $21,000. A's capital is twice B's, and B's is twice C's. What is the capital of each?

7. The difference between two numbers is 8, and their sum is 30. What are the numbers?

8. Divide 98 into three parts such that the second is twice the first and the third is twice the second.

9. A number, plus twice itself, plus 4 times itself, is equal to 56. What is the number?

10. The sum of three numbers is 160; two of these numbers are equal, and the third is twice either of the others. Find the numbers.

11. In a fishing party consisting of 4 boys, 2 of the boys caught the same number of fish, another caught 2 more than this number, and another 1 less; if the total number of fish caught was 29, how many did each catch?

12. If a locomotive weighs 3 times as much as a car, and the difference between their weights is 50 tons, what does the locomotive weigh?

13. Of two numbers, twice the first is seven times the second, and their difference is 75; find the numbers.

SUGGESTION. Let $7x =$ the first number, then $2x =$ the second.

14. An estate of $19,600 was so divided between two heirs that 5 times what one received was equal to 9 times what the other received; what was the share of each?

15. A horse, harness, and carriage together cost $340; if the horse cost 3 times as much as the harness, and the carriage cost $1\frac{1}{2}$ times as much as the horse and harness together, what was the cost of each?

16. A, B, C, and D together buy $16,000 worth of railroad stock. B buys three times as much as A, C twice as much as A and B together, and D one third as much as A, B, and C together. How much does each buy?

17. What number added to $\frac{1}{3}$ of itself equals 20?

Solution

Let $x =$ the number.

Then $x + \frac{1}{3}x = 20,$

i.e., $\frac{4}{3}x = 20,$

$\therefore$ $x = 20 \div \frac{4}{3} = 15.$

18. If $\frac{1}{3}$ of a number is added to the number, the sum is 120; what is the number?

19. If $\frac{1}{3}$ of a number is added to twice the number, the sum is 35; what is the number?

20. The difference between 4 times a certain number and $\frac{1}{4}$ of that number is 30; what is the number?

21. Three times A's age is four times B's, and the sum of their ages exceeds $\frac{1}{4}$ of A's age by 24 years; what is the difference between their ages?

22. A merchant owes a certain sum of money to A, $\frac{1}{3}$ as much to B, and twice as much to C as he owes A; various persons owe him 12 times as much as he owes B, and if all these debts were paid, the merchant would have $4000. What is the total amount that the merchant owes?

23. A boy found that he had the same number of 5, 10, and 25 cent pieces, and that the total amount of his money was $3.20; how many coins of each kind had he?

24. Of a family of seven children each child is 2 years older than the next younger; if the sum of their ages is 84 years, how old is the youngest child?

25. In a number consisting of two digits, the digit in units' place is 3 times that in tens' place, and if these digits be interchanged, the number will be increased by 36; what is the number (cf. Ex. 6, § 5)?

26. The president of a stock company owns 3 times as many shares as the vice president, and the secretary owns 6 shares less than the vice president; if these three men together own 539 shares, how many shares does each own?

27. Three newsboys sold a total of 191 papers in an afternoon; if the second sold 5 more than twice as many as the first, and the third sold three times as many as the second, how many did each sell?

28. A tree, whose height was 150 feet, was broken off by the wind, and it is found that 3 times the length of the part left standing is the same as 7 times that of the part broken off; how long is each part?

29. In a yachting party consisting of 36 persons, the number of children is 3 times the number of men, and the number of women is one half that of the men and children combined; how many women are there in this party?

30. If two boys together solved 65 problems, and if 8 times the number solved by the first boy equals 5 times the number solved by the second boy, how many did each boy solve?

31. An estate valued at $24,780 is to be divided among a family consisting of the mother, 2 sons, and 3 daughters; if the daughters are to receïve equal shares, each son twice as much as a daughter, and the mother twice as much as all the children together, what will be the share of each?

32. A library contains 17 times as many scientific books, and 6 times as many historical books, as books of fiction; if the books of fiction number 220 less than the scientific and historical books together, how many books are there in this library?

33. A, B, and C enter into a business partnership in which A furnishes 6 times as much capital as C, and B furnishes $\frac{2}{7}$ as much as A and C together; if the total capital is $13,500, how much is furnished by each partner?

7. Operations with literal numbers. As is pointed out in § 5, the *reasoning* employed with numbers represented by letters is precisely the same as if those numbers were represented by the Arabic characters. It may be worth while, however, to examine the fundamental operations a little more closely.

(i) *Addition.* Just as $3 + 7$ means that 7 is to be added to 3, so too, if a and b stand for any two numbers whatever, $a + b$ means that b is to be added to a.

Similarly, $a + x + p$ means that x is to be added to a, and that p is then to be added to that sum; and so in general.

(ii) *Subtraction.* Just as $15 - 9$ means that 9 is to be subtracted from 15, so $x - y$ means that y is to be subtracted from x, whatever the numbers represented by x and y.

NOTE. Observe that, while addition is always possible, the indicated subtraction $x - y$ is *arithmetically* possible only when the number represented by x is at least as great as that represented by y.

This restriction upon the relative values of the two numbers in such an expression as $x - y$ is often very inconvenient; in Chapter II the meaning of *number* is so extended as to make this subtraction possible even when y is greater than x.

(iii) *Multiplication.* Just as 6×5 means that 6 is to be multiplied by 5, so $b \times 3$ means that b is to be multiplied by 3. Again, $a \times y \times n$ means that a is to be multiplied by y, and that their product is then to be multiplied by n; and so in other cases.

Instead of the oblique cross ($\times$), a center point ($\cdot$) placed between two numbers (a little above the line to distinguish it from a decimal point) is frequently used as a sign of multiplication.

E.g., instead of 4×6, $3 \times n$, $a \times k$, etc., it is usual to write $4 \cdot 6$, $3 \cdot n$, $a \cdot k$, etc.

And even the center point is usually omitted in cases where its omission causes no misunderstanding.

E.g., $3 \times n = 3 \cdot n = 3\,n$, and $a \times k = a \cdot k = ak$; but, while $4 \times 6 = 4 \cdot 6$, it can not be written "46," for in that case it would be confused with $40 + 6$.

(iv) *Powers, exponents, etc.* Products in which all the factors are identical with one another are usually written in an abbreviated form. This form consists of the repeated factor written only once and having attached to it (at the right and slightly above) the number which tells how many times the given factor is to be repeated.

E.g., $2 \cdot 2 \cdot 2$ is written 2^3, $a \cdot a \cdot a \cdot a \cdot a$ is written a^5, and the product of n factors each of which is x is written x^n.

The expression x^n is called the **nth power** of x, and is usually read "x nth"; the number n is called the **exponent** of the power, and x is called the **base.** In particular, 2^3 is the third power of 2, the exponent is 3, and the base is 2.

A power is called **odd** or **even** according as its exponent is odd or even.

Similarly, a product in which the factor 2 is repeated 3 times, and the factor 5 is repeated 2 times, is written $2^3 \cdot 5^2$. And, more generally, the expression $a^m b^n c^p$ is the product of a repeated m times, b repeated n times, and c repeated p times; it is read "the mth power of a, multiplied by the nth power of b, multiplied by the pth power of c."

NOTE. Under the definition of a power given above, it is evident that a^1 has the same meaning as a, and the exponent 1, therefore, need not be written.

The second and third powers of numbers are, for geometric reasons, often called by the special names of *square* and *cube*, respectively; thus, a^2 is known as the "second power of a," the "square of a," and also as "a squared"; and x^3 is known as the "third power of x," the "cube of x," and also as "x cubed." Corresponding to the other powers there are no such special names.

(v) *Division.* Just as $40 \div 5$ indicates that 40 is to be divided by 5, so $a \div b$ indicates that a is to be divided by b, whatever the numbers represented by a and b; that is, $(a \div b) \cdot b = a$ for all values of a and b [cf. § 3 (iv), note 1].

Other forms of writing $a \div b$ are: $\frac{a}{b}$, $a : b$, and a/b.

In algebra, as in arithmetic, if the divisor is not exactly contained in the dividend, the indicated division is called a **fraction.***

E.g., $\frac{2}{3}$, $\frac{16}{5}$, $\frac{m}{n}$, and $\frac{a+x}{y}$ are fractions.

It is to be remarked, in passing, that literal numbers may be *fractional in form* and yet have integral *values*, and *vice versa.*

E.g., $\frac{a}{b}$, though fractional in form, has the integral value 3 if $a = 12$ and $b = 4$; and $m + 3n$, though integral in form, has the value $\frac{19}{20}$ if $m = \frac{1}{5}$ and $n = \frac{1}{4}$.

8. The order in which arithmetical operations are to be performed. Signs of aggregation. When there is no express statement to the contrary, a succession of multiplications and divisions is understood to mean that these operations are to be performed in the order in which they are written from left to right. The same rule applies in the case of a succession of additions and subtractions.

E.g., $9 \cdot 8 \div 6 \cdot 2$ means that 9 is to be multiplied by 8, that product to be divided by 6, and the resulting quotient to be multiplied by 2; it does *not* mean that the product of 9 by 8 is to be divided by the product of 6 by 2: the result is 24, and not 6.

So, too, $7 + 9 - 6 + 3$ means that 9 is to be added to 7, then 6 subtracted from that sum, and finally 3 added to this remainder; it does *not* mean that $6 + 3$ is to be subtracted from $7 + 9$: the result is 13, and not 7.

Again, by a succession of the operations of addition, subtraction, multiplication, and division, when the contrary is not expressly stated, it is customary to mean that *all* the operations of multiplication and division are to be performed in the order in which

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* A fraction is usually defined as "one or more of the equal parts into which a unit has been divided," but this definition is only a special case of the one given above; it is meaningless when the denominator is not an integer.

they are written from left to right, before *any* of those of addition and subtraction are performed; the resulting expression will then contain only the operations of addition and subtraction, and these operations are then to be performed in the order in which they occur.

E.g., the expression $2+6\cdot5-8\div2$ means $2+30-4$, which is 28.

Should the writer of such an expression desire that a different meaning be given to the expression (*e.g.*, that one or more of the additions and subtractions be performed before some of the multiplications and divisions are performed), he would indicate his meaning by employing one or more of the so-called **signs of aggregation**; among these are the **parenthesis** ( ), the **brace** { }, the **bracket** [ ], and the **vinculum** $\overline{\qquad}$. An expression, included in the parenthesis, brace, or bracket, or under the vinculum, is to be regarded as a whole, and is to be treated as though it were represented by a single symbol.

E.g., $(2+6)\cdot5\div3-(7+8\div2)=8\cdot5\div3-11$, *i.e.*, $2\frac{1}{3}$. So, too, $(4+6)\div2=5$. while without the parenthesis its value would be 7.

It may even be useful sometimes to employ one sign of aggregation within another.

E.g., $$72\div\{252-(24\cdot\overline{4+6})\}.$$

In such a case the innermost sign of aggregation is, of course, to be attended to first; the value of the above expression is 6.

EXERCISES

Find the value of each of the following expressions:

1. $38-6+14-12-2$. **2.** $38-(6+14)-(12-2)$.

3. $9\cdot6-4(36\div3\cdot2)+54-(17-\overline{12-5})$.

4. $12\cdot3\div(9+3-6)\cdot18\div\overline{6-3}$.

5. $\{4\cdot9-16\div2-(12-8)\div(4+6\div3)\}\div(6-2)$.

6. Give a definition of a fraction that will include cases in which the denominator is such a number as $3\frac{2}{5}$.

7. May an expression be fractional in *form*, but integral in value? Give three examples of this kind.

Read each of the following expressions, then tell in what order the indicated operations are to be performed, and finally find the numerical values of these expressions when $a = 8$, $b = 3$, $c = 12$, and $d = \frac{1}{2}$:

8. $a - \frac{b}{c} + \frac{d}{5\,a} \div \frac{a}{d} \cdot \frac{4\,ac}{bd} - \frac{ab}{cd}$.

9. $4\,a + 3\,b - c - \frac{abc}{d} \div \left(\frac{b}{a} + \frac{c}{d}\right)$.

10. $(a + b)^2 - (a - b)^2 - 4\,ab$.

11. $(abc + b) - (4\,cd + d) \div [c - (a + 4\,d)]$.

12. $\frac{ab^2d^3 + c^2 - b^2c}{cd \div ad^2}$.

13. $a(c - b) + b(a - d) + d(a - b)$.

14. $\{\overline{6\,a \div 2\,c} - 2\,d^2\} + \frac{5\,a}{4\,d^3} \div (2\,b \cdot d)$.

9. Advantages of using letters to represent numbers. Attention has already been called (§ 6) to one of the many advantages which result from the use of letters to represent numbers; two further advantages will now be considered.

(i) Suppose it to have been noticed, in a few particular cases, that half the sum of two numbers plus half their difference equals the greater of these numbers, and suppose that it is required to ascertain whether this is true for a certain few pairs of numbers only, or whether it is true for *all* possible pairs of numbers.

For any particular pair of numbers that may be under consideration, 15 and 7 for example, its correctness is easily *verified*, thus

$$\frac{15 + 7}{2} + \frac{15 - 7}{2} = 11 + 4 = 15,$$

but after having made this verification one is still in doubt about every *untried* pair of numbers.

If, on the other hand, letters are employed, it may be *proved*, once for all, that the above property belongs to *every* pair of numbers, and no further verifications are needed. Thus, let a and b represent any two numbers whatever, and let a be greater than b; then

$$\frac{a + b}{2} + \frac{a - b}{2} = \frac{a}{2} + \frac{b}{2} + \frac{a}{2} - \frac{b}{2} = \frac{a}{2} + \frac{a}{2} + \frac{b}{2} - \frac{b}{2} = \frac{a}{2} + \frac{a}{2} = a,$$

which *proves* that half the sum of *any two numbers whatever*, plus half their difference, equals the greater of these numbers. The literal notation has here served to prove a *general law*.

(ii) Another advantage of the literal notation may be illustrated by comparing the solutions of the two following problems.

Prob. 1. If A can do a piece of work in 15 days, and B can do it in 10 days, in how many days can both working together do it?

Prob. 2. If A can do a piece of work in a days, and B can do it in b days, in how many days can both working together do it?

SOLUTION OF PROBLEM 1

Since A can do all of the work in 15 days, therefore he can do $\frac{1}{15}$ of it in one day; similarly, B can do $\frac{1}{10}$ of it in one day, and both together can therefore do $\frac{1}{10} + \frac{1}{15}$, that is, $\frac{1}{6}$, of it in one day; hence it will take both together $1 \div \frac{1}{6}$, *i.e.*, 6, days to do the work.

SOLUTION OF PROBLEM 2

Since A can do the work in a days, therefore he can do $\frac{1}{a}$ of it in one day; similarly, B can do $\frac{1}{b}$ of the work in one day, and both together can do $\frac{1}{a} + \frac{1}{b}$, *i.e.*, $\frac{a+b}{ab}$, of it in one day; hence it will take both together $1 \div \frac{a+b}{ab}$, that is, $\frac{ab}{a+b}$, days to do the work.

The *reasoning* in the two solutions just given is exactly the same; it is to be observed, however, that while in the course of the first solution the numbers given in that problem (viz., 15 and 10) have, by combining, completely lost their identity before the result is reached, yet the numbers given in the second problem (viz., a and b) preserve their identity to the end.

Because of this fact the answer to the second problem may be used as a *formula* by means of which the answer to any other *like* problem may be immediately written down. Thus, if $a = 15$ and $b = 10$, then the second problem becomes exactly like the first, and its answer, viz., $\frac{ab}{a+b}$, becomes $\frac{15 \cdot 10}{15 + 10}$, which is 6 as before. In other words, the solution of the second problem includes the solution of every other similar problem; numerical problems like the first are merely *particular cases* of the second.

10. Recapitulation. Two things mentioned in this chapter must be carefully kept in mind when reading the following pages; they are: (1) the somewhat broader, and at the same time more precise, definitions * of the fundamental arithmetical operations; and (2) the advantages connected with the use of letters to represent numbers.

While the Arabic characters, 1, 2, 3, · · ·, always represent the same numbers, wherever they occur, a letter may be chosen to represent one number in one problem, and a different number in another problem; a letter may also represent a number to which no *specific* value is assigned (cf. § 9), as well as a number whose value is at first *unknown* and is to be found in the course of the solution of the problem (cf. § 6).

EXERCISES

1. Express the following indicated products by means of the exponent notation: $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$; $a \cdot a \cdot a \cdot a$; $x \cdot x \cdot x \cdots$ to 12 factors; $5 \cdot 5 \cdot 5 \cdots$ to n factors; $ax \cdot ax \cdot ax \cdots$ to k factors.

2. Define the expressions: power, base, and exponent, and illustrate your meaning by means of exercise 1.

3. Express the following numbers as products of powers of prime numbers: 48, 200, 972, and 1183.

When $a = \frac{2}{3}$ and $b = \frac{3}{5}$, verify the following statements:

4. $a(a + 2b) = a^2 + 2ab.$

5. $(a + b)^2 = a^2 + 2ab + b^2.$

6. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$

7. $(a + b)(a - b) = a^2 - b^2.$

Find the numerical value of each of the following expressions when $a = 3$, $b = \frac{2}{3}$, $c = \frac{1}{2}$, $x = 4$, $y = 2$, $m = 5$, and $n = 2$:

8. $\dfrac{a^2b^n - c^2x}{xy^m - ax^n}.$

9. $\dfrac{a + x}{b + y} + \dfrac{c + m}{x + n} - \dfrac{c^m}{n^a}.$

10. $\left[a^4 + \left(\frac{1}{b}\right)^2\right] \cdot \left(\frac{1}{c}\right)^n - \left(\frac{ax}{y^n}\right)^y.$

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* These definitions pave the way for the proofs of some fundamental laws to be given later.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

11. General remarks. As already pointed out, an important use of numbers is to enable man to express, in a brief and simple way, the relations of the *things* which are everywhere round about him. At first he used only the natural numbers, *i.e.*, the integers, to express these relations, but as his need and desire for precision and conciseness increased, he found it necessary to extend his number system so as to include in it, not only fractions, but also other kinds of numbers, some of which will presently be studied.

E.g., when he wished to express even so simple a relation as that between the lengths of two lines, he found that integers alone are not sufficient unless the lengths of these lines *happen* to be such that the longer can be divided into parts *each* of which will be just as long as the shorter; thus, if the given lines are 12 ft. and 5 ft. long, respectively, then the relation between their lengths can not be *exactly* expressed by an integer, *because* $12 \div 5$ *is not an integer.*

In order to meet this and other like needs, man extended his number system so as *to make the arithmetical operation of division always possible, i.e.*, he included common fractions in his number system (§ 3, note 2). Before fractions were introduced, division was possible only in the comparatively few cases in which the dividend *happened* to be a multiple of the divisor.

12. Need of negative numbers. In § 11 it is shown that a number system consisting of integers only is not sufficient for man's needs, but that if the system be so enlarged *as to make division always possible, i.e.,* so as to include fractions also, this enlarged system will serve him far better — indeed this enlarged system serves all the purposes of ordinary arithmetic.

In the study of algebra, however, there are many considerations which make it very advantageous to enlarge the number system still further.

To illustrate: on every hand there are found *things* which stand in a relation of opposition to each other — *e.g.*, assets and liabilities in business, latitude north and latitude south of the equator, temperature above zero and temperature below zero, etc. — and if the relations between these opposite things are to be expressed *in the simplest possible way*, then there must be numbers which stand in this same relation of opposition to each other.

How to enlarge the number system — which now consists of integers and fractions (§ 11) — so that it will meet the requirements just now pointed out, becomes evident if it be observed that all such cases of opposition as those mentioned on the preceding page, may be arrived at by subtracting a number from one that is less than itself.

E.g., if a business man whose assets are $5000 loses $6000, *i.e.*, if $6000 be subtracted from his $5000 of assets, it leaves him not only without any *assets*, but with $1000 of *liabilities*, *i.e.*, he has $1000 less than nothing; if from latitude 40° *north* 50° be subtracted (counted off), the result is latitude 10° *south*; if the thermometer records 5° *above* zero and the temperature *falls* 8°, it will then record 3° *below* zero; etc.

Hence, if the number system be so enlarged *as to make subtraction always possible*, even when the subtrahend is greater than the minuend, this enlarged system of numbers will provide for all such cases of opposition as those above mentioned. The nature of these new numbers will be more closely examined in the next article.

NOTE. The considerations mentioned in §§ 11 and 12 demand, respectively, that the natural number system be extended so as to make division and subtraction always possible, *i.e.*, so as to give a meaning to the expressions $a \div b$ and $a - b$, whatever the *relative* values of a and b.

There are, however, other important considerations which lead to the same conclusions; *e.g.*, algebra makes extensive use of letters to represent numbers, and it often happens, as in the problems of § 6, that the number represented by a given letter may be unknown until *after* the problem is solved; if then the number system consists of integers only, and if a and b represent two numbers whose values are not yet known, then, should the combination $a \div b$ present itself in a problem, one would not know whether or not it could be treated as a number (because it would be a number of the given system only if a happened to be a multiple of b), and further progress with the problem must necessarily cease. A much wiser plan is, of course, to extend the number system so as to make $a \div b$ represent a number, whatever the relative values of a and b (*i.e.*, to include fractions in the number system); then the solution may be continued and the proper *interpretation* given at the end. A similar argument applies to such an expression as $a - b$.

13. Negative numbers introduced. The natural numbers arranged in a series increasing by one from left to right, and therefore decreasing by one from right to left, are

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots;$$

addition is performed by counting toward the right (cf. § 3), and subtraction by counting toward the left, in this series. Moreover, addition is always possible *because* this series extends without end toward the right, and subtraction is arithmetically possible only when the subtrahend is not greater than the minuend *because* this series is limited at the left.

What has just been said shows that to make subtraction with natural numbers always possible, it is only necessary to add to the present number system such numbers as will continue the above series indefinitely toward the left.

Let the result of subtracting 1 from 1 be designated by 0; of subtracting 1 from 0, by ${}^{-}1$; of subtracting 1 from ${}^{-}1$, by ${}^{-}2$; of subtracting 1 from ${}^{-}2$, by ${}^{-}3$, etc.; with these new numbers included, and arranged as before, the series becomes

$$\cdots,\ {}^{-}6,\ {}^{-}5,\ {}^{-}4,\ {}^{-}3,\ {}^{-}2,\ {}^{-}1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \cdots,$$

which extends without end toward the left as well as toward the right.

Since in this enlarged series each number is less by *one* than the next number at its right (and therefore greater by one than the next number at its left), therefore addition and subtraction with natural numbers may, as before, be performed by counting toward the right and left respectively.

E.g., to subtract 8 from 5, *i.e.*, to find the number which is 8 less than 5, we begin at 5 and count 8 toward the left, arriving at ${}^{-}3$; hence, $5 - 8 = {}^{-}3$.

Similarly, $4 - 6 = {}^{-}2$, $4 - 9 = {}^{-}5$, ${}^{-}2 - 3 = {}^{-}5$, etc.; hence, besides indicating a particular place in the enlarged number series, ${}^{-}5$ also indicates *that the subtrahend is* 5 *greater than the minuend.** Similarly in general.

Again, to add 7 to ${}^{-}4$, *i.e.*, to find the number which is 7 greater than ${}^{-}4$, we begin at ${}^{-}4$ and count 7 toward the right, arriving at 3. Similarly in general.

14. Negative numbers defined. Numbers less than 0 are called **negative numbers**, and are written thus: ${}^{-}1, {}^{-}2, {}^{-}3, \cdots$; while numbers greater than 0 are, for distinction, called **positive numbers,**

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* Such an expression as $4 - 9 = {}^{-}5$ is, of course, not to be understood to mean that 9 actual units of any kind can be subtracted from 4 such units; 4 of the 9 units may be *immediately* subtracted, leaving the other 5 units to be subtracted later if there is anything from which to subtract; in this sense the number ${}^{-}5$ may be said to indicate a *postponed* subtraction, and thus to have a *subtractive quality*; hence the appropriateness of attaching the minus sign to such numbers.

and are written either $^{+}1$, $^{+}2$, $^{+}3$, ⋯, or, when there is no danger of confusion, simply 1, 2, 3, ⋯.

Positive and negative numbers taken together are sometimes called **algebraic numbers**, while positive numbers alone are called **arithmetical numbers.** The signs $^{+}$ and $^{-}$ employed in the algebraic numbers above are called **signs of quality**, or simply the **signs**, of these numbers. Two algebraic numbers, one of which is positive and the other negative, are said to be of *opposite quality*, or to have *unlike signs*, while if both numbers are positive, or both negative, they are of the *same quality*, *i.e.*, they have *like signs.* A number written without a sign is understood to be positive; *the negative sign, however, is never omitted.*

The numbers $^{-}1$, $^{-}2$, $^{-}3$, ⋯, are read: *negative one, negative two, negative three*, etc., and also *minus one, minus two*, etc.; and the numbers $^{+}1$, $^{+}2$, $^{+}3$, ⋯, *i.e.*, 1, 2, 3, ⋯, are read: *positive one, positive two*, etc., also *plus one, plus two, plus three*, etc., or simply *one, two, three*, etc.

By the **absolute value** of a number is meant its mere magnitude irrespective of its quality; thus, $^{-}2$ and $^{+}2$ have the same absolute value, so too in general have ^{-}a and ^{+}a, whatever the number represented by a.

Two numbers which have the same absolute value, but which are of opposite quality, are called **opposite numbers**; thus, 5 and $^{-}5$ are opposite numbers, so too are ^{+}a and ^{-}a, whatever the number represented by a.

15. Interpretation of negative numbers. The interpretation of a negative number depends upon the nature of the problem which gives rise to it.

E.g., a lady with $15 in her purse goes shopping and makes purchases amounting to $12; how much money has she left?

Here the answer is clearly 15 — 12 dollars, that is, 3 dollars. Had the purchases amounted to $19, the answer would have been 15 — 19 dollars, that is, $^{-}4$ dollars; and the $^{-}4$ dollars would mean that she not only had no money left, but that she was 4 *dollars in debt.*

In this case then, when *possessions* are under consideration, the negative number means *indebtedness.*

The student should now re-read § 12; he should also show that if in a certain problem temperature *above* zero is under consideration, then a negative number means temperature *below* zero; similarly, if positive numbers are used to represent degrees of *north* latitude, then negative numbers will mean degrees of *south* latitude, etc.; in other words, negative numbers must in all cases be interpreted as representing things *opposite in character* to those dealt with in the problem.

EXERCISES

[The following questions should be supplemented by others asked by the teacher.]

1. If temperature above zero be regarded as positive, interpret the following temperature record taken from a U. S. Weather Bureau report: Albany, $^{+}8^\circ$; Bismarck (S.D.), $^{-}11^\circ$; Buffalo, $^{-}2^\circ$; Chattanooga, $^{+}26^\circ$; Denver, $^{-}5^\circ$; Galveston, $^{+}34^\circ$; Marquette, $^{-}9^\circ$; Oswego, $^{+}1^\circ$.

2. How much warmer is it at Albany than at Bismarck in the above record? at Buffalo than at Denver? at Buffalo than at Chattanooga?

3. Answer the questions in Ex. 2 if the word "colder" be put in place of "warmer."

4. The value of all the available property of a merchant is a dollars, and his total indebtedness is b dollars, hence the value of his estate is $(a - b)$ dollars. In such a case is it possible that b is greater than a? If so, what kind of a number is $a - b$? In this case how should this negative number be interpreted? Can one *actually* pay out more money than he has?

5. If assets are represented by positive numbers, how may indebtedness be represented? Interpret the financial conditions represented by the following numbers: $\$^{+}783$; $\$^{-}2568$; $\$^{-}374.20$; and $\$(856 - 1232)$.

Also interpret these conditions if indebtedness be represented by positive numbers.

6. A boy who weighs 54 lb. is playing with a toy balloon which pulls upward with a force of 6 lb.; if the boy were weighed while holding the balloon, what would be the combined weight? If $^{+}54$ lb. represents the weight of the boy, what would represent the *weight* of the balloon?

7. In Ex. 6 the *combined* weight of the boy and the balloon may be represented as $(^{+}54 + {}^{-}6)$ lb., hence *adding* the negative number cancels part of the positive number; is this true in general for additions of positive and negative numbers? Illustrate your answer.

8. If distances upstream on a river be indicated by positive numbers, what would $^{-}5$ miles along this stream mean? Indicate by a number and sign the distance and direction that a boat would *float* on this stream, in $1\frac{1}{2}$ hours, if the river flows $2\frac{1}{3}$ miles an hour.

9. An oarsman who can row 4 miles an hour in still water is rowing upstream on the river in Ex. 8; show that the distance he will go in one hour is $(4 + {}^{-}2\frac{1}{3})$ miles. Here too adding a negative number to a positive number cancels it in part. How far upstream can he row in 7 hours?

10. An ocean steamer is in 12° east longitude; if east longitude be indicated by positive numbers, and if the vessel moves westward through 7° of longitude per day, indicate by a number and sign the longitude of the vessel 4 days hence; $1\frac{1}{2}$ days hence; 2 days ago.

11. If the vessel in Ex. 10 sails westward for 2 days and then, being disabled, drifts $1\frac{1}{2}°$ eastward, what is its longitude?

12. What is meant by the absolute value of a number? Which is the greater, 8 or $^{-}12$? Why?* Which of these numbers has the greater absolute value?

16. Addition of negative numbers. In order to understand just what is meant by adding a negative number to any given number, one has only to recall the essential meaning of a negative number. The symbol $^{-}3$, for example, means (and may always be replaced by) a subtraction in which the subtrahend exceeds the minuend by 3 units, *i.e.*, it is equivalent to an unperformed (postponed) subtraction of 3 units.† Hence, to *add* $^{-}3$ to any number whatever means to *subtract* $^{+}3$ from that number.

E.g., $8 + {}^{-}3 = 8 - 3 = 5$; $4 + {}^{-}10 = 4 - 10 = {}^{-}6$; $^{-}9 + {}^{-}5 = {}^{-}9 - 5 = {}^{-}14$; etc.

Manifestly the above reasoning applies to any negative numbers whatever, hence *the sum of two or more negative numbers is a negative number whose absolute value is the sum of the absolute values of the given numbers;*

And the sum of a negative and a positive number is a number whose absolute value is the difference of the absolute values of the two given numbers, and whose sign is that of the larger of these numbers.

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* Compare § 117.

† Compare footnote, p. 20.

EXERCISES

Find the value of each of the following expressions:

1. $13 + {}^{-}4$.

2. ${}^{-}8 + {}^{-}3$.

3. ${}^{-}6 + 10 + {}^{-}7$.

4. $3\frac{1}{2} + {}^{-}9\frac{1}{4} + {}^{-}5\frac{3}{4}$.

5. ${}^{-}6\frac{5}{8} + 10 + {}^{-}11\frac{7}{8}$. -19

6. ${}^{-}2 + {}^{-}13 + 8 + {}^{-}4 + 6$.

7. Regarding a negative number as a postponed subtraction, show that the result in Ex. 6, and in all others like it, might be found by adding the positive numbers separately, and the negative numbers separately, and then uniting these two sums.

8. If money in hand, or to be received, is represented by a positive number, then how should money owed (a postponed subtraction), or to be paid out, be represented?

Indicate by a sum of positive and negative numbers that a man had \$20 and received \$15 more, and that he paid out for various things \$8, \$3, and \$7.50; also show in two ways that he then had \$16.50 left.

9. If distances westward from a certain point be indicated by positive numbers, how should distances to the eastward be indicated?

A wheelman after riding 37 miles westward from a certain point rides back 12 miles; show that $37 + {}^{-}12$ miles indicates both his direction and distance from the starting point.

10. Indicate by a sum of positive and negative numbers what temperature is now registered by a thermometer which stood at 4° above zero, then rose 2°, later fell 9°, and then rose $2\frac{1}{2}$° (cf. Ex. 9).

11. Make up exercises similar to 8, 9, and 10 to *illustrate* exercises 1–6; observe, however, that the demonstration given in § 16 relies wholly upon the *definition* of a negative number, and is in no way dependent upon any *illustration*.

12. From the reasoning in § 16 it follows that in adding a positive and a negative number, negative units and positive units cancel each other; show that this is true in all the illustrations above.

17. Subtraction of negative numbers. Since subtraction is the inverse of addition, *i.e.*, since to subtract any number, a, from another number, b, means to find the number to which a must be added to produce b,* therefore the results of § 16 may be used to show how to subtract negative numbers.

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* Definition of subtraction, § 3 (iii).

Thus, to subtract $^-3$ from 8 means to find the number to which $^-3$ must be added to produce 8, and by § 16 this number is 11, hence

$$8 - {}^-3 = 11;$$

but

$$8 + 3 = 11,$$

$\therefore$

$$8 - {}^-3 = 8 + 3.$$

Similarly, $15 - {}^-2 = 15 + 2$; $4 - {}^-9 = 4 + 9$; ${}^-8 - {}^-3 = {}^-8 + 3$;

and, in general, ${}^+a - {}^-b = {}^+a + {}^+b$, and ${}^-a - {}^-b = {}^-a + {}^+b$,

whatever the numbers represented by a and b; *i.e., subtracting a negative number from any given number (positive or negative) gives the same result as adding a positive number of the same absolute value to the given number.*

NOTE. If three or more algebraic numbers are to be combined by addition and subtraction, the order in which these operations are to be performed, when there is no express indication to the contrary (parenthesis, bracket, etc.), is understood to be from left to right as in § 8. *E.g.*, ${}^+11 - {}^+4 + {}^-2 = {}^+7 + {}^-2 = {}^+5$.

Moreover, since the subtraction of an algebraic number is equivalent to the addition of its opposite, such an expression as ${}^+11 - {}^+4 + {}^-2$ (above) is usually spoken of as an **algebraic sum**.

EXERCISES

1. To what number must $^-5$ be added to produce 12? What then is the value of $12 - {}^-5$? Answer these questions if 12 is replaced by 3; by $^-2$; by x; by $4 + n$.

Find the value of each of the following expressions:

2. $9 - {}^-6$. **3.** ${}^-4 - {}^-12$. **4.** $26\frac{2}{3} - {}^-4\frac{5}{6}$.

5. A "rule" for subtracting one number from another is often stated thus: "reverse the sign of the subtrahend and proceed as in addition." By means of § 17 establish the correctness of this rule when the subtrahend is a negative number.

6. Using positive numbers to represent money in hand or receivable, illustrate the fact that subtracting a negative number from a positive number increases that number. Does subtracting a negative number *always* enlarge the minuend? Is it so in ${}^-7 - {}^-3$?

7. In the extended number series of § 13, viz., $\cdots$, $^-3$, $^-2$, $^-1$, 0, 1, 2, 3, 4, $\cdots$, how by counting may we add 5 to 3? to $^-2$? to $^-8$? Do we count forward or backward when adding a positive integer? Since subtraction is the inverse of addition, which way should we count when subtracting a positive integer? State and explain the corresponding facts for adding and subtracting negative integers.

Simplify each of the following expressions, that is, find the value of each of these algebraic sums:

8. $137 + {}^-86 - {}^-7 + {}^-26 - 8$. **10.** $\frac{432}{7} - {}^-54\frac{1}{2} + {}^-38\frac{3}{4} - 28$.

9. ${}^-54 + {}^-864 + 732 - {}^-413 - 36$. **11.** $18 - {}^-4\frac{7}{8} - 13\frac{1}{2} + {}^-6 - {}^-17\frac{1}{2}$.

12. Mount Washington is 6290 feet above the sea level, Pikes Peak is 14,083 feet above the sea level, and a place near Haarlem, in Holland, is $16\frac{1}{2}$ feet below the sea level. Find by subtraction how much higher Pikes Peak is than Mount Washington; and also how much higher Mount Washington is than the place near Haarlem.

13. An engineer when making measurements for the grade of a street indicates the distances of points *above* a certain horizontal reference plane by positive numbers, and those that are *below* this plane by negative numbers. Show that the difference of level between any two points may always be found by subtraction. Also draw figures to illustrate several different cases.

18. Product of two algebraic numbers. Rule of signs. The product of any two algebraic numbers is readily obtained from the definition of a product, which is given in § 3 (iii), viz., the product of any two numbers is the result obtained by performing upon the multiplicand the same operation that must be performed upon the *positive unit* to get the multiplier.

E.g., since $3 = 1 + 1 + 1$,

therefore $8 \cdot 3 = 8 + 8 + 8 = 24$;

and ${}^-8 \cdot 3 = {}^-8 + {}^-8 + {}^-8 = {}^-24$.

Again, to get ${}^-3$ from 1, this positive unit must be increased 3-fold and then have its quality sign reversed; therefore, to multiply any number by ${}^-3$, first increase that number 3-fold and then reverse the quality sign.

E.g., since ${}^-3 = {}^-(1 + 1 + 1)$,

therefore $8 \cdot {}^-3 = {}^-(8 + 8 + 8) = {}^-24$;

similarly, ${}^-8 \cdot {}^-3$ means that ${}^-8$ is to be increased 3-fold and then have its quality sign reversed, but ${}^-8$ increased 3-fold is ${}^-24$, therefore

$${}^-8 \cdot {}^-3 = {}^+24.$$

From what has just been said, ${}^-8 \cdot 3 = {}^-(8 \cdot 3)$, $8 \cdot {}^-3 = {}^-(8 \cdot 3)$, and ${}^-8 \cdot {}^-3 = {}^+(8 \cdot 3)$; by the same reasoning as that employed in these particular cases, it follows that, whatever the numbers represented by a and b,

$$\begin{aligned} {}^{+}a \cdot {}^{+}b &= {}^{+}(a \cdot b), \\ {}^{-}a \cdot {}^{+}b &= {}^{-}(a \cdot b), \\ {}^{+}a \cdot {}^{-}b &= {}^{-}(a \cdot b), \\ \text{and } {}^{-}a \cdot {}^{-}b &= {}^{+}(a \cdot b). \end{aligned}$$

These results may be formulated in words thus: *the absolute value of the product of any two numbers is equal to the product of their absolute values, and this product is positive if the factors have like quality signs, otherwise it is negative.*

NOTE 1. Since a succession of multiplications* is to be performed by first getting the product of the first two numbers, then multiplying this product by the next number, and so on (cf. § 8), therefore, by the successive application of the principle established for the product of *two* numbers, it follows that *the absolute value of a continued product is the product of the absolute values of the factors, and this product is negative if it contains an odd number of negative factors, otherwise it is positive.*

E.g., $5 \cdot {}^{-}3 \cdot {}^{-}2 \cdot 7 = {}^{-}15 \cdot {}^{-}2 \cdot 7 = 30 \cdot 7 = 210 = {}^{+}(5 \cdot 3 \cdot 2 \cdot 7)$.

NOTE 2. From Note 1 it follows that *odd* powers (*i.e.*, powers whose exponents are odd numbers) of negative numbers are negative, while *even* powers of negative numbers are positive, and *all* powers of positive numbers are positive.

E.g., $(-2)^2 = +4$, $(-2)^3 = -8$, $(-2)^4 = +16$, etc.

EXERCISES

Find the value of each of the following indicated products:

1. $5 \cdot 3$.

2. ${}^{-}6 \cdot 4$.

3. ${}^{-}7 \cdot {}^{-}2$.

4. $12 \cdot {}^{-}9$.

5. ${}^{-}7\frac{2}{3} \cdot {}^{-}6$.

6. ${}^{-}m \cdot {}^{-}5$.

7. ${}^{-}4\,a \cdot 3$.

8. ${}^{-}12 \cdot {}^{-}3\,x$.

9. ${}^{-}2\,c \cdot 3\,c$.

10. ${}^{-}3 \cdot 4 \cdot {}^{-}6 \cdot {}^{-}2$.

11. $3 \cdot {}^{-}k \cdot {}^{-}x \cdot 4\,a$.

12. $({}^{-}3)^2 \cdot 5 \cdot {}^{-}2$.

13. In the above products, how does the absolute value of the product compare with the product of the absolute values of the factors? What is meant by the absolute value of a number?

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* A succession of multiplications such as $3 \cdot 5 \cdot 9 \cdot 4 \cdots$ is often called a **continued product.**

14. If two numbers have like signs (both plus, or both minus), what is the sign of their product? If they have unlike signs, what is the sign of their product?

15. In the continued product of Ex. 10 above, what is the sign of the product of the first two factors? of this product multiplied by the next factor? of this product by the next factor?

16. Can the *sign* of a continued product be ascertained without actually performing the multiplication? How? What is the sign of the result in Ex. 10 above? in Ex. 11? in Ex. 12? If a continued product has five negative factors, what is the sign of the result?

17. Define the *product* of two numbers, and on the basis of your definition *prove* that the sign of the product $^{-}4 \cdot 7$ is negative. Also that the sign of the product $^{-}4 \cdot {}^{-}7$ is positive.

18. How is $^{-}5$ obtained from the positive unit? How then is the product $8 \cdot {}^{-}5$ obtained? the product $^{-}8 \cdot {}^{-}5$? Show that $^{-}2 \cdot {}^{-}2 \cdot {}^{-}2 \cdot {}^{-}2$, *i.e.*, $(^{-}2)^4$, is 16; also that $(^{-}2)^5 = {}^{-}32$. What is the sign of $(^{-}6)^3$? of $(^{-}2)^4 \cdot (^{-}3)^2$? of $(^{-}1)^{10}$?

19. Define a continued product, and state the order in which its multiplications are to be performed. What is an odd power of a number (cf. § 7)? an even power?

Find the value of $(a + b) \cdot (x - y)$:

20. When $a = 2$, $b = {}^{-}3$, $x = {}^{-}4$, and $y = 6$.

21. When $a = \frac{3}{4}$, $b = {}^{-}2\,a$, $x = {}^{-}6$, and $y = {}^{-}10$.

22. When $a = {}^{-}4$, $b = 6$, $x = ab$, and $y = {}^{-}12$.

23. When $a = {}^{-}4$, $b = a^2$, $x = 3\,a$, and $y = 2\,a^3$.

19. Division of algebraic numbers. Since division is the inverse of multiplication [cf. § 3 (iv)], therefore the results of § 18 may be used to show how to divide algebraic numbers.

For example, to divide $^{+}24$ by $^{-}3$ means to find the number which being multiplied by $^{-}3$ will produce $^{+}24$; but, by § 18, this number is $^{-}8$; hence

$$^{+}24 \div {}^{-}3 = {}^{-}8.$$

And, in general, whatever the numbers represented by a and b,

$$^{+}(a \cdot b) \div {}^{+}b = {}^{+}a,$$
$$^{+}(a \cdot b) \div {}^{-}b = {}^{-}a,$$
$$^{-}(a \cdot b) \div {}^{+}b = {}^{-}a,$$
$$\text{and } {}^{-}(a \cdot b) \div {}^{-}b = {}^{+}a.$$

These results may be formulated in words thus: *the absolute value of the quotient of two numbers is the quotient of their absolute values, and this quotient is positive if the dividend and divisor have like signs, otherwise it is negative.*

EXERCISES

Find the value of each of the following indicated quotients:

1. $14 \div 2$.

2. $14 \div {}^{-}2$.

3. ${}^{-}18 \div 4\frac{1}{2}$.

4. ${}^{-}3\frac{1}{4} \div {}^{-}1\frac{5}{8}$.

5. ${}^{-}24 \div 9$.

6. $({}^{-}6)^2 \div ({}^{-}2)^3$.

7. $15 \div {}^{-}1$.

8. ${}^{-}365 \div {}^{-}9\frac{1}{8}$.

9. ${}^{-}63\,a^2 \div {}^{-}7$.

10. Of what operation is division the inverse? What is an inverse operation? In an exercise in division, what is it that corresponds to the *product* in multiplication? How may the correctness of an exercise in division be tested?

11. If the dividend is positive, and the divisor negative, what is the sign of the quotient? If the dividend is positive, how do the signs of divisor and quotient compare? if the dividend is negative?

Find the value of each of the following expressions:

12. $24 - 28 \div {}^{-}7 + {}^{-}16 \div {}^{-}4 \cdot {}^{-}3$.

13. ${}^{-}8 \cdot {}^{-}6 \div 24 - 27 \div {}^{-}6 \div 3$.

14. $\{28 \div {}^{-}7 - 2 \cdot ({}^{-}4 - 2) + 24\} \div ({}^{-}2)^3$.

Verify that $\dfrac{a+b}{x+y} \cdot \dfrac{a-b}{x-y} = \dfrac{a^2-b^2}{x^2-y^2}$:

15. When $a = 6$, $b = 2$, $x = 10$, and $y = 6$.

16. When $a = {}^{-}8$, $b = 12$, $x = {}^{-}9$, and $y = 7$.

20. Small quality signs (${}^{+}$and^{-}) dispensed with. To distinguish sharply between the positive and the negative quality of numbers, and at the same time to avoid confusing signs of *quality* with the signs of the *operations* of addition and subtraction, the small plus and minus signs (${}^{+}$and^{-}) have thus far been employed.

In order to simplify this notation, which is manifestly somewhat cumbersome, the larger plus and minus signs (+ and —) may in future be employed to indicate both the quality of numbers, and also the operations of addition and subtraction. A number without a quality sign attached to it will continue to mean a positive number, while a negative number will be indicated by writing the minus sign before the numeral, and inclosing both the numeral and its sign in a parenthesis when the parenthesis is necessary to avoid ambiguity: *the quality sign — is never omitted.*

With this simpler notation: 5 means the same as $^{+}5$; a the same as ^{+}a; -8, or (-8), the same as $^{-}8$; $9-5-(-3)$* the same as $^{+}9-{}^{+}5-{}^{-}3$, etc.

In general it may be said that the sign prefixed to a number indicates an *operation* unless that number stands alone, or stands first among several which are to be united, or is inclosed, together with its sign, in a parenthesis.

EXERCISES

1. In the expression $^{+}5+{}^{+}3-{}^{+}4$, which are signs of quality and which are signs of operation?

2. Rewrite the expression in Ex. 1, omitting the quality signs. Has this change in the *writing* really made any change in the *quality* of the numbers?

3. Answer questions 1 and 2 with regard to the expression $^{+}5-{}^{+}3+{}^{+}4$.

4. Could *all* the quality signs in the expression $^{+}15-{}^{+}3+{}^{-}8$ be omitted without changing the meaning of the expression? Which of these signs might be omitted? When no quality sign is written, what is the quality of the number?

5. If the expression in Ex. 4 be written so as to use only the larger signs, is a parenthesis necessary to preserve the meaning? Write the expression so. Also answer the same questions with regard to the expression $x-{}^{-}5+{}^{-}8$.

6. Show that the expression $x-{}^{-}5+{}^{-}8$ is equal to $x+5-8$, wherein both 5 and 8 are positive numbers, and the signs + and − indicate *operations.*

21. Algebraic expressions. Terms. In the course of operations with algebraic numbers, it often happens that the expression for a number does not consist of a single symbol, but rather of a combination of such symbols.

E.g., if a and b represent numbers, then ab, $a+b$, and a^2-3ab^4 also represent numbers.

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* By §§ 16 and 17 this expression equals $9-5+3$, which is 7. In this connection attention may also be called to the fact that since $a+(-b)=a-b$ (§ 16), therefore such an expression as $a-b$ may be understood as meaning either that b is subtracted from a, or that $-b$ is added to a.

Such expressions for numbers as

$$a+b,\ \ 3\,xy,\ \ m^2+2\,n^2-5\,x,\ \ 9\,ax^2+\frac{7\,c}{m^3}-10\frac{bz^2}{m-k}+8\,axy^3,\ \text{etc.},$$

are called **algebraic expressions.***

The parts of an algebraic expression which are connected by the signs + and − (or, rather, these parts together with the signs preceding them) are called the **terms** of the expression. Terms preceded by the plus sign are called **positive terms,** while those preceded by the minus sign are called **negative terms.**

E.g., in the expression $5\,a^2+3\,b-10\,c^3x^2$, there are three terms, viz.: $5\,a^2$, $+3\,b$, and $-10\,c^3x^2$; the first two are positive, and the third is negative.

EXERCISES

1. How many terms are there in the expression

$$5\,a^2x+2\,axy^3-7\,mx^2-26?$$

What are they? Which are positive? Which negative?

2. Answer the same questions as in Ex. 1 with regard to the expression

$$-12+7\,m^2x^3-5\,ay^3\div 3\,x^2-\tfrac{2}{3}\,a^2m^4.$$

3. The sum of two times a number and three times the same number is how many times that number? Unite the two terms $3\,x+5\,x$ into one. What single term is equal to $\frac{1}{2}\,x-\frac{1}{3}\,x$? Is $5\,x+13\,x-9\,x$ equal to $(5+13-9)\,x$? Why?

22. Recapitulation. In this chapter it has been shown that, in order to express in a simple way the relations between assets and liabilities, latitude north and latitude south of the equator, temperature above zero and temperature below zero, in fact, between any of the things which bear a relation of opposition to each other, and which are everywhere met with in one's daily intercourse, it is advantageous to extend the number system so as to make subtraction always possible.

Further considerations have shown that the numbers needed to make subtraction always possible are the so-called negative numbers, and in §§ 15–19 it has been shown how to interpret these numbers, and also how to operate with and upon them. A rapid re-reading of these paragraphs is recommended.

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* An algebraic expression is spoken of as an *expression* or as a *number* according as the thought is of the combined symbol, or of the numerical value which that symbol represents.

CHAPTER III

THE EQUATION

23. Definitions. Although a discussion of the fundamental principles relating to equations must be postponed until more of the theory connected with algebraic expressions has been developed (see Chapter X), yet the importance of the equation as an instrument of investigation demands that it be presented as early as possible.

An equation has already been defined [§ 3 (i)] as a statement that each of two expressions has the same value as the other, *i.e.*, it is a statement that each of these expressions represents the same number. These two expressions are called the **members** of the equation, and that expression which is written at the left of the sign of equality is known as the **first member**, while the other is known as the **second member**.

E.g., $8x - 21 = 3x + 4$ is an equation of which $8x - 21$ is the first member, and $3x + 4$, the second member.

Manifestly the two members of the equation just written do not represent equal numbers for *all* values that may be assigned to the unknown number represented by x: indeed there is only one value of x for which they are equal; viz., for $x = 5$. Hence such an equation is called a **conditional equation**; it is an equation only on condition that $x = 5$.

An equation which is true for *all* values that may be assigned to its letters is called an **identical equation** or, more briefly, an **identity**. To indicate that an equation is an identity, rather than a conditional equation, the sign $\equiv$ may be used instead of $=$ to connect the two members.

E.g., $3x + 5 - x \equiv 2x + 7 - 2$ and $ax^2 + b - ax^2 \equiv b$ are identities. Many other examples of identities will present themselves in the following pages.

The process of deducing from any conditional equation the values that must be substituted for the unknown number to make the two members equal, is called **solving the equation**, and these values themselves are called the **solutions** or **roots** of the equation.

NOTE. The final test as to whether a number is or is not a root of a given equation is to substitute that number for the letter representing the unknown number in the equation; if this substitution satisfies the equation, *i.e.*, if it makes the two members reduce to the same number, then it is a root, otherwise it is not. *E.g.*, 5 is a root of the equation $8x - 21 = 3x + 4$, because substituting 5 for x satisfies this equation.

24. Some axioms and their use. The following principles, usually called axioms, are useful in solving equations.

(1) *If equals be added to or subtracted from equals, the results will be equal.*[*]

(2) *If equals be multiplied or divided by equals, the results will be equal.*[†]

The application of these axioms to the solution of equations is illustrated by the following examples:[‡]

Ex. 1. If $8x - 21 = 3x + 4$, find the value of x; *i.e.*, solve this equation.

SOLUTION

Since $8x - 21 = 3x + 4,$

therefore $8x - 21 + 21 = 3x + 4 + 21,$ [Axiom (1)

i.e., $8x = 3x + 25,$

and therefore $8x - 3x = 3x + 25 - 3x,$ [Axiom (1)

i.e., $5x = 25,$

whence $x = 5.$ [Axiom (2)

VERIFICATION. Substituting 5 for x in the original equation, each member reduces to 19; that is, the substitution of 5 for x satisfies this equation, and 5 is therefore a root of it.

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[*] *Equal numbers* are really the *same* number; such numbers may, of course, be expressed in different ways (*e.g.*, $19 + 5$, $3 \cdot 8$, and $5 \cdot 5 - 1$ each express 24), but they are, nevertheless, the same number, and the self-evidence of these axioms rests upon that fact.

[†] It is not permissible, however, to divide by zero.

[‡] See footnote, p. 6.

Ex. 2. Solve the equation $\frac{2}{3}x + 12 + 7x = \frac{1}{2}x - 10\frac{1}{3} - 4x$.

SOLUTION

Since $\frac{2}{3}x + 12 + 7x = \frac{1}{2}x - 10\frac{1}{3} - 4x,$

therefore, multiplying each member by 6,

$$4x + 72 + 42x = 3x - 62 - 24x, \qquad \text{[Axiom (2)}$$

i.e., $46x + 72 = -21x - 62,$

and therefore, subtracting 72 from each member,

$$\begin{aligned} 46x &= -21x - 62 - 72 \qquad \text{[Axiom (1)} \\ &= -21x - 134, \end{aligned}$$

and, adding $21x$ to each member,

$$67x = -21x - 134 + 21x = -134,$$

whence $x = -2.$ [Axiom (2)

VERIFICATION. Since the substitution of -2 for x satisfies the original equation, therefore -2 is a root of that equation.

EXERCISES

3. Define an equation. Also distinguish between a conditional equation and an identity. Give an illustrative example of each of these two kinds of equations. Is $2ax + 3a = a(4x + 3) - 2ax$ a conditional equation or an identity?

4. What are the members of an equation? Which is called the first member? What is the other member called? What is meant by a root of an equation? Illustrate your answers by suitable examples.

5. What is meant by solving an equation? Describe briefly the process of solving an equation. State the axioms which have thus far been employed in solving equations. Illustrate your answers by suitable examples.

6. How may the correctness of a solution (root) be verified? Show that 4 is a root of $7x - 10 = 4x + 2$. Is 2 a root of $x^2 - 5x + 6 = 0$? Is 3 also a root of this last equation?

Solve the following equations, give the reasons for each step of the work, and test the correctness of the roots:

7. $3x + 2 = x + 30.$

8. $7x - 55 = 18 - 2x - 1.$

9. $2x + \frac{x}{3} = \frac{35}{6}.$

10. $5x - 3\frac{1}{2}x = 17 - x.$

11. If the second member of an equation be multiplied by any number, say 4, what must be done to the first member in order to preserve the equality? If any given number be added to either member, what must be done to the other member? Why?

12. If $2a$ be subtracted from each member of the equation $5x + 2a = 3x + 4b$, what is the resulting equation? What does this show with reference to removing a term from the first to the second member of an equation? Is the same thing true when a term is removed from the second member to the first? Show this by adding $-3x$ to each member of the given equation.

25. Transposition; directions for solving equations. Removing a term from one member of an equation to the other is spoken of as **transposing** that term. It has doubtless been observed, in the solutions of the equations of § 24, that *a term may be transposed from one member of an equation to the other by merely reversing its sign.*

This fact may be formally proved as follows: let any term of either member (*e.g.*, the first) of any given equation be represented by k, — this term may be positive or negative, and may contain any number of letters, — and let the remaining terms of the first member of this equation be represented by M, and its second member by N; then the equation is

$$M + k = N.$$

Subtracting k from each member of this equation, it becomes, by axiom (1),

$$M = N - k,$$

i.e., the term k has disappeared from the first member of the given equation, but has reappeared, with its sign reversed, in the second member.

The following simple directions may now be given for solving such equations as those considered in § 24.

(1) *If the equation contains fractions, multiply both of its members by the least common multiple of the denominators of these fractions* (axiom 2); this is usually spoken of as **clearing the equation of fractions.**

(2) *Transpose all the terms containing the unknown number to the first member of the equation, and all other terms to the second member.*

(3) *Unite the terms of each member, and then divide both members by the coefficient* of the unknown number.*

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* The *coefficient* of the unknown number is the factor which multiplies it.

(4) *Substitute the value thus found for the unknown number in the given equation; if this satisfies the equation, then it is a root of the equation, otherwise it is not.*

These directions may be illustrated by solving again Ex. 2 of § 24, thus:

Given $\frac{2}{3}x + 12 + 7x = \frac{1}{2}x - 10\frac{1}{3} - 4x;$

multiplying the given equation by 6 to clear it of fractions, it becomes

$4x + 72 + 42x = 3x - 62 - 24x,$ [Axiom (2)

whence, transposing, $4x + 42x - 3x + 24x = -62 - 72,$

i.e., $67x = -134;$ [Uniting terms

therefore, dividing by 67, $x = -2;$

and this value of x proves, on substitution, to be a root of the given equation.

EXERCISES

Solve the following conditional equations, and verify the results:

1. $12x + 5x + 20 - 8x = 48 + 3x - 4.$

2. $3(x - 5)$* $+ 4x + 8 = 5(4x - 20).$

3. $5(2x - 10) + 7x - 15 = 20x.$

4. $\frac{x+1}{3} + \frac{x+1}{7} = 4.$

5. $\frac{3x}{4} + 5 = 91 - 10x.$

6. $7y + 2 - y = 17.$

7. $8 + 2y + \frac{y}{4} = 1\frac{3}{4} + \frac{2y}{3}.$

8. $4k - 15 = 2k + 11.$

9. $14k - 20 + 7k - 2 = 6k + \frac{37}{2}.$

10. $2v + \frac{v}{2} - \frac{3v}{4} + 14 = 7v - \frac{v}{4} + \frac{2v}{7} - \frac{15}{2}.$

26. Problems leading to equations. A **problem** is a question proposed for solution; it always asks to find one or more numbers which at the beginning are unknown, and it states certain relations (conditions) between these numbers, by means of which their values may be determined.

The process of solving problems has already been illustrated in § 6, — which should now be re-read. The important steps are:

(1) *Represent one of the unknown numbers involved in the problem by some letter, as x.*

(2) *From the verbal conditions of the problem find algebraic expressions for the other unknown numbers, and form two such expressions that are equal to each other.*

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* That $3(x - 5) = 3x - 15$ may for the present be assumed; it is proved in § 39.

(3) *With these two equal expressions, form an equation,*— called the **equation of the problem.**

(4) *Solve this equation and verify the correctness of the result.*

These steps are illustrated in the solutions of the following problems:

Prob. 1. The sum of the ages of a father and son is 54 years, and the father is 24 years older than the son. How old is each?

SOLUTION

The conditions of this problem, stated in verbal language, are:

(1) The number of years in the father's age plus the number of years in the son's age is 54.

(2) The number of years in the son's age plus 24 equals the number of years in the father's age.

To translate these conditions into symbolic language, let x represent the number of years in the son's age,* then by the second condition the number of years in the father's age is $x + 24$, and by the first condition

$$x + 24 + x = 54,$$

which is the *equation of the problem.*

From this equation it is found that $x = 15$, which is the number of years in the son's age, and $x + 24 = 39$, the number of years in the father's age. By substituting these numbers it is found that they satisfy the two given conditions of the problem and are, therefore, its solution.

NOTE. It may be worth remarking that it was not *necessary*, but only *convenient*, to let x stand for the number of years in the son's age.

Thus, if x represents the number of years in the father's instead of in the son's age, then the given conditions translated into algebraic language become:

(1) $54 - x =$ the number of years in the son's age, and

(2) $54 - x + 24 = x$,

which is the equation of the problem.

From this equation it is found that $x = 39$, whence $54 - x = 15$; these are the same numbers as obtained before.

Again, if $3x$ were chosen to represent the number of years in the son's age, then the equation of the problem would be

$$3x + 24 + 3x = 54,$$

whence $x = 5$ and $3x = 15$, the son's age, and $3x + 24 = 39$, the father's age.

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* It is to be carefully noted that x represents a *number;* it does not represent the son's age, but represents the *number of years* in the son's age.

Prob. 2. A boy was given 39 cents with which to purchase 3-cent and 5-cent postage stamps, and was told to purchase 5 more of the former than of the latter. How many of each kind should he purchase?

Solution

The conditions of this problem, stated in verbal language, are:

(1) The total expenditure is 39 cents.

(2) There are to be 5 more 3-cent stamps than 5-cent stamps.

To translate these conditions into symbolic language, let x stand for the number of 5-cent stamps purchased; their cost is then $5x$ cents: then, by the second condition, the number of 3-cent stamps is $x+5$, and their cost is $(3x+15)$ cents; hence, by the first condition,

$$5x + 3x + 15 = 39,$$

which is the equation of this problem.

The solution of this equation gives $x = 3$, the number of 5-cent stamps, and $x + 5 = 8$, the number of 3-cent stamps; and it is easily verified by substitution that these two numbers do, in fact, satisfy both the conditions of the problem; hence they are the numbers sought.

Prob. 3. If a certain number be diminished by 6, and 2 times this difference be added to 5 times the number, the result will equal 88 minus 3 times the number. What is the number?

Solution

To form the equation of this problem, let x represent the given number; then 5 times the number is $5x$, the number diminished by 6 is $x-6$, etc., and the given condition becomes

$$5x + 2(x - 6) = 88 - 3x,$$

whence $$5x + 2x - 12 = 88 - 3x,$$

and, transposing, $$5x + 2x + 3x = 88 + 12,$$

i.e., $$10x = 100,$$

and, therefore, $$x = 10,$$

which, on verification, proves to be the required number.

Prob. 4. A number consists of two digits whose sum is 5; if the digits be interchanged, the number will be diminished by 9. What is the number?

Solution

To form the equation of this problem, let x represent the digit in units' place; then, by the first condition, $5 - x$ will represent the digit in tens' place; therefore, the number is $10(5-x)+x$, — compare Ex. 6, § 5, — and the number formed by interchanging the digits is $10x+(5-x)$. The second condition then gives

$$10x+(5-x)=10(5-x)+x-9,$$

whence $x=2$, the digit in units' place,

and $5-x=3$, the digit in tens' place.

These two digits are found to satisfy both the conditions of the problem, hence the number sought is 32.

PROBLEMS

5. Divide 28 into two parts whose difference is 4.

6. The sum of two numbers is 63, and the larger exceeds the smaller by 17. What are the numbers?

7. If $\frac{1}{3}$ of a certain number exceeds $\frac{1}{5}$ of that number by 8, what is the number?

8. Divide 48 into two parts such that twice the larger part equals 5 times the smaller part.

9. A man who is 32 years old has a son who is 8 years old; how many years hence will the father be 3 times as old as his son?

10. On being asked his age, a gentleman replies that his age 5 years hence will be twice as great as it was 20 years ago; how old is he?

11. How old is a person if 20 years hence his age will be less by 5 years than twice his present age?

12. If 16 be added to a certain number, the result will be the same as it would be if 7 times the number were subtracted from 56; what is the number?

13. If 6 times a certain number is as much less than 62 as 3 times this number exceeds 19, what is the number?

14. Of four given numbers each exceeds the next below it by 3, and the sum of these numbers is 58; find the numbers.

15. Mary is 25 years younger than her mother, but if she were one year older than she is she would be $\frac{1}{3}$ as old as her mother; what is the age of each?

16. The sum of three numbers is 25; the first of these numbers is greater by 5 than the third, but only $\frac{1}{3}$ as great as the second; find the numbers.

17. Divide $2200 among A, B, and C in such a way that B shall have twice as much as A, and C $200 more than B.

18. Divide $351 among three persons in such a way that for every dime the first receives, the second shall receive 25 cents, and the third a dollar.

19. Three boys together have 140 marbles; if the second has twice as many as the first, but only half as many as the third, how many marbles has each boy?

20. After taking 3 times a certain number from 11 times that number, and then adding 12 to the remainder, the result is less than 117 by 7 times the number; what is the number?

21. A number consists of two digits whose sum is 8, and if 36 be subtracted from this number the order of its digits will be reversed; what is the number?

22. In a certain two-digit number the tens' digit is twice the units' digit, and the number formed by interchanging the digits equals the given number diminished by 18; what is the number?

23. In a three-digit number the tens' digit exceeds the hundreds' digit by 3, the units' digit is 4 less than twice the hundreds' digit, and interchanging the units' and tens' digits decreases the number by 45; what is the number?

24. A two-digit number is equal to 7 times the sum of its digits, and the tens' digit exceeds the units' digit by 3; what is the number?

25. A merchant owes A three times as much as he owes B, he owes C twice as much as he owes A, and he owes D as much as he owes A and B together; if the sum of his indebtedness to A, B, C, and D is $28,000, how much does he owe each?

26. Two clerks, A and B, have the same salary; A saves $\frac{1}{3}$ of his, but B, by spending $150 more than A each year, saves only $350 in 7 years; what is the salary of each?

27. A merchant bought some eggs at the rate of 2 for 3 cents, he then bought $\frac{1}{4}$ as many more at the rate of 6 for 5 cents, and later sold them all at the rate of 3 for 4 cents, thereby losing 6 cents; how many did he buy?

28. If $\frac{5}{6}$ of a number is as much less than the number itself as $\frac{3}{8}$ of the number is less than 65, what is the number?

29. The sum of three consecutive integers is 51; what are these three numbers (cf. Ex. 8, § 5)? Show that the sum of any three consecutive integers is 3 times the second of these integers.

30. The sum of four consecutive odd integers is 80; what are these four numbers? Prove that the sum of any four odd integers is an even integer.

31. M can do a certain piece of work in 8 days, and N can do it in 12 days; in how many days can both do it when working together [cf. § 9 (ii)]?

32. If M begins the work mentioned in Prob. 31, and, after working a certain number of days at it, turns it over to N to finish, and the entire piece of work is done in 10 days, how long did each work at it?

33. A country club consisting of 200 members, having decided to build a new club house, assessed each of its members a certain sum for that purpose; meanwhile the membership was increased by 50, and it was then found that the assessment could be reduced by $10; what was the cost of the proposed house?

34. A real estate dealer purchased three houses, paying $1\frac{1}{2}$ times as much for the second as for the first, and $1\frac{4}{5}$ times as much for the third as for the first; if the difference between the cost of the second and third was $1500, what was the cost of each?

35. A gentleman left his property, valued at $300,000, to be divided among three colleges; if the first was to receive $30,000 more than the second, and the third half as much as the other two together, how much was each to receive?

36. Five boys had agreed to purchase a pleasure-boat, but one of them withdrew, and it was then found that each of the remaining boys had to pay $2 more than would have been necessary under the original plan; how much did the boat cost?

37. A lady having already spent $10 more than $\frac{2}{5}$ of her money made further purchases amounting to $10 more than $\frac{2}{5}$ of what then remained, and found that she had only $2 left; how much had she at first?

38. A laborer was engaged to do a certain piece of work on condition that he was to receive $2 for every day that he worked, and to forfeit 50 cents for every day that he was idle; at the end of 18 days he received $28.50. How many days did he work?

39. A certain number being subtracted from 50, and also from 34, it is found that $\frac{3}{4}$ of the first of these remainders exceeds $\frac{2}{5}$ of the second by 47; what is the number?

CHAPTER IV

ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS — PARENTHESES

I. ADDITION

27. Monomials, binomials, etc.; coefficients. An algebraic expression consisting of but one term* is called a **monomial**, while one consisting of two or more terms is called a **polynomial**. A polynomial consisting of only two terms is usually called a **binomial**, and one consisting of three terms, a **trinomial**; but to polynomials consisting of more than three terms it is not customary to give special names corresponding to binomial and trinomial.

E.g., $2ax^3$, $-7m^3p^2$, and $3bx^2y^5$ are monomials; x^2+3y, $5m-2z^2$, and $-3ab^2-\frac{2}{3}k^3y^4$ are binomials; and $2x^3+4ay-5b^2$, $2s^4-6y+3m^2x^3$, and $x+3t-\frac{2}{7}abx^2$ are trinomials.

If a term is composed of several factors, any one of its factors, or the product of two or more of them, is called the **coefficient** of the product of the remaining factors.

E.g., in the term $5axy^2$, the coefficient of axy^2 is 5, the coefficient of xy^2 is $5a$, the coefficient of $5xy^2$ is a, etc.

A coefficient consisting of Arabic characters only is a **numerical coefficient**, while one that contains one or more literal factors is a **literal coefficient**.

E.g., in the term $-3ax^2y^4$, the numerical coefficient of ax^2y^4 is -3, but $-3a$ and $3ay^4$ are literal coefficients of x^2y^4 and $-x^2$ respectively.

NOTE. The word "coefficient" is usually understood to mean "numerical coefficient," and the sign (+ or −) written before a term is usually regarded as belonging to the numerical coefficient. When no numerical coefficient is written, the term is understood to have the coefficient 1.

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* For the definition of an "algebraic expression," and of a "term," see § 21.

28. Positive and negative terms; like and unlike terms. A term whose sign is + is called a **positive term**, and one whose sign is − is called a **negative term**. If the first term of an algebraic expression is positive, its sign is usually omitted, but the sign of a negative term is never omitted.

Note. As has already been pointed out, the letters in an algebraic expression may represent any numbers whatever, — they may be positive or negative, even or odd, integers or fractions, — and therefore an algebraic expression which is fractional in *appearance* may have an integral value, and *vice versa*; so too a term which is positive in *appearance* may still, for certain values of the letters involved in it, have a negative value, and *vice versa*.

Terms which either do not differ at all, or which differ only in their numerical coefficients, or in their quality signs, are called **like terms**, and also **similar terms**; terms which differ in other respects are called **unlike terms**, and also **dissimilar terms**.

E.g., $3x^2y$, $5x^2y$, and $-\frac{1}{2}x^2y$ are like terms, while $2ax$, $-5b^3x^2y$, and $3xy^2$ are unlike terms.

Like terms must contain the same letters, and these letters must be affected with the same exponents, but they may differ in their signs and also in their coefficients.

EXERCISES

1. What is the coefficient of a^2x in each of the following expressions:

$$3a^2x,\quad -5a^2x,\quad a^2x,\quad 4a^2bx,\quad -\tfrac{3}{8}a^2x,\quad \frac{12a^2bx}{7m},\ \text{and}\ \ -9a^3x?$$

2. Which of the above coefficients are literal and which numerical? Which of the terms in Ex. 1 are positive and which negative?

3. Do the positive terms in Ex. 1 necessarily represent positive numbers for all values that may be assigned to the letters involved? Try $a = 3$ and $x = -2$.

4. What is the coefficient of $x - y$ in each of the following expressions: $13(x-y)$, $-a(x-y)$, $\frac{5}{7}m(x-y)$, and $(4-a^3)(x-y)$? Which of these coefficients are numerical? Which literal? Which of these expressions are positive and which negative? Try various values for the letters and see whether the negative expressions necessarily represent negative numbers.

5. Consult a good dictionary for the derivation of the words "monomial," "binomial," "trinomial," and "polynomial." Write three monomials, three binomials, three trinomials, and three polynomials.

6. Distinguish carefully between the meanings of 5 in the expressions $5x$ and x^5. What name is given to the 5 in each of these expressions?

7. What are like terms? By what other name are they known? In what respects may they differ and still be like terms? Are $3\,x^2y$, $-2\,x^2y$, and $\frac{5}{8}\,x^2y$ similar? Are $4\,ax^3$ and $-6\,bx^3$ similar? Are these last two terms similar if $4\,a$ and $-6\,b$ are regarded as their respective coefficients?

8. Write three sets of like terms, some terms being positive and some negative, and each set containing at least four terms.

29. Addition of monomials. That the sum of several similar monomials may be united into a single term has already been illustrated in some of the exercises and problems in the preceding pages; this subject will now be considered in greater detail.

Since 5 times any given number, plus 2 times that number, is 7 times the given number, *i.e.*, $(5+2)$ times the given number, therefore $5\,a+2\,a=(5+2)\,a=7\,a$, whatever the number represented by a. So too $3\,mx^2y+8\,mx^2y=(3+8)\,mx^2y=11\,mx^2y$.

Observe that this reasoning applies to any two similar monomials whatever.

Since the sum of three or more numbers is obtained by adding the third to the sum of the first and second, the fourth to the sum of the first three, etc., therefore, *to add any number of similar monomials, add their coefficients, and to this result annex the common literal factors.*

It is usually most convenient to write the terms to be added under one another, as in arithmetic, thus:

$$\begin{array}{r}3\,xy^2\\ 8\,xy^2\\ \hline 11\,xy^2\end{array}\qquad\qquad \begin{array}{r}153\,a^2mx^3\\ 74\,a^2mx^3\\ \hline 227\,a^2mx^3\end{array}\qquad\qquad \begin{array}{r}18\,ak^2s\\ -7\,ak^2s\\ \hline 11\,ak^2s^{*}\end{array}$$

If the monomials to be added are *dissimilar*, they cannot be united into a single term, but their sum may be *indicated* in the usual way; *e.g.*, the sum of $5\,a$ and $2\,cx^2$ is $5\,a+2\,cx^2$.

EXERCISES

1. If 6 times any number whatever be added to 13 times that number, the result is how many times the given number?

2. To 6 times any given number add 13 times that number, and to this sum add –8 times the given number; what is the result?

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* Since $18+(-7)=11$; compare § 16.

3. State in words a convenient rule for adding any number of like terms. Does your rule apply to cases in which some of these terms are negative?

4. Find the sum of $6\,n$, $7\,n$, $-3\,n$, $18\,n$, and $-11\,n$.

5. Find the sum of $4\,a^2x^3$, $5\,a^2x^3$, $-2\,a^2x^3$, and $-6\,a^2x^3$.

Simplify the following expressions, *i.e.*, unite similar terms:

6. $3\,mxy^2 + (-4\,mxy^2) + (-12\,mxy^2) + 5\,mxy^2$.

7. $14\,abx^3 + 32\,abx^3 + (-19\,abx^3) + 5\,abx^3$.

8. $3\,mp^2 + 7\,mp^2 + 13\,a^2x - 4\,mp^2 + (-5\,a^2x) - 2\,a^2x$.

9. $4(a-b) + 3(a-b) - 2(a-b) + (a-b)$.

10. $4(ax)^2 + 11(ax)^2 - 3(ax)^2 + [-6(ax)^2]$.

11. $7(x+y+z) + 19(x+y+z) + 4(x+y+z) - 8(x+y+z)$.

12. $-15(ax^2+3) + 27(ax^2+3) - 9(ax^2+3)$.

Add the following terms, uniting as far as possible, and indicating the addition where necessary:

13. $3\,mp^2$, $-8\,mp^2$, $5\,a^2x$, $-4\,mp^2$, $-3\,a^2x$, and $2\,a^2x$.

14. $23\,a^2$, $5\,b^2$, $-8\,a^2b^2$, $-13\,b^2$, $24\,a^2b^2$, and $-19\,a^2$.

15. $-5(a-b)$, $2(ax)^2$, $-8(ax)^2$, $12(a-b)$, and $-4(ax)^2$.

16. $16\,x$, $-y$, $4\,x$, $-x$, $4\,z$, $5\,y$, x, $2\,x$, and $-3\,z$.

17. $mxy + nxy$ equals how many times xy?

18. $ax^2 + bx^2 - cx^2 - lx^2$ is how many times x^2?

30. Addition of polynomials. The explanation given in § 29 for the addition of monomials is easily extended so as to apply to the addition of polynomials also.

E.g., $7\,b^2y^3 - 3\,ax^2 + 6\,abc$ and $4\,b^2y^3 + 5\,ax^2 - 12\,abc$ may be added thus ·

$$\begin{array}{r} 7\,b^2y^3 - 3\,ax^2 + 6\,abc \\ 4\,b^2y^3 + 5\,ax^2 - 12\,abc \\ \hline 11\,b^2y^3 + 2\,ax^2 - 6\,abc \end{array}$$

Similarly in general, hence:

To add two or more polynomials, write them under one another so that similar terms shall stand in the same column, and then add each column separately as in § 29.

EXERCISES

Find the sum of each of the following groups of polynomials:

1. $6a-5b+3c$, $7a+10b-6c$, $8a-9b-10c$, and $19a+8b+2c$.

2. $2c-7d+6n$, $8d-3n-9c$, $4d+16n-4c$, and $3c-4n+d$.

3. $2c-7d-x+6n$, $8y-14n-3z$, $18z+10n+8d+3x$, $4n-18c-5x+6d$, $19c+4x+8n-6d$, and $5c+2d-10c-4z$.

4. $2x^3+7bx^2-4b^2x+3b^3$, $8b^2x-15bx^2-5b^3-10x^3$, $3x^3-6bx^2$, $4bx^2-6b^3+10x^3$, and $-bx^2+x^3-4b^3$.

Simplify the following polynomials, *i.e.*, unite their similar terms:

5. $8mx-5x^2+3m^2+2x^2-8m^2+13m^2-18mx+6x^2-9m^2$.

6. $3a^2-6ab-8b^2+7a^2-3a^2+2ab-14b^2-6ab+8b^2$.

7. $4xy-xy+10x^3-4y^2-8x^3-4x^3+3y^2-15xy+23xy$.

8. $4a^2-6a+4-3a^2+a+1.5a^2-2+5a-3.4a^2-3.75-2a$.

9. $ax^2-4x^2+by^3-cx^2+14x^2-by^3+ay^3-3y^3$.

[Collect all the x^2 terms and all the y^3 terms.]

II. SUBTRACTION

31. Subtraction of monomials. Since 5 times any given number, minus 2 times that number, is 3 times the given number, *i.e.*, (5−2) times the given number, therefore $5a-2a=(5-2)a=3a$, whatever the number represented by a. So too $13mx^2y^3-8mx^2y^3=(13-8)mx^2y^3=5mx^2y^3$.

Observe that the reasoning just now given applies to any two similar monomials whatever, hence:

To subtract one of two similar monomials from the other, subtract the coefficient of the subtrahend from that of the minuend, and to this remainder annex the common literal factors.

Here, as in arithmetic, it is usually most convenient to write the subtrahend under the minuend, thus:

$$\begin{array}{r} 126\,az^2 \\ 92\,az^2 \\ \hline 34\,az^2 \end{array} \qquad \begin{array}{r} 13\,mx^2y^3 \\ 8\,mx^2y^3 \\ \hline 5\,mx^2y^3 \end{array} \qquad \begin{array}{r} 53\,bcx^3 \\ -9\,bcx^3 \\ \hline 62\,bcx^3 \end{array}*$$

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* Since $53-(-9)=53+9=62$; compare § 17.

Note. Since algebraic expressions represent numbers, the rule just now given may be stated thus:

To subtract one of two similar monomials from the other, reverse the quality sign of the subtrahend and proceed as in addition (cf. § 17, Ex. 5).

In order to avoid confusion when reviewing one's work, it is usually best not actually to change the sign of the subtrahend, but only to *conceive* it to be changed, or at most to write the changed sign *below* the term, thus:

$$\begin{array}{r} 13\,mx^2y^3 \\ 8\,mx^2y^3 \\ - \quad\quad\; \\ \hline 5\,mx^2y^3 \end{array} \qquad\qquad \begin{array}{r} 53\,bcx^3 \\ -9\,bcx^3 \\ + \quad\quad\; \\ \hline 62\,bcx^3 \end{array}$$

EXERCISES

In the following exercises subtract the number written below from the one above it:

1. $$\begin{array}{rrrrrr} 18 & \quad -18 & \quad -18 & \quad 18 & \quad -9 & \quad 9 \\ 5 & 5 & -5 & -5 & 9 & -9 \\ \hline \end{array}$$

2. $$\begin{array}{rrrrr} 7\,a & \quad 16\,bx^2 & \quad -18\,m^3 & \quad -18\,r^2x^3 & \quad 26\,v^3y^5 \\ 4\,a & -3\,bx^2 & 5\,m^3 & -5\,r^2x^3 & -7\,v^3y^5 \\ \hline \end{array}$$

3. $$\begin{array}{rrrr} 15\,cx^n & \quad 6\,m^2p^x & \quad -34.7\,k^2x^3y^r & \quad 5\tfrac{2}{3}\,a^2m^4 \\ 3\,cx^n & -5\,m^2p^x & 6.8\,k^2x^3y^r & -2\tfrac{1}{2}\,a^2m^4 \\ \hline \end{array}$$

4. Are the signs written in the above exercises signs of operation or signs of quality?

5. Define subtraction, and from your definition show how to verify the correctness of the above exercises.

6. Show that "changing the sign of the subtrahend and proceeding as in addition" will give the remainder in each of the above exercises.

7. From $5(a-2\,b^3)$ subtract $-11(a-2\,b^3)$; also subtract $15\,m^2(x-y)$ from $-23\,m^2(x-y)$; and $-2\,x(1+5\,a^2y)$ from $14\,x(1+5\,a^2y)$.

8. From the sum of $6\,ax^3$, $-3\,ax^3$, and $11\,ax^3$, subtract the sum of $-4\,ax^3$, $9\,ax^3$, and $-7\,ax^3$.

9. Re-read §§ 16 and 17, and then *prove* that, in any subtraction, the remainder may be obtained by adding the subtrahend, with its sign changed, to the minuend.

32. Subtraction of polynomials. From the reasoning already given, it is evident that one polynomial may be subtracted from another by writing the subtrahend under the minuend, similar terms under one another, and subtracting term by term, thus:

$$\begin{array}{r}
7\,b^2y^3 - 3\,ax^2 + 6\,abc \\
4\,b^2y^3 + 5\,ax^2 - 12\,abc \\
- \qquad - \qquad + \\
\hline
3\,b^2y^3 - 8\,ax^2 + 18\,abc
\end{array}$$

EXERCISES

1. From $12\,a - 3\,b$ subtract $6\,a - 5\,b$.

2. From $3\,x - 2\,y + 5\,z$ subtract $5\,y - z - 8\,x$.

3. From $4\,a^2xy^3 - 9\,x^2y + 10\,a^3y^2$ take $7\,a^3y^2 - 3\,a^2xy^3 - 12\,x^2y$.

4. From $8\,b^2 - 7\,m^3 - 13\,ax^2$ take $4\,m^3 - 8\,ax^2$.

5. From $5\,x^2 + 4\,a^3b^2$ take $13\,a^3b^2 - 2\,x^2 + 5\,abx$.

6. From $x^4 + 1$ take $1 - 2\,x + x^4 + 3\,x^2 - 4\,x^3$.

7. From $2\,a - 3\,x + z$ take the sum of $9\,x + z - 4\,a$ and $10\,z - 5\,x + a$.

8. From $7.42\,x^2 - 3\frac{1}{2}\,xy + 10\,y^2$ take $2.5\,xy - y^2 + 3.02\,x^2$.

9. From $34\,a^2x^3 - 10\,m^4y^n$ take $15\,y^6 + 10\,a^2x^3 + m^4y^n$.

10. Subtract $-7\,c^3r^2 + 3\,a^2 - r^3$ from $5\,a^2 + 2\,r^2 + s^4 - 3\,c^3r^2$.

11. Subtract $1 - 3\,x + 10\,x^2$ from $2\,x^2 + 5$; from 4; from 0.

12. Subtract $-8\,a + 3\,b - 13\,x^2$ from $5\,b$; from $-6\,x^2 + 2\,a$; from -7.

13. Subtract $3\,b^2 - 10\,ax + 5\,x^2$ from the sum of $5\,ax - 2\,x^2$ and $10\,b^2 - 13\,ax$.

14. Subtract the sum of $6\,a - 4\,b + 3\,c$ and $5\,b - 2\,a - c$ from $8\,a - 3\,b$.

15. Subtract $3\,x - 10\,ay^2 - 2\,a^3$ minus $x - 6\,ay + a^3$ from $4\,x^2 - x$ plus $5\,a^3 - 3\,ay^2$.

16. From $\frac{5}{8}\,x^3 - \frac{2}{3}\,x^2 + \frac{7}{4}\,x - 3$ subtract $\frac{1}{2}\,x^3 - 2\frac{1}{3} - \frac{1}{2}\,x - \frac{4}{5}\,x^2$.

17. Subtract $1 + 3(x - y) - 5(a^2 + b)$ from $a^3 + 2(a^2 + b) - 8(x - y)$.

18. Subtract the sum of $5\,a - 3\,b^2 + 2\,x$ and $-4\,x + 2\,b^2$ from $3\,x^2 + 4\,a - 12\,b^2$ minus $3\,a - 7\,b^2 + 2\,x^2$.

19. Subtract $4.5\,m - 1.3\,y^r + 10\,a^2c^4$ from $1.4\,y^r - 8\,a^2c^4$ plus $6.3\,y^r - 18\frac{1}{2}\,a^2c^4$.

20. From the sum of $x^2 - 1$, $3\,x + 2$, and $-8\,x^2 - 5\,x$, subtract $4\,x - 3\,x^2$ plus $4 - 2\,x$ minus $6\,x + 3\,x^2 - 8$.

III. PARENTHESES

33. Removal of parentheses. That one expression is to be subtracted from another may be indicated by inclosing the subtrahend in a parenthesis and writing the minus sign before it.

E.g., $6x-(2x-y)$ means that $2x-y$ is to be subtracted from $6x$.

Moreover, since a subtraction is *performed* by changing the sign of each term of the subtrahend and then adding it to the minuend (§§ 32 and 31), therefore *a parenthesis preceded by the minus sign may be removed by simply changing the sign of each term inclosed by it.**

E.g., $a-(-bc+mp)=a+bc-mp$; $3kx^2-(2by-7a^2)=3kx^2-2by+7a^2$; $x^2+2bx-(b^3-bx+3x^2)=x^2+2bx-b^3+bx-3x^2=-2x^2+3bx-b^3$; and $-(-4k^2+5ax-8by^3)=4k^2-5ax+8by^3$.

NOTE. If a parenthesis is preceded by the *plus* sign, it may be removed without changing the signs of the terms inclosed by it, because the expression within such a parenthesis is to be *added* to whatever precedes it.

34. Parenthesis within parenthesis. It often happens that a sign of aggregation may inclose one or more other signs of aggregation, thus:

$$3a^2x-\{2mb+[a^2x-(-4s^2t+5mb)+s^2t]\}.$$

In such cases it may be best for the *beginner*, after removing all those signs of aggregation which are preceded by the plus sign (§ 33, note), to remove the *innermost* of those signs of aggregation which are preceded by the minus sign, then the next innermost, and so on until all are removed.

E.g., omitting the square bracket in the above expression, since it is preceded by the plus sign, that expression becomes

$$3a^2x-\{2mb+a^2x-(-4s^2t+5mb)+s^2t\};$$

now removing the parenthesis, this expression becomes

$$3a^2x-\{2mb+a^2x+4s^2t-5mb+s^2t\};$$

and, removing the brace, we obtain

$$3a^2x-2mb-a^2x-4s^2t+5mb-s^2t,$$

i.e.,

$$2a^2x+3mb-5s^2t.$$

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* Compare also § 39, Ex. 19.

Note. The work of removing parentheses in such expressions as that just given may be somewhat shortened by removing the *outermost* negative parenthesis first, then the next outermost, and so on, instead of beginning with the innermost. The expression within an inner parenthesis is, of course, to be regarded as a single term of the next outer parenthesis. Parentheses preceded by the sign + should be dropped whenever they occur. The student may simplify the above expression by this method and then compare his work with that above.

The essential thing in both plans is that on removing a negative parenthesis the sign of every term inclosed by it must be reversed.

EXERCISES

Simplify the following expressions:

1. $7x - 3ac + (x - 2ac)$.

2. $7x - 3ac - (x - 2ac)$.

3. $4a - 2b - (c + 3a) - (2c + 3b - 2a)$.

4. $5x^2 + (7ax - 10y) + 3y - (4ax - 5y + 3x^2)$.

5. $3xy + 2y^2 - (-x^2 + y^2 + xy)$.

6. $mx^2 - [8y + (6a - mx) - 2a]$.

7. $-(a + b - c) + 4a - (c + 3b)$.

8. $3x - 2y + f - g - \{2x - (3y + 3z - 2b) + 2f - 2g\}$.

9. $a - y - \{a - (-y - \overline{a - 2})\}$.

10. $15 - (6 - x) - [13 - \{x - (y + 2) + 2y\} + 2x]$.

11. $x - \{3x - [-(-3x + 2y) + 5y] - 3y\}$.

12. $-\{-[-(x - y)]\}$.

13. $8a - 2b - \{(3c - d) - [4c - d - (-8a + 2b)] - 2d\}$.

14. $4 - [5y - \{3 - (2x - 2) - 4x\}] - \{x + 5y - \overline{x + 3}\}$.

15. $5a^2x^2y^2 - \{2a^2x^2y^2 - [a^2 + (3x^2y^2 - a^2 - 3a^2x^2y^2) + 4a^2] - 3x^2y^2\}$.

16. $a^n - n^a - (3a^n - 2n^a) + (-5a^n - 2n^a) - \{-[-(-\overline{a^n - n^a})]\}$.

17. $-3ax - (5xy - 3z) + 2z - [(4xy + 6z + ax) + 3xy]$.

18. $2a - [a - \{b - (3b - \overline{2a - b}) - 3a\} + 4b] - (b - a)$.

35. Inserting parentheses. From §§ 33 and 34 it follows that the value of a polynomial is not altered by inclosing any number of its terms in a parenthesis, provided only that if this parenthesis is preceded by the minus sign, the sign of each inclosed term be reversed.

EXERCISES

1. Indicate by means of a parenthesis that $a + b + c$ is to be subtracted from $a - b + c$; then remove the parenthesis and simplify the expression.

2. Inclose the last two terms of $x^2 + y^2 - z^2$ in a parenthesis preceded by the plus sign; by the minus sign.*

3. Inclose the last three terms of $ax - 4\,y + 3\,a - 8\,x$ in a parenthesis preceded by the plus sign; by the minus sign.

4. In the expression $3\,m - 4\,a + 10\,x^2 - 5\,y + 3\,ab^2 - 8\,ax$, inclose the 4th and 5th terms in a parenthesis preceded by the minus sign; then inclose this parenthesis, together with the two preceding terms, in a bracket preceded by the minus sign.

5. Make the changes asked for in Ex. 4, in the expressions $3\,m + 4\,a - 10\,x^2 - 5\,y + 3\,ab^2 - 8\,ax$, $3\,m - 4\,a - 10\,x^2 + 5\,y - 3\,ab^2 + 8\,ax$, and $-5\,x^2 + 3\,y^2 - 4\,a - 14\,bc + 8\,m^2$.

6. Inclose the first three terms of each of the expressions in Ex. 5 in a parenthesis preceded by the plus sign; preceded by the minus sign.

7. When terms are inclosed in a parenthesis preceded by the plus sign, are any changes in the signs of these terms made? Why? Explain why the signs are changed when the parenthesis is preceded by the minus sign.

8. Just as $5\,x + 3\,x = (5 + 3)x$, so $ax + bx = (a + b)x$, and similarly, $mx - nx + px = (m - n + p)x = -(-m + n - p)x$.

Similarly combine the terms of $5\,x - mx - nx$.

9. Combine all the x-terms, and also all the y-terms, in the following expressions: $ax - by - dy - ex - cx + fy$, $mx - cx + py - ay + gx$, and $3\,cx + 4\,dy - 2\,ax - 5\,mx - 7\,by + ax$.

10. Arrange the letters within the parentheses in the expressions of Ex. 9 in their alphabetical order, and give to each parenthesis the sign of the first letter it contains.

11. Group together the like powers of y in the following expressions: $ay^4 - 2\,by - 3\,cy^2 - my^4 - ny + dy^2$, $y^3 - ay^2 - 3\,ry^2 + ny^3 - ly^2$, and $-3\,y^3 - cy^4 + ay - dy + by^3 - 2\,ay^4 + ny^4 - y$.

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* In such exercises it is, of course, understood that the value of the expression is to be left unchanged.

CHAPTER V

MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

I. MULTIPLICATION

36. Some fundamental laws. Before going farther it is perhaps well to point out that thus far in this book, as well as in the arithmetic previously studied, it has been silently assumed that, whatever the numbers represented by a, b, and c,

$$a + b + c = a + c + b = b + c + a, \text{ etc.};$$

i.e., it has been assumed that the sum of several numbers is not changed by changing the *order* in which these numbers are added. This is known as the **commutative law** of addition.

This assumption was based upon the fact that with any particular set of numbers, such as 2, 5, and 8, the correctness of these statements (equations) is easily verified.

E.g., $2 + 5 + 8 = 2 + 8 + 5$. [Each member being 15

It has also been assumed that the sum of several numbers is not changed by grouping together any two or more of the summands, and replacing them by their sum. This is known as the **associative law** of addition.

E.g., $a + b + c = a + (b + c)$.

The **commutative** and **associative** laws of multiplication are expressed by such equations as

$$a \cdot b \cdot c = a \cdot c \cdot b = b \cdot c \cdot a,$$

and

$$a \cdot b \cdot c = a \cdot (b \cdot c),$$

respectively; their correctness has also thus far been assumed.

While attention is now expressly called to the fact that mere verifications, however numerous, cannot *prove* the generality of a law, the *proofs* of the above laws are deferred till Chapter VI; until then their correctness will continue to be assumed.

37. Law of exponents in multiplication. The words "power," "base," and "exponent," as used in connection with *arithmetical* numbers, were defined and illustrated in § 7 (iv). The definitions there given apply also when *algebraic* numbers are under consideration, though it is to be carefully noted that, while the base and the power may be negative or fractional, the *exponent* (under the present definition) *is necessarily a positive integer.*

It follows directly from these definitions that, if a represents any number whatever, then

$$a^3 \cdot a^2 = (a \cdot a \cdot a) \cdot (a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \quad \text{[Associative law}$$

$$= a^5,$$

i.e., $$a^3 \cdot a^2 = a^5.$$

Similarly in general, if m, n, and p are any positive integers whatever, then

$$a^m \cdot a^n = (a \cdot a \cdot a \cdot \cdots \text{ to } m \text{ factors}) \cdot (a \cdot a \cdot a \cdot \cdots \text{ to } n \text{ factors})$$

$$= a \cdot a \cdot a \cdot \cdots \text{ to } (m+n) \text{ factors} \quad \text{[Associative law}$$

$$= a^{m+n}.$$

So, too, $$a^m \cdot a^n \cdot a^p = a^{m+n+p}.$$

The **law of exponents**, expressed by these equations, may be formulated into words thus: *the product of two or more powers of any number is that power of the given number whose exponent is the sum of the exponents of the factors.*

38. Product of two or more monomials. The product of two or more monomials may be obtained as a simple extension of § 37.

E.g., if a, b, and x represent any numbers whatever, then

$$(2\,ax^3) \cdot (3\,b^2x) = 2 \cdot a \cdot x^3 \cdot 3 \cdot b^2 \cdot x \quad \text{[Associative law}$$

$$= 2 \cdot 3 \cdot a \cdot b^2 \cdot x^3 \cdot x \quad \text{[Commutative law}$$

$$= 6\,ab^2x^4. \quad \text{[Associative law}$$

Similarly, $$(3\,a^2x^3) \cdot (-2\,abx^2) \cdot (5\,ab^2x^4) = 3 \cdot (-2) \cdot 5 \cdot a^2 \cdot a \cdot a \cdot b \cdot b^2 \cdot x^3 \cdot x^2 \cdot x^4$$

$$= -30\,a^4b^3x^9.$$

And, manifestly, the product of any number of monomials may be obtained in the same way.

This method of obtaining the product of several monomials may be formulated into the following rule: *to the product of the numerical coefficients of the several monomials, annex each of the letters which they contain, and give to each letter an exponent equal to the sum of the exponents of that letter in the several monomials.*

EXERCISES

1. Define and illustrate the meaning of exponent, base, and power.

2. May the base be a negative number? a fraction? May the exponent be either negative or fractional?

3. If the base is a fraction, what is the power? If the base is negative and the exponent is 3, is the power positive or negative? Why?

4. If the base is negative, what is the sign of the power when the exponent is 4? when it is 5? when it is 6? when the exponent is even? when it is odd?

5. What is the meaning of x^3? of x^5? How many times is x used as a factor in $x^3 \cdot x^5$? How then may this product be represented? State the law of exponents for multiplication.

6. If x stands for a negative number, is x^5 positive or negative? Why? How does 3^4 compare with $(-3)^4$? 2^6 with $(-2)^6$? 2^5 with $(-2)^5$? State the general law of which these are particular cases.

7. What is the meaning of a^2y^3? of a^5y^2? How many times is a used as a factor in the product $a^2y^3 \cdot a^5y^2$? How many times is y so used? In what simpler form may this product be written? Why?

	8.	**9.**	**10.**	**11.**	**12.**
Multiply	$4\,a^2x^3$	$2\,m^3t^2$	$-8\,an^3y^2$	$-4\,a^2b^3x^2$	$5\,b^2x^3$
by	$3\,ax^2$	$-3\,mt^4$	$5\,ay^3$	$-6\,a^4bx$	$-7\,a^4by^3$

	13.	**14.**	**15.**	**16.**
Multiply	$7\,p^2w^3y^5$	m^3x^2	xy^3z^2	$\frac{3}{4}\,a^5m^2x^4$
by	$-9\,pw^2x^4$	$-3\,p^4xy^3$	$-x^3y^2z$	$-\frac{2}{9}\,bm^3y^2$

17. Write a carefully worded rule for finding the product of two monomials—it should, of course, make special mention of the coefficient, the letters, the exponents, and the sign of the product.

18. Find the product of $4\,ax^2$, $-2\,a^3xy^4$, and $5\,aby^2$.

19. What is the product of $3\,m^2pw^4$, $-2\,ap^3w^2$, $-6\,mp^2$, and $-aw^3$?

20. What is the product of $2\frac{1}{2}\,ab^2x$, $1.2\,b^3x^2$, and $-\frac{2}{3}\,a^3b$?

21. What is the meaning of x^n? May x represent any number whatever here? may n? How may the product of x^n and x^5 be represented? of a^3, a^m, and a^r? What are the restrictions upon m and r in this last question?

22. What is the meaning of y^{n-2}? What are the limitations upon n here? What is the product of $4\,a^3$ and $-3\,a^{n-2}$? of $2\,a^mx^n$ and $-a^{r-2}x^4$? Does the answer given to Ex. 17 apply to such multiplications as these?

23. What is meant by $(a^2)^4$? by $(x^3)^2$? by $(-3\,a^2y)^2$? Write each of these expressions in its simplest form.

24. Without actually performing the following indicated operations, tell by inspection what the sign of the result is in each case, and why: $(-3)^4$; $(-2)^9$; $(-11)^{40}$; 5^9; 7^{24}; $(-5)^n$ when n is an even positive integer, and when n is an odd positive integer; $(-3)^{2n}$ and $(-3)^{2n+1}$, when n is *any* positive integer.

25. As in Ex. 24, determine the sign of the result in each of the following indicated operations if $a=2$ and $b=-4$: $(a-b)^3$; $(a-b)^4$; $(a+b)^3$; $(ab^2)^5$; $(a-4\,b)^5$; $(a^3b^2)^3$; and $(a^3b)^{ab^2}$.

26. Tell what is meant by the commutative and associative laws of addition and multiplication. Illustrate your answer in each case.

39. Product of a polynomial by a monomial. Since the product of two numbers is obtained from the multiplicand in the same way as the multiplier is obtained from the positive unit [§ 3 (iii)], therefore $5\cdot(2+6)=5\cdot 2+5\cdot 6$, because the multiplier $2+6$ is obtained by first taking the unit 2 times, then 6 times, and adding the two results.

Similarly, whatever the numbers or expressions represented by $a, b, c, d, \cdots$,

$$a(b+c+d+\cdots)=ab+ac+ad+\cdots;$$

and, applying the commutative law to each member of this equation, it becomes

$$(b+c+d+\cdots)\cdot a=ba+ca+da+\cdots.$$

These last two equations state what is known as the **distributive law** * of multiplication as to addition; it may be put into words

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* The multiplication of a sum is "distributed" over the parts of that sum.

thus: *the product of a polynomial by a monomial is obtained by multiplying each term of the polynomial by the monomial and adding the partial products.*

E.g., $5x(3a^2 - 2b + c^2) = (3a^2 - 2b + c^2) \cdot 5x = 15a^2x - 10bx + 5c^2x$. The actual work may be conveniently arranged thus:

$$\begin{array}{l} 3a^2 - 2b + c^2 \\ 5x \\ \hline 15a^2x - 10bx + 5c^2x, \end{array}$$

each term of the multiplicand being multiplied by the multiplier, and the partial products added.

EXERCISES

1. How is $a + b - c$ obtained from $+1$? How then is the product $3 \cdot (a + b - c)$ to be obtained from 3?

2. Is $3 \cdot (a + b - c)$ equal to $(a + b - c) \cdot 3$? Why?

3. What is the product of 365 by 2? of $(300 + 60 + 5) \cdot 2$? Show that this illustrates the distributive law.

4. Since $a(b + c + d + \cdots) = (b + c + d + \cdots) \cdot a = ab + ac + ad + \cdots$, whatever the numbers represented by $a, b, c, d, \cdots$, what is the product of $2ax$ and $3x^2 - 4a^2x^3 + 5ax^4$?

5. Multiply $3a^2b^2 - 7ax$ by $2abx$. Also $5mx^3 - 7ay^2 - 4a^3m$ by $-2am^2$. Write a rule for multiplying a polynomial by a monomial.

6. When an indicated multiplication has been performed, and the result is expressed by an equation, is that equation an identity or merely a conditional equation? *E.g.*, is $(3a^2b^2 - 7ax) \cdot 2abx = 6a^3b^3x - 14a^2bx^2$ a conditional equation or an identity?

7. The fact that the equation in Ex. 6 is an *identity* may be used as a *partial* check upon the correctness of the multiplication. Are the two members equal when $a = b = x = 1$? If they were not equal when these *special* values are assigned to the letters, could the multiplication be correct? Does the equality of the two members for this set of values *prove* that the multiplication is correct, or does it merely increase the probability of its correctness? Is it then a "complete" or only a "partial" check?

	8.	**9.**	**10.**
Multiply	$8a^2 - 4ax + 3m^2$	$-3x^2z - 5x^3 + 4xz^3$	$2a - 3b + c$
by	$-4am^3$	$-2xz^2$	$-abc$

11. Check Exs. 8, 9, and 10 by the method of Ex. 7. Could other special values for the letters than those there given be employed for such a check? Why?

Multiply (and check the work):

12. $5m^2 - 2k^3$ by $3mk^2$.

13. $-8.5h^3x^2y + 5\frac{2}{3}hy^4$ by $\frac{5}{17}xy$.

14. $25a^3 - 17a^5 - a^6$ by $-3a^4$.

15. $xy^6 - 2x^2y^5 - 15x^4y^3 + 4x^6y$ by $-x^{n-1}y^{m-2}$.

Perform the following multiplications and check the work:

16. $-2x^2 \cdot (x^4 - 5x^3y - 16x^2y^2 + 24xy^3 - y^4 - xy - 4)$.

17. $(a^3b^2c^3 - 3ab^3c^4 - 4a^4b^2c + abc) \cdot 2abc^2$.

18. $-1 \cdot (3mx - 4m^2 - 2x^2)$.

19. Since $-1 \cdot (3mx - 4m^2 - 2x^2) = -(3mx - 4m^2 - 2x^2)$, derive from Ex. 18 a new proof that a parenthesis preceded by the minus sign may be removed if the sign of each term inclosed by it be reversed (cf. § 33).

40. Product of two polynomials. Since $m + n$ is obtained from the positive unit by adding n times this unit to m times the unit, therefore, by the definition of multiplication,

$$\begin{aligned}(a+b+c)\cdot(m+n) &= (a+b+c)m + (a+b+c)n\\ &= am+bm+cm+an+bn+cn. \qquad [\S\ 39.\end{aligned}$$

Similarly for any polynomials whatever; *i.e.*, *the product of two polynomials is obtained by multiplying each term of the multiplicand by each term of the multiplier, and adding the partial products.*

If any two or more terms of a product are similar, they should, of course, be united.

The actual work of such a multiplication, and its check, may be conveniently arranged thus:

$$\begin{array}{rll} & & \text{CHECK}\\ & a^2+2ab-b^2 & =+2, \text{ when } a=b=1\\ & \underline{a+b} & \underline{=+2}, \text{ when } a=b=1\\ (a^2+2ab-b^2)\cdot a = & a^3+2a^2b-ab^2 & \\ (a^2+2ab-b^2)\cdot b = & \underline{\qquad a^2b+2ab^2-b^3} & \\ & a^3+3a^2b+ab^2-b^3 & =+4, \text{ when } a=b=1 \end{array}$$

NOTE. The product of three or more polynomials may be obtained by multiplying the product of the first two by the third, this product by the fourth, and so on.

EXERCISES

Multiply (and check the work):

1. $4ax + 5a^2 - 2x^2$ by $3a - 4x$.

2. $2x^3 - 7xy + 3a^2x$ by $-5x + 3y$.

3. $4m^2 - 3mp$ by $3p^2 - 2m + m^2$.

4. $5s - 3t$ by $2s - 3r + t$.

5. $ax^2 - by^2$ by $bx + ay$.

6. $a^2 - 2ax + x^2$ by $a - x$.

7. $2a^2 - 6ab + 3b^2$ by $a + b + ab$.

8. $x - 5x^2 + 10$ by $2 - 7x + x^2$.

9. $ax^4 - 2a^3x + 5$ by $a - x - 3$.

10. $m^2 + 2mn + n^2$ by $m + n - mn$.

11. $a + b - c + d$ by $a - b + c - d$.

12. $3a - 5t^2 + atx$ by $-t + 2a - 3x^2$.

13. $a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$ by $a - b - c$.

14. $x^n + y^n$ by $x - y$.

15. $x^n + y^n$ by $x^2 - y^2$.

16. $x^n + y^n$ by $x^n - y^n$.

17. $x^n + y^n$ by $x^r - y^r$.

18. $3a^3 - 4a^2b + 2ab^2 - b^3$ by $5a^2 - 3ab + b^2$.

19. $1.8x^2 - 2xy - 2.3y^2$ by $1\frac{1}{2}x - 3\frac{2}{5}y$.

20. $2.5a^2x^2 - 1.4axy + 3\frac{1}{2}y^2$ by $-3ax - 4y - 1.2a$.

41. Integral expressions, degree and arrangement of expressions, etc. In multiplications with polynomials, and elsewhere, it is often advantageous to arrange the terms of a polynomial in a particular order; such arrangements will now be explained.

A term is said to be **integral** if it contains no letters in its denominator;* it is *integral with regard to a particular one of its letters* if that letter does not appear in its denominator. A polynomial is integral, or integral with regard to a particular letter, if each of its terms is so.

E.g., $3ax^2 + \frac{2bmy^2}{a} - \frac{5axy}{3}$ is integral with regard to b, m, x, and y; it is fractional with regard to a; its first and last terms are altogether integral, while its second term is integral only with regard to b, m, and y.

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* It may contain *numerical* denominators and still be called integral.

By the **degree** of an integral term is meant the number of literal factors which that term contains, *i.e.*, it is the sum of the exponents of all the letters of that term.

E.g., $5\,ax$ is of the 2d degree, and $3^2\,a^4cy^3$ is of the 8th degree.

An integral polynomial is said to be of the same degree as its highest term; if all of its terms are of the same degree, it is said to be **homogeneous.**

E.g., $6\,aby^2 - 2\,bmx + 5\,a^2x^3y$ is of degree 6, and $2\,ax^2 - 6\,xyz + 5\,abx - y^3$ is homogeneous, and of degree 3.

One is often concerned with the degree of a polynomial (or of a term) with regard to *some* rather than *all* of its letters; in such a case only those letters are considered in determining the degree.

E.g., $5\,a^2x^2y - 3\,ab^3xy^2 + 2\,x^3$ is homogeneous, and of degree 3, with regard to the letters x and y; it is of degree 2 in y alone, and of degree 3 in x alone, and non-homogeneous; its degree in all the letters is 7.

A polynomial is said to be **arranged according to ascending powers** of some one of its letters if the exponents of that letter, in going from term to term toward the right, increase, and that letter is then called the **letter of arrangement**; it is **arranged according to descending powers** of the letter of arrangement if taken in the reverse order.

E.g., $2\,x^3 - 5\,ax^2y - 7\,b^2xy^2 + 3\,m^2y^3$ is arranged according to descending powers of x, and ascending powers of y.

42. Multiplication in which the polynomials are arranged. If each of two polynomials be arranged according to powers of some letter which is contained in each, then their product will arrange itself according to powers of that letter, and the actual multiplication will take on an orderly appearance.

E.g., to get the product of $7\,x - 2\,x^2 + 5 + x^3$ by $3\,x + 4\,x^2 - 2$, arrange the work thus:

$$\begin{array}{rrrrrrl}
 & & x^3 & -\,2\,x^2 & +\,7\,x & +\,5 & \quad \text{CHECK} \\
 & & & 4\,x^2 & +\,3\,x & -\,2 & \quad = 11, \text{ when } x = 1 \\
\hline
4\,x^5 & -\,8\,x^4 & +\,28\,x^3 & +\,20\,x^2 & & & \quad \underline{=\ \ 5}, \text{ when } x = 1 \\
 & 3\,x^4 & -\,6\,x^3 & +\,21\,x^2 & +\,15\,x & & \\
 & & -\,2\,x^3 & +\,4\,x^2 & -\,14\,x & -\,10 & \\
\hline
4\,x^5 & -\,5\,x^4 & +\,20\,x^3 & +\,45\,x^2 & +\,x & -\,10 & \quad = 55, \text{ when } x = 1
\end{array}$$

EXERCISES

1. Is the monomial $\frac{4}{3}a^2x^3$ integral or fractional? With regard to what letters is $\frac{5a^2x^3y}{bc^2}$ integral? With regard to what letters is it fractional?

2. What is meant by the degree of an integral algebraic expression? When is such an expression said to be homogeneous?

3. Arrange the expression $4ax^3 - 7x^2 + 5x^4 - 2bx - 8a^2$ according to descending powers of x. Also according to ascending powers of x. Of what degree is its present first term?

4. Arrange the expression $3x^2y^5 + xy^6 - 8x^4y^3 - 6x^5y^2 + x^6y$ according to descending powers of x. How is it then arranged with reference to y? Of what degree is this expression? Is it homogeneous?

In the following exercises arrange both multiplier and multiplicand according to some letter contained in each, then multiply and observe that the product has a corresponding arrangement.

Multiply:

5. $6x^2 - 2 + 5x + 3x^3$ by $x^2 + 5 - x$.

6. $2a + a^3 - a^2 - 1$ by $4 - a^2 + a$.

7. $3a^2x - 4ax^2 + x^3 - a^3$ by $a^2 - ax + x^2$.

8. $3xy^2 - y^3 - 3x^2y + x^3$ by $-2xy + x^2 + y^2$.

9. $x^2y^2 - xy^3 + y^4 - x^3y + x^4$ by $x^2 + xy - y^2$.

10. $4h^2r - hr^2 - h^3 + 2r^3$ by $h - 2r$.

11. In the product of two homogeneous polynomials, one of degree 5 and the other of degree 2, what is the degree of each term? Why? Is then this product homogeneous? Show that this consideration may be used as a partial check upon the correctness of such a product. Compare also Exs. 7–10.

12. Find the product of $ax^2 + b^2x + a^2b$, $a + b + x$, and $a - x$. Should this product be homogeneous? Why?

13. Find the product of $2m^2 - 5mn + 3n^2$, $3m - 2n$, and $1 - m - n$. Should this product be homogeneous?

Expand,* and check, the following indicated multiplications:

14. $(3a + 2b)(2ax - a^2 - x^2)(bx - 2a)$.

15. $(x^4 - 3x^3y + y^4 - 3xy^3)(x^2 - 2xy + y^2)$.

---

* An indicated product is said to have been expanded when the multiplication has been performed.

16. $(3t^2 - 5r + 2s)(s - 2t + r)(3 - s - t)$.

17. $[3x + 2y - 3(y + 2x) - z][2 - 5(x - z + 3y)]\ (x + y - 1)$.

18. $(x + y)^3$, *i.e.*, $(x + y)(x + y)(x + y)$.

19. $(x - y)^2(x + y)^2$.

20. $(a - 2b)^3(2a - b)(2a + b)$.

21. $(x^2 + xy + y^2)(x - y)$.

22. $(a^3 + a^2b + ab^2 + b^3)(a - b)$.

23. $(x^2 + xy + y^2)(x^2 - xy + y^2)(x - y)(x + y)$.

24. If the multiplier and multiplicand are each arranged according to the descending powers of some particular letter, how will the product arrange itself? From what two terms is the highest term in the product obtained? The lowest term in the product?

25. The results in Exs. 21–23 show that some of the terms of a product may cancel each other, and that the number of terms in a product of polynomials may be as small as two. Show that there must be *at least* two terms in such a product (cf. Ex. 24).

26. When both multiplier and multiplicand are arranged according to the powers of some letter, the actual work of multiplying may be somewhat shortened, thus:

Multiply $3x^4 - 2x^3 - 5x^2 + 6x - 4$ by $7x^2 - 3x + 2$.

ORDINARY PROCESS

$$\begin{array}{rrrrrrr}
 & & 3x^4 & -\ 2x^3 & -\ 5x^2 & +\ 6x & -\ 4 \\
 & & 7x^2 & -\ 3x & +\ 2 & & \\
\hline
21x^6 & -14x^5 & -35x^4 & +42x^3 & -28x^2 & & \\
 & -\ 9x^5 & +\ 6x^4 & +15x^3 & -18x^2 & +12x & \\
 & & +\ 6x^4 & -\ 4x^3 & -10x^2 & +12x & -8 \\
\hline
21x^6 & -23x^5 & -23x^4 & +53x^3 & -56x^2 & +24x & -8
\end{array}$$

SHORTER PROCESS

$$\begin{array}{rrrrrrr}
 & & 3x^4 & -\ 2x^3 & -\ 5x^2 & +\ 6x & -\ 4 \\
 & & 7x^2 & -\ 3x & +\ 2 & & \\
\hline
21x^6 & -14x^5 & -35x^4 & +42x^3 & -28x^2 & & \\
 & -\ 9 & +\ 6 & +15 & -18 & +12x & \\
 & & +\ 6 & -\ 4 & -10 & +12 & -8 \\
\hline
21x^6 & -23x^5 & -23x^4 & +53x^3 & -56x^2 & +24x & -8
\end{array}$$

Perform Exs. 5–9 by this shorter process, and check the work.

27. Since the powers of the letter of arrangement in the multiplication in Ex. 26 follow one another in regular order, in each partial product, the process may be still further abridged by omitting the letters until the very end. This is known as the method of **detached coefficients.**

Thus, to multiply $3x^4 - 2x^3 - 5x^2 + 6x - 4$ by $7x^2 - 3x + 2$, write only the coefficients:

$$\begin{array}{rrrrrrr}
 & & 3 & -\ 2 & -\ 5 & +\ 6 & -\ 4 \\
 & & 7 & -\ 3 & +\ 2 & & \\
\hline
21 & -14 & -35 & +42 & -28 & & \\
 & -\ 9 & +\ 6 & +15 & -18 & +12 & \\
 & & +\ 6 & -\ 4 & -10 & +12 & -8 \\
\hline
21 & -23 & -23 & +53 & -56 & +24 & -8
\end{array}$$

i.e., the product is $21x^6 - 23x^5 - 23x^4 + 53x^3 - 56x^2 + 24x - 8$.

Perform Exs. 5–9 by the method of detached coefficients.

28. Since, for example, $7325 = 7(10)^3 + 3(10)^2 + 2(10) + 5$, is not ordinary arithmetical multiplication performed by means of detached coefficients? Only the coefficients of the various powers of 10 are used.

29. Any absent term, in the regular order of arrangement of a polynomial to be multiplied by using detached coefficients, should be inserted, with zero for its coefficient.

Thus, multiply $3x^4 - 2x^3 + 6x - 4$, *i.e.*, $3x^4 - 2x^3 + 0x^2 + 6x - 4$, by $5x - 2$.

Compare this with such multiplications in arithmetic (see Ex. 28).

30. Multiply $2a^3 - 5a + 1$ by $4a - 2$, using detached coefficients.

31. Multiply $6x^4 - 2x^2 - 5$ by $3x^2 + 5x$, using detached coefficients.

II. DIVISION

43. Law of exponents in division. Assuming for the present, as in arithmetic, that the quotient is not changed if equal factors be cancelled from dividend and divisor, the law of exponents in division is easily discovered.

For example, $$\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a}$$ [Definition of exponent

$$= \frac{a \cdot a \cdot \cancel{a} \cdot \cancel{a} \cdot \cancel{a}}{\cancel{a} \cdot \cancel{a} \cdot \cancel{a}} = a^2;$$

i.e., $$a^5 \div a^3 = a^{5-3}.$$

Similarly, $$x^7 \div x^4 = x^{7-4} = x^3;$$

and $$s^2 \div s^8 = \frac{1}{s^{8-2}} = \frac{1}{s^6}.$$

In precisely the same way, it follows that if m and n are any two positive integers, then

$$a^m \div a^n = a^{m-n}, \text{ when } m > n,^*$$

$$a^m \div a^n = 1 \quad, \text{ when } m = n,$$

and $$a^m \div a^n = \frac{1}{a^{n-m}}, \text{ when } m < n.$$

---

* The symbols > and < stand, respectively, for "is greater than" and "is less than"; thus, $m > n$ is read: "m is greater than n."

44. Zero and negative exponents defined. Thus far the symbol a^n has been defined only when n is a positive integer; we are therefore still free to say what we shall mean by such symbols as a^{-3} and a^0. It will be found advantageous to agree that, when such symbols present themselves in any operation, a^0 shall be interpreted to mean 1, and a^{-k} shall mean $\frac{1}{a^k}$.*

Under this definition of a^0 and a^{-k}, the *three* expressions for the quotient of $a^m \div a^n$, which are given in § 43, may be replaced by the *single* expression

$$a^m \div a^n = a^{m-n},$$

whether $m > n$, $m = n$, or $m < n$.

For, when $m = n$, then $a^m \div a^n = a^{m-n}$, because then $a^m \div a^n$ is manifestly 1, and a^{m-n} is a^0, which is also 1. Again, when $m < n$, then $a^m \div a^n = \frac{1}{a^{n-m}}$ (§ 43), but by the above definition $\frac{1}{a^{n-m}} = a^{-(n-m)}$ $= a^{m-n}$, so that even in this case $a^m \div a^n = a^{m-n}$.

Hence, with this extended meaning of an exponent, *the quotient of any two powers of a given number is that power of the number whose exponent is the exponent of the dividend minus that of the divisor.*

EXERCISES

1. What is the meaning of x^7? of x^3?

2. How many x's in the product of x^7 by x^3? How, then, may this product be most simply written?

3. How is the exponent of the product of two or more powers of any given number obtained? Why?

4. Since $x^7 \cdot x^3 = x^{10}$, what is the quotient when x^{10} is divided by x^3? Why? What is the quotient of x^{10} divided by x^7?

5. What is the quotient of N^8 divided by N^6? of y^{13} divided by y^5? of p^{18} by p^7? of x^n by x^r?

6. How is the exponent of the quotient of two powers of any given number obtained? Why?

---

* In *extending* the meaning of any symbol already in use, there is one principle that should always be observed, viz., the extended meaning should be such that any rules of operation already established for the symbol in question shall not be disturbed (cf. Ex. 9, below).

7. With exponents restricted to positive integers, could one say that $x^n \div x^r = x^{n-r}$ without knowing the relative values of n and r?

8. What meaning is it *necessary* to give to zero and negative exponents so that $x^n \div x^r$ may equal x^{n-r}, even when $n = r$ and when $n < r$? Why?

9. In § 37 it is shown that $a^m \cdot a^n = a^{m+n}$ when a represents any number whatever, and m and n are any two positive integers; show that this equation is still true if m, or n, or both m and n, have zero or negative values (cf. footnote, p. 63).

10. What is the meaning of m^{-4}? of x^0? of $\left(\frac{a}{b}\right)^{-2}$? Is a^0 equal to x^0 even when a is not equal to x? Why?

11. What is the product of x^5 by x^{-3}? Is it $x^{5+(-3)}$? Why? What is the quotient of a^3 divided by a^5? Is it a^{3-5}? Why? What is the quotient of $N^5 \div N^{-2}$? Is it $N^{5-(-2)}$? Why?

45. Division of monomials. Since division is the inverse of multiplication, *i.e.*, since the quotient multiplied by the divisor equals the dividend [§ 3 (iv), note 1], therefore it follows from the method of multiplying monomials (§ 38) that *the coefficient of the quotient of two monomials is the coefficient of the dividend divided by that of the divisor, and the exponent of every letter in the quotient is the exponent of that letter in the dividend diminished by its exponent in the divisor.*

E.g., $12\,a^5x^8 \div 4\,a^2x^3 = \frac{12}{4}\,a^{5-2}x^{8-3}$, *i.e.*, $3\,a^3x^5$; $-18\,a^4b^7 \div 6\,a^3b^2 = \frac{-18}{6}a^{4-3}b^{7-2} = -3\,ab^5$; and $5\,m^6x^2 \div 10\,m^4x^5 = \frac{5}{10}\,m^{6-4}x^{2-5} = \frac{1}{2}\,m^2x^{-3} = \frac{1}{2}\,\frac{m^2}{x^3}$ (§ 44).

Dividing one monomial by another may also be accomplished by cancellation, as in § 43. To test the correctness of a quotient, multiply it by the divisor; the product should be the dividend.

8

EXERCISES

1. What is the quotient of $6\,a^3$ divided by $2\,a$? of $15\,a^4x^7$ divided by $3\,a^2x^4$? of $12\,m^2x^5$ divided by $-4\,x^2$?

2. How is the *sign* of a quotient determined? the *coefficient?* the *letters?* their *exponents?*

3. How may the correctness of a quotient be tested? Perform the following indicated divisions, and test the result in each case: $18\,a^3x^5 \div 3\,ax^2$; $15\,h^4p^3 \div (-6\,hp^5)$; and $-\frac{2}{3}\,m^8z^9$ divided by $-\frac{5}{6}\,m^5z^7$.

4. What is the sign of the product of two monomials each of which is positive? Of their quotient?

5. Answer the same questions as in Ex. 4 if each of the monomials is negative; also if one is positive and the other negative.

6. If two monomials have *like* signs, what is the sign of their product? of their quotient? In *multiplication* how is the exponent of any particular letter in the product obtained? in division?

7. Multiply $5a^2b^3$ by $2a^4c^2$; $3^2m^4x^3$ by $2mx^3y$; $2^3a^5x^{-2}$ by $-6a^2x^5z$; and $-\frac{5}{2^4}b^3k^{-2}p^4$ by $\frac{8}{15}ab^{-1}k^4$.

8. Divide $18m^4r^2$ by $-3m^4r^3$; $-\frac{8}{3^2}a^5x^{-3}$ by $\frac{16}{27}a^{-1}x^2$; and $(\frac{2}{5})^2n^4z^2$ by $(-\frac{3}{10})^3a^2z$. Also test the correctness of the result in each case.

9. Divide $\frac{1}{2}h^8k^3l^{-2}$ by $-\frac{1}{3}h^5k^{-4}$; $-27a^2m^{-5}x^{-3}$ by $-4\frac{1}{2}m^2xy^3$; $2x^{m+3}$ by $6x^m$; and $15a^4x^py^n$ by $5ax^{p+2}y^r$. Also test the correctness of the results.

10. Show that even when some of the exponents are negative, as in Exs. 8 and 9, the exponent of any letter in the quotient, of one monomial divided by another, is the exponent of that letter in the dividend diminished by its exponent in the divisor.

11. Based upon the definition of such a symbol as x^{-3}, given in § 44, show that $x^5y^{-3}=\frac{x^5}{y^3}$; that $6a^2x^{-3}y^{-4}=\frac{6a^2}{x^3y^4}$; that $\frac{3c^5x^{-2}}{2a^{-3}y^4}=\frac{3a^3c^5}{2x^2y^4}$; and that $\frac{m^2n^3x^{-1}}{a^4x^4y^3}=\frac{a^{-4}m^2n^3}{x^5y^3}$.

12. Following the suggestion of Ex. 11, show that a *factor* may be transferred from the numerator to the denominator of a fraction, and *vice versa*, by merely changing the sign of its exponent.

46. Division of a polynomial by a monomial. Since the quotient multiplied by the divisor always equals the dividend [§ 3 (iv)], therefore the quotient of a polynomial divided by a monomial must, by § 39, be a *polynomial* whose separate terms being multiplied by the divisor produce the separate terms of the dividend; hence this quotient is obtained by dividing each term of the dividend by the divisor.

E.g., $(15a^2x^3-10bx^4y+c^2x^2)\div 5x^2=3a^2x-2bx^2y+\frac{1}{5}c^2.$

EXERCISES

1. What is meant by saying that division is the *inverse* of multiplication?

2. Since $(a + b - c + d) \cdot s = as + bs - cs + ds$, what must be the quotient of $(as + bs - cs + ds)$ divided by s? Why?

3. What is the quotient of $15\,ax^2 - 6\,a^4bx + 21\,a^2x^3y^2$ divided by $3\,ax$? Why?

4. How may any polynomial whatever be divided by a monomial? How are the signs of the several quotient terms determined? their coefficients? their letters? their exponents?

5. Divide $6\,a^2x^3 - 9\,ab^2x^2 - 15\,a^3c^2x^4$ by $3\,ax^2$; also by $-3\,ax^2$.

6. Divide $-x + 4\,ax^2 - 3\,m^3x - 6\,amx$ by $-x$; also by $2\,x$.

7. Divide $-m - n + x - a$ by -1.

8. Divide $26\,a^3m^2 - 52\,a^2bm^3 - 39\,a^4m^2x^3$ by $-13\,a^2m^2$; also by $13\,a^2m^2$.

9. Divide $-10\,r^4s^3y^5 - 25\,k^2r^3s^2 + 15\,ad^2r^4s^3$ by $5\,r^3s$; also by $-5\,r^2s^2$.

10. Divide $4\,am^3 - 6\,a^2x^2 + 3\,a^{-2}mx$ by $\frac{2}{3}\,a^2x^{-1}$.

11. What is the meaning of a negative exponent? of a zero exponent? How may the correctness of an exercise in division be verified?

Perform the following indicated divisions and verify the results:

✓ **12.** $(-a^2m^3 - 4\,a^0b^3x^{-1} + 6\,a^{-2}b^0m^4x^{-3}) \div (-\frac{1}{2}\,ab^0c^0x^2)$.

[What is the effect of such a factor as a^0 in any term?]

13. $(-\frac{2}{3}\,m^2x^{-2} + \frac{7}{8}\,c^3m^4x^2 - \frac{4}{3}\,a^0m^{-2}x^4) \div \frac{4}{5}\,a^0m^{-1}x^3$.

14. $\{x(x + y)^4 - x^2(x + y)^3 + x^3(x + y)^2\} \div \{-x(x + y)^2\}$.

15. $\{-(a - b) - 2(a - b)^2 + 3(a - b)^4\} \div \{-(a - b)\}$.

16. $(a^m - 2\,a^{m+1} - 5\,a^{m+2} + 3\,a^{m-1}) \div \frac{2}{3}\,a^m$.

✓ **17.** $(z^{n+4} - 3\,z^{n-1} + 4\,az^2) \div (-\frac{1}{2}\,z^{n-1})$.

18. $(a^nb^n - \frac{2}{3}\,a^{n-1}b^{n+1} + \frac{1}{10}\,a^{n-2}b^{n+2}) \div \frac{2}{5}\,a^nb^{-n}$.

47. Division of a polynomial by a polynomial. Since (see § 42)

$$(4\,x^2+3\,x-2) \cdot (x^3-2\,x^2+7\,x+5)=4\,x^5-5\,x^4+20\,x^3+45\,x^2+x-10,$$

therefore, with this last expression as dividend, and $x^3-2\,x^2+7\,x+5$ as divisor, the quotient must be $4\,x^2 + 3\,x - 2$, *i.e.*,

$$(4\,x^5-5\,x^4+20\,x^3+45\,x^2+x-10)\div(x^3-2\,x^2+7\,x+5)=4\,x^2+3\,x-2.$$

The *process* of obtaining this quotient from the given dividend and divisor will now be explained.

Since the dividend is the product of the divisor by the quotient, therefore *the highest term in the dividend is the product of the highest term in the divisor multiplied by the highest term in the quotient,* and therefore, if $4x^5$, the highest term in the dividend, be divided by x^3, the highest term in the divisor, the result, $4x^2$, is the highest term in the quotient.

Moreover, since the dividend is the algebraic *sum* of the several products obtained by multiplying the divisor by *each term* of the quotient, therefore, if $4x^5 - 8x^4 + 28x^3 + 20x^2$, the product of the divisor by the highest term of the quotient, be subtracted from the dividend, the remainder, viz., $3x^4 - 8x^3 + 25x^2 + x - 10$, is the sum of the products of the divisor multiplied by each of the other terms of the quotient except this one.

For the same reason as that given above, if $3x^4$, the highest term of this remainder, be divided by x^3, the highest term of the divisor, the result, $3x$, is the next highest term of the quotient.

By continuing this process all of the terms of the quotient may be found. It is convenient to arrange the work as follows:

$$
\begin{array}{rr|l}
 & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{\textbf{DIVIDEND}} & \overbrace{\qquad\qquad\quad}^{\textbf{DIVISOR}} \\
 & 4x^5 - 5x^4 + 20x^3 + 45x^2 + x - 10 & x^3 - 2x^2 + 7x + 5 \\
(x^3 - 2x^2 + 7x + 5) \cdot 4x^2 = & 4x^5 - 8x^4 + 28x^3 + 20x^2 & \underbrace{4x^2 + 3x - 2}_{\textbf{QUOTIENT}} \\ \cline{2-2}
 & 3x^4 - 8x^3 + 25x^2 + x - 10 & \\
(x^3 - 2x^2 + 7x + 5) \cdot 3x = & 3x^4 - 6x^3 + 21x^2 + 15x & \\ \cline{2-2}
 & -2x^3 + 4x^2 - 14x - 10 & \\
(x^3 - 2x^2 + 7x + 5) \cdot (-2) = & -2x^3 + 4x^2 - 14x - 10 & \\ \cline{2-2}
 & 0 &
\end{array}
$$

CHECK

When $x = 1$, dividend $= 55$, divisor $= 11$, and quotient $= 5$, as it should.

Even if it is not known beforehand that the dividend was actually obtained as the product of two polynomials, the *process of division* may still be applied as above.

The method just now explained, which may be employed to solve any example whatever of this kind, may be formulated into the following rule:

(1) *Arrange both dividend and divisor according to the descending (or ascending) powers of some one of the letters*

involved in each, and write the divisor at the right of the dividend.*

(2) *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

(3) *Multiply the entire divisor by this first quotient term, and subtract the result from the dividend.*

(4) *Treat this remainder as a new dividend, arranging as before, and repeat this process until a zero remainder is reached, or until the remainder is of lower degree in the letter of arrangement than the divisor.*

Note. Since each remainder is of lower degree in the letter of arrangement, than the preceding one, therefore it is always possible to comply with (4) in the rule just given. If a zero remainder is reached, then the division is said to be *exact;* otherwise the complete quotient consists of an entire algebraic expression plus a fraction whose numerator is the last remainder and whose denominator is the given divisor.

EXERCISES

Divide (and check your results):

1. $x^2 + 7x + 12$ by $x + 3$. **2.** $x^2 - x - 20$ by $x - 5$.

3. $b^2 - 6b - 16$ by $b + 2$.

4. $p^4 + 4p^3 + 6p^2 + 5p + 2$ by $p^2 + p + 1$.

5. $2x^4 + 6x^2 - 4x - 5x^3 + 1$ by $x^2 - x + 1$.

6. $3a^4 + 3a^2 + 3 + 3a + a^5 + 5a^3$ by $1 + a$.†

7. $4xy^2 + 8x^3 + y^3 + 8x^2y$ by $y + 2x$.*

8. $6a^2x^2 - 4a^3x - 4ax^3 + a^4 + x^4$ by $a^2 + x^2 - 2ax$.

9. $2a^4 + k^4 - 5a^3k - 4ak^3 + 6a^2k^2$ by $k^2 + a^2 - ak$.

10. If the quotient be multiplied by the divisor, how must the result compare with the dividend? What must the result be if the dividend be divided by the quotient?

---

* If there is more than one letter involved in the given polynomials, then the expression "highest term" in the explanation on p. 67 is to be replaced by "term of highest degree in the letter of arrangement."

† Just as in "long division" in arithmetic, so here, some labor may be saved by bringing down only so much of the remainder at any stage of the work as is needed in the next step.

11. If the partial quotient, at any stage of the process of division, be multiplied by the divisor, and the corresponding remainder added, how must the result compare with the dividend?

12. Could the principles involved in Exs. 10 and 11 be employed as a check upon the correctness of an exercise in division? Is this check more or less conclusive than that given in connection with the solution on p. 67? Why?

13. Is it *necessary* or merely *convenient* to arrange both dividend and divisor according to the descending or the ascending powers of some letter contained in each? Could not the highest term of the dividend be divided by the highest term of the divisor in whatever order the terms of these expressions are written?

14. Divide $2x^5 + x^4 + 49x^2 - 13x - 12$ by $x^3 - 2x^2 + 7x + 3$.*

Divide (and check the results):

15. $v^6 - v^4 - 1 + 2v + v^3 - v^2$ by $v - 1 + v^2$.

16. $a^5 - 41a - 120$ by $a^2 + 4a + 5$.

17. $m^4 + 16 + 4m^2$ by $2m + m^2 + 4$.

18. $cd - d^2 + 2c^2$ by $c + d$.

19. $x^3 - y^3$ by $x - y$.

20. $a^4 - 16b^4$ by $a - 2b$.

21. $h^8 - k^8$ by $h^2 + k^2$.

22. $a^{2n} - x^{2n}$ by $a^n - x^n$.

23. $u^{2n} + 11u^n + 30$ by $u^n + 6$.

24. $x^{m+n}y^n - 4x^{m+n-1}y^{2n} - 27x^{m+n-2}y^{3n} + 42x^{m+n-3}y^{4n}$ by $x^m + 3x^{m-1}y^n - 6x^{m-2}y^{2n}$.

25. $\frac{9}{16}x^4 - \frac{7}{8}x^3y + \frac{19}{36}x^2y^2 + \frac{1}{6}xy^3$ by $\frac{3}{2}x + \frac{1}{3}y$.

26. $1.2ax^4 - 5.494a^2x^3 + 4.8a^3x^2 + 0.4a^4x - .478a^5$ by $6ax - 2a^2$.

27. $(3x^4 - 1 + 3x + 6x^2 + 7x^3)(1 + x^2 - x)$ by $x + 1 + x^2$.

28. $a^5 - b^5$ by $(a^3 + b^3)(a + b) + a^2b^2$.

29. $10x^3y^2 + x^5 - 10x^2y^3 + 5xy^4 - 5x^4y - y^5$ by $x^2 + y^2 - 2xy$.

30. $2x^2 - 2y^2 - 3z^2 - 3xy - 5xz - 5yz$ by $x - 2y - 3z$.

31. $x^4 - 3x^3 + x^2 + 2x - 1$ by $x^2 - x - 2$.

[In Ex. 31 the complete quotient is

$$x^2 - 2x + 1 + \frac{-x+1}{x^2 - x - 2}; \qquad \text{compare § 47, note.]}$$

32. $x^3 + x - 25$ by $x - 3$.

33. $a^5 - 1$ by $a + 1$.

34. $2s^3 - 3s + 8$ by $s^2 - 4$.

---

* Since there is no term in x^3 in the dividend, care must be used to keep the remainders properly arranged (cf. Ex. 29, p. 62).

35. $4\,m^3x^3 - 8\,m^5x^5 + 40\,m^2x^2 + 25$ by $5 + 3\,mx - m^3x^3$.

36. $abc + ax^2 + x^3 + abx + bx^2 + cx^2 + acx + bcx$ by $x^2 + ab + ax + bx$.

Since x occurs in more terms than any other letter, it will be best to arrange the work in Ex. 36 thus:

$$\begin{array}{r|l}
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc & x^2+(a+b)x+ab \\ \cline{2-2}
x^3+(a+b)x^2+abx\qquad\qquad\quad & x+c \\ \hline
cx^2+(ac+bc)x+abc & \\
\underline{cx^2+(ac+bc)x+abc} & \\
0 &
\end{array}$$

37. $adx^4+cf+bfx+bex^2+ecx+bdx^3+(af+cd)x^2+aex^3$ by ax^2+bx+c.

38. $ay^2 - aby + y^3 - by^2 - acy - cy^2 + bcy + abc$ by $y^2 - ab - by + ay$.

39. $14\,xy^2 + 6\,x^3 - 4\,y^3 - 16\,x^2y - 2\,x^2 - 2\,y^2 + 4\,xy$ by $3\,x - 1 - 2\,y$.

40. $7\,x^3 + x^5 + 2\,x^4 - 46\,x + 6\,x^2 - 120$ by $4\,x + 5 + x^2$.

41. $7\,a^2 - 6\,a^3 + a^4 - 4\,a - 12$ by $3 - 2\,a + a^2$.

42. $(4\,m^4 - 5\,m^2b^2 + b^4)(5\,x^2y + x^3 + y^3 + 5\,xy^2)$
by $(2\,m^2 - 3\,mb + b^2)(x^2 + y^2 + 4\,xy)$.

43. $a^3 - b^3 + c^3 + 3\,abc$ by $a^2 + b^2 + c^2 + ab - ac + bc$.

44. $x^6 - 6\,x + 5$ by $x^2 - 2\,x + 1$.

45. Divide $3\,ab + a^2 + b^2$ by $a + b$, arranging according to descending powers of a. Perform this division also with the expressions arranged according to descending powers of b, and compare the two results.

46. Divide $2\,xy^3 + 3\,x^4 - 4\,x^2y^2 - 7\,x^3y + y^4$ by $x^2 + y^2 - xy$, arranging first according to powers of x, then according to powers of y, and compare the results.

47. As has just been seen, in Exs. 45 and 46, the *form* of the quotient depends upon the choice of the letter of arrangement *when the division is not exact;* prove that this is not the case when the division is exact.

48. Divide $p^6 + q^6$ by $p + q$, until 4 quotient terms are obtained.

49. Divide a by $a - x$, to 5 quotient terms.

50. Divide 1 by $1 - r$, to 8 quotient terms.

51. Divide 1 by $1 - mx$, to 4 quotient terms.

52. Divide $x^n - y^n$ by $x + y$, to 8 quotient terms. What does this quotient become when $n = 2, 3, 4, \dots$? What is the remainder when $n = 2, 4, 6, 8, \dots$? when $n = 3, 5, 7, \dots$?

53. Examine the quotient $(x^n - y^n) \div (x - y)$ under the same circumstances as in Ex. 52. Also $(a^n + b^n) \div (a + b)$, and $(p^n + q^n) \div (p - q)$.

54. Some labor may often be saved in an exercise in division by using the coefficients only, and omitting the letters until the end.

Thus, $(4x^5-5x^4+20x^3+45x^2+x-10) \div (x^3-2x^2+7x+5)$, with letters omitted, becomes

$$
\begin{array}{r|l}
4-5+20+45+1-10 & 1-2+7+5 \\ \cline{2-2}
4-8+28+20 & 4+3-2,\ \textit{i.e.},\ 4x^2+3x-2. \\ \hline
3-8+25+1 & \\
3-6+21+15 & \\ \hline
-2+4-14-10 & \\
-2+4-14-10 & \\ \hline
0 &
\end{array}
$$

This example has already been solved on p. 67; the student should carefully compare the two methods. He should also note that this last method — called the method of **detached coefficients** — is altogether similar to "long division" in arithmetic, and analogous to that employed in Ex. 27, p. 61.

By the method of detached coefficients, perform Exs. 1, 4, 5, 6, 8, and 9.

55. In using the method of detached coefficients, if any powers of the letter of arrangement are absent they must be supplied, giving them zero coefficients; compare this with Ex. 29, p. 62. Solve Ex. 14 by this method, writing the dividend thus: $2x^5+x^4+0x^3+49x^2-13x-12$.

56. Solve Exs. 16 and 17, using detached coefficients.

57. Divide x^3+4x^2-7x+2 by $x-a$, and show that *the remainder is what would be obtained by substituting a for x in the dividend.*

58. Divide $5m^3-8m+3$ by $m-r$, and compare the remainder with the dividend. Similarly, divide $z^5-8z^3+z^2-1$ by $z-b$; v^3-3v+1 by $v-2$; and $2x^4+5x^3-x+10$ by $x-c$.

48. Remainder theorem. In Ex. 57 on this page it is shown that when x^3+4x^2-7x+2 is divided by $x-a$ the remainder is a^3+4a^2-7a+2; *i.e.*, the remainder is what would be obtained by substituting a for x in the dividend.

To show that this relation between dividend and remainder is not accidental, but that it is always true when a polynomial in x is divided by $x-a$, let $Ax^n+Bx^{n-1}+Cx^{n-2}+\cdots+Hx+K$ represent any such polynomial whatever, arranged according to descending powers of x, and let Q and R, respectively, represent the quotient and remainder when this polynomial is divided by $x - a$; then, since the dividend equals the quotient times the divisor, plus the remainder,

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Hx + K = Q(x - a) + R.$$

Moreover, since the second member of this equation, when multiplied out, must be exactly like the first member, therefore this equation is true for *all* values that may be assigned to x; but if the value a be given to x, the equation becomes

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ha + K = R,\text{*}$$

hence, in every such division, the remainder may be immediately written down by substituting a for x in the dividend.

It also follows from this theorem that if

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ha + K = 0,$$

then $Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Hx + K$ is exactly divisible by $x - a$, for in that case the remainder is zero; and conversely.

EXERCISES

1. What is the remainder when $3x^4 - 2x + 1$ is divided by $x - c$? by $x - a$? by $x - 2$? Answer these questions by means of § 48.

2. What is the remainder when $y^3 + 2y^2 - 14y - 3$ is divided by $y - a$? by $y - k$? by $y + 2$, *i.e.*, by $y - (-2)$? by $y - 3$? Try the last two cases.

3. Is $x - 3$ an exact divisor of $x^4 - 4x^3 + 5x + 12$? Answer without actually performing the division.

REVIEW QUESTIONS—CHAPTERS I-V

1. Define the following operations: addition; subtraction; multiplication; division. Which of these are inverse operations? Explain.

2. Point out at least one advantage which the definition of multiplication as given in § 3 (iii) has over the usual arithmetical definition.

3. In a number system consisting of positive integers only, is division always a possible operation? How must this number system be enlarged so that division may be always possible?

Answer these questions with regard to subtraction also.

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* Since, in that case, $Q(x - a)$ becomes $Q(a - a)$, *i.e.*, 0.

4. Point out at least two advantages of using letters to represent numbers.

5. Define and illustrate a negative number. How may a negative number be subtracted from any given number? State and prove the "law of signs" for multiplication of negative numbers. Also for division.

6. How may a parenthesis which incloses several terms, and which is preceded by the minus sign, be removed without affecting the value of the expression? Why?

7. Define an algebraic expression; a term; a binomial; a polynomial; a coefficient; an exponent; the degree of a term, and of an integral polynomial.

8. State the several steps in solving an algebraic problem. What axioms are frequently used in such solutions? What is meant by "checking the work"?

9. How are two or more similar monomials added? State a rule for subtracting one polynomial from another.

10. Prove that $a^m \cdot a^n \cdot a^p = a^{m+n+p}$ if a is any number whatever and m, n, and p are positive integers.

11. How may the product of two or more monomials be obtained?

12. Give a rule for dividing one polynomial by another. Also explain a device for abbreviating the work. State two ways of checking the correctness of an exercise in division.

13. Are negative numbers ever used as exponents? Is zero so employed? What is the interpretation of such symbols as 5^{-2}, a^0, and x^{-n}? What is the advantage of such exponents?

14. Prove that, under a proper interpretation, negative and zero exponents conform to all the laws previously established for positive integral exponents.

15. Prove that any factor may be transferred from the numerator of a fraction to the denominator, or *vice versa*, by merely reversing the sign of its exponent—whether the given exponent be positive or negative.

CHAPTER VI

COMBINATORY PROPERTIES OF NUMBERS*

49. Introductory. Some combinatory properties of numbers, the correctness of which has thus far in this book, and also in arithmetic, been *assumed,* deserve to be somewhat carefully studied. This further study is not so much needed to give the student confidence in their correctness as it is to *justify* the confidence he already feels; it is designed to guard the student against drawing conclusions which are not fully warranted.

To illustrate: since by actual counting $3+5=8$ and $5+3=8$, therefore $3+5=5+3$; similarly it is found that $9+2=2+9$, $15+7=7+15$, etc.; but merely *verifying* this fact in particular cases does not warrant the conclusion that $a+b=b+a$, when a and b represent an untried pair of numbers. So far as the above reasoning is concerned, the very next pair of numbers that is tried may prove to be an exception.

If a large number of verifications could establish a general law, then the conclusion that $a^b=b^a$, for every pair of numbers, would be valid to one who had *happened* to try only those pairs of numbers for which this is true; *e.g.*, $2^4=4^2$.†

50. Commutative law of addition. In § 49 it was *verified* that $3+5=5+3$, $9+2=2+9$, etc. These are particular cases of a general principle which is known as the **commutative law of addition.** This law may be stated thus: *the sum of two or more numbers is not changed by changing the order in which these numbers are added.*

That this law is true for every set of numbers without exception will now be shown, not by verifying it in particular cases, — that

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* This chapter may, if the teacher prefers, be omitted on a first reading.

† Admitting fractional exponents, which are introduced later (§ 153), the number of pairs of numbers for which $a^b=b^a$ is infinitely large.

method would not really prove anything for any untried set of numbers (§ 49), — but by fundamental considerations based upon the primary meaning of number.

(i) *The numbers positive integers.* To show that $a + b$ equals $b + a$, whatever positive integers are represented by a and b, let there be a objects * in one group and b objects in another, then $a + b$ means the number of objects in the group formed by adding the objects of the second group to those of the first, and $b + a$ means the number of objects in the group formed by adding the objects of the first group to those of the second; but manifestly the total number of objects in the two groups † is the same whether the second group be added to the first or the first to the second, and therefore $a + b = b + a.$

Similarly, the correctness of this law is shown for any number of positive integers.

(ii) *The numbers negative integers.* Since the sum of any number of negative integers is found by getting the sum of the absolute values of these numbers and prefixing to this sum the minus sign (§ 16), therefore, by (i) above, the commutative law is true for any number of negative integers.

(iii) *The numbers integers, some positive and some negative.* Such a sum as $2 + (-6) + 7$ is obtained by first adding -6 to 2 and then adding 7 to that result; but $2 + (-6) = -4$ (§ 16), and $-4 + 7 = 3$; *i.e.*, two of the negative units in -6 are cancelled by the 2, and the -4 that remains cancels four of the positive units in 7. Similarly in general.

In other words, in adding positive and negative numbers one negative unit cancels one positive unit and but one, and *vice versa.* Now neither the number of positive units nor the number of negative units is changed by changing the order in which the addition is performed [(i) and (ii) above]; therefore the sum (the number of uncancelled units) is not changed by changing the order in which the additions are made.

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* These may be any objects whatever, and need not even be of the same kind; for the purpose of mere counting any object may take the place of any other.

† This assumes merely that an object may be removed from one position to another without destroying its individuality.

(iv) *The numbers fractions.* It will presently be shown that any given fractions can always be reduced to equivalent fractions having a common denominator, and such that the numerators and denominators are integers; it will also be shown (§ 54) that such a fraction as $\frac{m}{n}$ is equal to m times $\frac{1}{n}$. Assuming this for the present, it follows that the commutative law is true for this case also, for, if the simplified fractions are $\frac{m}{n}, \frac{p}{n}, \frac{q}{n}$, etc., then $\frac{m}{n}+\frac{p}{n}+\frac{q}{n}+\cdots$ means m times $\frac{1}{n}+p$ times $\frac{1}{n}+q$ times $\frac{1}{n}+\cdots$, *i.e.*, if $\frac{1}{n}$ be called the fractional unit, then $\frac{m}{n}+\frac{p}{n}+\frac{q}{n}+\cdots$ means $m+p+q+\cdots$ times this fractional unit; but, by (i), (ii), and (iii) above, the sum $m+p+q+\cdots$ is independent of the order in which the addition is performed; therefore $\frac{m}{n}+\frac{p}{n}+\frac{q}{n}+\cdots$ is independent of the order in which the fractions are added.

Hence the commutative law of addition is true for positive and negative integers and fractions.

51. Associative law of addition. Another law of the same general character as that given in § 50 above, is known as the **associative law of addition,** and may be stated thus: *the sum of three or more numbers is not changed by grouping together two or more of the summands, and replacing them by their sum.*

E.g., $3+6+2=3+(6+2)=3+8$, [Each member being 11

and $5+3+6+8=5+(3+6)+8=5+9+8$. [Each member being 22

To show that this law, which has just been verified in two particular cases, is true for any set of numbers whatever (positive or negative, integers or fractions), let a, b, c, and d represent any four such numbers;

then $a+b+c+d=b+d+a+c$ [§ 50

$=(b+d)+a+c$ [§ 8

$=a+(b+d)+c,$ [§ 50

i.e., the numbers b and d may be grouped together and replaced by their sum; similarly for any two or more of the summands.

Observe that the process employed in the proof just given is entirely general, *i.e.*, that it applies to any number of summands and to any desired grouping of them; it consists in first bringing the numbers which it is desired to group together into the leading places in the sum (§ 50), then grouping them together (§ 8), and then putting the group (which is a number) into any desired place (§ 50).

52. Commutative law of multiplication. Another principle which the student has already used freely, and which is of the same general character as those given in §§ 50 and 51, is known as the **commutative law of multiplication.** This principle may be stated thus: *the product of two or more numbers is not changed by changing the order in which the multiplications are performed.*

E.g., $5 \cdot 8 = 8 \cdot 5.$ [Each member is 40

So, too, $3 \cdot 4 \cdot 9 = 4 \cdot 3 \cdot 9 = 9 \cdot 3 \cdot 4$, etc. [Each member is 108

Although the law which has just been stated and illustrated is true for any numbers whatever, its complete proof necessarily divides itself into several parts; the proof of its correctness when some or all of the numbers are fractions is given in § 54 (iii), while the part of the proof which concerns integers only will now be given.

(i) *Proof for three positive integers; also for two.* Let a, b, and c represent any three positive integers whatever,* and let a rectangular array containing b rows and c columns of groups of a objects each, be formed, thus:

$$\begin{array}{rl} & \quad c\text{ columns} \\ & \overbrace{\hphantom{a,\ \ a,\ \ a,\ \ \cdots,\ \ a}} \\ b\text{ rows} & \left\{\begin{array}{lllll} a, & a, & a, & \cdots, & a \\ a, & a, & a, & \cdots, & a \\ a, & a, & a, & \cdots, & a \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a, & a, & a, & \cdots, & a \end{array}\right. \end{array}$$

Since there are a objects in each group and b groups in each column, therefore the number of objects in a column is $a \cdot b$; and since there are $a \cdot b$ objects in each column and c columns, there-

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[*] When reading this proof for the first time, it may be best for the student to use a set of particular numbers such as 3, 5, and 6 instead of a, b, and c.

fore the number of objects in the entire array is $(a \cdot b) \cdot c$, *i.e.*, $a \cdot b \cdot c$.*

Again, the number of objects in a row is $a \cdot c$; and, since there are b rows, the number of objects in the entire array is $(a \cdot c) \cdot b$, *i.e.*, $a \cdot c \cdot b$.

But the number of objects in the entire array is manifestly the same when they are counted in one order as it is when they are counted in another; therefore

$$a \cdot b \cdot c = a \cdot c \cdot b, \tag{1}$$

i.e., *the product of any three positive integers is not changed by interchanging the order of the second and third.*

If $a = 1$, then equation (1) becomes

$$b \cdot c = c \cdot b, \tag{2}$$

i.e., *the product of any two positive integers is not changed by interchanging their order.*

REMARK. Since multiplier and multiplicand may be interchanged, each is called a **factor** of the product; and, in general, the numbers which multiplied together produce a certain product are called the factors of that product.

(ii) *Proof for any number of positive integers.* By means of the proof given in (i), it is easily shown that any two *consecutive* factors, in a product of two or more integers, may be interchanged without changing that product.

E.g., that $k \cdot m \cdot n \cdot p \cdot s = k \cdot m \cdot p \cdot n \cdot s$,

may be shown as follows:

$$k \cdot m \cdot n \cdot p = (k \cdot m) \cdot n \cdot p \qquad [\S\ 8$$

$$= (k \cdot m) \cdot p \cdot n \qquad [\text{(i) above}$$

$$= k \cdot m \cdot p \cdot n, \qquad [\S\ 8$$

i.e., $k \cdot m \cdot n \cdot p = k \cdot m \cdot p \cdot n$;

whence $k \cdot m \cdot n \cdot p \cdot s = k \cdot m \cdot p \cdot n \cdot s$, [Multiplying each member by s

i.e., the product $k \cdot m \cdot n \cdot p \cdot s$ is not changed by interchanging the two *consecutive* factors n and p. Similarly in general.

---

* The order of multiplication being from left to right (§ 8), $a \cdot b \cdot c$ means the same as $(a \cdot b) \cdot c$.

Moreover, by successive interchanges of two consecutive factors, all the factors of a product may be arranged in any desired order.

Therefore, *the product of any number of positive integers is not changed by any change whatever in the order of the factors.*

(iii) *Proof when some factors are negative.* The proof just given applies also to products in which some of the factors are negative, because the absolute value of such a product is the same as though all of its factors were positive; and its quality is determined by the number of its negative factors (§ 18, note 1); hence neither the quality nor the absolute value of a product of two or more integers is changed by merely changing the *order* of its factors.

Therefore, *the product of any number of integers is not changed by any change whatever in the order of the factors.*

53. Associative law of multiplication. As might be inferred from its name (cf. § 51), this law asserts that *the product of any number of factors is not changed by grouping together two or more of these factors and replacing them by their product.*

E.g., $2 \cdot 5 \cdot 3 \cdot 7 = 2 \cdot (5 \cdot 3) \cdot 7 = 2 \cdot 15 \cdot 7.$ [Each member is 210

The proof* of this law is as follows: the factors to be grouped together may, by successive applications of the commutative law (§ 52), be brought together into the leading places, in which position they may be grouped together and replaced by their product (§ 8); if it is desired to group together still other factors, they may now be treated in the same way.

To illustrate: if a, b, c, d, and e represent any integers whatever,

then $$a \cdot b \cdot c \cdot d \cdot e = a \cdot (b \cdot e) \cdot c \cdot d;$$

for $$a \cdot b \cdot c \cdot d \cdot e = b \cdot e \cdot a \cdot c \cdot d \qquad [\S\ 52$$

$$= (b \cdot e) \cdot a \cdot c \cdot d \qquad [\S\ 8$$

$$= a \cdot (b \cdot e) \cdot c \cdot d, \qquad [\S\ 52$$

i.e., $$a \cdot b \cdot c \cdot d \cdot e = a \cdot (b \cdot e) \cdot c \cdot d,$$

which was to be shown.

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* The proof is here limited to the case of integers because it depends upon § 52, which is thus limited; in § 54 (iv) the case involving fractions will be considered.

54. Some fundamental principles involved in operations with fractions.* The way to *use* fractions has already been taught in arithmetic, but the underlying principles upon which such use is based should also be carefully mastered by the student.

Among these principles are:

(i) *The product of two simple fractions† is a simple fraction whose numerator is the product of the numerators of the given fractions, and whose denominator is the product of their denominators;*

i.e., if p, q, r, and s represent any four integers whatever,

then $$\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}.$$

In order to simplify the proof of (i), let it first be observed that:

(*a*) If $P\cdot n=Q\cdot n$, then $P=Q$; for if P is not equal to Q, let $P=Q+R$ (wherein R is positive or negative), then $P\cdot n=(Q+R)\cdot n=Q\cdot n+R\cdot n$ (§ 39), *i.e.*, $P\cdot n$ is not equal to $Q\cdot n$, which is contrary to the hypothesis.

(*b*) It follows from the different ways of counting the a's in the rectangular array in § 52 (i) that, whatever the number represented by a, so long as b and c are integers,
$$a\cdot b\cdot c=a\cdot c\cdot b=a\cdot(b\cdot c).$$

(*c*) To multiply any number by the simple fraction $\frac{r}{s}$ means first to multiply that number by r, and then to divide the product by s, for the fraction $\frac{r}{s}$ is obtained from the unit in this way [cf. § 7 (v)].

The proof that $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$ is as follows:

$$\frac{p}{q}\cdot\frac{r}{s}\cdot s\cdot q=\frac{p}{q}\cdot r\div s\cdot s\cdot q \qquad \text{[By (}c\text{) above}$$

$$=\frac{p}{q}\cdot r\cdot q \qquad \text{[§ 7 (v)}$$

$$=\frac{p}{q}\cdot q\cdot r \qquad \text{[By (}b\text{) above}$$

$$=pr;$$

---

* Observe carefully that, in the following proofs, a fraction is always regarded as an indicated division.

† By a "simple fraction" is here meant one whose numerator and denominator are integers.

and $$\frac{pr}{qs}\cdot s\cdot q=\frac{pr}{qs}\cdot qs$$ [By (*b*) above

$$=pr;$$ [§ 7 (v)

hence $$\frac{p}{q}\cdot\frac{r}{s}\cdot s\cdot q=\frac{pr}{qs}\cdot s\cdot q,$$ [Each member being equal to *pr*

and therefore $$\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs},$$ [By (*a*) above

which was to be proved.

(ii) *The product of any number of simple fractions is a simple fraction whose numerator is the product of the numerators of the given fractions, and whose denominator is the product of their denominators.*

For, since $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$, which is a simple fraction, therefore $\frac{p}{q}\cdot\frac{r}{s}\cdot\frac{u}{v}=\frac{pr}{qs}\cdot\frac{u}{v}=\frac{pru}{qsv}$, and similarly for the product of any number of simple fractions.

(iii) *The product of two or more simple fractions is not changed by changing the order in which the multiplications are performed* (commutative law).

E.g., $$\frac{p}{q}\cdot\frac{r}{s}\cdot\frac{u}{v}\cdot\frac{x}{y}=\frac{prux}{qsvy}$$ [By (ii) above

$$=\frac{purx}{qvsy}$$ [§ 52

$$=\frac{p}{q}\cdot\frac{u}{v}\cdot\frac{r}{s}\cdot\frac{x}{y};$$ [By (ii) above

i.e., the product of these fractions remains unchanged by interchanging the factors $\frac{r}{s}$ and $\frac{u}{v}$; similarly in general for any number of factors, and for any desired order.

NOTE. Since $\frac{m}{1}$ is the same as m, and since in the above demonstrations any of the denominators may be 1, therefore those proofs remain valid when some of the factors are fractions and some are integers.

In particular, it follows from (iii) that

$$\frac{m}{1}\cdot\frac{1}{n}=\frac{m}{n}=\frac{1}{n}\cdot\frac{m}{1};\ \textit{i.e.}\text{, that } m\cdot\frac{1}{n}=\frac{1}{n}\cdot m=\frac{m}{n}.$$

(iv) *The product of any number of fractions (and integers) is not changed by grouping together any two or more of them and replacing them by their product* (associative law).

For the factors to be grouped may, by (iii) above, and note, be brought together into the leading places, in which position they may be grouped together and replaced by their product (§ 8); if it is desired to group together still other factors, they may now be treated in the same way.

(v) *The value of any simple fraction is not changed by multiplying both numerator and denominator by any integer whatever, or by dividing both by any integer factor of each.*

For, since $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs},$ [By (i) above

whatever integers are represented by the letters,

therefore $\frac{p}{q}\cdot\frac{r}{r}=\frac{pr}{qr},$ *i.e.*, $\frac{p}{q}=\frac{pr}{qr};$ [Since $\frac{r}{r}=1$

and, since this last equation may be read either way, the proposition is proved.

This theorem enables one to reduce fractions to their "lowest terms," and also to reduce two or more given fractions to equivalent fractions having a "common denominator."

(vi) *To divide by a simple fraction gives the same result as to multiply by this fraction inverted.*

For, let N represent any integer or simple fraction, and let $\frac{r}{s}$ represent any simple fraction; then

$$N\div\frac{r}{s}=N\div\frac{r}{s}\cdot\left(\frac{r}{s}\cdot\frac{s}{r}\right)$$ [Since $\frac{r}{s}\cdot\frac{s}{r}=\frac{rs}{sr}=1$

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$$=N\div\frac{r}{s}\cdot\frac{r}{s}\cdot\frac{s}{r}$$ [By (iv) above

$$=N\cdot\frac{s}{r},$$ [Since $N\div\frac{r}{s}\cdot\frac{r}{s}=N,$ [§ 7 (v)]

i.e., $N\div\frac{r}{s}=N\cdot\frac{s}{r},$ which was to be proved.

REMARK. If a represents any number whatever, then $1 \div a$ is called the **reciprocal** of a. From this definition it follows that the reciprocal of a simple fraction is that fraction inverted; for, if $N = 1$ in the proof just given, then

$$1 \div \frac{r}{s} = 1 \cdot \frac{s}{r} = \frac{s}{r}.$$

(vii) *The sum of two or more simple fractions which have the same denominator is a fraction whose numerator is the sum of the numerators of the given fractions, and whose denominator is the common denominator of the given fractions.*

For, let $\frac{a}{d}$, $\frac{b}{d}$, and $\frac{c}{d}$ represent any simple fractions having a common denominator; then

$$\frac{a}{d} + \frac{b}{d} + \frac{c}{d} = a \cdot \frac{1}{d} + b \cdot \frac{1}{d} + c \cdot \frac{1}{d} \qquad \text{[By (iii) above, note}$$

$$= (a + b + c) \cdot \frac{1}{d} \qquad \text{[Distributive law,* § 39}$$

$$= \frac{a + b + c}{d}, \qquad \text{[By (iii) above, note}$$

i.e., $$\frac{a}{d} + \frac{b}{d} + \frac{c}{d} = \frac{a + b + c}{d};$$

and similarly for any number of such fractions. If the given fractions have not a common denominator, they must be reduced to equivalent fractions having a common denominator [see (v) above] before they can be added.

(viii) *Complex fractions.* A complex fraction is usually understood to mean a fraction whose numerator or denominator or both are themselves fractions, *i.e.*, *it is an indicated division* in which the dividend and divisor may themselves be fractions.

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* In § 39 it was proved that multiplication is distributive as to addition; the student is advised to re-read that proof, and to observe that the reasoning there employed makes no restriction upon the numbers involved, — these numbers may be integers or fractions, and positive or negative. It follows then that division also is distributive as to addition, because dividing by any number d is the same as multiplying by $\frac{1}{d}$ [(iii), note, and (iv)].

By the foregoing principles, and especially by (vi) above, complex fractions may always be reduced to equivalent simple fractions, and may then be replaced by these simple fractions; hence the commutative and associative laws, which were demonstrated above for integers and simple fractions, apply to complex fractions also; *i.e., these laws, as well as the distributive law* (§ 39), *apply to any integers and fractions whatever.*

55. Zero; operations involving zero. Zero may be defined as the result of subtracting any number from itself; it is represented by the symbol 0.

E.g., $a - a = 0,$

whatever the number represented by a.

By replacing 0 by $a - a$ it is easily shown that

$$n + 0 = n = n - 0; \quad 0 \cdot n = n \cdot 0 = 0; \quad \text{and} \quad 0 \div n = 0,$$

where n represents any finite number whatever.

Again, since $n \div d$ stands for the number which, being multiplied by d, will produce n [§ 3 (iv)], therefore $0 \div 0$ *may have any finite value whatever*, because any finite number multiplied by 0 equals 0; and $n \div 0$ (wherein n is any finite number) *has no finite value whatever*, because no finite number multiplied by 0 equals n.* From what has just been said, it is clear that 0 *must not be used as a divisor.*

EXERCISES

1. What are the values of the expression $2n+1$ when $n = 1, 2, 3, \cdots, 15$? Are these values even or odd?

2. Do the answers of Ex. 1 warrant the conclusion that $2n + 1$ represents an odd number for *every* integer value of n (cf. § 49)? Prove [illegible] both $2t + 1$ and $2t - 1$ represent odd numbers for all integer values [illegible]

3. Show also that *any* odd number whatever may be represented by $2n + 1$ by giving a suitable integer value to n.

4. What are the values of $n^2 + n + 17$ when $n = 1, 2, 3, \cdots, 9$? Are these values prime or composite?

---

* Compare note to Ex. 15 below.

5. Do the answers of Ex. 4 warrant the conclusion that $n^2 + n + 17$ represents a prime number for every integer value of n (cf. § 49)? Is not 17 a factor of $n^2 + n + 17$ when $n = 17$?

6. Do the expressions $x^2 + x + 41$ and $2\,x^2 + 29$ represent prime or composite numbers when $x = 1, 2, 3, \cdots$? Are their values prime for *all* integral values of x?

NOTE. The above questions are designed to emphasize § 49 by showing the kind of errors into which some distinguished mathematicians have been led by basing *general* conclusions upon more or less numerous verifications. The celebrated mathematician Fermat concluded from a certain number of verifications that $2^n + 1$ is always prime when $n = 2, 2^2, 2^3, 2^4, \cdots$; Euler, however, discovered later that $2^{32} + 1$ is a composite number.

7. What is meant by saying that addition is a commutative operation (cf. § 50)? That it is an associative operation?

Is subtraction commutative? Multiplication? Division? Illustrate your answer in each case.

8. What is meant by saying that multiplication is distributive with reference to addition (cf. § 39, and footnotes, pp. 55 and 83)? Can you name another instance in which one operation is distributive with reference to another?

9. Regarding the expression $-(a+b-c+\cdots)$ as $-1 \cdot (a+b-c+\cdots)$, apply the distributive law of multiplication as to addition to prove the correctness of the principle given in § 33 for removing a sign of aggregation preceded by the minus sign.

10. By means of the commutative and associative laws of multiplication, show that $(3 \cdot 2)^4 = 3^4 \cdot 2^4$. So, too, show that $(a \cdot b)^n = a^n \cdot b^n$.

Is the raising of the product of several factors to a power a distributive operation with reference to the factors?

11. Is $(2 + 5)^2$ equal to $2^2 + 5^2$? Compare this with Ex. 10, and then state the operations over which an exponent is distributive, and those over which it is not distributive.

12. Which of the combinatory laws discussed in the present cha[illegible] is it usually necessary to employ when a polynomial is simplifie[illegible] uniting similar terms? When a polynomial is arranged according to powers of one of its letters? When an equation is cleared of fractions?*

13. Give the proofs which are taught in arithmetic of the principles given in § 54. Compare the arithmetical treatment with that given here, and note the advantages of the present proofs.

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* Compare (1) and Ex. 10, of § 25.

14. Under the arithmetical definition is $\frac{5}{7\frac{1}{2}}$ a fraction, *i.e.*, is it "one or more of the equal parts into which a unit has been divided"? How is a fraction defined in the preceding pages of this book? Is $\frac{5}{7\frac{1}{2}}$ a fraction under this definition?

15. Write down the successive values which the fraction $\frac{5}{x}$ takes when the values $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ are assigned to x. How do these successive values of the fraction compare? Can you name a number so large that none of these values of the fraction will exceed it? Can you name a number so near 0 that none of the series of numbers $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ will be still nearer to 0?

NOTE. Ex. 15 illustrates the fact that in mathematical operations numbers may arise which are greater, and others which are less, than any numbers which we can name or even think of; such numbers are usually called **infinitely large** and **infinitely small** numbers, respectively, — all other numbers being classed together as **finite** numbers. An infinitely large number is usually represented by the symbol ∞.

16. Having defined 0 as $a - a$, wherein a is any finite number, *prove* that $0 \cdot n = 0$ for every finite value of n.

SUGGESTION. Substitute $a - a$ for 0, then apply § 39, and finally the definition of zero.

17. Point out the fallacy in the following reasoning:

If $\quad x = a,$

then $\quad x^2 = ax,$

and $\quad x^2 - a^2 = ax - a^2,$

[Subtracting a^2 from each member

i.e., $\quad (x + a)(x - a) = a(x - a);$

therefore $\quad 2a(x - a) = a(x - a),$ [Since $x = a$

and, therefore, $\quad 2 = 1.$ [Dividing by $a(x - a)$

CHAPTER VII

TYPE FORMS IN MULTIPLICATION—FACTORING

I. SOME TYPE FORMS IN MULTIPLICATION

56. Type forms. Although all exercises in multiplication and division of integral algebraic expressions can be readily solved by § 40 and § 47, yet there are a few special cases of these operations which occur so frequently in practice that it is well worth one's while to be able to perform them by inspection; they are often spoken of as **type forms.** Some of these type forms are considered in the next few paragraphs.

57. Square of a binomial. This may be divided into two cases, according as the binomial is the *sum* or the *difference* of two numbers.

(i) *The square of the sum of two numbers.* Let a and b represent any two algebraic numbers; then by actual multiplication (§ 40),

$$(a+b)(a+b)=a^2+2\,ab+b^2,\ i.e.,\ (a+b)^2=a^2+2\,ab+b^2.*$$

This formula may be translated into words thus: *the square of the sum of two numbers equals the square of the first number, plus twice the product of the two numbers, plus the square of the second number.*

E.g., $(x+3)^2=x^2+6\,x+9$; $(y+p)^2=y^2+2\,yp+p^2$; etc.

(ii) *The square of the difference of two numbers.* By actual multiplication, as before,

$$(x-3)^2=x^2-6\,x+9,$$

and, in general, $(a-b)^2=a^2-2\,ab+b^2.*$

The student may translate this formula into words.

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* **This second member is called the expansion of the binomial.**

NOTE. If either or both of the terms of the binomial are represented by more than a single symbol, they may be inclosed in parentheses (to preserve their individuality) and the simplified result may then be written as a third member of the equation.

E.g., $(2x+3y)^2=(2x)^2+2(2x)(3y)+(3y)^2=4x^2+12xy+9y^2$.

With a little practice and care, this intermediate step may, however, be safely omitted.

EXERCISES

Expand the following expressions:

1. $(x+y)^2$.
2. $(m+n)^2$.
3. $(h+k)^2$.
4. $(u+w)^2$.
5. $(a-p)^2$.
6. $(c-h)^2$.
7. $(x+3)^2$.
8. $(a-5)^2$.
9. $(7-v)^2$.
10. $(2x-3b)^2$.
11. $(4a+7x)^2$.
12. $(3m^4-2n)^2$.
13. $(\frac{2}{5}x^2-\frac{3}{4})^2$.
14. $(2a^2x+3by^3)^2$.
15. $\left(\frac{x}{a}+\frac{a}{x}\right)^2$.
16. $\left(\frac{3x^2}{2a}-\frac{4a^3}{3x}\right)^2$.
17. $(9abc+bcd)^2$.
18. $\{(a+b)+c\}^2$.
19. $\{(a+b)-c\}^2$.

20. Compare the fully expanded form of Ex. 18 with $(a+b+c)^2$, and state, if you can, a general rule for writing down the square of any trinomial (see also § 61).

21. Expand $(x-y+z+s)^2$.

SUGGESTION. $x-y+z+s=(x-y)+(z+s)$.

22. Since $a-b=a+(-b)$, show that case (ii), p. 87, is included under case (i).

23. Expand $(x^n+y^n)^2$. Also $(3a^n-2s^m)^2$.

24. What must be added to x^2+6x to make it the square of $x+3$?

25. What must be added to $t^2+\frac{3}{2}t$ to make it the square of $t+\frac{3}{4}$?

26. What must be added to $a^4+a^2b^2+b^4$ to make it the square of a^2+b^2?

27. What must be added to $x^8+2x^4y^3+4y^6$ to make it the square of x^4+2y^3?

28. Find what must be added to each of the following expressions to make them exact squares; also give the expressions of which they are then the squares:

$m^4-8m^2n^2+4n^4$; a^2-2ab; $x^2y^2+12xyz^3$; x^2+ax; and $A^2+\frac{m}{n}AB$.

29. Find, by the method of § 57, the square of 53, *i.e.*, of 50 + 3.

30. Write down the squares of the following numbers: 18 (*i.e.*, 20 − 2), 39, 71, 83, and 34.

58. Product of sum and difference. If a and b represent any two numbers whatever, then, by actual multiplication,

$$(a+b)(a-b)=a^2-b^2,$$

i.e., the product of the sum of any two numbers, by the difference of these numbers, is the square of the first number minus the square of the second.*

E.g., $(x+3)(x-3)=x^2-9$; $(5+m)(5-m)=25-m^2$; etc.

NOTE. Here, too, as in § 57, complex terms may be inclosed in parentheses, thus:

$$(3x^2+5y)(3x^2-5y)=(3x^2)^2-(5y)^2=9x^4-25y^2.$$

EXERCISES

Without actually performing the following indicated multiplications, write down the products by inspection:

1. $(x+y)(x-y)$.

2. $(m+n)(m-n)$.

3. $(3x+y)(3x-y)$.

4. $(7x-2y)(7x+2y)$.

5. $(14a+15b)(14a-15b)$.

6. $(6p-5q)(6p+5q)$.

7. $(4m^2-3n^3)(4m^2+3n^3)$.

8. $(x^3+y^2)(x^3-y^2)$.

9. $(10lmn-6p^2q^2)(10lmn+6p^2q^2)$.

10. $\{(x-y)+z\}\{(x-y)-z\}$.

11. $\{(a^2+b^2)-ab\}\{(a^2+b^2)+ab\}$.

12. $(a+b+c)(a+b-c)$.

13. $(a-b+c)(a-b-c)$.

14. $(a-b+x)(a+b-x)$.

15. $(m-2n+s-t)(m-t+2n-s)$.

16. Show that $x^2+2xy+y^2-z^2$ is the product of the sum and difference of $x+y$ and z.

17. Show that $a^2+2ab+b^2-c^2-2cd-d^2$ is the product of the sum and difference of $a+b$ and $c+d$.

18. $(9x^2-4y^2)\div(3x-2y)=$? Why?

19. $(16a^2-25b^2)\div(4a+5b)=$? Why?

20. $(x^4-y^4)\div(x^2-y^2)=$? Why?

21. $(x^6-y^4)\div(x^3-y^2)=$?

22. $(x^{18}-y^8)\div(x^9+y^4)=$?

23. Find, by the above method, the product of 22 by 18.

SUGGESTION. $22=20+2$ and $18=20-2$.

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* The order in which these numbers are written being the same in both factors.

24. By this method find the following products: 63 by 57; 48 by 52; 34 by 26.

NOTE. The identity $(a+b)(a-b)=a^2-b^2$, *i.e.*, $a^2=(a+b)(a-b)+b^2$, furnishes a very practical device for mentally squaring any number consisting of two digits.

E.g., to square 73 mentally, let $a=73$ and $b=3$; then the last formula above becomes

$$(73)^2=76\cdot 70+9=5329.$$

Similarly, to square 58, let $a=58$ and $b=2$; then the formula becomes

$$(58)^2=60\cdot 56+4=3364.$$

25. By the method given in the above note, write down the square of 47; of 82; of 29; of 53; of 98; and of 61.

59. Product of binomials having common term. By actual multiplication,

$$(x+3)(x+5)=x^2+8x+15=x^2+(3+5)x+15;$$

and $$(x+3)(x-5)=x^2-2x-15=x^2+(3-5)x-15.$$

So, too, in general, $(x+a)(x+b)=x^2+(a+b)x+ab$;

i.e. the product of two binomials having a term in common equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the product of the unlike terms.

EXERCISES

Without actually performing the following multiplications, write down the products by inspection:

1. $(a+5)(a+7)$.

2. $(a-5)(a-7)$.

3. $(a+5)(a-7)$.

4. $(a-5)(a+7)$.

5. $(y-c)(y+2c)$.

6. $(x^2+4)(x^2+5)$.

7. $(x^2+4)(x^2-5)$.

8. $(x^2-4)(x^2-5)$.

9. $(x^2-4)(x^2+5)$.

10. $(a+b)(a+c)$.

11. $(a-b)(a+c)$.

12. $(2x+3)(2x-5)$.

13. $(3a+4)(3a-6)$.

14. $(4a^2-5)(4a^2+1)$.

15. $(xy-4)(xy+16)$.

16. $(l^2m^2n^3+2)(l^2m^2n^3-8)$.

17. $\{(l+m)-2\}\{(l+m)-5\}$.

18. $\{(l+m)+8\}\{(l+m)-15\}$.

60. Product of two binomials which contain the same letters. The product of two binomials containing the same letters is a trinomial which, by a little practice, may be written down without writing the intermediate steps.

E.g., the product of $3x+5$ and $2x-7$ may be arranged as in the margin: the term $6x^2$ is the product of the first terms of the binomials, the term $-11x$ is the algebraic sum of the "cross products" ($2x$ by 5 and $3x$ by -7), and -35 is the product of the last terms of the binomials. This final product may, with a little practice, be easily written down, omitting the intermediate steps.

$$\begin{array}{rrr} 3x & +5 & \\ 2x & -7 & \\ \hline 6x^2 & +10x & \\ & -21x & -35 \\ \hline 6x^2 & -11x & -35 \end{array}$$

Similarly, in the product of $3x+4y$ by $5x-2y$, the product of the first terms is $15x^2$, the algebraic sum of the cross products is $14xy$, and the product of the last terms is $-8y^2$; hence $(3x+4y)(5x-2y)=15x^2+14xy-8y^2$. So, too, $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$.

EXERCISES

Write down the following products by inspection:

1. $(3x+2)(4x-3)$.
2. $(3x+2y)(4x+3y)$.
3. $(x-3y)(5x+6y)$.
4. $(2a-4b^2)(5a-6b^2)$.
5. $(7a^2+b^2)(3a^2+8b^2)$.
6. $(9x-2y)(x+y)$.
7. $(x+a)(x+b)$.

61. The square of any polynomial. By actual multiplication it is found that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc,$$

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc$$
$$+2bd+2cd,$$

$$(a+b+c+d+e)^2=a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad+2ae$$
$$+2bc+2bd+2be+2cd+2ce+2de,$$

etc. This may be formulated into words, thus: *the square of any polynomial whatever equals the sum of the squares of all the terms of the polynomial, plus twice the product of each term by all the terms that follow it.**

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* The formal proof of this theorem is given in Chapter XVIII.

EXERCISES

Expand the following expressions by inspection:

1. $(m+n-s)^2$.

2. $(a-b-c)^2$.

3. $(2x+y+z)^2$.

4. $(2x+3y-z)^2$.

5. $(2x-3y+z)^2$.

6. $(3a+4b+c)^2$.

7. $(3a-4b-2c)^2$.

8. $(a-b+c-d)^2$.

9. $(ax+by+cz)^2$.

10. $(abx-acy-bcz)^2$.

11. $(l+m+n+p+q+r+s)^2$.

12. $(2x-3y+4z-5a+3b-4)^2$.

13. $(x^4+2x^3-3x^2+4x-5)^2$.

62. Higher powers of binomials — binomial theorem. By actual multiplication it is found that

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3,$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4,$$
$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5,$$
$$(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6,$$

etc.; and that

$$(x-y)^3 = x^3-3x^2y+3xy^2-y^3,$$
$$(x-y)^4 = x^4-4x^3y+6x^2y^2-4xy^3+y^4,$$
$$(x-y)^5 = x^5-5x^4y+10x^3y^2-10x^2y^3+5xy^4-y^5; \text{ etc.}$$

A careful study of the second members of the above equations will show that they all follow the same laws, and that they may, therefore, be written down by the same rules. In fact, such a study will show that:

(1) *The number of terms in the expansion is in every case greater by 1 than the exponent of the binomial.*

(2) *The x* appears in every term of the expansion except the last, and the y appears in every term of the expansion except the first.*

(3) *The exponent of x in the first term of the expansion is the same as the exponent of the binomial, and it decreases by 1 from term to term in passing to the right, while the*

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* In applying these rules to other binomials, observe that x is here used for "the first term of the binomial" and y for "the second term of the binomial."

exponent of y in the second term of the expansion is 1, and it increases by 1 from term to term in passing toward the right.

(4) *The coefficient of the first term of the expansion is 1; the coefficient of the second term is the same as the exponent of the binomial; and if the coefficient of any term be multiplied by the exponent of x in that term, and this product be divided by the number of the term (i.e., by this term's exponent of y increased by 1), the result will be the coefficient of the next term.*

(5) *The signs of the terms of the expansion are all positive if each term of the binomial is positive, but if the second term of the binomial is negative, then the terms of the expansion are alternately positive and negative—the first term being positive.*

NOTE. It is proved later (Chap. XVIII) that the above laws apply to all positive integral powers of any binomial whatever; hence such powers may be expanded without actually performing the multiplications.

Ex. 1. Expand $(a-b)^8$.

SOLUTION. By (1), (2), and (3) above, the letters and exponents in the several terms of this expansion are:

$$a^8 \quad a^7b \quad a^6b^2 \quad a^5b^3 \quad a^4b^4 \quad a^3b^5 \quad a^2b^6 \quad ab^7 \quad b^8;$$

by (4), the coefficients are:

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1;$$

and by (5), the signs are:

$$+ \quad - \quad + \quad - \quad + \quad - \quad + \quad - \quad +;$$

hence, combining these results,

$$(a-b)^8 = a^8 - 8\,a^7b + 28\,a^6b^2 - 56\,a^5b^3 + 70\,a^4b^4 - 56\,a^3b^5 + 28\,a^2b^6 - 8\,ab^7 + b^8.$$

Ex. 2. Expand $(2\,x-a^2)^3$.

SOLUTION. Letters and exponents,

	$(2\,x)^3$	$(2\,x)^2(a^2)$	$(2\,x)(a^2)^2$	$(a^2)^3$;	[Cf. (1), (2), (3)
coefficients,	1	3	3	1;	[Cf. (4)
signs,	$+$	$-$	$+$	$-$;	[Cf. (5)

combined result, $(2\,x-a^2)^3 = (2\,x)^3 - 3\,(2\,x)^2(a^2) + 3(2\,x)\,(a^2)^2 - (a^2)^3$;

simplified result, $(2\,x-a^2)^3 = 8\,x^3 - 12\,x^2a^2 + 6\,xa^4 - a^6$.

With a little practice the combined result may be written down at once instead of making several steps of the work.

EXERCISES

Expand the following expressions:

3. $(a+b)^3$. **6.** $(u-v)^6$. **9.** $(x-y)^{10}$.

4. $(a-x)^4$. **7.** $\left(x+\frac{2}{x}\right)^4$. **10.** $(x-2a)^7$.

5. $(m-t)^3$. **8.** $(3a^2-2b^5)^3$. **11.** $(m^2+3n)^6$.

12. Write the first 4 terms of $(a+x)^{25}$.

13. Write the first 3 terms, and also the 7th term, of $(x-y)^{32}$.

14. Write the first 5 terms of $(2ax-3k^2)^9$.

II. FACTORING

63. Definitions. In a broad sense, any two or more numbers whose product is a given number are factors of that number.

Thus, since $\frac{1}{3}\cdot\frac{6}{5}\cdot 10=4$, therefore $\frac{1}{3}$, $\frac{6}{5}$, and 10 are factors of 4; so also are $\frac{5}{12}$, 18, and $\frac{8}{15}$.

In this sense, however, the problem of finding the factors of any given number, or algebraic expression, is manifestly indeterminate; it is therefore customary, when speaking of **factors,** to mean only the *rational* and integral exact divisors of a given number or expression.*

E.g., ± 1,† ± 2, ± 3, ± 4, ± 6, and ± 12 are factors of 12; and ± 1, ± 5, $\pm(2x+y)$, $\pm(2x-y)$, as well as products of any two or more of these, are factors of $20x^2-5y^2$. Every number is a factor of itself, and 1 is a factor of every number.

A number, or an algebraic expression, is said to be **prime** if it has no exact divisor (*i.e.*, factor) except itself and unity; otherwise it is **composite.**

A factor is **prime** or **composite** according as the expression for it is prime or composite; and it is **integral** with regard to any given

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* An expression is *rational* with regard to a particular letter if it contains no indicated root of that letter (see § 130).

† The sign ± is called the **double sign,** and is read "plus or minus"; it is used to combine two expressions into one: thus the expression $\pm a$ means both $+a$ and also $-a$.

letter if the algebraic expression for it is integral with regard to that letter (cf. § 41).

It will appear later that the writing of an expression as the product of its prime factors often greatly simplifies algebraic work; and it is therefore important that the student should early master those cases of factoring which present themselves most frequently. Some of these cases will now be considered.

64. Factors of a monomial. This is the simplest of all the exercises in factoring, and can be done by inspection.

E.g., $30\,ax^2y = 2 \cdot 3 \cdot 5 \cdot a \cdot x \cdot x \cdot y$, which exhibits the given monomial as the product of its prime factors; the product of any two or more of these prime factors is a composite factor of the given monomial (cf. § 63).

A rule for this kind of factoring may be stated thus: *by inspection, or by trial, find the prime factors of the numerical coefficient of the given monomial, and to their indicated product annex each of the literal factors as many times as there are units in its exponent.*

EXERCISES

Separate the following monomials into their prime factors:

1. $6\,a^2x^3$. **2.** $15\,m^2p^4z^3$. **3.** $36\,s^4t^3$.

4. $420\,m^3x^4y^2$. **5.** $572\,a^3c^2uv^2$.

65. Monomial and polynomial factors of a polynomial. If a polynomial contains a monomial factor, the latter can usually be discovered by mere inspection.

E.g., in $12\,a^2x^3 + 4\,abx^2y - 8\,ax^2y^2$, it is seen that each term contains the factor $4\,ax^2$, hence

$$12\,a^2x^3 + 4\,abx^2y - 8\,ax^2y^2 = 4\,ax^2 \cdot (3\,ax + by - 2\,y^2).$$

In order to factor a polynomial *completely*, it is then only necessary to consider further how to factor a polynomial which contains no monomial factor. This problem, however, is in general very difficult, and only its simplest cases will at present be considered. Fortunately it is these simpler cases which present themselves most frequently in practice.

EXERCISES

Separate the following expressions into their monomial and polynomial factors:

1. $5a - 10b$.

2. $17x^2 - 289x^3$.

3. $4x^3 - 8x^2y$.

4. $10m^4n^2 - 15m^3n^3$.

5. $16x^2 - 2abx$.

6. $4a^3b^2 - 24a^2b^3$.

7. $15x^4 - 10x^3 + 5x^2$.

8. $3a^4 - 6a^4b + a^4b^2$.

9. $x^{12}y^{12} + x^{11}y^{11} + x^{10}y^8 - x^8$.

10. $3m^5 - 12m^3n^2 + 6mn^4$.

11. $ac - bc - cd - abcd$.

12. $13x^3y^3 - 13x^2y^2 + 12xy$.

13. $14x^2y^3z^3 - 7x^3y^2z^2 + 8xy^2z^2$.

14. $60m^2n^3r^2 - 45m^3n^2r^3 + 90m^4n^3r^2$.

15. $12x^2b^2y - 18xy^3b + 24x^4b^4y^4$.

16. $14a^2mn^2 - 21a^3m^2n^3 - 49a^4mn^2$.

17. $25c^2dx^3 + 35c^3d^2x^4 - 55c^2d^2x^5$.

18. $51xy^2z^3 - 68x^3y^2z^2 + 85x^4y^3z^4$.

19. $52a^2b^3c^4 - 65a^3b^2c^2 + 91a^2b^2c^2$.

20. $44a^4x^3y^2 + 66a^3x^4y^3 + 88a^2x^5y^4$.

66. Use of type forms in factoring. Since finding the factors of a given number or expression is, in a certain sense, the undoing of a multiplication, therefore the type forms in multiplication already studied (§§ 57–62) may be advantageously employed in separating certain types of expressions into their factors; some of these will now be given.

(i) *Trinomials of the type* $x^2 \pm 2xy + y^2$.* In § 57 (i) and (ii) it is shown that, whatever the numbers or expressions represented by a and b,

$$(a+b)^2 = a^2 + 2ab + b^2 \text{ and } (a-b)^2 = a^2 - 2ab + b^2;$$

therefore $a + b$ and $a + b$ are the factors of $a^2 + 2ab + b^2$, and $a - b$ and $a - b$ are the factors of $a^2 - 2ab + b^2$.

Similarly in general, *if in a trinomial two terms are exact squares, and the remaining term is the double product of their square roots,*† *then the given trinomial is the square of a binomial.*

E.g., $m^2 + 6mn + 9n^2$ is a trinomial of this type, and its factors are $m + 3n$ and $m + 3n$; so, too, is $4x^2 + 25 - 20x$, of which the factors are $2x - 5$ and $2x - 5$.

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* $x^2 \pm 2xy + y^2$ means both $x^2 + 2xy + y^2$ and also $x^2 - 2xy + y^2$ (cf. § 63, footnote).

† The square root of a number is that number which, being multiplied by itself, will produce the given number. Cf. § 122.

EXERCISES

Factor the following expressions:

1. $x^2 - 6x + 9$.

2. $225 + 30x + x^2$.

3. $1 - 10y + 25y^2$.

4. $x^2 + 4xy + 4y^2$.

5. $x^6 - 4x^3 + 4$.

6. $a^2b^2 + 2ab + 1$.

7. What first suggests to you that $x^2 + 9y^2 + 6xy$ may be the square of a binomial? How do you test the correctness of this supposition? When is a trinomial the square of a binomial?

8. Write out a carefully worded rule for factoring expressions of the type $x^2 \pm 2xy + y^2$. How are the *terms* of the binomial obtained? How determine the sign by which they are to be connected?

9. Is $a^4 + 2a^2b^3 - b^6$ the square of a binomial? Why?

10. Is $(x+y)^2 + (a+b)^2 + 2(a+b)(x+y)$ the square of a binomial?

Separate the following expressions into their *prime* factors, and check your work by assigning simple numerical values to the letters involved (cf. Ex. 7, § 39):

11. $a^2b^2 + 6abcd + 9c^2d^2$.

12. $4x^4 - 64x^2 + 256$.

13. $9x^2 - 12xyz + 4y^2z^2$.

14. $81x^2 - 18ax + a^2$.

15. $196a^2b^2c^2 + 112ab^2c^2d + 16b^2c^2d^2$.

16. $y^2 - 4y(x+y) + 4(x+y)^2$.

17. $(x+y)^2 - 10(x+y)(y+z) + 25(y+z)^2$.

18. $16(a+x)^2 - 32(a+x)(x-y) + 16(x-y)^2$.

19. $25(x+y)^2 - 50(x+y)z^4 + 25z^8$.

20. $4(a+3b)^2 - 24(a+3b)(b-c) + 36(b-c)^2$.

21. $9a^{2n} - 12a^nb^{2n} + 4b^{4n}$.

22. $-x^6 - 16y^4 - 8x^3y^2$.

23. $-x^4 + 2a^2x^3 - a^4x^2$.

24. $(x^2+y^2)^2 - 2(x^2+y^2)z^2 + z^4$.

(ii) *Expressions of the type* $x^2 - y^2$. In § 58 it was shown that, whatever the numbers or expressions represented by a and b,

$$(a+b)(a-b) = a^2 - b^2;$$

therefore the factors of $a^2 - b^2$ are $a + b$ and $a - b$.

Similarly the difference of the squares of any two numbers or expressions may be factored.

E.g., $25n^2 - 9t^2$ is of this type, and its factors are $5n + 3t$ and $5n - 3t$; so, too, $a^2 + b^2$ and $a^2 - b^2$ are factors of $a^4 - b^4$, but $a^2 - b^2$ is not prime; the *prime* factors of $a^4 - b^4$ are $a^2 + b^2$, $a + b$, and $a - b$.

EXERCISES

Factor the following expressions:

1. $y^2 - z^2$.

2. $y^2 - 9z^2$.

3. $4y^2 - 25b^2$.

4. $225a^2b^2 - 16$.

5. $9y^2 - 1$.

6. $25x^4 - 9y^6$.

7. $a^{16} - 4b^8$.

8. $a^2x - b^2x$.

9. $36a^2e^2 - 81d^2$.

10. $x^{2n} - 4$.

11. $121a^4 - 36b^4$.

12. $64x^2y^{2n} - 144z^2$.

13. $(x+y)^2 - (a+c)^2$.

14. $49 - 36x^2y^2$.

15. $m^{2n} - n^{2m}$.

16. $169x^2y^5z^6 - 16y^3d^8$.

17. $324x^2y^4z^6 - 81$.

18. $289x^6z^9 - y^{8n}z$.

19. $16d^2 - 9(x-y)^2$.

20. For what values of a and b is $(a+b)(a-b)$ equal to $a^2 - b^2$? Is this equation true even when $a = b$? (Cf. § 55.)

21. Factor $a^2 + 2ab - c^2 + b^2$.

SUGGESTION. $a^2 + 2ab - c^2 + b^2 = a^2 + 2ab + b^2 - c^2 = (a+b)^2 - c^2$.

By rearranging and grouping the terms as in Ex. 21, factor the following:

22. $b^2 + 2bc + c^2 - d^2$.

23. $a^2 - 6ax + 9x^2 - 4c^2$.

24. $a^2 + 2ab - d^2 + b^2$.

25. $(x+y+z)^2 - a^2 - 2ab - b^2$.

26. $x^2 - b^2 - 2xy + y^2$.

27. $x^2 + 4xy - 4z^2 + 4y^2$.

28. $4a^2 + 9b^2 - 16c^2 - 12ab$.

29. $9x^2 - 25z^2 + 16y^2 + 24xy$.

30. $4b^2 - x^2 + 4xy + a^2 + 4ab - 4y^2$.

31. $1 - x^2 - 2xy - y^2$.

32. $1 - 4x + 4x^2 - 1 + 6x - 9x^2$.

33. $25b^2 - 1 - 9b^2x^2 - 10ab + a^2 + 6bx$.

(iii) *Expressions of the type* $x^2 + (a+b)x + ab$. In § 59 it was shown that, whatever the numbers or expressions represented by a, b, and x,

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

This formula is helpful in factoring trinomials of the above type.

E.g., $x^2 + 7x + 12$* may be written $x^2 + (4+3)x + 4 \cdot 3$; it is therefore of this type, and its factors are $x + 4$ and $x + 3$.

Observe that the plan of factoring this trinomial is first to find all the pairs of numbers whose product is 12, then to select from among these that pair whose sum is 7; from which the required factors are manifest.

In the same way it may be shown that the factors of $m^2 - 6m + 8$ are $m - 4$ and $m - 2$; so, too, $x^2 + 2x - 15 = (x+5)(x-3)$; $9y^2 - 18y - 7$, *i.e.*, $(3y)^2 - 6(3y) - 7 = (3y-7)(3y+1)$; and $x^2 - 3ax - 28a^2 = (x+4a)(x-7a)$.

This method of factoring expressions of the form $x^2 + ax + b$ is, however, advantageous only when the number of pairs of factors of b is not large; another method is given in § 164, Ex. 69.

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* Such an expression is usually called a **quadratic trinomial**.

EXERCISES

1. If the expression $x^2 + 5x - 36$ is the product of two binomial factors, what is the product of the unlike terms in these binomials? Have these terms like or unlike signs? Why? What is the sum of these unlike terms? Is the larger of them positive or negative? Why?

2. Based upon such considerations as those given in Ex. 1, write out a carefully worded rule for factoring trinomials of this type.

Separate each of the following expressions into its prime factors:

3. $x^2 - 3x + 2$.

4. $x^2 + x - 6$.

5. $x^2 - x - 2$.

6. $y^2 - 6y + 5$.

7. $a^2 + 7a - 30$.

8. $n^2 - 4n - 60$.

9. $p^2 - 12p + 35$.

10. $4 - 6x + 2x^2$.

11. $6y - y^2 - y^3$.

12. $x^3 - 17x^2 + 72x$.

13. $13x - 30 + x^2$.

14. $x^4 - 24x^2 + 63$.

15. $3y^6 + 39y^3 + 66$.

16. $x^2 + (3a - 2b)x - 6ab$.

17. $ax^2 + 7a^2x + 6a^3$.

18. $x^{-2} - 26x^{-1} + 69$.

19. $a^2b^2 - 7ab + 10$.

20. $(x + y)^2 + 7(x + y) + 6$.

21. $9x^2 + 6x - 8$. SUGGESTION. $9x^2+6x-8=(3x)^2+2(3x)-8$.

22. $4x^2 + 4xy - 3y^2$.

23. $16x^2 + 32x + 15$.

24. $15x^2 + 32x^2y + 16x^2y^2$.

25. $x^{2n} + 5x^n + 6$.

26. Can $x^2 + x + 6$ be separated into two binomial factors like those found for the other exercises above? Explain.

(iv) *Expressions of the type* $acx^2 + (ad + bc)x + bd$. The foregoing method is easily extended so as to include many trinomials which are not of type (iii).

From § 60 it follows that if the trinomial $6x^2 - 11x - 35$, for example, is the product of two binomial factors, then the first terms of these binomials are factors of $6x^2$, and the last terms are factors of -35; hence the *possible* pairs of binomial factors are:

$$\begin{Bmatrix}6x-5\\ x+7\end{Bmatrix}, \begin{Bmatrix}6x+5\\ x-7\end{Bmatrix}, \begin{Bmatrix}6x-7\\ x+5\end{Bmatrix}, \begin{Bmatrix}6x+7\\ x-5\end{Bmatrix}, \begin{Bmatrix}3x-5\\ 2x+7\end{Bmatrix}, \begin{Bmatrix}3x-7\\ 2x+5\end{Bmatrix},$$

etc.; and from among these the pair to be selected is that one for which the algebraic sum of the "cross products" is $-11x$; this pair is $3x+5$ and $2x-7$, hence $6x^2-11x-35=(3x+5)(2x-7)$.

Similarly it is found that $12x^2 + 8x - 15 = (6x - 5)(2x + 3)$, and that $15a^2 + 14ab - 8b^2 = (3a + 4b)(5a - 2b)$.

EXERCISES

Factor each of the following expressions:

1. $3x^2+x-10.$

2. $4x^2+16x+15.$

3. $8y^2-10xy-3x^2.$

4. $8A^2+23AB-3B^2.$

5. $16x^5+4x^3y^2-30xy^4.$

6. $4ab^2-73abc+18ac^2.$

7. $90xyz^2-98a^2xyz+8a^4xy.$

8. $15M^{4x}+16M^{2x}N^2+4N^4.$

9. $3(a+b)^2+10(a+b)(a+2b)-8(a+2b)^2.$

(v) *Other types; exact powers.* The formulas of §§ 61 and 62 may also be employed to factor polynomials of the types to which they belong. When such polynomials present themselves for factoring, which is comparatively seldom, the student need only arrange them properly and observe whether all the requirements stated in § 61 or § 62 are satisfied; if so, the given polynomial is an exact power, and its factors are written by inspection.

E.g., to factor the expression $x^2+z^2-4yz+2xz+4y^2-4xy$, observe that it consists of three square terms, and of three double products, hence it *may* belong to the type considered in § 61. A slight rearrangement of the terms shows that it is of this type, viz., $x^2+4y^2+z^2-4xy+2xz-4yz=(x-2y+z)^2$. Similarly for expressions which belong to the type considered in § 62, namely, powers of binomials.

EXERCISES

Factor the following expressions, and check your results:

1. $m^2-2ms-2ns+s^2+2mn+n^2.$

2. $y^2+4xy+4x^2+4xz+2yz+z^2.$

3. $m^3-t^3+3mt^2-3m^2t.$

4. $x^4+8x^2+24+\dfrac{32}{x^2}+\dfrac{16}{x^4}.$

5. $9a^2+4c^2-12ac+16bc-24ab+16b^2.$

6. $9m^4+30m^3+25m^2-12m^2n+4n^2-20mn.$

67. Factoring by means of the remainder theorem. In § 48 it was proved that if $Ax^n+Bx^{n-1}+\cdots+Hx+K$ is exactly divisible by $x-a$, then $Aa^n+Ba^{n-1}+\cdots+Ha+K=0$, and conversely; on this fact is based a simple method for finding binomial factors of a large number of algebraic expressions.

E.g., to ascertain whether $x-2$ is a factor of x^2-5x+6, it is only necessary to substitute 2 for x in x^2-5x+6, and observe whether or not the result is 0; this result is 0, and therefore $x-2$ is a factor of x^2-5x+6.

So, too, $x-6$ is a factor of $x^2-8x+12$ because $6^2-8\cdot 6+12=0$; and $x+1$, *i.e.*, $x-(-1)$, is a factor of x^2+7x+6 because $(-1)^2+7(-1)+6=0$.

Again, if $x-a$ is a divisor of $x^3-2x^2-9x+18$, then 18 is the product of a by the last quotient term; hence, in *seeking* this class of factors of $x^3-2x^2-9x+18$, only numbers which are factors of 18 need be tried in the place of a. The factors of 18 are: $+1, -1, +2, -2, +3, -3, +6, -6, +9, -9, +18$, and -18; if these numbers be substituted in turn for x in the given expression, it will be found that $+2$ is the first one that reduces that expression to 0, therefore neither $x-1$ nor $x+1$ are factors, but $x-2$ is a factor; further trial will show that $x-3$ and $x+3$ are also factors of the given expression.

When any factor of an expression has been discovered, by any process whatever, that factor may be divided out of the given expression, and the remaining factors may then be more easily found.

EXERCISES

1. If $x^4+6x^2-12x+5$ be divided by $x-a$, what will be the remainder? Without performing the division, find the remainder when the divisor is $x-2$ (cf. § 48); also when it is $x+1$, and when it is $x-1$. Which of these divisors is a factor of the given expression?

2. If the expression x^3-3x^2-x+3 has a factor of the form $x-a$, what are the four possible values of a? Find *all* such binomial factors of x^3-3x^2-x+3.

By the above method, find all the factors you can of the following expressions:

3. x^3-7x+6.

4. $x^3-9x^2+23x-15$.

5. $x^3+14x^2+35x+22$.

6. $x^3-11x^2+31x-21$.

7. $k^3+4k^2-11k-30$.

8. $w^4-15w^2+10w+24$.

9. $a^3+7a^2+2a-40$.

10. $c^3-5c^2-29c+105$.

11. $x^4-x^3-7x^2+x+6$.

12. $y^5-10y^4+40y^3-80y^2+80y-32$.

13. If $x-k$ is a factor of any given expression, what does the value of that expression become when $x=k$? Why? Prove that the converse of this is also true.

14. By means of the remainder theorem show that $a-b$, $b-c$, and $c-a$ are factors of $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

15. What is the remainder when $(2x-3a)^2+(3x-a)^3$ is divided by $x-a$? When $(x-y+z)^3-y^3+x^3$ is divided by $x-y$? by $x+y$?

16. Find the factors of $4x^3-4x^2-9x+9$.

Suggestion. $4x^3-4x^2-9x+9=4(x^3-x^2-\frac{9}{4}x+\frac{9}{4})$; now apply the above method to the expression within the parenthesis.

Find the factors of:

17. $4x^2-16x+15$.

18. $2y^3+5y^2-2y-5$.

19. What value of x will reduce to zero any expression which contains $2x-3$ as a factor? How then may the remainder theorem be used to detect the factor $2x-3$ in any given expression? Use this suggestion to solve Exs. 17 and 18.

20. What is the remainder when x^n-a^n is divided by $x-a$? Why? When x^n-a^n is divided by $x+a$ and n is an even positive integer?

21. Prove that $x-1$ is a factor of every expression of the form $Ax^n+Bx^{n-1}+Cx^{n-2}+\cdots+Hx+K=0$, in which the sum of the *positive* coefficients (among $A, B, C, \cdots K$) equals the sum of the negative coefficients. Compare Exs. 3, 4, and 6, above.

68. Binomial factors of $x^n \pm a^n$. The method of the preceding article may be used to find binomial factors of the expressions x^n-a^n and x^n+a^n, wherein x and a represent any numbers whatever, and n is a positive integer.

(i) Thus x^n-a^n is exactly divisible by $x-a$, whatever integer n may be, because if a be substituted for x, the expression x^n-a^n becomes a^n-a^n, *i.e.*, 0.

Hence, *the difference of like positive integral powers of two numbers is exactly divisible by the difference of the numbers.*

By actual division, it is found that

$$\frac{x^2-a^2}{x-a}=x+a; \quad \frac{x^3-a^3}{x-a}=x^2+xa+a^2; \quad \frac{x^4-a^4}{x-a}=x^3+x^2a+xa^2+a^3;$$

$$\frac{x^5-a^5}{x-a}=x^4+x^3a+x^2a^2+xa^3+a^4; \text{ etc.}$$

Binomials of the form x^n-a^n can always be separated into at least two factors, both of which may be written down by inspection; one of these factors is $x-a$ and the other is $x^{n-1}+x^{n-2}a+x^{n-3}a^2+\cdots+xa^{n-2}+a^{n-1}$; this last factor is homogeneous, of degree $n-1$, in the two numbers, and contains n terms, all of which are positive.

(ii) Again, $x + a$, *i.e.*, $x - (-a)$, is a factor of $x^n - a^n$ when n is *even*, because then $(-a)^n - a^n = a^n - a^n = 0$ (§ 18, note 2).

Hence, *the difference of like even positive powers of two numbers is exactly divisible by the sum of the numbers.*

By actual division, it is found that

$$\frac{s^2 - t^2}{s + t} = s - t; \quad \frac{s^4 - t^4}{s + t} = s^3 - s^2t + st^2 - t^3; \quad \frac{s^6 - t^6}{s + t} = s^5 - s^4t + s^3t^2 - s^2t^3 + st^4 - t^5; \text{ etc.}$$

The student may make a verbal statement of this case of factoring similar to the last paragraph in (i) above.

(iii) Again, $x + a$, *i.e.*, $x - (-a)$, is a factor of $x^n + a^n$ when n is *odd*, for in that case $(-a)^n + a^n = -a^n + a^n = 0$ (§ 18, note 2).

Hence, *the sum of like odd positive powers of two numbers is exactly divisible by the sum of these numbers.*

By actual division, it is found that

$$\frac{x^3 + y^3}{x + y} = x^2 - xy + y^2; \quad \frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4;$$

$$\frac{x^7 + y^7}{x + y} = x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6; \text{ etc.}$$

The student may formulate this principle into words, — see last paragraph in (i) above.

(iv) Finally, $x - a$ is never a factor of $x^n + a^n$; for if a be substituted for x in this expression it becomes $a^n + a^n$, which is not 0 either when n is even or when n is odd, and therefore $x^n + a^n$ is not exactly divisible by $x - a$ (§ 48).

Note. Principles (i) to (iv), above, may be briefly recapitulated thus:

$x^n - a^n$ is *always* divisible by $x - a$,
$x^n - a^n$ is divisible by $x + a$ only when n is *even*,
$x^n + a^n$ is divisible by $x + a$ only when n is *odd*,
$x^n + a^n$ is *never* divisible by $x - a$.

EXERCISES

1. Show by means of the remainder theorem that $x^5 - a^5$ is exactly divisible by $x - a$; also that $x^5 + a^5$ is exactly divisible by $x + a$.

2. Prove that $x - a$ is a factor of $x^n - a^n$ for every positive integral value of n.

3. Prove that $x + a$ is a factor of $x^n + a^n$ for odd positive integral values of n, and of $x^n - a^n$ for even positive integral values of n.

4. Prove that neither $x - a$ nor $x + a$ is a factor of $x^n + a^n$ when n is an even positive integer.

Write out the following quotients by inspection, and then verify them by actual division:

5. $\frac{x^2 - y^2}{x - y}$.

6. $\frac{x^3 - y^3}{x - y}$.

7. $\frac{a^4 - b^4}{a - b}$.

8. $\frac{u^8 - v^8}{u - v}$.

9. $\frac{v^4 - w^4}{v + w}$.

10. $\frac{m^6 - n^6}{m + n}$.

11. $\frac{u^8 - v^8}{u + v}$.

12. $\frac{x^3 + y^3}{x + y}$.

13. $\frac{x^5 + y^5}{x + y}$.

14. $\frac{m^7 + s^7}{m + s}$.

15. $\frac{a^9 + b^9}{a + b}$.

16. $\frac{(x^2)^5 + (y^2)^5}{x^2 + y^2}$.

17. $\frac{(2a)^4 - x^4}{2a - x}$.

18. $\frac{m^5 - 32}{m - 2}$.

19. $\frac{4k^2 - 9}{2k - 3}$.

20. $\frac{16p^4 - 81}{2p + 3}$.

21. $\frac{x^4 - y^4}{x^2 + y^2}$.

22. $\frac{81a^4 - 16}{3a + 2}$.

23. $\frac{64 - r^6}{r + 2}$.

24. $\frac{27x^3 + 64a^3}{3x + 4a}$.

25. $\frac{32x^5 + 1}{2x + 1}$.

26. $\frac{x^6 + y^6}{x^2 + y^2}$.

27. $\frac{a^{10} + b^{10}}{a^2 + b^2}$.

28. $\frac{32x^{10} + y^{15}}{2x^2 + y^3}$.

29. Compare the quotients in Exs. 5–15 with the corresponding powers of a binomial (§ 62), with reference to coefficients, exponents, signs, etc.

30. Of what is x^6 the square? Of what is it the cube?
Write $x^6 - y^6$ as the difference of two squares; of two cubes.
Is $x^2 - y^2$ a factor of $x^6 - y^6$? Why? Is $x^3 - y^3$? Is $x^3 + y^3$? Why?
Find the *prime* factors of $x^6 - y^6$.

31. When seeking the prime factors of $x^6 - y^6$ show that it is better not to divide out the factor $x - y$ at once, but rather to separate $x^6 - y^6$ first into the factors $x^3 - y^3$ and $x^3 + y^3$, and then to separate each of these factors further. Is a similar plan advisable in general, *e.g.*, with $x^8 - y^8$ and $p^{20} - q^{20}$?

32. Find the prime factors of $m^{12} - n^{12}$; compare Ex. 31.

33. Find the prime factors of $x^9 - y^9$; also of $64a^6 - 1$.

34. Prove that $p^n - r^n$ is exactly divisible by $p^2 - r^2$, if n is an even positive integer.

35. For what positive integral values of n between 1 and 9 has $x^n + y^n$ no binomial factor? Is $x^2 + y^2$ a factor of $x^8 + y^8$?

Resolve the following expressions into their prime factors:

36. $x^4 - y^4$.

37. $a^6 - b^6$.

38. $a^8 - b^8$.

39. $m^8 - 1$.

40. $a^{10}x^{10} - y^{10}$.

41. $p^9 + 1$.

42. $16\,a^4b^4 - 81\,x^4y^4$.

43. $a^{12}x^{13} - b^{12}xy^{12}$.

44. $3\,as^{12} - 3\,at^{12}$.

45. $x^9 + y^9$.

46. $64\,x^6 + y^6$.

69. Factoring by rearranging and grouping terms. A rearrangement and grouping of the terms of an expression will often reveal a factor which could not be easily seen before.

Ex. 1. Find the factors of $ax - 3\,by + bx - 3\,ay$.

SOLUTION.
$$\begin{aligned} ax - 3\,by + bx - 3\,ay &= ax + bx - 3\,by - 3\,ay \\ &= x(a + b) - 3\,y(a + b) \\ &= (a + b)(x - 3\,y). \end{aligned}$$

Ex. 2. Find the factors of $x(x + 4) - y(y + 4)$.

SOLUTION.
$$\begin{aligned} x(x + 4) - y(y + 4) &= x^2 + 4\,x - y^2 - 4\,y \\ &= x^2 - y^2 + 4(x - y) \\ &= (x - y)(x + y + 4). \end{aligned}$$

NOTE. The factor $x - y$ could also have been detected by means of § 67, because the given expression is zero when $x = y$.

EXERCISES

Find the factors of the following expressions:

3. $ax^3 + 1 + a + x$.

4. $a^2b^2 + a^2 + b^2 + 1$.

5. $ac + bd - ad - bc$.

6. $ac^2 + bd^2 - ad^2 - bc^2$.

7. $m^2 - n^2 - (m - n)^2$.

8. $x^3 + x^2 - 4\,x - 4$.

9. $5\,x^3 + 1 - x^2 - 5\,x$.

10. $a^2 - 9\,x^2 + 4\,c^2 - 4\,ac$.

SUGGESTION. The first, third, and fourth terms of the expression in Ex. 10 are together $(a - 2\,c)^2$, *i.e.*, the given expression equals $(a - 2\,c)^2 - 9\,x^2$, of which the factors are obvious.

11. $x^4 - xy^3 - ax^2 + ay^2$.

12. $1 + bx - (a^2 + ab)x^2$.

13. $a^2c^2 + acd + abc + bd$.

14. $x^4 - 4\,x^2y^2 + 2\,x^3 - 16\,y^3$.

15. $ab + bx^n - x^ny^m - ay^m$.

16. $3\,xy(x + y) + 16\,x^3 + 16\,y^3$.

17. $(p^2 - q^2)^2 - (p^2 - pq)^2$.

18. $(x + y)^2 + 12\,(x + y) - 85$.

19. $a^2x + abx + ac + b^2y + aby + bc$.

20. $(x^2 + 6\,x + 9)^2 - (x^2 + 5\,x + 6)^2$.

21. $x^3 + (a + b - c)x^2 + (ab - ac - bc)x - abc$.

22. $m^2 + n^2 + m + mn + n + mn.$

23. $14\,a(x - y) + 49\,a^2 + (x - y)^2.$

24. $x^2 - a^2 + y^2 - b^2 + 2\,xy - 2\,ab.$

25. $h^2 - m^2 + 10\,m + k^2 - 25 - 2\,hk.$

26. $9\,a^2 + 12\,ab + 4\,b^2 - 15\,a - 10\,b - 24.$

27. $a^2 + b^2 + c^2 + 2\,(ab + ac + bc) + 5\,(a + b + c).$

28. $x^2 + y^2 + z^2 + 2\,(xy + xz + yz) + 5\,(x + y + z) + 6.$

29. $4\,x^2 + 10\,x + 6 - 5\,a - 4\,ax + a^2.$

70. Factoring by means of other devices. It often happens that the factors of an expression will become apparent by adding a certain number to, and subtracting the same number from, the given expression; this, of course, leaves the value of the expression unchanged.

Ex. 1. Find the factors of $x^4 + x^2 + 1$.

Solution. If the second term in this expression were $2\,x^2$ instead of x^2, then [§ 66 (i)] the expression could be written $(x^2 + 1)^2$; this suggests that x^2 be both added and subtracted, which gives

$$\begin{aligned} x^4 + x^2 + 1 &= x^4 + 2\,x^2 + 1 - x^2 \\ &= (x^2 + 1)^2 - x^2 \\ &= (x^2 + 1 + x)\,(x^2 + 1 - x), \qquad [\S\ 66\ (ii) \end{aligned}$$

i.e.,

$$x^4 + x^2 + 1 = (x^2 + x + 1)\,(x^2 - x + 1).$$

Ex. 2. Find the factors of $a^4 + a^2b^2 + b^4$.

Solution. This expression may be treated in the same way as Ex. 1, thus:

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2\,a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + ab + b^2)\,(a^2 - ab + b^2). \end{aligned}$$

Ex. 3. Find the factors of $x^2 - 4\,x - 32$.

Solution. Here the first two terms, plus 4, are an exact square, and this suggests the following arrangement:

$$\begin{aligned} x^2 - 4\,x - 32 &= x^2 - 4\,x + 4 - 32 - 4 \\ &= (x - 2)^2 - 36 \\ &= (x - 2 + 6)(x - 2 - 6), \end{aligned}$$

i.e.,

$$x^2 - 4\,x - 32 = (x + 4)(x - 8).$$

Note. Observe the superiority of the method of Ex. 3 over the method of § 66 (iii) for factoring the same expression.

EXERCISES

Factor the following expressions:

4. $m^4 + m^2n^2 + n^4$.

5. $p^4 + 4q^4$.

6. $x^2 + 6x + 5$.

7. $9s^2 + 30st + 16t^2$.

8. $x^4 + a^2x^2 + a^4$.

9. $x^8 + x^4y^4 + y^8$.

10. $4a^8 - 21a^4b^4 + 9b^8$.

11. $a^4b^4 + a^2b^2c^2d^2 + c^4d^4$.

12. $9x^4 + 8x^2y^2 + 4y^4$.

13. $5x^4 - 70x^2y^2 + 5y^4$.

14. $9a^4 + 26a^2b^2 + 25b^4$.

15. $a^2 + 2ab - d^2 - 2bd$.

16. $x^4 + 64y^4$.

17. $4a^4 + 81$.

18. $x^5y^2 + 4xy^2$.

19. $m^5 + 4mn^4$.

20. $a^4 + 8a^2 - 128$.

21. $5nx^4 - 70nx^2 + 200n$.

22. What must be added to $x^4 + 3x^2 + 4$ to make it an exact square? What must then be subtracted to leave the result unchanged? Factor this expression.

23. What must be added to $x^4 - 3x^2 + 4$ to make it an exact square, and what must then be subtracted so as not to change the value of the given expression? If the given expression is written in the form $(x^2-2)^2 + x^2$, can it be factored by any of the preceding methods (cf. § 68)?

24. Can the sum of two squares be factored (cf. § 68)? Is not the expression in Ex. 5 above the sum of two squares? Could this expression be written $(p^2 + 2q^2)^2 - (2pq)^2$?

25. Factor the expression $3x^2 + x - 10$ [cf. Ex. 1, § 66 (iv)].

SOLUTION.

$$\begin{aligned}3x^2+x-10 &= \frac{12(3x^2+x-10)}{12}\\ &= \frac{36x^2+12x-120}{12}\\ &= \frac{36x^2+12x+1-121}{12}\\ &= \frac{(6x+1)^2-(11)^2}{12}\\ &= \frac{(6x+12)(6x-10)}{12}\\ &= (x+2)(3x-5).\end{aligned}$$

NOTE. The above method is more direct than that given in § 66 (iv); it consists in multiplying the given expression by such a number as will make *its highest term an exact square*, and the next highest term *exactly divisible by twice the square root of the highest term*, then factoring the resulting expression as explained in § 70, and finally dividing the whole by the number first used as a multiplier, so as not to change the value of the expression.

Factor the following expressions:

26. $5m^2 - 2m - 3$.

27. $6a^2 - 11a - 35$.

28. $18x^2 - 3x - 36$.

29. $6R^2 - 2R - 20$.

30. $8A^2 + 23AB - 3B^2$.

31. $4N^2 + 16NM^3 + 15M^6$.

32. $2x^2 + 5xy + 2y^2$.

33. $3x^2 - 10xy + 3y^2$.

71. General plan for factoring a polynomial. Based upon §§ 65–70, the following suggestions for separating a polynomial into its prime factors may be made. By inspection find the monomial factors of the given polynomial, if there are any such, and then write this polynomial as the indicated product of the monomial and the corresponding polynomial factor; then, by rearrangement of the terms, or by some one of the other methods given above, separate this polynomial factor into two factors, and replace it by their indicated product; then further separate each of these factors into two others, if possible, and so continue until all of the factors are prime.

EXERCISES

Factor the following expressions:

1. $m^2x^5 + m^2y^5$.

2. $c^2 - 5c - 14$.

3. $21m^2 - ma - 10a^2$.

4. $x^2 + ax - ay - yx$.

5. $x^4 - 8x^8 + 15x^2$.

6. $m^4n^4 - 5m^2n^2 + 4$.

7. $25a^2 + y^2 + 10x^2 + 10ay - 35ax - 7xy$.

8. $m^2 + 6m - x^2 + 9 - 4xy - 4y^2$.

9. $2(a^2b^2 - a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$.

10. $x^{12} - y^{12}$.

11. $a^{12}x^{12}y^{12} + r^{12}s^{12}$.

12. $4ax^2 + 4ay^2$.

13. $a^2b^2x^2 + 4ab^2xy + 4b^2y^2$.

14. $32a - ax^5$.

15. $a^9 + 4a$.

16. $a^{15} + 1$.

17. $(a^2+5a+4)^2-(a^2-5a-6)^2$.

18. $x^{2n-2} + b^2y^2 + 2x^{n-1}by$.

19. $x^6 - y^6 - 3x^4y^2 + 3x^2y^4$.

20. $x^3y - 15x^2y + 38xy - 24y$.

21. $b^8 + b^4y^2 + y^4$.

22. $m^8n^3 + 2m^6n^7r^2s^8 + m^4n^{11}r^4s^6$.

23. $x^2 + 9y^2 + 25z^2 - 6xy - 10xz + 30yz$.

24. $x^5 + 5x^4az^2 + 10x^3a^2z^4 + 10x^2a^8z^6 + 5xa^4z^8 + a^5z^{10}$.

25. $a^2 - 2ab + b^2 - 2ac + 2bc + c^2 - 2ad + 2bd + 2cd + d^2$.

72. Solving equations by factoring. If all the terms of an equation be transposed to its first member, factoring that member will always simplify the finding of the roots of the given equation; this is illustrated by the following examples.

Ex. 1. Given $x^2 - 5x + 6 = 0$; to find its roots, *i.e.*, to find those values of x for which this equation is satisfied (cf. § 23).

SOLUTION. By § 66 (iii) the first member of this equation is the product of $x - 3$ and $x - 2$, and the given equation may, therefore, be written

$$(x - 2)(x - 3) = 0.$$

It is manifest, moreover, that a product is 0 if, and only if, at least one of its factors is 0; hence $(x - 2)(x - 3) = 0$ if, and only if,

$$x - 2 = 0 \text{ or } x - 3 = 0,$$

i.e., if, and only if, $x = 2$ or $x = 3$;

hence the roots of the given equation are 2 and 3.

Ex. 2. Given $x^2 = 3x + 4$; to find its roots.

SOLUTION. On transposing, this equation becomes

$$x^2 - 3x - 4 = 0,$$

i.e., $(x - 4)(x + 1) = 0$; [§ 66 (iii)

hence either $x - 4 = 0$ or $x + 1 = 0$, *i.e.*, $x = 4$ or $x = -1$,

and therefore the roots are 4 and -1.

Ex. 3. Solve the equation $6x^2 - 11x = 35$.

SOLUTION. Transposing and factoring [§ 66 (iv)], this equation may be written

$$(3x + 5)(2x - 7) = 0;$$

hence $3x + 5 = 0$ or $2x - 7 = 0$, *i.e.*, $x = -\frac{5}{3}$ or $x = \frac{7}{2}$,

and therefore the roots are $-\frac{5}{3}$ and $\frac{7}{2}$.

NOTE. Since the roots of the equation $(x - a)(x - b) = 0$ are a and b, therefore an equation which shall have any given numbers as roots may be immediately written down; thus the equation whose roots are 3 and 8 is

$$(x - 3)(x - 8) = 0, \text{ i.e., } x^2 - 11x + 24 = 0.$$

Similarly, the equation whose roots are 2, -1, and 5 is

$$(x - 2)(x + 1)(x - 5) = 0, \text{ i.e., } x^3 - 6x^2 + 3x + 10 = 0.$$

EXERCISES

4. What is meant by a root of an equation? May an equation have more than one root?

5. Find the roots of $x^2 - 4x - 21 = 0$. Verify their correctness by substituting them, in turn, for x in the given equation.

6. Solve the equation $y^2 - 6y + 5 = 0$, and verify the solution.

7. What values of x will satisfy the equation $(x-2)(x-3) = 0$? If $x \neq 2$,* will $x - 2$ be 0? If $x \neq 3$, will $x - 3$ be 0? If, then, x is neither 2 nor 3, can the given equation be satisfied? This equation has then how many roots?

8. Write the equation whose roots are 5 and 1. Also one whose roots are 3, 2, and 7.

9. Write the equation whose roots are: 1 and -5; $\frac{2}{3}$ and 6; a and b; 3, -1, and 5; a, $-a$, and $2a$; 1, 2, 3, and 4.

Solve the following equations, and verify the correctness in each case:

10. $x^2 - 2x = 15$.

11. $6x^2 - x - 1 = 0$.

12. $3y^2 + y = 10$.

13. $8y^2 + 15 = -26y$.

14. $5x^2 - 7x = 0$.

15. $12z^2 = 4z$.

16. $2x^3 + 5x^2 = 2x + 5$.

17. $x^2 - 4 = 0$.

18. $x^4 - 13x^2 + 36 = 0$.

19. $x^3 + x^2 - x = 1$.

20. $(x-1)(x+1)(x-2) = 0$.

21. Can the roots of the equation in Ex. 20 be determined by mere inspection? Can the roots of the equation

$$(3x-2)(x+1) = 2$$

be so determined? What are these roots?

22. Write out a rule for solving such equations as those given in the above examples.

PROBLEMS

By the method of § 26 † solve the following problems:

1. If the product of the two remainders obtained by first subtracting 3 from a certain number, and then 5 from the same number, is 24, what is that number? How many solutions has this problem? Explain.

2. If the sum of two numbers is 12 and one of these numbers is x, what is the other number? Find two numbers whose sum is 12 and of which the square of the larger is 1 less than 10 times the smaller.

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* The expression $x \neq 2$ is read "x is not equal to 2."

† § 26 should now be re-read.

3. The difference between two numbers is 2, and the sum of their squares is 130. What are these numbers?

4. One side of a rectangle is 3 feet longer than the other. If the longer side be diminished by 1 foot and the shorter side increased by 1 foot, the area of the rectangle will then be 30 square feet. How long is this rectangle?

5. A rectangular orchard contains 2800 trees, and the number of trees in a row is 10 less than twice the number of rows. How many trees are there in a row?

6. If the dimensions of a certain rectangular box, which contains 120 cubic inches, were increased by 2, 3, and 4 inches, respectively, the new box would be cubical in form. Find the dimensions of this box.

7. How may $128 be divided equally among a certain number of persons so that the number of dollars received by each person shall exceed the number of persons by 8?

8. A certain club banquet is to cost $75, and it is found that this will require each member of the club to pay 50 cents more than $\frac{1}{10}$ as many dollars as there are members in the club. How much must each pay, and how many members are there in the club?

CHAPTER VIII

HIGHEST COMMON FACTORS — LOWEST COMMON MULTIPLES

I. HIGHEST COMMON FACTORS

73. Definitions. A factor of each of two or more numbers or algebraic expressions is called a **common factor** of these numbers or expressions; the **highest common factor** — usually designated by the letters H. C. F. — of two or more numbers or expressions is the product of all the prime factors (§ 63) that are common to these numbers or expressions.

E.g., the H. C. F. of $12\,a^3b^2cx^4$ and $6\,ab^3x^2y$ is $6\,ab^2x^2$, because when this factor is removed from the given expressions they have no *common* factor left; $6\,ab^2x^2$ is then the product of all the common prime factors of the given expressions.

Similarly, $3\,a(x-1)^2(x-2)$ is the H. C. F. of $6\,a^2x(x-1)^4(x-2)(a-y)$ and $15\,ab(x-y)(x-1)^2(x-2)^3$.

NOTE. It is evident from the above definition that no common factor of two or more expressions is of higher degree in any letter than their H. C. F.

Two or more numbers or algebraic expressions which have no common factor except unity are said to be **prime to each other.**

74. Highest common factor of two or more monomials. From the definition and illustration given above, it is clear that the H. C. F. of two or more monomials can be found by inspection.

E.g., to find the H. C. F. of $12\,a^3b^2xy$, $6\,ab^3x^2$, and $9\,ab^2x^4$.

Inspection shows that these monomials have the prime factors 3, a, b, b, and x in common, and that, when these are removed, there are no other factors common to the given monomials; hence their H. C. F. is $3 \cdot a \cdot b \cdot b \cdot x$, *i.e.*, $3\,ab^2x$.

A rule for writing down the H. C. F. of several monomial expressions may be formulated thus: *to the H. C. F. of the numerical coefficients annex those letters that are found in each one of the given monomials, and give to each of these letters the lowest exponent which it has in any of the monomials.*

EXERCISES

Find the H. C. F. of the following sets of monomials:

1. $3a^2b^3cd$ and $6ab^4c^2d^3$.

2. $15x^3y^2z$, $24x^2y^4z^3$, and $18x^4y^4$.

3. $16x^2y^3z^3m^3$, $169y^4z^6m$, and $39x^7y^8m^4$.

4. $2041a^4b^6c^7$ and $8476a^5bc^4d$.

5. $292x^5y^7z^3$, $1022x^3y^2z^5$, and $1095x^4y^6z^4$.

6. $364x^{2m}y^{3n}z^{4r}$ and $455x^my^{2n}z^{3r}$.

7. Is the H. C. F., as above defined, the same as the *greatest* common divisor (G. C. D.) in the arithmetical sense? What is the H. C. F. of a^3x^2y and a^4xy^3? Is this H. C. F also the G. C. D. when $a = \frac{1}{2}$, $x = 6$, and $y = 4$?

NOTE. Observe that *highest* refers to *degree*, while *greatest* refers to *value*. If c is any proper fraction, then $c > c^2 > c^3 \cdots$, but c^5 is always higher than c^2.

Find the H. C. F. of the following sets of expressions:

8. $24a^3x(y-z)^2(w+3)$ and $56a^2bx^3(y-z)^4(w+3)^2$.

9. $473h^2s^3v(x-1)^2(3-2y)^3$ and $319a^4hs^4(x-1)(x-2)^2(3-2y)^4$.

75. H. C. F. of two or more polynomials whose prime factors are known. The H. C. F. of several polynomials whose prime factors are known may be written down by inspection as is done for monomials in § 74.

EXERCISES

Find the H. C. F. of each of the following sets of expressions:

1. $4(a+b)^3(a-b)$ and $b(a+b)^2(a-b)^2$.

2. $6(a+b)^2(a-b)^2$ and $15(a-b)^2(a+b)$.

3. $4ax^2 - 20ax + 24a$ and $6abx^2 + 24abx - 126ab$.

SOLUTION

Since $4ax^2 - 20ax + 24a = 4a(x^2 - 5x + 6) = 2 \cdot 2a(x-2)(x-3)$,
and $6abx^2 + 24abx - 126ab = 6ab(x^2 + 4x - 21) = 3 \cdot 2ab(x+7)(x-3)$,
therefore the H. C. F. is $2a(x-3)$.

4. $a^2 - b^2$, $a(a+b)$, and $a^2 + 2ab + b^2$.

5. $5 - 19x - 4x^2$ and $2x^2 + 7x - 15$.

6. x^2+5x+6, $x^2+7x+10$, and $x^2+12x+20$.

7. a^2-a-12 and $a^2-4a-21$.

8. $15(yz-z)$ and $35(y^4z-yz)$.

9. $x^4+x^2y^2+y^4$ and $(x^2-xy+y^2)^2$.

10. Of what is the H. C. F. of two or more expressions composed? State a rule for finding the H. C. F. of two or more expressions which are already separated into their prime factors, or which may be easily so separated.

11. What is the H. C. F. of $x^2(x-1)^2$ and $x(x^2-1)$? Is this also the G. C. D. of these expressions for all values of x? Try $x=3$, and also $x=4$. Compare Ex. 7, § 74.

Find the H. C. F. of the following sets of expressions:

12. $4ab^2x^2+12ab^2x-40ab^2$, $6a^2x^2y-6a^2xy-12a^2y$, and $18a^2mx^2-54a^2mx+36a^2m$.

13. $15a^4x^2+15a^2b^4x^2+15b^8x^2$ and $3(a^2-ab^2+b^4)$.

14. x^3+a^3 and $3a^3+3a^2x-5ax^2-5x^3$.

15. $2x^2-x-3$ and $2x^3+11x^2-x-30$.*

16. $(x+3)(x^2-4)$, $x^4+4x^3+2x^2-x+6$, and $2x^3+9x^2+7x-6$.

17. a^3+1, $3a^3-4a^2+4a-1$, and $2a^3+a^2-a+3$.

76. H. C. F. of two polynomials neither of which can be readily factored. Although it is only in exceptional cases that the factors of a polynomial can be found (such cases were examined in Chapter VII), yet the *common* factors of any two given polynomials can *always* be found.

The method for finding the H. C. F. of two polynomials neither of which can be readily factored, is precisely the same as that used in arithmetic for finding the G. C. D. of two numbers, neither of which can be easily factored.

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* Since the second of these expressions is not easily factored, — although the first is, — find by trial whether the factors of the first expression are also factors of the second.

This method may be employed whenever any *one* of a given set of expressions is easily separated into its prime factors.

To illustrate, let it be required to find the G. C. D. of 1183 and 2639.

$$\begin{array}{l}
1183)2639(2 \\
\qquad\;\underline{2366} \\
\qquad\;\;273)1183(4 \\
\qquad\qquad\quad\;\underline{1092} \\
\qquad\qquad\qquad\;91)273(3 \\
\qquad\qquad\qquad\qquad\;\underline{273} \\
\qquad\qquad\qquad\qquad\quad\;0
\end{array}$$

The last divisor, 91, is the G. C. D. of the given numbers. This work may be more compactly arranged thus:

$$\begin{array}{r|c|r}
 & \text{QUOTIENTS} & \\
1183 & 2 & 2639 \\
 & & \underline{2366} \\
\underline{1092} & 4 & 273 \\
91 & 3 & \underline{273} \\
 & & 0
\end{array}$$

Similarly, the H. C. F. of $x^4+3x^3+2x^2-3x-3$ and x^3+x^2-2 may be found thus:

$$\begin{array}{r|c|l}
 & \text{QUOTIENTS} & \\
x^4+3x^3+2x^2-3x-3 & x+2 & x^3+x^2-2 \\
\underline{x^4+x^3-2x} & & \underline{x^3-x^2} \\
2x^3+2x^2-x-3 & & 2x^2-2 \\
\underline{2x^3+2x^2-4} & & \underline{2x^2-2x} \\
-x+1 & -x^2-2x-2 & 2x-2 \\
 & & \underline{2x-2} \\
 & & 0
\end{array}$$

and $-x+1$, which is the last divisor, is the H. C. F. of the given polynomials.*

The procedure illustrated above may be formulated in words thus:

Arrange the given polynomials according to the descending powers of some common letter, and divide the higher expression by the lower, continuing the division until the remainder is of lower degree than the divisor; then using this remainder as a divisor, with the preceding divisor as a dividend (and with the same letter of arrangement), divide as before; continue this process until the remainder is either zero, or free from the letter of arrangement:—if it is zero, the last divisor is the H. C. F. sought; and (cf. § 77) *if it is free from the letter of arrangement, the given expressions have no common factor containing that letter.*

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* The H. C. F. of these polynomials may also be regarded as $x-1$. Why?

EXERCISES

By the above method, find the H. C. F. of the following pairs of expressions:

1. $x^2 + 5x + 6$ and $4x^3 + 21x^2 + 30x + 8$.

2. $12x^4 - 8x^3 - 55x^2 - 2x + 5$ and $6x^3 - x^2 - 29x - 15$.

3. $6a^2 - 13a - 5$ and $18a^3 - 51a^2 + 13a + 5$.

4. $5n^4 - 10n^3 + 11n^2 - 6n + 1$ and $10n^5 - 5n^4 - 7n^3 + 19n^2 - 14n + 2$.

77. Fundamental principle. The success of the method employed in § 76 for finding the H. C. F., whether in arithmetic or algebra, is due to the following principle:

If an integral algebraic expression be divided by another such expression which is of the same or of a lower degree in the letter of arrangement, and if there be a remainder, then the H. C. F. of this remainder and the divisor is also the H. C. F. of the two given expressions.*

To prove the correctness of this principle, let E_1 and E_2 represent any two given integral expressions, and let the degree of E_2, in the letter of arrangement, be at least as low as that of E_1; also let q_1 and R_1 represent, respectively, the quotient and remainder when E_1 is divided by E_2; then (§ 47, Ex. 11),

$$E_1 = q_1E_2 + R_1, \tag{1}$$

whence,

$$R_1 = E_1 - q_1E_2. \tag{2}$$

Now since any factor of *each term* of an expression is a factor of the whole expression, therefore any factor common to E_2 and R_1 is also a factor of $q_1E_2 + R_1$, and therefore, by equation (1), of E_1; *i.e.*, all the factors common to R_1 and E_2 are also factors of E_1, and therefore common to E_2 and E_1.

But, by exactly the same reasoning, equation (2) shows that all the factors common to E_1 and E_2 are also common to E_2 and R_1;

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* "Integral expression" as here used includes arithmetical numbers also.

i.e., the factors common to R_1 and E_2 are precisely those which are common to E_1 and E_2. Hence the H. C. F. of R_1 and E_2 is also the H. C. F. of E_1 and E_2.

From the proof just given it follows: (1) that if E_2 be now divided by R_1, giving a remainder R_2, then the H. C. F. of R_2 and R_1 is also the H. C. F. of E_2 and R_1, and therefore of E_1 and E_2. So, too, if R_1 be divided by R_2, giving a remainder R_3, then the H. C. F. of R_2 and R_3 is also the H. C. F. of E_1 and E_2, and so on; *i.e.*, the H. C. F. of E_1 and E_2 is also the H. C. F. of any two consecutive remainders in this succession of divisions.

But these successive remainders are of lower and lower degrees,* hence a remainder R_n which is either 0, or free from the letter of arrangement, must finally be reached; if $R_n = 0$, then R_{n-1} is the H. C. F. of R_{n-1} and R_{n-2}, and therefore of E_1 and E_2, but if R_n is merely free from the letter of arrangement, then R_{n-1} and R_{n-2} can have no common factor containing this letter, and therefore E_1 and E_2 have no common factor which contains that letter.

Note. It follows directly from the definition (§ 73) that *the H. C. F. of two entire expressions is not altered by multiplying or dividing either of them by any number which is not a factor of the other.* By introducing and suppressing suitable factors during the divisions above described, fractional coefficients, which might otherwise arise, may always be avoided.

To illustrate, let it be required to find the H. C. F. of $3x^3+8x^2+3x-2$ and x^3-2x^2+x+4.

Since these expressions are of the same degree, either one may be used as divisor; the work may be arranged thus:

$$\begin{array}{r|c|r}
3x^3+8x^2+3x-\ \ 2 & 3 & x^3-2x^2+x+4 \\
\underline{3x^3-6x^2+3x+12} & & \underline{x^3 \qquad\qquad -x \qquad\quad} \\
14)\underline{14x^2-14} & & -2x^2+2x+4 \\
x^2-1 & x-2 & \underline{-2x^2 \qquad\quad +2} \\
\underline{x^2+x} & & 2x+2(2 \\
-x-1 & x-1 & x+1 \\
\underline{-x-1} & & \\
0 & &
\end{array}$$

Before beginning the second division the factor 14 is suppressed (see note above), and later 2 is also suppressed; fractional coefficients are thus avoided.

and $x+1$, which is the last divisor, is the H. C. F. of the given expressions.

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* If E_1 and E_2 represent arithmetical numbers, then R_1, R_2, and R_3, ⋯, represent smaller and smaller numbers.

As a further illustration, let us find the H. C. F. of

$$x^4+4x^3+2x^2-x+6 \text{ and } 2x^3+9x^2+7x-6.$$

$$\begin{array}{r|c|r}
x^4+4x^3+2x^2-x+6 & x+1 & 2x^3+9x^2+7x-6 \\
\underline{2} & & \underline{2x^3+10x^2+12x} \\
2x^4+8x^3+4x^2-2x+12 & & -x^2-5x-6 \\
\underline{2x^4+9x^3+7x^2-6x} & & \underline{-x^2-5x-6} \\
-x^3-3x^2+4x+12 & & 0 \\
\underline{-2} & & \\
2x^3+6x^2-8x-24 & & \\
\underline{2x^3+9x^2+7x-\ \ 6} & & \\
-3)\underline{-3x^2-15x-18} & & \\
x^2+5x+6 & 2x-1 &
\end{array}$$

Before beginning the division the factor 2 is introduced so as to avoid fractional coefficients in the quotient (cf. note above); later -2 is introduced for the same purpose; and finally -3 is rejected.

x^2+5x+6, which is the last divisor, is the H. C. F. of the given expressions.

EXERCISES

By the above method find the H. C. F. of the following pairs of expressions:

1. x^3-3x^2+3x-1 and $x^4-2x^3+2x^2-2x+1$.

2. $8x^3-22x^2+17x-3$ and $6x^3-17x^2+14x-3$.

3. $x^5-4x^4+5x^3-3x^2+3x-2$ and $2x^3-5x^2+x+2$.

4. $x^6-4x^4+5x^2-2$ and $3x^4+5x+2$.

5. $x^5-2x^4-2x^3-11x^2-x-15$ and $2x^5-7x^4+4x^3-15x^2+x-10$.

6. $x^6+x^4+x^3-x-2$ and $3x^6+x^5-x^2-1$.

7. a^3+3a^2-2a-6 and $4a^2-a+a^5+4a^4-12+4a^3$.

8. $1-4m^3+3m^4$ and $1-5m^3+4m^4+m-m^2$.

9. $x^5-3x^4-3x^3-15-19x$ and $3x^4-3x^3+x^5-15+9x^2-x$.

10. What is meant by the H. C. F. of two expressions E_1 and E_2? If a is not a factor of E_1, how does the H. C. F. of E_1 and $a \cdot E_2$ compare with the H. C. F. of E_1 and E_2? Why? Compare § 77, note.

11. If a is a factor of E_1, but not of E_2, how does the H. C. F. of E_1 and $a \cdot E_2$ compare with the H. C. F. of E_1 and E_2? In introducing and suppressing factors during the process of division (§ 77), what special precaution must be exercised, and why?

12. Suppose that, at some stage of the work in an exercise like those above, the divisor is $2x^2-4x+2$, and the dividend x^3-3x^2+3x+1; what would be the effect on the final result if the factor 2 were introduced into the dividend to avoid fractional coefficients? What should be done in this case instead of introducing the factor 2? Why?

13. Show that every factor common to A and B is also common to $A-B$ and $A+B$; and also to $mA+nB$ and $mA-nB$. Is the H. C. F. of A and B necessarily the H. C. F. of $A-B$ and $A+B$?

78. Supplementary to §§ 76 and 77. (i) *H. C. F. of polynomials which contain monomial factors.* The problem of finding the H. C. F. of a pair of polynomials, either of which contains monomial factors, is usually much simplified by setting aside these monomial factors before the division process is begun. Factors which are *common* to the given polynomials must, of course, be reserved as factors of their H. C. F.; all others may be rejected.

Thus, to find the H. C. F. of

$$6x^5+18x^4+12x^3-18x^2-18x \text{ and } 3ax^4+3ax^3-6ax,$$

remove the monomial factors $6x$ and $3ax$ from the given expressions, and the remaining polynomial factors are, respectively, $x^4+3x^3+2x^2-3x-3$ and x^3+x^2-2; the H. C. F. of the monomial factors is $3x$, and the H. C. F. of the polynomial factors is $x-1$ (see illustrative example, § 76); hence the H. C. F. of the given polynomials is $3x(x-1)$.

(ii) *H. C. F. of polynomials which involve several letters.* Although the examples given in § 77 involve only one letter, yet it should be especially observed that the demonstration there given applies to expressions involving any number of letters.

Thus, if the given expressions involve several letters, then, to find whether they have a common factor containing any particular one of these letters, they need only be arranged according to the descending powers of that letter, and divided as above described. If, therefore, the given expressions be successively arranged according to each of the several letters which they have in common, and divided as above, then *all* their common factors (*i.e.*, their H. C. F.) will be found.

Manifestly, however, any common factor which contains two or more letters will be found when the given expressions are arranged according to any one of these letters.

(iii) *H. C. F. of three or more polynomials.* Since the H. C. F. of three polynomials is a factor of each of them, it is also a factor of the H. C. F. of any two of them; therefore the H. C. F. of three polynomials is found by first finding the H. C. F. of any two of them, and then the H. C. F. of that result and the third polynomial. By continuing this process the H. C. F. of any number of polynomials may be found.

EXERCISES

Find the H. C. F. of:

1. $21\,ax - 17\,ax^2 - 5\,ax^3 + ax^4$ and $5\,ax^3 - 34\,ax^2 - 7\,ax$.

2. $7\,m^2x^3 - 49\,m^2x + 42\,m^2$ and $14\,a^2mx^3 + 14\,a^2mx^2 - 56\,a^2mx - 56\,a^2m$.

3. $48\,s^3tx^4 - 162\,s^3tx^2 + 54\,s^3t$ and $18\,s^2t^2u - 9\,s^2t^2ux - 48\,s^2t^2ux^2 + 24\,s^2t^2ux^3$.

4. $6\,cx^3(1 + y^2) - 18\,cx^3z + 2\,cy^3 - 4\,cy^2z + 12\,cz^2 - 2\,cz(3\,y + 2) + 2\,cy$ and $2\,ay^4 + 2\,ax^2(y^2 - 3\,z) - 6\,ay^2z + 2\,a(x^2 - y^2) + 4\,a(3\,z - 1)$.

5. $4\,x^4 - 12\,x^3y + 5\,x^2y^2 + 12\,xy^3 - 9\,y^4$ and $12\,x^4 - 36\,x^3y + 11\,x^2y^2 + 48\,xy^3 - 36\,y^4$.

6. $mn(x^2 + y^2) + xy(m^2 + n^2)$ and $mn(x^3 + y^3) + xy(m^2y + n^2x)$.

7. $3\,ax^2 - 6\,a^2x + 9\,a^3 - 3\,x^2 + 6\,ax - 9\,a^2$ and $6\,a^2x^2 + 24\,a^3x + 6\,a^4 - 6\,x^2 - 24\,ax - 6\,a^2$.

8. Show that the proof given in § 77 applies to expressions containing any number of letters.

9. Explain fully the method of finding the H. C. F. of more than two expressions.

10. Why must the H. C. F. of any number of expressions be a factor of the H. C. F. of any two of these expressions? Must it be the H. C. F. itself of any two of the given expressions? Explain.

Find the H. C. F. of:

11. $a^4 + 4\,a^3 + 4\,a^2$, $a^3b - 4\,ab$, and $a^4b + 5\,a^3b + 6\,a^2b$.

12. $x^3 - 6\,x^2 + 11\,x - 6$, $x^3 - 9\,x^2 + 26\,x - 24$, and $x^3 - 8\,x^2 + 19\,x - 12$.

13. $a^3 + a^2x - 2\,x^3$, $a^3 + 3\,a^2x + 4\,ax^2 + 2\,x^3$, and $2\,a^3 + 3\,a^2x + 2\,ax^2 - 2\,x^3$.

14. $ax + b^2x + c^3x - acy - b^2cy - c^4y$, $a^2 + 2\,ab + ab^2 + 2\,b^3 + ac^3 + 2\,bc^3$, and $2\,a^2 + 2\,ab^2 + c^3b + 2\,ac^3 + b^3 + ab$.

79.* **Other important consequences of § 77.** Some further important conclusions may be easily drawn from such a series of divisions as that described in §§ 76 and 77; thus, if M and N are any two integers, of which M is the greater, and if M be divided by N, giving a quotient Q_1 and a remainder R_1, and if N be then divided by R_1, giving a quotient Q_2 and a remainder R_2, and so on, — subsequent quotients and remainders, all of which are, of course,

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* This article may be omitted on a first reading.

integers, being designated by Q_3, Q_4, Q_5, $\cdots$, and R_3, R_4, R_5, $\cdots$, respectively, — then (§ 47, Ex. 11)

$$M=Q_1N+R_1,\ \ N=Q_2R_1+R_2,\ \ R_1=Q_3R_2+R_3,\ \ R_2=Q_4R_3+R_4, \text{ etc.}$$

From this series of equations it is easy to express the several remainders R_1, R_2, R_3, $\cdots$ in terms of M, N, and the quotients Q_1, Q_2, Q_3, $\cdots$.

Thus, by transposing, the first equation becomes $R_1=M-Q_1N$; transposing in the second equation, and then substituting this value of R_1, gives

$$R_2=N-Q_2R_1=N-Q_2(M-Q_1N)=-Q_2M+(1+Q_1Q_2)N;$$

similarly, from the third equation,

$$\begin{aligned}R_3=R_1-Q_3R_2&=(M-Q_1N)-Q_3\{(1+Q_1Q_2)N-Q_2M\}\\&=(1+Q_2Q_3)M-(Q_1+Q_3+Q_1Q_2Q_3)N;\end{aligned}$$

and so on for the later remainders; *i.e.*, *the successive remainders may each be expressed in the form* $aM+bN$, *wherein* a *and* b *are integers* (one positive and the other negative), *which involve the successive quotients, but not the given numbers, nor the remainders.*

Again, if M and N are prime to each other, then (§ 77) the last remainder is 1, and therefore, by what has just been said, two integers a and b can be found such that

$$aM+bN=1.$$

From this last equation it is easy to establish the following important principle: *if* M *is a factor of* NL, *but is prime to* N, *then it is a factor of* L.

To prove this it is only necessary to multiply the above equation by L; this gives

$$aML+bNL=L,$$

wherein the first member is manifestly divisible by M (M being a factor of NL by hypothesis); therefore the second member, viz., L, is also divisible by M, which was to be proved.

EXERCISES

The following direct consequences of the principle just now established may be proved by the student:

1. If M is prime to N and also to L, then it is prime to the product NL.

2. If M is prime to $N, L, P, \cdots$, then it is prime to the product $NLP \cdots$.

3. A number can be separated into but one set of prime factors.

4. If M is a prime to N, then it is prime to any integral power of N.

5. Show that, with slight verbal modifications, the principles proved above apply also to integral expressions involving one or more letters.

II. LOWEST COMMON MULTIPLES

80. Multiples of algebraic expressions. A multiple of an algebraic expression* is another algebraic expression that is exactly divisible by the given one, *i.e.*, it is an algebraic expression that contains all the prime factors of the given expression.

A **common multiple** of two or more algebraic expressions is a multiple of each of these expressions.

E.g., $12a^4x^3(y^2-1)$ is a common multiple of $3a^2x^3(y+1)$ and $2a^4x(y-1)$.

The **lowest common multiple** — usually written L. C. M. — of two or more algebraic expressions is that algebraic expression of lowest degree which is exactly divisible by each of the given expressions; it is that expression which contains all the prime factors of each of the given expressions, but no superfluous factors.

From these definitions, it is easy to find a common multiple of any two or more algebraic expressions whose prime factors are known.

E.g., a common multiple of $a^3b^2x^5$ and $a^2x^3y^4$ may be found thus:

Since a^3 is the highest power of a that is found in either of these expressions, therefore any common multiple of the given expressions *must* contain the factor a^3; it *may*, of course, contain a still higher power of a. Similarly, a common multiple of these two expressions must contain b^2, x^5, and y^4 as factors. Moreover, *any* expression which contains among its factors a^3, b^2, x^5, and y^4, is exactly divisible by each of the given expressions, and is, therefore, a common multiple of them.

The L. C. M. of these expressions is that one of their common multiples which contains no factor that is superfluous; it is $a^3b^2x^5y^4$.

Similarly, $6a^2x^3(x-2)^4(x-1)^3$ is a common multiple of $a^2x(x-2)^2(x-1)$ and $x^3(x-2)(x-1)^3$, but it is not their L. C. M., because it contains the factor $(x-2)^4$ when only $(x-2)^2$ is needed, and it contains the further superfluous factor 6; the L. C. M. of these given expressions is $a^2x^3(x-2)^2(x-1)^3$.

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* "Algebraic expressions" as here used include arithmetical numbers also.

A rule for writing down the L. C. M. of two or more monomials, or of any two or more entire algebraic expressions *whose prime factors are either known, or can easily be found*, may be formulated thus: *write down the indicated product of the different prime factors that enter into any of the given expressions, giving to each of these factors the highest exponent which that factor has in any of the given expressions.*

EXERCISES

Find the L. C. M. and H. C. F. of

1. $8a^2b^2$, $24a^4b^2c^2$, and $18abc^3$.

2. $15a^3b^4$, $20a^2b^2c^2$, and $30ac^3$.

3. $16a^2b^3c$, $24a^3dc$, and $36a^4b^2d^2$.

4. $18a^2br^2$, $12p^2q^2r$, and $54ab^2p^3$.

5. x^2-y^2 and $x^2+2xy+y^2$.

6. $21x^3$ and $7x^2(x+1)$.

7. x^2-1 and x^2+x.

8. $4x^2y-y$ and $2x^2+x$.

Find the L. C. M. of:

9. $a+b$, $a-b$, a^2+b^2, and a^4+b^4.

10. $3+a$, $9-a^2$, $3-a$, and $5a+15$.

11. x^3-y^3, x^2+xy+y^2, and x^2-xy.

12. $4a+4b$, $6a^2-24b^2$, and $a^2-3ab+2b^2$.

13. x^3+y^3, x^3y-y^4, and x^6-y^6.

14. y^2-5y+6 and $y^2-7y+10$.

15. $x^2-(a+b)x+ab$ and $x^2-(a-b)x-ab$.

16. Is $12a^3b^4(x^2-y^2)$ a common multiple of $2a^2b(x-y)$ and $3ab^3(x-y)$? Is it their L. C. M.?

17. What is the essential requirement in order that one expression may be a common multiple of two or more others? that it may be their L. C. M.?

Find the L. C. M. of

18. $3x^2+7x+2$ and x^2-x-6.

19. a^2+4a+4, a^2-4, and a^4-16.

20. $(a+b)^2-c^2$ and $(a+b+c)^2$.

21. $x^{2n}-y^{2n}$ and $(x^n-y^n)^2$.

22. $x^3+6x^2+5x-12$ and $x^3-8x^2+19x-12$.

SUGGESTION. Use § 67 to find one factor of each of these expressions.

23. $x^3-6x^2+11x-6$ and $x^3-9x^2+26x-24$.

24. a^3+2a^2-4a-8, $a^3-a^2-8a+12$, and $a^3+4a^2-3a-18$.

81. The L. C. M. of two entire algebraic expressions found by means of their H. C. F. The use of the H. C. F. in finding the L. C. M. may be better understood if a particular example be first worked out before the general discussion is given.

Let it be required to find the L. C. M. of $3x^4 - x^3 - x^2 + x - 2$ and $2x^3 - 3x^2 - 2x + 3$.

By § 76 it is found that the H. C. F. of these expressions is $x^2 - 1$; they may, therefore, be written thus:

$$3x^4 - x^3 - x^2 + x - 2 = (x^2 - 1)(3x^2 - x + 2),$$

and
$$2x^3 - 3x^2 - 2x + 3 = (x^2 - 1)(2x - 3),$$

wherein $3x^2 - x + 2$ and $2x - 3$ have no common factor. Hence the L. C. M. of the given expressions is

$$(x^2 - 1)(3x^2 - x + 2)(2x - 3).*$$

Similarly, in general, let E_1 and E_2 be any two entire algebraic expressions, and let their H. C. F. be F; then they may be written:

$$E_1 = FQ_1,$$

and
$$E_2 = FQ_2,$$

wherein Q_1 and Q_2 have no common factor, since F is the H. C. F. of E_1 and E_2. Hence the L. C. M. of E_1 and E_2 is the product of F, Q_1, and Q_2, *i.e.*, it is FQ_1Q_2.

Moreover, since $E_1 \cdot E_2 = FQ_1 \cdot FQ_2 = F(FQ_1Q_2)$, therefore the product of any two entire algebraic expressions is equal to the product of their H. C. F by their L. C. M.

Hence: *to find the L. C. M. of any two entire algebraic expressions, divide the product of the given expressions by their H. C. F.*

EXERCISES

Find the L. C. M. of:

1. $x^3 - 6x^2 + 11x - 6$ and $x^3 - 9x^2 + 26x - 24$.

2. $x^3 - 5x^2 - 4x + 20$ and $x^3 + 2x^2 - 25x - 50$.

3. $2y^3 - 11y^2 + 18y - 14$ and $2y^3 + 3y^2 - 10y + 14$.

4. $6a^3x - 5a^2x - 18ax - 8x$ and $6a^3b - 13a^2b - 6ab + 8b$.

5. $4x^4 - 17x^2y^2 + 4y^4$ and $2x^4 - x^3y - 3x^2y^2 - 5xy^3 - 2y^4$.

6. $2x^4 - 9x^3 + 18x^2 - 18x + 9$ and $3x^4 - 11x^3 + 17x^2 - 12x + 6$.

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* This is the L. C. M. because it contains all the necessary factors, and none that are superfluous.

82. The L. C. M. of three or more expressions. The L. C. M. of three or more entire algebraic expressions, whose factors are not easily determined, may be found by first finding the L. C. M. of two of the given expressions (§ 81), then the L. C. M. of that result and another of the given expressions, and so on.

EXERCISES

Find the L. C. M. of:

1. $x^4 - 2x^3 + x^2 - 1$, $x^4 - x^2 + 2x - 1$, and $x^4 - 3x^2 + 1$.

2. $x^3 + 3x^2 - 6x - 8$, $x^3 - 2x^2 - x + 2$, and $x^2 + x - 6$.

3. $x^2 - 4a^2$, $x^3 + 2ax^2 + 4a^2x + 8a^3$, and $x^3 - 2ax^2 + 4a^2x - 8a^3$.

4. If A, B, and C stand for any three given expressions, and if M_1 is the L. C. M. of A and B, while M_2 is the L. C. M. of M_1 and C, *prove* that M_2 is the L. C. M. of A, B, and C.

Find the L. C. M. of:

5. $a^3 + 7a^2 + 14a + 8$, $a^3 + 3a^2 - 6a - 8$, and $a^3 + a^2 - 10a + 8$.

6. $k^3 - 9k^2 + 23k - 15$, $k^3 + k^2 - 17k + 15$, and $k^3 + 7k^2 + 7k - 15$.

CHAPTER IX

ALGEBRAIC FRACTIONS

83. Definitions. An **algebraic fraction** is an indicated division in which the divisor, or both dividend and divisor, are algebraic expressions, and the dividend is not a multiple of the divisor.

E.g., $5 \div (x - 2y)$, $x^2 \div y$, and $3ax \div (a^2 - x^2)$ are algebraic fractions.

Fractions in algebra are written in the same form as that used in arithmetic, and the parts are called by the same names, *i.e.*, the dividend is called the **numerator**, the divisor is called the **denominator**, the numerator and denominator taken together are called the **terms** of the fraction, and the numerator is usually written above the denominator, from which it is separated by a line.

E.g., the fractions $5 \div (x - 2y)$, $x^2 \div y$, and $3ax \div (a^2 - x^2)$ are usually written as $\frac{5}{x - 2y}$, $\frac{x^2}{y}$, and $\frac{3ax}{a^2 - x^2}$, respectively.

An algebraic fraction is called a **proper fraction** if its numerator is of lower degree than its denominator, otherwise it is called an **improper fraction.**

E.g., $\frac{x - 1}{x^2 - 2x + 4}$ is a proper fraction, while $\frac{x^2 - 2x + 4}{x - 1}$ is an improper fraction.

An expression which consists of a part that is fractional and a part that is integral is called a **mixed expression.**

E.g., $m + \frac{n}{p}$, $a + \frac{2b - a^2}{a + c}$, and $x + y - \frac{2a}{x - y}$ are mixed expressions.

Observe the difference in writing a mixed number in arithmetic and a mixed expression in algebra: $5\frac{2}{3}$ means $5 + \frac{2}{3}$ in arithmetic, while in algebra $m\frac{n}{p}$ means $m \cdot \frac{n}{p}$, and not $m + \frac{n}{p}$.

It is sometimes desirable to write an integral expression in the *form* of a fraction; this is done by using 1 as the denominator; *e.g.*, $x^2 - 2x$, in the form of a fraction, is $\frac{x^2 - 2x}{1}$.

Attention is again called to the fact that algebraic expressions may be fractional in form and yet, for certain values of the letters involved, represent integers, and *vice versa* [cf. § 7, (v)].

84. Operations with algebraic fractions. As in arithmetic, so in algebra, it is often necessary to reduce fractions to their "lowest terms" and to a "common denominator," and also to change mixed expressions to improper fractions, and *vice versa.* The operations of addition, subtraction, multiplication, and division must also often be performed with algebraic fractions.

Moreover, since algebraic expressions represent *numbers,* therefore the principles which were demonstrated in § 54 apply to algebraic as well as to arithmetical fractions, and all of the above operations are therefore essentially the same in algebra as in arithmetic; the student should carefully observe this similarity in the next few articles.

85. Converting an improper fraction into a mixed expression. This change in form is made in precisely the same way as the corresponding case was treated in arithmetic.

E.g., just as $\frac{10}{3} = 3\frac{1}{3}$, *i.e.*, $3 + \frac{1}{3}$, so, too, *since a fraction is an indicated division,* $\frac{x^3+2x^2+5}{x^2+x+1} = x+1+\frac{4-2x}{x^2+x+1}$.

EXERCISES

Reduce each of the following improper fractions to an equivalent mixed expression, and explain your procedure:

1. $\frac{a^2 - 2ab + c}{a}$.

2. $\frac{3x^2 + 9x + 2}{3x}$.

3. $\frac{2x^2 + ax - 3a^2}{x + a}$.

4. $\frac{x^4 + 16}{x + 2}$.

5. $\frac{x^5 - x^3 - 2x^2 - 2x - 1}{x^2 - x - 1}$.

6. $\frac{a^4 + a^2 + 1}{a + 1}$.

7. $\frac{8x^3 - 10x^2 - 3x + 5}{4x^2 - 3}$.

8. $\frac{3x^6 + 2x - 5}{x^3 + 2x + 1}$.

9. $\frac{7x^5 - 1}{x^3 + x + 1}$.

10. $\frac{18x^4 - x^3 - 2x^2 - 7}{x^2 - 3x + 1}$.

11. Is $\frac{a^3 - 2a + 1}{5a^3 - 8a + 3}$ a proper or an improper fraction? Why?

Reduce the following mixed expressions to equivalent improper fractions, and check the correctness of your work (cf. Ex. 7, § 39):

12. $2x+\frac{x^2-6xy}{x^3+3y}$.

13. $6y-x+\frac{x^2}{4y^3+x}$.

14. $x+y+z-\frac{x-y-z}{x^2-y-z}$.

15. $3a-2b+c-\frac{4+b-c}{a-5b+2c}$.

86. Reduction of fractions to lowest terms. In § 54 (v), it was shown that any factor which is found in both terms of a fraction may be rejected (canceled) without changing the value of the fraction.

E.g., $\frac{3ax^2}{4bxy}=\frac{3ax}{4by}$; and $\frac{x^2-1}{x^2-2x+1}=\frac{(x+1)(x-1)}{(x-1)(x-1)}=\frac{x+1}{x-1}$.

In algebra, as in arithmetic, a fraction is said to be in its **lowest terms** when the numerator and denominator have no common factor; hence, *a fraction may always be reduced to an equivalent fraction in its lowest terms by dividing both its numerator and denominator by their H. C. F.*

E.g., to reduce $\frac{6a^2xy^3}{9ax^3y^4}$ to its lowest terms, divide both numerator and denominator by $3axy^3$, which (§ 73) is their H. C. F.

Instead of dividing both terms of a fraction by their H. C. F., and thus reducing the fraction to its lowest terms in a single operation, the same result may, of course, be accomplished by canceling *any* common factor as soon as it is discovered, and continuing this process until the resulting numerator and denominator are prime to each other. Recourse to the H. C. F. is necessary only when no common factors can be found by other methods. Observe that it is only equal *factors*, and not equal *parts*, that may be canceled. *E.g.*, $\frac{3a+x}{5bc+x}$ is not equal to $\frac{3a}{5bc}$; nor is $\frac{2m+xy}{6s-5n^2}$ equal to $\frac{m+xy}{3s-5n^2}$.

EXERCISES

Reduce each of the following fractions to its lowest terms:

1. $\frac{a^2-ab}{a^2-b^2}$.

2. $\frac{34a^2b^2c^4}{51a^4b^2c}$.

3. $\frac{a^2-b^2}{(a-b)^2}$.

4. $\frac{a^2+2ab+b^2}{a^3+b^3}$.

5. $\frac{2x^2+3x+1}{x^2+5x+4}$.

6. $\frac{x^3+y^3}{y^4-x^4}$.

7. $\frac{a^3+b^3}{a^4+a^2b^2+b^4}$.

8. $\frac{3a^2-2a-1}{1+a-a^3-a^2}$.

9. $\frac{a^4-a^2-20}{a^4-9a^2+20}$.

10. $\dfrac{x^2+2xy+y^2-z^2}{z^2+x^2+y^2+2xy+2xz+2yz}$.

11. $\dfrac{a^6-b^6}{a^5+a^4b+a^3b^2+ab^4+b^5}$.

12. $\dfrac{x^3+3x^2+4x+2}{x^3-3x^2-8x-10}$.

13. $\dfrac{x^3+x^2-22x-40}{x^3-7x^2+2x+40}$.

14. $\dfrac{1-2x-5x^2+6x^3}{1+5x+2x^2-8x^3}$.

15. $\dfrac{3a^3+20a^2-a-2}{3a^3+17a^2+21a-9}$.

16. $\dfrac{x^5-2x^4-2x^3-11x^2-x-15}{2x^5-7x^4+4x^3-15x^2+x-10}$.

17. $\dfrac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$.

18. $\dfrac{x^3-6x^2y+2xy^2+3y^3}{x^3+6x^2y-2xy^2-5y^3}$.

19. $\dfrac{a^2+b^2+2c^2+2ab+3ac+3bc}{a^2+b^2+c^2+2ab+2ac+2bc}$.

20. May the factor $5ax$ be canceled from the first two terms of the numerator and denominator of $\dfrac{5ax^2-10a^2x+3b(x-2a)}{15a^3x^4-30a^4x^3+6b(x-2a)}$ without changing the value of this fraction? Why?

21. Is the value of a fraction changed by canceling equal *factors* from both numerator and denominator? Is it changed by canceling equal *parts* or equal factors of parts of the numerator and denominator?

87. Changing fractions to equivalent fractions having given denominators. Since multiplying both terms of a fraction by the same number does not change its value, therefore any given fraction may be reduced to an equivalent fraction whose denominator is any desired multiple of the given denominator.

E.g., to reduce $\dfrac{3a}{4x^2}$ to an equivalent fraction whose denominator shall be $12cx^2y$, multiply both terms of the given fraction by $3cy$.

EXERCISES

1. If the denominator of a fraction be multiplied by any given expression, what must be done to the numerator in order to preserve the value of the fraction?

2. How find the expression by which it is necessary to multiply both terms of a given fraction in order that the new equivalent fraction shall have a given denominator? A given numerator?

3. Reduce $\dfrac{3a-5}{2x(3+ax)}$ to an equivalent fraction whose denominator is $18x-2a^2x^3$. Also to one whose numerator is $12ax+18ay-20x-30y$.

Find the required numerator in each of the following equations:

4. $\frac{3x-2a}{x^2-3ax+2a^2}=\frac{?}{2x^3-7ax^2+7a^2x-2a^3}$.

5. $\frac{4}{(y-a)(a-x)(3-4y)}=\frac{?}{(a-y)(a-x)(4y-3)(3-7y)}$.

6. $\frac{3m-8}{2x-5}=\frac{?}{-2x+5}$.

7. $\frac{3x}{1}=\frac{?}{7x^2-3x+5}$.

88. Reduction of fractions to common denominators. In § 87 it is shown that any given fraction may be reduced to an equivalent fraction whose denominator is *any desired multiple* of the given fraction; if then any *common* multiple of the denominators of two or more given fractions be chosen as the new denominator, it is clear that these fractions may be reduced to equivalent fractions having this denominator in common.

E.g., since $12a^2x^2$ is a common multiple of the denominators of $\frac{3}{2x}$, $\frac{2a}{3x^2}$, and $\frac{5m}{6a}$, therefore these fractions may be reduced to the equivalent fractions $\frac{18a^2x}{12a^2x^2}$, $\frac{8a^3}{12a^2x^2}$, and $\frac{10amx^2}{12a^2x^2}$, which have the common denominator $12a^2x^2$. Similarly for any given fractions whatever.

In practice it is usually desirable to keep the denominators of fractions as small as possible, and therefore, instead of choosing *any* common multiple, as above, it is best to choose the L. C. M. of the given denominators.

E.g., the L. C. M. of the denominators of $\frac{2a}{(x-1)(x+1)}$ and $\frac{5x}{(x+1)(x+3)}$ is $(x-1)(x+1)(x+3)$, and these fractions are respectively equal to $\frac{2a(x+3)}{(x-1)(x+1)(x+3)}$ and $\frac{5x(x-1)}{(x-1)(x+1)(x+3)}$; moreover, the given fractions can not be reduced to equivalent fractions having a lower common denominator.

To reduce two or more given fractions to equivalent fractions having the lowest possible common denominator, *divide the L. C. M. of the given denominators by the denominator of one of the given fractions, and then multiply both terms of that fraction by the resulting quotient; do the same with each of the given fractions.*

EXERCISES

Reduce the following fractions to equivalent fractions having the lowest possible common denominator:

1. $\frac{3a+1}{4}$ and $\frac{3x+4}{6}$.

2. $\frac{9-3a}{16b}$ and $\frac{3+5x}{20b^2}$.

3. $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

4. $\frac{x-y}{x^3-y^3}$ and $\frac{x+y}{x^3+y^3}$.

5. $\frac{3}{(m-1)(m-2)}$ and $\frac{5}{(2-m)(m-3)}$.

6. $\frac{x+y}{x^2+xy+y^2}$ and $\frac{x-y}{x^2-xy+y^2}$.

7. $\frac{x-y}{x+y}$, $\frac{x+y}{x-y}$, and $\frac{x^2+y^2}{x^2-y^2}$.

8. $\frac{x-y}{x^3-y^3}$ and $\frac{x^2-x+1}{x^4+x^2y^2+y^4}$.

9. $\frac{1}{1+x}$, $\frac{2}{1-x^2}$, and $\frac{3}{x^3-1}$.

10. $\frac{3(a+b)}{a^2-2ab+b^2}$, $\frac{4(a-b)}{a^2+2ab+b^2}$, and $\frac{5a}{a^2-b^2}$.

11. $\frac{6x}{15-13x+2x^2}$, $\frac{3a}{x^2-8x+15}$, and $\frac{3a-6x}{x^2-2x-15}$.

12. $7x$, $\frac{b-x}{x^2-a^2}$, $\frac{a-x}{x^2-b^2}$, and $\frac{3}{x^2-(a+b)x+ab}$.

13. $\frac{2x-5}{x^2+7x+10}$, $\frac{3}{x^2+x-2}$, and $\frac{2(x+1)}{x^2+4x-5}$.

14. $\frac{a+5}{a^2-4a+3}$, $\frac{a-2}{a^2-8a+15}$, and $\frac{a+1}{a^2-6a+5}$.

15. $\frac{5(u-3v)}{u-2v}$, $\frac{8}{u^2-5uv+6v^2}$, and $\frac{2(u-2v)}{u-3v}$.

89. Addition and subtraction of fractions. As in arithmetic, so in algebra, *the sum (or difference) of two given fractions which have a common denominator is a fraction whose numerator is the sum (or difference) of the given numerators, and whose denominator is the common denominator of the given fractions* [cf. § 54 (vii)].

E.g., $$\frac{x^2+x+5}{x^2-2x+1}-\frac{x^2+3}{x^2-2x+1}=\frac{x^2+x+5-(x^2+3)}{x^2-2x+1}=\frac{x+2}{x^2-2x+1}.$$

NOTE 1. The minus sign before the second fraction means that *all* of that fraction is to be subtracted, hence the necessity for the parenthesis in the numerator of the next fraction.

Note 2. Since a fraction is a *quotient*, therefore its sign (*i.e.*, the sign written *before* the dividing line) is governed by the laws of signs in division. Thus, if E_1 and E_2 are any algebraic expressions whatever, then $-\frac{E_1}{E_2}=+\frac{-E_1}{E_2}=+\frac{E_1}{-E_2}$.

Hence the above example may also be arranged thus :

$$\frac{x^2+x+5}{x^2-2x+1}-\frac{x^2+3}{x^2-2x+1}=\frac{x^2+x+5}{x^2-2x+1}+\frac{-x^2-3}{x^2-2x+1}=\frac{x+2}{x^2-2x+1}.$$

If the given fractions have *not* a common denominator, they must be reduced to equivalent fractions which have a common denominator (§ 88) before they can be added or subtracted.

E.g., $$\frac{1}{x-1}-\frac{3}{x}+\frac{2}{x+1}=\frac{x(x+1)}{x(x-1)(x+1)}-\frac{3(x-1)(x+1)}{x(x-1)(x+1)}+\frac{2x(x-1)}{x(x-1)(x+1)}$$
$$=\frac{x(x+1)-3(x-1)(x+1)+2x(x-1)}{x(x-1)(x+1)}$$
$$=\frac{3-x}{x(x-1)(x+1)};$$

and this result, viz., $\frac{3-x}{x(x-1)(x+1)}$, is called the "algebraic sum" of the given fractions.

EXERCISES

Simplify the following expressions:

1. $\frac{a+3}{5}+\frac{a+5}{7}$.
2. $\frac{x-1}{2}+\frac{x+3}{5}+\frac{x+7}{10}$.
3. $\frac{a+x}{a-x}+\frac{a-x}{a+x}$.
4. $\frac{1}{2x-3y}+\frac{x+y}{4x^2-9y^2}$.
5. $\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$.*
6. $\frac{1}{x+y}+\frac{1}{x-y}$.
7. $\frac{3}{x-6}-\frac{1}{x-2}$.
8. $\frac{x}{1-x^2}-\frac{x}{1+x^2}$.
9. $\frac{x+7}{x^2-3x-10}-\frac{x+2}{x^2+2x-35}$.
10. $\frac{1}{1+x-2x^2}-\frac{3}{6x^2-x-2}$.
11. $\frac{1}{x^2-1}+\frac{1}{x+2-x^2}$.
12. $\frac{1}{(x-y)^2}-\frac{1}{(x+y)^2}$.
13. $\frac{a+b}{a^2-2ab+b^2}-\frac{a-b}{a^2+2ab+b^2}$.
14. $\frac{a^2-ax+x^2}{a^2+ax+x^2}-\frac{a-x}{a+x}$.
15. $\frac{a+b+c}{a^2-(b+c)^2}-\frac{a-b+c}{(a-b)^2-c^2}$.
16. $\frac{a-x}{x}+\frac{a+x}{x}-\frac{a^2-x^2}{2ax}$.

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* Compare example under Note 2, above.

17. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ac}.$

18. $\frac{2}{xy}-\frac{3y^2-x^2}{xy^3}+\frac{xy+y^2}{x^2y^2}.$

19. $\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}+3x.$

SUGGESTION. $3x=\frac{3x}{1}.$

20. $\frac{1}{x+y}+\frac{x-y}{x^2-xy+y^2}-\frac{x^2-xy}{x^3+y^3}.$

21. $\frac{1+x}{1-x}+\frac{1-x}{1+x}+x.$

22. $\frac{a}{x-a}-\frac{a^2}{x^2-a^2}-2x.$

23. $\frac{1}{x(x-y)}+\frac{1}{y(x+y)}-\frac{1}{xy}.$

24. $\frac{1}{x^2-7x+12}-\frac{1}{x^2-5x+6}.$

25. $\frac{1}{2x^2-x-1}+\frac{1}{3-x-2x^2}.$

26. $\frac{1}{a+b}-\frac{1}{a-b}-\frac{2a}{b^2-a^2}.$

27. $\frac{a}{a-1}-1-\frac{1}{a(a-1)}.$

28. $\frac{1}{a}-\frac{2}{a+1}+\frac{1}{a+2}.$

29. $\frac{1}{x-1}-\frac{1}{2(x+1)}.$

30. $\frac{a}{a-1}-\frac{2}{a+1}+\frac{a}{a+2}-\frac{1}{a}.$

31. $\frac{1}{1-x}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}.$

32. $\frac{x}{x-1}+x-\frac{x^2}{1-x}.$

33. $\frac{2}{x+4}-\frac{x-3}{x^2-4x+16}+\frac{x^3}{x^3+64}.$

34. $\frac{b}{a+b}-\frac{ab}{(a+b)^2}-\frac{ab^2}{(a+b)^3}.$

35. $\frac{x^3+ax^2}{ax^2-a^3}-\frac{x(x-a)}{a(x+a)}-\frac{2ax}{x^2-a^2}.$

36. $\frac{2b-a}{x-b}-\frac{3x(a-b)}{b^2-x^2}+\frac{b-2a}{x+b}.$

37. $\frac{-1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}.$

Is $(a-b)(a-c)(b-c)$ a common multiple of these three denominators? Is $(a-b)(b-c)(c-a)$?

38. $\frac{1}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}-\frac{1}{x+2}.$

39. $\frac{5}{1+2x}-\frac{3x}{1-2x}+\frac{4-13x}{4x^2-1}.$

40. $\frac{x}{x^3+y^3}-\frac{y}{x^3-y^3}+\frac{x^3y+xy^3}{x^6-y^6}.$

41. $\frac{1}{x+a}+\frac{4a}{x^2-a^2}-\frac{1}{a-x}-\frac{2a}{x^2+a^2}.$

42. $\frac{1}{x^2-5x+6}+\frac{2}{3x-2-x^2}+\frac{3}{4x-3-x^2}.$

43. $\frac{x-1}{(x-2)(x-3)}+\frac{2(x-2)}{(3-x)(x-1)}-\frac{x-3}{(x-1)(2-x)}.$

44. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}.$

45. $\dfrac{bc}{(a-c)(a-b)}+\dfrac{ca}{(b-c)(b-a)}+\dfrac{ab}{(c-a)(c-b)}$.

46. $\dfrac{1}{x^2-5xy+6y^2}-\dfrac{2}{x^2-4xy+3y^2}+\dfrac{1}{x^2-3xy+2y^2}$.

47. $\dfrac{a^2+2a+1}{a^2-2a+1}-2+\dfrac{a^2-2a+1}{a^2+2a+1}$.

48. $\dfrac{x^3+x^2+x+1}{x^2-x+1}-\dfrac{3}{x-1}-1$.

90. Reducing mixed expressions to improper fractions. Since an entire expression may be written in the fractional form with the denominator 1, therefore reducing mixed expressions to improper fractions is merely a special case of addition.

E.g., $x+1-\dfrac{2a}{x-1}=\dfrac{x+1}{1}-\dfrac{2a}{x-1}=\dfrac{(x+1)(x-1)}{x-1}-\dfrac{2a}{x-1}=\dfrac{x^2-1-2a}{x-1}$.

EXERCISES

By the above method simplify the following expressions:

1. $x-1+\dfrac{x^2}{x^2-1}$.

2. $x+1-\dfrac{2x}{x-1}$.

3. $3-x^2+\dfrac{x^4+6x-3}{x^2-2x+1}$.

4. $1-y-y^2-\dfrac{y-y^2}{1-y^4}$.

5. $a^2-ab+b^2-\dfrac{b^3}{a+b}$.

6. $3a-6b-\dfrac{16b^2-5c^2}{a+2b}$.

7. $x-x^2-x^3-\dfrac{x^4+x^3-x+1}{1+x+x^2}$.

8. $y^3-2y+4-\dfrac{4+4y^2+y^3}{1-2y+y^2}$.

9. $2a-3b-\dfrac{4a^2+9b^2}{2a+3b}$.

10. $1-ax-bx-\dfrac{ax+bx+ab}{1-ab+x^2}$.

11. Prove that any mixed expression may be reduced to an improper fraction by multiplying the integral part by the denominator of the fraction, adding or subtracting the numerator as the case may be, and placing this result over the denominator. Also compare § 47, Ex. 11.

91. The product of two or more fractions. In algebra, as in arithmetic, *the product of two or more fractions is a fraction whose numerator is the product of the numerators of the given fractions, and whose denominator is the product of their denominators* [cf. § 54 (ii)].

E.g., $\dfrac{3x^2}{4by^2}\cdot\dfrac{2a^2y}{3x^2}\cdot\dfrac{5ab^2}{2x-3y}=\dfrac{30a^3b^2x^2y}{12bx^2y^2(2x-3y)}=\dfrac{5a^3b}{2y(2x-3y)}$.

EXERCISES

Find the product of:

1. $\frac{abc}{b^2d^2c^2}$ and $\frac{b^3c^2}{ab^2}$.

2. $\frac{3\,xy}{8\,yz}$ and $\frac{16\,y^2z^2}{9\,x^2y^2}$.

3. $\frac{a^{m+1}}{b^{m+2}}$ and $\frac{b^{m+1}}{a^m}$.

4. $\frac{a}{a+b}$ and $\frac{b}{a-b}$.

5. $\frac{a^2-ab}{x^2-xy}$ and $\frac{x^2+xy}{a^2+ab}$.

6. $\frac{a^2-2\,ab+b^2}{x^4+x^2y^2+y^4}$ and $\frac{x^2-xy+y^2}{(a-b)^2}$.

7. Simplify $\left(x+2\,y-\frac{5}{y}\right)\frac{3\,y}{a+x}$, making use of the distributive law.

8. Simplify $\left(x+2\,y-\frac{5}{y}\right)\frac{3\,y}{a+x}$ by first reducing the multiplicand to an improper fraction (cf. Ex. 7).

9. Simplify $\left(y+3-\frac{5}{y-3}\right)\left(2\,y+3-\frac{5}{2\,y-3}\right)$ by the method of Ex. 7, and also by that of Ex. 8, and compare results.

10. Give a convenient rule for multiplication when one or more factors are mixed expressions.

11. Prove that $\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$, and show that the proof is still valid when some or all of the letters represent algebraic expressions (cf. § 54 and § 84).

12. How may an integral expression be multiplied by a fraction (cf. § 54)? Is $n\cdot\frac{p}{q}$ equal to $\frac{np}{q}$? Is it equal to $\frac{p}{q\div n}$?

13. How does the identity $\left(\frac{E_1}{E_2}\right)^n=\frac{E_1^{\,n}}{E_2^{\,n}}$ follow from § 54 (ii)?

14. Prove that $\frac{p}{q}=\frac{pn}{qn}$, and thus prove that $\frac{pr}{qs}\cdot\frac{st}{rw}=\frac{p\cancel{r}}{q\cancel{s}}\cdot\frac{\cancel{s}t}{\cancel{r}w}=\frac{pt}{qw}$.

15. Based upon Ex. 14, give a convenient rule for multiplying two or more fractions together by cancellation.

Find the product of:

16. $\frac{(a-x)^3}{x^3-y^3}$ and $\frac{x^2+xy+y^2}{a^2-2\,ax+x^2}$.

17. $\frac{(a-b)^2-1}{(a+b)^2-1}$ and $\frac{a+b+1}{a-b-1}$.

18. $\frac{a^4-1}{x^3+1}$ and $\frac{(a^2+1)(x+1)^2}{(x+1)^3}$.

19. $\frac{x^6+y^6}{a^6+b^6}$ and $\frac{a^4-a^2b^2+b^4}{x^4+2\,x^2y^2+y^4}$.

20. $\frac{x^2+xy}{x^2+y^2}$ and $\left(\frac{x}{x-y}-\frac{y}{x+y}\right)$.

21. $a+\frac{ab}{a-b}$ and $b-\frac{ab}{a+b}$.

22. $\dfrac{x^2 - 7x + 12}{x^2 - 9x + 20}$ and $\dfrac{x^2 - 10x + 25}{x^2 - 6x + 9}$.

23. $\dfrac{(a^3 - 1)(a^3 + 1)}{x^4 + x^2y^2 + y^4}$ and $\dfrac{(a - 1)^3(a + 1)^3}{(a^2 - 1)(a^4 - 2a^2 + 1)}$.

24. $\dfrac{a^3 - 8b^3}{a^2 - 4b^2}$ and $\dfrac{a + 2b}{a^2 + 2ab + 4b^2}$.

25. $\dfrac{x^2 - y^2}{2xy}$, $\dfrac{x(x + 2y)}{xy + y^2}$, and $\dfrac{3y^2}{x^2 - xy}$.

26. $\dfrac{a^2 + b^2 - c^2 + 2ab}{a^2 - b^2 - c^2 - 2bc}$ and $\dfrac{c^2 - b^2 + a^2 - 2ac}{c^2 - a^2 + b^2 - 2bc}$.

27. $\dfrac{a + b}{a - b} + \dfrac{a - b}{a + b}$ and $\dfrac{a + b}{a - b} - \dfrac{a - b}{a + b}$.

28. $\dfrac{a}{bc} - \dfrac{b}{ac} - \dfrac{c}{ab} - \dfrac{2}{a}$ and $1 - \dfrac{2c}{a + b + c}$.

92. Division of fractions. In algebra, as in arithmetic, *to divide by any fraction gives the same result as to multiply by the reciprocal of that fraction* [cf. § 54 (vi)].

E.g., $\dfrac{a^2x}{b^2y^2} \div \dfrac{cx}{by} = \dfrac{a^2\cancel{x}}{\cancel{b^2y^2}} \cdot \dfrac{\cancel{by}}{c\cancel{x}} = \dfrac{a^2}{bcy}$.

Note. If the divisor is an integral expression, it should be first written in a fractional form, and if it is a mixed expression, it should be first reduced to an improper fraction, before proceeding as above.

EXERCISES

1. Prove that $\dfrac{p}{q} \div \dfrac{r}{s} = \dfrac{p}{q} \cdot \dfrac{s}{r}$ (cf. § 54 and § 84).

Perform the following indicated operations, and simplify the results:

2. $\dfrac{6x^5y}{14a^3b^4} \div \dfrac{2x^3}{2a^2b^2}$.

3. $\dfrac{a^2 - 121}{a^2 - 4} \div \dfrac{a + 11}{a + 2}$.

4. $\dfrac{x^3 - a^3}{x^3 + a^3} \div \dfrac{(x - a)^2}{x^2 - a^2}$.

5. $\dfrac{14x^2 - 7x}{12x^3 + 24x^2} \div \dfrac{2x - 1}{x^2 + 2x}$.

6. $\dfrac{a^4 - b^4}{a^4 + a^2b^2 + b^4} \div \dfrac{(a - b)^6}{a^6 - b^6}$.

7. $\left(a + \dfrac{3x^2}{a}\right)\left(\dfrac{a^2}{3x^2} - 1\right) \div \dfrac{a}{x^2}$.

8. $\dfrac{(a - b)^2 - 9}{(a + b)^2 - 9} \div \dfrac{a - b + 3}{a + b + 3}$.

9. $\dfrac{x^2 - 1}{x^2 - 3x - 10} \div \dfrac{x^2 - 12x + 35}{x^2 + 3x + 2}$.

10. $\dfrac{a^2 + x^2 - 1 + 2ax}{x^2 + y^2 - 9 + 2xy} \div \dfrac{a + 1 + x}{x + 3 + y}$.

11. $\dfrac{x^3 - 6x^2 + 36x}{x^2 - 49} \div \dfrac{x^4 + 216x}{x^2 - x - 42}$.

12. $\dfrac{5\,m^6n - 5\,n^7}{m^2n + 2\,mn^2 + n^3} \div \dfrac{m^2 - mn + n^2}{m + n}.$

13. $\dfrac{2\,x^2 + 13\,x + 15}{4\,x^2 - 9} \div \dfrac{2\,x^2 + 11\,x + 5}{4\,x^2 - 1}.$

14. $\dfrac{x^4 - 17\,x^2 + 16}{9\,a^4 - 34\,a^2 + 25} \div \dfrac{x^2 - 3\,x - 4}{3\,a^2 + 8\,a + 5}.$

15. $\dfrac{4\,a^2 + b^2 - c^2 + 4\,ab}{4\,a^2 - b^2 - c^2 - 2\,bc} \div \dfrac{2\,a + b + c}{2\,a - b - c}.$

16. $\dfrac{a^2 + ab + ac + bc}{ax - ay - x^2 + xy} \cdot \dfrac{a^2 - ax + ay - xy}{a^2 + ac + ax + cx} \div \dfrac{a^2 - a(y - b) - by}{x^2 - x(y - a) - ay}$

17. $\dfrac{x^4 - 3\,x^3 - 23\,x^2 + 75\,x - 50}{x^4 - 5\,x^3 - 21\,x^2 + 125\,x - 100} \div \dfrac{x^3 - 12\,x^2 + 45\,x - 50}{x^3 - 10\,x^2 + 29\,x - 20}.$

18. $\dfrac{p^4 - q^4}{(p - q)^2} \div \dfrac{p^2 + pq}{p - q} + \dfrac{p^2}{p^2 + q^2} \cdot \dfrac{p^2 - 2\,pq + q^2}{p^2 + 2\,pq + q^2}.$

93. Complex fractions. In algebra, as in arithmetic, a fraction whose numerator or denominator, or both, are themselves fractional expressions, is called a **complex fraction.**

E.g., $\dfrac{\frac{1}{a} - a}{1 + a}$ and $\dfrac{x - \frac{1}{x}}{x + 2 + \frac{1}{x}}$ are complex fractions.

A complex fraction, like a simple one, is primarily an indicated quotient, but it usually also involves some of the other fundamental operations already studied; performing these operations is spoken of as *simplifying the fraction.*

E.g., the above complex fractions are simplified as follows:

$$\frac{\frac{1}{a} - a}{1 + a} = \left(\frac{1}{a} - a\right) \div (1 + a) = \frac{1 - a^2}{a} \div \frac{1 + a}{1} = \frac{1 - a^2}{a} \cdot \frac{1}{1 + a} = \frac{1 - a}{a};$$

and
$$\frac{x - \frac{1}{x}}{x + 2 + \frac{1}{x}} = \frac{\frac{x^2 - 1}{x}}{\frac{x^2 + 2\,x + 1}{x}} {}^{*} = \frac{x^2 - 1}{x} \cdot \frac{x}{x^2 + 2\,x + 1} = \frac{x - 1}{x + 1}.$$

NOTE. Multiply both numerator and denominator of this last fraction by x, and reduce to lowest terms. How does this method compare with that used above?

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* To avoid ambiguity, the principal dividing line in a complex fraction is best made somewhat heavier than the others.

EXERCISES

Simplify each of the following expressions:

1. $$\left\{\left(\frac{1}{a}+\frac{1}{b}\right)^2-\frac{1}{ab}-\frac{4}{(a-b)^2}\right\}\div\left\{\left(\frac{1}{a}-\frac{1}{b}\right)^2-\frac{1}{ab}\right\}.$$

2. $$\left(\frac{x}{x-2}+\frac{5}{x-8}\right)\div\left\{1\div\left(\frac{x-3}{3x-8}-\frac{2}{x+2}\right)\right\}.$$

3. $$\frac{\left\{1+\dfrac{c}{a+b}+\dfrac{c^2}{(a+b)^2}\right\}\left\{1-\dfrac{c^2}{(a+b)^2}\right\}}{\left\{1-\dfrac{c^3}{(a+b)^3}\right\}\left\{1+\dfrac{c}{a+b}\right\}}.$$

4. $$\frac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}}\cdot\frac{m^2-n^2}{m^3-n^3}.$$

5. $$\left\{1+\frac{2b^2}{a(a-3b)}\right\}\left\{1+\frac{b}{2b-a}\right\}.$$

6. $$\left\{x-y-\frac{1}{x-y+\dfrac{xy}{x-y}}\right\}\frac{x^3+y^3}{x^2-y^2}.$$

7. $$\left\{x+y-\frac{1}{x+y-\dfrac{xy}{x+y}}\right\}\frac{x^3-y^3}{x^2-y^2}.$$

8. $$\frac{x-2-\dfrac{1}{x-2}}{x-2-\dfrac{4}{x-5}}\cdot\frac{x-4-\dfrac{4}{x-4}}{x-4-\dfrac{1}{x-4}}.$$

9. $$\left(\frac{a^2+b^2}{2ab}-1\right)\frac{ab^2}{a^3+b^3}\div\frac{4ab(a+b)}{a^2-ab+b^2}.$$

10. $$\frac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}.$$

11. $$\frac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$$

12. $$\frac{\dfrac{m-n}{m+n}+\dfrac{m^2+n^2}{m^2-n^2}}{\dfrac{m^2}{m-n}+\dfrac{m^2n+n^3}{(m-n)^2}}.$$

13. $$\frac{x^3-y^3}{x^2-xy+y^2}\div\frac{x^3+y^3}{x^2+xy+y^2}\cdot\left(1+\frac{y}{x-y}\right).$$

14. $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}.$

15. $\dfrac{\dfrac{1}{3x-2}-\dfrac{1}{3x+2}}{9-\dfrac{4}{x^2}}.$

16. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}{\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}}.$

17. $\dfrac{9x^2-64}{x-1-\dfrac{1}{1-\dfrac{x}{4+x}}}.$

18. $\dfrac{1}{1+\dfrac{1}{1+\dfrac{x}{1-x}}}.$

19. $\dfrac{x+y}{x+y+\dfrac{1}{x-y+\dfrac{1}{x+y}}}.$

20. Recalling the meaning of a negative exponent (§ 44), show that $\dfrac{a^4x^{-3}}{b^2}=\dfrac{a^4\cdot\dfrac{1}{x^3}}{b^2}=\dfrac{a^4}{b^2x^3}.$

21. As in Ex. 20, show that $\dfrac{a^2x^{-5}y^3}{m^4s^{-3}w^2}=\dfrac{a^2s^3y^3}{m^4w^2x^5}.$ Show also that $\dfrac{a^ms^k}{b^rp^n}=\dfrac{a^mp^{-n}}{b^rs^{-k}}.$

22. Prove that any factor whatever of the numerator of a fraction may be transferred to the denominator by merely reversing the sign of the exponent of that factor. Also show how a factor may be transferred from denominator to numerator.

23. Is $\dfrac{a^2+b^{-3}c^4}{4x}$ equal to $\dfrac{a^2+c^4}{4xb^3}$? Why? Observe carefully that a *factor*, but not a *part*, may be transferred as in Ex. 22.

Clear the following expressions of negative exponents and simplify them as far as possible; in any case of doubt employ the definition of § 44, viz., $a^{-k}=\dfrac{1}{a^k}$:

24. $\dfrac{3m^{-1}n^2}{2(a+x)}.$ **25.** $\dfrac{a^2x^5}{3\cdot 2^{-1}b^3y^{-2}}.$ **26.** $\dfrac{3a+2bc^{-3}}{5x^{-1}-8y}.$

REVIEW QUESTIONS—CHAPTERS VI-IX

1. Define and illustrate: even numbers; odd numbers; prime numbers; composite numbers; finite numbers; and infinite numbers.

2. What is the value of $\frac{0}{0}$? Of $\frac{3}{0}$? Explain.

3. Show that the absurdity in Ex. 17, § 55, arises from dividing zero by zero.

4. By applying the distributive law show that $-(a+x-5)=-a-x+5$.

5. State the binomial theorem. Apply this theorem to expand $(2a-3x^2)^5$.

6. If $Ax^n+Bx^{n-1}+\cdots+Hx+K$ is divided by $x-a$, prove that the remainder is $Aa^n+Ba^{n-1}+\cdots+Ha+K$.

7. By means of Ex. 6, and without actually performing the division, show that $x-1$ and $x+2$ are factors of $x^4+2x^2+7x-10$.

8. As in Ex. 7, show that x^n-y^n, wherein n is any positive integer is exactly divisible by $x-y$.

9. By means of factoring, find the roots of $x^2-7x+12=0$, and explain.

10. Form the equation whose roots are 3 and -7, and explain.

11. What is meant by the L. C. M. of two or more expressions? How may it, in general, be found?

12. How may the L. C. M. of three or more given expressions be found?

13. Simplify $\dfrac{1+\dfrac{x}{1+x}}{x+\dfrac{1}{x+1}}\div\dfrac{(x-1)^2-x^2}{x^2+x+1}$.

14. Is $\dfrac{2ax^5y^2}{3b^3}$ equal to $\dfrac{ax^5}{3\cdot 2^{-1}b^3y^{-2}}$? Explain.

CHAPTER X

SIMPLE EQUATIONS

I. INTEGRAL EQUATIONS

94. Introductory remarks and definitions. Some preliminary work in simple equations has already been given in Chapter III; the text of that chapter should now be rapidly reread. In the present chapter it is proposed to treat this subject in a somewhat more careful and rigorous manner.

Every algebraic problem involves one or more numbers whose values are at first unknown, and which are to be found from given relations which they bear to other numbers whose values are known; to distinguish between these two kinds of numbers the first are called **unknown numbers**, and are usually represented by some of the later letters of the alphabet, as x, y, and z (cf. § 26), while the others are called **known numbers**, and are represented either by the Arabic characters, 1, 2, 3, ⋯, or by some of the early letters of the alphabet, as a, b, and c.

If any of the known numbers in an equation are represented by letters, then it is called a **literal equation**, otherwise it is called a **numerical equation.** If its members are integral expressions so far as the unknown numbers are concerned (§ 41), then it is called an **integral equation**; known numbers may appear as divisors and the equation still be integral.

E.g., $3x^2+5xy-10y^2=8$, $4-\dfrac{2x}{3}=7x$, and $5(x^2+y^2)=\dfrac{3x}{a}$ are integral equations; the first two are numerical, while the third is literal.

By the **degree** of an integral algebraic equation is meant the highest number of *unknown* factors which it contains in any one term. If all of its terms are of the same degree, the equation is **homogeneous.**

E.g., $3x+7=13$ and $2+4y-5x=0$ are numerical equations of the first degree, while $x^3+10x=\frac{1}{2}x-\frac{1}{5}$, $4xy^2=3ax^3-7y^3$, and $axy^2-x=3y$ are of the third degree; of these last three equations the first is numerical, the second and third are literal, and the second is homogeneous.

Special names are often given to equations of the lower degrees; thus an equation of the first degree is known as a **simple equation** and also as a **linear equation**;* one of the second degree is also called a **quadratic equation**; one of the third degree, a **cubic equation**; etc.

EXERCISES

1. What is meant by a root (or solution) of an equation? Is 2 a root of $x^2-7x+10=0$? What then are the factors of $x^2-7x+10$ (cf. § 67)? What other root has this equation?

2. Verify that $x=4$ and $y=3$ constitute a solution of the equation $7x+2y=34$. If $x=2$ in this equation, what must be the corresponding value of y? If $x=a$, what is y? If $y=6$, what is x? Find four other solutions of this equation.

3. How many solutions has the equation in Ex. 1? How many solutions has the equation in Ex. 2?

4. Is the equation in Ex. 1 homogeneous? integral? literal? numerical? simple? Define each of these kinds of equations.

5. Show that $x^3+10x^2y+8y^3=3xy^2$ is a homogeneous equation. What is its degree? Can a homogeneous equation have a term free from the unknown number?

6. Is $3x^2-5y^2=2a^2$ homogeneous? Why? Write a homogeneous linear equation in two unknown numbers; also an integral, literal, quadratic, non-homogeneous equation in two unknown numbers.

Solve the following equations, using the methods of Chapter III, and also § 72:

7. $\frac{x}{2}-\frac{2x}{3}+\frac{2x-3}{6}+5=0.$

8. $x-3x+4-\left(3x+2-\frac{x}{4}\right)=0.$

9. $\frac{2x-4}{5}-\frac{3x-7}{7}+2=0.$

10. $2ax=2c-3bx.$

11. $\frac{x}{2a}-\frac{a}{4b}=c.$

12. $(a-x)(a-b)-a(b-x)=0.$

13. $x^2-x=6.$

14. $x^2+(a-b)x=ab.$

15. $x^3+2x^2=x+2.$

16. Find three solutions of $5x-3y=7$.

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* The appropriateness of this name will be seen in § 115.

95. Equivalent equations. Two equations are said to be **equivalent** if every root of either is also a root of the other.

The methods thus far employed for solving equations (in Chapter III, and elsewhere) consist in clearing equations of fractions, transposing and collecting terms, etc., *i.e.*, these methods consist in deducing from any given equation a succession of *new* equations whose roots are more and more easily found, and then finding the root of the *simplest* of these new equations, — compare Exs. 1 and 2, § 24.

That the root of this final simplest equation *happens* also to be a root of the given equation depends upon the following principles:

(1) *Adding*[*] *the same number to each member of any given equation, forms a new equation which is equivalent to the first* (cf. § 24, Ax. 1).

(2) *Multiplying*[*] *each member of an equation by the same number or algebraic expression, which does not involve the unknown number, and which has a finite value different from zero, forms a new equation which is equivalent to the first* (cf. § 24, Ax. 2).

To prove Principle (1) let the members of any given equation be represented by E_1 and E_2 respectively, *i.e.*, let the equation be

$$E_1 = E_2 \qquad (1)$$

This does not mean that E_1 and E_2 represent the same number for *every* value that may be substituted for the unknown number, but that they represent the same number only when a *root* of the equation is substituted for the unknown number.

But manifestly, if N represents any number whatever, then

$$E_1 + N = E_2 + N \qquad (2)$$

whenever $E_1 = E_2$; *i.e., every root of Eq.* (1) *is also a root of Eq.* (2).

By precisely the same reasoning, every root of Equation (2) is also a root of

$$(E_1 + N) + (-N) = (E_2 + N) + (-N), \qquad (3)$$

i.e., of

$$E_1 = E_2.$$

Hence, *every root of Equation* (1) *is a root of Equation* (2), *and vice versa;* therefore these equations are equivalent.

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[*] Since adding a negative number is the same as subtracting a positive number of the same absolute value, and since dividing by any number is the same as multiplying by its reciprocal, therefore subtraction and division are included in these statements.

To prove Principle (2), it is simpler first to write Equation (1) in the form

$$E_1 - E_2 = 0, \tag{4}$$

which, by Principle (1), is equivalent to Equation (1).

If now N represents any finite number that does not contain the unknown number, and is not zero, then manifestly

$$N(E_1 - E_2) = 0 \tag{5}$$

for every value of the unknown number which makes $E_1 = E_2$, and for no others, *i.e., every root of Equation* (1) *is also a root of Equation* (5), *and vice versa ; i.e.,* Equation (1) and Equation (5) are equivalent.

NOTE 1. That the multiplier in Principle (2) above must not contain the unknown number, and that it must not be zero, becomes evident on examining any given equation, *e.g.*, $3x - 4 = 2$. On multiplying each member of this equation by $x - 3$, and simplifying, it becomes $3x^2 - 15x + 18 = 0$; but since 3 is a root of this equation, and not a root of $3x - 4 = 2$, therefore the two equations are not equivalent.

So, too, if each member of the given equation be multiplied by zero it becomes $(3x - 4) \cdot 0 = 2 \cdot 0$, of which any finite number whatever is a root, and hence the new equation is not equivalent to the given one.

NOTE 2. The *language* in this discussion applies to equations containing only one unknown number, though it is evident that the same argument is applicable however many unknown numbers may be involved.

EXERCISES

1. Apply Principles (1) and (2) of § 95 in solving the equation $\frac{x-1}{2} + \frac{x-3}{5} + 3x = 0$; and show in detail that each derived equation is equivalent to the one preceding, and thus to the given equation.

2. Show that the equation $6x - 30 = \frac{18x - 12}{7} + 36$ is equivalent to $x - 5 = \frac{3x - 2}{7} + 6$; and that each of these is equivalent to $7x - 35 = 3x - 2 + 42$, and therefore to $4x = 75$, *i.e.*, to $x = 18\frac{3}{4}$.

3. Provided that no error has been made in the transformations in Ex. 2, do we really know, without verifying, that $18\frac{3}{4}$ is a root of the given equation?* Why?

4. Show that Principle (1) above includes the principle of transposition (§ 25); and that Principle (2) is far more useful in solving equations than Axiom 2, § 24 (cf. Note 1, § 95).

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* Though it is no longer *necessary* to verify that the root of the last of such a set of equations as those in Ex. 2 is also a root of the given equation — because of the principles of § 95 — yet verifying serves as a check upon the correctness of the actual work, and is still recommended.

Apply Principles (1) and (2) of § 95 in solving the following equations; and in particular point out the equivalence of the several equations involved in each exercise, and the reason for this equivalence:

5. $\frac{x}{2} - \frac{2x-3}{4} + \frac{3x-15}{8} - 2x = 15.$

6. $3x - 3(2x+15) + 2(x-2) - 14 = 0.$

7. $(3x-5)(x-2) - 4x^2 + 14x - 12 = 0.$

8. $\frac{4x-5}{3} = \frac{7x-15}{4} - \frac{4(3x-2)}{2}.$

9. $\frac{13x-3}{4} - \frac{7x-2}{2} = -15x + 16.$

10. $1 = \frac{2x-5}{5} - \frac{3x-2}{7} + \frac{x+2}{6}.$

11. $\frac{x-1}{2} - \frac{x-2}{3} = \frac{2}{3} - \frac{x-3}{4}.$

12. $\frac{7x+1}{5} = 1 + 2x - \frac{4x+7}{7}.$

13. $\frac{\frac{2x}{3}+4}{2} = \frac{7\frac{1}{2}-x}{3} + \frac{x}{2}\left(\frac{6}{x}-1\right).$

14. $1.75x + \frac{3+.5x}{.25} = \frac{.25x-2.375}{1.125}.$

15. $\frac{2y-9}{3} + 1 - \frac{3y-4}{7} = 0.$

96. Literal equations. The same method that has been followed in the solution of numerical equations, and the same principles as are there involved, apply also to literal equations.

E.g., given the equation $ax + b = cx + d$; to find x. This equation is equivalent to

$$ax - cx = d - b, \qquad [\S\ 95\ (1)$$

i.e., to

$$x(a-c) = \frac{d-b}{a-c},$$

and hence to

$$x = \frac{d-b}{a-c}, \qquad [\S\ 95\ (2)$$

which is the required root.

Show that the root just found will serve as a formula for solving any equation of this kind [cf. § 9 (ii)].

97. A simple equation in one unknown number has one and but one root. By transposing and uniting terms, etc., every equation of the first degree, which is not an identity, and which contains only one unknown number, is easily reduced to an equivalent equation of the form $ax = b$ (§ 95); but this last equation has, manifestly, one and but one root, viz., $b \div a$, hence the given equation has one and but one root, which was to be proved.

EXERCISES

1. What is a literal equation? A numerical equation? To which class does $2x - 13 + ax = 14x$ belong?

2. Find the roots of $x^2 - 5x + 6 = 0$. How many roots has this equation? By factoring its first member *prove* that this equation has the two roots 3 and 2, and that it has no other root whatever.

3. How many roots has $3x - 1 = x + 3$? How do you know that it really has one root? Prove that it has only one.

4. By the formula of § 96, solve the equation in Ex. 3, and explain fully.

Solve the following literal equations; show in detail that the steps you employ always yield equivalent equations; verify the correctness of your solution in Exs. 5–10 by actual substitution of the roots. Also solve Exs. 5–8 by using the formula of § 96:

5. $bx - (a + b)x + 20 - cx = d.$

6. $c^3 - x + n^3x = n - c - x.$

7. $\frac{x}{b} - \frac{x + 2b}{a} = \frac{a}{b} - 3.$

8. $\frac{x - 2ab}{c} - 1 = \frac{x - 3c}{ab}.$

9. $7x + 5\left(1 - \frac{3x}{b}\right) = a(x - a).$

10. $\frac{x - 4a}{2b} + \frac{x}{a} = \frac{a^2 + 4b^2}{ab}.$

11. $d(3x - 9c + 14b) = c(c - x).$

12. $(a - b)(x - c) - (b - c)(x - a) = (c - a)(x - b).$

13. $(x - a)(a - b + c) = (x + a)(b - a + c).$

14. $b(c - x) + a(b - x) - b(b - x) = 0.$

15. $\frac{4(3 - 2x)}{m - n} - \frac{x}{n^2 - m^2} = \frac{3}{m + n}.$

16. $a^3x + b^3x + 3x(a^2b + ab^2) = 3ab.$

17. $\frac{4x - 3a}{b} + \frac{5x}{a} = 5 + \frac{15b}{a}.$

18. $\frac{x + a}{b} + \frac{x + c}{a} + \frac{x + b}{c} = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 1.$

19. $\frac{ax + bx - a^2 - ab}{a} = \frac{2bx - 2b^2 + 2ax - 2ab}{b}.$

20. If an equation is an identity (cf. § 23), how many roots has it? If the equation $ax = b$ is an identity, what is the value of a? Of b? Of the root $b \div a$ (cf. § 55)? Show that this is entirely consistent.

II. EQUATIONS INVOLVING FRACTIONS

98. Fractional equations. Equations containing expressions which are fractional with regard to an unknown number (§ 41) are usually called **fractional equations.** Such equations frequently present themselves in connection with practical problems, and the process of solving them will now be illustrated; the demonstration of the principles involved is given in the next article.

Ex. 1. Given the equation $\frac{3}{x}-\frac{1}{2}=\frac{5}{3x}+\frac{1}{6}$; to find the value of x.

Solution. If each member of the given equation be multiplied by $6x$ (the L. C. M. of the denominators), it becomes

$$18-3x=10+x,$$

whence, by § 95, $x=2$;

moreover, by substituting 2 for x, it is found that the given equation is satisfied, hence 2 is a root of this equation.

Ex. 2. Given $\frac{3}{2(x-1)}-\frac{1}{7(x+1)}=\frac{8}{x+1}-\frac{10}{7(x^2-1)}$; to find x.

Solution. On multiplying this equation by $2\cdot 7\cdot(x+1)\cdot(x-1)$, which is the L. C. M. of the denominators, it becomes

$$3\cdot 7\cdot(x+1)-2(x-1)=8\cdot 2\cdot 7\cdot(x-1)-20,$$

i.e., $21x+21-2x+2=112x-112-20,$

whence, $x=\frac{5}{3}$;

and, on being substituted for x in the given equation, $\frac{5}{3}$ proves to be a root of that equation.

Ex. 3. Given $\frac{7}{6}+\frac{x^3-1}{x^2-1}=x$; to find x.

Solution. On multiplying this equation by $6(x^2-1)$, it becomes

$$7(x^2-1)+6(x^3-1)=6x(x^2-1),$$

i.e., $7x^2-7+6x^3-6=6x^3-6x,$

whence, $7x^2+6x-13=0,$

i.e., $(x-1)(7x+13)=0,$

and the roots of this equation are 1 and $-\frac{13}{7}$. [§ 72

But by trial it is found that $-\frac{13}{7}$ is a root of the given fractional equation, and that 1 is *not* a root of that equation.

Note. Clearing an equation of fractions *may* bring in **extraneous roots,** *i.e.*, roots which do not belong to the given equation; this is illustrated in Ex. 3 where the extraneous root 1 was brought in by multiplying by the unnecessary factor $x-1$ in clearing the given equation of fractions; the fraction $\frac{x^3-1}{x^2-1}$ might first have been reduced to its lowest terms.

The method employed for solving fractional equations in the examples given above may be stated thus: (1) *clear the given equation of fractions by multiplying it by the L.C.M. of its denominators,* (2) *solve the resulting integral equation, and* (3) *substitute the roots of this integral equation in the given fractional equation, and reject those which prove to be extraneous.*

EXERCISES

Solve the following equations:

4. $\frac{2x}{3}-\frac{5}{6}=\frac{x}{4}.$

5. $\frac{x-3}{7}+\frac{x+5}{3}-\frac{x+2}{6}=4.$

6. $\frac{x-1}{2}+\frac{x-2}{3}+\frac{x-3}{4}-\frac{1-5x}{6}=0.$

7. $\frac{4}{x}-\frac{13}{16}=1+\frac{3}{8x}.$

8. $\frac{3}{x+1}-\frac{2}{x-1}=0.$

9. $\frac{y-1}{y+1}=1-\frac{1}{y}.$

10. $\frac{7}{10}-\frac{1}{4y}=\frac{3}{5y}-1.$

11. $\frac{2x^2}{x^2-1}+\frac{x}{x-1}=\frac{x}{x+1}+3.$

12. Define a fractional equation. Are the equations in Exs. 4–6 fractional? Are Exs. 7–11 fractional? Why?

13. In solving the equations in Exs. 4–6, are the successive equations equivalent? Why? Is this true with reference to Exs. 7–11 also?

14. If in Ex. 11 we clear of fractions and simplify, we obtain the equation $x^2-2x-3=0$, *i.e.*, $(x+1)(x-3)=0$, whose roots are -1 and 3. Is 3 a root of the given equation? Is -1 a root (cf. § 55)? Was the factor $x+1$ necessary to clear of fractions? Compare also Ex. 3, which is solved above.

Solve the following equations, and test your results:

15. $\frac{x-1}{x-2}+\frac{x+2}{x+1}=\frac{1}{x^2-x-2}.$

16. $\frac{y}{3-y}-\frac{7}{8}+\frac{2y-3}{y+3}=\frac{y}{8(y+3)}.$

17. $\frac{z-5}{z+5}-\frac{z-10}{z+10}=\frac{z-4}{z+4}-\frac{z-9}{z+9}.$

SUGGESTION. Simplify each member before clearing of fractions.

18. $\frac{x+1}{x+2}-\frac{x+2}{x+3}=\frac{x+5}{x+6}-\frac{x+6}{x+7}.$

19. $\frac{x-1}{x-2}+\frac{x-7}{x-8}=\frac{x-5}{x-6}+\frac{x-3}{x-4}.$

20. $\frac{x^3+2}{x+1}-\frac{x^3-2}{x-1}=\frac{10}{x^2-1}-2x.$

21. $\frac{x}{2}(2-x)-\frac{x}{4}(3-2x)=\frac{x+10}{6}.$

22. $\frac{1}{1-x}+\frac{6}{x}=6+\frac{x^2}{1-x}.$

23. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}.$

24. $\frac{2x}{3}\cdot\frac{(5x-3)}{10x^2-1}-\frac{1}{3}=0.$

25. $\frac{2x\left(1-\frac{5}{x}\right)}{3}+\frac{3x\left(1-\frac{4}{x}\right)}{4}=\frac{x-4}{\frac{4}{5}}.$

26. $\frac{17+\frac{3}{x}}{3}+\frac{1+\frac{18}{x}}{5}=\frac{\frac{21}{x}-1}{9}+\frac{\frac{100}{x}+\frac{5}{3}}{15}.$

27. $1+\frac{1}{1+\frac{1}{x}}=\frac{2+\frac{10}{x}}{1+\frac{6}{x}}.$

28. $\frac{2c}{a}+\frac{b}{x}=\frac{c}{2-x}-\frac{2cx}{a(2-x)}.$

29. $\frac{x^2-a^2}{x+b}=a-b+\frac{x^2-b^2}{x+a}.$

30. $\frac{x^2-ax}{x^2+cx-ax-ac}+\frac{a-x}{x-c}=0.$

31. $\frac{x+7a}{x+6a}+\frac{x-a}{x-3a}=\frac{x+7a}{x+a}-\frac{a-x}{2a+x}.$

32. $\frac{1}{a(b-x)}+\frac{1}{b(c-x)}-\frac{1}{a(c-x)}=0.$

99. Demonstration of principles involved in § 98. The success of the method employed in § 98 for solving fractional equations is due to the fact that, in the great majority of cases, the integral equation obtained by clearing an equation of fractions is equivalent (§ 95) to the given fractional equation; the exceptions, as the student may already have observed, are those in which an unnecessary factor is used to clear of fractions (cf. Exs. 3 and 11, § 98).

To prove the above, let it first be recalled that transposing and uniting terms (whether those terms are integral or fractional) leads to an equivalent equation (§ 95). Hence, by performing these operations, every fractional equation may be reduced to an equivalent equation of the form

$$\frac{N}{D}=0, \quad (1)$$

wherein N and D represent integral expressions in the unknown number (say x), and D is the L. C. M. of the denominators of the fractions in the given equation.

If now N and D have no common factor (as usually happens), then (§§ 72 and 48) there is no value of x for which both N and D will become zero, and therefore Eq. (1) is equivalent to

$$N = 0, \tag{2}$$

i.e., the *given equation* is equivalent to Eq. (2); but Eq. (2) is the result of clearing the given equation of fractions, hence, *in all such cases*, clearing an equation of fractions leads to an equivalent integral equation.

If, on the other hand, N and D have a common factor — which rarely happens — then Eq. (2) is not equivalent to Eq. (1); for, if $N = F \cdot N'$ and $D = F \cdot D'$, where F is the H. C. F. of N and D, then only those values of x for which

$$N' = 0 \tag{3}$$

are roots of Eq. (1), while Eq. (2) has all of these roots, and also those for which

$$F = 0; \tag{4}$$

these extraneous roots were brought in by using the unnecessary factor F to clear the given equation of fractions, — *they are those roots of Eq. (2) which will make $D = 0$, and are, therefore, easily detected.*

EXERCISES

1. Show that clearing the equation $\frac{x}{x-7} - \frac{7}{x+2} + \frac{55}{9(x+2)} = \frac{55}{7(x-7)}$ of fractions, by the usual method, produces an integral equation which is equivalent to the given fractional equation, *i.e.*, show that multiplying this equation by the L. C. M. of its denominators introduces no extraneous root.

2. Show that while 2 is a root of the integral equation resulting from clearing $\frac{3x}{x+5} + \frac{42}{(x+5)(x-2)} = 8 + \frac{6}{x-2}$ of fractions, it is not a root of the fractional equation itself. What is the value of D (see demonstration above) for this equation when $x = 2$? How may extraneous roots be most easily detected?

Solve the following equations, and tell, by mere inspection, *i.e.*, without substituting in the original equation, which of the roots of the integral equations, if any, are extraneous; also state your reasons in full:

3. $\frac{20}{x+3}+\frac{40}{x^2+4x+3}+7=\frac{4x}{x+1}.$

4. $\frac{3x}{x+1}-\frac{15}{3x^2+x-2}=\frac{10}{3x-2}-5.$

5. $\frac{2}{x-5}-\frac{5x}{3x+2}=\frac{x+29}{(x-5)(3x+2)}-3.$

6. $\frac{12x}{3x-7}-\frac{2x+6}{x-3}=1.$

PROBLEMS

By the method of § 26 solve the following problems, applying also the principles of the present chapter:

1. If $\frac{5}{6}$ of a certain number is diminished by $\frac{1}{4}$ of that number, and the result is 3 more than $\frac{1}{3}$ of the number, what is the number?

2. B's present age is 18 years, which is $\frac{1}{3}$ of A's age; after how many years will B's age be $\frac{2}{5}$ of A's age?

3. The tail of a fish is 4 inches long. Its head is as long as its tail and $\frac{1}{7}$ of its body, and its body is as long as the head and $\frac{1}{7}$ of its tail; how long is its body?

4. Mary, who is now 24 years old, is twice as old as Ann was when Mary was as old as Ann now is. How old is Ann?

5. A boy bought some apples for 24 cents; had he received 4 more for the same sum, the cost of each would have been 1 cent less. How many did he buy?

6. A reservoir is fitted with three pipes, one of which can empty it in 4 hours, another in 3 hours, and the third in $1\frac{5}{7}$ hours. If the reservoir is half full, and the three pipes are opened, in what time will it be emptied?

7. A man's age is such that $\frac{2}{5}$ of it, less $\frac{1}{9}$ of what it will be a year hence, is equal to $\frac{1}{3}$ of what it was 5 years ago; how old is he?

8. An orchard has twice as many trees in a row as it has rows. By increasing the number of trees in a row by 2, and the number of rows by 3, the whole number of trees will be increased by 126. How many trees are there in the orchard?

9. What number must be added to each term of the fraction $\frac{7}{11}$ so that the resulting fraction shall be $\frac{3}{4}$?

10. If a certain number be added to, and also subtracted from, each term of the fraction $\frac{5}{8}$, the first result will exceed the second by $\frac{1}{2}$. What is the number? How many solutions has this problem?

11. The combined cost of a table and a chair is $11, of the table and a picture, $14, and the chair and the picture together cost 3 times as much as the table. What is the cost of each?

12. A field is twice as long as it is wide, and increasing its length by 20 rods and its width by 30 rods, increases its area by 2200 square rods. What are the dimensions of this field?

13. In a certain quantity of gunpowder, which is a mixture of saltpeter, sulphur, and charcoal, the saltpeter weighs 6 lb. more than $\frac{1}{2}$ the whole, the sulphur 5 lb. less than $\frac{1}{3}$ of the whole, and the charcoal 3 lb. less than $\frac{1}{4}$ of the whole. How many pounds of each of these constituents are contained in this quantity of gunpowder?

14. An officer in forming his men into a solid square, with a certain number on a side, finds that he has 49 men left over, and if he puts 1 more man on a side he lacks 50 men of completing the square. How many men has he?

15. A regiment drawn up in the form of a solid square lost 60 men in battle, and when the men were rearranged with 1 less on a side, there was 1 man left over. How many men were there in this regiment?

16. In a regiment which is drawn up in the form of a solid square, it is found that the number of men in the outside 5 rows, counted all around, is $\frac{9}{25}$ of the entire regiment. How many men are there in this regiment? Has the *equation of this problem* [cf. § 26 (3)] one or two solutions? Is each also a solution of the problem itself?

17. A boy was engaged at 15 cents a day, to deliver a daily paper to those of its subscribers who live in a certain part of the city, with the added condition, however, that he was to forfeit 5 cents for every day he failed to perform this service; at the end of 60 days he received $7. How many days did he serve?

18. A man was hired for 30 days on the following terms: for every day he worked he was to receive $2.50 and his board, while for every day he was idle he was not only to receive nothing, but was charged 75 cents for his board. If at the end of the period he received $49, how many days did he work?

19. A steamer can sail 20 miles an hour in still water. If it can sail 72 miles with the current in the same time that it can sail 48 miles against the current, what is the velocity of the current?

20. A steamer now goes 5 miles downstream in the same time that it takes to go 3 miles upstream, but if its rate each way is diminished by 4 miles an hour, its downstream rate will be twice its upstream rate. What is its present rate in each direction?

21. A man rows downstream at the rate of 6 miles an hour, and returns at the rate of 3 miles an hour. How far downstream can he go and return if he has 9 hours at his disposal? At what rate does the stream flow?

22. The sum of two numbers is 18, and the quotient of the less divided by the greater is equal to $\frac{1}{5}$. What are the numbers?

23. Divide the number 25 into two such parts that the square of the greater part exceeds by 75 the square of the lesser part.

24. Divide 72 into four parts, such that if the first part be divided by 2, the second multiplied by 2, the third increased by 2, and the fourth diminished by 2, the four results will all be equal.

25. What number must be subtracted from each of the four numbers, 20, 24, 16, and 27, so that the product of the first two remainders shall equal the product of the second two?

26. Find a number such that its square being diminished by 9, and this remainder being divided by 10, the quotient is greater by 3 than the number itself. How many solutions has this problem?

27. A line 28 inches long is divided into two parts of which the length of the shorter is $\frac{3}{4}$ that of the longer. What is the length of each part?

28. An automobile runs 10 miles an hour faster than a bicycle, and it takes the automobile 6 hours longer to run 255 miles than it does the bicycle to run 63 miles. Find the rate of each. How many solutions has the equation of this problem? Is each of these also a solution of the problem itself?

29. At what time between 2 and 3 o'clock are the hands of a clock together?

SUGGESTION. Let x represent the number of minute spaces over which the minute hand passes from 2 o'clock on, until it overtakes the hour hand between 2 and 3 o'clock, then show that $\frac{x}{12}+10$ represents the same number, and thus form an equation and find the value of x.

30. At what time between 3 and 4 o'clock is the minute hand 15 minute spaces ahead of the hour hand?

31. At what time between 8 and 9 o'clock are the hands of a clock together?

32. At what time between 4 and 5 o'clock do the hands of a watch extend in opposite directions?

33. At what time between 9 and 10 o'clock is the hour hand 20 minute spaces in advance of the minute hand?

34. In an alloy of silver and copper weighing 90 oz., there is 6 oz. of copper; find how much silver must be added in order that 10 oz. of the new alloy shall contain but $\frac{2}{5}$ of an ounce of copper.

35. If 80 lb. of sea water contains 4 lb. of salt, how much fresh water must be added in order that 45 lb. of the new solution may contain $1\frac{2}{3}$ lb. of salt?

36. What *percentage* of evaporation must take place from a 6% solution of salt and water (salt water of which 6% by weight is salt) in order that the remaining portion of the mixture may be a 12% solution? That it may be an 8% solution?

37. How many minutes is it before 4 o'clock, if $\frac{3}{4}$ of an hour ago it was twice as many minutes past 2 o'clock?

38. If the specific gravity of brass is $8\frac{2}{5}$,* while that of iron is $7\frac{1}{2}$, and if an alloy of brass and iron, which weighs 57 lb., displaces 7 lb. of water when it is immersed, what is the weight of each of these metals in the alloy?

39. If, on being immersed in water, 97 oz. of gold displaces 5 oz. of water, and 21 oz. of silver displaces 2 oz. of water, how many ounces of gold and of silver are there in an alloy of these metals which weighs 320 oz., and which displaces 22 oz. of water? What is the specific gravity of each of these metals and of the alloy?

40. Two boat builders, A and B, working together, can build a boat of a certain size in 12 days, and A, working alone, can build such a boat in 18 days. In how many days can B alone build such a boat (cf. Prob. 31, § 26)?

41. A, B, and C together can do a piece of work in $3\frac{1}{5}$ days; B can do $\frac{1}{2}$ as much as A, and C can do $\frac{3}{4}$ as much as B. In how many days can each do this work alone?

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* This means that a given volume of brass weighs $8\frac{2}{5}$ times as much as an equal volume of water.

42. A can do a certain piece of work in 6 days, and B can do the same work in 14 days. A, having begun this work, had later to abandon it, when B took his place and finished the work in exactly 10 days from the time it was begun by A. How many days did B work at it?

43. A and B can dig a certain trench in 10 days, B and C can dig it in 6 days, and A and C in $7\frac{1}{2}$ days. How long would it take each working alone to do this work?

44. The first of three outlet pipes can empty a certain cistern in 2 hr. and 40 min., the second in 3 hr. and 15 min., and the third in 4 hr. and 25 min. If the cistern is $\frac{3}{4}$ full, and all three pipes are opened at the same time, how long will it take to empty it?

45. A gentleman invested $\frac{1}{6}$ of his capital in 4% bonds,* $\frac{3}{4}$ of it in $3\frac{1}{2}$% bonds, and the remainder in 6% bonds, purchasing all these bonds at par. If his total annual income is $2100, what is the amount of his capital?

46. A gentleman made two investments amounting together to $4330; on the first he lost 5% and on the second he gained 12%. If his *net* gain was $251, what was the amount of each investment?

47. An estate was divided among four heirs, A, B, C, and D. The amounts received by A and B were, respectively, $\frac{1}{3}$ and $\frac{1}{4}$ of an amount $1000 less than the estate; and C and D received, respectively, $\frac{1}{5}$ and $\frac{1}{6}$ of an amount greater than the estate by $\frac{1}{5}$ of it. How much did each receive?

48. A wheelman and a pedestrian start at the same time for a place 54 miles distant, the former going 3 times as fast as the latter; the wheelman, after reaching the given place, returns and meets the pedestrian $6\frac{3}{4}$ hours from the time they started. At what rate did each travel?

49. A girl found that she could buy 12 apples with her money and have 5 cents left, or 10 oranges and have 6 cents left, or 6 apples and 6 oranges and have 2 cents left. How much money had she?

50. Find a fraction whose numerator is greater by 3 than one half of its denominator, and whose value is $\frac{2}{3}$.

51. The numerator of a certain fraction is less by 8 than its denominator, and if each of its terms be decreased by 5, its value will be $\frac{1}{4}$; what is the fraction?

52. The tens' digit of a certain two-digit number is $\frac{1}{2}$ the units' digit, and if this number, increased by 27, be divided by the sum of its digits, the quotient will be $6\frac{1}{4}$. What is the number (cf. Prob. 4, § 26)?

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* Bonds yielding 4% interest per annum.

53. A certain number is increased by 1, and also diminished by 1, and it is then found that 3 times the reciprocal of the first result, being increased by $\frac{1}{4}$, equals 2 times the reciprocal of the second. What is this number? How many solutions has this problem?

54. A steamer's speed is such that, on a certain stream, it takes as long to go 3 miles upstream as it does to go 5 miles downstream, but if its rate in still water were 4 miles less per hour, its downstream rate would be twice its upstream rate. What is its rate in still water?

55. A physician having a 6% solution of a certain kind of medicine wishes to dilute it to a $3\frac{1}{2}$% solution. What percentage of water must he add to the present mixture?

56. A physician having a 6% solution, and also a 3% solution, of a certain kind of medicine, mixes these in such proportions as to form a $3\frac{3}{4}$% solution. What percentage of the new mixture is taken from each of the given mixtures?

57. A tourist ascends a certain mountain at an average rate of $1\frac{1}{4}$ miles an hour, and descends by the same path at an average rate of $4\frac{1}{2}$ miles an hour. If it takes him $6\frac{2}{3}$ hours to make the round trip, how long is the path?

58. If a father takes 3 steps while his son takes 5, and if 2 of the father's steps are equal in length to 3 of the son's, how many steps will the son have to take before he overtakes his father, who is 36 of his own steps ahead?

Solution. The simplest way to form the equation of this problem is to compare two *lengths*. To do this

let $l =$ the number of feet in the son's step,

then $\frac{3l}{2} =$ the number of feet in the father's step;

also let $x =$ the number of steps the son must take,

then $\frac{3x}{5} =$ the number of steps the father will take;

and the equation of the problem is

$$xl = \frac{3x}{5} \cdot \frac{3l}{2} + 36 \cdot \frac{3l}{2}, \qquad \text{(why ?)}$$

i.e., $x = \frac{9x}{10} + 54$; whence $x = 540$.

Observe that fractions could have been avoided by letting $5x$ and $2l$, respectively, stand for the number and length of the son's steps.

59. A hare pursued by a hound takes 4 leaps while the hound takes 3, but 2 of the hound's leaps are equal in length to 3 of the hare's. If the hare has a start equal to 60 of her own leaps, how many leaps must the hound take to catch the hare?

60. Solve Prob. 59 if all its conditions are unchanged except that the hare's start is equal to 60 of the hound's leaps.

61. A merchant added annually to his capital an amount equal to $\frac{1}{3}$ of it, but deducted at the end of each year $2000 for personal expenses. If after taking out the $2000 at the end of the third year, he finds that he has just twice his original capital, what was the original capital?

62. A pedestrian finds that his uphill rate of walking is 3 miles an hour, while his downhill rate is 4 miles an hour. If he walked 60 miles in 17 hours, how much of this distance was uphill?

63. A hound is 39 of his leaps behind a rabbit that takes 7 leaps while the hound takes 8. If 6 leaps of the rabbit are equal to 5 leaps of the hound, how many leaps must the hound take to catch the rabbit?

64. A picture whose length lacks 2 inches of being twice its width, is inclosed in a frame 4 inches wide. If the length of the frame divided by its width, plus the length of the picture divided by its width, is $3\frac{1}{3}$, what are the dimensions of the picture? How many solutions has the equation of this problem? Is each of these a solution of the problem also?

III. GENERAL PROBLEMS

100. General problems. Interpretations of their solutions. A problem in which the known numbers are represented by letters, instead of by the Arabic characters, is often called a **general problem,** because it includes all those particular problems which may be obtained by giving particular values to these letters — compare § 9, and also the illustrations given below.

Prob. 1. A yacht was chartered for a pleasure party consisting of p persons, the expense to be shared equally by those participating; q of the proposed party being unable to go, it was found necessary for each person who did go to pay d dollars more than would otherwise have been necessary. How much was paid for the yacht? How much was each to pay under the original arrangement?

Solution

Let $x =$ the number of dollars each member of the original party was to have paid, then $x + d$ is the number of dollars that each participant actually did pay, while px and $(p-q)\cdot(x+d)$ are two expressions, each of which represents the number of dollars charged for the yacht; hence

$$px = (p-q)(x+d) = px + pd - qx - qd;$$

whence $x = \frac{d(p-q)}{q}$, the amount each was to pay,

and $px = p \cdot \frac{d(p-q)}{q}$, the price of the yacht.

The student may solve this problem independently if $p = 12$, $q = 3$, and $d = 2$, and compare the results with those obtained by substituting these values for p, q, and d in the above general solution (formula).

Prob. 2. Divide m golf balls into two groups in such a way that the first group shall contain n balls more than the second group.

Solution

Let $x =$ the number of balls in the first group.

Then $m - x =$ the number of balls in the second group,

and, therefore, by the condition of the problem,

$$x = m - x + n;$$

whence $x = \frac{m+n}{2}$, the number of balls in the first group, and $m - x = m - \frac{m+n}{2} = \frac{m-n}{2}$, the number of balls in the second group.

As in the previous problem, so here, the general solution just obtained may be employed to obtain the solution of any particular problem of the same kind. For example, if $m = 30$ and $n = 4$, then the two groups contain, respectively, $\frac{30+4}{2}$ and $\frac{30-4}{2}$ balls, *i.e.*, 17 and 13; while, if $m = 10$ and $n = 2$, then the two groups contain 6 and 4 balls, respectively.

If, however, $m = 10$ and $n = 14$, then the number of balls in the two groups, as given by the above solution, is $\frac{10+14}{2}$ and $\frac{10-14}{2}$, respectively, *i.e.*, 12 and -2; but since there can not be an actual group containing -2 golf balls, therefore this last problem is impossible, and the impossibility is indicated by the negative result.

Note 1. Some problems admit of negative results, and some do not, just as some problems admit of fractional results, while others do not. The nature of the *things* with which any particular problem is concerned will always make it evident whether or not fractional or negative solutions are admissible.

For example, let it be required to find the temperature at Chicago on a certain day, it being known that on that day the sum of the thermometer readings at New York and Chicago is 10°, their difference 14°, and that it is colder in Chicago than in New York.

Let the reading at Chicago be x degrees. Then it is $(10-x)$ degrees at New York, and the other condition of the problem becomes $x=(10-x)-14$, whence $x=-2$, *i.e.*, the reading at Chicago is 2° below zero. The negative result is admissible in this problem.

NOTE 2. Observe also that two algebraic problems which differ widely with regard to the things with which the problems are concerned may yet give rise to the same equation, and the solution of this equation may be a solution of one of the problems, while it merely shows that the other problem demands what is impossible of fulfilment.

Thus, if the head of a certain fish is $7\frac{1}{2}$ inches long, the tail as long as the head and $\frac{1}{3}$ of the body, and the body as long as the head and tail together, how long is the body of the fish?

If $x=$ number of inches in the length of the body, then the second condition of the problem becomes $x=7\frac{1}{2}+7\frac{1}{2}+\frac{x}{3}$, *i.e.*, $x=15+\frac{x}{3}$, whence $x=22\frac{1}{2}$.

This number is found to satisfy all the conditions of the given problem, and is, therefore, not only the solution of the *equation*, but is also the solution of the *problem*.

Again, let it be required to find how many sparrows a certain flock must contain if $\frac{1}{12}$ of their number, plus $\frac{1}{4}$ of their number, plus 15, equals the whole number.

If $x=$ their number, then the given condition becomes $x=\frac{x}{12}+\frac{x}{4}+15$, *i.e.*, $x=15+\frac{x}{3}$, which is the same as the equation in the former problem, but the solution of this equation, viz., $x=22\frac{1}{2}$, is not now a solution of the *problem*, but merely shows the impossibility of fulfilling the conditions of the problem.

Prob. 3. Two boys, A and B, are running along the same road, A at the rate of a rods per minute, and B at the rate of b rods per minute; if B is m rods in advance of A, and if they continue running at the same rates, in how many minutes will A overtake B?

SOLUTION

Let $x =$ the number of minutes that must elapse before A overtakes B. Then, by the conditions of the problem,

$$ax = bx + m,$$

whence $x = \dfrac{m}{a-b}$, the number of minutes before A overtakes B.

As in the two previous problems, so here the general solution just obtained may be employed to find the solution of any particular problem of the same kind.

E.g., if $a=60$, $b=50$, and $m=90$, then $x=\frac{90}{60-50}=9$, *i.e.*, A will overtake B in 9 minutes.

Again, if $a = 50$, $b = 50$, and $m = 90$, then $x = \frac{90}{50 - 50} = \frac{90}{0}$, *i.e.*, an infinite number of minutes will elapse before A overtakes B; in other words, A will *never* overtake B. Compare § 55, and also the note to Ex. 15, of § 55.

But if $a = 50$, $b = 60$, and $m = 90$, then $x = \frac{90}{50 - 60} = -9$, *i.e.*, the two boys are together -9 minutes from the moment they were observed, and since adding -9 minutes to the present time is the same as *subtracting* 9 minutes from the present time, therefore the two boys *were* together 9 minutes ago.

This interpretation of the negative result accords fully with the physical conditions of the actual problem, because if B is already 90 rods in advance of A, and if he is running 10 rods per minute *faster* than A, he will not only keep getting farther and farther ahead of A, but he must also have passed him 9 minutes ago.

Prob. 4. The present ages of a father and son are respectively 50 and 20 years; after how many years will the father be 4 times as old as the son?

SOLUTION

Let x = the number of years from now to the time when the father's age shall be 4 times that of the son. Then, by the conditions of the problem,

$$50 + x = 4(20 + x),$$

whence

$$x = -10.$$

This means that *10 years ago* the age of the father *was* 4 times that of the son.

N.B. The general problem, which includes Prob. 4 as a particular case, may be stated thus: The present ages of a father and son are, respectively, m and n years; after how many years will the father's age be p times that of the son?

EXERCISES AND PROBLEMS

5. Is Prob. 25 of § 99 a particular or a general problem? Why? Formulate a general problem which shall include this one as a particular case. Solve the new problem and thus find a formula by which Prob. 25 may be solved.

6. Answer the questions in Ex. 5 above, supposing them to have been asked with regard to Prob. 24, p. 153.

7. Answer the questions in Ex. 5 above, supposing them to have been asked with regard to Prob. 10, p. 152.

8. Does Prob. 24, p. 153, admit of a fractional result? Of a negative result? Explain your answers.

9. By a slight change in the wording of Prob. 4, § 100, make an *equivalent* problem of which the answer shall be positive. This should agree with the *interpretation* there given of the negative result.

10. By slightly changing the wording of the last particular case under Prob. 3, § 100, make an *equivalent* problem whose answer shall be positive.

11. A farmer can plow a certain field in a days, and his son can plow the same field in b days. In how many days can both working together plow the field?

12. Is Prob. 11 a particular or a general problem? Make several examples of which it is the generalization. Solve one of these particular examples independently, and then show that its answer could have been obtained from the answer to Prob. 11.

13. At what time between n and $n+1$ o'clock will the hands of a clock be together? At what time between these hours will they be pointing in opposite directions, if $n<6$? If $n>6$? If $n=6$?

14. A father is m times as old as his son, and in p years he will be n times as old. Find their respective ages.

Interpret your result when $m<n$. Is p positive or negative in this case?

15. A merchant has two kinds of sugar worth, respectively, a and b cents a pound. How many pounds of each kind must be taken to make a mixture of n pounds worth c cents a pound?

Interpret the result if $a=b$, and c is less than a; also when $a=b=c$. Do these interpretations of the results agree with the conditions of the problem under the same suppositions?

16. An alloy of two metals is composed of m parts (by weight) of one to n parts of the other. How many pounds of each of the metals are there in a pounds of the alloy?

Show that the problem just stated is the generalization of such a problem as this: Bell metal is an alloy of 5 parts (by weight) of tin to 16 of copper; how many pounds of tin and of copper in a bell weighing 4200 lb.?

17. A wheelman sets out from a certain place at m miles an hour, and is pursued by a second wheelman, who starts from the same place a hours later, and rides p miles an hour. How far from the starting point will the second wheelman overtake the first? What does this result become if $m=10$, $p=12$, and $a=4$?

18. Two wheelmen, A and B, are observed passing a certain point, A being c hours in advance of B, and traveling at the rate of a miles in b hours, while B travels p miles in q hours. How far will A travel before he is overtaken by B?

Under what conditions is this solution positive? Negative? Zero? Infinite? *Interpret* the result in each case.

CHAPTER XI

SIMULTANEOUS SIMPLE EQUATIONS

I. TWO UNKNOWN NUMBERS

101. Indeterminate equations. Although a simple equation in one unknown number has one and but one solution (cf. § 97), yet it is easy to see that an equation which involves two or more unknown numbers has an infinite number of solutions.

E.g., in the equation $x + 3y = 5$, which is equivalent to

$$y = \frac{5 - x}{3}, \qquad \text{[§ 95}$$

there is a perfectly definite value of y corresponding to *every* value that one may choose to assign to x; thus, if $x = 1$, then $y = \frac{4}{3}$, if $x = 2$, $y = 1$, if $x = 3$, $y = \frac{2}{3}$, if $x = -1$, $y = 2$, and so on indefinitely; *i.e.*, each of these pairs of numbers, viz., 1 and $\frac{4}{3}$, 2 and 1, 3 and $\frac{2}{3}$, etc., constitutes a solution of the given equation, because, when substituted for x and y respectively, they satisfy that equation.

An equation, such as the one just now considered, which has an infinite number of solutions, is, for that reason, called an **indeterminate equation.**

102. Positive integral solutions of indeterminate equations. Although the number of solutions of an indeterminate equation, as has just been illustrated, is unlimited, yet it often happens that only solutions of a particular kind are sought, — *e.g.*, those that are positive integers, — and the number of these may be finite.

In practice the positive integral solutions of an indeterminate equation can usually be found by mere inspection, or by trial.

E.g., to find the positive integral solutions of the equation $2x + 3y = 7$, it is only necessary to assign to one of the unknown numbers, say x, the values 1, 2, 3, ⋯ in turn, and to find the corresponding values of the other unknown number, which are $\frac{5}{3}$, 1, $\frac{1}{3}$, ⋯; moreover, if $x = 4$, or any greater number, then y is negative, hence the *only* positive integral solution of the given equation is $x = 2$ and $y = 1$.

Many problems lead to indeterminate equations which, from the nature of the things involved, demand solutions that are positive integers.

E.g., a farmer spent \$22 purchasing two kinds of lambs, the first kind costing him \$3 each, and the second kind \$5 each. How many of each kind did he buy?

SOLUTION. Let $x =$ the number of the first kind,

and $y =$ the number of the second kind.

Then one condition of the problem is that

$$3x + 5y = 22,$$

and the other condition is that x and y shall be positive integers.*

By § 95, this equation is equivalent to $x = \frac{22 - 5y}{3}$,

and, if the values 1, 2, 3, and 4 be assigned to y, the corresponding values of x are found to be $\frac{17}{3}$, 4, $\frac{7}{3}$, and $\frac{2}{3}$; moreover, if $y = 5$, or more, then x is negative, and therefore the *only* positive integral solution of the above equation is $x = 4$ and $y = 2$; *i.e.*, 4 and 2 are, respectively, the numbers of lambs purchased.

103. Positive integral solutions: another method. Another method of finding the positive integral solutions of an indeterminate equation will now be illustrated.

Given the equation $7x + 4y = 46$; to find its positive integral solutions.

By transposing and dividing, this equation becomes

$$y = \frac{46 - 7x}{4} = 11 - x + \frac{2 - 3x}{4},$$

i.e.,

$$y - 11 + x = \frac{2 - 3x}{4},$$

and, since x and y are integers, therefore the first member of this equation represents an integer, and therefore the second member, viz., $\frac{2 - 3x}{4}$, also represents an integer.

Again, since $\frac{2 - 3x}{4}$ represents an integer, therefore the product obtained by multiplying it by any integer whatever also represents an integer; moreover, if this multiplier be so chosen that the new coefficient of x shall exceed some multiple of the denominator by 1 (cf. § 79), then the integral values of x and y may be easily determined as follows:

Since $\frac{2-3x}{4}$ represents an integer, therefore $\frac{3(2-3x)}{4} = \frac{6-9x}{4} = 1 - 2x + \frac{2-x}{4}$ represents an integer, and therefore $\frac{2 - x}{4}$ represents an integer. If this last integer be designated by p, then $\frac{2 - x}{4} = p$,

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* Although this condition is not expressible by means of an equation, yet it is none the less vital on that account.

whence $$x = 2 - 4p,$$

and, on substituting this value of x in the given equation, it becomes

$$y = 8 + 7p.$$

In these last two equations x and y are *positive* integers, and p is an integer, though not necessarily positive. This shows that p is either -1 or 0 (in order that x and y may be positive), whence $x = 6$ and $y = 1$, or $x = 2$ and $y = 8$; and there are no other positive integral values of x and y which satisfy the given equation.

EXERCISES

Find five solutions to each of the following equations:

1. $3x - 4y = 8$. **2.** $2w = 5 + 3z$. **3.** $3r + 6s = 20$.

4. How many solutions has each of the above equations? Why? What are such equations called?

5. If possible solve the equations in Exs. 1, 2, and 3 above, in positive integers. How many such solutions has each?

Find the positive integral solutions of the following equations:

6. $\frac{x}{2} + \frac{y}{3} = 5$. **7.** $5x + 7y = 52$. **8.** $13u + 5v = 229$.

Show that the following equations have no positive integral solutions:

9. $2x - 4y = 1$. **10.** $3x + 6y = 5$. **11.** $9x + 3y = 17$.

12. Show that the indeterminate equation $ax + by = c$ can not be solved in positive integers when $a + b > c$; nor when a and b have a common factor which is not a factor of c.

13. Find three solutions of the equation $2x - 5y + 3z = 6$.

14. If a man spends $300 for cows and sheep, which cost respectively $45 and $6 a head, how many of each does he purchase?

15. In how many and what ways may a 19-pound package be weighed with 5-pound and 2-pound weights?

16. How many pineapples, at 25 cents each, and watermelons, at 15 cents each, can be purchased for $2.15?

17. Divide a line which is 100 feet long into two parts, one of which shall be a multiple of 11, and the other of 6.

18. Find the least number which when divided by 4 gives a remainder of 3, but when divided by 5 gives a remainder of 4.

19. A man selling eggs to a grocer counted them out of his basket 4 at a time and had 1 egg left over, and the grocer counted them into his box 5 at a time and there were 3 left over. If the man had between 6 and 7 dozen eggs, how many must there have been?

104. Definitions: simultaneous equations, etc. Although a single equation which involves two unknown numbers has just been shown to be indeterminate, *i.e.*, to have an indefinite number of solutions, yet if *two* such simple equations be given, it usually happens that one, and only one, pair of numbers can be found which will satisfy each of them, *i.e.*, be a solution of each.

E.g., the equations $4x + 3y = 5$ and $2x - 5y = 9$ are *each* satisfied by $x = 2$ and $y = -1$, and by no other pair of numbers.

Two or more equations which are satisfied by the same set (or sets) of numbers are called **simultaneous equations** (also called **consistent equations**), while two equations which have no solution whatever in common are called **inconsistent equations** (also called **incompatible equations**); *e.g.*, $x + y = 4$ and $2x + 2y = 9$ are inconsistent equations.

Two or more equations which express different relations between the unknown numbers, and therefore can not be reduced to the same form, are called **independent equations**.

Two or more equations taken together are often called a **system of equations**; and any set of numbers which satisfies every equation of the system is called a **solution of the system.**

105. Solving simultaneous equations. The process of finding a solution of a system of simultaneous equations is called **solving the equations**; this process will now be illustrated by some easy examples.

Ex. 1. Solve the equations $\begin{cases} x + y = 4, & (1) \\ x - y = 2. & (2) \end{cases}$

SOLUTION. Adding these two equations, member to member, gives

$$2x = 6,$$

whence $$x = 3.$$

Substituting this value of x in Eq. (1) gives

$$3 + y = 4,$$

whence $$y = 1.$$

That these numbers, viz., $x = 3$ and $y = 1$, really constitute a solution of the given equations is verified by substituting them for x and y in those equations.

Ex. 2. Solve the equations $\begin{cases} 3x - 4y = 7, & (1) \\ x + 2y = 9. & (2) \end{cases}$

Solution. On multiplying Eq. (2) by 2, it becomes

$$2x + 4y = 18, \qquad (3)$$

and adding Eq. (3) to Eq. (1) gives

$$5x = 25,$$

whence $$x = 5;$$

and the corresponding value of y may be found by substituting this value of x in either of the equations which contain both x and y. *E.g.*, by this substitution Eq. (2) becomes

$$5 + 2y = 9,$$

whence $$y = 2;$$

and it is easily verified as in Ex. 1 that $x = 5$ and $y = 2$ is a solution of each of the given equations.

Ex. 3. Solve the equations $\begin{cases} 3x + 2y = 26, & (1) \\ 5x + 9y = 83. & (2) \end{cases}$

Solution. On multiplying both members of Eq. (1) by 5, and of Eq. (2) by 3, they become, respectively,

$$15x + 10y = 130, \qquad (3)$$

$$15x + 27y = 249; \qquad (4)$$

and subtracting Eq. (3) from Eq. (4) gives

$$17y = 119,$$

whence $$y = 7.$$

Substituting this value of y in any one of the equations containing both x and y gives

$$x = 4;$$

and it is readily verified that $x = 4$ and $y = 7$ is a solution of the given system of equations.

Observe that if Eq. (1) had been multiplied by 9, and Eq. (2) by 2, and if one of the two resulting equations had been subtracted from the other, then y would have disappeared, and the value of x would have been found before that of y.

Ex. 4. Solve the equations

$$\begin{cases} \dfrac{x-2}{3} - 1\tfrac{3}{4} = -\dfrac{y}{4}, & (1) \\[2ex] \dfrac{x}{2} + \dfrac{2\,y}{3} = 4\tfrac{1}{2}. & (2) \end{cases}$$

Solution. Multiplying both members of Eq. (1) by 12, and of Eq. (2) by 6, gives

$$4\,x - 8 - 21 = -3\,y, \qquad (3)$$

and

$$3\,x + 4\,y = 27\,; \qquad (4)$$

and, on transposing and simplifying, Eq. (3) becomes

$$4\,x + 3\,y = 29. \qquad (5)$$

Equations (4) and (5) may now be solved by the method employed in Ex. 3; and it is easily verified that their solution, viz., $x = 5$ and $y = 3$ is, at the same time, a solution of equations (1) and (2).

106. Elimination. Any process of deducing from two or more simultaneous equations other equations which contain fewer unknown numbers is called **elimination.** Such a process eliminates (*i.e.*, gets rid of) one or more of the unknown numbers, and thus makes the finding of a solution easier.

That particular plan of elimination which was followed in the examples given in § 105 is known as **elimination by addition and subtraction.** It is evident, moreover, that this method is applicable to any pair of such equations. The procedure may be formulated thus:

(1) *Unless each of the given equations is already in the form* $ax + by = c$, *wherein* a, b, *and* c *are integers, reduce them to this form.*

(2) *Multiply these equations by such numbers as will make the coefficient of the letter to be eliminated the same (in absolute value) in both equations.*

(3) *Subtract or add these last two equations (according as the terms to be eliminated have like or unlike signs), solve the resulting equation for the unknown number which it contains, and substitute that value in any one of the earlier equations to find the other unknown number.*

(4) *Verify that these two numbers really satisfy the two given equations.*

Note. If the coefficients which are to be made of equal absolute value are prime to each other, then each may be used as a multiplier for the other equation; if, however, these coefficients are not prime, their least common multiple should be divided by each in turn, and these quotients used as the multipliers.

EXERCISES

Solve each of the following systems of equations, and check the results:

1. $$\begin{cases} 15x + 77y = 92, \\ 5x - 3y = 2. \end{cases}$$

2. $$\begin{cases} 6y - 5x = 18, \\ 12x - 9y = 0. \end{cases}$$

3. $$\begin{cases} 5x + 6y = 17, \\ 6x + 5y = 16. \end{cases}$$

4. $$\begin{cases} 5p + 3q = 68, \\ 2p + 5q = 69. \end{cases}$$

5. $$\begin{cases} 22x - 8y = 50, \\ 26x + 6y = 175. \end{cases}$$

6. $$\begin{cases} 28x - 23y = 33, \\ 63x - 25y = 199. \end{cases}$$

7. $$\begin{cases} 4s - \frac{1}{3}(v - 3) = 5s - 3, \\ 2v + 5s = 69. \end{cases}$$

8. $$\begin{cases} \dfrac{x}{2} - \frac{1}{3}(y - 2) - \frac{1}{4}(x - 3) = 0, \\ x - \frac{1}{2}(y - 1) - \frac{1}{3}(x - 2) = 0. \end{cases}$$

9. $$\begin{cases} \dfrac{x}{3} + \dfrac{y}{4} = 3x - 7y - 37, \\ \dfrac{x+3}{5} - \dfrac{8-y}{4} = \dfrac{3(x+y)}{8}. \end{cases}$$

10. $$\begin{cases} \dfrac{x}{4} + \dfrac{y}{2} = 12, \\ \dfrac{x}{4} - \dfrac{y}{2} = -2. \end{cases}$$

Suggestion. Eliminate without first clearing of fractions. When is it advantageous to do this?

11. $$\begin{cases} \dfrac{x}{3} + \dfrac{y}{3} = 7, \\ \dfrac{x}{6} + \dfrac{y}{2} = 6\frac{1}{2}. \end{cases}$$

12. What is meant by saying that two equations are simultaneous? Consistent? Independent? Inconsistent? Show the appropriateness of these names. What is a *system* of equations?

Which of these names apply to the systems of equations in the above exercises?

107. Other methods of elimination. Besides the method of elimination which is explained in § 106, there are several other methods that serve the same purpose; two of these, which are often useful, will now be explained.

(i) *Elimination by substitution.*

Ex. 1. Solve the system of equations $\begin{cases} 3x - 4y = 7, & (1) \\ 2x + 3y = 16. & (2) \end{cases}$

SOLUTION

From Eq. (1) $x = \frac{7 + 4y}{3}$;

on substituting this expression for x, Eq. (2) becomes

$$2\left(\frac{7 + 4y}{3}\right) + 3y = 16; \quad (3)$$

whence $y = 2$,

and, by substituting this value in either of the given equations,

$$x = 5.$$

It is easily verified that these values, viz., $x = 5$ and $y = 2$, constitute a solution of the given system of equations.

The method of elimination which has just now been illustrated is known as **elimination by substitution**; it is manifestly applicable to any such system of equations as the above.

The student may solve, by this method, the system

$$\begin{cases} 3u - 4v = 19, \\ 5u + 2v = 10, \end{cases}$$

being careful to check the result, and then write out a "rule" for applying this method to all such exercises.

(ii) *Elimination by comparison.*

Ex. 2. Solve the system of equations $\begin{cases} 3x - 4y = 7, & (1) \\ 2x + 3y = 16. & (2) \end{cases}$

SOLUTION

From Eq. (1) $x = \frac{7 + 4y}{3}$, and from Eq. (2) $x = \frac{16 - 3y}{2}$. Now, since x is to have the same value in each of these equations,

therefore $\frac{7 + 4y}{3} = \frac{16 - 3y}{2}$. (3)

Solving Eq. (3) gives $y = 2$,

whence, substituting this value in either of the given equations,

$$x = 5.$$

It is easily verified that these values, viz., $x = 5$ and $y = 2$, constitute a solution of the given system of equations.

The method of elimination which has just now been illustrated is called **elimination by comparison**; it is manifestly applicable to all such systems of equations.

The student may solve, by this method, the system

$$\begin{cases} 8r+5s=3, \\ 12r-7s=48, \end{cases}$$

and then write out a "rule" for applying this method to all such exercises.

EXERCISES

Solve each of the following systems of equations, using first the method of elimination by substitution, and then that by comparison, and observe which method is easier in the different exercises:

3. $\begin{cases} 2x+3y=23, \\ 5x-2y=10. \end{cases}$

4. $\begin{cases} 4x+y=34, \\ 4y+x=16. \end{cases}$

5. $\begin{cases} 2x+7y=34, \\ 5x+9y=51. \end{cases}$

6. $\begin{cases} 8x-21y=33, \\ 6x+35y=177. \end{cases}$

7. $\begin{cases} 21y+20x=165, \\ 77y-30x=295. \end{cases}$

8. $\begin{cases} 11t-10v=14, \\ 5t+7v=41. \end{cases}$

9. $\begin{cases} 2x+y=50, \\ \frac{x}{6}+\frac{y}{7}=5. \end{cases}$

10. $\begin{cases} \frac{x}{2}+\frac{y}{3}=7, \\ \frac{x}{3}+\frac{y}{4}=5. \end{cases}$

11. $\begin{cases} \frac{x}{5}+\frac{y}{6}=18, \\ \frac{x}{2}-\frac{y}{4}=21. \end{cases}$

12. $\begin{cases} -\frac{5r}{8}+\frac{7t}{4}=13, \\ \frac{11r}{12}-\frac{5t}{8}=12. \end{cases}$

13. Show that elimination by comparison is merely a special case of elimination by substitution.

14. State such suggestions as occur to you for determining, by mere inspection, which of the three methods of elimination thus far considered will be most advantageous to use in any given exercise.

108. Principles involved in elimination. Two systems of equations (§ 104) are said to be **equivalent** when every solution of either system is also a solution of the other.

The methods already given (§§ 106 and 107) for the solution of a system consisting of two independent equations, each containing two unknown numbers, consist in replacing a given system of equations by an *equivalent* system whose solution may be more easily obtained. Those methods are based upon the following principles:*

(i) *If any equation of a system be replaced by an equivalent equation* (§ 95), *the new system thus formed will be equivalent to the given system.*

The truth of this principle follows at once from the definition of equivalent systems, because if the new equation has the same solutions, and only those, as the equation which it replaces, then the new system will have those solutions, and only those, which the given system has; in other words, the two systems are equivalent.

(ii) *If any equation of a given system be replaced by the equation formed by adding* (*or subtracting*) *any other equation of the system to it, member to member, then the new system thus formed will be equivalent to the given system.*

Proof. Suppose the given system of equations to be

$$\text{(I)} \begin{cases} P = 0, \\ Q = 0, \\ R = 0, \end{cases}$$

wherein P, Q, and R represent polynomials, — this is allowable, because if the equations are not in the above form, they may be brought to that form by transposition, which produces equivalent equations (§ 95), and the *systems* also are equivalent [(i) above]; — then it is required to prove that the above system is equivalent to the system

$$\text{(II)} \begin{cases} P + Q = 0, \\ \qquad Q = 0, \\ \qquad R = 0. \end{cases}$$

Now mere inspection of the two systems shows that every set of values of the unknown numbers which satisfies system (I), *i.e.*, which makes P, Q, and R each separately 0, also satisfies system (II), and that every solution of system (II) is also a solution of system (I); therefore these systems are equivalent.

Similarly in general.

(iii) *If any equation of a given system be solved for one of its unknown numbers, say x, in terms of the other unknown numbers which it involves, it may be written in*

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* Observe that these principles apply to systems of any number of equations in any number of unknown numbers.

the form $x = R$, and this equation will be equivalent to the one which was solved to obtain it (§ 95). *If now the expression R be substituted for x in each of the other equations, the system of equations thus formed, together with the equation $x = R$, will be equivalent to the given system.*

To prove this principle, it need only be remarked that the only difference between the two systems of equations is that, in the second system, every x is replaced by R, but, by virtue of the equation $x = R$, every solution of either system makes the expression R represent exactly the same number as does x; hence the two systems are equivalent.

109. Applications of the principles of the preceding article. The solutions of the exercises given in §§ 106 and 107 are all based upon one or more of the principles given in § 108; this will now be illustrated by reconsidering the solution of Ex. 4, § 105.

Given the system of Eqs.

$$\text{(I)}\ \begin{cases} \dfrac{x-2}{3} - 1\tfrac{3}{4} = -\dfrac{y}{4}, & (1) \\[2ex] \dfrac{x}{2} + \dfrac{2y}{3} = 4\tfrac{1}{2}. & (2) \end{cases}$$

On multiplying Eq. (1) by 12, and Eq. (2) by 6, (I) becomes

$$\text{(II)}\ \begin{cases} 4x - 8 - 21 = -3y, & (3) \\ 3x + 4y = 27; & (4) \end{cases}$$

and, on replacing Eq. (3) by its simplified form, (II) becomes

$$\text{(III)}\ \begin{cases} 4x + 3y = 29, & (5) \\ 3x + 4y = 27; & (6) \end{cases}$$

multiplying Eq. (5) by 4, and Eq. (6) by 3, (III) becomes

$$\text{(IV)}\ \begin{cases} 16x + 12y = 116, & (7) \\ 9x + 12y = 81; & (8) \end{cases}$$

replacing Eq. (7) by the result of subtracting Eq. (8) from Eq. (7), (IV) becomes

$$\text{(V)}\ \begin{cases} 7x = 35, & (9) \\ 9x + 12y = 81, & (10) \end{cases}$$

and simplifying, (V) becomes

$$\text{(VI)}\ \begin{cases} x = 5, & (11) \\ 3x + 4y = 27. & (12) \end{cases}$$

But if $x = 5$, then Eq. (12) shows that $y = 3$, and hence $x = 5$ and $y = 3$ is a solution of system (VI); moreover, (I), (II), (III), and (IV) are equivalent, by § 108 (i); (IV) is equivalent to (V), by § 108 (ii); and (V) is equivalent to (VI), by § 108 (i); hence (I) and (VI) are equivalent, and therefore $x = 5$ and $y = 3$ is a solution of (I) also.

EXERCISES

Solve each of the following systems of equations. In the solution of the first ten, give detailed explanations like those given in § 109:

1. $\begin{cases} 7x + 4y = 1, \\ 9x + 4y = 3. \end{cases}$

2. $\begin{cases} 3x + 5y = 19, \\ 5x - 4y = 7. \end{cases}$

3. $\begin{cases} x - 11y = 1, \\ 111y - 9x = 99. \end{cases}$

4. $\begin{cases} 8u - 21v = 5, \\ 6u + 14v = -26. \end{cases}$

5. $\begin{cases} 34x - 15y = 4, \\ 51x + 25y = 101. \end{cases}$

6. $\begin{cases} 39x - 15y = 93, \\ 65x + 17y = 113. \end{cases}$

7. $\begin{cases} 19s + 85t = 350, \\ 17s + 119t = 442. \end{cases}$

8. $\begin{cases} 8s - 11w = 0, \\ 25s - 17w = 139. \end{cases}$

9. $\begin{cases} 3x - 11y = 0, \\ 19x - 19y = 8. \end{cases}$

10. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 1, \\ \dfrac{x}{4} - \dfrac{2y}{3} = 3. \end{cases}$

11. $\begin{cases} \dfrac{x}{3} - \dfrac{y}{6} = \dfrac{1}{2}, \\ \dfrac{x}{5} - \dfrac{3y}{10} = -\dfrac{1}{2}. \end{cases}$

12. $\begin{cases} \dfrac{x}{3} + 3y + 14 = 0, \\ \dfrac{x}{5} + 5y + 4 = 0. \end{cases}$

13. $\begin{cases} \dfrac{x}{5} + 5z = -4, \\ \dfrac{z}{5} + 5x = 4. \end{cases}$

14. $\begin{cases} \dfrac{x+2}{3} + 4y = 2, \\ \dfrac{y+11}{11} - \dfrac{x+1}{2} = 1. \end{cases}$

15. $\begin{cases} \dfrac{2r+3t}{5} + \dfrac{t+6}{7} = 2, \\ \dfrac{2r-5t}{3} + \dfrac{r+7}{4} = 1. \end{cases}$

16. $\begin{cases} \dfrac{m-2}{3} - \dfrac{n+2}{4} = 0, \\ \dfrac{2m-5}{5} - \dfrac{11-2n}{7} = 0. \end{cases}$

17. $\begin{cases} \dfrac{h-2}{3} - \dfrac{k+5}{2} = 0, \\ \dfrac{2h-7}{3} - \dfrac{13-k}{6} = 10. \end{cases}$

18. $\begin{cases} \dfrac{x}{2} - 12 = \dfrac{y+32}{4}, \\ \dfrac{y}{8} + \dfrac{3x-2y}{5} = 25. \end{cases}$

19. $\begin{cases} \dfrac{.2y+.5}{1.5} = \dfrac{.49x-.7}{4.2}, \\ \dfrac{.5x-.2}{1.6} = \dfrac{41}{16} - \dfrac{1.5y-11}{8}. \end{cases}$

20. $$\begin{cases} v + \tfrac{1}{2}(3\,v - w - 1) = \tfrac{1}{4} + \tfrac{3}{4}(w-1), \\ \tfrac{1}{5}(4\,v + 3\,w) = \tfrac{1}{10}(7\,w + 24). \end{cases}$$

21. $$\begin{cases} \left(\dfrac{x}{7} + \dfrac{y}{4} + 1\tfrac{1}{3}\right) - \left(4\,x - \dfrac{y}{8} - 25\right) = \dfrac{5}{6}, \\ \dfrac{3\,x - 5\,y}{3} - \dfrac{2\,x - 8\,y - 9}{12} = \dfrac{31}{12}. \end{cases}$$

110. Simultaneous fractional equations. By first clearing the given equations of fractions (§§ 98 and 99), the foregoing methods become applicable to the solution of fractional equations, — the following examples will illustrate this.

Ex. 1. Given the system of equations $\left\{\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{x} \\ \dfrac{6}{x} - \dfrac{1}{y} = \dfrac{1}{xy} \end{array}\right\}$; to find x and y.

Solution

On multiplying each of these equations by xy, they become, respectively,

$$y + x = 3\,y,$$

and $$6\,y - x = 1.$$

The solution of these integral equations is (§ 106) $x = \frac{1}{2}$ and $y = \frac{1}{4}$; and it is easily verified that these numbers constitute a solution of the given system of fractional equations also.

Ex. 2. Given the system of equations $\left\{\begin{array}{l} \dfrac{1}{x - 3\,y} + \dfrac{4}{x} = \dfrac{16}{x(x - 3\,y)} \\ \dfrac{x}{3} - 1 - y = 0 \end{array}\right\}$; to find x and y.

Solution

On multiplying these equations by $x(x - 3\,y)$ and 3, respectively, they become $$x + 4\,(x - 3\,y) = 16,$$

and $$x - 3 - 3\,y = 0.$$

By § 106 the solution of this system of integral equations is $x = 4$ and $y = \frac{1}{3}$, and these two numbers prove also to be a solution of the given system of fractional equations.

Ex. 3. Given the system of equations $\left\{\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} = 3 \\ \dfrac{2}{x} - \dfrac{3}{y} = 1 \end{array}\right\}$; to find its solution.

SOLUTION

On multiplying each of these equations by xy, they become, respectively,

$$y + x = 3\,xy,$$

and
$$2\,y - 3\,x = xy;$$

if the first of these be subtracted from three times the second, the result will be

$$5\,y - 10\,x = 0,$$

i.e.,
$$y = 2\,x.$$

On substituting this value of y in the first of the given equations, it becomes

$$\frac{1}{x} + \frac{1}{2\,x} = 3,$$

whence, multiplying by $2\,x$, $\quad 2 + 1 = 6\,x,$

i.e.,
$$x = \tfrac{1}{2},$$

and therefore, since $y = 2\,x$, $\quad y = 1.$

It is, moreover, easily verified that $x = \frac{1}{2}$ and $y = 1$ constitutes a solution of the given fractional equations.

NOTE. Solve Ex. 3 by eliminating before clearing of fractions (*e.g.*, subtract the second equation from twice the first), and compare the two methods.

Ex. 4. Given the system of equations $\left\{\begin{aligned} &\frac{2}{x} + y = 4 \\ &\frac{1}{x-1} + \frac{1}{y-2} = 2 \end{aligned}\right\}$; to find x and y.

SOLUTION

On multiplying the first of these equations by x and the second by $(x - 1)(y - 2)$, and simplifying, they become, respectively,

$$xy - 4\,x + 2 = 0,$$

and
$$2\,xy - 5\,x - 3\,y + 7 = 0.$$

To eliminate the term containing xy subtract twice the first of these integral equations from the second; the result is

$$3\,x - 3\,y + 3 = 0,$$

i.e.,
$$y = x + 1.$$

On substituting this value of y in the first of the integral equations, it becomes

$$x(x+1) - 4x + 2 = 0,$$

i.e., $$x^2 - 3x + 2 = 0,$$

i.e., $$(x-1)(x-2) = 0,$$

whence $$x = 1 \text{ or } x = 2;$$

and since $y = x + 1$, therefore the *corresponding* values of y are

$$y = 2 \text{ and } y = 3.$$

While each of these pairs of corresponding values, viz., $x = 1$, $y = 2$ and $x = 2$, $y = 3$, is a solution of the system of integral equations obtained by clearing the given system of fractions, yet it is easily verified that the second pair is a solution of the given system of fractional equations and that the first pair is not a solution of this system.

Observe that extraneous solutions, here as in § 99, reduce one or more of the denominators to zero.

EXERCISES

Solve the following systems of equations, and check the results; eliminate before clearing of fractions when that is possible, as in Exs. 8–11, 13, etc.:

5. $$\begin{cases} \dfrac{3x + 2y + 6}{4x - 2y} = 1, \\[2ex] \dfrac{3 - 7y}{2x + 1} = 2. \end{cases}$$

6. $$\begin{cases} \dfrac{15 + y - 2x}{4x - 5y - 2} = 5, \\[2ex] \dfrac{3x - 2y + \frac{3}{2}}{x - y} = \dfrac{16}{3}. \end{cases}$$

7. $$\begin{cases} \dfrac{8}{5x + 16y} = \dfrac{15}{3x - 4y}, \\[2ex] 8y - 2x = 7. \end{cases}$$

8. $$\begin{cases} \dfrac{4}{x} + \dfrac{3}{y} = 3, \\[2ex] \dfrac{2}{x} - \dfrac{3}{y} = 1. \end{cases}$$

9. $$\begin{cases} \dfrac{5}{x} + \dfrac{6}{y} = 20, \\[2ex] \dfrac{6}{x} + \dfrac{5}{y} = 10. \end{cases}$$

10. $$\begin{cases} \dfrac{2}{s} - \dfrac{3}{r} = 5, \\[2ex] \dfrac{5}{s} - \dfrac{2}{r} = 7. \end{cases}$$

11. $$\begin{cases} \dfrac{3}{2v} - \dfrac{1}{w} + 3 = 0, \\[2ex] \dfrac{5}{2v} + \dfrac{3}{w} = 23. \end{cases}$$

12. $$\begin{cases} \dfrac{1}{x - 2} + \dfrac{2}{3 - y} = 0, \\[2ex] 3x + y = 9.\text{*} \end{cases}$$

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* Show that the equations in Ex. 12 are inconsistent.

13. $$\begin{cases} \dfrac{3}{2x-5} - \dfrac{2}{3y+2} = \dfrac{13}{5}, \\[2ex] \dfrac{5}{2x-5} - \dfrac{3}{2+3y} = 8. \end{cases}$$

14. $$\begin{cases} \dfrac{3}{4u+v} - \dfrac{5}{2u-v} = 2, \\[2ex] \dfrac{3}{2u-v} + \dfrac{4}{v+4u} = \dfrac{23}{5}. \end{cases}$$

15. $$\begin{cases} \dfrac{x}{2} - 3 = \dfrac{5y+2x}{4-x} + \dfrac{x-3}{2}, \\[2ex] \dfrac{2y-3x}{y+1} + y = 12 - \dfrac{7-2y}{2}. \end{cases}$$

16. $$\begin{cases} 4x + \dfrac{2y - \frac{x}{2}}{17-3x} = \dfrac{16x+19}{4}, \\[2ex] 50 - \dfrac{y-1}{\frac{5}{3}(x-2)} = 8y + \dfrac{147-24y}{3}. \end{cases}$$

17. $$\begin{cases} \dfrac{3}{x-2} + \dfrac{5}{3y+8} = 2, \\[2ex] 5y + 3\frac{1}{3} = \dfrac{5y}{2x-7}.\text{*} \end{cases}$$

18. $$\begin{cases} \dfrac{3}{u} = 2 - v, \\[2ex] \dfrac{1}{u-1} = \dfrac{2}{v+3}. \end{cases}$$

19. $$\begin{cases} x - 20 - \dfrac{2y-x}{23-x} = \dfrac{2x-59}{2}, \\[2ex] y - \dfrac{3-y}{x-18} = 30 + \dfrac{3y-73}{3}. \end{cases}$$

20. $$\begin{cases} \frac{3}{4}(2x+3) + \dfrac{3x+5y}{2(2x-3)} = 3\frac{1}{4} + \dfrac{3x+4}{2}, \\[2ex] \dfrac{8y+7}{10} + \dfrac{6x-3y}{2(y-4)} = 4 + \dfrac{4y-9}{5}. \end{cases}$$

111. Literal equations. Literal equations of the first degree, and involving but one unknown number, have already been discussed (§ 97); the present article will be devoted to the consideration of a pair of simultaneous, independent, literal equations of the first degree, each involving two unknown numbers.

Since, by transposing and collecting terms, every first degree equation in two unknown numbers may be reduced to an equivalent equation of the form $ax + by = c$, wherein a, b, and c represent known numbers, therefore the two given equations will be assumed as already in that form. It is then proposed to solve the system of equations

$$\begin{cases} a_1x + b_1y = c_1, & (1) \\ a_2x + b_2y = c_2. & (2) \end{cases}$$

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* Compare Ex. 4 above, and use § 66 (iv) if necessary.

To eliminate y multiply Eq. (1) by b_2, and Eq. (2) by b_1, and then subtract; this gives

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2, \tag{3}$$

whence
$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}. \tag{4}$$

Similarly, by eliminating x,

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1, \tag{5}$$

whence
$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{6}$$

It is also easily verified (by substitution) that these expressions for x and y satisfy the given system of equations. Hence the given system of equations has *at least one* solution, provided only that $a_1b_2 - a_2b_1 \neq 0$.*

Moreover, by § 108, the system consisting of Eqs. (4) and (6) is equivalent to the given system; but, for any given set of values of the coefficients, Eqs. (4) and (6) have manifestly but one solution, and hence the given system has *but one* solution.

Hence, *any system consisting of two independent and consistent first degree equations, involving two unknown numbers, has one solution, and but one.*

NOTE. It may also be stated here that *three or more independent equations of the first degree, involving only two unknown numbers, can not all be satisfied by the same values of the unknown numbers.*

For, if the solution of the first two of these equations is a solution of the third equation also, then

$$a_3\left(\frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}\right) = c_3;$$

i.e., in this case, there is a definite relation (equation) connecting the coefficients of the given equations, and these equations are, therefore, not independent.

Similarly in general.

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* If $a_1b_2 - a_2b_1 = 0$, then $x \left(= \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}\right)$ is infinite, unless it happens that $b_2c_1 - b_1c_2$ is also 0, in which case $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, and the given equations are not independent, for either of them may then be obtained by multiplying the other by a suitable factor; *i.e.*, in this case there is really only *one* equation and the number of solutions is infinite (§ 101).

EXERCISES

Solve the following systems of equations, and check the results; eliminate without clearing of fractions when possible:

1. $$\begin{cases} ax + by = m, \\ bx + ay = n. \end{cases}$$

2. $$\begin{cases} x - y = a - b, \\ ax + by = a^2 - b^2. \end{cases}$$

3. $$\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = \dfrac{1}{c}, \\ \dfrac{c}{x} - \dfrac{a}{y} = \dfrac{1}{b}. \end{cases}$$

4. $$\begin{cases} \dfrac{1}{ax} + \dfrac{1}{by} = \dfrac{1}{c^2}, \\ \dfrac{1}{bx} + \dfrac{1}{cy} = \dfrac{1}{a^2}. \end{cases}$$

5. $$\begin{cases} \dfrac{a}{a+x} + \dfrac{b}{b+y} = \dfrac{c}{c+1}, \\ \dfrac{b}{a+x} - \dfrac{c}{b+y} = \dfrac{a}{a+1}. \end{cases}$$

6. $$\begin{cases} \dfrac{x+y}{a} - \dfrac{x-y}{b} = 0, \\ \dfrac{x-a}{b} - \dfrac{y-b}{a} = 0. \end{cases}$$

7. $$\begin{cases} 2\,a^2 - ax = 2\,b^2 + by, \\ \dfrac{y}{a-b} = \dfrac{x}{a+b} + \dfrac{a+b}{ab}. \end{cases}$$

8. $$\begin{cases} (a+b)x+(a+c)y=a+b, \\ (a+c)x+(a+b)y=a+c. \end{cases}$$

9. $$\begin{cases} \dfrac{x+1}{y+1} = \dfrac{a+b+1}{a-b+1}, \\ x - y = 2\,b. \end{cases}$$

10. $$\begin{cases} hx + ky = 4\,h^2, \\ \dfrac{1}{x-k} + \dfrac{1}{y-h} = \dfrac{h}{k(y-h)}. \end{cases}$$

11. Under what circumstances has Ex. 1 above no finite solution? Answer this question with regard to Ex. 2 also; and with regard to Ex. 7.

12. What relation among the coefficients is needed in order that Ex. 1 shall have more than one solution? If this relation exists, how many solutions has this system of equations?

13. What relation among the coefficients is required in order that the three equations $ax + by = c$, $bx + cy = a$, and $cx + ay = b$ may have one solution in common?

PROBLEMS

1. Find two numbers whose difference is $\frac{1}{35}$ of their sum, and such that 5 times the smaller minus 4 times the larger is 39.

SOLUTION

Let $x =$ the larger number,

and $y =$ the smaller number.

Then, by the conditions of the problem,

$$x - y = \frac{x+y}{35},$$

and $$5y - 4x = 39.$$

Solving these equations, we obtain

$$x = 54 \text{ and } y = 51;$$

and these numbers, which constitute a solution of *the equations of the problem*, also satisfy all the conditions of the *problem itself*, and are, therefore, the numbers sought.

2. Find two numbers such that 3 times the greater exceeds twice the less by 29, and twice the greater exceeds 3 times the less by 1.

3. A lady purchased 20 yds. of one kind of goods, and 50 yds. of another, for $30; she could have purchased 30 yds. of the first kind, and 20 of the second, for $23. What was the price of each?

4. If A's money were increased by $4000, he would have twice as much as B. If B's money were increased by $5500, he would have 3 times as much as A. How much money has each?

5. One eleventh of A's age is greater by 2 years than $\frac{1}{7}$ of B's, and twice B's age equals what A's age was 13 years ago. Find the ages of each.

6. If 45 bushels of wheat and 37 bushels of rye together cost $62.70, and 37 bushels of wheat and 25 bushels of rye, at the same prices, cost $48.30, what is the price of each per bushel?

7. A pound of tea and 6 lb. of sugar together cost 72 cents; if sugar were to advance 50%, and tea 10%, then 2 lb. of tea and 12 lb. of sugar would cost $1.68. Find the present price of tea, and also of sugar.

8. A man having $45 to distribute among a group of children, finds that he lacks $1 of being able to give $3 to each girl and $1 to each boy, but that he has just enough to give $2.50 to each girl and $1.50 to each boy. How many boys and how many girls are there in this group?

9. John said to James, "Give me 8 cents and I shall have as much as you have left." James said to John, "Give me 16 cents and I shall have 4 times as much as you have left." How much money had each?

10. A boy bought some oranges at the rate of 3 for 5 cents, and another kind at 4 for 5 cents, and paid for the whole $4.60. He afterwards sold them all at 2 cents apiece, clearing thereby $1.54. How many of each kind did he buy?

11. A fishing rod consists of two parts; the length of the upper part is $\frac{5}{7}$ that of the lower part; the sum of 9 times the length of the upper part and 13 times the length of the lower part exceeds 11 times the length of the whole rod by 36 inches. Find the length of the rod.

12. If a certain rectangular floor were 2 ft. broader and 3 ft. longer, its area would be increased by 64 sq. ft., but if it were 3 ft. broader and 2 ft. longer, its area would be 68 sq. ft. greater than it now is. Find its length and breadth.

13. Three rectangles are equal in area; the second is 6 meters longer and 4 meters narrower than the first, and the third is 2 meters longer and 1 meter narrower than the second. What are the dimensions of each?

14. The sum of the ages of a father and son will be doubled in 25 years, and 20 years hence the difference of their ages will just equal $\frac{1}{3}$ of their sum at that time. What is the present age of each?

15. If 1 be added to each term of a certain fraction, its value will be $\frac{2}{3}$; but if 1 be subtracted from each of its terms, its value will be $\frac{1}{3}$. What is the fraction?

16. The sum of the digits of a two-digit number is 12, and if its digits be interchanged, the number thus formed will lack 12 of being the double of what it now is. What is the number?

17. If a certain two-digit number is divided by the sum of its digits, the quotient is 8, and when the tens' digit is diminished by 3 times the units' digit, the remainder is 1. What is the number?

18. The tickets of admission to an entertainment were 50 cents for adults and 35 cents for children. If the proceeds from the sale of 100 tickets was $39.50, how many tickets of each kind were sold?

Solve this problem also by using but one letter to represent an unknown number.

19. A capitalist invested $4000, part of it at 5% and the balance at 4%, and found that his annual income from this investment was $175. How much was invested at 5%, and how much at 4%?

Can this problem be solved without using two letters to represent unknown numbers? How?

20. A boat crew can row 4 miles downstream and back again in $1\frac{1}{2}$ hours, or 6 miles downstream and halfway back in the same time. What is the rate of rowing in still water, and what is the rate of the current?

21. A capitalist invested $\$A$, part at $p\%$ and the balance at $q\%$, and found that his annual income from this investment was $\$B$. How much was invested at $p\%$?

Show that this problem includes Prob. 19 as a special case — it is the generalization of Prob. 19 (cf. § 100).

22. Generalize Prob. 14. Find the solution of the generalized problem, and then show that the answer to the particular problem (14) may be found by merely substituting in the answer to the generalized problem.

23. Generalize Prob. 20, solve, etc., as in Prob. 22.

24. A man rows 15 miles downstream and back in 11 hours. If he can row 8 miles downstream in the same time as it takes him to row 3 miles upstream, what is his rate of rowing in still water? and what is the velocity of the current?

25. Divide the number N into two such parts that $\frac{1}{m}$ of the first part, plus $\frac{1}{n}$ of the second, shall exceed the first part by M.

Specialize this problem, and find the solution of the special problem by substituting in the general solution.

26. Three cities, A, B, and C, are situated at the vertices of a triangle; the distance from A to C by way of B is 50 miles, from A to B by way of C is 70 miles, and from B to C by way of A is 60 miles. How far apart are these cities?

Solve this problem by first generalizing it, and then substituting the particular numbers 50, 70, and 60 in the general solution.

27. Two boats which are d miles apart will meet in a hours if they sail toward each other, and the second will overtake the first in b hours if they sail in the same direction. Find the respective rates at which these boats sail. Also discuss fully your solution, *i.e.*, interpret the results when the rate of the second boat is greater than, equal to, and less than, the rate of the first — compare Prob. 3 of § 100.

28. Two men, A and B, had a certain distance to row and alternated in the work; A rowed at a rate sufficient to cover the entire distance in 10 hours, while B's rate would require 14. If the journey was completed in 12 hours, how many hours did each row?

29. A mine which is to be emptied of water has two pumps which together can discharge 1250 gallons an hour. The larger pump can do the work alone in 5 hours, but with the help of the smaller pump only 4 hours are needed. How many gallons an hour does each pump discharge?

Solve this problem by first generalizing it, as in Prob. 26 above.

30. Two trains are scheduled to leave the cities A and B, m miles apart, at the same time, and to meet in h hours; but, the train leaving A being a hours late in starting, they met k hours later than the scheduled time. What is the rate at which each train runs?

From the solution of this problem find, by substitution, the solution of the special problem in which $m = 800$, $h = 10$, $a = 1\frac{2}{3}$, and $k = \frac{7}{10}$.

31. Two boys, A and B, run a race of 400 yards, A giving B a start of 20 seconds and winning by 50 yards. On running this race again, A, giving B a start of 125 yards, wins by 5 seconds. What is the speed of each? Generalize this problem.

32. A and B working together can build a wall in $5\frac{5}{11}$ days; finding it impossible to work at the same time, A works 5 days, and later B takes up the work, finishing it in 6 days. In how many days could each have built this wall alone? Generalize this problem.

33. A railway train, after running 1 hour and 36 minutes, was detained 40 minutes by an accident, after which it proceeded at $\frac{6}{5}$ of its former rate, and reached its destination 16 minutes late. Under the same circumstances, had the accident occurred 10 miles farther on, the train would have arrived 20 minutes late. At what rate did the train move before the accident, and what was the entire distance traveled?

II. THREE OR MORE UNKNOWN NUMBERS

112. Equations containing more than two unknown numbers. It is easy to see that the methods employed in § 105 for solving a system of two simultaneous integral equations, each containing two unknown numbers, may also be employed for solving a system of three or more such equations involving as many unknown numbers as there are independent equations. (Cf. Exs. 1 and 2 below.)

Ex. 1. Given
$$\begin{cases} x + 3y - z = 5, & (1)\\ 3x + 6y + 2z = 3, & (2)\\ 2x - 3y - 3z = 6; & (3)\end{cases}$$
to find the solution of this system of equations.

Solution. Adding 2 times Eq. (1) to Eq. (2), member to member, gives
$$5x + 12y = 13, \qquad (4)$$
and subtracting Eq. (3) from 3 times Eq. (1) gives
$$x + 12y = 9. \qquad (5)$$

Now subtracting Eq. (5) from Eq. (4) gives
$$4x = 4,$$
whence
$$x = 1. \qquad (6)$$

On substituting this value of x, Eq. (5) becomes
$$1 + 12y = 9,$$
whence
$$y = \tfrac{2}{3}; \qquad (7)$$

and substituting these values of x and y in Eq. (1) gives

$$1 + 2 - z = 5,$$

whence $$z = -2. \tag{8}$$

That these numbers, viz., $x = 1$, $y = \frac{2}{3}$, and $z = -2$, really constitute a solution of the given system of equations is easily verified by substituting them for x, y, and z in these equations.

NOTE. It should be carefully observed that, by principles (i) and (ii) of § 108, Eq. (2) of the given system of equations may be replaced by Eq. (4), — which is derived from Eq. (1) and (2), — and the new system thus formed will be equivalent to the given system, *i.e.*, the system of Eqs. (1), (3), and (4) is equivalent to the system of Eqs. (1), (2), and (3).

So too Eq. (3) may be replaced by Eq. (6), making the system formed of Eqs. (1), (4), and (6) equivalent to the given system; and this last system, being readily solved, furnishes a solution of the given system.

The foregoing is another illustration of the fact to which attention has already been called (§ 108), viz., that solving a system of simultaneous equations is accomplished by first replacing the given system by an equivalent system whose solution is more easily obtained.

Ex. 2. Given
$$\begin{cases} 2x - 3y - 2z = -1, & (1) \\ 3x + z = 6, & (2) \\ x + y + z = 3; & (3) \end{cases}$$

to find the solution of these equations.

SOLUTION. Since the second of these equations is already free from the unknown number y, therefore it is best to combine Eqs. (1) and (3) so as to eliminate y, and thus obtain another equation involving only x and z. Adding Eq. (1) to 3 times Eq. (3) gives

$$5x + z = 8, \tag{4}$$

and subtracting Eq. (2) from Eq. (4) gives

$$2x = 2,$$

whence $$x = 1. \tag{5}$$

Substituting this value of x in Eq. (2) gives

$$z = 3;$$

and substituting these two values in Eq. (3) gives

$$y = -1.$$

Moreover, it is easily verified that $x = 1$, $y = -1$, and $z = 3$ constitute a solution of the given equations.

Ex. 3. Show that Eqs. (2), (3), and (5), in Ex. 2, form a system which is equivalent to the given system.

113. Formulation of the method of procedure of § 112. The process of finding a solution of three independent integral equations of the first degree and containing three unknown numbers, which is illustrated in § 112, may be stated thus:

Combine any two of the three given equations in such a way as to eliminate some one of the unknown numbers, thus deriving from them an equation containing but two unknown numbers; then combine the remaining equation of the given system with either one of the other two in such a way as to eliminate the same unknown number as before, thus deriving another equation which contains the same two unknown numbers as does the first derived equation; next combine these two derived equations so as to eliminate one of the unknown numbers, thus deriving another equation which contains but one unknown number; from this last equation the value of the unknown number which it contains can be found, and then, by successively substituting in earlier equations, the values of the other two unknown numbers can be found.

Similarly for the solution of a system of n independent integral equations of the first degree and containing n unknown numbers. When n is greater than 3 the eliminating should be done very systematically, since otherwise the derived equation may not be independent; the procedure may be stated thus:

So combine some one of the given equations (the first, for example) with each of the others, as to eliminate the same unknown number in each case, thus forming what may be called a first derived system of $n-1$ equations, which will be independent, integral, and of the first degree, and which will contain $n-1$ unknown numbers; by proceeding with the first derived system just as with the given system, a second derived system containing $n-2$ equations involving $n-2$ unknown numbers is obtained; by continuing this process, there is finally obtained a single equation with but one unknown number; from this equation the value of that unknown number is found, and then, by *successive substitutions in earlier equations, the values of all the other unknown numbers are found.*

NOTE. It may be remarked that any one of the *given* equations, together with the $n-1$ equations of the first derived system, constitute a system which is equivalent to the given system; also that any one of the given system, together with any one of the first derived system, and the $n-2$ equations of the second derived system, are equivalent to the given system, and so on; finally, that the system composed of any one of the given equations, any one of the first derived system, any one of the second derived system, and so on including the single equation of the last derived system, is equivalent to the given system.

EXERCISES

Solve each of the following systems of equations:

1. $$\begin{cases} 2x + 3y + 4z = 20, \\ 3x + 4y + 5z = 26, \\ 3x + 5y + 6z = 31. \end{cases}$$

2. $$\begin{cases} 4x - y - z = 5, \\ 3x - 4y + 16 = 6z, \\ 3y + 2(z - 1) = x. \end{cases}$$

3. $$\begin{cases} 7x + 3y - 2z = 16, \\ 2x + 5y + 3z = 39, \\ 5x - y + 5z = 31. \end{cases}$$

4. $$\begin{cases} 5x - 6y + 4z = 15, \\ 7x + 4y - 3z = 19, \\ 2x + y + 6z = 46. \end{cases}$$

5. $$\begin{cases} 2x + 4y + 5z = 19, \\ -3x + 5y + 7z = 8, \\ 8x - 3y + 5z = 23. \end{cases}$$

6. $$\begin{cases} 5x + 6y - 12z = 5, \\ 2x - 2y - 6z = -1, \\ 4x - 5y + 3z = 7\tfrac{1}{2}. \end{cases}$$

7. $$\begin{cases} y + z - 86 = 72 - 5x, \\ 93 - \tfrac{1}{2}x - \tfrac{1}{4}y = \tfrac{3}{4}y - 2z, \\ \tfrac{1}{4}x + \tfrac{1}{3}y + \tfrac{1}{2}z = 58. \end{cases}$$

8. $$\begin{cases} \tfrac{1}{2}x + \tfrac{1}{3}y = 12 - \tfrac{1}{6}z, \\ \tfrac{1}{2}y + \tfrac{1}{3}z = 8 + \tfrac{1}{6}x, \\ \tfrac{1}{2}x + \tfrac{1}{3}z = 10. \end{cases}$$

9. $$\begin{cases} 2x - 5y + 19 = 0, \\ 3y - 4z + 7 = 0, \\ 2z - 5x - 2 = 0. \end{cases}$$

10. $$\begin{cases} \dfrac{x}{6} - \dfrac{y}{5} + \dfrac{z}{4} = 3, \\ \dfrac{x}{5} - \dfrac{y}{4} + \dfrac{z}{5} = 1, \\ \dfrac{x}{4} - \dfrac{y}{3} + \dfrac{z}{2} = 5. \end{cases}$$

11. $$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 6, \\ \dfrac{1}{y} + \dfrac{1}{z} = 10, \\ \dfrac{1}{z} + \dfrac{1}{x} = 8. \end{cases}$$

12. $$\begin{cases} \dfrac{3}{x} + \dfrac{2}{y} + \dfrac{1}{z} = 1, \\ -\dfrac{4}{x} - \dfrac{4}{y} + \dfrac{3}{z} = 10, \\ \dfrac{1}{x} + \dfrac{4}{y} = 1. \end{cases}$$

13. $$\begin{cases} x + y - z = a, \\ x - y = 2\,b, \\ x + z = 3\,a + b. \end{cases}$$

14. $$\begin{cases} \dfrac{xy}{x + y} = \dfrac{1}{a}, \\ \dfrac{yz}{y + z} = \dfrac{1}{b}, \\ \dfrac{xz}{x + z} = \dfrac{1}{c}. \end{cases}$$

SUGGESTION. If $\dfrac{xy}{x+y} = \dfrac{1}{a}$, then $\dfrac{x+y}{xy} = a$, *i.e.*, $\dfrac{1}{y} + \dfrac{1}{x} = a$.

15. $$\begin{cases} 2\,v + 3\,x + y - z = 0, \\ 3\,y - 2\,x + z - 4\,v = 21, \\ 2\,z - 3\,v - y + x = 6, \\ v + 4\,x + 2\,y - 3\,z = 12. \end{cases}$$

16. $$\begin{cases} v + x + y = 15, \\ x + y + z = 18, \\ v + y + z = 17, \\ v + x + z = 16. \end{cases}$$

SUGGESTION. Adding these equations and dividing the sum by 3 gives $v + x + y + z = 22$.

17. $$\begin{cases} y + z + v - x = 22, \\ z + v + x - y = 18, \\ v + x + y - z = 14, \\ x + y + z - v = 10. \end{cases}$$

18. $$\begin{cases} y + z - 3\,x = 2\,a, \\ x + z - 3\,y = 2\,b, \\ x + y - 3\,z = 2\,c, \\ 2\,x + 2\,y + v = 0. \end{cases}$$

19. $$\begin{cases} 3\,u + 5\,v - 2\,x + 3\,z = 2, \\ 2\,u + 4\,x - 3\,y - z = 3, \\ u + v + z = 2, \\ 6\,y + 4\,v + u = 2, \\ 5\,z + 4\,x - 7\,v = 0. \end{cases}$$

20. $$\begin{cases} \dfrac{a}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 2, \\ \dfrac{1}{x} + \dfrac{b}{y} + \dfrac{2}{z} = 5, \\ yz + xz + cxy = 3\,xyz. \end{cases}$$

SUGGESTION. Carefully compare the last equation with either of the other two.

21. $$\begin{cases} abxyz + cxy - ayz = bxz, \\ bcxyz + ayz - bxz = cxy, \\ acxyz + bxz - cxy = ayz. \end{cases}$$

22. $$\begin{cases} 5\,xy + 6(x + y) = 0, \\ 5\,yz - 2(y + z) = 0, \\ 14\,xz - 3(x + z) = 0. \end{cases}$$

23. From the considerations presented in § 113, prove that a system consisting of n independent and consistent equations of the first degree, and containing n unknown numbers, has one and only one solution. (Cf. also § 111.)

24. If there are more unknown numbers than independent equations in any given system, how many solutions has that system? Why? (Such a system is usually called an **indeterminate system.**)

25. If there are more consistent equations than unknown numbers in a system, prove that these equations can not all be independent. (Cf. § 111, note.)

26. Prove that there is no unique solution of the system

$$\begin{cases} 11x + 8y - 4z = 9, \\ 5x + 2y - 2z = 5, \\ 3x + 3y - z = 2. \end{cases}$$

Is this system indeterminate (cf. Ex. 24)? Explain.

PROBLEMS

1. A grain dealer sold to one customer 5 bushels of wheat, 2 of corn, and 3 of rye, for $6.60; to another, 2 of wheat, 3 of corn, and 5 of rye, for $5.80; and to another, 3 of wheat, 5 of corn, and 2 of rye, for $5.60. What was the price per bushel of each of these kinds of grain?

2. A quantity of water, which is just sufficient to fill three jars of different sizes, will fill the smallest jar exactly 4 times; or the largest jar twice, with 4 gallons to spare; or the second jar 3 times, with 2 gallons to spare. What is the capacity of each of these jars?

3. If A and B can do a certain piece of work in 10 days, A and C in 8 days, and B and C in 12 days, how long will it take each to do the work alone?

4. Divide 800 into three parts such that the first, plus $\frac{1}{2}$ of the second, plus $\frac{2}{3}$ of the third, shall equal the second, plus $\frac{3}{4}$ of the first, plus $\frac{1}{4}$ of the third: each of these sums being 400.

5. A merchant having three kinds of tea, sold to one customer 2 lb. of the first kind, 3 of the second, and 4 of the third, for $4.70; and to another he sold 4 lb. of the first kind, 3 of the second, and 2 of the third, for $4.30. If a pound of the third kind is worth 5 cents more than $\frac{3}{4}$ lb. of the first kind and $\frac{1}{2}$ lb. of the second kind taken together, what is the price of each per pound?

6. Divide 90 into three parts such that $\frac{1}{2}$ of the first, plus $\frac{1}{3}$ of the second, plus $\frac{1}{4}$ of the third, shall be 30; and that the first part shall equal twice the third part diminished by twice the second part.

7. The sum of the digits of a 3-digit number is 11; the double of the second digit exceeds the sum of the first and third by 1; and if the first and second digits be interchanged, the number will be diminished by 90. What is the number?

8. The third digit of a 3-digit number is as much larger than the second as the second is larger than the first; if the number be divided by the sum of its digits, the quotient will be 15; and the number will be increased by 396 if the order of its digits be reversed. What is the number?

9. The sum of the digits of a 4-digit number is 11; if the order of the digits be reversed, the number will be increased by 819; if 9 be subtracted from the number, the units' and tens' digits will be interchanged; and the sum of the units' and tens' digits equals the hundreds' digit. What is the number?

10. Of three alloys, the first contains 35 parts of silver, to 5 of copper, to 4 of tin; the second, 28 parts of silver, to 2 of copper, to 3 of tin; and the third, 25 parts of silver, to 4 of copper, to 4 of tin. How many ounces of each of these alloys melted together will form 600 oz. of an alloy consisting of 8 parts of silver, to 1 of copper, to 1 of tin?

11. If Problem 10 merely demanded that the alloy should contain 8 parts of silver to 1 of copper, how many ounces of each of the given alloys would then be required? Why is this problem indeterminate?

12. A tank whose capacity is 1600 gallons is *supplied* by two pipes, and has one *outlet* pipe. If the tank is empty, and all three pipes are opened, it will be filled in 80 hours; if it is $\frac{3}{4}$ full, and all the pipes are opened for 10 hours, and if the larger supply pipe is then closed, leaving the other two open 10 hours longer, the tank will then be $\frac{5}{8}$ full; and it can be filled by the larger pipe alone in $26\frac{2}{3}$ hours. Find the number of gallons discharged per hour by each of the three pipes, assuming the flow to be uniform.

13. Find an expression of the form $ax^2 + bx + c$ whose value will be 6 when $x = 2$, 3 when $x = -1$, and 10 when $x = 4$.

SUGGESTION. $4a + 2b + c$ is the value of $ax^2 + bx + c$ when $x = 2$; therefore, $4a + 2b + c = 6$, etc.

14. Can such an expression as that in Prob. 13 be found which shall take four prescribed values when four particular values are assigned to x? Why? What letters represent unknown numbers in Prob. 13?

III. GRAPHIC REPRESENTATION OF EQUATIONS*

114. Preliminary remarks. Although an equation in two unknown numbers has an infinitely large number of solutions, and is in that sense entirely indeterminate (§ 101), yet, by a beautiful device, due to a celebrated mathematician and philosopher named Descartes, a perfectly definite geometric picture of such an equation may be made. The method by which this is done will be explained in this and the next article.

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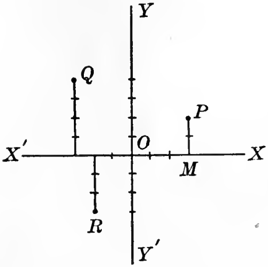
* This subject is discussed in detail in a later course in mathematics, — in Analytic Geometry.

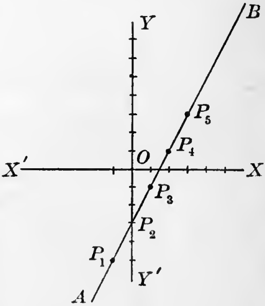
Let two indefinite straight lines $X'X$ and $Y'Y$ be drawn at right angles to each other and intersecting in the point O — as in the figure. If now it be agreed that distances measured to the right, or upward, be represented by positive numbers, while distances to the left, or downward, are represented by negative numbers, then the position of any point whatever, in the plane of this page, is completely determined by merely giving the distances of that point from the lines $X'X$ and $Y'Y$. It will be observed that this is similar to locating a place on a map by means of its latitude and longitude.



E.g., to locate a point P, whose distances from $Y'Y$ and $X'X$ are respectively 3 inches and 2 inches, measure 3 inches to the right from O, to the point M say, and then measure 2 inches up from M. This point is usually represented by the symbol (3, 2), *i.e.*, by $P \equiv (3, 2)$; the numbers 3 and 2 are called the **coördinates** of the point P, and the lines $X'X$ and $Y'Y$ are called the **axes of coördinates**. Similarly, the point $Q \equiv (-3, 4)$ is located by measuring 3 units toward the *left* from O, and then 4 units upward. The point $R \equiv (-2, -3)$ is also represented in the figure.

The student may draw a figure and locate accurately the following points upon it:* $(5, -1)$, $(4, 7)$, $(-4, 2)$, $(3\frac{1}{2}, -4)$, $(-2\frac{1}{3}, -5\frac{2}{7})$, and $(8, -6\frac{3}{4})$.

115. Geometric picture, or graph, of an equation. By the geometric picture (or map) of an equation — usually called the **locus** or **graph** of the equation — is meant the totality of all those points whose coördinates satisfy that equation.



E.g., since the numbers -1 and -5, when substituted for x and y, respectively, satisfy the equation $2x - y = 3$, therefore the point $P_1 \equiv (-1, -5)$ lies on the graph of this equation; so, too, the points $P_2 \equiv (0, -3)$, $P_3 \equiv (1, -1)$, $P_4 \equiv (2, 1)$, $P_5 \equiv (3, 3)$, etc., are on the graph of this equation, because each of these pairs of numbers satisfies the equation.

If these points are located, by the method of § 114, it is found that they are not scattered

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* It is recommended that cross-section paper be used for this purpose; such paper may be obtained from all stationers.

indiscriminately over the page, but that they all lie upon the line AB; this line is the graph of the given equation.* It is due to this fact that such equations are often called *linear equations* (cf. § 94).

The points P_2, P_3, P_4, ⋯ were found by assigning the values 0, 1, 2, 3, ⋯ to x, and then finding the corresponding values of y from the equation; other points between any two of these may be found by assigning intermediate values to x.

The above method of finding the graph of any given equation in two unknown numbers may be stated thus: by assigning to x a succession of values, such as 0, 1, 2, 3, ⋯, -1, -2, -3, ⋯, find the corresponding values of y, *i.e.*, find as many solutions of the given equation as may be desired; locate the points whose coördinates are these solutions, and draw a line connecting these points in regular order; this line will represent the required graph.

EXERCISES

Draw a pair of axes, as in §§ 114 and 115, and locate the following points:

1. $(5, 4)$; $(3, 7)$; $(4, -2)$; $(-3, 1)$; and $(-4, -6)$.

2. $(3, 0)$; $(-5, 0)$; $(0, 8)$; $(0, 0)$; and $(0, -2)$.

3. Where are all points whose second number is 0? Where are those whose first number is 0? Where are all those whose second number is $3\frac{1}{2}$? Draw a line through this last class of points.

4. Where are those points whose second number is the same as its first number? Where are those whose second number is the opposite of its first number? Draw a line through each of these two classes of points.

5. What is meant by the graph of an equation? Find ten pairs of numbers, each of which satisfies the equation $2x + y = 12$. Carefully locate the points determined by these pairs of numbers.

6. How many solutions has such an equation as that given in Ex. 5? Show that its graph may be regarded as a *record* of *all* of its solutions.

7. Show that the equation $3x = 2$ (*i.e.*, $3x + 0 \cdot y = 2$) is satisfied by each of the following pairs of numbers: $\frac{2}{3}, 1$; $\frac{2}{3}, 2$; $\frac{2}{3}, 3$; $\frac{2}{3}, 4$; etc., $\frac{2}{3}, 0$; $\frac{2}{3}, -1$; $\frac{2}{3}, -2$; etc., *i.e.*, by *every* pair of numbers of which the first is $\frac{2}{3}$.

Where do all these points lie (cf. Ex. 3)? What, then, is the graph of the equation $3x = 2$? Draw it.

8. As in Ex. 7, construct the graph of $2y = 5$. Of $x = -1$. Of $y = 4x$. Of $x^2 = 9$.

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* Students who are acquainted with the theory of similar triangles will find no difficulty in proving that all these points lie on the same straight line (AB), and also that the coördinates of every point on AB will satisfy the given equation.

Assuming the graph of a first degree equation in two unknown numbers to be a straight line, construct the graph of each of the following equations by finding two of its points and drawing a straight line through them:

9. $2x + y - 4 = 0.$

10. $3y - 4x + 2 = 0.$

11. $\frac{2}{5}x - y = 3.$

12. $\frac{3}{x} - \frac{4}{y} = \frac{2}{xy}.$

116. Intersection of two graphs. Since any two numbers which satisfy an equation are the coördinates of some point on the graph of that equation (§ 115), therefore a pair of numbers which satisfies *each* of two given equations must be the coördinates of a point which is on the graph of *each* of these equations, *i.e.*, these numbers are the coördinates of a point in which these graphs intersect.

Hence, to find the coördinates of the point in which the graphs of two equations intersect each other, it is only necessary to solve these equations, regarding them as simultaneous.

On the other hand, instead of solving two simultaneous equations in the ordinary way, one may accurately draw the graph of each of these equations, using the same axes for both, and carefully *measure* the coördinates of their point of intersection; these coördinates will constitute an *approximate* solution of the given equations.

EXERCISES

1. Find the coördinates of the point of intersection of the graphs of $x + y = 5$ and $2x - y = 4$, both by solving these equations and also by measurement, and compare the results.

2. Solve the system of equations $3x + 4y = 7$ and $2x - 3y = 16$ by the graphic method, *i.e.*, by *measuring* the coördinates of the point in which their graphs intersect.

Find the coördinates of the point of intersection (as in Ex. 1) of the graphs of each of the following pairs of equations:

3. $\begin{cases} 4y + 3x = 5, \\ 4x - 3y = 3. \end{cases}$

4. $\begin{cases} 3x - \frac{2}{3}y = 3, \\ \frac{2}{5}x - 2y = 4. \end{cases}$

5. $\begin{cases} 2x - 3y = 7, \\ 5x - 7\frac{1}{2}y = 11. \end{cases}$

6. Show that the two equations in Ex. 5 are algebraically inconsistent. How are their graphs related to each other? Where is their intersection?

7. In how many points can two straight lines intersect each other? Does this agree with § 111? Explain.

CHAPTER XII

INEQUALITIES

117. Definitions. Expressed in algebraic language, the conditions of the problems thus far met with have led to *equations;* but there are many other problems whose conditions lead only to a *statement that one of two expressions is greater or less than the other.* A correct analysis of such a statement is often of great importance, and may afford all the desired information concerning the numbers involved in the given problem.

The symbols $>$ and $<$ are called the **symbols of inequality**, and are read "is greater than," and "is less than," respectively.

Thus, $a > b$ is read "a is greater than b," and $a < b$ is read "a is less than b."

One number is said to be **greater** than another when the result of subtracting the second from the first is a positive number, and one number is said to be **less** than another when the result of subtracting the second from the first is a negative number.

Thus, if $a-b$ is positive then $a > b$, while if $a-b$ is negative, then $a < b$. Again: since $5-2=3$, therefore $5 > 2$; also, since $2-(-6)=8$, therefore $2 > -6$; and since $8-15=-7$, therefore $8 < 15$.

The statement that one of two numbers or expressions is greater or less than the other is called an **inequality.** The number or expression which stands at the left of the symbol of inequality is called the **first member** of the inequality, while the number or expression which stands at the right of this symbol is called the **second member,** — the opening of the symbol being toward the greater number.

Thus, $a > b$ is an inequality of which a is the first member and b the second; it is read, "a is greater than b."

Two inequalities are said to be of the **same species** (or to *subsist in the same sense*) if the first member is the greater in each, or if the first member is the lesser in each; otherwise they are of **opposite species.**

Thus the inequalities $a > b$ and $c+d > c$ are of the same species, while $x^2+y^2 > z^2$ and $m^2 < n^2+mn$ are of opposite species.

118. General principles in inequalities.

(i) *If the same number be added to, or subtracted from, each member of an inequality, the result will be an inequality of the same species as the given one.*

E.g., $10 > 8$, and so, also, $10+5 > 8+5$, and $10-5 > 8-5$.

To prove this principle generally, let the given inequality be $a < b$, and let c be any number whatever; then $(a+c)-(b+c)$, which equals $a-b$, is negative, since $a < b$, and therefore, by definition,

$$a+c < b+c.$$

Similarly, $$a-c < b-c.$$

Manifestly the proof would have been just the same if the given inequality had been $a > b$.

From the principle just proved it follows that *terms may be transposed in an inequality, just as in an equation, viz., by reversing their signs;* for subtracting any given term from each member will cause that term to disappear from one member, and to reappear, with its sign reversed, in the other.

(ii) *If several inequalities of the same species be added, member to member, the result will be an inequality of the same species.*

E.g., adding the inequalities $3 < 7$, $21 < 30$, and $-2 < 1$, member to member, we obtain $22 < 38$.

To prove this principle generally let $a > b$, $c > d$, $e > f$, $\cdots$, $h > k$ be any number of given inequalities, all of the same species; then each of the differences $a-b$, $c-d$, $e-f$, $\cdots$, $h-k$ is positive, hence their sum is positive,

i.e., $(a-b)+(c-d)+(e-f)+\cdots+(h-k)$ is positive,

hence $(a+c+e+\cdots+h)-(b+d+f+\cdots+k)$ is positive,

and therefore, $a+c+e+\cdots+h > b+d+f+\cdots+k$; which was to be proved.

It should be carefully noted that if two or more inequalities which are not of the same species are added, the result may or may not be an inequality.

The student may illustrate this statement by means of some numerical examples.

(iii) *If an inequality be subtracted from an equation, or from an inequality of opposite species, member from member, the result will be an inequality whose species is opposite to that of the subtrahend.*

The proof of this principle is similar to that of (ii) above, and is left as an exercise for the student.

The student may also illustrate, by appropriate examples, that if one inequality be subtracted from another inequality of the same species, the result may be an inequality of the same or of opposite species, or it may be an equation.

(iv) *If each member of an inequality be multiplied or divided by the same positive number, the result will be an inequality of the same species.*

E.g., $24 > 20$, and so, also, $24 \div 4 > 20 \div 4$; again, $3 < 5$, and so also $3 \cdot 7 < 5 \cdot 7$.

To prove this principle, let $a > b$ be any inequality, and let c be any *positive* number whatever; then $(a - b)c$ is positive, since each factor is positive, *i.e.*, $ac - bc$ is positive, and hence by definition,

$$ac > bc,$$

which was to be proved.

Similarly it is proved that, under the above conditions,

$$\frac{a}{c} > \frac{b}{c}.$$

The principle just proved enables one to clear an inequality of fractions, and also to remove any factors that are common to both members.

(v) *If each member of an inequality be multiplied or divided by the same negative number, the result will be an inequality of opposite species.*

To prove this principle, let $a > b$ be any inequality, and let c be any *negative* number whatever; then $(a - b)c$ is negative, *i.e.*, $ac - bc$ is negative, and hence

$$ac < bc,$$

which was to be proved.

Similarly it is proved that, under the given conditions,

$$\frac{a}{c} < \frac{b}{c}.$$

(vi) *If the signs of all the terms of an inequality be reversed, then the symbol of inequality must also be reversed.*

E.g., if $2a - 4c + 3x > 2d + 5y - 7b$, then $4c - 2a - 3x < 7b - 2d - 5y$.

The proof of this principle follows directly from (v) by putting -1 for the multiplier c.

(vii) *If the first of three numbers is greater than the second, and the second is greater than the third, then the first is greater than the third; and conversely.*

E.g., $10 > 7$ and $7 > 3$, and $10 > 3$ also.

To prove this principle, let $a > b$ and $b > c$ be the given inequalities; then $a - b$ is positive, as is also $b - c$, and hence their sum $(a - b) + (b - c)$, *i.e.*, $a - c$, is positive, and therefore $a > c$, which was to be proved.

Similarly it is proved that if $a < b$ and $b < c$, then $a < c$.

(viii) *If two inequalities which are of the same species, and whose members are all positive, be multiplied together, member by member, the result will be an inequality of the same species.*

E.g., $5 > 3$ and $4 > 2$, and $5 \cdot 4 > 3 \cdot 2$ also.

To prove this principle, let $a > b$ and $c > d$ be two such inequalities; then by (iv) $ac > bc$, but by (iv) $bc > bd$, whence by (vii) $ac > bd$, which was to be proved.

By proceeding step by step, it is clear that principle (viii) holds for any number of (and not merely for two) such inequalities.

The student may modify the above statement and proof so as to apply to the case in which some of the members are negative.

EXERCISES

1. When is the first of two numbers said to be greater than the second? When is it said to be less?

2. By the definitions of "greater" and "less" given in § 117, show that $5 > 2$; that $-23 < -12$; and that $2 > -9$.

3. If $a \neq b$, show that $a^2 + b^2 > 2ab$. This is a very important relation, and well worth remembering.

Suggestion. $(a - b)^2$ is positive whether $a > b$ or $a < b$.

4. If two or more inequalities of the same species are added, what is the species of the resulting inequality? Prove your answer. Is it necessary that the members of these inequalities be *positive* numbers?

5. If an inequality is subtracted from another inequality of the same species, member from member, what is the result? Prove your answer.

6. If two inequalities of the same species are multiplied together, member by member, what is the result? Prove your answer. Is it necessary in this case that the members of these inequalities be positive numbers?

7. What happens if the signs of the terms of each member of an inequality are reversed? Why?

8. May terms be transposed from one member of an inequality to the other? If so, how and why?

9. What other operations may be performed with or upon inequalities, producing results whose relations are known?

10. Name and illustrate some operations with inequalities that give results about whose relations there is doubt. *E.g.*, the quotient of two inequalities of the same species, divided member by member, may be an equality or an inequality of the same or of opposite species.

119. Unconditional and conditional inequalities. An **unconditional inequality** is one which is true for all values of the letters involved — *e.g.*, $a + 4 > a$; while a **conditional inequality** is one which is true only on condition that the values to be assigned to the letters involved shall be somewhat restricted — *e.g.*, $x + 4 < 3x - 2$ only on condition that the values assigned to x shall be greater than 3.*

To solve a conditional inequality means to find those values of its letters for which the inequality is true; this may be done by means of the principles which were proved in the preceding article — for illustrations see Exs. 1 and 2 which follow.

Ex. 1. Given $3x - \frac{25}{3} > \frac{11}{3} - x$, to find the possible values of x.

SOLUTION. On multiplying each member of the given inequality by 3, it becomes

$$9x - 25 > 11 - 3x, \qquad [\S\ 118\ (iv)$$

whence

$$9x + 3x > 11 + 25, \qquad [\S\ 118\ (i)$$

i.e.,

$$12x > 36,$$

whence

$$x > 3;. \qquad [\S\ 118\ (iv)$$

therefore, if the given inequality is true, x must be greater than 3.

By means of the principles established in § 118 the student may show that each step in the reasoning of Ex. 1 is reversible, and hence that the converse of that example is also true; viz., that if $x > 3$, then $3x - \frac{25}{3} > \frac{11}{3} - x$.

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* Let it be observed that conditional and unconditional inequalities are respectively analogous to conditional and identical equations; the student may also note the analogy between solving an inequality and solving an equation.

Ex. 2. Given the two relations $\left\{\begin{array}{c} 2x+3y>5 \\ x+4y=6 \end{array}\right\}$; to find those values of x and y that will satisfy them *both*.

Solution. On multiplying each member of the inequality by 4, and each member of the equation by 3, they become, respectively,

$$8x+12y>20,$$

and $$3x+12y=18;$$

whence, subtracting, $$5x>2, \qquad [\S\ 118\ \text{(i)}$$

and therefore $$x>\tfrac{2}{5}. \qquad [\S\ 118\ \text{(iv)}$$

Now substitute for x any number greater than $\frac{2}{5}$, in the above equation, and find the corresponding value of y; these values of x and y, taken together, will satisfy both the equation and the inequality.

EXERCISES

3. Distinguish between a conditional and an unconditional inequality. To which of these classes does $a^2+b^2+1>2ab$ belong? Why?

4. Is the expression $6x-5>3x+10$ true for *all* values of x? If not, what is the least value that x may have in this inequality? To which class does this inequality belong?

5. What is meant by "solving" a conditional inequality? Describe the procedure. Illustrate what you have said by solving the inequality in Ex. 4.

6. From the inequality in Ex. 4 above it is found that $x>5$, *i.e.*, the *range* of values that x may have in this inequality is from *just above* 5 upward; 5 may here be called the *lower limit*, or *minimum*, of the possible values of x. Find the minimum value of x in $3x<5x-9$.

7. Show that the range of values of x in $x^2+24<11x$ is between 3 and 8, *i.e.*, that 3 is the lower limit, or minimum, and that 8 is the upper limit or maximum.

Suggestion. In order that $(x-3)(8-x)$, *i.e.*, $11x-x^2-24$, may be positive, both factors must be positive or both negative.

Find the range of values of x in each of the following inequalities:

8. $x^2>9$.

9. $x^2+24>11x$.

10. $30>x+\dfrac{3x}{2}>25$.

11. $28>3x+x^2$.

12. $x^2>9x-18$.

13. $x^2+5x>24$.

14. $\left\{\begin{array}{l} 4x-11>\dfrac{x}{3}, \\ 10-x>5. \end{array}\right.$

15. $\left\{\begin{array}{l} 3-4x<7, \\ x+2<4. \end{array}\right.$

16. By the definitions of "greater" and "less" given in § 117, show that $n + \frac{1}{n} \not< 2$, when n is any positive number,* *i.e.*, show that *the sum of any positive number and its reciprocal is not less than 2.*

17. Show that $4x^2 + 9 \not< 12x$.*

18. Show that $2b(6a - 5b) \not> (2a + b)(2a - b)$.

If a, b, and c are positive and unequal, prove the correctness of the following statements:

19. $a^2 + b^2 + c^2 > ab + bc + ac$.

20. $a^3 + b^3 > a^2b + ab^2$.

21. $a^3 + b^3 + c^3 > 3abc$.

22. If $a^2 + b^2 = 1$, and $c^2 + d^2 = 1$, prove that $ab + cd \not> 1$.*

23. If m and n are both positive, which of the expressions $\frac{m+n}{2}$ or $\frac{2mn}{m+n}$ is the greater?

Solve the following systems:

24. $\begin{cases} 2x - 3y < 2, \\ 2x + 5y = 6. \end{cases}$

25. $\begin{cases} 3x + 2y = 42, \\ 3x - \frac{y}{7} > 16. \end{cases}$

26. $\begin{cases} x + y = 10, \\ 4x < 3y. \end{cases}$

27. $\begin{cases} y - x > 9, \\ \frac{7x}{20} + \frac{y}{15} = 9. \end{cases}$

28. $\begin{cases} x > y + 4, \\ x - 2y = 8. \end{cases}$

Find the integral values of x and y in the following systems:

29. $\begin{cases} 5x + y > 51, \\ 3x - y < 21. \end{cases}$

30. $\begin{cases} 9x + \frac{5y}{7} > 31, \\ 13x - y < 33. \end{cases}$

31. If 16 more than 3 times the number of sheep in a certain flock exceeds 27 plus twice their number, and if 45 less than 4 times their number is less than their number diminished by 6, how many sheep are there in the flock?

32. Find the smallest integer fulfilling the condition that $\frac{1}{3}$ of it decreased by 7 is greater than $\frac{1}{4}$ of it increased by 6.

33. Find a simple fraction (in its lowest terms) which, when 2 is added to its numerator and subtracted from its denominator, shall be greater than $\frac{2}{3}$, while if 2 is subtracted from its numerator and added to its denominator, it shall be less than $\frac{1}{3}$.

34. Three times A's money and 4 times B's is \$1 more than 6 times A's; and if A gives \$5 to B, then B will have more than 6 times as much as A will have left. Find the range of values of A's money and B's.

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* Compare also Ex. 3, p. 196. The symbol $\not<$ stands for "is not less than."

REVIEW QUESTIONS — CHAPTERS X-XII

1. Define and illustrate: conditional equations; equivalent equations; integral equations; the degree of an equation; literal equations.

2. Outline the plan for solving a conditional equation in one unknown number, and state the principles upon which this plan rests.

3. How may a fractional equation in one unknown number be solved?

4. Under what circumstances are extraneous roots introduced by clearing an equation of fractions? How may such roots be detected?

5. By means of the equation $\frac{3x}{x+5}+\frac{42}{x^2+3x-10}=\frac{2(4x-5)}{x-2}$, illustrate your answer to Ex. 4.

6. Define and illustrate what is meant by: an indeterminate equation; an indeterminate system of equations; consistent equations; independent equations; simultaneous equations.

7. Outline three methods of elimination.

8. Prove that the system of equations $a_1x+b_1y=c_1$ and $a_2x+b_2y=c_2$ has one solution, and only one, if $a_1b_2 \neq a_2b_1$.

9. Outline the procedure for solving a system consisting of n independent simple equations in n unknown numbers.

10. Find an expression of the form $ax^2+bx+\frac{c}{x}$ whose value is 16 when $x=-1$, 2 when $x=1$, and 40 when $x=2$.

11. What is meant by the graph of an equation? Illustrate your answer.

12. How may the graph of an equation be constructed? Construct the graph of $5y=3x+10$; also of $2y^2=8x+1$.

13. How may a pair of equations, such as that given in Ex. 8, be solved graphically? Illustrate your answer.

14. Define a conditional inequality, also an unconditional inequality. Illustrate each.

15. How may a conditional inequality be solved? Illustrate your answer by finding the range of values of x in the inequality $x-3<\frac{10}{x}$.

16. If $x-3<\frac{10}{x}$, does it follow that $x^2-3x<10$?

17. Prove that a positive proper fraction is increased by adding the same positive number to both its numerator and its denominator.

CHAPTER XIII

INVOLUTION AND EVOLUTION

I. INVOLUTION

120. Definitions. If a represents any number* whatever, then it has been *agreed* that the product $a \cdot a \cdot a \cdots$ (to n factors), which is called the **nth power of a**, shall, for brevity, be represented by the symbol a^n, which is usually read "a nth." The number a is called the **base**, and n the **exponent**, of the power [cf. § 7 (iv)].

The operation of raising a number to any given power is called **involution**. It consists merely in a succession of multiplications; thus, $4^3 = 4 \cdot 4 \cdot 4 = 64$, $(-2)^5 = -32$, $(a+b)^2 = a^2 + 2ab + b^2$, etc.

Under the above definition the symbol a^n has been appropriated only when n is a positive integer; that definition assigns no meaning whatever to such expressions as a^{-3}, a^0, and $a^{\frac{2}{3}}$. In § 44† it was shown, however, that in operating with such symbols as a^n it is often advantageous to make the further *agreement* that a^{-k}, where k is any positive integer, shall mean $\frac{1}{a^k}$, and that a^0 shall mean 1. In Chap. XIV such symbols as $a^{\frac{2}{3}}$ will have a meaning assigned to them, and will receive detailed consideration.

121. The exponent laws. Under the above agreements as to the meaning of a^n, the following laws for exponents are easily established.

(i) *First exponent law.* If a is any base, and m and n are integers (positive or negative), or zero, then

$$a^m \cdot a^n = a^{m+n}. \ddagger$$

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* The word number is here used to include *algebraic expression* also.

† This article should now be reread.

‡ Compare also § 37.

For, if m and n are positive integers, then

$$a^m \cdot a^n = (a \cdot a \cdot a \cdots \text{ to } m \text{ factors}) \cdot (a \cdot a \cdot a \cdots \text{ to } n \text{ factors})$$

$$= a \cdot a \cdot a \cdots \text{ to } (m+n) \text{ factors} \qquad [\text{Associative law}$$

$$= a^{m+n}.$$

If either m or n is a negative integer, say $n = -k$, where k is a positive integer, then

$$a^m \cdot a^n = a^m \cdot a^{-k} = a^m \cdot \frac{1}{a^k} \qquad \left[a^{-k} = \frac{1}{a^k}\right.$$

$$= \frac{a \cdot a \cdot a \cdots \text{ to } m \text{ factors}}{a \cdot a \cdot a \cdots \text{ to } k \text{ factors}}$$

$$= a^{m-k} \text{ or } \frac{1}{a^{k-m}},$$

according as $m > k$, or $m < k$;

but (since $n = -k$) $\quad a^{m-k} = a^{m+n}$,

and $\quad \frac{1}{a^{k-m}} = a^{-(k-m)} = a^{m-k} = a^{m+n}$;

therefore $\quad a^m \cdot a^n = a^{m+n}$,

even if one of the exponents is a negative integer.

Similarly the student may prove the correctness of this law if both m and n are negative, or if either or both of them are 0.

By successive applications of the foregoing law, and with the same limitations upon the exponents, it follows that

$$a^m \cdot a^n \cdot a^p \cdot a^r \cdots = a^{m+n+p+r+\cdots}.$$

(ii) *Second exponent law.* If a is any base, and m and n are integers (positive or negative), or zero, then

$$(a^m)^n = a^{mn}.$$

For, if m and n are positive integers, then

$$(a^m)^n = (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})^n$$

$$= a \cdot a \cdot a \cdots \text{ to } mn \text{ factors} \qquad [\text{Associative law}$$

$$= a^{mn};$$

and if either m or n is a negative integer, say $m = -k$, where k is a positive integer, then

$$(a^m)^n = (a^{-k})^n = \left(\frac{1}{a^k}\right)^n = \frac{1}{a^k} \cdot \frac{1}{a^k} \cdot \frac{1}{a^k} \cdots \text{ to } n \text{ factors}$$

$$= \frac{1}{(a^k)^n} = \frac{1}{a^{kn}} = a^{-kn} = a^{mn}. \qquad [-k = m$$

If both m and n are negative, or if either or both of them are zero, the proof is similar to that just given; hence, for all these cases,

$$(a^m)^n = a^{mn}.$$

(iii) *Third exponent law.* If a and b are any two bases, and n is a positive or negative integer, or zero, then

$$a^n \cdot b^n = (ab)^n.$$

For, if n is a positive integer, then

$$a^n \cdot b^n = (a \cdot a \cdot a \cdots \text{ to } n \text{ factors}) \cdot (b \cdot b \cdot b \cdots \text{ to } n \text{ factors})$$

$$= ab \cdot ab \cdot ab \cdots \text{ to } n \text{ factors} \qquad \left[\begin{array}{l}\text{Commutative and}\\ \text{associative laws}\end{array}\right.$$

$$= (ab)^n;$$

if n is a negative integer, say $n = -k$, where k is a positive integer, then

$$a^n \cdot b^n = a^{-k} \cdot b^{-k} = \frac{1}{a^k} \cdot \frac{1}{b^k} = \frac{1}{a^k \cdot b^k} = \frac{1}{(ab)^k} = (ab)^{-k} = (ab)^n,$$

as before;

and if $n = 0$, then $\qquad a^n \cdot b^n = 1 = (ab)^n; \qquad$ [Since $x^0 = 1$

hence, for all these cases, $a^n \cdot b^n = (ab)^n$.

By successive applications of the above law it follows that

$$a^n \cdot b^n \cdot c^n \cdot d^n \cdots = (abcd \cdots)^n.$$

(iv) *Fourth exponent law.* If a is any base and m and n are any integers, or zero, then

$$a^m \div a^n = a^{m-n}.$$

The proof of the correctness of this law rests directly upon the first exponent law [(i) above], and the definition of a quotient [§ 3 (iv)], for, since $a^{m-n} \cdot a^n = a^{m-n+n} = a^m$, [(i) above

therefore $a^m \div a^n = a^{m-n}$. [§ 3 (iv)

EXERCISES

1. Write a carefully worded statement of each of the four exponent laws above, — *e.g.*, the third law may be stated thus: "The product of like powers of any two or more numbers is the like power of the product of those numbers."

2. How is the sign of the power in such a case as $(-b^2)^5$ determined? State, illustrate, and prove a law which shall cover all such cases, bearing in mind that the exponents may be positive or negative integers, or zero (cf. § 18, especially note 2).

3. Tell what the *sign* of the result in each of the following expressions is, and explain your answer:

$(-a)^3$; $(-a)^4$; $(a)^{-3}$; $(-a)^{-3}$; $(-4)^0$; $\left(-\frac{a^{-1}}{b^{-3}}\right)^{-2}$; $(-6)^{280}$; $(-x)^{2n}$; and $(-x)^{2n-1}$.

What is the value of $(-2)^3 \cdot 2^{-2}$? Of $3^{-2} \cdot (-2)^3$? Of $3^2 \cdot 2^{-2}$?

4. How is a fraction raised to a power? Why? Give four illustrations of your answer. Read again the second paragraph of § 120.

5. What does a represent in the proofs of § 121? May it represent any polynomial whatever, as well as any number? What does it represent in Ex. 3?

6. By § 62 expand the following expressions: $(x+y)^2$, $(x+y)^3$, and $(x+y)^5$; then multiply the first two expanded forms together, and thus verify that $(x+y)^2 \cdot (x+y)^3 = (x+y)^5$.

7. To what kind of numbers were exponents originally limited? To what extent has this limitation now been removed? What is the meaning of such an expression as x^{-3}? Of x^0? Read again § 44, and the third paragraph of § 120.

Simplify the following expressions (free them from negative and zero exponents where such occur, etc.) and explain each step of your work fully, always referring to the appropriate exponent laws:

8. $a^3b^3c^3$; $a^{-3}b^{-3}c^{-3}$; and $(a^{-2})^3$.

9. $(a^2x^3y^{-2})^4$; $(m^3xy^{-4})^2$; and $(a^2y^{-3})^{-2}$.

10. $(a^2x^2)^3 \div (-ax^2)^2$; $(6\,a^0)^2 \div (2\,x^0)^2$; and $(-12^2\,a^{-3}x^4y^0)^2 \div (-3^2\,a^{-5}x^2)^3$.

11. $\left(\dfrac{2\,x^2y^3z^4}{4\,a^3b^5c^6}\right)^4.$ **13.** $\left(\dfrac{-5\,a^2b^0}{a^{-2}b^3c}\right)^3.$* **15.** $\left(\dfrac{a^{n-1}c^n}{x^{m+n-1}}\right)^p.$

12. $\left(-\dfrac{a^n}{b^{n+1}}\right)^5.$ **14.** $\left(-\dfrac{3\,b^2x^{-3}}{5\,ay^2}\right)^4.$ **16.** $\left(-\dfrac{x^{n-1}c^{n+2}}{x^{m+n-2}z^{n^2}}\right)^{2n}.$

17. State the binomial theorem (§ 62).

18. How many terms are there in the expansion of $(m+n)^5$? How many in $(a-b)^8$? How many in $(3\,s-2\,t)^n$?

19. What are the signs of the terms in $(a-b)^8$? Compare $(a-b)^8$ with $[a+(-b)]^8$, and explain why the alternate terms of the expansion are negative.

Write down the expansions of the following expressions, and remove negative exponents where they present themselves:

20. $(2\,a-3\,b)^4$.†

21. $(x^2-y^4)^3$.

22. $(x^{-2}+3\,y^{-1})^4$.

23. $(3\,a+2\,b^2)^5$.

24. $(2\,m+3\,x)^6$.

25. $(a+b+c)^3$, *i.e.*, $[a+(b+c)]^3$.

26. $(3\,x^2y^2-4\,y^3z^5)^4$.

27. $(x^3-2\,y^{-3})^5$.

28. $(a^{-2}-bc^{-1})^3$.

29. $(2\,x^3+3\,x^2-5)^4$.

30. Is $(a\cdot b\cdot c\cdot d)^2$ equal to $a^2\cdot b^2\cdot c^2\cdot d^2$? Is $(a+b+c+d)^2$ equal to $a^2+b^2+c^2+d^2$? Explain your answer.

Is involution distributive over a product (cf. § 39)? over a sum?

31. Translate the following symbolic statement into a verbal one:

$$(a+x)^n \neq a^n + x^n.$$

32. Is $[(-2)^3]^2$ equal to $[(-2)^2]^3$? What is the sign of each result? Why?

33. Prove that $(a^m)^n=(a^n)^m$, wherein a is any number or algebraic expression, and m and n are integers (positive or negative) or zero [cf. law (ii) above]. Also state this principle in words.

II. EVOLUTION

122. Definitions. A number whose nth power is a given number (n being any positive integer) is called an ***n*th root** of the given number; thus, if $a^n=b$, then a is an nth root of b.‡

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* Compare Exs. 20–26, § 93.

† Compare note, § 57.

‡ As here used the word number includes *algebraic expression* also.

E.g., 2 is a 3d root of 8 *because* $2^3 = 8$; so also $2\,ab^2$ is a 5th root of $32\,a^5b^{10}$; either $+3$ or -3 (*i.e.*, ± 3) is a 2d root of 9; $\pm \frac{2}{3}$ is a 4th root of $\frac{16}{81}$; a 3d root of $x^3 - 3\,x^2y + 3\,xy^2 - y^3$ is $x - y$; etc.

The special names **square root** and **cube root** are usually employed instead of 2d root and 3d root, respectively [cf. § 7 (iv), note].

The operation of finding any root of a given number is called **evolution,** or **extraction of roots.** Evolution is then the inverse of involution,* just as subtraction is the inverse of addition, and division the inverse of multiplication.

The radical sign, $\sqrt{\ }$, is placed before a number to indicate that a root of the given number is required, and a small figure, called the **index** of the root, is placed in the opening of the radical sign to indicate what particular root is to be extracted.

The number whose root is required is called the **radicand**; and an indicated root is said to be an **even root** or an **odd root** according as its index is an even or an odd number.

Thus, $\sqrt[3]{27}$ stands for the cube root of 27; this is an odd root since its index, 3, is an odd number, and 27 is the radicand. $\sqrt[5]{}(32\,a^5b^{10})$ is the 5th root of $32\,a^5b^{10}$, and $\sqrt[n]{a}$ is the nth root of a. If no index is written, the index is understood to be 2, *i.e.*, $\sqrt{4}$ stands for the square root of 4.

The radical sign is a modification of the letter r — the initial letter of the Latin word *radix*, meaning root.

In practice the radical sign is usually combined with a vinculum (§ 8) to indicate clearly just how much of the expression following the radical sign is to be affected by that sign; thus $\sqrt{9+16}$ means the square root of the sum of 9 and 16, while $\sqrt{9}+16$ indicates that 16 is to be added to the square root of 9.

Instead of the vinculum a parenthesis may be used for the same purpose, in connection with a radical sign, thus: $\sqrt{}(9+16) = \sqrt{9+16}$, $\sqrt{a^3b^6}\cdot c = \sqrt{}(a^3b^6)\cdot c$, etc.

123. Roots of monomials. If a monomial is an exact power, the corresponding root can usually be written down by inspection.

E.g., $\sqrt[3]{8\,a^6x^3} = 2\,a^2x$, because $(2\,a^2x)^3 = 8\,a^6x^3$ (§ 121); $\sqrt{9\,x^4y^6} = +3\,x^2y^3$ or $-3\,x^2y^3$, because $(+3\,x^2y^3)^2 = (-3\,x^2y^3)^2 = 9\,x^4y^6$; $\sqrt[5]{-32\,x^{10}} = -2\,x^2$, because $(-2\,x^2)^5 = -32\,x^{10}$; $\sqrt[3]{\frac{8\,m^3}{x^3y^6}} = \frac{2m}{xy^2}$, because $\left(\frac{2m}{xy^2}\right)^3 = \frac{8\,m^3}{x^3y^6}$; etc. See also Exs. 5 and 21 below.

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* It is to be remarked, however, that while raising a number to a power always produces a single result, extracting a root may lead to more than one result; *e.g.*, $3^2 = 9$, but the square root of $9 = +3$ or -3.

This is often expressed by saying that involution is a *unique* operation, while evolution is *non-unique*.

EXERCISES

1. What is meant by the square root of a number? Of what two equal *positive* factors is 25 the product? What, then, is a square root of 25? Has 25 another square root? Why?

2. What are the square roots of 49? Why? The fourth roots of 81? Why? Prove that if a is any *even* root of a number, then $-a$ is also a root (with the same index) of that number.

3. What is the cube root of 27? Why? Of -27? Why? Of 64 and of -64? Why? How does $\sqrt[5]{32}$ compare with $\sqrt[5]{-32}$? Why?

Compare the *signs* of odd roots of numbers with the signs of the numbers themselves, and give your reasons in full. Is this also true for even roots?

4. What is the sign of any *even* power of any positive or negative number? Why? Can, then, an even root of a negative number be an integer or a fraction, positive or negative? Why?

5. What is the nth power of $a^3b^2x^{k}y^{-5}$? What, then, is $\sqrt[n]{a^{3n}b^{2n}x^{kn}y^{-5n}}$? Why? What is the *sign* of this root? Why? How do the exponents of the root compare with those of the number itself? Why?

6. Is $\sqrt{9 \cdot 16}$ equal to $\sqrt{9} \cdot \sqrt{16}$? Why? Is $\sqrt{9+16}$ equal to $\sqrt{9}+\sqrt{16}$? Compare Ex. 30, § 121, and give a verbal statement of your general conclusion.

Find the following indicated roots, and verify your answers. Also tell which are even and which are odd roots, and name the radicand and the index in each case:

7. $\sqrt[3]{a^8b^6c^{15}}$.

8. $\sqrt{16\,a^4x^6y^{-2}}$.

9. $\sqrt[5]{32\,x^5y^{10}}$.

10. $\sqrt[n]{a^{4n}x^{-3n}y^{n^2}}$.

11. $\sqrt[3]{-64\frac{x^6}{y^9}}$.

12. $\sqrt[5]{-243\,a^{10}x^{-5}}$.

13. $\sqrt[7]{128\,a^{7x}b^{-14y}}$.

14. $\sqrt[3]{-\frac{125\,x^{12}y^6}{1728\,a^6z^9}}$.

15. $\sqrt[7]{\frac{-(x-y)^{14}}{128\,x^{14}}}$.

16. $\sqrt[5]{\frac{-32\,a^5x^{40}}{243\,y^{25}}}$.

17. $\sqrt[8]{\frac{256\,m^8p^{16}}{6561\,h^{32}z^{-8}}}$.

18. $\sqrt[3]{\frac{.027\,a^3x^6}{.064\,b^3z^{-6}}}$.

19. $\sqrt[2n]{\frac{a^{2n^2-2n}b^{4n}}{x^{6n}y^{4n}}}$.

20. $\sqrt[x]{\frac{a^{5x}b^{x^2}x^{3x}}{2^{2x}y^{4x}z^{x}}}$.

21. Write a rule for the extraction of such roots as the above, and emphasize particularly the matter of exponents and signs. Does your rule apply to roots of polynomials also?

124. Roots of polynomials extracted by inspection. If a polynomial is an exact power of a *binomial*, a little study will usually reveal the corresponding root; this is illustrated by the following examples.

Ex. 1. Find the square root of $m^6 + 4m^3n + 4n^2$.

Solution. This expression is easily seen to be $(m^3 + 2n)^2$; therefore $\sqrt{m^6 + 4m^3n + 4n^2} = \pm(m^3 + 2n)$.

Ex. 2. Find the cube root of $8a^3 - 36a^2b - 27b^3 + 54ab^2$.

Solution. Since the given polynomial has four terms, two of which, viz., $8a^3$ and $-27b^3$, are exact cubes, therefore it *may* be the cube of a binomial (§ 62); if it is the cube of a binomial, that binomial must be $2a - 3b$ (why?), which, on further examination, proves to be the required cube root.

Hence $\sqrt[3]{8a^3 - 36a^2b - 27b^3 + 54ab^2} = 2a - 3b$.

A polynomial which is the *square* of another polynomial may also sometimes be recognized as such (cf. § 61), and its square root may then be written down by inspection.

Ex. 3. Find the square root of $a^2 + b^2 - 2ab - 4bc + 4c^2 + 4ac$.

Solution. Since the given polynomial consists of six terms, three of which are exact squares, and three of which are double products, therefore (§ 61) it *may* be the square of a trinomial whose terms are the square roots of the square terms; by a little further examination it is seen that

$$\sqrt{a^2 + b^2 - 2ab - 4bc + 4c^2 + 4ac} = \pm(a - b + 2c).$$

EXERCISES

Extract the following indicated roots by inspection, and verify:

4. $\sqrt{4x^2 + 12x + 9}$.

5. $\sqrt{25y^2 - 40y + 16}$.

6. $\sqrt{(m+n)^2 - 4(m+n) + 4}$.

7. $\sqrt{x^2 + 2xy + y^2 - 2xz - 2yz + z^2}$.

8. $\sqrt[3]{8h^3 - 84h^2k + 294hk^2 - 343k^3}$.

9. $\sqrt[4]{x^4 - 4x^3y + y^4 - 4xy^3 + 6x^2y^2}$.

10. $\sqrt[3]{8u^3 - 12u^2v - v^3 + 6uv^2}$.

11. $\sqrt[5]{a^5 - b^5 - 5a^4b + 5ab^4 + 10a^3b^2 - 10a^2b^3}$.

12. $\sqrt{a^2 + 9b^2 - 6ab + 6(x - 2y)(a - 3b) + 9(x^2 - 4xy + 4y^2)}$.

13. $\sqrt[6]{x^6 - 6abx^5 + 15a^2b^2x^4 - 20a^3b^3x^3 + 15a^4b^4x^2 - 6a^5b^5x + a^6b^6}$.

125. Square roots of polynomials. Since it is not always easy to find the square root of a polynomial by the method illustrated in § 124, another method, which is always applicable, will now be given. This method will be better understood by first squaring a polynomial and carefully observing its formation, and then reversing that process.

(i) Consider first the binomial $A+B$; its square is $A^2+2AB+B^2$, therefore the square root of $A^2+2AB+B^2$ is $A+B$; and the question now to be investigated is: *given the power $A^2+2AB+B^2$, how may the root $A+B$ be found from it?*

Since the first term of the power is the square of the first term of the root, therefore the first term of the root is the square root of the first term of the power; *i.e.*, the first term of the root is $\sqrt{A^2}$, viz., A.*

If the square of the root term just found be subtracted from the given power, then the first term of the remainder, viz., $2AB$, will be the double product of the first and second terms of the root, therefore the second term of the root is found by dividing the first term of the remainder by twice the root already found.

Twice the root already found at any stage of the work is usually called the **trial divisor**, and the trial divisor plus the next root term is called the **complete divisor.**

The work of finding the square root just considered may be put into the following form:

$$\begin{array}{lll|l}
 & & A^2+2AB+B^2 & \underline{A+B} \\
 & & \underline{A^2} & \\
\text{Trial divisor,} & 2A & 2AB+B^2 & \\
\text{Complete divisor,} & 2A+B & \underline{2AB+B^2} & =(2A+B)\cdot B \\
 & & 0 &
\end{array}$$

Observe that the first and second subtractions are together equivalent to the subtraction of $(A+B)^2$ from the given power.

Similarly, to find the square root of $9m^2-42mx^3+49x^6$, the work may be arranged thus (the student should fully explain each step of the process):

$$\begin{array}{lll|l}
 & & 9m^2-42mx^3+49x^6 & \underline{3m-7x^3} \\
 & & \underline{9m^2} & \\
\text{Trial divisor,} & 6m & -42mx^3+49x^6 & \\
\text{Complete divisor,} & 6m-7x^3 & \underline{-42mx^3+49x^6} & =(6m-7x^3)(-7x^3) \\
 & & 0 &
\end{array}$$

(ii) The above plan for extracting the square root of a trinomial power is easily extended so as to apply to polynomial powers of any number of terms.

Consider, for example, the expression $A+k+B$, wherein A stands for the first n terms, k for the next term, and B for all the remaining terms of any poly-

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* For the consideration of the negative root (viz., $-A$), see note 2, page 211.

nomial whatever; and let *all* of the terms of this polynomial be regarded as already arranged according to the descending powers of some one of its letters.

The square of this polynomial is $A^2+2Ak+k^2+2AB+2Bk+B^2$, and the question is: *given the power* $A^2+2Ak+k^2+2AB+2Bk+B^2$, *how may the root* $A+k+B$ *be found from it?*

Let it be assumed that the terms represented by A have already been found,* —by (i) above or any method whatever,—then it is clear that when A^2 has been subtracted from the power, the highest term in the remainder is the highest term in $2Ak$, hence *the next term in the root* (viz., k) may be found by dividing the *highest term* in this remainder by the *highest term* in $2A$, *i.e.*, by the highest term in the trial divisor. But since A stands for the terms of the root already found, therefore what has just been said shows how to find the *next* term of the root *at any stage of the work*, *i.e.*, it shows how to find *all* the terms of the root.

This work may be arranged thus:

$$\begin{array}{lll|l}
 & & A^2+2Ak+k^2+2AB+2Bk+B^2 & \underline{A+k} \\
 & & A^2 & \\ \cline{3-3}
\text{Trial divisor,} & 2A & \multicolumn{1}{|l}{2Ak+k^2+2AB+2Bk+B^2} & \\
\text{Complete divisor,} & 2A+k & \multicolumn{1}{|l}{2Ak+k^2} & =(2A+k)\cdot k \\ \cline{3-3}
 & & \qquad\qquad 2AB+2Bk+B^2 &
\end{array}$$

Observe that the two subtractions here made are together equivalent to subtracting $(A+k)^2$ from the given power; *i.e.*, by proceeding as above explained, the remainder at any stage of the work is the same as that obtained by subtracting the square of the root found at that stage of the work from the given power.

Similarly, to find the square root of $9x^4+6x^3y-11x^2y^2-4xy^3+4y^4$, the work may be arranged thus (the student should, however, explain each step):

$$\begin{array}{lll|l}
 & & 9x^4+6x^3y-11x^2y^2-4xy^3+4y^4 & \underline{3x^2+xy-2y^2} \\
 & & 9x^4 & \\ \cline{3-3}
\text{1st trial div.,} & 2(3x^2)=6x^2 & \multicolumn{1}{|l}{6x^3y-11x^2y^2-4xy^3+4y^4} & \\
\text{1st comp. div.,} & 6x^2+xy & \multicolumn{1}{|l}{6x^3y+\quad x^2y^2} & =(6x^2+xy)\cdot xy \\ \cline{3-3}
\text{2d trial div.,} & 2(3x^2+xy)=6x^2+2xy & \multicolumn{1}{|l}{-12x^2y^2-4xy^3+4y^4} & \\
\text{2d comp. div.,} & 6x^2+2xy-2y^2 & \multicolumn{1}{|l}{-12x^2y^2-4xy^3+4y^4} & =(6x^2+2xy-2y^2)\cdot 2y^2 \\ \cline{3-3}
 & & \qquad\qquad\qquad 0 &
\end{array}$$

The above method for extracting the square root of a polynomial may be stated thus:

(1) *Arrange the terms of the given polynomial according to the descending powers of some one of its letters, and write the square root of its first term as the first term of the required root.*

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* The first term at least may always be found as in (i) above.

(2) *Subtract the square of the root term just found from the given polynomial, and divide the first term of the remainder by twice the first term of the root; write the quotient as the next term of the required root, and also annex it to the trial divisor to form the complete divisor.*

(3) *Multiply the complete divisor by the last root term, which has just been found, and subtract the product from the preceding remainder.*

(4) *Divide the first term of this new remainder by the first term of the new trial divisor; write the quotient as the next term of the required root, and also add it to the trial divisor to form the complete divisor.*

(5) *Repeat the steps* (3) *and* (4) *until all the terms of the root are found.*

Note 1. Observe that if polynomials are arranged according to *ascending* instead of to *descending* powers of the letter of arrangement, the above demonstration still applies; it requires only the verbal change of *lowest term* for *highest term.*

Note 2. If the negative value, instead of the positive value, of the square root of the first term of the polynomial had been used in the above demonstration, the sign of each term of the result would have been changed, *i.e.*, the result would have been the negative square root of the given polynomial.

Note 3. It has been shown above how to find the square root of a polynomial *which is an exact square; i.e.*, if the above process be continued until a zero remainder is reached, then the square of the expression thus found will be the given polynomial. If, however, the same process be applied to a polynomial which is not an exact square, then as many root terms as desired may be found, and the square of this root, at any stage of the work, will equal the result of subtracting the corresponding remainder from the given polynomial — such a root is usually called an **approximate** root, and also the **root to** n **terms.**

EXERCISES

Find the square root of each of the following expressions, and verify the correctness of your result:

1. $x^4 - 4x^3 + 8x + 4$.

2. $4m^4 - 4m^3 - 3m^2 + 2m + 1$.

3. $1 - 6y + 5y^2 + 12y^3 + 4y^4$.

4. $25x^8y^6 - 40a^2b^3x^4y^3 + 16a^4b^6$.

5. $4x^6 + 17x^2 - 22x^3 + 13x^4 - 24x - 4x^5 + 16$.

6. $4a^4 + 64b^4 - 20a^3b + 57a^2b^2 - 80ab^3$.

7. $6x^5y + 2x^3y^3 - 28xy^5 + 9x^6 + 4y^6 + 45x^2y^4 + 43x^4y^2$.

8. $3x^4 - 2x^5 - x^2 + 2x + 1 + x^6$.*

9. $48a^4 + 12a^2 + 1 - 4a - 32a^3 + 64a^6 - 64a^5$.

10. $46x^2 + 25x^4 - 44x^3 - 40x + 4x^6 + 25 - 12x^5$.

11. $x^4 - 2x^2y + 2x^2z^2 - 2yz^2 + y^2 + z^4$.

12. $x^8 - 2a^2x^6 - 3a^4x^4 + 4a^6x^2 + 4a^8 - 16a^7x + 32a^5x^3 - 20a^3x^5 + 4ax^7$.

13. $\frac{r^4}{4} + r^3s + \frac{4r^2s^2}{3} + \frac{2rs^3}{3} + \frac{s^4}{9}$.

14. $\frac{x^4}{a^2} + 16a^2y^6 + 8x^2y^3$.

15. $x^2 + 2x - 1 - \frac{2}{x} + \frac{1}{x^2}$.†

16. $9x^2 - 24x + 28 - \frac{16}{x} + \frac{4}{x^2}$.

17. $n^4 + 4n^3 + \frac{1}{n^2} + 2n + 4 + 4n^2$.

18. $x^4 + \frac{1}{x^4} + 4x^3 + \frac{1}{x^3} + 6x^2 + \frac{9}{4x^2} + 5 + 5x + \frac{5}{x}$.

19. $4 + \frac{9a^2}{b^2} - \frac{6a}{b} - \frac{b}{a} + \frac{b^2}{4a^2}$.

20. $(x - y)^2 - 2(xy + xz - y^2 - yz) + (y + z)^2$.

21. $x^{2r}y^{2s} - 6x^{r+1}y^{s+1} - 30x^ry^{s+2} + 10x^{2r-1}y^{2s+1} + 25x^{2r-2}y^{2s+2} + 9x^2y^2$.

22. $1 + x$, to 4 terms. See note 3.

23. $a^2 + 1$, to 3 terms.

24. $1 + x - x^2$, to 4 terms.

25. $x^4 + 2x^3y + y^4 + xy^3 + x^2y^2$, to 4 terms.

26. By extracting the square root until a numerical remainder is reached, show that $x^4 + 4x^3 + 8x^2 + 8x - 5$ equals $(x^2 + 2x + 2)^2 - 9$, and thus find the factors of $x^4 + 4x^3 + 8x^2 + 8x - 5$.

27. Similarly, find the factors of $x^4 + 6x^3 + 11x^2 + 6x - 8$ and $a^6 - 6a^4 + 10a^3 + 9a^2 - 30a + 9$.

---

* Check Exs. 8–21 by the method of Ex. 7, § 39.

† Show first that this expression is already arranged according to descending powers of x.

126. Square roots of arithmetical numbers. Arithmetical numbers are merely disguised polynomials — *e.g.*, $3862 = 3(10)^3 + 8(10)^2 + 6(10) + 2$ — and their square roots are extracted by virtually the same process as that given in the preceding article.

Although it is not *necessary* to do so, yet it is more systematic to find the several digits of these roots in their order from left to right, just as the *terms* are found in the case of polynomials; to do this the given number is first separated into *periods* of two figures each, to the right and left of the decimal point.

The reason for the separation into periods lies in this: the square of any number of tens ends in two ciphers, and hence the first two digits at the left of the decimal point are useless when finding the tens' digit of the root; they are therefore set aside until needed to find the units' digit of the root. So, too, the square of any number of hundreds ends in four ciphers, and hence, for a like reason, two periods are set aside when the hundreds' digit of the root is being found, and so on. Similarly for the periods at the right of the decimal point.

The application of the method of § 125 to extracting square roots of arithmetical numbers may be best understood in general by first considering some particular examples.

Let it be required, for instance, to find the square root of 1156.

Since this number consists of two *periods*, therefore its square root will consist of two integer places, *i.e.*, of tens and units.

Moreover, since $30^2 < 1156 < 40^2$, therefore the required root lies between 30 and 40, *i.e.*, the tens' digit is 3, the square root of the greatest square integer in the left-hand period of the given number.

The units' digit may now be found as follows: let k represent the part of the root already *known* (viz. 30), and let u represent the unknown part of the root;

then $$1156 = (k + u)^2 = k^2 + 2ku + u^2,$$

and, therefore, $$2ku + u^2 = 1156 - k^2 = 256. \qquad [k^2 = 900$$

Again, since k represents tens while u represents units, therefore $2ku$ is much greater than u^2; hence the last equation above shows that $2ku$ (though somewhat less than 256) is *approximately* equal to 256, and hence that $256 \div 2k$ (though somewhat too great) is approximately equal to u, *i.e.*, $256 \div 2k$ will *suggest* a value for u, which must then be tested by the above equation.*

---

* Since $256 = (2k + u)u$, therefore $256 \div (2k + u) = u$, *i.e.*, the *complete* divisor is $2k + u$, and $2k$ is merely a *trial* divisor; hence the appropriateness of these names. Since $256 \div 2k$ gives too great a quotient, therefore the *units' digit* in the required square root is either 4 or a smaller number; hence if the units' digit is not 4 (*i.e.*, if it is 3, 2, 1, or 0), then $(k + 4)^2 > 1156$, *i.e.*, $1156 - (k + 4)^2$ is negative, and the next smaller number must be tried. This shows that the *first* one of these numbers (4, 3, ⋯) which leaves a positive remainder in the above subtraction is the units' digit in $\sqrt{1156}$. Similarly in general.

Finally, since k is already known to be 30, therefore $256 \div 2k = 256 \div 60 = 4+$, hence u is *probably* equal to 4; substituting this value of u in the equation $256 = 2ku + u^2$, proves that $u = 4$, and hence that $\sqrt{1156} = 34$.

The work may be arranged as follows:

$$\begin{array}{lrll}
 & (k+u)^2 = k^2 + 2ku + u^2 = & 11'56 & \underline{30+4} = 34 \\
 & k^2 = (30)^2 = & \underline{900} & \\
\text{trial divisor is} & 2k = 60 & 256 = 2ku + u^2 & \\
\text{complete divisor is} & 2k + u = 64 & \underline{256} = (2k+u)\cdot u & \\
 & & 0 &
\end{array}$$

Again, let it be required to find the square root of 315844.

Since this number consists of three periods, therefore its square root will consist of three integer places. The work may be arranged as follows (the student should fully explain each step):

$$\begin{array}{lrll}
 & & 31'58'44 & \underline{500+60+2} = 562 \\
 & & \underline{250000} & \\
\text{1st trial divisor,} & 2\cdot 500 = 1000 & 65844 & \\
\text{1st complete divisor,} & 1000 + 60 = 1060 & \underline{63600} = 1060\cdot 60 & \\
\text{2d trial divisor,} & 2\cdot 560 = 1120 & 2244 & \\
\text{2d complete divisor,} & 1120 + 2 = 1122 & \underline{2244} = 1122\cdot 2 & \\
 & & 0 &
\end{array}$$

Note. When some familiarity with the above process has been gained, the work may be abridged by omitting unnecessary ciphers, and annexing to each remainder the two digits which compose the next period in the given number, thus:

$$\begin{array}{lll}
 & 31'58'44 & \underline{562} \\
 & \underline{25} & \\
\text{1st complete divisor, 106} & 658 & \\
 & \underline{636} & \\
\text{2d complete divisor, 1122} & 2244 & \\
 & \underline{2244} & \\
 & 0 &
\end{array}$$

Finally, let it be required to extract the square root of 10.5625.

The work may be arranged thus:

$$\begin{array}{lll}
 & 10.'56'25 & \underline{3.25} \\
 & \underline{9} & \\
\text{1st complete divisor, 62} & 156 & \\
 & \underline{124} & \\
\text{2d complete divisor, 645} & 3225 & \\
 & \underline{3225} & \\
 & 0 &
\end{array}$$

The results of the discussion of the present article may be stated thus:

(1) *Separate the given number into periods of two digits each, beginning at the decimal point and counting both toward the right and toward the left, completing the right-hand decimal period by annexing a cipher if necessary.*

(2) *By inspection find the greatest square integer in the left-hand period, and write its square root as the first digit of the required root.*

(3) *Subtract the square of the root digit already found from the left-hand period of the given number, and bring down the next period as part of the remainder.*

(4) *Divide this remainder, exclusive of its right-hand digit, by twice the root digit already found, i.e., by the trial divisor, and annex the quotient digit to the root and also to the trial divisor, thus forming the complete divisor.*

(5) *Multiply the complete divisor by the last digit in the root, subtract the product from the former remainder, and bring down the next period of the given number as part of this new remainder.*

(6) *Repeat* (4) *and* (5) *above until all the periods of the given number are exhausted.*

(7) *If a negative remainder presents itself in the above work, it indicates that the corresponding trial root digit is too great, and the one next lower must be tried.*

(8) *For a given number which is not a perfect square as many decimal figures as desired in the root may be found by annexing the necessary number of periods of ciphers to the number* (cf. § 125, note 3).

EXERCISES

Extract the square root of each of the following numbers:

1. 1296.

2. 841.

3. 7396.

4. 12.96.

5. 667489.

6. 1664.64.

7. 17424.

8. 101.0025.

9. How may the square root of a fraction be found? Why? What is the square root of $\frac{9}{64}$? Why?

10. Find the square root of $\frac{625}{729}$. Is $-\frac{25}{27}$ also a square root of this fraction? Why?

11. If a number contains 3 decimal places, how many decimal places does the square of this number contain? Why? Generalize this relation.

12. Extract the square root of 2 to three decimal places. How many decimal ciphers must be annexed to 2 for this purpose? Why?

Find the square root of each of the following numbers, correct to three decimal places:

13. 13.5. **14.** .017. **15.** $\frac{5}{8}$. **16.** $4\frac{2}{3}$.

17. Show by actual trial that, having found the square root of 35.8 correct to 3 decimal places, the next 2 decimal figures of the root may be found by simply dividing the remainder at that stage of the work by the corresponding trial divisor.

18. If the square root of a number is desired, correct to $2n+1$ figures, prove that when the first $n+1$ figures have been found in the usual way, the remaining n figures may be found by ordinary division (cf. Ex. 17).

Suggestion. Let N stand for any number whatever, k for the first $n+1$ figures of its square root (with n ciphers annexed), and r for the remaining n figures of the root.

Then $$N=(k+r)^2=k^2+2kr+r^2,$$

whence $$\frac{N-k^2}{2k}=r+\frac{r^2}{2k}, \quad \text{in which } \frac{r^2}{2k} \text{ is a proper fraction (why?);}$$

i.e., merely dividing $N-k^2$ (which is the remainder when the first $n+1$ figures have been found) by the trial divisor at that stage of the work (viz., $2k$) gives the next n figures of the root, together with a proper fraction.

19. Find $\sqrt{84256}$ to 5 figures, $\sqrt{3.642}$ to 3 figures, and $\sqrt{6048274}$ to 3 decimal places. How many root figures must be found by the usual process, in each of these cases, before the ordinary division may begin?

127. Cube root of polynomials. The general method for extracting the square root of a polynomial, which is given in § 125, may easily be extended so as to apply to cube root also — and indeed to the higher roots as well. The process is in all cases the inverse of that employed in raising a polynomial to a power. The several steps are indicated below.*

Since $$(k+u)^3=k^3+3k^2u+3ku^2+u^3 \qquad (1)$$

$$=k^3+(3k^2+3ku+u^2)u, \qquad (2)$$

therefore:

---

* To avoid needless repetition here the student is referred for fuller statement of reasons to the detailed explanation already given in § 125.

(1) *Arrange the terms of the given polynomial according to the descending powers of some one of its letters.*

(2) *The highest term of the required root is the cube root of the highest term of the given power; i.e., the highest term in the above root is* $\sqrt[3]{k^3}$, *viz.,* k.

(3) *If the cube of the part of the root already found be subtracted from the given polynomial, the remainder will be* $3k^2u+3ku^2+u^3$, *and the next term of the root may be found by dividing the first term of this remainder by three times the square of the first term of the root (which is already known); i.e., the second term of the root is* $3k^2u \div 3k^2$, *viz.,* u.

The *trial divisor* here is $3 \cdot k^2$, *i.e.*, it is three times the square of the root already known; and, from Eq. (2) above, it is clear that the *complete divisor* is $3k^2+3ku+u^2$, *i.e.*, it is the trial divisor, plus three times the product of the last term of the root by the preceding part of the root, plus the square of the last term of the root.

The work may be put in the following form:

$$\begin{array}{lll} & & k^3+3k^2u+3ku^2+u^3 \,\big|\, \underline{k+u} \\ & & \underline{k^3} \\ \text{Trial divisor,} & 3\cdot k^2 & \big|\, 3k^2u+3ku^2+u^3 \\ \text{Complete divisor,} & 3k^2+3ku+u^2 & \big|\, \underline{3k^2u+3ku^2+u^3}=(3k^2+3ku+u^2)\cdot u \\ & & \qquad\qquad 0 \end{array}$$

Observe that the two subtractions just performed are together equivalent to the subtraction of $(k+u)^3$ from the given polynomial.

(4) By proceeding as in § 125 (ii) it is easy to show that, *having found any number of terms of the required root, and having subtracted the cube of this part of the root from the given polynomial, the next root term may be found by dividing the first term of the remainder by the first term of the trial divisor,—the trial divisor being three times the square of the part of the root already found. By continuing this process all the terms of the required root may be found.*

The work of finding the cube root of $x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1$ may be arranged as follows:

$$\begin{array}{lr|l}
 & & x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1 \;\underline{|\, x^2 - 3x + 1} \\
 & (x^2)^3 = & x^6 \\ \cline{3-3}
\text{1st trial divisor,} & & \\
 & 3(x^2)^2 = 3x^4 & -9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1 \\
\text{1st complete divisor,} & & \\
 & 3x^4 - 9x^3 + 9x^2 & -9x^5 + 27x^4 - 27x^3 \\ \cline{3-3}
\text{2d trial divisor,} & & \\
 & 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2 & 3x^4 - 18x^3 + 30x^2 - 9x + 1 \\
\text{2d complete divisor,} & & \\
 & \left\{\begin{array}{r} 3x^4 - 18x^3 + 27x^2 \\ 3x^2 - 9x + 1 \\ \hline 3x^4 - 18x^3 + 30x^2 - 9x + 1 \end{array}\right. & \begin{array}{r} 3x^4 - 18x^3 + 30x^2 - 9x + 1 \\ \hline 0 \end{array}
\end{array}$$

The student may now solve this example by arranging the terms according to the ascending powers of x and compare his result with the above.

EXERCISES

Find the cube root of each of the following expressions, and verify the correctness of your results:

1. $8x^3 - 12x^2 + 6x - 1$.

2. $27x^3 - 189x^2y + 441xy^2 - 343y^3$.

3. $125n^3 - 150mn^2 - 8m^3 + 60m^2n$.

4. $675u^2v + 1215uv^2 + 125u^3 + 729v^3$.

5. $x^6 - 20x^3 - 6x + 15x^4 - 6x^5 + 15x^2 + 1$.

6. $3x^5 + 9x^4 + x^6 + 8 + 12x + 13x^3 + 18x^2$.

7. $342x^2 - 108x - 109x^3 + 216 + 171x^4 - 27x^5 + 27x^6$.

8. $156x^4 - 144x^5 - 99x^3 + 64x^6 + 39x^2 - 9x + 1$.

9. $54x + \frac{8}{x^3} - 112 - \frac{48}{x^2} + \frac{108}{x} + x^3 - 12x^2$.*

10. $20 + \frac{15}{c^2} + 15c^2 + c^6 + \frac{6}{c^4} + c^{-6} + 6c^4$.

11. $30y^{-1} + 8y^{-3} + 8y^3 + 30y - 12y^2 - 25 - 12y^{-2}$.

12. $6a^5x^4 - 4a^3x^6 - 2a^6x^3 + 6a^2x^7 + 3a^8x + a^9 + x^9 - 3ax^8$.

13. $108y^5z - 27y^6 - 90y^4z^2 + 8z^6 - 80y^3z^3 + 60y^2z^4 + 48yz^5$.

---

* Compare § 125, Ex. 15.

14. $\frac{a^3c^6}{b^3} - \frac{3\,a^3c^5}{b} + 3\left(ab^3 + \frac{b}{a}\right)c^2 - ab(a^2b^2 + 6)c^3 + 3\left(a^3b + \frac{a}{b}\right)c^4 + \frac{b^3}{a^3} - \frac{3\,b^3c}{a}.$

15. $x^6 + a^3b^3 - 3\,a^3b^3x - 3\,abx^5 + 3\,ab\,(1 + ab)\,x^4 + 3\,a^2b^2(1 + ab)\,x^2 - a^2b^2(6 + ab)\,x^3.$

16. $x^3y^{-3} + x^{-3}y^3 + 3\,xy^{-1}(y^{-2} - 1) + 3\,x^{-1}y(x^{-2} - 1) + 3\,x^{-1}y^{-1}(1 + x^{-2} + y^{-2}) - 3\,x^3y^{-1} - 3\,x^{-1}y^3 - x^3y^3 + x^{-3}y^{-3} + 3\,xy(x^2 + y^2 - 1).$

17. $64\,v^{3n} + 117\,v^{3n-3} + 12\,v^{3n-2} - 6\,v^{3n-4} - 36\,v^{3n-5} - 144\,v^{3n-1} - 8\,v^{3n-6}.$

18. Find the first 3 terms of $\sqrt[3]{1 + x}$.

19. Find the first 4 terms of $\sqrt[3]{1 - 3x + x^2}$.

128. Cube root of arithmetical numbers. To extract the cube root of an arithmetical number, proceed as follows:*

(1) *Separate the given number into periods of three digits each, beginning at the decimal point and counting both toward the right and toward the left, completing the right-hand decimal period by annexing one or two ciphers if necessary.*

(2) *By inspection (or by trial) find the greatest cube integer in the left-hand period, and write its cube root as the first digit of the required root.*

(3) *Subtract the cube of the root digit just found from the left-hand period of the given number, and bring down the next period as part of the remainder.*

(4) *To three times the square of the root digit already found annex two ciphers, thus forming the trial divisor; divide the above remainder by this trial divisor, and annex the first quotient digit to the root.*

(5) *To the trial divisor add three times the product of the last root digit multiplied by the part of the root previously found with a cipher annexed, and also the square of the last root digit, thus forming the complete divisor. Multiply the complete divisor by the last root digit, and subtract the product from the above remainder, bringing down the next period as part of the new remainder.*

---

* The reasoning here is similar to that given in § 126, and should be given by the student.

(6) *Repeat* (4) *and* (5) *above until all the periods of the given number are exhausted.*

Note. As in the case of square root (§ 126), so here, if a negative remainder presents itself in the course of the above work, it indicates that the corresponding trial root digit is too great, and the next lower digit must be tried.

As many decimal figures as desired in the root may be obtained by annexing the necessary number of periods of ciphers to a number which is not a perfect cube.

The work of finding the cube root of 9800344 may be arranged as follows:

		9'800'344	214	
		8		
1st trial divisor,	1200	1800		[1800 ÷ 1200 = 1+
1st correction,	60			
2d correction,	1			
1st complete divisor,	1261	1261	= 1261 · 1	
2d trial divisor,	132300	539344		[539344 ÷ 132300 = 4 +
1st correction,	2520			
2d correction,	16			
2d complete divisor,	134836	539344	= 134836 · 4	
		0		

Verification of the correctness of the above root: $(214)^3 = 9800344$.

Again, let it be required to find the cube root of 43614208.

		43'614'208	36	
		27		
Trial divisor,	2700	16614		[16614 ÷ 2700 = 6+
1st correction,	540			
2d correction,	36			
Complete divisor,	3276	19656		

Since the remainder would be negative, therefore the trial digit 6 is too great, and 5 must be tried.

		43'614'208	352	
		27		
1st trial divisor,	2700	16614		
1st correction,	450			
2d correction,	25			
1st complete divisor,	3175	15875	= 3175 · 5	
2d trial divisor,	367500	739208		[739208 ÷ 367500 = 2+
1st correction,	2100			
2d correction,	4	739208	= 369604 · 2	
2d complete divisor,	369604	0		

Verification of the correctness of this root: $(352)^3 = 43614208$.

EXERCISES

Extract the cube root of each of the following numbers:

1. 1728.
2. 571787.
3. 31855.013.
4. 148877.
5. 39304.
6. 426.957777.
7. 305.909539272.
8. 34.7, to 2 decimal places.
9. .04, to 3 decimal places.
10. $3\frac{7}{8}$, to 2 decimal places.

11. If the cube root of a number consists of $2n+2$ figures, show that when $n+2$ of these figures have been obtained by the ordinary method, the remaining n figures may then be found by simple division (cf. Ex. 18, § 126).

12. By the method of Ex. 11, find $\sqrt[3]{.0783259}$ correct to 6 decimal figures.

129. Higher roots of polynomials and of numbers. The methods for extracting the square and cube roots of polynomials which are given in §§ 125 and 127, respectively, may be easily extended so as to apply to the higher roots.

E.g., the identity $(k+u)^4 = k^4 + 4k^3u + 6k^2u^2 + 4ku^3 + u^4$ shows that the first term of the fourth root is the *fourth root of the first term of the power, i.e.*, of the given polynomial; again, if k and u represent respectively the known and unknown parts of the root at any stage of the work, and if k^4 be subtracted from the power, the remainder may be written thus: $(4k^3+6k^2u+4ku^2+u^3)u$, which shows that the *trial divisor* is $4k^3$, and that there are three corrections, viz., $6k^2u$, $4ku^2$, and u^3, which must be added to the trial divisor to give the *complete divisor*. From here on the work proceeds as in the case of cube root.

Similarly, in extracting the fifth root the trial divisor is $5k^4$, and there are *four* corrections to be added to the trial divisor to form the complete divisor; in the nth root (where n is any positive integer) the trial divisor is nk^{n-1}, and there are $n-1$ corrections.

The method of extracting any root of a polynomial is easily adapted to the extraction of the corresponding root of an arithmetical number, as has already been illustrated in §§ 126 and 128.

NOTE. If a number be separated into two equal factors, and each of these two factors be further separated into three equal factors, the given number will then really have been separated into 6 (*i.e.*, $3 \cdot 2$) equal factors; from this it follows that if N represents a number which can be separated into 6 equal factors, then $\sqrt[6]{N} = \sqrt[3]{\sqrt{N}}$.

Similarly, in general, if N represents a number which can be separated into $p \cdot q$ equal factors, then $\sqrt[pq]{N} = \sqrt[p]{\sqrt[q]{N}} = \sqrt[q]{\sqrt[p]{N}}$. This fact simplifies the extraction of the higher roots whenever the index of the required root is a composite number (cf. also § 136).

EXERCISES

Find the indicated roots of the following expressions — both directly, and also by the method given in the preceding note:

1. $\sqrt[4]{x^4 - 8x^3 + 24x^2 - 32x + 16}$.

2. $\sqrt[4]{81y^4 + 54x^2y^2 + x^4 + 12x^3y + 108xy^3}$.

3. $\sqrt[6]{x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64}$.

4. $\sqrt[6]{15a^4c^4x^2 + a^6c^6 + 6a^5c^5x + 20a^3c^3x^3 + 15a^2c^2x^4 + x^6 + 6acx^5}$.

5. Find the fifth root of $32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$.

Find the following indicated roots:

6. $\sqrt[5]{u^{10} + 243v^{10} + 15u^8v^2 + 405u^2v^8 + 90u^6v^4 + 270u^4v^6}$.

7. $\sqrt[4]{50625}$. **8.** $\sqrt[6]{531441}$. **9.** $\sqrt[8]{5764801}$. **10.** $\sqrt[4]{1874161}$.

CHAPTER XIV

IRRATIONAL AND IMAGINARY NUMBERS — FRACTIONAL EXPONENTS

I. IRRATIONAL NUMBERS

130. Preliminary considerations and definitions. While such roots as $\sqrt{4}$, $\sqrt[3]{-\frac{8}{27}}$, $\sqrt[5]{32\,a^5x^{10}}$, etc., can be exactly expressed by means of integers and fractions, many others which frequently present themselves in algebraic investigations can not be so represented; *e.g.*, $\sqrt{2}$ and $\sqrt{-5}$.

These new numbers, and their laws of combination, will now be examined, and they will henceforth be included in the number system, which heretofore has comprised only positive and negative integers and fractions.

NOTE 1. That $\sqrt{2}$ is neither an integer nor a fraction may be shown as follows: By the definition of a root (§ 122), $\sqrt{2}$ means a number whose square is 2, and since $(+1)^2 < 2$ and $(+2)^2 > 2$, therefore the number whose square is 2 must, in absolute value, lie between 1 and 2, and therefore can not be an integer. Moreover, $\sqrt{2}$ can not be a fraction such as $\frac{m}{n}$ because if it were, then $\frac{m^2}{n^2}$ would equal 2, but if $\frac{m}{n}$ is a fraction, it may be supposed to be in its lowest terms, and then $\frac{m^2}{n^2}$ is also a fraction in its lowest terms and can not be equal to the integer 2. It is then proved that $\sqrt{2}$ is neither an integer nor a fraction.

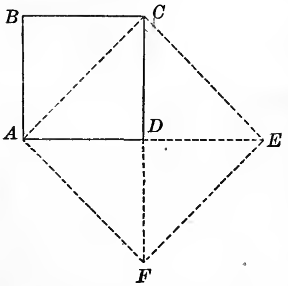
NOTE 2. Although, as has just been shown, such numbers as $\sqrt{2}$ can not be *exactly* represented by integers or by fractions, yet they can be *approximately* represented, and to any required degree of accuracy, by means of these numbers.

E.g., squaring 1, 2, 3, ⋯ in turn shows that $1 < \sqrt{2} < 2$, then squaring 1.1, 1.2, 1.3, ⋯ in turn shows that $1.4 < \sqrt{2} < 1.5$, then squaring 1.41, 1.42, 1.43, ⋯ in turn shows that $1.41 < \sqrt{2} < 1.42$, etc.

Thus it is shown that $1 < \sqrt{2} < 2$, $1.4 < \sqrt{2} < 1.5$, $1.41 < \sqrt{2} < 1.42$, $1.414 < \sqrt{2} < 1.415$, etc.; and since a number which lies between two other numbers differs from either of them by less than they differ from each other, therefore $\sqrt{2}$ differs from 1 or 2 by less than 1, from 1.4 or 1.5 by less than 0.1, from 1.41 or 1.42 by less than 0.01, etc. If, then, the numbers 1, 1.4, 1.41, 1.414, ⋯ be taken as successive approximations to the value of $\sqrt{2}$, the errors will be less than 1, 0.1, 0.01, 0.001, ⋯ respectively; hence it is clear that, by continuing the above process, a number can be found which can be expressed by means of integers, and which will represent $\sqrt{2}$ to any required degree of accuracy.

Furthermore, it is evident from the nature of the argument just given that it applies equally well to *any* indicated root of a *positive* number, and also to *odd* roots of negative numbers.

Note 3. Although such numbers as $\sqrt{2}$ can not be exactly expressed by means of integers and fractions, they are just as *definite* and *precise* as are integers and fractions, and they are also necessary in human affairs.



E.g., let the figure $ABCD$ be a square whose side AB is 1 foot long, and let the figure $ACEF$ be another square whose side AC is the diagonal of the first square; then it is easily proved by geometry that the area of the square $ACEF$ is 2 times that of $ABCD$, and hence, if x is the number of feet in AC, then $x^2=2$; *i.e.*, if the length of the side of a square is 1 foot, then the length of the diagonal of that square is precisely $\sqrt{2}$ feet.

This illustration shows also that such numbers are *necessary* in human affairs, *e.g.*, $\sqrt{2}$ is the *only* number which exactly expresses the length of the diagonal of a unit square, — the numbers 1, 1.4, 1.41, 1.414, 1.4142, ⋯ are successive *approximations* to the length of this diagonal, but its *exact* length is a number whose square is exactly 2, and which is represented by the symbol $\sqrt{2}$.

Note 4. That the other root indicated above, viz., $\sqrt{-5}$, can not be expressed, even approximately, by means of integers and fractions follows directly from the law of signs in multiplication; if it could be so expressed it must be either a positive or a negative number, and its square would then be a positive number and not -5. The same argument applies to every indicated even root of a negative number.

Numbers that involve indicated roots which can not be *exactly* expressed by means of integers and fractions, but which may be expressed to any required degree of accuracy by means of these numbers, are called **irrational numbers**, while integers and fractions are classed together as **rational numbers**.

E.g., $\sqrt{2}$, $4-\sqrt[3]{7}$, and $\sqrt{2}+\sqrt[3]{5}$ are irrational numbers.

Numbers which involve indicated *even* roots of *negative* numbers are called **imaginary numbers**,* and all other numbers are, for distinction, called **real numbers**.

E.g., $\sqrt{-3}$, $2+\sqrt[4]{-5}$, and $3\sqrt{-2}$ are imaginary numbers.

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* The name "imaginary" is rather an unhappy one because these numbers are just as *real*, under their proper interpretation, as any other numbers.

For present purposes it seems best to define irrational and imaginary numbers as above, and thus to separate them; the name "irrational" is, however, often employed to include the imaginary numbers also.

For a broader definition of imaginary numbers see Appendix B.

Although the *language* employed in defining a root of a number in § 122 is general, and includes the irrational and imaginary roots as well as the rational roots, yet the student's *conception* of a root has doubtless heretofore been limited to those roots which happened to be rational; it is therefore worth while especially to emphasize here that the symbol $\sqrt[n]{a}$ stands for a number whose nth power is a,

$$(\sqrt[n]{a})^n = a,^*$$

where a is any number whatever, and the only limitation upon the symbol is that n must be a positive integer.

NOTE 5. Having now further enlarged the number concept, it may be worth while to recapitulate briefly what has already been said upon this subject in the preceding pages.

The first numbers which man invented to express the relations of the things about him were the positive integers; with these he found it necessary to perform certain fundamental operations (addition, subtraction, etc.), and later he found it necessary to enlarge his idea of number so as to make these operations always possible (cf. § 12, note). Thus fractions arose from generalizing the operation of division (cf. § 11); negative numbers arose from generalizing the operation of subtraction (cf. §§ 12–14); and in the present article it appears that generalizing the operation of extracting roots introduces two further new kinds of numbers, viz., the irrational and the imaginary.

In other words: while the *direct* operations (viz., addition, multiplication, and involution) with positive integers always produce results that are positive integers, the *inverse* operations (viz., subtraction, division, and evolution) lead respectively to negative, fractional, and irrational and imaginary numbers, and demand for their accommodation that the primitive idea of number be so enlarged as to include these new kinds of numbers along with the positive integers.

EXERCISES

1. What is an irrational number? Show that $\sqrt{-5}$ is not an irrational number. To what class of numbers does $\sqrt{-5}$ belong?

2. Is $\sqrt[3]{8}$ an irrational number? Why? Show that $\sqrt{5}$ is neither an integer nor a fraction. To what class of numbers does $\sqrt{5}$ belong? Why?

3. Find three successive approximations to the value of $\sqrt{5}$ (cf. note 2 above). Compare these approximations with the result of extracting the square root of 5 by the method of § 126.

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* It may be remarked that, under this definition, $\sqrt[1]{a}$ means the same as a [cf. § 7 (iv) note].

4. Find two approximate values of $\sqrt{3}$, one larger and the other smaller than the true value, which differ from $\sqrt{3}$ by less than .001.

5. A fruit grower has 16 plum trees and wishes to plant them in rows in a rectangular plot of ground, and to have the number of trees in each row exceed the number of rows by 2. How many trees shall he plant in a row?

Suggestion. If x represents the number of trees to be planted in a row, show that $x^2 - 2x = 16$. From this equation it follows that $(x-1)^2 - 17 = 0$, *i.e.*, that $(x - 1 + \sqrt{17})(x - 1 - \sqrt{17}) = 0$; whence $x = 1 + \sqrt{17}$, or $x = 1 - \sqrt{17}$.

Does the fact that one can not plant $1 + \sqrt{17}$ trees in a row show that there is no such number as $1 + \sqrt{17}$? Or does it merely show that the present problem demands what is impossible?

6. Show how to construct a line which shall be exactly $1 + \sqrt{17}$ times as long as a given line.

7. Can $\sqrt{-8}$ be expressed by means of an integer or a fraction? Is it then an irrational number? Why not? What kind of number is it?

8. Is the number $21 + \sqrt{17}$ rational or irrational? Why? What kind of number is $84\sqrt{5} - \sqrt[4]{-8}$? Why?

131. Further definitions. An indicated root of a number is usually called a **radical**; if this root is irrational, but the radicand rational, the expression is also called a **surd**.

E.g., $\sqrt{2}$, $\sqrt[3]{8}$, $\sqrt[7]{5+\sqrt{10}}$, and $6\sqrt[3]{45}$ are radicals; and of these $\sqrt{2}$ and $6\sqrt[3]{45}$ alone are called surds.

The **coefficient** of a radical is the factor which multiplies it, and the **order** of the radical is determined by the root index. Two radicals which have the same root index are said to be of the **same order.**

E.g., the surds $12\sqrt[7]{5ax^2}$ and $m^2\sqrt[7]{674}$ are of the same order, viz., the 7th, and their coefficients are 12 and m^2, respectively.

Surds of the second and third orders are usually called *quadratic* and *cubic* surds, respectively.

If two or more radicals are of the same order, and have their radicands (cf. § 122) exactly alike — or if they can be reduced to such — they are called **similar radicals** and also **like radicals**; otherwise they are **dissimilar (unlike).**

Expressions which involve radicals, in any way whatever, are called **radical expressions**; they are **monomial, binomial,** etc. (cf. § 27), depending upon the number of their terms.

E.g., $\sqrt{5}$ and $3\sqrt{5}$ are similar quadratic surds, while $6\sqrt[3]{a^2+2bx+y}$* and $(m+2n)\sqrt[3]{a^2+2bx+y}$ are similar cubic surds. The four examples just given are monomial surds, while $5a+3\sqrt{7}$ and $2\sqrt[5]{9}+3\sqrt{x}$ are binomial surds.

132. Principal roots. It has already appeared that a number has *two* square roots (*e.g.*, $\sqrt{9}$ is $+3$ or -3), and it will be seen later that every number has *three* cube roots, *four* fourth roots, *five* fifth roots, etc.

E.g., $\sqrt[3]{8}=2$, $-1+\sqrt{-3}$, or $-1-\sqrt{-3}$, since the cube of each of these numbers is 8 (cf. Ex. 23, § 170); and $\sqrt[4]{16}=2$, -2, $2\sqrt{-1}$, or $-2\sqrt{-1}$.

Although, as has just been said, a number has 3 cube roots, 4 fourth roots, etc., some of these roots are *imaginary*, and when there are two *real* roots, they are equal in absolute value and of opposite sign.†

By the **principal root** of a number is meant its *real root*, if there is but *one* real root, and its *real positive root* if there are *two* real roots.

E.g., if attention is confined to principal roots, $\sqrt{9}=3$ (and not -3), $\sqrt[3]{-8}=-2$, $\sqrt[3]{125}=5$, $\sqrt[4]{16}=2$, etc.

That irrational and imaginary numbers obey the fundamental combinatory laws (commutative, associative, etc.) which have already been established in the case of rational numbers is proved in the appendix; logically this proof for irrational numbers should now be read, but it may be deferred until later if the reader will carefully bear in mind that the following discussion *assumes* that irrational numbers are subject to these laws, and that the results are therefore to be regarded as *tentative* until this fact is *proved*.

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* Such expressions are said to be surd in *form* even though values may be assigned to the letters involved which make them rational in *value*.

† It should be especially observed that a number can not have two real roots of *unequal* absolute value. For suppose $\sqrt[n]{a}=r_1$ and also r_2, where r_1 and r_2 are real, and $r_1>r_2$ in absolute value; from this it follows that $r_1^n>r_2^n$ in absolute value, and therefore, if $r_1^n=a$, then $r_2^n\neq a$, *i.e.*, $\sqrt[n]{a}\neq r_2$.

EXERCISES

1. What is a radical expression? A surd? Give examples to illustrate your answer. Are all radicals surds? Are all surds radicals?

2. What is the coefficient of a surd? Give an example. May this coefficient be a negative number? May it be a fraction? Are there any restrictions upon it?

3. What is meant by the order of a surd? Illustrate by examples. May the order of a surd as now defined be negative or fractional?

4. Define similar surds, and illustrate your definition by several examples. May the coefficients differ and the surds still be similar?

5. What factor have any two similar surds necessarily in common? What kind of number, then, is the quotient of two similar surds? Illustrate your answer.

6. What is an imaginary number? Give several illustrations. For what values of n is $\sqrt[n]{-5}$ an imaginary number? Give a reason for calling these numbers "imaginary."

7. Illustrate by examples: monomial and trinomial surds; quadratic and cubic surds; and the order of a surd.

8. How many values has $\sqrt{16}$? What are they? What is the principal square root of 16? What is the principal fifth root of -32? Define the principal root of a number.

9. Show that $\sqrt[3]{343}$ is 7. Under what conditions is $\sqrt[n]{K}$ equal to p? How, in general, is the correctness of a root tested?

10. Show that under the definition given in § 132 no number can have more than *one* principal root of any specified order.

133. Product of two or more radicals of the same order.*

Just as $\sqrt{9}\cdot\sqrt{25}=\sqrt{225}$, *i.e.*, $\sqrt{9\cdot 25}$,

and $\sqrt[3]{-8}\cdot\sqrt[3]{27}=\sqrt[3]{-216}$;

[Each member of the first of these equations being 15, and of the second, -6.]

so, too, if x and y are any numbers whatever (cf. footnote, p. 229), and n is any positive integer,

$$\sqrt[n]{x}\cdot\sqrt[n]{y}=\sqrt[n]{xy}.$$

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* In §§ 133–145 imaginary numbers are excluded, and the proofs are further limited to "principal roots."

For, since $(\sqrt[n]{x}\sqrt[n]{y})^n = (\sqrt[n]{x}\sqrt[n]{y}) \cdot (\sqrt[n]{x}\sqrt[n]{y}) \cdots$ to n factors

$$= (\sqrt[n]{x})^n \cdot (\sqrt[n]{y})^n \qquad [\S\S\ 52 \text{ and } 53$$

$$= xy;$$

i.e., since the nth power of $\sqrt[n]{x} \cdot \sqrt[n]{y}$ is xy, therefore (§ 130)

$$\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy}. \tag{1}$$

Similarly, it is easily shown that

$$\sqrt[n]{x} \cdot \sqrt[n]{y} \cdot \sqrt[n]{z} \cdots = \sqrt[n]{xyz \cdots}, \tag{2}$$

which may be formulated in words thus: *the product of the nth roots of two or more numbers* is the nth root of the product of those numbers.*

EXERCISES

Express each of the following indicated products as a single radical:

1. $\sqrt{5} \cdot \sqrt{7}$.

2. $\sqrt{3} \cdot \sqrt{7} \cdot \sqrt{2}$.

3. $\sqrt[5]{2} \cdot \sqrt[5]{6} \cdot \sqrt[5]{5} \cdot \sqrt[5]{10}$.

4. $\sqrt{3\,a} \cdot \sqrt{10\,bx}$.

5. $\sqrt[3]{4\,ax^2} \cdot \sqrt[3]{5\,ab^2y} \cdot \sqrt[3]{21\,c^2y}$.

6. $\sqrt{x+y} \cdot \sqrt{x-y}$.

7. Verify that $\sqrt{x+y} \cdot \sqrt{x-y} = \sqrt{x^2-y^2}$ when $x = 5$ and $y = 4$.

8. Is the equation in Ex. 7 true for *all* values of x and y, or only for certain particular values, such as $x = 5$ and $y = 4$? Why?

9. Is $\sqrt{a} \cdot \sqrt{b}$ equal to $\sqrt{ab}$? Why? If $\sqrt{a} \cdot \sqrt[3]{b}$ were also equal to $\sqrt{ab}$, how would $\sqrt[3]{b}$ and $\sqrt{b}$ compare?

10. Is $\sqrt{b}$ equal to $\sqrt[3]{b}$ when $b \neq 1$? Is then $\sqrt{a} \cdot \sqrt[3]{b}$ equal to $\sqrt{ab}$ or to $\sqrt[3]{ab}$ for all values of a and b?

11. When may the product of two or more radicals be expressed as a single radical?

134. Special cases of § 133. If $x = y$, then Eq. (1) of § 133,

viz., $$\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy},$$

becomes $$\sqrt[n]{x} \cdot \sqrt[n]{x} = \sqrt[n]{xx},$$

i.e., $$(\sqrt[n]{x})^2 = \sqrt[n]{x^2}.$$

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* If n is *even*, these numbers must be positive, since imaginary numbers are excluded from the present discussion.

Similarly, if $x = y = z = \cdots$, then Eq. (2) of § 133,

viz., $$\sqrt[n]{x} \cdot \sqrt[n]{y} \cdot \sqrt[n]{z} \cdots = \sqrt[n]{xyz \cdots},$$

becomes $$(\sqrt[n]{x})^p = \sqrt[n]{x^p}, \qquad (1)$$

where p is any positive integer, *i.e.*, *the pth power of the nth root of a number is equal to the nth root of the pth power of that number.*

Again, if either x or y is itself the nth power of some number, say $x = a^n$, then Eq. (1) of § 133,

viz., $$\sqrt[n]{x} \cdot \sqrt[n]{y} = \sqrt[n]{xy},$$

becomes $$\sqrt[n]{a^n} \cdot \sqrt[n]{y} = \sqrt[n]{a^n y},$$

i.e., $$a\sqrt[n]{y} = \sqrt[n]{a^n y}; \qquad (2)$$

hence, *a coefficient of a radical may be inserted (as a factor) under the radical sign by first raising it to a power corresponding in degree to the index of the root;* and (reading Eq. (2) from right to left) *a factor of the radicand, which is an exact power corresponding in degree with the indicated root, may be placed outside of the radical sign (as a coefficient) by merely extracting the indicated root.*

EXERCISES

1. What is the value of $(\sqrt{4})^3$? Of $\sqrt{4^3}$? How, then, does $(\sqrt{4})^3$ compare with $\sqrt{4^3}$? Does this agree with Eq. (1) above?

2. Is $(\sqrt[3]{7})^5$ equal to $\sqrt[3]{7^5}$? Why?

3. What is the value of $5\sqrt[3]{8}$? Of $\sqrt[3]{5^3 \cdot 8}$, *i.e.*, of $\sqrt[3]{1000}$? How, then, does $5\sqrt[3]{8}$ compare with $\sqrt[3]{5^3 \cdot 8}$? Does this agree with Eq. (2) above?

4. Is $3\sqrt{5}$ equal to $\sqrt{3^2 \cdot 5}$? Why?

5. Using the *method* by which Eq. (2) above was established, *prove* the correctness of your answer in Ex. 4.

In the following expressions insert the coefficients under the radical signs, and explain your work in each case:

6. $3\sqrt{5}$.

7. $2\sqrt{10}$.

8. $2\sqrt[4]{7}$.

9. $5\sqrt[3]{4}$.

10. $\frac{2}{3}\sqrt{6}$.

11. $\frac{2}{5}\sqrt{\frac{35}{46}}$.

12. $\frac{5}{8}\sqrt{3\,ax}$

13. $\frac{3\,a}{4}\sqrt[3]{12\,a^2x}$.

14. $\frac{4}{5}\sqrt{2\frac{5}{8}}$.

15. $\frac{x+1}{x-1}\sqrt{1-\frac{3}{x+1}}$.

16. $\frac{1}{ax}\sqrt[3]{a^2x(x-\frac{1}{2})}$.

17. State in words how a coefficient of a radical may be inserted under the radical sign.

Write each of the following radicals in a form having the radicand as small as possible:

18. $\sqrt{45}$.

SUGGESTION. $\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{3^2 \cdot 5}$, — compare Eq. (2) above.

19. $\sqrt{180}$.

20. $\sqrt{162}$.

21. $\sqrt[3]{320}$.

22. $\sqrt[3]{-54}$.

23. $\sqrt[3]{-192}$.

24. $\sqrt{392\,a^3x^5}$.

25. $\sqrt[5]{160\,a^7b^3x^{12}}$.

26. $\sqrt[5]{-486\,m^2v^8}$.

27. $\sqrt{12\,a^3(x+y)^5}$.

28. $\sqrt[3]{16\,a^2x^4 - 24\,bx^5}$.

29. $\sqrt{18\,a - 9}$.

30. $\sqrt{3\,x^2 - 6\,xy + 3\,y^2}$.

31. $12\sqrt[3]{-8\,m^4 + 24\,m^3n}$.

32. Is $\sqrt{x^2y^2}$ equal to xy? Why?

33. Is $\sqrt{x^2+y^2}$ equal to $x+y$? Why?

34. Verify your answer to Ex. 33 when $x = 3$ and $y = 4$.

35. Is the extraction of roots distributive over a sum? Over a product? Compare Exs. 32 and 33.

135. Quotient of two radicals of the same order.

Just as
$$\frac{\sqrt[3]{8}}{\sqrt[3]{27}} = \sqrt[3]{\frac{8}{27}}, \qquad \text{[Each being } \tfrac{2}{3}$$

so, too, if x and y are any numbers whatever (cf. footnote, p. 229), and n is any positive integer,

$$\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}.$$

To prove this it is only necessary to remember that

$$\left(\frac{\sqrt[n]{x}}{\sqrt[n]{y}}\right)^n = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \cdot \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \cdot \frac{\sqrt[n]{x}}{\sqrt[n]{y}} \cdots \text{ to } n \text{ factors}$$

$$= \frac{\sqrt[n]{x} \cdot \sqrt[n]{x} \cdot \sqrt[n]{x} \cdots \text{ to } n \text{ factors}}{\sqrt[n]{y} \cdot \sqrt[n]{y} \cdot \sqrt[n]{y} \cdots \text{ to } n \text{ factors}} \qquad [\S\ 54\ \text{(ii)}$$

$$= \frac{(\sqrt[n]{x})^n}{(\sqrt[n]{y})^n} = \frac{x}{y};$$

i.e., the nth power of $\frac{\sqrt[n]{x}}{\sqrt[n]{y}}$ is $\frac{x}{y}$, and therefore, by the definition of a root (§ 130), $\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$, — which was to be proved.

The student may state in words what has just been proved (cf. § 133).

EXERCISES

Express each of the following quotients by means of a single radical:

1. $\sqrt{35} \div \sqrt{5}$.

2. $\sqrt{216} \div \sqrt{12}$.

3. $\sqrt[3]{216} \div \sqrt[3]{12}$.

4. $\sqrt[3]{16\,a^2x^4} \div \sqrt[3]{2\,ax^2}$.

5. $\sqrt{x^2 - y^2} \div \sqrt{x + y}$.

6. $\sqrt[3]{16\,a^4b^5 - 32\,a^2x^2} \div \sqrt[3]{4\,a^2}$.

7. Verify that $\sqrt{a^2 - b^2} \div \sqrt{a - b} = \sqrt{a + b}$ when $a = 5$ and $b = 3$.

8. Is the equation in Ex. 7 true for *all* values of a and b, or only for certain particular values of these letters? Why?

9. Is $\sqrt[3]{24\,a^2x} \div \sqrt{6\,ax}$ equal to $\sqrt[3]{\frac{24\,a^2x}{6\,ax}}$? Why? Compare also Ex. 9, § 133.

10. If two radicals are of *different* orders, can their quotient be expressed as a radical of the same order as either one?

11. $\sqrt[3]{\frac{16\,a^2}{9\,x}} \div \sqrt[3]{4\frac{x}{3\,a}} = ?$

12. $\sqrt[5]{\frac{6\,a^2x^3y^2}{35\,b^3}} \div \sqrt[5]{\frac{2\,a}{7\,b^2x^2y^3}} = ?$

136. Radicals whose indices are composite numbers.

Just as $\sqrt[6]{64} = \sqrt[3]{\sqrt{64}} = \sqrt{\sqrt[3]{64}}$, [Each being 2

so, too, if x is any number whatever (cf. footnote, p. 229), and n and p are positive integers,

$$\sqrt[np]{x} = \sqrt[n]{\sqrt[p]{x}} = \sqrt[p]{\sqrt[n]{x}}.$$

This principle may be proved as follows (cf. §§ 133 and 135):

$$\begin{aligned}
\left(\sqrt[p]{\sqrt[n]{x}}\right)^{np} &= \sqrt[p]{\sqrt[n]{x}} \cdot \sqrt[p]{\sqrt[n]{x}} \cdot \sqrt[p]{\sqrt[n]{x}} \cdots \text{to } np \text{ factors} \\
&= \left\{ \sqrt[p]{\sqrt[n]{x}} \cdot \sqrt[p]{\sqrt[n]{x}} \cdot \sqrt[p]{\sqrt[n]{x}} \cdots \text{to } p \text{ factors} \right\}^{n} \\
&= \left\{ \left(\sqrt[p]{\sqrt[n]{x}}\right)^{p} \right\}^{n}, \\
&= \left\{ \sqrt[n]{x} \right\}^{n} \qquad \left[\text{Since } \left(\sqrt[p]{\sqrt[n]{x}}\right)^{p} = \sqrt[n]{x}\right. \\
&= x;
\end{aligned}$$

i.e., the npth power of $\sqrt[p]{\sqrt[n]{x}}$ is x, and therefore $\sqrt[p]{\sqrt[n]{x}} = \sqrt[np]{x}$. In the same way it may be shown that $\sqrt[n]{\sqrt[p]{x}} = \sqrt[np]{x}$.

This principle is useful in extracting roots whose indices are composite numbers (cf. § 129, note).

EXERCISES

1. What is meant by the symbol $\sqrt[k]{N}$ (cf. §§ 122 and 130)? Point out two places in the above proof where this definition is employed.

2. Using § 136, show that $\sqrt[6]{125} = \sqrt{5}$, and $\sqrt[10]{36} = \sqrt[5]{6}$; also state verbally the general principle which is involved in these equations.

Reduce each of the following radicals to an equivalent radical of lower order:

3. $\sqrt[4]{36}$.

4. $\sqrt[6]{81}$.

5. $\sqrt[6]{343}$.

6. $\sqrt[6]{27x^3}$.

7. $\sqrt[10]{32a^5b^{10}x^5}$.

8. $\sqrt[6]{a^4b^2x^8y^2}$.

9. $\sqrt[4]{121a^6y^4}$.

10. $\sqrt[4]{a^2 - 2ax + x^2}$.

137. Changing the order of a radical. It follows directly from the principle established in § 136 that

$$\sqrt[n]{a^t}=\sqrt[np]{a^{tp}},$$

wherein a is any number whatever (cf. footnote, p. 229), and n, p, and t are positive integers;

for, $\sqrt[np]{a^{tp}}=\sqrt[np]{(a^t)^p}$ [§ 121 (ii)

$=\sqrt[n]{\sqrt[p]{(a^t)^p}}$ [§ 136

$=\sqrt[n]{a^t}$, [Since $\sqrt[s]{N^s}=N$

that is, $\sqrt[n]{a^t}=\sqrt[np]{a^{tp}}$;

hence, *multiplying both the index of the radical and the exponent of the radicand by any positive integer, or dividing them both by any positive integral factor which they may contain, leaves the value of the expression unchanged.**

EXERCISES

1. Is $\sqrt[6]{a^2}$ equal to $\sqrt[3]{a}$? Why? Employ § 136 to prove the correctness of your answer. Show also that it follows from § 137.

2. Is $\sqrt[5]{3a^2x}$ equal to $\sqrt[10]{9a^4x^2}$? Why? Show that the correctness of this equation follows from § 136; also from § 137.

3. Reduce $\sqrt[4]{a^2n^6}$ to an equivalent radical of the second order, and explain. Also $\sqrt[6]{x^3y^{15}}$.

4. Change $\sqrt[3]{2x^4}$ to an equivalent radical of the 12th order, and explain. Also $\sqrt[4]{a^2y^3}$.

5. Reduce $\sqrt[8]{25m^4z^6}$ and $\sqrt[12]{8a^9b^6x^3}$, respectively, to equivalent radicals of the fourth order, and explain.

6. Reduce $\sqrt{3ax}$, $\sqrt[3]{2m^2n}$, and $\sqrt[4]{a^2n^8x^2}$, respectively, to equivalent radicals of the 12th order; of the 24th order.

7. Write out a carefully worded rule (from the principle of § 137) for reducing given radicals to equivalent radicals of higher or lower orders, and state the necessary limitations.

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* This property at once suggests that the exponent of the radicand and the index of the root bear to each other a relation similar to that of the numerator and denominator of a fraction; this relation will be more fully considered in § 153.

138. Reduction of radicals to the same order. Comparison of radicals. From § 137 it follows that any two or more radicals (real numbers) may be reduced to radicals which are of the same order, and which are, respectively, equivalent to the given radicals.

E.g., $\sqrt[3]{5}$ and $\sqrt[4]{7}$ are respectively equivalent to $\sqrt[3\cdot4]{5^4}$ and $\sqrt[4\cdot3]{7^3}$, *i.e.*, to $\sqrt[12]{625}$ and $\sqrt[12]{343}$.

The student may, by this method, reduce $\sqrt[m]{A}$ and $\sqrt[n]{B}$ to equivalent radicals of the same order; he may then formulate the procedure into a rule.

Reducing any two given radicals (real numbers) to the same order furnishes a means for *comparing* the values of these radicals; thus, in the above illustration, $\sqrt[3]{5} > \sqrt[4]{7}$ because their respective equivalents, viz., $\sqrt[12]{625}$ and $\sqrt[12]{343}$, stand in this relation.

EXERCISES

Reduce the following to equivalent radicals of the same order, and thus compare their values:

1. $\sqrt{5}$ and $\sqrt[3]{11}$.

2. $\sqrt[4]{7}$ and $\sqrt{3}$.

3. $\sqrt[6]{10}$, $\sqrt{2}$, and $\sqrt[3]{5}$.

4. $\sqrt[5]{4}$, $\sqrt[3]{2}$, and $\sqrt[6]{5}$.

Reduce the following to equivalent radicals of the same order:

5. $\sqrt{3ab}$, $\sqrt[3]{2ax^2}$, and $\sqrt[6]{5a^2b^3x^2}$.

6. $\sqrt[4]{2x^3}$, $\sqrt{ax}$, and $\sqrt[3]{2m^2z}$.

7. $\sqrt[3]{x}$, $\sqrt{xy}$, and $\sqrt[n]{x^2y^2}$.

8. $\sqrt{a+b}$, $\sqrt[4]{a^2+b^2}$, and $\sqrt{a-b}$.

9. Can the radicals in Ex. 5 be reduced to equivalent radicals of the 6th order? Of the 12th order? Of the 9th order? Give the reasons for your answer in each case.

10. What is the lowest *common* order to which the radicals in Ex. 6 can be reduced? Those in Ex. 7? Those in Ex. 8?

11. Compare the rule, asked for in § 138, with the procedure in solving Exs. 1 to 8, and see whether it meets all the requirements.

12. Which is greater, $3\sqrt{5}$ or $2\sqrt[3]{11}$? Compare §§ 134 and 138.

13. Which is greater, $2\sqrt[3]{9}$ or $3\sqrt{3\sqrt{2}}$? Why?

14. How may the values of any two numerical radicals (real numbers) whatever be compared?

139. Reduction of radicals to their simplest forms. A radical is said to be in its **simplest form** when the radicand is integral, when the index of the root is as small as possible, and when no factor of the radicand is a perfect power corresponding in degree with the indicated root.

The following examples may serve to illustrate the application of the foregoing principles to the reduction of any given radical to its simplest form.

Ex. 1. Reduce $\sqrt[3]{\frac{2}{5}}$ to its simplest form.

Solution. $$\sqrt[3]{\frac{2}{5}}=\sqrt[3]{\frac{2\cdot 5^2}{5\cdot 5^2}}=\sqrt[3]{\frac{1}{5^3}\cdot 50}=\tfrac{1}{5}\sqrt[3]{50}. \qquad [\S\ 134$$

Ex. 2. Reduce $\sqrt[6]{4\,a^2x^4y^4}$ to its simplest form.

Solution. $$\sqrt[6]{4\,a^2x^4y^4}=\sqrt[6]{(2\,ax^2y^2)^2}=\sqrt[3]{2\,ax^2y^2}. \qquad [\S\ 137$$

Ex. 3. Reduce $\sqrt{8\,a^3x^5y^2}$ to its simplest form.

Solution. $$\sqrt{8\,a^3x^5y^2}=\sqrt{4\,a^2x^4y^2}\cdot\sqrt{2\,ax} \qquad [\S\ 133$$
$$=2\,ax^2y\,\sqrt{2\,ax}.$$

EXERCISES

4. Is $\sqrt{3\,ax}$ in its simplest form? Why?

5. Is $\sqrt{12\,ax}$ in its simplest form? Why?

6. Is $5\sqrt[3]{4\,a^5m^2}$ in its simplest form? Why?

7. Is $12\sqrt{\frac{5}{3}\,ax^3}$ in its simplest form? Reduce it to its simplest form.

8. What is meant by saying that a radical is in its simplest form?

Reduce each of the following radicals to its simplest form:

9. $\sqrt{12}$.

10. $\sqrt{162}$.

11. $\sqrt[3]{16}$.

12. $\sqrt[3]{250}$.

13. $\sqrt[3]{81}$.

14. $\sqrt[3]{189}$.

15. $\sqrt{128}$.

16. $\sqrt[4]{32}$.

17. $\sqrt[5]{640}$.

18. $\sqrt{\frac{1}{2}}$.

19. $\sqrt{\frac{3}{8}}$.

20. $\sqrt{\frac{x^3}{2\,y}}$.

21. $\sqrt[3]{\frac{5}{4}\,x^7}$.

22. $\sqrt[4]{\frac{a^2x^6}{9\,y^2}}$.

23. $\sqrt{\frac{3\,ab^3}{50\,x^3y}}$.

24. $\sqrt{\frac{x+y}{x-y}}$.

25. $3\,a\sqrt{\frac{a-2\,bx}{2\,a}}$.

26. $3\sqrt[4]{25\,a^2b^2x^6}$.

27. $\sqrt[n]{a^{2n}x^{n+1}}$.

28. $\sqrt[r]{a^{r+3}b^{2r}y^{r-4}}$.

29. $\sqrt{a^{2n}x^{n+5}}$.

30. $\sqrt[3]{-40\,x^{5n+8}y^{14}}$.

31. $4\sqrt[n]{a^{3n}-a^{4n}x^n}$.

32. $\sqrt[3]{16-\frac{24}{x^3}}$.

140. Addition and subtraction of radicals. Similar radicals (cf. § 131) may evidently be added and subtracted just as rational numbers are added and subtracted, *i.e.*, by regarding the common radical factor as the unit of addition.

E.g., just as $3+10-4=9$, in which 1 is the unit, and $3a+10a-4a=9a$, in which a may be regarded as the unit of addition, so $3\sqrt{2}+10\sqrt{2}-4\sqrt{2}=9\sqrt{2}$, in which $\sqrt{2}$ may be regarded as the unit of addition.

If the radicals are in their simplest forms and are dissimilar, then their sum or difference can only be indicated, and this is done by connecting them with the proper signs.

E.g., the sum of $7\sqrt[3]{15}$, $3a\sqrt[5]{2xy^2}$, and $3\sqrt{2}$ is *indicated* thus:

$$7\sqrt[3]{15}+3a\sqrt[5]{2xy^2}+3\sqrt{2}.$$

If the radicals which are to be added are not in their simplest forms, they should first be reduced; the following examples may serve to illustrate the procedure:

Ex. 1. Find the sum of $\sqrt{75}$ and $3\sqrt{12}$.

SOLUTION. $\sqrt{75}+3\sqrt{12}=\sqrt{25\cdot 3}+3\sqrt{4\cdot 3}=5\sqrt{3}+6\sqrt{3}=11\sqrt{3}.$

Ex. 2. Find the sum of $5\sqrt{18}$, $-\sqrt{0.5}$, and $\sqrt{\frac{1}{8}}$.

SOLUTION. $$5\sqrt{18}-\sqrt{0.5}+\sqrt{\tfrac{1}{8}}=5\sqrt{9\cdot 2}-\sqrt{\tfrac{1}{4}\cdot 2}+\sqrt{\tfrac{1}{16}\cdot 2}\,*$$
$$=15\sqrt{2}-\tfrac{1}{2}\sqrt{2}+\tfrac{1}{4}\sqrt{2}=14\tfrac{3}{4}\sqrt{2}.$$

Ex. 3. Find the sum of $\sqrt{9x-18}$, $6\sqrt{4x+8}$, $\sqrt{36x-72}$, and $-\sqrt{25x+50}$.

SOLUTION. $$\sqrt{9x-18}+6\sqrt{4x+8}+\sqrt{36x-72}-\sqrt{25x+50}$$
$$=3\sqrt{x-2}+12\sqrt{x+2}+6\sqrt{x-2}-5\sqrt{x+2}$$
$$=9\sqrt{x-2}+7\sqrt{x+2}.$$

EXERCISES

Find the sum of:

4. $\sqrt{50}$, $\sqrt{18}$, and $\sqrt{98}$.

5. $\sqrt{12}$, $\sqrt{75}$, and $\sqrt{27}$.

6. $\sqrt{28}$, $\sqrt{63}$, and $\sqrt{700}$.

7. $\sqrt[3]{250}$, $\sqrt[3]{16}$, and $\sqrt[3]{54}$.

8. $\sqrt[3]{500}$, $\sqrt[3]{108}$, and $\sqrt[3]{-32}$.

---

* Since $0.5=\frac{1}{2}=\frac{2}{4}$, and $\frac{1}{8}=\frac{2}{16}$.

9. What is the sum of a, $2b$, and c? Of $3x$, $4y$, a, $2x$, and $-5y$?

10. What is the sum of $3\sqrt{2}$ and $5\sqrt[3]{7}$? Of $3\sqrt{2}$, $5\sqrt[3]{7}$, $-2\sqrt{7}$, and $\sqrt{2}$?

11. Write out a carefully worded rule for the addition and subtraction of radicals; provide both for those cases in which the given radicals are similar and for those in which they are dissimilar.

Simplify the following expressions as far as possible, and explain your work in each case:

12. $\sqrt[3]{135} + \sqrt[3]{625} - \sqrt[3]{320}.$

13. $\sqrt[3]{40} + \sqrt{28} + \sqrt{175} + \sqrt[6]{25}.$

14. $\sqrt[3]{375} - \sqrt{44} - \sqrt[3]{192} + \sqrt{99}.$

15. $\sqrt{\frac{1}{3}} + \sqrt{75} - \sqrt{12} + \frac{2}{3}\sqrt{3}.$

16. $\sqrt{147} - \sqrt{\frac{3}{4}} + \frac{1}{3}\sqrt{3} + \frac{7}{6}\sqrt[4]{9}.$

17. $6\sqrt[3]{\frac{40}{27}} + 4\sqrt[3]{\frac{10}{16}} - 8\sqrt[3]{\frac{675}{320}}.$

18. $\sqrt[3]{128x} + \sqrt[3]{375x} - \sqrt[3]{54x}.$

19. $\sqrt{\frac{a}{x^2}} + \sqrt{\frac{a}{y^2}} - \sqrt{\frac{a}{z^2}}.$

20. $\sqrt[4]{\frac{a^2x^4}{b^2y^4}} - \sqrt{\frac{16ax^2}{by^2}} + \sqrt{\frac{4ax^2}{by^2}}.$

21. $\sqrt{(a+b)c} - \sqrt{(a-b)c}.$

22. $\sqrt[6]{192x^4} - 2\sqrt[6]{3x^4} - \sqrt[3]{5x} + \sqrt[3]{40x^4}.$

23. $\sqrt[3]{abx} + \sqrt[6]{a^2b^2x^2} - \sqrt[9]{8a^3b^3x^3}.$

24. $\sqrt{3x^3 + 30x^2 + 75x} - \sqrt{3x^3 - 6x^2 + 3x}.$

25. $\sqrt{5a^5 + 30a^4 + 45a^3} - \sqrt{5a^5 - 40a^4 + 80a^3}.$

26. $\sqrt{50} + \sqrt[6]{9} - 4\sqrt{\frac{1}{2}} + \sqrt[3]{-24} + \sqrt[9]{27} - \sqrt[4]{64}.$

27. $\sqrt{\frac{2}{3}} + 6\sqrt{\frac{5}{4}} - \frac{1}{5}\sqrt{18} + \sqrt[4]{36} - \sqrt[8]{\frac{16}{81}} + \sqrt[6]{125} - \sqrt{\frac{2}{25}}.$

28. $\sqrt{a^3 - a^2x} - \sqrt{ax^2 - x^3} - \sqrt{(a+x)(a^2 - x^2)}.$

141. Multiplication of monomial radicals. In § 133 it is shown how to get the product of two or more radicals *which are of the same order*, and in § 138 it is shown how to reduce any given radicals to the same order; therefore the product of *any* two or more monomial radicals (real numbers) may now be found.

Ex. 1. Multiply $\sqrt[3]{5}$ by $\sqrt{2}$.

SOLUTION. $\sqrt[3]{5} \cdot \sqrt{2} = \sqrt[6]{5^2} \cdot \sqrt[6]{2^3}$ [§ 138

$= \sqrt[6]{5^2 \cdot 2^3} = \sqrt[6]{200}.$ [§ 133

Note. The student should observe that, although a root remains to be extracted in this result, viz., $\sqrt[6]{200}$, the result is simpler in form than the *indicated* product, viz., $\sqrt[3]{5} \cdot \sqrt{2}$, and also that the arithmetical work of finding the approximate numerical value is much easier in the final than in the original form.

Ex. 2. Find the product of $5\sqrt{2}$ by $8\sqrt[3]{7}$.

Solution. $5\sqrt{2} \cdot 8\sqrt[3]{7} = 5 \cdot 8 \cdot \sqrt{2} \cdot \sqrt[3]{7}$ [§ 52

$$= 40\sqrt[6]{2^3} \cdot \sqrt[6]{7^2} = 40\sqrt[6]{392}.$$

EXERCISES

3. Multiply $\sqrt{3}$ by $\sqrt{6}$, and simplify the result.

4. Multiply $\sqrt{3}$ by $\sqrt[3]{2}$.

5. How may the product of two or more radicals which are of the same order be found (cf. § 133)?

6. How may the product of two or more radicals which are of different orders be found?

Find the following products, and simplify the results:

7. $\sqrt{3}$ by $\sqrt{15}$.

8. $2\sqrt{5}$ by $3\sqrt{10}$.

9. $5\sqrt{2}$ by $4\sqrt[3]{5}$.

10. $\sqrt[3]{3}$ by $3\sqrt{3}$.

11. $2\sqrt[4]{5}$ by $7\sqrt[6]{10}$.

12. $2\sqrt[5]{2}$ by $\sqrt[10]{512}$.

13. $\sqrt[3]{\frac{2}{3}}$ by $2\sqrt{\frac{1}{2}}$.

14. $\sqrt[3]{2} \cdot \sqrt[6]{\frac{1}{3}} \cdot \sqrt[3]{3}$.

15. $\sqrt{x^3y^2} \cdot \sqrt{12\,x} \cdot \sqrt{75\,xy^2}$.

16. $\sqrt{2\,ab} \cdot \sqrt[3]{abc} \cdot \sqrt[4]{4\,a^2b^2}$.

17. $\sqrt{x^{-1}y} \cdot \sqrt[3]{x^{-2}y^2} \cdot \sqrt{x^{-3}y^3}$.

18. $\sqrt[3]{4\,a^2}$ by $\sqrt{8\,a^3}$.

19. $\sqrt[n]{a^t}$ by $\sqrt[m]{a^s}$.

20. $3\sqrt[3]{2}$ by $3\sqrt[3]{2}$, *i.e.*, $(3\sqrt[3]{2})^2$.

21. $(2\sqrt[4]{5\,ax^2})^3$.

22. $(\sqrt[6]{12\,a^{-2}x^4y})^3$.

142. Multiplication of polynomials containing radicals. The product of two polynomials containing radicals is obtained by multiplying each term of the multiplicand by each term of the multiplier and adding the partial products, just as in the case of rational polynomials.

Ex. 1. Multiply $5\sqrt{2}-2\sqrt{3}$ by $3\sqrt{2}+4\sqrt{3}$.

SOLUTION.

$$\begin{array}{r}5\sqrt{2}-2\sqrt{3}\\ 3\sqrt{2}+4\sqrt{3}\\ \hline 30-6\sqrt{6}\\ +20\sqrt{6}-24\\ \hline 30+14\sqrt{6}-24\end{array}=6+14\sqrt{6}.$$

Ex. 2. Expand $(2\sqrt{3}-\sqrt[3]{2})^2$ by the binomial theorem.

SOLUTION.
$$(2\sqrt{3}-\sqrt[3]{2})^2=(2\sqrt{3})^2-2(2\sqrt{3})\sqrt[3]{2}+(\sqrt[3]{2})^2 \qquad [\S\ 57$$
$$=12-4\sqrt[6]{108}+\sqrt[3]{4}. \qquad [\S\ 140$$

EXERCISES

Perform the following multiplications, and simplify the results:

3. $\sqrt{5}-5$ by $\sqrt{5}+1$.

4. $2\sqrt{2}+\sqrt{3}$ by $\sqrt{2}+4\sqrt{3}$.

5. $\sqrt[3]{2}+3\sqrt[4]{2}$ by $\sqrt{\frac{1}{2}}$.

6. $5+\sqrt[3]{4}-2\sqrt[4]{5}$ by $\sqrt{5}+\sqrt{6}$.

7. $2\sqrt{3}+\sqrt[3]{2}$ by $2\sqrt{3}-\sqrt[3]{4}$.

8. $a^2-ab\sqrt{2}+b^2$ by $a^2+ab\sqrt{2}+b^2$.

9. $x-\sqrt{xyz}+yz$ by $\sqrt{x}+\sqrt{yz}$.

10. $-\sqrt{a}+\sqrt{x}$ by $-\sqrt{a}-\sqrt{x}$.

Expand the following expressions, and simplify the results:

11. $(\sqrt{2}-3\sqrt[3]{3})^2$.

12. $(\sqrt{2\,xy}-\sqrt{3\,xy})^2$.

13. $(a+\sqrt{b}-\sqrt{c})^2$.

14. $(\sqrt{m-n}+\sqrt{m+n})^2$.

15. $(\sqrt[3]{2\,m}-\sqrt[3]{3\,x^2})^3$.

16. $(\sqrt[3]{a}+2\sqrt{3})^5$.

17. $\left(x+\frac{p}{2}-\sqrt{\frac{p^2-4\,q}{4}}\right)\cdot\left(x+\frac{p}{2}+\sqrt{\frac{p^2-4\,q}{4}}\right)$.

18. $(\sqrt{2\,a-\sqrt{b}}+\sqrt{2\,a+\sqrt{b}})^2$.

143. Division of monomial radicals. By means of §§ 135 and 138 the quotient of any two given monomial radicals (real numbers) may be expressed as a single radical (cf. § 141).

Ex. 1. Divide $\sqrt[6]{4\,ax^3y^2}$ by $\sqrt[4]{2\,a^3x}$.

SOLUTION.
$$\frac{\sqrt[6]{4\,ax^3y^2}}{\sqrt[4]{2\,a^3x}}=\frac{\sqrt[12]{16\,a^2x^6y^4}}{\sqrt[12]{8\,a^9x^3}} \qquad [\S\ 138$$
$$=\sqrt[12]{\frac{16\,a^2x^6y^4}{8\,a^9x^3}} \qquad [\S\ 135$$
$$=\sqrt[12]{\frac{2\,x^3y^4}{a^7}}=\frac{1}{a}\sqrt[12]{2\,a^5x^3y^4}. \qquad [\S\ 139,\ \text{Ex. } 1$$

EXERCISES

2. What is the quotient of $\sqrt{50}$ divided by $\sqrt{8}$?

3. What is the quotient of $4\sqrt{5}$ divided by $\sqrt{40}$?

4. What is the quotient of $7\sqrt[3]{54}$ divided by $2\sqrt[3]{686}$?

5. How is the quotient of two monomial radicals obtained if these radicals are of the same order?

Express each of the following indicated quotients in its simplest form:

6. $2\sqrt[3]{12} \div \sqrt{8}$.

7. $2\sqrt[3]{6} \div \sqrt[6]{2}$.

8. $\sqrt{18} \div \sqrt[3]{500}$.

9. $8 \div 3\sqrt[3]{2}$.

10. $\sqrt{\frac{3}{4}} \div 3\sqrt[3]{\frac{2}{3}}$.

11. $\sqrt[3]{ax} \div \sqrt{xy}$.

12. $\sqrt{2\,ax} \div \sqrt[3]{5\,a^2x^2}$.

13. $2\sqrt[3]{9\,a^2b^2} \div 3\sqrt{3\,ab}$.

14. $a\sqrt[4]{4\,x^2y^2} \div 2\,b\sqrt[3]{2\,xy}$.

15. $3\,a\sqrt[2n]{2\,x^{2n-1}} \div 2\,b\sqrt[n]{3\,x^{n-5}}$.

16. How is the quotient of two monomial radicals obtained if these radicals are of different orders?

17. Apply the answer of Ex. 16 to show that

$$\sqrt[3]{x^2-y^2} \div \sqrt{x+y} = \frac{1}{x+y}\sqrt[6]{(x-y)^2(x+y)^5} = \frac{1}{x+y}\sqrt[6]{(x^2-y^2)^2(x+y)^3}.$$

Verify this equation when $x = 64$ and $y = 0$. Is this equation true for *all* values of x and y, or merely for certain particular values of these letters? What other name is given to such equations (cf. § 23)?

144. Division of polynomials containing radicals. If the divisor is a monomial, then, manifestly, the quotient may be obtained by dividing each term of the dividend by the divisor — just as in the case of rational expressions.

E.g.,

$$\frac{3\sqrt{2}+4\sqrt{3}-2\sqrt[3]{4}}{\sqrt{2}} = 3+4\sqrt{\frac{3}{2}}-2\sqrt[6]{\frac{4^2}{2^3}} \qquad [\S\ 138$$

$$= 3+2\sqrt{6}-2\sqrt[6]{2}. \qquad [\S\ 139$$

Instead of dividing directly by a radical, it is usually advantageous first to multiply both dividend and divisor by an expression which will make the new divisor rational — indeed, it is frequently *necessary* to do so.

E.g., since $(3\sqrt{2}-\sqrt{13})\cdot(3\sqrt{2}+\sqrt{13})=(3\sqrt{2})^2-(\sqrt{13})^2=5,$

therefore $5\div(3\sqrt{2}-\sqrt{13})=3\sqrt{2}+\sqrt{13},$

but one could not easily obtain this quotient by dividing directly. It may be obtained thus:

$$\frac{5}{3\sqrt{2}-\sqrt{13}}=\frac{5(3\sqrt{2}+\sqrt{13})}{(3\sqrt{2}-\sqrt{13})(3\sqrt{2}+\sqrt{13})}$$ [Multiplying numerator and denominator by $3\sqrt{2}+\sqrt{13}$

$$=\frac{15\sqrt{2}+5\sqrt{13}}{5}=3\sqrt{2}+\sqrt{13}.$$

This method of dividing (usually called *division by means of rationalizing the divisor*) will often be found very advantageous even when it is not strictly necessary.

E.g., $$\frac{3\sqrt{2}+4\sqrt{3}}{\sqrt{2}}=\frac{(3\sqrt{2}+4\sqrt{3})\cdot\sqrt{2}}{(\sqrt{2})^2}=\frac{6+4\sqrt{6}}{2}=3+2\sqrt{6}.$$

The factor by which a given radical is multiplied to obtain a rational product is called its **rationalizing factor.**

E.g., of $\sqrt[3]{4}$ and $\sqrt[3]{2}$ each is a rationalizing factor of the other (why?); so also are $\sqrt[n]{a^p}$ and $\sqrt[n]{a^{n-p}}$ (why?), and $a\sqrt{x}+b\sqrt{y}$ and $a\sqrt{x}-b\sqrt{y}$ (why?).*

Of two such binomial quadratic surds as $a\sqrt{x}+b\sqrt{y}$ and $a\sqrt{x}-b\sqrt{y}$, which differ from each other only in the quality sign of one of their terms, each is called the **conjugate** of the other.

EXERCISES

1. Divide $\sqrt{15}-\sqrt{3}$ by $\sqrt{3}$.

2. Divide $\sqrt{6}+2\sqrt{3}$ by $\sqrt{2}$.

3. Divide $\sqrt[3]{12}-4\sqrt{5}+2\sqrt[3]{6}$ by $\sqrt{3}$.

4. Perform the divisions in Exs. 1–3 by first rationalizing the divisors, and show whether or not there is any advantage here in rationalizing.

5. Show that $2\sqrt{3}-\sqrt{5}$ is a rationalizing factor of $2\sqrt{3}+\sqrt{5}$.

6. Is $\sqrt{5}-2\sqrt{3}$ a rationalizing factor of $2\sqrt{3}+\sqrt{5}$? Why? Are these surds conjugate to each other?

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* The question of finding rationalizing factors for given expressions is further considered in § 161.

Find the simplest rationalizing factor of each of the following surds:

7. $\sqrt{2a}$.

8. $\sqrt{4ax^2}$.

9. $\sqrt[3]{4ax^2}$.

10. $\sqrt{a+b}$.

11. $5\sqrt[3]{\frac{2a^2m}{3}}$.

12. $\sqrt{2}-\sqrt{7}$.

13. $2\sqrt{3}+\sqrt{6}$.

14. $4+5\sqrt{3}$.

15. $3a-2\sqrt{5x}$.

16. $5x-\sqrt{2ay}$.

17. $\sqrt{a^3}+2\sqrt{3b}$.

18. $\sqrt{\frac{14x}{a}}+\frac{3}{4}\sqrt{ax^3}$.

19. Divide 31 by $7+3\sqrt{2}$.

20. Divide $2\sqrt{6}$ by $\sqrt{5}-\sqrt{3}$.

21. Divide $5\sqrt[3]{12}-2\sqrt{6}+4$ by $\sqrt[3]{4}$. What is the smallest multiplier that will rationalize $\sqrt[3]{4}$?

22. Divide $3\sqrt{2}-4\sqrt{5}$ by $2\sqrt{3}+\sqrt{7}$.

23. Divide $4\sqrt{3}+5\sqrt{2}$ by $3\sqrt{2}-2\sqrt{3}$.

24. If the result of Ex. 21 were wanted correct to 4 decimal places, say, show in detail that it is far simpler first to rationalize the divisor than to extract roots and divide by the ordinary arithmetical method.

25. What is the product of $(2+\sqrt{3})-\sqrt{5}$ by $(2+\sqrt{3})+\sqrt{5}$? Of this result by $2-4\sqrt{3}$? What then is a rationalizing factor of $2+\sqrt{3}-\sqrt{5}$? Of $2+\sqrt{3}+\sqrt{5}$?

26. Divide $2-\sqrt{3}$ by $1+\sqrt{3}-\sqrt{2}$.

Reduce the following to equivalent fractions having rational denominators:

27. $\frac{a+\sqrt{a^2+x}}{a-\sqrt{a^2+x}}$.

28. $\frac{\sqrt{x+y}-\sqrt{x-y}}{\sqrt{x+y}+\sqrt{x-y}}$.

29. $\frac{a^2}{\sqrt{a^2+x^2}-x}$.

30. Simplify $\frac{2}{\sqrt[3]{2}-1}+\frac{3}{\sqrt[3]{2}+1}$.

31. Simplify $\frac{(\sqrt{2}+3)(\sqrt{5}-2)}{(3-\sqrt{2})(2+\sqrt{5})}$.

32. Find the value of $\frac{\sqrt{5}}{2-\sqrt{3}}+\frac{\sqrt{2}-1}{\sqrt{2}+1}$ correct to 3 decimal places.

145. An important property of quadratic surds. Neither the sum nor the difference of two dissimilar *quadratic* surds (§ 131) can be a rational number; for, if possible, let

$$\sqrt{x}+\sqrt{y}=r, \qquad (1)$$

$\sqrt{x}$ and $\sqrt{y}$ being dissimilar surds, and r rational, and not zero.

From Eq. (1) $$\sqrt{y} = r - \sqrt{x}, \tag{2}$$

whence, squaring, $$y = r^2 - 2r\sqrt{x} + x, \tag{3}$$

and, solving for $\sqrt{x}$, $$\sqrt{x} = \frac{r^2 + x - y}{2r},$$

i.e., if Eq. (1) were true, then the surd $\sqrt{x}$ would equal the rational number $\frac{r^2 + x - y}{2r}$, which is impossible; hence Eq. (1) can not be true.

Similarly, $$\sqrt{x} - \sqrt{y} \neq r.$$

From what has just been shown it at once follows that *if* $x + \sqrt{y} = a + \sqrt{b}$, *where* x *and* a *are rational, and* $\sqrt{y}$ *and* $\sqrt{b}$ *are quadratic surds, then* $x = a$ *and* $y = b$.

For, if $$x + \sqrt{y} = a + \sqrt{b},$$

then $$\sqrt{y} - \sqrt{b} = a - x;$$

which, by the above proof, can be true only if each member is zero, *i.e.*, if $a = x$ and $\sqrt{y} = \sqrt{b}$. In other words, the equation $x + \sqrt{y} = a + \sqrt{b}$ is equivalent to the two equations $x = a$ and $y = b$.

II. IMAGINARY NUMBERS

146. Imaginary numbers. In solving the equations of the next chapter, indicated square roots of negative numbers frequently appear; such numbers have already been defined (§ 130) as **imaginary numbers**; if they present themselves in the form $\sqrt{-b}$, where b is a positive number, they are called **pure imaginary numbers**, while if they present themselves in the mixed binomial form $a + \sqrt{-b}$, where a and b are real and b is positive, they are usually called **complex numbers.***

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* A broader definition of imaginary numbers is given in appendix B, where it is shown that every such number can be expressed in the form $a + b\sqrt{-1}$, and where it is *proved* that these numbers obey the laws already established for real numbers (commutative, associative, etc.). Logically this proof should now be read, but it may be deferred until later if the reader will carefully bear in mind that the following discussion *assumes* that imaginary numbers are subject to those laws, and is therefore to be regarded as *tentative* until this fact is *proved*. The very elementary discussion which is given in the next few pages will suffice for present needs.

E.g., $\sqrt{-5}$, $2\sqrt{-6}$, and $\sqrt{-\frac{1}{3}}$ are pure imaginary numbers, while $2-\sqrt{-3}$ and $7+2\sqrt{-5}$ are complex numbers.

Operations with imaginary numbers are greatly simplified by observing that, by the definition of $\sqrt[n]{a}$, § 130,

$$(\sqrt{-b})^2=-b, \tag{1}$$

and also (cf. method of § 133, and apply §§ 52 and 53) that

$$\sqrt{-b}=\sqrt{b}\cdot\sqrt{-1}. \tag{2}$$

The symbol $\sqrt{-1}$ is called the **imaginary unit**, and is often represented by the letter i.

147. Positive integral powers of $\sqrt{-1}$. As a special case of Eq. (1), § 146,

$$(\sqrt{-1})^2=-1;$$

consequently, $(\sqrt{-1})^3$, *i.e.*, $(\sqrt{-1})^2\cdot\sqrt{-1}=-\sqrt{-1}$.

Similarly,

$$(\sqrt{-1})^4=(\sqrt{-1})^3\cdot\sqrt{-1}=-\sqrt{-1}\cdot\sqrt{-1}=-(\sqrt{-1})^2=1,$$

$$(\sqrt{-1})^5=(\sqrt{-1})^4\cdot\sqrt{-1}=1\cdot\sqrt{-1}=\sqrt{-1},$$

$$(\sqrt{-1})^6=(\sqrt{-1})^4\cdot(\sqrt{-1})^2=-1,$$

$$(\sqrt{-1})^7=(\sqrt{-1})^4\cdot(\sqrt{-1})^3=-\sqrt{-1},$$

and so on for the higher powers, *i.e.*, the positive integral powers of $\sqrt{-1}$ have only these four values: $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1; see also Exs. 5, 6, and 7 below.

EXERCISES

1. Define an imaginary number; compare § 130.

2. Which of the following are imaginary numbers: $\sqrt{-3}$, $\sqrt[4]{-2}$, $\sqrt[3]{-6}$, $\sqrt{5}$, $\sqrt[6]{-a^2}$, $3\sqrt[4]{-7}$, $4a\sqrt{\frac{-1}{3}}$ and $\frac{2}{3}+\frac{1}{2}\sqrt{-5}$?

3. Is $\sqrt{-x}$ imaginary when x represents a positive number? When x represents a negative number?

4. Show that if $i=\sqrt{-1}$, then $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$, $i^6=-1$, $i^7=-i$, $i^8=1$, and $i^9=i$.

5. Since any *even* number may be written in the form $2\,n$, where n is an integer, and since $a^{2n} = (a^2)^n$, show that every *even* power of i is real.

6. As in Ex. 5, show that every *odd* power of i is either i or $-i$.

7. Since $x^{a+b} = x^a \cdot x^b$, and since any positive integer whatever can be represented by one of the following expressions, viz., $4\,n+1$, $4\,n+2$, $4\,n+3$, or $4\,n$, show that the positive integer powers of i can have no other values than i, -1, $-i$, and $+1$, and that these values always recur in this order.

8. Distinguish between pure and complex imaginary numbers, and give three examples of each.

148. Addition and subtraction of imaginary numbers. By first writing the imaginary numbers in the form $a + b\sqrt{-1}$, these numbers may be added and subtracted exactly as though they were real; this is illustrated below.

Ex. 1. Find the sum of $\sqrt{-4}$, $4\sqrt{-9}$, and $\sqrt{-25}$.

Solution

$$\sqrt{-4}+4\sqrt{-9}+\sqrt{-25}=2\sqrt{-1}+4\cdot 3\sqrt{-1}+5\sqrt{-1} \qquad [\S\ 146, \text{Eq. } (2)$$

$$=(2+12+5)\sqrt{-1} \qquad [\text{Footnote, p. } 83$$

$$=19\sqrt{-1}.$$

Ex. 2. Find the sum of $3+\sqrt{-16}$, $\sqrt{-4}$, and $5-\sqrt{-9}$.

Solution. $3+\sqrt{-16}+\sqrt{-4}+5-\sqrt{-9}$

$$=3+5+\sqrt{-16}+\sqrt{-4}-\sqrt{-9} \qquad [\S\ 50$$

$$=(3+5)+(\sqrt{-16}+\sqrt{-4}-\sqrt{-9}) \qquad [\S\ 51$$

$$=8+3\sqrt{-1}. \qquad [\text{Ex. } 1$$

Ex. 3. Simplify the expression $x\sqrt{-4}+\sqrt{-x^2-2\,x-1}-\sqrt{-32}$.

Solution. Since $x\sqrt{-4}=2\,x\sqrt{-1}$,

$$\sqrt{-x^2-2\,x-1}=\sqrt{-(x+1)^2}=(x+1)\sqrt{-1},$$

and

$$-\sqrt{-32}=-\sqrt{32}\cdot\sqrt{-1}=-4\sqrt{2}\cdot\sqrt{-1},$$

therefore the given expression becomes

$$\{2\,x+(x+1)-4\sqrt{2}\}\cdot\sqrt{-1},\ i.e.,\ (3\,x+1-4\sqrt{2})\cdot\sqrt{-1}.$$

Similarly in general.

EXERCISES

Simplify each of the following expressions:

4. $3+\sqrt{-36}-(1+2\sqrt{-25})+3\sqrt{-16}$.

5. $\sqrt{-49}+5\sqrt{-4}-(6+2\sqrt{-9})$.

6. $\sqrt{-8}-3\sqrt{-2}+6\sqrt{-18}-2\sqrt{-27}+8+\sqrt{-12}$.

7. $\sqrt{-\frac{1}{2}}-(2\sqrt{-\frac{2}{3}}+5-3\sqrt{-24})+3\sqrt{-18}$.

8. $\sqrt{-16a^2x^2}+\sqrt{1-5}+2\sqrt{5-30}-\sqrt{-9a^2x^2}+\sqrt{-a^2x^2}$.

149. Multiplication of imaginary numbers. Multiplication of imaginary numbers is also performed by first writing these numbers in the form $a+b\sqrt{-1}$; this is illustrated below.

Ex. 1. Multiply $\sqrt{-2}$ by $\sqrt{-5}$.

SOLUTION.
$$\begin{aligned}\sqrt{-2}\cdot\sqrt{-5}&=\sqrt{2}\cdot\sqrt{-1}\cdot\sqrt{5}\cdot\sqrt{-1} && [146, \text{Eq. } (2)\\ &=(\sqrt{2}\cdot\sqrt{5})(\sqrt{-1}\cdot\sqrt{-1}) && [\S\S\ 52 \text{ and } 53\\ &=\sqrt{10}\cdot(-1)=-\sqrt{10}.\end{aligned}$$

Similarly in general: $\sqrt{-a}\cdot\sqrt{-b}=-\sqrt{ab}$.

NOTE. The student should carefully observe that (§ 133) the law for the product of two radicals, *i.e.*, principal roots, does *not* apply to the product of two imaginary numbers; according to that law the product of $\sqrt{-a}\cdot\sqrt{-b}$ would be $\sqrt{(-a)\cdot(-b)}$, *i.e.*, $\sqrt{ab}$, and not $-\sqrt{ab}$. Errors of this kind are easily avoided by writing an imaginary number in the form $a+b\sqrt{-1}$ before operating with it.

Ex. 2. Multiply $3+\sqrt{-5}$ by $2-\sqrt{-3}$.

SOLUTION. Writing these imaginary numbers in terms of the imaginary unit, the work may be arranged thus:

$$\begin{array}{l}3+\sqrt{5}\cdot\sqrt{-1}\\ 2-\sqrt{3}\cdot\sqrt{-1}\\ \hline 6+2\sqrt{5}\cdot\sqrt{-1}\\ \quad -3\sqrt{3}\cdot\sqrt{-1}-\sqrt{15}(\sqrt{-1})^2\\ \hline 6+(2\sqrt{5}-3\sqrt{3})\cdot\sqrt{-1}+\sqrt{15}.\end{array}$$

Similarly in general:

$$(a+\sqrt{-b})\cdot(c+\sqrt{-d})=ac-\sqrt{bd}+(a\sqrt{d}+c\sqrt{b})\sqrt{-1}.$$

EXERCISES

Find the product of:

3. $3\sqrt{-6}$ by $5\sqrt{-12}$. **6.** $\sqrt{-6}+\sqrt{-3}$ by $\sqrt{-6}-\sqrt{-3}$.

4. $5\sqrt{-8}$ by $2\sqrt{-6}$. **7.** $3+2\sqrt{-9}$ by $5-4\sqrt{-1}$.

5. $2\sqrt{-4}$ by $\sqrt{-4a^2x^3}$. **8.** $\sqrt{-50}-2\sqrt{-12}$ by $\sqrt{-8}-5\sqrt{-3}$.

9. Show that the sum and also the product of $a+bi$ and $a-bi$ (wherein a and b are real) is real.* Show that this is also true for $\sqrt{-4}-3$ and $-\sqrt{-4}-3$.

10. Prove that both the sum and also the product of *any* two conjugate complex numbers is real.

11. Multiply $\sqrt{-a}+\sqrt{-b}+\sqrt{-c}$ by $\sqrt{-a}-\sqrt{-b}+\sqrt{-c}$.

12. $(1+\sqrt{-5})^2=?$ **13.** $(2-3i)^3=?$ **14.** $(2a-3x\sqrt{-1})^2=?$

15. Find the product of $a\sqrt{-b}+b\sqrt{-a}$, $a\sqrt{-a}+b\sqrt{-b}$, and $b\sqrt{-b}-a\sqrt{-1}$.

16. Show that $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ are conjugates of each other, and also that the square of either is equal to the other.

17. Write a rule for multiplying one pure imaginary number by another, and compare it with the rule for getting the product of two monomial surds of the same order. Wherein do the two rules differ?

18. Reduce $\dfrac{2-3\sqrt{-1}}{3-\sqrt{-4}}+\dfrac{2+3i}{3+2i}$ to its simplest form.

150. Division of imaginary numbers. The simpler cases of division of imaginary numbers are illustrated by the following examples:

Ex. 1. Divide $\sqrt{-6}$ by $\sqrt{-2}$.

Solution. $$\frac{\sqrt{-6}}{\sqrt{-2}}=\frac{\sqrt{6}\cdot\sqrt{-1}}{\sqrt{2}\cdot\sqrt{-1}}=\frac{\sqrt{6}}{\sqrt{2}}=\sqrt{\frac{6}{2}}=\sqrt{3}. \qquad [\S\S\ 146,\ 135$$

Similarly in general:

$$\frac{\sqrt{-a}}{\sqrt{-b}}=\sqrt{\frac{a}{b}},\quad \frac{\sqrt{a}}{\sqrt{-b}}=-\sqrt{-\frac{a}{b}},\ \text{and}\ \frac{\sqrt{-a}}{\sqrt{b}}=\sqrt{-\frac{a}{b}}.$$

---

* Of two complex numbers which differ only in the sign of the imaginary term each is called the conjugate of the other (cf. § 144).

Ex. 2. Divide $12+\sqrt{-25}$ by $3-\sqrt{-4}$.

Solution. Such divisions are easily performed by rationalizing the divisor (cf. § 144), thus:

$$\frac{12+\sqrt{-25}}{3-\sqrt{-4}}=\frac{12+5\sqrt{-1}}{3-2\sqrt{-1}}=\frac{(12+5\sqrt{-1})(3+2\sqrt{-1})}{(3-2\sqrt{-1})(3+2\sqrt{-1})}$$

$$=\frac{36+39\sqrt{-1}+10(\sqrt{-1})^2}{9-4(\sqrt{-1})^2}$$

$$=\frac{26+39\sqrt{-1}}{9+4}$$

$$=2+3\sqrt{-1}=2+\sqrt{-9}.$$

Similarly in general: $\dfrac{a+b\sqrt{-1}}{c+d\sqrt{-1}}=\dfrac{(a+b\sqrt{-1})(c-d\sqrt{-1})}{(c+d\sqrt{-1})(c-d\sqrt{-1})}$

$$=\frac{ac+bd+(bc-ad)\sqrt{-1}}{c^2+d^2}.$$

EXERCISES

3. Verify the correctness of the result in Ex. 2 above by multiplying the quotient by the divisor.

4. Divide $\sqrt{-6}+2\sqrt{-8}$ by $\sqrt{-2}$.

5. Divide 4 by $1+i$.

6. Divide 2 by i^4+i^8.

Simplify the following:

7. $\dfrac{2-\sqrt{-3}}{3+\sqrt{-2}}$.

9. $\dfrac{\sqrt{2x}-3ai}{\sqrt{2x}+2bi}$.

8. $\dfrac{5+\sqrt{-4}}{5-2i}$.

10. $\dfrac{\sqrt{a}-i\sqrt{b}}{i\sqrt{b}+\sqrt{a}}$.

11. Write a rule for dividing one pure imaginary number by another, and compare it with the rule for finding the quotient of two monomial surds of the same order.

12. Divide $3-\sqrt{-5}+2i$ by $2+\sqrt{-5}-\sqrt{-4}$ (cf. § 144, Ex. 25).

151. Important property of imaginary numbers. Neither the sum nor the difference of two different pure imaginary numbers can be a real number (cf. also § 145); for, if possible, let

$$\sqrt{-a} - \sqrt{-b} = r;\text{*} \qquad (1)$$

then, transposing, $$\sqrt{-a} = r + \sqrt{-b},$$

and squaring, $$-a = r^2 + 2r\sqrt{-b} - b,$$

whence $$\sqrt{-b} = \frac{b - a - r^2}{2r};$$

i.e., if Eq. (1) were true, then the imaginary number $\sqrt{-b}$ would equal the real number $\frac{b-a-r^2}{2r}$, which is impossible, and hence Eq. (1) can not be true.

Similarly it may be shown that $\sqrt{-a} + \sqrt{-b} \neq r$.

From what has just been shown it follows that if

$$x + \sqrt{-y} = a + \sqrt{-b},$$

wherein a and x are real and $\sqrt{-y}$ and $\sqrt{-b}$ pure imaginary numbers, then

$$x = a \text{ and } y = b.$$

For, if $$x + \sqrt{-y} = a + \sqrt{-b},$$

then, transposing, $$\sqrt{-y} - \sqrt{-b} = a - x,$$

which, by the above proof, can be true only if each member is zero, *i.e.*, if $$y = b \text{ and } x = a,$$

which was to be proved.

In other words, the equation $x + \sqrt{-y} = a + \sqrt{-b}$ is equivalent to the two equations $x = a$ and $y = b$.

---

* The expressions $\sqrt{-a}$ and $\sqrt{-b}$ represent different pure imaginary numbers, and r is real, and not zero.

152. Complex factors. Solving equations by factoring. Since $(a+bi)(a-bi)=a^2+b^2$, wherein a and b may be any real numbers whatever, therefore the *sum* of any two real positive numbers may be separated into two *imaginary* factors.

E.g., $x^2+4=(x+2i)\cdot(x-2i)$; $a^2+3=(a+i\sqrt{3})(a-i\sqrt{3})$; x^2+2x+5 $=(x+1)^2+4=(x+1+2i)(x+1-2i)$; $x^4-x^2+1=x^4-2x^2+1+x^2$ $=(x^2-1)^2+x^2=(x^2-1+x\cdot i)(x^2-1-x\cdot i)$.

NOTE. Observe that the most important step in the above factoring is first to write the given expression as the sum of two squares; the plan for doing this is precisely that which is followed in § 70.

The following examples will illustrate the use of imaginary factors in solving certain kinds of equations; this method will be more fully treated, however, in Chapter XV.

Ex. 1. Solve the equation $x^2+2x+5=0$.

SOLUTION. Since this equation may be written in the following forms:

$$x^2+2x+1+4=0,$$

$$(x+1)^2+4=0,$$

$$(x+1+2i)(x+1-2i)=0,$$

therefore it is clear (§ 72) that the only values of x that satisfy it are those that make

$$x+1+2i=0 \text{ or } x+1-2i=0;$$

i.e., the given equation is satisfied if, and only if,

$$x=-1-2i \text{ or } x=-1+2i;$$

i.e., the roots of that equation are $-1-2i$ and $-1+2i$.

Ex. 2. Solve the equation $x^2=4x-22$.

SOLUTION. This equation may be written in the following forms:

$$x^2-4x+22=0,$$

$$(x-2)^2+18=0,$$

$$(x-2+i\sqrt{18})(x-2-i\sqrt{18})=0;$$

hence its roots are $2-i\sqrt{18}$ and $2+i\sqrt{18}$, *i.e.*, $2-3\sqrt{-2}$ and $2+3\sqrt{-2}$.

EXERCISES

3. By actual substitution verify the correctness of the roots found in Exs. 1 and 2 on page 251.

4. What must be added to $x^2 - 8x$ in order that the sum shall be the square of a binomial?

5. Write $x^2 - 8x + 25$ as the sum of two squares.

Solve the following equations and verify the correctness of your results:

6. $x^2 + 25 = 8x$.

7. $x^2 + x + 1 = 0$.

8. $x^2 - x + 1 = 0$.

9. $4x^2 + 9 = 0$.

10. $3x^2 - 5x + 21 = 0$.

11. $x^4 + a^2x^2 + a^4 = 0$.

12. Write an equation whose roots are 1, i, and $-i$ (see § 72, note).

13. Write an equation whose roots are 1, $-\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, and $-\frac{1}{2} - \frac{1}{2}i\sqrt{3}$.

14. If $s = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, show by substitution that $s^2 + s + 1 = 0$. What other root has this equation?

III. FRACTIONAL EXPONENTS

153. Fractional exponents.* In § 137 it is shown that the exponent of the radicand and the index of the root may both be multiplied by any integer, or both be divided by any factor which they may have in common, without changing the value of the expression. This property at once suggests that these numbers may bear to each other relations similar to those of the numerator and denominator of a fraction.

For this and other reasons, some of which will presently appear, it is customary to employ, when it is desired to indicate that roots are to be extracted, not only the radical sign, the use of which has already been explained, but also what is known as a **fractional exponent**. This new symbol may perhaps be best defined by the identity

$$A^{\frac{p}{r}} \equiv (\sqrt[r]{A})^p,$$

i.e., the symbol $A^{\frac{p}{r}}$ means the pth power of the rth root of A, and r must therefore necessarily represent a *positive* integer, while p may be positive or negative.

E.g., $9^{\frac{5}{2}} = (\sqrt{9})^5 = 3^5 = 243$, and $8^{\frac{-4}{3}} = (\sqrt[3]{8})^{-4} = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$.

---

* For a similar treatment of fractional exponents see Tannery's Arithmétique.

The expression $A^{\frac{p}{r}}$, whatever the value of A, is usually spoken of as a **fractional power** of A, just as A^5 is called a positive integral power, and A^{-2} a negative integral power.

In the next few articles it is shown how to *use* this new symbol in the various algebraic operations; these uses will further justify its adoption.

For the sake of simplicity, here, as in §§ 133–145, only the principal roots (§ 132) are considered, and for these roots it has already been shown that $(\sqrt[r]{A})^p = \sqrt[r]{A^p}$ [§ 134, Eq. (1)]; hence, in the following proofs, either $(\sqrt[r]{A})^p$ or $\sqrt[r]{A^p}$ may be used for $A^{\frac{p}{r}}$.

NOTE. Although $\frac{p}{r}$, in the expression $A^{\frac{p}{r}}$, is called a fractional exponent, and is written in the *form* of a fraction, and although it will presently appear that such exponents may often be dealt with as though they were really fractions, yet it must be carefully remembered that they are not fractions at all; *this fractional-exponent notation is merely another way of indicating that roots are to be extracted.*

EXERCISES

1. What is meant by the symbol $\frac{m}{n}$? Has it the same meaning when used as an exponent?

2. Is the exponent $\frac{m}{n}$, in the symbol $x^{\frac{m}{n}}$, really a fraction? What is the precise meaning of $x^{\frac{m}{n}}$?

3. Is it correct to say that the symbol $x^{\frac{m}{n}}$ is merely a convenient way of indicating the mth power of the nth root of x? Is this the same as the nth root of the mth power of x, when only the *principal* roots are under consideration?

Express each of the following radicals by means of the fractional-exponent notation:

4. $\sqrt[3]{a^2}$. **6.** $\sqrt[3]{a^2x}$. **8.** $\sqrt[12]{(a+2x)^7}$. **10.** $\sqrt[10]{a^{-3}b^7}$.

5. $(\sqrt[3]{m})^3$. **7.** $\sqrt[4]{2^3a^2}$. **9.** $3b^2\sqrt[5]{2^8x^4}$. **11.** $\sqrt[n]{2a^p(x+3y)^q}$.

Find the numerical value of each of the following expressions, and explain your work:

12. $4^{\frac{1}{2}}$. **14.** $25^{-\frac{3}{2}}$*. **16.** $3 \cdot 32^{-\frac{2}{5}}$. **18.** $\left(\frac{27}{8}x^0\right)^{\frac{2}{3}}$.

13. $9^{\frac{3}{2}}$. **15.** $4 \cdot 4^{-\frac{3}{2}} \cdot 9^{\frac{1}{2}}$. **17.** $(.09)^{-\frac{3}{2}}$. **19.** $169^{\frac{1}{2}} \cdot \left(\frac{8}{27}\right)^{-\frac{2}{3}} \cdot 16^{\frac{3}{2}}$.

---

* First write $\frac{-3}{2}$ for $-\frac{3}{2}$.

Translate the following into equivalent radical expressions:

20. $a^{\frac{m}{n}}$.

21. $a^{\frac{1}{3}} + b^{\frac{3}{8}}$.

22. $5 \cdot \left(\frac{2}{3}\right)^{\frac{1}{2}} + 2(ax)^{\frac{4}{7}}$.

23. $-2a^{\frac{2}{3}}x^{-\frac{5}{3}}$.

24. $\dfrac{3x^{\frac{2}{5}} - 7a^{\frac{1}{4}}b^{\frac{2}{3}}}{a^{-\frac{1}{2}} + bx^{\frac{1}{3}}}$.

25. Of the following expressions, which are integral and which are fractional powers (see § 153)? Which are positive and which negative powers? Give the reason for your answer in each case.

$$4^{\frac{1}{2}},\ \left(\frac{5}{8}\right)^2,\ \left(\frac{4}{15}\right)^{-\frac{3}{4}},\ \left(-\frac{a}{b}\right)^5,\ \frac{2^5}{8^{\frac{1}{3}}},\ \text{and}\ -\left(\frac{1}{32}\right)^{-\frac{3}{5}}.$$

154. Fractional exponents changed to lower and higher terms. Under the above definition of a fractional exponent it is easily verified that

$$16^{\frac{1}{2}} = 16^{\frac{2}{4}}, \qquad \text{[Each member being 4}$$

and that

$$9^{\frac{6}{4}} = 9^{\frac{3}{2}}. \qquad \text{[Each member being 27}$$

So, too, in general, *if A is any number whatever,* and if* $\frac{p}{r}$ *is any simple fraction in which r is positive, then*

$$A^{\frac{p}{r}} = A^{\frac{pm}{rm}}\dagger,$$

wherein m is any positive integer whatever.

The proof of this statement follows directly from the definition of a fractional exponent and from § 137, for

$$A^{\frac{p}{r}} = \sqrt[r]{A^p} \qquad \text{[§ 153}$$

$$= \sqrt[rm]{A^{pm}} \qquad \text{[§ 137}$$

$$= A^{\frac{pm}{rm}}, \qquad \text{[§ 153}$$

i.e., $A^{\frac{p}{r}} = A^{\frac{pm}{rm}}$, which was to be proved.

---

* If r is even A must be positive, since imaginaries are excluded from this discussion (cf. also footnote, p. 229).

† Observe that this equality can not be affirmed merely because $\frac{p}{r} = \frac{pm}{rm}$, considered as fractions.

Note. Observe that the proof of § 154 applies to real numbers only; if imaginary numbers present themselves, here or elsewhere, they must be dealt with in accordance with the principles given in §§ 146–152.

Ex. 1. By means of fractional exponents reduce $\sqrt[3]{a^2}$ and $\sqrt{x^5}$ to equivalent radicals of the same order.

Solution. The given radicals are respectively equivalent to $a^{\frac{2}{3}}$ and $x^{\frac{5}{2}}$, and these expressions are respectively equivalent to $a^{\frac{4}{6}}$ and $x^{\frac{15}{6}}$, *i.e.*, to $\sqrt[6]{a^4}$ and $\sqrt[6]{x^{15}}$, each of which is of order 6.

EXERCISES

2. Can $a^{\frac{2}{3}}$ and $x^{\frac{5}{2}}$ be reduced to equivalent expressions whose common order is any multiple whatever of 2 and 3? How?

3. State in detail how the principle proved in § 154 may be employed to reduce any two or more given radicals (real numbers) to equivalent radicals of a common order.

4. Solve Exs. 1–8 of § 138 by means of fractional exponents.

155. Product of fractional powers of any number. *If A is any number whatever* (cf. footnote, p. 254), *and if* $\frac{p}{r}$ *and* $\frac{p'}{r'}$ *are any two simple fractions in which r and r' are positive, then*

$$A^{\frac{p}{r}} \cdot A^{\frac{p'}{r'}} = A^{\frac{pr'+p'r}{rr'}}.$$

For, since $$A^{\frac{p}{r}} = A^{\frac{pr'}{rr'}} = \sqrt[rr']{A^{pr'}}, \qquad [\S\S\ 154 \text{ and } 153$$

and since $$A^{\frac{p'}{r'}} = A^{\frac{p'r}{r'r}} = \sqrt[rr']{A^{p'r}}, \qquad [\S\S\ 154 \text{ and } 153$$

therefore $$A^{\frac{p}{r}} \cdot A^{\frac{p'}{r'}} = \sqrt[rr']{A^{pr'}} \cdot \sqrt[rr']{A^{p'r}} = \sqrt[rr']{A^{pr'+p'r}} \qquad [\S\ 133$$

$$= A^{\frac{pr'+p'r}{rr'}}, \qquad [\S\ 153$$

which was to be proved.

Since $\frac{pr'+p'r}{rr'} = \frac{p}{r} + \frac{p'}{r'}$, we may write, instead of $A^{\frac{pr'+p'r}{rr'}}$, the simpler form $A^{\frac{p}{r}+\frac{p'}{r'}}$, if we are careful to remember that the symbol $A^{\frac{p}{r}+\frac{p'}{r'}}$ is to be interpreted by first adding the exponents as though they really were fractions. With this understanding the principle which has just been proved becomes

$$A^{\frac{p}{r}} \cdot A^{\frac{p'}{r'}} = A^{\frac{p}{r}+\frac{p'}{r'}}.$$

Similarly, $$A^{\frac{p}{r}} \cdot A^{\frac{p'}{r'}} \cdot A^{\frac{p''}{r''}} = A^{\frac{p}{r}+\frac{p'}{r'}} \cdot A^{\frac{p''}{r''}} = A^{\frac{p}{r}+\frac{p'}{r'}+\frac{p''}{r''}},$$

and so on for any number of factors; hence, under the above definitions, *fractional exponents conform to the exponent law*

$$A^m \cdot A^n \cdot A^p \cdots = A^{m+n+p+\cdots}$$

already demonstrated when m, n, p, $\cdots$ *are integers.*

EXERCISES

1. What is the numerical value of $16^{\frac{3}{2}} \cdot 16^{\frac{3}{4}} \cdot 16^{\frac{1}{2}}$? Of $16^{\frac{3}{2}+\frac{3}{4}+\frac{1}{2}}$? Is then $16^{\frac{3}{2}} \cdot 16^{\frac{3}{4}} \cdot 16^{\frac{1}{2}}$ equal to $16^{\frac{3}{2}+\frac{3}{4}+\frac{1}{2}}$?

2. Do the fractional exponents in Ex. 1 conform to the same law as if they were positive integers? State that law.

Without extracting any irrational roots, reduce the following expressions to their simplest forms:

3. $8^{\frac{2}{3}} \cdot 8^{\frac{5}{6}} \cdot 8^{\frac{1}{2}}$.

4. $8^{\frac{2}{3}} \cdot 8^{\frac{5}{6}} \cdot 8^{-\frac{1}{2}}$.

5. $24^{\frac{5}{2}} \cdot 24^{\frac{3}{8}} \cdot 24^{-\frac{3}{4}}$.

6. $5^{\frac{1}{2}} \cdot 5^{-\frac{1}{4}} \cdot 5^{\frac{2}{3}} \cdot 5^{\frac{1}{6}} \cdot 5^{-\frac{1}{12}}$.

7. $a^{\frac{2}{5}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{3}{4}}$.

8. $2\,x^{\frac{1}{2}} \cdot 3\,x^{\frac{2}{3}} \cdot \frac{5}{12}\,x^{\frac{5}{6}}$.

9. $a^{\frac{1}{2}}b^{\frac{1}{3}}x^{\frac{2}{5}} \cdot b^{\frac{1}{2}}x^{\frac{3}{10}} \cdot a^{\frac{2}{3}}b^{\frac{1}{6}}$.

10. $a^{\frac{3}{m}}x^{\frac{2}{n}}y^{\frac{4}{r}} \cdot x^{\frac{n}{m}}y^{\frac{m}{r}} \cdot a^{\frac{2}{r}}y^{\frac{4}{m}}$.

11. Show that every step in the proof of the above principle (§ 155) remains valid even if p' should be negative (cf. Ex. 4); and also if $p = r$, or if there is any other relation among p, r, p', and r'.

156. Quotient of fractional powers of any number. From the definition of a fractional exponent (§ 153) it follows directly that

$$64^{\frac{2}{3}} \div 64^{\frac{1}{2}} = 64^{\frac{1}{6}},\ \textit{i.e.},\ 64^{\frac{2}{3}-\frac{1}{2}}, \qquad \text{[Each being 2}$$

and that $$64^{\frac{1}{2}} \div 64^{\frac{2}{3}} = 64^{-\frac{1}{6}},\ \textit{i.e.},\ 64^{\frac{1}{2}-\frac{2}{3}}. \qquad \text{[Each being } \tfrac{1}{2}$$

So, too, in general, *if A is any number whatever* (cf. footnote, p. 254), *and if* $\frac{p}{r}$ *and* $\frac{p'}{r'}$ *are any two simple fractions in which* r *and* r' *are positive, then*

$$A^{\frac{p}{r}} \div A^{\frac{p'}{r'}} = A^{\frac{p}{r}-\frac{p'}{r'}}. \qquad \text{[Where } A^{\frac{p}{r}-\frac{p'}{r'}} \equiv A^{\frac{pr'-pr'}{rr'}}$$

For, since $$A^{\frac{p}{r}} = A^{\frac{pr'}{rr'}} = \sqrt[rr']{A^{pr'}}, \qquad \text{[§§ 154 and 153}$$

and since $$A^{\frac{p'}{r'}} = A^{\frac{p'r}{rr'}} = \sqrt[rr']{A^{p'r}}, \qquad \text{[§§ 154 and 153}$$

therefore $$A^{\frac{p}{r}} \div A^{\frac{p'}{r'}} = \sqrt[rr']{A^{pr'}} \div \sqrt[rr']{A^{p'r}} = \sqrt[rr']{A^{pr'-p'r}} \qquad \text{[§ 135}$$

$$= A^{\frac{pr'-p'r}{rr'}}. \qquad \text{[§ 153}$$

i.e., $$A^{\frac{p}{r}} \div A^{\frac{p'}{r'}} = A^{\frac{p}{r}-\frac{p'}{r'}},$$

which was to be proved. This proof shows that, under the above definitions, *fractional exponents also conform to the law*

$$A^m \div A^n = A^{m-n}$$

already demonstrated when m *and* n *are integers.*

EXERCISES

1. What is the numerical value of $16^{\frac{3}{4}} \div 16^{\frac{1}{2}}$? Of $16^{\frac{3}{4}-\frac{1}{2}}$? Is, then, $16^{\frac{3}{4}} \div 16^{\frac{1}{2}}$ equal to $16^{\frac{3}{4}-\frac{1}{2}}$?

2. Do the fractional exponents in Ex. 1 conform to the same law as though they were positive integers? State that law.

Simplify the following expressions:

3. $8^{\frac{4}{3}} \div 8^{\frac{5}{6}}$.

4. $8^{\frac{2}{3}} \cdot 8^{\frac{5}{6}} \div 8^{\frac{1}{2}}$.

5. $64^{\frac{5}{6}} \div 64^{\frac{2}{3}} \cdot 64^{\frac{1}{6}}$.

6. $12^{\frac{3}{4}} \cdot 12^{\frac{1}{2}} \div 12$.

7. $2\,x^{\frac{3}{8}} \div 4\,x^{\frac{1}{4}}$.

8. $x^{\frac{1}{2}} \div 3\,a^{\frac{2}{3}}x^{\frac{1}{4}}$.

9. Show that every step of the proof of the above principle (§ 156) remains valid even if $p = 0$, and thus prove that $1 \div a^{\frac{m}{n}} = a^{-\frac{m}{n}}$. Compare this result with § 44.

10. By means of Ex. 9, show that a *factor* may be transferred from numerator to denominator, or *vice versa*, by merely reversing the sign of its exponent, *even when the exponent is fractional* (cf. Exs. 22–26, § 93).

157. Product of like powers of different numbers. From § 153 it follows directly that

$$8^{\frac{2}{3}} \cdot 27^{\frac{2}{3}} = (8 \cdot 27)^{\frac{2}{3}},\ \textit{i.e.,}\ 216^{\frac{2}{3}}. \qquad \text{[Each being 36}$$

So, too, in general, *if A and B are any two numbers whatever* (cf. footnote, p. 254), *and if $\frac{p}{r}$ is any simple fraction in which r is positive, then*

$$A^{\frac{p}{r}} \cdot B^{\frac{p}{r}} = (AB)^{\frac{p}{r}}.$$

For, since $A^{\frac{p}{r}} = \sqrt[r]{A^p}$ and $B^{\frac{p}{r}} = \sqrt[r]{B^p}$, [§ 153

therefore

$$A^{\frac{p}{r}} \cdot B^{\frac{p}{r}} = \sqrt[r]{A^p} \cdot \sqrt[r]{B^p} = \sqrt[r]{A^p \cdot B^p} \qquad \text{[§ 133}$$

$$= \sqrt[r]{(AB)^p} \qquad \text{[§ 121 (iii)}$$

$$= (AB)^{\frac{p}{r}}; \qquad \text{[§ 153}$$

i.e.,

$$A^{\frac{p}{r}} \cdot B^{\frac{p}{r}} = (AB)^{\frac{p}{r}},$$

which was to be proved. This proof shows that, under the above definitions, *fractional exponents also conform to the law*

$$A^n \cdot B^n = (AB)^n$$

already demonstrated when n is an integer.

Moreover, by successive applications of the above proof it follows that

$$A^{\frac{p}{r}} \cdot B^{\frac{p}{r}} \cdot C^{\frac{p}{r}} \cdots = (ABC \cdots)^{\frac{p}{r}},$$

for any number of factors whatever.

EXERCISES

1. What is the numerical value of $16^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$? Of $144^{\frac{1}{2}}$, *i.e.*, of $(16 \cdot 9)^{\frac{1}{2}}$? Is, then, $16^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}$ equal to $(16 \cdot 9)^{\frac{1}{2}}$?

2. Does the fractional exponent in Ex. 1 conform to the same law as though it were a positive integer? State that law.

3. Does the law asked for in Ex. 2 apply to products of three or more factors as well as to products of only two factors? Verify it for the product $8^{\frac{2}{3}} \cdot 125^{\frac{2}{3}} \cdot .064^{\frac{2}{3}}$; and *prove* it for $a^{\frac{m}{n}}s^{\frac{m}{n}}x^{\frac{m}{n}}$.

158. A power of a power of a number. From § 153 it follows directly that

$$(64^{\frac{1}{2}})^{\frac{2}{3}} = 64^{\frac{1}{3}}, \textit{ i.e., } 64^{\frac{1}{2} \cdot \frac{2}{3}}. \qquad \text{[Each being 4}$$

So, too, in general, *if A is any number whatever* (cf. footnote, p. 254), *and if $\frac{p}{r}$ and $\frac{p'}{r'}$ are any two simple fractions in which r and r' are positive, then*

$$\left(A^{\frac{p}{r}}\right)^{\frac{p'}{r'}} = A^{\frac{p}{r}\cdot\frac{p'}{r'}}.$$

For, since $$A^{\frac{p}{r}} = \sqrt[r]{A^p}, \qquad [\S\ 153$$

therefore $$\left(A^{\frac{p}{r}}\right)^{\frac{p'}{r'}} = \sqrt[r']{\left(\sqrt[r]{A^p}\right)^{p'}} = \sqrt[r']{\sqrt[r]{A^{pp'}}} \qquad [\S\ 134$$

$$= \sqrt[rr']{A^{pp'}} \qquad [\S\ 136$$

$$= A^{\frac{pp'}{rr'}}, \qquad [\S\ 153$$

i.e., $$\left(A^{\frac{p}{r}}\right)^{\frac{p'}{r'}} = A^{\frac{p}{r}\cdot\frac{p'}{r'}},$$

which was to be proved. This proof shows that, under the above definitions, *fractional exponents also conform to the law*

$$(A^m)^n = A^{mn}$$

already demonstrated when m and n are integers.

EXERCISES

1. What is the numerical value of $(729^{\frac{1}{3}})^{\frac{1}{2}}$? Of $729^{\frac{1}{6}}$? Is, then, $(729^{\frac{1}{3}})^{\frac{1}{2}}$ equal to $729^{\frac{1}{3}\cdot\frac{1}{2}}$? Is it also equal to $(729^{\frac{1}{2}})^{\frac{1}{3}}$?

2. Do the fractional exponents in Ex. 1 conform to the same law as though they were positive integers? State that law.

3. *Read* the equation $\left(x^{\frac{r}{s}}\right)^{\frac{p}{q}} = x^{\frac{rp}{sq}}$; state what the several indicated operations are; mention the order in which they are to be performed; and *prove* the correctness of the equation.

159. Summary of exponent laws. As originally used, the symbol A^n was merely an abbreviation for the product $A \cdot A \cdot A \cdots$ to n factors [cf. § 7 (iv) and also § 37], and n was therefore necessarily a positive integer. Later on (§ 44) it was found desirable slightly to extend the meaning of an exponent, and it was agreed that A^0 should mean 1, and that A^{-k}, where k is a positive integer, should mean $\frac{1}{A^k}$. Under these interpretations, it was then proved

(§ 121) that when m and n represent any positive or negative integers whatever, including zero, then

$$A^m \cdot A^n = A^{m+n}, \tag{1}$$

$$(A^m)^n = A^{mn}, \tag{2}$$

$$A^n \cdot B^n = (AB)^n, \tag{3}$$

and

$$A^m \div A^n = A^{m-n}. \tag{4}$$

These formulas state the so-called "exponent laws." It has *now* been shown (§§ 155–158) that, under the definition given in § 153, *these exponent laws remain valid even when some or all of the exponents are simple fractions* (cf. § 154, note).

EXERCISES

1. Translate the first exponent law into a rule for multiplying together two different powers of any given number.

Find the following products and explain each; does the rule given in Ex. 1 apply in finding these products?

2. $5^3 \cdot 5^4$. **4.** $(\frac{1}{3})^4 \cdot (\frac{2}{3})^2$. **6.** $8^{\frac{2}{3}} \cdot 8^{\frac{3}{4}}$. **8.** $26^{-\frac{3}{4}} \cdot 26^{-\frac{2}{3}}$.

3. $12^6 \cdot 12^{-4}$. **5.** $6^4 \cdot 6^0$. **7.** $14^{\frac{5}{2}} \cdot 14^{-\frac{1}{3}}$. **9.** $.04^{\frac{3}{2}} \cdot .04^{-\frac{1}{6}}$.

10. State in detail the precise meaning that we have agreed to give to each of the different kinds of exponents used in Exs. 2–9, *i.e.*, the meaning of 5^3, 12^{-4}, 6^0, $14^{\frac{5}{2}}$, and $26^{-\frac{2}{3}}$.

11. State briefly the important steps by which law (1) was established when m and n are positive integers; when one or both are negative integers; and when they are simple fractions.

12. Prove that law (1) applies also to products of three or more powers of any given number, — *e.g.*, that $x^m \cdot x^n \cdot x^r = x^{m+n+r}$, where m, n, and r may be integers, fractions, or zeros.

13. Translate law (2) into a rule and employ it to simplify $(8^{\frac{2}{3}})^{\frac{5}{2}}$.

14. Make up 3 examples to illustrate the application of law (2) with the various kinds of exponents (cf. Exs. 2–9 above).

15. Is $[(x^m)^n]^r$ equal to x^{mnr}? Why? May m, n, and r be fractions as well as integers here? May one or more of them be negative? Zero?

16. Show that $(a^{-2})^{-3} = \left(\frac{1}{a^2}\right)^{-3} = \frac{1}{\left(\frac{1}{a^2}\right)^3} = \frac{1}{\frac{1}{a^6}} = a^6$. Is this the same as $a^{(-2)\cdot(-3)}$? [Cf. § 121 (ii)].

17. As in the first part of Ex. 16 show that $(m^{-\frac{3}{5}})^{-\frac{10}{3}} = m^2$.

18. Translate law (3) into a rule, and state what limitations, if any, are placed upon the value of n.

19. Prove that law (3) applies also to the product of three or more like powers, — *i.e.*, that $a^m \cdot b^m \cdot c^m \cdot d^m \cdots = (abcd \cdots)^m$, wherein m may be positive or negative, integral or fractional, or zero.

20. Make up 4 examples to illustrate the application of law (3) with the various kinds of exponents.

21. Translate law (4) into a rule, and illustrate its application.

22. What is the product of A^{m-n} by A^n? What, then, is the quotient of A^m divided by A^n [cf. definition of division, § 3 (iv)]?

23. By means of the suggestion contained in Ex. 22, prove law (4) from law (1′) and the definition of division, — independent of § 156.

160. Operations with polynomials involving fractional exponents. Since the operations with polynomials are merely combinations of the corresponding operations with monomials, therefore the principles already demonstrated (§§ 155–159) for monomials suffice for operations with polynomials also.

Moreover, since fractional exponents obey the familiar laws formerly established for integral exponents, and since any radical expression may be written in the fractional-exponent notation, therefore operations with radicals (real numbers) are usually greatly simplified by using fractional exponents;* this is illustrated below.

Ex. 1. Find the product of $3\sqrt{a} - 5\sqrt[3]{y}$ by $2\sqrt{a} + \sqrt[3]{y}$.

Solution. Since $3\sqrt{a} - 5\sqrt[3]{y} = 3a^{\frac{1}{2}} - 5y^{\frac{1}{3}}$, and $2\sqrt{a} + \sqrt[3]{y} = 2a^{\frac{1}{2}} + y^{\frac{1}{3}}$, therefore this product becomes

$$\begin{array}{r}
3a^{\frac{1}{2}} - 5y^{\frac{1}{3}} \qquad\qquad\qquad \\
2a^{\frac{1}{2}} + y^{\frac{1}{3}} \qquad\qquad\qquad \\
\hline
6a^{\frac{1}{2}+\frac{1}{2}} - 10a^{\frac{1}{2}}y^{\frac{1}{3}} \qquad\qquad\qquad \\
+ 3a^{\frac{1}{2}}y^{\frac{1}{3}} - 5y^{\frac{1}{3}+\frac{1}{3}} \\
\hline
6a - 7a^{\frac{1}{2}}y^{\frac{1}{3}} - 5y^{\frac{2}{3}}.
\end{array}$$

If it is desired, this product may, of course, be written in either of the following forms: $6a - 7\sqrt{a}\sqrt[3]{y} - 5\sqrt[3]{y^2}$ or $6a - 7\sqrt[6]{a^3y^2} - 5\sqrt[3]{y^2}$.

---

* Although the radical notation and the fractional-exponent notation are each equivalent to the other, and either may therefore replace the other, yet each is frequently met with, and it is desirable that the student should understand how to operate with each form without first converting it into the other.

Ex. 2. Divide $x^2 - y^3$ by $\sqrt[3]{x} + \sqrt{y}$.

Solution. Since $\sqrt[3]{x} + \sqrt{y} = x^{\frac{1}{3}} + y^{\frac{1}{2}}$, this solution may be put into the following form:

$$\begin{array}{l|l}
x^2 - y^3 & \underline{x^{\frac{1}{3}} + y^{\frac{1}{2}}} \\
\underline{x^2 + x^{\frac{5}{3}}y^{\frac{1}{2}}} & \underline{x^{\frac{5}{3}} - x^{\frac{4}{3}}y^{\frac{1}{2}} + xy - x^{\frac{2}{3}}y^{\frac{3}{2}} + x^{\frac{1}{3}}y^2 - y^{\frac{5}{2}}} \\
-x^{\frac{5}{3}}y^{\frac{1}{2}} - y^3 & \\
\underline{-x^{\frac{5}{3}}y^{\frac{1}{2}} - x^{\frac{4}{3}}y} & \\
\qquad x^{\frac{4}{3}}y - y^3 & \\
\qquad \underline{x^{\frac{4}{3}}y + xy^{\frac{3}{2}}} & \\
\qquad\quad -xy^{\frac{3}{2}} - y^3 & \\
\qquad\quad \underline{-xy^{\frac{3}{2}} - x^{\frac{2}{3}}y^2} & \\
\qquad\qquad\quad x^{\frac{2}{3}}y^2 - y^3 & \\
\qquad\qquad\quad \underline{x^{\frac{2}{3}}y^2 + x^{\frac{1}{3}}y^{\frac{5}{2}}} & \\
\qquad\qquad\qquad -x^{\frac{1}{3}}y^{\frac{5}{2}} - y^3 & \\
\qquad\qquad\qquad \underline{-x^{\frac{1}{3}}y^{\frac{5}{2}} - y^3} & \\
\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}$$

The above quotient may also be written thus:

$$\sqrt[3]{x^5} - \sqrt[3]{x^4}\,\sqrt{y} + xy - \sqrt[3]{x^2}\,\sqrt{y^3} + \sqrt[3]{x} \cdot y^2 - \sqrt{y^5}.$$

Note. To appreciate one of the advantages of fractional exponents the student has only to perform the division in Ex. 2, using the radical notation, and compare his work with the above solution.

Ex. 3. Extract the square root of $\sqrt[5]{x^4} - 2\sqrt[5]{x^3} + 5\sqrt[5]{x^2} - 4\sqrt[5]{x} + 4$.

Solution. This expression written in the equivalent fractional-exponent form is $x^{\frac{4}{5}} - 2x^{\frac{3}{5}} + 5x^{\frac{2}{5}} - 4x^{\frac{1}{5}} + 4$, and in this form its square root may be extracted just as though it were a rational expression (cf. § 125); thus:

$$\begin{array}{l|l|l}
 & x^{\frac{4}{5}} - 2x^{\frac{3}{5}} + 5x^{\frac{2}{5}} - 4x^{\frac{1}{5}} + 4 & \underline{x^{\frac{2}{5}} - x^{\frac{1}{5}} + 2} \\
 & \underline{x^{\frac{4}{5}} \qquad\qquad\qquad\qquad\qquad\qquad} & \\
2x^{\frac{2}{5}} - x^{\frac{1}{5}} & -2x^{\frac{3}{5}} + 5x^{\frac{2}{5}} & \\
 & \underline{-2x^{\frac{3}{5}} + x^{\frac{2}{5}}} & \\
2x^{\frac{2}{5}} - 2x^{\frac{1}{5}} + 2 & \qquad 4x^{\frac{2}{5}} - 4x^{\frac{1}{5}} + 4 & \\
 & \qquad \underline{4x^{\frac{2}{5}} - 4x^{\frac{1}{5}} + 4} & \\
 & \qquad\qquad\qquad\qquad 0 &
\end{array}$$

hence the square root of the given expression is $x^{\frac{2}{5}} - x^{\frac{1}{5}} + 2$, *i.e.*, $\sqrt[5]{x^2} - \sqrt{x} + 2$.

EXERCISES

Perform the following multiplications:

4. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ (cf. § 58).

5. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

6. $m^{\frac{2}{5}} - m^{\frac{1}{5}}n^{\frac{1}{5}} + n^{\frac{2}{5}}$ by $m^{\frac{1}{5}} + n^{\frac{1}{5}}$.

7. $m^{\frac{2}{5}} - m^{\frac{1}{5}}y^{-\frac{1}{5}} + n^{-\frac{2}{5}}$ by $m^{\frac{1}{5}} + n^{-\frac{1}{5}}$.

8. $\frac{1}{8}x^{\frac{3}{2}} - \frac{1}{12}xy^{\frac{1}{2}} + \frac{1}{18}x^{\frac{1}{2}}y - \frac{1}{27}y^{\frac{3}{2}}$ by $\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{3}y^{\frac{1}{2}}$.

9. $81\sqrt[5]{x^4} - 27\sqrt[5]{x^3}\sqrt[3]{y} + 9\sqrt[5]{x^2}\sqrt[3]{y^2} - 3y\sqrt[5]{x} + y\sqrt[3]{y}$ by $3\sqrt[5]{x} + \sqrt[3]{y}$.

10. $\sqrt{a} - 4\sqrt[8]{a^3x} + 6\sqrt[4]{ax} - 4\sqrt[8]{ax^3} + \sqrt{x}$ by $\sqrt[4]{a} - 2\sqrt[8]{ax} + \sqrt[4]{x}$.

11. $\sqrt[3]{x^2} + 2\sqrt{y^3} - \sqrt[3]{z^2} - \sqrt[3]{x}\sqrt{y} + 2\sqrt{x}\sqrt[3]{z} - \sqrt[3]{y}\sqrt{z}$
by $\sqrt{x} - 2\sqrt[8]{y} + \sqrt{z}$.

12. $m^{\frac{2}{3}} + m^{-\frac{2}{3}} - 2m^{\frac{1}{3}} + 4m^{-\frac{1}{3}}$ by $1 + 2m^{\frac{1}{3}} - \frac{1}{\sqrt[3]{m}}$.

13. $p^{-\frac{3}{2}} + q^{-1.6} - p^{-.75}q^{-\frac{4}{5}}$ by $p^{-.75} + q^{-\frac{4}{5}}$.

14. $1\frac{4}{5}n^{\frac{1}{2}}x\sqrt[5]{x} + 2n\sqrt{n} + \frac{2}{5}x^{1.8} + 6n\sqrt[5]{x^3}$ by $\sqrt{n} - 3x^{\frac{3}{5}} + \frac{5}{2a}x^{\frac{4}{5}}$.

15. $5a^{-3}x^{\frac{2}{3}} + 3a^{.2}b^{n}x^{-1} - b^{2-n}x^{\frac{1}{2}}$ by $x^{-p} - 3b^{n-\frac{1}{2}}x^{.5} + a^{\frac{7}{2}}$.

Perform the following divisions:

16. $a + x^2$ by $a^{\frac{1}{3}} + x^{\frac{2}{3}}$.

17. $m^{\frac{4}{5}} - n^{\frac{2}{3}}$ by $m^{\frac{1}{5}} - n^{\frac{1}{6}}$.

18. $x^{-1} + 3y^{-\frac{1}{2}} - 10xy^{-1}$ by $x^{-1}\sqrt{y} - 2$.

19. $a^{\frac{2}{5}} + 2\sqrt[5]{a}\,b^{-\frac{1}{2}} + \frac{1}{b}$ by $\sqrt[5]{a} + b^{-\frac{1}{2}}$.

20. $x^{\frac{5}{2}} + x^2\sqrt[3]{y} - x\sqrt{x}\,y^{\frac{2}{3}} - xy + \sqrt{x}\,y^{\frac{4}{3}} + y^{1\frac{2}{3}}$ by $\sqrt{x} + \sqrt[3]{y}$.

Simplify the following expressions:

21. $\left(\frac{\sqrt{x} + \sqrt[3]{y}}{\sqrt{x} - \sqrt[3]{y}}\right)^2 \cdot \frac{\sqrt[3]{x} - \sqrt{y}}{\sqrt[3]{x} + \sqrt{y}}$.

22. $\frac{x^m + y^n}{x^{-m} + y^{-n}} \cdot \frac{x^n - y^m}{x^{-n} - y^{-m}}$.

23. $\frac{a}{\sqrt[3]{a} - 1} - \frac{a^{\frac{2}{3}}}{\sqrt[3]{a} + 1} - \frac{1}{a^{\frac{1}{3}} - 1} + \frac{1}{a^{\frac{1}{3}} + 1}$.

24. $\frac{\sqrt{y}}{y + \sqrt{y} + 1} \div \frac{1}{y^{\frac{3}{2}} - 1}$.

25. $\frac{x - y}{\sqrt{x} - \sqrt{y}} - \frac{\sqrt{x^3} - y^{\frac{3}{2}}}{x - y}$.

Extract the square root of each of the following expressions:

26. $x^2 + 2x^{\frac{3}{2}} + 3x + 4x^{\frac{1}{2}} + 3 + 2x^{-\frac{1}{2}} + x^{-1}$.*

27. $a^{\frac{2}{3}} - 4a^{\frac{5}{6}} + 4a + 2a^{\frac{7}{6}} - 4a^{\frac{4}{3}} + a^{\frac{5}{3}}$.

28. $n^{\frac{8}{5}} - 2m^{-\frac{3}{5}}n^{\frac{11}{5}} + 2m^{\frac{4}{5}}n^{\frac{4}{5}} + m^{-\frac{6}{5}}n^{\frac{14}{5}} - 2m^{\frac{1}{5}}n^{\frac{7}{5}} + m^{\frac{8}{5}}$.

Write down, by inspection if possible, the square root of each of the following expressions:

29. $1 - 2u^{\frac{1}{3}} + u^{\frac{2}{3}}$.

30. $x^{\frac{4}{5}} + 4x^{\frac{2}{5}} + 4$.

31. $p^{\frac{1}{2}} - 4 + 4p^{-\frac{1}{2}}$.

32. $ax^{\frac{2}{3}} + 2a^{\frac{5}{6}}x^{\frac{5}{6}} + a^{\frac{2}{3}}x$.

33. $m + n + p - 2m^{\frac{1}{2}}n^{\frac{1}{2}} + 2n^{\frac{1}{2}}p^{\frac{1}{2}} - 2m^{\frac{1}{2}}p^{\frac{1}{2}}$.

Extract the cube root of each of the following expressions; write the results first with all the exponents positive, and then replace all fractional-exponent forms by radical signs:

34. $8 + 12x^{\frac{2}{3}} + 6x^{\frac{4}{3}} + x^2$.

35. $8x^{-1} - 12x^{-\frac{2}{3}}y + 6x^{-\frac{1}{3}}y^2 - y^3$.

36. $t^{-\frac{3}{2}} - 6t^{-1} + 15t^{-\frac{1}{2}} - 20 + 15t^{\frac{1}{2}} - 6t + t^{\frac{3}{2}}$.

37. $8a^3b^{-\frac{3}{2}} + 9ab^{\frac{1}{2}} + 13a^{\frac{3}{2}} + 3a^{\frac{1}{2}}b + 18a^2b^{-\frac{1}{2}} + b^{\frac{3}{2}} + 12a^{\frac{5}{2}}b^{-1}$.

161. Rationalizing factors of binomial surds. Another advantage of the fractional-exponent notation is that it furnishes an easy method for finding a rationalizing factor of any binomial surd whatever, — only *quadratic* binomial surds have thus far been rationalized (§ 144).

To illustrate this method, let it be required to rationalize the binomial surd $x^{\frac{1}{3}} + y^{\frac{1}{2}}$.

Since $(x^{\frac{1}{3}})^n - (y^{\frac{1}{2}})^n$ is exactly divisible by $x^{\frac{1}{3}} + y^{\frac{1}{2}}$ whenever n is an even positive integer [§ 68 (ii)], therefore, if n be given such an even integral value as will make both $(x^{\frac{1}{3}})^n$ and $(y^{\frac{1}{2}})^n$ *rational,* — *e.g.*, 6, 12, 18, ⋯, — then the quotient of $(x^{\frac{1}{3}})^n - (y^{\frac{1}{2}})^n$ divided by $x^{\frac{1}{3}} + y^{\frac{1}{2}}$ will be a rationalizing factor of $x^{\frac{1}{3}} + y^{\frac{1}{2}}$, because the product of $x^{\frac{1}{3}} + y^{\frac{1}{2}}$ by this quotient will be $(x^{\frac{1}{3}})^n - (y^{\frac{1}{2}})^n$, which is rational for all such values of n.

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* Observe that this expression is arranged according to descending powers of x.

In the present case, 6 is the smallest admissible value of n, and the required rationalizing factor is

$$\frac{(x^{\frac{1}{3}})^6-(y^{\frac{1}{2}})^6}{x^{\frac{1}{3}}+y^{\frac{1}{2}}}=\frac{x^2-y^3}{x^{\frac{1}{3}}+y^{\frac{1}{2}}}=x^{\frac{5}{3}}-x^{\frac{4}{3}}y^{\frac{1}{2}}+xy-x^{\frac{2}{3}}y^{\frac{3}{2}}+x^{\frac{1}{3}}y^2-y^{\frac{5}{2}}.$$

Again, a rationalizing factor of $x^{\frac{1}{3}}+y^{\frac{1}{5}}$ is the quotient $[(x^{\frac{1}{3}})^{15}+(y^{\frac{1}{5}})^{15}]\div(x^{\frac{1}{3}}+y^{\frac{1}{5}})$, *i.e.*, $(x^5+y^3)\div(x^{\frac{1}{3}}+y^{\frac{1}{5}})$; and a rationalizing factor of $a^{\frac{2}{3}}-b^{\frac{3}{4}}$ is the quotient $[(a^{\frac{2}{3}})^{12}-(b^{\frac{3}{4}})^{12}]\div(a^{\frac{2}{3}}-b^{\frac{3}{4}})$, *i.e.*, $(a^8-b^9)\div(a^{\frac{2}{3}}-b^{\frac{3}{4}})$.

The student may now, from the above examples, formulate a rule for finding a rationalizing factor for *any* binomial surd; he should distinguish three cases, viz., (1) when the binomial is a *difference;* (2) when it is a *sum* and the L. C. M. of the denominators of its fractional exponents is *odd;* and (3) when it is a *sum* and this L. C. M. is *even.*

EXERCISES

Find the simplest rationalizing factor for each of the following expressions:

1. $a^{\frac{2}{3}}-x^{\frac{1}{4}}$. **2.** $m^{\frac{1}{2}}+n^{\frac{3}{8}}$. **3.** $2x^{\frac{3}{2}}-3y^{\frac{1}{3}}$. **4.** $a^{\frac{1}{3}}b^{\frac{2}{5}}+3v^2$. **5.** $x^{-\frac{1}{3}}+2y^{\frac{3}{4}}$.

CHAPTER XV

QUADRATIC EQUATIONS

I. EQUATIONS CONTAINING BUT ONE UNKNOWN NUMBER

162. Introductory remarks. It has already been shown that the first step in solving an algebraic problem is to translate its conditions into algebraic language, and also that this translation leads to equations which contain one or more unknown numbers; the values of these unknown numbers are then found by solving the equations (§ 26).

Although nearly all of the problems thus far met with are such that their conditions give rise to equations of the *first* degree in the letters representing the unknown numbers,[*] yet there are many other problems which lead to equations of the *second* degree in those letters; the solution of equations of this kind will be investigated in the present chapter.

NOTE. It may be recalled, however, that some easy equations of the second degree have already been solved by means of factoring (§ 72); it will presently appear that *all* such equations may be solved by the same method.

163. Definitions. An integral algebraic equation which involves the second but no higher degree of a number, is called a **quadratic equation** in that number (cf. § 94).

E.g., $x^2+5=0$, $3x^2-4=7x$, and $ax^2+bx+c=0$ are quadratic equations in the number represented by x; $4c^2+2c=9$ and $a(c+4)^2-3c+8=0$ are quadratic equations in c; and $a(y-3)^2+b(y-3)-6=0$ is a quadratic equation in $y-3$, and also in y.

Unless the contrary is expressly stated, a quadratic equation is understood to mean a quadratic equation *in the unknown number*.

Every quadratic equation in one unknown number, say x, may evidently, by transposing and simplifying, be reduced to the **standard form**

$$ax^2+bx+c=0,$$

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[*] For the solution of first degree equations see Chapters X and XI.

wherein a, b, and c represent known numbers and are usually called the **coefficients of the equation**; the term free from x, viz., c, is also called the **absolute term.** Although b or c may be zero, a can not be zero, for if $a = 0$ the equation becomes $bx + c = 0$, which is not quadratic.

If neither b nor c is zero, the equation is called a **complete quadratic equation,** while if either b or c is zero, it is called an **incomplete quadratic equation.** If $b = 0$, the equation is also often called a **pure quadratic equation,** otherwise it is called an **affected quadratic equation.**

E.g., the equation $2x^2 + 5 - 3x = 7x - 8$ becomes, by transposing and uniting terms, $2x^2 - 10x + 13 = 0$, which is in the above standard form, — the coefficients a, b, and c of the general equation being for this particular case 2, -10, and 13, respectively; it is a complete, and also an affected, quadratic equation.

Again, the equation $8x^2 + 4 - 3x = \frac{5 - 4x}{2} - x + 3$ becomes, by clearing of fractions, transposing and uniting terms, $16x^2 - 3 = 0$, which is in the standard form, a, b, and c being 16, 0, and -3, respectively; it is an incomplete, and also a pure, quadratic equation.

In the same way *every* quadratic equation in one unknown number may be reduced to the standard form.

EXERCISES

1. What are the important steps in the solution of an algebraic problem (cf. § 26)? What is meant by the "equation of a problem"?

2. If the conditions of a problem, when translated into algebraic language, lead to a quadratic equation (such as $5x^2 - 8x + 10 = 0$), can that problem be solved by the methods given in Chapter III or Chapter X?

3. What is a numerical equation? a literal equation? a simple equation? a general equation? a particular equation? a root of an equation?

4. Is $3x^2 - 2x = 0$ a complete or an incomplete quadratic equation? Why? Is it pure or affected? Why?

5. Reduce $5x^2 + 2 - 8x = 4(8 - x)$ to the "standard form." What is its absolute term? Is this equation pure or affected? complete or incomplete? Why?

6. Clear the equation $2x - 3 + \frac{6}{x} = x + 2$ of fractions, then reduce it to the standard form, and classify it (pure, complete, etc.); also solve it by the method of § 72.

7. Is the equation in Ex. 6 a quadratic or a simple equation? Why?

8. If x and y stand for unknown numbers, tell which of the following equations are simple, which quadratic, and which of a still higher degree:

$$a^4x^2 + a^2x + a = 0; \quad \frac{x-4}{2} = \frac{5}{x}; \quad 5x - 7y = 11; \quad 5x + xy - 7y = 11;$$

$$2(x^2 - x) + 6 = 2x^2; \quad \frac{x}{y} - 4 = 5x + \frac{3}{y+2}; \quad 3x + 4a^2 - 2ax = 7.$$

9. What *particular* equation is obtained by substituting the values 2, – 7, and 5 for the coefficients in the *general* equation $ax^2 + bx + c = 0$?

10. By assigning different sets of values to the letters a, b, and c, how many particular quadratic equations can be formed from the general equation $ax^2 + bx + c = 0$?

Why is this last equation called a "general" equation, and one in which the coefficients are numerals a "particular" equation?

164. Solution of quadratic equations. Although the roots of any quadratic equation whatever may be found by the method of factoring (§§ 72 and 165), yet there are various other methods for solving these equations, and one of these, which will doubtless be more easily followed by the student, will now be explained.

Ex. 1. Find the roots of the equation $2x^2 - 3 - 5x = 7x + 11$.

Solution. By transposing and uniting terms, the given equation becomes

$$2x^2 - 12x = 14, \tag{1}$$

whence, dividing by 2,

$$x^2 - 6x = 7; \tag{2}$$

if now 9 be added to each member of Eq. (2), it becomes

$$x^2 - 6x + 9 = 16, \tag{3}$$

i.e. (see "remark" below),

$$(x-3)^2 = 16, \tag{4}$$

whence, taking square roots,

$$x - 3 = \pm 4, \tag{5}$$

i.e.,

$$x - 3 = +4, \text{ or } x - 3 = -4, \tag{6}$$

hence, transposing,

$$x = 7, \text{ or } x = -1,$$

and, on substituting these values of x in the given equation, it is found that they each satisfy that equation; they are, therefore, the roots of the given equation.

That this equation has no other roots is shown in Ex. 38 below.

REMARK. Since $(x \pm k)^2 = x^2 \pm 2kx + k^2$, therefore the expression $x^2 \pm 2kx$, whatever the value of k, lacks only the term k^2 of being the square of $x \pm k$, *i.e.*, *if the square of half the coefficient of the first power of x be added to an expression of the form $x^2 + bx$, the result will be an exact square.*[*]

E.g., if $\left(\frac{6}{2}\right)^2$ be added to $x^2 - 6x$, the expression becomes $(x-3)^2$, as in Eq. (3) above; if $\left(\frac{5}{2}\right)^2$ be added to $y^2 + 5y$, it becomes $\left(y + \frac{5}{2}\right)^2$; and if $\left(\frac{b}{2}\right)^2$ be added to $x^2 + bx$, it becomes $\left(x + \frac{b}{2}\right)^2$.

More generally, if $\left(\frac{28kx}{2\sqrt{4k^2x^2}}\right)^2$, *i.e.*, 7^2, be added to $4k^2x^2 + 28kx$, it becomes $(2kx+7)^2$; this may also be seen by first writing $4k^2x^2 + 28kx$ in the form $(2kx)^2 + 14(2kx)$.

Ex. 2. Solve the equation $x^2 + 11x + 1 = 8x$.

SOLUTION. On transposing, the given equation becomes

$$x^2 + 3x = -1, \tag{1}$$

whence, adding $(\frac{3}{2})^2$,
$$x^2 + 3x + (\tfrac{3}{2})^2 = -1 + (\tfrac{3}{2})^2, \tag{2}$$

i.e.,
$$(x + \tfrac{3}{2})^2 = \tfrac{5}{4}, \tag{3}$$

and hence
$$x + \tfrac{3}{2} = \pm\sqrt{\tfrac{5}{4}} = \pm\tfrac{1}{2}\sqrt{5}, \tag{4}$$

i.e.,
$$x = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5} = \frac{-3 \pm \sqrt{5}}{2}, \tag{5}$$

and each of these values of x, viz., $\frac{-3+\sqrt{5}}{2}$ and $\frac{-3-\sqrt{5}}{2}$, is found, on substitution, to satisfy the given equation; they are, therefore, the roots of that equation.

Ex. 3. Solve the equation $ax^2 + bx + c = 0$.

SOLUTION. On transposing and dividing by a, this equation becomes

$$x^2 + \frac{b}{a}x = -\frac{c}{a}; \tag{1}$$

whence
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}, \tag{2}$$

i.e.,
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}, \tag{3}$$

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[*] Making this addition to the given expression is usually spoken of as **completing the square.**

therefore
$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a}, \qquad (4)$$

i.e.,
$$x=-\frac{b}{2a}+\frac{\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \qquad (5)$$

and as before, each of these values of x, viz., $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, is a root of the given equation.

Note. Having now shown how to find the roots of any quadratic equation whatever, the method of § 67 may be employed to find the factors of any quadratic expression of the form ax^2+bx+c (cf. also § 165).

E.g., since 7 is a root of the *equation* $x^2-6x-7=0$ (see Ex. 1 above), therefore $x-7$ is a factor of the *expression* x^2-6x-7 (cf. § 67).

Similarly, from Ex. 2, the factors of x^2+3x+1 are $x-\frac{-3+\sqrt{5}}{2}$ and $x-\frac{-3-\sqrt{5}}{2}$; and $x-\frac{-b+\sqrt{b^2-4ac}}{2a}$ is a factor of the expression ax^2+bx+c.

EXERCISES

4. In Ex. 1 above, how was Eq. (1) obtained from the given equation? State also how Eq. (2) was obtained from Eq. (1); Eq. (3) from Eq. (2); Eq. (5) from Eq. (3). How many equations are expressed in (5)? How were the roots of the given equation finally found from Eq. (5)?

5. Show that the essential steps in the solution of Ex. 2, and of Ex. 3, are the same as those in Ex. 1, viz.,

(1) *Transposing and uniting terms, and dividing each member of the new equation by the coefficient of the second power of the unknown number, thus reducing the given equation to the form* $x^2+mx=n$; (2) *adding* $\left(\frac{m}{2}\right)^2$ *to each member, thus making the first member an exact square;* (3) *extracting the square root of each member (giving the double sign to the second member), and solving the two resulting simple equations.*

By the above method find the roots of the following equations, and verify the correctness of each:

6. $2x^2-27=9x-x^2+3$.

7. $x^2+5x=21+x$.

8. $y^2-5y-24=0$.

9. $2x^2-x=3$.

10. $2y^2-10y=y^2+10y-51$.

11. $z^2+z-150=4-2z$.

12. $5x=x^2-14$.

13. $19x+5x^2=15-5x^2$.

14. $2y^2-5y=3y+234$.

15. $22t+3t^2=4t^2-48$.

16. $5t^2-3=10t-3t^2$.

17. $9-5x^2=12x$.

18. Write a carefully worded rule for solving such equations as those given above; also show that by this rule any quadratic equation whatever, which contains but one unknown number, may be solved.

19. Show that the rule asked for in Ex. 18 will serve to solve such equations as $x^2 + 6x = 0$. What are the two roots of this equation? Verify your answer.

20. Show that while such equations as that given in Ex. 19 *may* be solved by the above method, they may be much more easily solved by the method given in § 72.

Prove that if an equation has no absolute term, one of its roots is necessarily zero.

21. Does the rule asked for in Ex. 18 apply to such equations as $4x^2 - 9 = 0$? What are the roots of this equation? Verify your answer.

Solve the following equations, and verify your results:

22. $5x^2 = 8x$.

23. $13x + 2x^2 = 5x + 4x^2$.

24. $3y^2 - 8y = 2y(y - 4) + 9$.

25. $ax^2 + bx = cx^2$.

26. $ax^2 + b = 0$.

27. $(m + n)x^2 + n^2 = m^2$.

28. What must be added to $x^2 + 8x$ to "complete the square"?

29. What must be added to $P^2 - 5P$ to complete the square?

30. What must be added to $(x + y)^2 - 4(x + y)$ to complete the square?

31. What must be added to $4M^2 + 8M$ to complete the square?

32. What must be added to $9a^2x^4 + 12ax^2$ to complete the square?

33. Show that the answer to each of the exercises 28–32 conforms to what is said in the "remark" under Ex. 1.

34. How many different equations are expressed by $P = \pm Q$? What are they? Write them separately.

35. How many different equations are expressed by $\pm P = \pm Q$? What are they? Write them separately. Do the equations $+P = +Q$ and $-P = -Q$ express the same or different relations between P and Q?

36. Show that the equation $P = \pm Q$ expresses *all* the relations between P and Q that are expressed by the equation $\pm P = \pm Q$; and hence show that the double sign ($\pm$) need be employed in only *one* member of an equation which is obtained by extracting the square root of each member of a given equation. Illustrate this in the solutions of Exs. 1 and 2 above.

37. Prove that the two equations $P = \pm Q$ are together equivalent (§ 95) to the equation $P^2 = Q^2$.

PROOF. The equation $$P^2 = Q^2$$
is equivalent to $$P^2 - Q^2 = 0, \qquad [§ 95 (1)$$
i.e., to $$(P - Q)(P + Q) = 0,$$
and, manifestly, this last equation is satisfied when, and only when,
$$P - Q = 0 \text{ or } P + Q = 0,$$
i.e., when $$P = \pm Q;$$
hence the equations $P^2 = Q^2$ and $P = \pm Q$ are equivalent.

38. In the solution of Ex. 1 above, show that the given equation and Eqs. (1), (2), (3), and (4) are all equivalent to each other, and that each is equivalent to the two equations (5), *i.e.*, to the two in (6). Hence show that the given equation has two roots, and only two.

39. By the method of Ex. 38, show that the equation given in Ex. 2, above, has two roots, and only two.

40. Show that Ex. 3 has two solutions, and only two, and thus prove that *every* quadratic equation in one unknown number has two roots, and only two (cf. § 97).

Solve the following equations, and verify your results:

41. $3x^2 + 5x - 4 = x^2 - 2x + 3.$

42. $x^2 - \frac{4}{3}x - 2 = 0.$

43. $(2 - x)(x + 1) + 4 = x - 3.$

44. $(2y - 3)^2 = 6(y + 1) - 5.$

45. $2y^2 + 3 = 7y.$

46. $3x^2 - 10 = 7x.$

47. $6 + 5t = 6t^2.$

48. $\frac{1}{2}x - \frac{3x^2}{4} + 2 = 0.$

49. What are the roots of $x^2 - 3x - 2 = 0$? Are these roots rational or irrational numbers? Define rational and irrational numbers. Are the above roots real or imaginary?

50. What are the roots of $x^2 - 3x + 4 = 0$? Verify the correctness of your answer. Are these roots real or imaginary?

51. Solve the equation $3x^2 - 8x + 10 = 0.$

SUGGESTION. The method already explained for solving such equations gives rise to fractions; these fractions can be avoided by proceeding thus:

On multiplying the given equation by 3 (the coefficient of x^2), and transposing, it becomes
$$9x^2 - 24x = -30;$$
completing the square, $$9x^2 - 24x + 16 = -30 + 16 = -14,$$
i.e., $$(3x - 4)^2 = -14,$$
hence $$3x - 4 = \pm\sqrt{-14},$$
and $$x = \tfrac{1}{3}(4 + \sqrt{-14}) \text{ or } \tfrac{1}{3}(4 - \sqrt{-14}).$$

52. Solve the equation $3x^2 - 5x - 2 = 0$.

SUGGESTION. Multiply this equation by $4 \cdot 3$ and then proceed as in Ex. 51.

53. Solve the equation $ax^2 + bx + c = 0$.

Multiply by $4a$ and then proceed as in Ex. 51.

54. Solve the equation $mx^2 + 2nx + k = 0$.

Multiply by m and proceed as in Ex. 51.

55. By studying Exs. 51–54, especially 53 and 54, point out when it is necessary to multiply by 4 times the coefficient of the second degree term in order to avoid fractions in the solution of a quadratic equation; and also when multiplying by that coefficient alone will suffice.

Solve the following equations, avoiding fractions in completing the square:

56. $3x^2 + 2x = 7$.

57. $5x^2 + 6x = 8$.

58. $3y^2 + 4y = 95$.

59. $2y^2 + 3y = 27$.

60. $2t^2 + 7t = -6$.

61. $3x^2 - 5x = 2$.

62. $5x^2 - x - 3 = 0$.

63. $15y^2 - 7y - 2 = 0$.

64. Is 8 a root of $x^2 - 5x - 24 = 0$? Why? What is the corresponding factor of $x^2 - 5x - 24$ (cf. Ex. 3, note)? What is the other factor of this quadratic expression? What root of the given equation corresponds to this other factor?

65. Since $x^2 - 7x + 10 \equiv (x-2)(x-5)$, what are the roots of the equation $x^2 - 7x + 10 = 0$? Why (cf. § 72)?

66. Since 2 and 7 are the roots of $x^2 - 9x + 14 = 0$, what are the factors of $x^2 - 9x + 14$? Why (cf. § 67)?

67. Since $\frac{1}{2}$ and $\frac{2}{3}$ are the roots of $6x^2 - 7x + 2 = 0$, what are the factors of $6x^2 - 7x + 2$? Are these the only factors, or is there also a numerical factor?

68. By first finding the roots of the *equation* $15x^2 - 4x - 3 = 0$, find *all* the factors of the *expression* $15x^2 - 4x - 3$.

69. Based upon the note under Ex. 3, and upon Exs. 64–68, write a carefully worded rule for factoring quadratic expressions.

Apply the rule asked for in Ex. 69 in finding all the factors of the following expressions, and verify their correctness:

70. $5x^2 + 12x - 9$.

71. $8t^2 - 10t - 3$.

72. $\frac{3x^2}{4} - \frac{x}{2} - 5$.

73. $(x+1)(2-x) + 9 - x$.

74. $(2y-3)^2 - 6(y+1) + 8$.

75. $ax^2 + bx + c$.

76. Are the expressions in Exs. 70–75 equal to 0? What justification have we then for writing them so?

77. Write an equation whose roots are 3 and 8 (cf. § 72).

78. Write an equation whose roots are $-\frac{3}{2}$ and 12; 7 and -1; $\frac{2}{3}$ and $\frac{15}{8}$; $1+\sqrt{3}$ and $1-\sqrt{3}$; i and i; $2+3i$ and $2-3i$.

79. By first finding the factors of $x^2-3x-10$, prove that the roots of $7(x^2-3x-10)=0$ are also roots of $x^2-3x-10=0$, and *vice versa.* Prove this also from § 95 (2).

80. Is there any number which is a root of $x^2-3x-10=0$ and also of $3x^2+x-10=0$; *i.e.*, have these equations a root in common?

SUGGESTION. Solve either of these equations and substitute its roots in the other equation. Also solve by means of § 76.

81. Find the common roots, if any, of $2x^3-33x^2-5x+6=0$ and $6x^3+7x^2+4x+1=0$.

82. Find *all* the roots of the equations in Ex. 81.

165. Solution of quadratic equations by factoring. In § 72 it was shown how factoring may be employed to solve algebraic equations; it will now be shown that any quadratic equation whatever may be solved by this method.

Ex. 1. Solve the equation $x^2+6x+8=0$.

SOLUTION. The *expression* x^2+6x+8

$$\begin{aligned}
&= x^2+6x+(\tfrac{6}{2})^2-(\tfrac{6}{2})^2+8 \qquad \text{[cf. §§ 70 and 164}\\
&= x^2+6x+9-9+8\\
&= (x+3)^2-1\\
&= \{(x+3)+1\}\cdot\{(x+3)-1\}\\
&= (x+4)(x+2);
\end{aligned}$$

hence the given *equation* is equivalent to

$$(x+4)(x+2)=0,$$

which, in turn, is equivalent to the two equations

$$x+4=0 \text{ and } x+2=0,$$

whose roots are -4 and -2, respectively; therefore, the roots of the given equation are -4 and -2.

NOTE. Observe that the *plan* of the above solution is first to transform the expression x^2+6x+8 into the difference of two squares, one of which shall contain all the terms involving x, and then to factor the resulting expression by the formula $A^2-B^2=(A-B)(A+B)$.

Ex. 2. Solve the equation $x^2 - 3x + 1 = 0$.

Solution. $$x^2 - 3x + 1 = x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1$$
$$= \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}$$
$$= \left(x - \frac{3}{2} - \frac{\sqrt{5}}{2}\right) \cdot \left(x - \frac{3}{2} + \frac{\sqrt{5}}{2}\right)$$
$$= \left(x - \frac{3 + \sqrt{5}}{2}\right) \cdot \left(x - \frac{3 - \sqrt{5}}{2}\right);$$

hence the roots of the given equation are the same as the roots of

$$\left(x - \frac{3 + \sqrt{5}}{2}\right) \cdot \left(x - \frac{3 - \sqrt{5}}{2}\right) = 0,$$

i.e., they are $\frac{3 + \sqrt{5}}{2}$ and $\frac{3 - \sqrt{5}}{2}$.

Ex. 3. Solve the equation $ax^2 + bx + c = 0$.

Solution. The expression $ax^2 + bx + c$, whatever the values of a, b, and c, may be factored as follows:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\left\{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right\}$$
$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}$$
$$= a\left\{x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right\} \cdot \left\{x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right\}$$
$$= a\left\{x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right\} \cdot \left\{x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right\};$$

hence the roots of the given equation are the same as those of

$$a\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right) \cdot \left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right) = 0,$$

i.e., they are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Since every quadratic equation is reducible to the standard form $ax^2 + bx + c = 0$, therefore the solution of Ex. 3 shows not only how to *factor* any *expression* of the form $ax^2 + bx + c$, but also that *every quadratic equation has two roots, and only two;* compare also § 164, Ex. 40.

EXERCISES

4. By first finding the factors of the expression $x^2 - 9x + 14$, solve the equation $x^2 - 9x + 14 = 0$.

5. By first finding the factors of $15x^2 - 4x - 3$, find the roots of the equation $15x^2 - 4x - 3 = 0$.

6. Factor $3y^2 - 2y - 20$, and thus solve the equation $3y^2 - 2y - 20 = 0$.

Factor the following expressions, both by the method of § 164 and also by that of § 165; also point out which method is simpler, and why:

7. $8t^2 - 10t - 3$.

8. $(x-1)(2-x) + 9 - x$.

9. $3y^2 + 4y - 1$.

10. $5m^2 + 6m + 2$.

11. $x^2 + (m+n)x + mn$.

12. $x^2 + px + q$.

166. Solution of quadratic equations by means of a formula. Since every quadratic equation in one unknown number may be reduced to an equivalent equation of the form $ax^2 + bx + c = 0$ (§ 163), and since the roots of this equation are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, *whatever the numbers represented by a, b, and c* (§ 165, Ex. 3, and § 164, Ex. 3), therefore the roots of any *particular* quadratic equation may be found by merely substituting for a, b, and c, in the expressions for the roots of the above *general* equation, those values which these coefficients have in the particular equation under consideration.

E.g., since the roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, therefore the roots of $3x^2 + 10x - 8 = 0$ (in which $a = 3$, $b = 10$, and $c = -8$) are

$$\frac{-10 \pm \sqrt{10^2 - 4 \cdot 3 \cdot (-8)}}{2 \cdot 3}, \textit{ i.e., } \frac{-10 \pm 14}{6}, \textit{ i.e., } \frac{2}{3} \text{ and } -4.$$

So, too, the roots of $6y^2 + 19y - 7 = 0$ are

$$\frac{-19 \pm \sqrt{19^2 - 4 \cdot 6(-7)}}{2 \cdot 6}, \textit{ i.e., } \frac{1}{3} \text{ and } -\frac{7}{2}.$$

And the roots of $x^2 - 3x + 5 = 0$ are $\frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$, *i.e.*, $\frac{3 \pm \sqrt{-11}}{2}$.

Note. While the student should, of course, be able to solve quadratic equations without the use of the formula (by the method of § 164, or of § 165), he is advised to commit this formula carefully to memory, and henceforth to employ it freely as in the illustrative examples above; he will find this well worth his while, because roots of quadratic equations are so very frequently required in mathematical investigations.

EXERCISES

1. Write down the formula for the roots of $ax^2+bx+c=0$. How many values has this expression? Write two expressions which are together equivalent to this formula.

2. Do these two expressions represent the roots of $ax^2+bx+c=0$ for *all* values of the coefficients a, b, and c, or only for *particular* values of these letters?

By means of the above formula, write down the roots of each of the following equations, verify their correctness in each case, and point out which are real, which imaginary, which rational, and which irrational:

3. $x^2-5x+6=0$.

4. $3u^2-4u-10=0$.

5. $v^2=8v+\frac{5}{6}$.

6. $(3v+1)(2-v)=v(3-v)$.

7. $mx^2+nx+p=0$.

8. $\frac{3}{a}t^2=\frac{m}{n}t-\frac{am}{2n}$.

9. If the numbers represented by p and q are such that $p^2>4q$, are the roots of $x^2+px+q=0$ real or imaginary? What are they if $p^2<4q$?

10. What are the roots of $36m^2x^2+36m^2nx-n^2=m^2(1-9n^2)$? Show that each of these roots is real whatever integers or fractions (positive or negative) may be represented by m and n.

167. Character of the roots. It has already been shown (§ 165) that the roots of the equation $ax^2+bx+c=0$ are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a};$$

hence, if a, b, and c represent real and rational numbers, these roots can be imaginary or irrational only if $\sqrt{b^2-4ac}$ is imaginary or irrational. *E.g.*, both roots are imaginary if b^2-4ac is negative.

The conditions for discriminating the character of the roots may be summarized thus:

if $b^2-4ac>0$, *the roots are real, and unequal,*
if $b^2-4ac=0$, *the roots are real, and equal,*
if $b^2-4ac<0$, *both roots are imaginary,*
and the roots are rational only when b^2-4ac *is an exact square.**

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* The expression b^2-4ac is, for this reason, usually called the **discriminant** of the quadratic equation.

The *character* of the roots of any particular quadratic equation may, therefore, be determined by merely finding the value of the expression b^2-4ac for that equation.

E.g., the roots of $3x^2-5x-1=0$ are real, irrational, and unequal, because here $b^2-4ac=37$ (since $a=3$, $b=-5$, and $c=-1$), and $\sqrt{37}$ is real and irrational;

The roots of $3x^2-5x-2=0$ are real, rational, and unequal, because in this equation $\sqrt{b^2-4ac}=\sqrt{49}=\pm 7$, *i.e.*, it is rational;

The roots of $2x^2+5x-8=4x-11$ are imaginary, because in this equation $\sqrt{b^2-4ac}=\sqrt{-23}$;

And the roots of $4x^2-12x+9=0$ are real, rational, and equal, because in this equation $b^2-4ac=0$.

EXERCISES

1. If $b^2=4ac$, what is the value of b^2-4ac? of $\sqrt{b^2-4ac}$? of $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$? of $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$? How, then, do the two roots of $ax^2+bx+c=0$ compare when $b^2=4ac$?

2. State verbally the condition that must hold among the coefficients of a quadratic equation in order that the roots of that equation shall be equal,—instead of "a" say "the coefficient of the second power of the unknown number," etc.

3. For what value of k will the roots of $3x^2-10x+2k=0$ be equal?

Suggestion. The roots are equal if $(-10)^2=4\cdot 3\cdot 2k$. Why?

4. Find the value of m for which $mx^2-6x+3=0$ has equal roots.

5. Find the values of k for which the roots of $3x^2-4kx+2=0$ are equal.

6. For what values of a are the roots of $ax^2-5ax+11=a$ equal?

7. For what values of m are the roots of $x^2-3x-m(x+2x^2+4)=5x^2+3$ equal?

Without first solving, tell whether the roots of the following equations are real, imaginary, rational, equal, etc., and explain your answers:

8. $x^2-5x+6=0$.

9. $x^2-6x+9=0$.

10. $3t^2-11t-17=0$.

11. $3t^2+11t+17=0$.

12. $\dfrac{3x^2+2}{7}-\dfrac{1}{3}=\dfrac{x-5}{6}$.

13. $7u^2+4u+1=0$.

14. Are the roots of the equation in Ex. 13 related in any way (cf. Ex. 9, § 149)?

15. Show that if either root of a quadratic equation is imaginary, then the other root is also imaginary, and that each is the conjugate of the other.

16. For what values of k are the roots of $36x^2 - 24kx + 15k = -4$ imaginary?

SOLUTION. The roots of this equation (§ 166) are

$$\frac{24k + \sqrt{(-24k)^2 - 4 \cdot 36(15k+4)}}{2 \cdot 36} \text{ and } \frac{24k - \sqrt{(-24k)^2 - 4 \cdot 36(15k+4)}}{2 \cdot 36},$$

i.e.,

$$\frac{2k + \sqrt{4k^2 - 15k - 4}}{6} \text{ and } \frac{2k - \sqrt{4k^2 - 15k - 4}}{6};$$

and these roots are imaginary for those values of k for which the expression under the radical, viz., $4k^2 - 15k - 4$, is negative, and for those values only.

Now $4k^2 - 15k - 4$, which equals $(4k+1)(k-4)$ (§ 165), is negative for those values of k for which one of these factors is positive and the other negative, and for no others; hence the roots of the given equation are imaginary when k lies between $-\frac{1}{4}$ and 4.

17. From the solution of Ex. 16 point out those values of k for which the roots of the given equation are real, and explain your answer.

18. If $k = -\frac{1}{4}$, are the roots of the equation in Ex. 16 real or imaginary? How do they compare in value when $k = -\frac{1}{4}$? when $k = 4$?

19. Without actually solving the equation, find the values of m for which the roots of $4m^2x^2 + 12m^2x + 10 - m = 0$ are equal.

20. Without actually solving the equation, find the values of m for which the roots in Ex. 19 are real, and those for which these roots are imaginary.

21. Find the sum of the two roots of $ax^2 + bx + c = 0$; also the sum of the roots of $x^2 + px + q = 0$.

22. By means of the results of Ex. 21, and without first solving the equation, determine the sum of the roots of $x^2 + 5x - 2 = 0$; also the sum of the roots of $4x^2 - 6x = 3$. Verify your answers by actually adding the roots.

23. Find the product of the roots of $x^2 + px + q = 0$; also the product of the roots of $ax^2 + bx + c = 0$.

24. By means of the results of Ex. 23, determine the product of the roots of $x^2 - 10x + 16 = 0$; also of $4x^2 - 30x + 25 = 0$.

25. State *verbally* the relation between the sum of the roots of a quadratic equation and the coefficients of that equation; also make a similar statement concerning the product of the roots, — compare Exs. 21 and 23.

168. Sum and product of the roots. If r and r' be employed to represent the roots of the equation $ax^2 + bx + c = 0$, *i.e.*, if

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r' = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

then by adding, and by multiplying, it follows that

$$r + r' = -\frac{b}{a} \text{ and } r \cdot r' = \frac{c}{a}. \qquad \left[\text{cf. Exs. 21 and 23, § 167}\right.$$

The student should perform these operations in detail, and should also express the results in verbal language. Compare Ex. 25, § 167.

NOTE. Rationalizing the numerators in the above expressions for the roots of $ax^2 + bx + c = 0$, shows that

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-2c}{b + \sqrt{b^2 - 4ac}},$$

and

$$r' = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2c}{b - \sqrt{b^2 - 4ac}}.$$

Since $r \cdot r' = \frac{c}{a}$, therefore if c is very small as compared with a, *i.e.*, if $\frac{c}{a}$ is very small, then at least *one* of the roots (r or r') must be very small; to see which one this is, and also to see how large the other root is, it is only necessary to examine the above expressions for r and r'.

Thus as $c \doteq 0$,* $4ac \doteq 0$, and $b^2 - 4ac \doteq b^2$, *i.e.*, $\sqrt{b^2 - 4ac} \doteq b$, and the first expression for r shows that $r \doteq 0$, — since $\frac{0}{2a} = 0$.

Similarly it may be shown, from the *first* expression for r', that when $c \doteq 0$, then $r' \doteq -\frac{b}{a}$, — observe that the *second* expression for r' becomes indeterminate when $c \doteq 0$, *i.e.*, it becomes $\frac{0}{0}$.

What has just been shown is usually expressed by saying "*if the absolute term of a quadratic equation is zero, then one root of that equation is also zero*" (cf. Ex. 20, § 164).

Again, if $a \doteq 0$, then the above expressions show that r' becomes $-\infty$ (cf. note to Ex. 15, § 55), and that r becomes $-\frac{c}{b}$, — the first expression for r becomes $\frac{0}{0}$, which is indeterminate, but the second shows its value to be $-\frac{c}{b}$.

What has just been shown may be expressed by saying "*$a = 0$ is the condition that one root of $ax^2 + bx + c = 0$ is infinitely large.*"

EXERCISES

1. Without solving the equations, write down the sum and also the product of the roots of each of the equations in Exs. 6–11 of § 164, and explain your answer in each case.

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* The symbol $\doteq$ is here used to mean "approaches indefinitely near to."

2. Give the sum and also the product of the roots of each equation in Exs. 22–27 of § 164, and verify your work.

3. If one root of the equation $x^2 + 5x - 24 = 0$ is known to be 3, how may the other root be found from the absolute term? from the coefficient of the first power of x? Do the results agree?

4. If one root of *any* given quadratic equation whatever be known, how may the other root be most easily found?

5. What is the sum of the roots of $3m^2x^2 + (8m - 1)x + 5 = 0$? For what value of m is this sum 3?

6. For what values of k will one of the roots of $2(k + 1)^2x^2 - 3(2k + 1)(k + 1)x + 9k = 0$ be the reciprocal of the other?

SUGGESTION. Equate one of the roots to the reciprocal of the other, and solve.

7. For what value of k will one root of the equation in Ex. 6 be zero? With this value of k, what will be the value of the other root?

8. For what value of k will one root of the equation in Ex. 6 be infinite (cf. note, § 168)?

9. For what values of n will one of the roots of $(n-3)y^2-(2n+1)y = 2 - 5n$ be double the other?

10. Prove that one of the roots of $ax^2 + bx + c = 0$, whatever the values of a, b, and c, will be double the other if $2b^2 = 9ac$.

11. If r and r' are the roots of $ax^2 + bx + c = 0$, find the value of $\frac{1}{r} + \frac{1}{r'}$ expressed in terms of a, b, and c.

12. It has already been shown that if r and r' are roots of the equation $ax^2 + bx + c = 0$, then $ax^2 + bx + c \equiv a(x - r)(x - r')$; from this fact prove that if r'' is not equal to r or to r', then r'' can not be a root of $ax^2 + bx + c = 0$ (cf. Ex. 40, § 164).

13. Show that the roots of $ax^2 + 2bx + c = 0$ are $\frac{-b + \sqrt{b^2 - ac}}{a}$ and $\frac{-b - \sqrt{b^2 - ac}}{a}$. How do these expressions compare with the expressions for r and r' above?

14. Apply the formulas of Ex. 13 to write down the roots of $3x^2 - 8x - 3 = 0$; also of $2x^2 + 10x = 7$. Compare these results with those obtained by the formulas of § 166; which of these formulas gives the simpler result when the coefficient of the first power of the unknown number is *even?*

15. Show that when a and c represent numbers having like signs, the roots of $ax^2 + bx + c = 0$ may be real, or may be imaginary, depending upon the relative values of a, b, and c; but that these roots are necessarily real when a and c represent numbers having unlike signs.

16. What relation exists between the roots of $ax^2 + bx + c = 0$ when $a = c$? when $a = - c$?

17. If r and r' represent the roots of $ax^2 + bx + c = 0$, form an equation whose roots are $- r$ and $- r'$.

Solution. The equation whose roots are $-r$ and $-r'$ is (§ 72)

$$(x+r)(x+r')=0, \text{ i.e., } x^2+(r+r')x+rr'=0;$$

but $r+r'=-\frac{b}{a}$ and $rr'=\frac{c}{a}$ (§ 168), hence the required equation is

$$x^2-\frac{b}{a}x+\frac{c}{a}=0, \text{ i.e., } ax^2-bx+c=0.$$

18. Find $r^2 + r'^2$ from $ax^2 + bx + c = 0$. Also find the sum of the reciprocals of the roots of $x^2 - 5x + 2 = 0$ (cf. Ex. 11).

19. If r and r' are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are r^2 and r'^2; also one whose roots are $\frac{1}{r}$ and $\frac{1}{r'}$.

20. What do the roots of $ax^2 + bx + c = 0$ become when $c \doteq 0$? when $c \doteq 0$ and $b \doteq 0$? when $a \doteq 0$? when $a \doteq 0$ and $b \doteq 0$? when $b \doteq 0$? Compare the note on p. 280.

169. Fractional equations which lead to quadratics. The general principles underlying the solution of fractional equations are discussed in § 99; manifestly those principles apply whatever the degree of the integral equation to which the fractional equation leads. The following solutions may illustrate the procedure.

Ex. 1. Solve the equation $\frac{x+5}{x+2}+1=3x$.

Solution. On clearing the given equation of fractions, it becomes

$$x + 5 + x + 2 = 3x^2 + 6x,$$

which reduces to $3x^2 + 4x - 7 = 0$,

whence $x = \frac{-4 \pm \sqrt{16+84}}{6}$ [§ 166

$$= \frac{-4 \pm 10}{6},$$

i.e., $x = 1$ or $-\frac{7}{3}$;

and since neither $x = 1$ nor $x = -\frac{7}{3}$ reduces to zero the multiplier which was used to clear of fractions, therefore they are the roots of the given equation (cf. § 99).

Ex. 2. Solve the equation $\frac{x}{1-x}+\frac{4x+3}{x+1}=\frac{2x^2}{x^2-1}$.

Solution. On clearing the given equation of fractions, it becomes

$$-x^2-x-4x-3+4x^2+3x=2x^2,$$

which reduces to $$x^2-2x-3=0,$$

whence $$x=\frac{2\pm\sqrt{4+12}}{2}=\frac{2\pm 4}{2},$$

i.e., $$x=3 \text{ or } -1;$$

but although both 3 and -1 are roots of the *integral* equation, yet 3 alone is a root of the given fractional equation. Observe that $x=-1$ reduces the multiplier x^2-1 to zero; compare also § 99.

EXERCISES

Solve the following fractional equations, being careful to exclude all extraneous roots:

3. $15x+\frac{2}{x}=11$.

4. $\frac{1}{x}-2+x=\frac{2}{x}$.

5. $\frac{3}{2(x^2-1)}-\frac{1}{4(x+1)}=\frac{1}{8}$.

6. $\frac{2x-2}{5x+5}=\frac{x-1}{x+1}$.

7. $\frac{x-2}{x+2}+\frac{x+2}{x-2}=2\left(\frac{x+3}{x-3}\right)$.

8. $\frac{1}{x-1}+(x-2)^{-1}=\frac{1}{3-x}$.

9. $\frac{3x}{x+5}+\frac{42}{(x+5)(x-2)}=8+\frac{6}{x-2}$.

10. $\frac{20}{x+3}+\frac{40}{x^2+4x+3}+7=\frac{4x}{x+1}$.

11. $\frac{2x+1}{1-2x}-\frac{5}{7}=\frac{x-8}{2}$.

12. $\frac{2a+x}{2a-x}+\frac{a-2x}{a+2x}=\frac{8}{3}$.

13. $\frac{bx}{a-x}+b=\frac{a(x+2b)}{a+b}$.

14. $\frac{2}{x-5}-\frac{5x}{3x+2}=\frac{x+29}{(3x+2)(x-5)}-3$.

15. $\frac{x}{x-1}-\frac{x}{x+1}=c$.

170. Irrational equations. Equations which contain indicated roots of the *unknown* numbers are usually called **irrational equations**; they are also sometimes spoken of as **radical equations.**

E.g., $\sqrt{x}-5=0$, $\sqrt{x+1}+x=8$, $\frac{x-6}{\sqrt{x}}+1=0$, and $3+\frac{\sqrt{x}}{2}=\sqrt[3]{x^2-1}$ are irrational equations, but $x-\sqrt{3}=5x$ is a rational equation.

The solution of irrational equations may be illustrated by the following examples:

Ex. 1. Solve the equation $\sqrt{x} - 5 = 0$.

SOLUTION. The given equation is (§ 95) equivalent to

$$\sqrt{x} = 5,$$

whence, squaring, $$x = 25.$$

On substituting 25 for x, the given equation is satisfied, provided that $\sqrt{x}$ is understood to mean the *positive* value of the square root; and in that case 25 is, therefore, a root of the given equation.

Ex. 2. Solve the equation $\sqrt{x+1} + x = 11$.

SOLUTION. The given equation is (§ 95) equivalent to

$$\sqrt{x+1} = 11 - x,$$

whence, squaring, $$x + 1 = 121 - 22x + x^2,$$

i.e., $$x^2 - 23x + 120 = 0,$$

whence $$x = \frac{23 \pm \sqrt{23^2 - 480}}{2},$$

i.e., $$x = 15 \text{ or } 8,$$

and, on substitution, it is found that 15 satisfies the given equation if $\sqrt{x+1}$ means the *negative* value of this root, while 8 satisfies it if the *positive* value of this root is intended.

Ex. 3. Solve the equation $\dfrac{6-x}{\sqrt{x}} + 1 = 0$.

SOLUTION. The given equation is equivalent to

$$6 - x = -\sqrt{x},$$

whence, squaring, $$36 - 12x + x^2 = x,$$

and therefore $$x = 9 \text{ or } 4;$$

of which 9 is a root of the given equation if the positive value of the square root is meant, otherwise 4 is a root.

The above procedure may be formulated thus: (1) *isolate the radical, or one of the radicals, if there are two or more,* (2) *by involution rationalize the given equation,* (3) *solve this rational equation, and* (4) *test the results by substituting them in the given equation.*

NOTE 1. Observe that a quadratic irrational equation is ambiguous unless it is stated which of the two values of the radical is intended.

E.g., the equation $\sqrt{x} - 5 = 0$ really contains in itself *two* equations, viz., $\overset{+}{\sqrt{x}} - 5 = 0$* and $\overset{-}{\sqrt{x}} - 5 = 0$; and the equation $\sqrt{x} + \sqrt{5-x} = 3$ contains in itself

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* Let $\overset{+}{\sqrt{\ }}$ and $\overset{-}{\sqrt{\ }}$ indicate the positive and negative values, respectively, of the roots.

four equations, viz., $\sqrt[+]{x}+\sqrt[+]{5-x}=3$, $\sqrt[+]{x}+\sqrt[-]{5-x}=3$, $\sqrt[-]{x}+\sqrt[+]{5-x}=3$, and $\sqrt[-]{x}+\sqrt[-]{5-x}=3$. Hence, in order to avoid ambiguity, it is always necessary to specify in connection with a radical equation *which root* is intended.

NOTE 2. It should also be observed that if both members of any given equation be raised to the same positive integral power, then every root of the given equation will be a root of the new equation thus formed, and the new equation will, in general, have one or more additional roots which were introduced by the involution.

To prove this, let the members of the given equation be represented by u and v respectively (where u and v may be expressions containing the unknown number x); then the given equation is $u=v$, and from this it follows by squaring that $u^2=v^2$, which is equivalent to $u^2-v^2=0$, *i.e.*, to $(u-v)(u+v)=0$; but every root of the given equation makes $u=v$, *i.e.*, makes $u-v=0$, and hence satisfies the equation $(u-v)(u+v)=0$, while the new equation is also satisfied by those *additional* values of x which make $u+v=0$; hence the correctness of the above statement.

Similarly if the members of the given equation had been raised to a higher power than the second.

Hence the roots of any given irrational equation are to be found among the roots of the equation resulting from rationalizing the given equation, and if *none* of the roots of the rational equation prove to be roots of the irrational equation, then that equation has no root whatever.

E.g., the equation $\sqrt[+]{3x+4}+2\sqrt[+]{x+5}-\sqrt[+]{x}=0$ leads to $3x^2+4x-64=0$, whose roots are 4 and $-\frac{16}{3}$, neither of which is a root of the given equation, hence that equation has no root whatever.

EXERCISES

4. Show that if the signs of the radicals are left unrestricted, then the equation $\sqrt{3x+4}+2\sqrt{x+5}-\sqrt{x}=0$ has two roots. What are these roots?

Solve the following equations, and show what restrictions, if any, must be made on the signs of the radicals in order that your results shall be roots of the equations:

5. $\sqrt{5-x}=x-5$,

6. $x+\sqrt{x}=4x-4\sqrt{x}$.

7. $y+y^{\frac{1}{2}}=20$.

8. $\sqrt{4y+17}+\sqrt{y+1}-4=0$.

9. $\sqrt{x+1}+(x+1)^{-\frac{1}{2}}=2$.

10. $\sqrt{3+x}+\sqrt{x}=\dfrac{5}{\sqrt{x}}$.

11. $\sqrt{4x+1}-\sqrt{x+3}=\sqrt{x-2}$.

12. $\sqrt{x+a}+\sqrt{x+b}=\sqrt{2x+a+b}$.

13. $\sqrt{x+3}+\sqrt{4x+1}=\sqrt{10x+4}$.

14. $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=2$.

15. $\dfrac{\sqrt{x}-2}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}+21}$.

16. $\sqrt{\dfrac{a^2}{x}+b}-\sqrt{\dfrac{a^2}{x}-b}=c$.

Find all the roots of the following restricted equations (cf. note 2, above), and verify your results:

17. $\sqrt[+]{x+4}+\sqrt[-]{x-4}=2.$

18. $\sqrt[+]{x+4}+\sqrt[-]{x-4}={}^{-}2.$

19. $\sqrt[-]{x+4}+\sqrt[-]{x-4}=2.$

20. $\sqrt[-]{3x-5}+\sqrt[+]{x-9}-2\sqrt[+]{x-1}=0.$

21. $\sqrt[-]{3x-5}+\sqrt[-]{x-9}-2\sqrt[-]{x-1}=0.$

22. $\sqrt[+]{3x-5}+\sqrt[-]{x-9}-2\sqrt[+]{x-1}=0.$

23. By first rationalizing the equation $x=\sqrt[3]{1}$, and then transposing and factoring, show (§ 72) that this equation has 3 solutions; *i.e.*, show that 1 has 3 distinct cube roots, viz.: 1, $\frac{1}{2}(-1+\sqrt{-3})$ and $\frac{1}{2}(-1-\sqrt{-3})$.

Similarly it may be shown that any number whatever has 3 cube roots (cf. § 132).

171. Problems which lead to quadratic equations. The directions already given for solving problems whose conditions lead to simple equations (§ 26) are also applicable to problems which lead to quadratic and still higher equations; the three important steps are:

(1) Translate the conditions of the problem into equations,
(2) Solve these equations,
(3) Test and interpret the results.

Special emphasis is to be laid upon the testing and interpreting of the results, because a *problem* often contains restrictions upon its numbers, expressed or implied, which are *not* translated into the *equations*, and therefore the solutions of the equations may or may not be solutions of the problem itself, — compare the illustrative problems which follow, and also § 100.

Prob. 1. A farmer purchased some sheep for \$168; later he sold all but 4 of them for the same sum, and found that his profit on each sheep sold was \$1. How many sheep did he purchase?

Solution

Let $x=$ the number of sheep purchased.

Then $\frac{168}{x}=$ the number of dollars each sheep cost,

and $\frac{168}{x-4}=$ the number of dollars received for each sheep,

and hence $\frac{168}{x-4}-\frac{168}{x}=1;$ [Since the profit is \$1 on each sheep

therefore (§ 169) $x=28$ or $-24.$

The first of these values, viz., 28, is found to be a solution of the *problem* as well as of the *equation*, but while the second satisfies the *equation* it can not satisfy the *problem*, because the number of sheep purchased is necessarily a positive integer.

Prob. 2. At a certain dinner party it is found that 6 times the number of guests exceeds the square of $\frac{2}{3}$ their number by 8; how many guests are there?

SOLUTION

Let $x =$ the number of guests.

Then the expressed condition of the problem is

$$6x - \left(\frac{2x}{3}\right)^2 = 8,$$

i.e., $$2x^2 - 27x + 36 = 0,$$

whence $$x = 12 \text{ or } \tfrac{3}{2}.$$

Here, too, an *implied* condition of the problem is that the answer must be a positive integer, and hence, although $\frac{3}{2}$ satisfies the equation, it is not a solution of the problem.

Prob. 3. If 4 times the square root of a certain number be subtracted from that number, the result will be 12; what is the number?

SOLUTION

Let $x =$ the required number.

Then the problem states that $x - 4\sqrt{x} = 12$,

i.e., $$x^2 - 40x + 144 = 0,$$

whence $$x = 36 \text{ or } 4.$$

If the above square root is understood to be *positive*, then 36 is the solution, but if the *negative* root is meant, then 4 is the solution.

Prob. 4. If the number of dollars that a certain man has, be multiplied by that number diminished by 4, the product will be 21. How much money has he?

SOLUTION

Let $x =$ the number of dollars he has.

Then the problem states that $x(x - 4) = 21$,

whence $$x = 7 \text{ or } -3.$$

Each of these numbers will satisfy the conditions of the problem, provided, in the case of the second, that a negative possession be regarded as an indebtedness; *i.e.*, the man may either *possess* \$7, or he may *owe* \$3.

Prob. 5. The sum of the ages of a father and his son is 100 years, and one tenth of the product of the number of years in their ages, minus 180, equals the number of years in the father's age; what is the age of each?

Solution

Let $x =$ the number of years in the father's age.

Then $100 - x =$ the number of years in the son's age,

and the condition of the problem states that

$$\frac{x(100 - x)}{10} - 180 = x,$$

whence $x = 60$ or 30.

Although each of these numbers is a positive integer, yet the second is not a solution of the problem, since it would make the son older than the father. Hence the father is 60, and the son 40 years old.

If, in the above problem, "two persons" be substituted for "a father and his son," then both solutions are admissible, and their ages are either 60 and 40, or 30 and 70 years.

PROBLEMS

6. Find two numbers whose difference is 11, and whose sum multiplied by the greater is 513.

7. A man purchased a flock of sheep for $75. If he had paid the same sum for a flock containing 3 more sheep they would have cost him $1.25 less per head. How many did he purchase?

Is each solution of the *equation* of this problem a solution of the problem itself? Why?

8. A clothier having purchased some cloth for $30 found that if he had received 3 yards more for the same money, the cloth would have cost him 50 cents less per yard. How many yards did he purchase? Has this problem more than one solution?

9. Divide 10 into two parts whose product is $22\frac{3}{4}$.

10. Find two numbers whose sum is 10 and whose product is 42. Are there any two *real* numbers which satisfy these requirements?

11. Find two consecutive integers the sum of whose squares is 61. How many solutions has the equation of this problem? Show that each of these solutions of the equation is also a solution of the problem itself.

12. Are there two consecutive integers the sum of whose squares is 118? Are there two *numbers* whose difference is 1, and the sum of whose squares is 118? What are they? How does the second of the above questions differ from the first?

13. Find three consecutive integers whose sum is equal to the product of the first two.

14. Is it possible to find three consecutive integers whose sum equals the product of the first and last? How is the impossibility of such a set of numbers shown?

15. If the number of eggs which can be bought for 25 cents is equal to twice the number of cents which 8 eggs cost, what is that number? How many solutions has the equation of this problem? Is each of these a solution of the problem itself? Explain.

16. A farmer, having taken some eggs to market, found that their price had risen $2\frac{2}{5}$ cents per dozen, and he also discovered that he had broken 6 eggs. He received $2.88 for his eggs, which was exactly what he would have received if he had broken none, and if the price had not risen. How many eggs did he take to the market?

Is each solution of the *equation* of the problem a solution of the problem itself? Explain.

17. Find two numbers whose sum is $\frac{5}{6}$, and whose difference is equal to their product. How many solutions has this problem?

18. The product of three consecutive integers is divided by each of them in turn, and the sum of the three quotients is 74. What are these integers? How many solutions has this problem? Explain.

19. If the product of two numbers is 6, and the sum of their reciprocals is $\frac{35}{36}$, what are the numbers? How many solutions has the *equation* of this problem? How many solutions has the problem itself? Explain.

20. A merchant who had purchased a quantity of flour for $96 found that if he had obtained 8 barrels more for the same money, the price per barrel would have been $2 less. How many barrels did he buy? How many solutions has this problem? Explain.

21. Why is it that the solutions of the equation of a problem are not always solutions of the problem itself? Compare the last paragraph in § 171.

22. The area of a rectangle is $55\frac{1}{4}$ sq. in., and the sum of its length and breadth is 15 in. How long is the rectangle?

23. Find the length of a rectangle whose area is 464 sq. in., and the sum of whose length and breadth is 16 in.

What is the interpretation of the imaginary result in this problem (cf. note 1, § 100)? Does an imaginary result *always* show that the conditions of the problem are impossible of fulfillment (cf. Prob. 10, above)?

24. A boating club on returning from a short cruise found that its expenses had been $90, and that the number of dollars each member had to pay was less by $4\frac{1}{2}$ than the number of men in the club. How many men were there in the club?

25. If in Prob. 24 the expense of the cruise had been $145, the other conditions remaining unchanged, how many members would the club contain?

What is the significance of the fractional and negative results in this problem? Do such results always indicate that the conditions of a problem are impossible of fulfillment?

26. The cost of an entertainment was $20, and was to have been shared equally by those present. Four of them, however, left without paying, and this made it necessary for each of the others to pay 25 cents extra. How many persons attended the entertainment?

27. The number of miles to a certain city is such that its square root, plus its half, equals 12. What is the distance?

Has this problem more than one solution? Explain.

28. When a certain train has traveled 5 hours it is still 60 miles from its destination; if it is also known that, by traveling 5 miles faster per hour, 1 hour could be saved on the whole trip, what is the entire distance? And what is the actual speed?

29. The diagonal and the longer side of a rectangle are together five times the shorter side, and the longer side exceeds the shorter by 35 yards. What is the area of the rectangle?

30. It took a number of men as many days to dig a trench as there were men. If there had been 6 more men, the work would have been done in 8 days. How many men were there?

31. A crew can row $5\frac{1}{2}$ miles downstream and back again in 2 hours and 23 minutes; if the rate of the current is $3\frac{1}{2}$ miles an hour, find the rate at which the crew can row in still water.

32. A crew can row a certain course upstream in $8\frac{4}{7}$ minutes, and if there were no current, they could row it in 7 minutes less than it takes them to drift down the stream. How long would it take them to row the course downstream?

33. The hypotenuse of a right-angled triangle is 10 inches, and one of the sides is 2 inches longer than the other; required the length of the sides.

34. From a thread whose length is equal to the perimeter of a square, one yard is cut off, and the remainder is equal to the perimeter of another square whose area is $\frac{4}{9}$ of that of the first. What is the length of the thread at first?

35. If one train by going 15 miles an hour faster than another, requires 12 minutes less than the other to run 36 miles, what is the speed of each train?

36. A tank can be filled by one of its two feed-pipes in 2 hours less time than by the other, and by both pipes together in $1\frac{7}{8}$ hours. How long will it take each pipe separately to fill the tank?

37. A man who owned a lot 56 rods long and 28 rods wide constructed a street of uniform width along its entire border, and thereby decreased the available area of the lot by 2 acres. What was the width of the street?

38. Of two casks, one contains a certain number of gallons of water, and the other $\frac{1}{2}$ as many gallons of wine; 6 gallons are drawn from each cask, and then emptied into the other, after which it is found that the percentage of wine is the same in the one cask as in the other. How many gallons of water did the first cask originally contain?

39. A and B together can do a given piece of work in a certain time; but if they each do one half of this work separately, A would have to work 1 day less, and B 2 days more, than when they work together. In how many days can they do the work together?

40. In going a mile, the hind wheel of a carriage makes 145 revolutions less than the front wheel, but if the circumference of the hind wheel were 16 inches greater, it would then make 200 revolutions less than the front wheel. What is the circumference of the front wheel?

172. Equations above second degree, but solved like quadratics. Two important classes of equations of higher degree than the second can be solved like quadratics; they are discussed below.

(i) *Equations in the quadratic form.* Equations which contain only two *different* powers of the unknown number, the exponent of one being twice that of the other, may all be reduced to equivalent equations of the form $ax^{2n} + bx^n + c = 0$; such equations are said to be in the **quadratic form,** and may be solved like quadratics.

Ex. 1. Solve the equation $2\,x^2(x^2+1)=5-x^2$.

Solution. The given equation is equivalent to $2\,x^4+3\,x^2-5=0$, and on putting y in place of the lower power of x, *i.e.*, putting $y=x^2$, this equation becomes

$$2\,y^2+3\,y-5=0,$$

whence $$y=\frac{-3\pm\sqrt{9+40}}{4}, \qquad \text{[§ 166}$$

i.e., $$y=1 \text{ or } -\tfrac{5}{2},$$

and therefore $$x^2=1 \text{ or } -\tfrac{5}{2},$$

whence $$x=\pm 1 \text{ or } \pm\sqrt{-\tfrac{5}{2}};$$

i.e., the roots of the given equation are: $+1$, -1, $+\frac{1}{2}\sqrt{-10}$, and $-\frac{1}{2}\sqrt{-10}$.

Ex. 2. Solve the equation $x^{\frac{2}{3}}+6\,x^{\frac{1}{3}}=3+x^{\frac{1}{3}}-x^{\frac{2}{3}}$.

Solution. The given equation is equivalent to $2\,x^{\frac{2}{3}}+5\,x^{\frac{1}{3}}-3=0$, or, on putting y for $x^{\frac{1}{3}}$, to $2\,y^2+5\,y-3=0$;

whence $$y=\frac{-5\pm\sqrt{25+24}}{4},$$

i.e., $$y=\tfrac{1}{2} \text{ or } -3,$$

and therefore $$x^{\frac{1}{3}}=\tfrac{1}{2} \text{ or } -3,$$

whence $$x=\tfrac{1}{8} \text{ or } -27.$$

Ex. 3. Solve the equation $\sqrt{x^2-5\,x+10}=2\,x^2-10\,x+14$.

Solution. Since the rational part of this equation is, so far as the terms containing x are concerned, simply a multiple of the part under the radical, therefore the equation may be easily transformed into the quadratic form; thus, the given equation is equivalent to

$$\sqrt{x^2-5\,x+10}=2\,(x^2-5\,x+10)-6;$$

and, on letting y stand for $\sqrt{x^2-5\,x+10}$, the given equation becomes

$$y=2\,y^2-6,$$

whence $$y=2 \text{ or } -\tfrac{3}{2},$$

i.e., $$\sqrt{x^2-5\,x+10}=2 \text{ or } -\tfrac{3}{2},$$

and therefore $$x^2-5\,x+10=4 \text{ or } \tfrac{9}{4},$$

whence $$x=2,\ 3,\ \frac{5+\sqrt{-6}}{2} \text{ or } \frac{5-\sqrt{-6}}{2}.$$

EXERCISES

4. Show that rationalizing the equation given in Ex. 3, leads to an equation of the 4th degree. Is this rational equation easily reduced to the quadratic form? Of the methods of §§ 170 and 172 which is preferable in such equations?

Solve the following equations:

5. $x^4 - 8x^2 + 12 = 0.$

6. $3v^6 - 4v^3 = 10.$

7. $x^2 + \dfrac{1}{x^2} = a^2 + \dfrac{1}{a^2}.$

8. $y^{\frac{1}{2}} - y^{\frac{1}{4}} = 6.$

9. $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24.$

10. $(x^2 + 1)^2 + 4(x^2 + 1) = 45.$

11. $x^2 - 5x + 2\sqrt{x^2 - 5x - 2} = 10.$

12. $x^{-\frac{2}{3}} + 5x^{-\frac{1}{3}} + 4 = 0.$

13. $\left(\dfrac{12}{u} - 1\right)^2 + 8\left(\dfrac{12}{u} - 1\right) = 33.$

14. $\dfrac{v^2}{v+1} - \dfrac{v+1}{v^2} = \dfrac{7}{12}.$

[Observe that $\dfrac{v+1}{v^2}$ is the reciprocal of $\dfrac{v^2}{v+1}.$]

15. $\dfrac{y+2}{y^2+4} + \dfrac{2(y^2+4)}{y+2} = \dfrac{51}{5}.$

16. $x^4 + 4x^3 - 8x + 3 = 0.$*

17. $y^4 + 2y^3 + 5y^2 + 4y = 60.$

18. $16x^4 - 8x^3 - 31x^2 + 8x + 15 = 0.$

19. $x^3 + 2x^2 - 9x = 18.$

(ii) *Reciprocal equations.* An equation which remains unchanged when, for the unknown number, its reciprocal is substituted, and the new equation is cleared of fractions, is called a **reciprocal equation.**

Reciprocal equations of the fifth and lower degrees are readily solved like quadratics, as is shown in the following examples:

Ex. 1. Solve the equation $ax^3 + bx^2 + bx + a = 0.$

Solution. This equation is equivalent to $a(x^3 + 1) + bx(x + 1) = 0,$

i.e., to
$$(x + 1) \cdot \{a(x^2 - x + 1) + bx\} = 0,$$
which is equivalent to the two equations,
$$x + 1 = 0 \text{ and } ax^2 - ax + bx + a = 0,$$
from which the values of x are easily found.

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* By extracting the square root of the first member, show that this equation may be written in the form $(x^2 + 2x - 2)^2 = 1$, from which the complete solution readily follows.

Ex. 2. Solve the equation $ax^4 + bx^3 + cx^2 + bx + a = 0$.

SOLUTION. This equation is equivalent to $ax^2 + bx + c + \frac{b}{x} + \frac{a}{x^2} = 0$,

i.e., to
$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0;$$

and, remembering that
$$x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2,$$

this equation becomes
$$a\left(x + \frac{1}{x}\right)^2 + b\left(x + \frac{1}{x}\right) + c - 2a = 0.$$

Now, on putting y for $x + \frac{1}{x}$, this last equation becomes
$$ay^2 + by + (c - 2a) = 0,$$

whence
$$y = \frac{-b \pm \sqrt{b^2 - 4a(c - 2a)}}{2a} = k_1 \text{ and } k_2, \text{ let us say;}$$

then
$$x + \frac{1}{x} = k_1, \text{ and } x + \frac{1}{x} = k_2,$$

i.e.,
$$x^2 - k_1x + 1 = 0, \text{ and } x^2 - k_2x + 1 = 0,$$

whence the four values of x are easily found when a, b, and c are known.

EXERCISES

3. Prove (from the definition) that if $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$ is a reciprocal equation, then $a = f$, $b = e$, and $c = d$, or $a = -f$, $b = -e$, and $c = -d$. Also generalize this result.

4. Show from Ex. 3, by grouping terms as in Ex. 1, that a reciprocal equation of odd degree contains the factor $x + 1$ or $x - 1$.

5. By comparing Ex. 3, show that every reciprocal equation of even degree may have its terms grouped as in Ex. 2.

Solve the following equations:

6. $2x^3 + 3x^2 + 3x + 2 = 0$.

7. $x^4 + x^3 - 4x^2 + x + 1 = 0$.

8. $y^4 - 3y^3 + 4y^2 = 3y - 1$.

9. $3x^5 + 6x^4 - 2x^3 - 2x^2 + 6x + 3 = 0$.

173. Maximum and minimum values of quadratic expressions. Evidently such an expression as $3 + 5x - x^2$ will, in general, have different values when different values are assigned to x; and it is often important to determine the greatest or the least value (*i.e.*, the **maximum** * or the **minimum** value) that such an expression may have, for real values of the letter or letters involved in the expression.

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* While this definition is somewhat narrow, it serves present purposes best.

Ex. 1. Find the maximum value of the expression $3 + 5x - x^2$, for *real* values of x.

Solution. Let m stand for the numerical value of the given expression, *i.e.*, let

$$3 + 5x - x^2 = m.$$

Then

$$x^2 - 5x + m - 3 = 0,$$

whence

$$x = \frac{5 \pm \sqrt{25 - 4(m-3)}}{2} = \frac{5 \pm \sqrt{37 - 4m}}{2}. \qquad [\S\ 166$$

From this last expression it is clear (§ 167) that x will be real only so long as $4m \ngtr 37$, *i.e.*, so long as $m \ngtr \frac{37}{4}$; hence the *greatest* value that the given expression may have, while x is real, is $\frac{37}{4}$. Moreover, since $x = \frac{5 \pm \sqrt{37 - 4m}}{2}$, therefore, $x = \frac{5}{2}$ when $m = \frac{37}{4}$; *i.e.*, $\frac{5}{2}$ is the value of x which gives the above expression its maximum value.

Ex. 2. Find the least positive value of $x + \frac{1}{x}$, for *real* values of x.

Solution. Let

$$x + \frac{1}{x} = m. \qquad [\text{Wherein } m \text{ is positive}$$

Then

$$x^2 - mx + 1 = 0,$$

whence

$$x = \frac{m \pm \sqrt{m^2 - 4}}{2}.$$

In order that x may be real, $m^2 - 4 \nless 0$, *i.e.*, $m \nless 2$; hence the least positive value of m is 2; and the corresponding value of x is 1.

Note. This exercise may also be solved thus: for any real value of x, $(x-1)^2 \nless 0$, *i.e.*, $x^2 - 2x + 1 \nless 0$, whence $x^2 + 1 \nless 2x$, whence $x + \frac{1}{x} \nless 2$—since the problem requires that x be positive (why?)—*i.e.*, 2 is then the *least* value of $x + \frac{1}{x}$; and the expression takes this value when $x = 1$.

Ex. 3. Find the range of values of the fraction $\frac{x^2 - 6x + 2}{x + 1}$, for real values of x.

Solution. Let

$$\frac{x^2 - 6x + 2}{x + 1} = m.$$

Then

$$x^2 - (6 + m)x + 2 - m = 0,$$

whence

$$x = \frac{6 + m \pm \sqrt{(6+m)^2 - 4(2-m)}}{2} = \frac{6 + m \pm \sqrt{m^2 + 16m + 28}}{2}.$$

Hence, in order that x may be real,

$$m^2 + 16m + 28 \nless 0,$$

i.e.,

$$(m + 14) \cdot (m + 2) \nless 0,$$

and, therefore, the factors $m + 14$ and $m + 2$ must both be positive or both be negative (in order that their product shall be positive); hence m

may have any value whatever from $-\infty$ to -14, and from -2 to $+\infty$, but it can not have a value *between* -14 and -2. In other words, for real values of x the given fraction has no value between -14 and -2.

Ex. 4. A window consisting of a rectangle surmounted by a semicircle, is to have a perimeter of 18 ft.; what shall be the dimensions of the rectangle in order that the window shall admit the maximum amount of light? And what will be the window's area?

SOLUTION. Let x stand for the number of feet in the width of the window;* then $\frac{x}{2}$ is the radius of the semicircular part, and $\pi\frac{x}{2}$ is the semicircle's length. And since the entire perimeter is 18 ft., therefore the height of the rectangular part must be $\frac{1}{2}\left(18 - x - \pi\frac{x}{2}\right)$, *i.e.*, $9 - \frac{\pi + 2}{4}x$.

From these dimensions it follows at once that the area of the window is

$$x\left(9 - \frac{\pi + 2}{4}x\right) + \tfrac{1}{2}\pi\left(\frac{x}{2}\right)^2, \text{ i.e., } 9x - \frac{\pi + 4}{8}x^2;$$

hence, if a represents the area,

$$9x - \frac{\pi + 4}{8}x^2 = a,$$

whence $$(\pi + 4)x^2 - 72x + 8a = 0.$$

Solving this equation gives

$$x = \frac{36 \pm \sqrt{(36)^2 - 8a(\pi + 4)}}{\pi + 4},$$

and hence, in order that x be real,

$$(36)^2 - 8a(\pi + 4) \nless 0, \text{ i.e., } a \ngtr \frac{(36)^2}{8(\pi + 4)}, \text{ which is 22.68 (nearly);}$$

hence the maximum area of the window is nearly 22.68 sq. ft.; and the width and height are, therefore, (nearly) 5.04 ft. and 2.52 ft., respectively.

EXERCISES

For real values of x, find the maximum, or the minimum, value of each of the following expressions; also the corresponding value of x:

5. $x^2 - 8x + 10$. **6.** $9 - 2x^2 + 16x$. **7.** $12 + x^2 - 2ax$.

8. Find the range of values of $\frac{x^2 + 2x - 3}{x^2 - 2x + 3}$.

9. Find the dimensions of the largest rectangular field that can be inclosed by 160 rods of fence. How many acres does this field contain?

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* The student should draw a figure to represent the window; it will make the solution easier to understand.

10. Solve Ex. 9 if a be substituted for 160.

11. Divide 20 into two parts such that the sum of their squares shall be a minimum.

12. A man who can row 4 miles per hour, and can walk 5 miles per hour, is in a boat 3 miles from the nearest point on a straight beach, and wishes to reach in the shortest time a place on the shore 5 miles from this point. Where must he land?

II. QUADRATIC EQUATIONS IN TWO OR MORE UNKNOWN NUMBERS

174. Introductory remarks. The really essential thing in solving any system of simultaneous equations, is first to combine the given equations so as to eliminate all but one of the unknown numbers, and then to solve the resulting equation containing that unknown number. When each equation of the given system is of the first degree, this elimination, as well as the solution of the resulting equation, is easily effected (§ 112); but these operations become much more difficult if one or more of the given equations is quadratic, or of a still higher degree.

The next few articles are devoted to a study of the procedure in cases where the given system consists of two equations one or both of which are quadratic.

175. One equation simple and the other quadratic. In this case elimination by substitution (cf. § 107) is usually advisable.

Ex. 1. Solve the following system of simultaneous equations:

$$\begin{cases} 3x - 2y = 3, & (1) \\ x^2 + 4y^2 = 13. & (2) \end{cases}$$

Solution. From Eq. (1), $x = \dfrac{3 + 2y}{3}$, (3)

whence, by substituting this value of x, Eq. (2) becomes

$$\left(\frac{3 + 2y}{3}\right)^2 + 4y^2 = 13, \quad (4)$$

and, on expanding and simplifying, Eq. (4) becomes

$$10y^2 + 3y - 27 = 0, \quad (5)$$

whence (§ 164) $y = \frac{3}{2}$ or $-\frac{9}{5}$. (6)

But Eq. (3) — also Eq. (1) — shows that to every value of y corresponds one, and only one, value of x; and that when $y = \frac{3}{2}$ then $x = 2$, and when $y = -\frac{9}{5}$ then $x = -\frac{1}{5}$. It is, moreover, easily verified that each of these pairs of numbers is a solution of the given system of equations.

Manifestly the above method is applicable whenever one equation of the given system is simple and the other quadratic.

EXERCISES

Solve the following systems of equations and verify the correctness of your results:

2. $\begin{cases} 4x + 3y = 9, \\ 2x^2 + 5xy = 3. \end{cases}$

3. $\begin{cases} x^2 + xy - 12 = 0, \\ x - y = 2. \end{cases}$

4. $\begin{cases} 3uv - v = 10u, \\ u + 2 = v. \end{cases}$

5. $\begin{cases} (x + 3)(y - 7) = 48, \\ x + y = 18. \end{cases}$

6. $\begin{cases} 2s + 3t = 10, \\ t(s + t) = 25. \end{cases}$

7. $\begin{cases} 2x^2 + y^2 = 3xy + 14, \\ 2x - y = 7. \end{cases}$

8. $\begin{cases} 16 + 4v + 2u^2 = 5uv, \\ 11v - 5u = 4. \end{cases}$

9. $\begin{cases} \dfrac{3x - 2}{6 + y} = 2, \\ \dfrac{x^2 + 2x + y}{y^2 - 5x + 3} = \dfrac{4}{9}. \end{cases}$

10. $\begin{cases} \dfrac{3}{xy} + 14 = \dfrac{2}{x^2} + \dfrac{1}{y^2}, \\ \dfrac{2}{x} - \dfrac{1}{y} = 7.* \end{cases}$

11. Write a rule for solving a pair of simultaneous equations one of which is simple and the other quadratic, and which contain two unknown numbers. Could two such equations containing three unknown numbers be solved? Compare § 111 note, and explain.

12. How many solutions has each of the above systems of equations (Exs. 2–10)? Has every such system two solutions, and only two? Why (see also § 176, Exs. 1 and 2)?

176. Principles involved in § 175. The success of the method of solution employed in § 175 depends upon the fact that, if X, Y,

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* Solve first for $\dfrac{1}{x}$ and $\dfrac{1}{y}$.

and Z represent any expressions whatever which contain either x or y, or both, then the system of equations

$$\begin{cases} X = 0, \\ Y \cdot Z = 0, \end{cases}$$

is equivalent to the two systems

$$\begin{cases} X = 0, \\ Y = 0, \end{cases} \quad \text{and} \quad \begin{cases} X = 0, \\ Z = 0. \end{cases}$$

To prove this equivalence, it need only be observed that every solution of either of the last two systems is evidently a solution of the first system; and every solution of the first system is found among the solutions of the last two systems, for it must make $X = 0$ and also either $Y = 0$ or $Z = 0$.*

EXERCISES

1. By means of the proof just given show that Ex. 1, § 175, has two solutions, and only two.

SUGGESTION. The given system of equations is equivalent to Eqs. (1) and (5) (Why ?), and Eq. (5) may be written in the form $(2y - 3)(5y + 9) = 0$. Compare also § 108 (iii) and § 111.

2. By means of the suggestion just given show that every system consisting of two equations, one of which is simple and the other quadratic, and containing two unknown numbers, has two solutions, and only two.

3. Show that the solutions mentioned in Ex. 2 may be imaginary (cf. Ex. 6, § 175), and also that one or both of these solutions may be infinite (cf. note, § 168).

4. In the solution of Ex. 1, § 175, are Eqs. (2) and (6) equivalent to the given system? May then the values of y from Eq. (6) be substituted in Eq. (2) to find the corresponding values of x? In which two equations may they be substituted? Why? Does your "rule" (Ex. 11, § 175) provide for this?

177. Both equations quadratic, — one homogeneous†. If both of the equations of a given system are quadratic, then elimination by substitution, as in § 176, leads to an equation of the 4th degree

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* Similarly it may be shown that the system $\begin{cases} W \cdot X = 0, \\ Y \cdot Z = 0, \end{cases}$ is equivalent to the four systems $\begin{cases} W = 0 \\ Y = 0 \end{cases}$, $\begin{cases} W = 0 \\ Z = 0 \end{cases}$, $\begin{cases} X = 0 \\ Y = 0 \end{cases}$, and $\begin{cases} X = 0 \\ Z = 0 \end{cases}$.

† An equation is said to be **homogeneous** if all of its terms are of the same degree in the unknown numbers (cf. § 41).

in one of the unknown numbers,* and this equation can not, in general, be solved by the methods already studied.

If, however, one of the given equations is *homogeneous,* then the solution of the system may always be made to depend upon the solution of a quadratic equation in one unknown number; this is illustrated below.

Ex. 1. Solve the following system of equations:

$$\begin{cases} 6x^2 + 5xy - 6y^2 = 0, & (1) \\ 2x^2 - y^2 + 5x = 9. & (2) \end{cases}$$

SOLUTION. On dividing Eq. (1) by y^2, it becomes

$$6\left(\frac{x}{y}\right)^2 + 5\left(\frac{x}{y}\right) - 6 = 0, \qquad (3)$$

whence (§ 164) $$\frac{x}{y} = \frac{2}{3}, \text{ or } \frac{x}{y} = -\frac{3}{2}, \qquad (4)$$

i.e., $$x = \tfrac{2}{3}y, \text{ or } x = -\tfrac{3}{2}y. \qquad (5)$$

On substituting the *first* of these two values of x, viz., $\frac{2}{3}y$, in Eq. (2), that equation becomes

$$2(\tfrac{2}{3}y)^2 - y^2 + 5(\tfrac{2}{3}y) = 9, \qquad (6)$$

i.e., $$y^2 - 30y + 81 = 0, \qquad (7)$$

whence (§ 164) $$y = 27 \text{ or } y = 3, \qquad (8)$$

and, since $x = \frac{2}{3}y$, the *corresponding* values of x are 18 and 2.

By substituting these pairs of numbers, viz., $x = 18$, $y = 27$, and $x = 2$, $y = 3$, in the given system of equations, it is easily verified that each pair is a solution of that system.

Similarly, if the *second* of the two values of x in Eq. (5), viz., $-\frac{3}{2}y$, be substituted in Eq. (2), two other solutions of the given system of equations will be found; these are: $x = -\frac{9}{2}$, $y = 3$, and $x = \frac{9}{7}$, $y = -\frac{6}{7}$.

It is, moreover, evident that every such system of equations may be solved by this method.

NOTE 1. The success of the method of solution here employed is due to the fact that the two systems of equations from which the values of x and y were finally found, are together equivalent to the given system.

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* For example, given the system $x^2 - 3x + 8y = 4$ and $3x^2 - 16y^2 + 20y = 9$. Solving the second of these equations for y gives $y = \frac{1}{8}(5 \pm \sqrt{12x^2 - 11})$, and on substituting this value of y, Eq. (1) becomes $x^2 - 3x + 5 \pm \sqrt{12x^2 - 11} = 4$, which, when rationalized, is $x^4 - 6x^3 - x^2 - 6x + 12 = 0$.

This equivalence may be seen by writing the given system thus:

$$\begin{cases}(3x-2y)(2x+3y)=0,\\ 2x^2-y^2+5x=9,\end{cases}$$

and recalling that, by § 176, this system is equivalent to the two systems

$$\left.\begin{cases}3x-2y=0,\\ 2x^2-y^2+5x=9,\end{cases}\right\} \text{ and } \left.\begin{cases}2x+3y=0,\\ 2x^2-y^2+5x=9,\end{cases}\right\}$$

from which the above solutions were obtained.

Moreover, since each of these systems has two solutions, and only two (§ 176), therefore the given system has four solutions, and only four.

NOTE 2. In practice the above method may be somewhat simplified by putting a single letter, say v, in place of the fraction $\frac{x}{y}$ in Eq. (3), *i.e.*, by putting $x = vy$ in the *homogeneous* equation. Thus, on substituting vy for x in Eq. (1), it becomes

$$6v^2y^2+5vy^2-6y^2=0,$$

and hence, dividing by y^2, $6v^2+5v-6=0,$

whence (§ 164) $v=\frac{2}{3}$ or $v=-\frac{3}{2}$;

and, since $x = vy$, therefore $x=\frac{2}{3}y$ and $x=-\frac{3}{2}y$. From here on the work is the same as that already given.

EXERCISES

Solve the following systems of equations and verify the correctness of your results:

2. $\begin{cases}5x^2+4xy=y^2,\\ x^2+3x=5+y.\end{cases}$

3. $\begin{cases}x^2+xy-14=y-x,\\ 2x^2-3y^2=xy.\end{cases}$

4. $\begin{cases}2(x^2+y^2)=5xy,\\ x^2-y^2=75.\end{cases}$

5. $\begin{cases}x^2-2xy-3y^2=0,\\ y(x+y)=4.\end{cases}$

6. Show that every such system of equations as those above has four solutions (real or imaginary, finite or infinite), and only four.

178. Both equations homogeneous in the terms containing the unknown numbers. The solution of a system consisting of two quadratic equations, each of which is homogeneous *in the terms which contain the unknown numbers,* is easily made to depend upon § 177.

Ex. 1. Solve the following system of equations:

$$\begin{cases}3x^2+3xy+2y^2=8, & (1)\\ x^2-xy-4y^2=2. & (2)\end{cases}$$

SOLUTION. On subtracting Eq. (1) from 4 times Eq. (2), the result is

$$x^2-7xy-18y^2=0, \qquad (3)$$

and the given system of equations is equivalent to the system consisting of Eq. (3) together with either Eq. (1) or Eq. (2); but of this last system Eq. (3) is homogeneous, and hence the system can be solved by the method of § 177.

EXERCISES

2. By the method of § 177 complete the solution of Ex. 1 above, *i.e.*, solve the equations $\begin{cases} x^2 - 7\,xy - 18\,y^2 = 0, \\ x^2 - xy - 4\,y^2 = \text{[illegible]} \end{cases}$ and verify the correctness of your results.

Solve the following systems of equations and verify your results:

3. $\begin{cases} 4\,x^2 - xy - 3\,y^2 = 2, \\ x^2 + 6\,xy - y^2 = -\,6. \end{cases}$

$\begin{cases} y^2 + 15 = 2\,xy, \\ x^2 + y^2 = 21 + xy. \end{cases}$

4. $\begin{cases} 2\,x^2 - xy = 28, \\ x^2 + 2\,y^2 = 18. \end{cases}$

6. $\begin{cases} x^2 + 5\,xy = 3 - 6\,y^2, \\ x^2 - 25 = 2\,y\,(y + 2\,x). \end{cases}$

7. Substitute vy for x in each of the equations of Ex. 6; then solve each of the resulting equations for y^2 in terms of v; from the first equation you will find $y^2 = \frac{3}{v^2+5\,v+6}$, and from the second, $y^2 = \frac{25}{v^2-4\,v-2}$; now equate these two values of y^2, solve the resulting equation in v, and from its values find the values of y, and thence the corresponding values of x.

8. Solve Exs. 4 and 5 above, by the method outlined in Ex. 7.

9. Is the method of Ex. 7 easier or more difficult than that outlined in Ex. 1? In what respect?

10. Is the method of Ex. 7 applicable to *all* such exercises as those given above?

11. Solve the system $\begin{cases} 3\,x^2 - 5\,xy - 4\,y^2 = 3\,x, \\ 9\,x^2 + xy - 2\,y^2 = 6\,x. \end{cases}$

Suggestion. Subtract the second of these equations from twice the first, and then proceed as in Exs. 1 and 2 above.

12. By the method of Ex. 11, solve the following system of equations, and verify your results:

$$\begin{cases} 4\,x^2 + 6\,xy - y^2 = \tfrac{1}{3}\,y, \\ 6\,x^2 - 9\,xy + 2\,y^2 = 2\,y. \end{cases}$$

13. Show that the method suggested in Ex. 11 may be successfully applied to any system of equations whatever of the form

$$\begin{cases} ax^2 + bxy + cy^2 = dx, \\ a'x^2 + b'xy + c'y^2 = d'x. \end{cases}$$

14. Could the method suggested in Ex. 7 be employed in such systems of equations as those given in Exs. 11, 12, and 13? Explain.

Solve the following systems of equations, and verify your results:

15. $\begin{cases} x^2 - xy - y^2 = 20, \\ x^2 - 3xy + 2y^2 = 8. \end{cases}$ **16.** $\begin{cases} u^2 + 3uv + v^2 = 61, \\ u^2 - v^2 = 31 - 2uv. \end{cases}$

17. $\begin{cases} [illegible] - \frac{5y}{2} + \frac{2x}{3} = x - 2, \\ \frac{4}{3} [illegible] \frac{4y^2}{x-1} = \frac{y^2 + 2xy}{2(1-x)}. \end{cases}$

179. Special devices. The kinds of systems of equations specified in §§ 175, 177, and 178 occur frequently, and, although they present themselves in a great variety of forms, they may *always* be solved by the methods there given.

It is worth remarking, however, that special devices of elimination sometimes give simpler and more elegant solutions, not only for the systems already considered, but also for many others which need not now be classified. Some of these special devices are illustrated in the following examples, where it is also shown that they apply to some exercises in which equations above the second degree are involved.

Facility in the use of these special devices can be acquired only by practice, but a little study of any particular problem will often suggest a suitable method for attacking it.

Ex. 1. Solve the equations $\begin{cases} x - y = 5, & (1) \\ xy = -6. & (2) \end{cases}$

SOLUTION. From Eq. (1), $x^2 - 2xy + y^2 = 25,$ (3)

from Eq. (2), $4xy = -24,$ (4)

adding Eq. (4) to Eq. (3), $x^2 + 2xy + y^2 = 1,$ (5)

whence $x + y = \pm 1;$ (6)

and from Eq. (1) and Eq. (6), $x = 3$ or $2.$

The *corresponding* values of y are $y = -2$ or $-3.$

Observe that this exercise belongs to the class of § 175, and could have been solved by the method there given.

Ex. 2. Solve the equations $\begin{cases} x^2 + 3xy = 54, & (1) \\ xy + 4y^2 = 115. & (2) \end{cases}$

SOLUTION. On adding Eqs. (1) and (2), we obtain

$$x^2 + 4xy + 4y^2 = 169,$$

i.e., $(x + 2y)^2 = 169,$ (3)

whence $x + 2y = \pm 13.$ (4)

From the first of the two equations in (4), and either Eq. (1) or Eq. (2), by § 175, it is found that $x = 3$, $y = 5$ and $x = 36$, $y = -11^1$ are solutions. Similarly, by using the second equation in (4), it is found that $x = -36$, $y = 11\frac{1}{2}$ and $x = -3$, $y = -5$ are also solutions of the given system of equations.

Observe that this exercise belongs to the class of § 178, and could have been solved by the method there given.

Ex. 3. Solve the equations
$$\begin{cases} x^2 + y^2 = 6, & (1) \\ xy = 2(x + y) - 5. & (2) \end{cases}$$

Solution. On adding 2 times Eq. (2) to Eq. (1), we obtain

$$x^2 + 2\,xy + y^2 = 4(x + y) - 4, \tag{3}$$

i.e.,
$$(x + y)^2 - 4(x + y) + 4 = 0; \tag{4}$$

$\{(x+y)-2\}^2$

whence
$$x + y = 2. \tag{5}$$

Substituting this value of $x + y$ in Eq. (2) gives

$$xy = 4 - 5 = -1; \tag{6}$$

and 2 times Eq. (6) subtracted from Eq. (1) gives

$$x^2 - 2\,xy + y^2 = 8, \tag{7}$$

whence
$$x - y = \pm 2\sqrt{2}. \tag{8}$$

From Eq. (5) and Eq. (8), it follows that $x = 1 + \sqrt{2}$, $y = 1 - \sqrt{2}$, and $x = 1 - \sqrt{2}$, $y = 1 + \sqrt{2}$ are solutions of the given equations.

Equations like those in Ex. 3, which are not changed by interchanging x and y, are usually said to be **symmetric** with regard to those letters.

If the equations of a given system are symmetric, or symmetric except for the *signs* of one or more terms, their solution is often facilitated by substituting $u+v$ for one of the letters and $u-v$ for the other; this method of solution is illustrated in Exs. 4–6 below.

Ex. 4. Solve the equations
$$\begin{cases} x^2 + y^2 = 6, & (1) \\ xy = 2(x + y) - 5. & (2) \end{cases}$$

Solution. On putting $x = u + v$ and $y = u - v$, the given equations become, respectively,

$$2\,u^2 + 2\,v^2 = 6, \text{ and } u^2 - v^2 = 4\,u - 5; \tag{3}$$

therefore, eliminating v^2 and simplifying,

$$u^2 - 2\,u + 1 = 0,$$

whence
$$u = 1.$$

Substituting this value of u in either one of Eqs. (3), gives

$$v = \pm\sqrt{2},$$

whence (since $x = u + v$, and $y = u - v$)

$$x = 1 \pm \sqrt{2}, \text{ and } y = 1 \mp \sqrt{2},$$

which agrees with the result found in Ex. 3 above.

Ex. 5. Solve the equations $\begin{cases} xy = -6, & (1) \\ x - y = 5. & (2) \end{cases}$

Solution. On putting $x = u + v$, and $y = u - v$, the given equations become, respectively, $u^2 - v^2 = -6$, and $2v = 5$. (3)

From the second of these, $v = \frac{5}{2}$,

and substituting this in the first gives

$$u = \pm \tfrac{1}{2},$$

whence $x = 3$ or 2, and $y = -2$ or -3 (cf. Ex. 1, above).

Ex. 6. Solve the equations $\begin{cases} x^3 + y^3 = xy - 5, \\ x + y + 1 = 0. \end{cases}$

Solution. On putting $x = u + v$ and $y = u - v$, the given equations become, respectively,

$$2u^3 + 6uv^2 - u^2 + v^2 + 5 = 0, \text{ and } 2u + 1 = 0.$$

From the second of these equations,

$$u = -\tfrac{1}{2},$$

and substituting this value in the first gives

$$v = \pm \tfrac{3}{2},$$

whence $x = 1$ or -2, and $y = -2$ or 1.

Ex. 7. Solve the equations $\begin{cases} x^4 + y^4 = 17, & (1) \\ x + y = 3. & (2) \end{cases}$

Solution. This example may be solved like Exs. 4, 5, and 6; another solution is as follows:

On raising each member of Eq. (2) to the 4th power, we obtain

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 81, \quad (3)$$

whence, by subtracting Eq. (1) from Eq. (3) and simplifying,

$$xy(2x^2 + 3xy + 2y^2) = 32; \quad (4)$$

from Eq. (2), $2x^2 + 3xy + 2y^2 = 18 - xy$, (5)

whence, on substituting from Eq. (5), Eq. (4) becomes

$$xy(18 - xy) = 32, \quad (6)$$

i.e., $(xy)^2 - 18(xy) + 32 = 0$, (7)

whence (§ 164) $xy = 2$ or 16. (8)

By combining Eq. (8) with Eq. (2) it is now easy to show that

$$x = 1,\ 2,\ \text{or}\ \frac{3 \pm \sqrt{-55}}{2},$$

and the corresponding values of y are

$$y = 2,\ 1,\ \text{and}\ \frac{3 \mp \sqrt{-55}}{2},\ \text{respectively.}$$

If one of two equations is exactly divisible by the other, member by member, their solution may often be greatly simplified, as is shown below.

Ex. 8. Solve the equations $\begin{cases} x^2 - y^2 = 3, & (1) \\ x - y = 1. & (2) \end{cases}$

SOLUTION. On dividing Eq. (1) by Eq. (2), member by member, we obtain

$$x + y = 3, \qquad (3)$$

whence, from Eqs. (2) and (3),

$$x = 2,\ \text{and}\ y = 1.$$

Ex. 9. Solve the equations $\begin{cases} x^3 - 8 = (x^2 - y^2)y, & (1) \\ x + y = 2. & (2) \end{cases}$

SOLUTION. By transposing, Eq. (2) becomes

$$x - 2 = -y, \qquad (3)$$

and, dividing Eq. (1) by Eq. (3), member by member, we obtain

$$x^2 + 2x + 4 = -x^2 + y^2, \qquad (4)$$

whence, from Eqs. (2) and (4), by § 175,

$$x = 0 \text{ or } -6, \text{ and } y = 2 \text{ or } 8.$$

NOTE. That this method of division must be applied with some caution is, however, evident from Ex. 9, for, while it is easily verified that the two pairs of numbers there found are solutions of the given system of equations, that system has another solution, viz., $x = 2$, and $y = 0$, which the above process has failed to reveal. This last solution is found by equating each member of Eq. (3) separately to zero.*

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* The general theory for such cases may be stated thus: if P, Q, R, and S represent any expressions whatever, which contain either x or y or both, then the system of equations $\left\{\begin{matrix} P \cdot Q = R \cdot S, \\ P = S, \end{matrix}\right\}$ is equivalent to the two systems $\left\{\begin{matrix} Q = R, \\ P = S, \end{matrix}\right\}$ and $\left\{\begin{matrix} P = 0, \\ S = 0; \end{matrix}\right\}$ because every solution of either of the last two systems is evidently a solution of the first system, and every solution of the first system is found among the solutions of the last two systems.

In Ex. 9 above, $P = x - 2$, $S = -y$, $Q = x^2 + 2x + 4$, and $R = -x^2 + y^2$.

Ex. 10. Solve the equations $\begin{cases} \dfrac{3}{x} - \dfrac{2}{y} = 3, \\ \dfrac{1}{x^2} + \dfrac{4}{y^2} = 13. \end{cases}$

SOLUTION. These equations being fractional, the first step toward their solution would ordinarily be to clear them of fractions; in cases like this it is, however, easier to regard $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown numbers, and to eliminate without first clearing of fractions.

If, for brevity, u and v be substituted for $\frac{1}{x}$ and $\frac{1}{y}$, respectively, the given equations become, respectively,

$$3u - 2v = 3,$$

and $$u^2 + 4v^2 = 13,$$

whence (§ 175) $$u = 2 \text{ or } -\tfrac{1}{5}, \text{ and } v = \tfrac{3}{2} \text{ or } -\tfrac{9}{5},$$

and therefore $$x = \tfrac{1}{2} \text{ or } -5, \text{ and } y = \tfrac{2}{3} \text{ or } -\tfrac{5}{9}.$$

EXERCISES

Solve the following systems of equations:

11. $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$

12. $\begin{cases} x^2 + y^2 = 1, \\ 25xy + 12 = 0. \end{cases}$

13. $\begin{cases} x^2 + y^2 + x = y + 26, \\ xy = 12. \end{cases}$

14. $\begin{cases} x^2 + y^2 = a, \\ x + y = b. \end{cases}$

15. $\begin{cases} u^2 + v^2 = 61, \\ u + v = 11. \end{cases}$

16. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}, \\ \dfrac{1}{xy} + \dfrac{1}{18} = 0. \end{cases}$

17. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 74, \\ \dfrac{1}{x} - \dfrac{1}{y} = 2. \end{cases}$

18. $\begin{cases} \dfrac{x}{y} - \dfrac{y}{x} = \dfrac{16}{15}, \\ x - y = 2. \end{cases}$

19. $\begin{cases} x^3 + y^3 = 26, \\ x + y = 2. \end{cases}$

20. $\begin{cases} r^3 - p^3 = 91, \\ r - p = 7. \end{cases}$

21. $\begin{cases} \dfrac{1}{x^3} + \dfrac{1}{y^3} = 91, \\ \dfrac{1}{x} + \dfrac{1}{y} = 7. \end{cases}$

22. $\begin{cases} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2, \\ x^{\frac{3}{2}} + y^{\frac{3}{2}} = 26. \end{cases}$

23. $\begin{cases} x^3 + y^3 = a, \\ x + y = b. \end{cases}$

24. $\begin{cases} x^4 + y^4 = 97, \\ x + y = -1. \end{cases}$

25. $\begin{cases} m^2n^2 = 96 - 4\,mn, \\ m + n = 6. \end{cases}$

26. $\begin{cases} x^2 + xy + y^2 = 84, \\ x - \sqrt{xy} + y = 6. \end{cases}$

27. $\begin{cases} s^3 - t^3 = 37, \\ st\,(s - t) = 12. \end{cases}$

28. $\begin{cases} x + y = 25, \\ \sqrt{x} + \sqrt{y} = 7. \end{cases}$

29. $\begin{cases} x^2 - 3\,xy + y^2 = 5, \\ x^4 + y^4 = 2. \end{cases}$

30. $\begin{cases} x + y + 2\sqrt{x + y} = 24, \\ x - y + 3\sqrt{x - y} = 10. \end{cases}$

31. $\begin{cases} x^2 + y^2 + 6\sqrt{x^2 + y^2} = 55, \\ x^2 - y^2 = 7. \end{cases}$

32. $\begin{cases} \dfrac{x + y}{x - y} + \dfrac{x - y}{x + y} = \dfrac{10}{3}, \\ x^2 + y^2 = 45. \end{cases}$

33. $\begin{cases} 2(x^2 + y^2) = 5\,xy, \\ x^{-1} + y^{-1} = 1.5. \end{cases}$

34. $\begin{cases} (2 + x)(y + 1) = 4, \\ (2 + x)^{\frac{1}{2}} - (y + 1)^{\frac{1}{2}} = \frac{1}{6}. \end{cases}$

35. $\begin{cases} \sqrt{x^2 - y^2} + 2 = \sqrt{x^2 + y^2}, \\ 2\sqrt{x + y} = 2\sqrt{x - y} + 3. \end{cases}$

36. $\begin{cases} 3\,x^{-2} - y^{-2} = 1, \\ 5\,x^{-2} - (xy)^{-1} + 2\,y^{-2} = 3. \end{cases}$

37. $\begin{cases} 3\,xy + 3\,xy^{-1} = 5, \\ 3\,xy + 3\,x^{-1}y = 2.5. \end{cases}$

38. $x - \dfrac{b^2}{y} = \dfrac{a^2}{x} - y = a - b.$

180. Systems containing three or more unknown numbers. Although the solution of a system consisting of three or more simultaneous quadratic equations (involving as many unknown numbers as there are equations in the system) can not in general be made to depend upon the solution of a quadratic equation in one unknown number, yet some solutions of special cases of such systems may be found in this way.

Ex. 1. Solve the equations

$$\begin{cases} x^2 + xy + xz = 2, & (1) \\ xy + y^2 + yz = -2, & (2) \\ xz + yz + z^2 = 4. & (3) \end{cases}$$

Solution. Since these equations may be written in the form

$$\begin{cases} x(x + y + z) = 2, & (4) \\ y(x + y + z) = -2, & (5) \\ z(x + y + z) = 4, & (6) \end{cases}$$

therefore, dividing Eqs. (5) and (6) by Eq. (4), member by member, we obtain

$$\frac{y}{x} = -1, \text{ and } \frac{z}{x} = 2, \qquad (7)$$

i.e.,

$$y = -x, \text{ and } z = 2\,x; \qquad (8)$$

substituting these values of y and z, in terms of x, Eq. (1) becomes

$$x^2 = 1,$$

whence
$$x = \pm 1;$$

and, substituting these values of x in Eq. (8), we obtain

$$x = 1,\ y = -1,\ z = 2, \text{ and also } x = -1,\ y = 1,\ z = -2,$$

as solutions of the given system of equations.

Ex. 2. Solve the equations
$$\begin{cases} 4\,xy - 3\,x - 2\,y = 0, & (1) \\ 2\,xz - 3\,x - 6\,z = 0, & (2) \\ 5\,yz + 3\,y - 4\,z = 0. & (3) \end{cases}$$

SOLUTION. On dividing these equations by xy, xz, and yz, respectively, they become

$$\begin{cases} 4 - \frac{3}{y} - \frac{2}{x} = 0, \\ 2 - \frac{3}{z} - \frac{6}{x} = 0, \\ 5 + \frac{3}{z} - \frac{4}{y} = 0. \end{cases}$$

These last equations, being of the first degree in the fractions $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$, may be readily solved for $\frac{1}{x}$, etc., and hence the values of x, y, and z themselves be found.

Ex. 3. Solve the equations
$$\begin{cases} 2\,x + 2\,y - z = 3, & (1) \\ x - 6\,y + z = 2, & (2) \\ x^2 - 8\,y^2 + 3\,yz = 16. & (3) \end{cases}$$

SOLUTION. From Eqs. (1) and (2), $y = \frac{3\,x - 5}{4}$ and $z = \frac{7\,x - 11}{2}$; on substituting these expressions for y and z in Eq. (3), and reducing, it becomes

$$5\,x^2 - 12\,x - 9 = 0,$$

whence
$$x = 3 \text{ or } -\tfrac{3}{5},$$

and the corresponding values of y and z are readily found.

EXERCISES

4. $\begin{cases} xy = 30, \\ yz = 60, \\ xz = 50. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 13, \\ y^2 + z^2 = 34, \\ x^2 + z^2 = 29. \end{cases}$

6. $\begin{cases} \dfrac{xyz}{x + y} = 1.2, \\ \dfrac{xyz}{y + z} = 1.5, \\ \dfrac{xyz}{z + x} = 2. \end{cases}$

7. $\begin{cases}(x+y)(x+z)=2,\\(y+z)(y+x)=3,\\(z+x)(z+y)=6.\end{cases}$

8. $\begin{cases}x^2+y^2+z^2=29,\\xy+yz+zx=-10,\\x+y+5=z.\end{cases}$

9. $\begin{cases}\dfrac{x^2+y^2}{xyz}=\dfrac{5}{6},\\[2ex]\dfrac{y^2+z^2}{xyz}=\dfrac{5}{3},\\[2ex]\dfrac{z^2+x^2}{xyz}=\dfrac{13}{6}.\end{cases}$

181. Square roots of binomial quadratic surds. Having now learned how to solve simultaneous quadratic equations, it is possible to deal with an interesting problem which was necessarily postponed from Chapter XIII; this problem is the extraction of the square root of a binomial quadratic surd.

Ex. 1. Find the square root of $8+\sqrt{60}$.

SOLUTION. Let $\sqrt{x}+\sqrt{y}=\sqrt{8+\sqrt{60}}$.

Then, by squaring, $x+2\sqrt{xy}+y=8+\sqrt{60}$,

i.e., $x+y+2\sqrt{xy}=8+\sqrt{60}$,

whence (§ 145) $x+y=8$ and $2\sqrt{xy}=\sqrt{60}$;

combining these last two equations—after squaring the second—easily leads (§ 175) to the solution

$$x=3,\ y=5;$$

therefore $\sqrt{8+\sqrt{60}}=\sqrt{3}+\sqrt{5}$,

as is easily verified by squaring each member of this last equation.

Ex. 2. Find the square root of $a-\sqrt{b}$.

SOLUTION. Let $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$.

Then, as before, $x+y=a$ and $4\,xy=b$,

whence (§ 175) $x=\frac{1}{2}(a+\sqrt{a^2-b})$ and $y=\frac{1}{2}(a-\sqrt{a^2-b})$,

and, therefore, $\sqrt{a-\sqrt{b}}=\sqrt{\dfrac{a+\sqrt{a^2-b}}{2}}-\sqrt{\dfrac{a-\sqrt{a^2-b}}{2}}$, as is easily verified.

NOTE. The above solution shows that although an expression can always be found whose square is $a-\sqrt{b}$, yet, unless a^2-b happens to be a perfect square, the expression so found is more complicated than $\sqrt{a-\sqrt{b}}$; in other words, the procedure of Exs. 1 and 2 is of advantage only when a^2-b is a perfect square.

EXERCISES

3. In Ex. 1 above, why is $x + y$ equal to 8, and $2\sqrt{xy}$ equal to $\sqrt{60}$?

Find the square root of each of the following expressions:

4. $25 + 10\sqrt{6}$. **5.** $11 + 6\sqrt{2}$. **6.** $47 - 12\sqrt{11}$. **7.** $18 - 6\sqrt{5}$.

8. If the numerical value of $\sqrt{21 + 8\sqrt{5}}$ is required, is it easier to find first the binomial whose square is $21 + 8\sqrt{5}$, or to begin by extracting the square root of 5? Explain. Also answer this question if $12 - 6\sqrt{7}$ be substituted for $21 + 8\sqrt{5}$.

182. Square roots of complex numbers. The square root of a complex number may be found by a process similar to that used in § 181.

E.g., to find the square root of $5 + 12\sqrt{-1}$,

let $$\sqrt{x} + \sqrt{y}\sqrt{-1} = \sqrt{5 + 12\sqrt{-1}}.$$

Then, by squaring, $$x + 2\sqrt{xy}\sqrt{-1} - y = 5 + 12\sqrt{-1},$$

whence (§ 151) $$x - y = 5 \text{ and } 2\sqrt{xy} = 12,$$

and therefore (§ 175) $$x = 9 \text{ and } y = 4,$$

whence $$\sqrt{5 + 12\sqrt{-1}} = 3 + 2\sqrt{-1},$$

as is easily verified.

Similarly in general.

Note. By means of extracting square roots of complex numbers every imaginary number may be reduced to the form $a + b\sqrt{-1}$, wherein a and b are real, and $b \neq 0$.

E.g., $$\sqrt[12]{-1} = \sqrt[4]{\sqrt[3]{-1}} = \sqrt[4]{-1} \qquad [\sqrt[3]{-1} = -1$$
$$= \sqrt{\sqrt{-1}} = \sqrt{0 + \sqrt{-1}}$$
$$= \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}\sqrt{-1}. \qquad \text{[As in above example}$$

Similarly in general; for, by definition, a number is imaginary only when it contains an expression of the type $\sqrt[n]{-1}$, wherein n is an even positive integer; moreover, if n contains any *odd* factors, let their product be p and let the other factor of n be 2^s; then

$$\sqrt[n]{-1} = \sqrt[2^s p]{-1} = \sqrt[2^s]{\sqrt[p]{-1}} = \sqrt[2^s]{-1}; \quad [p \text{ being odd, } \sqrt[p]{-1} = -1$$

but, by repeatedly extracting the *square* root of an imaginary number as above, the expression $\sqrt[2^s]{-1}$ may be brought to the form $a + b\sqrt{-1}$, and thus the given number may also be brought to this form.

EXERCISES

Find the square root of each of the following expressions:

1. $5-6\sqrt{-1}$.

2. $6\sqrt{-2}-17$.

3. $3+2\sqrt{-10}$.

4. $5.125-3.75\sqrt{-2}$.

5. Reduce $\sqrt[8]{-1}$ to an equivalent expression of the form $a+b\sqrt{-1}$.

PROBLEMS

1. The sum of two numbers is 14, and the difference of their squares is 28. What are the numbers?

2. Find two numbers whose difference is 15, and such that if the greater be diminished by 12, and the smaller increased by 12, the sum of the squares of the results will be 261.

3. Find two numbers whose difference is 80, and the sum of whose square roots is 10.

4. The sum of two numbers, their product, and also the difference of their squares, are all equal; find the numbers.

5. Find two numbers whose product is 8 greater than twice their sum, and 48 less than the sum of their squares.

6. If 5 times the sum of the digits of a certain two-digit number be subtracted from the number, its digits will be interchanged, and if the number be multiplied by the sum of its digits, the product will be 648. What is the number?

7. Find two numbers such that the square of either of them equals 112 diminished by 12 times the other.

8. If the length of the diagonal of a rectangular field, containing 30 acres, is 100 rods, how many rods of fence will be required to inclose the field?

9. Find the dimensions of a rectangular field whose perimeter is 188 rods, and whose area will remain unchanged if the length be diminished by 4 rods and the width increased by 2 rods.

10. The combined capacity of two cubical coal bins is 2728 cu. ft., and the sum of their lengths is 22 ft.; find the length of the diagonal of the smaller bin.

11. It took a number of men as many days to pave a sidewalk as there were men, but had there been three more workmen employed the work would have been done in 4 days. How many men were employed?

12. A farmer found that he could buy 16 more sheep than cows for \$100, and that the cost of 3 cows was \$15 greater than the cost of 12 sheep. What was the price of each?

13. If 5 be added to the numerator and subtracted from the denominator of a certain fraction, the result will be the reciprocal of the fraction; and if 2 be subtracted from the numerator, the result will be $\frac{7}{8}$ of the original fraction. What is the fraction?

14. A sum of money at interest for one year at a certain rate amounted to \$11,130. If the rate had been 1 % less and the principal \$100 more, the amount would have been the same. What was the principal and what the rate?

15. A certain kind of cloth loses 2 % in width and 5 % in length by shrinking. Find the dimensions of a rectangular piece of this cloth whose shrinkage in perimeter is 38 in., and in area 8.625 sq. ft.

16. A formal rectangular flower garden is to be enlarged by a border whose uniform width is 10 % of the length of the garden. If the area of the border is 900 sq. ft., and the width of the old garden is 75 % of the width of the new one, find the dimensions of the garden and the width of the border.

17. In going 40 yds. more than $\frac{1}{4}$ of a mile the fore wheel of a carriage revolves 24 times more than the hind wheel, but if the circumference of each wheel had been 3 ft. greater the fore wheel would have revolved 16 times more than the hind wheel. What is the circumference of the hind wheel?

18. A merchant paid \$125 for an invoice of two grades of sugar. By selling the first grade for \$91, and the second for \$36, he gained as many per cent on the first grade as he lost on the second. How much did he pay for each grade?

19. Two trains start at the same time from stations A and B, respectively, and travel toward each other. These stations are 320 miles apart, and it requires, from the time the trains meet, 6 hr. and 40 min. for the first train to reach B, and 2 hr. and 24 min. for the second to reach A. Find the rate at which each train runs.

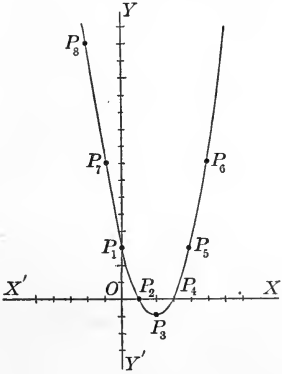
20. After traveling 2 hr., a railway train is detained 1 hr. by an accident, after which it proceeds at 60 % of its former rate, and arrives 7 hr. and 40 min. behind time. If the accident had occurred 50 miles farther on, the train would have saved 80 min. What was the entire distance traveled by the train?

21. The hundreds' digit of a 3-digit number equals the sum of the other two digits, the square of the tens' digit equals the units' digit multiplied by the sum of the units' and hundreds' digits, and if 594 be subtracted from the number, the order of its digits will be reversed. What is the number?

22. Find the dimensions of a room of which two adjacent side walls and the floor contain, respectively, $26\frac{2}{3}$, 20, and 48 square yards.

III. GRAPHIC REPRESENTATION OF EQUATIONS

183. Graphs of quadratic equations. The methods of §§ 114–116 (which should now be reread) are manifestly applicable to equations of any degree whatever, provided only that these equations contain two unknown numbers.



E.g., to find the graph of the equation $4x+y=x^2+3$, it is merely necessary to find a sufficient number of solutions of this indeterminate equation, to locate the points having these solutions as coördinates, and then to connect these successive points by a smooth curve.

Thus, on solving the above equation for y, it becomes $y=x^2-4x+3$, which shows that

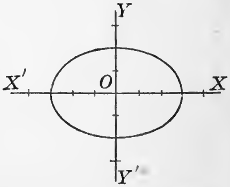
when $x = 0, 1, \ 2, 3, 4, 5, \cdots, -1, -2, -3, \cdots,$

then $y = 3, 0, -1, 0, 3, 8, \cdots, \ 8, 15, 24, \cdots;$

and therefore (§ 115) that the points $P_1 \equiv (0, 3)$, $P_2 \equiv (1, 0)$, $P_3 \equiv (2, -1)$, $P_4 \equiv (3, 0)$, $P_5 \equiv (4, 3)$, $P_6 \equiv (5, 8)$, $\cdots$, $P_7 \equiv (-1, 8)$, $P_8 \equiv (-2, 15)$, $P_9 \equiv (-3, 24)$, $\cdots$ are on the required graph.

If these points, and as many more as may be desired, are located by the method of § 114, it is easily seen that the required graph is approximately represented by the curved line $P_8P_3P_6$ in the above figure.

If the above equation is written in the form $y=(x-1)(x-3)$, it shows that as x increases from 3 to ∞, or decreases from 1 to $-\infty$, y increases from 0 to ∞, and that y is negative only for values of x between 1 and 3, *i.e.*, y is negative when $1<x<3$. And if the equation is solved for x, *i.e.*, written in the form $x=2\pm\sqrt{1+y}$, it shows that there are no points on the graph for which $y<-1$.



Again, let the graph of the equation $4x^2+9y^2=36$ be required. If this equation is solved for y, it becomes $y=\pm\frac{2}{3}\sqrt{9-x^2}$, which shows that y is real for all values of x from $x=-3$ to $x=3$, but imaginary for all other values of x, *i.e.*, this form of the equation shows that no part of the graph lies at the left of $x = -3$, nor at the right of $x = 3$. It also shows that

when $x = -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2$, and 3,

then $y = 0, \ \pm\frac{2}{3}\sqrt{5}, \ \pm\frac{4}{3}\sqrt{2}, \ \pm 2, \ \pm\frac{4}{3}\sqrt{2}, \ \pm\frac{2}{3}\sqrt{5}$, and 0.

If the points having these solutions as coördinates be located (§ 114) and connected in succession by a smooth curve (using approximate values for the square roots indicated above), this curve will represent the required graph. See accompanying figure.

EXERCISES

Construct the graphs of the following equations (cf. footnote, p. 190):

1. $y^2 = 8x$.

2. $16x^2 + y^2 = 64$.

3. $3x^2 + 4y^2 = 12$.

4. $3x^2 - 4y^2 = 12$.

5. $4x^2 + 54y = 8x + 9y^2 + 113$.

6. $4y^2 = x^2$.

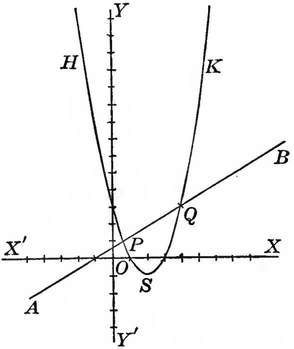
7. Show, from its *equation*, that no part of the graph of Ex. 1 lies to the left of the y-axis (the line $Y'Y$).

8. Show, from its equation, that no part of the graph of Ex. 2 lies outside of a certain rectangle whose length is 16 and whose width is 4.

9. Show from the equation of Ex. 4 that its graph consists of four infinitely long branches, one in each of the quarters into which the axes divide the plane, and that no part of it lies between $x = -2$ and $x = 2$.

10. Construct the graph of the equation $4x + y = x^2 + 5$, and show that it is the same as that given in the first figure of § 183, except that it is moved two divisions upward. Explain why this should be so.

184. Graphic solution of simultaneous equations. If the graph of one of two simultaneous equations is drawn across the graph of the other, *i.e.*, if the same axes are used for both graphs, then the coördinates of each of the points of intersection of the two graphs (these coördinates may be measured) constitute a simultaneous solution of the given equations (cf. § 116).



E.g., the graph of $3x - 5y = -3$, viz. AB, intersects the graph of $4x + y = x^2 + 3$, viz. HSK, in the points P and Q. The coördinates of Q, on being measured, are found to be 4 and 3, and those of P are approximately $\frac{19}{32}$ and $\frac{31}{32}$; and it is easily verified that each of these pairs of numbers constitutes an approximate simultaneous solution of the given equations.

REMARK. It should be observed that the longer the unit divisions on the axes are made, *i.e.*, the larger the scale on which the drawing is made, the greater the degree of accuracy with which the coördinates of any given point can be measured.

EXERCISES

By constructing their graphs, find the approximate simultaneous solutions of each of the following pairs of equations, and check the correctness of your results by the methods of §§ 174–180:

1. $\begin{cases} 9x^2 + 9y^2 = 289, \\ 4x^2 - 9y^2 = 36. \end{cases}$

2. $\begin{cases} 9x^2 + 64y^2 = 576, \\ xy = 11. \end{cases}$

3. $\begin{cases} x^3 + 9x = y + 7x^2 + 1, \\ y = -2. \end{cases}$

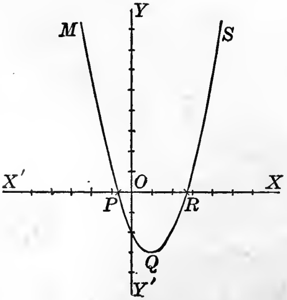
4. $\begin{cases} x^3 + 9x = y + 7x^2 + 1, \\ y = 0. \end{cases}$

185. Graphic solution of equations containing but one unknown number. Manifestly the roots of the equation $x^2 - 2x - 2 = 0$ are the values of x found by solving the pair of simultaneous equations

$$\begin{cases} y = x^2 - 2x - 2, \\ y = 0. \end{cases}$$

Now, by § 184, the solutions of this pair of simultaneous equations are the coördinates of the points in which their graphs intersect each other, and, since the graph of $y = 0$ is the line $X'X$, therefore the roots of $x^2 - 2x - 2 = 0$ may be found graphically by measuring the distances from O to the points in which the graph of $y = x^2 - 2x - 2$ intersects the line $X'X$.

Thus, the graph of the equation $y = x^2 - 2x - 2$ is the curve MQS in the figure, and the distances OR and OP are found to be approximately 2.75 and $-.75$; hence the roots of the equation $x^2 - 2x - 2 = 0$ are approximately 2.75 and $-.75$.



NOTE 1. Although the *measurement* of a root, OR for example, gives only a roughly approximate result, yet, assuming that the graph is continuous, which it really is, it is possible to find that result to any required degree of accuracy. Thus, by trial, it is found that y is negative when 0, 1, and 2 are substituted for x, but positive when $x = 3$; therefore the graph crosses the line $X'X$ between $x = 2$ and $x = 3$, *i.e.*, $2 < OR < 3$. Again, by substituting 2.1, 2.2, 2.3, ⋯, for x, it is found that $2.7 < OR < 2.8$; similarly that $2.73 < OR < 2.74$, $2.732 < OR < 2.733$, etc.

Note 2. Although a quadratic equation is used to illustrate the method for the graphic solution of numerical equations, yet it is only for equations above the second degree that this method is advantageous, — first and second degree equations can be more easily solved by other methods.

EXERCISES

1. Show that one root of $x^3 - 7x^2 + 9x = 1$ lies between 1 and 2.

2. By the above method find, correct to two decimal places, the root referred to in Ex. 1.

3. Between what two integers do each of the other two roots of the equation in Ex. 1 lie? Compare § 184, Ex. 4.

4. Corresponding to any given value of x, how does the value of y in $y = x^2 - 6x + 6$ compare with its value in $y = x^2 - 6x + 7$? Could, then, the graph of the second equation be obtained by merely moving that of the first vertically upward through one division?

5. Compare the graphs of $y = 2x^2 - 10x - 3$ and $y = 2x^2 - 10x + 1$; also those of $y = 3 + 4x - x^2$ and $y = 10 + 4x - x^2$.

6. By first constructing the graphs of $y = x^2 - 6x + 6$, $y = x^2 - 6x + 7$, etc., compare the roots of $x^2 - 6x + 6 = 0$, $x^2 - 6x + 7 = 0$, $x^2 - 6x + 8 = 0$, $x^2 - 6x + 9 = 0$, $x^2 - 6x + 10 = 0$, and $x^2 - 6x + 11 = 0$.

7. As in Ex. 6, compare the two smaller roots of $x^3 - 7x^2 + 9x - 1 = 0$ with those of $x^3 - 7x^2 + 9x - 3 = 0$ and $x^3 - 7x^2 + 9x - 5 = 0$.

[Exercises 6 and 7 illustrate how, by changing the absolute term in an equation, a pair of unequal roots can be made gradually to become equal and then imaginary.]

By means of graphs show how the following expressions vary in value as x varies gradually from $-\infty$ through 0 to $+\infty$:

8. $x^2 - 7x + 12$. **9.** $6 + 4x - x^2$. **10.** $x^3 - 18x + 2$.

CHAPTER XVI

RATIO, PROPORTION, AND VARIATION

I. RATIO

186. Definitions. The **ratio** of one of two numbers to the other is the quotient obtained by dividing the first of these numbers by the second. These numbers themselves are usually called the **terms** of the ratio, the first being the **antecedent**, and the second the **consequent**.

E.g., the ratio of 15 to 5 is $15 \div 5$, *i.e.*, 3; the ratio of 6 to 9 is $6 \div 9$, *i.e.*, $\frac{2}{3}$; and the ratio of a to b (whatever the numbers represented by a and b) is $a \div b$. The terms of this last ratio are a and b, of which a is the antecedent and b the consequent.

Each of the expressions $a \div b$, $a : b$, and $\frac{a}{b}$ is used to denote the ratio of a to b, and they may each be read "the ratio of a to b" or "a divided by b."

The **inverse ratio** of a to b is $b \div a$, *i.e.*, it is the reciprocal of the direct ratio of these numbers.

EXERCISES

1. What is the ratio of 6 to 2? of 15 to 3? of 12 to 18? of 4.9 to .7? of $\frac{5}{8}$ to $\frac{15}{16}$?

2. Read the expression $18 : 32$, and tell what it means. What is the inverse ratio of 18 to 32?

3. Write two other expressions which mean the same as $25 : 40$.

4. Does the antecedent of a ratio correspond to dividend or to divisor? In the ratio $5 : 8$ what is the antecedent? What is the other number called?

5. What is meant by the reciprocal of a number? Show that the inverse ratio of x to y is the direct ratio of the reciprocal of x to the reciprocal of y.

6. If the ratio of x to 5 equals 2, find x, and verify your work.

7. If the ratio of two numbers is $\frac{2}{3}$, and the consequent is 6, what is the antecedent?

Find x in each of the following ratios, and verify your result:

8. $x^2 : 2 = \frac{2}{9}$.

9. $x : 6 = x - 10$.

10. $25 : x^2 = 9$.

11. $36 : x = x$.

12. Given $x^2 + 6y^2 = 5xy$, find the two values of the ratio $x : y$.

13. The ratio of two numbers is $\frac{5}{6}$, and the ratio which their sum bears to the difference of their squares equals that of 1 to 7. Find these numbers and verify your result.

14. Prove that the value of a ratio is not changed by multiplying or by dividing each of its terms by any number whatever, except zero.

15. If the antecedent of a ratio be multiplied by any number, what effect will this have upon the value of the ratio? Why? What is the effect of multiplying the consequent? Why?

16. Prove that a ratio which is less than 1 is increased, and that a ratio which is greater than 1 is diminished, by adding the same positive number to each of its terms (cf. § 117, and Ex. 17, p. 200).

17. What number must be added to each term of the ratio $15 : 27$ in order that the resulting ratio shall be $2 : 3$? Has this addition increased or diminished the given ratio?

187. Ratio of like quantities. Commensurable and incommensurable numbers. If $A = n \cdot B$, where A and B are any two *quantities* of the same kind, and n is a *number*, then the quantity A is said to have the ratio n to the quantity B.

E.g., since a line 10 inches long equals 2 times a line 5 inches long, therefore the ratio of a 10-inch line to a 5-inch line is 2, *i.e.*, it is the same as the ratio of the *numbers* $10 : 5$.

Similarly the ratio of \$6 to \$9 is the same as $6:9$, *i.e.*, as $2:3$.

Since, by the above definition, the ratio of any two like *quantities* is the same as that of the *numbers* which represent these quantities, therefore it is sufficient for present purposes to study the ratios of numbers only.

If the ratio of two numbers (or quantities) is a rational number (§ 130), then the given numbers (or quantities) are said to be **commensurable* with each other**, but if this ratio is an irrational number, then they are said to be **incommensurable with each other.**

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* In this case the numbers have a common measure, hence the name.

E.g., since $\sqrt{5}:3$ is an irrational number, therefore $\sqrt{5}$ and 3 are incommensurable with each other; the diagonal and a side of a square are incommensurable with each other, their ratio being $\sqrt{2}$ (§ 130); but the two irrational numbers $3\sqrt{2}$ and $5\sqrt{2}$ are commensurable with each other, since their ratio is $3:5$.

Note. An irrational number is also often called an incommensurable number, since it is incommensurable with the unit 1.

EXERCISES

1. Show that the following ratios are all equal: 8 bu. oats : 6 bu. oats; 4 tons of coal : 3 tons of coal; \$12 : \$9; 10 qt. of milk : $7\frac{1}{2}$ qt. of milk; $4:3$; and $\frac{1}{5}:\frac{3}{20}$.

2. Find the value of each of the following ratios:
$8:6$; 32 lb. : 4 lb.; $4\sqrt{3}$ in. : $3\sqrt{2}$ in.; $2.7:9$; $9:2.7$; $4\sqrt{2}:\sqrt{2}$; $4\sqrt{2}:2$; 8.46 cm. : 2.35 cm.; and \$5.80 : 29 cents.

3. Which of the pairs of numbers (or quantities) in Ex. 2 are commensurable with each other? Which are incommensurable? Why?

4. Which of the individual terms in Ex. 2 are irrational?

II. PROPORTION

188. Definitions. An expression of the equality of two or more ratios is called a **proportion.**

E.g., if $a:b$ equals $c:d$, then the equation $a:b=c:d$ is a proportion, and the numbers a, b, c, and d are said to be *proportional* (also *in proportion*); thus, since $6:3=10:5$, therefore the numbers 6, 3, 10, and 5 are in proportion.

The proportion $a:b=c:d$ is sometimes written in the form $a:b::c:d$, which is read "a is to b as c is to d."

E.g., the proportion $6:3::10:5$ is read "6 is to 3 as 10 is to 5"; its meaning is the same as $6:3=10:5$, *i.e.*, the same as $\frac{6}{3}=\frac{10}{5}$.

The first and fourth terms of a proportion are called the **extremes**, while the second and third terms are called the **means**, and the fourth term is called the **fourth proportional** to the other three. The **antecedents** and **consequents** of a proportion are the antecedents and consequents of its two ratios.

E.g., in the proportion $a:b=c:d$, the extremes are a and d; the means, b and c; the antecedents, a and c; the consequents, b and d; and the fourth proportional to a, b, and c is d.

If the first of three numbers is to the second as the second is to the third, then the second is said to be a **mean proportional** between the other two, and the third is called the **third proportional** to the other two.

E.g., in the proportion $a : b = b : c$ the number b is a mean proportional between a and c, and c is the third proportional to a and b.

A succession of equal ratios in which the consequent of each is also the antecedent of the next, is called a **continued proportion.**

E.g., if $a : b = b : c = c : d = d : e = \cdots$, then this expression is a continued proportion.

EXERCISES

1. Is it true that $8 : 12 :: 10 : 15$? Why? How is this proportion read? What does it mean?

2. Is it true that $8 : 10 :: 12 : 15$? What are the means, and what the extremes, of this proportion? What is the fourth proportional to 8, 10, and 12? What are the antecedents? What are the consequents?

3. How does the proportion in Ex. 1 compare with that in Ex. 2? If any four numbers are in proportion, will they be in proportion after the means have been interchanged? Try several numerical cases, and also compare § 189, Prin. 5.

4. Show that the numbers 3, 4, 6, and 8 are proportional in the order in which they now stand. Arrange these numbers in three other ways in each of which they will be proportional.

5. Show that 6 is a mean proportional between 4 and 9; also between 2 and 18. Is -6 also a mean proportional between these numbers? What are the third proportionals in these cases?

189. Important principles of proportion. Since a proportion is merely an *equation* whose members are *fractions*, the principles of proportion may be easily derived (as is shown below) from those already demonstrated for equations and fractions.

Principle 1. *If four numbers are in proportion, then the product of the means equals the product of the extremes.*[*]

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[*] Before reading the *proofs* of these principles the student is urged to make several numerical illustrations of each, and also to try to make a general proof for himself, which he may then compare with that given in the text. Verbal statements of these principles should be committed to memory.

If the terms of a proportion are *quantities*, they may first be replaced by their representative numbers (cf. § 187), after which the above principle may be applied; the product of two quantities is meaningless.

For, let a, b, c, and d be any four numbers which are in proportion, then

$$a : b = c : d;$$

i.e.,

$$\frac{a}{b} = \frac{c}{d},$$

whence $ad = bc$, [Multiplying by bd

which was to be proved.

Principle 2. *If the product of two numbers equals the product of two others, then these four numbers form a proportion of which the two factors of either product may be made the means, and those of the other product the extremes.**

For, if

$$ad = bc,$$

then

$$\frac{a}{b} = \frac{c}{d},$$ [Dividing by bd

i.e.,

$$a : b = c : d.$$

In the same way it may be shown that, if $ad = bc$, then

$$b : a = d : c, \quad c : a = d : b, \text{ etc.};$$

hence the correctness of Principle 2.

Remark. From the proof just given it follows that *the correctness of a proportion is established when it is shown that the product of the means equals the product of the extremes;* this test is very useful.

Principle 3. *The products of the corresponding terms of two (or more) proportions are proportional.*

For, if $a : b = c : d$ and $e : f = g : h$,

i.e., if $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{h}$,

then (multiplying) $\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{g}{h}$, *i.e.*, $\frac{ae}{bf} = \frac{cg}{dh}$;

hence

$$ae : bf = cg : dh,$$

which was to be proved.

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* Principle 2 is the converse of Principle 1.

PRINCIPLE 4. *The quotients of the corresponding terms of two proportions are proportional.*

For, if $$a:b=c:d \text{ and } e:f=g:h,$$

then $$ad=bc \text{ and } eh=fg,$$

whence $$adfg=bceh; \qquad [\S\ 24\ (2)$$

on dividing each member of this last equation by $ehfg$, it becomes

$$\frac{ad}{eh}=\frac{bc}{fg}, \text{ i.e., } \frac{a}{e}\cdot\frac{d}{h}=\frac{b}{f}\cdot\frac{c}{g},$$

and from this last equation, by Principle 2,

$$\frac{a}{e}:\frac{b}{f}=\frac{c}{g}:\frac{d}{h},$$

which was to be proved.

PRINCIPLE 5. *If* $a:b=c:d$,

then (1) $b:a=d:c$;

(2) $a:c=b:d$;

(3) $(a+b):a(\text{or } b)=(c+d):c(\text{or } d)$;

(4) $(a-b):a(\text{or } b)=(c-d):c(\text{or } d)$;

and (5) $(a+b):(a-b)=(c+d):(c-d)$.

The correctness of these proportions [(1) to (5)] easily follows from the remark at the end of Principle 2; the detailed proofs are left as an exercise for the student.

REMARK. Proportion (1), above, is usually said to be formed from the given proportion by **inversion**; (2) by **alternation**; (3) by **composition**; (4) by **division** (or by **separation**); and (5) by **composition and division.**

The student should translate each part of the above principle into *verbal* language, and commit it to memory; *e.g.*, (3) thus translated is: *If four numbers are in proportion, then they are also in proportion when taken by composition; i.e., the sum of the first and second is to the first (or the second) as the sum of the third and fourth is to the third (or the fourth).*

Principle 6. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its own consequent.*

Thus, if $a : b = c : d = e : f = g : h = \cdots = x : y,$

then $(a + c + e + g + \cdots + x) : (b + d + f + h + \cdots + y) = e : f.$

To prove this theorem, let each of the given equal ratios be represented by a single letter, say r;

then $\frac{a}{b} = r,\ \frac{c}{d} = r,\ \frac{e}{f} = r,\ \frac{g}{h} = r,\ \cdots,$ and $\frac{x}{y} = r,$

hence $a = br,\ c = dr,\ e = fr,\ g = hr,\ \cdots,$ and $x = yr,$

and, adding these equations, member to member,

$$a + c + e + g + \cdots + x = (b + d + f + h + \cdots + y)r,$$

and therefore $\frac{a + c + e + g + \cdots + x}{b + d + f + h + \cdots + y} = r = \frac{e}{f},$

which proves the principle.

Note. As in the proof just given, so it will often be found advantageous to represent a ratio by a single letter.

Principle 7. *Like powers of proportional numbers are proportional; so also are like roots; i.e., if*

$$a : b = c : d, \textit{ then } a^n : b^n = c^n : d^n.$$

For, if $\frac{a}{b} = \frac{c}{d},$ then $\left(\frac{a}{b}\right)^n = \left(\frac{c}{d}\right)^n,$ *i.e.*, $\frac{a^n}{b^n} = \frac{c^n}{d^n};$

hence, if $a : b = c : d,$ then $a^n : b^n = c^n : d^n,$*

which was to be proved.

EXERCISES

1. Find the fourth term of the proportion of which the first three terms are 5, 12, and 15.

Suggestion. Let x represent the fourth term, and apply Principle 1.

2. Find a mean proportional between 4 and 25. How many answers has this problem?

3. Find the third proportional to 25 and 40.

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* According as n is an integer or its reciprocal, a^n is a power or a root of a.

4. If a line 18 inches long is divided into two parts whose ratio is $4:5$, how long is each part?

5. If $x:15=(x-1):12$, find x.

6. If $32:x^2=12:(x+2)$, find x.

7. Find the mean proportionals between am^3 and a^3m; also between $a+b$ and $a-b$.

8. If $a:b=c:d$, show that $am:bn=cm:dn$, wherein m and n are any numbers whatever; also translate this principle into verbal language.

9. Show that the product of the means of a proportion, divided by either extreme, equals the other extreme.

10. Show that the mean proportional between any two numbers is the square root of the product of these numbers.

11. Prove Principle 6 by means of the remark under Principle 2.

12. Prove Principle 4 by using a single letter to represent a ratio (compare proof of Principle 6).

13. Add 1 to each member of the equation $a:b=c:d$, write the result in the form of a proportion, and thus prove (3) of Principle 5.

14. If $a:b=c:d$, and if a is not equal to b nor to c, show that no number whatever can be added to each term of the proportion and leave the results in proportion.

If $p:q=r:s$, prove that:

15. $r:s=\frac{1}{q}:\frac{1}{p}$.

16. $5p:3r=5q:3s$.

17. $pr:qs=r^2:s^2$.

18. $(p+q):(r+s)=\sqrt{p^2+q^2}:\sqrt{r^2+s^2}$.

19. Given $\left\{\begin{array}{l}(x+y+1):(x+y+2)=6:7,\\ (y+2x):(y-2x)=(12x+6y-3):(6y-12x-1)\end{array}\right\}$; find x and y.

20. Given $x:27=y:9=2:(x-y)$; find x and y.

21. If $a:b=c:d=e:f=g:h=\cdots$, and $l, m, n, p, \cdots$ are any numbers whatever, prove that

$$(ma+lc-ne+pg+\cdots):(mb+ld-nf+ph+\cdots)=a:b.$$

22. If $a:x=b:y=c:z=d:w=\cdots$, show that

$$(a^n+b^n+c^n+\cdots):(x^n+y^n+z^n\cdots)=a^n\,.\,x^n.$$

23. If $(p+q+r+s)(p-q-r+s)=(p-q+r-s)(p+q-r-s)$, show that $p:q=r:s$.

24. If $a:b=c:d=e:f$, show that

$$c:d=\sqrt{a^2+4\,ac+5\,c^2}:\sqrt{b^2+4\,bd+5\,d^2}.$$

25. If $(x-y):(y-z):(z-x)=l:m:n$*, and $x\neq y\neq z$, show that $l+m+n=0$.

By the principles of proportion, solve the following equations:

26. $\dfrac{\sqrt{x+7}+\sqrt{x}}{\sqrt{x+7}-\sqrt{x}}=\dfrac{4+\sqrt{x}}{4-\sqrt{x}}.$

SUGGESTION. Apply Principle 5 (5).

27. $\dfrac{x+\sqrt{x-1}}{x-\sqrt{x-1}}=\dfrac{13}{7}.$

28. $(a-\sqrt{2\,ax-x^2}):(a-b)=(a+\sqrt{2\,ax-x^2}):(a+b).$

SUGGESTION. First apply Principle 5 (2).

29. If $\dfrac{ax+cy}{by+dz}=\dfrac{ay+cz}{bz+dx}=\dfrac{az+cx}{bx+dy}$, show that each of these ratios equals $\dfrac{a+c}{b+d}$.

30. The perimeter of a triangle, whose sides are in the ratio $5:6:8$, is 57 meters; find the lengths of the sides.

31. Divide 16 into two parts such that their product is to the sum of their squares as $3:10$.

32. Find two integers whose ratio is the same as $15\frac{3}{8}:9\frac{5}{9}$. Can the ratio of *any* two numbers whatever be expressed by means of two integers (cf. Ex. 2, p. 320)?

33. By the addition of new books, a certain circulating library was increased in the ratio of $12:11$; later 160 old books were discarded, and it was then found that the library remained increased only in the ratio $35:33$. How many books were there in the library originally?

34. If x, y, and z represent positive numbers, which of the following ratios is the greater, $\dfrac{2\,x+5\,y}{2\,x+7\,y}$ or $\dfrac{x+2\,y}{x+3\,y}$? $\dfrac{x-y+z}{x+y-z}$ or $\dfrac{x+y+z}{x-y-z}$?

35. If $a:b$, $c:d$, $e:f$, $g:h$, $\cdots$ are unequal ratios, in which $a, b, c, \cdots$ are positive numbers, and if $a:b$ is the greatest and $e:f$ the least among these ratios, show that $(a+c+e+g+\cdots):(b+d+f+h+\cdots)$ is less than $a:b$, but greater than $e:f$.

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* The expression $a:b:c=x:y:z$, means that $a:b=x:y$, $a:c=x:z$, and $b:c=y:z$. It may also be written $a:x=b:y=c:z$.

III. VARIATION

190. Definitions. Many questions in mathematics are concerned with numbers whose values are changing; such numbers are usually spoken of briefly as **variables**, while numbers whose values do not change are called **constants**.

Two variables may, also, be so related that a change in one of them necessarily produces a corresponding change in the other.

E.g., if w and v represent, respectively, the weight and volume (*i.e.*, the number of pounds, and the number of cubic feet) of the quantity of water in a certain tank, and if a cubic foot of water weighs 62.5 pounds, then $w = 62.5\,v$.

Moreover, while the water is flowing into this tank, both w and v will manifestly change (*i.e.*, they will be variables), but through all their changes the relation between these variables continues to be

$$w = 62.5\,v.$$

Of two variables which are so related that, during all their changes, their ratio remains constant, each is said to **vary as the other.***

E.g., if x and y are any two variables so related that, during all their changes, $x : y = k$, wherein k is a constant, then x varies as y, and y also varies as x.

The equation $x : y = k$, or, what is the same thing, $x = ky$, shows that when y is doubled, tripled, halved, etc., then x is also doubled, tripled, halved, etc.

The symbol employed to denote variation is $\propto$; it stands for the words "varies as," and the expression $a \propto b$ is read "a varies as b."

In the above example about the water, w varies as v, because their ratio is constant (*i.e.*, $w : v = 62.5$, whatever the quantity of the water); this is commonly expressed by saying that "the weight of water varies as its volume."

One of two numbers is said to **vary inversely** as the other if the ratio of the first to the reciprocal of the second is constant.

E.g., the time required for a railway train to travel a given route varies inversely as its speed; for, if t, r, and d represent, respectively, the time, rate, and distance, then

$$t \cdot r = d, \text{ that is, } t : \frac{1}{r} = d,$$

where d is constant. From the first of these equations it follows also that if the speed is doubled, then the time will be halved; if the speed is divided by 3, then the time will be trebled, etc.

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* Also "to vary *directly* as the other."

Again, if x, y, and z are variables such that $x = kyz$, where k is a constant, then x is said to **vary jointly as y and z**; and if $x = \frac{ky}{z}$, then x is said to **vary directly as y and inversely as z.**

Note. It should be remarked in passing that such an expression as $w \propto v$ above (*i.e.*, the weight of water varies as its volume) is merely an abbreviated form of the proportion

$$w : w' = v : v',$$

wherein w and w' stand for the respective weights, and v and v' for the volumes, of any two quantities of water.

The theory of variation is substantially included in that of ratio and proportion, and the only reason for even defining the expressions "varies as," "varies inversely as," etc., here, is that this convenient phraseology is so well established in physics, chemistry, etc.

EXERCISES AND PROBLEMS

1. Which of the following quantities are constants and which are variables: (1) the circumference of a growing orange? (2) the length of the shadow cast by a certain church steeple between sunrise and sunset? (3) the length of the steeple itself? (4) the time since the discovery of America? (5) the interest earned by a note? (6) the principal of the note?

2. What is meant by the expression, "the speed being constant, the distance traveled by a railway train varies as the time"? Express this fact by means of a proportion (cf. note, above).

3. What is meant by saying "the interest earned by a certain principal varies as the time"? Express this fact as a proportion; also as an equation.

4. What is meant by the expression $x \propto y$? Are x and y constants or variables here?

5. Express by means of an equation that $x \propto y$. Explain.

6. If $x \propto y$, and if $x = 12$ when $y = 3$, find the equation connecting x and y, and the value of x when $y = 7$.

Solution. Since $x \propto y$, therefore $x = ky$, where k is a constant. Moreover, if $x = 12$ when $y = 3$, then the equation $x = ky$ gives $12 = 3k$, from which we find $k = 4$; therefore, *under the given conditions*, $x = 4y$; and therefore $x = 28$ when $y = 7$.

7. If x varies inversely as y, and $x = 10$ when $y = 3$, what is the value of x when $y = 5$?

8. If m varies inversely as n, and is equal to 4 when $n = 2$, what is the value of n when $m = 1\frac{1}{3}$?

9. The area of a circle varies as the square of its radius, and the area of a circle whose radius is 10 ft. is 314.6 sq. ft. What is the area of a circle whose radius is 5 ft.? of one whose radius is 12 ft.?

10. Find the radius of a circle whose area is twice as great as that of a circle whose radius is 10 ft. (cf. Ex. 9).

11. If one of two numbers varies inversely as the other, show that their product is constant.

12. If $A \propto B$ and $B \propto C$, prove that $A \propto C$.

SUGGESTION. Show that $A = kC$, wherein k is some constant.

13. If $m \propto n$ and $p \propto n$, prove that $m \pm p \propto n$.

14. If p varies inversely as q and q varies inversely as r, prove that $p \propto r$.

15. If $3m^2 - 18 \propto 2n + 1$, and $m = 4$ when $n = 2$, what is the value of m when $n = 23.5$?

16. If x varies as y when z is constant, and as z when y is constant, prove that, when both y and z vary, $x \propto yz$; *i.e.*, that x *varies jointly* as y and z.

SUGGESTION. Let y and z vary separately, and write each variation as a proportion; thus from the change in y, $\frac{x}{x'} = \frac{y}{y'}$, and now letting z change, $\frac{x'}{x''} = \frac{z}{z'}$, whence $\frac{x}{x''} = \frac{yz}{y'z'}$, from which the conclusion is evident.

17. The area of a triangle varies as its altitude if its base is constant, and as its base if its altitude is constant. If the area of a triangle whose base and altitude are, respectively, 6 and 5 in., is 15 sq. in., what is the area when the base and altitude are 13 and 10 in. respectively?

18. If the volume of a pyramid varies jointly as its base and altitude, and if the volume is 20 cu. in. when the base is 12 sq. in. and the altitude is 5 in., what is the altitude of the pyramid whose base is 48 sq. in. and whose volume is 76 cu. in.?

19. The distance (in feet) fallen by a body from a position of rest varies as the square of the time (in seconds) during which it falls. If a body falls $257\frac{1}{3}$ ft. in 4 sec., how far will it fall in 5 sec.? how far during the 5th second? how far during the 7th second?

20. If the intensity of light varies inversely as the square of the distance from its source, how much farther from a lamp must a book, which is now 2 ft. away, be removed so as to receive just one third as much light?

21. A rectangle moves with its center on a given straight line and its plane perpendicular to that line. If one of its sides varies as the distance, and an adjacent side as the square of the distance, of the rectangle from a certain point on this line, and if at the distance 3 ft. the rectangle becomes a square 2 ft. on a side, what is its area when the distance is 5 ft.?

22. In order that two weights attached to a rod should balance each other when the support on which the rod rests is between them, their distances from the point of support should vary inversely as the weights. Find the point of support for a 12-foot plank on which two boys weighing 75 and 90 lb., respectively, wish to play see-saw.

23. The number of oscillations made by a pendulum in a given time varies inversely as the square root of its length. If a pendulum 39.1 inches long oscillates once a second, what is the length of a pendulum that oscillates twice a second?

24. The volume of a sphere varies as the cube of its radius, and the volume of a sphere whose radius is 1 ft. is 4.19 cu. ft. Find the volume of a sphere whose radius is 3 ft.

25. Three metal spheres whose radii are 3, 4, and 5 in. respectively, are melted and formed into a single sphere. Find the radius of this new sphere.

Suggestion. If S_1 and S_2 are the volumes of two spheres whose radii are r_1 and r_2, and if S is a sphere of radius r and equivalent to $S_1 + S_2$, then $S_1 = kr_1^3$, and $S = S_1 + S_2 = k(r_1^3 + r_2^3) = kr^3$.

CHAPTER XVII

SERIES — THE PROGRESSIONS

191. Definitions. A **series** is a succession of related numbers which conform to some definite law. The numbers which constitute the series are called its **terms.**

The law of a series may prescribe the way each of the terms, after a given term, is formed from those which precede it, or it may state how each term is related to the number of the place it occupies in the series.

E.g., in the series 1, 2, 3, 5, 8, 13, ⋯ each term, after the second, is the sum of the two preceding terms.

In the series 2, 6, 18, 54, ⋯ each term, after the first, is 3 times the preceding term; and 3, 7, 11, 15, 19, ⋯ is a series of which each term, after the first, is formed by adding 4 to the preceding term.

On the other hand, in the series 1, 4, 9, 16, 25, ⋯ each term is the square of the number of its place in the series; and the law of the series $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, $\frac{4}{9}$, $\frac{5}{11}$, ⋯ is expressed by $\frac{n}{1+2n}$, where n is the number of the term's place in the series.

If the number of terms of a series is unlimited, it is called an **infinite series,** otherwise it is a **finite series.**

E.g., in each of the five examples given above the series is infinite, but the series 1, 2, 3, 5, 8, 13, ⋯ 89 is finite, consisting of 10 terms.

Only the simplest kinds of series — the so-called "progressions" — will be studied in the present chapter.

I. ARITHMETICAL PROGRESSION

192. Definitions and notation. A series in which the difference found by subtracting any term from the next following term is the same throughout the series is an **arithmetical series**; it is also often called an **arithmetical progression,** and is designated by "A. P." This constant difference, which may be either positive or negative, is called the **common difference** of the series.

E.g., the numbers 2, 5, 8, 11, 14, ⋯ form an A. P. because $5-2=8-5=11-8=14-11=\cdots$; the common difference of this series is 3.

So, too, 18, 11, 4, -3, -10, ⋯ is an A. P. whose common difference is -7.

In any given A. P. it is customary to represent the first term, the last term, the common difference, the number of terms, and the sum of all the terms, by the letters a, l, d, n, and s, respectively; and these are called the **elements** of the series.

E.g., in the series 2, 5, 8, ⋯ 32, the elements are: $a=2$, $l=32$, $d=3$, $n=11$, and $s=187$.

EXERCISES

1. Define a series. If a row of numbers be written down quite at random, will they constitute a series? Explain.

2. Define an arithmetical series. Is 1, 7, 13, 19, 25, an A. P.? What are its elements?

3. If the series 7, 11, 15, 19 be continued toward the right, what is the next term? Why? Extend this series by writing the next four terms at the right, and also the next three at the left.

4. Do the numbers 7, 11, and 15 belong to the same A. P. as 27, 31, and 35? What is d for each of these two series? Write the series which includes both of these sets of numbers.

5. If the first, third, and fifth terms of an A. P. are 18, 24, and 30, respectively, find d and write 8 consecutive terms of this series. Also write 10 consecutive terms of the series of which 19, 9, and 4 are the first, fifth, and seventh terms, respectively. What is d for this last series?

6. Are the numbers 5, $5+3$, $5+6$, $5+9$, and $5+12$ an A. P.? What are the values of a, d, l, n, and s for this series? How may the second term of this series be formed from the first? the third from the second? any term from the one preceding?

7. Are the numbers x, $x+y$, $x+2y$, $x+3y$, $x+4y$, ⋯ an A. P.? Why? What is d in this series? How may the second term be formed from the first? the third from the first? the fourth from the first? the tenth from the first? the fifteenth from the first? How may any term whatever (say the nth) be formed from the first?

8. Show from the definition of an A. P. that such a series may be written in the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots\ l-2d,\ l-d,\ l,$$

wherein a, d, and l represent, respectively, the first term, common difference, and last term.

193. Formulas. The elements of an A. P. are connected by two fundamental equations (formulas), which will now be derived.

Since each term of an A. P. may be derived by adding d to the preceding term (cf. Exs. 6–8, § 192), therefore, if l stands for the nth term,

$$l = a + (n - 1)d. \qquad (1)$$

Again, since the sum of the terms of an A. P. may be written in each of the two following forms,

$$s = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l,$$

and $$s = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a,$$

therefore, by adding these equations, term by term,

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l);$$

i.e., $$2s = n(a + l), \qquad [n \text{ terms}$$

whence $$s = \frac{n(a + l)}{2}. \qquad (2)$$

Note. If any three of the five elements of an A. P. are given, the other two can always be found from formulas (1) and (2) above, because, in that case, the remaining two unknown elements will be connected by two independent equations (cf. Ex. 17, p. 334).

EXERCISES AND PROBLEMS

1. Verify formulas (1) and (2) above, in the case of the arithmetical series 7, 10, 13, 16, 19, 22, 25. What is the value of a in this series? of d? of n? of l?

2. Verify formulas (1) and (2) above, for the arithmetical series 26, 19, 12, 5, -2, -9, -16, -23, -30; also for the series -8, $-5\frac{2}{3}$, $-3\frac{1}{3}$, -1, $1\frac{1}{3}$, $3\frac{2}{3}$, 6, $8\frac{1}{3}$, $10\frac{2}{3}$, 13.

3. By means of formula (1) find the 17th term of 7, 11, 15, ⋯; then, using formula (2), and without writing all the terms, find the sum of the first 17 terms of this series.

4. Using formulas (1) and (2) find the 8th term, and also the sum of the first 8 terms of 1, 3.5, 6, 8.5, ⋯.

5. Find the 26th term, and also the sum of the first 18 terms of the series 1, 5, 9, ⋯.

6. Find the sum of 10 terms of 4, 11, 18, ⋯.

7. Find the sum of 30 terms of -2, -0.5, 1, 2.5, ⋯.

8. Find the sum of 19 terms of 2, 5, 8, ⋯; also find the sum of k terms of this series.

9. Find the sum of n terms of the series $5, 5 + k, 5 + 2k, 5 + 3k, \cdots$.

10. Find the sum of t terms of the series $h, 2h, 3h, \cdots$. What is this sum if $h = 2$ and $t = 50$?

11. Find the sum of the even numbers from 2 to 100 inclusive. Compare your result with that found in Ex. 10.

12. How many strokes does a clock make during the 24 hours of a day?

13. Suppose that 50 eggs were placed in a row, each 2 yds. from the next, and a basket 2 yds. beyond the last egg, how far would a boy, starting at the basket, walk in picking up these eggs and carrying them, one at a time, to the basket?

14. If a body falls 16.1 feet during the first second, 3 times as far during the next second, 5 times as far during the third second, etc., how far will it fall during the 8th second? how far during the first 8 seconds?

15. If the 6th and 11th terms of an A. P. are, respectively, 17 and 32, find the common difference, and also the sum of the first 11 terms.

Suggestion. Since the 6th term is 17, therefore $17 = a + 5d$. Similarly, $32 = a + 10d$. From these two equations find a and d, and then find s.

16. By means of formula (1) find the number of the terms in the series 2, 6, 10, ⋯, 66. Also find the sum of the series.

17. How many terms are there in the series $-1, 2, 5, \cdots$ if the sum of this series is 221?

Suggestion. Since in this series $a = -1$, $d = 3$, and $s = 221$, therefore formulas (1) and (2) of § 193 become, respectively, $l = -1 + (n - 1)3$ and $221 = \frac{n}{2}(-1 + l)$; and from these equations it is easy to determine n and l.

18. Determine the unknown elements in the series ⋯, 10, 13, 16, ⋯ if $s = 112$ and $n = 7$.

19. If s, n, and d are given, find a and l, *i.e.*, find a and l in terms of s, n, and d (cf. Ex. 18).

20. Find a and n in terms of d, l, and s. Make up and solve eight other examples of this kind.

21. Show that an A. P. is fully determined when any three of its elements are given.

22. Prove that the products obtained by multiplying each term of an A. P. by any given number are themselves in arithmetical progression.

If each term of an A. P. be divided by any given number, or be increased or diminished by any given number, will the results be in arithmetical progression? Explain.

194. Arithmetical means. The two end terms of an arithmetical series are called the **extremes** of the series, while all the other terms are called the **arithmetical means** between these two.

E.g., in the series 5, 9, 13, 17, 21, the extremes are 5 and 21, and 9, 13, and 17 are arithmetical means between 5 and 21.

Ex. 1. Insert 5 arithmetical means between 3 and 27.

SOLUTION. Since there are to be 5 means between 3 and 27, therefore the complete series will consist of 7 terms, and therefore, for this series, $a = 3$, $l = 27$, and $n = 7$; whence, from formula (1) of § 193, $d = 4$, and the series is: 3, 7, 11, 15, 19, 23, 27.

EXERCISES AND PROBLEMS

2. Insert 4 arithmetical means between 12 and 27.

3. Insert 15 arithmetical means between 19 and 131.

4. Insert 20 arithmetical means between 16 and -40.

5. If m arithmetical means are inserted between two given numbers, such as a and b, show that the common difference for the series thus formed is $d = (b - a) \div (m + 1)$.

6. If x is the (one) arithmetical mean between a and b, show, directly from the definition of an A. P., that $x = (a + b) \div 2$. Does this agree with the statement in Ex. 5? Explain.

7. Without actually finding the means asked for in Ex. 2, find the sum of the series formed by inserting them.

8. Find 3 numbers in A. P. whose sum is 15 and the sum of whose squares is 107.

SUGGESTION. Let $x - y$, x, and $x + y$ represent the required numbers.

9. The sum of 7 terms of an A. P. is 105, and the sum of its third and fifth terms is 10 times its first term. Find the series.

10. The product of the extremes of an A. P. of 3 terms is 4 less than the square of the mean, and the sum of the series is 24. Find the series.

11. The sum of 4 numbers in A. P. is 14, and the product of the means is 12. What are the numbers?

SUGGESTION. Let $x - 3y$, $x - y$, $x + y$, and $x + 3y$ represent the series.

12. The sum of an A. P. of 5 terms is 15, and the product of the extremes is 3 less than that of the second and fourth terms. Find the series.

13. How many arithmetical means must be inserted between 4 and 25 so that the sum of the series may be 116?

14. A number consists of 3 digits which are in A. P.; and the sum of the digits multiplied by 30.4 equals the number, but if 9 be added to the number, the units' and tens' digits will be interchanged. What is the number?

15. In the series 1, 3, 5, ⋯ what is the nth term? Prove that the sum of the first n odd numbers, beginning with 1, is n^2.

II. GEOMETRIC PROGRESSION

195. Definitions and notation. A series in which the quotient of any term (after the first) divided by the next preceding term is the same throughout the series is a **geometric series**; it is also often called a **geometric progression**, and is designated by "G. P." This constant quotient is called the **common ratio**, or simply the **ratio**, of the series.

E.g., the numbers 2, 6, 18, 54, ⋯ form a geometric series, whose ratio is 3; while $\frac{2}{3}$, -1, $\frac{3}{2}$, $-\frac{9}{4}$, $\frac{27}{8}$, ⋯ is a G. P. whose ratio is $-\frac{3}{2}$.

It is customary to represent the **elements** of a G. P., *i.e.*, the first term, the last term, the number of terms, the ratio, and the sum of all the terms, by the letters a, l, n, r, and s, respectively.

E.g., in the G. P. 2, -6, 18, -54, 162, -486, 1458, $a=2$, $l=1458$, $n=7$, $r=-3$, and $s=1094$.

EXERCISES

1. Is 7, 21, 63, 189, 567 a geometric series? Why? What are its elements?

2. Is 2, 8, 32, 96, 288 a geometric series? If not, why not?

3. Is -6, 12, -24, 48, -96, 192, -384, 768 a G. P.? What are its elements? How may the second term be obtained from the first? the third from the second? the sixth from the fifth?

4. If the series in Ex. 3 be continued toward the right, what is the next term? the next after that? Extend this series 5 terms toward the left also.

5. If a represents the first term of a G. P., and r the ratio, what is the second term? the third? the fourth? the fifth? the fourteenth? the twenty-third? the nth? Explain.

6. Show that x, xy, xy^2, xy^3, xy^4, ⋯ is a G. P. What are a and r in this series?

Answer these questions with regard to $\frac{p^4}{q^2}$, p^3, p^2q^2, pq^4, q^6, $\frac{q^8}{p}$ also.

7. What is r in the series $2, \frac{2}{3}, \frac{2}{9}, \cdots$? in the series $21, 7, \frac{7}{3}, \cdots$? Are these two series merely parts of the same series? Explain.

8. If the first, third, and sixth terms of a G. P. are 12, 3, and $\frac{3}{8}$, respectively, find r, and then write down the first 8 terms of this series.

196. Formulas. The elements of a G. P. are connected by two fundamental equations which will now be derived (cf. § 193).

Since each term of a G. P. may be obtained by multiplying the preceding term by r (cf. Exs. 5 and 6, § 195), therefore, if l represents the nth term of such a series, then

$$l = ar^{n-1}. \tag{1}$$

Again, if s represents the sum of a G. P. of n terms, then

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1},$$

whence $sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$, [multiplying by r

and therefore, by subtracting the second of these equations from the first, member from member,

$$s - sr = a - ar^n,$$

hence
$$s = \frac{a - ar^n}{1 - r}. \tag{2}$$

EXERCISES AND PROBLEMS

1. By means of formula (1) above, write down the fifth term of the G. P. 7, 21, 63, ⋯.

2. By formula (1) write down the seventh term of 3, 6, 12, ⋯, and then find the sum of the first 7 terms of this series by means of formula (2). Verify your answers by actually writing the first 7 terms of the given series.

3. Find the G. P. whose third term is 18 and whose eighth term is 4374.

SUGGESTION. Since the third term is 18, therefore, by formula (1), $18 = ar^2$; similarly, $4374 = ar^7$; therefore, by dividing the second of these equations by the first, $243 = r^5$, *i.e.*, $r = 3$; etc.

4. Find the G. P. whose fifth term is $\frac{8}{3}$ and whose ninth term is $\frac{128}{243}$. Also find the sum of this series.

5. Find the sum of the first 10 terms of the series 1, 2, 4, ⋯.

6. Find the sum of the first 6 terms of 1, 1.5, 2.25, ⋯.

7. Find the sum of the first 7 terms of $2, -\frac{2}{3}, \frac{2}{9}, \cdots$.

8. Find the sum of the first 7 terms of $1, -2x, 4x^2, \cdots$.

9. Find the sum of the first k terms of $-5, -2, -.8, \cdots$.

10. Find the sum of the first 9 terms of the series whose first term is 13.5 and whose fourth term is 4.

11. By actually dividing $a - ar^n$, *i.e.*, $a(1 - r^n)$, by $1 - r$, verify the correctness of formula (2) of § 196 [cf. § 68 (1)].

12. Show that the sum of n terms of a G.P. may be expressed in each of the following forms:

$$\frac{a - rl}{1 - r}, \quad \frac{rl - a}{r - 1}, \quad \frac{ar^n - a}{r - 1}, \quad \text{and} \quad \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

13. If r, n, and l are given, find a and s; *i.e.*, find a and s in terms of r, n, and l (cf. Ex. 19, § 193).

14. By means of formulas (1) and (2), § 196, show that a G.P. is fully determined when any three of its elements are given (cf. Ex. 21, § 193).

15. If $r = 3$, do the terms of the series $a, ar, ar^2, ar^3, \cdots ar^{n-1}$ increase or decrease in going toward the right? Can you name a number so large that it will exceed the nth term of this series for all values of n, however large?

16. If $r > 1$ (numerically), show that the terms of the series $a, ar, ar^2, ar^3, \cdots$ grow larger and larger in passing toward the right, and that, by taking n sufficiently large, the nth term, *i.e.*, ar^{n-1}, may be made to exceed any given finite number however large.

17. If $r < 1$ (numerically), show that the terms of the series $a, ar, ar^2, ar^3, \cdots$ grow smaller and smaller in passing toward the right, and that, by taking n sufficiently large, ar^{n-1} may be made to differ from zero by less than any given number however small.*

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* Suggestion on Exs. 16 and 17. Let h be any positive number, then since $(1+h)^s - (1+h)^{s-1} \equiv (1+h)^{s-1}\{(1+h)-1\} \equiv h(1+h)^{s-1}$, and since $h(1+h)^{s-1} > h$, when $s - 1$ is positive, therefore $(1+h)^2 - (1+h) > h$, $(1+h)^3 - (1+h)^2 > h$, $(1+h)^4 - (1+h)^3 > h$, $(1+h)^5 - (1+h)^4 > h$, $\cdots$ and $(1+h)^n - (1+h)^{n-1} > h$. Now adding these inequalities, and the equation $1 + h = 1 + h$, member to member, we have $(1+h)^n > 1 + nh$; but $1 + nh > Q$ (where Q is any given number however large) when $n > (Q - 1) \div h$, hence, for this or larger values of n, $(1+h)^n > Q$; and therefore, by taking n large enough, the nth power of any number greater than 1 can be made to exceed any number however large.

Again, let $p < 1$ and $p \cdot q = 1$, then $q (= 1 \div p) > 1$, and therefore q^n, *i.e.*, $1 \div p^n$, may be made larger than any given number however large, hence p^n may be made smaller than any given number however small.

18. Three numbers whose product is 216 form a G. P., and the sum of their squares is 189. What are the numbers?

SUGGESTION. Let $\frac{a}{r}$, a, and ar represent the required numbers.

19. If the population of the United States was 76,000,000 in 1900, and if it doubles itself every 25 years, what will it be in the year 2000?

20. Thinking $1 per bushel too high a price to pay for wheat, a man bought 10 bu., paying 3 cents for the first bushel, 6 cents for the second, 12 cents for the third, and so on. What did the tenth bushel cost him, and what was the average price per bushel?

21. A gentleman loaned a friend $250 at the beginning of each year for 4 years. If money is worth 5% compound interest, how much should be paid back to him at the end of the fourth year to discharge the obligation?

22. Divide 38 into three parts which are in G. P., and such that when 1, 2, and 1 are added to these parts, respectively, the result shall be in A. P.

197. Infinite decreasing geometric series. If $r < 1$ (numerically), the G. P. is called a **decreasing series**, while if $r > 1$ (numerically), it is an **increasing series.**

Formula (2) of § 196, which gives the sum of the first n terms of the series $a, ar, ar^2, ar^3, \cdots$ may evidently be written in the form

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Now, for a decreasing series the value of $\frac{ar^n}{1-r}$ becomes smaller and smaller, and approaches zero as a limit when n becomes infinite (cf. Ex. 17, p. 338); therefore the sum of the first n terms of an infinite decreasing G. P. may, by taking n sufficiently large, be made to differ from $\frac{a}{1-r}$ by less than any given number however small. This is usually expressed by saying that the **sum to infinity** of a decreasing G. P. is $\frac{a}{1-r}$; and if s_∞ stands for "limit of s_n when n becomes infinite," it may be written thus:

$$s_\infty = \frac{a}{1-r}.$$

EXERCISES AND PROBLEMS

1. From a line one foot long cut off one half, then one half of the remainder, then one half the next remainder, and so on; if this process were continued without end, show that, when expressed in inches, the parts cut off form the G. P.:

$$6,\ 3,\ \tfrac{3}{2},\ \tfrac{3}{4},\ \tfrac{3}{8},\ \tfrac{3}{16},\ \tfrac{3}{32},\ \tfrac{3}{64},\ \cdots.$$

2. By means of formula (2), § 196, find s_5 for the series in Ex. 1. Also find s_8, s_9, s_{10}, and s_n.

3. Based upon the manner in which the series in Ex. 1 was formed, show that $s_n < 12$, however large n may be. How near to 12 will s_n approach as n is made larger and larger? Explain. Also find s_∞ by § 197.

4. Find s_∞ for the series 0.6, 0.06, 0.006, ⋯, and thus show that $0.\dot{6}$, *i.e.*, that 0.666 ⋯, equals $\frac{2}{3}$.

Find s_∞ for each of the following series:

5. $1,\ -\frac{1}{2},\ \frac{1}{4},\ \cdots$.

6. $1,\ \frac{1}{3},\ \frac{1}{9},\ \cdots$.

7. $\frac{3}{2},\ -\frac{2}{3},\ \frac{8}{27},\ \cdots$.

8. $\sqrt{2},\ 1,\ \sqrt{0.5},\ \cdots$.

9. $0.\dot{3}$.

10. $0.\dot{1}\dot{2}$.

11. $1.36\dot{2}$.

12. $4.7\dot{5}2\dot{3}$.

13. $1,\ k,\ k^2,\ \cdots$ (wherein $k < 1$).

14. $x,\ \frac{1}{x},\ \frac{1}{x^8},\ \cdots$ (wherein $x > 1$).

15. If, in a G. P., r is positive and less than 0.5, show that any term of the series is greater than all the terms that follow it.

16. If a point moves from a given position, and along a straight line, with such a velocity that during any given second it moves 75 % as far as it did during the preceding second, and if it moved 24 feet during the first second, how far will it move before it comes to rest?

17. If a sled runs 80 feet during the first second after reaching the bottom of a hill, and if its distance decreases 20 % during each second thereafter, how far will it run on the level before coming to rest?

18. If a ball, on being dropped from a tower window 100 feet above the pavement rebounds 40 feet, then falls and rebounds 16 feet, and so on, how far will it move before coming to rest?

19. The president of a woman's charity organization starts a "letter chain" by writing 3 letters, each numbered 1, requesting each recipient to remit 10 cents to the society, and also to send out 3 other letters, each numbered 2, with a similar request, the chain to close with the letters numbered 20. If every one addressed complies with the requests, how much money will be realized for the society?

20. Although s_∞ for the series $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ⋯ is 1, show that for the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, ⋯, s_n grows larger beyond all bounds, by sufficiently increasing n.

SUGGESTION. Write the series thus: $s_n = \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots$, putting 8 terms in the next group, 16 in the next, and so on, and show that each group is greater than $\frac{1}{2}$.

198. Geometric means. The two end terms of a finite G. P. are called its **extremes**, while all the other terms are called the **geometric means** between these two.

E.g., in the series $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, and $\frac{4}{27}$ the extremes are $\frac{3}{4}$ and $\frac{4}{27}$, and $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{9}$ are geometric means between them.

Ex. **1.** Insert 4 geometric means between $\frac{4}{9}$ and $-\frac{27}{8}$.

SOLUTION. Since 4 means are to be inserted, therefore the complete series will consist of 6 terms, and therefore, for this series, $a = \frac{4}{9}$, $l = -\frac{27}{8}$, and $n = 6$; hence, by formula (1) of § 196,

$$-\tfrac{27}{8} = \tfrac{4}{9} \cdot r^5, \text{ therefore } r^5 = -\tfrac{243}{32}, \textit{ i.e.}, r = -\tfrac{3}{2},$$

and the series is: $\frac{4}{9}$, $-\frac{2}{3}$, 1, $-\frac{3}{2}$, $\frac{9}{4}$, and $-\frac{27}{8}$.

EXERCISES

2. Insert 4 geometric means between 3 and 96.

3. Insert 3 geometric means between 2 and $\frac{2}{81}$ (two answers).

4. Insert 5 geometric means between x^6 and y^6 (two answers).

5. If m geometric means are inserted between any two given numbers, such as a and b, show that the common ratio for the series thus formed is $\sqrt[m+1]{b \div a}$.

6. If x is the (one) geometric mean between a and b, show directly from the definition of a G. P. that $x = \sqrt{ab}$. Does this agree with the statement in Ex. 5? Explain.

7. Insert a geometric mean between 12 and 3. Give two solutions, and compare Ex. 6.

8. Insert a geometric mean between 0.5 and $3.\dot{5}$; also between $(a + b)^2$ and $(a - b)^2$; and between $3\,m^5x^2$ and $75\,m^{-2}x$.

9. If the difference between two numbers is 24, and if their arithmetical mean exceeds their geometric mean by 6, what are the numbers?

199. Arithmetico-geometric series. A series formed by multiplying corresponding pairs of terms of an A. P. and a G. P. is usually called an **arithmetico-geometric series.** The sum of n terms of such a series may be found by the method of § 196.*

Ex. 1. Find the sum of the series $1, 2r, 3r^2, 4r^3, 5r^4, \cdots nr^{n-1}$.

SOLUTION. Let $s = 1 + 2r + 3r^2 + 4r^3 + \cdots + nr^{n-1}$,

then $rs = r + 2r^2 + 3r^3 + \cdots + (n-1)r^{n-1} + nr^n$,

whence $s - rs = 1 + r + r^2 + r^3 + \cdots + r^{n-1} - nr^n$,

i.e., $$s(1-r) = \frac{1-r^n}{1-r} - nr^n,$$ [§ 196, formula (2)

and therefore $$s = \frac{1-r^n}{(1-r)^2} - \frac{nr^n}{1-r}.$$

EXERCISES

2. By the method of Ex. 1 find the sum of the n terms of the series obtained by multiplying the corresponding terms of the two series a, $a + d$, $a + 2d$, $\cdots$ $a + (n-1)d$ and $1, r, r^2, \cdots r^{n-1}$.

3. Find the sum of the series whose $(n+1)$th term is $(a+nb)x^n$, *i.e.*, find $a + (a+b)x + (a+2b)x^2 + \cdots + (a+nb)x^n$.*

III. HARMONIC PROGRESSION

200. Harmonic series. A series of numbers whose reciprocals form an A. P. is called an **harmonic series**; it is also often called an **harmonical progression**, and is designated by "H. P."

E.g., the series $1, \frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \cdots$ is an H. P. because the reciprocals of its terms are $1, 4, 7, 10, \cdots$, and these form an A. P.

It follows immediately from the above definition that questions concerning harmonic series, which admit of solution,† may be solved by treating the reciprocals of the terms of the given series as an A. P.

E.g., to extend the H. P. $\frac{3}{7}, \frac{1}{4}, \frac{3}{17}, \frac{3}{22}$ three terms further at each end it is only necessary to take the reciprocals of these numbers, which form the A. P. $\frac{7}{3}, 4, \frac{17}{3}, \frac{22}{3}$, in which $d = \frac{5}{3}$, and extend it three terms at each end, and write the reciprocals of its terms. Thus, the given series extended is $-\frac{3}{8}, -1, \frac{3}{2}, \frac{3}{7}, \frac{1}{4}, \frac{3}{17}, \frac{3}{22}, \frac{1}{9}, \frac{3}{32}$.

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* For an extension of this subject see Chrystal's Algebra, Part I, p. 489.

† There is no general formula for the sum of n terms of an H. P.

EXERCISES

1. If x is the harmonic mean between a and b, show, as above, that $\frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}$, and hence that $x=\frac{2ab}{a+b}$.

2. Insert 5 harmonic means between 2 and -3.

3. The arithmetical mean between two numbers is 5, and their harmonic mean is 3.2. What are the numbers?

4. The difference between two numbers is 2, and their arithmetical mean exceeds their harmonic mean by $\frac{1}{3}$. Find the numbers.

5. Given $(b-a):(c-b)=a:x$, prove that x equals a, b, or c, according as a, b, and c form an A. P., a G. P., or an H. P.

6. If the sixth term of an H. P. is $\frac{1}{3}$, and the seventeenth term is $\frac{2}{17}$, find the thirty-seventh term.

7. If a and b are any two unequal positive numbers, show that their arithmetical mean is greater than their geometric mean, and that this, in turn, is greater than their harmonic mean; also that the geometric mean is a mean proportional between their arithmetical and harmonic means.

CHAPTER XVIII

MATHEMATICAL INDUCTION — BINOMIAL THEOREM

201. Proof by induction. An elegant and powerful form of proof, and one that finds extensive application in almost every branch of mathematics, is what is known as "proof by induction."

Suppose it to have been found, by trial or otherwise, that $x-y$ is a factor of x^2-y^2, x^3-y^3, and x^4-y^4, and that one wishes to know whether it is a factor of x^5-y^5, x^6-y^6, $\cdots$ also. Actual trial with any one of these, say x^5-y^5, would show that it is exactly divisible by $x-y$, but, besides being somewhat tedious, this division gives no information as to whether $x-y$ is or is not a factor of x^6-y^6, $\cdots$ also; each successful trial increases the *probability* of the success of the next, but it really *proves* nothing beyond the single case on trial.

That $x-y$ is a factor of x^n-y^n, for every positive integral value of n, may be proved as follows:

Since $$x^n-y^n \equiv x(x^{n-1}-y^{n-1})+y^{n-1}(x-y),$$

therefore $x-y$ is a factor of x^n-y^n, *if it is a factor of* $x^{n-1}-y^{n-1}$. In other words: if $x-y$ is a factor of the difference of two like integral powers of x and y, then it is a factor of the difference of the *next higher* powers also.

But since, by actual trial, $x-y$ is already *known* to be a factor of x^4-y^4, therefore, by what has just been proved, it is also a factor of x^5-y^5; again, since it is *now* known to be a factor of x^5-y^5, therefore it is a factor of x^6-y^6; and so on without end: *i.e.*, $x-y$ is a factor of x^n-y^n for every positive integral value of n [cf. § 68 (*i*)].

The proof just given is an example of what is known as a proof by **mathematical induction**; it consists essentially of two steps, viz.:

(*a*) *Showing, by trial or otherwise, the correctness of a given proposition when applied to one or more particular cases,* and

(*b*) *Proving that if the proposition is true for any given case, then it is true for the next higher case also.*

From (*a*) and (*b*) it then follows that the proposition under consideration is true for all like cases.*

EXERCISES

1. Prove that the sum of the first n odd integers is n^2.

Solution. (*a*) By trial it is found that $1+3=2^2$ and $1+3+5=3^2$.

(*b*) Moreover, *if* $\quad 1+3+5+\cdots+(2k-1)=k^2, \qquad (1)$

then, by adding the next odd integer to each member of Eq. (1), we have

$$1+3+5+\cdots+(2k-1)+(2k+1)=k^2+(2k+1)=(k+1)^2;$$

i.e., *if* the law in question is true for the first k odd integers, then it is true for the first $k+1$ odd integers also.

But, by actual trial, this law is known to be true for the first 3 odd integers, hence it is true for the first 4; and, since it is *now* known to be true for the first 4, therefore it is true for the first 5, and so on without end; hence the sum of any number of consecutive odd integers beginning with 1 equals the square of that number.

By mathematical induction prove that:

2. $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$.

3. $2+4+6+\cdots+2n=n(n+1)$.

4. $1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

5. $1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2=(1+2+3+\cdots+n)^2$.

6. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$.

7. $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

8. Having established (*a*) and (*b*) in the inductive proof of any proposition, show the generality of the proposition by showing that there can be no *first* exception, and therefore no exception whatever.

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* The student should carefully distinguish between mathematical induction, as here defined, and what is known as inductive reasoning in the natural sciences; a proof by mathematical induction is, from its very nature, absolutely conclusive. On the other hand, the inductive method in physics, chemistry, etc., consists in formulating a statement of a law which will fit the particular cases that are known, and regarding it as a *law* only so long as it is not contradicted by other facts, not previously taken into account. From the nature of the case step (*b*) above can not be applied in physics, etc.

202. The binomial theorem. The method of induction (§ 201) furnishes a convenient proof of what is known as the **binomial theorem**; this theorem, which was presented without formal proof in § 62, may be symbolically stated thus:

$$(x+y)^n \equiv x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}y^3 + \cdots,^* \tag{1}$$

wherein $x+y$ represents any binomial whatever, and n is any positive integer.

To prove this theorem by mathematical induction, observe first that it is correct when $n=2$, for it then becomes

$$(x+y)^2 = x^2 + 2\,xy + \frac{2 \cdot 1}{1 \cdot 2}x^0y^2, \text{ i.e., } (x+y)^2 = x^2 + 2\,xy + y^2,$$

which agrees with the result of actual multiplication.

Again, if Eq. (1) is true for any particular value of n, say for $n=k$, *i.e.*, if

$$(x+y)^k = x^k + kx^{k-1}y + \frac{k(k-1)}{1 \cdot 2}x^{k-2}y^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}x^{k-3}y^3 + \cdots, \tag{2}$$

then, on multiplying each member of Eq. (2) by $x+y$, it becomes

$$\begin{aligned}(x+y)^{k+1} &= x^{k+1} + kx^ky + \frac{k(k-1)}{1 \cdot 2}x^{k-1}y^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3}x^{k-2}y^3 + \cdots \\ &\quad + x^ky + kx^{k-1}y^2 + \frac{k(k-1)}{1 \cdot 2}x^{k-2}y^3 + \cdots \\ &= x^{k+1} + (k+1)x^ky + \left\{\frac{k(k-1)}{1 \cdot 2} + k\right\}x^{k-1}y^2 \\ &\quad + \left\{\frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} + \frac{k(k-1)}{1 \cdot 2}\right\}x^{k-2}y^3 + \cdots,\end{aligned}$$

i.e.,

$$(x+y)^{k+1} = x^{k+1} + (k+1)x^ky + \frac{(k+1)k}{1 \cdot 2}x^{k-1}y^2 + \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3}x^{k-2}y^3 + \cdots, \tag{3}$$

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* The student should now re-read § 62, and observe that the second member of this identity conforms in every detail to the statement there given.

which is of precisely the same form as Eq. (2),* merely having $k+1$ wherever Eq. (2) has k. Moreover, Eq. (3) is obtained from Eq. (2) by actual multiplication, and is therefore true if Eq. (2) is true; hence, *if the theorem is true when the exponent has any particular value* (say k), *then it is also true when the exponent has the next higher value.**

But, by actual multiplication, the theorem is *known* to be true when $n=2$, hence, by what has just been proved, it is true when $n=3$; again, since it is *now* known to be true when $n=3$, therefore it is true when $n=4$;* and so on without end: hence the theorem is true for every positive integral exponent,* which was to be proved.

EXERCISES

1. In the expansion of $(x+y)^n$ what is the exponent of y in the 2d term? in the 3d term? in the 4th term? in the 12th term? in the rth term? What is the sum of the exponents of x and y in each term?

2. In the expansion of $(x+y)^n$ what is the highest factor in the denominator of the 3d term? of the 4th term? of the 10th term? of the rth term? How does this factor compare with the exponent of y in any given term?

3. What is subtracted from n in the last factor of the numerator in the 3d term of the expansion of $(x+y)^n$? in the 4th term? in the 5th term? in the 9th term? in the rth term?

4. Based upon your answers to Exs. 1–3, write down the 6th term of $(x+y)^n$. Also write the 10th term; the 17th term; and the rth term.

203. Binomial theorem continued. Strictly speaking, all that was really proved in § 202 is that, for every positive integral value of the exponent, the first *four terms* of the expansion follow the law expressed by Eq. (1); that *all* the terms follow this law will now be shown.

In multiplying Eq. (2) of § 202 by $x+y$ the 2d term of the product (3) is x times the 2d term plus y times the 1st term of (2); so, too, the 10th term of (3) would be found by adding x times the 10th term to y times the 9th term of (2), and the rth

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* Only the first four terms are given in Eqs. (2) and (3); see § 203 for complete proof.

term of (3) by adding x times the rth term to y times the $(r-1)$th term of (2).

But the $(r-1)$th and the rth terms of (2) are, respectively,

$$\frac{k(k-1)(k-2)\cdots(k-r+3)}{1\cdot 2\cdot 3\cdot\cdots(r-2)}x^{k-r+2}y^{r-2}$$

and $$\frac{k(k-1)(k-2)\cdots(k-r+3)(k-r+2)}{1\cdot 2\cdot 3\cdot\cdots(r-2)(r-1)}x^{k-r+1}y^{r-1},$$

therefore the rth term of (3) is

$$\left\{\frac{k(k-1)(k-2)\cdots(k-r+3)}{1\cdot 2\cdot 3\cdot\cdots(r-2)}\right.$$
$$\left.+\frac{k(k-1)(k-2)\cdots(k-r+3)(k-r+2)}{1\cdot 2\cdot 3\cdot\cdots(r-2)(r-1)}\right\}x^{k-r+2}y^{r-1},$$

i.e., $$\frac{(k+1)k(k-1)\cdots(k-r+3)}{1\cdot 2\cdot 3\cdot\cdots(r-1)}x^{k-r+2}y^{r-1},$$

which conforms to the law for the rth term expressed by (1) of § 202. Hence the rth term (*i.e.*, every term) in (3) conforms to the law expressed by (1), which was to be proved.

EXERCISES

1. Write down the expansion of $(a+b)^5$; also of $(p-q)^8$. Explain why the alternate terms in the expansion of $(p-q)^8$ are negative.

2. Write down the first 3 terms of $(x+y)^{11}$; also the 8th term.

3. Write down the 4th and 7th terms of $(a-x)^{13}$.

4. How many terms are there in the expansion of $(x+y)^{16}$? Write down the first three, and also the last three terms of this expansion, and compare their coefficients.

5. Write down the coefficient of the term containing a^4y^9 in $(a-y)^{13}$.

6. Expand $(3a^2-2xy^3)^5$; compare Ex. 2, p. 93.

7. Write down the 4th term of $(\frac{3}{2}x-\frac{2}{3}y)^{11}$; also the 9th term.

8. How many terms are there in $\left(x-\frac{1}{x}\right)^{18}$? Write down the 10th term. Also write the 5th term of $\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)^8$.

9. Write down the term of $(3x^4-2x^2)^7$, *i.e.*, of $(x^2)^7(3x^2-2)^7$, which contains x^{20}.

10. Write down the term of $\left(a^3 - \frac{2}{3a}\right)^9$ which contains a^{11}.

11. Expand $(a^{\frac{1}{2}} + 3a^2x^{-1})^6$, and write the result with positive exponents.

12. Expand $(1 - x + x^2)^4$ by means of the binomial theorem (cf. Ex. 25, p. 205).

13. By applying the law expressed in Eq. (1) of § 202, show that the coefficient of the $(n + 1)$th term of $(x + y)^n$ is 1; also show that the coefficient of every term thereafter contains a zero factor, and hence that $(x + y)^n$ contains only $n + 1$ terms.

14. Since $(a + b)^n \equiv (b + a)^n$, show that the coefficients equally distant from the ends of $(a + b)^n$ are equal; show this also by comparing the coefficient of the rth term from the beginning with that of the rth term from the end [*i.e.*, with the $(n - r + 2)$th term from the beginning].

15. Show that the sum of the binomial coefficients, *i.e.*, of 1, n, $\frac{n(n-1)}{2}$, $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$, ..., is 2^n.

SUGGESTION. Let $x = y = 1$, after expanding $(x + y)^n$.

16. Show that the sum of the even coefficients (*i.e.*, the 2d, 4th, ...) in Ex. 15 equals the sum of the odd coefficients, and that each sum is 2^{n-1}.

SUGGESTION. Let $x = 1$ and $y = -1$ in $(x + y)^n$.

17. Show that the coefficient of the rth term in $(x + y)^n$ may be obtained by multiplying that of the $(r - 1)$th term by $\frac{n - r + 2}{r - 1}$, and thus show that the binomial coefficients increase numerically in going from term to term toward the center (cf. also Ex. 14).

18. Show that the coefficient of the rth term is numerically greater than that of the $(r - 1)$th term so long as $r < \frac{1}{2}(n + 3)$; and thus write down the term whose coefficient is greatest in the expansion of $(x + y)^{11}$; and also in $(x + y)^{10}$.

204. Binomial theorem extended. It may be remarked in passing that the binomial theorem (§ 202), which has thus far been restricted to the case where the exponent is a positive integer, is greatly extended in Higher Algebra, where it is shown that, under certain restrictions, it admits negative and fractional exponents also. Although the *proof* of this fact is beyond the limits of this book, its correctness may be assumed in the following exercises.

EXERCISES

1. By means of the binomial theorem write the first four terms of $(1+x)^{\frac{1}{2}}$; the first five terms of $(a+b)^{-2}$; the 5th term of $(1-3x)^{\frac{2}{3}}$.

2. Show that the application of the binomial theorem to such cases as the above gives rise to an infinite series (cf. Ex. 13, § 203).

3. Expand $(1-x)^{-1}$ to 8 terms by the binomial theorem and compare the result with the first 8 terms of the quotient $1 \div (1-x)$.

4. Show that $(25+1)^{\frac{1}{2}} = 5 + \frac{1}{10} - \frac{1}{1000} + \frac{1}{50000} - \cdots$, when expanded by the binomial theorem and simplified; compare this result with $\sqrt{26}$ as found by the usual method.

5. By means of the expansion of $(9-2)^{\frac{1}{2}}$, show how to get an approximate value of the square root of 7.

205. The square of a polynomial. In § 61 it was pointed out that, by actual multiplication, the square of a polynomial consisting of 3, 4, or 5 terms, equals the sum of the squares of all the terms of the polynomial, plus twice the product of each term by all those that follow it. It will now be shown that if this theorem is true for polynomials of n terms, then it is also true for those of $n+1$ terms, and from this it will follow, as in § 201, that it is true for polynomials of any finite number of terms whatever, since it is already known to be true for polynomials of five terms.

Let $a+b+c+\cdots+p+q$ be a polynomial of n terms, and let

$$(a+b+c+\cdots+p+q)^2 \equiv a^2+b^2+\cdots+q^2+2ab+2ac+\cdots+2aq$$
$$+2bc+\cdots+2bq+\cdots+2pq.$$

In this identity replace a everywhere by $x+y$; then the number of terms in the polynomial in the first member will become $n+1$, and the second member will still consist of the sum of the squares of all the terms of the polynomial, plus twice the product of each term by all those that follow it (the student should work this out in detail); therefore, if the theorem is true for polynomials of n terms, then it is also true for those of $n+1$ terms, which was to be proved.

EXERCISES

Expand:

1. $(a+b-3x+2ab-1)^2$.

2. $(2-3a^2+4mx^2-3mx+3x-3a^2x)^2$.

3. $\left(x+\frac{2}{x}-3m+\frac{2}{m}-1\right)^2$.

APPENDIX A

IRRATIONAL NUMBERS

[Supplementary to § 132]

206. Irrational numbers are defined and illustrated in Chapter XIV, and it is there tentatively assumed, not only that the earlier definitions of sum, product, etc., apply to these numbers, but also that they are subject to the combinatory laws previously established for rational numbers.

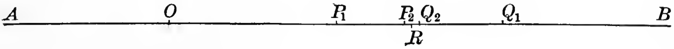
These definitions will now be restated from a somewhat broader point of view, and one from which the proofs of the combinatory laws are easily established.

As in § 130, note 2, two infinite series may be found such that the square of each term of the first series is less than 2, while the square of each term of the second series is greater than 2. These series may be conveniently written in the form

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \cdots < \sqrt{2} < 2,\ 1.5,\ 1.42,\ 1.415,\ \cdots; \qquad (1)$$

and the value of $\sqrt{2}$ may be thought of as *defined* by them.

For, let a point P move along a straight line AB in such a way that, at successive stages, its distances from O are: 1, 1.4, 1.41, ⋯ (shown in the figure by OP_1, OP_2, ⋯), and let another point Q move along this line so that its distances from O are successively: 2, 1.5, 1.42, ⋯ (shown in the figure by OQ_1, OQ_2, ⋯). Then clearly the point P will always be at the left of Q, — since each number of the first series is smaller than each number of the second, — and P and Q will approach each other infinitely closely, but will never meet, — since the distance between them at the nth stage of their progress is $\frac{1}{10^n}$, which may be made smaller than any assigned distance, however small, by making n sufficiently large, but which can not be made zero. In other words: *the points P and Q are each approaching, infinitely closely, a fixed common point R which lies between them.*



Moreover, there exists only *one* such fixed point, as R, between P and Q: for, if there be more than one, let R_1 be another point distinct from R, and approached infinitely closely by both P and Q, and let d be the distance between R and R_1; now the distance between P and Q is $\frac{1}{10^n}$, and this may be made smaller than d by sufficiently increasing n; therefore R and R_1 can not both be between P and Q, which was to be shown.

Now, there being *one, and only one*, fixed point, R, determined (defined) by the two infinite series in (1) above, therefore the distance OR may be said to be defined by these infinite series; and since these series are formed as above explained, therefore the distance OR may be appropriately represented by the symbol $\sqrt{2}$; hence the above series may be said to *define* the value of $\sqrt{2}$ (cf. § 130, note 3).

As in the particular example just now considered, so in general, *any two infinite series of rational numbers (expressed decimally or otherwise), one series increasing and the other decreasing, define an irrational number if the difference between the nth terms of the two series, while it can never be made zero, may be made smaller than any assigned number, however small, by sufficiently increasing* n. *Moreover, every irrational number may be represented in this way* (cf. § 130).

207. Equality, sum, product, etc., of irrational numbers. Let k and k' be two given positive irrational numbers, and let them be defined by infinite series of rational numbers as explained in § 206;

i.e., let $$a_1,\ a_2,\ a_3,\ \cdots\ a_n,\ \cdots < k < b_1,\ b_2,\ b_3,\ \cdots\ b_n,\ \cdots, \qquad (1)$$

and $$a'_1,\ a'_2,\ a'_3,\ \cdots\ a'_n,\ \cdots < k' < b'_1,\ b'_2,\ b'_3,\ \cdots\ b'_n,\ \cdots, \qquad (2)$$

wherein $a_n - b_n$ and $a'_n - b'_n$ may each be made smaller than any assigned number, however small, by sufficiently increasing n.

Then k is said to be **equal** to k' if, and only if, every one of the a's is less than every one of the b''s, and every a' is less than every b.

And k is said to be **greater** than k' if, and only if, some of the a's are greater than some of the b''s.

Again, the **sum, difference, product**, and **quotient** of k and k' may be defined, respectively, by the following pairs of infinite series:

$$a_1 + a'_1,\ a_2 + a'_2,\ a_3 + a'_3,\ \cdots\ a_n + a'_n,\ \cdots < k + k' < b_1 + b'_1,$$
$$b_2 + b'_2,\ b_3 + b'_3,\ \cdots\ b_n + b'_n,\ \cdots, \qquad (3)$$

$$a_1 - b'_1,\ a_2 - b'_2,\ a_3 - b'_3,\ \cdots\ a_n - b'_n,\ \cdots < k - k' < b_1 - a'_1,$$
$$b_2 - a'_2,\ b_3 - a'_3,\ \cdots\ b_n - a'_n,\ \cdots, \qquad (4)$$

$$a_1 \cdot a'_1,\ a_2 \cdot a'_2,\ a_3 \cdot a'_3,\ \cdots a_n \cdot a'_n,\ \cdots < k \cdot k' < b_1 \cdot b'_1,\ b_2 \cdot b'_2,\ b_3 \cdot b'_3,\ \cdots b_n \cdot b'_n,\ \cdots, \tag{5}$$

and $$a_1 \div b'_1,\ a_2 \div b'_2,\ a_3 \div b'_3,\ \cdots a_n \div b'_n,\ \cdots < k \div k' < b_1 \div a'_1,\ b_2 \div a'_2,\ b_3 \div a'_3,\ \cdots b_n \div a'_n,\ \cdots. \tag{6}$$

Note 1. Observe that if $k = k'$, as defined above, then these two irrational numbers have the *same decimal expressions*, however far they may be carried out. For suppose that some decimal figure, say the 14th, in k is greater than the corresponding figure in k', then the corresponding a would be equal to, or greater than, the corresponding b', and k would *not* equal k' under the above definition.

Note 2. In applying the above definitions, say that of the sum, it may happen that $a_1 + a'_1 = a_2 + a'_2 = \cdots = a_n + a'_n = \cdots = b_n + b'_n = \cdots$; in this case $k + k' = a_n + a'_n = b_n + b'_n$, *i.e.*, this sum is a *rational* number. To illustrate this fact numerically, let $k = \sqrt{2}$ and $k' = 5 - \sqrt{2}$.

Note 3. The above definitions [inequalities (3)–(6)] apply also when negative irrational numbers are involved: those of sum and difference apply directly, and those of product and quotient apply by regarding the numbers as positive and attaching the proper sign to the result.

208. Comparisons and operations between rational and irrational numbers. A given rational number r is said to be less than k (see § 207) if, and only if, some of the a's are greater than r, otherwise it is greater than k.

The sum of a rational and an irrational number, say $k + r$, is defined by the series

$$a_1+r,\ a_2+r,\ a_3+r,\ \cdots a_n+r < k+r < b_1+r,\ b_2+r,\ b_3+r,\ \cdots b_n+r,\ \cdots;$$

and the difference, product, and quotient of a rational and an irrational number are defined in a similar manner.

209. Combinatory laws of irrational numbers. That the irrational numbers are subject to the same combinatory laws as are the rational numbers follows easily from the definitions given in §§ 207 and 208. Thus, by (3) of § 207,

$$a_1+a'_1,\ a_2+a'_2,\ a_3+a'_3,\ \cdots < k+k' < b_1+b'_1,\ b_2+b'_2,\ b_3+b'_3,\ \cdots, \tag{1}$$

and $$a'_1+a_1,\ a'_2+a_2,\ a'_3+a_3,\ \cdots < k'+k < b'_1+b_1,\ b'_2+b_2,\ b'_3+b_3,\ \cdots; \tag{2}$$

but since the addition of rational numbers is commutative, *i.e.*, since $a_1 + a'_1 = a'_1 + a_1$, etc., therefore the two infinite series which define $k + k'$ are exactly the same as those which define $k' + k$; but, by § 206, two such series define *one, and only one*, irrational number, therefore $k + k' = k' + k$.

In the same way it may be shown that the sum of any number of irrational numbers is independent of the order in which the summands are arranged; *i.e., irrational numbers are subject to the commutative law of addition.*

That this law holds also when rational numbers are added to irrational numbers, and *vice versa,* follows from § 208.

Moreover, by means of (5) of § 207, and by reasoning altogether similar to that which has just now been employed, the commutative law of multiplication may be established.

The associative law of addition, and also that of multiplication, is proved from the commutative law in precisely the same way as that employed for integers in §§ 51 and 53.

And finally, since $(l+m)n = ln + mn$ for all rational numbers, therefore, by reasoning altogether similar to that employed to prove the commutative law of addition and of multiplication, it is easily proved that $(k+k')k'' = k \cdot k'' + k' \cdot k''$, wherein k, k', and k'' are any three irrational numbers which are defined by infinite series of rational numbers as in § 207; hence, even for irrational numbers, multiplication is distributive with regard to addition.

REMARK. For a more extended treatment of irrational numbers see Tannery's Arithmétique, Chapter XII; or Weber's Encyklopädie der Elementar-Mathematik, Chapter IV.

APPENDIX B

COMPLEX NUMBERS

[Supplementary to § 146]

210. Complex numbers. In the treatment of complex numbers given in the preceding pages, considerations of simplicity demanded that the proofs of their combinatory laws be postponed; accordingly these laws were there tentatively assumed to hold, — compare footnote, p. 244.

The following definition of a complex number, while it may at first sight seem somewhat arbitrary, is fully justified by the beautiful results to which it leads, and it serves at the same time to illustrate a *means* of defining numbers which has not hitherto been employed in this book.

A **complex number** is a combination of two *real* numbers, such as a and b, which will be temporarily represented by the symbol (a, b), and which satisfies the following defining equations:

$$(a, b) = (a', b'), \text{ if, and only if, } a = a' \text{ and } b = b', \tag{1}$$

$$(a, b) + (a', b') = (a + a', b + b'), \tag{2}$$

and

$$(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b); \tag{3}$$

these equations merely define what is meant by *equals*, *sum*, and *product*, for complex numbers.

Moreover, in order immediately to connect complex numbers more closely with real numbers, and to make the latter a special case of the former, let

$$(a, 0) = a, \tag{4}$$

which may be done since it is consistent with each of the above defining equations.

211. Immediate consequences of the definitions in § 210. It will now be shown that if (a, b) is any combination whatever of two real numbers which satisfies the defining equations in § 210, then

$$(a, b) \equiv a + bi,$$

wherein $i^2 = -1$; and hence that the complex number defined in § 210 is none other than the complex number $a + bi$, already considered in Chapter XIV.

Thus, by (3) and (4) of § 210,

$$(0, 1) \cdot (0, 1) = (-1, 0) = -1,$$

i.e., $(0, 1)^2 = -1$, and therefore $(0, 1) = \sqrt{-1} = i$.

Again, by (3) of § 210, $(0, b) = (b, 0) \cdot (0, 1)$,

i.e., $(0, b) = bi.$

And finally, by (2) of § 210,

$$(a, b) = (a, 0) + (0, b),$$

i.e., $(a, b) = a + bi$

which was to be proved.

212. Combinatory laws of complex numbers. That the commutative law of addition, already established for real numbers, holds for complex numbers also may be easily proved as follows.

By (2) of § 210,

$$(a, b) + (a', b') = (a + a', b + b'), \text{ and } (a', b') + (a, b) = (a' + a, b' + b),$$

but, since a, a', b, and b' are real numbers,

therefore $a + a' = a' + a$, and $b + b' = b' + b$,

and therefore $(a, b) + (a', b') = (a', b') + (a, b)$;

i.e., the commutative law holds for the sum of two complex numbers.

Moreover, it is evident that the proof just now given for *two* complex numbers may be easily extended to any number of such numbers; and since (a, b) is a real number when $b = 0$, and a pure imaginary number when $a = 0$, therefore this proof applies also when real numbers and complex numbers are added together.

Again, by means of (3) of § 210, and by reasoning altogether similar to that which has just been employed in the proof of the commutative law of addition, it is easily shown that multiplication is also subject to the commutative law.

The associative law of addition and of multiplication is proved from the commutative law in precisely the same way as that employed for integers, §§ 51 and 53.

And finally, it is easily proved from the definitions of § 210 that

$$\overline{(a, b) + (a', b')} \cdot (a'', b'') = (a, b) \cdot (a'', b'') + (a', b') \cdot (a'', b'');*$$

---

* The details of this proof are left as an exercise for the student; he may establish this equality by showing that each member is equal to the complex number $(aa'' - bb'' + a'a'' - b'b'', ab'' + a''b + a'b'' + a''b')$.

i.e., multiplication with complex numbers is distributive with regard to addition.

213. Subtraction and division with complex numbers. Here, as with real numbers, subtraction and division are defined, respectively, as the inverses of addition and multiplication (cf. § 3); and, based upon this definition, it will now be shown that any two given complex numbers have a unique difference and a unique quotient, which may be easily written down from the given numbers. To show this, let (a, b) and (a', b') be any two given complex numbers, and let

$$(a, b) - (a', b') = (x, y);$$

then, by § 3 (ii), $$(x, y) + (a', b') = (a, b),$$

whence, from (2) and (1) of § 210,

$$x + a' = a \text{ and } y + b' = b;$$

therefore $$x = a - a' \text{ and } y = b - b',$$

i.e., $$(a, b) - (a', b') = (a - a', b - b').$$

Again, let $$(a, b) \div (a', b') = (x, y);$$

then, by § 3 (iv), $$(x, y) \cdot (a', b') = (a, b),$$

and therefore, from (3) and (1) of § 210,

$$a'x - b'y = a \text{ and } a'y + b'x = b,$$

whence $$x = \frac{aa' + bb'}{a'^2 + b'^2} \text{ and } y = \frac{a'b - ab'}{a'^2 + b'^2},$$

i.e., $$\frac{(a, b)}{(a', b')} = \left(\frac{aa' + bb'}{a'^2 + b'^2}, \frac{a'b - ab'}{a'^2 + b'^2}\right).$$

On recalling the conclusion of § 211, the two results just obtained may be written, respectively, as

$$a + bi - (a' + b'i) = a - a' + (b - b')i,$$

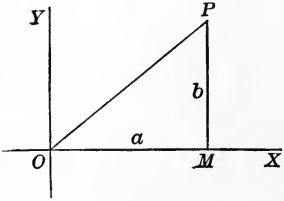
and $$\frac{a + bi}{a' + b'i} = \frac{aa' + bb' + (a'b - ab')i}{a'^2 + b'^2}.$$

214. Powers and roots of complex numbers. Raising a complex number to a positive integral power is merely a special case of multiplication, and is therefore fully provided for in (3) of § 210.

The method of extracting the square root of a complex number* is illustrated by means of a particular example in § 182; and it is evident, from what is there said, that this same process may be applied to any complex number whatever.

Moreover, by the method employed in the note of § 182, it is now evident that any even root of any negative number whatever can be expressed in the form $a + bi$, wherein a and b are real, and $i^2 = -1$.

215. Graphic representation of complex numbers. A complex number, such as $a + bi$, may be graphically represented by the point (P) whose coördinates (§ 114) are a and b. In this scheme of representation it is evident that to every complex number there corresponds one and only one point in the plane, and conversely, to every point in the plane there corresponds one and only one complex number,—if $a=0$ the corresponding point lies on the line OY, while if $b = 0$ it lies on OX.



This method of graphically representing a complex number was introduced by Argand in 1806, and is known as the **Argand diagram.**

Instead of representing $a + bi$ by the point P, it may also be represented by the line OP; each of these methods is, in fact, often employed.

The length of the line OP, which is $\sqrt{a^2 + b^2}$, is called the **modulus** (also the **absolute value**) of the number $a + bi$, and the angle XOP is called its **argument** (also its **amplitude**).

Not only may given complex numbers be represented by the Argand diagram, but the sum, product, etc., of two or more of them, being itself a complex number, may also be represented by such a diagram.

E.g., in the following diagram, OP represents $9 + 2i$, OQ represents $2 + 7i$, and OR represents their sum, viz., $11 + 9i$.

Observe that PR is equal and parallel to OQ (why?), and hence that $OPRQ$ is a parallelogram. From this it follows that if any two

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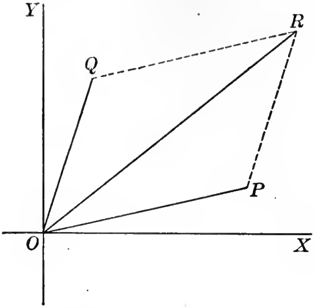
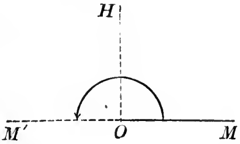
* Higher roots of complex numbers can not in general be extracted by the elementary methods thus far studied.



complex numbers are represented by the Argand diagram, then their sum is represented by the diagonal of the parallelogram of which the given numbers are a pair of adjacent sides.

NOTE 1. From a physical point of view, it is also quite appropriate to call *OR* the *sum* of *OP* and *OQ*. Thus, if two forces which are represented in amount and direction by *OP* and *OQ*, respectively, act simultaneously upon a body situated at *O*, the result is the same as if a single force represented in amount and direction by *OR* were acting on this body.

NOTE 2. The fact that $i \cdot i = -1$ is also consistent with the Argand diagram. *E.g.*, the effect of multiplying any given line as *OM* by -1 is to reverse its quality, and this may be thought of as accomplished by rotating *OM* through an angle of 180° to the position *OM'*, as shown in the figure; now, since multiplying *OM* by $i \cdot i$ also reverses its quality, therefore multiplying *OM* by i alone should rotate it through 90° to the position *OH*. Hence if *OM* represents any real number, then the number represented by $i \cdot OM$ should be laid off on a line perpendicular to *OM*, as it is in the Argand diagram.



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