Florian Cajori

ELEMENTARY ALGEBRA

BY

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PREFACE

In preparing a new algebra for secondary schools, the author has provided *a new set of examples throughout the book.* These examples have been selected and graded with great care. There are nearly four thousand of them, surely a sufficient number for practice in fixing in the minds of pupils the principles of elementary algebra.

At the request of many teachers a sufficiently full treatise on graphs and several pages of exercises in physics have been introduced.

The first chapter contains the necessary definitions and illustrations of the commutative, associative, and distributive laws of algebra. This chapter should be *read* carefully at first, and later particular attention should be given to the principal definitions. The second chapter treats of simple equations and is designed to lead the beginner to see the advantages of algebraic methods before he encounters negative numbers. Only *positive* numbers are involved in the first two chapters, and the recognition of the fact that the true nature of subtraction is counting backward, and that the true nature of multiplication is forming the product from the multiplicand in the same way as the multiplier is formed from unity, leads to an easy explanation in the third chapter of all the elementary processes with negative numbers. All the rules of this chapter are illustrated and enforced by examples that involve *simple algebraic expressions only.*

The more common operations with *compound expressions,* including resolution into factors and the treatment of fractions, follow the third chapter. The immediate succession of topics that require similar work is of the greatest importance to the beginner, and it is expected that the exercises in compound expressions will give sufficient practice in the use of symbols.

The chapter on factors has been made full in order to shorten subsequent work. The easy methods of resolving trinomials into factors and the explanation of the Factor Theorem will be found of great service in abridging many algebraic processes. Examples showing short methods of finding the highest common factor of compound expressions, and of solving quadratic equations by resolution into factors, should receive special attention when these subjects are reached.

Many examples have been worked out in full in order to exhibit the best methods of dealing with different classes of problems and the best arrangement of the work. In the statement of problems such assistance has been given as seemed to be required for the best results.

Short and easy chapters on Limits, Series, Four-Place Logarithms, and Permutations and Combinations have been introduced.

The author is under obligations to Professor B. F. Yanney, Mount Union College, Alliance, Ohio, who has read the proofs and given valuable suggestions.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH

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NOTICE TO TEACHERS

Pamphlets containing the answers will be furnished without charge to teachers for their classes, *on application of the teachers to GINN & COMPANY, Publishers.*

ELEMENTARY ALGEBRA

CHAPTER I

DEFINITIONS AND NOTATION

1. Magnitudes. Whatever admits of increase or decrease is called a **magnitude.** Every magnitude must therefore admit of comparison with another magnitude of the same kind in such a way as to determine whether the first is greater than, less than, or equal to the other.

A measurable magnitude is a magnitude that admits of being considered as made up of equal parts.

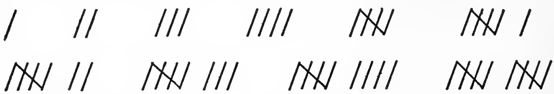
To **measure** any given measurable magnitude, we take for a standard of reference a definite magnitude of the same kind as the magnitude to be measured and determine how many magnitudes, each equal to the standard of reference, will together constitute the given magnitude.

2. Units. In counting separate objects or in measuring magnitudes, the *standards* by which we count or measure are called **units.**

Thus, in counting the boys in a school, the unit is a boy; in selling eggs by the dozen, the unit is a dozen eggs; in selling bricks by the thousand, the unit is a thousand bricks; in expressing the measure of short distances, the unit is an inch, a foot, or a yard; in expressing the measure of long distances, the unit is a rod or a mile.

3. Numbers. *Repetitions of the unit* are expressed by **numbers.**

A single unit and groups of units formed by successive additions of a unit may be represented as follows:



These groups represent **numbers** which are named one, two, three, four, five, six, seven, eight, nine, ten. It is obvious that these representative groups have the same meaning, whatever units are counted.

4. Quantities. A measurable magnitude expressed as a *magnitude actually measured* is called a **quantity.** Hence, a quantity consists of two components. One of these components is the *name* of the unit; the other component is the *number* of units taken to make the quantity.

Note. Quantities are often called *concrete numbers*, the adjective *concrete* being transferred from the units counted to the numbers that count them; but a number signifies the *times* a unit is taken whether the unit is expressed or understood, and *is always abstract.*

Thus, 4 barrels of flour means 4 times 1 barrel of flour; and 10 cords of wood means 10 times 1 cord of wood.

5. Algebra. Algebra, like Arithmetic, treats of numbers.

6. Number Symbols in Arithmetic. Instead of groups of straight marks, we use in Arithmetic the arbitrary symbols 1, 2, 3, 4, 5, 6, 7, 8, 9, called **Arabic numerals,** for the numbers one, two, three, four, five, six, seven, eight, nine.

The next number, ten, is indicated by writing the figure 1 in a different position, so that it shall signify not *one,* but *ten.* This change of position is effected by introducing a new symbol, 0, called **nought** or **zero,** and signifying *none.*

Each succeeding number up to the number consisting of 10 tens is expressed by writing the figure for the number of tens it contains in the second place from the right, and the figure for the number of units besides in the first place. The *hundreds* of a number are written in the *third place* from the right; the *thousands* are written in the *fourth place* from the right; and so on.

7. Number Symbols in Algebra. Algebra employs the *letters of the alphabet* in addition to the figures of Arithmetic to represent numbers. The letters of the alphabet are used as *general* symbols of numbers to which *any particular values* may be assigned. In any problem, however, a letter is understood to have the same value throughout the problem.

8. In Arithmetic the figures that represent numbers are generally themselves called numbers; and similarly in Algebra the symbols that stand for numbers are themselves called numbers. Letter symbols are called *literal expressions,* and figure symbols are called *numerical expressions.*

The number which a letter represents is called its *value,* and if represented *arithmetically,* its *numerical* value.

9. Names Common to Arithmetic and Algebra. Names common to Arithmetic and Algebra, as addition, sum, subtraction, minuend, subtrahend, difference, etc., have the same meaning in both, or an *extended meaning* in Algebra consistent with the sense attached to them in Arithmetic.

Principal Signs used in Algebra

10. Signs of Operations. The principal operations in Algebra are, as in Arithmetic, Addition, Subtraction, Multiplication, Division, Involution, and Evolution. A mark used to denote that one of these operations is to be performed on a number is called a **sign of operation.**

11. The Sign of Addition, +. The sign + is read *plus.*

Thus, $4 + 3$ is read 4 plus 3, and indicates that the number 3 is to be added to the number 4; $a + b$ is read a plus b, and indicates that the number b is to be added to the number a.

12. The Sign of Subtraction, −. The sign − is read *minus.*

Thus, $4 - 3$ is read 4 minus 3, and indicates that the number 3 is to be subtracted from the number 4; $a - b$ is read a minus b, and indicates that the number b is to be subtracted from the number a.

13. The Sign of Multiplication, ×. The sign × is read *times*, or *multiplied by*.

Thus, 4×3 is read 4 times 3, and indicates that the number 3 is to be multiplied by 4; $a \times b$ is read a times b, and indicates that the number b is to be multiplied by the number a.

A dot is sometimes used for the sign of multiplication. Thus, $2 \cdot 3 \cdot 4 \cdot 5$ means the same as $2 \times 3 \times 4 \times 5$. Either sign is read *multiplied by* when followed by the multiplier.

Thus, $\$a \times b$, or $\$a \cdot b$, is read a dollars multiplied by b.

14. The Sign of Division, ÷. The sign ÷ is read *divided by*.

Thus, $4 \div 2$ is read 4 divided by 2, and indicates that the number 4 is to be divided by 2; $a \div b$ is read a divided by b, and indicates that the number a is to be divided by the number b.

Division is also indicated by writing the dividend above the divisor with a horizontal line between them; or by separating the dividend from the divisor by an oblique line, called the solidus.

Thus, $\frac{a}{b}$, or a/b, means the same as $a \div b$.

NOTE. The operation of adding b to a, of subtracting b from a, of multiplying a by b, or of dividing a by b is *algebraically complete* when the two letters are connected by the proper sign.

15. The Signs of Relation.

$=$, read *equals, is equal to, will be equal to,* etc.
$\neq$, read *is not equal to,* etc.
$>$, read *is greater than,* thus $9 > 4$.
$<$, read *is less than,* thus $4 < 9$.
$\ngtr$, read *is not greater than.*
$\nless$, read *is not less than.*
:, : :, the signs of proportion, as in Arithmetic.

Thus, $a : b :: c : d$, or $a : b = c : d$, is read a is to b as c is to d.

16. The Signs for Words.

$\therefore$, read *therefore, hence.*

$\because$, read *because, since.*

Thus, $\because a = b$ and $b = c$, $\therefore a = c$, is read since a equals b and b equals c, therefore a equals c.

17. The Sign of Continuation, $\cdots$ The sign $\cdots$ is read *and so on.*

Thus, 1, 2, 3, 4, $\cdots$ is read one, two, three, four, *and so on.* a_1, a_2, a_3, $\cdots$, a_n is read a sub one, a sub two, a sub three, *and so on* to a sub n. a', a'', a''', $\cdots$ is read a prime, a second, a third, *and so on.*

18. The Signs of Aggregation. The signs of aggregation are the parenthesis ( ), the bracket [ ], the brace { }, the vinculum —, and the bar |.

These signs mean that the operations indicated in the expressions affected by them are to be performed first, and the result treated as a single number.

Thus, $(a + b) \times c$ means that b is to be added to a and the sum multiplied by c; $(a - b) \times c$ means that b is to be subtracted from a and the difference multiplied by c.

The vinculum is written over the expression that is to be treated as a single number.

Thus, $a - \overline{b + c}$ means the same as $a - (b + c)$, and signifies that c is to be added to b and the sum subtracted from a.

Factors, Powers, Roots

19. Factors. When a number is the product of two or more numbers, each of these numbers, or the product of two or more of them, is called a **factor** of the given number.

Thus, 2, a, b, $2a$, $2b$, ab are factors of $2ab$.

Factors that contain letters are called **literal** factors; factors expressed by figures are called **numerical** factors.

20. The sign $\times$ is omitted between factors if the factors are letters, or a numerical factor and a literal factor.

Thus, we write abc for $a \times b \times c$; we write $63\,ab$ for $63 \times a \times b$.

The product abc must not be confounded with the sum $a + b + c$.

If $$a = 2,\ b = 3,\ c = 4,$$
then $$abc = 2 \times 3 \times 4 = 24;$$
but $$a + b + c = 2 + 3 + 4 = 9.$$

NOTE. When a sign of operation is omitted in the notation of Arithmetic, it is always the *sign of addition;* but when a sign of operation is omitted in the notation of Algebra, it is always the *sign of multiplication.* Thus, 456 means $400 + 50 + 6$, but $4\,ab$ means $4 \times a \times b$.

21. If one factor of a product is equal to 0, the product is equal to 0, whatever the values of the other factors. A factor equal to 0 is called a **zero factor.**

Thus, $abcd = 0$, if a, b, c, or $d = 0$.

22. Coefficients. Any factor of a product may be considered as the **coefficient** of the remaining factors; that is, the **co-factor** of the remaining factors.

Coefficients expressed by letters are called *literal* coefficients; expressed by Arabic numerals, *numerical* coefficients.

Thus, in $7\,x$, 7 is the *numerical coefficient* of x; in ax, a is the *literal coefficient* of x.

If no numerical coefficient is written, 1 is understood.

23. Powers. A **power** of a number is the product obtained by using that number a certain number of times as a multiplier, starting with *unity as first multiplicand.* The operation of forming a power is called **involution**; the number used as a multiplier is called the **base** of the power; the *number of successive multiplications by the base* is called the **degree** of the power; the number indicating the degree of the power is called the **exponent** or **index** of the power and is written in small characters to the right and a little above the line of the base.

Thus, $1 \times a \times a$ is represented by a^2 (read a square); here a is the *base*, 2 is the *exponent* (or *index*), and a^2 is the second *power* of a.

$1 \cdot c \cdot c \cdot c$ is represented by c^3 (read c cube); here c is the base, 3 is the exponent, and the number c^3 is the third power of c.

In x^5 (read x to the fifth), x is the base, 5 is the exponent, and the number x^5 is the fifth power of x.

24. Since the exponent denotes the number of multiplications by the base to be made, the first to be performed on unity, it follows that a^1, the first power of a, represents $1 \times a$, or simply a.

Hence, also, a^0, the zero power of a, denotes that *no* multiplication by a is to be made, or, in other words, that the unit multiplicand is not to be multiplied by a. Therefore, $a^0 = 1$ for any value of a whatsoever.

25. In writing a power at full length as a product it is usual to omit the unit multiplicand, just as it is usual to omit a unit coefficient where such occurs.

Thus, instead of writing $x^3 = 1 \times x \times x \times x$, we write $x^3 = x \times x \times x$.

In this method of expressing a power *the exponent denotes the number of times the base is taken as a factor.*

26. Comparing powers, the second power is said to be *higher* than the first, the third higher than the second, and so on.

The difference of meaning between the terms *coefficient* and *exponent* must be carefully distinguished.

Thus,
$$4a = a + a + a + a;$$
$$a^4 = a \times a \times a \times a.$$

If $a = 3$,
$$4a = 3 + 3 + 3 + 3 = 12;$$
$$a^4 = 3 \times 3 \times 3 \times 3 = 81.$$

27. Roots. The inverse of involution, or the operation of finding a **root** of a number, is called **evolution.** A **root** is one of the equal factors of the number. If the number is resolved into *two equal factors,* each factor is called a **square root**; if into three equal factors, each factor is called a **cube root**; if into four equal factors, each factor is called a **fourth root**; and so on.

The root sign is $\sqrt{\ }$. Except for the square root, a number symbol is written over the root sign to show into how many equal factors the given number is to be resolved. This number symbol is called the **index of the root.**

Thus, $\sqrt{64}$ means the square root of 64; $\sqrt[3]{64}$ means the cube root of 64.

Algebraic Expressions

28. Algebraic Expressions. An algebraic expression is a number written with algebraic symbols. An algebraic expression may consist of one symbol, or of several symbols connected by signs.

Thus, a, $3abc$, $5a + 2b - 3c$, are algebraic expressions.

29. Terms. A term is an algebraic expression of one symbol, or of several symbols *not separated by the sign + or —.*

Thus, a, $5xy$, $2ab \times 4cd$, $\frac{3ab}{4cd}$ are algebraic expressions of one term each. A term may be separated into parts by the sign $\times$ or the sign $\div$.

30. Similar Terms. If terms have the *same letters,* and each letter has the *same exponent in all the terms,* these terms are called *like terms* or *similar terms.*

Thus, $3x^2y^3$, $5x^2y^3$, and $7x^2y^3$ are similar terms.

31. Simple Expressions. An algebraic expression of *one term* is called a **simple expression** or a **monomial.**

Thus, $5xy$, $7a \times 2b$, $7a \div 2b$, are simple expressions.

32. Compound Expressions. An algebraic expression of *two or more terms* is called a **compound expression** or a **polynomial.**

Thus, $5xy + 7a$, $2x - y - 3z$, are compound expressions.

33. A polynomial of two terms is called a **binomial**; of three terms, a **trinomial.** A polynomial is often called a **multinomial.**

Thus, $3a - b$ is a binomial; and $3a - b + c$ is a trinomial.

34. Positive and Negative Terms. A term preceded by the sign + is called a **positive term**; and a term preceded by the sign − is called a **negative term.** The sign + before a single term and before the first term of a polynomial is omitted.

35. A positive term and a negative term **cancel each other** when combined, *if both terms stand for the same number.*

36. Substitution. Two quantities, two numbers, or two operations are equal if either can be substituted for the other in algebraic expressions without changing the values of the expressions.

37. The Numerical Value of an Expression. The result obtained by putting particular values for the letters of an expression and performing the indicated operations is called the *numerical value* of the expression.

Numerical Values of Simple Expressions

1. If $a = 5$, find the numerical values of $4a$ and a^4.

$$4a = 4 \times a = 4 \times 5 = 20,$$

and
$$a^4 = a \times a \times a \times a = 5 \times 5 \times 5 \times 5 = 625.$$

2. If $a = 3$, $b = 4$, $c = 5$, find the numerical value of the expression $\frac{7}{15}abc$.

$$\tfrac{7}{15}abc = \tfrac{7}{15} \times 3 \times 4 \times 5 = 28.$$

3. If $x = 3$, $y = 4$, find the numerical value of $2x^3y^2$.

$$2x^3y^2 = 2 \times 3^3 \times 4^2 = 2 \times 27 \times 16 = 864.$$

4. If $x = 4$, $y = 5$, find the numerical value of $\frac{3}{4}xy^2$.

$$\tfrac{3}{4}xy^2 = \tfrac{3}{4} \times 4 \times 5^2 = \tfrac{3}{4} \times 4 \times 25 = 75.$$

5. If $a = 2$, $b = 3$, $c = 4$, find the numerical value of $8a^2b \div 3c^3$.

$$\frac{8a^2b}{3c^3} = \frac{8 \times 2 \times 2 \times 3}{3 \times 4 \times 4 \times 4} = \frac{1}{2}.$$

6. If $x = 3$, find the numerical value of

$$\text{(i) } \sqrt{4x^2}; \quad \text{(ii) } 2\sqrt{(9x^2)}; \quad \text{(iii) } \sqrt{4}\,x^2.$$

$$\text{(i)} \quad \sqrt{4x^2} = \sqrt{4 \times 3^2} = \sqrt{36} = 6.$$
$$\text{(ii)} \quad 2\sqrt{(9x^2)} = 2\sqrt{(9 \times 3^2)} = 2 \times 9 = 18.$$
$$\text{(iii)} \quad \sqrt{4}\,x^2 = \sqrt{4} \times 3^2 = 2 \times 9 = 18.$$

NOTE. When no vinculum or parenthesis is used, a radical sign affects only the symbol immediately following it.

EXERCISE 1

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $x = 5$, $y = 6$, $z = 0$, find the numerical value of:

1. $7x$.
2. $8b^2$.
3. $7c^3$.
4. $5bc$.
5. $6c^2d$.
6. $2c^2dx$.
7. $14c^3yz$.
8. b^3c^2y.
9. $3a^4b^2x$.
10. $\frac{3}{8}d^2x$.
11. $\frac{2}{9}c^3d$.
12. $\frac{3}{10}b^2xy$.
13. $\frac{5}{18}a^3dy^2$.
14. $\frac{2}{3}x^2y^3z$.
15. $\frac{4}{15}c^2x^2$.
16. $\frac{5}{12}xy^2$.
17. $\frac{7}{10}d^2x^2$.
18. $\frac{13}{12}c^2d^2$.
19. $\dfrac{4b^4x^2}{5d^3}$.
20. $\dfrac{7c^3x^2}{5by^2}$.
21. $\dfrac{x^2y^2}{5c^2d^2}$.
22. $\dfrac{9cdz}{dy}$.
23. $\dfrac{5a^5d^3}{4b^3x^2}$.
24. $\sqrt{b^2dx^2}$.
25. $\sqrt{}dy$.
26. $\sqrt[3]{27}\,d^2$.
27. $\sqrt{(bcy)}$.
28. $\sqrt{x^2y^3z^3}$.
29. $\sqrt[3]{9bcd}$.
30. $2\sqrt{c^2dx^2}$.
31. $cd\sqrt{dy^2}$.
32. $abc\sqrt{2cx^2y}$.
33. $\frac{3}{4}d\sqrt{dy^2}$.
34. $abcdxyz$.
35. $cx\sqrt[3]{b^2cdy^2}$.
36. $b^2c\sqrt[3]{d^2z^2}$.
37. b^2dx^2y.

Numerical Values of Compound Expressions

38. *The operations indicated in a term must be performed before the operation indicated by the sign prefixed to the term.*

Note. Each term should be written in the algebraic form by omitting the sign × between two literal factors or between a numerical factor and a literal factor.

39. The **parts of a term** are combined in the order of the signs × and ÷ *from left to right.*

The **terms** of an expression are combined in the order of the signs + and − from left to right.

Thus, $60 - 40 \div 5 \times 3 - 20 = 60 - \frac{40}{5} \times 3 - 20 = 16.$

40. *The sum of two numbers is the same whether the second number is added to the first or the first is added to the second.*

In symbols, $a + b = b + a.$

This is called the **commutative law for addition.**

41. *The sum of three numbers is the same whether the sum of the second and third is added to the first number, or the third number is added to the sum of the first and second numbers.*

In symbols, $a + (b + c) = (a + b) + c.$

This is called the **associative law for addition.**

1. If $a = 2$, $b = 10$, $x = 3$, $y = 5$, find the numerical value of $6b \div (b - y) - 3x + 2bxy \div 10a.$

$$6b \div (b - y) - 3x + 2bxy \div 10a = \frac{6 \times 10}{10 - 5} - 3 \times 3 + \frac{2 \times 10 \times 3 \times 5}{10 \times 2}$$
$$= 12 - 9 + 15 = 18.$$

2. If $a = 6$, $b = 4$, find the numerical value of

$$(a + b)(a - b) + \frac{a + b}{a - b}.$$

$$(a + b)(a - b) + \frac{a + b}{a - b} = (6 + 4)(6 - 4) + \frac{6 + 4}{6 - 4}$$
$$= 10 \times 2 - \frac{10}{2} = 20 - 5 = 15.$$

Exercise 2

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, $f = 0$, find the numerical value of:

1. $9a + 2b + 3c - 2f$.

2. $4e - 3a - 3b + 5c$.

3. $8abc - bcd + 9cde - def$.

4. $\frac{4ac}{b} + \frac{8bc}{d} - \frac{5cd}{e}$.

5. $7e + bcd - \frac{3bde}{2ac}$.

6. $abc^2 + bcd^2 - dea^2 + f^3$.

7. $e^4 + 6e^2b^2 + b^4 - 4e^3b - 4eb^3$.

8. $\frac{8a^2 + 3b^2}{a^2b^2} + \frac{4c^2 + 6b^2}{c^2 - b^2} - \frac{c^2 + d^2}{e^2}$.

9. $\frac{d^c}{b^e}$.

10. $\frac{e^c + b^a}{c^b - b^c}$.

11. $\frac{b^c + d^c}{b^2 + d^2 - bd}$.

12. $\frac{e^c - d^c}{e^2 + ed + d^2}$.

If $a = 2$, $b = 10$, $x = 3$, $y = 5$, find the numerical value of:

13. $xy + 4a \times 2$.

14. $xy + 15b \div 5$.

15. $3x + 7y \div 7 + a \times y$.

16. $6b - 8y \div 2y \times b - 2b$.

17. $(6b - 8y) \div 2y \times b + 2b$.

18. $(6b - 8y) \div (2y \times b) + 2b$.

19. $6b - (8y \div 2y) \times b - 2b$.

Parentheses

42. A Parenthesis preceded by the Sign +. If a man has 10 dollars and afterwards collects 3 dollars and then 2 dollars, it makes no difference whether he puts the 3 dollars and the 2 dollars together and adds their sum to his 10 dollars, or adds the 3 dollars to his 10 dollars, and then the 2 dollars.

The first process is represented by $10+(3+2)$.
The second process is represented by $10+3+2$.

Hence, $$10+(3+2)=10+3+2. \quad (1)$$

If a man has 10 dollars and afterwards collects 3 dollars and then pays a bill of 2 dollars, it makes no difference whether he pays the 2 dollars from the 3 dollars collected and adds the remainder to his 10 dollars, or adds the 3 dollars collected to his 10 dollars and pays from this sum his bill of 2 dollars.

The first process is represented by $10+(3-2)$.
The second process is represented by $10+3-2$.

Hence, $$10+(3-2)=10+3-2. \quad (2)$$

If we use general symbols in (1) and (2), we have,

$$a+(b+c)=a+b+c,$$

and $$a+(b-c)=a+b-c.$$ Hence,

The general rule for a parenthesis preceded by +:

If an expression within a parenthesis is preceded by the sign +, the parenthesis may be removed without making any change in the signs of the terms of the expression.

Instead of a parenthesis, any other sign of aggregation may be used and the same rule will apply.

43. A Parenthesis preceded by the Sign −. If a man with 10 dollars has to pay two bills, one of 3 dollars and one of 2 dollars, it makes no difference whether he takes 3 dollars and 2 dollars at one time, or takes 3 dollars and 2 dollars in succession, from his 10 dollars.

The first process is represented by $10-(3+2)$.
The second process is represented by $10-3-2$.

Hence, $$10-(3+2)=10-3-2. \quad (1)$$

If a man has 10 dollars consisting of two 5-dollar bills, and has a debt of 3 dollars to pay, he can pay his debt by giving a 5-dollar bill and receiving 2 dollars.

This process is represented by $10 - 5 + 2$.

Since the debt paid is 3 dollars, that is, $(5 - 2)$ dollars, the number of dollars he has left can be expressed by

$$10 - (5 - 2).$$

Hence, $$10 - (5 - 2) = 10 - 5 + 2. \qquad (2)$$

If we use general symbols in (1) and (2), we have,

$$a - (b + c) = a - b - c,$$

and $$a - (b - c) = a - b + c.$$ Hence,

The general rule for a parenthesis preceded by —:

If an expression within a parenthesis is preceded by the sign —, the parenthesis may be removed, provided the sign before each term within the parenthesis is changed, the sign + to — and the sign — to +.

NOTE. If the *vinculum* is used, the sign prefixed to the first term under the vinculum must be understood as the sign before the vinculum.

Thus, $a + \overline{b - c}$ has the same meaning as $a + (b - c)$,
and $a - \overline{b - c}$ has the same meaning as $a - (b - c)$.

EXERCISE 3

Remove the parentheses and combine:

1. $7 + (5 + 3)$.
2. $7 + (5 - 3)$.
3. $8 + (6 + 2)$.
4. $8 + (6 - 2)$.
5. $7 - (5 - 3)$.
6. $9 - (5 + 3)$.
7. $8 - (6 + 2)$.
8. $8 - (6 - 2)$.
9. $(12 - 8) - (7 - 4)$.
10. $(10 - 4) - (2 + 3)$.
11. $(14 - 6) + (8 - 6)$.
12. $(7 + 3) - (4 - 2)$.

If $a = 8$, $b = 5$, $c = 6$, $d = 3$, find the value of:

13. $(a + b) + (c + d)$.

14. $(a + b) - (c + d)$.

15. $(a + b) + (c - d)$.

16. $(a + b) - (c - d)$.

17. $(a - b) + (c - d)$.

18. $(a - b) - (c - d)$.

19. $(a - b) + (c + d)$.

20. $(c + d) - (a - b)$.

Product of a Compound by a Simple Factor

44. In finding the product of $4(5 + 3)$, it makes no difference in the result whether we multiply the sum of 5 and 3 by 4, or multiply 5 by 4 and 3 by 4 and add the products.

By the first process, $4(5 + 3) = 4 \times 8 = 32$.
By the second process, $4(5 + 3) = (4 \times 5 + 4 \times 3) = 32$.
In like manner, $4(5 - 3) = 4 \times 2 = 8$,
and $4(5 - 3) = (4 \times 5 - 4 \times 3) = 8$.

In general symbols, $a(b + c) = ab + ac$,
and $a(b - c) = ab - ac$. Hence,

Multiplying the several terms of a compound expression by any number multiplies the expression by that number.

This is called the **distributive law for multiplication.**

45. *The order of the factors is immaterial.*

Thus, $4(5 + 3) = 4 \times 5 + 4 \times 3 = 32$,
and $(5 + 3)4 = 5 \times 4 + 3 \times 4 = 32$.
In general symbols, $ab = ba$.

This is called the **commutative law for multiplication.**

Perform the indicated operations:

1. $x + 3(a - b)$.
2. $x - 3(a - b)$.

1. $x + 3(a - b) = x + (3a - 3b) = x + 3a - 3b$.
2. $x - 3(a - b) = x - (3a - 3b) = x - 3a + 3b$.

Exercise 4

Perform the indicated operations, and find the numerical value of each expression if $a = 4$, $b = 3$, $c = 2$:

1. $5(ab + c)$.
2. $4(ac + b)$.
3. $3(a + bc)$.
4. $7(ab - c)$.
5. $6(ac - b)$.
6. $5a(b - c)$.
7. $2ab - a(bc - a)$.
8. $c + 2ab(ac - b)$.
9. $3ac - c(b + c)$.
10. $5ab - (b^2 + b)$.
11. $6bc - 4(ab - 3c)$.
12. $ab + b(a - c)$.
13. $(a - c)b - bc$.
14. $(a - b)c + 3ac$.
15. $(2a + 3b)b - 2ab$.
16. $(a^2 - b^2)c - (a^2 - c^2)$.
17. $(a^2 - c^2)b - (b^2 - c^2)$.
18. $2(bc + ac) - c(b^2 + c^2)$.

Quotient of a Compound by a Simple Expression

46. In finding the quotient of $(8 + 4) \div 2$ it makes no difference in the result whether we divide the sum of 8 and 4 by 2, or divide 8 by 2 and 4 by 2 and add the quotients.

By the first process, $(8 + 4) \div 2 = 12 \div 2 = 6.$

By the second process, $(8 + 4) \div 2 = (8 \div 2 + 4 \div 2) = 6.$

In general symbols, $\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c},$

and $\frac{a - b}{c} = \frac{a}{c} - \frac{b}{c}.$ Hence,

Dividing each term of a compound expression by any number divides the expression by that number.

This is called the **distributive law for division.**

1. Perform the indicated operations in the expression

$$x + (3a + 3b) \div 3.$$

$$x + (3a + 3b) \div 3 = x + (a + b) = x + a + b.$$

2. Perform the indicated operations in the expression

$$x - (3a + 3b) \div 3.$$

$$x - (3a + 3b) \div 3 = x - (a + b) = x - a - b.$$

Exercise 5

Perform the indicated operations, and find the numerical value of each quotient if $a = 8$, $b = 4$, $c = 2$:

1. $(a + b) \div b.$	5. $(ac + b) \div b.$	9. $(b^2 + c^2) \div a.$
2. $(a + c) \div c.$	6. $(ab + c) \div c.$	10. $(b^3 - ac) \div b.$
3. $(a - b) \div b.$	7. $(ac - b) \div b.$	11. $(a^2 - bc) \div c.$
4. $(a - b) \div c.$	8. $(ab - c) \div c.$	12. $(b^2 + ac) \div bc.$

CHAPTER II

SIMPLE EQUATIONS

47. Equations. A statement in symbols that two expressions stand for the same number is called an **equation.**

Thus, the equation $3x + 2 = 8$ states that $3x + 2$ and 8 stand for the same number.

48. Members. That part of the equation which precedes the sign of equality is called the **first member,** or **left side**; and that part of the equation which follows the sign of equality is called the **second member,** or **right side.**

49. Identities and Equations of Condition. An equation containing letters, if true for all values of the letters involved, is called an **identical equation,** or an **identity**; but if an equation is true only for certain particular values of the letters involved, it is called an **equation of condition,** or simply an **equation.**

Thus, $a + b = b + a$, which is true for *all values* of a and b, is an *identical equation*; and $3x + 2 = 8$, which is true only when x stands for 2, is an *equation of condition.*

In an identical equation it is customary to use the sign $\equiv$, called the **sign of identity,** instead of the sign of equality.

Thus, the two expressions $(a + b)^2$ and $a^2 + 2ab + b^2$ have the same value for all values of a and b, and we accordingly write the identity

$$(a + b)^2 \equiv a^2 + 2ab + b^2.$$

This is read $(a + b)^2$ is identically equal to $a^2 + 2ab + b^2$.

50. We often employ an equation to discover an *unknown number* from its relation to known numbers. We usually represent the unknown number by one of the *last* letters of the alphabet, as x, y, z; and the known numbers by the *first* letters, a, b, c, and by the Arabic numerals.

51. Simple Equations. Equations which, when reduced to their simplest form, contain only the *first power* of the unknown numbers are called *simple equations*, or *equations of the first degree.*

Thus, $7x + 5 = 4x + 14$, and $ax + b = c$ are simple equations in x.

52. Combining Like Terms. Two or more like terms may be combined to form a single like term by uniting their *numerical coefficients.*

Thus, $3ax + ax = 4ax$; and $5ax - 3ax = 2ax$.

53. To Solve an Equation with One Unknown Number is to find the unknown number; that is, to find the number which, when substituted for its symbol in the given equation, renders the equation an identity. This number is said to **satisfy** the equation and is called the **root** of the equation.

54. Axioms. In solving equations we make use of the following truths, called **axioms**, which are admitted to be true without proof.

Ax. 1. If equal numbers are added to equal numbers, the sums are equal.

Ax. 2. If equal numbers are subtracted from equal numbers, the remainders are equal.

Ax. 3. If equal numbers are multiplied by equal numbers, the products are equal.

Ax. 4. If equal numbers are divided by equal numbers, the quotients are equal.

Ax. 5. If two numbers are equal to the same number, they are equal to each other.

55. Transposition of Terms. It is convenient in solving a simple equation to bring all the terms that contain the symbols for the unknown numbers to one side of the equation, and all the other terms to the other side. This process is called **transposing the terms.**

56. *Any term may be transposed from one side of an equation to the other provided the sign of the term is changed.*

1. Find the number for which x stands when $x - b = a$.

Add b to each side, $x - b + b = a + b$. (Ax. 1)
Now $-b + b = 0$ (p. 9, § 35), $\therefore x = a + b$.

The result is the same as if we had transposed $-b$ from the left side to the right side and changed its sign.

2. Find the number for which x stands when $x + b = a$.

Subtract b from each side, $x + b - b = a - b$. (Ax. 2)
Now $+b - b = 0$ (p. 9, § 35), $\therefore x = a - b$.

In this case we have transposed b from the left side to the right side and changed its sign.

We may proceed in like manner in any other case.

57. *The sign of every term of an equation may be changed without destroying the equality.*

If we transpose each term of the equation,

$$c - x = a - b, \quad (1)$$

we have $$b - a = x - c,$$
that is, $$x - c = b - a. \quad (2)$$

Now (2) is the same as (1) with the sign of each term changed.

58. Numerical Equations. An equation in which all the known numbers are expressed by Arabic numerals is called a *numerical equation.*

59. To Solve a Simple Numerical Equation in x,

Transpose all the terms that contain x *to the left side, and all the other terms to the right side. Combine similar terms, and divide both members by the coefficient of* x.

1. Solve the equation $5x - 7 = 37 - 6x$.

Transpose $-6x$ and -7, $5x + 6x = 37 + 7$. (§ 56)
Combine, $11x = 44$. (p. 19, § 52)
Divide by 11, $x = 4$. (Ax. 4)

2. Solve the equation $1 - 2(4x + 1) = 5x - 2(5x + 8)$.

Multiply the compound factor by the simple factor in each side,

$$1 - (8x + 2) = 5x - (10x + 16).$$

Remove the parentheses, $1 - 8x - 2 = 5x - 10x - 16$.

Transpose, $10x - 8x - 5x = 2 - 1 - 16$.

Change the sign of every term,

$$8x + 5x - 10x = 16 - 2 + 1.$$

Combine, $3x = 15$.

Divide by 3, $x = 5$.

60. Verification. If the value found for x is substituted for x in the *original equation*, and the equation reduces to an *identity*, the *value* of x, that is, *the root of the equation*, is said to be **verified**.

NOTE. In verifying a solution, as in solving an equation, it is important to notice that the signs of all the terms may be changed.

Show that x stands for 4 in the equation

$$4x - 11 = 29 - 6x.$$

Put 4 for x, $4 \times 4 - 11 = 29 - 6 \times 4$,

or $16 - 11 = 29 - 24$,

that is, $5 = 5$.

EXERCISE 6

Find the value of x and verify the answer:

1. $3x + 1 = 19$.
2. $2x + 5 = 11$.
3. $17x = 7x + 10$.
4. $5x + 2 = 17$.
5. $4x + 7 = 6x + 1$.
6. $3x - 7 = 2x + 2$.
7. $7x - 12 = 3x + 4$.
8. $21x - 4 = 14x + 17$.
9. $3x + 13 = 4x + 6$.
10. $7x - 9 = 6x + 1$.
11. $8x - 14 = 6x + 6$.
12. $x - 3 = 15 - 2x$.
13. $6x - 6 = 8 + 4x$.
14. $x - 2 = 2x - 7$.

15. $42 - 21x + 15 - 5x = 68 - 11 - 4x.$

16. $60 - 4x - 3x + 6x = 77 - 7x - 11.$

17. $28x - 21 + 42 - 28x = 22 - 4x.$

18. $9(13 - x) - 4x = 5(21 - 2x) + 9x.$

19. $6(x - 5) + 2x = 8x - 2(x + 10).$

20. $199 + 15x - (2x - 5) = 17(x + 17) - 13x - 22.$

21. $8(3x - 2) - 7x - 5(12 - 3x) + 28 = 8(3x + 2) - 32.$

22. $7(3x - 6) + 5(x - 3) + 4(17 - x) = 77.$

23. $364 - 15(15x - 7) - 3(13 - 2x) + 8 = 6(x - 2).$

24. $111 + 39x = 7(18 - 3x) + 3(20 - 5x).$

25. $8(x - 10) - 9 + 3(15 - x) + 2(18 - x) - 22 = 0.$

26. $3(x + 5) - 5(x - 4) + 22 = 2(x + 6) + 3(11 - x) + 4.$

27. $2(16 - x) + 3(5x - 4) = 12(3 + x) - 2(12 - x).$

Statement and Solution of Problems

61. To express in algebraic language the conditions of a problem that are stated in common language is generally difficult for the beginner. We will therefore give an exercise on translating common language into algebraic language before proceeding to the solution of problems.

Exercise 7

1. Write in symbols: a diminished by b; a increased by b; a multiplied by b; a divided by b; the square of a; the square root of a; the cube root of a; the square of a multiplied by the fourth power of b.

2. A man sold a horse for a dollars and lost b dollars on the cost. What did the horse cost him?

3. If a is an integral number, what integral number immediately precedes it? What integral number immediately follows it?

4. A man bought a horse for m dollars and sold it for n dollars. If m is greater than n, how much did he lose?

5. A man sold a cow for c dollars and gained a dollars on the cost. What did he pay for the cow?

6. If the length of a day is t hours, what is the length of the night?

7. A rectangular field is a rods long and b rods wide. What is the area of the field?

8. A farmer receives c cents for one bushel of corn. What does he receive for a bushels of corn?

9. A man pays b dollars for one cord of wood. What must he pay for a cords of wood?

10. A boy runs a yards in one second. How many yards will he run in t seconds?

11. The area of a rectangular field is a square rods and the length of the field is b rods. What is the breadth?

12. A train travels c hours at the rate of a miles an hour. Find the distance traveled.

13. A horse goes c miles in b hours. How many miles does he go in one hour?

14. The product is p and the multiplier is m. Find the multiplicand.

15. The divisor is a, the quotient b, and the remainder c. Find the dividend.

16. The divisor is a, the dividend b, and the remainder c. Find the quotient.

17. The difference between two numbers is a and the larger number is b. Find the smaller number.

18. The difference between two numbers is a and the smaller number is b. Find the larger number.

19. Write the excess of a over b.

20. What is the excess of $5x + 10$ over $3x - 7$?

21. What is the excess of $3x - 16$ over $40 - 7x$?

22. Express in cents the value of a quarters and b dimes.

23. A man has a dollars, b half dollars, and c quarters. He spends d half dollars and e quarters. How many cents has he left?

24. George is a years old to-day. How old was he b years ago? How old will he be c years hence?

25. Frank is a years old to-day. In how many years will he be b times as old?

26. A boy can run a yards in b seconds. How many yards can he run in one second?

27. A man has oats enough for n horses m days. How many days would the oats last a horses?

28. A cubical box is a inches long. Find the contents of the box.

29. A gallon contains 231 cubic inches. How many gallons will a rectangular tank hold, a inches long, b inches wide, and c inches high?

30. How many bricks a inches long, b inches wide, and c inches high can be placed in a room l inches long, m inches wide, and n inches high?

62. In stating problems x must not be put for money, length, time, weight, etc., but for the required *number* of *specified units* of money, length, time, weight, etc.

Each statement must be made in algebraic symbols, and the meaning of each algebraic statement should be written out in full in common language.

After the algebraic statements are written it is necessary and sufficient, in problems involving only one unknown number, to select two expressions that stand for the same number and to make them the members of the required equation (Ax. 5).

Problems Stated and Solved

1. Three times a certain number is equal to the number increased by 12. Find the number.

Let $x =$ the number.
Then $3x =$ three times the number,
and $x + 12 =$ the number increased by 12.

But the last two numbers are equal.

$$\therefore 3x = x + 12.$$

Transpose, $3x - x = 12.$
Combine, $2x = 12.$
Divide by 2, $x = 6.$

Therefore, the required number is 6.

2. The sum of two numbers is 36 and twice the greater number exceeds three times the smaller by 2. Find the numbers.

Let $x =$ the greater number.

Then, since 36 is the sum and x one of the numbers, the other number must be the sum minus x. Hence,

$$36 - x = \text{the smaller number.}$$

Now twice the greater number is $2x$ and three times the smaller number is $3(36 - x)$; and $2x - 3(36 - x)$ is equal to the excess of twice the greater number over three times the smaller number.

But $2 =$ this excess.

$$\therefore 2x - 3(36 - x) = 2.$$

Remove parenthesis, $2x - 108 + 3x = 2.$
Transpose, $2x + 3x = 2 + 108.$
Combine, $5x = 110.$
Divide by 5, $x = 22.$
$36 - x = 14.$

Therefore, the numbers are 22 and 14.

3. James and John together have \$32 and James has \$12 more than John. How many dollars has each?

Let $x =$ the number of dollars John has.
Then $x + 12 =$ the number of dollars James has,
and $x + (x + 12) =$ the number of dollars they together have.

But $32 =$ the number of dollars they together have.

$$\therefore x + (x + 12) = 32.$$

Remove the parenthesis, $x + x + 12 = 32.$
Transpose, $x + x = 32 - 12.$
Combine, $2x = 20.$
Divide by 2, $x = 10.$
Add 12 to each side, $x + 12 = 22.$

Therefore, John has \$10 and James has \$22.

Note. The beginner must avoid the mistake of writing

Let $x =$ John's money.

We are required to find the *number* of dollars John has, and therefore x must represent this required number.

4. A man is now twice as old as his son; 10 years ago he was three times as old. Find the age of each.

Let $x =$ the number of years in the son's age.
Then $2x =$ the number of years in the father's age,
$x - 10 =$ the number of years in the son's age 10 years ago,
and $2x - 10 =$ the number of years in the father's age 10 years ago.

But 10 years ago three times the son's age was equal to the father's age.

$$\therefore 3(x - 10) = 2x - 10.$$

Remove the parenthesis, $3x - 30 = 2x - 10.$
Transpose, $3x - 2x = 30 - 10.$
Combine, $x = 20.$
$2x = 40.$

Therefore, the son is 20 years old and the father 40 years old.

Note. The beginner should note carefully that we let x equal the *number* of years in the son's age.

5. A has \$9 in half dollars and quarters. If he has 31 coins in all, how many are halves and how many quarters?

Let $x =$ the number of halves.

Then $31 - x =$ the number of quarters,

$50x =$ the number of cents in the halves,

and $25(31 - x) =$ the number of cents in the quarters.

Therefore, $50x + 25(31 - x) =$ the number of cents in his money.

But $900 =$ the number of cents in his money.

$$\therefore 50x + 25(31 - x) = 900.$$

Remove the parenthesis,

$$50x + 775 - 25x = 900.$$

Transpose, $50x - 25x = 900 - 775.$

Combine, $25x = 125.$

Divide by 25, $x = 5.$

$$31 - x = 26.$$

Therefore, there are 5 half dollars and 26 quarters.

Exercise 8

1. The sum of two numbers is 60 and the greater is four times the less. Find the numbers.

2. The difference between two numbers is 12 and their sum is 76. Find the numbers.

3. The difference between two numbers is 11 and their sum is 97. Find the numbers.

4. Three times a certain number is equal to the number increased by 24. Find the number.

5. The sum of two numbers is 39 and the larger exceeds the smaller by 9. Find the numbers.

6. The sum of two numbers is 63 and the larger exceeds twice the smaller by 3. Find the numbers.

7. The sum of two numbers is 49 and twice the smaller exceeds the larger by 14. Find the numbers.

8. Find three consecutive numbers whose sum is 39.

9. Find five consecutive numbers whose sum is 70.

10. If a certain number is multiplied by 12, the product is 168. Find the number.

11. The difference of two numbers is 22 and three times the smaller exceeds twice the larger by 6. Find the numbers.

12. Find a number such that when 9 is added to three times the number the sum is 42.

13. A man sold a quantity of wood for $49, half of it at $3 a cord and the other half at $4 a cord. How many cords of wood did he sell?

14. The sum of the ages of a father and son is 42 years and the father is five times as old as the son. What is the age of each?

15. A tree 120 feet high was broken so that the length of the part broken off was four times the length of the part left standing. Find the length of each part.

16. Two men start from the same place and travel in *opposite* directions, one 35 miles a day and the other 25 miles a day. In how many days will they be 360 miles apart?

17. Two men start from the same place and travel in the *same* direction, one 35 miles a day and the other 25 miles a day. In how many days will they be 360 miles apart?

18. Divide $21 among A, B, and C so that A and B shall each receive three times as much as C.

19. A, B, and C buy a summer cottage for $3000. B pays twice as much as A, and C pays as much as A and B together. How much does each pay?

20. Divide 48 into two parts such that one part shall exceed the other by 6.

21. A father is twice as old as his son; 11 years ago he was three times as old as his son. Find the age of each.

22. A man is four times as old as his son; in 18 years he will be only twice as old. Find the age of each.

23. A dealer bought 25 dozen oranges for $9.25. For a part he paid 40 cents a dozen and for the remainder 35 cents a dozen. How many dozen of each kind did he buy?

24. I have $3.85 in quarters and ten-cent pieces and I have three times as many ten-cent pieces as quarters. How many coins of each kind have I?

25. Divide 75 into two parts such that one part shall exceed twice the other by 9.

26. Two trains start at the same time from two cities 450 miles apart and travel towards each other, one at the rate of 50 miles an hour and the other at the rate of 40 miles an hour. In how many hours will the trains meet?

27. A dealer buys 30 tons of coal and 40 cords of wood for $400. The coal costs twice as much per ton as the wood costs per cord. How much does he pay for a ton of coal and how much for a cord of wood?

28. A man was hired for 50 days. Each day he worked he was to receive $2 and each day he was idle he was to pay 50 cents. At the end of the 50 days he received $80. How many days did he work?

29. Ten men agreed to buy a camp together, but two declined to take their share, and each of the others had to pay $5 more for his share. What was the cost of the camp?

30. Divide 37 into two parts such that the sum of twice the greater part and three times the smaller shall be 87.

31. The sum of two numbers is 35 and three times the larger number is equal to four times the smaller. Find the numbers.

32. The sum of two numbers is 72 and twice the larger number exceeds four times the smaller by 12. Find the numbers.

33. Four years ago a man was seven times as old as his son and 16 years hence he will be only twice as old as his son. Find the age of each.

34. Ten years ago A was three times as old as B and ten years hence he will be 18 years less than twice as old as B. Find the age of each.

35. Four times the excess of a certain number over 9 is equal to 6 less than twice the number. Find the number.

36. A can hoe 4 rows in an hour; B, 3 rows; and C, 2 rows. How many hours will it take the three together to hoe 126 rows?

37. A cistern that will hold 900 gallons has three pipes. The first lets in 7 gallons a minute, the second 15 gallons, and the third 23 gallons. In how many minutes will the three pipes fill the cistern?

38. A tree 90 feet high was broken so that the length of the part broken off was five times the length of the part left standing. Find the length of each part.

39. A man has $1.10 in dimes and nickels and he has 15 coins in all. How many coins of each kind has he?

40. A man has 9 dollars in half dollars and quarters and he has four times as many quarters as half dollars. How many coins of each kind has he?

41. A man has 12 hours for an excursion. How far into the country can he ride a bicycle at the rate of 15 miles an hour so as to return in time, driving a horse at the rate of 5 miles an hour?

42. A, whose horse travels at the rate of 10 miles an hour, starts 2 hours after B, whose horse travels at the rate of 8 miles an hour. How many miles must A drive to overtake B?

43. If a certain number is diminished by 9 and the remainder multiplied by 9, the result is the same as if the number were diminished by 6 and the remainder multiplied by 6. Find the number.

44. If 5 times a certain number is diminished by 3 and the remainder multiplied by 23, the result is the same as if 7 times the number were increased by 9 and the sum multiplied by 7. Find the number.

45. A and B have together \$20; A and C, \$22; B and C, \$26. How much has each?

46. The sum of the ages of a father and his son is 60 years, and the father's age is 3 years more than twice the age of his son. Find the age of each.

47. A farmer bought 16 sheep. If he had bought 4 sheep more for the same money, each sheep would have cost him one dollar less. How much did he pay for a sheep?

48. A man has \$64 in five-dollar bills and one-dollar bills. He has three times as many five-dollar bills as one-dollar bills. How many bills of each kind has he?

49. The current of a river runs 2 miles an hour. A man can paddle a canoe a certain distance up the river in 7 hours and the same distance down in 3 hours. How many miles an hour can he paddle in still water?

50. An express train, which travels 45 miles an hour, starts 2 hours after an accommodation train, which it overtakes in 4 hours. What is the rate of the accommodation train?

51. A sum of money was divided among A, B, and C in such a way that A received four times as much as B, and B twice as much as C. If A received \$35 more than C, how great a sum of money was divided?

52. At an election there were 2 candidates and 1280 votes were cast. The successful candidate had a majority of 40. How many votes were cast for each candidate?

53. Three times the excess of a certain number over 8 is equal to twice the number plus 1. Find the number.

54. A man has $56 in ten-dollar bills and one-dollar bills. He has four times as many one-dollar bills as ten-dollar bills. How many bills of each kind has he?

55. Nine times a certain number exceeds 56 by as much as five times the number is less than 56. Find the number.

56. Divide 39 into two parts such that the greater part exceeds twice the smaller by 1 less than twice the smaller part.

57. Ten years ago A was three times as old as B, and 20 years ago A was five times as old as B. Find the age of each.

58. A flag pole 105 feet high was broken so that the length of the part broken off was six times the length of the part left standing. Find the length of each part.

59. A river flows at the rate of 3 miles an hour. A man rows a certain distance up the river in 12 hours and rows back to the starting point in 3 hours. How many miles an hour can he row in still water?

60. A man finds that it takes his naphtha launch 2 hours to go 24 miles with the tide, and 4 hours to go 8 miles against the tide. Find the rate of the tide in miles an hour.

61. A merchant has two kinds of coffee, one kind costing 35 cents and the other 40 cents a pound. He makes a mixture of 100 pounds. If a pound of the mixture costs him 39 cents, how many pounds of each kind does he take?

62. A had $27 and B had $21. A paid B a certain sum and then B had twice as much as A had left. How many dollars did A pay B?

63. A man has a certain number of dollars, half dollars, and quarters. The number of quarters is twice the number of half dollars, and the number of half dollars is twice the number of dollars. If he has $45 in all, how many coins of each kind has he?

CHAPTER III

POSITIVE AND NEGATIVE NUMBERS

63. Positive and Negative Quantities. If a person is engaged in trade, his capital will be *increased* by his *gains* and *diminished* by his *losses.*

Increase in temperature is measured by the number of degrees the mercury *rises* in a thermometer, and *decrease* in temperature by the number of degrees the mercury *falls.*

In considering any quantity whatever, a quantity that *increases* the quantity under consideration is called a **positive quantity**; and a quantity that *decreases* the quantity under consideration is called a **negative quantity.**

64. The Natural Series of Numbers. If from a given point, marked 0, we draw a straight line to the right, and beginning from the *zero* point lay off units of length on this line, the successive repetitions of the unit will be expressed by the *natural series of numbers,* 1, 2, 3, 4, ⋯ Thus:

In this series, if we wish to *add* 2 to 5, we begin at 5, count 2 units *forwards,* and arrive at 7. If we wish to *subtract* 2 from 5, we begin at 5, count 2 units *backwards,* and arrive at 3. If we wish to subtract 5 from 5, we count 5 units backwards from 5, and arrive at 0. If we wish to subtract 5 from 2, we cannot do it, because when we have counted backwards from 2 as far as 0, *the natural series of numbers comes to an end.*

65. Positive and Negative Numbers. In order to subtract a greater number from a smaller it is necessary to *assume* a new series of numbers, beginning at zero and extending backwards. If the natural series advances from zero to the right, by repetitions of the unit, the new series must recede from zero to the left, by *repetitions of the unit;* and the *opposition* between the right-hand series and the left-hand series must be clearly marked.

This opposition is indicated by calling every number in the right-hand series a **positive number** and prefixing to it, when written, the sign +; and by calling every number in the left-hand series a **negative number** and prefixing to it the sign –. The two series of numbers will be written thus:

$$\cdots -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \cdots$$

and may be considered as forming but a single series consisting of a positive portion or branch, a negative portion or branch, and zero. The complete series thus formed is called the **scalar series of numbers**; and the different numbers of this series are called **scalar numbers.**

If, now, we wish to subtract 4 from 2, we begin at 2 in the positive series, count 4 units in the *negative direction* (to the left), and arrive at -2 in the negative series; that is, $2-4=-2$.

The result obtained by subtracting a greater number from a less, when both numbers are positive, is *always a negative number.*

In general, if a and b represent any two numbers of the positive series, the expression $a-b$ is a positive number when a is greater than b; is equal to zero when a is equal to b; is a negative number when a is less than b.

Numbers counting from left to right in the scalar series *increase* in magnitude; counting from right to left, numbers *decrease* in magnitude. Thus, -3, -1, 0, $+2$, $+4$ are arranged in *ascending* order of magnitude.

66. The Absolute Value of a Number is its value independent of its sign. Numbers regarded without reference to the signs + or − are called **absolute numbers.**

67. Every scalar number consists of a *sign* + or − and the *absolute value* of the number. The sign shows whether the number belongs to the positive or the negative series of numbers; the absolute value shows the place the number occupies in the positive or the negative series.

When no sign stands before a number, the sign + is always understood; but *the sign − is never omitted.*

Thus, 4 means the same as + 4; a means the same as $+ a$.

68. Two scalar numbers that have one the sign + and the other the sign − are said to have **unlike signs.**

Two scalar numbers that have the same absolute values, but unlike signs, cancel each other when combined.

Thus, $+ 4 - 4 = 0$; $+ a - a = 0$.

69. Double Meanings of the Signs + and −. The use of the signs + and − to indicate addition and subtraction must be carefully distinguished from the use of the signs + and − to indicate in which series, the positive or the negative, a given number belongs. In the first sense they are signs of *operation,* and are common to Arithmetic and Algebra; in the second sense they are signs of *opposition,* and are employed in Algebra alone.

NOTE. In Arithmetic, if the things counted are *whole units,* the numbers that count them are called **whole numbers, integral numbers,** or **integers,** the adjective being transferred from the things counted to the numbers that count them. If the things counted are *parts of units,* the numbers that count them are called **fractional numbers,** or **fractions,** the adjective being transferred from the things counted to the numbers that count them.

Likewise in Algebra, if the units counted are *negative,* the numbers that count them are called **negative numbers,** the adjective that defines the nature of the units counted being transferred to the numbers that count them.

70. Addition of Scalar Numbers. A scalar number is often inclosed in a parenthesis, in order that the signs + and −, which are used to distinguish positive and negative numbers, may not be confounded with the + and − signs that denote the operations of addition and subtraction.

Thus, $+4+(-3)$ expresses the sum of the numbers $+4$ and -3, and $+4-(-3)$ expresses the difference of the numbers $+4$ and -3.

71. In order to add two scalar numbers, we begin at the place in the series which the first number occupies, and count, *in the direction indicated by the sign of the second number*, as many units as there are in the absolute value of the second number.

$$\cdots -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad +3 \quad +4 \cdots$$

Thus, the sum of $+2+(+3)$ is found by counting from $+2$ three units in *the positive direction;* that is, *to the right*, and is therefore $+5$.

The sum of $+2+(-3)$ is found by counting from $+2$ three units in *the negative direction;* that is, *to the left*, and is therefore -1.

The sum of $-2+(+3)$ is found by counting from -2 three units in the positive direction, and is therefore $+1$.

The sum of $-2+(-3)$ is found by counting from -2 three units in the negative direction, and is therefore -5.

72. If a and b represent any two scalar numbers, we have

$$+a+(+b)=a+b; \qquad -a+(+b)=-a+b;$$
$$+a+(-b)=a-b; \qquad -a+(-b)=-a-b.$$

Therefore, from these four cases, we have the following

Rule for Adding Two Scalar Numbers:

1. *If the numbers have like signs, find the sum of their absolute values and prefix the common sign to the result.*

2. *If the numbers have unlike signs, find the difference of their absolute values and prefix the sign of the greater number to the result.*

73. Algebraic Sum. The result obtained by adding two or more scalar numbers is called the **algebraic sum** in distinction from the *arithmetical sum;* that is, the sum of the absolute values of the numbers.

NOTE. If there are more than two numbers to be added, add two of the numbers, and then this sum to a third number, and so on; or find the sum of the positive numbers and the sum of the negative numbers, then the difference between the absolute values of these two sums, and prefix the sign of the greater sum to the result.

EXERCISE 9

Perform mentally the indicated addition:

1.	2.	3.	4.	5.	6.
+ 19	− 22	− 34	− 17	+ 17	+ 33
− 12	− 14	+ 19	+ 24	+ 24	− 48

7.	8.	9.	10.	11.	12.
+ 27	− 16	− 25	− 17	+ 26	− 42
− 32	− 24	+ 48	− 23	− 11	+ 86
+ 45	+ 14	− 11	− 10	− 15	− 29

13.	14.	15.	16.	17.	18.
− 19	+ 22	− 36	+ 29	+ 22	− 17
− 29	− 16	+ 32	− 17	+ 27	− 34
+ 39	+ 8	+ 12	− 12	+ 36	+ 50

19.	20.	21.	22.	23.	24.
28	− 46	28	− 14	− 12	80
− 19	− 19	19	22	− 17	− 60
− 17	− 24	− 12	− 38	− 22	− 40
12	16	− 7	15	− 29	24

Addition of Similar Monomials

1. Find the sum of $4a$, $7a$, a, $2a$, $3a$.

The sum of the coefficients is $4 + 7 + 1 + 2 + 3 = 17$.
Hence, the sum of the monomials is $17a$.

2. Find the sum of $-2b$, $-5b$, $-b$, $-4b$, $-6b$.

The sum of the coefficients is $-2 - 5 - 1 - 4 - 6 = -18$.
Hence, the sum of the monomials is $-18b$.

3. Find the sum of $5c$, $-4c$, $-7c$, $-8c$, $3c$, $6c$.

The sum of the positive coefficients is $5 + 3 + 6 = 14$.
The sum of the negative coefficients is $-4 - 7 - 8 = -19$.
The difference between 14 and 19 is 5, and the sign of the greater is negative.
Hence, the sum of the monomials is $-5c$. Therefore,

74. To Find the Sum of Similar Monomials,

Find the algebraic sum of the coefficients and annex to this sum the letters common to the terms.

Exercise 10

Perform mentally the indicated addition:

1.	2.	3.	4.	5.
$7a$	$-4xy$	$11a^2b$	$3ac$	$12x^2$
$5a$	$-5xy$	$-5a^2b$	$-2ac$	$-18x^2$
$-4a$	$6xy$	$8a^2b$	$5ac$	$6x^2$
$-6a$	$2xy$	$-6a^2b$	$-4ac$	$-3x^2$

6.	7.	8.	9.	10.
$-17a^2$	$12xyz$	$-7ab$	$4y^2$	$-2abc$
$-15a^2$	$-17xyz$	$-4ab$	$6y^2$	$-3abc$
$-16a^2$	$-8xyz$	$-2ab$	$8y^2$	$-4abc$
$32a^2$	$10xyz$	$8ab$	$-12y^2$	$-5abc$

Exercise 11

Find the algebraic sum of:

1. $4a, -7a, a, 8a, -12a.$
2. $7c, 5c, 8c, -14c, -2c.$
3. $8ab, -7ab, -4ab, -ab.$
4. $7xy, 4xy, -8xy, -5xy.$
5. $4abc, 6abc, -8abc, 10abc.$
6. $14b, -16b, -2b, 8b.$
7. $8yz, -7yz, -3yz, -2yz.$
8. $-6x^2, 8x^2, -4x^2, 2x^2.$
9. $-4a^2b, -7a^2b, 16a^2b, -8a^2b.$
10. $-b^2m^3, 7b^2m^3, -4b^2m^3, 3b^2m^3.$
11. $5a^2b, -3a^2b, 4a^2b, -7a^2b.$
12. $4a^2c, -10a^2c, 6a^2c, -9a^2c.$
13. $2mn, 5mn, -mn, -4mn.$
14. $3yz, 2yz, -10yz, 3yz.$
15. $20xy, -7xy, -8xy, 5xy.$
16. $-8z^4, 18z^4, -15z^4, 11z^4.$

Express in one term:

17. $-7x^2 - 18x^2 + 7x^2 + 7x^2 + 20x^2 + 3x^2.$
18. $4a^2 + a^2 - 9a^2 + 11a^2 - 6a^2 - 7a^2.$
19. $-3ab^2 - 5ab^2 - 29ab^2 + 4ab^2 + 5ab^2.$
20. $3a^2x + 6a^2x + 4a^2x - 9a^2x - 13a^2x.$
21. $-5a^2b^2 - 10a^2b^2 + 9a^2b^2 + 3a^2b^2 - 5a^2b^2.$
22. $5a^3b - 21a^3b - 11a^3b + 18a^3b + 17a^3b.$
23. $41y^2z^2 - 43y^2z^2 + 61y^2z^2 - 2y^2z^2 - 27y^2z^2.$
24. $-3bc - 4bc - 31bc + 28bc + 7bc - bc.$
25. $b^2d - 8b^2d + 7b^2d - 19b^2d + 7b^2d + 4b^2d.$
26. $14ac - 7ac + 6ac - 9ac + ac + 4ac.$
27. $4xy + 3xy - 10xy - xy + 5xy - 7xy.$
28. $14ac^2 + 17ac^2 - 7ac^2 - 3ac^2 - 4ac^2 - 15ac^2.$
29. $4x^2 - 11x^2 - 7x^2 + 5x^2 - 9x^2 + 3x^2 + 5x^2.$
30. $16abcx - 11abcx - 2abcx + 3abcx + 4abcx.$
31. $13cy + 12cy - 24cy + 2cy - 3cy + 8cy.$
32. $2ax - 10ax - 7ax + 3ax - 4ax + 6ax.$

SUBTRACTION OF SCALAR NUMBERS

75. In order to subtract one scalar number from another, we begin at the place in the series which the minuend occupies, and count, *in the direction opposite to that indicated by the sign of the subtrahend*, as many units as there are units in the absolute value of the subtrahend.

Thus, the result of subtracting $+1$ from $+3$ is found by counting from $+3$ one unit in the *negative direction;* that is, in the direction *opposite to that indicated by the sign + before* 1, and is therefore $+2$.

The result of subtracting -1 from $+3$ is found by counting from $+3$ one unit in the *positive direction*, and is therefore $+4$.

The result of subtracting $+1$ from -3 is found by counting from -3 one unit in the *negative direction*, and is therefore -4.

The result of subtracting -1 from -3 is found by counting from -3 one unit in the *positive direction*, and is therefore -2.

If a and b represent any two scalar numbers, we have

$$+a-(+b)=a-b; \qquad -a-(+b)=-a-b;$$
$$+a-(-b)=a+b; \qquad -a-(-b)=-a+b.$$

76. From these four cases we see

1. *Subtracting a positive number* is equivalent to *adding an equal negative number;*

2. *Subtracting a negative number* is equivalent to *adding an equal positive number.* Therefore,

77. To Subtract One Scalar Number from Another,

Change the sign of the subtrahend and add the result to the minuend.

Exercise 12

Perform mentally the indicated subtraction:

1.	2.	3.	4.	5.
14	-14	-14	-16	-4
-8	8	-8	-4	16
6.	**7.**	**8.**	**9.**	**10.**
-8	16	15	-15	-15
18	20	-20	20	-5
11.	**12.**	**13.**	**14.**	**15.**
-12	-18	18	-18	18
-16	-24	24	-18	-18

Subtraction of Similar Monomials

1. From $13\,ax$ take $6\,ax$.

$$13\,ax - 6\,ax = 7\,ax.$$

2. From $18\,a^2b^2$ take $-4\,a^2b^2$.

$$18\,a^2b^2 - (-4\,a^2b^2) = 18\,a^2b^2 + 4\,a^2b^2 = 22\,a^2b^2.$$

Hence,

78. To Subtract a Monomial from a Similar Monomial,

Change the sign of the coefficient of the subtrahend; then add the coefficients and annex the common letters to the result.

Exercise 13

Perform mentally the indicated subtraction:

1.	2.	3.	4.	5.
$4\,ab$	$-8\,a^2c$	$15\,xyz$	$-15\,x^2y^2$	$14\,abc$
$-6\,ab$	$8\,a^2c$	$-8\,xyz$	$-7\,x^2y^2$	$8\,abc$
6.	**7.**	**8.**	**9.**	**10.**
$-18\,cd$	$14\,a^2b$	$-14\,a^2b$	$8\,ac^2$	$-4\,ab$
$-7\,cd$	$-9\,a^2b$	$9\,a^2b$	$-10\,ac^2$	$-8\,ab$

If $a = 5$, $b = -3$, $c = -2$, find the value of:

11. $a + (-b) + c$.	15. $a - (-b) + (-c)$.
12. $-a - (-b) + c$.	16. $-a + (-b) + (-c)$.
13. $a - b + (-c)$.	17. $-a + (-b) - (-c)$.
14. $-(-a) - (-b) - (-c)$.	18. $-(-a) + b + c$.

Multiplication of Scalar Numbers

79. Multiplication is generally defined in Arithmetic as the process of finding the result when one number (the multiplicand) is taken as many times as there are units in another number (the multiplier). This definition fails when the *multiplier is a fraction.* In multiplying by a fraction, we divide the multiplicand into as many **equal parts** as there are units in the denominator, and take as many of these parts as there are units in the numerator.

If, for example, we multiply 6 by $\frac{2}{3}$, we divide 6 into *three equal parts* and take *two* of these parts, obtaining 4 for the product. The multiplier, $\frac{2}{3}$, is $\frac{2}{3}$ of 1, and the product, 4, is $\frac{2}{3}$ of 6; that is, *the product is obtained from the multiplicand precisely as the multiplier is obtained from 1.*

80. Multiplication may be defined, therefore, as

The process of obtaining the product from the multiplicand as the multiplier is obtained from unity.

81. Every *extension of the meaning of a term* in Algebra must be consistent with the sense previously attached to the term, and with the general laws of numbers.

This extension of the *meaning of multiplication* is consistent with the sense attached to multiplication when the multiplier is a positive whole number.

Thus,	$5 \times 7 = 35$.
The multiplier, 5,	$= 1 + 1 + 1 + 1 + 1$,
and the product, 35,	$= 7 + 7 + 7 + 7 + 7$.

Law of Signs in Multiplication

82. By the definition of multiplication (p. 42, § 80),

since $$+3 = +1 + 1 + 1,$$
then $$3 \times (+8) = +8 + 8 + 8 = +24,$$
and $$3 \times (-8) = -8 + (-8) + (-8) = -24.$$

Again, since $$-3 = -1 - 1 - 1,$$
then $$(-3) \times 8 = -8 - 8 - 8 = -24,$$
and $$(-3) \times (-8) = -(-8) - (-8) - (-8)$$
$$= +8 + 8 + 8 = +24.$$

The *minus sign before the multiplier*, 3, signifies that the repetitions of the multiplicand are to be *subtracted.*

If a and b stand for any two scalar numbers, we have

$$(+a) \times (+b) = +ab, \qquad (-a) \times (+b) = -ab,$$
$$(+a) \times (-b) = -ab, \qquad (-a) \times (-b) = +ab.$$

That is, if two numbers have **like** signs, the product has the **plus** sign; if **unlike** signs, the product has the **minus** sign. Hence,

The Law of Signs in Multiplication:

Like signs give + and unlike signs give —.

The Index Law in Multiplication

83. Since $$a^2 = aa, \text{ and } a^3 = aaa,$$
$$a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3};$$
$$a^4 \times a = aaaa \times a = a^5 = a^{4+1}.$$

If then m and n are *positive integers,*

$$a^m \times a^n = a^{m+n}.$$

In like manner, $a^m \times a^n \times a^p = a^{m+n+p}$. Hence,

The Index Law in Multiplication:

The exponent of a letter in the product is equal to the sum of the exponents of the letter in the factors of the product.

Multiplication of Monomials

1. Find the product of $4\,x^2y^4$ and $5\,xy^2z$.

Since the order of the factors is immaterial, (p. 15, § 45)

$$4\,x^2y^4 \times 5\,xy^2z = 4 \times 5 \times x^2 \times x \times y^4 \times y^2 \times z = 20\,x^3y^6z.$$

2. Find the product of $-2\,xy$ and $5\,x^2y$.

$$-2\,xy \times 5\,x^2y = -2 \times 5 \times x \times x^2 \times y \times y = -10\,x^3y^2.$$

3. Find the product of a^m and a^2, and of a^m and a^m.

$$a^m \times a^2 = a^{m+2}.$$

$$a^m \times a^m = a^{m+m} = a^{2m}.$$

Hence,

84. To Find the Product of Two Monomials,

Find the product of the numerical coefficients, and to this product annex the letters, giving to each letter an exponent equal to the sum of its exponents in the factors.

85. A product of three or more factors is called **the continued product** of the factors.

1. Find the continued product of $(-x) \times (-y) \times (-z)$.

By the law of signs (p. 43, § 82), $(-x) \times (-y) = xy,$

and $(xy) \times (-z) = -xyz.$

$$\therefore (-x) \times (-y) \times (-z) = -xyz.$$

2. Find the continued product of

$$(-w) \times (-x) \times (-y) \times (-z).$$

By the law of signs,

$$(-w) \times (-x) = wx,$$
$$(wx) \times (-y) = -wxy,$$
$$(-wxy) \times (-z) = wxyz.$$
$$\therefore (-w) \times (-x) \times (-y) \times (-z) = wxyz.$$

86. From Examples 1 and 2 (§ 85), we see that an **odd number of negative factors** gives a **negative product**; and an **even number of negative factors** gives a **positive product.**

EXERCISE 14

NOTE. The beginner should first write the *sign* of the product, then the product of the numerical coefficients after the sign, and, lastly, the letters in alphabetical order, giving to each letter the proper exponent.

Find mentally the product of:

1.	2.	3.	4.	5.	6.
$4c$	$2c$	$5ab$	$-2xy$	$4ac$	$8x^2$
$5c$	c	$-2ab$	$-2xy$	$-2ac$	$-2x$
7.	**8.**	**9.**	**10.**	**11.**	**12.**
$-5x^2$	$5ab^2$	$8ac$	$-8x^2y$	$-a^3b^2$	$9ab^2$
$-2x^2$	$-2a^2b$	$2a^2c^2$	$-3xy^2$	$-ab^5$	$-2ab^2$
13.	**14.**	**15.**	**16.**	**17.**	**18.**
$-6ax^2$	$5xy^3$	$9x^6$	$-16a^3b$	$-x^2y^2$	$7a^2b$
$-2a^2x^2$	$-4x^2y^2$	$-2x^4$	$-4ab^3$	$-3xy^5$	$2a^5b^3$
19.	**20.**	**21.**	**22.**	**23.**	**24.**
$11a^2y$	$-x^{m+1}$	a^2b^m	$-x^nb^n$	a^{m+3}	a^{m+n-1}
$-3bx^4$	$-x^{2m-1}$	$-a^nb^2$	$-x^ny^n$	a^{m-4}	$-a^{m-n+1}$

25. $9xy$, $-7x^2y^3$, and $-10xy^3$.

26. $-7a^2b$, $6ab^2$, and $-4a^2b^2$.

27. $4a^3b$, $-6a^2b$, and $5a^3b^3$.

28. $2a^4$, a^2b^2, $-6a^3b$, and $2a^5b^7$.

29. x^2, $-x^2y$, y^2z^2, and x^2y^3.

30. x^{2n}, $-x^{n+m}$, $-x^{n-3}$, and x^{m-n}.

31. a^{2m+1}, a^{2n-1}, a^{m+n}, a^n, and a^{m-3n}.

32. x^{c+d}, x^{d-c}, x^{c-d}, x^4, and x^{2c-d}.

Division of Scalar Numbers

87. **Division** is the operation of finding one of two factors when their product and the other factor are given.

88. With reference to this operation the product is called the **dividend**, the given factor the **divisor**, and the required factor the **quotient**.

Law of Signs in Division

89. Since $(+a)\times(+b)=+ab, \therefore +ab\div(+a)=+b.$
Since $(+a)\times(-b)=-ab, \therefore -ab\div(+a)=-b.$
Since $(-a)\times(+b)=-ab, \therefore -ab\div(-a)=+b.$
Since $(-a)\times(-b)=+ab, \therefore +ab\div(-a)=-b.$

That is, if the dividend and divisor have **like** signs, the quotient has the **plus** sign; and if they have **unlike** signs, the quotient has the **minus** sign. Hence,

The Law of Signs in Division:

Like signs give + and unlike signs give −.

The Index Law in Division

90. The quotient contains the factors of the dividend that are not found in the divisor.

1. Divide x^6 by x^2.

$$\frac{x^6}{x^2}=\frac{xxxxxx}{xx}=xxxx=x^4=x^{6-2}.$$

2. Divide c^3 by c.

$$\frac{c^3}{c}=\frac{ccc}{c}=cc=c^2=c^{3-1}.$$

If m and n are positive integers, and n not greater than m,

$$a^m\div a^n=a^{m-n}.$$

Hence,

The Index Law in Division:

The exponent of a letter in the quotient is equal to the exponent of the letter in the dividend minus the exponent of the letter in the divisor.

Division of Monomials

1. Divide $15\,a^6$ by $5\,a^4$.

$$\frac{15\,a^6}{5\,a^4} = 3\,a^{6-4} = 3\,a^2.$$

We obtain the factor 3 of the quotient by dividing 15 by 5, and the factor a^2 of the quotient by writing a with an exponent equal to the exponent of a in the dividend minus the exponent of a in the divisor.

2. Divide $14\,a^4b^5$ by $-2\,a^2b^2$.

$$\frac{14\,a^4b^5}{-2\,a^2b^2} = -7\,a^{4-2}b^{5-2} = -7\,a^2b^3.$$

3. Divide $-50\,a^4b^5c^6$ by $-30\,ab^4c^3$.

$$\frac{-50\,a^4b^5c^6}{-30\,ab^4c^3} = \frac{5}{3}a^3bc^3.$$

4. Divide $77\,a^{2m}b^nc^x$ by $11\,a^mb^nc^3$.

$$\frac{77\,a^{2m}b^nc^x}{11\,a^mb^nc^3} = 7\,a^{2m-m}b^{n-n}c^{x-3}$$
$$= 7\,a^mb^0c^{x-3}.$$
$$= 7\,a^mc^{x-3}.$$

NOTE. Since by division $\frac{b^n}{b^n} = 1$, and by the index law $\frac{b^n}{b^n} = b^0$, it follows that $b^0 = 1$. Hence, any letter in the quotient *with zero for an exponent* may be omitted without affecting the quotient.

91. To Find the Quotient of Two Monomials,

Divide the numerical coefficient of the dividend by the numerical coefficient of the divisor, and to the result annex the letters, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

Exercise 15

Perform mentally the indicated division:

1. $\frac{54\,a^2y}{6\,y}$

2. $\frac{108\,m^2c}{9\,m^2}$

3. $\frac{17\,x^8y}{17\,x^4}$

4. $\frac{72\,x^5y^3}{12\,x^4y}$

5. $\frac{6\,x^7y^8}{3\,x^2y^5}$

6. $\frac{-21\,a^4x^2y^5}{-7\,a^3x^2y^2}$

7. $\frac{8\,a^5bc^4}{-2\,a^2c^3}$

8. $\frac{121\,a^7x^8}{-11\,a^5x^2}$

9. $\frac{-96\,a^2y^7}{8\,ay^6}$

10. $\frac{-15\,a^7b^4}{-3\,a^3b^3}$

11. $\frac{-12\,m^4n^5}{3\,m^4n}$

12. $\frac{3\,ab^7y^8}{ab^4y^3}$

13. $\frac{-21\,a^2b^5c^5}{-7\,ab^4c^2}$

14. $\frac{3\,m^8y^6z^7}{m^3y^5z^4}$

15. $\frac{-56\,a^2cd^x}{-8\,acd^x}$

16. $\frac{92\,a^5b^6c^7}{46\,a^3b^2c}$

17. $\frac{14\,x^5y^4z^8}{-7\,x^5y^4z^4}$

18. $\frac{12\,x^ny^m}{6\,x^2y^3}$

19. $\frac{-14\,b^3c^8d^9}{-2\,c^5d^4}$

20. $\frac{36\,l^2m^5}{12\,l^2m^2}$

21. $\frac{40\,a^2x^3y^4}{-8\,x^3y^2}$

22. $\frac{-28\,c^3d^4}{-7\,c^2d^2}$

23. $\frac{-36\,x^{n-2}y^{n-1}}{9\,x^2y^2}$

24. $\frac{x^3y^3}{x^n}$

25. $\frac{-y^7z^{12}}{-y^5z^4}$

26. $\frac{9\,a^3b^2c^n}{3\,a^2b^2c^{n-1}}$

27. $\frac{3\,a^{n+3}}{-a^{n-3}}$

28. $\frac{10\,a^{12}x^7}{5\,a^8x^5}$

29. $\frac{-3\,x^{m+1}y^{m+2}}{-x^{m-1}y^{m-2}}$

30. $\frac{4\,a^m}{2\,a^{m-4}}$

31. $\frac{-18\,x^5y^6z}{9\,x^3y^8z}$

32. $\frac{-x^{22}y^{18}}{-x^{14}y^{14}}$

CHAPTER IV

ADDITION AND SUBTRACTION

92. Integral Expressions. An algebraic expression is **integral** if no one of its terms contains a letter in its denominator.

Thus, $a^4 - 5a^2b^2 + 7ab^3 - 4b^4$ and $x^3 - \frac{1}{2}c^2 + \frac{3}{4}d - \frac{5}{8}y^2$ are integral expressions.

The numerical value of an integral expression may be a fraction for some values of the letters.

Thus, if x stands for 1 and y for $\frac{1}{2}$, the integral expression $5x - 9y$ stands for $5 - 4\frac{1}{2}$, or $\frac{1}{2}$.

NOTE. Integral expressions, therefore, are so named on account of the *form of the expressions*, with no reference whatever to the numerical value of the expressions when definite numbers are put in place of the letters.

ADDITION OF INTEGRAL COMPOUND EXPRESSIONS

93. The addition of two integral compound algebraic expressions may be represented by connecting the second expression with the first by the sign +. If there are no similar terms in the two expressions, the operation is *algebraically complete* when the two expressions are thus connected (p. 4, § 14, note).

If, for example, it is required to add $m + n - p$ to $a + b - c$, the result is $a + b - c + (m + n - p)$; or, removing the parenthesis (p. 12, § 42), $a + b - c + m + n - p$.

94. If there are like terms in the expressions, every set of like terms may be replaced by a single term with a coefficient equal to the algebraic sum of the coefficients of the like terms.

1. To $7x^2 - 4xy + 5y^2$ add $2x^2 - 3xy - 4y^2$.

$$\begin{aligned}
&7x^2 - 4xy + 5y^2 + (2x^2 - 3xy - 4y^2)\\
&\quad = 7x^2 - 4xy + 5y^2 + 2x^2 - 3xy - 4y^2 \quad \text{(p. 12, § 42)}\\
&\quad = 7x^2 + 2x^2 - 4xy - 3xy + 5y^2 - 4y^2 \quad \text{(p. 11, § 41)}\\
&\quad = 9x^2 - 7xy + y^2.
\end{aligned}$$

It is convenient to arrange the terms in columns, so that like terms shall stand in the same column.

2. Add $x^4 + 3x^3y - 6x^2y^2 + 2xy^3 - y^4$; $3x^4 - 2x^3y - 4x^2y^2 + 3xy^3 - 2y^4$; $2x^4 - 4x^3y + 9x^2y^2 - 4xy^3 - 3y^4$; and $5x^4 + 5x^3y - 3x^2y^2 - xy^3 - 4y^4$.

$$\begin{array}{rrrrr}
x^4 & + 3x^3y & - 6x^2y^2 & + 2xy^3 & - y^4\\
3x^4 & - 2x^3y & - 4x^2y^2 & + 3xy^3 & - 2y^4\\
2x^4 & - 4x^3y & + 9x^2y^2 & - 4xy^3 & - 3y^4\\
5x^4 & + 5x^3y & - 3x^2y^2 & - xy^3 & - 4y^4\\
\hline
11x^4 & + 2x^3y & - 4x^2y^2 & & - 10y^4
\end{array}$$

In the result the coefficient of x^4 is $1+3+2+5$, or $+11$; the coefficient of x^3y is $3-2-4+5$, or $+2$; the coefficient of x^2y^2 is $-6-4+9-3$, or -4; the coefficient of xy^3 is $2+3-4-1$, or 0, and therefore the term xy^3 does not appear in the sum (p. 9, § 35); the coefficient of y^4 is $-1-2-3-4$, or -10.

95. Check for Addition. To test the accuracy of the work in addition an easy check is obtained by substituting for the letters any convenient numerical values both in the expressions to be added and in the sum. If the two results agree, the work may be considered accurate.

1. Add $3a^2 + 2ab + b^2$; $2a^2 - 4ab + 5b^2$; $-a^2 + 2ab - 4b^2$; and check the result.

Check. Put 2 for a and 1 for b.

Then

$$\begin{array}{rrrcrrrcr}
3a^2 & + 2ab & + b^2 & = & 12 & + 4 & + 1 & = & 17\\
2a^2 & - 4ab & + 5b^2 & = & 8 & - 8 & + 5 & = & 5\\
-a^2 & + 2ab & - 4b^2 & = & -4 & + 4 & - 4 & = & -4\\
\hline
4a^2 & & + 2b^2 & = & 16 & & + 2 & = & 18
\end{array}$$

Beginners should use a similar check in all examples in addition, subtraction, multiplication, and division.

EXERCISE 16

Add, and check the result:

1. $2a + 3b - 5c$; $3a - 2b + 3c$; $a - 2b + 3c$.

2. $5x^3 - 3x^2y + 3xy^2 - y^3$; $-3x^3 + x^2y + 7y^3$; $2x^3 - 5x^2y - 8xy^2 + y^3$; $-3x^3 + 4x^2y + 7xy^2 - y^3$.

3. $7a - b + c - d$; $-5a + 4b - 8c + 4d$; $-2a + 5b + 3c - 7d$; $a - 8b + 4c - 4d$.

4. $7a - 3b + 2c - 3d$; $5a - 4b - 5c + 7d$; $9a + 3b - 4c + 8$; $-7a - 3b - 2c - 17$.

5. $8a^3 - a^2b + 7ab^2 + 3b^3$; $-9a^3 + 4a^2b - 7ab^2 - 5b^3$; $a^3 + a^2b - ab^2 + b^3$; $a^3 - a^2b - ab^2 + 3b^3$.

6. $7a - 3b + 5c - 10d$; $2b - 3c + d - 4e$; $5c - 6a - 4e + 2d$; $-3b - 8c + 7a - e$; $21e - 16c + a - 5d$.

7. $3ab^2 - 4a^2b + a^3$; $-4ac^2 + 5ab^2 - c^3$; $-7b^3 + 2a^2b - 6ac^2$; $5a^3 - 11ab^2 - 12ac^2$.

8. $a^3 + 3a^2b + 3ab^2 + b^3$; $-5ab^2 + 3a^2b - b^3 + 3a^3$; $3ab^2 - 5a^2b + 3b^3 - 3a^3$; $-5b^3 + 2a^2b - 4a^3 + 3ab^2$; $7a^3 + 6b^3 - 5a^2b + 5ab^2$.

9. $4c^2d^2 - 3c^3d + 17c^4 - 8c^5$; $14c^2d^2 + 4c^3d + 5c^4 - 3c^5$; $-c^2d^2 - 2c^3d + 4c^4 + 19c^5$; $2c^2d^2 + 5c^3d - 7c^4 + 9c^5$; $3c^2d^2 + 41c^3d - c^4 + 7c^5$.

10. $a^4 + 6a^3b + 10a^2b^2 + 6ab^3 + b^4$; $-7b^4 - 3a^2b^2 + 7ab^3 + 8a^4 + 4a^3b$; $8a^2b^2 + 3ab^3 - 5a^4 + 3b^4 + 6a^3b$; $-3a^3b + 5a^2b^2 + 3a^4 - 5b^4 - 3ab^3$.

11. $4x^4y^5z^6 - 3x^3y^4z^5 + 17x^2y^3z^4 - 8xy^2z^3$; $14x^2y^3z^4 + 4xy^2z^3 + 5x^3y^4z^5 - 3x^4y^5z^6$; $-x^4y^5z^6 - 2x^3y^4z^5 + 4xy^2z^3 + 19x^2y^3z^4$; $2x^3y^4z^5 + 5xy^2z^3 - 7x^4y^5z^6 + 9x^2y^3z^4$; $-12xy^2z^3 + 4x^4y^5z^6 - 15x^2y^3z^4 - x^3y^4z^5$; $3x^4y^5z^6 + 41x^2y^3z^4 - x^3y^4z^5 + 7xy^2z^3$.

SUBTRACTION OF COMPOUND EXPRESSIONS

96. The subtraction of one compound algebraic expression from another may be represented by connecting the subtrahend with the minuend by the sign $-$.

If, for example, it is required to subtract $a + b - c$ from $m + n - p$, the result is $m + n - p - (a + b - c)$; or, removing the parenthesis (p. 13, § 43), $m + n - p - a - b + c$.

97. If there are like terms in the expressions, each set of like terms may be replaced by a single term.

1. Subtract $2x^2 - 3xy - 4y^2$ from $7x^2 - 4xy + 5y^2$.

$$
\begin{aligned}
&7x^2 - 4xy + 5y^2 - (2x^2 - 3xy - 4y^2) \\
&\quad = 7x^2 - 4xy + 5y^2 - 2x^2 + 3xy + 4y^2 \qquad \text{(p. 13, § 43)} \\
&\quad = 7x^2 - 2x^2 - 4xy + 3xy + 5y^2 + 4y^2 \qquad \text{(p. 11, § 41)} \\
&\quad = 5x^2 - xy + 9y^2.
\end{aligned}
$$

The easiest method of performing the actual process of subtraction is as follows:

Write the subtrahend under the minuend, mentally change the sign of each term of the subtrahend, and add.

2. From $5x^4 + 7x^3y - 9x^2y^2 - 6xy^3 + 2y^4$
take $2x^4 - 4x^3y - 9x^2y^2 - 8xy^3 + y^4$.

$$
\begin{array}{rrrrr}
5x^4 & + \;\; 7x^3y & - 9x^2y^2 & - 6xy^3 & + 2y^4 \\
2x^4 & - \;\; 4x^3y & - 9x^2y^2 & - 8xy^3 & + \;\; y^4 \\
\hline
3x^4 & + 11x^3y & & + 2xy^3 & + \;\; y^4
\end{array}
$$

In the result the coefficient of x^4 is $5 - 2$, or 3; the coefficient of x^3y is $7 + 4$, or 11; the coefficient of x^2y^2 is $-9 + 9$, or 0, and therefore the term containing x^2y^2 will not appear; the coefficient of xy^3 is $-6 + 8$, or 2; the coefficient of y^4 is $2 - 1$, or 1.

98. **Check for Subtraction.** An easy test of the accuracy of the work in subtraction is obtained by substituting for the letters any convenient arbitrary values, as in the check for addition (p. 50, § 95).

1. Subtract $a^3 - 4ab^2 - 3a^2b + 3b^3$ from $3a^3 - a^2b + ab^2 + 2b^3$ and check the result.

Write like terms in the same column.

Check. Put 2 for a and 1 for b.

$$\begin{array}{rcl} 3a^3 - a^2b + ab^2 + 2b^3 &=& 24 - 4 + 2 + 2 = 24 \\ a^3 - 3a^2b - 4ab^2 + 3b^3 &=& 8 - 12 - 8 + 3 = -9 \\ \hline 2a^3 + 2a^2b + 5ab^2 - b^3 &=& 16 + 8 + 10 - 1 = 33 \end{array}$$

The two results agree; hence the work may be considered accurate.

Exercise 17

Check the result in each example:

1. From $4a - 6b + 2c$ take $3a - 3b - 3c$.
2. From $2a - 2b - 4c$ take $3a - 3b - 5c$.
3. From $4x^2 + 2xy + 3y^2$ take $x^2 - xy + 2y^2$.
4. From $a^3 + 3a^2b + 3ab^2 + b^3$ take $a^3 - 3a^2b + 3ab^2 - b^3$.
5. From $x^4 + 3ax^3 - 2bx^2 + 3cx - 4d$ take $3x^4 + ax^3 - 4bx^2 - 3cx + d$.
6. From $72x^4 - 78x^3y - 10x^2y^2 + 17xy^3 + 3y^4$ take $-x^4 + 36x^3y - 17xy^3 - 34y^4 + 10x^2y^2$.
7. From $6x^2 + 7xy - 5y^2 - 12xyz - 8yz - 5z^2$ take $-8xy - 7yz + 3x^2 - 4y^2 + 5xyz$.
8. From $4x^4 + 12x^3 - x^2 - 27x - 18$ take $4x^4 + 4x^3 - 17x^2 - 9x + 18$.
9. From $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$ take $3ax^3 - 7a^2x^2 + 3a^3x - 2a^4$.
10. From $6x^2 - 13xy + 6y^2 - 12xz - 13yz + 6z^2$ take $2x^2 - 5xy + 2y^2 - xz - yz - z^2$.
11. From $2x^4 - 6x^3 - x^2 + 15x - 10$ take $4x^4 + 6x^3 - 4x^2 - 15x - 15$.

INSERTION OF PARENTHESES

99. We have the following equivalent expressions:

$$a+(b+c)=a+b+c, \quad \therefore a+b+c=a+(b+c);$$
$$a+(b-c)=a+b-c, \quad \therefore a+b-c=a+(b-c);$$
$$a-(b+c)=a-b-c, \quad \therefore a-b-c=a-(b+c);$$
$$a-(b-c)=a-b+c, \quad \therefore a-b+c=a-(b-c).$$

Hence, a parenthesis preceded by the sign + may not only be removed *without changing the sign of any term*, but may also be inserted, inclosing any number of terms, *without changing the sign of any term.*

And a parenthesis preceded by the sign − may not only be removed, *provided the sign of every term within the parenthesis is changed*, namely, + to − and − to +, but may also be inserted, inclosing any number of terms, *provided the sign of every term inclosed is changed.*

100. Expressions may occur having parentheses within parentheses. In such cases signs of aggregation of different shapes are used, and the beginner, when he meets with one branch of a parenthesis (, or bracket [, or brace {, must look carefully for the other branch; and all that is included between the two branches must be treated as the + or − sign before the sign of aggregation directs. It is best to remove each parenthesis in succession, generally beginning with the innermost.

1. $$\begin{aligned} & a+[b-(c-d)+e] \\ &= a+[b-c+d+e] \\ &= a+b-c+d+e. \end{aligned}$$

2. $$\begin{aligned} & a-\{b-[c-(d-\overline{e-f})+g]\} \\ &= a-\{b-[c-(d-e+f)+g]\} \\ &= a-\{b-[c-d+e-f+g]\} \\ &= a-\{b-c+d-e+f-g\} \\ &= a-b+c-d+e-f+g. \end{aligned}$$

Exercise 18

Simplify the following expressions by removing the parentheses and combining like terms:

1. $(7a - 2b) - [(3a - c) - (2b - 3c)]$.

2. $(9a - 4c) - \{[5a - (3b - 4c)] - 3b\}$.

3. $2a - (3b + 2c) - \{5b - 3a - (a + b) + 5c - [2a - (c + 2b)]\}$.

4. $a - \{2b + [3x - 3a - (a + b)] + [2a - (b + c)]\}$.

5. $16 - x - \{7x - [x - 9x - (3 - 6x)]\}$.

6. $2a - \{3b + (2b - c) - 4c + [2a - (3b - \overline{c - 2b})]\}$.

7. $x^4 - \{4x^3 - [6x^2 - (4x - 1)]\} - (x^4 + 4x^3 + 6x^2 + 4x + 1)$.

8. $ab - [(3bce - 2ab) - (5bce - bef) + (3ab - 3bef)]$.

9. $1 - [1 - (2 - x)] + [4x - (3 - 6x)] + [4 - (6x - 5)]$.

10. $4x - 3y - \{(2x + 4y) + 3x + [y - 9x - (2y - x) + (x - y)]\}$.

11. $11x - \{7x - [8x - (9x - \overline{12a - 6x})]\}$.

12. $2a - \{7b + [4a - 7b - (2a - \overline{6b + 4a})] - 3a\}$.

13. $a + 2b + 1 - (3a - 2b) - [a - \{2b + 1 + (3a + 2b) - 2b\}]$.

14. $x^3 + (3xy + y^3 - 1) - [x^3 - (y^3 - 1)]$.

15. $a + 1 - [a^2 - (2a - 3) + a^3 - (5a^2 - \overline{8a - 7})]$.

16. $3x - \{2y + [3x - (4y - 5x + \overline{2x - 3y}) + 4y] - 2x\}$.

17. $3a^2 - ab + [4b^2 - \{3ab - 2a^2 + (2b^2 - \overline{3a^2 - 2ab}) - b^2\}$
$+ a^2 + ab]$.

18. $3x^2 - 2xy + 3y^2 - \{2x^2 - [4x^2 - \overline{5xy + 2y^2} + (3x^2 + 2y^2)]\}$
$+ \{3x^2 - [2xy + 3y^2 - (2x^2 + 3xy + y^2)]\}$.

19. $44x + [48y - (6z + \overline{3y - 7x}) + 4z]$
$- [48y - \overline{8x + 2z} - (4x + y)]$.

Exercise 19

Inclose the last three terms in a parenthesis preceded by $-$:

1. $a^{12} - b^{12} + 6a^9b^3 - 6b^9a^3 + 8b^6a^6.$
2. $x^3 - y^3 - 5xy - 2y^2 - 2x^2.$
3. $25z^2 + 10y^2 + 25y^2z^2 - 15x^2 + 10x^2z^2.$
4. $10x^4 + 5x^3 + 18x^2 + 30x - 360.$

Inclose the last three terms in a parenthesis preceded by $+$:

5. $x^4 + 5x^3 - 16x^2 + 20x + 16.$
6. $x^4 + 4x^3y + x^2y^2 + 12xy^3 + 9y^4.$
7. $4x^4 + 4x^3y - 65x^2y^2 - 10xy^3 + 25y^4.$
8. $9a^4 + 18a^3b - 52a^2b^2 - 12ab^3 - 4b^4.$

Collect in parentheses the coefficients of x, y, and z:

9. $a^2x + 2a^2y + a^2z + abx - abz + b^2x + b^2y + b^2z.$
10. $2ax - by + 3cz - dx + 3ey - 5fz + x - y - z.$
11. $ax + bx - cz + 4ay - bz + cy + cx - by - az.$
12. $3x - 4y + 5z + 3ax + 2by + 2cz - 5dx - 3ey - 4fz.$
13. $mx + ny + pz + 3x - 4z + 5y - nx - my.$
14. $7ax - 3ay - 4az + 4bx - 2by + 3bz - 2cx + 2cy - 5cz.$
15. $-5a^2x + 6a^3y + 3abz + 3ax + 5bx - 4b^2y + 6ay + 12bz.$
16. $a^2x - (b^2 - c^2)y + (c^2 - a^2)z + b^2x - c^2z - a^2y - b^2z.$
17. $12a^3x - 4b^3y + 2c^3z - 5a^3z + 7b^3x - 3c^3y + x - y.$
18. $3ax - 2by + 5cz - 3ay - 2bx - 3bz + 2cy - 11cx - az.$
19. $az + 3y - 4bx + 2by - 3ax - z + 6bz + 2ay - 3x.$

EXERCISE 20. REVIEW

1. Add $2 + 3x + 5x^2 + 9x^3 + 17x^4 + 33x^5$; $1 + x - 7x^3 + 33x^4 - 130x^5$; $1 - 4x + 9x^2 - 16x^3 + 25x^4 - 36x^5$; $2 + 5x + 10x^2 + 17x^3 + 26x^4 + 37x^5$.

2. Add $3a^2x^2 + 7ax^3 + 2x^4 - 7a^3x + 2a^4$; $5a^4 - 2a^3x - 3a^2x^2 - 5ax^3 + 2x^4$; $a^4 + 2a^3x - 4a^2x^2 + 2ax^3 + x^4$; $5a^2x^2 - 3x^4 - 4a^4 + 7a^3x + 5ax^3$.

NOTE. *Similar compound expressions* are added in precisely the same way as simple expressions, by finding the sum of their coefficients.
Thus, $2(a - b) + 4(a - b) - 3(a - b) = 3(a - b)$.

3. Add $7(b - c)$; $4(b - c)$; $2a(b - c)$; $-3a(b - c)$; $-3(b - c)$; $-5(b - c)$.

4. Add $(a + b)x + (b + c)y + (a + c)z$; $(b - c)x + (c - a)y + (a - b)z$; $(a + c)x + (a + b)y + (b + c)z$.

5. Add $6m(x^2 + 1) - 8m(x - 1) + 2m(x + 1)$; $-3m(x^2 + 1) - 2m(x - 1) + 3m(x + 1)$; $2m(x^2 + 1) + 9m(x - 1) - 4m(x + 1)$.

6. From $a^4 - b^4$ take $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

7. From $3y^3 - 6xy^2 - 4x^2y + x^3$ take $x^3 + 2x^2y - 5xy^2 + 2y^3$.

8. From $a^2b^2 + 3b^2c^2 + 4c^2a^2$ take $2a^2b^2 - 4bc + 2b^2c^2$.

9. From $(b + c)x^2 + (a + c)xy - (a + b)y^2$ take $(a + c)x^2 - (b - a)xy + (a - b)y^2$.

10. From $(a + b)^2 - (a + b + c) + (b + c)^2$ take $(b + c)^2 - 2(a + b + c) - (a + b)^2$.

11. Simplify $4a - [3a - b - \{2a + b - (a - \overline{b - a})\} - 2b]$.

12. Simplify $x^2 - y^2 - [2x^2 + \{3x^2 - (2x^2 - y^2) - 2y^2\} + 2x^2]$.

Arrange according to the descending powers of x, and inclose the coefficients of like powers of x in parentheses:

13. $ax - x^2 - bx^2 + x^3 - ax^3 + bx - x + bx^3 - 2\,cx + ax^2$.

14. $3\,ax^2 - 3\,bx + 2\,cx^3 - bx^3 + 2\,bx^2 - 2\,ax + 2\,cx^2 - 3\,cx + 3\,ax^3$.

15. From $3\,c + 4\,d - 5\,a - 2\,b$ take $2\,c - 3\,a + b - 2\,d$ and check the result when $a = 3$, $b = 4$, $c = 5$, $d = 6$.

16. If $a = -2$, $b = -3$, $c = 4$, $d = 0$, find the value of $a^3 - 2\,b^3 + c^3 - 3\,abc + 4\,bcd + 2\,ac^2$.

If $a = 5$, $b = 6$, $c = 7$, and $2\,s = a + b + c$, find the value of:

17. $s^2 - (s - a)(s - b) - (s - b)(s - c) - (s - a)(s - c)$.

18. $s^2 - (s - a)(s - b)(s - c) + (s - a)^2(s - b) + (s - b)^2(s - c)$.

19. $s^2 + (s - a)^2(s - b) + (s - b)^2(s - c) + (s - c)^2(s - a)$.

20. What number must be added to $x^2 + 2\,xy + y^2$ that the sum may be $2\,x^2 - y^2$?

21. What number must be added to $a^3 - 3\,a^2b + 3\,ab^2 - b^3$ that the sum may be $3\,a^3 - a^2b + 2\,b^3$?

22. What number must be subtracted from $2\,a^3 - 5\,b^3 + 3\,a^2b$ that the remainder may be $a^3 - 3\,a^2b + 3\,ab^2 - b^3$?

If $A = a^2 + b^2 + c^2$, $B = a^2 + b^2 - c^2$, $C = a^2 - b^2 + c^2$, and $D = b^2 + c^2 - a^2$, find the value of:

23. $A + B + C + D$.

24. $A - B - C + D$.

25. $A - B + C - D$.

26. $A - B - C - D$.

If $A = 3\,a^2 - 2\,ab + 5\,b^2$, $B = 7\,a^2 - 8\,ab + 5\,b^2$, $C = 9\,a^2 - 5\,ab + 3\,b^2$, $D = 11\,a^2 - 3\,ab - 4b^2$, find the value of:

27. $A + B + C + D$.

28. $A - B - C + D$.

29. $B - (A + C - D)$.

30. $A + B - (C + D)$.

31. $A - (B - C - D)$.

32. $B - \{C - [A - (B + D)]\}$.

CHAPTER V

MULTIPLICATION AND DIVISION

101. Degree of a Term. The **degree** of a *term* is the number of *literal* factors the term contains, and each literal factor is called a **dimension** of the term.

Thus, $2ab^2c^3$ is a term of the *sixth* degree. The term $2ab^2c^3$ is said also to be of *one* dimension in a, of *two* dimensions in b, and of *three* dimensions in c.

102. Degree of a Polynomial. The **degree** of a *polynomial* is the degree of that term of the polynomial which is of the *highest degree.*

Thus, $1 + 2ax + 3a^2x^2$ is of the *fourth* degree since $3a^2x^2$, the term of highest degree, is of the fourth degree.

103. A Homogeneous Polynomial. A polynomial is said to be **homogeneous** when all its terms are of the *same* degree.

Thus, $a^3 + 3a^2b - 4ab^2 + b^3 - 6abc$ is homogeneous, since every term is of the third degree.

104. Dominant Letter. If there is one letter in a polynomial of more importance than the rest, that letter is called the **dominant letter**; and the degree of the polynomial is called by the degree of the *dominant letter.*

Thus, $a^2x^2 + bx + c$ is of the *second degree in* x.

105. Arrangement of a Polynomial. A polynomial is said to be *arranged* according to the powers of some letter when the exponents of that letter descend or ascend, from left to right, in *the order of magnitude.*

Thus, $2ax^3 + 3bx^2 - 4cx + 5d$ is arranged according to the descending powers of x; and $5d - 4cx + 3bx^2 + 2ax^3$ is arranged according to the ascending powers of x.

Multiplication of Polynomials by Monomials

106. By p. 15, § 44, we have $a(b + c) = ab + ac$,
and $a(b - c) = ab - ac$. Hence,

To Multiply a Polynomial by a Monomial,

Multiply each term of the polynomial by the monomial, and connect the partial products by their proper signs.

1. Find the product of $x^2 - 2xy + y^2$ and $2xy$.

$$\begin{array}{l} x^2 - 2xy + y^2 \\ \underline{2xy \qquad\qquad\qquad} \\ 2x^3y - 4x^2y^2 + 2xy^3 \end{array}$$

We multiply x^2, the first term of the multiplicand, by $2xy$, and work to the right.

Exercise 21

Multiply:

1. $2x^2 - 3xy + y^2$ by $2x$.
2. $a^2 - 2ab + 3b^2$ by ab.
3. $1 + 2a - 3a^2$ by a^2.
4. $2a^2 - 3ab + b^2$ by $3ab$.
5. $x^7 - x^4 - 2x$ by $2x^2$.
6. $x^8 - 15x - 10$ by $3ax$.
7. $2a^8 - 16a + 6$ by $3a$.
8. $4x^4 - x^2y^2 - 9y^4$ by $-xy$.
9. $6x^2 - 7x + 2$ by $-3x^2$.
10. $-5x^5y - 2x^4y^2 - x^2y^4$ by $-3y^2$.
11. $a^2 - 3a + 10$ by $4a^2b$.
12. $9x^4 + 3x^2y^2 + 4y^4$ by $2xy$.
13. $3x^3 - x^2 + 13x - 30$ by $2x^2$.
14. $x^2 + 2ax + a^2 - b^2$ by $-3ab$.
15. $6ac + 9bc - 2ad - 3bd$ by $2ad$.
16. $3a^3 - 3a^2b + ab^2 - b^3$ by $-5ab$.
17. $x^4 - 6x^3 + 13x^2 - 12x + 4$ by $4x^2$.
18. $2a^5 - 4a^4 + 8a^3 - 12a^2 + 6a$ by $-3a$.
19. $a^6 - a^2b^4 - a^4b + b^5$ by $-2ab$.

Multiplication of Polynomials by Polynomials

107. If we have $m + n + p$ to be multiplied by $a + b + c$, we may substitute M for the multiplicand.

Then $$(a + b + c)M = aM + bM + cM. \qquad \text{(p. 15, § 44)}$$

If now we substitute for M its value $m + n + p$, we have

$$\begin{aligned}(a + b + c)(m + n + p) \\ &= a(m + n + p) + b(m + n + p) + c(m + n + p) \\ &= am + an + ap + bm + bn + bp + cm + cn + cp.\end{aligned}$$

To Multiply a Polynomial by a Polynomial,

Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.

108. In multiplying polynomials, it is a convenient arrangement to write the multiplier under the multiplicand and place like terms of the partial products in columns. With a view to bringing like terms of the partial products in columns, the terms of the multiplicand and multiplier should be arranged in the *same order*.

1. Multiply $3x - 2y$ by $2x - 4y$.

$$\begin{array}{lll} 3x - 2y & & \\ 2x - 4y & & \\ \hline 6x^2 - 4xy & & \\ \quad\;\; - 12xy + 8y^2 & & \\ \hline 6x^2 - 16xy + 8y^2 & & \end{array}$$

We multiply $3x$, the first term of the multiplicand, by $2x$, the first term of the multiplier, and obtain $6x^2$; then we multiply $-2y$, the second term of the multiplicand, by $2x$, the first term of the multiplier, and obtain $-4xy$. The first line of partial products is $6x^2 - 4xy$. In multiplying the multiplicand by $-4y$, the second term of the multiplier, we obtain for the second line of partial products $-12xy + 8y^2$, and this we put one place to the right, so that the like terms $-4xy$ and $-12xy$ may stand in the same column. We then add the two lines of partial products, and obtain the complete product in its simplest form, $6x^2 - 16xy + 8y^2$.

2. Multiply $3a^2 - 2 + 2a^3 + 4a$ by $2a^2 - 4a - 3$.

Arrange both multiplicand and multiplier according to the descending powers of a.

$$
\begin{array}{rrrrrr}
2a^3 & +3a^2 & +4a & -2 & & \\
2a^2 & -4a & -3 & & & \\
\hline
4a^5 & +6a^4 & +8a^3 & -4a^2 & & \\
 & -8a^4 & -12a^3 & -16a^2 & +8a & \\
 & & -6a^3 & -9a^2 & -12a & +6 \\
\hline
4a^5 & -2a^4 & -10a^3 & -29a^2 & -4a & +6
\end{array}
$$

3. Multiply $3x^2 - 4x^3 - 2x + 1$ by $3 - 4x + 8x^3$.

Arrange both multiplicand and multiplier according to the ascending powers of x.

$$
\begin{array}{rrrrrrr}
1 & -2x & +3x^2 & -4x^3 & & & \\
3 & -4x & +8x^3 & & & & \\
\hline
3 & -6x & +9x^2 & -12x^3 & & & \\
 & -4x & +8x^2 & -12x^3 & +16x^4 & & \\
 & & & +8x^3 & -16x^4 & +24x^5 & -32x^6 \\
\hline
3 & -10x & +17x^2 & -16x^3 & & +24x^5 & -32x^6
\end{array}
$$

Exercise 22

Multiply:

1. $3a - 4b$ by $a - b$.
2. $4b - 5c$ by $3b + 4c$.
3. $4x - 9y$ by $x - 5y$.
4. $3a - 2b$ by $2a - b$.
5. $a + b$ by $x + y$.
6. $2a - 3b$ by $5a + 7b$.
7. $a - 5b$ by $a + 3b$.
8. $3x - 5$ by $3x + 5$.
9. $7x - 3$ by $5x - 4$.
10. $7a - 5b$ by $6a + 5b$.
11. $a - b$ by $x + y$.
12. $3a + 5b$ by $7b - 4a$.
13. $5x + 1$ by $7y - 2$.
14. $2a + 3b$ by $2a - 3b$.
15. $8x - 7y$ by $7x + 6y$.
16. $x + y$ by $x + y$.
17. $x - y$ by $x - y$.
18. $x - 1$ by $x - 2$.
19. $x^2 - 2ax + a^2$ by $x - a$.
20. $a^2 + 2ab + b^2$ by $a + b$.

21. $3x^2 - 4x + 7$ by $5x^2 - x - 4$.

22. $5a^3 - 2a^2x + ax^2$ by $2a^2 - ax + x^2$.

23. $5a^2 - 3ab - 2b^2$ by $a^2 + 2ab$.

24. $x^2 + 7x - 5$ by $x^2 - 3x + 7$.

25. $3a^3b - 2a^2b^2 + ab^3$ by $2a^2 - ab - 5b^2$.

26. $x^2 - 2x + 1$ by $x^3 - 3x + 2$.

27. $x^2 - 5ax - 2a^2$ by $x^2 + 2ax + 3a^2$.

28. $7x^2 + y^2 - 3xy$ by $2x^3 + y - x$.

29. $1 + 2x + x^2$ by $1 - 2y + y^2$.

30. $x^2 + bx - c$ by $x^2 - ax + b$.

31. $x^2 - ax - b^2$ by $px + x^2 - q^2$.

32. $3x^3 - 2x^2 + x - 1$ by $5x^2 - 4x - 1$.

33. $3a + 4b - 5c$ by $4a - 3b + c$.

34. $a^2 - 2ab + b^2$ by $5a^2 - 5ab + 4b^2$.

35. $2x^3 - 4x^2y - 5xy^2$ by $4x^3 + 8x^2y + 5xy^2$.

36. $a^2 - 3a + 1$ by $2a^2 + 4a - 3$.

37. $a^m + 2a^{m+1} - 3a^{m+2} - 1$ by $a + 1$.

38. $a^n - 4a^{n-1} - 5a^{n-2} + a^{n-3}$ by $a - 1$.

39. $a^n - a^{n-1}b + a^{n-2}b^2$ by $a - b$.

40. $1 - ax + bx^2 - cx^3$ by $1 - x^2 + x$.

41. $4b^3 - 2b + 1 - 3b^2$ by $4b^2 + 5b + 2b^3$.

42. $a^3 + 3a - 2a^2 + 1$ by $4a^2 + 3 - 2a$.

43. $bc + ac + ab$ by $a - b + c$.

44. $2b^2d^2 + 3b^4 - 7bd^3 - 5b^3d + d^4$ by $3b^2 + 3d^2 - 5bd$.

45. $x^{m+1} - 3x^{m+2} + x^{m+3} - 2x^{m+4}$ by $2x^{m-1} + 3x^{m-2} - 4x^{m-3}$.

46. $a^n + 3a^{n-2} - 2a^{n-1}$ by $2a^{n+1} + a^{n+2} - 3a^n$.

47. $x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + x^{n-5}y^4$ by $x + y$.

48. $a^p - 3a^{p-1} + 4a^{p-2} - 6a^{p-3} + 5a^{p-4}$ by $2a^3 - a^2 + a$.

109. Detached Coefficients. In multiplying two polynomials that involve but one letter, or are homogeneous (p. 59, § 103) and involve but two letters, we shall considerably shorten the work if we write only the coefficients.

1. Multiply $3x^2 - 4x^3 - 2x + 1$ by $3 - 4x + 8x^3$.

Since the x^2 term of the multiplier is missing, we supply a zero coefficient, and arrange both expressions according to the ascending powers of x.

$$\begin{array}{l}
1 - 2 + 3 - 4 \\
\underline{3 - 4 + 0 + 8} \\
3 - 6 + 9 - 12 \\
\quad - 4 + 8 - 12 + 16 \\
\qquad\quad + 8 - 16 + 24 - 32 \\
\hline
3 - 10 + 17 - 16 + 0 + 24 - 32
\end{array}$$

Write in the powers of x, and the product is

$$3 - 10x + 17x^2 - 16x^3 + 24x^5 - 32x^6.$$

2. Multiply $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 + b^2 - 2ab$.

Arrange both expressions according to the descending powers of a.

$$\begin{array}{l}
1 - 3 + 3 - 1 \\
\underline{1 - 2 + 1} \\
1 - 3 + 3 - 1 \\
\quad - 2 + 6 - 6 + 2 \\
\qquad\quad + 1 - 3 + 3 - 1 \\
\hline
1 - 5 + 10 - 10 + 5 - 1
\end{array}$$

Hence, the product is $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

Exercise 23

Multiply by the method of detached coefficients:

1. $2x^3 + 4x^2y - 3xy^2 - y^3$ by $5x^2 - 2xy - y^2$.
2. $a^5 - 2a^4b + 3a^3b^2 - 2a^2b^3 + ab^4 - b^5$ by $2a^2 - ab + b^2$.
3. $5c^8 - 3c^6 + 9c^4 - 10c^2 + 4$ by $9c^6 + 5c^4 - 13c^2$.
4. $x^4 - 3x^3 + 2x^2 - x + 5$ by $3x^3 - 2x^2 - 5x + 3$.

5. $2 + 7a - 2a^2 + 8a^3 - 4a^4$ by $2 + 5a + 3a^2 - 7a^3$.

6. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^2 + 2xy + y^2$.

7. $3a^4b^2 - 6a^3b^3 + 4a^2b^4 + ab^5$ by $2a^2b - 5ab^2 + 2b^3$.

8. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$ by $x^3 - 3x^2y + 3xy^2 - y^3$

9. $3x^4 + 6x^3y - 9xy^3 + 12y^4$ by $2x^2 - 3xy + y^2$.

10. $3a^3x - 2a^2x^2 + 5ax^3 - 8x^4$ by $-2a^2 + 6ax - 4x^2$.

11. $2by^5 - 3b^2y^4 + 4b^3y^3 - 5b^4y^2$ by $y^3 - 4by^2 + 6b^2y$.

12. $c^3 - 8c^2z + 5cz^2 + 11z^3$ by $6c^2 - 2cz + 12z^2$.

13. $a^4 - 7a^3b + 6a^2b^2 + 8ab^3 - 2b^4$ by $a^2 - 3ab + 2b^2$.

14. $2x - x^2 + 4x^3 - 10 + 7x^5$ by $2x^4 - 3x^2 + 2x^3 + 2 + 3x$.

15. $2x^4 - 3x^3y + 2y^4 - 2xy^3 + 4x^2y^2$ by $2xy^2 - y^3 + x^3 - 3x^2y$.

16. $3y^m - 2y^{m+1} - 5y^{m+2} + y^{m+3}$ by $3y^{m-3} + 2y^{m-2} - 5y^{m-1} - y^m$.

17. $a^n - a^{n+1} + a^{n+2} - a^{n+3}$ by $a^3 - a^2 - a + 1$.

18. $a^6 - 3a^5 + 6a^4 - 7a^3 + 6a^2 - 3a + 1$ by $a^2 - a + 1$.

19. $2m^4 - 6m^3 + 3m^2 - 3m + 1$ by $m^2 - 3m + 1$.

20. $6a^3x - 17a^2x^2 + 14ax^3 - 3x^4$ by $2a - 3x$.

21. $x^6 - 5x^4 + 3x^3 + 6x^2 - 7x + 2$ by $x^3 - 3x^2 + 3x - 1$.

22. $8ab^6 - 8a^6b + 24a^2b^5 + a^5b^2 + 14a^3b^4 + 12a^4b^3 + 5a^7$
by $7a^2b + 2ab^2 + 5a^3$.

23. $x^5 + 4x^3 - 92x - 3x^4 + 26x^2 + 5$ by $x^2 - 3x + 4$.

24. $5x^{p-3}y^{r+3} - 2x^{p-1}y^{r+1} - x^{p-2}y^{r+2}$
by $3x^{p+4}y^{r-1} + 4x^{p+5}y^{r-2} - x^{p+3}y^r$.

25. $3x^{4n+1} - 4x^{3n} + 2x^{2n-1} - x^{n-2}$
by $2x^{4n+1} - 5x^{3n} - 2x^{2n-1} + x^{n-2}$.

26. $a^{p+4}b^{1-q} + a^{p+3}b^{2-q} - 2a^{p+2}b^{3-q} - 4a^{p+1}b^{4-q} + a^pb^{5-q}$
by $2a^{5-p}b^q - 2a^{4-p}b^{q+1} + 6a^{3-p}b^{q+2} - 2a^{2-p}b^{q+3}$.

Division of a Polynomial by a Monomial

110. By the *distributive law for division* (p. 16, § 46),

1. $$\frac{4a^3b^2x^2 - 6a^2bx^2 + 4ax^2}{2ax^2} = \frac{4a^3b^2x^2}{2ax^2} - \frac{6a^2bx^2}{2ax^2} + \frac{4ax^2}{2ax^2}$$
$$= 2a^2b^2 - 3ab + 2.$$

2. $$\frac{9a^{4m+1} - 6a^{3m+2}}{3a^{2m+1}} = \frac{9a^{4m+1}}{3a^{2m+1}} - \frac{6a^{3m+2}}{3a^{2m+1}} = 3a^{2m} - 2a^{m+1}.$$

Here we have $(4m + 1) - (2m + 1) = 4m + 1 - 2m - 1 = 2m$, and $(3m + 2) - (2m + 1) = 3m + 2 - 2m - 1 = m + 1$, as the exponents of a in the first and second terms of the quotient respectively.

To Divide a Polynomial by a Monomial,

Divide each term of the dividend by the divisor, and connect the partial quotients by their proper signs.

Exercise 24

Divide:

1. $3x^2 - 4x^3$ by x.
2. $4a^2 + 2ab$ by $2a$.
3. $5x^9 - 10x^6$ by $5x^3$.
4. $4x^6 - 6x^5y + 8x^4y^2$ by $2x^3$.
5. $9x^4y + 6x^3y^2 - 6x^2y^3 - 3xy^4$ by $3xy$.
6. $5a^3b^3 - 35a^2b^2c^2 + 20abc^4$ by $-5ab$.
7. $8x^4y^2 - 12x^2 - 16x$ by $4x$.
8. $-3x^4y + 5x^3y^2 - 6x^2y^3 - xy^4$ by $-xy$.
9. $5x^3y - 25x^2y^2 + 10xy^3$ by $-5xy$.
10. $17a^3b^2 - 51a^4b^3 + 34a^5b^4 - 68a^6b^5$ by $17a^2b^2$.
11. $2x^6 - 4x^5 + 6x^4 - 4x^3 + 2x^2$ by $2x^2$.
12. $2a^5b^4c^3 - 2a^4b^3c^4 + 4a^3b^2c^5 - 6a^2bc^6$ by $2a^2bc^2$.
13. $3y^m - 2y^{m+1} - 5y^{m+2} + y^{m+3}$ by y^4.
14. $4x^n + 3x^{n+1} - 2x^{n+2} + x^{n+3}$ by x^{n-1}.
15. $2a^p - 4a^{p+1} - 6a^{p+2} + 8a^{p+3}$ by $2a^{p-2}$.
16. $c^{r+3} + c^{r+4} - 2c^{r+5} - 3c^{r+6}$ by c^{r+2}.

Division of One Polynomial by Another

111. If the divisor (one factor) $= a + b + c,$
and the quotient (other factor) $= n + p + q,$

then the dividend (product) $= \begin{cases} + an + bn + cn \\ + ap + bp + cp \\ + aq + bq + cq. \end{cases}$

The first term of the dividend is an; that is, the product of a, the first term of the divisor, by n, the first term of the quotient. The first term n of the quotient is therefore found by dividing an, the first term of the dividend, by a, the first term of the divisor.

If the partial product formed by multiplying the entire divisor by n is subtracted from the dividend, the first term of the remainder ap is the product of a, the first term of the divisor, by p, the second term of the quotient; that is, the second term of the quotient is obtained by dividing the first term of the remainder by the first term of the divisor. In like manner, the third term of the quotient is obtained by dividing the first term of the new remainder by the first term of the divisor; and so on.

To Divide a Polynomial by a Polynomial,

Arrange both the dividend and divisor in ascending or descending powers of some common letter.

Divide the first term of the dividend by the first term of the divisor.

Write the result as the first term of the quotient.

Multiply all the terms of the divisor by the first term of the quotient.

Subtract the product from the dividend.

If there is a remainder, consider it as a new dividend and proceed as before.

112. It is of the greatest importance to arrange the dividend and the divisor *in the same order* with respect to a common letter, and *to keep this order throughout the operation.*

The beginner should study carefully the processes in the following examples:

1. Divide $x^2 + 5xy + 6y^2$ by $x + 2y$.

$$\begin{array}{l|l} x^2 + 5xy + 6y^2 & x + 2y \\ \underline{x^2 + 2xy} & x + 3y \\ \quad 3xy + 6y^2 & \\ \quad \underline{3xy + 6y^2} & \end{array}$$

The beginner will notice that by this process we have in effect separated the dividend into two parts, $x^2 + 2xy$ and $3xy + 6y^2$, and have divided each part by $x + 2y$, and that the complete quotient is the sum of the partial quotients x and $3y$. Thus,

$$x^2 + 5xy + 6y^2 = x^2 + 2xy + 3xy + 6y^2 = (x^2 + 2xy) + (3xy + 6y^2).$$

$$\therefore \frac{x^2 + 5xy + 6y^2}{x + 2y} = \frac{x^2 + 2xy}{x + 2y} + \frac{3xy + 6y^2}{x + 2y} = x + 3y.$$

2. Divide $x^2 - 2xy + y^2$ by $x - y$.

$$\begin{array}{l|l} x^2 - 2xy + y^2 & x - y \\ \underline{x^2 - \quad xy} & x - y \\ \quad - xy + y^2 & \\ \quad \underline{- xy + y^2} & \end{array}$$

3. Divide $a^5 + 28a^3b^2 + 64ab^4 - 8a^4b - 56a^2b^3 - 32b^5$ by $a^2 - 2ab + 4b^2$.

Arrange according to the descending powers of a.

$$\begin{array}{l|l} a^5 - 8a^4b + 28a^3b^2 - 56a^2b^3 + 64ab^4 - 32b^5 & a^2 - 2ab + 4b^2 \\ \underline{a^5 - 2a^4b + 4a^3b^2} & a^3 - 6a^2b + 12ab^2 - 8b^3 \\ \quad - 6a^4b + 24a^3b^2 - 56a^2b^3 & \\ \quad \underline{- 6a^4b + 12a^3b^2 - 24a^2b^3} & \\ \qquad\qquad 12a^3b^2 - 32a^2b^3 + 64ab^4 & \\ \qquad\qquad \underline{12a^3b^2 - 24a^2b^3 + 48ab^4} & \\ \qquad\qquad\qquad - 8a^2b^3 + 16ab^4 - 32b^5 & \\ \qquad\qquad\qquad \underline{- 8a^2b^3 + 16ab^4 - 32b^5} & \end{array}$$

4. Divide $2x^4 - 6x^3y + 3x^2y^2 - 3xy^3 + y^4$ by $x^2 - 3xy + y^2$.

$$\begin{array}{l|l}
2x^4 - 6x^3y + 3x^2y^2 - 3xy^3 + y^4 & x^2 - 3xy + y^2 \\
\underline{2x^4 - 6x^3y + 2x^2y^2} & 2x^2 + y^2 \\
\qquad\qquad x^2y^2 - 3xy^3 + y^4 & \\
\qquad\qquad \underline{x^2y^2 - 3xy^3 + y^4} &
\end{array}$$

Check. Put 1 for x and 1 for y. Then the dividend is equal to $2 - 6 + 3 - 3 + 1 = -3$; the divisor, $1 - 3 + 1 = -1$; and the quotient, $2 + 1 = 3$. Since $(-3) \div (-1) = 3$, the work appears to be correct.

5. Divide $1 - a + 2a^2 - a^3 + 2a^4 - a^5 + a^6$ by $1 - 2a + 3a^2 - 2a^3 + a^4$.

$$\begin{array}{l|l}
1 - a + 2a^2 - a^3 + 2a^4 - a^5 + a^6 & 1 - 2a + 3a^2 - 2a^3 + a^4 \\
\underline{1 - 2a + 3a^2 - 2a^3 + a^4} & 1 + a + a^2 \\
\quad a - a^2 + a^3 + a^4 - a^5 & \\
\quad \underline{a - 2a^2 + 3a^3 - 2a^4 + a^5} & \\
\qquad a^2 - 2a^3 + 3a^4 - 2a^5 + a^6 & \\
\qquad \underline{a^2 - 2a^3 + 3a^4 - 2a^5 + a^6} &
\end{array}$$

6. Divide $20x^{4n} - x^{4n-3} - 15x^{4n-1} + 2x^{4n+2} + 7x^{4n-2} - 11x^{4n+1}$ by $2x^{3n+1} + x^{3n-1} - 5x^{3n}$.

Arrange according to the descending powers of x.

$$\begin{array}{l|l}
 & 2x^{3n+1} - 5x^{3n} + x^{3n-1} \\
 & x^{n+1} - 3x^n + 2x^{n-1} - x^{n-2} \\
2x^{4n+2} - 11x^{4n+1} + 20x^{4n} - 15x^{4n-1} + 7x^{4n-2} - x^{4n-3} & \\
\underline{2x^{4n+2} - 5x^{4n+1} + x^{4n}} & \\
\quad -6x^{4n+1} + 19x^{4n} - 15x^{4n-1} & \\
\quad \underline{-6x^{4n+1} + 15x^{4n} - 3x^{4n-1}} & \\
\qquad 4x^{4n} - 12x^{4n-1} + 7x^{4n-2} & \\
\qquad \underline{4x^{4n} - 10x^{4n-1} + 2x^{4n-2}} & \\
\qquad\quad -2x^{4n-1} + 5x^{4n-2} - x^{4n-3} & \\
\qquad\quad \underline{-2x^{4n-1} + 5x^{4n-2} - x^{4n-3}} &
\end{array}$$

We find the index of x in the first term of the quotient by subtracting the index of x in the first term of the divisor from the index of x in the first term of the dividend. In the same way the other indices of x in the quotient are found.

Exercise 25

Divide:

1. $35x^3 + 47x^2 + 13x + 1$ by $5x + 1$.
2. $2a^4 - 6a^3 + 3a^2 - 3a + 1$ by $a^2 - 3a + 1$.
3. $42a^4 + 41a^3 - 9a^2 - 9a - 1$ by $7a^2 + 8a + 1$.
4. $2x^4 - 4x^3 - 3x^2 - x + 1$ by $x^2 - 3x + 1$.
5. $6a^3x - 17a^2x^2 + 14ax^3 - 3x^4$ by $2a - 3x$.
6. $4x^2 - 28xy + 49y^2$ by $2x - 7y$.
7. $4a^3 + 4a^2 - 29a + 21$ by $2a - 3$.
8. $6a^3 - 13a^2b + 4ab^2 + 3b^3$ by $2a - 3b$.
9. $6x^4 - 13ax^3 - 13a^3x + 13a^2x^2 - 5a^4$ by $2x^2 - 3ax - a^2$.
10. $45x^4 + 35x^2 - 4 + 18x^3 + 4x$ by $9x^3 + 7x - 2$.
11. $6x^4 + 13xy^3 - x^3y + 4y^4 + 2x^2y^2$ by $2x^2 - 3xy + 4y^2$.
12. $2x^4 + x^3y - 3xy^3 - 13x^2y^2 + y^4$ by $x^2 - 2xy - y^2$.
13. $15x^5 - 3xy^4 + 4x^3y^2 + 10x^4y + 6x^2y^3$ by $5x^3 + 3xy^2$.
14. $21a^4 - 16a^3b - 5ab^3 + 16a^2b^2 + 2b^4$ by $3a^2 - ab + b^2$.
15. $-73x^2 - 25 + 56x^4 + 95x - 59x^3$
    by $-11x + 7x^3 - 3x^2 + 5$.
16. $7a^4b^3 - 19a^2b^5 + a^7 - 9a^5b^2 + 13a^3b^4 - b^7 + 8ab^6$
    by $b^3 - 2ab^2 + a^3$.
17. $4x^5y - 4x^3y^3 + 4x^2y^4 - xy^5$ by $2x^2 - 2xy + y^2$.
18. $4x^4 - x^2y^2 + 6xy^3 - 9y^4$ by $2x^2 - xy + 3y^2$.
19. $x^6 - 5x^4 + 3x^3 + 6x^2 - 7x + 2$ by $x^3 - 3x^2 + 3x - 1$.
20. $a^4 - b^4$ by $a^3 + a^2b + ab^2 + b^3$.
21. $3x^5 + 4x^3y^2 + 4x^2y^3 + xy^4 + 4y^5$ by $x^3 + x^2y + xy^2 + y^3$.
22. $a^5 - 5a^3b^2 - 5a^2b^3 + b^5$ by $a^2 - 3ab + b^2$.

23. $a^4 + 4\,ab^3 - 4\,abc + 3\,b^4 + 2\,b^2c - c^2$ by $a^2 - 2\,ab + 3\,b^2 - c$.

24. $4\,x^4 - 5\,x^2y^2 - 8\,x^2 + y^4 - 4\,y^2 + 4$ by $2\,x^2 - 3\,xy + y^2 - 2$.

25. $a - 4\,a^3 + 12\,a^4 - 9\,a^5$ by $1 + 2\,a - 3\,a^2$.

26. $a^{m+2} - a^{m+3} + 37\,a^{m+5} - 55\,a^{m+6} + 50\,a^{m+7}$
by $a^2 - 3\,a^3 + 10\,a^4$.

27. $6\,d^{x+3} - 5\,d^{x+2} - 6\,d^{x+1} + 19\,d^x - 21\,d^{x-1} + 4\,d^{x-2}$
by $2\,d^3 + d^2 - 4\,d$.

28. $4\,a^{x+1} - 30\,a^x + 19\,a^{x-1} + 5\,a^{x-2} + 9\,a^{x-4}$
by $a^{x-3} - 7\,a^{x-4} + 2\,a^{x-5} - 3\,a^{x-6}$.

29. $6\,x^{m-n+2} + x^{m-n+1} - 22\,x^{m-n} + 19\,x^{m-n-1} - 4\,x^{m-n-2}$
by $3\,x^{3-n} - 4\,x^{2-n} + x^{1-n}$.

30. $6\,b^{x+y+2} + b^{x+y+1} - 9\,b^{x+y} + 11\,b^{x+y-1} - 6\,b^{x+y-2} + b^{x+y-3}$
by $2\,b^{y+2} + 3\,b^{y+1} - b^y$.

31. $a^{m+n}b^n - 4\,a^{m+n-1}b^{2n} - 27\,a^{m+n-2}b^{3n} + 42\,a^{m+n-3}b^{4n}$
by $a^nb^n - 7\,a^{n-1}b^{2n}$.

32. $x^{3n} - 3\,x^{2n}y^n + 3\,x^ny^{2n} - y^{3n}$ by $x^n - y^n$.

33. $x^2y^2z + 3\,x^2y^3 - 3\,xyz^3 + yz^4 - x^2z^3 + y^5 - 2\,y^3z^2 + 3\,xy^3z$
$- 3\,x^2yz^2$ by $y^2 - z^2$.

113. Detached Coefficients. In division, as in multiplication, it is convenient to use only the coefficients when the dividend and the divisor are expressions involving but one letter, or homogeneous expressions involving but two letters.

Thus, the work of Example 3, § 112, p. 68, may be arranged as follows:

$$\begin{array}{l|l}
1 - 8 + 28 - 56 + 64 - 32 & 1 - 2 + 4 \\
\underline{1 - 2 + 4} & \overline{1 - 6 + 12 - 8} \\
\quad - 6 + 24 - 56 & \\
\quad \underline{- 6 + 12 - 24} & \\
\qquad\quad 12 - 32 + 64 & \\
\qquad\quad \underline{12 - 24 + 48} & \\
\qquad\qquad - 8 + 16 - 32 & \\
\qquad\qquad \underline{- 8 + 16 - 32} &
\end{array}$$

Hence, the quotient is $a^3 - 6\,a^2b + 12\,ab^2 - 8\,b^3$.

Exercise 26

Divide by the method of detached coefficients:

1. $2a^4 - 3a^3b + 6a^2b^2 - ab^3 + 6b^4$ by $a^2 - 2ab + 3b^2$.

2. $2x^4 + 3x^3y + x^2y^2 - 4xy^3 - 2y^4$ by $x^2 + 2xy + 2y^2$.

3. $a^5 - 2a^4b - a^3b^2 + 5a^2b^3 - 4ab^4 + b^5$ by $a^3 - 2ab^2 + b^3$.

4. $c^5 + 2c^4d - 5c^3d^2 + 15c^2d^3 - 6cd^4 + 9d^5$ by $c^2 - 2cd + 3d^2$.

5. $9x^5 + 5x^3y^2 - 4x^2y^3 - 8xy^4 - 2y^5$
by $3x^3 + 2x^2y + 4xy^2 + 2y^3$.

6. $2a^6 - 5a^5 + 6a^4 - 6a^3 + 6a^2 - 4a + 1$
by $a^4 - a^3 + a^2 - a + 1$.

7. $x^7 - 2x^6 + x^5 + 2x^3 - 3x^2 + 1$ by $x^5 + 2x + 1$.

8. $1 + 2b - 2b^2 - 3b^3 - b^4 - 4b^5 - b^7$ by $1 - 2b^2 - b^4$.

9. $x^4y^4 - 4x^2y^2z^2 + 12xyz^3 - 9z^4$ by $x^2y^2 - 2xyz + 3z^2$.

Hint. In this example xy is used as a single letter.

10. $6a^5 - 17a^4b + 14a^2b^3 + 3ab^4 + 4b^5$ by $3a^3 - 4a^2b - b^3$.

11. $26x^3y^3 + x^6 + 6y^6 - 17xy^5 - 5x^5y - 2x^4y^2 - x^2y^4$
by $x^2 - 3y^2 - 2xy$.

12. $48x^2y^3 + 26xy^4 - 17x^3y^2 + 8x^5 - 22x^4y - 8y^5$
by $2x^2 - 4y^2 - 3xy$.

13. $8ab^6 - 8a^6b + 24a^2b^5 + a^5b^2 + 14a^3b^4 + 12a^4b^3 + 5a^7$
by $7a^2b + 2ab^2 + 5a^3$.

14. $2x^8 + 5x^7y + 3x^5y^3 - 3x^4y^4 - 5x^3y^5 - 2x^2y^6 - 2xy^7 + 4y^8$
by $2x^5 - x^4y + 3x^3y^2 - 2x^2y^3 + xy^4 - 2y^5$.

15. $6a^{n+6} - 20a^{n+5} + 25a^{n+4} - 19a^{n+3} + 10a^{n+2} - 4a^{n+1} + a^n$
by $3a^4 - 7a^3 + 4a^2 - 2a + 1$.

16. $6x^{m+5} + 11x^{m+4}y - 21x^{m+3}y^2 - 5x^{m+2}y^3 + 10x^{m+1}y^4 - x^{m-1}y^6$
by $3x^m - 2x^{m-1}y - x^{m-2}y^2$.

114. Integral expressions may have *fractional numerical coefficients,* since an algebraic expression is integral if it has no *letter* in the denominator. The processes with fractional coefficients are precisely the same as with integral coefficients, as will be seen by the solutions of the following examples.

1. Add $\frac{3}{8}a^3-\frac{4}{9}a^2b+\frac{2}{7}ab^2-\frac{2}{5}b^3$ and $\frac{1}{2}a^3+\frac{2}{3}a^2b-\frac{5}{14}ab^2-\frac{7}{10}b^3$.

$$\begin{array}{l}\frac{3}{8}a^3-\frac{4}{9}a^2b+\frac{2}{7}ab^2-\frac{2}{5}b^3\\ \underline{\frac{1}{2}a^3+\frac{2}{3}a^2b-\frac{5}{14}ab^2-\frac{7}{10}b^3}\\ \frac{7}{8}a^3+\frac{2}{9}a^2b-\frac{1}{14}ab^2-\frac{11}{10}b^3\end{array}$$

2. From $\frac{1}{2}a^3+\frac{1}{3}a^2b-\frac{3}{14}ab^2-\frac{7}{10}b^3$ take $\frac{3}{8}a^3-\frac{4}{9}a^2b+\frac{2}{7}ab^2-\frac{2}{5}b^3$.

$$\begin{array}{l}\frac{1}{2}a^3+\frac{1}{3}a^2b-\frac{3}{14}ab^2-\frac{7}{10}b^3\\ \underline{\frac{3}{8}a^3-\frac{4}{9}a^2b+\frac{2}{7}ab^2-\frac{2}{5}b^3}\\ \frac{1}{8}a^3+\frac{7}{9}a^2b-\frac{1}{2}ab^2-\frac{3}{10}b^3\end{array}$$

3. Multiply $\frac{2}{3}a^2-\frac{1}{2}ab+\frac{2}{5}b^2$ by $\frac{1}{3}a-\frac{3}{5}b$.

$$\begin{array}{llll}\frac{2}{3}a^2 & -\frac{1}{2}ab & +\frac{2}{5}b^2 & \\ \frac{1}{3}a & -\frac{3}{5}b & & \\ \hline \frac{2}{9}a^3 & -\frac{1}{6}a^2b & +\frac{2}{15}ab^2 & \\ & -\frac{2}{5}a^2b & +\frac{3}{10}ab^2 & -\frac{6}{25}b^3 \\ \hline \frac{2}{9}a^3 & -\frac{17}{30}a^2b & +\frac{13}{30}ab^2 & -\frac{6}{25}b^3\end{array}$$

4. Divide $\frac{3}{5}a^3-\frac{37}{40}a^2b+\frac{1}{3}ab^2-\frac{2}{9}b^3$ by $\frac{1}{2}a-\frac{2}{3}b$.

$$\begin{array}{l|l}\frac{3}{5}a^3-\frac{37}{40}a^2b+\frac{1}{3}ab^2-\frac{2}{9}b^3 & \frac{1}{2}a-\frac{2}{3}b\\ \hline \frac{3}{5}a^3-\frac{4}{5}a^2b & \frac{6}{5}a^2-\frac{1}{4}ab+\frac{1}{3}b^2\\ \hline -\frac{1}{8}a^2b+\frac{1}{3}ab^2 & \\ -\frac{1}{8}a^2b+\frac{1}{6}ab^2 & \\ \hline \frac{1}{6}ab^2-\frac{2}{9}b^3 & \\ \frac{1}{6}ab^2-\frac{2}{9}b^3 & \\ \hline \end{array}$$

Exercise 27

1. Add $\frac{3}{8}x^2+\frac{2}{3}xy+\frac{3}{10}y^2$; $\frac{1}{4}x^2+\frac{2}{9}xy-\frac{2}{5}y^2$; $\frac{1}{2}x^2-\frac{5}{6}xy-\frac{1}{2}y^2$.

2. Add $\frac{2}{7}a^2-\frac{1}{5}ab+\frac{2}{3}b^2$; $\frac{3}{14}a^2+\frac{7}{10}ab-\frac{1}{9}b^2$; $-\frac{8}{21}a^2-\frac{3}{5}ab-\frac{17}{27}b^2$; $\frac{13}{28}a^2+\frac{4}{15}ab+\frac{4}{3}b^2$.

3. From $\frac{1}{2}x-\frac{13}{6}a+\frac{1}{3}b-\frac{1}{4}y$ take $-\frac{1}{2}y-\frac{3}{5}b+\frac{1}{6}a+\frac{11}{4}x$.

4. From $\frac{5}{2}a - 76 - 3bc + \frac{1}{2}c$ take $\frac{3}{4}a - 56\frac{4}{5} + \frac{1}{4}c - 3bc$.

5. From $\frac{5}{3}a^6 + \frac{7}{2}a^5 - \frac{9}{4}a^4 - \frac{11}{6}a^3 + \frac{13}{3}a^2 - \frac{5}{8}a + \frac{1}{4}$
take $\frac{7}{2}a^6 - \frac{9}{4}a^5 - \frac{11}{6}a^4 + \frac{13}{3}a^3 - \frac{5}{3}a^2 - \frac{7}{12}a - \frac{5}{12}$.

6. Multiply $24x^5 - 36x^4y - 16x^3y^2 + 20x^2y^3 - 16xy^4 + \frac{1}{2}y^5$
by $\frac{1}{4}x^3 + 8x^2y + 7xy^2 - 11y^3$.

7. Multiply $\frac{1}{3}a^3 + \frac{3}{5}a^2x - \frac{5}{6}ax^2 + \frac{1}{2}x^3$ by $\frac{1}{2}a^2 - \frac{8}{11}ax - \frac{6}{11}x^2$.

8. Multiply $\frac{1}{4} - \frac{1}{4}x^3 - \frac{1}{6}x + \frac{1}{5}x^2$ by $\frac{1}{4} + \frac{1}{4}x + \frac{1}{2}x^2$.

9. Divide $\frac{2}{3}a^3 - \frac{443}{270}a^2d + \frac{11}{8}ad^2 - \frac{5}{12}d^3$ by $\frac{3}{2}a - \frac{5}{3}d$.

10. Divide $x^5 - \frac{23}{30}x^4 + \frac{31}{10}x^3 + \frac{5}{3} - \frac{181}{18}x - \frac{7}{3}x^2$ by $x^2 - \frac{1}{6}x + 5$.

11. Divide $\frac{1}{3}a^5 + \frac{11}{45}a^4b - \frac{73}{30}a^3b^2 - \frac{31}{15}a^2b^3 + \frac{7}{10}ab^4 - 3b^5$
by $\frac{2}{3}a^2 - \frac{2}{5}ab - 4b^2$.

12. Divide $\frac{16}{15}a^7 + \frac{2}{15}a^6b - \frac{91}{30}a^5b^2 + \frac{233}{180}a^4b^3 + \frac{5}{3}a^3b^4 - \frac{7}{12}a^2b^5$
$- \frac{5}{8}ab^6$ by $\frac{8}{9}a^3 + \frac{2}{3}a^2b - \frac{5}{3}ab^2$.

13. Multiply $\frac{5}{2}x^5 + \frac{13}{4}x^4 - \frac{13}{2}x^3 + \frac{3}{5}x^2 - \frac{1}{3}x + \frac{2}{7}$
by $\frac{4}{5}x^2 - \frac{2}{3}x + 2$.

14. Multiply $\frac{6}{5}a^4 - \frac{1}{4}a^3b - \frac{2}{9}a^2b^2 + \frac{3}{2}ab^3 - b^4$
by $-\frac{5}{2}a^3 - \frac{3}{4}a^2b + \frac{1}{2}ab^2 + 6b^3$.

15. Multiply $\frac{9}{16}a^4 - \frac{7}{8}a^3b + \frac{9}{14}a^2b^2 - \frac{6}{11}ab^3 + \frac{9}{17}b^4$
by $\frac{2}{3}a^2 + \frac{4}{9}ab + \frac{1}{3}b^2$.

16. Divide $0.08a^4 + 0.22a^3b - 0.29a^2b^2 + 0.57ab^3 - 0.3b^4$
by $0.2a^2 + 0.7ab - 0.5b^2$.

17. Divide $0.2m^6 - 0.46m^5n + 0.32m^4n^2 - 0.08m^3n^3 - 2m^2n^4$
$+ 0.2mn^5 + 1.12n^6$ by $0.5m^4 - 0.4m^3n + 1.2m^2n^2$
$+ 0.8mn^3 - 1.4n^4$.

18. Divide $\frac{6}{5}x^5 + \frac{11}{144}a^4x + \frac{299}{480}a^2x^3 - \frac{7}{8}ax^4 - \frac{3}{2}a^3x^2 + \frac{5}{9}a^5$
by $\frac{5}{8}ax - \frac{3}{2}x^2 + \frac{2}{3}a^2$.

CHAPTER VI

SPECIAL RULES

MULTIPLICATION

115. Square of the Sum of Two Numbers.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

Hence,

RULE 1. *The square of the sum of two numbers is the sum of their squares plus twice their product.*

116. Square of the Difference of Two Numbers.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a(a-b)-b(a-b)\\ &= a^2-ab-ab+b^2\\ &= a^2-2ab+b^2.\end{aligned}$$

Hence,

RULE 2. *The square of the difference of two numbers is the sum of their squares minus twice their product.*

117. Product of the Sum and Difference of Two Numbers.

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ab-b^2\\ &= a^2-b^2.\end{aligned}$$

Hence,

RULE 3. *The product of the sum and difference of two numbers is the difference of the squares of the two numbers.*

118. The following rule for raising a monomial to any required power is useful in solving examples in multiplication:

Raise the numerical coefficient to the required power and multiply the exponent of each letter by the exponent of the required power.

Thus, the square of $5xy^2z^3$ is $25x^2y^4z^6$.

Exercise 28

Write the product of:

1. $(x+a)^2$.
2. $(y-b)^2$.
3. $(1+2x^2)^2$.
4. $(2x-a)^2$.
5. $(2a-5b)^2$.
6. $(4z-3)^2$.
7. $(3+2d)^2$.
8. $(3x+5y)^2$.
9. $(-2m+3n)^2$.
10. $(2+4m^4)^2$.
11. $(2a-5a^2)^2$.
12. $(x+a)(x-a)$.
13. $(1+x)(1-x)$.
14. $(3-y)(3+y)$.
15. $(3x-2y)(3x+2y)$.
16. $(3ab+1)(3ab-1)$.
17. $(5x^2-3y^3)(5x^2+3y^3)$.
18. $(a^2+3x)(a^2-3x)$..
19. $(5-bx^3)(bx^3+5)$.
20. $(6m+7n^4)(7n^4-6m)$.
21. $(8-ab^2)(ab^2+8)$.
22. $(a^4x+ax^4)(ax^4-a^4x)$.
23. $(3ab^2-4a^2b)(3ab^2+4a^2b)$.
24. $(2x^2y+5xy^2)(2x^2y-5xy^2)$.

119. If we are required to multiply $a+b+c$ by $a+b-c$, we may abridge the ordinary process as follows:

$$(a+b+c)(a+b-c)=\{(a+b)+c\}\{(a+b)-c\}$$

By Rule 3, $\quad =(a+b)^2-c^2$

By Rule 1, $\quad =a^2+2ab+b^2-c^2$.

If we are required to multiply $a + b - c$ by $a - b + c$, we may put the expressions in the following forms, and perform the operation:

$$(a + b - c)(a - b + c) = \{a + (b - c)\}\{a - (b - c)\}$$

By Rule 3, $\quad = a^2 - (b - c)^2$

By Rule 2, $\quad = a^2 - (b^2 - 2bc + c^2)$

By p. 13, § 43, $\quad = a^2 - b^2 + 2bc - c^2.$

Exercise 29

Write the product of:

1. $(x + y + z)(x - y - z)$.
2. $(x - y + z)(x - y - z)$.
3. $(x^2 + xy + y^2)(x^2 - xy + y^2)$.
4. $(a^2 + 2ab + b^2)(a^2 - 2ab + b^2)$.
5. $(a^2 + a - 1)(a^2 - a + 1)$.
6. $(2x - y - 3z)(2x - y + 3z)$.
7. $(x - 2y + 3z)(x + 2y + 3z)$.
8. $(bc + ac + ab)(bc - ac - ab)$.
9. $(ax^2 + bx + c)(ax^2 - bx - c)$.
10. $(a^3 + 3a^2 - 4a)(a^3 - 3a^2 + 4a)$.
11. $(3x^2 - 2xy + y^2)(3x^2 + 2xy - y^2)$.
12. $(x^2 + 2ax + 3b)(x^2 - 2ax - 3b)$.
13. $(x^2 + 6x - 7)(x^2 - 6x + 7)$.
14. $(a^3b - 2a^2b^2 - 2ab^3)(a^3b + 2a^2b^2 - 2ab^3)$.

120. Square of any Polynomial. If we put x for a, and $y + z$ for b, in the identity

$$(a + b)^2 = a^2 + 2ab + b^2,$$

we have

$$\{x + (y + z)\}^2 = x^2 + 2x(y + z) + (y + z)^2,$$

or

$$(x + y + z)^2 = x^2 + 2xy + 2xz + y^2 + 2yz + z^2$$
$$= x^2 + y^2 + z^2 + 2xy + 2xz + 2yz.$$

The product is the sum of the squares of the several terms of the given expression and twice the products obtained by multiplying every term into each of the terms that follows it.

Again, if we put $a - b$ for a, and $c - d$ for b, in the same identity, we have

$$\{(a-b)+(c-d)\}^2$$
$$= (a-b)^2 + 2(a-b)(c-d) + (c-d)^2$$
$$= (a^2 - 2ab + b^2) + 2a(c-d) - 2b(c-d) + (c^2 - 2cd + d^2)$$
$$= a^2 - 2ab + b^2 + 2ac - 2ad - 2bc + 2bd + c^2 - 2cd + d^2$$
$$= a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd.$$

Here the same law holds as before, the sign of each double product being + or −, according as the factors composing it have *like* or *unlike* signs. The same is true for any polynomial. Hence,

RULE 4. *The square of a polynomial is the sum of the squares of the several terms and twice the products obtained by multiplying every term into each of the terms that follows it.*

EXERCISE 30

Write the square of:

1. $3a - 4b$.
2. $x - y + z$.
3. $x + y - z$.
4. $y^2 - 3yz + 6z^2$.
5. $m^2 + 2mn + n^2$.
6. $a^2 - 2ab + b^2$.
7. $x^2 - 3x - 4$.
8. $x^2 - x - 1$.
9. $x^2 - 3a^2 + 2b^2$.
10. $a^2 - 5a + 2$.
11. $a^3b - 2a^2b^2 - 2ab^3$.
12. $x^2 - 3ax - 2a^2$.
13. $a + 2b + 3c + d$.
14. $a - 2b + 3c - d$.
15. $1 + x + 2x^2 + 3x^3$.
16. $1 - x - 2x^2 - 3x^3$.
17. $2 - a^2 + 3a^3 - d^2$.
18. $\frac{1}{2}x^2 - 2y + \frac{2}{3}y^2 + 2z^3$.

121. Product of Two Binomials. The product of two binomials which have the form $x + a$, $x + b$ should be carefully noticed and remembered.

1. $(x+3)(x+2) = x(x+2) + 3(x+2)$
$= x^2 + 2x + 3x + 6$
$= x^2 + 5x + 6.$

2. $(x-3)(x-2) = x(x-2) - 3(x-2)$
$= x^2 - 2x - 3x + 6$
$= x^2 - 5x + 6.$

3. $(x+3)(x-2) = x(x-2) + 3(x-2)$
$= x^2 - 2x + 3x - 6$
$= x^2 + x - 6.$

4. $(x-3)(x+2) = x(x+2) - 3(x+2)$
$= x^2 + 2x - 3x - 6$
$= x^2 - x - 6.$

Each of these products has three terms.

The first term of each product is the product of the first terms of the binomials.

The last term of each product is the product of the second terms of the binomials.

The middle term of each product has for the coefficient of x the *algebraic sum* of the second terms of the binomials.

The intermediate step given above may be omitted and the products written at once by *inspection*. Thus,

1. Multiply $x + 6$ by $x + 5$.

$$6 + 5 = 11,$$

and $$6 \times 5 = 30.$$

$$\therefore (x+6)(x+5) = x^2 + 11x + 30.$$

2. Multiply $x - 6$ by $x - 5$.

$$(-6) + (-5) = -11,$$

and $$(-6)(-5) = +30.$$

$$\therefore (x-6)(x-5) = x^2 - 11x + 30.$$

3. Multiply $x - 6y$ by $x + 4y$.

$$(-6y) + (4y) = -2y,$$

and

$$(-6y)(4y) = -24y^2.$$

$$\therefore (x - 6y)(x + 4y) = x^2 - 2xy - 24y^2.$$

4. Multiply $x^2 + 5(y + z)$ by $x^2 - 3(y + z)$.

$$5(y + z) + \{-3(y + z)\} = 2(y + z),$$

and

$$\{5(y + z)\}\{-3(y + z)\} = -15(y + z)^2.$$

$$\therefore \{x^2 + 5(y + z)\}\{x^2 - 3(y + z)\} = x^4 + 2(y + z)x^2 - 15(y + z)^2.$$

Exercise 31

Write by inspection the product of:

1. $(x + 1)(x + 2)$.
2. $(c - 3)(c - 2)$.
3. $(a - 7)(a + 4)$.
4. $(x - 11)(x + 5)$.
5. $(a - 6)(a - 11)$.
6. $(x - 3y)(x + 7y)$.
7. $(a + 5b)(a - 8b)$.
8. $(c - 3d)(c + 2d)$.
9. $(x^2 - a^3)(x^2 + 3a^3)$.
10. $(a + 1)(a + 4)$.
11. $(c - 3d)(c + d)$.
12. $(a^2 + 2b^2)(a^2 - 7b^2)$.
13. $(x^4 - 2)(x^4 + 5)$.
14. $(x^2 + 3yz)(x^2 - 2yz)$.
15. $(a^2 + 2ab)(a^2 + 5ab)$.
16. $(x^2 + 4ax)(x^2 - 3ax)$.
17. $(a^6 - 3b^6)(a^6 - 5b^6)$.
18. $(x - 3y)(x - 7y)$.
19. $(x^2 + 2y^2)(x^2 + y^2)$.
20. $(a - 4b)(a - 5b)$.
21. $(x + y^2)(x + 7y^2)$.
22. $(ax + 5by)(ax - 4by)$.
23. $(x^2 + 3xy)(x^2 - 5xy)$.
24. $(xy - 7yz)(xy + 6yz)$.
25. $\{a - (2b + c)\}\{a - 2(2b + c)\}$.
26. $\{x + 3(y + 2z)\}\{x - (y + 2z)\}$.
27. $\{a + 4(b + c)\}\{a - 7(b + c)\}$.

122. In like manner, the product of *any* two binomials may be written.

1. Multiply $3x - y$ by $2x + 5y$.

$$(3x - y)(2x + 5y) = 6x^2 + 15xy - 2xy - 5y^2$$
$$= 6x^2 + 13xy - 5y^2.$$

2. Multiply $3x + 5y$ by $2x - 3y$.

The middle term is

$$(3x)(-3y) + (5y)(2x) = -9xy + 10xy = xy.$$
$$\therefore (3x + 5y)(2x - 3y) = 6x^2 + xy - 15y^2.$$

Exercise 32

Write by inspection the product of:

1. $(3a - 4b)(a - b)$.
2. $(4b - 5c)(3b + 2c)$.
3. $(4x + 9y)(x - 5y)$.
4. $(3a - 2b)(2a - b)$.
5. $(5a + 2b)(3a - 4b)$.
6. $(a - 2b)(3a + 4b)$.
7. $(a^3 - 4a)(3a^3 - 6a)$.
8. $(a + 3)(3a + 5)$.
9. $(7a - 2b)(3a - b)$.
10. $(9a - 4c)(3a - 4c)$.
11. $(3b + 2c)(c + 2b)$.
12. $(3x - a)(2x - 3a)$.
13. $(8 - 9x)(3 - 6x)$.
14. $(2x + 4y)(y - 9x)$.
15. $(9x - 2y)(4x - 3y)$.
16. $(7x - 8a)(2x - 3a)$.
17. $(3a + 2b)(5a + 9b)$.
18. $(2a + 5c)(4a + c)$.
19. $(7c - 4d)(8c + 4d)$.
20. $(9x - 3y)(5x - 2y)$.
21. $(1 + 2x^2)(3 - 5x^2)$.
22. $(9m^3 - 5n)(m^3 - n)$.
23. $(10 - x^2)(6 + 5x^2)$.
24. $(7a - 2)(8a - 9)$.
25. $(3x - 2y)(3x + y)$.
26. $(5x^2 - 2y^2)(3x^2 - 7y^2)$.
27. $(a^2 - 5d)(3a^2 + 9d)$.
28. $(4m^2 - y^4)(7m^2 - 8y^4)$.
29. $(3m - a)(2m + 5a)$.
30. $(\frac{1}{5}a + \frac{1}{4}b)(\frac{1}{8}a - \frac{1}{5}b)$.

Division

123. The following rule for finding any required root of a monomial will be found useful in solving examples in division:

Find the required root of the numerical coefficient and divide the exponent of each letter by the index of the required root.

Thus, the square root of $64\,a^2b^6$ is $8\,ab^3$.

124. Difference of Two Squares. By performing the division, we find that

$$\frac{a^2-b^2}{a+b}=a-b,$$

and

$$\frac{a^2-b^2}{a-b}=a+b.$$

Hence,

RULE 1. *The difference of the squares of two numbers is divisible by the sum of the numbers, and the quotient is the difference of the numbers.*

The difference of the squares of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the numbers.

Exercise 33

Write by inspection the quotient of:

1. $\dfrac{x^4-64}{x^2+8}$.
2. $\dfrac{x^2-4\,y^2}{x-2\,y}$.
3. $\dfrac{a^8-81\,b^4}{a^4-9\,b^2}$.
4. $\dfrac{a^6-b^6}{a^3+b^3}$.
5. $\dfrac{16\,y^2-4\,x^2}{4\,y+2\,x}$.
6. $\dfrac{64\,a^2b^2-49\,b^2c^2}{8\,ab-7\,bc}$.
7. $\dfrac{4\,b^2-9\,m^2n^2}{2\,b-3\,mn}$.
8. $\dfrac{4\,a^2b^2-25\,c^2d^2}{2\,ab+5\,cd}$.
9. $\dfrac{16\,m^2n^2-49\,a^2b^2}{4\,mn+7\,ab}$.
10. $\dfrac{25\,a^2b^2c^2-36\,x^2y^4z^6}{5\,abc-6\,xy^2z^3}$.

125. Sum and Difference of Two Cubes. By performing the division, we find that

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2,$$

and

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

Hence,

RULE 2. *The sum of the cubes of two numbers is divisible by the sum of the numbers, and the quotient is the sum of the squares of the numbers minus their product.*

RULE 3. *The difference of the cubes of two numbers is divisible by the difference of the numbers, and the quotient is the sum of the squares of the numbers plus their product.*

EXERCISE 34

Write by inspection the quotient of:

1. $\dfrac{8a^3 - 27b^3}{2a - 3b}$.

2. $\dfrac{8a^3 + 27b^3}{2a + 3b}$.

3. $\dfrac{a^9 - 27b^3}{a^3 - 3b}$.

4. $\dfrac{8a^3 - 125b^3}{2a - 5b}$.

5. $\dfrac{216 + 125a^9}{6 + 5a^3}$.

6. $\dfrac{343c^3 + 64d^3}{7c + 4d}$.

7. $\dfrac{1 - 64c^6}{1 - 4c^2}$.

8. $\dfrac{216m^3 + 343n^9}{6m + 7n^3}$.

9. $\dfrac{27x^3y^3 + 125a^3b^3}{3xy + 5ab}$.

10. $\dfrac{b^3x^9 - 125}{bx^3 - 5}$.

11. $\dfrac{27y^6 - 512z^9}{3y^2 - 8z^3}$

12. $\dfrac{64x^3 + 729y^6}{4x + 9y^2}$

13. $\dfrac{125d^3 - 1331b^3}{5d - 11b}$.

14. $\dfrac{8a^6b^3 + 125x^3y^9}{2a^2b + 5xy^3}$.

126. Sum and Difference of any Two Like Powers. By performing the division, we find that

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3;$$

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3;$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4;$$

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

We find by trial that

$$a^2 + b^2,\ a^4 + b^4,\ a^6 + b^6, \cdots$$

are *not* divisible by $a + b$ or by $a - b$. Hence,

When n is a positive integer, it is proved in Chap. VII,

1. $\mathrm{a^n + b^n}$ *is divisible by* $\mathrm{a + b}$ *if* n *is odd, and by neither* $\mathrm{a + b}$ *nor* $\mathrm{a - b}$ *if* n *is even.*

2. $\mathrm{a^n - b^n}$ *is divisible by* $\mathrm{a - b}$ *if* n *is odd, and by both* $\mathrm{a + b}$ *and* $\mathrm{a - b}$ *if* n *is even.*

NOTE. It is important to notice in the above examples that the terms of the quotient are all *positive* when the divisor is $a - b$, and *alternately positive and negative* when the divisor is $a + b$; also, that the quotient is homogeneous, the exponent of a decreasing and of b increasing by 1 for each successive term.

EXERCISE 35

Write by inspection the quotient of:

1. $\dfrac{a^4b^4 - x^4y^4}{ab - xy}$.
2. $\dfrac{b^5x^5 + c^5y^5}{bx + cy}$.
3. $\dfrac{32\,a^5 - 243\,b^5}{2\,a - 3\,b}$.
4. $\dfrac{x^4 - 81\,y^4}{x + 3\,y}$.
5. $\dfrac{16\,x^4 - 81\,y^4}{2\,x - 3\,y}$.
6. $\dfrac{x^6 - 729\,y^6}{x - 3\,y}$.
7. $\dfrac{x^6 - 729\,y^6}{x + 3\,y}$.
8. $\dfrac{625\,x^4 - 81\,a^4}{5\,x - 3\,a}$.
9. $\dfrac{625\,x^4 - 81\,a^4}{5\,x + 3\,a}$.

CHAPTER VII

FACTORS

127. Rational Integral Expressions. An expression is *rational* if none of its terms contains square or other roots; and an expression is *integral* if none of its terms contains a letter in the denominator.

Thus, $ax^3 - \frac{1}{2}bx^2 + \frac{1}{3}cx$ is a rational integral expression.

128. Surds. If an indicated root of a rational number cannot be obtained exactly, the indicated root is called a **surd**.

Thus, $\sqrt{2}$ and $\sqrt[3]{5}$ are surds.

129. Factors of Rational Integral Expressions. By factors of a *given integral number* in Arithmetic we mean integral numbers that will exactly divide the given number.

Likewise, by factors of a *rational integral expression* in Algebra we mean rational integral expressions that will exactly divide the given expression.

130. Simple Factors and Quadratic Factors. Factors of the first degree are called *simple factors.* Factors of the second degree are called *quadratic factors.*

131. Factors of Monomials. The factors of a monomial may be found by inspection.

Thus, the factors of $15xy^2$ are 3, 5, x, y, and y.

132. Factors of Polynomials. Not every polynomial can be resolved into factors; for, as we have *prime* numbers in Arithmetic, so in Algebra we have expressions *that are not the product of any factors.*

The *form* of a polynomial that can be resolved into factors often suggests the process of finding the factors.

133. When the Terms have a Common Monomial Factor.

Resolve into factors $3x^2 + 6xy + 9xz$.

Since 3 and x are factors of each term, we have

$$\frac{3x^2 + 6xy + 9xz}{3x} = \frac{3x^2}{3x} + \frac{6xy}{3x} + \frac{9xz}{3x} = x + 2y + 3z.$$

$$\therefore 3x^2 + 6xy + 9xz = 3x(x + 2y + 3z).$$

Hence, the required factors are $3x$ and $x + 2y + 3z$.

Exercise 36

Resolve into factors:

1. $x^3 + x^2 - 5x$.
2. $a^2x^2 - 3a^2x + 5a^2$.
3. $a^2x^2 + 3ay + 4az^2$.
4. $a^2cx - abcy$.
5. $a^3 - 2a^2b - a^2$.
6. $2abxy - 3acxz$.
7. $6a^3x - a^2x^2 - 9a^4$.
8. $x^2yz + xy^2z + xyz^2$.
9. $ab^2 + abc - a^2b$.
10. $b^2c^2x^2 - bcd^2x$.
11. $abx^4 + 2ax^3 + 6acx$.
12. $acx^2 - bc^2x + b^2c$.
13. $3x^3y^2 + 4x^2y^3 + 5xy^4$.
14. $2x^5y^2 + 4x^4y^3 + 6x^3y^4$.
15. $a^4bc + a^3b^2c^2 + 2a^2b^3c^3$.
16. $8x^3y^2z^2 + 4x^2y^3z^3 + 8xy^4z^4$.
17. $3a^4x^6 - 2a^3x^4 + 7a^2x^2$.
18. $5abx^3 - 4a^2b^2x^2 - 4a^3b^3x$.
19. $6a^4b + 12a^3b^2 - 2a^2b^3$.
20. $9ady + 60ad^2 - 12a^2dx$.

134. When the Terms can be Grouped so as to Show a Common Compound Factor.

1. Resolve into factors $ax + ay + bx + by$.

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (a + b)(x + y). \end{aligned}$$

Hence, the required factors are $a + b$ and $x + y$.

Note. The given expression must be so grouped that the expressions in each group after the monomial factor is taken out shall be *exactly* alike. If this cannot be done, the given expression cannot be factored by this method.

2. Resolve into factors $ax - bx - ay + by$.

$$\begin{aligned} ax - bx - ay + by &= (ax - bx) - (ay - by) \\ &= x(a - b) - y(a - b) \\ &= (x - y)(a - b). \end{aligned}$$

NOTE. Here the last two terms, $-ay + by$, being put within a parenthesis preceded by the sign $-$, have their signs changed.

3. Resolve into factors $2x^2 + ax - 6bx - 3ab$.

$$\begin{aligned} 2x^2 + ax - 6bx - 3ab &= (2x^2 + ax) - (6bx + 3ab) \\ &= x(2x + a) - 3b(2x + a) \\ &= (x - 3b)(2x + a). \end{aligned}$$

4. Resolve into factors $3a^3 + a - 6a^2b - 2b$.

$$\begin{aligned} 3a^3 + a - 6a^2b - 2b &= (3a^3 - 6a^2b) + (a - 2b) \\ &= 3a^2(a - 2b) + (a - 2b) \\ &= (3a^2 + 1)(a - 2b). \end{aligned}$$

5. Resolve into factors

$$axy - 2ay^2 - 2a^2x + 4a^2y + 3ayz - 6a^2z.$$

$$\begin{aligned} &axy - 2ay^2 - 2a^2x + 4a^2y + 3ayz - 6a^2z \\ &\quad = a(xy - 2y^2 - 2ax + 4ay + 3yz - 6az) \\ &\quad = a[(xy - 2y^2 + 3yz) - (2ax - 4ay + 6az)] \\ &\quad = a[y(x - 2y + 3z) - 2a(x - 2y + 3z)] \\ &\quad = a(y - 2a)(x - 2y + 3z). \end{aligned}$$

EXERCISE 37

Resolve into factors:

1. $ax - ay + bx - by$.
2. $2ac + bc - 2ad - bd$.
3. $ax - 2ay - 3bx + 6by$.
4. $8mx - 7my - 8nx + 7ny$.
5. $3mx + 2cy - 3cx - 2my$.
6. $mx + nx + m + n$.
7. $y^2 + b^2 + ay^2 + ab^2$.
8. $ax - bx - 2ay + 2by - 2cy + cx$.
9. $3b - a^2 - a^2x^4 + 3bx^4$.
10. $y - z^2 + xz^2 - xy$.

11. $3x^3 - 6ax^2 - x + 2a.$

12. $2a^2x - 9 - 6x + 3a^2.$

13. $3x^2 + ax + 6bx + 2ab.$

14. $6a^2 - 3ab - 4ac + 2bc.$

15. $4y + 9x^3 - 6x^2y - 6x.$

16. $a^2c^2 - 2a^3b - 2c^3d + 4abcd.$

17. $2xy + 3yz + 6y^2 + xz.$

18. $c^2m - abm + abn - c^2n.$

19. $a^2 + 3ad - 2ab - 6bd.$

20. $12bx + 3ay - 9by - 4ax.$

21. $3x^2 + bx + 9ax + 3ab.$

22. $6a^2 - 3ab - 2bm + 4am.$

23. $xy - xz - 2z^2 + 2yz.$

24. $6a^2 + 9ac - 6bc - 4ab.$

25. $x^4 + x^2y^2 + x^2z^2 - x^3y - xyz^2 + y^2z^2.$

26. $3x^3 + 5axy - 5b^2y - 3ax^2 + 3b^2x - 5x^2y.$

27. $a^2b + by^2 + 6ax^2 - 3xy^2 - 3a^2x - 2abx.$

28. $a^2 - 4cx - 2ax + 2ac + 2bx - 3ad + 6dx - ab.$

29. $6a^2x + a^2xy + 9a^2bz - 3abxz - 2ax^2 - 3a^3y.$

135. When a Trinomial is a Perfect Square. A trinomial is a perfect square if two of its terms are perfect squares and positive, and the other term is plus or minus twice the product of their square roots.

Thus, $4x^2 + 12xy + 9y^2$ and $4x^2 - 12xy + 9y^2$ are perfect squares.

The rule for extracting the square root of a perfect trinomial square is as follows:

Extract the square root of the terms that are perfect squares and connect these square roots by the sign of the other term.

Thus, the square root of $4x^2 + 12xy + 9y^2$ is $2x + 3y$,
and the square root of $4x^2 - 12xy + 9y^2$ is $2x - 3y$.

1. Resolve into factors $25x^2 + 20xy + 4y^2$.

$$25x^2 + 20xy + 4y^2 = (5x + 2y)(5x + 2y) = (5x + 2y)^2.$$

2. Resolve into factors $9a^2b^2 - 42abcd + 49c^2d^2$.

$$9a^2b^2 - 42abcd + 49c^2d^2 = (3ab - 7cd)(3ab - 7cd) = (3ab - 7cd)^2.$$

Exercise 38

Resolve into factors:

1. $x^2 + 2abx + a^2b^2$.
2. $x^2 + 6xy + 9y^2$.
3. $x^2 - 4ax + 4a^2$.
4. $x^2 - 10xy + 25y^2$.
5. $9x^2 + 48xy + 64y^2$.
6. $25x^2 - 70xy + 49y^2$.
7. $16x^2 + 40x + 25$.
8. $4x^2 - 12x + 9$.
9. $9a^2 + 30ab + 25b^2$.
10. $1 - 16x + 64x^2$.
11. $4a^4 + 20a^2b + 25b^2$.
12. $36x^4 + 60x^2 + 25$.
13. $x^6y^6 - 6x^3y^3z + 9z^2$.
14. $x^6 - 8x^3y^2 + 16y^4$.
15. $25x^2 + 40xy + 16y^2$.
16. $49a^4 - 42a^2b^2c^2 + 9b^4c^4$.
17. $9x^4 + 24x^2y^2 + 16y^4$
18. $4x^4y^4 - 20x^3y^3 + 25x^2y^2$.
19. $25a^{10}b^{10} - 20a^5b^5c^2d^2 + 4c^4d^4$.
20. $9m^4 - 66m^2n + 121n^2$.
21. $49n^{16} + 84m^3n^8 + 36m^6$.
22. $4x^4 + 52ax^2 + 169a^2$.
23. $144x^2y^2 - 168ab^2xy + 49a^2b^4$.
24. $1 - 8abc + 16a^2b^2c^2$.
25. $49x^2 + 126xy^2 + 81y^4$.
26. $64a^4b^2 - 16a^2bx^3 + x^6$.
27. $121a^2 + 176ab + 64b^2$.
28. $144x^2 + 600xy + 625y^2$.
29. $(a+b)^2 - 6c(a+b) + 9c^2$.
30. $(x-y)^2 + 12z(x-y) + 36z^2$.

136. When a Binomial is the Difference of Two Squares. The difference of two squares is the product of two factors which may be determined as follows:

Extract the square root of the first term and the square root of the second term.

The sum of these roots will form the first factor.

The difference of these roots will form the second factor.

1. Resolve into factors $4x^2 - 25y^2$.

$$4x^2 - 25y^2 = (2x + 5y)(2x - 5y).$$

2. Resolve into factors $9a^2b^2 - 16c^2$.

$$9a^2b^2 - 16c^2 = (3ab + 4c)(3ab - 4c).$$

Exercise 39

Resolve into factors:

1. $a^2 - b^2$.
2. $1 - a^2$.
3. $4x^2 - 81y^2$.
4. $1 - 36a^2$.
5. $4 - 49x^2$.
6. $49x^2 - 64$.
7. $36a^2 - 25b^2$.
8. $4m^2 - 1$.
9. $9 - 16n^2$.
10. $25x^2 - 4$.
11. $y^2 - 25z^2$.
12. $49a^2 - 25b^2$.
13. $64x^2 - 25$.
14. $4a^4 - 9x^2$.
15. $1 - a^4b^4c^2$.
16. $256x^2y^2 - 625$.
17. $a^{10} - 4b^4$.
18. $64a^4b^2 - x^6$.
19. $121a^4 - 64b^2$.
20. $49x^2 - 100y^4$.
21. $144x^2y^2 - 49a^2b^4$.
22. $49x^2 - 9b^4c^2$.
23. $256x^2y^2 - 1$.
24. $a^4b^2 - 4c^2d^4$.
25. $169x^2y^2 - 144a^2$.
26. $4a^2b - 25b^3$.
27. $4x^4 - 9y^2z^2$.
28. $a^8 - b^6$.
29. $144x^2 - 625y^2$.
30. $4a^2b^2 - 169c^2$.
31. $4 - 361a^4b^6c^2$.
32. $9m^4 - 121n^2p^4$.
33. $25a^{10}b^{10} - 4c^4d^4$.
34. $36x^4y^8 - 25a^4b^6$.
35. $81x^2y^6 - 49c^4$.
36. $a^{2m} - b^{2n}$.
37. $4x^{2m} - 9y^{2n}$.
38. $81a^{4m} - 49b^{2m}$.
39. $a^{4c} - 36b^{2c}c^{4c}$.
40. $121x^2 - 144z^{2m}$.
41. $9a^{10n} - 64b^{12n}$.
42. $49x^{2m} - 36y^{4m}z^{6m}$.

137. The same method may be employed if either square is a compound expression or if both squares are compound expressions.

1. Resolve into factors $(a + 2b)^2 - 9c^2$.

The square roots of the terms are $a + 2b$ and $3c$.
The sum of these roots is $a + 2b + 3c$.
The difference of these roots is $a + 2b - 3c$.
Therefore, $(a + 2b)^2 - 9c^2 = (a + 2b + 3c)(a + 2b - 3c)$.

2. Resolve into factors $(a + b)^2 - (2x - 3y)^2$.

The square roots of the terms are $a + b$ and $2x - 3y$.
The sum of these roots is $a + b + 2x - 3y$.
The difference of these roots is $(a + b) - (2x - 3y)$, or $a + b - 2x + 3y$.
Therefore, $(a + b)^2 - (2x - 3y)^2 = (a + b + 2x - 3y)(a + b - 2x + 3y)$.

If the factors have like terms, these terms should be collected so as to give the results in the simplest form.

3. Resolve into factors $(2x + 3y)^2 - (x - 4y)^2$.

The square roots of the terms are $2x + 3y$ and $x - 4y$.
The sum of these roots is $(2x + 3y) + (x - 4y)$,

or $$2x + 3y + x - 4y = 3x - y.$$

The difference of these roots is $(2x + 3y) - (x - 4y)$,

or $$2x + 3y - x + 4y = x + 7y.$$

Therefore, $(2x + 3y)^2 - (x - 4y)^2 = (3x - y)(x + 7y)$.

Exercise 40

Resolve into factors:

1. $(x + 3y)^2 - 4z^2$.
2. $(x - 5y)^2 - (a - 2b)^2$.
3. $(4a + 5)^2 - (2x - 3)^2$.
4. $(3a + 5b)^2 - (1 + 8x)^2$.
5. $(x^3 - 4y^2)^2 - (3a^4 - 2b^2)^2$.
6. $(x + y)^2 - (x - 3y)^2$.
7. $(2a - 5b)^2 - (a + 4b)^2$.
8. $(7a - 3bc)^2 - (4x - 5yz)^2$.
9. $(2x^2 + 13a)^2 - (3y + 7b)^2$.
10. $(3x - y)^2 - (2x - 3y)^2$.
11. $(4a + 3b)^2 - (3a + 2b)^2$.
12. $(2x - 1)^2 - (3x + 1)^2$.
13. $(x + y - z)^2 - (x - y - z)^2$.
14. $36(a + b)^2 - 25(c - d)^2$.
15. $49(a - b)^2 - 36(c + d)^2$.
16. $(5x^2 - 4y)^2 - (3x^2 - 2y)^2$.
17. $64(x - y)^2 - 81(y - z)^2$.
18. $49(x + y)^2 - 25(x - y)^2$.

138. By properly grouping the terms, compound expressions may often be written as the difference of two squares, and the factors readily found.

1. Resolve into factors $4x^2 + 4xy + y^2 - a^2$.

$$\begin{aligned} 4x^2 + 4xy + y^2 - a^2 &= (4x^2 + 4xy + y^2) - a^2 \\ &= (2x + y)^2 - a^2 \\ &= (2x + y + a)(2x + y - a). \end{aligned}$$

2. Resolve into factors $a^2 - x^2 - 9y^2 + b^2 + 2ab + 6xy$.

Here $2ab$ shows that it is the middle term of the expression which has for its first and last terms a^2 and b^2; and $6xy$ shows that it is the middle term of the expression that has in its first and last terms x^2 and $9y^2$, and the minus sign before x^2 and $9y^2$ shows that these terms must be put in a parenthesis with the minus sign before it in order that they may be made positive. Hence,

$$\begin{aligned} & a^2 - x^2 - 9y^2 + b^2 + 2ab + 6xy \\ &= a^2 + 2ab + b^2 - x^2 + 6xy - 9y^2 \\ &= (a^2 + 2ab + b^2) - (x^2 - 6xy + 9y^2) \\ &= (a+b)^2 - (x - 3y)^2 \\ &= (a + b + x - 3y)(a + b - x + 3y). \end{aligned}$$

3. Resolve into factors $6c^2d^2 + 4b^2 - 9c^4 - 4a^2b^2 - d^4 + a^4$.

$$\begin{aligned} & 6c^2d^2 + 4b^2 - 9c^4 - 4a^2b^2 - d^4 + a^4 \\ &= a^4 - 4a^2b^2 + 4b^4 - 9c^4 + 6c^2d^2 - d^4 \\ &= (a^4 - 4a^2b^2 + 4b^4) - (9c^4 - 6c^2d^2 + d^4) \\ &= (a^2 - 2b^2)^2 - (3c^2 - d^2)^2 \\ &= (a^2 - 2b^2 + 3c^2 - d^2)(a^2 - 2b^2 - 3c^2 + d^2). \end{aligned}$$

Exercise 41

Resolve into factors:

1. $x^2 - 2xy + y^2 - 16z^2$.
2. $a^2 - 4ab + 4b^2 - 9c^2$.
3. $x^2 - 6xy + 9y^2 - 16z^2$.
4. $x^2 - 2ax + a^2 - 49y^2$.
5. $36b^2 - 4 + 20a - 25a^2$.
6. $4c^2 - a^4 - 9b^2 - 6a^2b$.
7. $4x^2 - 25y^2 + 9 - 12x$.
8. $25x^2 - 4a^2 + 16y^2 + 40xy$.
9. $16a^4 - x^2 - y^2 + 2xy$.
10. $9x^2 - a^2 - 4ab - 4b^2$.
11. $a^2 + 2ab + b^2 - c^2 - 2cd - d^2$.
12. $a^2 + b^2 - c^2 - d^2 - 2ab + 2cd$.
13. $a^2 - c^2 + 4b^2 - 4d^2 - 4ab - 4cd$.
14. $x^2 - 9b^2 + 9y^2 - a^2 + 6ab - 6xy$.

15. $49x^2 + 4y^2 - 9a^2 - 16b^2 - 28xy - 24ab.$

16. $16a^2 - 9x^2y^2 - 25z^4 + 25b^2 + 30xyz^2 + 40ab.$

17. $4a^2 + 9x^2 - 12ax + 12by - 9b^2 - 4y^2.$

18. $4x^4 + 9y^6 - 12x^2y^3 - 64a^2b^4c^6.$

19. $9x^2 + 16y^2 - 49a^2 - 4b^2 + 28ab + 24xy.$

20. $81x^2 - m^2n^2 - 64a^4 + 126xy + 16a^2mn + 49y^2.$

21. $121 - 110x^2 - y^2 - 625z^2 - 50yz + 25x^4.$

22. $x^6 + 225 - 30x^3 + 28abc - 49a^2b^2 - 4c^2.$

23. $4x^{2m} - p^{2n} - a^{2n} + 9 - 12x^m - 2a^np^n.$

24. $36a^{2n+2} - 4y^{2n+4} - 12x^ny^{n+2} + 16b^{2n-2} - 48a^{n+1}b^{n-1} - 9x^{2n}.$

139. Some expressions which at first sight do not seem to belong to any known form may be represented as the difference of two squares by adding and subtracting the same number. No general rule for factoring such expressions can be laid down, but the following examples will illustrate the general principle.

1. Resolve into factors $a^4 + a^2b^2 + b^4$.

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2). \end{aligned}$$

2. Resolve into factors $9a^4 - 34a^2b^2 + 25b^4$.

Twice the product of the square roots of $9a^4$ and $25b^4$ is $30a^2b^2$. We may, therefore, separate the term $-34a^2b^2$ into two terms, $-30a^2b^2$ and $-4a^2b^2$. Hence,

$$\begin{aligned} 9a^4 - 34a^2b^2 + 25b^4 &= 9a^4 - 30a^2b^2 + 25b^4 - 4a^2b^2 \\ &= (9a^4 - 30a^2b^2 + 25b^4) - 4a^2b^2 \\ &= (3a^2 - 5b^2)^2 - (2ab)^2 \\ &= (3a^2 - 5b^2 + 2ab)(3a^2 - 5b^2 - 2ab) \\ &= (3a^2 + 2ab - 5b^2)(3a^2 - 2ab - 5b^2). \end{aligned}$$

3. Resolve into factors $9a^4 + 26a^2b^2 + 25b^4$.

Twice the product of the square roots of $9a^4$ and $25b^4$ is $30a^2b^2$. We may add $4a^2b^2$ to the given expression and subtract $4a^2b^2$ from the sum.

$$\begin{aligned} 9a^4 + 26a^2b^2 + 25b^4 &= (9a^4 + 30a^2b^2 + 25b^4) - 4a^2b^2 \\ &= (3a^2 + 5b^2)^2 - (2ab)^2 \\ &= (3a^2 + 5b^2 + 2ab)(3a^2 + 5b^2 - 2ab) \\ &= (3a^2 + 2ab + 5b^2)(3a^2 - 2ab + 5b^2). \end{aligned}$$

4. Resolve into factors $x^4 + 4y^4$.

$$\begin{aligned} x^4 + 4y^4 &= (x^4 + 4x^2y^2 + 4y^4) - 4x^2y^2 \\ &= (x^2 + 2y^2)^2 - (2xy)^2 \\ &= (x + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy) \\ &= (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2). \end{aligned}$$

Many expressions may be resolved into more than two factors.

5. Resolve into factors $x^8 - y^8$.

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

Exercise 42

Resolve into factors:

1. $x^4 + 4x^2 + 16$.
2. $9x^4 + 2x^2y^2 + y^4$.
3. $4a^4 + 11a^2b^2 + 25b^4$.
4. $a^4 + 64b^4$.
5. $4x^4 - 16a^2x^2 + 9a^4$.
6. $4a^4 - 13a^2 + 1$.
7. $9x^4 + 15x^2y^2 + 16y^4$.
8. $25a^4 + 71a^2x^2 + 64x^4$.
9. $49x^2 - 44x^2y^2 + 4y^4$.
10. $81a^4 + 45a^2b^2 + 49b^4$.
11. $4a^4 - 29a^2x^2 + 25x^4$.
12. $121a^4 + 54a^2y^2 + 81y^4$.
13. $49 - 221x^2 + 100x^4$.
14. $49a^4 + 6a^2c^4 + 9c^8$.
15. $81c^4 - 115c^2 + 25$.
16. $121x^4 + 29x^2y^2 + 25y^4$.
17. $144a^4 + 47a^2c^2 + 49c^4$.
18. $36x^4 - 141x^2y^2 + 25y^4$.
19. $49x^4 - 121x^2y^2 + 64y^4$.
20. $49a^4b^4 + 61a^2b^2y^2 + 169y^4$.
21. $121x^4 - 302x^2y^2 + 169y^4$.
22. $4a^4 + 4a^2x^2 + 25x^4$.

23. $a^4 - 16b^4$.

24. $x^8 - 256y^8$.

25. $16c^4 - 81d^4$.

26. $256x^4 - 625y^4$.

27. $x^4 + 324y^4$.

28. $81a^4 - 207a^2b^2 + 49b^4$.

29. $64a^4b^4 + 81x^4 + 63a^2b^2x^2$.

30. $100a^4 + 9b^4x^4 + 11a^2b^2x^2$.

31. $81a^8b^8 + 25x^4 - 171a^4b^4x^2$.

32. $121a^4 + 144x^8y^4 + 260a^2x^4y^2$.

33. $49a^4 - 193a^2b^2 + 64b^4$.

34. $4b^4x^4 - 133b^2x^2y^4 + 9y^8$.

35. $36x^4y^4 + 11x^2y^2z^2 + 25z^4$.

36. $64x^4y^8 + 79x^2y^4z^2 + 100z^4$.

37. $49a^4b^4 - 53a^2b^2x^2 + 4x^4$.

38. $121x^4 - 295x^2y^2 + 169y^4$.

39. $289x^4y^4 + 21x^2y^2d^2 + 9d^4$.

40. $361x^4 - 92a^2c^2x^2 + 4a^4c^4$.

41. $4m^4n^8 + 3a^2m^2n^4 + 9a^4$.

42. $9a^4 + 5a^2x^2 + 25x^4$.

43. $49m^4 + 110m^2n^2 + 81n^4$.

44. $49x^4 - 15x^2y^2 + 121y^4$.

45. $64a^4 + 128a^2b^2c^2 + 81b^4c^4$.

46. $4m^4n^4 - 37m^2n^2x^2 + 9x^4$.

140. The factors of the sum of the squares of any two numbers may be found by the processes of § 139, but *surds* are often involved in the factors.

1. Resolve into two factors $x^2 + y^2$.

$$\begin{aligned} x^2 + y^2 &= (x^2 + 2xy + y^2) - 2xy \\ &= (x + y)^2 - (\sqrt{2xy})^2 \\ &= (x + y + \sqrt{2xy})(x + y - \sqrt{2xy}) \\ &= (x + \sqrt{2xy} + y)(x - \sqrt{2xy} + y). \end{aligned}$$

Exercise 43

Resolve into two factors:

1. $a^2 + 4b^2$.
2. $4x^2 + 9y^2$.
3. $25m^2 + 16n^2$.
4. $64c^2 + 49d^2$.
5. $36a^2 + 49b^2c^2$.
6. $9x^2 + 25y^2$.
7. $81m^2 + 4n^4$.
8. $9c^2 + 121d^2$.
9. $169a^2 + 4b^2c^2$.
10. $144x^2 + 49y^2z^2$.

141. When a Trinomial has the Form $x^2 + ax + b$.

Where a is the *algebraic sum* of two numbers, and is either positive or negative; and b is the *product* of these two numbers, and is either positive or negative.

Since $(x+3)(x+2) = x^2 + 5x + 6$,
the factors of $x^2 + 5x + 6$ are $x + 3$ and $x + 2$.

Since $(x-3)(x-2) = x^2 - 5x + 6$,
the factors of $x^2 - 5x + 6$ are $x - 3$ and $x - 2$.

Since $(x+3)(x-2) = x^2 + x - 6$,
the factors of $x^2 + x - 6$ are $x + 3$ and $x - 2$

Since $(x-3)(x+2) = x^2 - x - 6$,
the factors of $x^2 - x - 6$ are $x - 3$ and $x + 2$.

Hence, if a trinomial of the form $x^2 + ax + b$ is such an expression that it may be resolved into two binomial factors, the first term of each factor is x; and the second terms of the factors are two numbers *whose product is b*, the last term of the trinomial, and *whose algebraic sum is a*, the coefficient of x in the middle term of the trinomial.

1. Resolve into factors $x^2 + 9x + 20$.

We are required to find two numbers whose product is 20 and whose sum is 9.

Two numbers whose product is 20 are 1 and 20, 2 and 10, 4 and 5; and the sum of the last two numbers is 9. Hence,

$$x^2 + 9x + 20 = (x+4)(x+5).$$

2. Resolve into factors $x^2 - 11x + 24$.

We are required to find two numbers whose product is 24 and whose algebraic sum is -11.

Since the product is $+24$, the two numbers are *both positive* or *both negative;* and since their sum is -11, both numbers must be negative.

Two negative numbers whose product is 24 are -1 and -24, -2 and -12, -3 and -8, -4 and -6; and the sum of the third pair is -11. Hence,

$$x^2 - 11x + 24 = (x-3)(x-8).$$

3. Resolve into factors $x^2 + 5x - 24$.

We are required to find two numbers whose product is -24 and whose algebraic sum is $+5$.

Since the product is -24, one of the numbers is positive and the other negative; and since the sum is $+5$, the larger number is positive.

Two numbers whose product is -24, and the larger number positive, are 24 and -1, 12 and -2, 8 and -3, 6 and -4; and the sum of the third pair of numbers is $+5$. Hence,

$$x^2 + 5x - 24 = (x + 8)(x - 3).$$

4. Resolve into factors $x^2 - x - 20$.

We are required to find two numbers whose product is -20 and whose algebraic sum is -1.

Since the product is -20, one of the numbers is positive and the other negative; and since the sum is -1, the larger number is negative.

Two numbers whose product is -20, and the larger number negative, are -20 and 1, -10 and 2, -5 and 4; and the sum of the last two numbers is -1. Hence,

$$x^2 - x - 20 = (x - 5)(x + 4).$$

5. Resolve into factors $x^2 - 9xy + 20y^2$.

We are required to find two numbers whose product is $20y^2$ and whose algebraic sum is $-9y$.

Since the product is $+20y^2$ and the sum $-9y$, the second terms of the factors must both be negative.

Two negative numbers whose product is $20y^2$ are $-y$ and $-20y$, $-2y$ and $-10y$, $-4y$ and $-5y$; and the sum of the last two numbers is $-9y$. Hence,

$$x^2 - 9xy + 20y^2 = (x - 4y)(x - 5y).$$

142. From these examples it will be seen that the following statements are true:

1. If the *third* term of a trinomial of the form $x^2 + ax + b$ is *negative*, the *second* terms of the binomial factors have *unlike signs*.

2. If the *third* term of a trinomial of the form $x^2 + ax + b$ is *positive*, the *second* terms of the binomial factors have the *same sign*, and this sign is the *sign of the middle term* of the trinomial.

Exercise 44

Resolve into factors:

1. $x^2 + 11x + 24$.
2. $x^2 + 5x - 14$.
3. $a^2 - 3a - 40$.
4. $c^2 - 12c + 35$.
5. $y^2 - 4y - 12$.
6. $x^2 - 10x + 16$.
7. $a^2 + 10a + 24$.
8. $a^2 + 2a - 15$.
9. $a^2 - 3a - 28$.
10. $x^2 - 10x + 21$.
11. $x^2 - x - 30$.
12. $b^2 + 11b + 30$.
13. $d^2 + 11d + 28$.
14. $a^2 - 15a + 54$.
15. $x^2 + 3x - 54$.
16. $x^2 - 4x - 77$.
17. $c^2 + 3c - 40$.
18. $c^2 - 14c + 45$.
19. $x^2 + 13x + 40$.
20. $x^2 - 3x - 54$.
21. $x^2 - 16x + 63$.
22. $a^2 + 4a - 60$.
23. $a^2 - 7a - 18$.
24. $x^2 + 15x + 44$.
25. $x^2 - 20xy + 96y^2$.
26. $x^2 + 8xy - 65y^2$.
27. $x^2 + 17xy + 66y^2$.
28. $c^2 - 4cd - 96d^2$.
29. $c^2 + 17cd + 72d^2$.
30. $x^2 - xy - 72y^2$.
31. $y^2 - 18yz + 72z^2$.
32. $m^2 + 6mn - 27n^2$.
33. $a^2 + 3ab - 154b^2$.
34. $x^2 - 20xy + 91y^2$.
35. $a^2 + 18ad + 56d^2$.
36. $x^2 - 4xz - 45z^2$.
37. $x^2 + 20xy + 75y^2$.
38. $x^2 - 38xy + 105y^2$.
39. $m^2 + 10mn - 24n^2$.
40. $c^2 - 8cd - 105d^2$.
41. $a^2 + 13ab - 140b^2$.
42. $a^2 + 22ac + 96c^2$.
43. $x^2 - 13xyz - 90y^2z^2$.
44. $a^2 - 23abc + 120b^2c^2$.
45. $a^2 - 17abc + 72b^2c^2$.
46. $m^2 + 5mnp - 84n^2p^2$.
47. $x^2y^2 + 30xyz + 209z^2$.
48. $a^2b^2 - abc - 110c^2$.

143. When a Trinomial has the Form $ax^2 + bx + c$.

Where b and c are either positive or negative.

1. Resolve into factors $6x^2 - 11x - 10$.

Multiply the given trinomial by 6, the coefficient of x^2, and write the result in the following form:

$$(6x)^2 - 11(6x) - 60.$$

Put z for $6x$, $\qquad z^2 - 11z - 60.$

Resolve this trinomial into two binomial factors,

$$(z - 15)(z + 4). \qquad \text{(p. 96, § 141)}$$

Since we have multiplied by 6, and put z for $6x$, we must reverse this process. Hence, put $6x$ for z and divide by 6, and we have

$$\frac{(6x - 15)(6x + 4)}{6}.$$

Since 3 is a factor of $6x - 15$ and 2 is a factor of $6x + 4$, we divide by 6 by dividing the first factor by 3 and the second factor by 2.

Then $\qquad \frac{(6x - 15)(6x + 4)}{6} = (2x - 5)(3x + 2).$

Therefore, $\qquad 6x^2 - 11x - 10 = (2x - 5)(3x + 2).$

2. Resolve into factors $35x^2 - 83xy + 36y^2$.

Multiply by 35, $\qquad (35x)^2 - 83y(35x) + 1260y^2.$
Put z for $35x$, $\qquad z^2 - 83yz + 1260y^2.$
Resolve into factors, $\qquad (z - 20y)(z - 63y).$ $\qquad$ (p. 96, § 141)
Put $35x$ for z, $\qquad (35x - 20y)(35x - 63y).$
Divide by 5×7, $\qquad (7x - 4y)(5x - 9y).$

Therefore, $35x^2 - 83xy + 36y^2 = (7x - 4y)(5x - 9y).$

Exercise 45

Resolve into factors:

1. $6x^2 - 19x + 15.$
2. $6x^2 + 17x + 12.$
3. $12x^2 + 7x - 12.$
4. $15x^2 - 29x - 14.$
5. $12x^2 + 8x - 15.$
6. $20x^2 - 41x + 20.$

7. $6x^2 - 17x - 14$.

8. $6x^2 + 13x + 6$.

9. $8x^2 + 2x - 15$.

10. $14x^2 + 31xy - 10y^2$.

11. $6x^2 + 11xy - 35y^2$.

12. $21x^2 + 26xy - 15y^2$.

13. $42x^2 - 59xy + 20y^2$.

14. $21x^2 - 37xy + 12y^2$.

15. $14x^2 + 53xy + 14y^2$.

16. $21x^2 - 40xy + 16y^2$.

17. $30x^2 + 43xy + 15y^2$.

18. $21x^2 + 41xy + 10y^2$.

19. $21x^2 + 29xy - 10y^2$.

20. $10x^2 + 19xy - 15y^2$.

21. $6a^2 + 11ab - 35b^2$.

22. $12c^2 - 11cd - 15d^2$.

23. $12m^2 + mn - 20n^2$.

24. $12y^2 + 25yz + 12z^2$.

25. $15a^2 - 14ab - 8b^2$.

26. $15a^2 - 16ab + 4b^2$.

27. $30c^2 - cd - 20d^2$.

28. $30c^2 - 11cd - 30d^2$.

29. $10a^2 - 51ab + 56b^2$.

30. $15b^2 + 2bc - 45c^2$.

31. $12a^2 - 25ab - 50b^2$.

32. $6m^2 + 11mn - 72n^2$.

33. $15x^2 + 44xy - 20y^2$.

34. $20x^2 - xy - 99y^2$.

144. A trinomial of the form $ax^2 + bx + c$ can sometimes be more easily resolved into factors by the method of *completing the square.* As will be explained in Chap. XX, the binomial $x^2 + mx$, where m is any number, positive or negative, integral or fractional, becomes a perfect trinomial square if we add $(\frac{1}{2}m)^2$, *the square of half the coefficient of* x.

1. Resolve into factors $6x^2 - 11x - 10$.

Divide the given trinomial by 6, the coefficient of x^2, and we have

$$x^2 - \tfrac{11}{6}x - \tfrac{5}{3}.$$

We now *complete the square* of the first two terms. Half the coefficient of x is $\frac{1}{2}$ of $\frac{11}{6}$, or $\frac{11}{12}$; and the square of $\frac{11}{12}$ is $\frac{121}{144}$. Add $\frac{121}{144}$ to the first two terms; it is necessary, therefore, to *subtract* $\frac{121}{144}$ from the third term.

Then we have $(x^2 - \frac{11}{6}x + \frac{121}{144}) - (\frac{5}{3} + \frac{121}{144})$.

This may be written $(x - \frac{11}{12})^2 - (\frac{19}{12})^2$.

We now have the difference of two squares, and resolve the expression into two factors,

$$(x - \tfrac{11}{12} + \tfrac{19}{12})(x - \tfrac{11}{12} - \tfrac{19}{12}),$$

or

$$(x + \tfrac{2}{3})(x - \tfrac{5}{2}).$$

Since we have divided by 6, we must multiply by 6, multiplying the first factor by 3 and the second factor by 2.

Then we have $(3x + 2)(2x - 5)$.

Therefore, $6x^2 - 11x - 10 = (3x + 2)(2x - 5)$.

Exercise 46

Resolve into factors:

1. $6x^2 - 19x + 15$.
2. $6x^2 + 17x + 12$.
3. $12x^2 + 7x - 12$.
4. $12x^2 + 8x - 15$.
5. $20x^2 - 41x + 20$.
6. $6x^2 + 13x + 6$.
7. $8x^2 + 2x - 15$.
8. $6x^2 + 11xy - 35y^2$.
9. $30x^2 + 43xy + 15y^2$.
10. $10x^2 + 19xy - 15y^2$.
11. $12c^2 - 11cd - 15d^2$.
12. $12m^2 + mn - 20n^2$.
13. $12y^2 + 25yz + 12z^2$.
14. $15a^2 - 16ab + 4b^2$.
15. $30c^2 - cd - 20d^2$.
16. $30c^2 - 11cd - 30d^2$.
17. $10a^2 - 51ab + 56b^2$.
18. $15x^2 + 44xy - 20y^2$.

145. When a Binomial is the Sum or the Difference of Two Cubes.

Since $\dfrac{a^3 + b^3}{a + b} = a^2 - ab + b^2$, (p. 83, § 125)

therefore, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$. (1)

Since $\dfrac{a^3 - b^3}{a - b} = a^2 + ab + b^2$, (p. 83, § 125)

therefore, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. (2)

Therefore, *the sum of two perfect cubes is divisible by the sum of their cube roots, and the difference of two perfect cubes is divisible by the difference of their cube roots.*

1. Resolve into factors $27x^3 + 125y^3$.

The cube root of $27x^3$ is $3x$, and the cube root of $125y^3$ is $5y$.
Put $3x$ for a, and $5y$ for b in (1),

$$27x^3 + 125y^3 = (3x + 5y)(9x^2 - 15xy + 25y^2).$$

2. Resolve into factors $343x^3 - 64y^3$.

The cube root of $343x^3$ is $7x$, and the cube root of $64y^3$ is $4y$.
Put $7x$ for a, and $4y$ for b in (2),

$$343x^3 - 64y^3 = (7x - 4y)(49x^2 + 28xy + 16y^2).$$

Exercise 47

Resolve into factors:

1. $27x^3 + 343y^3$.
2. $216a^3 - 1331b^3$.
3. $125a^3b^3 + 216c^3d^3$.
4. $729x^3y^3 + 512a^3b^3$.
5. $125x^3 - 343y^3z^3$.
6. $729a^3 - 343b^3$.
7. $8x^6 + 27y^9$.
8. $27x^6 - 64y^6z^9$.
9. $1000x^3 - 1331y^3$.
10. $1331m^3 + 1728n^3$.
11. $27 + a^3b^6c^9$.
12. $216x^3y^3z^3 - 125$.
13. $(a + 2b)^3 - 1$.
14. $(a - 2b)^3 + 1$.
15. $1 - (x - y)^3$.
16. $x^6 + y^6$.
17. $a^9 - y^9$.
18. $27a^3 - (a - b)^3$.
19. $64a^3 + (a + b)^3$.
20. $(x^2 - 5)^3 - y^6$.
21. $(x^3 + 4)^3 + y^3$.
22. $(a + b)^3 - (a - b)^3$.
23. $64(x + y)^3 + 27(x - y)^3$.
24. $(5a - 2b)^3 + 8c^3$.
25. $x^{21}y^{18} + a^{15}b^{12}$.
26. $(2a - 3b)^3 - (x - 4y)^3$.
27. $x^3 - (y - 2z)^3$.
28. $(x + y)^3 + (x - y)^3$.

Theory of Divisors

146. Theorem. *The expression* x − y *is an exact divisor of* $x^n - y^n$ *when* n *is any positive integer.*

Since $-x^{n-1}y + x^{n-1}y = 0,$ (p. 9, § 35)

$$x^n - y^n = x^n - x^{n-1}y + x^{n-1}y - y^n.$$

Taking out x^{n-1} from the first two terms of the right side, and y from the last two terms, we have

$$x^n - y^n = x^{n-1}(x - y) + y(x^{n-1} - y^{n-1}).$$

Now $x - y$ is an exact divisor of the right side, if it is an exact divisor of $x^{n-1} - y^{n-1}$; and if $x - y$ is an exact divisor of the right side, it is an exact divisor of the left side; that is, $x - y$ is an exact divisor of $x^n - y^n$ *if* it is an exact divisor of $x^{n-1} - y^{n-1}$.

Therefore, *if* x − y *is an exact divisor of the difference of any two like powers of* x *and* y, *it is an exact divisor of the difference of the next higher powers of* x *and* y.

But $x - y$ *is* an exact divisor of $x^3 - y^3$ (p. 83, § 125), therefore it is an exact divisor of $x^4 - y^4$; and since it is an exact divisor of $x^4 - y^4$, it is an exact divisor of $x^5 - y^5$; and so on, indefinitely.

Therefore, $x - y$ is an exact divisor of $x^n - y^n$.

The method employed in proving this Theorem is called **Proof by Mathematical Induction**

147. The Factor Theorem. *If a rational and integral expression in* x *vanishes, that is, becomes equal to* 0, *when* r *is put for* x, *then* x − r *is an exact divisor of the expression.*

Given $ax^n + bx^{n-1} + \cdots + hx + k.$ (1)

By supposition, $ar^n + br^{n-1} + \cdots + hr + k = 0.$ (2)

By subtracting the left member of (2) from (1), the given expression becomes $a(x^n - r^n) + b(x^{n-1} - r^{n-1}) + \cdots + h(x - r).$

But $x - r$ is an exact divisor of $x^n - r^n$, $x^{n-1} - r^{n-1}$, and so on. (§ 146)

Therefore, $x - r$ is an exact divisor of the given expression.

Note. If $x - r$ is an exact divisor of the given expression, r is an exact divisor of k; for k, the last term of the dividend, is equal to r, the last term of the divisor, multiplied by the last term of the quotient.

Therefore, in searching for numerical values of x that will make the given expression vanish, only exact divisors of the last term of the expression need be tried.

1. Resolve into factors $x^3 + 2x^2 - 11x - 12$.

The exact divisors of -12 are $1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12$.

If we put 1 for x in $x^3 + 2x^2 - 11x - 12$, the expression does not vanish. If we put -1 for x, the expression vanishes.

Therefore, $x - (-1)$, that is $x + 1$, is a factor.

Divide the given expression by $x + 1$, and we have

$$x^3 + 2x^2 - 11x - 12 = (x+1)(x^2 + x - 12)$$
$$= (x+1)(x-3)(x+4).$$

2. Resolve into factors $x^3 - 48x - 7$.

By trial we find that the only exact divisor of -7 that makes the expression vanish is $+7$.

Therefore, divide by $x - 7$, and we have

$$x^3 - 48x - 7 = (x-7)(x^2 + 7x + 1).$$

As neither $+1$ nor -1, the exact divisors of $+1$, will make $x^2 + 7x + 1$ vanish, this expression cannot be resolved into factors.

Exercise 48

Resolve into factors:

1. $x^3 + 4x^2 + x - 6$.
2. $a^3 - 7a^2 + 16a - 12$.
3. $c^3 - 8c + 3$.
4. $2x^3 - x^2 - 13x - 6$.
5. $x^3 - 19x + 12$.
6. $6x^3 - 11x^2 + 6x - 1$.
7. $x^3 - 9x^2 + 26x - 24$.
8. $2c^3 + 9c^2 + 12c + 4$.
9. $a^3 - 5a^2 - 2a + 24$.
10. $x^3 - 2x^2 - 15x + 36$.
11. $m^3 + m^2 - 22m - 40$.
12. $n^3 - n^2 - 8n + 12$.
13. $6x^5 + 19x^4 - 20x^3 - 65x^2 + 24x + 36$.
14. $6x^5 + x^4 - 111x^3 + 66x^2 + 448x - 480$.
15. $4x^5 - 22x^4 + 17x^3 + 83x^2 - 152x + 60$.

148. A compound expression involving x and y is divisible by $x-y$ if the expression vanishes when $+y$ is put for x; and is divisible by $x+y$ if the expression vanishes when $-y$ is put for x.

If n is a positive integer, prove by the Factor Theorem:

1. $x^n + y^n$ is never divisible by $x - y$.

Put y for x in $x^n + y^n$; then $x^n + y^n = y^n + y^n = 2y^n$.
As $2y^n$ is not zero, $x^n + y^n$ is not divisible by $x - y$.

2. $x^n - y^n$ is divisible by $x + y$, **if n is even.**

Put $-y$ for x in $x^n - y^n$, then $x^n - y^n = (-y)^n - y^n$.
If n is even, $(-y)^n = y^n$, and $(-y)^n - y^n = y^n - y^n$.
As $y^n - y^n = 0$, $x^n - y^n$ is divisible by $x + y$, **if n is even.**

3. $x^n + y^n$ is divisible by $x + y$, **if n is odd.**

Put $-y$ for x in $x^n + y^n$, then $x^n + y^n = (-y)^n + y^n$.
If n is odd, $(-y)^n = -y^n$, and $(-y)^n + y^n = -y^n + y^n$.
As $-y^n + y^n = 0$, $x^n + y^n$ is divisible by $x + y$, **if n is odd.**

From p. 103, § 146, and these three cases, we have

1. For all positive integral values of n,

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}).$$

2. For all positive *even* integral values of n,

$$x^n - y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - y^{n-1}).$$

3. For all positive *odd* integral values of n,

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1}).$$

4. $x^n + y^n$ is never divisible by $x - y$; and is not divisible by $x + y$, *if* n *is even.*

Note. In applying the preceding rules for resolving an expression into factors, *if the terms have a common monomial factor*, this factor should first be removed.

When an expression can be expressed as the *difference of two perfect squares*, the method of the difference of two perfect squares should be employed in preference to any other.

Exercise 49. Review

Resolve into factors:

1. $a^3bc^2 - a^2bc^2x - a^2b^2c^2$.
2. $6ax - 15by + 9bx - 10ay$.
3. $9a^2 - 24ac + 16c^2$.
4. $49x^2 - 81y^2$.
5. $a^2 - 25c^2 + 9b^2 + 6ab$.
6. $16a^4 + 25b^4 + 24a^2b^2$.
7. $x^2 + 2xy - 99y^2$.
8. $24x^2 + 2xy - 35y^2$.
9. $x^6 + 8y^3$.
10. $x^3 + 8x^2 + 20x + 16$.
11. $15x^2 - 76xy + 77y^2$.
12. $a^4y^2 + a^3xy^2 + a^2y^4$.
13. $4x^2 + 28xy + 49y^2$.
14. $36x^4 + 49y^4 - 100x^2y^2$.
15. $4x^2 + 25y^2 - 16z^2 - 20xy$.
16. $36a^2 - 121b^2$.
17. $x^2 - 20x + 51$.
18. $6x^2 - 13xy - 63y^2$.
19. $289a^4 + 151a^2b^2 + 64b^4$.
20. $x^2 + 19xy + 70y^2$.
21. $25a^2 - 40abc + 16b^2c^2$.
22. $64x^4y^4 - 49y^4z^4$.
23. $81a^4 + 64b^4 - 225a^2b^2$.
24. $729x^3 - 1000y^3$.
25. $25m^2 + 16n^2 - 64c^2 - 40mn$.
26. $144m^4 + 68m^2n^4 + 49n^8$.
27. $a^4 - 5a^3 - 84a^2$.
28. $4x^2 + 10ab - 5bx - 8ax$.
29. $32a^2 + 4ab - 15b^2$.
30. $144m^2n^2 - 121p^2$.
31. $c^3 + 23c^2 + 102c$.
32. $36m^2 - 31mn - 56n^2$.
33. $b^2x^5 - b^2x^3 + 2b^2x^2$.
34. $121c^4 + 144d^4 + 183c^2d^2$.
35. $49x^4 - 48yz - 9y^2 - 64z^2$.
36. $81a^4 - 256b^4$.
37. $m^2 - 16mn + 60n^2$.
38. $2y^2 + 3dy - 14cy - 21cd$.
39. $27a^3 - 1331y^3$.
40. $49x^4 - 263x^2y^2 + 169y^4$.
41. $y^2 - 9yz - 112z^2$.
42. $36x^2 - 84xyz + 49y^2z^2$.
43. $49a^4 + 3a^2b^2 + 4b^4$.
44. $9a^{4n} - 25b^{2n}$.
45. $343x^3 + 125y^3$.
46. $y^5 + 5y^4 - 36y^3$.
47. $x^4y^2 + x^3y^3 - x^3y^2z$.
48. $81a^2b^2 + 90abc + 25c^2$.

49. $64b^4 - 176b^2y^2 + 49y^4$.

50. $c^2 + cd - 110d^2$.

51. $500a^3 + 864b^3$.

52. $d^6 + 21d^5 + 90d^4$.

53. $22c^2 + 37cd - 45d^2$.

54. $x^2 - 17xy + 60y^2$.

55. $c^2 + 6abdx - 2acx - 3bcd$.

56. $36a^2 + 42bc - 9b^2 - 49c^2$.

57. $25x^4 - 81y^2$.

58. $64b^2c^2 + 144bcx^2 + 81x^4$.

59. $256x^4 - 705x^2z^2 + 225z^4$.

60. $x^2 - 7xy - 44y^2$.

61. $1728a^3 - 125x^3$.

62. $36a^4b^4 - 121b^4c^4$.

63. $a^2 + 13ab + 30b^2$.

64. $4m^4 + 8m^2n^2 + 121n^4$.

65. $14x^2 + 61xy + 42y^2$.

66. $144x^2 - 264xy^2 + 121y^4$.

67. $20a^2 - 49ac + 30c^2$.

68. $a^5b - 3a^4b^2 + 3a^3b^3$.

69. $25a^4 - 211a^2y^2 + 81y^4$.

70. $144a^2b^2 - 25c^2d^2$.

71. $x^9 - 512y^9$.

72. $2mx + 4nx - 10an - 5am$.

73. $x^2 - 5xy - 36y^2$.

74. $49x^2 - 169y^2z^2$.

75. $99x^2 - 17xy - 12y^2$.

76. $81x^2 - 121a^2 + 176ab - 64b^2$.

77. $64x^6 + 125y^9$.

78. $81x^2y^2 - 180xy^2z + 100y^2z^2$.

79. $x^2 - 12xy + 35y^2$.

80. $225a^3b - 256ab^3$.

81. $64x^4 - 73x^2y^2 + 9y^4$.

82. $6x^3 - 8ax + 20ab - 15bx^2$.

83. $24x^2 - 86xy + 42y^2$.

84. $100a^4 + 36a^2b^2 + 81b^4$.

85. $a^2 + 8ab - 33b^2$.

86. $6a^2 + 25ab - 91b^2$.

87. $16x^6y^2 - 40x^4y^4 + 25x^2y^6$.

88. $343a^3 + (3x + 5y)^3$.

89. $196x^4y^2 - 225x^2y^4$.

90. $a^4b^2y + a^3b^3x + ab^4z$.

91. $81x^4 - 18x^2y^2 + 49y^4$.

92. $49x^2 + 56xy + 16y^2 - 16a^2$.

93. $121a^2mn - 169mny^2$.

94. $x^2 + 16xy + 28y^2$.

95. $49a^5b - 154a^3b^3 + 121ab^5$.

96. $8a^6 - 6a^4b^2 - 35a^2b^4$.

97. $20ax + 6by - 15ay - 8bx$.

98. $169a^8 + 9a^4b^2 + 81b^4$.

99. $16a^4c^2 + 88a^3c^3 + 121a^2c^4$.

100. $x^2 + xy - 30y^2$.

101. $100a^3x - 484ax^3$.

102. $a^4x^2 - 3a^3x^3 + 6a^2x^4$.

103. $a^{2m+2} - a^2c^{2n}$.

104. $12x^4 - x^2y^2 - 35y^4$.

105. $x^4 - 15x^2y + 56y^2$.

106. $729 - 8(4y - 7z)^3$.

107. $a^7b - ab^5$.

108. $x^3y - 2x^2y^2 - 99xy^3$.

109. $x^2 - 18xy + 45y^2$.

110. $24x^2 + 118xy + 45y^2$.

111. $b^4c^2 - 7b^3c^3 + 4b^2c^4$.

112. $144x^3y - 1089xy^3$.

113. $25x^4y^2z^2 + 110x^2yz^3 + 121z^4$.

114. $196x^4 - 221x^2y^2 + 25y^4$.

115. $10ax + 45ay - 4bx - 18by$.

116. $16x^2 + 72a^2b^2x + 81a^4b^4$.

117. $18m^4 - 53m^2n^2 - 35n^4$.

118. $(2x + 5y)^3 - (3y - 7z)^3$.

119. $625a^4 - 1300a^2b^2 + 324b^4$.

120. $36x^6 - 156x^4y^2 + 169x^2y^4$.

121. $x^4 + 21x^2y + 38y^2$.

122. $144x^2 - 144ac - 81a^2 - 64c^2$.

123. $21ax - 28ay + 24by - 30bz + 35az - 18bx$.

124. $484x^8 - 664x^4y^2 + 225y^4$.

125. $x^4 + 4x^3y - 13x^2y^2 - 40x^3y + 48y^4$.

126. $x^2 - 4n^2 + 4mn - 6xy - m^2 + 9y^2$.

127. $(3a - 4b)^3 + (2x + 3y)^3$.

128. $12a^4b^2 + a^3b^3 - 35a^2b^4$.

129. $6x^4 + 49x^3y + 146x^2y^2 + 189xy^3 + 90y^4$.

130. $9a^4x^3 - 35a^2b^2xy^2 - 21a^4xy^2 + 15a^2b^2x^3$.

131. $9x^2 - 9a^2 + 64y^2 - 64b^2 + 48xy + 48ab$.

132. $15a^6 - 89a^5 + 152a^4 - 44a^3 - 48a^2$.

133. $16a^2 - 34ab + 15b^2$.

134. $(5a - 6b)^3 - (3x - 2y)^3$.

135. $16x^4 - 81b^2 - 25a^4 + 56x^2y - 90a^2b + 49y^2$.

136. $x^4 - 10x^3 + 18x^2 + x - 10$.

137. $6a^2 - 9ay - 15by - 20bx - 12ax + 10ab$.

138. $x^5 - 6x^4 - 3x^3 + 88x^2 - 204x + 144$.

139. $25a^2 - 36x^2 + 121b^2 - 49y^2 - 110ab - 84xy$.

CHAPTER VIII

COMMON FACTORS AND MULTIPLES

Highest Common Factor

149. Common Factors. A **common factor** of two or more integral and rational expressions is an integral and rational expression that divides each of them without a remainder.

Thus, $3xy$ is a common factor of $6x^3y$, $9x^2y^2$, and $12xy^3$.

150. Prime Expressions. Two expressions that have no common factor except 1 are said to be *prime* to each other.

Thus, $3xy$ and $5ab$ are prime to each other.

151. Highest Common Factor. The **highest common factor** of two or more integral and rational expressions is the integral and rational expression *of highest degree and greatest numerical coefficient* that will divide each of them without a remainder.

Thus, $5x^2$ is the highest common factor of $15x^3$, $20x^4$, and $25x^2$; $4x^2y^2$ is the highest common factor of $12x^3y^2$, $16x^2y^3$, and $20x^3y^3$.

For brevity we use H.C.F. for highest common factor.

1. Find the H.C.F. of $35a^3b^2$ and $21a^2b^3$.

$$35a^3b^2 = 5 \times 7 \times aaa \times bb;$$
$$21a^2b^3 = 3 \times 7 \times aa \times bbb.$$
$$\therefore \text{ the H.C.F.} = 7 \times aa \times bb = 7a^2b^2.$$

2. Find the H.C.F. of $a^2 + 2ax$ and $a^2 + 4ax + 4x^2$.

$$a^2 + 2ax = a(a + 2x);$$
$$a^2 + 4ax + 4x^2 = (a + 2x)(a + 2x).$$
$$\therefore \text{ the H.C.F.} = a + 2x.$$

3. Find the H.C.F. of $2x^3+2x^2-24x$; $3x^3-15x^2+18x$; x^3-2x^2-3x; $5x^3-30x^2+45x$.

$$2x^3+2x^2-24x = 2x(x^2+x-12) = 2x(x-3)(x+4);$$
$$3x^3-15x^2+18x = 3x(x^2-5x+6) = 3x(x-3)(x-2);$$
$$x^3-2x^2-3x = x(x^2-2x-3) = x(x-3)(x+1);$$
$$5x^3-30x^2+45x = 5x(x^2-6x+9) = 5x(x-3)(x-3).$$
$$\therefore \text{ the H.C.F.} = x(x-3).$$

Therefore,

152. To Find the H.C.F. of Two or More Expressions,

Resolve each expression into its prime factors.

The product of all the common factors, each factor being taken the least number of times it occurs in any of the given expressions, is the highest common factor required.

NOTE. The *highest common factor* in Algebra corresponds to the *greatest common measure*, or *greatest common divisor*, in Arithmetic. We cannot apply the terms *greatest* and *least* to algebraic expressions in which particular values have not been given to the letters contained in the expressions. Thus, a is *greater* than a^2, if a stands for $\frac{1}{2}$; but a is of *lower degree* than a^2.

EXERCISE 50

Find the H.C.F. of:

1. $27x^3$ and $45x^4$.
2. $50a^2b^3$ and $75a^3b^2$.
3. $24a^3x^4$ and $36ax^3$.
4. $108a^2c^4$ and $84ac^3$.
5. $99x^4y^4$ and $66x^3y^2$.
6. $144x^5$ and $108x^7$.
7. x^2-y^2 and $x+y$.
8. $6(x+y)^2$ and $9(x+y)^3$.
9. $a+b$ and a^3+b^3.
10. a^3-8x^3 and $a-2x$.
11. x^2-9 and x^2-4x+3.
12. a^2-16 and a^2-2a-8.
13. $x^2-7x+12$ and x^2-5x+6.
14. a^2+a-6 and $a^2+8a+15$.

15. $a^2 + ab - 12\,b^2$ and $a^2 + 6\,ab + 8\,b^2$.

16. $x^3 + ax^2 - 6\,a^2x$ and $x^4 + 5\,ax^3 + 6\,a^2x^2$.

17. $8\,x^2 - 2\,xy - 3\,y^2$ and $12\,x^2 - xy - 6\,y^2$.

18. $6\,a^2 - ab - 2\,b^2$ and $4\,a^2 - 4\,ab - 3\,b^2$.

19. $12\,a^2 - 23\,ax + 10\,x^2$ and $8\,a^2 - 22\,ax + 15\,x^2$.

20. $12\,x^2 - 34\,x + 14$ and $42\,a - 4\,ax - 6\,ax^2$.

21. $x^4 - a^4$; $x^3 + a^3$; $x^2 - a^2$.

22. $x^4 + a^3x - ax^3 - a^4$ and $x^3 - a^3$.

23. $2\,x^4 - 11\,x^2y^2 + 12\,y^4$ and $3\,x^6 - 48\,x^2y^4$.

24. $36\,x^6 - 18\,x^5 - 27\,x^4 + 9\,x^3$ and $27\,x^2y^2 - 18\,x^4y^2 - 9\,x^3y^2$.

25. $x^5 + y^5$; $x^4 - y^4$; $x^3 + y^3$.

26. $x^2 - 2\,a^2 - ax$; $x^2 - 6\,a^2 + ax$; $x^2 - 8\,a^2 + 2\,ax$.

27. $30\,a^2b^4 - 25\,a^3b^3 + 5\,a^4b^2$ and $9\,ab^3 - 9\,a^2b^2 + 2\,a^3b$.

28. $x^3 + 3\,x^2y + 3\,xy^2 + y^3$ and $x^2 + 2\,xy + y^2$.

29. $6\,x^4y - 13\,x^3y^2 + 6\,x^2y^3$ and $8\,x^3y^2 - 14\,x^2y^3 + 3\,xy^4$.

Lowest Common Multiple

153. Common Multiples. A **common multiple** of two or more integral and rational expressions is an integral and rational expression that is exactly divisible by each of the expressions.

Thus, $15\,x^3y^3$ is a common multiple of $5\,x^2y$ and $3\,xy^2$.

154. Lowest Common Multiple. The **lowest common multiple** of two or more integral and rational expressions is the integral and rational expression *of lowest degree and of smallest numerical coefficient* that is exactly divisible by each of the given expressions.

Thus, $10\,x^2y^2$ is the lowest common multiple of $5\,x^2y$ and $10\,xy^2$.

For brevity we use L.C.M. for lowest common multiple.

1. Find the L.C.M. of $10\,a^3b$; $15\,a^2b^2$; $25\,ab^3$.

$$10\,a^3b = 2 \times 5 \times aaa \times b;$$
$$15\,a^2b^2 = 3 \times 5 \times aa \times bb;$$
$$25\,ab^3 = 5 \times 5 \times a \times bbb.$$

The L.C.M. must evidently contain each factor the greatest number of times that it occurs in any expression.

$$\therefore \text{ the L.C.M.} = 2 \times 3 \times 5 \times 5 \times aaa \times bbb = 150\,a^3b^3.$$

2. Find the L.C.M. of $6\,ax^2 - 18\,ax + 12\,a$; $2\,x^3 - 8\,x^2 + 6\,x$; $4\,bx^2 - 20\,bx + 24\,b$.

$$6\,ax^2 - 18\,ax + 12\,a = 6\,a\,(x^2 - 3\,x + 2) = 2 \times 3\,a\,(x-1)(x-2);$$
$$2\,x^3 - 8\,x^2 + 6\,x = 2\,x\,(x^2 - 4\,x + 3) = 2\,x\,(x-1)(x-3);$$
$$4\,bx^2 - 20\,bx + 24\,b = 4\,b\,(x^2 - 5\,x + 6) = 2 \times 2\,b\,(x-2)(x-3).$$
$$\therefore \text{ the L.C.M.} = 2^2 \times 3\,abx\,(x-1)(x-2)(x-3)$$
$$= 12\,abx\,(x-1)(x-2)(x-3).$$

Therefore,

155. To Find the L.C.M. of Two or More Expressions,

Resolve each expression into its prime factors.

The product of all the different factors, each factor being taken the greatest number of times it occurs in any of the given expressions, is the lowest common multiple required.

Exercise 51

Find the L.C.M. of:

1. a^3b^2 and a^2b^4.
2. $5\,xy^2$ and $3\,x^2y$.
3. $6\,a^2c^3$ and $9\,a^3c^4$.
4. $28\,x^2y^3$ and $35\,xy^4$.
5. $49\,a^4b^3$ and $56\,a^2b^2$.
6. $25\,a^6b^9$ and $60\,a^3b^5$.
7. $x - 1$ and $x + 1$.
8. $a^2 + 2\,b$ and $a^2 - 2\,b$.
9. a^3 and $a^3 + a^2$.
10. $x^2 + xy$ and $xy + y^2$.
11. $1 - y^2$ and $1 + y^3$.
12. $a^3 - b^3$ and $a^2 - b^2$.

13. $x^2 - 7x + 12$ and $x^2 - 5x + 6$.

14. $x^2 - x - 6$ and $x^2 + 6x + 8$.

15. $x^2 + 15x + 36$ and $x^2 - 6x - 27$.

16. $x^4y - 2x^3y^2 - 35x^2y^3$ and $x^3y^2 - 5x^2y^3 - 14xy^4$.

17. $x^2 + ax - 6a^2$ and $x^2 + 2ax - 8a^2$.

18. $6x^2 + xy - 12y^2$ and $6x^2 - 17xy + 12y^2$.

19. $15a^2 + 2ab - 24b^2$ and $10a^2 + 13ab - 30b^2$.

20. $36a^3b + 39a^2b^2 - 42ab^3$ and $48a^2b + 156ab^2 + 126b^3$.

21. $a^3 - x^3$; $a - x$; $a^2 - 2ax + x^2$; $a^2 - x^2$.

22. $(x^2 + 2ax)^2$; $x^3 - 4a^2x$; $(x - a)^3$; $(x^2 + ax - 2a^2)^2$.

23. $x(xy - y^2)$; $x^4y^2 - x^2y^4$; $y(x^2 - xy)^2$.

24. $x^4 + ax^3 - a^2x^2 + 2a^3x$ and $2x^3 + 6ax^2 + 2a^2x - 4a^3$.

25. $25x^3y - 20x^2y + 4xy$; $10x^2 - 29x + 10$; $125x^3 - 8$.

26. $x^3 - 8y^3$; $x^2 - 4y^2$; $x^2 + 3xy + 2y^2$; $x^2 - xy - 2y^2$.

27. $12x^3-16x^2-16x$; $12x^3+6x^2-36x$; $12x^3+24x^2+4x+8$.

156. In finding the H.C.F. and the L.C.M. of two or more algebraic expressions the chief difficulty consists in resolving the given expressions into factors.

When it is required to find the H.C.F. of two algebraic expressions that cannot readily be resolved into factors, we arrange the two given expressions in descending powers of a common letter, and divide the expression which is of higher degree in the common letter by the other expression. After the first division we take the remainder for a new divisor and the divisor for a new dividend, and so proceed until there is no remainder. The last divisor is the highest common factor.

NOTE. If the two expressions are of the same degree in the common letter, either expression may be taken for the divisor.

1. Find the H.C.F. of $x^2 - 3x + 2$ and $2x^3 - 6x^2 + 5x - 2$.

$$
\begin{array}{l}
x^2 - 3x + 2)\,2x^3 - 6x^2 + 5x - 2\,(2x \\
\qquad\qquad\quad \underline{2x^3 - 6x^2 + 4x} \\
\qquad\qquad\qquad\qquad x - 2)\,x^2 - 3x + 2\,(x - 1 \\
\qquad\qquad\qquad\qquad\qquad\; \underline{x^2 - 2x} \\
\qquad\qquad\qquad\qquad\qquad\quad -x + 2 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{-x + 2}
\end{array}
$$

Therefore, the H.C.F. is $x - 2$.

NOTE. Each division is continued until the first term of the remainder is of lower degree than the first term of the divisor.

157. This method is of use only to obtain the *compound factor* of the H.C.F. *Monomial factors* of the given expressions should be taken out and the highest common factor of these monomial factors reserved to be multiplied into the compound factor obtained. Also at any stage of the operation a monomial factor of *either* expression may be taken out without affecting the compound factor. In many cases modifications of this method are needed.

1. Find the H.C.F. of

$$16a^3 + 12a^2 - 40a \text{ and } 24a^4 + 42a^3 - 18a^2 - 90a.$$

$$16a^3 + 12a^2 - 40a = 4a(4a^2 + 3a - 10).$$

$$24a^4 + 42a^3 - 18a^2 - 90a = 6a(4a^3 + 7a^2 - 3a - 15).$$

The H.C.F. of $4a$ and $6a$ is $2a$. Hence, we reserve $2a$ as a factor of the highest common factor sought.

$$
\begin{array}{l}
4a^2 + 3a - 10)\,4a^3 + 7a^2 - 3a - 15\,(a + 1 \\
\qquad\qquad\qquad \underline{4a^3 + 3a^2 - 10a} \\
\qquad\qquad\qquad\qquad 4a^2 + 7a - 15 \\
\qquad\qquad\qquad\qquad \underline{4a^2 + 3a - 10} \\
\qquad\qquad\qquad\qquad\qquad 4a - 5)\,4a^2 + 3a - 10\,(a + 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4a^2 - 5a} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 8a - 10 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{8a - 10}
\end{array}
$$

Therefore, the H.C.F. is $2a(4a - 5)$.

2. Find the H.C.F. of

$$6a^3 + a^2 - 5a - 2 \text{ and } 6a^3 + 5a^2 - 3a - 2.$$

$$\begin{array}{r} 6a^3 + a^2 - 5a - 2)\,6a^3 + 5a^2 - 3a - 2\,(1 \\ \underline{6a^3 + a^2 - 5a - 2} \\ 4a^2 + 2a \end{array}$$

The first division ends here, for $4a^2$ is of lower degree than $6a^3$. We take out the *simple factor* $2a$ from $4a^2 + 2a$, for $2a$ is not a factor of the given expressions, and its rejection can in no way affect the compound factor sought. We then proceed with $2a + 1$ for a divisor.

$$\begin{array}{r} 2a + 1)\,6a^3 + a^2 - 5a - 2\,(3a^2 - a - 2 \\ \underline{6a^3 + 3a^2} \\ -2a^2 - 5a \\ \underline{-2a^2 - a} \\ -4a - 2 \\ \underline{-4a - 2} \end{array}$$

Therefore, the H.C.F. is $2a + 1$.

3. Find the H.C.F. of

$$2a^3 + a^2 - 12a + 9 \text{ and } 2a^3 - 7a^2 + 12a - 9.$$

$$\begin{array}{r} 2a^3 + a^2 - 12a + 9)\,2a^3 - 7a^2 + 12a - 9\,(1 \\ \underline{2a^3 + a^2 - 12a + 9} \\ -8a^2 + 24a - 18 \end{array}$$

We remove the factor 2 from the remainder $-8a^2 + 24a - 18$, and will find it more convenient to divide by -2, leaving $4a^2 - 12a + 9$.

For the *signs* of all the terms of the remainder may be changed, since if a number A is divisible by $+B$, it is also divisible by $-B$.

Then to avoid the inconvenience of fractions we *multiply* the expression $2a^3 + a^2 - 12a + 9$ by the simple factor 2 to make its first term exactly divisible by $4a^2$.

The *introduction* of such a factor can in no way affect the H.C.F. sought, for 2 is not a factor of either of the given expressions; and if we multiply only one of the expressions by 2 we do not introduce a *common* factor.

Therefore, we continue the process by dividing the remainder by -2, and multiplying the divisor by 2.

$$
\begin{array}{lr}
 & 4a^2 - 12a + 9)\,4a^3 + 2a^2 - 24a + 18\,(a \\
 & \underline{4a^3 - 12a^2 + 9a} \\
 & 14a^2 - 33a + 18 \\
\text{Multiply by 2,} & \underline{2\qquad\qquad\qquad} \\
 & 28a^2 - 66a + 36\,(7 \\
 & \underline{28a^2 - 84a + 63} \\
\text{Divide by 9,} & 9)\,\underline{18a - 27} \\
 & 2a - 3
\end{array}
$$

$$
\begin{array}{r}
2a - 3)\,4a^2 - 12a + 9\,(2a - 3 \\
\underline{4a^2 - 6a} \\
-6a + 9 \\
\underline{-6a + 9}
\end{array}
$$

Therefore, the H.C.F. is $2a - 3$.

The following arrangement of the work is convenient :

$$
\begin{array}{r|r|l}
2a^3 + a^2 - 12a + 9 & 2a^3 - 7a^2 + 12a - 9 & 1 \\
\underline{2\qquad\qquad\qquad\qquad} & \underline{2a^3 + a^2 - 12a + 9} & \\
4a^3 + 2a^2 - 24a + 18 & -2)\,-8a^2 + 24a - 18 & \\
\underline{4a^3 - 12a^2 + 9a} & 4a^2 - 12a + 9 & a + 7 \\
14a^2 - 33a + 18 & \underline{4a^2 - 6a} & 2a - 3 \\
\underline{2\qquad\qquad\qquad} & -6a + 9 & \\
28a^2 - 66a + 36 & \underline{-6a + 9} & \\
\underline{28a^2 - 84a + 63} & & \\
9)\,\underline{18a - 27} & & \\
2a - 3 & &
\end{array}
$$

In practice the work is performed as follows :

$$
\begin{array}{r|r|l}
2 + 1 - 12 + 9 & 2 - 7 + 12 - 9 & 1 \\
\underline{2\qquad\qquad\qquad} & \underline{2 + 1 - 12 + 9} & \\
4 + 2 - 24 + 18 & -2)\,-8 + 24 - 18 & \\
\underline{4 - 12 + 9} & 4 - 12 + 9 & 1 + 7 \\
14 - 33 + 18 & \underline{4 - 6} & 2 - 3 \\
\underline{2\qquad\qquad} & -6 + 9 & \\
28 - 66 + 36 & \underline{-6 + 9} & \\
\underline{28 - 84 + 63} & & \\
9)\,\underline{18 - 27} & & \\
2 - 3 & &
\end{array}
$$

158. In the examples worked out we have *assumed* that the divisor which is contained in the corresponding dividend without a remainder is the H.C.F. required.

The *proof* may be given as follows:

Let A and B stand for two expressions which have no monomial factors, and which are arranged according to the descending powers of a common letter, the degree of B being not higher than that of A in the common letter.

Let A be divided by B, and let Q stand for the quotient and R for the remainder. Then, since the dividend is equal to the product of the divisor and quotient plus the remainder, we have

$$A = BQ + R. \qquad (1)$$

Since the remainder is equal to the dividend minus the product of the divisor and quotient, we have

$$R = A - BQ. \qquad (2)$$

Now a factor of each of the terms of an expression is a factor of the expression. Hence, any common factor of B and R is a factor of $BQ + R$, and by (1) a factor of A. That is, a common factor of B and R is also a common factor of A and B.

Also, any common factor of A and B is a factor of $A - BQ$, and by (2) a factor of R. That is, a common factor of A and B is also a common factor of B and R.

Therefore, the common factors of A and B are *the same* as the common factors of B and R; and consequently the H.C.F. of A and B is *the same* as the H.C.F. of B and R.

The proof for each succeeding step in the process is precisely the same; so that the H.C.F. of *any* divisor and the corresponding dividend is the H.C.F. required.

If at any step there is no remainder, the divisor is a factor of the corresponding dividend, and is therefore the H.C.F. of itself and the corresponding dividend. Hence, *this divisor* is the H.C.F. required.

159. The methods of resolving expressions into factors, given in the last chapter, often enable us to shorten the work of finding the H.C.F. required.

1. Find the H.C.F. of x^4+x^3+2x-4 and x^3-2x^2-5x+6.

Each of the given expressions vanishes when we put 1 for x.
Therefore, each expression is divisible by $x-1$. (p. 103, § 147)
The first quotient is $x^3+2x^2+2x+4=(x^2+2)(x+2)$.
The second quotient is $x^2-x-6=(x-3)(x+2)$.
Therefore, the H.C.F. is $(x-1)(x+2)$.

2. Find the H.C.F. of

$4x^3+7x^2-3x-15$ and $8x^4-6x^3-x^2+15x-25$.

$$
\begin{array}{l|rrrrr|l}
4x^3+7x^2-3x-15 & 8x^4 & -\ 6x^3 & -\ \ x^2 & +\ 15x & -\ 25 & 2x-5 \\
 & 8x^4 & +\ 14x^3 & -\ 6x^2 & -\ 30x & & \\
\cline{2-6}
 & & -\ 20x^3 & +\ 5x^2 & +\ 45x & -\ 25 & \\
 & & -\ 20x^3 & -\ 35x^2 & +\ 15x & +\ 75 & \\
\cline{3-6}
 & & 10) & 40x^2 & +\ 30x & -\ 100 & \\
\cline{4-6}
 & & & 4x^2 & +\ 3x & -\ 10 &
\end{array}
$$

The remainder $4x^2+3x-10$ vanishes when -2 is put for x.

Therefore, $x+2$ is a factor of the remainder and the remainder $4x^2+3x-10=(x+2)(4x-5)$.

Since $+2$ is not an exact divisor of -15, $x+2$ is not a factor of $4x^3+7x^2-3x-15$; but $4x-5$ is found by trial to be a factor of $4x^3+7x^2-3x-15$.

Therefore, $4x^3+7x^2-3x-15=(4x-5)(x^2+3x+3)$.
Hence, the H.C.F. is $4x-5$.

3. Find the H.C.F. of

$2a^4-6a^3-a^2+15a-10$ and $4a^4+6a^3-4a^2-15a-15$.

$$
\begin{array}{l|rrrrr|l}
2a^4-6a^3-a^2+15a-10 & 4a^4 & +\ 6a^3 & -\ 4a^2 & -\ 15a & -\ 15 & 2 \\
 & 4a^4 & -\ 12a^3 & -\ 2a^2 & +\ 30a & -\ 20 & \\
\hline
 & & 18a^3 & -\ 2a^2 & -\ 45a & +\ 5 &
\end{array}
$$

Now
$$
\begin{aligned}
18a^3-2a^2-45a+5 &= (18a^3-2a^2)-(45a-5)\\
&= 2a^2(9a-1)-5(9a-1)\\
&= (2a^2-5)(9a-1).
\end{aligned}
$$

The factor $2a^2-5$ is found to be the H.C.F. required.

4. Find the H.C.F. and the L.C.M. of

$2a^4 + 3a^3x - 9a^2x^2$ and $6a^4x - 3ax^4 - 17a^3x^2 + 14a^2x^3$.

Arrange according to the descending powers of a.

$$\begin{array}{l|l}
\begin{array}{r}
a^2)\,2a^4 + 3a^3x - 9a^2x^2 \\
2a^2 + 3ax - 9x^2 \\
2a^2 - 3ax \\
\hline
6ax - 9x^2 \\
6ax - 9x^2 \\
\hline
\end{array}
&
\begin{array}{rl}
ax)\,6a^4x - 17a^3x^2 + 14a^2x^3 - 3ax^4 & \\
6a^3 - 17a^2x + 14ax^2 - 3x^3 & 3a \\
6a^3 + 9a^2x - 27ax^2 & \\
\hline
-26a^2x + 41ax^2 - 3x^3 & -13x \\
-26a^2x - 39ax^2 + 117x^3 & \\
\hline
40x^2)\,80ax^2 - 120x^3 & \\
\hline
2a - 3x & a + 3x
\end{array}
\end{array}$$

$\therefore$ the H.C.F. $= a(2a - 3x)$.

To find the L.C.M., factor each expression by dividing by the H.C.F.

$$2a^4 + 3a^3x - 9a^2x^2 = a^2(2a - 3x)(a + 3x).$$

$$6a^4x - 17a^3x^2 + 14a^2x^3 - 3ax^4 = ax(2a - 3x)(3a^2 - 4ax + x^2)$$

$$= ax(2a - 3x)(a - x)(3a - x).$$

$\therefore$ the L.C.M. $= a^2x(2a - 3x)(a + 3x)(a - x)(3a - x)$.

Exercise 52

Find the H.C.F. and the L.C.M. of:

1. $6x^4 - 13x^3 + 6x^2$; $8x^4 - 36x^3 + 54x^2 - 27x$.
2. $6x^3 - 7x^2 - 16x + 12$; $4x^3 - 8x^2 - 9x + 18$
3. $x^3 - 6x^2 + 11x - 6$; $x^3 - 9x^2 + 26x - 24$.
4. $4x^4 + 7x^2 + 16$; $4x^4 + 12x^3 + 9x^2 - 16$.
5. $2x^3 + x^2 - 4x - 3$; $2x^3 + x^2 - 9$.
6. $n^3 + 4n^2 + 5n + 2$; $n^3 + 2n^2 - n - 2$.
7. $6x^3 - 13x^2 + 19x - 7$; $9x^3 - 27x^2 + 41x - 28$.
8. $2x^3 - 9x^2 + 11x - 3$; $4x^3 - 4x^2 - 5x + 3$.
9. $x^3 - 3x^2y + 3xy^2 - 2y^3$; $x^3 + 3xy^2 - 2x^2y - 6y^3$.
10. $7x^4 - 10x^3 + 3x^2 - 4x + 4$; $8x^4 - 13x^3 + 5x^2 - 3x + 3$.

11. $x^4 - x^3y - x^2y^2 + xy^3$; $x^5 - x^4y - xy^4 + y^5$.

12. $x^4 + 2x^2 + 9$; $x^4 - 4x^3 + 8x - 21$.

13. $2x^4 + x^3 - 9x^2 + 8x - 2$; $2x^4 - 7x^3 + 11x^2 - 8x + 2$.

14. $6x^4 - 10x^3 + x^2 - 31x - 21$; $6x^4 - 4x^3 + 3x^2 - 68x - 42$.

15. $6x^3 - 7ax^2 - 11a^2x + 12a^3$; $8x^3 - 14ax^2 - 7a^2x + 15a^3$.

16. $42x^4 + 41x^3 - 9x^2 - 9x - 1$; $35x^3 + 47x^2 + 13x + 1$.

17. $c^5 - 2c^4 - 6c^3 + 4c^2 + 13c + 6$; $c^5 + 3c^4 - 8c^2 - 9c - 3$.

18. $x^3 + 8x^2 + 19x + 12$; $x^3 + 9x^2 + 26x + 24$.

19. $6x^4 - x^3 - x^2 + 17x - 6$; $9x^4 - 30x^3 + 46x^2 - 41x + 10$.

20. $4x^4 + 2x^3 - 18x^2 + 3x - 5$; $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$.

21. $2x^4 + 3x^3 + 5x^2 + 9x - 3$; $3x^4 - 2x^3 + 10x^2 - 6x + 3$.

22. $x^4 - 6x^3 + 13x^2 - 12x + 4$; $x^5 - 4x^4 + 8x^3 - 16x^2 + 16x$.

23. $6x^5 - 9x^4y + 19x^3y^2 - 12x^2y^3 + 19xy^4 + 15y^5$;
$4x^4 - 2x^3y + 10x^2y^2 + xy^3 + 15y^4$.

24. $a^6 - 3a^5 + 6a^4 - 7a^3 + 6a^2 - 3a + 1$;
$a^6 - a^5 + 2a^4 - a^3 + 2a^2 - a + 1$.

25. $15x^5 + 10x^4y + 4x^3y^2 + 6x^2y^3 - 3xy^4$;
$12x^3y^2 + 38x^2y^3 + 16xy^4 - 10y^5$.

26. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$; $x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5$.

27. $4a^5b - 4a^3b^3 + 4a^2b^4 - ab^5$;
$24a^5 + 16a^4b - 36a^3b^2 - 12a^2b^3 + 8ab^4$.

28. $a^6 + 4a^5b - 3a^4b^2 - 16a^3b^3 + 11a^2b^4 + 12ab^5 - 9b^6$;
$6a^5 + 20a^4b - 12a^3b^2 - 48a^2b^3 + 22ab^4 + 12b^5$.

29. $2x^5 - 4x^4 + 8x^3 - 12x^2 + 6x$; $3x^5 - 3x^4 - 6x^3 + 9x^2 - 3x$.

30. $4x^5 + 20x^4y + 13x^3y^2 - 16x^2y^3 - 17xy^4 - 4y^5$;
$6x^5 + 11x^4y - 54x^3y^2 - x^2y^3 + 30xy^4 + 8y^5$.

31. $6x^5 + 2x^4y - 46x^3y^2 + 38x^2y^3 - 16xy^4 - 32y^5$;
$4x^5 - 34x^3y^2 + 28x^2y^3 - 14xy^4 - 24y^5$.

160. *The product of the H.C.F. and the L.C.M. of two expressions is equal to the product of the given expressions.*

Let A and B stand for any two expressions; and let F stand for their H.C.F. and M for their L.C.M.

Let a and b be the quotients when A and B respectively are divided by F. Then

$$A = aF$$

and $$B = bF.$$

Therefore, $$AB = F \times abF. \tag{1}$$

Since F stands for the H.C.F. of A and B, F contains *all the common factors* of A and B. Therefore, a and b have no common factor, and abF is the L.C.M. of A and B.

Put M for its equal, abF, in equation (1), and we have

$$AB = FM.$$

161. Since $$FM = AB, \qquad (§\ 160)$$

$$M = \frac{AB}{F} = \frac{A}{F} \times B = \frac{B}{F} \times A.$$ That is,

The lowest common multiple of two expressions may be found by dividing their product by their highest common factor, or by dividing either expression by their highest common factor and multiplying the quotient by the other expression.

162. The H.C.F. of three or more expressions may be obtained by finding the H.C.F. of two of them; then the H.C.F. of this result and the third expression; and so on.

For, if A, B, and C stand for three expressions,
and D for the highest common factor of A and B,
and E for the highest common factor of D and C,
then D contains every factor common to A and B,
and E contains every factor common to D and C;
that is, every factor common to A, B, and C.

163. The L.C.M. of three or more expressions may be obtained by finding the L.C.M. of two of them; then the L.C.M. of this result and the third expression; and so on.

For, if A, B, and C stand for three expressions,
and L for the lowest common multiple of A and B,
and M for the lowest common multiple of L and C,
then L is the expression of lowest degree that is exactly divisible by A and B,
and M is the expression of lowest degree that is exactly divisible by L and C;
that is, M is the expression of lowest degree that is exactly divisible by A, B, and C.

Exercise 53

Find the H.C.F. and the L.C.M. of:

1. $a^3+a^2b-ab^2-b^3$; $a^3-3ab^2+2b^3$; $a^3-2a^2b-ab^2+2b^3$.
2. $a^2+3ab+2b^2$; $a^2+4ab+3b^2$; $a^2+5ab+6b^2$.
3. $a^2+5ab+10b^2$; $a^3+3a^2b-20b^3$; $a^3-15ab^2-50b^3$.
4. x^2+2x-3; x^3+3x^2-x-3; x^3+4x^2+x-6.
5. $6x^2-13x+6$; $6x^2+5x-6$; $9x^2-4$.
6. $x^3-6x^2+11x-6$; $x^3-9x^2+26x-24$; $x^3-8x^2+19x-12$.
7. $3a^2+14a-5$; $6a^2+39a+45$; $6a^3+7a^2-3a$.
8. $14x^2+5xy-y^2$; $21x^2-17xy+2y^2$; $6x^2-xy-2y^2$.
9. $x^4-2x^3-2x^2+7x-10$; $x^4-7x^3+18x^2-23x+15$; $2x^4-11x^3+24x^2-28x+15$.
10. $3x^4+2x^3y-15x^2y^2+13xy^3-15y^4$; $3x^4-19x^3y+34x^2y^2-29xy^3+20y^4$; $6x^4+x^3y-23x^2y^2+20xy^3-25y^4$.
11. $20x^4-31x^3y+2x^2y^2+13xy^3-4y^4$; $4x^4+x^3y-5x^2y^2-xy^3+y^4$; $4x^4-11x^3y+4x^2y^2+5xy^3-2y^4$.

CHAPTER IX

FRACTIONS

164. Algebraic Fractions. An **algebraic fraction** is the indicated quotient of two algebraic expressions. An algebraic fraction is generally written in the form $\frac{a}{b}$.

165. Terms. The dividend, a, is called the **numerator** of the fraction; the divisor, b, is called the **denominator** of the fraction.

The numerator and the denominator are called the **terms** of the fraction.

166. Fundamental Principle of Fractions.

Let $$\frac{a}{b} = x. \quad (1)$$

Multiply by b, $$a = bx.$$

Multiply by c, $$ac = bcx.$$

Divide by bc, $$\frac{ac}{bc} = x. \quad (2)$$

From (1) and (2), $$\frac{a}{b} = \frac{ac}{bc}.$$

Now $\frac{ac}{bc}$ may be obtained from $\frac{a}{b}$ by multiplying each term of the fraction by c; and $\frac{a}{b}$ may be obtained from $\frac{ac}{bc}$ by dividing each term of the fraction by c. Hence,

If both the numerator and the denominator of a fraction are multiplied by the same number, or divided by the same number, the value of the fraction is not changed.

Reduction of Fractions to Lowest Terms

167. Lowest Terms. A fraction is expressed in **lowest terms** when the numerator and denominator have no common factor.

168. To Reduce a Fraction to Lowest Terms,

Resolve the numerator and the denominator into their prime factors, and cancel all the factors common to both terms; or, *Divide the numerator and the denominator by their highest common factor.*

Reduce to lowest terms

$$\frac{26\,x^3y^3z^3}{39\,x^3y^4z^2},\ \frac{x^3-y^3}{x^4-y^4},\ \frac{x^2+7\,xy+12\,y^2}{x^2-2\,xy-15\,y^2},\ \frac{x^3-4\,x^2-4\,x+1}{x^3+6\,x^2+4\,x-1}.$$

$$\frac{26\,x^3y^3z^3}{39\,x^3y^4z^2}=\frac{2\times 13\,x^3y^3z^3}{3\times 13\,x^3y^4z^2}=\frac{2\,z}{3\,y}.$$

$$\frac{x^3-y^3}{x^4-y^4}=\frac{(x-y)(x^2+xy+y^2)}{(x+y)(x-y)(x^2+y^2)}=\frac{x^2+xy+y^2}{(x+y)(x^2+y^2)}.$$

$$\frac{x^2+7\,xy+12\,y^2}{x^2-2\,xy-15\,y^2}=\frac{(x+3\,y)(x+4\,y)}{(x+3\,y)(x-5\,y)}=\frac{x+4\,y}{x-5\,y}.$$

$$\frac{x^3-4\,x^2-4\,x+1}{x^3+6\,x^2+4\,x-1}.$$

Since we can determine no common factor easily by inspection, we find the H.C.F. of the numerator and the denominator to be $x+1$. Now $x^3-4\,x^2-4\,x+1$ divided by $x+1$ is $x^2-5\,x+1$, and $x^3+6\,x^2+4\,x-1$ divided by $x+1$ is $x^2+5\,x-1$.

$$\therefore \frac{x^3-4\,x^2-4\,x+1}{x^3+6\,x^2+4\,x-1}=\frac{x^2-5\,x+1}{x^2+5\,x-1}.$$

Exercise 54

Reduce to lowest terms:

1. $\dfrac{10\,a^4b^3}{15\,a^3b^4}$.
2. $\dfrac{6\,x^4y^5z^6}{12\,x^6y^5z^4}$.
3. $\dfrac{64\,a^6b^3c}{48\,a^4bc^5}$.
4. $\dfrac{36\,x^2y^4z^6}{60\,x^4y^2z^5}$.
5. $\dfrac{60\,a^7b^6c^5d^4}{80\,a^8b^4c^4d^6}$.
6. $\dfrac{75\,a^4x^5y^4z^5}{50\,a^2x^7y^5z^3}$.

7. $\dfrac{a^2 - 2ab}{ab - 2b^2}$.

8. $\dfrac{10a^2 - 2ab}{15ab - 3b^2}$.

9. $\dfrac{14a^5 + 7a^4b}{10a^4c + 5a^3bc}$.

10. $\dfrac{12a^4 + 27a^3b}{16a^3b + 36a^2b^2}$.

11. $\dfrac{45a^3x^3 - 105a^5x^2}{33ab^3x - 77a^3b^3}$.

12. $\dfrac{49a^7b^2x^6 - 81ab^{12}x^2}{56a^6b^4x^5 + 72a^3b^9x^3}$.

13. $\dfrac{x^3 + x^2y}{x^2 + 2xy + y^2}$.

14. $\dfrac{x^3 + 2x^2}{x^2 + 4x + 4}$.

15. $\dfrac{x^4 - y^4}{x^4 + 2x^2y^2 + y^4}$.

16. $\dfrac{7x^2 + 3}{245x^5 + 210x^3 + 45x}$.

17. $\dfrac{15a^2 - 27ab}{25a^2 - 90ab + 81b^2}$.

18. $\dfrac{x - xy + z - zy}{1 - 3y + 3y^2 - y^3}$.

19. $\dfrac{x^3 + y^3}{2(x + y)^2}$.

20. $\dfrac{x^8y - x^2y^5z^2}{x^3y^4 - y^6z}$.

21. $\dfrac{x^2 + (a + b)x + ab}{x^2 + (a + c)x + ac}$.

22. $\dfrac{4a^2 - 7ab + 3b^2}{5a^2 - 3ab - 2b^2}$.

23. $\dfrac{3x^2 - 10xy + 8y^2}{5x^2 - 13xy + 6y^2}$.

24. $\dfrac{2x^3 + 5x^2 - 12x}{7x^3 + 25x^2 - 12x}$.

25. $\dfrac{a^4 - a^3 - a + 1}{a^4 - 2a^3 - a^2 - 2a + 1}$.

26. $\dfrac{(x + a)^2 - (b + c)^2}{(x + b)^2 - (a + c)^2}$.

27. $\dfrac{(x + y)^7 - x^7 - y^7}{(x + y)^5 - x^5 - y^5}$.

28. $\dfrac{a^5 - a^4b - ab^4 + b^5}{a^4 - a^3b - a^2b^2 + ab^3}$.

29. $\dfrac{x^4 + 4x + 3}{x^5 - 5x - 4}$.

30. $\dfrac{(a^6 - b^6)(a - b)}{(a^3 - b^3)(a^4 - b^4)}$.

31. $\dfrac{x^2(y - z) + y^2(z - x) + z^2(x - y)}{x^2z - xz^2 + xy^2 - x^2y + yz^2 - y^2z}$.

32. $\dfrac{x^3 - 5x^2y - 2xy^2 + 24y^3}{x^3 + 3x^2y - 10xy^2 - 24y^3}$.

33. $\dfrac{2x^4 - 3x^3 - 47x^2 - 45x + 18}{3x^4 - 13x^3 - 45x^2 + 94x - 24}$.

169. There are *three signs* to consider in a fraction: the sign before the fraction, the sign of the numerator, and the sign of the denominator.

Since $$\frac{a}{b}=\frac{-a}{-b}=-\frac{-a}{b}=-\frac{a}{-b},\qquad \text{(p. 46, § 89)}$$

any *two* of the three signs may be changed without changing the value of the fraction.

The sign of a compound expression is changed by changing the *sign of every term* of the expression. Hence,

1. *We may change the sign of every term of the numerator and denominator of a fraction without changing the value of the fraction.*

2. *We may change the sign before a fraction and the sign of every term of either the numerator or denominator without changing the value of the fraction.*

170. From the Law of Signs, therefore,

1. We may change the signs of an **even number of factors** of the numerator, or of the denominator, or of both taken together, without changing the sign of the fraction.

2. We may change the signs of an **odd number of factors** of the numerator, or of the denominator, or of both taken together, *if we change the sign before the fraction.*

Reduce to lowest terms $\dfrac{(a-b)(c-d)}{(b-a)(c+d)}$.

Change the sign of the factor $(b-a)$ of the denominator and the sign before the fraction, and we have

$$\frac{(a-b)(c-d)}{(b-a)(c+d)}=-\frac{(a-b)(c-d)}{(a-b)(c+d)}=-\frac{c-d}{c+d}.$$

In the last fraction change the sign of the numerator, the sign of the fraction, and the *order* of the terms of the denominator, and we have

$$-\frac{c-d}{c+d}=\frac{d-c}{d+c}.$$

NOTE. Factors and terms must not be confounded.

Exercise 55

Reduce to lowest terms:

1. $\frac{a^2 - 4ab + 4b^2}{2b^2 + ab - a^2}$.

2. $\frac{(4y - 12)^3}{27y^2 - 3y^4}$.

3. $\frac{6y^2z - 3x^2y^2}{x^4 - 4x^2z + 4z^2}$.

4. $\frac{a^8 - x^4y^4}{15x^2y^2 - 15a^2xy}$.

5. $\frac{(9a^2x^2 - 15ax^3)^3}{675a^4x^8 - 243a^6x^6}$.

6. $\frac{6x^2 - 3y - 2x^4 + x^2y}{3y^2 + 3x^4 - 9x^2 - x^2y^2}$.

7. $\frac{100a^3b - 64a^7b}{48a^4b^2 - 120a^2b^2 + 75b^2}$.

8. $\frac{2a^2c^2 - 3a^2b^2 - 2b^2c^2 + 3b^4}{3b^2c^2 - 2a^2b^2 - 3a^2c^2 + 2b^4}$.

9. $\frac{48a^4b^3x^4 - 144a^2c^2x^5}{720ab^2c^6x^4 - 80a^5b^8c^2x^2}$.

10. $\frac{6a^2c^2 + 18bc^2 - 2a^4 - 6a^2b}{4a^4 + 2a^2b^2 - 12a^2c^2 - 6b^2c^2}$.

Reduction of Fractions to Integral or Mixed Expressions

171. Mixed Expressions. A **mixed expression** is an integral expression and a fraction.

Thus, $a + \frac{b}{c}$ and $a - \frac{a-b}{a+b}$ are mixed expressions.

172. If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to the form of an integral or a mixed expression *by dividing the numerator by the denominator.*

The quotient is the integral expression; the remainder (if any) is the numerator and the divisor the denominator, of the fractional expression.

1. Reduce $\frac{6a^2 + 13ab + 7b^2}{2a + 3b}$ to an integral or a mixed expression.

$$\frac{6a^2 + 13ab + 7b^2}{2a + 3b} = 3a + 2b + \frac{b^2}{2a + 3b}.$$

2. Reduce $\dfrac{x^3 + 4x^2 - 3x - 7}{x^2 + 2x - 5}$ to an integral or a mixed expression.

Divide the numerator by the denominator.

$$\begin{array}{r|l} x^3 + 4x^2 - 3x - 7 & x^2 + 2x - 5 \\ \underline{x^3 + 2x^2 - 5x \phantom{{}-7}} & x + 2 \\ +2x^2 + 2x - 7 & \\ \underline{+2x^2 + 4x - 10} & \\ -2x + 3 & \end{array}$$

The remainder $-2x + 3$ is the numerator of a fraction, and the divisor $x^2 + 2x - 5$ is its denominator, to be added to the integral quotient $x + 2$. Thus, the complete mixed expression required is

$$x + 2 + \frac{-2x + 3}{x^2 + 2x - 5}.$$

Therefore, we change the sign of each term of the numerator and the sign before the fraction (p. 126, § 169), and the required expression becomes

$$x + 2 + \frac{-2x + 3}{x^2 + 2x - 5} = x + 2 - \frac{2x - 3}{x^2 + 2x - 5}.$$

The last form of the expression is the form usually written.

Exercise 56

Reduce to an integral or a mixed expression:

1. $\dfrac{12x^2 - 5y}{6x}$.

2. $\dfrac{25x^3 - 3a + 5c}{5x}$.

3. $\dfrac{x^2 + 3x + 2}{x + 3}$.

4. $\dfrac{x^2 + 3y^2}{x + y}$.

5. $\dfrac{a^4 - b^4}{a^2 + b^2}$.

6. $\dfrac{a^2 + 6a + 12b - 5b^2}{a - 2b + 6}$.

7. $\dfrac{4ab - 2b^2 - a^2}{2a - b}$.

8. $\dfrac{axy + 4x + 5y}{x + y}$.

9. $\dfrac{a^2 + 2ab + b^2}{a - 2b}$.

10. $\dfrac{x^3 - 3x^2y + 3xy^2 - y^3}{x^2 - y^2}$.

Reduction of Mixed Expressions to Fractions

173. The value of a number is not changed if the number is both multiplied and divided by the same number. Hence,

To Reduce a Mixed Expression to a Fraction,

Multiply the integral expression by the denominator, to the product add the numerator, and under the result write the denominator.

Reduce $x + 2y - \dfrac{x^2 - 2xy + 4y^2}{x + 2y}$ to a fraction.

$$x + 2y - \frac{x^2 - 2xy + 4y^2}{x + 2y} = \frac{(x + 2y)^2 - (x^2 - 2xy + 4y^2)}{x + 2y}$$

$$= \frac{x^2 + 4xy + 4y^2 - x^2 + 2xy - 4y^2}{x + 2y}$$

$$= \frac{6xy}{x + 2y}.$$

The dividing line between the terms of a fraction has the force of a vinculum affecting the numerator. Therefore, if a *minus sign* precedes the dividing line, as in the preceding example, and this line is removed, the numerator of the given fraction must be inclosed in a parenthesis preceded by the minus sign, or *the sign of every term of the numerator must be changed.*

Exercise 57

Reduce to a fraction:

1. $a - b + \dfrac{a^2 + b^2}{a + b}.$
2. $a - b - \dfrac{a - b - c}{2}.$
3. $a^2 - ax + x^2 - \dfrac{a^3}{a + x}.$
4. $3a + 4 + \dfrac{28 - 5a}{2a - 7}.$
5. $\dfrac{a - 7b}{a + 4b} - a + 4b.$
6. $\dfrac{a + b + c}{a + c} - 1.$
7. $a^2 + ax + x^2 + \dfrac{2x^3}{a - x}.$
8. $1 - \dfrac{a^2 - 2ax + x^2}{a^2 + x^2}.$

Reduction of Fractions to Equivalent Fractions Having the Lowest Common Denominator

174. 1. Reduce $\frac{7x}{6a^2}$, $\frac{5y}{4a}$, $\frac{3}{a^3}$ to equivalent fractions having the lowest common denominator.

The L.C.M. of the denominators is $12a^3$.

Divide the L.C.M. by the denominators $6a^2$, $4a$, a^3.

The respective quotients are $2a$, $3a^2$, 12.

Multiply both terms of the given fractions taken in order by the respective quotients $2a$, $3a^2$, 12.

We have for the required fractions

$$\frac{14ax}{12a^3}, \frac{15a^2y}{12a^3}, \frac{36}{12a^3}.$$

2. Reduce $\frac{2a+3}{a^2-4a+3}$ and $\frac{3a-2}{a^2-5a+6}$ to equivalent fractions having the lowest common denominator.

Express the denominators in their prime factors.

$$\frac{2a+3}{a^2-4a+3}, \frac{3a-2}{a^2-5a+6} = \frac{2a+3}{(a-1)(a-3)}, \frac{3a-2}{(a-2)(a-3)}.$$

The lowest common denominator (L.C.D.) is $(a-1)(a-2)(a-3)$.

The respective quotients are $a-2$, $a-1$.

The respective products are $(a-2)(2a+3)$, $(a-1)(3a-2)$.

Therefore, the required fractions are

$$\frac{(a-2)(2a+3)}{(a-1)(a-2)(a-3)}, \frac{(a-1)(3a-2)}{(a-1)(a-2)(a-3)}.$$ Hence,

175. To Reduce Fractions to Equivalent Fractions Having the Lowest Common Denominator,

Find the lowest common multiple of the denominators of the given fractions for the common denominator. Divide this common denominator by each of the given denominators; and multiply the given numerators each by its corresponding quotient for the required numerators.

Note. Each of the given fractions should be in its lowest terms before the common denominator is found.

Exercise 58

Express with lowest common denominator:

1. $\frac{x}{a-b};\ \frac{y}{a+b}.$

2. $\frac{4x-5}{3};\ \frac{2x+7}{4}.$

3. $\frac{2a-3b}{2x};\ \frac{3a-4b}{3x}.$

4. $\frac{4a-5b}{7ac};\ \frac{2a+3b}{5a^2c^2}.$

5. $\frac{2}{x-2};\ \frac{3}{x-3}.$

6. $\frac{5a}{x+4};\ \frac{3a}{x-2}.$

7. $\frac{1}{1+2c};\ \frac{1}{1-4c^2}.$

8. $\frac{1}{x-2a};\ \frac{1}{x-3a}.$

9. $\frac{1}{x+y};\ \frac{2y}{x^2-y^2}.$

10. $\frac{a}{a-b};\ \frac{a^2}{a+b};\ \frac{a^3}{a^2-b^2}.$

11. $\frac{2}{a-b};\ \frac{2}{b-c};\ \frac{2}{c-a}.$

12. $\frac{7a+4}{x-3};\ \frac{5a+2}{3x-9};\ \frac{4a-5}{4x-12}.$

13. $\frac{a}{a+1};\ \frac{a^2}{(a+1)^2};\ \frac{a^3}{(a+1)^3}.$

14. $\frac{4}{9-a^2};\ \frac{3-a}{3+a};\ \frac{3+a}{3-a}.$

15. $\frac{1}{x^2-5x+6};\ \frac{1}{x^2-7x+12};\ \frac{1}{x^2-6x+8}.$

16. $\frac{7x}{8x^2-10xy+3y^2};\ \frac{4x}{12x^2-xy-6y^2};\ \frac{3x}{6x^2+xy-2y^2}.$

17. $\frac{3a+2b}{6a^2+ab-35b^2};\ \frac{2a-3b}{12a^2-25ab-7b^2};\ \frac{4a+5b}{8a^2+22ab+5b^2}.$

Addition and Subtraction of Fractions

176. 1. Find the algebraic sum of $\frac{x}{a}+\frac{y}{a}-\frac{3}{a}.$

$$\frac{x}{a}+\frac{y}{a}-\frac{3}{a}=\frac{x+y-3}{a}.$$

2. Find the algebraic sum of $\frac{a+x}{a-x} - \frac{a-x}{a+x} + \frac{a^2+x^2}{a^2-x^2}$.

The L.C.D. $= a^2 - x^2 = (a+x)(a-x)$.

The multipliers, that is, the quotients obtained by dividing the L.C.D. by $a - x$, $a + x$, and $a^2 - x^2$, are $a + x$, $a - x$, and 1 respectively.

Hence, the sum of the fractions is

$$\frac{(a+x)^2}{a^2-x^2} - \frac{(a-x)^2}{a^2-x^2} + \frac{a^2+x^2}{a^2-x^2} = \frac{a^2+2ax+x^2}{a^2-x^2} - \frac{a^2-2ax+x^2}{a^2-x^2} + \frac{a^2+x^2}{a^2-x^2}$$

$$= \frac{a^2+2ax+x^2-a^2+2ax-x^2+a^2+x^2}{a^2-x^2}$$

$$= \frac{a^2+4ax+x^2}{a^2-x^2}.$$

The work may be arranged as follows:

The L.C.D. $= (a+x)(a-x)$.

The multipliers are $a + x$, $a - x$, and 1 respectively.

$$\begin{aligned} (a+x)(a+x) &= a^2 + 2ax + x^2 = \text{1st numerator.} \\ -(a-x)(a-x) &= -a^2 + 2ax - x^2 = \text{2d numerator.} \\ 1(a^2+x^2) &= a^2 \qquad\quad + x^2 = \text{3d numerator.} \\ & \quad\; a^2 + 4ax + x^2 = \text{sum of numerators.} \end{aligned}$$

Therefore, the sum of the fractions $= \frac{a^2+4ax+x^2}{a^2-x^2}$. Hence,

177. To Add Fractions,

Reduce the fractions, if they have different denominators, to equivalent fractions having the lowest common denominator; and write the algebraic sum of the numerators of these fractions over the common denominator. Reduce the resulting fraction to lowest terms.

Exercise 59

Find the algebraic sum of:

1. $\frac{4a-23b}{4} - \frac{4a-25b}{6} + \frac{19b-3a}{12}$.

2. $\frac{29a}{90} - \frac{3a-20b}{60} + \frac{3a-10b+5c}{20} - \frac{4a-2b+3c}{12}$.

3. $\frac{2}{3a} - \frac{1}{2b} - \frac{2a+3}{6a^2} - \frac{1}{2x^2} + \frac{3a-2b}{6ab}$.

4. $\frac{5}{2(x+1)} - \frac{1}{10(x-1)} - \frac{24}{5(2x+3)}$.

5. $\frac{2+3x-5x^2}{x^3-3x^2+3x-1} - \frac{2x+5}{x^2-2x+1} + \frac{1}{x-1}$.

6. $\frac{1}{a^2-7a+12} + \frac{2}{a^2-4a+3} - \frac{3}{a^2-5a+4}$.

7. $\frac{x+1}{x^2+x+1} + \frac{x-1}{x^2-x+1} + \frac{2}{x^4+x^2+1}$.

8. $\frac{x^2+ax+a^2}{x^3-a^3} - \frac{x^2-ax+a^2}{x^3+a^3}$.

9. $\frac{x^2+y^2}{xy} - \frac{x^2}{xy+y^2} - \frac{y^2}{x^2+xy}$.

10. $\frac{x^2-2x+3}{x^3+1} + \frac{x-2}{x^2-x+1} - \frac{1}{x+1}$.

11. $\frac{x-y}{x^2-xy+y^2} + \frac{1}{x+y} + \frac{xy}{x^3+y^3}$.

12. $\frac{1}{x-y} + \frac{x-y}{x^2+xy+y^2} + \frac{xy-2x^2}{x^3-y^3}$.

13. $\frac{1}{x-3} + \frac{x-1}{x^2+3x+9} + \frac{x^2+x-3}{x^3-27}$.

14. $\frac{2a}{a^4-a^2+1} - \frac{1}{a^2-a+1} + \frac{1}{a^2+a+1}$

15. $\frac{2}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{2}{x-1} - \frac{1}{x}$.

16. $\frac{x-1}{x^2-7x+10} - \frac{x+2}{x^2-9x+14} - \frac{x-3}{x^2-12x+35}$.

17. $\frac{2+x}{3+x^2} - \frac{3-x}{9+6x+x^2} + \frac{1-2x^2}{(3+x)(3+x^2)}$.

18. $\dfrac{a^2 - bc}{(a+b)(a+c)} + \dfrac{b^2 - ac}{(b+c)(a+b)} + \dfrac{c^2 - ab}{(a+c)(b+c)}.$

19. $\dfrac{5x - 2}{x^2 - 3x - 4} + \dfrac{2x + 1}{x^2 - x - 12} - \dfrac{3x - 1}{x^2 + x - 20}.$

20. $\dfrac{7 + 3x^2}{4 - x^2} - \dfrac{5 - 2x^2}{4 + 4x + x^2} - \dfrac{3 - 2x + x^2}{4 - 4x + x^2}.$

21. $\dfrac{a - 5x}{4a^2 - 20ax + 25x^2} - \dfrac{3a^2 - 4ax + x^2}{24a^3 - 180a^2x + 450ax^2 - 375x^3}.$

22. $\dfrac{x + a}{x^2 - (b+c)x + bc} + \dfrac{x + b}{x^2 - (a+c)x + ac} + \dfrac{x + c}{x^2 - (a+b)x + ac}.$

23. $\dfrac{1}{a^2 + 5a + 6} + \dfrac{2a}{a^2 + 4a + 3} + \dfrac{1}{a^2 + 3a + 2} + \dfrac{2a + 1}{a + 3}.$

24. $\dfrac{x^2 - 7xy + 12y^2}{4x^2 - 11xy + 3y^2} - \dfrac{2x^2 + 7xy - 4y^2}{8x^2 - 6xy + y^2}.$

25. $\dfrac{a^2 + ab}{a^2 + ab + b^2} + \dfrac{a^2 - ab}{a^2 - ab + b^2} - \dfrac{a^4 - a^2b^2 - b^4}{a^4 + a^2b^2 + b^4}.$

26. $\dfrac{x^4 - 2x^2 - 3}{15x^6 - 17x^2 - 18 + 25x^4} - \dfrac{x^2 - 4x + 1}{12x^4 - x^2 - 6}.$

27. $\dfrac{2 - 3x}{6 - 4x} + \dfrac{3 - 4x}{6 + 4x} - \dfrac{1 - 2x + x^2}{9 - 12x + 4x^2} - \dfrac{2 - 5x^2}{36 - 16x^2}.$

28. $\dfrac{x - y}{x + y} + \dfrac{y - z}{y + z} + \dfrac{z - x}{z + x} + \dfrac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}.$

29. $\dfrac{y - z}{x^2 - (y - z)^2} + \dfrac{z - x}{y^2 - (z - x)^2} + \dfrac{x - y}{z^2 - (x - y)^2}.$

30. $\dfrac{2}{x - y} + \dfrac{2}{y - z} + \dfrac{2}{z - x} + \dfrac{(x-y)^2 + (y-z)^2 + (z-x)^2}{(x-y)(y-z)(z-x)}.$

31. $\dfrac{a^3}{(a-b)(a-c)} - \dfrac{b^3}{(a-b)(b-c)} + \dfrac{c^3}{(a-c)(b-c)}.$

178. When the denominators of the fractions are polynomials not arranged in the same order, we first write the fractions so that the denominators shall be arranged in the same order (p. 126, § 169).

1. Find the algebraic sum of $\frac{2}{x}-\frac{3}{2x-1}+\frac{2x-3}{1-4x^2}$.

Change the signs before the terms of the denominator of the third fraction and change the sign before that fraction. We now have

$$\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}.$$

The L.C.D. $= x(2x+1)(2x-1)$.

The multipliers are $(2x+1), (2x-1)$, $x(2x+1)$, and x respectively.

$$\begin{aligned}
(2x+1)(2x-1)(2) &= 8x^2 \qquad -2 = \text{1st numerator.}\\
-x(2x+1)(3) &= -6x^2-3x \qquad = \text{2d numerator.}\\
-x(2x-3) &= \underline{-2x^2+3x} \qquad = \text{3d numerator.}\\
& \qquad\qquad -2 = \text{the sum of the numerators.}
\end{aligned}$$

Therefore, the required sum $= \frac{-2}{x(2x+1)(2x-1)} = \frac{2}{x(1+2x)(1-2x)}$.

Exercise 60

Find the algebraic sum of :

1. $\frac{1}{(x-2)(x-3)}+\frac{2}{(x-1)(3-x)}+\frac{1}{(x-1)(x-2)}$.

2. $\frac{1}{x^2(x^2+y^2)}-\frac{1}{2x^3(y-x)}+\frac{1}{2x^3(x+y)}$.

3. $\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}$.

4. $\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}$.

5. $\frac{a+b}{b}-\frac{2a}{a+b}+\frac{a^2b-a^3}{a^2b-b^3}$.

6. $\frac{x^4}{(x-y)(x-z)}+\frac{y^4}{(y-z)(y-x)}+\frac{z^4}{(z-x)(z-y)}$.

7. $\frac{a(b+c-a)}{(a-b)(a-c)}+\frac{b(c+a-b)}{(b-c)(b-a)}+\frac{c(a+b-c)}{(c-a)(c-b)}.$

8. $$\frac{a^2}{(a-b)(a-c)(1+ax)}+\frac{b^2}{(b-c)(b-a)(1+bx)}+\frac{c^2}{(c-a)(c-b)(1+cx)}.$$

9. $\frac{3}{(x-y)(y-z)}-\frac{4}{(y-x)(z-x)}-\frac{6}{(x-z)(z-y)}.$

10. $\frac{7x-13}{2x-1}+\frac{28-13x}{3-2x}+\frac{28x+43}{4x^2-8x+3}.$

11. $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(a-c)^2}{(a+b)^2-c^2}-\frac{c^2-(a-b)^2}{a^2-(b+c)^2}.$

Multiplication of Fractions

179. Find the product of $\frac{a}{b}\times\frac{c}{d}$.

Let $$\frac{a}{b}\times\frac{c}{d}=x. \qquad (1)$$

Multiply each of these equals by $b \times d$.

Since the order of the factors is immaterial, (p. 15, § 45)

$$\left(\frac{a}{b}\times b\right)\times\left(\frac{c}{d}\times d\right)=b\times d\times x.$$

Or $$a\times c=b\times d\times x.$$

Divide by $b \times d$, $$\frac{a\times c}{b\times d}=x. \qquad (2)$$

From (1) and (2), $$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d}.$$

Likewise, $$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{a\times c}{b\times d}\times\frac{e}{f}=\frac{a\times c\times e}{b\times d\times f};$$

and so on for any number of fractions. Hence,

180. To Find the Product of Two or More Fractions,

Find the product of the numerators for the numerator of the product, and the product of the denominators for the denominator of the product.

In applying the rule, reduce every mixed expression to a fraction, and every integral expression to a fraction with 1 for the denominator.

Cancel every factor common to a numerator and a denominator, as the canceling of a common factor *before* the multiplication is equivalent to canceling it *after* the multiplication.

1. Find the product of $\frac{3a^3d}{5b^2c^2} \times \frac{5b^3d^4}{2a^4c^3} \times \frac{2a^2b^2c^3}{7d^2}$.

$$\frac{3a^3d}{5b^2c^2} \times \frac{5b^3d^4}{2a^4c^3} \times \frac{2a^2b^2c^3}{7d^2} = \frac{3 \times 5 \times 2a^5b^5c^3d^5}{5 \times 2 \times 7a^4b^2c^5d^2} = \frac{3ab^3d^3}{7c^2}.$$

2. Find the product of $\frac{2ax^3 + 2a^3x}{(x-a)^2(x+a)^2} \times \frac{x^2 - a^2}{2(x^2+a^2)} \times \frac{x+a}{ax}$.

Express the numerators and denominators in prime factors.

$$\frac{2ax(x^2 + a^2)}{(x-a)^2(x+a)^2} \times \frac{(x+a)(x-a)}{2(x^2+a^2)} \times \frac{x+a}{ax} = \frac{1}{x-a}.$$

The common factors canceled are 2, a, x, $x^2 + a^2$, $x + a$, $x - a$, and $x + a$.

Exercise 61

Find the product of:

1. $\frac{12x^2}{5y^2} \times \frac{10xy}{9z^2}$.

2. $\frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab}$.

3. $\frac{a^2b}{x^2y} \times \frac{b^2c}{y^3z} \times \frac{x^2y^2z}{abc}$.

4. $\frac{5a^2b}{3cd} \times \frac{4b^2c}{10a^2} \times \frac{9c^2d}{16b^3}$.

5. $\frac{x^2 - y^2}{x^2 + y^2} \times \frac{4x}{x + y}$.

6. $\frac{x^2 - y^2}{x^2 + y^2} \times \frac{3x^2}{4x + 4y}$.

7. $\frac{ab + ac}{bd - cd} \times \frac{ab - ac}{bd + cd}$.

8. $\frac{4x + 4y}{3x^2 - 3y^2} \times \frac{5x^2 - 5y^2}{7x + 7y}$.

9. $\dfrac{3x^2 - x}{a} \times \dfrac{2a}{2x^2 - 4x}$.

11. $\dfrac{(x-y)^2}{x+y} \times \dfrac{y}{x(x-y)}$.

10. $\dfrac{3x^2}{5x - 10} \times \dfrac{3x - 6}{4x^3}$.

12. $\dfrac{x^4 - y^4}{x^2 - 2xy + y^2} \times \dfrac{x - y}{x^2 + xy}$.

13. $\dfrac{x(a-x)}{a^2 + 2ax + x^2} \times \dfrac{a(a+x)}{a^2 - 2ax + x^2}$.

14. $\dfrac{a^6 - b^6}{a^4 + 2a^2b^2 + b^4} \times \dfrac{a^2 + b^2}{a^2 - ab + b^2} \times \dfrac{a + b}{a^3 - b^3}$.

15. $\dfrac{a^3 - b^3}{a^3 + b^3} \times \dfrac{a + b}{a - b} \times \left(\dfrac{a^2 - ab + b^2}{a^2 + ab + b^2}\right)^2$.

16. $\dfrac{a^3 + 3a^2b + 3ab^2 + b^3}{a^2 - b^2} \times \dfrac{a + b}{a^2 + 2ab + b^2}$.

17. $\dfrac{2a^3 + 13a^2x - 15ax^2 - 126x^3}{a^2x + 6ax^2 - 7x^3} \times \dfrac{2a^3 + 19a^2x + 35ax^2}{a^3 - a^2x - 4ax^2 - 6x^3}$.

18. $\dfrac{a^2 - x^2}{a + b} \times \dfrac{a^2 - b^2}{ax + x^2} \times \left(a + \dfrac{ax}{a - x}\right)$.

19. $\left(\dfrac{x}{a} - \dfrac{a}{x} + \dfrac{y}{b} - \dfrac{b}{y}\right)\left(\dfrac{x}{a} - \dfrac{a}{x} - \dfrac{y}{b} + \dfrac{b}{y}\right)$.

20. $\left(\dfrac{a}{bc} + \dfrac{b}{ac} - \dfrac{c}{ab} - \dfrac{2}{c}\right)\left(1 - \dfrac{2c}{a - b + c}\right)$.

21. $\left(\dfrac{x^2}{a^2} + \dfrac{a^2}{x^2} - \dfrac{x}{a} - \dfrac{a}{x} + 1\right)\left(\dfrac{x}{a} - \dfrac{a}{x}\right)$.

22. $\dfrac{x^2 + 2xy + y^2 - 4z^2}{4x^2 - (y + 2z)^2} \times \dfrac{(2x - y)^2 - 4z^2}{x^2 - (y - 2z)^2} \times \dfrac{(x - y)^2 - 4z^2}{(x + y)^2 - 4z^2}$.

23. $\dfrac{x^2 - 3ax + 2a^2}{x^2 - 2ax - 3a^2} \times \dfrac{x^2 - 7ax + 12a^2}{x^2 - 8ax + 12a^2} \times \dfrac{x^2 - 11ax + 30a^2}{x^2 - 5ax + 4a^2}$.

24. $\frac{(x+3y)^2-z^2}{x^2-(3y+z)^2} \times \left(1-\frac{6y}{x+3y-z}\right) \times \frac{x^2-4y^2}{x^3-8y^3}$.

25. $\frac{2a^2b^3c}{3x^2y^3} \times \frac{a^m b^n c^r}{x^m y^n} \times \frac{6x^{m-1}y^{n-2}}{a^{m+1}b^{n+2}c^{r+3}}$.

26. $\frac{a^{x-1}b^{x-2}c^{x-3}}{x^{n-1}y^{n+2}z^{n-3}} \times \frac{x^{n+1}y^{n-2}z^{n+4}}{a^{x-2}b^{x+1}c^{x-1}} \times \frac{a^2c^2}{b^2x^2}$.

27. $\frac{a^{m-n}b^{n-p}c^{p-m}}{x^{n-p}y^{p-m}z^{m-n}} \times \frac{a^{n-p}b^{p-m}c^{m-n}}{x^{p-1}y^{m-2}z^{n-3}} \times \frac{x^n y^p z^m}{a^m b^n c^p}$.

Division of Fractions

181. Reciprocals. If the product of two numbers is equal to 1, each of the numbers is called the **reciprocal** of the other.

The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$; for $\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ab} = 1$.

The reciprocal of a fraction, therefore, is the fraction inverted.

182. Find the quotient of $\frac{a}{b} \div \frac{c}{d}$.

Let $$\frac{a}{b} \div \frac{c}{d} = x. \quad (1)$$

Since the dividend is the product of the divisor and quotient,

$$\frac{a}{b} = \frac{c}{d} \times x.$$

Multiply each of these equals by $\frac{d}{c}$.

Then $$\frac{a}{b} \times \frac{d}{c} = \frac{c}{d} \times \frac{d}{c} \times x = x. \quad (2)$$

From (1) and (2), $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$. Therefore,

183. To Divide by a Fraction,

Multiply the dividend by the reciprocal of the divisor.

Simplify $\frac{x^2 - y^2}{x^2 - 3xy + 2y^2} \div \frac{x^2 + xy}{xy - 2y^2} \times \frac{x^2 - xy}{(x - y)^2}$.

$$\frac{x^2 - y^2}{x^2 - 3xy + 2y^2} \div \frac{x^2 + xy}{xy - 2y^2} \times \frac{x^2 - xy}{(x - y)^2}$$

$$= \frac{x^2 - y^2}{x^2 - 3xy + 2y^2} \times \frac{xy - 2y^2}{x^2 + xy} \times \frac{x^2 - xy}{(x - y)^2}$$

$$= \frac{(x + y)(x - y)}{(x - y)(x - 2y)} \times \frac{y(x - 2y)}{x(x + y)} \times \frac{x(x - y)}{(x - y)^2} = \frac{y}{x - y}.$$

The common factors canceled are $x + y$, $x - y$, $x - 2y$, x, and $x - y$.

Exercise 62

Find the quotient of:

1. $\frac{18xy}{25uz} \div \frac{6mn}{35x^2y^2}$.

2. $\frac{14a^2b^3c}{39x^2y^5z^6} \div \frac{35x^7y^4z^8}{9a^4b^5z^2}$.

3. $\frac{25a^4b^5c^6}{49x^4y^5z^6} \div \frac{30a^4bc^8}{77xy^7z^2}$.

4. $\frac{3p - 3q}{5p + 5q} \div \frac{9q - 9p}{10q + 10p}$.

5. $\frac{x^2 + y^2}{x^2 - y^2} \div \frac{3x^2 + 3y^2}{x + y}$.

6. $\frac{6ab - 6b^2}{a(a + b)} \div \frac{2b^2}{a(a^2 - b^2)}$.

7. $\frac{a^2 - 4x^2}{a^2 + 4ax} \div \frac{a^2 - 2ax}{ax + 4x^2}$.

8. $\frac{8a^3}{a^3 - b^3} \div \frac{4a^3}{a^2 + ab + b^2}$.

9. $\frac{x^2 + y^2 + 2xy - z^2}{z^2 - x^2 - y^2 + 2xy} \div \frac{x + y + z}{y + z - x}$.

10. $\frac{a^2 - 3ab + 2b^2}{a^2 - 6ab + 9b^2} \div \frac{a^2 - 2ab + b^2}{a^2 - 5ab + 6b^2}$.

11. $\frac{a^4 - 3a^3x + 3a^2x^2 - ax^3}{a^3b - b^4} \div \frac{a^4 - 2a^3x + a^2x^2}{a^2b^2 + ab^3 + b^4}$.

12. $\frac{64a^3 - 96a^2b + 36ab^2}{36a^2b - 729b^3} \div \frac{48a^2b - 27b^3}{8a^3 - 72a^2b + 162ab^2}$.

13. $\left(\frac{1+x}{1-x}-\frac{1-y}{1+y}\right)\div\left[1+\frac{(1+x)(1-y)}{(1-x)(1+y)}\right].$

14. $\left(\frac{a+b}{1-ab}-a\right)\frac{a}{b}\div\left[1+\frac{a(a+b)}{1-ab}\right].$

15. $\left(2x-\frac{x^2-y^2}{x}\right)\left(3y+\frac{x^2+y^2}{y}\right)\div\left(\frac{x^2}{y^2}+5+\frac{4y^2}{x^2}\right).$

16. $\frac{x^2-3ax+2a^2}{x^2-2ax-3a^2}\times\frac{x^2-8ax+15a^2}{x^2-8ax+12a^2}\div\frac{x^2-6ax+5a^2}{x^2-11ax+30a^2}.$

17. $\left(\frac{4}{x+1}-\frac{2}{x+2}\right)\div\left[\frac{(x+1)^2-4}{x^2+x-2}\right].$

18. $\frac{y^2+(a+c)y+ac}{y^2+(b+c)y+bc}\div\frac{y^2-a^2}{y^2-b^2}.$

19. $\frac{(x+y)^2-(a+b)^2}{(x+a)^2-(y+b)^2}\div\frac{(x-a)^2-(b-y)^2}{(x-y)^2-(b-a)^2}.$

Complex Fractions

184. Complex Fractions. A **complex fraction** is a fraction that has one or more fractions in either or both of its terms.

185. Simplify the complex fraction $\dfrac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x+1}}.$

$$\frac{1+\dfrac{1}{x-1}}{1-\dfrac{1}{x+1}}=\frac{\dfrac{x-1+1}{x-1}}{\dfrac{x+1-1}{x+1}}=\frac{\dfrac{x}{x-1}}{\dfrac{x}{x+1}}=\frac{x}{x-1}\times\frac{x+1}{x}=\frac{x+1}{x-1}.$$ Hence,

186. To Simplify a Complex Fraction,

Divide the numerator by the denominator

187. The shortest way to simplify a complex fraction is often to multiply both numerator and denominator of the complex fraction by the L.C.D. of the simple fractions (p. 123, § 166).

1. Simplify $\dfrac{\dfrac{a}{a-x}-\dfrac{a}{a+x}}{\dfrac{x}{a-x}+\dfrac{x}{a+x}}$.

The L.C.D. of the simple fractions is $(a-x)(a+x)$.

Multiply both numerator and denominator of the given complex fraction by $(a-x)(a+x)$. Then we have

$$\frac{a(a+x)-a(a-x)}{x(a+x)+x(a-x)}=\frac{a^2+ax-a^2+ax}{ax+x^2+ax-x^2}=\frac{2\,ax}{2\,ax}=1.$$

2. Simplify $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}$.

Multiply both terms of the *last* complex fraction $\dfrac{1}{x+\dfrac{1}{x}}$ by x. We have $\dfrac{x}{x^2+1}$, and this put in place of the last complex fraction changes the given fraction to the form $\dfrac{1}{x-\dfrac{x}{x^2+1}}$. Multiply both terms of this fraction by x^2+1. We have $\dfrac{x^2+1}{x^3+x-x}=\dfrac{x^2+1}{x^3}$.

EXERCISE 63

Simplify:

1. $\dfrac{1+\dfrac{1}{a-1}}{1-\dfrac{1}{a+1}}$.

2. $\dfrac{1+\dfrac{y}{x-y}}{1-\dfrac{y}{x+y}}$.

3. $\dfrac{m+\dfrac{mn}{m-n}}{m-\dfrac{mn}{m+n}}$.

4. $\dfrac{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}{\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}}$.

5. $\dfrac{x+\dfrac{y-x}{1+xy}}{1-\dfrac{(y-x)x}{1+xy}}$.

6. $\dfrac{\dfrac{2\,m+n}{m+n}-1}{1-\dfrac{n}{m+n}}$.

7. $$\frac{a-1+\dfrac{6}{a-6}}{a-2+\dfrac{3}{a-6}}.$$

8. $$\frac{1}{1+\dfrac{c}{1+c+\dfrac{2c^2}{1-c}}}.$$

9. $$\frac{1}{a+\dfrac{1}{1+\dfrac{a+1}{3-a}}}.$$

10. $$\frac{a}{b+\dfrac{c}{d+\dfrac{e}{f}}}.$$

11. $$\frac{\dfrac{a^2+b^2}{b}-a}{\dfrac{1}{b}-\dfrac{1}{a}}\times\frac{a^2-b^2}{a^3+b^3}.$$

12. $$\frac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\times\left(1+\frac{b^2+c^2-a^2}{2bc}\right).$$

13. $$\frac{\dfrac{x^3-y^3}{x^2+y^2}\times\dfrac{x^2-y^2}{x^3+y^3}\times\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)\div\left(\dfrac{1}{y}-\dfrac{1}{x}\right)}{\dfrac{(x+y)^2-xy}{(x-y)^2+xy}}.$$

14. $$\frac{\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1-x^4}-\dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}}.$$

15. $$\frac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}+\frac{\dfrac{1}{1+x}}{\dfrac{x}{1-x}}+\frac{\dfrac{1}{1-x}}{\dfrac{x}{1+x}}.$$

16. $$\frac{\left[\dfrac{(a+b)^2}{4ab}-1\right]\left[\dfrac{(a-b)^2}{4ab}+1\right]}{(a+b)^3-3a^2b-3ab^2}$$
$$\times\frac{[(a+b)^2-ab][(a-b)^2+ab]}{(a-b)^3+3ab(a-b)}$$
$$\times 4\left[\left(\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}\right)\div\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right)\right].$$

CHAPTER X

FRACTIONAL EQUATIONS

Reduction of Equations Containing Fractions

188. To solve an equation that contains fractions the work is generally made easier by *clearing the equation of fractions.*

1. Solve $1 - \frac{7 - 3x}{5} = \frac{x+1}{2} - \frac{3 - 7x}{10}$.

Multiply by 10, the L.C.M. of the denominators.

Then $\qquad 10 - 14 + 6x = 5x + 5 - 3 + 7x.$

Transpose, $\qquad 6x - 5x - 7x = 5 - 3 - 10 + 14.$

Combine, $\qquad -6x = -6.$

$$\therefore x = 1.$$

Note. When a minus sign precedes a fraction, in removing the denominator the sign of every term of the numerator is changed.

2. Solve $\frac{x-9}{x-5} + \frac{x-5}{x-8} = 2$.

The L.C.D. $= (x-5)(x-8)$.

Multiply by the L.C.D.

Then $\qquad x^2 - 17x + 72 + x^2 - 10x + 25 = 2x^2 - 26x + 80.$

Transpose,

$$x^2 + x^2 - 2x^2 - 17x - 10x + 26x = 80 - 72 - 25.$$

Combine, $\qquad -x = -17.$

$$\therefore x = 17.$$

Hence,

189. To Clear an Equation of Fractions,

Multiply each term by the L.C.M. of the denominators.

If a fraction is preceded by a **minus sign,** *the sign of every term of the numerator must be changed when the denominator is removed.*

Solve $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$.

NOTE. The solution of this problem and of similar problems will be much easier by combining the fractions on the left side and the fractions on the right side instead of following the rule.

$$\frac{(x-1)(x-3)-(x-2)^2}{(x-2)(x-3)}=\frac{(x-5)(x-7)-(x-6)^2}{(x-6)(x-7)}.$$

Simplify the numerators.

Then $$\frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}.$$

Since the numerators are equal, the denominators are equal.

Hence, $(x-2)(x-3)=(x-6)(x-7)$.

Expand, $x^2-5x+6=x^2-13x+42$.

Transpose and combine, $8x=36$.

$\therefore x=4\frac{1}{2}$.

EXERCISE 64

Solve:

1. $\frac{4x-1}{3}-\frac{3}{4}=\frac{x-4}{6}+\frac{3x+5}{4}$.

2. $\frac{4x+9}{10}-\frac{x+5}{5}=\frac{7x-1}{25}-\frac{x+3}{20}$.

3. $\frac{7}{x}+\frac{1}{3}=\frac{23-x}{3x}+\frac{7}{12}-\frac{1}{4x}$.

4. $\frac{5}{x+3}+\frac{3}{2(x+3)}=\frac{1}{2}-\frac{7}{2(x+3)}$.

5. $\frac{3x-5}{5x-5}+\frac{5x-1}{7x-7}+\frac{x-4}{x-1}=2$.

6. $\frac{8x+2}{x-2}-\frac{2x-1}{3x-6}+\frac{3x+2}{5x-10}=10$.

7. $\frac{3x-1}{2x-6}+\frac{5x-7}{3x-9}+\frac{7x+1}{4x-12}=11$.

8. $\frac{2x+1}{3}+\frac{3x+1}{4}=28-\frac{5x-2}{7}.$

9. $\frac{5(x+3)}{12}+\frac{5}{9}-\frac{11-3x}{36}=5x-20-\frac{13-x}{12}-\frac{21-3x}{18}.$

10. $\frac{x-3}{4}-\frac{2x-5}{6}-\frac{41}{60}-\frac{3x-8}{5}+\frac{5x+6}{15}=0.$

11. $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}.$

12. $\frac{4x}{15}-\frac{9}{10}-\frac{16x-81}{24}=\frac{2x-3}{15}-\frac{4x-9}{20}+\frac{9}{40}.$

13. $\frac{5x-16}{x-5}=\frac{5x-3}{x+5}.$

14. $\frac{3x+7}{4x+5}=\frac{3x+5}{4x+3}.$

15. $\frac{29-10x}{9-5x}=\frac{5+36x}{18x}.$

16. $\frac{47+6x}{11+2x}=\frac{2(8+3x)}{3+2x}.$

17. $\frac{5-2x}{x-1}-\frac{2-7x}{x+1}=\frac{5x^2+4}{x^2-1}.$

18. $\frac{2x+1}{2x-1}-\frac{8}{4x^2-1}=\frac{2x-1}{2x+1}.$

19. $\frac{1+3x}{5+7x}-\frac{9-11x}{5-7x}=\frac{14(2x-3)^2}{25-49x^2}$

20. $\frac{11x-13}{14}-\frac{22x-75}{28}=\frac{13x+7}{2(3x+7)}.$

21. $\frac{4x+3}{15x-35}-\frac{11x-5}{9x+21}=\frac{375x-86x^2-35}{10(9x^2-49)}.$

22. $\frac{2}{x-14}-\frac{5}{x-13}=\frac{2}{x-9}-\frac{5}{x-11}.$

23. $\frac{9}{x-51}-\frac{9}{x-15}=\frac{2}{x-81}-\frac{2}{x+81}.$

24. $\frac{x-8}{x-3}+\frac{x-3}{x-5}+\frac{x-9}{x-7}=\frac{x-1}{x-3}+\frac{x-13}{x-5}+\frac{x-6}{x-7}.$

25. $\frac{x+2}{x+7}+\frac{x+7}{x+5}+\frac{x+1}{x+3}=\frac{x+9}{x+7}+\frac{x-3}{x+5}+\frac{x+4}{x+3}.$

190. If the denominators contain both simple and compound expressions, it is generally best to remove the simple expressions first, and then each compound expression in turn. After each multiplication the result should be reduced to the simplest form.

1. Solve $\frac{6x+5}{8x-15} - \frac{1+8x}{15} = \frac{1-x}{3} + \frac{3-x}{5}$.

Multiply by 15, $\frac{90x+75}{8x-15} - 1 - 8x = 5 - 5x + 9 - 3x$.

Transpose and combine, $\frac{90x+75}{8x-15} = 15$.

Multiply by $8x - 15$, $90x + 75 = 120x - 225$.

Transpose, $90x - 120x = -75 - 225$.

Combine, $-30x = -300$.

$\therefore x = 10$.

Exercise 65

Solve:

1. $\frac{4x+3}{9} - \frac{8x+19}{18} + \frac{7x-29}{5x-12} = 0$.

2. $\frac{16x+7}{24} - \frac{2x+1}{3} = \frac{x-16}{3(3x-59)}$.

3. $\frac{2x-3}{2x-4} - 6 = \frac{x+5}{3x-6} - \frac{11}{2}$.

4. $\frac{2x-13}{2x-16} + \frac{2(x-6)}{x-8} = \frac{7}{8} + \frac{2(5x-39)}{3x-24}$.

5. $\frac{x+1}{4(x+2)} + \frac{x+4}{5x+13} = \frac{9}{20}$.

6. $\frac{x-2}{2x+1} + \frac{x-1}{3(x-3)} = \frac{5}{6}$.

7. $\frac{10-x}{3} + \frac{13+x}{7} = \frac{7x+26}{x+21} - \frac{17+4x}{21}$.

8. $\frac{7}{3}+\frac{13}{5x}=\frac{13x-24}{3x}-\frac{37}{20}+\frac{10}{x}$.

9. $\frac{7x-6}{35}-\frac{x-5}{6x-101}=\frac{x}{5}$.

10. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{41}{56}$.

11. $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{10x-11}{30}+\frac{1}{105}$.

12. $\frac{18x-22}{39-6x}+2x+\frac{1+16x}{24}=\frac{53}{12}-\frac{101-64x}{24}$.

13. $\frac{x}{x-1}-\frac{5}{2x-2}=\frac{8}{3x-3}-\frac{x}{x-1}+\frac{5}{18}$.

14. $\frac{13}{8}-\frac{5x}{4x+8}=\frac{2}{x+2}+\frac{1}{2}$.

Literal Equations

191. Literal equations are equations in which some or all of the given numbers are represented by letters; known numbers are represented by the *first* letters of the alphabet.

Solve $(a-x)(a+x)=2a^2+2ax-x^2$.

Expand, $a^2-x^2=2a^2+2ax-x^2$.

Transpose and combine, $-2ax=a^2$.

$\therefore x=-\frac{a}{2}$.

Exercise 66

Solve:

1. $\frac{c+d}{x}-a=\frac{c-d}{x}+b$.

2. $\frac{c+d}{ab+bx}=\frac{m-x}{an+nx}$.

3. $\frac{a+bx}{a+b}=\frac{c+dx}{c+d}$.

4. $\frac{6x+a}{4x+b}=\frac{3x-b}{2x-a}$.

5. $\frac{3(x-a)}{b}-\frac{2(x-b)}{a}=1.$

6. $\frac{x-a}{x-m}+\frac{x-b}{x-n}=2.$

7. $\frac{ax}{b}-\frac{cx}{d}+\frac{fx}{g}=h.$

8. $\frac{mx+n}{mx-n}=\frac{a}{b}.$

9. $\frac{5x}{3a+b}-2=\frac{8b}{5a}.$

10. $\frac{x-2a}{x+3a}=3-\frac{2x^2-13a^2}{x^2-9a^2}.$

11. $\frac{m}{x+m}+\frac{n}{x+n}=\frac{m+n}{x+p}.$

12. $\frac{3x}{a+3b}+\frac{18bx}{a^2-9b^2}=1.$

13. $\frac{1}{a}+\frac{a}{x+a}=\frac{x+a}{ax}.$

14. $\frac{mx-a-b}{nx-c-d}=\frac{mx-a-c}{nx-b-d}.$

15. $\frac{a(x-a)}{a+2b}+\frac{b(x-b)}{2a+b}=\frac{x}{2}.$

16. $\frac{c-d}{ab-by}=\frac{m+y}{an-ny}.$

17. $\frac{ax-b}{c}-\frac{bx+c}{a}=abc.$

18. $\frac{5ax}{a-b}-3a=8x.$

19. $\frac{3a+x}{5a-b}=\frac{4x-b}{3a+2b}-1.$

20. $\frac{m}{x-m}-\frac{n}{x-n}=\frac{m-n}{x-p}.$

21. $\frac{x-c}{c-d}-\frac{x+c}{c+d}=\frac{2cx}{c^2-d^2}.$

22. $\frac{x}{a+b}+abx=a+b+\frac{1}{ab}.$

23. $\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{x^2-ab}.$

24. $\frac{7x}{9m^2}-\frac{3}{2m}=\frac{9n+14x}{18m^2-24mn}.$

25. $\frac{a(3-2x)}{b}+\frac{b(3x-2)}{a}-\frac{a-bx}{2(a+b)}=2.$

26. $\frac{(a+1)x}{b}+\frac{(b+1)x}{a}+\frac{2ab}{a+b}=a+b+1.$

27. $\frac{a+c}{a-b}-\frac{(3a-5c)x}{2a-3b}+\frac{(3a-2b)(x-1)}{a-b}=\frac{(5c-2b)x}{2a-3b}-\frac{a-c}{a-b}.$

Problems Involving Fractional Equations

Exercise 67

1. The sum of the fourth and fifth parts of a certain number is one less than twice the difference between the third and tenth parts. Find the number.

Let $x =$ the number.

Then $\frac{x}{4} + \frac{x}{5} =$ the sum of the fourth and fifth parts,

$\frac{x}{3} - \frac{x}{10} =$ the difference of the third and tenth parts,

$2\left(\frac{x}{3} - \frac{10}{x}\right) =$ twice the difference of the third and tenth parts,

and $2\left(\frac{x}{3} - \frac{x}{10}\right) - \left(\frac{x}{4} + \frac{x}{5}\right) =$ the given excess.

But $1 =$ the given excess.

$$\therefore 2\left(\frac{x}{3} - \frac{x}{10}\right) - \left(\frac{x}{4} + \frac{x}{5}\right) = 1.$$

Multiply by 60, the L.C.D. of the fractions,

$$40x - 12x - 15x - 12x = 60.$$

Combine, $x = 60.$

Therefore, the required number is 60.

2. The difference between the seventh and ninth parts of a certain number is 4. Find the number.

3. One third of a certain number is 4 less than the sum of its fourth and sixth parts. Find the number.

4. The sum of the third and fourth parts of a certain number exceeds the difference between the fifth and sixth parts by 33. Find the number.

5. There are two consecutive numbers, x and $x + 1$, such that half of the larger number exceeds one fifth of the smaller by 8. Find the numbers.

6. The sum of two numbers is 63, and if the greater number is divided by the smaller the quotient is 2 and the remainder 9. Find the numbers.

Let $x =$ the greater number.
Then $63 - x =$ the smaller number.
Since the quotient $= \dfrac{\text{dividend} - \text{remainder}}{\text{divisor}}$,
and since in this problem the dividend is x, the remainder is 9, and the divisor is $63 - x$, we have

$$\frac{x - 9}{63 - x} = 2.$$

Solving, $x = 45.$

$$\therefore 63 - x = 18.$$

Therefore, the two numbers are 45 and 18.

7. The sum of two numbers is 75, and if the greater number is divided by the smaller the quotient is 3 and the remainder 7. Find the numbers.

8. The sum of two numbers is 137, and if the greater number is divided by the smaller the quotient is 4 and the remainder 2. Find the numbers.

9. The difference between two numbers is 19, and if the greater number is divided by the smaller the quotient is 2 and the remainder 8. Find the numbers.

10. The difference between two numbers is 63, and if the greater number is divided by the smaller the quotient is 2 and the remainder 19. Find the numbers.

11. Divide 227 into two parts such that the smaller part is contained in the larger 5 times with a remainder of 5.

12. Divide 367 into two parts such that the smaller part is contained in the larger 4 times with a remainder of 52.

13. Divide 421 into two parts such that the smaller part is contained in the larger 7 times with a remainder of 5.

14. Seven years ago a boy was half as old as he will be one year hence. How old is he now?

Let $x =$ the number of years old he is now.
Then $x - 7 =$ the number of years old he was 7 years ago,
and $x + 1 =$ the number of years old he will be 1 year hence.

$$\therefore x - 7 = \frac{x+1}{2}.$$

Solving, $x = 15$.

Therefore, the boy is 15 years old.

15. A son is two fifths as old as his father. In 10 years he will be half as old as his father. Find the age of each.

16. A is one third as old as B. In 4 years A will be half as old as B. Find the age of each.

17. The sum of the ages of A and B is 70 years, and 15 years hence A's age will be one third of B's. Find their ages.

18. A is 50 years old and B is half as old as A. In how many years will B be two thirds as old as A?

19. A is 72 years old and B is two thirds as old as A. How many years ago was A five times as old as B?

20. The sum of the ages of a father and son is 80 years. The son's age increased by 10 years is one half of the father's age. Find their ages.

21. B is one half as old as A. Seven years ago B was two fifths as old as A. Find their ages.

22. B's age is one sixth of A's age. In six years B's age will be one fourth of A's age. Find their ages.

23. A's age is two thirds of B's age. Eight years ago A's age was three fifths of B's age. Find their ages.

24. A is sixteen years older than B. Sixteen years ago A was three times as old as B. Find their ages.

25. A rectangle has its length 14 feet more and its width 10 feet less than the side of the equivalent square. Find the dimensions of the rectangle.

Let $x =$ the number of feet in a side of the square.
Then $x + 14 =$ the number of feet in the length of the rectangle,
and $x - 10 =$ the number of feet in the width of the rectangle.

Since the number of square units in the area of a rectangle is equal to the product of the number of linear units in the length multiplied by the number of linear units in the width of the rectangle,

$(x + 14)(x - 10) =$ the number of square feet in the area of the rectangle,
and $x \times x =$ the number of square feet in the area of the square.

But these areas are equal.

$$\therefore (x + 14)(x - 10) = x^2.$$

Solving, $x = 35.$

$$\therefore x + 14 = 49,$$

and $x - 10 = 25.$

Therefore, the dimensions of the rectangle are 49 feet and 25 feet.

26. A rectangle has its length 21 feet more and its width 12 feet less than the side of the equivalent square. Find the dimensions of the rectangle.

27. A rectangle has its length 9 feet more and its width 8 feet less than the side of the equivalent square. Find the dimensions of the rectangle.

28. The length of a floor exceeds the width by 7 feet. If each dimension were 3 feet more, the area would be 102 square feet more. Find the dimensions.

29. The length of a rectangle exceeds the width by 9 inches. If the length were diminished by 5 inches and the width by 4 inches, the area would be diminished by 205 square inches. Find the dimensions.

30. The length of a rectangular field exceeds the width by 82 yards. If the length were diminished by 60 yards and the width increased by 40 yards, the area would be diminished by 1480 square yards. Find the dimensions.

31. A can do a piece of work in 3 days, and B can do the work in 4 days. How many days will it take them to do it working together?

Let x = the number of days it will take them working together.

Then $\frac{1}{x}$ = *the part both together can do in one day,*

$\frac{1}{3}$ = the part A can do in one day,

$\frac{1}{4}$ = the part B can do in one day,

and $\frac{1}{3}+\frac{1}{4}$ = *the part both together can do in one day.*

$$\therefore \frac{1}{3}+\frac{1}{4}=\frac{1}{x}.$$

Solving, $x = 1\frac{5}{7}$.

Therefore, A and B together can do the work in $1\frac{5}{7}$ days.

32. A can do a piece of work in 5 days, B in 6 days, and C in 7 days. How many days will it take them to do it working together?

33. A can do a piece of work in 9 days, B in 8 days, and C in $7\frac{1}{2}$ days. How many days will it take them to do it working together?

34. A can do a piece of work in $3\frac{1}{2}$ days, B in $4\frac{2}{3}$ days, and C in $5\frac{1}{4}$ days. How many days will it take them to do it working together?

35. A and B together can mow a field in 8 hours, A and C in 10 hours, and A alone in 15 hours. In how many hours can B and C together mow the field?

36. A can do a piece of work in $2a$ days, A and B together in b days, and A and C together in $a+\frac{b}{2}$ days. In how many days can the three do the work together?

37. A and B together can dig a ditch in 40 days, B and C together in 48 days, and A and C together in 36 days. How many days will it take each alone to dig the ditch, and how many days if they all work together?

HINT. By working *two days each* they dig $\frac{1}{40}+\frac{1}{48}+\frac{1}{36}$ of the ditch.

38. A cistern can be filled by two pipes in 3 and 4 hours respectively, and can be emptied by a waste pipe in 6 hours. In how many hours will it be filled if all the pipes together are open?

Let $x =$ the number of hours if all the pipes are open.

Then $\frac{1}{x} =$ the part filled in one hour if all the pipes are open,

$\frac{1}{3} =$ the part filled in one hour by the first pipe,

$\frac{1}{4} =$ the part filled in one hour by the second pipe,

$\frac{1}{6} =$ the part emptied in one hour by the waste pipe,

and $\frac{1}{3} + \frac{1}{4} - \frac{1}{6} =$ the part filled in one hour if all the pipes are open.

$$\therefore \frac{1}{3} + \frac{1}{4} - \frac{1}{6} = \frac{1}{x}.$$

Solving, $x = 2\frac{2}{5}$.

Therefore, if all the pipes are open, the cistern will be filled in $2\frac{2}{5}$ hours.

39. A cistern can be filled by three pipes in 6, 10, and 15 hours respectively. In how many hours will it be filled by all the pipes together?

40. A cistern can be filled by three pipes in 3 hours 30 minutes, 2 hours 48 minutes, and 2 hours 20 minutes respectively. In what time will the cistern be filled if all the pipes are running together?

41. A cistern can be filled by two pipes in $3\frac{1}{3}$ and $3\frac{3}{4}$ hours respectively, and can be emptied by a waste pipe in $2\frac{1}{2}$ hours. In how many hours will it be filled if all the pipes together are open?

42. A cistern has three pipes. The first pipe will fill it in $5\frac{1}{2}$ hours, the second in $7\frac{1}{3}$ hours, and the three pipes together will fill it in $1\frac{5}{6}$ hours. In how many hours will the third pipe alone fill the cistern?

43. A cistern can be filled by three pipes in $4\frac{1}{4}$, $5\frac{2}{3}$, and $3\frac{2}{5}$ hours respectively. In how many hours will it be filled by the three pipes together?

44. An accommodation train which goes 32 miles an hour is followed, after 2 hours, by an express train which goes 48 miles an hour. In how many hours will the express train overtake the accommodation train?

Let $x =$ the number of hours the express train goes.
Then $x + 2 =$ the number of hours the accommodation train goes,
$48x =$ the number of miles the express train goes,
and $32(x + 2) =$ the number of miles the accommodation train goes.
But these distances are equal.

$$\therefore\ 48x = 32(x + 2).$$

Solving, $x = 4$.

Therefore, the express train will overtake the accommodation train in 4 hours.

45. A starts from Boston and travels at the rate of 8 miles an hour. Four hours afterwards B starts from the same place and follows A at the rate of 10 miles an hour. How many miles from Boston will B overtake A?

46. A can run 10 yards a second and B can run 9 yards a second. If A gives B a start of 2 seconds, in how many seconds will A overtake B?

47. A freight train which goes 18 miles an hour is followed after 4 hours by a passenger train which goes 24 miles an hour. In how many hours will the passenger train overtake the freight train?

48. In going a certain distance a train that makes 42 miles an hour takes 2 hours less than a train that makes 35 miles an hour. Find the distance.

49. In traveling a certain distance a horse that goes 12 miles an hour takes 1 hour less than a horse that goes 10 miles an hour. Find the distance.

50. In traveling a certain distance an automobilist who travels 28 miles an hour takes $5\frac{1}{4}$ hours less than a bicyclist who travels 16 miles an hour. Find the distance.

51. Find the time between 3 and 4 o'clock when the hands of a clock are together.

At 3 o'clock the hour-hand is 15 minute-spaces ahead of the minute-hand.

Let $x =$ *the number of spaces the minute-hand moves over.*

Then $x - 15 =$ the number of spaces the hour-hand moves over.

Now, as the minute-hand moves 12 times as fast as the hour-hand,

$12(x-15) =$ *the number of spaces the minute-hand moves over.*

$$\therefore 12(x-15) = x.$$

Solving, $x = 16\frac{4}{11}$.

Therefore, the required time is $16\frac{4}{11}$ minutes past 3 o'clock.

52. Find the time between 6 and 7 o'clock when the hands of a clock are together.

53. Find the time between 9 and 10 o'clock when the hands of a clock are at right angles to each other.

Hint. In this case the minute-hand is 15 minute-spaces behind the hour-hand.

54. Find the time between 4 and 5 o'clock when the hands of a clock point in opposite directions.

55. Find the times between 5 and 6 o'clock when the hands of a clock make right angles with each other.

Hint. In this case the minute-hand is 15 minute-spaces behind the hour-hand, or 15 minute-spaces ahead of the hour-hand.

56. Find the time between 4 and 5 o'clock when the hands of a clock are together.

57. Find the time between 11 and 12 o'clock when the hands of a clock point in opposite directions.

58. Find the times between 7 and 8 o'clock when the hands of a clock make right angles with each other.

59. Find the time between 1 and 2 o'clock when the hands of a watch point in opposite directions.

60. A hound takes 2 leaps while a rabbit takes 3; but 3 of the hound's leaps are equivalent to 5 of the rabbit's. The rabbit has a start of 70 of her own leaps. How many leaps must the hound take to catch the rabbit?

Let $2x =$ the number of leaps taken by the hound.
Then $3x =$ the number of leaps of the rabbit in the same time.
Also let $a =$ the number of feet in one leap of the rabbit.
Then $\frac{5a}{3} =$ the number of feet in one leap of the hound,
and $2x \times \frac{5a}{3} = \frac{10ax}{3} =$ the number of feet the hound runs.

As the rabbit has a start of 70 leaps, and takes $3x$ leaps more before being caught, and as each leap is a feet,

$(70 + 3x)a =$ the number of feet the hound runs.

$$\therefore \frac{10ax}{3} = (70 + 3x)a.$$

Solving, $2x = 420.$

Therefore, the hound must take 420 leaps.

61. A hare takes 5 leaps to 3 of a hound; and 3 of the hound's leaps are equivalent to 7 of the hare's. The hare has a start of 60 of her own leaps. How many leaps must the hound take to catch the hare?

62. A hound takes 4 leaps while a hare takes 5; but 5 of the hound's leaps are equivalent to 8 of the hare's. The hare has a start of 84 of her own leaps. How many leaps will the hare take before she is caught?

63. A hound takes 3 leaps while a hare takes 4; but 5 of the hound's leaps are equivalent to 9 of the hare's. The hare has a start of 189 of *the hound's leaps*. How many leaps will each take before the hare is caught?

64. A merchant adds each year to his capital one fifth of it, and deducts, at the end of each year, $1000 for expenses. In four years, after deducting the last $1000, he has $3160 less than twice his original capital. How much had he at first?

Let x = the number of dollars the merchant had at first.

Then $\frac{6x}{5} - 1000$, or $\frac{6x - 5000}{5}$

= the number of dollars he had at the end of the first year,

$\frac{6}{5}\left(\frac{6x - 5000}{5}\right) - 1000$, or $\frac{36x - 55000}{25}$

= the number of dollars he had at the end of the second year,

$\frac{6}{5}\left(\frac{36x - 55000}{25}\right) - 1000$, or $\frac{216x - 455000}{125}$

= the number of dollars he had at the end of the third year,

and $\frac{6}{5}\left(\frac{216x - 455000}{125}\right) - 1000$, or $\frac{1296x - 3355000}{625}$

= the number of dollars he had at the end of the fourth year.

But $2x - 3160$ = the number of dollars he had at the end of the fourth year.

$$\therefore \frac{1296x - 3355000}{625} = 2x - 3160.$$

Solving, $x = 30000$.

Therefore, the merchant had $30,000 at first.

65. A merchant adds each year to his capital one fourth of it, but takes from it, at the end of each year, $1200 for expenses. At the end of the third year, after deducting the last $1200, he has $950 less than $1\frac{1}{2}$ times his original capital. How much had he at first?

66. A trader maintained himself for three years at an expense of $2000 a year, and each year increased that part of his stock that was not so expended by one third of it. At the end of three years his original stock was doubled. How much did he have at first?

67. A merchant adds each year to his capital one third of it, but takes from it, at the end of each year, $1250 for expenses. At the end of the third year, after he has deducted the last $1250, he finds that he has twice his original capital. What was his original capital?

68. The sum of the ages of three brothers is 24 years, and there is a difference of two years between their successive ages. Find the age of each.

69. An uncle is 10 years older than his nephew, and 15 years ago the uncle's age was twice that of the nephew. Find the age of each.

70. In an election 1240 votes were cast for two candidates. Of these votes 47 were defective and were thrown out. The successful candidate received 153 votes more than his opponent. How many votes did each candidate receive?

71. A man and a woman together can do a piece of work in 12 hours and the woman alone can do it in 30 hours. How long will it take the man alone?

72. A can do a piece of work in 50 days, B in 60 days, and C in 75 days. How long will it take the three working together?

73. A can dig a ditch in half the time it takes B, and B can dig it in two thirds the time it takes C. If the three work together they can dig the ditch in 6 days. How long will it take each alone?

74. Divide 451 into two parts such that if 14 is added to the first part and the second part is divided by 14 the results are the same.

75. A farmer agreed to give a farm hand in payment for a year's work $200 in money and a cow. The farm hand was obliged to leave after working 10 months and received $160 in money and the cow. At what price was the value of the cow estimated?

76. If three times the difference between a third and a fourth of a certain number is subtracted from three fourths of the number, the remainder is equal to 6 more than one fifth of the number. Find the number.

77. The denominator of a certain fraction is 5 greater than the numerator. If 8 is added to each term, the resulting fraction is equal to $\frac{3}{4}$. Find the fraction.

78. A number is composed of two digits whose difference is 5. If the digits are interchanged, the resulting number is three eighths of the original number. Find the original number.

79. A does $\frac{7}{30}$ of a piece of work in 5 hours, B does $\frac{5}{16}$ of the remainder in $2\frac{1}{2}$ hours, and C completes the work in 8 hours. How many hours would it take the three working together to do the work?

80. If a certain number is divided by 4 and by 9 and the two quotients multiplied together, the product is equal to the product obtained by multiplying the number by the difference between the number and 35. What is the number?

81. The length of a rectangular garden exceeds the breadth by 4 yards. If each dimension is increased by 1 yard, the area is increased by 27 square yards. Find the dimensions.

82. A and B have the same income. A saves 5 per cent of his income, while B, who spends each year $175 more than A, finds at the end of 4 years that he is $200 in debt. What income does each receive?

83. Find three numbers whose sum is 70 such that the second number divided by the first gives 2 for a quotient and 1 for a remainder, while the third number divided by the second gives 3 for a quotient and 3 for a remainder.

84. A railway train which travels at the rate of 32 miles an hour is 45 minutes in advance of a train which travels at the rate of 42 miles an hour. In how long a time will the second train overtake the first?

85. At 12 o'clock the hands of a watch point in the same direction. At what time do the hands next point in the same direction?

86. An express train which makes 42 miles an hour starts 50 minutes after an accommodation train, and overtakes it in 2 hours 5 minutes. How many miles an hour does the accommodation train make?

87. A starts on a trip, making 4 miles an hour, and 15 minutes afterwards B sets out from the same place at the rate of $4\frac{3}{4}$ miles per hour. How many miles will B walk to overtake A and how long will it take him?

88. Three men, A, B, and C, set out from the same place and travel over the same route at the rates of 4, 5, and 6 miles respectively per hour. B starts two hours after A. How long after B should C start that B and C may overtake A at the same moment?

89. At what times between 6 and 7 o'clock do the hands of a watch make right angles with each other?

90. At what time between 3 and 4 o'clock do the hands of a watch point in opposite directions?

91. A man is 565 of his own steps ahead of a horse. If 4 of the man's steps are equal to 3 of the horse's, and the man takes 5 steps while the horse takes 4, how many steps must the horse take to overtake the man?

92. A father 50 years old has two sons whose ages respectively are 14 years and 11 years. In how many years will the sum of the ages of the sons equal the age of the father?

93. Four workmen do a piece of work together. A could do the work in 8 days, B in 10 days, C in 12 days, and D in 15 days. In how many days do they finish the work?

94. A messenger starts to carry a dispatch; $4\frac{1}{2}$ hours after his departure a second messenger is sent, who ought to overtake the first in 9 hours, and to do this he is obliged to travel 3 more miles per hour. How many miles per hour does the first messenger travel?

95. A, B, and C have together \$1000. A has \$70 less than a third more than B, and C has \$20 more than three sevenths of the amount A and B together have. How many dollars has each?

96. A man has 9 hours at his disposal. How far into the country may he ride in an automobile at $22\frac{1}{2}$ miles an hour that he may return in time, driving a horse at the rate of $6\frac{1}{2}$ miles an hour?

97. In going from New York to Buffalo, a fast express train at 55 miles an hour takes 3 hours less than a passenger train at 40 miles an hour. Find the distance from New York to Buffalo.

98. A boy buys apples at the rate of 5 for 2 cents. He sells half of them 2 for a cent and the other half 3 for a cent, and gains 10 cents by the transaction. How many apples does he buy?

99. When a body of men is formed into a solid square, there are 54 men over; but when formed in a column with 5 men more in front and 4 men fewer in depth, there are lacking 4 men to complete it. How many men are there?

100. Divide the number 628 into two parts such that the smaller part is contained in the larger part twice, with a remainder of 7.

101. The difference between two numbers is 154, and if the greater number is divided by the less the quotient is 3 and the remainder 10. Find the numbers.

102. The length of a room is two thirds the width. If the width was 4 feet more and the length 4 feet less, the room would be square. Find the dimensions.

103. A walks to the top of a mountain at the rate of $1\frac{3}{4}$ miles an hour, and back again at the rate of $2\frac{1}{3}$ miles an hour, and is out 7 hours. How far is it to the top of the mountain?

FORMULAS AND RULES

192. Formulas and Rules. When the given numbers of a problem are represented by *letters,* the result obtained from solving the problem is a general expression which includes all problems of that class. Such an expression is called a **formula,** and the translation of this formula into words is called a **rule.**

We will illustrate by two examples.

1. The sum of two numbers is s, and the difference between them is d. Find the numbers.

Let $x =$ the smaller number.

Then $x + d =$ the larger number.

$$\therefore x + x + d = s.$$

$$2x = s - d.$$

$$\therefore x = \frac{s-d}{2},$$

and

$$x + d = \frac{s - d + 2d}{2}$$

$$= \frac{s+d}{2}.$$

Therefore, the numbers are $\frac{s+d}{2}$ and $\frac{s-d}{2}$.

Since these formulas hold true whatever numbers s and d represent, we have the general rule for finding two numbers when their sum and difference are given:

Add the difference to the sum and take half the result for the greater number.

Subtract the difference from the sum and take half the result for the smaller number.

2. If A can do a piece of work in m days, and B can do the same work in n days, in how many days can both together do the work?

Let $x =$ the required number of days.

Then $\frac{1}{x} =$ the part both together can do in one day,

$\frac{1}{m} =$ the part A can do in one day,

$\frac{1}{n} =$ the part B can do in one day,

and $\frac{1}{m} + \frac{1}{n} =$ the part both together can do in one day.

$$\therefore \frac{1}{m} + \frac{1}{n} = \frac{1}{x}.$$

$$nx + mx = mn.$$

$$(m + n)x = mn.$$

$$\therefore x = \frac{mn}{m + n}.$$

The translation of this formula gives the following rule for finding the time required by two agents together to produce a given result, when the time required by each agent separately is known:

Divide the product of the numbers that express the units of time required by each agent separately to do the work by the sum of these numbers. The quotient is the number of units of time required by both agents acting together.

Exercise 68

1. A and B together can do a piece of work in m days, A and C together in n days, and B and C together in p days. How many days will it take each alone to do the work, and how many days if the three work together?

2. Three partners, A, B, and C, are to divide a profit of d dollars made in a partnership. A furnished a dollars for m months; B, b dollars for n months; and C, c dollars for p months. What will be the share of each?

3. The population of a city increases each year t per cent of that of the preceding year. The population is a inhabitants. What was the population n years ago?

4. A number is divided by d, and the sum of the quotient, dividend, and divisor is b. What is the number?

5. A traveler wishes to accomplish a journey of k miles in n days. At the end of d days he receives news which obliges him to reach the end of his journey a days sooner. How many more miles must he travel each day?

6. The circumference of a fore wheel of a carriage is a feet, and that of a hind wheel is b feet. Find the distance the carriage has traveled when the fore wheel has made n revolutions more than the hind wheel.

7. A boatman, rowing with the tide, makes n miles in t hours. In returning he rows against a tide m times as strong as the first and makes the distance in t' hours. Find the velocity of the second tide.

Interest Formulas

193. The elements involved in the computation of interest are the *principal, rate, time, interest,* and *amount.*

Let p = the number of dollars in the principal,

r = the number of dollars in the interest of $1 for 1 year,

t = the time expressed in years,

i = the interest on the given principal for the given time at the given rate,

a = the amount (sum of principal and interest).

194. Given the Principal, Rate, and Time; to find the Interest. Since r is the interest on $1 for 1 year, pr is the interest on $$p$ for 1 year, and prt is the interest on $$p$ for t years.

Therefore, $\mathbf{i = prt}$. (Formula 1)

195. Given the Interest, Rate, and Time; to find the Principal.

By Formula 1, $prt = i.$

Divide by rt, $\mathbf{p} = \dfrac{\mathbf{i}}{\mathbf{rt}}.$ (Formula 2)

196. Given the Amount, Rate, and Time; to find the Principal.

By § 193, $p + i = a.$

By Formula 1, $p + prt = a.$

Therefore, $p(1 + rt) = a.$

Divide by $1 + rt$, $\mathbf{p} = \dfrac{\mathbf{a}}{\mathbf{1 + rt}}.$ (Formula 3)

197. Given the Amount, Principal, and Rate; to find the Time.

By § 196, $p + prt = a.$

Transpose, $prt = a - p.$

Divide by pr, $\mathbf{t} = \dfrac{\mathbf{a - p}}{\mathbf{pr}}.$ (Formula 4)

198. Given the Amount, Principal, and Time; to find the Rate.

By § 196, $p + prt = a.$

Transpose, $prt = a - p.$

Divide by pt, $\mathbf{r} = \dfrac{\mathbf{a - p}}{\mathbf{pt}}.$ (Formula 5)

Exercise 69

Solve by the preceding formulas:

1. Find the interest on \$1100 for 3 years 4 months at 5 per cent.

2. Find the interest on \$1275 for 3 years 2 months 15 days at 8 per cent.

3. Find the amount of \$6460 for 1 year 10 months 22 days at $4\frac{1}{4}$ per cent.

4. Find the amount of \$1250 for 3 months 16 days at 5 per cent.

5. Find the rate per cent when the interest on \$997.75 is \$199.55 for 5 years 4 months.

6. Find the rate per cent when \$350 amounts to \$406.70 in 3 years 7 months 6 days.

7. Find the time in which the interest on \$487.50 will amount to \$39 at 4 per cent.

8. Find the time in which the sum of \$1587.75 will amount to \$1611.68 at $5\frac{1}{2}$ per cent.

9. Find the principal that will produce \$1746.60 interest in 3 years 5 months at 6 per cent.

10. Find the principal that will produce \$1339.28 interest in 2 years 7 months 24 days at 6 per cent.

11. Find the principal that will amount to \$6378.75 in 1 year 1 month at 5 per cent.

12. Find the principal that will amount to \$21,047.95 in 1 year 7 months 21 days at $4\frac{1}{2}$ per cent.

13. If A can do a piece of work in $3\frac{1}{2}$ days, and B can do the same work in $4\frac{1}{2}$ days, in how many days can both together do the work?

14. A and B together can do a piece of work in 12 days, A and C together in 15 days, and B and C together in 16 days. How many days will it take each alone to do the work, and how many days if the three work together?

15. Three graziers, A, B, and C, hire a pasture for which they pay \$132.50. A puts in 10 oxen for 3 months; B, 12 oxen for 4 months; and C, 14 oxen for 2 months. How much rent ought each to pay?

16. A traveler wishes to accomplish a journey of 240 miles in 8 days. At the end of 3 days he receives news which obliges him to reach the end of his journey 2 days sooner. How many more miles must he travel each day?

17. The circumference of a fore wheel of a carriage is 11 feet 3 inches, and that of a hind wheel is 12 feet 2 inches. Find the distance the carriage has traveled when the fore wheel has made 150 more revolutions than the hind wheel.

18. A boatman, rowing with the tide, makes 27 miles in 4 hours. In returning he rows against a tide twice as strong as the first and makes the distance in 9 hours. Find the velocity of the second tide.

19. The population of a city increases each year 2 per cent of that of the preceding year. The population now is 54,121 inhabitants. What was the population 4 years ago?

20. A number is divided by 16, and the sum of the quotient, dividend, and divisor is 50. What is the number?

21. Find the time required for the interest at 4 per cent on a sum of money to be equal to the principal.

22. The sum of two numbers is 325 and the difference between them is 93. Find the numbers.

23. Three partners in a restaurant furnish respectively $500 for 7 months, $600 for 8 months, and $900 for 9 months. If they lose $410, what is each one's share of the loss?

24. Find the time in which the interest on $8520 will amount to $1746.60 at 6 per cent.

25. Find the principal that will produce $1312.65 interest in 2 years 3 months at 6 per cent.

26. Find the time required for $4000 to amount to $4625 at $5\frac{1}{2}$ per cent.

27. Find the time required for $3904.92 to amount to $4568.76 at 5 per cent.

28. A and B go into partnership. A furnishes $5000 for 13 months and B furnishes $7000 for 9 months. Their profits are $1700. What is the share of each?

CHAPTER XI

SIMULTANEOUS SIMPLE EQUATIONS

199. If we have *two* unknown numbers and but *one relation* between them, we may find an *unlimited number of pairs of values* for which the given relation will hold true.

Thus, if x and y are unknown numbers, and we have given only the one relation $x + y = 6$, we may *assume* any value for x, and then from the relation $x + y = 6$ find the corresponding value of y. For from $x + y = 6$ we find $y = 6 - x$. If x stands for 1, y stands for 5; if x stands for 2, y stands for 4; if x stands for -1, y stands for 7; if x stands for -2, y stands for 8; and so on without end.

200. Independent Equations. We may, however, have two equations that express *different* relations between the two unknowns. Such equations are called **independent equations.**

Thus, $x + y = 6$ and $x - y = 2$ are independent equations, for they evidently express *different* relations between x and y. But $x + y = 6$ and $2x + 2y = 12$ are not independent equations, for both express the *same* relation between the unknown numbers.

201. An equation is said to be **satisfied** by a number, if we may substitute that number for one of the unknowns in the equation without destroying the equality.

202. Simultaneous Equations. Independent equations that involve the *same* unknowns are called **simultaneous equations.**

If we have *two* unknowns, and have given *two* independent equations involving them, there is but *one* pair of values which holds true for both equations.

Thus, for the equations $x + y = 6$ and $x - y = 2$ the only pair of values for which *both* equations hold true is the pair $x = 4$, $y = 2$.

203. Elimination. Simultaneous simple equations are solved by combining the equations so as to obtain a single equation with one unknown number. This process is called **elimination.**

There are three methods of elimination in general use :

I. By Addition or Subtraction.
II. By Substitution.
III. By Comparison.

204. Elimination by Addition or Subtraction.

1. Solve

$$\left.\begin{aligned} 5x + 2y &= 36 \\ 3x - 5y &= 3 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by 5, $25x + 10y = 180.$ (3)
Multiply (2) by 2, $6x - 10y = 6.$ (4)
Add (3) and (4), $31x = 186.$
$\therefore x = 6.$

Substitute the value of x in (1), and we obtain $y = 3$.
Therefore, $x = 6$, and $y = 3$.

In this solution y is eliminated by *addition.*

Check. Substitute in (1) and (2) the values of x and y, and we have

in (1) $30 + 6 = 36,$
in (2) $18 - 15 = 3.$

2. Solve

$$\left.\begin{aligned} 2x + 3y &= 43 \\ 5x + 2y &= 36 \end{aligned}\right\} \quad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

Multiply (1) by 2, $4x + 6y = 86.$ (3)
Multiply (2) by 3, $15x + 6y = 108.$ (4)
Subtract (3) from (4), $11x = 22.$
$\therefore x = 2.$

Substitute the value of x in (1), and we obtain $y = 13$.
Therefore, $x = 2$, and $y = 13$.

In this solution y is eliminated by *subtraction.*

Check. Substitute in (1) and (2) the values of x and y, and we have

in (1) $4 + 39 = 43,$
in (2) $10 + 26 = 36.$

205. To Eliminate by Addition or Subtraction, therefore,

Multiply the equations by such numbers as will make the coefficients of one of the unknown numbers numerically equal in the resulting equations.

Add the resulting equations, or subtract one from the other, according as these equal coefficients have unlike or like signs.

NOTE. It is generally best to select as the letter to be eliminated that which requires the smaller multipliers to make the coefficients equal; and the smaller multiplier for each equation is found by dividing the L.C.M. of the coefficients of this letter by the given coefficient in that equation.

Sometimes the solution may be simplified by first adding the given equations or by subtracting one from the other.

1. Solve

$$\left.\begin{array}{r} x + 27y = 29 \\ 27x + y = 55 \end{array}\right\} \qquad \begin{array}{r}(1)\\(2)\end{array}$$

Add (1) and (2), $28x + 28y = 84.$ (3)

Divide (3) by 28, $x + y = 3.$ (4)

Subtract (4) from (1), $26y = 26.$

$\therefore y = 1.$

Subtract (4) from (2), $26x = 52.$

$\therefore x = 2.$

Therefore, $x = 2$, and $y = 1$.

EXERCISE 70

Solve by addition or subtraction:

1. $\left.\begin{array}{r} 4x + 9y = 3 \\ 3x + 7y = 2 \end{array}\right\}$

2. $\left.\begin{array}{r} 8x + 3y = 38 \\ 2x - 7y = -6 \end{array}\right\}$

3. $\left.\begin{array}{r} x + 4y = 31 \\ 4x + y = 19 \end{array}\right\}$

4. $\left.\begin{array}{r} 5x - 3y = 20 \\ 3x - 4y = 1 \end{array}\right\}$

5. $\left.\begin{array}{r} 8x + 4y = 100 \\ 9x - 5y = 8 \end{array}\right\}$

6. $\left.\begin{array}{r} 5x - 2y = 23 \\ 13x - 3y = 51 \end{array}\right\}$

7. $\left.\begin{aligned} 6x + 7y &= 31 \\ 3x + 2y &= 5 \end{aligned}\right\}$

8. $\left.\begin{aligned} 11x - 12y &= 31 \\ 10x + 7y &= 64 \end{aligned}\right\}$

9. $\left.\begin{aligned} 4x - 3y &= 29 \\ 2x + y &= 47 \end{aligned}\right\}$

10. $\left.\begin{aligned} 2x - 11y &= 67 \\ 5x - 6y &= 60 \end{aligned}\right\}$

11. $\left.\begin{aligned} 7x - 3y &= 22 \\ 3x + 5y &= 66 \end{aligned}\right\}$

12. $\left.\begin{aligned} 2x - 3y &= 3 \\ 5x + 7y &= 109 \end{aligned}\right\}$

13. $\left.\begin{aligned} 10x + 7y &= 75 \\ 12x - 5y &= 23 \end{aligned}\right\}$

14. $\left.\begin{aligned} 7x - 5y &= 1 \\ 9x - 10y &= 12 \end{aligned}\right\}$

206. Elimination by Substitution.

Solve

$$\left.\begin{aligned} 9x - 5y &= 52 \quad (1) \\ 8y - 3x &= 8 \quad (2) \end{aligned}\right\}$$

Transpose (1), $9x = 52 + 5y.$ (3)

Divide (3) by 9, $x = \dfrac{52 + 5y}{9}.$ (4)

Substitute the value of x in (2),

$$8y - 3\left(\frac{52 + 5y}{9}\right) = 8.$$

Simplify, $8y - \dfrac{52 + 5y}{3} = 8.$

Clear of fractions, $24y - 52 - 5y = 24.$

Transpose and combine, $19y = 76.$

$\therefore y = 4.$

Substitute the value of y in (4), $x = 8.$

Therefore, $x = 8$, and $y = 4$.

Check. Substitute in (1) and (2) the values of x and y, and we have

in (1) $72 - 20 = 52,$

in (2) $32 - 24 = 8.$

207. To Eliminate by Substitution, therefore,

From one of the equations obtain the value of either of the unknown numbers in terms of the other.

Substitute for this unknown number its value in the other equation, and reduce the resulting equation.

Exercise 71

Solve by substitution:

1. $\left.\begin{aligned} x+5y&=25 \\ 7x+3y&=47 \end{aligned}\right\}$

2. $\left.\begin{aligned} 3x-2y&=15 \\ 5x-9y&=8 \end{aligned}\right\}$

3. $\left.\begin{aligned} 10x-3y&=116 \\ 3x-10y&=53 \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x+y&=11 \\ 4x-9y&=99 \end{aligned}\right\}$

5. $\left.\begin{aligned} 3x-y&=16 \\ 5x-12y&=-25 \end{aligned}\right\}$

6. $\left.\begin{aligned} 7x-2y&=91 \\ 2x+5y&=143 \end{aligned}\right\}$

7. $\left.\begin{aligned} 2x+17y&=61 \\ 8x-y&=37 \end{aligned}\right\}$

8. $\left.\begin{aligned} 9x-14y&=14 \\ 11x+12y&=119 \end{aligned}\right\}$

9. $\left.\begin{aligned} 8x+3y&=113 \\ 4x+7y&=29 \end{aligned}\right\}$

10. $\left.\begin{aligned} 2x-9y&=6 \\ 5x+4y&=68 \end{aligned}\right\}$

11. $\left.\begin{aligned} 3y-2x&=30 \\ 2y-5x&=42 \end{aligned}\right\}$

12. $\left.\begin{aligned} 5x+7y&=125 \\ 7x-y&=13 \end{aligned}\right\}$

13. $\left.\begin{aligned} 3x+8y&=71 \\ 2x+3y&=31 \end{aligned}\right\}$

14. $\left.\begin{aligned} 5x-3y&=27 \\ 7y-3x&=15 \end{aligned}\right\}$

15. $\left.\begin{aligned} 7x-4y&=26 \\ 12x+7y&=197 \end{aligned}\right\}$

16. $\left.\begin{aligned} 19x-5y&=36 \\ 12x+25y&=248 \end{aligned}\right\}$

208. Elimination by Comparison.

Solve
$$\left.\begin{aligned} 12x+7y&=176 \quad (1) \\ 3y-19x&=3 \quad (2) \end{aligned}\right\}$$

Transpose (1), $7y=176-12x.$ (3)

Transpose (2), $3y=3+19x.$ (4)

Divide (3) by 7, $y=\dfrac{176-12x}{7}.$ (5)

Divide (4) by 3, $y=\dfrac{3+19x}{3}.$ (6)

Equate the values of y, $\dfrac{3+19x}{3}=\dfrac{176-12x}{7}.$

Multiply by 21, $21 + 133x = 528 - 36x.$

Transpose and combine, $169x = 507.$

$\therefore x = 3.$

Substitute the value of x in (6), $y = 20.$

Therefore, $x = 3$, and $y = 20$.

Check. Substitute in (1) and (2) the values of x and y, and we have

in (1) $36 + 140 = 176,$

in (2) $60 - 57 = 3.$

209. To Eliminate by Comparison, therefore,

From each equation obtain the value of the same unknown number in terms of the other.

Form an equation from these equal values and solve.

Exercise 72

Solve by comparison:

1. $\left.\begin{array}{l} 2x - 5y = 8 \\ 3x - 7y = 13 \end{array}\right\}$

2. $\left.\begin{array}{l} 9x - 2y = 69 \\ 7x - 3y = 32 \end{array}\right\}$

3. $\left.\begin{array}{l} 5x + 7y = 70 \\ 3x + 8y = 61 \end{array}\right\}$

4. $\left.\begin{array}{l} 4x - 11y = -32 \\ 8x + 9y = 60 \end{array}\right\}$

5. $\left.\begin{array}{l} 15x + 4y = 3 \\ 10x - 7y = 31 \end{array}\right\}$

6. $\left.\begin{array}{l} 6x + 7y = -80 \\ 9x - 5y = 4 \end{array}\right\}$

7. $\left.\begin{array}{l} 17x - 6y = 43 \\ 13x - 9y = 2 \end{array}\right\}$

8. $\left.\begin{array}{l} 11x + 4y = 56 \\ 5x - 6y = 2 \end{array}\right\}$

9. $\left.\begin{array}{l} 14x - 17y = 38 \\ 21x - 19y = 96 \end{array}\right\}$

10. $\left.\begin{array}{l} 9x - 15y = 30 \\ 14x + 25y = 385 \end{array}\right\}$

11. $\left.\begin{array}{l} 12x - 7y = 59 \\ 9x - 13y = 83 \end{array}\right\}$

12. $\left.\begin{array}{l} 4x + 9y = 56 \\ 6x + 5y = 50 \end{array}\right\}$

13. $\left.\begin{array}{l} 18x + 13y = 74 \\ 12x - 7y = 112 \end{array}\right\}$

14. $\left.\begin{array}{l} 16x - 5y = 40 \\ 12x + 7y = 116 \end{array}\right\}$

15. $\left.\begin{array}{l} 3x + 14y = 125 \\ 19x - 21y = 24 \end{array}\right\}$

16. $\left.\begin{array}{l} 41x + 33y = 42 \\ 31x + 22y = 17 \end{array}\right\}$

210. Each equation must be simplified, if necessary, before the elimination.

Solve
$$\left.\begin{aligned}\frac{x+3}{x+1}&=\frac{y+8}{y+5} \qquad (1)\\ \frac{2x-3}{2(y+1)}&=\frac{5x-6}{5y+7} \qquad (2)\end{aligned}\right\}$$

Clear (1) of fractions,
$$xy+5x+3y+15=xy+8x+y+8. \qquad (3)$$

Clear (2) of fractions,
$$10xy-15y+14x-21=10xy+10x-12y-12. \qquad (4)$$

Simplify (3), $3x-2y=7.$ (5)

Simplify (4), $4x-3y=9.$ (6)

Subtract (5) from (6), $x-y=2.$ (7)

$\therefore x=y+2.$ (8)

Substitute the value of x in (5),
$$3y+6-2y=7.$$

Transpose and combine, $y=1.$

Substitute the value of y in (8), $x=3.$

Therefore, $x=3$, and $y=1$.

Exercise 73

Solve:

1. $\left.\begin{aligned}\frac{x}{4}+\frac{y}{6}&=\frac{7}{2}\\ \frac{x}{3}-\frac{y}{8}&=\frac{1}{2}\end{aligned}\right\}$

2. $\left.\begin{aligned}\frac{x}{4}+\frac{y}{4}&=14\\ \frac{x}{3}-\frac{y}{6}&=\frac{7}{6}\end{aligned}\right\}$

3. $\left.\begin{aligned}\frac{x-4}{x-3}&=\frac{y+4}{y+7}\\ \frac{x+5}{x+2}&=\frac{y-1}{y-2}\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{2(5-11x)}{11(x-1)}+\frac{11-7y}{3-y}&=5\\ \frac{7+2x}{3-x}-\frac{125-144y}{36(y+5)}&=2\end{aligned}\right\}$

5. $\left.\begin{aligned}x-\frac{2y-x}{23-x}&=20+\frac{2x-59}{2}\\ y-\frac{y-3}{x-18}&=30-\frac{73-3y}{3}\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{7-6x}{10y-19}&=\frac{4-3x}{5y-11}\\ \frac{6x-10y-17}{3x-5y+2}&=\frac{4x-14y-5}{2x-7y+12}\end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{x+y}{3}+x&=15 \\ \frac{x-y}{5}+y&=6 \end{aligned}\right\}$

10. $\left.\begin{aligned} \frac{x+y}{8}+\frac{x-y}{6}&=5 \\ \frac{x+y}{4}-\frac{x-y}{3}&=10 \end{aligned}\right\}$

8. $\left.\begin{aligned} \frac{5}{x+2y}&=\frac{7}{2x+y} \\ \frac{7}{3x-2}&=\frac{5}{6-y} \end{aligned}\right\}$

11. $\left.\begin{aligned} \frac{x+3y+13}{4x+5y-25}&=3 \\ \frac{8x+y+6}{5x+3y-23}&=5 \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{3x-2}{5x-1}&=\frac{3y+7}{5y+16} \\ \frac{3x-1}{x+5}&=\frac{6y-5}{2y+3} \end{aligned}\right\}$

12. $\left.\begin{aligned} \frac{9x-7y+73}{13x-15y+17}&=2 \\ \frac{12x-2y+89}{13x-15y+17}&=3 \end{aligned}\right\}$

13. $\left.\begin{aligned} 2+\frac{5x-6y}{13}&=4y-3x \\ 12+\frac{5x-6y}{6}&=2y+\frac{3x-2y}{4} \end{aligned}\right\}$

14. $\left.\begin{aligned} \frac{3x-4y+2}{3}+\frac{7x-2y+7}{5}&=3 \\ \frac{6x+y-7}{11}+\frac{5x+3y+4}{13}&=5 \end{aligned}\right\}$

15. $\left.\begin{aligned} \frac{6y+5}{8}-\frac{4x-5y+5}{4x-2y}&=\frac{9y-4}{12} \\ \frac{8x+3}{4}-\frac{x-9y}{3-x}&=2x-2 \end{aligned}\right\}$

16. $\left.\begin{aligned} 8y-\frac{4(4+15y)}{3x-1}&=\frac{16xy-107}{2x+5} \\ 2+6x+9y&=\frac{27y^2-12x^2+38}{3y-2x+1} \end{aligned}\right\}$

17. $\left.\begin{aligned} x-\frac{2xy}{2y+5}&=\frac{15x+4y}{6y-2x}+\frac{5x^2+4y^2+105}{(x-3y)(2y+5)} \\ 3-\frac{7x+2y}{5x}&=5-\frac{5y+9}{3x} \end{aligned}\right\}$

211. Literal Simultaneous Equations.

Solve
$$\left.\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned}\right\}$$

NOTE. The letters a', b' are read *a prime*, *b prime*. In like manner a'', a''', a^{iv} are read *a second*, *a third*, *a fourth*; and a_1, a_2, a_3 are read *a sub one*, *a sub two*, *a sub three*. It is often convenient to represent different numbers that have a common property by the same letter marked by *accents* or *suffixes*. In this example a and a' have a common property as coefficients of x.

$$ax + by = c. \qquad (1)$$
$$a'x + b'y = c'. \qquad (2)$$

To find the value of y:

Multiply (1) by a', $\quad aa'x + a'by = a'c. \qquad (3)$

Multiply (2) by a, $\quad aa'x + ab'y = ac'. \qquad (4)$

Subtract (4) from (3), $\quad a'by - ab'y = a'c - ac'.$

Divide by $a'b - ab'$, $\quad y = \dfrac{a'c - ac'}{a'b - ab'}.$

To find the value of x:

Multiply (1) by b', $\quad ab'x + bb'y = b'c. \qquad (5)$

Multiply (2) by b, $\quad a'bx + bb'y = bc'. \qquad (6)$

Subtract (5) from (6), $\quad a'bx - ab'x = bc' - b'c.$

Divide by $a'b - ab'$, $\quad x = \dfrac{bc' - b'c}{a'b - ab'}.$

Therefore, $x = \dfrac{bc' - b'c}{a'b - ab'}$, and $y = \dfrac{a'c - ac'}{a'b - ab'}.$

EXERCISE 74

Solve:

1. $\left.\begin{aligned} x + y &= m \\ x - y &= n \end{aligned}\right\}$

2. $\left.\begin{aligned} x + y &= 5a - 4b \\ x - y &= 4a - 5b \end{aligned}\right\}$

3. $\left.\begin{aligned} ax - by &= c \\ a'x + b'y &= c' \end{aligned}\right\}$

4. $\left.\begin{aligned} ax + by &= 2ab \\ ay - bx &= a^2 - b^2 \end{aligned}\right\}$

5. $\left.\begin{aligned} bx + ay &= 2ab \\ a^2x + b^2y &= a^3 + b^3 \end{aligned}\right\}$

6. $\left.\begin{aligned} b(x - c) &= a(y - c) \\ x - y &= a - b \end{aligned}\right\}$

7. $\left.\begin{aligned} ax + by &= 4a + b \\ bx + ay &= 4a - b \end{aligned}\right\}$

8. $\left.\begin{aligned} ax + by &= abc \\ a'x + b'y &= a'b'c' \end{aligned}\right\}$

9. $\left.\begin{array}{l}(a+b)x+(a-b)y=2(a^2+b^2)\\(a-b)x+(a+b)y=2(a^2-b^2)\end{array}\right\}$

10. $\left.\begin{array}{l}x+y=\dfrac{2(a^2+b^2)}{a^2-b^2}\\x-y=\dfrac{4ab}{a^2-b^2}\end{array}\right\}$

11. $\left.\begin{array}{l}\dfrac{x-y+1}{x-y-1}=a\\\dfrac{x+y+1}{x+y-1}=b\end{array}\right\}$

12. $\left.\begin{array}{r}\dfrac{x-a+c}{y-a+b}=\dfrac{b}{c}\\\dfrac{x+c}{y+b}=\dfrac{a+b}{a+c}\end{array}\right\}$

13. $\left.\begin{array}{l}\dfrac{x+1}{y+1}=\dfrac{a+b+c}{a-b+c}\\\dfrac{x-1}{y-1}=\dfrac{a+b-c}{a-b-c}\end{array}\right\}$

14. $\left.\begin{array}{r}\dfrac{x}{a+b}+\dfrac{y}{a-b}=a+b\\\dfrac{x}{a}+\dfrac{y}{b}=2a\end{array}\right\}$

15. $\left.\begin{array}{l}\dfrac{x+c}{a+b}+\dfrac{y+b}{a+c}=2\\\dfrac{x-b}{a-c}+\dfrac{y-c}{a-b}=2\end{array}\right\}$

16. $\left.\begin{array}{l}\dfrac{x}{b^2-1}-\dfrac{y}{a^2-1}=a^2-b^2\\\dfrac{x}{a^2+1}+\dfrac{y}{b^2+1}=a^2+b^2-2\end{array}\right\}$

17. $\left.\begin{array}{l}ax+by=a^3+2a^2b+b^3\\bx+ay=a^3+2ab^2+b^3\end{array}\right\}$

18. $\left.\begin{array}{l}(a+c)x-(a-c)y=2ab\\(a+b)y-(a-b)x=2ac\end{array}\right\}$

19. $\left.\begin{array}{r}(a+b-c)x-(a-b+c)y=4a(b-c)\\\dfrac{x}{y}=\dfrac{a+b-c}{a-b+c}\end{array}\right\}$

20. $\left.\begin{array}{l}\dfrac{x+y}{x-y}=\dfrac{a}{b-c}\\\dfrac{x+c}{y+b}=\dfrac{a+b}{a+c}\end{array}\right\}$

21. $\left.\begin{array}{r}\dfrac{x-a}{y-a}=\dfrac{a-b}{a+b}\\\dfrac{x}{y}=\dfrac{a^3-b^3}{a^3+b^3}\end{array}\right\}$

22. $\left.\begin{array}{r}\dfrac{1+x}{2}=\dfrac{y-1}{2}+\dfrac{(a-b)^2-2b^2}{a^2-b^2}\\by-ax=\dfrac{(3a+b)ab}{a^2-b^2}+\dfrac{ab}{a+b}-(a+b)\end{array}\right\}$

212. Fractional simultaneous equations, with denominators which are simple expressions containing the unknown numbers, may be solved as follows:

Solve
$$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}=c \quad (1)\\ \frac{a'}{x}+\frac{b'}{y}=c' \quad (2)\end{aligned}\right\}$$

To find the value of x:

Multiply (1) by b', $$\frac{ab'}{x}+\frac{bb'}{y}=b'c. \quad (3)$$

Multiply (2) by b, $$\frac{a'b}{x}+\frac{bb'}{y}=bc'. \quad (4)$$

Subtract (4) from (3), $$\frac{ab'-a'b}{x}=b'c-bc'.$$

Multiply by x and divide by $b'c-bc'$,
$$x=\frac{ab'-a'b}{b'c-bc'}.$$

To find the value of y:

Multiply (1) by a', $$\frac{aa'}{x}+\frac{a'b}{y}=a'c. \quad (5)$$

Multiply (2) by a, $$\frac{aa'}{x}+\frac{ab'}{y}=ac'. \quad (6)$$

Subtract (5) from (6), $$\frac{ab'-a'b}{y}=ac'-a'c.$$

Multiply by y and divide by $ac'-a'c$,
$$y=\frac{ab'-a'b}{ac'-a'c}.$$

Therefore, $x=\dfrac{ab'-a'b}{b'c-bc'}$, and $y=\dfrac{ab'-a'b}{ac'-a'c}$.

Exercise 75

Solve:

1. $\left.\begin{aligned}\frac{5}{3x}+\frac{2}{5y}=7\\ \frac{7}{6x}-\frac{1}{10y}=3\end{aligned}\right\}$

2. $\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}=a\\ \frac{1}{x}-\frac{1}{y}=b\end{aligned}\right\}$

3. $$\left.\begin{aligned}\frac{24}{x}+\frac{12}{y}&=10\\ \frac{16}{x}+\frac{9}{y}&=7\end{aligned}\right\}$$

4. $$\left.\begin{aligned}\frac{25}{x}+\frac{14}{y}&=7\\ \frac{20}{x}-\frac{21}{y}&=1\end{aligned}\right\}$$

5. $$\left.\begin{aligned}\frac{33}{x}-\frac{12}{y}&=1\\ \frac{44}{x}+\frac{30}{y}&=9\end{aligned}\right\}$$

6. $$\left.\begin{aligned}\frac{1}{5x}+\frac{1}{4y}&=\frac{1}{12}\\ \frac{1}{4x}+\frac{1}{3y}&=\frac{11}{120}\end{aligned}\right\}$$

7. $$\left.\begin{aligned}\frac{2}{3x}+\frac{3}{4y}&=1\tfrac{7}{12}\\ \frac{3}{4x}+\frac{2}{5y}&=1\tfrac{3}{20}\end{aligned}\right\}$$

8. $$\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=c\\ \frac{b}{x}+\frac{a}{y}&=d\end{aligned}\right\}$$

9. $$\left.\begin{aligned}\frac{a}{bx}+\frac{b}{ay}&=c\\ \frac{b}{ax}+\frac{a}{by}&=d\end{aligned}\right\}$$

10. $$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{1}{a}\\ \frac{1}{x}-\frac{1}{y}&=\frac{1}{b}\end{aligned}\right\}$$

11. $$\left.\begin{aligned}\frac{1}{3x}+\frac{1}{4y}&=a\\ \frac{1}{4x}+\frac{1}{3y}&=b\end{aligned}\right\}$$

12. $$\left.\begin{aligned}\frac{9}{x}-\frac{7}{y}&=6\tfrac{3}{7}\\ \frac{7}{x}-\frac{9}{y}&=3\tfrac{8}{9}\end{aligned}\right\}$$

13. $$\left.\begin{aligned}\frac{a}{bx}+\frac{b}{ay}&=a+b\\ \frac{b}{x}+\frac{a}{y}&=a^2+b^2\end{aligned}\right\}$$

14. $$\left.\begin{aligned}\frac{20}{3x}+\frac{18}{5y}&=2\tfrac{8}{15}\\ \frac{28}{5x}-\frac{10}{3y}&=\frac{2}{225}\end{aligned}\right\}$$

15. $$\left.\begin{aligned}\frac{a+b}{x}+\frac{a-b}{y}&=a^2+b^2\\ \frac{a-b}{x}+\frac{a+b}{y}&=a^2-b^2\end{aligned}\right\}$$

16. $$\left.\begin{aligned}\frac{a+b}{x}-\frac{a-b}{y}&=c-d\\ \frac{a-b}{x}-\frac{a+b}{y}&=c+d\end{aligned}\right\}$$

Three or More Simultaneous Simple Equations

213. If three simultaneous equations are given involving three unknown numbers, one of the unknown numbers must be eliminated between *two pairs* of the equations; then a second between the two resulting equations.

Solve
$$\left.\begin{aligned} 5x + 2y - 3z &= 160 \\ 3x + 9y + 8z &= 115 \\ 2x - 3y - 5z &= 45 \end{aligned}\right\} \qquad \begin{matrix}(1)\\(2)\\(3)\end{matrix}$$

Eliminate y between equations (1) and (3).

Multiply (1) by 3, $\quad 15x + 6y - 9z = 480.$ (4)

Multiply (3) by 2, $\quad 4x - 6y - 10z = 90.$ (5)

Add (4) and (5), $\quad 19x - 19z = 570.$

Divide by 19, $\quad x - z = 30.$ (6)

Eliminate y between equations (2) and (3).

Multiply (3) by 3, $\quad 6x - 9y - 15z = 135.$ (7)

Add (2) and (7), $\quad 9x - 7z = 250.$ (8)

We now have two equations (6) and (8) involving the two unknowns x and z.

Multiply (6) by 7, $\quad 7x - 7z = 210.$ (9)

Subtract (9) from (8), $\quad 2x = 40.$

$\therefore x = 20.$

Substitute the value of x in (6), $\quad 20 - z = 30.$

$\therefore z = -10.$

Substitute the values of x and z in (1),

$$100 + 2y + 30 = 160.$$

Transpose and combine, $\quad 2y = 30.$

$\therefore y = 15.$

Therefore, $x = 20$, $y = 15$, and $z = -10$.

214. Likewise, if four or more equations are given involving four or more unknown numbers, we must eliminate one of the unknown numbers from three or more pairs of the equations, using every equation at least once in forming the pairs from which to eliminate; then eliminate a second unknown number from the pairs that may be formed of the resulting equations; and so on.

Exercise 76

Solve:

1. $\left.\begin{array}{l}4x+3y+2z=16\\3x+5y+3z=22\\2x+4y+5z=25\end{array}\right\}$

2. $\left.\begin{array}{l}x+y+z=5\\2x+5y+6z=39\\3x+4y+5z=28\end{array}\right\}$

3. $\left.\begin{array}{l}7x-11y+5z=5\\5x-7y+11z=21\\11x-5y+7z=43\end{array}\right\}$

4. $\left.\begin{array}{l}6x-5y+9z=27\\5x+8y-3z=7\\7x-3y+6z=32\end{array}\right\}$

5. $\left.\begin{array}{l}12x-5y+8z=2\\6x+10y-5z=13\\9x-15y+4z=-80\end{array}\right\}$

6. $\left.\begin{array}{l}3x-5y-2z=-37\\2x+3y+z=63\\3x+6y+2z=115\end{array}\right\}$

7. $\left.\begin{array}{l}10x-y+3z=103\\3x+3y-2z=-6\\4x+5y+4z=94\end{array}\right\}$

8. $\left.\begin{array}{l}7x+8y-4z=15\\5x-12y+6z=55\\4x-4y-5z=51\end{array}\right\}$

9. $\left.\begin{array}{l}3x+2y+3z=99\\5x+4y+5z=171\\7x+8y+6z=247\end{array}\right\}$

10. $\left.\begin{array}{l}5x-4y+3z=51\\7x+6y-6z=64\\9x-8y-9z=47\end{array}\right\}$

11. $\left.\begin{array}{l}x+12y+2z=86\\5y-4x-4z=-43\\x+y+10z=103\end{array}\right\}$

12. $\left.\begin{array}{l}6x+2y-7z=15\\7x-7y-11z=-4\\3x+9y+8z=35\end{array}\right\}$

13. $\left.\begin{array}{l}15x-18y+8z=15\\25x+15y+16z=91\\20x-21y-12z=-27\end{array}\right\}$

14. $\left.\begin{array}{l}ax+by-cz=2ab\\by+cz-ax=2bc\\cz+ax-by=2ac\end{array}\right\}$

15. $\left.\begin{array}{l}x+ay+a^2z+a^3=0\\x+by+b^2z+b^3=0\\x+cy+c^2z+c^3=0\end{array}\right\}$

16. $\left.\begin{array}{l}x+y+z=a+b+c\\bx+cy+az=a^2+b^2+c^2\\cx+ay+bz=a^2+b^2+c^2\end{array}\right\}$

17. $\left.\begin{array}{l}(a+b)x+(a-b)z=2bc\\(b+c)y+(b-c)x=2ac\\(c+a)z+(c-a)y=2ab\end{array}\right\}$

18. $\left.\begin{array}{l}x+y+z=a+b+c\\ax+by+cz=ab+bc+ac\\(b-c)x+(c-a)y=(b-a)z\end{array}\right\}$

19. $$\left.\begin{aligned} \frac{x}{a}+\frac{y}{b}+\frac{z}{c}&=1 \\ \frac{x}{a}+\frac{y}{c}+\frac{z}{b}&=1 \\ \frac{x}{b}+\frac{y}{a}+\frac{z}{c}&=1 \end{aligned}\right\}$$

20. $$\left.\begin{aligned} \frac{a}{x}+\frac{b}{y}-\frac{c}{z}&=m \\ \frac{a}{x}-\frac{b}{y}+\frac{c}{z}&=n \\ -\frac{a}{x}+\frac{b}{y}+\frac{c}{z}&=p \end{aligned}\right\}$$

21. $$\left.\begin{aligned} 21x+15y+16z&=-13 \\ 14x+25y-13z&=15\tfrac{1}{12} \\ 17x-10y-19z&=4\tfrac{7}{12} \end{aligned}\right\}$$

22. $$\left.\begin{aligned} \frac{x}{m+n}+\frac{y}{n-p}+\frac{z}{m+p}&=2p \\ \frac{x}{m-n}-\frac{y}{n-p}+\frac{z}{p-m}&=2m \\ \frac{x}{m-n}-\frac{y}{n-p}-\frac{z}{m+p}&=2m-2p \end{aligned}\right\}$$

23. $$\left.\begin{aligned} \frac{x}{n+p}+\frac{y}{p+m}&=n-m \\ \frac{x}{n+p}+\frac{z}{m+n}&=m-p \\ \frac{y}{p+m}+\frac{z}{m+n}&=p-n \end{aligned}\right\}$$

24. $$\left.\begin{aligned} \frac{6}{x}+\frac{4}{y}+\frac{5}{z}&=4 \\ \frac{3}{x}+\frac{8}{y}+\frac{5}{z}&=4 \\ \frac{9}{x}+\frac{12}{y}-\frac{10}{z}&=4 \end{aligned}\right\}$$

25. $$\left.\begin{aligned} x+2y+3z+4u&=2 \\ 2x+3y+4z+5u&=3 \\ 6x+7y-8z+9u&=3 \\ 64x+16y+4z+u&=25 \end{aligned}\right\}$$

26. $$\left.\begin{aligned} 2x-9y+3z-10u&=121 \\ x+7y-z-u&=583 \\ 3x+2y+5z+2u&=255 \\ 4x-6y-2z-9u&=516 \end{aligned}\right\}$$

27. $$\left.\begin{aligned} 3x+4y+5z+6u&=67 \\ 2x-5y+9z-8u&=-127 \\ 4x-3y+3z+4u&=73 \\ 6x-10y-6z+3u&=75 \end{aligned}\right\}$$

28. $$\left.\begin{aligned} \frac{x}{3}+\frac{y}{5}+\frac{2z}{7}&=58 \\ \frac{5x}{4}+\frac{y}{6}+\frac{z}{3}&=76 \\ \frac{x}{2}+\frac{3z}{8}+\frac{u}{5}&=79 \\ \frac{y}{5}+\frac{z}{2}+\frac{u}{25}&=92 \end{aligned}\right\}$$

CHAPTER XII

PROBLEMS PRODUCING SIMULTANEOUS SIMPLE EQUATIONS

215. In the solution of problems it is often necessary to employ two or more letters to represent the numbers to be found. In every case the conditions must be sufficient to give just as many equations as there are unknown numbers.

If there are *more* equations than unknown numbers, some of these equations are superfluous or inconsistent; if there are *fewer* equations than unknown numbers, the problem is indeterminate.

Exercise 77

1. If A gives B \$17, B will have three times as much money as A. If B gives A \$17, A will have \$5 more than four times as much money as B. How much has each?

Let $x =$ the number of dollars A has,
and $y =$ the number of dollars B has.

Then, if A gives B \$17,

$x - 17 =$ the number of dollars A will have,
and $y + 17 =$ the number of dollars B will have.

$$\therefore y + 17 = 3(x - 17). \quad (1)$$

If B gives A \$17,

$x + 17 =$ the number of dollars A will have,
and $y - 17 =$ the number of dollars B will have.

$$\therefore x + 17 = 4(y - 17) + 5. \quad (2)$$

From the solution of equations (1) and (2), $x = 32$, and $y = 28$.

Therefore, A has \$32, and B has \$28.

2. If the smaller of two numbers is divided by the larger, the quotient is 0.32 and the remainder 0.012; but if the larger number is divided by the smaller, the quotient is 3.11 and the remainder 0.0042. Find the numbers.

Let $x =$ the smaller number,

and $y =$ the larger number.

$$\therefore \frac{x - 0.012}{y} = 0.32, \qquad (1)$$

and

$$\frac{y - 0.0042}{x} = 3.11. \qquad (2)$$

Multiply (1) by y, $x - 0.012 = 0.32\,y.$ (3)

Multiply (2) by x, $y - 0.0042 = 3.11\,x.$ (4)

Transpose (3), $x - 0.32\,y = 0.012.$ (5)

Transpose (4), $-3.11\,x + y = 0.0042.$ (6)

Multiply (6) by 0.32, $-0.9952\,x + 0.32\,y = 0.001344.$ (7)

Add (5) and (7), $0.0048\,x = 0.013344.$

$\therefore x = 2.78.$

Substitute the value of x in (5), $2.78 - 0.32\,y = 0.012.$

Transpose and combine, $-0.32\,y = -2.768.$

$\therefore y = 8.65.$

Therefore, the required numbers are 2.78 and 8.65.

3. If B gave A $27, they would have equal sums of money; but if A gave B $43, A would then have half as much money as B. How much has each?

4. If the larger of two numbers is divided by the smaller, the quotient is 4 and the remainder 9; but if twenty times the smaller number is divided by twice the larger, the quotient is 2 and the remainder 152. Find the numbers.

5. If the smaller of two numbers is divided by the larger, the quotient is 0.37 and the remainder 0.01413; but if the larger number is divided by the smaller, the quotient is 2.69 and the remainder 0.00077. Find the numbers.

6. If A gives B $17, A will have twice as much money as B; but if A gives B $33, B will have one dollar less than A. How much has each?

7. The value of a certain fraction becomes $\frac{3}{4}$ when 7 is added to the numerator, and becomes $\frac{1}{2}$ when 2 is subtracted from the denominator. Find the fraction.

Let $x =$ the numerator of the fraction,

and $y =$ the denominator of the fraction.

Then $\frac{x}{y} =$ the fraction.

$$\therefore \frac{x+7}{y} = \frac{3}{4}, \quad (1)$$

and

$$\frac{x}{y-2} = \frac{1}{2}. \quad (2)$$

From the solution of equations (1) and (2), $x = 11$, and $y = 24$.

Therefore, the required fraction is $\frac{11}{24}$.

8. A certain fraction becomes equal to $\frac{8}{9}$ when 2 is added to the numerator and 1 is subtracted from the denominator, and equal to $\frac{1}{2}$ when 9 is added to the denominator. Find the fraction.

9. A certain fraction becomes equal to $\frac{1}{4}$ when 4 is added to the denominator, and equal to $\frac{1}{2}$ when 5 is added to the numerator. Find the fraction.

10. A certain fraction becomes equal to $\frac{3}{7}$ when 6 is added to the numerator, and equal to $\frac{1}{3}$ when 8 is subtracted from the denominator. Find the fraction.

11. A certain fraction becomes equal to $\frac{4}{5}$ when 1 is subtracted from the numerator, and equal to $\frac{7}{8}$ when 1 is subtracted from the denominator. Find the fraction.

12. There are two fractions, with numerators 3 and 11 respectively, whose sum is $1\frac{2}{3}$. If the denominators are interchanged, the sum is 3. Find the fractions.

13. A certain fraction becomes equal to $\frac{1}{3}$ when 4 is subtracted from each term, and equal to $\frac{1}{2}$ when 4 is added to the numerator and 3 subtracted from the denominator. Find the fraction.

14. The sum of the three digits of a number is 16; the sum of the first and second digits is equal to the third digit; and if 594 is added to the number, the order of the digits is reversed. Find the number.

Let $x =$ the digit in the hundreds' place,
$y =$ the digit in the tens' place,
and $z =$ the digit in the units' place.
Then $100x + 10y + z =$ the number.

$$\therefore x + y + z = 16, \quad (1)$$

$$x + y = z, \quad (2)$$

and

$$100x + 10y + z + 594 = 100z + 10y + x. \quad (3)$$

From the solution of (1), (2), and (3), $x = 2$, $y = 6$, and $z = 8$.

Therefore, the required number is 268.

15. The sum of the two digits of a number is 8, and if 36 is added to the number, the order of the digits is reversed. Find the number.

16. The sum of the two digits of a number is 14, and if 36 is subtracted from the number, the order of the digits is reversed. Find the number.

17. The sum of the three digits of a number is 18, and the sum of the first and third digits is equal to the second digit. If the hundreds' and units' digits are interchanged, the number is diminished by 99. Find the number.

18. If a certain number is divided by the sum of its two digits, the quotient is 4 and the remainder 6; if the digits are interchanged and the resulting number divided by the sum of the digits, the quotient is 6 and the remainder 7. Find the number.

19. A certain number, expressed by three digits, is equal to thirty-eight times the sum of the digits. If 198 is subtracted from the number, the order of the digits is reversed. The sum of the first and third digits is 2 greater than the middle digit. Find the number.

20. A boat's crew rows 14 miles down a river and back in 6 hours 40 minutes. The crew finds that it takes two and a third times as long to row a mile against the current as to row a mile with the current. Find the time the crew was rowing down and rowing up respectively.

Let x = the number of miles per hour the crew can row in still water,

and y = the number of miles per hour the current flows.

Then $\frac{14}{x+y}$ = the number of hours the crew was rowing down,

and $\frac{14}{x-y}$ = the number of hours the crew was rowing up.

$$\therefore \frac{14}{x+y} + \frac{14}{x-y} = 6\tfrac{2}{3}, \quad (1)$$

and
$$\frac{2\frac{1}{3}}{x+y} = \frac{1}{x-y}. \quad (2)$$

From the solution of equations (1) and (2), $x = 5$, and $y = 2$.

Hence, $\frac{14}{x+y} = 2$, and $\frac{14}{x-y} = 4\frac{2}{3}$.

Therefore, it takes the crew 2 hours to row down and 4 hours 40 minutes to row up.

21. A boatman rows down a river, which flows at the rate of $1\frac{1}{2}$ miles an hour, for a certain distance in 2 hours 15 minutes; it takes him 4 hours 30 minutes to return. Find the distance he rowed down the stream and his rate of rowing in still water.

22. A man rows 12 miles down a river and back again in 6 hours 15 minutes. He finds he can row 4 miles with the current in the same time he can row 3 miles against it. Find the time of his rowing down and of his rowing up the river, and the rate of the current.

23. A crew finds that it can row down a river at the rate of 9 miles an hour, and that it takes half as long again to row a mile up the river as to row a mile down. Find the rate of rowing in still water and the rate of the current.

24. A cistern has three pipes, A, B, and C. A and B together will fill the cistern in 12 hours, A and C together in 10 hours, and B and C together in 15 hours. How long will it take each pipe alone to fill the cistern?

Let $x =$ the number of hours required for A alone,
$y =$ the number of hours required for B alone,
and $z =$ the number of hours required for C alone.

Then $\frac{1}{x}, \frac{1}{y}, \frac{1}{z} =$ the parts filled in one hour by A, B, C respectively,

and $\frac{1}{x} + \frac{1}{y} =$ the part filled in one hour by A and B together.

But $\frac{1}{12} =$ the part filled in one hour by A and B together.

$$\therefore \frac{1}{x} + \frac{1}{y} = \frac{1}{12}. \qquad (1)$$

Likewise,
$$\frac{1}{x} + \frac{1}{z} = \frac{1}{10}, \qquad (2)$$

and
$$\frac{1}{y} + \frac{1}{z} = \frac{1}{15}. \qquad (3)$$

From equations (1), (2), and (3), $x = 17\frac{1}{7}$, $y = 40$, and $z = 24$.

Therefore, A will fill the cistern in $17\frac{1}{7}$ hours, B in 40 hours, and C in 24 hours.

25. A and B together can do a piece of work in $13\frac{1}{3}$ days, A and C together in 15 days, B and C together in 20 days. How long will it take each alone to do the work?

26. A and B together can mow a field in 12 days, A and C together in 15 days, B and C together in 20 days. How long will it take each alone to mow the field?

27. A cistern has three pipes, A, B, and C. A and B together will fill it in 35 minutes, A and C together in 42 minutes, B and C together in 70 minutes. How long will it take each alone to fill the cistern?

28. A and B together can do a piece of work in 24 days, A and C together in 32 days, B and C together in 30 days. How long will it take each alone to do the work?

29. A sum of money at simple interest amounted to \$2996.40 in 3 years, and to \$3234 in 5 years. Find the sum and the rate of interest.

Let $x =$ the number of dollars in the principal,
and $y =$ the rate of interest.

The interest for one year is $\frac{y}{100}$ of the principal, or $\frac{y}{100}$ of x, or $\frac{xy}{100}$; the interest for 3 years is $\frac{3xy}{100}$; for 5 years $\frac{5xy}{100}$.

$$\therefore x + \frac{3xy}{100} = 2996.40, \qquad (1)$$

and

$$x + \frac{5xy}{100} = 3234. \qquad (2)$$

Multiply (1) by 100, $100x + 3xy = 299640.$ (3)
Multiply (2) by 100, $100x + 5xy = 323400.$ (4)
Multiply (4) by 3 and divide by 5,

$$60x + 3xy = 194040. \qquad (5)$$

Subtract (5) from (3), $40x = 105600.$

$$\therefore x = 2640.$$

Substitute the value of x in (4), $y = 4\frac{1}{2}$.

Therefore, the sum is \$2640, and the rate is $4\frac{1}{2}\%$.

30. A sum of money at simple interest amounted to \$8910 in 2 years, and to \$9240 in 3 years. Find the sum and the rate of interest.

31. A sum of money at simple interest amounted to \$12,607.50 in 6 months, and to \$12,710 in 8 months. Find the sum and the rate of interest.

32. A sum of money at simple interest amounted to \$5326.80 in 2 years 7 months 18 days, and to \$5552.20 in 3 years 5 months 12 days. Find the sum and the rate of interest.

33. A man has invested the sum of \$10,000, for a part of which he receives 4 per cent interest, and for the remainder $4\frac{1}{2}$ per cent. The income from the 4 per cent investment is \$9 more than that from the $4\frac{1}{2}$ per cent. How much has he in each investment?

34. In running a quarter mile race A gives B a start of 32 yards and beats him by 1 second. In a second trial A gives B a start of 6 seconds and is beaten by $8\frac{4}{5}$ yards. Find the number of yards each runs a second.

Let $x =$ the number of yards A runs a second,
and $y =$ the number of yards B runs a second.

Since a quarter mile is 440 yards,

$$\frac{440}{x} = \text{the number of seconds it takes A to run the quarter mile.}$$

Since B has a start of 32 yards, he runs 408 yards the first trial; and since he was running 1 second longer than A,

$$\frac{440}{x} + 1 = \text{the number of seconds B was running.}$$

But $\frac{408}{y} =$ the number of seconds B was running.

$$\therefore \frac{440}{x} + 1 = \frac{408}{y}. \qquad (1)$$

In the second trial A runs $(440 - 8\frac{4}{5})$ yards $= 431\frac{1}{5}$ yards.

$$\therefore \frac{440}{y} = \frac{431\frac{1}{5}}{x} + 6. \qquad (2)$$

From the solution of equations (1) and (2), $x = 8\frac{4}{5}$, and $y = 8$.

Therefore, A runs $8\frac{4}{5}$ yards a second, and B 8 yards a second.

35. Two athletes, A and B, run a mile race, and A wins by 6 seconds. In a handicap race A gives B a start of 40 yards and is beaten by half a second. Find the number of yards each runs in a second and the time required by each to run a mile.

36. In running a mile race A beats B by $8\frac{4}{5}$ seconds. In a handicap race A gives B a start of 50 yards and beats him by 5 yards. Find the number of yards each runs in a second and the time required by each to run a mile.

37. A train, after traveling 2 hours from A towards B, meets with an accident which detains it 40 minutes. After the accident the train proceeds at three fourths its usual rate, and arrives 2 hours late. If the accident had happened 60 miles further on, the train would have been only an hour and a half late. Find the usual rate of the train per hour and the distance from A to B.

Let $4x$ = the number of miles the train travels per hour,
and y = the number of miles from A to B.
Then $3x$ = the number of miles the train travels per hour after the accident,
$y - 8x$ = the number of miles the train travels after the accident,

$\frac{y-8x}{4x}$ = the number of hours usually required,

and $\frac{y-8x}{3x}$ = the number of hours actually required.

Since the train was detained 40 minutes, or $\frac{2}{3}$ hour, and arrived 2 hours late, the *running time* was $1\frac{1}{3}$ hours more than usual;
that is, $1\frac{1}{3}$ = the number of hours of running time lost.

But $\frac{y-8x}{3x} - \frac{y-8x}{4x}$ = the number of hours of running time lost.

$$\therefore \frac{y-8x}{3x} - \frac{y-8x}{4x} = 1\tfrac{1}{3}. \qquad (1)$$

If the accident had happened 60 miles further on, the remainder of the run would have been $(y - 8x - 60)$ miles, and the loss in running time would have been $\frac{5}{6}$ of an hour.

$$\therefore \frac{y-8x-60}{3x} - \frac{y-8x-60}{4x} = \frac{5}{6}. \qquad (2)$$

From the solution of equations (1) and (2), $x = 10$, and $y = 240$.
Hence, $4x = 40$.

Therefore, the usual rate of the train is 40 miles an hour, and the distance from A to B is 240 miles.

38. A train, after traveling 1 hour 20 minutes from A towards B, meets with an accident which detains it 30 minutes. After the accident the train proceeds at two thirds its usual rate, and arrives 2 hours 20 minutes late. If the accident had happened 55 miles further on, the train would have been only an hour and a half late. Find the usual rate of the train per hour and the distance from A to B.

39. The sum of the three angles of any triangle is 180°. If one angle of a triangle exceeds half the sum of the other two angles by 15°, and their difference by 30°, find the angles.

40. An angle of a triangle is $\frac{5}{13}$ the sum of the other two angles and is $\frac{5}{6}$ their difference. Find the angles.

41. A number of young men purchase a camp. If there had been two more men in the company, each would have paid $12 less; and if there had been three men less, each would have paid $24 more. How many men were there and how much did each pay?

42. A sum of money at simple interest amounted in a years to m dollars, and in b years to n dollars. Find the sum and the rate of interest.

43. A sum of money at simple interest amounted in a months to m dollars, and in b months to n dollars. Find the sum and the rate of interest.

44. A boat's crew can row 15 miles an hour downstream. The crew can row a certain distance in still water in 15 minutes and requires 20 minutes to row the same distance up the stream. Find the rate of rowing in still water and the rate of the stream.

45. If the length of a rectangle was 7 feet greater and the breadth 2 feet less, the area would be 159 square feet greater. If the length was 5 feet less and the breadth 4 feet greater, the area would be 33 square feet less. Find the dimensions.

46. In running a mile race A gives B a start of 44 yards and is beaten by one second. In a second trial A gives B a start of 7 seconds and beats him by $4\frac{8}{9}$ yards. Find the number of yards each runs per second.

47. A courtyard, whose area is 1300 square feet, is paved with 50 stones of one size and 195 stones of another size. Ten stones of the first size and seven of the second size cover together 100 square feet. Find the area of each kind.

48. The sum of the three digits of a number is 16, and the sum of the first digit and the third digit is equal to the second. If the units' digit and the tens' digit are interchanged, the resulting number is less than the original number by 27. Find the number.

49. The report of an explosion traveled 1039 feet per second against the wind and 1061 feet per second with the wind. Find the velocity of the sound in still air and the velocity of the wind.

50. The sum of the ages of A, B, and C is 171 years. Fifteen years ago A's age was equal to the sum of the ages of B and C; and 34 years ago A's age was four times the difference between the ages of B and C. Find the age of each.

51. A, B, and C together subscribed $100. If A had subscribed a tenth less and B a tenth more, C must have subscribed $2 more to make up the sum; but if A had subscribed an eighth more, and B an eighth less, C must have subscribed $17.50. What was the subscription of each?

52. A and B together do a piece of work in 16 days, A and C together in 18 days, and B and C together in 20 days. In how many days can each alone do the work?

53. A boat travels $10\frac{1}{2}$ miles an hour down a river. It takes the boat three times as long to go a certain distance up the river as to go the same distance down the river. Find the rate at which the river flows and the rate of the boat in still water.

54. A sum of money at simple interest amounted in 7 months to $2118.30 and in 10 months to $2139. Find the sum and the rate of interest.

55. A boy bought 570 oranges, some at 16 for 25 cents, and the remainder at 18 for 25 cents. He sold them all at the rate of 15 for 25 cents, and made a profit of 75 cents. How many oranges at each price did he buy?

56. A person has a certain capital invested at a certain rate per cent. Another person has \$4000 more capital invested at one per cent better than the first, and receives \$300 more income. A third person has \$6000 more capital invested at two per cent better than the first, and receives \$560 more income. Find the capital of each and the rate at which it is invested.

57. A dealer sold a certain quantity of eggs. If he had sold 125 dozen more at 1 cent a dozen less, or 120 dozen less at 1 cent a dozen more, he would have received the same amount of money for his eggs. Find the number of dozen he sold and the price per dozen.

58. C sets out from the town A and walks toward B at the rate of 3 miles an hour. A quarter of an hour later D sets out from the town B and walks toward A at the rate of $3\frac{3}{4}$ miles an hour. If D walks 2 miles beyond the half-way point between B and A before he meets C, find the distance between A and B.

59. A and B together can do a piece of work in 12 days. After working together 9 days, however, they call on C to aid them, and the three finish the work in 2 days. C finds that he can do as much work in 5 days as A does in 6 days. In how many days can each alone do the work?

60. An income of \$670 a year is obtained from two investments, one in $4\frac{1}{2}$ per cent stock and the other in 5 per cent stock. If the $4\frac{1}{2}$ per cent stock should be sold at 110, and the 5 per cent stock at 125, the sum realized from the sales would be \$16,600. How much of each stock is there?

61. A box contains a mixture of 12 bushels of oats and 18 of corn, and another box contains a mixture of 12 bushels of oats and 4 of corn. How many bushels must be taken from each box in order to have a mixture of 14 bushels, half oats and half corn?

216. Discussion of a Problem. The **discussion** of a problem consists in making various suppositions as to the relative values of the given numbers, and explaining the results.

We will illustrate by the following example:

Two automobiles are traveling along the same road in the same direction. A travels m miles an hour and B travels n miles an hour. At 12 o'clock B is d miles in advance of A. When will the automobiles be together?

Suppose that the automobiles will be together x hours *after* 12. Then A has traveled mx miles and B has traveled nx miles; and as A has traveled d miles more than B,

$$mx - nx = d.$$

$$\therefore x = \frac{d}{m-n}.$$

Discussion. 1. If m is greater than n, the value of x is positive, and A will overtake B *after* 12 o'clock.

2. If m is less than n, the value of x is negative. In this case B travels faster than A, and as B is d miles ahead of A at 12 o'clock, A cannot overtake B *after* 12 o'clock; but B passed A *before* 12 o'clock. The supposition, therefore, that the automobiles are together *after* 12 o'clock is incorrect, and the *negative* value of x points to an **error in the supposition.**

In general a negative solution indicates a fault in the statement of the problem.

3. If m equals n, the value of x assumes the form $\frac{d}{0}$. Now, if the automobiles are d miles apart at 12 o'clock, and if they travel at equal rates, it is obvious that they *never* will be together; so that the solution $\frac{d}{0}$ may be regarded as the **symbol of impossibility.**

The symbol ∞, usually read *infinity*, is in general used for the symbol of impossibility.

4. If m equals n and d is 0, then $\frac{d}{m-n}$ becomes $\frac{0}{0}$. Now, if the automobiles are together at 12 o'clock, and if they travel at equal rates, it is obvious that they will be together *all the time*, so that x may have *any value whatever.* Hence, the solution $\frac{0}{0}$ may be regarded as the **symbol of indetermination.**

Exercise 78

1. At the present time A is a years old and B is b years old. In how many years will A be c times as old as B?

Ans. $\dfrac{a - bc}{c - 1}$.

Discuss the result (1) when b is greater than a; (2) when c is less than unity; (3) when $a = b$; (4) when $c = 1$.

2. A boatman, rowing with the tide, rows n miles in t hours. In returning he rows against a tide m times as strong as the first and makes the distance in t' hours. Find the velocity of the second tide in miles per hour.

Ans. $\dfrac{mn(t' - t)}{tt'(m + 1)}$.

Discuss the result (1) when $m = 1$; (2) when $t' = t$; (3) when t' is less than t; (4) when m is negative; (5) when n is negative.

3. A merchant owns two kinds of sugar that cost him, one a cents a pound and the other b cents a pound. He wishes to make a mixture of c pounds that shall cost him d cents a pound. How many pounds of each kind shall he take?

Ans. $\dfrac{(d - b)c}{a - b}$ of the first; $\dfrac{(a - d)c}{a - b}$ of the second.

Discuss the result (1) when $a = b$; (2) when $a = d$ or $b = d$; (3) when $a = b = d$; (4) when a is greater than b and less than d; (5) when a is greater than b and b is greater than d.

4. A party of a young men hired a camp, agreeing to share the expense equally; but, as b of the party were unable to go, each one who went paid c dollars more than he expected to pay. What was the rent of the camp?

Ans. $\dfrac{(a - b)ac}{b}$ dollars.

Discuss the result (1) when $b = 0$; (2) when $b = a$; (3) when b is greater than a; (4) when c is negative.

CHAPTER XIII

GRAPHS

217. Graphs. Diagrams, called **graphs**, are often used to show in a concise manner variations in temperature, in population, in prices, etc., etc.

218. Variables and Constants. A number that, under the conditions of the problem into which it enters, may take *different values* is called a **variable**.

A number that, under the conditions of the problem into which it enters, has a *fixed value* is called a **constant**.

NOTE. Variables are represented generally by the last letters of the alphabet, x, y, z, etc.; constants, by the Arabic numerals and by the first letters of the alphabet, a, b, c, etc.

219. Algebraic Functions. A *function* of a variable is an expression that changes in value when the variable changes in value. In general, any expression that involves a variable is a function of that variable. If x is involved only in a finite number of powers and roots, the expression is called an **algebraic function of x.**

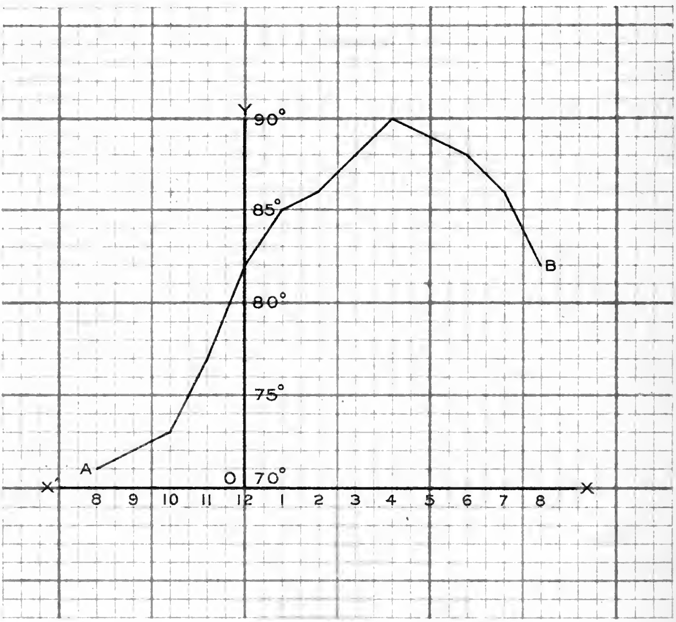
An algebraic function of x is **rational** and **integral** as regards x, if x is involved only in *positive integral powers;* that is, in powers and numerators, but not in roots or denominators.

Thus, x^2, $\sqrt[3]{x^2+x}$, $\frac{1}{x^3+4}$ are algebraic functions of x; but a^x, $\sqrt[x]{c}$ are not algebraic functions of x. Of $\frac{1}{x^2}$, $\frac{x}{x^2+a^2}$, $\sqrt{x}$, $2x+a$, $\frac{x^2}{a+b}$, ax^2+bx+c, the last three only are rational integral functions of x.

For brevity a function of x is represented by $f(x)$, $F(x)$, $\phi(x)$, each of which is read *function x.*

220. As an easy example we may illustrate by a graph the changes in temperature for a day from 8 A.M. to 8 P.M.

The official temperatures for Boston, July 17, 1905, were as follows: 8 A.M., 71°; 9 A.M., 72°; 10 A.M., 73°; 11 A.M., 77°; 12 M., 82°; 1 P.M., 85°; 2 P.M., 86°; 3 P.M., 88°; 4 P.M., 90°; 5 P.M., 89°; 6 P.M., 88°; 7 P.M., 86°; 8 P.M., 82°.

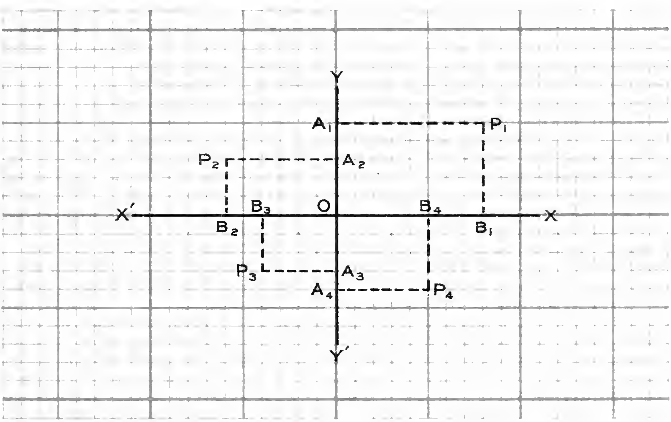


Draw a horizontal line XX' and a line OY perpendicular to XX'. Using any convenient units of length, lay off on XX' equal distances to represent the hours and on OY equal distances to represent degrees of temperature from 70° to 90°. At each point of division on XX' draw a perpendicular of sufficient length to represent the temperature at that hour. Through the upper ends of these perpendiculars draw the line AB. This line, or *graph*, presents to the eye a complete view of the changes in temperature for the day.

221. Coördinates. Let XX' be a horizontal straight line, and let YY' be a straight line perpendicular to the line XX' at the point O. Any point in the plane of the lines XX' and YY' is determined by its *distance* and *direction* from each of the perpendiculars XX' and YY'.

The distance of a point from YY' is measured from O on the line XX', and is called the **abscissa** of the point. The distance of a point from XX' is measured from O on the line YY' and is called the **ordinate** of the point.

Thus, the abscissa of P_1 is OB_1, the ordinate of P_1 is OA_1;
the abscissa of P_2 is OB_2, the ordinate of P_2 is OA_2;
the abscissa of P_3 is OB_3, the ordinate of P_3 is OA_3;
the abscissa of P_4 is OB_4, the ordinate of P_4 is OA_4.



The abscissa and the ordinate of a point are called the **coördinates** of the point. The lines XX' and YY' are called the **axes of coördinates**, or the **axes of reference**; the line XX' is called the **axis of abscissas**, or the **axis of *x***; and the line YY' is called the **axis of ordinates**, or the **axis of *y***. The point O is called the **origin**.

In general, an abscissa is represented by x, and an ordinate by y. The coördinates of a point whose abscissa is x and ordinate y are written (x, y). In this notation the abscissa is always written first and the ordinate second.

Thus, the point (4, 7) is the point whose abscissa is 4 and ordinate 7.

Abscissas measured to the *right* of YY' are called **positive**, to the *left* of YY' are called **negative**; ordinates measured *above* XX' are called **positive**, *below* XX' are called **negative**.

Thus, in the figure on page 201 the point P_1 is (8, 5), the point P_2 is (– 6, 3), the point P_3 is (– 4, – 3), and the point P_4 is (5, – 4).

222. Quadrants. The axes of coördinates divide the plane of the axes into four parts called **quadrants**. The quadrant XOY is called Quadrant I, the quadrant $X'OY$ is called Quadrant II, the quadrant $X'OY'$ is called Quadrant III, and the quadrant XOY' is called Quadrant IV.

Every point in Quadrant I has a positive abscissa and a positive ordinate; every point in Quadrant II has a negative abscissa and a positive ordinate; every point in Quadrant III has a negative abscissa and a negative ordinate; every point in Quadrant IV has a positive abscissa and a negative ordinate.

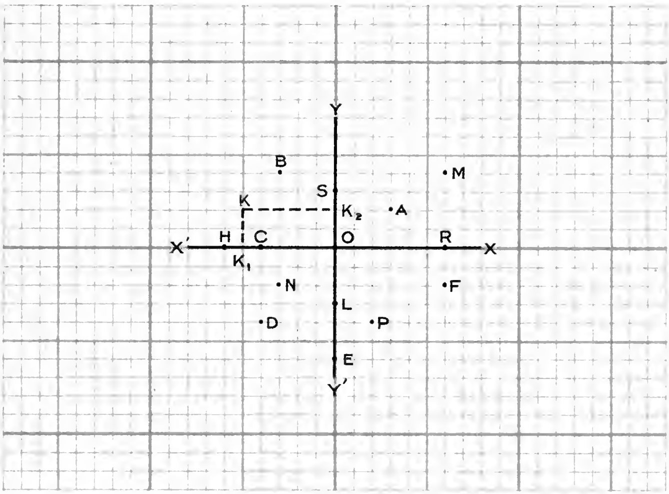
Hence, the signs of the coördinates of a point show at a glance in what quadrant the point is situated.

223. Plotting Points. It is evident that if the location of a point is known, the coördinates of that point referred to given axes may be found easily by measurement; and if the coördinates of a point are given, the point may be readily constructed, or **plotted**.

Thus, a convenient length is taken as the unit, and the point P is found by measurement to lie 2 units to the *right* of YY' and 4 units *below* XX', and is, therefore, the point (2, – 4).

Again, to plot the point (–5, 2), a distance of 5 units is laid off on XX' to the *left* from O to K_1, and a distance of 2 units on YY' *upwards* from O to K_2. The intersection of the perpendiculars erected at K_1 and K_2 determines the point K, which is the required point (– 5, 2).

NOTE. Coördinate paper is paper ruled in small squares. In plotting points and graphs the student will find coördinate paper of much help in giving accuracy and in saving time.



EXERCISE 79

1. In the figure determine the coördinates of B; of M; of N; of R; of S; of H; of L; of A; of F; of D; of C.

2. What is the abscissa of a point on the axis of y? What is the ordinate of a point on the axis of x?

3. Where must a point lie if its ordinate is zero? if its abscissa is zero? if both abscissa and ordinate are zero?

4. Plot the following points: (2, 5), (– 3, 6), (– 2, – 4), (3, – 5), (7, 0), (– 5, 0), (0, 0), (0, – 3), (– 4, – 5), (7, 2).

5. In what quadrant does a point lie if its coördinates are both positive? if both are negative? if the ordinate is positive and the abscissa negative? if the abscissa is positive and the ordinate negative?

6. Plot the points $(-2, -8)$, $(-1, -6)$, $(0, -4)$, $(1, -2)$, $(2, 0)$, $(3, 2)$, $(4, 4)$. Do these points lie in a straight line? Is the equation $2x - y = 4$ satisfied if the abscissas are substituted in turn for x, and the corresponding ordinates for y?

224. Graph of a Function. Let $f(x)$ be any algebraic function of x, where x is a variable. If $y = f(x)$, then y is a new variable connected with x by the relation of $y = f(x)$. If $f(x)$ is rational and integral, it is evident that to every value of x there corresponds one value, and only one value, of y.

If different values of x are laid off as abscissas, and the corresponding values of $f(x)$ as ordinates, a series of points will be obtained. A line, straight or curved, may be drawn through all these points. This line is called the **graph of the function $f(x)$**; it is also called the **graph of the equation $y = f(x)$**.

Plot the graph of the equation $x - 2y - 4 = 0$.

Transpose, $$2y = x - 4.$$
$$\therefore y = \frac{x-4}{2}.$$

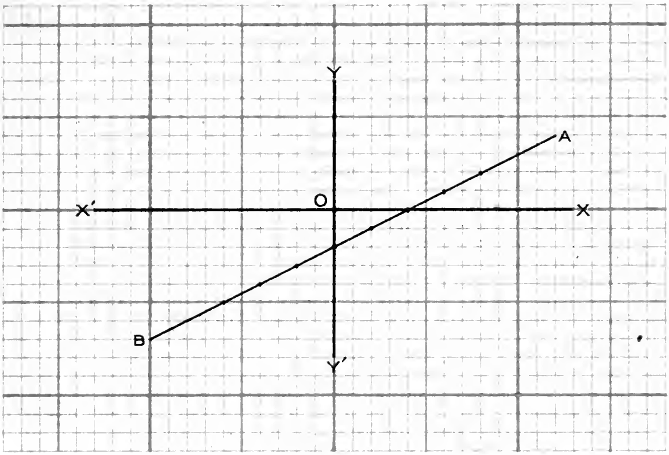
The following table may be computed readily.

If $x = 0$,	$y = -2$;		If $x = +2$,	$y = -1$;
$x = -2$,	$y = -3$;		$x = +4$,	$y = 0$;
$x = -4$,	$y = -4$;		$x = +6$,	$y = +1$;
$x = -6$,	$y = -5$.		$x = +8$,	$y = +2$.

These points are plotted in the figure on page 205 and all lie on the straight line AB. If, in the given equation $x - 2y - 4 = 0$, the abscissa of any point in the line AB is substituted for x and the corresponding ordinate for y, the equation is satisfied. The line AB extends indefinitely in either direction and is the *graph of the equation* $x - 2y - 4 = 0$.

If any two points of a straight line are known, the position of the line is definitely determined.

225. Linear Equations. The graph of every equation of the form $ax + by + c = 0$ is a straight line. For this reason such an equation is often called a **linear equation**.



EXERCISE 80

Plot the graph of:

1. $3x - 2y = 6$.
2. $5x + 2y = 10$.
3. $4x - y + 4 = 0$.
4. $-x + 3y = 6$.
5. $3x + 2y = 12$.
6. $x - 5y = 5$.

Plot the graphs of the following equations by finding the points in which the graphs cut the axes:

7. $7x + 2y - 14 = 0$.
8. $5x - 3y - 15 = 0$.
9. $3x - 4y - 24 = 0$.
10. $4x + 3y + 12 = 0$.
11. $x - 8y + 8 = 0$.
12. $5x + 4y + 30 = 0$.

Plot the graphs of the following equations by finding any two points:

13. $x + y = 0$.
14. $x - y = 0$.
15. $x - 5y = 0$.
16. $2x = 6y$.
17. $5x + 4y = 0$.
18. $7x - 5y = 0$.

19. In what respect do the equations of Examples 1–12 differ from the equations of Examples 13–18?

20. Does the graph of the equation $ax \pm by = 0$ pass through the origin? Why?

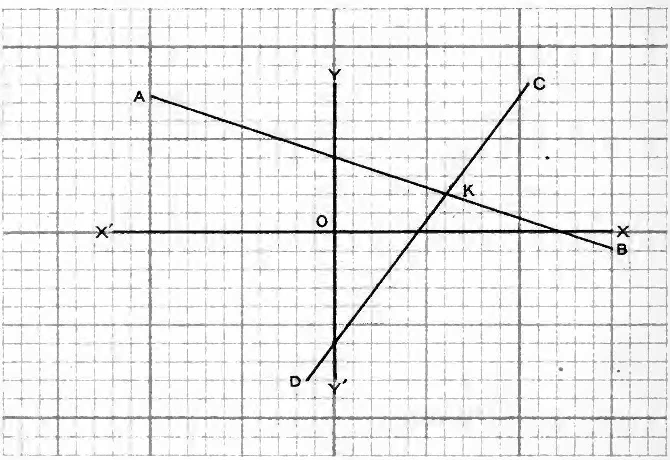
21. The equation of XX' is $y = 0$. Find the equation of YY'.

22. What is the position of a graph if its equation does not contain x? if its equation does not contain y?

Plot the graph of:

23. $3x = 6$. 25. $x = -\frac{7}{2}$. 27. $5x = 30$.

24. $2y = 5$. 26. $y = -\frac{4}{5}$. 28. $6y = -42$.



226. Graph of the Solution of a Pair of Simultaneous Linear Equations. In the figure the straight line AB is the graph of the equation $x + 3y = 12$, and the straight line CD is the graph of the equation $4x - 3y = 18$. It is evident that the coördinates of K, the point of intersection of the lines AB and CD, must satisfy both equations.

By solving the equations as simultaneous equations we find that $x = 6$ and $y = 2$, which are the coördinates of the point K.

Hence, it is evident that it is possible by the use of graphs to solve two simultaneous linear equations that contain only two unknown numbers. In some cases exact values of the unknown numbers may be found; in other cases only approximate values. The larger the scale used in plotting the graphs, the closer will be the approximations obtained.

Exercise 81

Find by graphs exact values of x and y in the following equations and verify by solving the equations:

1. $\left.\begin{aligned} 2x - 5y &= 0 \\ 4x + 2y &= 24 \end{aligned}\right\}$

2. $\left.\begin{aligned} 7x - 2y &= 14 \\ 5x + y &= 10 \end{aligned}\right\}$

3. $\left.\begin{aligned} 5x + 4y &= 30 \\ x - y &= -3 \end{aligned}\right\}$

4. $\left.\begin{aligned} 11x - 2y &= 21 \\ 2x + 4y &= -18 \end{aligned}\right\}$

5. $\left.\begin{aligned} 5x + 8y &= 20 \\ 2x - 3y &= -23 \end{aligned}\right\}$

6. $\left.\begin{aligned} 3x + 4y &= 30 \\ 5x - 6y &= 12 \end{aligned}\right\}$

Find by graphs approximate values of x and y:

7. $\left.\begin{aligned} 4x - 5y &= 10 \\ 2x + 3y &= 9 \end{aligned}\right\}$

8. $\left.\begin{aligned} 8x + y &= 20 \\ 2x - 5y &= 10 \end{aligned}\right\}$

9. $\left.\begin{aligned} 7x - 2y &= 14 \\ 5x + 3y &= 15 \end{aligned}\right\}$

10. $\left.\begin{aligned} 9x - 4y &= 18 \\ 2x + 5y &= 20 \end{aligned}\right\}$

11. The graphs of the equations $2x + 3y = 4$, $2x - y = 12$, and $x + 3y = -1$ meet in a point. Are the equations simultaneous? Give reason.

12. Do the graphs of the equations $4x - y = 2$, $3x + y = 10$, and $x - 6y = 5$ meet in a point? Are the equations simultaneous? Are the equations inconsistent?

13. Are the equations $2x - 3y = 5$ and $2x - 3y = 8$ simultaneous? What is shown by their graphs?

CHAPTER XIV

SIMPLE INDETERMINATE EQUATIONS

227. Indeterminate Equations. If a single equation involving two unknown numbers is given, and no other condition is imposed, the number of solutions of the equation is unlimited; for if one of the unknown numbers is assumed to have *any* particular value, a *corresponding* value of the other may be found. Such an equation is called an **indeterminate equation.**

228. The values of the unknown numbers in an indeterminate equation are *dependent upon each other;* so that they are confined to a particular range.

This range may be further limited by requiring these values to satisfy some given condition; as, for instance, that they shall be *positive integers.*

1. Solve $4x + 9y = 53$ in positive integers.

Transpose, $$4x = 53 - 9y.$$

Divide by 4, $$x = 13 - 2y + \frac{1-y}{4}.$$

Transpose, $$x + 2y - 13 = \frac{1-y}{4}.$$

Since x, $2y$, and 13 are integers, $\frac{1-y}{4}$ is an integer.

Let $$\frac{1-y}{4} = m, \text{ an integer.}$$

Then $$y = 1 - 4m. \qquad (1)$$

Put this value for y in the given equation, and reduce.

Then $$x = 11 + 9m. \qquad (2)$$

Equation (1) shows that m may be 0, or have any negative integral value, but cannot have a positive integral value. Equation (2) shows that m may be 0, but cannot have a negative integral value greater in absolute value than 1. Therefore, m may be 0 or -1.

Therefore, $x = 11$, and $y = 1$; or $x = 2$, and $y = 5$.

2. Solve $5x - 17y = 23$ in positive integers.

Transpose, $$5x = 23 + 17y.$$

Divide by 5, $$x = 4 + 3y + \frac{3 + 2y}{5}. \quad (1)$$

Multiply (1) by the smallest integer that will cause the coefficient of y in the fraction to be one greater than some multiple of the denominator.

Multiply (1) by 3, $$3x = 12 + 9y + \frac{9 + 6y}{5}.$$

$$3x = 12 + 9y + 1 + y + \frac{4 + y}{5}.$$

Transpose and combine,

$$3x - 10y - 13 = \frac{4 + y}{5}. \quad (2)$$

Since the left member of (2) is integral, the right member is integral.

Let $$\frac{4 + y}{5} = m, \text{ an integer.}$$

Then $$y = 5m - 4.$$

Substitute the value of y in (1), $x = 17m - 9$.

Both x and y are positive integers if m is a positive integer.

Hence, for $m = 1$, $x = 8$ and $y = 1$; for $m = 2$, $x = 25$ and $y = 6$; for $m = 3$, $x = 42$ and $y = 11$; and so on indefinitely.

Note. It will be seen, from the solutions of Problems 1 and 2, that when only positive integers are required the number of solutions is *limited* or *unlimited* according as the sign connecting x and y is *positive* or *negative*.

Exercise 82

Solve in positive integers:

1. $x + y = 9$.
2. $x + 9y = 27$.
3. $3x + 7y = 46$.
4. $4x + 5y = 33$.
5. $8x + 7y = 112$.
6. $7x + 10y = 100$.
7. $3x + 8y = 25$.
8. $15x + 9y = 87$.
9. $5x + 9y = 34$.
10. $8x + 13y = 413$.

Solve in least positive integers:

11. $4x - 9y = 53$.
12. $15x - 47y = 1$.
13. $13y - 10x = 19$.
14. $7x - 3y = 17$.

15. $11x - 9y = 31$.

16. $13x - 5y = 113$.

17. $14x - 9y = 229$.

18. $13x - 23y = 42$.

19. Find two numbers which, multiplied respectively by 3 and by 20, have for the sum of their products 368.

20. Divide 76 into two parts such that one part shall be a positive multiple of 8 and the other a positive multiple of 9.

21. How may a man pay a debt of $59, giving only bills of $5 and $2?

22. A man paid $126 for calves at $7 each, and pigs at $3 each. How many calves and how many pigs did he buy?

23. Solve $11x - 21y = 7$, so that x and y may both be positive integers, and x a positive multiple of y.

24. A woman paid $106 for silk at $2.50 a yard, and velvet at $3.50 a yard. How many yards of each did she buy?

25. A man paid $72 for lambs at $5.80 each, and geese at $1.40 each. How many lambs and how many geese did he buy?

26. The diameter of a five-franc piece is 3.7 centimeters, and that of a one-franc piece 2.3 centimeters. How many coins of each kind must a man use to obtain a length of 1 meter?

27. An excursion cost for each adult $7.15, and for each child $3.10. The total expense was $97.25. How many adults and how many children went on the excursion?

28. A box contains between 300 and 400 pears. If the pears are divided among some children, 13 pears being given to each child, 9 pears are left. If 15 pears are given to each child, 4 pears are left. How many pears are there in the box?

29. A farmer sells 12 calves, 16 lambs, and 9 pigs, and receives $187. At the same price he sells 14 calves, 10 lambs, and 15 pigs, and receives $197. What is the price of each?

30. Divide 151 into three parts such that the first part is divisible by 3, the second by 5, the third by 7, and twice the first part, three times the second, and four times the third is 483.

CHAPTER XV

INEQUALITIES

229. The number a is said to be *greater than* the number b when the number $a - b$ is positive; the number a is said to be *less than* the number b when the number $a - b$ is negative.

230. Inequalities. A statement in symbols that two expressions do not stand for the same number is called an **inequality.**

231. The Sign of Inequality. The sign of inequality is $>$, and is always placed with the point toward the smaller number and the opening toward the larger number.

Thus, $a > b$ is read *a is greater than b*, and $a < b$ is read *a is less than b.*

232. Members. That part of an inequality which precedes the sign of inequality is called the **first member,** or **left side**; and that part which follows the sign of inequality is called the **second member,** or **right side.**

233. The symbol $\ngtr$ is used for the words *is not greater than,* and the symbol $\nless$ for the words *is not less than.*

234. Two inequalities are said to **subsist in the same sense** if the signs of inequality are placed in the same way; and two inequalities are said to be the **reverse of each other** if the signs are placed in reverse ways.

Thus, $a > b$ and $c > d$ subsist in the same sense, but $a > b$ and $c < d$ are the reverse of each other.

Note. Letters in this chapter are understood to stand for *positive* numbers, unless the contrary is expressly stated.

235. *If the members of an inequality are interchanged, the inequality is reversed.*

Thus, if $a > b$, then $b < a$; and if $b < a$, then $a > b$.

236. *An inequality will continue to subsist in the same sense if each member is increased by, diminished by, multiplied by, or divided by the* **same positive number.**

Thus, if $a > b$, and c is a positive number,

then $a + c > b + c$; $a - c > b - c$; $ac > bc$; $a \div c > b \div c$.

Also, if $a < b$, and c is a positive number,

then $a + c < b + c$; $a - c < b - c$; $ac < bc$; $a \div c < b \div c$.

237. **A term may be transposed** *from either member of an inequality to the other member, provided the sign of the term is changed.*

Thus, if $a - c > b$,

by adding c to each member, $a > b + c$. (§ 236)

238. *An inequality will be reversed if its members are subtracted from equal numbers; or if its members are multiplied by or divided by the* **same negative number.**

Thus, if $c = d$ and $a > b$, then $c - a < d - b$;
and if $a > b$, then $-ac < -bc$, and $a \div (-c) < b \div (-c)$.

239. **The sums or the products** *of the corresponding members of two inequalities that subsist in the same sense form an inequality in the same sense.*

Thus, if $a < b$ and $c < d$, then $a + c < b + d$, and $ac < bd$.

240. **The differences or the quotients** *of the corresponding members of two inequalities that subsist in the same sense may form an inequality in the same sense, or in the reverse sense, or may form an equality.*

Thus,	$9 > 6$	$9 > 7$	$9 > 7$
	$5 > 4$	$5 > 2$	$5 > 3$
By subtraction,	$4 > 2$	$4 < 5$	$4 = 4$
Again,	$9 > 4$	$9 > 6$	$9 > 6$
	$4 > 2$	$6 > 3$	$6 > 4$
By division,	$2\frac{1}{4} > 2$	$1\frac{1}{2} < 2$	$1\frac{1}{2} = 1\frac{1}{2}$

1. Find one limit of the values of x, if

$$3x - 2 > \tfrac{1}{2}x + 3.$$

Multiply by 2, $6x - 4 > x + 6.$
Transpose and combine, $5x > 10.$
Divide by 5, $x > 2.$

Therefore, the lower limit of the values of x is 2.

2. Find the lower and upper limits of the values of x,

given $$6x + 1 > 4x + 7, \quad (1)$$
and $$7x - 12 < 6x - 8. \quad (2)$$

Transpose and combine (1), $2x > 6.$
Divide by 2, $x > 3.$
Transpose and combine (2), $x < 4.$

Therefore, the lower limit of the values of x is 3 and the upper limit is 4.

Exercise 83

Find one limit of x, given:

1. $7x - 12 > 3x + 4.$
2. $21x - 4 < 14x + 17.$
3. $3x + 13 > 4x + 6.$
4. $\dfrac{3x+7}{4x+5} > \dfrac{3x+5}{4x+3}.$
5. $\dfrac{1}{a} + \dfrac{a}{x+a} < \dfrac{x+a}{ax}.$

6. Find the limits of x, given

$$\frac{9x+5}{14} + \frac{8x-7}{6x+2} < \frac{9x+14}{14},$$

and $$\frac{2x+1}{2x-1} - \frac{8}{4x^2-1} > \frac{2x-1}{2x+1}.$$

7. Find the integral value of x, given

$$\tfrac{1}{3}(4x-1) - \tfrac{3}{4} > \tfrac{1}{6}(x-4) + \tfrac{1}{4}(3x+5),$$

and $$\tfrac{1}{3}(2x+1) + \tfrac{1}{4}(3x+1) < 5\tfrac{1}{12} + \tfrac{1}{7}(5x-2).$$

8. Three times a certain integral number increased by 5 is not greater than 23; and five times the number diminished by 2 is not less than 28. Find the number.

241. Theorem. *If* a *and* b *are unequal,* $a^2 + b^2 > 2ab$.

Whatever are the values of a and b, $(a - b)^2$ is positive.

That is, $(a - b)^2 > 0$.

Expand, $a^2 - 2ab + b^2 > 0$.

Transpose $-2ab$, $a^2 + b^2 > 2ab$.

1. Show that if a and b are positive $a^3 + b^3 > a^2b + ab^2$.

If $a^3 + b^3 > a^2b + ab^2$,

Dividing by $a + b$, $a^2 - ab + b^2 > ab$.

Transposing and combining, $a^2 + b^2 > 2ab$.

But $a^2 + b^2 > 2ab$. (§ 241)

$\therefore a^3 + b^3 > a^2b + ab^2$.

Exercise 84

Show that if the letters are unequal and positive:

1. $a^2 + b^2 + c^2 > ab + ac + bc$.
2. $(ab + cd)(ac + bd) > 4abcd$.
3. $ab + ac + bc < (a + b - c)^2 + (a + c - b)^2 + (b + c - a)^2$.
4. $(a + b + c)^3 > 3(a + b)(a + c)(b + c)$.
5. $a^2b + a^2c + ab^2 + b^2c + ac^2 + bc^2 > 6abc$.
6. $(a^2 + b^2)(a^4 + b^4) > (a^3 + b^3)^2$.
7. Show that if $x^2 = a^2 + b^2$, and $y^2 = c^2 + d^2$,
$$xy \not< ac + bd, \text{ or } ad + bc.$$
8. Which is the greater, $(a^2 + b^2)(c^2 + d^2)$ or $(ac + bd)^2$?
9. Which is the greater, $a^4 - b^4$ or $4a^3(a - b)$ when $a > b$?
10. Which is the greater, $\dfrac{a + b}{2}$ or $\dfrac{2ab}{a + b}$?
11. Which is the greater, $\dfrac{a + 4b}{a + 5b}$ or $\dfrac{a + 6b}{a + 7b}$?
12. Show that the sum of any positive fraction and its reciprocal is greater than 2.

CHAPTER XVI

INVOLUTION AND EVOLUTION

Involution

242. Involution. The operation of raising an expression to any required *power* is called **involution** (p. 6, § 23).

Every case of involution is, therefore, merely an example of *multiplication* in which all the factors are equal.

243. Index Law. If m is a positive integer, by definition

$$a^m = a \times a \times a \cdots \text{to } m \text{ factors.} \qquad \text{(p. 7, § 25)}$$

Hence, if m and n are both positive integers,

$$\begin{aligned}(a^n)^m &= a^n \times a^n \times a^n \cdots \text{to } m \text{ factors}\\ &= (a \times a \cdots \text{to } n \text{ factors})\,(a \times a \cdots \text{to } n \text{ factors})\\ &\quad\ (a \times a \cdots \text{to } n \text{ factors}) \cdots \text{to } m \text{ groups of factors}\\ &= a \times a \times a \cdots \text{to } mn \text{ factors}\\ &= a^{mn}.\end{aligned}$$

Similarly, $(a^m)^n = a^{mn} = (a^n)^m$. Hence,

Any required power of a given power of a number is found by multiplying the exponent of the given power by the exponent of the required power.

244. $$\begin{aligned}(ab)^n &= ab \times ab \times ab \cdots \text{to } n \text{ factors}\\ &= (a \times a \cdots \text{to } n \text{ factors})\,(b \times b \cdots \text{to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

In like manner, $(abc)^n = a^n b^n c^n$; and so on. Hence,

Any required power of a product is found by taking the product of its factors each raised to the required power.

245. In a similar way it may be shown that

Any required power of a fraction is found by taking the required power of the numerator and of the denominator.

246. If the exponent of the required power is a composite number, the exponent may be resolved into prime factors, the power denoted by one of these factors found, and the result raised to a power denoted by another factor; and so on.

Thus, the sixth power is the second power of the third power.

247. From the *Law of Signs* in multiplication (p. 43, § 82), all **even powers** of a scalar number are **positive**; all **odd powers** of a scalar number have the **same sign as the number itself.**

Hence, no *even power* of any scalar number can be *negative;* and the *same even* power of two compound expressions that have the same terms with opposite signs are identical.

Thus, $(b - a)^2 = \{-(a - b)\}^2 = (a - b)^2$.

248. Binomials. By actual multiplication, we obtain:

$$(a + b)^2 = a^2 + 2\,ab + b^2;$$
$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3;$$
$$(a + b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4;$$
$$(a + b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5.$$

In these results it will be observed that:

1. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

2. In the first term the exponent of a is the same as the exponent of the power to which the binomial is raised, and it decreases by one in each succeeding term.

3. b appears in the second term with 1 for an exponent, and its exponent increases by one in each succeeding term.

4. The coefficient of the first term is 1.

5. The coefficient of the second term is the same as the exponent of the binomial.

6. The coefficient of each succeeding term is found from the next preceding term by multiplying the coefficient of that term by the exponent of a and dividing the product by a number greater by one than the exponent of b.

249. If b is negative, the terms in which the **odd** powers of b occur are **negative**.

Thus, $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$;
$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.

By the above rules, any power of a binomial of the form $a \pm b$ may be written at once.

NOTE. The double sign $\pm$ is read *plus or minus*, and signifies the sum or the difference of the numbers between which it is placed. The double sign $\mp$ is read *minus or plus*, and signifies the difference or the sum of the numbers between which it is placed.

250. The same method may be employed when the terms of a binomial have *coefficients* or *exponents*.

1. Find the second power of $\frac{3}{4}x^6 + \frac{2}{3}y^2$.

Now $(a+b)^2 = a^2 + 2ab + b^2$.

Put $\frac{3}{4}x^6$ for a, and $\frac{2}{3}y^2$ for b,

$$(\tfrac{3}{4}x^6 + \tfrac{2}{3}y^2)^2 = (\tfrac{3}{4}x^6)^2 + 2(\tfrac{3}{4}x^6)(\tfrac{2}{3}y^2) + (\tfrac{2}{3}y^2)^2$$
$$= \tfrac{9}{16}x^{12} + x^6y^2 + \tfrac{4}{9}y^4.$$

2. Find the third power of $2x^3 - 3y^2$.

Now $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Put $2x^3$ for a, and $3y^2$ for b,

$$(2x^3 - 3y^2)^3 = (2x^3)^3 - 3(2x^3)^2(3y^2) + 3(2x^3)(3y^2)^2 - (3y^2)^3$$
$$= 8x^9 - 36x^6y^2 + 54x^3y^4 - 27y^6.$$

3. Find the fourth power of $\frac{1}{2}x^2 + \frac{1}{3}y$.

Now $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Put $\frac{1}{2}x^2$ for a, and $\frac{1}{3}y$ for b,

$$(\tfrac{1}{2}x^2 + \tfrac{1}{3}y)^4 = (\tfrac{1}{2}x^2)^4 + 4(\tfrac{1}{2}x^2)^3(\tfrac{1}{3}y) + 6(\tfrac{1}{2}x^2)^2(\tfrac{1}{3}y)^2 + 4(\tfrac{1}{2}x^2)(\tfrac{1}{3}y)^3 + (\tfrac{1}{3}y)^4$$
$$= \tfrac{1}{16}x^8 + \tfrac{1}{6}x^6y + \tfrac{1}{6}x^4y^2 + \tfrac{2}{27}x^2y^3 + \tfrac{1}{81}y^4.$$

251. In like manner, a *polynomial* of three or more terms may be raised to any power by inclosing its terms in parentheses, so as to give the polynomial the *form of a binomial.*

1. $(1-2x+3x^2)^3$
$=[(1-2x)+3x^2]^3$
$=(1-2x)^3+3(1-2x)^2(3x^2)+3(1-2x)(3x^2)^2+(3x^2)^3$
$=1-6x+12x^2-8x^3+9x^2-36x^3+36x^4+27x^4-54x^5+27x^6$
$=1-6x+21x^2-44x^3+63x^4-54x^5+27x^6.$

2. $(x^3-2x^2+3x-4)^2$
$=[(x^3-2x^2)+(3x-4)]^2$
$=(x^3-2x^2)^2+2(x^3-2x^2)(3x-4)+(3x-4)^2$
$=x^6-4x^5+4x^4+6x^4-20x^3+16x^2+9x^2-24x+16$
$=x^6-4x^5+10x^4-20x^3+25x^2-24x+16.$

Exercise 85

Perform the indicated operation:

1. $(x^3)^5$.
2. $(x^2y^3)^6$.
3. $\left(\frac{2a^3b^2}{3x^3y^4}\right)^3$.
4. $(-3x^2y^3)^4$.
5. $(-5a^3b^5)^3$.
6. $\left(-\frac{3x^2y^3}{4ab^2}\right)^5$.
7. $(2a^2b^3yz^4)^5$.
8. $(-4c^3dx^2)^4$.
9. $\frac{(-5ab^3)^5}{(10a^2xy^3)^3}$.
10. $(a-3)^5$.
11. $(a^2+2ab)^4$.
12. $(2x+3y)^5$.
13. $(3a-2b)^4$.
14. $(a+b)^7$.
15. $(xy-ab)^5$.
16. $(3n^2-2)^3$.
17. $(ab-3)^6$.
18. $(1-2x)^5$.
19. $(3a+5b)^4$.
20. $(x+2y)^8$.
21. $(1-x+x^2)^3$.
22. $\left(\frac{3}{2x}-\frac{2x}{3}\right)^5$.
23. $\left(1-\frac{2x^2}{3}\right)^6$.
24. $\left(\frac{2}{x}-\frac{3x^2}{2}\right)^7$.
25. $(1-2x+3x^2+4x^3)^3$.
26. $(1-3x+x^2)^4$.
27. $\left(3a^2b^m-\frac{a^mb^2}{3}\right)^4$.
28. $(a^4-a^3b+a^2b^2-ab^3+b^4)^2$.
29. $\left(a+\frac{a^2}{2}-\frac{a^3}{3}\right)^3$.
30. $(a^mb^n-x^py^q)^4$.

EVOLUTION

252. Evolution. The operation of finding any required root of an expression is called **evolution** (p. 7, § 27).

253. Index Law for Evolution. If k, m, and n are positive integers, we have $(a^m)^n = a^{mn}$. (p. 215, § 243)

Therefore, a^m is an nth root of a^{mn}.

That is, $a^m =$ one value of $\sqrt[n]{a^{mn}}$.

Also, $(a^k b^m)^n = a^{kn} b^{mn}$. (p. 215, § 244)

Therefore, $a^k b^m =$ one value of $\sqrt[n]{a^{kn} b^{mn}}$. Hence,

254. *Any required root of a monomial is found by dividing the exponent of each factor by the index of the required root and taking the product of the resulting factors*, the numerical coefficient, if other than 1, being first expressed as the product of its prime factors.

The root thus obtained is called the **principal root** of the monomial for the given index.

Thus, $2a^2$ is the principal third root of $8a^6 (= 2^3 a^6)$.

255. By the Law of Signs for Multiplication (p. 43, § 82),

$$(+a) \times (+a) = +a^2,$$

and

$$(-a) \times (-a) = +a^2.$$

Therefore, $\sqrt{+a^2}$ may be either $+a$ or $-a$,

but $\sqrt{-a^2}$ can be neither $+a$ nor $-a$. Hence,

1. Every positive number has *two* square roots, equal in absolute value but opposite in sign, one being positive, the other negative.

2. No scalar number can be the square root of a negative number.

3. An odd-indexed root of a scalar number has the same sign as the number itself.

256. If n is a positive integer,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}. \qquad \text{(p. 216, § 245)}$$

Conversely, $$\sqrt[n]{\frac{a^n}{b^n}} = \frac{a}{b} = \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}}.$$ Hence,

Any required root of a fraction is found by taking the required root of the numerator and of the denominator.

Thus, $\sqrt{\dfrac{16x^2}{81y^2}} = \pm\dfrac{4x}{9y}$; $\sqrt[4]{\dfrac{16x^4}{81y^4}} = \pm\dfrac{2x}{3y}$.

257. Imaginary Numbers. The indicated even root of a negative number is called an **imaginary** or **orthotomic number.**

Thus, $\sqrt{-2}$, $\sqrt{-x}$, and $\sqrt[4]{-x^2y^3}$ are orthotomic numbers.

258. If the root of a number expressed in figures is not readily found, first resolve the number into its prime factors.

Thus, $\sqrt{7683984} = \sqrt{2^4 \times 3^4 \times 7^2 \times 11^2} = 2^2 \times 3^2 \times 7 \times 11 = 2772$.

Exercise 86

Simplify:

1. $\sqrt{4a^4b^2}$.
2. $\sqrt[3]{27a^6b^9}$.
3. $\sqrt[4]{16x^8y^{12}}$.
4. $\sqrt[5]{32a^{10}b^{15}}$.
5. $\sqrt[3]{-27a^3x^6}$.
6. $\sqrt{625a^4b^4}$.
7. $\sqrt[4]{625a^8b^8}$.
8. $\sqrt[6]{64x^{18}}$.
9. $\sqrt[5]{-32a^{15}b^{20}}$.
10. $\sqrt[3]{512a^{12}b^{18}}$.
11. $\sqrt[6]{729a^{12}b^{18}}$.
12. $\sqrt[3]{-216x^3y^6}$.
13. $\sqrt[3]{1728c^6}$.
14. $\sqrt[4]{81a^{16}y^8}$.
15. $\sqrt[3]{-1331x^9y^{12}}$.
16. $\sqrt[3]{x^{3n}y^{6n}}$.
17. $\sqrt[4]{a^{4m}b^{8n}}$.
18. $\sqrt[3]{-27a^{3p}c^{3q}}$.
19. $\sqrt[5]{-\dfrac{32x^5}{243a^{10}}}$.
20. $\sqrt[3]{-\dfrac{8x^3y^9}{27a^6}}$.
21. $\sqrt[4]{\dfrac{16a^8}{81b^{12}c^{16}}}$.
22. $\dfrac{\sqrt[3]{216a^6x^3}}{\sqrt[5]{32a^{15}x^{10}}}$.
23. $\sqrt[3]{\dfrac{125a^3b^6c^9}{216x^9y^6z^3}}$.
24. $\sqrt[6]{\dfrac{729x^6y^{12}}{64a^{12}b^{12}}}$.

Square Roots of Polynomials

259. Since the square of $a + b$ is $a^2 + 2\,ab + b^2$, the square root of $a^2 + 2\,ab + b^2$ is $a + b$.

It is required to devise a method for extracting the square root $a + b$ when the square $a^2 + 2\,ab + b^2$ is given.

$$\begin{array}{r|l} & a^2 + 2\,ab + b^2 \;\underline{|\, a + b} \\ & a^2 \\ \hline 2\,a + b & 2\,ab + b^2 \\ & \underline{2\,ab + b^2} \end{array}$$

The first term, a, of the root is obviously the square root of the first term, a^2, of the expression.

If the a^2 is subtracted from the given expression, the remainder is $2\,ab + b^2$. Therefore, the second term, b, of the root is obtained when the first term of this remainder is divided by $2\,a$, that is, by *double the part of the root already found.* Also, since

$$2\,ab + b^2 = (2\,a + b)\,b,$$

the divisor *is completed by adding to the trial divisor the new term of the root.*

Find the square root of $25\,x^2 - 20\,x^3y + 4\,x^4y^2$.

$$\begin{array}{r|l} & 25\,x^2 - 20\,x^3y + 4\,x^4y^2 \;\underline{|\, 5\,x - 2\,x^2y} \\ & 25\,x^2 \\ \hline 10\,x - 2\,x^2y & -20\,x^3y + 4\,x^4y^2 \\ & \underline{-20\,x^3y + 4\,x^4y^2} \end{array}$$

The expression is *arranged* according to the ascending powers of x.

The square root of the first term is $5\,x$, and $5\,x$ is placed at the right of the given expression, for the first term of the root.

The second term of the root, $-2\,x^2y$, is obtained by dividing $-20\,x^3y$ by $10\,x$, the double of the part of the root already found, and this new term of the root is also annexed to the divisor, $10\,x$, to complete the divisor.

260. The same method applies to longer expressions, if care is taken to obtain the *trial divisor* at each stage of the process by *doubling the part of the root already found*, and to obtain the *complete divisor by annexing the new term of the root to the trial divisor.*

Find the square root of

$$5\,x^2 - 29\,x^4 + 26\,x^3 - 10\,x^5 - 12\,x + 25\,x^6 + 4.$$

Arrange the expression in descending powers of x.

$$
\begin{array}{r|l}
 & 25\,x^6 - 10\,x^5 - 29\,x^4 + 26\,x^3 + 5\,x^2 - 12\,x + 4 \;\big|\; \underline{5\,x^3 - x^2 - 3\,x + 2} \\
 & 25\,x^6 \\
10\,x^3 - x^2 & -10\,x^5 - 29\,x^4 \\
 & -10\,x^5 + x^4 \\
10\,x^3 - 2\,x^2 - 3\,x & -30\,x^4 + 26\,x^3 + 5\,x^2 \\
 & -30\,x^4 + 6\,x^3 + 9\,x^2 \\
10\,x^3 - 2\,x^2 - 6\,x + 2 & 20\,x^3 - 4\,x^2 - 12\,x + 4 \\
 & 20\,x^3 - 4\,x^2 - 12\,x + 4
\end{array}
$$

It will be noticed that each successive trial divisor may be obtained by taking the preceding complete divisor with its *last term doubled.*

EXERCISE 87

Find the square root of:

1. $x^4 - 6\,x^3 + 13\,x^2 - 12\,x + 4.$
2. $1 - 2\,a^3 + a^4 - 2\,a + 3\,a^2.$
3. $9\,x^4 - 12\,x^3y + 34\,x^2y^2 - 20\,xy^3 + 25\,y^4.$
4. $49\,a^4 - 42\,a^3b + 37\,a^2b^2 - 12\,ab^3 + 4\,b^4.$
5. $x^4 + 10\,x^3y + 33\,x^2y^2 + 40\,xy^3 + 16\,y^4.$
6. $4\,x^4 + 37\,x^2y^2 - 30\,xy^3 - 20\,x^3y + 9\,y^4.$
7. $16\,a^4 - 40\,a^3x + a^2x^2 + 30\,ax^3 + 9\,x^4.$
8. $16\,a^4 + 56\,a^3x + a^2x^2 - 84\,ax^3 + 36\,x^4.$
9. $9\,c^4 - 48\,c^3d + 34\,c^2d^2 + 80\,cd^3 + 25\,d^4.$
10. $13\,x^4 + 13\,x^2 + 4\,x^6 - 14\,x^3 + 4 - 4\,x - 12\,x^5.$
11. $4x^6 - 14\,x^2y^4 + 25\,y^6 - 7\,x^4y^2 + 12\,x^5y + 40xy^5 - 44\,x^3y^3.$
12. $4\,a^6 - 20\,a^5b - 3\,a^4b^2 + 82\,a^3b^3 + 19\,a^2b^4 - 42\,ab^5 + 9\,b^6.$

261. If an expression contains powers and reciprocals of powers of the same letter, the order of arrangement in descending powers of the letter is as follows:

$$\cdots, x^4, x^3, x^2, x, 1, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \cdots$$

Find the square root of

$$\frac{10}{9}+\frac{9a^2}{25x^2}+\frac{25x^2}{36a^2}-\frac{5x}{9a}-\frac{2a}{5x}.$$

Arrange in descending powers of a.

$$\begin{array}{l|l|l}
 & \frac{9a^2}{25x^2}-\frac{2a}{5x}+\frac{10}{9}-\frac{5x}{9a}+\frac{25x^2}{36a^2} & \frac{3a}{5x}-\frac{1}{3}+\frac{5x}{6a} \\
 & \frac{9a^2}{25x^2} & \\
\frac{6a}{5x}-\frac{1}{3} & -\frac{2a}{5x}+\frac{10}{9} & \\
 & -\frac{2a}{5x}+\frac{1}{9} & \\
\frac{6a}{5x}-\frac{2}{3}+\frac{5x}{6a} & 1-\frac{5x}{9a}+\frac{25x^2}{36a^2} & \\
 & 1-\frac{5x}{9a}+\frac{25x^2}{36a^2} &
\end{array}$$

262. An approximate value of the square root of an imperfect square may be found to any required number of terms.

Find to three terms the square root of x^2+px.

$$\begin{array}{l|l|l}
 & x^2+px & x+\frac{p}{2}-\frac{p^2}{8x}+\cdots \\
 & x^2 & \\
2x+\frac{p}{2} & px & \\
 & px+\frac{p^2}{4} & \\
2x+p-\frac{p^2}{8x} & -\frac{p^2}{4} & \\
 & -\frac{p^2}{4}-\frac{p^3}{8x}+\frac{p^4}{64x^2} &
\end{array}$$

Exercise 88

Find the square root of:

1. $\frac{a^4}{4} - \frac{a^2b^3}{3} + \frac{b^6}{9} + \frac{a^2c^4}{4} + \frac{c^8}{16} - \frac{b^3c^4}{6}$.

2. $\frac{x^2}{y^2} - \frac{4\,xz}{ay} + \frac{4\,z^2}{a^2} + \frac{6\,qx}{by} + \frac{9\,q^2}{b^2} - \frac{12\,qz}{ab}$.

3. $\frac{9\,a^6b^4}{25\,x^6y^8} - \frac{12\,a^5b^5}{35\,x^7y^9} - \frac{332\,a^4b^6}{735\,x^8y^{10}} + \frac{16\,a^3b^7}{63\,x^9y^{11}} + \frac{16\,a^2b^8}{81\,x^{10}y^{12}}$.

4. $x^6y^6 - \frac{2\,x^3y^{12}}{3\,a^3} + \frac{y^{18}}{9\,a^6} + \frac{4\,x^3y^{18}}{a^9} - \frac{4\,y^{24}}{3\,a^{12}} + \frac{4\,y^{30}}{a^{18}}$.

5. $\frac{9\,y^2}{16\,x^2} + 25\frac{3}{7} - \frac{20\,x}{7\,y} - \frac{15\,y}{2\,x} + \frac{4\,x^2}{49\,y^2}$.

6. $\frac{a^4}{4} + \frac{a^3b}{3} - \frac{5\,a^2b^2}{36} - \frac{ab^3}{6} + \frac{b^4}{16}$.

7. $\frac{9\,a^2b^2}{49\,x^2y^2} - \frac{15\,ab}{28\,xy} + \frac{203}{192} - \frac{35\,xy}{36\,ab} + \frac{49\,x^2y^2}{81\,a^2b^2}$.

8. $\frac{9\,a^2b^2}{x^2y^2} + \frac{4\,x^2y^2}{9\,a^2b^2} - \frac{4\,xy}{15\,ab} + \frac{101}{25} - \frac{6\,ab}{5\,xy}$.

9. $\frac{9}{25} + \frac{a^8}{25} + \frac{m^6}{36} - \frac{6\,a^4}{25} - \frac{m^3}{5} + \frac{4\,n^8}{49} + \frac{12\,n^4}{35} + \frac{a^4m^3}{15} - \frac{4\,a^4n^4}{35} - \frac{2\,m^3n^4}{21}$.

10. $\frac{4\,a^4}{9\,x^4} - \frac{2\,a^3}{3\,x^3} + \frac{25\,a^2}{36\,x^2} - \frac{11\,a}{9\,x} + \frac{13}{9} - \frac{17\,x}{18\,a} + \frac{7\,x^2}{9\,a^2} - \frac{2\,x^3}{3\,a^3} + \frac{x^4}{4\,a^4}$.

11. $\frac{1189\,a^2b^2}{2205\,x^2y^2} + \frac{3662}{6237} + \frac{25\,x^2y^2}{121\,a^2b^2} + \frac{4\,a^4b^4}{25\,x^4y^4} + \frac{172\,ab}{231\,xy} + \frac{40\,xy}{99\,ab} + \frac{12\,a^3b^3}{35\,x^3y^3}$.

Find to three terms the square root of:

12. $a^2 + b$.

13. $x^4 - \frac{1}{4}\,y^2$.

14. $b^2 - c^2$.

15. $x^2 - 2\,a$.

16. $x^2 + 4\,b$.

17. $x^2 + 5$.

18. $4\,x^2 + 2$.

19. $16\,a^4 - 3\,a^2$.

20. $25\,x^2 - 10\,x$.

Square Roots of Arithmetical Numbers

263. In extracting the square root of a number expressed by figures, the first step is to separate the figures into groups.

Since $1 = 1^2$, $100 = 10^2$, $10{,}000 = 100^2$, and so on, it is evident that the square root of any integral square number between 1 and 100 lies between 1 and 10; the square root of any integral square number between 100 and 10,000 lies between 10 and 100; and so on. In other words, the square root of any integral square number expressed by *one* or *two* figures is a number of *one* figure; the square root of any integral square number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, an integral square number is divided into groups of two figures each, from the right to the left, the number of figures in the square root is equal to the number of groups of figures. The last group to the left may consist of only one figure.

Find the square root of 4761.

We first separate the figures 4761 into two groups, 47 and 61, beginning at the right and counting from right to left.

```
     47 61(69
     36
129)11 61
    11 61
```

In this case a in the typical form $a^2 + 2ab + b^2$ represents 6 *tens*, that is, 60, and b represents 9. The 36 subtracted is really 3600, that is, a^2, and the complete divisor $2a + b$ is $2 \times 60 + 9 = 129$.

264. The same method applies to numbers of more than two groups by considering that a in the typical form represents at each step *the part of the root already found*, and that a represents *tens* with reference to the next figure of the root.

Find the square root of 10,265,616.

```
      10 26 56 16(3204
       9
  62)1 26
     1 24
6404)2 56 16
     2 56 16
```

We first separate the figures 10265616 into four groups, 10, 26, 56, and 16, beginning at the right and counting from right to left.

265. If the square root of a number contains decimal places, the number itself contains *twice* as many.

Thus, if 0.21 is the square root of some number, this number is $(0.21)^2 = 0.21 \times 0.21 = 0.0441$; and if 0.121 is the square root, the number is $(0.121)^2 = 0.121 \times 0.121 = 0.014641$.

Therefore, the number of *decimal* places in every square decimal is *even*, and the number of decimal places in the square root is *half* as many as in the given number itself.

Hence, if the given square number contains a decimal, we divide it into groups of two figures each, by beginning at the decimal point and proceeding toward the left for the integral number and toward the right for the decimal. The last group on the right of the decimal point must contain *two* figures, a cipher being annexed when necessary.

Find the square root of 28.7296; of 679.6449.

```
     28.72 96(5.36              6 79.64 49(26.07
     25                         4
 103)372                     46)2 79
      309                       2 76
1066)63 96                 5207)364 49
     63 96                      364 49
```

266. If a number contains an *odd* number of decimal places, or if any number gives a *remainder* when as many figures in the root may have been obtained as the given number has groups, then its exact root cannot be found. We may, however, approximate to its exact root as near as we please by annexing ciphers and continuing the operation.

The square root of a common fraction whose denominator is not a perfect square may be found approximately by reducing the fraction to a decimal and then extracting the root; or by reducing the fraction to an equivalent fraction whose denominator is a perfect square, and extracting the square root of both terms of this equivalent fraction.

Thus, $\sqrt{\frac{3}{8}} = \sqrt{0.375} = 0.61237$;

or $$\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{2.44948}{4} = 0.61237.$$

Find the square root of 7; the square root of 4.367.

```
          7.00(2.6457 ...                4.36 70(2.08973 ...
          4                              4
     46)3 00                       408)36 70
        2 76                           32 64
    524)24 00                     4169)4 06 00
        20 96                          3 75 21
   5285)3 04 00                  41787)30 79 00
        2 64 25                        29 25 09
  52907)39 75 00                417943)1 53 91 00
         37 03 49                       1 25 38 29
          2 71 51                         28 52 71
```

Exercise 89

Find the square root of:

1. 576.
2. 18,769.
3. 494,209.
4. 755,161.
5. 18,090.25.
6. 150.0625.
7. 4076.8225.
8. 432.2241.
9. 709,469.29.
10. 35.545444.
11. 2,611,456.
12. 279,100.89.
13. 351.112644.
14. 95,765,796.
15. 69,384.8281.
16. 1590.015625.
17. 179,301.4336.
18. 322,499.0521.

Find to four decimal places the square root of:

19. 2.
20. 3.
21. 5.
22. 6.
23. 7.
24. 8.
25. 0.3.
26. 0.4.
27. 0.5.
28. 0.6.
29. 0.8.
30. 0.9.
31. 0.307.
32. 0.635.
33. 0.375.
34. 0.869.
35. $\frac{7}{15}$.
36. $\frac{8}{19}$.
37. $\frac{2}{3}$.
38. $\frac{5}{8}$.

Cube Roots of Polynomials

267. Since the cube of $a + b$ is $a^3 + 3\,a^2b + 3\,ab^2 + b^3$, the cube root of $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ is $a + b$.

It is required to devise a method for extracting the cube root $a + b$ when the cube $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ is given.

The first term, a, of the root is obviously the cube root of the first term, a^3, of the given expression.

$$\begin{array}{r|l}
 & a^3 + 3\,a^2b + 3\,ab^2 + b^3 \,\underline{|\, a + b} \\
3\,a^2 \qquad & a^3 \\
+\,3\,ab + b^2 & 3\,a^2b + 3\,ab^2 + b^3 \\
\hline
3\,a^2 + 3\,ab + b^2 & \underline{3\,a^2b + 3\,ab^2 + b^3}
\end{array}$$

If a^3 is subtracted, the remainder is $3\,a^2b + 3\,ab^2 + b^3$; therefore, the second term, b, of the root is obtained by dividing the first term of this remainder by *three times the square of* a.

Also, since $3\,a^2b + 3\,ab^2 + b^3 = (3\,a^2 + 3\,ab + b^2)b$, the *complete divisor* is obtained by adding $3\,ab + b^2$ to the *trial divisor* $3\,a^2$.

Find the cube root of $27\,x^3 - 54\,x^2y + 36\,xy^2 - 8\,y^3$.

$$\begin{array}{rr|l}
 & & 27\,x^3 - 54\,x^2y + 36\,xy^2 - 8\,y^3 \,\underline{|\, 3\,x - 2\,y} \\
3\,(3\,x)^2 = 27\,x^2 & & 27\,x^3 \\
(9\,x - 2\,y)(-2\,y) = & -18\,xy + 4\,y^2 & -54\,x^2y + 36\,xy^2 - 8\,y^3 \\
\cline{2-2}
 & 27\,x^2 - 18\,xy + 4\,y^2 & \underline{-54\,x^2y + 36\,xy^2 - 8\,y^3}
\end{array}$$

The cube root of the first term is $3\,x$, and $3\,x$ is therefore the first term of the root. $27\,x^3$, the cube of $3\,x$, is subtracted.

The second term of the root, $-2\,y$, is obtained by dividing $-54\,x^2y$ by $3\,(3\,x)^2 = 27\,x^2$, which corresponds to $3\,a^2$ in the typical form, and the divisor is completed by annexing to $27\,x^2$ the expression

$$\{3\,(3\,x) - 2\,y\}\,(-2\,y) = -18\,xy + 4\,y^2.$$

268. The same method applies to longer expressions by considering a in the typical form $3\,a^2 + 3\,ab + b^2$ to represent at each stage of the process *the part of the root already found.*

Thus, if the part of the root already found is $x + y$, then $3\,a^2$ of the typical form is represented by $3\,(x + y)^2$; and if the third term of the root is $+ z$, the $3\,ab + b^2$ is represented by $3(x+y)z + z^2$. Hence, the complete divisor, $3\,a^2 + 3\,ab + b^2$, is represented by $3(x+y)^2 + 3(x+y)z + z^2$.

Find the cube root of $x^6 - 3\,x^5 + 5\,x^3 - 3\,x - 1$.

$$\begin{array}{rl|l}
 & & x^6 - 3x^5 + 5x^3 - 3x - 1 \,\underline{|\, x^2 - x - 1} \\
3(x^2)^2 = & 3x^4 & x^6 \\
(3x^2 - x)(-x) = & \underline{\quad -3x^3 + x^2} & -3x^5 + 5x^3 \\
 & 3x^4 - 3x^3 + x^2 & \underline{-3x^5 \qquad + 3x^4 - x^3} \\
3(x^2 - x)^2 = & 3x^4 - 6x^3 + 3x^2 & -3x^4 + 6x^3 - 3x - 1 \\
(3x^2 - 3x - 1)(-1) = & \underline{\qquad -3x^2 + 3x + 1} & \\
 & 3x^4 - 6x^3 \qquad + 3x + 1 & \underline{-3x^4 + 6x^3 - 3x - 1}
\end{array}$$

The first term of the root, x^2, is obtained by taking the cube root of the first term of the given expression; and the first trial divisor, $3\,x^4$, is obtained by taking three times the square of this term.

The first complete divisor is found by annexing to the trial divisor $(3\,x^2 - x)(-x)$, which corresponds to $(3\,a + b)\,b$ in the typical form.

The part of the root already found, a, is now represented by $x^2 - x$; therefore, $3\,a^2$ is represented by $3(x^2 - x)^2 = 3\,x^4 - 6\,x^3 + 3\,x^2$, the second trial divisor; and $(3\,a + b)\,b$ by $(3\,x^2 - 3\,x - 1)(-1)$. Therefore, in the second complete divisor, $3\,a^2 + (3\,a + b)\,b$ is represented by $(3\,x^4 - 6\,x^3 + 3\,x^2) + (3\,x^2 - 3\,x - 1)(-1) = 3\,x^4 - 6\,x^3 + 3\,x + 1$.

Exercise 90

Find the cube root of:

1. $27\,x^3 - 189\,x^2y + 441\,xy^2 - 343\,y^3$.
2. $300\,ab^2 - 240\,a^2b - 125\,b^3 + 64\,a^3$.
3. $a^6 - 3\,a^5b + 6\,a^4b^2 - 7\,a^3b^3 + 6\,a^2b^4 - 3\,ab^5 + b^6$.
4. $8\,x^6 - 12\,x^5y + 30\,x^4y^2 - 25\,x^3y^3 + 30\,x^2y^4 - 12\,xy^5 + 8\,y^6$.
5. $a^6 - 6\,a^5b + 21\,a^4b^2 - 44\,a^3b^3 + 63\,a^2b^4 - 54\,ab^5 + 27\,b^6$.
6. $8\,c^6 - 60\,c^5 + 114\,c^4 + 55\,c^3 - 171\,c^2 - 135\,c - 27$.
7. $27\,a^6 + 108\,a^5 + 198\,a^4 + 208\,a^3 + 132\,a^2 + 48\,a + 8$.
8. $64\,a^6 - 144\,a^5b + 60\,a^4b^2 + 45\,a^3b^3 - 15\,a^2b^4 - 9\,ab^5 - b^6$.

9. $125x^9 - 300x^8 + 465x^7 - 424x^6 + 279x^5 - 108x^4 + 27x^3$.

10. $27c^{12} + 189c^{11} + 198c^{10} - 791c^9 - 594c^8 + 1701c^7 - 729c^6$.

11. $a^6 - 9a^5b + 18a^4b^2 + 27a^3b^3 - 54a^2b^4 - 81ab^5 - 27b^6$.

12. $\frac{27a^{12}}{x^9} - \frac{135a^{11}}{x^8} + \frac{198a^{10}}{x^7} - \frac{35a^9}{x^6} - \frac{66a^8}{x^5} - \frac{15a^7}{x^4} - \frac{a^6}{x^3}$.

13. $\frac{8a^3}{27x^3} + \frac{4a^2}{3x^2} + \frac{3a}{x} + 4 + \frac{27x}{8a} + \frac{27x^2}{16a^2} + \frac{27x^3}{64a^3}$.

14. $8x^9 - 36x^8y + 6x^7y^2 + 141x^6y^3 - 84x^5y^4 - 186x^4y^5$
$+ 104x^3y^6 + 60x^2y^7 - 48xy^8 + 8y^9$.

Cube Roots of Arithmetical Numbers

269. In extracting the cube root of a number expressed by figures, the first step is to separate the figures into groups.

Since $1 = 1^3$, $1000 = 10^3$, $1{,}000{,}000 = 100^3$, and so on, it is evident that the cube root of any integral cube number between 1 and 1000, that is, of any integral cube number of *one*, *two*, or *three* figures, is a number of *one* figure; and that the cube root of any integral cube number between 1000 and 1,000,000, that is, of any integral cube number that has *four*, *five*, or *six* figures, is a number of *two* figures; and so on.

If, therefore, an integral cube number is divided into groups of three figures each, from right to left, the number of figures in the cube root is equal to the number of groups of figures. The last group to the left may consist of one, two, or three figures.

270. If the cube root of a number contains decimal places, the number itself contains *three times* as many.

Thus, if 0.11 is the cube root of some number, this number is $(0.11)^3 = 0.11 \times 0.11 \times 0.11 = 0.001331$.

Hence, if the given cube number contains a decimal, we divide it into groups of three figures each, by beginning at the decimal point and proceeding toward the left for the integral number and toward the right for the decimal. The last group on the right of the decimal point must contain *three* figures, ciphers being annexed when necessary.

271. If a given number is not a perfect cube, a number approximate to its cube root may be found by annexing ciphers and continuing the operation of extracting the cube root when all the figures have been used.

272. In the typical form the *first complete divisor* is

$$3\,a^2 + 3\,ab + b^2,$$

and the *second trial divisor* is $3\,(a + b)^2$, that is,

$$3\,a^2 + 6\,ab + 3\,b^2,$$

which may be obtained from the preceding complete divisor by adding to it *its second term and twice its third term.*

Extract the cube root of 5 to five places of decimals.

	5.000(1.70997
	1
$3 \times 10^2 = 300$	4000
$3\,(10 \times 7) = 210$	
$7^2 = 49$	
559	3913
259	87 000 000
$3 \times 1700^2 = 8670000$	
$3\,(1700 \times 9) = 45900$	
$9^2 = 81$	
8715981	78 443 829
45981	8 556 1710
$3 \times 1709^2 = 8762043$	7 885 8387
	670 33230
	613 34301

After the first two figures of the root are found, the next trial divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three numbers connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule is that, if the given number is not a perfect cube, *two less* than the number of figures already obtained may be found without error by division, the divisor being three times the square of the part of the root already found.

Exercise 91

Find the cube root of:

1. 103,823.
2. 262,144.
3. 8741.816.
4. 410.172407.
5. 6,148,602.368.
6. 634,725.648584.

Find to four decimal places the cube root of:

7. 100.
8. 206.
9. 6.3.
10. 0.375.
11. 0.5.
12. 0.98.
13. $\frac{2}{3}$.
14. $\frac{4}{5}$.
15. $\frac{3}{7}$.
16. $\frac{5}{16}$.

273. Since the fourth power is the square of the square, and the sixth power is the square of the cube, the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root.* In similar manner, the eighth, ninth, twelfth, ··· roots may be found.

Exercise 92

Find the fourth root of:

1. $256\,c^{16} - 768\,ac^{13}x^5 + 864\,a^2c^{10}x^{10} - 432\,a^3c^7x^{15} + 81\,a^4c^4x^{20}$.
2. $1 + 16\,x + 108\,x^2 + 400\,x^3 + 886\,x^4 + 1200\,x^5 + 972\,x^6 + 432\,x^7 + 81\,x^8$.

Find the sixth root of:

3. $729\,x^6 - 2916\,x^5y + 4860\,x^4y^2 - 4320\,x^3y^3 + 2160\,x^2y^4 - 576\,xy^5 + 64\,y^6$.
4. $\dfrac{64\,x^6}{15625} - \dfrac{576\,x^5y}{21875} + \dfrac{432\,x^4y^2}{6125} - \dfrac{864\,x^3y^3}{8575} + \dfrac{972\,x^2y^4}{12005} - \dfrac{2916\,xy^5}{84035} + \dfrac{729\,y^6}{117649}$.
5. Find the eighth root of $6561\,x^{16} + 17{,}496\,x^{14} + 20{,}412\,x^{12} + 13{,}608\,x^{10} + 5670\,x^8 + 1512\,x^6 + 252\,x^4 + 24\,x^2 + 1$.
6. Find the ninth root of $a^{27} - 9\,a^{24}y^2 + 36\,a^{21}y^4 - 84\,a^{18}y^6 + 126\,a^{15}y^8 - 126\,a^{12}y^{10} + 84\,a^9y^{12} - 36\,a^6y^{14} + 9\,a^3y^{16} - y^{18}$.

CHAPTER XVII

THEORY OF EXPONENTS

274. Positive Integral Exponents. If a is any definite number or any algebraic expression having one value and only one value, and m and n are positive integers, we have

$$a^n = a \times a \times a \times \cdots \text{ to } n \text{ factors}, \qquad \text{(p. 6, § 23)}$$

and $(\sqrt[n]{a^n}) = a.$ (p. 7, § 27)

We also know that $a^n = a^{n-1} \times a,$ (p. 43, § 83)

and $a^0 = 1.$ (p. 7, § 24)

We now easily deduce the following Laws of Calculation:

If a is any definite number or any algebraic expression of one definite value, and m and n are positive integers,

$$\text{I.}\quad \mathbf{a^m \div a^n = a^{m-n}}, \text{ if } \mathbf{n < m}, \text{ or if } \mathbf{n = m}.$$

$$\text{II.}\quad \mathbf{a^m \div a^n = \frac{1}{a^{n-m}}}, \text{ if } \mathbf{n > m}.$$

$$\text{III.}\quad \mathbf{(a^m)^n = a^{mn}}.$$

$$\text{IV.}\quad \mathbf{(\sqrt[n]{a^m})^n = a^m}.$$

275. Meaning of Negative Integral Exponents. To obtain an interpretation of the meaning of negative integral exponents we extend Law I to include the case when $n > m$; that is, we assume that Law I holds true for all integral values of $m - n$, negative as well as positive, and interpret the result so that it shall be consistent with Law II.

276. Meaning of Fractional Exponents. To obtain an interpretation of the meaning of fractional exponents we extend Law III to include all cases in which mn is integral.

That is, we assume that Law III holds true for all integral values of n and mn, negative or positive, and interpret the results so that they shall be consistent with Laws II and IV.

277. Negative Integral Exponents. If we divide a^3 successively by a in the ordinary manner, we have

$$a^3,\ a^2,\ a,\ 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \cdots \qquad (1)$$

If we divide a^3 by a successively by subtracting 1 from the exponent of the dividend, we have, since Law II holds true,

$$a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ \cdots \qquad (2)$$

If we compare series (1) and series (2), we see that

$$a^0 = 1,\ a^{-1} = \frac{1}{a},\ a^{-2} = \frac{1}{a^2},\ a^{-3} = \frac{1}{a^3},\ \cdots$$

From the preceding we see at once that we may interpret a^{-n} as equivalent to $\frac{1}{a^n}$ consistently with Law II.

Hence, $\qquad a^n = a \times a \times a \times \cdots$ to n factors,

and $\qquad a^{-n} = \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a} \times \cdots$ to n factors.

278. Positive Fractional Exponents. If n is a positive integer, we have, by the extended interpretation of Law III,

$$(a^{\frac{1}{n}})^n = a^{\frac{1}{n} \times n} = a^1 = a.$$

Take the nth root, $\qquad a^{\frac{1}{n}} = \sqrt[n]{a}$; $\qquad$ (p. 219, § 253)

that is, $a^{\frac{1}{n}}$ may be taken as denoting *any* number which when raised to the nth power produces a, and this is exactly what $\sqrt[n]{a}$ denotes.

Thus, $9^{\frac{1}{2}} = \sqrt{9} = \pm 3$; $(-8)^{\frac{1}{3}} = \sqrt[3]{-8} = -2$.

Again, if m and n are both positive integers, by the extended interpretation of Law III,

$$(a^{\frac{m}{n}})^n = a^{\frac{m}{n}\times n} = a^m.$$

But $\qquad (\sqrt[n]{a^m})^n = a^m.$

Therefore, $\qquad a^{\frac{m}{n}} = \sqrt[n]{a^m}.$ $\qquad$ Hence,

The numerator of a fractional exponent indicates a power and the denominator indicates a root.

279. Negative Fractional Exponents. If n is a positive integer, $-\frac{1}{n}$ is a negative fraction, and we have, by the extended interpretations of Laws I and III,

$$(a^{-\frac{1}{n}})^n = a^{-\frac{1}{n}\times n} = a^{-1} = \frac{1}{a}.$$

Take the nth root, $\qquad a^{-\frac{1}{n}} = \frac{1}{\sqrt[n]{a}} = \frac{1}{a^{\frac{1}{n}}}.$ $\qquad$ (p. 219, § 253)

Again, if m and n are both positive integers, by the extended interpretations of Laws I and III,

$$(a^{-\frac{m}{n}})^n = a^{-\frac{m}{n}\times n} = a^{-m} = \frac{1}{a^m}.$$

Take the nth root, $\qquad a^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{a^m}} = \frac{1}{a^{\frac{m}{n}}}.$

Hence, whether the exponent is integral or fractional, we have always $a^{-m} = \frac{1}{a^m}$.

280. It is worthy of notice that while by the definitions given $(a^{\frac{1}{n}})^n = a$, it does not necessarily follow that $(a^n)^{\frac{1}{n}} = a$.

An illustration will make this statement plain.

Thus, $\qquad (9^{\frac{1}{2}})^2 = (\pm 3)^2 = 9;$

but $\qquad (9^2)^{\frac{1}{2}} = 81^{\frac{1}{2}} = \pm 9.$

Hence, if $a^n = b^n$, it does not necessarily follow that $a = b$; all we are entitled to say is that if b takes in succession all its possible values, one of these values must be a.

281. It remains to be shown that the index laws established for division, involution, and evolution apply to *fractional* and *negative* exponents.

282. Index Law of Division for all Values of *m* and *n*. To *divide* by a number *is to multiply the dividend by the reciprocal of the divisor.*

Therefore, for all values of m and n,

$$\frac{a^m}{a^n} = a^m \times \frac{1}{a^n} = a^m \times a^{-n} = a^{m-n}.$$

283. Index Law of Involution and Evolution for all Values of *m* and *n*.

To prove $(a^m)^n = a^{mn}$ *for all values of* m *and* n.

CASE 1. Let m have *any* value, and let n be a *positive integer.*

Then
$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \cdots \text{to } n \text{ factors}\\ &= a^{m+m+m+\cdots \text{ to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

CASE 2. Let m have *any* value, and $n = \frac{p}{q}$, p and q being positive integers.

Then
$$(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}}.$$

CASE 3. Let m have *any* value, and $n = -r$, r being a positive integer or a positive fraction.

Then
$$(a^m)^{-r} = \frac{1}{(a^m)^r} = \frac{1}{a^{mr}} = a^{-mr}.$$ (p. 235, § 279)

Therefore, $(a^m)^n = a^{mn}$ for all values of m and n.

284. *To prove* $(ab)^n = a^n b^n$ *for any value of* n.

CASE 1. Let n be a positive integer.

Then
$$\begin{aligned}(ab)^n &= ab \times ab \times ab \times \cdots \text{to } n \text{ factors}\\ &= (a \times a \times \cdots \text{to } n \text{ factors})(b \times b \times \cdots \text{to } n \text{ factors})\\ &= a^n b^n.\end{aligned}$$

CASE 2. Let $n = \frac{p}{q}$, p and q being positive integers.

Then, by Case 1, § 283, since q is a positive integer,

$$\begin{aligned}[(ab)^{\frac{p}{q}}]^q &= (ab)^{\frac{p}{q}} \times (ab)^{\frac{p}{q}} \times \cdots \text{to } q \text{ factors} \\ &= (ab)^{\frac{p}{q}+\frac{p}{q}+\cdots \text{ to } q \text{ terms}} \\ &= (ab)^p \\ &= a^p b^p. \qquad \text{(By Case 1)}\end{aligned}$$

Also, by Case 1, § 283, since q is a positive integer,

$$\begin{aligned}(a^{\frac{p}{q}} b^{\frac{p}{q}})^q &= (a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{to } q \text{ factors})(b^{\frac{p}{q}} \times b^{\frac{p}{q}} \times \cdots \text{to } q \text{ factors}) \\ &= a^p b^p.\end{aligned}$$

But $\qquad [(ab)^{\frac{p}{q}}]^q = (ab)^p = a^p b^p.$

That is, $\qquad [(ab)^{\frac{p}{q}}]^q = [a^{\frac{p}{q}} b^{\frac{p}{q}}]^q.$

Take the qth root, $\qquad (ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$

CASE 3. Let $n = -r$, r being a positive integer or fraction.

Then $\qquad (ab)^{-r} = \frac{1}{(ab)^r} = \frac{1}{a^r b^r} = a^{-r} b^{-r}.$

Therefore, $(ab)^n = a^n b^n$ for any value of n.

EXERCISE 93

Express with positive exponents:

1. $a^{-2} b^{-3} c^2$.
2. $x^{-\frac{2}{3}} y^{-\frac{1}{3}} z^{-2}$.
3. $a^{\frac{1}{3}} b^{-\frac{1}{3}} c^{-\frac{2}{3}}$.
4. $8 a^{-4} b^{\frac{3}{2}} c^{-\frac{1}{2}}$.
5. $4^{-1} a^{-3} c^{-2}$.
6. $2 axy^{-5}$.
7. $\frac{a^{-4} b^{-5} c^{-6}}{a^{-2} b^{-7} c^{-5}}$.
8. $\frac{3 a^{-1} b^{-2} c^{-3}}{5 x^{-2} a^{-3} y^{-5}}$.

Write without denominators:

9. $\frac{3 x^2 y^3}{a^{-2} b^{-3}}$.
10. $\frac{5 a^2 b^5 c^3}{a^{-1} b^{-2} c^{-3}}$.
11. $\frac{a^{-1} b^{-2} c^{-3}}{a^{-2} b^{-2} c^{-2}}$.

Express with fractional exponents:

12. $\sqrt[4]{\dfrac{1}{a^3}}$.	13. $\sqrt[3]{(x-a^2)^2}$.	14. $\sqrt{\left(\dfrac{a^2-b^2}{a^2+b^2}\right)^3}$.

Express with radical signs:

15. $a^{-\frac{1}{3}}x^{-3}$.	16. $b^{-\frac{2}{3}}c^{\frac{3}{4}}d^{-\frac{3}{10}}$.	17. $a^{-2}b^{\frac{2}{3}}c^{-3}$.

Perform the indicated operations:

18. $(\frac{25}{36})^{-\frac{1}{2}}$.	26. $(-32)^{\frac{3}{5}}$.	34. $64^{\frac{2}{3}} \times 27^{\frac{4}{3}}$.
19. $(\frac{8}{27})^{-\frac{1}{3}}$.	27. $81^{\frac{3}{4}}$.	35. $32^{\frac{1}{5}} \times 81^{\frac{3}{4}}$.
20. $(3\frac{3}{8})^{-\frac{2}{3}}$.	28. $(5\frac{1}{16})^{-\frac{3}{4}}$.	36. $(\frac{1}{4})^{\frac{1}{2}} \times (\frac{1}{8})^{-\frac{1}{3}}$.
21. $81^{-\frac{1}{2}}$.	29. $(\frac{16}{81})^{\frac{3}{4}}$.	37. $(\frac{8}{27})^{\frac{1}{3}} \times 64^{-\frac{1}{6}}$.
22. $16^{\frac{5}{4}}$.	30. $(\frac{9}{16})^{-\frac{3}{2}}$.	38. $32^{\frac{4}{5}} \times 81^{\frac{5}{4}}$.
23. $(-0.064)^{\frac{2}{3}}$.	31. $(0.16)^{-\frac{1}{2}}$.	39. $(a^{-\frac{1}{3}}b^{-\frac{2}{3}}c^{-1})^{-\frac{1}{3}}$.
24. $(7\frac{19}{32})^{\frac{3}{5}}$.	32. $(\frac{1}{64})^{-\frac{2}{3}}$.	40. $(x^{-\frac{3}{4}}y^{-\frac{1}{2}}z^{-\frac{1}{4}})^{\frac{1}{4}}$.
25. $(-\frac{32}{243})^{-\frac{1}{5}}$.	33. $(-3\frac{3}{8})^{-\frac{1}{3}}$.	41. $(a^{\frac{3}{2}}b^{\frac{2}{3}}c^{-\frac{1}{3}})^{-\frac{2}{3}}$.

285. The following examples illustrate the use of negative and fractional exponents.

1. Multiply $x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$ by $x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

$$\begin{array}{llll}
x^{-1} & + x^{-\frac{1}{2}}y^{-\frac{1}{2}} & + y^{-1} & \\
x^{-1} & - x^{-\frac{1}{2}}y^{-\frac{1}{2}} & + y^{-1} & \\
\hline
x^{-2} & + x^{-\frac{3}{2}}y^{-\frac{1}{2}} & + x^{-1}y^{-1} & \\
 & - x^{-\frac{3}{2}}y^{-\frac{1}{2}} & - x^{-1}y^{-1} & - x^{-\frac{1}{2}}y^{-\frac{3}{2}} \\
 & & + x^{-1}y^{-1} & + x^{-\frac{1}{2}}y^{-\frac{3}{2}} + y^{-2} \\
\hline
x^{-2} & & + x^{-1}y^{-1} & \qquad\quad + y^{-2}
\end{array}$$

2. Divide $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}}$ by $x^{\frac{1}{3}} - 2 - 3x^{-\frac{1}{3}}$.

$$\begin{array}{l|l}
x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 1 + 6x^{-\frac{1}{3}} & x^{\frac{1}{3}} - 2 - 3x^{-\frac{1}{3}} \\
\cline{2-2}
x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 3 & x^{\frac{1}{3}} - 2 \\
\cline{1-1}
\quad - 2x^{\frac{1}{3}} + 4 + 6x^{-\frac{1}{3}} & \\
\quad - 2x^{\frac{1}{3}} + 4 + 6x^{-\frac{1}{3}} & \\
\cline{1-1}
\end{array}$$

3. Find the square root of $9a - 12a^{\frac{1}{2}} + 10 - 4a^{-\frac{1}{2}} + a^{-1}$.

$$\begin{array}{r|l}
 & 9a - 12a^{\frac{1}{2}} + 10 - 4a^{-\frac{1}{2}} + a^{-1} \quad \underline{|\, 3a^{\frac{1}{2}} - 2 + a^{-\frac{1}{2}}} \\
 & 9a \\
6a^{\frac{1}{2}} - 2 & -12a^{\frac{1}{2}} + 10 \\
 & -12a^{\frac{1}{2}} + 4 \\
6a^{\frac{1}{2}} - 4 + a^{-\frac{1}{2}} & 6 - 4a^{-\frac{1}{2}} + a^{-1} \\
 & 6 - 4a^{-\frac{1}{2}} + a^{-1}
\end{array}$$

Exercise 94

Multiply:

1. $8x^{\frac{3}{2}} - 4xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $2x^{\frac{1}{2}} - 3y^{\frac{1}{2}}$.
2. $x - 6a^{\frac{1}{3}}x^{\frac{2}{3}} + 12a^{\frac{2}{3}}x^{\frac{1}{3}} - 8a$ by $x^{\frac{2}{3}} - 4a^{\frac{1}{3}}x^{\frac{1}{3}} + 4a^{\frac{2}{3}}$.
3. $7a - 2 + 8a^{-1} - 4a^{-2}$ by $5a + 3 - 7a^{-1}$.
4. $x^{\frac{1}{2}} - 4y^{\frac{1}{2}} + 6x^{-\frac{1}{2}}y - 4x^{-1}y^{\frac{3}{2}} + x^{-\frac{3}{2}}y^2$ by $x^{\frac{1}{2}}y^{-\frac{1}{2}} + 2 + x^{-\frac{1}{2}}y^{\frac{1}{2}}$.
5. $2x - 4x^{-1} - 4x^{-2} - x^{-3}$ by $2x - 4x^{-1} - 4x^{-2} - x^{-3}$.
6. $x^{\frac{3}{2}} - 6a^{\frac{1}{2}}x + 12ax^{\frac{1}{2}} - 8a^{\frac{3}{2}}$ by $x - 4a^{\frac{1}{2}}x^{\frac{1}{2}} - 4a$.

Divide:

7. $a^2 - 3a^{\frac{5}{3}} + 6a^{\frac{4}{3}} - 7a + 6a^{\frac{2}{3}} - 3a^{\frac{1}{3}} + 1$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1$.
8. $\frac{9}{16}a^2 - \frac{7}{8}a^{\frac{3}{2}}b^{\frac{1}{2}} + \frac{19}{36}ab + \frac{1}{6}a^{\frac{1}{2}}b^{\frac{3}{2}}$ by $\frac{3}{2}a^{\frac{1}{2}} + \frac{1}{3}b^{\frac{1}{2}}$.
9. $x^{\frac{7}{3}} - 3x^2 + x^{\frac{5}{3}} - 4 + 12x^{\frac{1}{3}} - 4x^{\frac{2}{3}}$ by $x^{\frac{5}{3}} - 4$.
10. $6x - 13a^{\frac{1}{4}}x^{\frac{3}{4}} + 13a^{\frac{1}{2}}x^{\frac{1}{2}} - 13a^{\frac{3}{4}}x^{\frac{1}{4}} - 5a$ by $2x^{\frac{1}{2}} - 3a^{\frac{1}{4}}x^{\frac{1}{4}} - a^{\frac{1}{2}}$.
11. $62x^{-5}y^{-3} + 43x^{-7}y^{-5} - 55x^{-6}y^{-4} - 55x^{-4}y^{-2} - 6x^{-2}$
$- 21x^{-8}y^{-6} + 28x^{-3}y^{-1}$ by $4x^{-2}y^{-1} - 2x^{-1} - 3x^{-3}y^{-2}$.

Find the square root of:

12. $a^{\frac{4}{3}}b^{-1} + 4ab^{-\frac{1}{2}} - 2a^{\frac{2}{3}} - 12a^{\frac{1}{3}}b^{\frac{1}{2}} + 9b$.
13. $60mn^{\frac{5}{4}} - 4m^{\frac{4}{3}}n^{\frac{3}{2}} - 48m^{\frac{5}{3}}n^{\frac{7}{4}} + 16m^2n^2 + 25m^{\frac{2}{3}}n$.
14. $9x^{-\frac{4}{3}} + 24x^{-1}y^{-\frac{1}{3}} + 46x^{-\frac{2}{3}}y^{-\frac{2}{3}} + 40x^{-\frac{1}{3}}y^{-1} + 25y^{-\frac{4}{3}}$.
15. $4x^{-\frac{6}{5}} - 20x^{-1}y^{-1} + 41x^{-\frac{4}{5}}y^{-2} - 52x^{-\frac{3}{5}}y^{-3} + 46x^{-\frac{2}{5}}y^{-4}$
$- 24x^{-\frac{1}{5}}y^{-5} + 9y^{-6}$.

CHAPTER XVIII

RADICAL EXPRESSIONS

286. Radical Expressions. When a root of an expression is indicated by a radical sign, the indicated root is called a **radical expression,** or simply a **radical.** The expression affected by the radical sign is called the **radicand.**

Thus, $\sqrt{x}$, $\sqrt[3]{a^2b}$, $\sqrt[5]{92}$, $\sqrt[3]{a+b}$, $\sqrt[4]{16}$ are radicals; and x, a^2b, 92, $a+b$, 16 are the respective radicands.

287. Surds. If an indicated root of a rational number cannot be obtained exactly, the indicated root is called a **surd.**

The index of the required root shows the **order** of a surd; and a surd is named *quadratic, cubic, biquadratic,* according as the *second, third, fourth* root is required.

The product of a rational factor and a surd factor is called a **mixed surd**; as $3\sqrt{2}$. The rational factor of a mixed surd is called the **coefficient** of the mixed surd.

A surd that contains no rational factor not affected by the radical sign is said to be an **entire surd**; as $\sqrt{2}$, $\sqrt[3]{5}$.

288. Reduction of Radicals. To *reduce a radical* is to change the *form* of the radical without changing the *value.*

A surd is said to be in its **simplest form** when the radicand is *integral and as small as possible,* or *integral and of as low degree as possible.*

Surds which, when reduced to simplest form, have the same surd factor are said to be **similar surds.**

NOTE. Hereafter in this chapter by $\sqrt[n]{a}$, where a and n are positive and n is integral, is meant the number which taken n times as a factor gives a for the product. In operations with surds, arithmetical numbers contained in the surds should be expressed in their prime factors.

289. To Reduce Radicals to Simplest Form.

1. Reduce $\sqrt{16\,a^4}$, $\sqrt[4]{25\,a^2b^2}$, and $\sqrt[6]{36\,a^4b^2}$ to simplest form.

$$\sqrt{16\,a^4} = \sqrt{(4\,a^2)^2} = (4\,a^2)^{\frac{2}{2}} = (4\,a^2)^1 = 4\,a^2.$$
$$\sqrt[4]{25\,a^2b^2} = \sqrt[4]{(5\,ab)^2} = (5\,ab)^{\frac{2}{4}} = (5\,ab)^{\frac{1}{2}} = \sqrt{5\,ab}.$$
$$\sqrt[6]{36\,a^4b^2} = \sqrt[6]{(6\,a^2b)^2} = (6\,a^2b)^{\frac{2}{6}} = (6\,a^2b)^{\frac{1}{3}} = \sqrt[3]{6\,a^2b}.$$

Hence,

If the radicand is a perfect power and for an exponent has a factor of the index of the root, divide the exponent of the power by the index of the root.

Since $\sqrt[n]{a^{kn}b^m} = \sqrt[n]{a^{kn}} \times \sqrt[n]{b^m} = a^k\sqrt[n]{b^m}$, (p. 219, § 253)

If the root of any factor of the radicand may be taken, that factor may be removed from the radical, and its root used as a factor of the coefficient of the radical.

2. Reduce $\sqrt{50\,a^6b^3}$ and $4\sqrt[3]{72\,a^7b^8}$ to simplest form.

$$\sqrt{50\,a^6b^3} = \sqrt{25\,a^6b^2 \times 2\,b} = \sqrt{25\,a^6b^2} \times \sqrt{2\,b} = 5\,a^3b\sqrt{2\,b}$$
$$4\sqrt[3]{72\,a^7b^8} = 4\sqrt[3]{8\,a^6b^6 \times 9\,ab^2} = 4\sqrt[3]{8\,a^6b^6} \times \sqrt[3]{9\,ab^2} = 4 \times 2\,a^2b^2 \times \sqrt[3]{9\,ab^2}$$
$$= 8\,a^2b^2\sqrt[3]{9\,ab^2}.$$

3. Reduce $\frac{3}{4}\sqrt{\frac{2}{3}}$ and $2\sqrt[3]{\frac{2\,a^3b^2c}{3\,x^2yz^3}}$ to simplest form.

$$\frac{3}{4}\sqrt{\frac{2}{3}} = \frac{3}{4}\sqrt{\frac{2 \times 3}{3 \times 3}} = \frac{3}{4}\sqrt{\frac{6}{9}} = \frac{3}{4}\sqrt{\frac{1}{9} \times 6} = \frac{3}{4} \times \frac{1}{3}\sqrt{6} = \frac{1}{4}\sqrt{6}.$$

$$2\sqrt[3]{\frac{2\,a^3b^2c}{3\,x^2yz^3}} = 2\sqrt[3]{\frac{2\,a^3b^2c \times 9\,xy^2}{3\,x^2yz^3 \times 9\,xy^2}} = 2\sqrt[3]{\frac{18\,a^3b^2cxy^2}{27\,x^3y^3z^3}} = 2\sqrt[3]{\frac{a^3}{27\,x^3y^3z^3} \times 18\,b^2cxy^2}$$
$$= \frac{2\,a}{3\,xyz}\sqrt[3]{18\,b^2cxy^2}.$$

Hence,

If the radicand contains a fraction, multiply both terms of the fraction by a number that will make the denominator a perfect power of the same degree as the index of the radical, and proceed as before.

Exercise 95

Reduce to simplest form:

1. $\sqrt[4]{36x^2y^2}$.
2. $\sqrt[4]{49a^2b^2}$.
3. $\sqrt[6]{27a^3b^3}$.
4. $\sqrt[6]{64a^3b^3}$.
5. $\sqrt[8]{16x^4y^4}$.
6. $\sqrt{72a^3b^4}$.
7. $\sqrt[3]{72a^3b^4}$.
8. $4\sqrt[3]{27a^2b^3}$.
9. $\sqrt[4]{112x^5y^6}$.
10. $2\sqrt[4]{81a^6b^2}$.
11. $5a\sqrt{4a^2b}$.
12. $2\sqrt[3]{56x^3y^2}$.
13. $\sqrt[4]{567c^4d}$.
14. $2\sqrt{405y^4z^3}$.
15. $5\sqrt{80a^4b^3}$.
16. $5\sqrt[4]{144c^5d^9}$.
17. $\sqrt[5]{2048a^5b^7}$.
18. $\sqrt[4]{1024a^5x^3}$.
19. $2\sqrt[3]{7290a^5}$.
20. $3\sqrt[5]{160b^6y^7}$.
21. $5\sqrt[3]{500a^4b^5}$.
22. $2ab\sqrt[3]{108ab^2}$.
23. $3ab\sqrt[3]{192x^5y^7}$.
24. $2\sqrt[3]{-1029a^4}$.
25. $3a^2\sqrt[3]{-648a^7}$.
26. $2\sqrt[3]{\dfrac{3a^2b^4}{4x^4y}}$.
27. $2a\sqrt[3]{\dfrac{5a^3x^2}{18b^4}}$.
28. $3c\sqrt[4]{\dfrac{9x^2y^5}{8c^4d^2}}$.
29. $3m\sqrt[3]{\dfrac{125m^2}{81n^5}}$.
30. $2a^2\sqrt{\dfrac{49c^3d^4}{50a^4b}}$.
31. $3ab\sqrt[3]{\dfrac{a^2b^3}{5x^4}}$.
32. $2a\sqrt[5]{\dfrac{18x^5y^4}{3125a^7}}$.
33. $2\sqrt[5]{-\dfrac{3a^7x^4}{512c^8d^9}}$.
34. $\sqrt[3]{\dfrac{(x+y)^4z^3}{512}}$.
35. $3x^2y\sqrt[3]{\dfrac{9x^5z^4}{80y^7}}$.
36. $3a\sqrt[6]{3645a^7b^6c^5}$.
37. $\sqrt[3]{\dfrac{(a-b)^5(c-d)^4}{(x-y)^2}}$.
38. $\sqrt[4]{\dfrac{(a-x)^5(b+y)}{243(a+b)^7}}$.

290. To Reduce a Mixed Surd to an Entire Surd.

Since $$a^k\sqrt[n]{b^m} = \sqrt[n]{a^{kn}} \times \sqrt[n]{b^m} = \sqrt[n]{a^{kn}b^m},$$

To reduce a mixed surd to an entire surd, raise the coefficient of the surd to a power of the same degree as the index of the radical, multiply this power by the given radicand, and indicate the required root of the product.

Reduce $2a\sqrt[3]{a^2b}$ and $-3x^2\sqrt[4]{2x}$ to entire surds.

$$2a\sqrt[3]{a^2b} = \sqrt[3]{(2a)^3 \times a^2b} = \sqrt[3]{8a^3 \times a^2b} = \sqrt[3]{8a^5b}.$$

$$-3x^2\sqrt[4]{2x} = -\sqrt[4]{(3x^2)^4 \times 2x} = -\sqrt[4]{81x^8 \times 2x} = -\sqrt[4]{162x^9}.$$

Exercise 96

Reduce to an entire surd:

1. $3a^3\sqrt{3a}$.
2. $-\frac{3}{4}\sqrt[3]{\frac{8}{9}a^2}$.
3. $\frac{2}{3}b\sqrt[3]{\frac{9}{16}b^2c}$.
4. $-3a\sqrt[3]{bd}$.
5. $-5a\sqrt[4]{a^2c^3}$.
6. $4x^3y^2\sqrt[4]{4x^3y^2}$.
7. $-\frac{1}{2}\sqrt[4]{8x^3y^5}$.
8. $8b\sqrt[3]{4a^2b}$.
9. $-2a\sqrt[5]{\frac{27}{16}a^4b^2}$.

291. To Reduce Radicals to a Common Index.

Since $$\sqrt[m]{a^k} = a^{\frac{k}{m}} = a^{\frac{kn}{mn}} = \sqrt[mn]{a^{kn}},$$

and $$\sqrt[n]{b^h} = b^{\frac{h}{n}} = b^{\frac{hm}{mn}} = \sqrt[mn]{b^{hm}};$$

To reduce radicals to a common index, write the radicals with fractional exponents, and change these fractional exponents to equivalent exponents having the lowest common denominator. Raise each radical to the power denoted by the numerator, and indicate the root denoted by the lowest common denominator.

Reduce $\sqrt[3]{9}$ and $\sqrt[4]{25}$ to surds of the same order.

$$\sqrt[3]{9} = \sqrt[3]{3^2} = 3^{\frac{2}{3}} = 3^{\frac{4}{6}} = \sqrt[6]{3^4} = \sqrt[6]{81}.$$

$$\sqrt[4]{25} = \sqrt[4]{5^2} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}.$$

292. Surds of different orders may be reduced to surds of the same order, and then compared with respect to magnitude.

Arrange $\sqrt[4]{8}$, $\sqrt{3}$, and $\sqrt[3]{5}$ in order of magnitude.

$$\sqrt[4]{8} = 2^{\frac{3}{4}} = 2^{\frac{9}{12}} = \sqrt[12]{2^9} = \sqrt[12]{512}.$$

$$\sqrt{3} = 3^{\frac{1}{2}} = 3^{\frac{6}{12}} = \sqrt[12]{3^6} = \sqrt[12]{729}.$$

$$\sqrt[3]{5} = 5^{\frac{1}{3}} = 5^{\frac{4}{12}} = \sqrt[12]{5^4} = \sqrt[12]{625}.$$

Therefore, the required order is $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[4]{8}$.

Exercise 97

Reduce to surds of the same order:

1. $\sqrt{3}$ and $\sqrt[3]{5}$.
2. $\sqrt[3]{7}$ and $\sqrt[5]{6}$.
3. $\sqrt[3]{a^2}$ and $\sqrt[4]{b^3}$.
4. $\sqrt[3]{4}$ and $\sqrt[4]{8}$.
5. $2\sqrt[3]{6}$, $3\sqrt{5}$, and $4\sqrt[4]{3}$.
6. $2\sqrt{2}$, $5\sqrt[3]{5}$, and $6\sqrt[4]{6}$.
7. $\sqrt[4]{7}$, $2\sqrt[5]{5}$, and $\sqrt[10]{120}$.
8. $2\sqrt[3]{a^2}$, $a\sqrt[5]{b^2}$, and $b\sqrt[6]{a^3}$.

Arrange in order of magnitude:

9. $\sqrt{18}$ and $\sqrt[3]{79}$.
10. $2\sqrt[3]{3}$ and $3\sqrt[4]{2}$.
11. $\sqrt{33}$ and $\sqrt[3]{185}$.
12. $3\sqrt{\frac{2}{5}}$, $2\sqrt[3]{\frac{2}{3}}$, and $3\sqrt[3]{\frac{2}{9}}$.
13. $\sqrt[6]{81}$, $\sqrt[3]{10}$, and $\sqrt{8}$.
14. $2\sqrt{2}$, $2\sqrt[3]{3}$, and $2\sqrt[4]{5}$.

293. Addition and Subtraction of Radicals. Reduce each surd to its simplest form; then if the resulting surds are similar,

Find the algebraic sum of the coefficients, and to this sum annex the common surd factor.

If the resulting surds are not similar,

Connect them with the proper signs.

1. Simplify $\sqrt{243} - \sqrt{75}$.

$$\sqrt{243} = \sqrt{81 \times 3} = \sqrt{9^2 \times 3} = 9\sqrt{3}.$$
$$\sqrt{75} = \sqrt{25 \times 3} = \sqrt{5^2 \times 3} = 5\sqrt{3}.$$
$$\therefore \sqrt{243} - \sqrt{75} = (9-5)\sqrt{3} = 4\sqrt{3}.$$

2. Simplify $2\sqrt{\frac{5}{3}} + \sqrt{\frac{3}{5}} - \sqrt{\frac{1}{2}}$.

$$2\sqrt{\frac{5}{3}} = 2\sqrt{\frac{5 \times 3}{3^2}} = 2\sqrt{\frac{15}{9}} = \frac{2}{3}\sqrt{15}.$$
$$\sqrt{\frac{3}{5}} = \sqrt{\frac{3 \times 5}{5^2}} = \sqrt{\frac{15}{25}} = \frac{1}{5}\sqrt{15}.$$
$$\sqrt{\frac{1}{2}} = \sqrt{\frac{1 \times 2}{2^2}} = \sqrt{\frac{2}{4}} = \frac{1}{2}\sqrt{2}.$$
$$\therefore 2\sqrt{\tfrac{5}{3}} + \sqrt{\tfrac{3}{5}} - \sqrt{\tfrac{1}{2}} = \tfrac{2}{3}\sqrt{15} + \tfrac{1}{5}\sqrt{15} - \tfrac{1}{2}\sqrt{2} = \tfrac{13}{15}\sqrt{15} - \tfrac{1}{2}\sqrt{2}.$$

EXERCISE 98

Simplify:

1. $8\sqrt{a} + 5\sqrt{x} - 7\sqrt{a} + 4\sqrt{a} - 6\sqrt{x} - 3\sqrt{a}$.

2. $\sqrt{a} + 3\sqrt{2a} - 2\sqrt{3a} + \sqrt{4a} - \sqrt{8a} + \sqrt{12a}$.

3. $7\sqrt{4a} + 4\sqrt{9a} + 3\sqrt{45a} - 5\sqrt{36a} - 2\sqrt{80a}$.

4. $3\sqrt{8} + 4\sqrt{32} - 5\sqrt{50} - 7\sqrt{72} + 6\sqrt{98} + \sqrt{18}$.

5. $7\sqrt{12} - 5\sqrt{27} + 8\sqrt{48} - 6\sqrt{75} + 2\sqrt{108} + \sqrt{27}$.

6. $5\sqrt[3]{16} + 3\sqrt[3]{-54} - 6\sqrt[3]{-128} + 7\sqrt[3]{-250} + 2\sqrt[3]{432}$.

7. $7\sqrt[3]{24} + 5\sqrt[3]{81} + 4\sqrt[3]{-192} + 2\sqrt[3]{-375} - \sqrt[3]{1029}$.

8. $\sqrt{(a+b)^2x} + \sqrt{(a-b)^2x} - \sqrt{a^2x} + \sqrt{(1-a)^2x} - \sqrt{x}$.

9. $3\sqrt{7} + 2\sqrt{54} - 5\sqrt{63} + 3\sqrt{150} + 4\sqrt{252} + 7\sqrt{24}$.

10. $\sqrt{275} + \sqrt{1300} - \sqrt{44} + \sqrt{52} - 2\sqrt{99} - 2\sqrt{208}$.

11. $3\frac{1}{2}\sqrt{24} - 5\frac{3}{4}\sqrt{54} + 13\frac{1}{3}\sqrt{99} + 2\frac{17}{24}\sqrt{216} - 21\sqrt{44}$.

12. $\sqrt[3]{\frac{4}{27}} - \sqrt[3]{5} - 3\sqrt[3]{256} + \sqrt[3]{625} + \sqrt[3]{\frac{32}{27}} - \sqrt[3]{\frac{125}{54}}$.

13. $\sqrt{9} + \sqrt{20} - \sqrt{\frac{1}{16}} - \sqrt{\frac{9}{5}} + \sqrt{\frac{9}{16}} + \sqrt{\frac{25}{81}} + \sqrt{\frac{20}{9}}$.

14. $\sqrt{11\frac{1}{4}} + 3\sqrt{8} - \sqrt{20} - \frac{1}{6}\sqrt{245} - 5\sqrt{\frac{1}{18}} - \sqrt{24\frac{1}{2}}$.

15. $\frac{3}{2}\sqrt[3]{\frac{2}{49}} + 0.8\sqrt{\frac{8}{3}} - \frac{1}{15}\sqrt{96} + \frac{3}{2}\sqrt[3]{\frac{7}{4}} - \frac{11}{28}\sqrt[3]{1750} + 8\sqrt{\frac{3}{2}}$.

294. Multiplication of Radicals. Since $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$,

$$4\sqrt{3} \times 2\sqrt[3]{9} = 4\sqrt[6]{3^3} \times 2\sqrt[6]{9^2} = 4\sqrt[6]{3^3} \times 2\sqrt[6]{3^4}$$
$$= 8\sqrt[6]{3^7} = 8 \times 3\sqrt[6]{3} = 24\sqrt[6]{3}.$$

Hence,

To multiply radicals, express the radicals with a common index. Find the product of the coefficients for the required coefficient, and the product of the surd factors for the required surd factor. Reduce the result to its simplest form.

Exercise 99

Find the product of:

1. $\sqrt{3} \times \sqrt{12}$.
2. $\sqrt{5} \times \sqrt{20}$.
3. $\sqrt{20} \times \sqrt{30}$.
4. $\sqrt[3]{5} \times \sqrt[3]{50}$.
5. $\sqrt[3]{2a^2} \times \sqrt[3]{4a}$.
6. $5\sqrt{3} \times 3\sqrt{5}$.
7. $\sqrt[3]{3} \times \sqrt[3]{36}$.
8. $\sqrt[3]{6} \times \sqrt{18}$.
9. $\sqrt[3]{9} \times 2\sqrt[3]{3}$.
10. $2\sqrt{\frac{2}{3}} \times 3\sqrt{\frac{5}{6}}$.
11. $3\sqrt[3]{16} \times 5\sqrt[3]{4}$.
12. $\frac{1}{2}\sqrt[4]{8} \times \frac{3}{4}\sqrt[4]{2}$.
13. $\sqrt[3]{-108} \times \sqrt[3]{40}$.
14. $\sqrt[5]{-343} \times \sqrt[5]{-1568}$.
15. $\sqrt{6} \times \sqrt{12} \times \sqrt{18}$.
16. $\frac{2}{3}\sqrt[3]{4} \times \frac{3}{4}\sqrt[3]{-3} \times \frac{4}{5}\sqrt[3]{9}$.
17. $\sqrt[3]{16} \times 5\sqrt[3]{4} \times 3\sqrt[3]{\frac{1}{2}}$.
18. $5\sqrt[3]{24} \times 3\sqrt[3]{32} \times 2\sqrt[3]{36}$.
19. $\sqrt[3]{\frac{5}{6}} \times \sqrt[4]{\frac{2}{3}} \times \sqrt[3]{\frac{1}{2}}$.
20. $5\sqrt{2} \times \frac{2}{5}\sqrt{72} \times \frac{2}{3}\sqrt{48}$.
21. $(3\sqrt[3]{24} + 4\sqrt[3]{81} - 5\sqrt[3]{375} + \frac{1}{2}\sqrt[3]{192}) \times \sqrt[3]{2}$.
22. $(2\sqrt{6} - \sqrt{12} - 3\sqrt{24} + \frac{1}{4}\sqrt{48}) \times 3\sqrt{2}$.

295. Compound radicals are multiplied as follows:

Multiply $2\sqrt{3} + 3\sqrt{x}$ by $3\sqrt{3} - 4\sqrt{x}$.

$$\begin{array}{l} 2\sqrt{3} + 3\sqrt{x} \\ \underline{3\sqrt{3} - 4\sqrt{x}} \\ 18 + 9\sqrt{3x} \\ \underline{\quad\;\; - 8\sqrt{3x} - 12x} \\ 18 + \quad \sqrt{3x} - 12x \end{array}$$

Exercise 100

Find the product of:

1. $(\sqrt{7 + \sqrt{18}})(\sqrt{7 - \sqrt{18}})$.
2. $(\sqrt{3 + 2\sqrt{5}})(\sqrt{3 + 2\sqrt{5}})$.
3. $(\sqrt{8} + \sqrt{7})(\sqrt{7} - \sqrt{2})$.
4. $(7 + 3\sqrt{7})(2\sqrt{7} - 7)$.
5. $(8 + 3\sqrt{5})(2 - \sqrt{5})$.
6. $(2a + 3\sqrt{x})(3a - 2\sqrt{x})$.
7. $(5\sqrt{7} - 2\sqrt{5})(3\sqrt{7} + 10\sqrt{5})$.
8. $(4\sqrt{a} - \sqrt{3x})(\sqrt{a} + 2\sqrt{3x})$.

9. $(2\sqrt{30}-3\sqrt{5}+5\sqrt{3})(\sqrt{8}+\sqrt{3}-\sqrt{5})$.

10. $(3+\sqrt{6}+\sqrt{15})(2+\sqrt{6}-\sqrt{10})$.

11. $(2\sqrt{7}-8\sqrt{28}-5\sqrt{63})(2\sqrt{7}-8\sqrt{28}-5\sqrt{63})$.

12. $(2\sqrt{2}-3\sqrt{3}+4\sqrt{5})(3\sqrt{5}+\sqrt{2}-\sqrt{3})$.

13. $(3\sqrt{3}-4\sqrt{8}+5\sqrt{5})(5\sqrt{2}+\sqrt{3}+\sqrt{5})$.

14. $(\sqrt[3]{6}-2\sqrt[3]{2}+4\sqrt[3]{4})(5\sqrt[3]{4}+3\sqrt[3]{6}-2\sqrt[3]{2})$.

15. $(2\sqrt{6}+5\sqrt{3}-7\sqrt{2})(\sqrt{6}-2\sqrt{3}+4\sqrt{2})$.

16. $(\sqrt[3]{9}-7\sqrt[3]{72}+6\sqrt[3]{1125})(3\sqrt[3]{2}+8\sqrt[3]{3}-4\sqrt[3]{\frac{1}{9}})$.

17. $(5\sqrt{112}+\sqrt{176}-\sqrt{4375})(3\sqrt{396}+\sqrt{175}-2\sqrt{539})$.

18. $(\sqrt{x}+\sqrt{y})(x^2+xy+y^2)(\sqrt{x}-\sqrt{y})$.

19. $(a+b+\sqrt[3]{a^2b}+\sqrt[3]{ab^2})(\sqrt[3]{a}-\sqrt[3]{b})$.

296. Division of Radicals.

Since $\sqrt[n]{ab}\div\sqrt[n]{a}=(\sqrt[n]{a}\times\sqrt[n]{b})\div\sqrt[n]{a}=\sqrt[n]{b}$,

$$\frac{4\sqrt[3]{3}}{2\sqrt{2}}=\frac{4\sqrt[6]{3^2}}{2\sqrt[6]{2^3}}=\frac{4\sqrt[6]{3^2\times 2^3}}{2\sqrt[6]{2^6}}=\frac{4\sqrt[6]{72}}{2\times 2}=\sqrt[6]{72}.$$ Hence,

To divide one radical by another, express the radicals with a common index. Find the quotient of the coefficients for the required coefficient, and the quotient of the surd factors for the required surd factor. Reduce the result to its simplest form.

EXERCISE 101

Divide:

1. $\sqrt[3]{56}$ by $\sqrt[3]{7}$.
2. $\sqrt[3]{1080}$ by $\sqrt[3]{5}$.
3. $\sqrt[4]{243}$ by $\sqrt[4]{3}$.
4. $\sqrt{\frac{2}{9}}$ by $\sqrt{\frac{1}{2}}$.
5. $\sqrt{a^5}$ by $\sqrt{a^3}$.
6. $\sqrt[3]{96a^8}$ by $\sqrt[3]{6a^5}$.
7. $\sqrt[4]{125a^7}$ by $\sqrt[4]{\frac{1}{5}a^2}$.
8. $\sqrt[4]{\frac{2}{5}m^3}$ by $\sqrt[4]{\frac{5}{8}m}$.
9. 6 by $\sqrt{72}$.
10. $\sqrt{a^3}$ by $\sqrt[9]{a^8}$.
11. ax^5 by $\sqrt{a^2x}$.
12. $\sqrt[4]{8a^3b^2}$ by $\sqrt{2a^2b^3}$.

13. $15\sqrt{105} - 36\sqrt{10} + 30\sqrt{3}$ by $3\sqrt{15}$.

14. $56\sqrt{30} - 84\sqrt{10} + 100\sqrt{14}$ by $4\sqrt{35}$.

15. $10\sqrt{14} - 5\sqrt{63} + 4\sqrt{28}$ by $2\sqrt{7}$.

16. $18\sqrt{22} + 9\sqrt{66} - 3\sqrt{44}$ by $3\sqrt{11}$.

17. $9\sqrt[3]{9} - 6\sqrt[3]{36} + 4\sqrt[3]{72}$ by $3\sqrt[3]{3}$.

18. $16\sqrt[3]{25} - 8\sqrt[3]{75} + 4\sqrt[3]{225}$ by $4\sqrt[3]{5}$.

297. Rationalizing the Divisor. The quotient of one surd divided by another may be found by *rationalizing the divisor;* that is, by multiplying both dividend and divisor by a factor that will free the divisor from surds.

Thus,
$$\frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2} = \frac{3}{2}\sqrt{2}.$$

298. It is easy to rationalize the denominator of a fraction when the denominator is a *binomial* involving only quadratic surds. The multiplier required is the terms of the given denominator connected by the opposite sign.

Thus,
$$\frac{7 - 3\sqrt{5}}{6 + 2\sqrt{5}} = \frac{(7 - 3\sqrt{5})(6 - 2\sqrt{5})}{(6 + 2\sqrt{5})(6 - 2\sqrt{5})} = \frac{72 - 32\sqrt{5}}{16} = \frac{9}{2} - 2\sqrt{5}.$$

299. By two operations the denominator of a fraction may be rationalized when that denominator consists of *three* terms of quadratic surds.

Thus,
$$\frac{\sqrt{3} + \sqrt{2}}{\sqrt{6} + \sqrt{3} - \sqrt{2}} = \frac{(\sqrt{3} + \sqrt{2})(\sqrt{6} - \sqrt{3} + \sqrt{2})}{(\sqrt{6} + \sqrt{3} - \sqrt{2})(\sqrt{6} - \sqrt{3} + \sqrt{2})}$$
$$= \frac{3\sqrt{2} + 2\sqrt{3} - 1}{2\sqrt{6} + 1} = \frac{(3\sqrt{2} + 2\sqrt{3} - 1)(2\sqrt{6} - 1)}{(2\sqrt{6} + 1)(2\sqrt{6} - 1)}$$
$$= \frac{10\sqrt{3} + 9\sqrt{2} - 2\sqrt{6} + 1}{23}.$$

Exercise 102

Divide:

1. $\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$.
2. 12 by $4-\sqrt{7}$.
3. $7-\sqrt{5}$ by $3+\sqrt{5}$.
4. $\sqrt{3}+\sqrt{2}$ by $\sqrt{3}-\sqrt{2}$.
5. $9-5\sqrt{3}$ by $7-3\sqrt{3}$.
6. $a+b\sqrt{x}$ by $c+d\sqrt{x}$.
7. $\sqrt{x}-\sqrt{y}$ by $\sqrt{x}+\sqrt{y}$.
8. $3+\sqrt{6}$ by $\sqrt{3}+\sqrt{2}$.
9. $5\sqrt{3}-3\sqrt{5}$ by $\sqrt{5}-\sqrt{3}$.
10. $7\sqrt{5}+5\sqrt{7}$ by $\sqrt{7}+\sqrt{5}$.
11. $2\sqrt{3}+\sqrt{6}$ by $\sqrt{3}+\sqrt{6}$.
12. $3+2\sqrt{x}$ by $5+3\sqrt{x}$.
13. $2\sqrt{6}$ by $\sqrt{2}+\sqrt{3}+\sqrt{5}$.
14. $2\sqrt{15}$ by $\sqrt{3}+\sqrt{5}+2\sqrt{2}$.
15. $a\sqrt{x}-b\sqrt{y}$ by $c\sqrt{x}-d\sqrt{y}$.
16. $25\sqrt{14}-4\sqrt{2}$ by $8\sqrt{2}+2\sqrt{7}$.
17. $1+3\sqrt{2}-2\sqrt{3}$ by $\sqrt{6}+\sqrt{3}+\sqrt{2}$.
18. $3-\sqrt{5}-\sqrt{2}$ by $3+\sqrt{5}+\sqrt{2}$.
19. $60\sqrt{2}+12\sqrt{3}$ by $5\sqrt{6}+3\sqrt{2}-2\sqrt{3}$.
20. $7-2\sqrt{3}+3\sqrt{2}$ by $3+3\sqrt{3}-2\sqrt{2}$.
21. $2+3\sqrt{6}-4\sqrt{2}$ by $2-\sqrt{6}+2\sqrt{2}$.
22. $2\sqrt{3}-3\sqrt{6}$ by $2+\sqrt{2}+\sqrt{3}+\sqrt{6}$.

300. Involution and Evolution of Radicals. Any power or root of a radical may be found easily by using fractional exponents.

1. Find the square of $3\sqrt[3]{a^2}$; the cube of $3\sqrt{a}$.

$$(3\sqrt[3]{a^2})^2 = (3a^{\frac{2}{3}})^2 = 3^2 \times a^{\frac{4}{3}} = 9a^{\frac{4}{3}} = 9a\sqrt[3]{a}.$$

$$(3\sqrt{a})^3 = (3a^{\frac{1}{2}})^3 = 3^3 \times a^{\frac{3}{2}} = 27a^{\frac{3}{2}} = 27a\sqrt{a}.$$

2. Find the square root and the cube root of $4a\sqrt{x^3y^3}$.

$$(4a\sqrt{x^3y^3})^{\frac{1}{2}} = (4ax^{\frac{3}{2}}y^{\frac{3}{2}})^{\frac{1}{2}} = 4^{\frac{1}{2}}a^{\frac{1}{2}}x^{\frac{3}{4}}y^{\frac{3}{4}} = 4^{\frac{1}{2}}a^{\frac{2}{4}}x^{\frac{3}{4}}y^{\frac{3}{4}} = 2\sqrt[4]{a^2x^3y^3}.$$

$$(4a\sqrt{x^3y^3})^{\frac{1}{3}} = (4ax^{\frac{3}{2}}y^{\frac{3}{2}})^{\frac{1}{3}} = 4^{\frac{1}{3}}a^{\frac{1}{3}}x^{\frac{1}{2}}y^{\frac{1}{2}} = 4^{\frac{2}{6}}a^{\frac{2}{6}}x^{\frac{3}{6}}y^{\frac{3}{6}} = \sqrt[6]{16a^2x^3y^3}.$$

EXERCISE 103

Perform the operations indicated:

1. $(\sqrt[4]{a})^5$.
2. $(\sqrt[8]{x^2y^3})^7$.
3. $(\sqrt[3]{2\,a^4})^2$.
4. $(\sqrt[3]{ax^4})^6$.
5. $(-\sqrt[3]{xy^2})^3$.
6. $(-\sqrt[3]{4})^6$.
7. $(-\sqrt[4]{127})^4$.
8. $(-\sqrt[3]{9})^4$.
9. $(\sqrt[8]{a})^4$.
10. $\sqrt[3]{\sqrt{b}}$.
11. $\sqrt[4]{\sqrt[3]{a}}$.
12. $\sqrt[6]{\sqrt[3]{a^6}}$.
13. $\sqrt[5]{\sqrt[6]{2\,a}}$.
14. $\sqrt[4]{\sqrt{3\,a^2}}$.
15. $\sqrt[5]{\sqrt[3]{a^{10}}}$.
16. $\sqrt[4]{\sqrt[3]{a^5}}$.
17. $\sqrt[5]{\sqrt[3]{(m+n)^{30}}}$.
18. $\sqrt{\sqrt[5]{(1-x^2)^{20}}}$.
19. $\sqrt[6]{\sqrt{512\,a^{18}}}$.
20. $\sqrt[13]{6\sqrt[5]{a^{65}}}$.
21. $\sqrt[11]{\sqrt[3]{a^{33}b^{22}}}$.
22. $\sqrt[5]{a\sqrt[3]{b^2}}$.
23. $\sqrt[5]{a\sqrt[8]{b}}$.
24. $a\sqrt[4]{a^{-3}\sqrt[4]{a^{-3}}}$.

301. Quadratic Surds. *The product or the quotient of two dissimilar quadratic surds is a quadratic surd.*

For every quadratic surd, when simplified, has in the radicand one or more factors of the first power; and two *dissimilar* surds cannot have *all* these factors alike.

302. *A quadratic surd cannot be equal to the sum of a rational number and a surd.*

For, if $\sqrt{a}$ could be equal to $c+\sqrt{b}$, then

squaring, $a = c^2 + 2\,c\sqrt{b} + b$;

transposing, $2\,c\sqrt{b} = a - b - c^2$.

This is impossible, for a surd cannot equal a rational number.

303. *The sum or the difference of two unequal quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

If the surds are similar, the theorem is evident.

If $\sqrt{a} \pm \sqrt{b}$ could be equal to a rational number c, then

squaring, $a \pm 2\sqrt{ab} + b = c^2$;

transposing, $\pm 2\sqrt{ab} = c^2 - a - b$.

Now, as the right member is rational, the left member must be rational; but $\sqrt{ab}$ cannot be rational (§ 301).

Therefore, $\sqrt{a} \pm \sqrt{b}$ cannot be rational.

Similarly, $\sqrt{a} \pm \sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

304. *If* $a + \sqrt{b} = x + \sqrt{y}$, *then* a *equals* x, *and* b *equals* y.

For, transposing, $\sqrt{b} - \sqrt{y} = x - a$; and if b was not equal to y, the difference of two unequal surds would be rational, which is impossible (§ 303).

$$\therefore b = y, \text{ and } a = x.$$

In like manner, if $a - \sqrt{b} = x - \sqrt{y}$, then $a = x$, and $b = y$.

305. Binomial Surds. An expression of the form $a + \sqrt{b}$, where b is a surd, is called a **binomial surd.**

306. Square Root of a Binomial Surd.

1. Extract the square root of $a + \sqrt{b}$.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$.

Square, $a + \sqrt{b} = x + 2\sqrt{xy} + y$.

$\therefore x + y = a,$ (1)

and $2\sqrt{xy} = \sqrt{b}$. (§ 304)

$\therefore a - \sqrt{b} = x - 2\sqrt{xy} + y$.

Extract the root, $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

$\therefore (\sqrt{a + \sqrt{b}})(\sqrt{a - \sqrt{b}}) = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$.

$\therefore \sqrt{a^2 - b} = x - y.$ (2)

From (1) and (2), $x = \frac{1}{2}(a + \sqrt{a^2 - b})$, and $y = \frac{1}{2}(a - \sqrt{a^2 - b})$.

2. Extract the square root of $7 + 4\sqrt{3}$.

Let $\sqrt{x} + \sqrt{y} = \sqrt{7 + 4\sqrt{3}}$. (1)

Then $\sqrt{x} - \sqrt{y} = \sqrt{7 - 4\sqrt{3}}$. (2)

Multiply (1) by (2), $x - y = \sqrt{49 - 48}$.

$\therefore x - y = 1.$ (3)

Square (1), $x + 2\sqrt{xy} + y = 7 + 4\sqrt{3}$.

By § 304, $x + y = 7.$ (4)

From (3) and (4), $x = 4$, and $y = 3$.

$$\therefore \sqrt{7 + 4\sqrt{3}} = \sqrt{4} + \sqrt{3} = 2 + \sqrt{3}.$$

307. When a given binomial surd can be written in the form $a + 2\sqrt{b}$, its square root may be found by inspection by finding two numbers that have their sum equal to a and their product equal to b.

Find by inspection the square root of $11 + 6\sqrt{2}$.

$$11 + 6\sqrt{2} = 11 + 2\sqrt{2 \times 3^2}.$$

Two numbers whose sum is 11 and product 2×3^2 are 9 and 2.

Then $$11 + 2\sqrt{2 \times 3^2} = 9 + 2\sqrt{9 \times 2} + 2$$
$$= (\sqrt{9} + \sqrt{2})^2.$$

Therefore, the square root of $11 + 6\sqrt{2}$ is $\sqrt{9} + \sqrt{2}$, or $3 + \sqrt{2}$.

Exercise 104

Find the square root of:

1. $17 + 12\sqrt{2}$.
2. $52 - 30\sqrt{3}$.
3. $102 - 28\sqrt{2}$.
4. $30 + 12\sqrt{6}$.
5. $220 - 30\sqrt{35}$.
6. $23 - 4\sqrt{15}$.
7. $57 - 12\sqrt{15}$.
8. $95 - 24\sqrt{14}$.
9. $314 - 28\sqrt{30}$.
10. $62 - 24\sqrt{3}$.
11. $68 + 16\sqrt{15}$.
12. $139 + 24\sqrt{21}$.
13. $66 - 36\sqrt{2}$.
14. $69 - 28\sqrt{5}$.
15. $207 + 40\sqrt{14}$.
16. $180 + 80\sqrt{2}$.
17. $735 + 300\sqrt{6}$.
18. $162 + 108\sqrt{2}$.

308. Equations containing Radicals. An equation that contains a *single* radical may be solved by arranging the terms so as to have the radical alone on one side, and then raising both members to a power equal to the index of the radical. If an equation contains *two* radicals, two steps may be necessary.

1. Solve $x = 7 - \sqrt{x^2 - 7}$.

Transpose, $$\sqrt{x^2 - 7} = 7 - x.$$
Square, $$x^2 - 7 = 49 - 14x + x^2.$$
Transpose, $$x^2 - x^2 + 14x = 49 + 7.$$
Combine, $$14x = 56.$$
$$\therefore x = 4.$$

2. Solve $\sqrt{x-7}=\sqrt{x+1}-2$.

Square, $x-7=x+1-4\sqrt{x+1}+4$.

Transpose and combine, $4\sqrt{x+1}=12$.

Divide by 4, $\sqrt{x+1}=3$.

Square, $x+1=9$.

$\therefore x=8$.

Exercise 105

Solve:

1. $9+4\sqrt{x}=11$.
2. $6-3\sqrt{x}=4$.
3. $\sqrt[5]{80x-43}=-3$.
4. $3\sqrt{16x-9}=6\sqrt{4x}-9$.
5. $\sqrt{22-x}-2=\sqrt{10-x}$.
6. $\sqrt{12+x}=6-\sqrt{x}$.
7. $7\sqrt{3x}-1=5\sqrt{3x}+5$.
8. $a\sqrt{x}-b=c\sqrt{x}-d$.
9. $\sqrt{3x}-\sqrt{2x}=1$.
10. $\sqrt{3x}-\sqrt{x}=2$.
11. $\sqrt{ax}+\sqrt{bx}=c$.
12. $\sqrt{x}+\sqrt{2x}=1$.
13. $5\sqrt{x}+\sqrt{3x}=22$.
14. $2\sqrt{x}-\sqrt{2x}=2+\sqrt{2}$.
15. $b(1+\sqrt{x})=a(1-\sqrt{x})$.
16. $7+\sqrt{x^2-11x+4}=x$.
17. $\frac{2}{3}(7\sqrt{x}+5)-5=\frac{3}{2}(3\sqrt{x}-1)$.
18. $\frac{1}{2}(16-\sqrt{x})-\frac{1}{3}(10-\sqrt{x})=\sqrt{x}$.
19. $\sqrt{x+60}=2\sqrt{x+5}+\sqrt{x}$.
20. $\sqrt{9x+7}+\sqrt{4x+1}=\sqrt{25x+14}$.
21. $\sqrt{4x+9}-\sqrt{x-1}=\sqrt{x+6}$.
22. $2\sqrt{x+5}+3\sqrt{x-7}=\sqrt{25x-79}$.
23. $3\sqrt{x+3}-2\sqrt{x-12}=5\sqrt{x-9}$.
24. $\sqrt{x-9}+\sqrt{x+12}=\sqrt{x-4}+\sqrt{x+3}$.
25. $\sqrt{x-7}+\sqrt{x-2}-\sqrt{x-10}=\sqrt{x+5}$.
26. $\sqrt{x+15}+\sqrt{x-24}-\sqrt{x-13}=\sqrt{x}$.

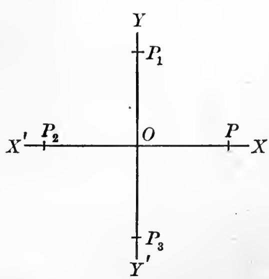
CHAPTER XIX

IMAGINARY NUMBERS

309. Orthotomic Numbers. For the complete treatment of the solution of equations of degree higher than the first, it is necessary to take into account the square roots of negative scalar numbers. Since these roots are not scalar numbers (p. 219, § 255), it is necessary to assume a *new series of numbers* distinct from the scalar series, but such that the square of every number in the new series is a number in the negative branch of the scalar series. These new numbers are called **orthotomic numbers** or **imaginary numbers.**

An orthotomic number is any indicated square root of a negative scalar number or any scalar multiple thereof.

310. Graphic Representation of Orthotomic Numbers. Let the straight lines XX' and YY' intersect at right angles at the point O. Take OP, OP_1, OP_2, and OP_3 all equal to a given length a. A rotation counter-clockwise through a right angle would convert OP into OP_1, OP_1 into OP_2, OP_2 into OP_3, and OP_3 into OP. Hence, if *direction* as well as *length* is taken into account,



$$\frac{OP_1}{OP} = \frac{OP_2}{OP_1} = \frac{OP_3}{OP_2} = \frac{OP}{OP_3}.$$

Let i denote this common ratio.

Then $$\frac{OP_1}{OP} = i,$$

and $$\frac{OP_2}{OP} = \frac{OP_2}{OP_1} \times \frac{OP_1}{OP} = i^2.$$

But $$\frac{OP_2}{OP} = -1.$$

$$\therefore\ i^2 = -1.$$

Again, $$\frac{OP_3}{OP} = \frac{OP_3}{OP_2} \times \frac{OP_2}{OP_1} \times \frac{OP_1}{OP} = i^3,$$

and $$\frac{OP_3}{OP} = \frac{OP_3}{OP_2} \times \frac{OP_2}{OP} = -i.$$

$$\therefore\ i^3 = -i.$$

Also, $$\frac{OP}{OP} = \frac{OP}{OP_3} \times \frac{OP_3}{OP_2} \times \frac{OP_2}{OP_1} \times \frac{OP_1}{OP} = i^4.$$

$$\therefore\ i^4 = +1.$$

Hence, if direction as well as length is taken into account,

$$OP_1 = i \times OP = (\sqrt{-1})OP,$$

$$OP_2 = i^2 \times OP = (-1)OP,$$

and $$OP_3 = i^3 \times OP = (-\sqrt{-1})OP.$$

Therefore, if the point P represents the positive scalar number a, the point P_2 represents the negative scalar number $-a$, the point P_1 represents the positive orthotomic number $a\sqrt{-1}$, and the point P_3 represents the negative orthotomic number $-a\sqrt{-1}$.

Thus, exactly as all scalar numbers may be represented by points on the axis XX', so all orthotomic numbers may be represented by points on the axis YY', which cuts the axis XX' at right angles, or *orthotomically*.

The line XX' is called the **axis of scalars** and the line YY' is called the **axis of orthotomics**; the point O is called the **origin.**

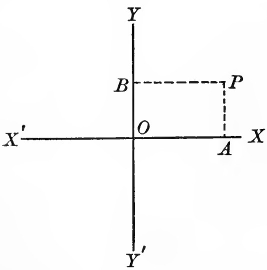
The only point on both axes is O. This agrees with the fact that zero is the only number that may be considered either scalar or orthotomic.

Again, a and ai are measured on different lines. This agrees with the fact that a and ai are *different in kind.*

311. Complex Numbers. The sum of any two scalar numbers is a scalar number, and it will be shown (p. 258, § 316) that the sum of any two orthotomic numbers is an orthotomic number; but the sum of a scalar number and an orthotomic number is evidently neither a scalar number nor an orthotomic number. Such a number is called a **complex number.**

A complex number is the indicated sum or difference of a scalar number and an orthotomic number.

312. Graphic Representation of Complex Numbers. To determine the point that represents the complex number $a+b\sqrt{-1}$, determine on the axis of scalars the point A that represents a, and on the axis of orthotomics the point B that represents $b\sqrt{-1}$. Through the points A and B draw lines perpendicular to the axes. These perpendiculars intersect in a point P which represents the number $a+b\sqrt{-1}$ in the scale in which $OA=a$, and $OB=b$.



It is evident that the point $a-b\sqrt{-1}$ lies in Quadrant IV, the point $-a+b\sqrt{-1}$ in Quadrant II, and the point $-a-b\sqrt{-1}$ in Quadrant III.

313. The introduction of orthotomic and complex numbers requires the meanings of the four elementary operations to be made more general in the Algebra of orthotomic and complex numbers than they are in the Algebra of scalar numbers. These enlarged meanings, however, must be consistent with the older meanings of scalar Algebra and include them as special cases.

A full statement of these enlarged meanings of the four elementary operations, with illustrative applications, will be given in the chapter on Complex Numbers.

314. Square Roots of Orthotomic Numbers. If a and b are positive scalar numbers, we have

$$+\sqrt{ab}=(+\sqrt{a})(+\sqrt{b})=(-\sqrt{a})(-\sqrt{b}),$$
$$-\sqrt{ab}=(-\sqrt{a})(+\sqrt{b})=(+\sqrt{a})(-\sqrt{b}).$$ (p. 236, § 284)

In extending this law to orthotomic numbers it is assumed that the law still holds when either factor of the radicand is negative, or when both factors are negative.

Thus, $$(+\sqrt{-a})(+\sqrt{-b})=(\sqrt{a}\times\sqrt{-1})(\sqrt{b}\times\sqrt{-1})$$
$$=\sqrt{a}\times\sqrt{b}\times(\sqrt{-1})^2$$
$$=\sqrt{ab}\times(-1)$$
$$=-\sqrt{ab}.$$

In similar manner,

$$(-\sqrt{-a})(-\sqrt{-b})=-\sqrt{ab},$$
$$(-\sqrt{-a})(+\sqrt{-b})=+\sqrt{ab},$$
$$(+\sqrt{-a})(-\sqrt{-b})=+\sqrt{ab}.$$ Hence,

Two orthotomic factors with like signs give a negative scalar product; two orthotomic factors with unlike signs give a positive scalar product.

Two factors, one scalar and one orthotomic, give a positive orthotomic product if the factors have like signs, and a negative orthotomic product if the factors have unlike signs.

315. The successive powers of $\sqrt{-1}$ are:

$$(\sqrt{-1})^2=-1;$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=(-1)\sqrt{-1}=-\sqrt{-1};$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=+1;$$
$$(\sqrt{-1})^5=(\sqrt{-1})^4\sqrt{-1}=(+1)\sqrt{-1}=+\sqrt{-1};$$
. . . .

The successive powers of $\sqrt{-1}$ form the repeating series $+\sqrt{-1}, -1, -\sqrt{-1}, +1$.

316. Every orthotomic number is of the form $\pm a\sqrt{-b}$, where a and b are positive scalar numbers.

Now $$\pm a\sqrt{-b} = \pm a\sqrt{b}(\sqrt{-1}).$$

Since the factor $\pm a\sqrt{b}$ is a scalar number, every orthotomic number may be written in the form $a\sqrt{-1}$, where a is a scalar number; and conversely, if a is a scalar number, then $a\sqrt{-1}$ is an orthotomic number.

Hence, the sum of two orthotomic numbers $a\sqrt{-1}$ and $b\sqrt{-1}$ is an orthotomic number or is zero, for

$$a\sqrt{-1} + b\sqrt{-1} = (a+b)\sqrt{-1};$$

and $a+b$ is a scalar number, a and b being scalar numbers, or is zero if $a+b=0$.

317. Every complex number may be made to take the form $a+b\sqrt{-1}$, where a and b are scalar numbers, whether integers, fractions, or surds.

The form $\mathbf{a}+\mathbf{b}\sqrt{-1}$ is the *typical form of complex numbers.*

Reduce $4+\sqrt{-12}$ to the typical form.

$4+\sqrt{-12} = 4+\sqrt{12}\sqrt{-1} = 4+2\sqrt{3}\sqrt{-1}$; here $a=4$, and $b=2\sqrt{3}$.

318. Sum of Two Complex Numbers. The algebraic sum of two complex numbers is in general a complex number.

Add $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$.

$$\begin{array}{r} a+b\sqrt{-1} \\ c+d\sqrt{-1} \\ \hline (a+c)+(b+d)\sqrt{-1} \end{array}$$

The sum is $(a+c)+(b+d)\sqrt{-1}$

This sum is in general a complex number. If $a+c=0$, the sum becomes $(b+d)\sqrt{-1}$, an orthotomic number; if $b+d=0$, the sum becomes $a+c$, a scalar number.

319. Product of Two Complex Numbers. The product of two complex numbers is in general a complex number.

Multiply $a + b\sqrt{-1}$ by $c + d\sqrt{-1}$.

$$\begin{array}{r} a + b\sqrt{-1} \\ c + d\sqrt{-1} \\ \hline ac + bc\sqrt{-1} \qquad\quad \\ + ad\sqrt{-1} - bd \end{array}$$

The product is $(ac - bd) + (bc + ad)\sqrt{-1}$

This product is in general a complex number. If $ac - bd = 0$, the product becomes $(bc + ad)\sqrt{-1}$, an orthotomic number; if $bc + ad = 0$, the product becomes $ac - bd$, a scalar number.

320. Quotient of Two Complex Numbers. The quotient of two complex numbers is in general a complex number.

Divide $a + b\sqrt{-1}$ by $c + d\sqrt{-1}$.

$$\frac{a + b\sqrt{-1}}{c + d\sqrt{-1}} = \frac{(a + b\sqrt{-1})(c - d\sqrt{-1})}{(c + d\sqrt{-1})(c - d\sqrt{-1})}$$

$$= \frac{(ac + bd) + (bc - ad)\sqrt{-1}}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\sqrt{-1}.$$

This quotient is a complex number in the typical form. If $bc - ad = 0$, the quotient is scalar; if $ac + bd = 0$, the quotient is orthotomic.

321. Conjugate Numbers. Two numbers of the forms $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are called **conjugate numbers.**

Now $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a$,

and $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 - ab\sqrt{-1} + ab\sqrt{-1} + b^2$

$$= a^2 + b^2.$$

Hence, the *sum* or the *product* of two conjugate numbers is a scalar number.

322. *An imaginary number cannot be equal to a scalar number.*

For, if possible, let $a + b\sqrt{-1} = c.$

Transpose, $b\sqrt{-1} = c - a.$

Square, $-b^2 = (c - a)^2.$

Since b^2 and $(c - a)^2$ are both positive, we have a negative number equal to a positive number, which is impossible.

323. *If two complex numbers are equal, the scalar parts are equal and the orthotomic parts are equal.*

For, let $a + b\sqrt{-1} = c + d\sqrt{-1}.$

Transpose, $b\sqrt{-1} - d\sqrt{-1} = c - a.$

Factor, $(b - d)\sqrt{-1} = c - a.$

Square, $-(b - d)^2 = (c - a)^2.$

This equation is impossible unless $b = d$, and $a = c$.

324. *If* a *and* b *are scalar and* $a + b\sqrt{-1} = 0$, *then* $a = 0$, *and* $b = 0$.

For, let $a + b\sqrt{-1} = 0.$

Transpose, $b\sqrt{-1} = -a.$

Square, $-b^2 = a^2.$

This equation is impossible unless $a = 0$, and $b = 0$.

Exercise 106

Reduce to the typical form $a\sqrt{-1}$:

1. $\sqrt{-9}$.
2. $\sqrt{-25}$.
3. $\sqrt{-36}$.
4. $2\sqrt{-49}$.
5. $3\sqrt{-625a^2}$.
6. $2\sqrt{-81x^2y^2}$.
7. $\sqrt{-x^2-y^2}$.
8. $\sqrt{-\frac{1}{4}}$.
9. $ax\sqrt{-a^8x^4}$.
10. $a^2\sqrt{-729a^6}$.
11. $\sqrt{-a^6b^{-8}}$.
12. $\sqrt[4]{-a^{-4}b^{-8}c^2}$.

Find the sum of:

13. $\sqrt{-25}+\sqrt{-36}+\sqrt{-49}+\sqrt{-64}+\sqrt{-81}$.

14. $2\sqrt{-a^4}+3\sqrt{-a^6}-4\sqrt[4]{-16a^8}+5\sqrt{-25a^4}$.

15. $(3+7\sqrt{-1})+(2-6\sqrt{-1})-(3+4\sqrt{-1})$.

16. $(a+b\sqrt{-1})+(c-d\sqrt{-1})-(a-d\sqrt{-1})$.

Find the product of:

17. $(3+5\sqrt{-1})(7+4\sqrt{-1})$.

18. $(5-2\sqrt{-7})(6-2\sqrt{-7})$.

19. $(2\sqrt{7}+3\sqrt{-8})(3\sqrt{7}-10\sqrt{-2})$.

20. $(3\sqrt{3}+2\sqrt{-2})(3\sqrt{3}-2\sqrt{-2})$.

21. $(4+7\sqrt{-1})(2+3\sqrt{-1})$.

22. $(a+b\sqrt{-1})(a-c\sqrt{-1})$.

23. $(a\sqrt{-1}-b)(c\sqrt{-1}-d)$.

24. $(2\sqrt{5}+5\sqrt{-2})(2\sqrt{5}-3\sqrt{-5})$.

Perform the divisions indicated:

25. $\dfrac{4}{1+\sqrt{-3}}$.

26. $\dfrac{21}{4+3\sqrt{-6}}$.

27. $\dfrac{5}{\sqrt{2}-\sqrt{-3}}$.

28. $\dfrac{1+\sqrt{-3}}{1-\sqrt{-3}}$.

29. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$.

30. $\dfrac{5-29\sqrt{-5}}{7-3\sqrt{-5}}$.

31. $\dfrac{3}{\sqrt{2}+\sqrt{-1}}$.

32. $\dfrac{\sqrt{3}+\sqrt{-2}}{\sqrt{3}-\sqrt{-2}}$.

33. $\dfrac{\sqrt{-a}+\sqrt{-b}}{\sqrt{-a}-\sqrt{-b}}$.

34. $\dfrac{a-b\sqrt{-1}}{a\sqrt{-1}+b}$.

35. $\dfrac{x+i\sqrt{1-x^2}}{x-i\sqrt{1-x^2}}$.

36. $\dfrac{\sqrt{a-b}+\sqrt{b-a}}{\sqrt{a-b}-\sqrt{b-a}}$.

Extract the square root of:

37. $5+12\sqrt{-1}$.

38. $-21+20\sqrt{-1}$.

39. $16-30\sqrt{-1}$.

40. $-40-42\sqrt{-1}$.

CHAPTER XX

QUADRATIC EQUATIONS

325. Quadratic Equations. An equation which when reduced to simplest form contains the *second power*, but no higher power, of the unknown numbers is called a **quadratic equation.**

326. A quadratic equation that involves but one unknown number can contain only:

1. Terms involving the square of the unknown number.
2. Terms involving the first power of the unknown number.
3. Terms that do not involve the unknown number.

If similar terms are combined, every quadratic equation in one unknown may be made to assume the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are known numbers, and x is the unknown number.

If a, b, and c are numbers expressed by figures, the equation is called a **numerical quadratic.** If a, b, and c are numbers represented wholly or in part by letters, the equation is called a **literal quadratic.**

Thus $3x^2 - 2x + 4 = 0$ is a numerical quadratic,
and $ax^2 - 2x + bd = 0$ is a literal quadratic.

In the equation $ax^2 + bx + c = 0$, the numbers a, b, and c are called the **coefficients** of the equation. The third term c is called the **constant term.**

327. Pure and Affected Quadratics. If the first power of x is missing, the equation is a **pure quadratic**; in this case $b = 0$.

If the first power of x is present, the equation is an **affected quadratic,** or a **complete quadratic.**

328. Solution of Pure Quadratic Equations.

1. Solve the equation $7x^2 - 36 = 3x^2$.

We have	$7x^2 - 36 = 3x^2.$
Transpose and combine,	$4x^2 = 36.$
Divide by 4,	$x^2 = 9.$
Extract the square root,	$x = \pm 3.$

It will be observed that there are two roots, which are numerically equal, but one is positive and one negative. There are but two roots, since there are but two square roots of any number.

It may seem as though we ought to write the sign $\pm$ before the x as well as before the 3. If we do this, we have

$$+x = +3, \quad -x = -3, \quad +x = -3, \quad -x = +3.$$

From the first and second, $x = 3$; from the third and fourth, $x = -3$; these values of x are both given by $x = \pm 3$. Hence it is *unnecessary*, although *perfectly correct*, to write the sign $\pm$ on *both* sides of the reduced equation.

2. Solve the equation $5x^2 - 9 = 2x^2 + 24$.

We have	$5x^2 - 9 = 2x^2 + 24.$
Transpose and combine,	$3x^2 = 33.$
Divide by 3,	$x^2 = 11.$
Extract the square root,	$x = \pm\sqrt{11}.$

The roots cannot be found exactly, since the square root of 11 cannot be found exactly; it can, however, be found to any required degree of accuracy; for example, it lies between 3.31662 and 3.31663.

3. Solve the equation $7x^2 + 35 = 0$.

We have	$7x^2 + 35 = 0.$
Transpose,	$7x^2 = -35.$
Divide by 7,	$x^2 = -5.$
Extract the square root,	$x = \pm\sqrt{-5}.$

The solutions are imaginary numbers.

329. A root that can be found exactly is called an **exact root**, or a **rational root**. Such a root is either a whole number or a fraction.

A root that involves an indicated root of a positive rational number which cannot be found exactly is called a **surd root**, or an **irrational root.**

Exact roots and surd roots together are called **real roots.**

A root that may be indicated but cannot be found as a scalar number, positive or negative, either exactly or approximately, is called an **imaginary root.** Such a root involves the even root of a negative number.

Exercise 107

Solve:

1. $x^2 - 36 = 0.$
2. $x^2 - 144 = 0.$
3. $4x^2 - 144 = 0.$
4. $x^2 - a^2 = 0.$
5. $x^2 - a = 0.$
6. $4x^2 - a^2 = 0.$
7. $4x^2 - a = 0.$
8. $a^2x^2 - b^2 = 0.$
9. $a^2x^2 - b = 0.$
10. $x^2 - 5 = 0.$
11. $4x^2 - 15 = 0.$
12. $x^2 + 5 = 0.$
13. $4x^2 + 15 = 0.$
14. $5x^2 + 3x^2 = 72.$
15. $6x^2 + 5x^2 = 176.$
16. $7x^2 - 2x^2 = 125.$
17. $9x^2 - 4x^2 = 80.$
18. $12x^2 - 5x^2 = 343.$
19. $3x^2 + 12 = 7x^2 - 88.$
20. $9x^2 - 53 = 6x^2 + 94.$
21. $13x^2 - 19 = 7x^2 + 5.$
22. $17x^2 - 7 = 418.$
23. $4x^2 + 7 = 6x^2 - 3.$
24. $6x^2 - 12 = 3x^2 - 27.$
25. $(x + \frac{1}{2})(x - \frac{1}{2}) = \frac{5}{16}.$
26. $(3x - 7)(3x + 7) = 32.$
27. $mx^2 = a^2 - nx^2.$
28. $ax^2 - b = cx^2 + d.$
29. $(a + x)(x - b) = (a - x)(x + b).$
30. $(ax + b)(c + dx) = (cx + d)(a + bx).$
31. $(a - x)(b - x) = (1 - ax)(1 - bx).$
32. $ax^2 + a = bx^2 - b.$
33. $(x + 3)(x - 3) = 66 - 2x^2.$
34. $(x + 2)(x + 3) = 2x^2 + 5x.$
35. $x^2 - 5x + 9 = 3x^2 - 5x - 9.$
36. $(a + x)(b - x) = (a - x)(b + x).$

37. $\dfrac{25+x}{9+x}=\dfrac{13+x}{47-x}.$

38. $\dfrac{35+3x}{1+x}=\dfrac{x-55}{3x-53}.$

39. $\dfrac{x-2}{3x+14}=\dfrac{3(8-x)}{28-x}.$

40. $\dfrac{x+5a+b}{x-3a+b}=\dfrac{x-a+b}{a-x+3b}.$

41. $\dfrac{3a-2b+3x}{a-2b+x}=\dfrac{x-7a+8b}{3x-5a+4b}.$

42. $\dfrac{x+a-b}{x-a+b}=\dfrac{a(x+a+5b)}{b(x+5a+b)}.$

43. $(x+a)(x-a)+(x+b)(x-b)-(x+c)(x-c)=0.$

44. $(a+bx)(b-ax)+(b+cx)(c-bx)+(c+ax)(a-cx)=0.$

45. $\dfrac{x^2-x+1}{x+1}+6=\dfrac{x^2+x+1}{x-1}.$

46. $\dfrac{7a-b+x}{7b-a+x}=\dfrac{a(a+5b+x)}{b(5a+b+x)}.$

47. $\dfrac{17a+b-x}{a+17b-x}=\dfrac{a^2(a+17b+x)}{b^2(17a+b+x)}.$

48. $(2x+3)(3x+4)(4x+5)-(2x-3)(3x-4)(4x-5)=184.$

49. $(a+5b+x)(5a+b+x)=3(a+b+x)^2.$

50. $\dfrac{x^2-2bx+2ax-b^2}{x^3-a^3}+\dfrac{x+2b}{x^2+ax+a^2}=\dfrac{1}{x-a}.$

51. $\dfrac{3a}{x-5a}+\dfrac{x+4a}{x+3a}=\dfrac{7a^2+2ax-x^2}{x^2-2ax-15a^2}.$

52. $(a+x)(b-x)+(1+ax)(1-bx)=(a+b)(1+x^2).$

330. Solution of Affected Quadratic Equations. Since $(x \pm b)^2$ is identical with $x^2 \pm 2bx + b^2$, it is evident that the expression $x^2 \pm 2bx$ lacks only the third term b^2 of being a perfect square.

This third term is the square of half the coefficient of x. Every affected quadratic may be made to assume the form $x^2 \pm 2bx = c$ by dividing the equation through by the coefficient of x^2.

331. To **solve** such an equation three steps are required.

1. Add to each member *the square of half the coefficient of* x. This process is called **completing the square.**

2. *Extract the square root* of each member of the resulting equation.

3. *Solve* the two resulting simple equations.

1. Solve the equation $x^2 - 4x = 12$.

We have $\quad x^2 - 4x = 12.$

Complete the square, $\quad x^2 - 4x + 4 = 16.$

Extract the square root, $\quad x - 2 = \pm 4.$

Solve, $\quad x = 2 + 4 = 6,$

or $\quad x = 2 - 4 = -2.$

Therefore, the roots required are 6 and -2.

Verify by putting these numbers for x in the given equation.

$x = 6.$
$6^2 - 4(6) = 12.$
$36 - 24 = 12.$

$x = -2.$
$(-2)^2 - 4(-2) = 12.$
$4 + 8 = 12.$

2. Solve the equation $\dfrac{x+7}{9-4x^2} - \dfrac{1-x}{2x+3} = \dfrac{4}{2x-3}.$

Clear of fractions by multiplying by $9 - 4x^2$,

$$x + 7 - (1 - x)(3 - 2x) = -4(3 + 2x).$$

Simplify, $\quad x + 7 - 3 + 5x - 2x^2 = -12 - 8x.$

Transpose and combine, $\quad 2x^2 - 14x = 16.$

Divide by 2, $\quad x^2 - 7x = 8.$

Complete the square, $\quad x^2 - 7x + \frac{49}{4} = \frac{81}{4}.$

Extract the square root, $\quad x - \frac{7}{2} = \pm \frac{9}{2}.$

Solve, $\quad x = \frac{7}{2} + \frac{9}{2} = 8,$

or $\quad x = \frac{7}{2} - \frac{9}{2} = -1.$

Therefore, the roots required are 8 and -1.

Verify by putting these numbers for x in the given equation.

$x = 8.$
$$\frac{8+7}{9-256} - \frac{1-8}{16+3} = \frac{4}{16-3}.$$
$$-\tfrac{15}{247} + \tfrac{7}{19} = \tfrac{4}{13}.$$

$x = -1.$
$$\frac{-1+7}{9-4} - \frac{1+1}{-2+3} = \frac{4}{-2-3}.$$
$$\tfrac{6}{5} - 2 = -\tfrac{4}{5}.$$

Exercise 108

Solve:

1. $x^2 + 2x = 63$.
2. $x^2 - 8x + 15 = 0$.
3. $x^2 - 6x + 4 = 0$.
4. $x^2 - 10x + 32 = 0$.
5. $x^2 + 2x - 1 = 0$.
6. $x^2 + 6x - 91 = 0$.
7. $x^2 - 7x - 30 = 0$.
8. $x^2 - 12x + 35 = 0$.
9. $x^2 + x - 56 = 0$.
10. $x^2 + 2x - 15 = 0$.
11. $x^2 - 4x - 77 = 0$.
12. $x^2 - 14x + 45 = 0$.
13. $x^2 - 3x - 54 = 0$.
14. $x^2 + 6x + 9 = 0$.
15. $x^2 + 4x - 60 = 0$.
16. $12x^2 + 8x - 15 = 0$.
17. $2x^2 + 6x + 15 = 0$.
18. $3x^2 + 5x - 22 = 0$.
19. $\dfrac{5x-7}{9} + \dfrac{14}{2x-3} = x - 1$.
20. $\dfrac{6x+4}{5} - \dfrac{15-2x}{x-3} = \dfrac{7(x-1)}{5}$.
21. $\dfrac{5x-1}{9} + \dfrac{3x-1}{5} = \dfrac{2}{x} + x - 1$.
22. $\dfrac{16-x}{4} - \dfrac{2(x-11)}{x-6} = \dfrac{x-4}{12}$.
23. $2x^2 - 7x + 3 = 0$.
24. $2x^2 - 5x - 3 = 0$.
25. $3x^2 - 17x + 10 = 0$.
26. $6x^2 - 5x - 6 = 0$.
27. $4x^2 - 3x - 2 = 0$.
28. $3x^2 + 4x + 5 = 0$.
29. $\dfrac{x-5}{x+3} + \dfrac{x-8}{x-3} = \dfrac{3(1-x)}{x^2-9}$.
30. $\dfrac{3x+4}{2x-1} + \dfrac{5x+12}{2x+1} = \dfrac{4x+3}{4x^2-1} + 7$.
31. $\dfrac{3x-3}{x+1} + \dfrac{3x-1}{x+2} - \dfrac{3x-11}{x-1} = 3$.

332. When the Coefficient of x^2 is a Perfect Square. Since $(ax \pm \frac{1}{2}b)^2$ is identical with $a^2x^2 \pm abx + \frac{1}{4}b^2$, it is evident that the expression $a^2x^2 \pm abx$ lacks only the third term $\frac{1}{4}b^2$ of being a perfect square.

This third term is *the square of the quotient obtained by dividing the second term by twice the square root of the first term.*

Hence, if the coefficient of x^2 is a perfect square, we may complete the square by adding to each member of the equation the square of the quotient obtained by dividing the second term by twice the square root of the first term.

Solve the equation $9x^2 - 9x = 10$.

The square root of $9x^2$ is $3x$, and $-9x$ divided by twice $3x$ is $-\frac{3}{2}$.
Add the square of $-\frac{3}{2}$ to each member,

$$9x^2 - 9x + (-\tfrac{3}{2})^2 = 10 + (-\tfrac{3}{2})^2,$$

or $$9x^2 - 9x + \tfrac{9}{4} = \tfrac{49}{4}.$$

Extract the square root, $$3x - \tfrac{3}{2} = \pm\tfrac{7}{2}.$$

Transpose and combine, $$3x = 5, \text{ or } -2.$$

$$\therefore x = \tfrac{5}{3}, \text{ or } -\tfrac{2}{3}.$$

Check. If $x = \frac{5}{3}$, then $9(\frac{5}{3})^2 - 9(\frac{5}{3}) = 10$, or $25 - 15 = 10$.
If $x = -\frac{2}{3}$, then $9(-\frac{2}{3})^2 - 9(-\frac{2}{3}) = 10$, or $4 + 6 = 10$.

If the coefficient of x^2 is not a perfect square, we may multiply the equation by a number that will make the coefficient of x^2 a perfect square, and proceed as before.

Solve the equation $3x^2 + 7x = 6$.

Here the coefficient of x^2 is 3, and to make this coefficient a perfect square we multiply the equation by 3.

Then $$9x^2 + 21x = 18.$$

Complete the square, $$9x^2 + 21x + \tfrac{49}{4} = \tfrac{121}{4}.$$

Extract the square root, $$3x + \tfrac{7}{2} = \pm\tfrac{11}{2}.$$

Transpose and combine, $$3x = 2, \text{ or } -9.$$

$$\therefore x = \tfrac{2}{3}, \text{ or } -3.$$

Check. If $x = \frac{2}{3}$, then $3(\frac{2}{3})^2 + 7(\frac{2}{3}) = 6$, or $\frac{4}{3} + \frac{14}{3} = 6$.
If $x = -3$, then $3(-3)^2 + 7(-3) = 6$, or $27 - 21 = 6$.

EXERCISE 109

Solve:

1. $25x^2 - 30x - 27 = 0.$
2. $4x^2 - 4x - 15 = 0.$
3. $4x^2 + 24x + 35 = 0.$
4. $4x^2 - 24x + 35 = 0.$
5. $9x^2 + 15x - 6 = 0.$
6. $16x^2 - 32x + 15 = 0.$
7. $16x^2 + 8x - 35 = 0.$
8. $16x^2 - 8x - 35 = 0.$
9. $4x^2 + 24x + 27 = 0.$
10. $3x^2 - 22x + 35 = 0.$
11. $3x^2 - 7x - 16 = 0.$
12. $2x^2 - 8 = 3x + 12.$
13. $2x^2 - x - 15 = 0.$
14. $6x^2 - 17x + 7 = 0.$
15. $3x^2 + 14x + 15 = 0.$
16. $4x^2 - 5x - 9 = 0.$
17. $5x^2 + 6x - 27 = 0.$
18. $3x^2 - 22x + 21 = 0.$
19. $(2x - 5)^2 - (x - 6)^2 = 80.$
20. $(1 - 3x)(x - 6) = 5(x + 2).$
21. $4x^2 + 24x - 35 = 10.$
22. $(3x - 5)^2 - (2x + 1)^2 + 32 = 0.$
23. $(2x - 3)^2 - 5(2x - 5)^2 - 4 = 0.$
24. $(5x + 2)^2 - 5(3x - 4) + 331 = 0.$
25. $\dfrac{15}{3x - 1} - \dfrac{3}{4} = \dfrac{220}{9x^2 - 1} - \dfrac{11}{3x + 1}.$
26. $\dfrac{2x - 1}{x - 2} + \dfrac{3x + 1}{x - 3} - \dfrac{11(x + 1)}{(x - 2)(x - 3)} = 0.$
27. $\dfrac{1}{x - 3} + \dfrac{7}{x + 3} = \dfrac{4}{x^2 - 9} - \dfrac{16 - 3x}{x - 3}.$
28. $\dfrac{2x + 1}{x + 3} - \dfrac{x - 1}{x^2 - 9} = \dfrac{x + 3}{3 - x} + \dfrac{9x + 20}{2(3 + x)}.$
29. $\dfrac{x + 7}{2x + 3} - \dfrac{x + 1}{2x - 3} = \dfrac{4x - 5}{4x^2 - 9} - \dfrac{x - 3}{2x - 3}.$

333. Another Method of Completing the Square. When the coefficient of x^2 is not unity, we may proceed as in § 332, or we may complete the square by another method.

Since $(ax \pm b)^2$ is identical with $a^2x^2 \pm 2\,abx + b^2$, it is evident that the expression $a^2x^2 \pm 2\,abx$ lacks only the third term b^2 of being a perfect square.

The third term is the square of the quotient obtained by dividing the second term by twice the square root of the first term.

Any affected quadratic of the form $px^2 \pm qx = s$ may be made to assume the form $a^2x^2 \pm 2\,abx = c$ by multiplying the given quadratic through by $4\,p$. Hence,

To complete the square, *multiply the given quadratic by four times the coefficient of* x^2, *and add to each member the square of the coefficient of* x.

NOTE. In general this method is the best to follow, for all fractions are avoided in completing the square.

Solve the equation $3\,x^2 - 5\,x = 2$.

Complete the square by multiplying by 12 and adding 5^2 to each side,

$$36\,x^2 - 60\,x + 25 = 49.$$

Extract the square root, $6\,x - 5 = \pm 7.$

Transpose and combine, $6\,x = 12, \text{ or } -2.$

$$\therefore\ x = 2, \text{ or } -\tfrac{1}{3}.$$

Check. If $x = 2$, then $3\,(2)^2 - 5\,(2) = 2$, or $12 - 10 = 2$.
If $x = -\frac{1}{3}$, then $3\,(-\frac{1}{3})^2 - 5\,(-\frac{1}{3}) = 2$, or $\frac{1}{3} + \frac{5}{3} = 2$.

334. If the coefficient of x is an *even number,* we may multiply by the *coefficient of* x^2, and add to each member the square of *half the coefficient of* x in the given equation.

Solve the equation $5\,x^2 - 6\,x = 27$.

Complete the square, $25\,x^2 - (\ ) + 9 = 144.$

Extract the square root, $5\,x - 3 = \pm 12.$

Transpose and combine, $5\,x = 15, \text{ or } -9.$

$$\therefore\ x = 3, \text{ or } -\tfrac{9}{5}.$$

NOTE. If a trinomial is a *perfect square*, its root is found by taking the roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to *write* the middle term, but its place may be indicated as in this example.

EXERCISE 110

Solve:

1. $6x^2 - 19x - 20 = 0.$
2. $6x^2 + 17x + 12 = 0.$
3. $6x^2 - 11x - 10 = 0.$
4. $8x^2 + 2x - 15 = 0.$
5. $6x^2 - 17x - 14 = 0.$
6. $6x^2 + 13x + 6 = 0.$
7. $10x^2 + 19x - 15 = 0.$
8. $3x^2 - 7x + 4 = 0.$
9. $4x^2 - 11x - 45 = 0.$
10. $3x^2 + 14x + 15 = 0.$
11. $2x^2 - 14x - 36 = 0.$
12. $3x^2 - 5x + 7 = 0.$
13. $5x^2 - 2x + 14 = 0.$
14. $3x^2 + 8x - 21 = 0.$
15. $6x^2 - 5x + 42 = 0.$
16. $5x^2 + 12x - 17 = 0.$
17. $3x^2 + 7x - 76 = 0.$
18. $5x^2 - 8x - 189 = 0.$
19. $\frac{2x-3}{x-2} + \frac{x+1}{x-1} = \frac{3x+11}{x+1}.$
20. $\frac{5+x}{3-x} - \frac{8-3x}{x} = \frac{2x}{x-2}.$
21. $\frac{1}{x+6} + \frac{1}{x+30} + \frac{1}{3x} = \frac{1}{x}.$
22. $\frac{2}{x-2} + \frac{3}{x-3} = \frac{8}{x-4}.$
23. $\frac{3}{x+2} + \frac{2}{x+4} - \frac{1}{x+6} = \frac{4}{x}.$
24. $\frac{3x}{2} - \frac{3x-20}{18-2x} = 2 + \frac{3x^2-80}{2(x-1)}.$
25. $\frac{x}{2x-1} + \frac{24}{4x^2-1} = \frac{2(x-4)}{2x+1} - \frac{1}{9}.$

335. As was explained on page 100, § 144, any **quadratic trinomial** of the form $ax^2 + bx + c$ may be resolved into two factors by writing a given trinomial as *the difference of two squares.*

Resolve $6x^2 - 7x - 20$ into two factors.

$$\begin{aligned}
6x^2 - 7x - 20 &= 6\left(x^2 - \tfrac{7}{6}x - \tfrac{10}{3}\right)\\
&= 6\left[\left(x^2 - \tfrac{7}{6}x + \tfrac{49}{144}\right) - \tfrac{49}{144} - \tfrac{10}{3}\right]\\
&= 6\left[\left(x - \tfrac{7}{12}\right)^2 - \tfrac{529}{144}\right]\\
&= 6\left(x - \tfrac{7}{12} + \tfrac{23}{12}\right)\left(x - \tfrac{7}{12} - \tfrac{23}{12}\right)\\
&= 6\left(x + \tfrac{4}{3}\right)\left(x - \tfrac{5}{2}\right)\\
&= (3x + 4)(2x - 5).
\end{aligned}$$

Another method of resolving a quadratic trinomial into two factors will prove to be simpler in many cases.

Resolve $6x^2 - 7x - 20$ into two factors.

$$\begin{aligned}
6x^2 - 7x - 20 &= 6\left(x^2 - \tfrac{7}{6}x - \tfrac{20}{6}\right)\\
&= 6\left(x^2 - \tfrac{7}{6}x - \tfrac{120}{36}\right)\\
&= 6\left(x + \tfrac{8}{6}\right)\left(x - \tfrac{15}{6}\right)\\
&= 6\left(x + \tfrac{4}{3}\right)\left(x - \tfrac{5}{2}\right)\\
&= (3x + 4)(2x - 5).
\end{aligned}$$

Divide by the coefficient of x^2 without reducing to lowest terms; multiply both terms of the last fraction by the denominator of that fraction.

To factor the trinomial $x^2 - \frac{7}{6}x - \frac{120}{36}$, find two numbers whose product is -120 and whose sum is -7. Two such numbers are $+8$ and -15.

Hence, the two factors are $x + \frac{8}{6}$ and $x - \frac{15}{6}$, or $x + \frac{4}{3}$ and $x - \frac{5}{2}$.

Multiplying by 6, we have $3x + 4$ and $2x - 5$ for the required factors.

Exercise 111

Resolve into two factors:

1. $6x^2 - 7x + 2$.
2. $12x^2 - 7x - 10$.
3. $21x^2 - 13x + 2$.
4. $3x^2 + 8x + 4$.
5. $6x^2 - 11x + 3$.
6. $3x^2 + x - 10$.
7. $8x^2 + 22x + 5$.
8. $9x^2 - 3x - 2$.
9. $6x^2 + 13x - 5$.
10. $6x^2 + x - 15$.
11. $9x^2 - 12x - 1$.
12. $2x^2 + 10x - 9$.
13. $3x^2 - 7x - 5$.
14. $6x^2 - 5x + 5$.

336. Solutions by Factoring. A quadratic equation that has been reduced to its simplest form, and has all its terms written in one member, may often have that member resolved into factors *by inspection.*

In this case the roots of the quadratic are seen at once without completing the square.

In fact, if a quadratic may be resolved into factors readily, this method of obtaining the roots is far shorter and more satisfactory than any other method.

1. Solve the equation $x^2 - 2x - 24 = 0$.

We have $x^2 - 2x - 24 = 0.$
Factor, $(x + 4)(x - 6) = 0.$

It will be observed that if *either* of the factors $x + 4$ or $x - 6$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $x + 4 = 0$, or $x - 6 = 0.$
$\therefore x = -4$, or 6.

2. Solve the equation $x^2 - 5x = 0$.

We have $x^2 - 5x = 0.$
Factor, $x(x - 5) = 0.$
The equation is satisfied if $x = 0$, or if $x - 5 = 0.$
$\therefore x = 0$, or 5.

3. Solve the equation $x^3 - 5x^2 + 6x = 0$.

The equation $x^3 - 5x^2 + 6x = 0$
may be written $x(x^2 - 5x + 6) = 0,$
or $x(x - 2)(x - 3) = 0.$
The equation is satisfied if $x = 0$, $x - 2 = 0$, or $x - 3 = 0.$

Therefore the equation has *three* roots, 0, 2, 3.

4. Solve the equation $x^4 - 5x^2 + 4 = 0$.

The equation $x^4 - 5x^2 + 4 = 0$
may be written $(x^2 - 1)(x^2 - 4) = 0,$
or $(x - 1)(x + 1)(x - 2)(x + 2) = 0.$
The equation is satisfied if $x - 1 = 0$, $x + 1 = 0$, $x - 2 = 0$, or $x + 2 = 0.$

Therefore, the equation has *four* roots, 1, -1, 2, -2.

5. Solve the equation $x^3 - 3x^2 - 13x + 15 = 0$.

We find that $x - 1$ is a factor of the left member (p. 103, § 147). Hence, the given equation may be written

$$(x-1)(x^2-2x-15)=0,$$

or

$$(x-1)(x+3)(x-5)=0.$$

Therefore, the three roots are 1, -3, 5.

6. Solve the equation $9x^2 - 6x - 4 = 0$.

Divide by 9, $x^2 - \frac{2}{3}x - \frac{4}{9} = 0.$

Add and subtract the square of half the coefficient of x,

$$x^2 - \tfrac{2}{3}x + (\tfrac{1}{3})^2 - \tfrac{1}{9} - \tfrac{4}{9} = 0,$$

or

$$(x - \tfrac{1}{3})^2 - \tfrac{5}{9} = 0.$$

The square root of $\frac{5}{9} = \sqrt{\frac{1}{9} \times 5} = \frac{1}{3}\sqrt{5}$.
Hence, the equation becomes

$$(x - \tfrac{1}{3} + \tfrac{1}{3}\sqrt{5})(x - \tfrac{1}{3} - \tfrac{1}{3}\sqrt{5}) = 0.$$

$$\therefore x = \tfrac{1}{3} - \tfrac{1}{3}\sqrt{5}, \text{ or } \tfrac{1}{3} + \tfrac{1}{3}\sqrt{5}.$$

7. Solve the equation $2x^2 - x + 1 = 0$.

Divide by 2, $x^2 - \frac{1}{2}x + \frac{1}{2} = 0.$

Add and subtract the square of half the coefficient of x,

$$x^2 - \tfrac{1}{2}x + \tfrac{1}{16} - \tfrac{1}{16} + \tfrac{1}{2} = 0,$$

or

$$(x - \tfrac{1}{4})^2 + \tfrac{7}{16} = 0.$$

In order to make the left member the difference of two squares, write it

$$(x - \tfrac{1}{4})^2 - (-\tfrac{7}{16}) = 0.$$

The square root of $-\frac{7}{16} = \sqrt{\frac{1}{16} \times (-7)} = \frac{1}{4}\sqrt{-7}$.
Hence, the equation becomes

$$(x - \tfrac{1}{4} + \tfrac{1}{4}\sqrt{-7})(x - \tfrac{1}{4} - \tfrac{1}{4}\sqrt{-7}) = 0.$$

$$\therefore x = \tfrac{1}{4} - \tfrac{1}{4}\sqrt{-7}, \text{ or } \tfrac{1}{4} + \tfrac{1}{4}\sqrt{-7}.$$

NOTE. The student will observe from the solutions of the foregoing examples that in solving any equation it is best to reduce the equation to its simplest form, and then resolve the left member into as many linear factors as is *conveniently* possible. When all the linear factors are found, the nature of the remaining factor, if any factor remains, will determine the easiest way of finding the remaining roots.

Exercise 112

Find all the roots of:

1. $x^2 + 11x + 30 = 0.$
2. $x^2 - 2x - 24 = 0.$
3. $x^2 - 10x + 21 = 0.$
4. $x^2 - 2x - 3 = 0.$
5. $36x^2 - 35x + 6 = 0.$
6. $(7x - 8)(2x - 3) = 0.$
7. $(3x + 2)(5x + 9) = 0.$
8. $(10 - x)(6 + 5x) = 0.$
9. $x^2 - 16x = 0.$
10. $x^2 + 23x = 0.$
11. $4x^2 + 28x + 49 = 0.$
12. $x^3 - 1 = 0.$
13. $x^3 + 1 = 0.$
14. $x^3 - 27 = 0.$
15. $x^4 - 16 = 0.$
16. $8x^3 - 27 = 0.$
17. $8x^3 + 27 = 0.$
18. $216 + 125x^3 = 0.$
19. $x^4 + 4x^2 + 16 = 0.$
20. $x^4 + 64 = 0.$
21. $4x^4 - 29x^2 + 25 = 0.$
22. $35x^2 - 83x + 36 = 0.$
23. $42x^2 - 59x + 20 = 0.$
24. $16x^4 + 25 + 24x^2 = 0.$
25. $x^3 - 23x^2 + 102x = 0.$
26. $y^5 + 5y^4 - 36y^3 = 0.$
27. $x^3 + x^2 - 22x - 40 = 0.$
28. $9x^4 + 6x^3 + 3x^2 + 2x = 0.$
29. $4x^3 - 12x^2 + 9x - 1 = 0.$
30. $6x^4 + 25x^3 + 5x^2 - 60x - 36 = 0.$
31. $6x^4 + 13x^3 - 85x^2 - 104x + 240 = 0.$
32. $x^4 - 17x^2 - 36x - 20 = 0.$
33. $6x^4 + 49x^3 + 146x^2 + 189x + 90 = 0.$
34. $2x^4 - 3x^3 - 4x^2 + 3x + 2 = 0.$
35. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$
36. $x^5 - 3x^4 - 17x^3 - x^2 + 24x + 4 = 0.$
37. $(x^2 - 5x + 7)^2 - (x - 2)(x - 3) - 1 = 0.$
38. $8x^5 - 46x^4 + 47x^3 + 47x^2 - 46x + 8 = 0.$

337. Solutions by a Formula. Every quadratic equation may be reduced to the form $ax^2 + bx + c = 0$, in which a, b, and c represent numbers, positive or negative, integral or fractional.

Solve the equation $ax^2 + bx + c = 0$.

Complete the square,

$$4a^2x^2 + (\ ) + b^2 = b^2 - 4ac.$$

Extract the root, $$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

By this formula, the values of x in an equation of the form $ax^2 + bx + c = 0$ may be written at once.

Solve the equation $\dfrac{2x+7}{2x-3} + \dfrac{3x-2}{x+1} = 5.$

Clear of fractions,

$$2x^2 + 9x + 7 + 6x^2 - 13x + 6 = 10x^2 - 5x - 15.$$

Transpose and combine, $2x^2 - x - 28 = 0.$

Here $a = 2,\ b = -1,\ c = -28.$

Substitute these values for the letters in the formula,

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-28)}}{2(2)} = \frac{1 \pm \sqrt{225}}{4} = \frac{1 \pm 15}{4} = 4, \text{ or } -\frac{7}{2}.$$

Exercise 113

Solve by the formula:

1. $x^2 - 9x + 20 = 0.$
2. $2x^2 - 9x + 10 = 0.$
3. $4x^2 - 4x - 3 = 0.$
4. $6x^2 + 29x + 35 = 0.$
5. $20x^2 - 23x + 6 = 0.$
6. $6x^2 + 35x - 6 = 0.$
7. $14x^2 - 55x + 21 = 0.$
8. $10x^2 - 21x - 10 = 0.$
9. $24x^2 - 2x - 15 = 0.$
10. $15x^2 + 19x + 6 = 0.$
11. $5y^2 + 13y - 6 = 0.$
12. $3x^2 - 14x + 24 = 0.$
13. $x^2 - 2x - 4 = 0.$
14. $3x^2 - 4x - 22 = 0.$
15. $x^2 - 2x + 2 = 0.$
16. $x^2 - 6x + 7 = 0.$
17. $x^2 + x + 1 = 0.$
18. $8x^2 - 4x + 3 = 0.$

338. Literal Equations are solved by the same general methods as are numerical equations.

1. Solve the equation $\frac{5ab - 3b^2 - ax}{2a - x} = \frac{2a + x}{3}$.

Clear of fractions by multiplying by $3(2a - x)$,

$$15ab - 9b^2 - 3ax = 4a^2 - x^2.$$

Transpose, $x^2 - 3ax = 4a^2 - 15ab + 9b^2.$

Complete the square, $4x^2 - (\ ) + 9a^2 = 25a^2 - 60ab + 36b^2.$

Extract the square root, $2x - 3a = \pm(5a - 6b).$

Transpose and combine, $2x = 8a - 6b, \text{ or } 6b - 2a.$

$$\therefore x = 4a - 3b, \text{ or } 3b - a.$$

2. Solve the equation $\frac{2ab}{3x+1} + \frac{(3x-1)b^2}{2x+1} = \frac{(2x+1)a^2}{3x+1}$.

Clear of fractions by multiplying by $(2x+1)(3x+1)$,

$$2ab(2x+1) + (9x^2 - 1)b^2 = (2x+1)^2a^2.$$

Remove parentheses,

$$4abx + 2ab + 9b^2x^2 - b^2 = 4a^2x^2 + 4a^2x + a^2.$$

Transpose and combine,

$$9b^2x^2 - 4a^2x^2 + 4abx - 4a^2x = a^2 - 2ab + b^2,$$

or $(9b^2 - 4a^2)x^2 + 4a(b - a)x = (a - b)^2,$

or $(9b^2 - 4a^2)x^2 - 4a(a - b)x = (a - b)^2.$

Complete the square,

$$(9b^2 - 4a^2)^2x^2 - (\ ) + 4a^2(a-b)^2 = (9b^2 - 4a^2)(a - b)^2 + 4a^2(a - b)^2,$$

or $(9b^2 - 4a^2)^2x^2 - (\ ) + 4a^2(a-b)^2 = 9b^2(a - b)^2.$

Extract the square root,

$$(9b^2 - 4a^2)x - 2a(a - b) = \pm 3b(a - b).$$

Transpose and combine,

$$(9b^2 - 4a^2)x = (2a + 3b)(a - b) \text{ or } (2a - 3b)(a - b).$$

$$\therefore x = \frac{(2a+3b)(a-b)}{(3b+2a)(3b-2a)}, \text{ or } \frac{(2a-3b)(a-b)}{(3b+2a)(3b-2a)}.$$

$$\therefore x = \frac{a-b}{3b-2a}, \text{ or } \frac{b-a}{3b+2a}.$$

NOTE. Observe that the left member of the simplified equation must be expressed in two terms, simple or compound, the first term involving x^2 and the second involving x.

Exercise 114

Solve:

1. $x^2 + ax = 6a^2$.
2. $x^2 - 3bx = 28b^2$.
3. $x^2 - 4abx = 5a^2b^2$.
4. $6x^2 + cx = 12c^2$.
5. $12x^2 + 28a^2 = 29ax$.
6. $6x^2 = 25c(x - c)$.
7. $4x^2 + ab = 2(a + b)x$.
8. $x(x - b) = b(x - a)$.
9. $bx^2 - ax = a^2$.
10. $x^2 + b^2 = ax$.
11. $x^2 - ax = b$.
12. $x^2 - ab = (a - b)x$.
13. $x^2 + 3bx = a^2 + ab - 2b^2$.
14. $4x^2 - 4ax + a^2 - b^2 = 0$.
15. $(a - x)^2 + (x - b)^2 = (a - b)^2$.
16. $ax^2 - (a^2 + 1)x + a = 0$.
17. $abx^2 - (a^2 + b^2)x + ab = 0$.
18. $bcx - bd = adx - acx^2$.
19. $x^3 = a^3$.
20. $x^3 = -b^3$.
21. $4ax^2 + 20a^2x + 9a^2 = 0$.
22. $3x^2 + ax = b$.
23. $ax^2 - 2bx = c$.
24. $ax^2 + 2bx + c = 0$.
25. $x^2 - (a + b)x + (a + c)(b - c) = 0$.
26. $x^2 - (a - b)x - (a - 1)(b - 1) = 0$.
27. $x^2 + 2ab(a^2 + b^2) = (a + b)^2x$.
28. $(a + b + c)x^2 - (2a + b + c)x + a = 0$.
29. $x + \frac{1}{x} = \frac{a - b}{a + b} + \frac{a + b}{a - b}$.
30. $abx^2 - (a + b)(ab + 1)x + (ab + 1)^2 = 0$.
31. $(a - x)^2 + (b - x)^2 = \frac{5}{2}(a - x)(b - x)$.
32. $\frac{x^3 + 3x^2 + 3x + 1}{x^3 - x^2 - x + 1} = \frac{a^2}{b^2}$.
33. $\frac{(a - x)^2 + (x - b)^2}{(a - x)^2 - (x - b)^2} = \frac{a^2 + b^2}{a^2 - b^2}$.
34. $\frac{(a - x)^3 + (x - b)^3}{(a - x)^2 + (x - b)^2} = \frac{a^3 - b^3}{a^2 + b^2}$.

35. $x^2 - a^2 = \frac{3}{2}ax.$

36. $a^2(b-x)^2 = b^2(a-x)^2.$

37. $\frac{a}{x} + \frac{x}{a} = \frac{m}{n}.$

38. $x^4 + (a-x)^4 = c.$

39. $2a^3x^3 = (ax-6)^2.$

40. $\frac{a-x}{x-b} + \frac{x-b}{a-x} = \frac{c}{d}.$

41. $(a-1)^2x^2 - a(x+b) = ax^2(a-2) - bx.$

42. $(a-b)x^2 + \frac{ab}{a-b} = (a+b)x.$

43. $(x^2+ax+a^2)(x+a) = (x^2+a^2)(x-2a) + 6a^3.$

44. $4(a-x)^4 - 17(a-x)^2(x-b)^2 + 4(x-b)^4 = 0.$

45. $x(x-2b) + a(b-x) + 2c(a-2c) + b^2 = 0.$

46. $(a-x)(b+x) = x(b-a+2x) - \frac{ab}{3}.$

47. $\frac{a-x}{x-b} + \frac{x-b}{a-x} = \frac{a}{b} + \frac{b}{a}.$

48. $\frac{bx}{c}(acx+b) + \frac{3a^2x}{c} = \frac{6a^2+ab-2b^2}{c^2}.$

49. $\frac{ax+b}{a+bx} + \frac{cx+d}{c+dx} = \frac{ax-b}{a-bx} + \frac{cx-d}{c-dx}.$

50. $\frac{a}{x-a} + \frac{b}{x-b} - \frac{2c}{x-c} = 0.$

51. $\frac{a-x}{a-b} + \frac{a-b}{a-x} = \frac{b-x}{6(a-b)} + \frac{6(a-b)}{b-x}.$

52. $(x+a)(3x-b) = (3b-a)x - (a-b)^2 - ab.$

53. $\frac{a+b+(a-b)x^2}{2x} - \frac{2x}{(a-b)+(a+b)x^2} = 0.$

54. $b(a+x)^2 = (a+b)^2 - x^2 - \frac{b^2(2a+b)}{b+1}.$

55. $x^4 + (ab+1)^2 = (a^2+b^2)(x^2+1) + 2(a^2-b^2)x + 1.$

339. Character of the Roots of a Quadratic Equation. By solving the general quadratic equation

$$ax^2 + bx + c = 0,$$

we obtain (p. 276, § 337) for its two roots

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are two roots and only two roots, since the expression $b^2 - 4ac$ *has two square roots and only two square roots.*

The *character* of the two roots depends wholly upon the value of $b^2 - 4ac$, the expression *affected by the radical sign.* Hence, the expression $b^2 - 4ac$ is called the **discriminant** of the general quadratic equation $ax^2 + bx + c = 0$.

There are three cases to be considered:

1. **$b^2 - 4ac$ positive and not zero.** In this case the roots are *real* and *different.* That the roots are different appears by writing them as follows:

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, \qquad -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

These roots cannot be equal, since $b^2 - 4ac$ is not zero.

If $b^2 - 4ac$ is a perfect square, the roots are *rational.* If $b^2 - 4ac$ is not a perfect square, the roots are *irrational.*

2. **$b^2 - 4ac$ equal to zero.** In this case the two roots are *real* and *equal,* since each is equal to $-\frac{b}{2a}$.

Hence, the roots of $ax^2 + bx + c = 0$ are real, if $b^2 \nless 4ac$.

3. **$b^2 - 4ac$ negative.** In this case both roots have a real part and an imaginary part, and are therefore *imaginary.*

If we write the roots in the form

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, \qquad -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a},$$

we see that if a quadratic has imaginary roots, these roots cannot be equal, since $b^2 - 4ac$ is not zero. They have the same real part, $-\frac{b}{2a}$, and the same imaginary parts with opposite signs.

Hence, if the roots of a quadratic are imaginary, the roots are *conjugate numbers* (p. 259, § 321).

The three cases may be summarized as follows:

1. $b^2 - 4ac > 0$, **roots real and different.**
2. $b^2 - 4ac = 0$, **roots real and equal.**
3. $b^2 - 4ac < 0$, **roots imaginary.**

340. By calculating the value of $b^2 - 4ac$ we may determine the character of the roots of a given quadratic equation without solving the equation.

1. $6x^2 + 11x - 10 = 0.$

Here $a = 6,\ b = 11,\ c = -10.$

Therefore, $b^2 - 4ac = 11^2 - 4 \times 6 \times (-10) = 121 + 240 = 361.$

Since $b^2 - 4ac$ is greater than 0, the roots are real and different. Since $b^2 - 4ac$ is a perfect square, the roots are rational.

2. $2x^2 - 9x + 3 = 0.$

Here $a = 2,\ b = -9,\ c = 3.$

Therefore, $b^2 - 4ac = (-9)^2 - 4 \times 2 \times 3 = 81 - 24 = 57.$

Since $b^2 - 4ac$ is greater than 0, the roots are real and different. Since $b^2 - 4ac$ is not a perfect square, the roots are irrational.

3. $9x^2 - 30x + 25 = 0.$

Here $a = 9,\ b = -30,\ c = 25.$

Therefore, $b^2 - 4ac = (-30)^2 - 4 \times 9 \times 25 = 900 - 900 = 0.$

Since $b^2 - 4ac$ is equal to 0, the roots are real and equal.

4. $5x^2 + 4x + 3 = 0.$

Here $a = 5,\ b = 4,\ c = 3.$

Therefore, $b^2 - 4ac = 4^2 - 4 \times 5 \times 3 = 16 - 60 = -44.$

Since $b^2 - 4ac$ is less than 0, the roots are imaginary.

5. Find the values of m for which the equation

$$mx^2 + 2x^2 + 2m = 3mx - 9x + 10$$

has its two roots equal.

We have $\quad mx^2 + 2x^2 + 2m = 3mx - 9x + 10.$

Transpose and combine,

$$(m+2)x^2 - (3m-9)x + (2m-10) = 0.$$

Here $\quad a = m+2,\ b = -(3m-9),\ c = 2m - 10.$

$\therefore\ b^2 - 4ac = [-(3m-9)]^2 - 4(m+2)(2m-10) = m^2 - 30m + 161.$

If the roots are to be equal, we must have $b^2 - 4ac = 0$,

or $\quad m^2 - 30m + 161 = 0.$

Factor, $\quad (m-7)(m-23) = 0.$

$\therefore\ m = 7$, or 23.

Exercise 115

Determine, without solving, the character of the roots of each of the following equations:

1. $x^2 + 7x + 12 = 0.$
2. $x^2 + 2x + 5 = 0.$
3. $x^2 - 4x - 2 = 0.$
4. $5x^2 - 3x - 2 = 0.$
5. $9x^2 + 12x + 4 = 0.$
6. $4x^2 - 4x - 1 = 0.$
7. $x^2 - 2x - 4 = 0.$
8. $7x^2 - 9x - 10 = 0.$
9. $25x^2 - 20x + 4 = 0.$
10. $3x^2 - 7x - 6 = 0.$
11. $x^2 + 7x + 18 = 0.$
12. $2x^2 - 13x + 15 = 0.$
13. $3x^2 - 4x - 22 = 0.$
14. $2x^2 + 3x + 8 = 0.$

Determine the values of m for which the two roots of each of the following equations are equal:

15. $(2m+1)x^2 + (7m+2)x + 6m + 1 = 0.$
16. $(3m+1)x^2 + (11m+1)x + 8m + 9 = 0.$
17. $2mx^2 + 7x^2 - 8mx + 2x + 5m + 4 = 0.$
18. $mx^2 + 4m + 1 = 2x + 3mx + 2x^2.$
19. $3m + 2x + 5x^2 = 11mx + 3 - 11mx^2.$

341. Equivalent Equations. Two equations that involve the same unknown number are called **equivalent equations,** if the solutions of each include *all the solutions of the other.*

Thus, $5x - 3b = 3x + 5b$ and $2x = 8b$ are equivalent equations, for the solution of each is $x = 4b$.

A single equation is often equivalent to two or more equations.

Thus, the equation $x^3 + 1 = 0$ may be written

$$(x+1)(x^2 - x + 1) = 0;$$

and this equation is equivalent to the two equations

$$x + 1 = 0 \quad \text{and} \quad x^2 - x + 1 = 0.$$

In solving the equation $x^3 + 1 = 0$, we should write it as $x + 1 = 0$ and $x^2 - x + 1 = 0$, and solve each of these equations.

342. *If each member of an equation is multiplied by the same factor and this factor involves an unknown number of the equation, in general new solutions are introduced.*

Thus, if we multiply $x - 2 = 0$ by $x - 3$, we obtain $(x-2)(x-3) = 0$, and introduce the solution of $x - 3 = 0$.

If, however, the multiplying factor is a denominator of a fraction of the equation, new solutions are, in general, not introduced.

Thus, $\dfrac{5}{x-1} = 3 + x$ becomes, when multiplied by $x - 1$,

$$5 = (x-1)(3+x), \text{ or } x^2 + 2x - 8 = 0.$$

Factoring, we have $(x+4)(x-2) = 0$. Whence $x = -4$, or 2.

Therefore, the solution $x = 1$ is not introduced, and this solution is the only solution that could be introduced by the factor $x - 1$.

In general, new solutions are not introduced in clearing an equation of fractions, if we proceed as follows:

1. Combine fractions that have a common denominator.
2. Reduce fractions to their lowest terms.
3. Use the L.C.M. of the denominators for the multiplier.

343. *If each member of an equation is raised to the same power, new solutions are, in general, introduced.*

Thus, if we square each member of the equation $x = 3$, we have $x^2 = 9$, or $x^2 - 9 = 0$; that is, $(x+3)(x-3) = 0$.

Therefore, the solution of $x + 3 = 0$ is introduced in squaring both members of the equation $x = 3$.

344. *In solving an equation, if each member is raised to any power, such solutions of the resulting equation must be rejected as do not satisfy the original equation.*

Solve $\sqrt{x+3} + \sqrt{x+8} = 5\sqrt{x}$.

Square, $\quad x + 3 + 2\sqrt{(x+3)(x+8)} + x + 8 = 25x$.

Transpose and combine, $\quad 2\sqrt{x^2 + 11x + 24} = 23x - 11$.

Square, $\quad 4x^2 + 44x + 96 = 529x^2 - 506x + 121$.

Transpose and combine, $\quad 525x^2 - 550x + 25 = 0$.

Divide by 25, $\quad 21x^2 - 22x + 1 = 0$.

Factor, $\quad (x-1)(21x-1) = 0$.

$$\therefore x = 1, \text{ or } \tfrac{1}{21}.$$

Of these values only the value 1 will satisfy the given equation.

Squaring both members of the original equation is equivalent to transposing $5\sqrt{x}$ to the left member and then multiplying by the rationalizing factor $\sqrt{x+3} + \sqrt{x+8} + 5\sqrt{x}$.

The result reduces to

$$2\sqrt{x^2 + 11x + 24} - (23x - 11) = 0.$$

Transposing and squaring again is equivalent to multiplying by

$$(\sqrt{x+3} - \sqrt{x+8} - 5\sqrt{x})(\sqrt{x+3} - \sqrt{x+8} + 5\sqrt{x}).$$

Therefore, the equation $21x^2 - 22x + 1 = 0$ is really obtained from

$$\begin{aligned} &(\sqrt{x+3} + \sqrt{x+8} - 5\sqrt{x}) \\ \times &(\sqrt{x+3} + \sqrt{x+8} + 5\sqrt{x}) \\ \times &(\sqrt{x+3} - \sqrt{x+8} - 5\sqrt{x}) \\ \times &(\sqrt{x+3} - \sqrt{x+8} + 5\sqrt{x}) = 0. \end{aligned}$$

This equation is satisfied by any value of x that will make any one of the *four* factors of its left member equal to zero. The first factor is 0 for $x = 1$, and the last factor is 0 for $x = \frac{1}{21}$, while no value can be found for x that will make the second factor or the third factor vanish.

Since $\frac{1}{21}$ does not satisfy the given equation but is introduced by multiplying by another equation, it is called an *extraneous value of* **x**.

Exercise 116

Solve:

1. $\sqrt{x+7}+\sqrt{5(x-2)}=3.$

2. $\sqrt{14x-11}+\sqrt{3(2x-1)}=2\sqrt{2x+1}.$

3. $\sqrt{a^2-x}+\sqrt{b^2+x}=a+b.$

4. $\sqrt{a-x}+\sqrt{b-x}=\sqrt{a+b-2x}.$

5. $2\sqrt{5+2x}-\sqrt{13-6x}=\sqrt{37-6x}.$

6. $\sqrt{2(x-3)}=\sqrt[4]{x^2-4x+11}.$

7. $\sqrt{2x+1}+\sqrt{x-3}=2\sqrt{x}.$

8. $\sqrt{5x-1}-\sqrt{8-2x}=\sqrt{x-1}.$

9. $\sqrt{4x-3}+\sqrt{5x+1}=\sqrt{15x+4}.$

10. $x\sqrt{x-a}+a\sqrt{x+a}=\sqrt{x^3+a^3}.$

11. $\sqrt{a(x-b)}+\sqrt{b(x-a)}=x.$

12. $a\sqrt{c+x}-b\sqrt{c-x}=\sqrt{c(a^2+b^2)}.$

13. $\sqrt{2a-b+2x}-\sqrt{10a-9b-6x}=4\sqrt{a-b}.$

14. $\sqrt{(a+x)(x+b)}+\sqrt{(a-x)(x-b)}=2\sqrt{ax}.$

15. $2x^2+a\sqrt{b(b+4x)}=a(b+2x).$

16. $\sqrt{x-a}-\sqrt{x-b}=\sqrt{b-a}.$

17. $2\sqrt{2a+b+2x}+\sqrt{10a+b-6x}=\sqrt{10a+9b-6x}.$

18. $\sqrt{a+b+x}-\sqrt{a+b-x}=\dfrac{x}{\sqrt{a}}.$

19. $\dfrac{a-x}{\sqrt{a-x}}+\dfrac{x-b}{\sqrt{x-b}}=\sqrt{a-b}.$

20. $\sqrt{a-x}+\sqrt{-(a^2+ax)}=\dfrac{a}{\sqrt{a-x}}.$

345. Equations in the Quadratic Form. An equation is in the *quadratic form* if it contains but two powers of the unknown number, and the exponent of one power is exactly twice that of the other.

346. Equations in the quadratic form may be solved by the methods for solving quadratics.

1. Solve $x^2 - 3x + 2 = 6\sqrt{x^2 - 3x - 3}$.

Transpose, $x^2 - 3x + 2 - 6\sqrt{x^2 - 3x - 3} = 0.$

Add -5 to each member,

$$(x^2 - 3x - 3) - 6\sqrt{x^2 - 3x - 3} = -5.$$

Complete the square, $(x^2 - 3x - 3) - (\ ) + 9 = 4.$

Extract the root, $\sqrt{x^2 - 3x - 3} - 3 = \pm 2.$

Transpose and combine, $\sqrt{x^2 - 3x - 3} = 5$, or 1.

Square, $x^2 - 3x - 3 = 25$, or 1.

Transpose and combine,

$$x^2 - 3x - 28 = 0, \text{ or } x^2 - 3x - 4 = 0.$$

Factor, $(x - 7)(x + 4) = 0$, or $(x + 1)(x - 4) = 0.$

$$\therefore x = 7, -4, -1, \text{ or } 4.$$

2. Solve $4x^4 - 12x^3 + 5x^2 + 6x - 15 = 0$.

Begin by attempting to extract the square root.

$$\begin{array}{r|l}
 & 4x^4 - 12x^3 + 5x^2 + 6x - 15 \;\big|\; 2x^2 - 3x - 1 \\
 & 4x^4 \\
4x^2 - 3x & -12x^3 + 5x^2 \\
 & -12x^3 + 9x^2 \\
4x^2 - 6x - 1 & -4x^2 + 6x - 15 \\
 & -4x^2 + 6x + 1 \\
 & -16
\end{array}$$

Hence, we see that the given equation may be written

$$(2x^2 - 3x - 1)^2 - 16 = 0.$$

Transpose, $(2x^2 - 3x - 1)^2 = 16.$

Extract the root, $2x^2 - 3x - 1 = \pm 4.$

Transpose and combine, $2x^2 - 3x = 5$, or -3.

Solving these equations, we obtain for the four values of x,

$$-1, \tfrac{5}{2}, \tfrac{1}{4}(3 \pm \sqrt{-15}).$$

EXERCISE 117

Solve:

1. $\sqrt{x^2 - 7} = x^2 - 13.$
2. $x + c + \sqrt{x + c} = a.$
3. $x^4 - 40x^2 + 144 = 0.$
4. $x^{\frac{3}{2}} + 8x^{\frac{1}{2}} = 9x.$
5. $2x^{\frac{1}{3}} - 3x^{\frac{2}{3}} + x = 0.$
6. $\sqrt[4]{x^3} - 2\sqrt{x} + x = 0.$
7. $10x^4 - 156 = x^2.$
8. $\sqrt{x^3} - 3\sqrt[4]{x^3} = 40.$
9. $8x^6 + 63x^3 = 8.$
10. $8x^{-6} + 999x^{-3} = 125.$
11. $(3x - 5)^2 - 8(3x - 5) + 7 = 0.$
12. $(2x - 3)^2 - 4(2x - 3) + 5 = 0.$
13. $(5x + 2)^2 - 6(5x + 2) + 9 = 0.$
14. $x^2 + 5 = 8x + 2\sqrt{x^2 - 8x + 40}.$
15. $2x^2 + 3\sqrt{x^2 - x + 1} = 2x + 3.$
16. $3\sqrt{3x^2 - 2x + 4} = 3x^2 - 2x - 6.$
17. $2x^2 + 3x - 7\sqrt{2x^2 + 3x - 2} + 8 = 0.$
18. $39 - 8x + \sqrt{7x^2 + 8x - 19} - 7x^2 = 0.$
19. $(2x^2 - 3x + 1)^2 = 22x^2 - 33x + 1.$
20. $x + 2a\sqrt{2(a^2 + b^2) - x} = 3a^2 + b^2.$
21. $x^4 - 6x^3 + 7x^2 + 6x - 8 = 0.$
22. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0.$
23. $32x^4 - 48x^3 - 10x^2 + 21x + 5 = 0.$
24. $x^4 - 4(a + b)x^2 + 16(a - b)^2 = 0.$
25. $x^4 - 2(a^2 + 4ab - b^2)x^2 + (a - b)^4 = 0.$

347. Relations of the Roots and the Coefficients. If we divide the general equation of the second degree $ax^2 + bx + c = 0$ by a, we obtain the equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. This equation may be written $\mathbf{x^2 + px + q = 0}$, which is called the **equation in the *p* form.**

348. By solving $x^2 + px + q = 0$, and denoting the first value of x by r_1 and the second value by r_2, we have

$$r_1 = -\frac{p}{2} + \frac{\sqrt{p^2 - 4q}}{2},$$

and
$$r_2 = -\frac{p}{2} - \frac{\sqrt{p^2 - 4q}}{2}.$$

Add, $\quad r_1 + r_2 = -p.$

Multiply, $\quad r_1 r_2 = q.$

It appears, then, that if any quadratic equation is made to take the form $x^2 + px + q = 0$, the following relations hold between the coefficients and the roots of the equation.

1. The **sum** of the two roots is equal to the coefficient of x with its sign changed.
2. The **product** of the two roots is equal to the constant term.

349. To Form the Equation when the Roots are Given. If r_1 and r_2 are the roots of the equation $x^2 + px + q = 0$, the equation may be written
$$(x - r_1)(x - r_2) = 0.$$

Form the equation of which the roots are 4 and $-\frac{5}{2}$.

The equation is $\quad (x - 4)(x + \frac{5}{2}) = 0,$

or $\quad (x - 4)(2x + 5) = 0,$

or $\quad 2x^2 - 3x - 20 = 0.$

Exercise 118

Form the equation of which the roots are:

1. 5, 4.	7. $\frac{2}{9}, -\frac{4}{9}$.	13. $a + b, a - b$.
2. 6, -3.	8. 8, $-\frac{5}{3}$.	14. $a + 2b, a - 3b$.
3. $-3, -5$.	9. $\frac{7}{11}, -\frac{3}{11}$.	15. $2 + \sqrt{3}, 2 - \sqrt{3}$.
4. $\frac{5}{2}, \frac{8}{3}$.	10. 2, 4, -3.	16. $\frac{8}{3} + \frac{2}{3}\sqrt{37}, \frac{8}{3} - \frac{2}{3}\sqrt{37}$.
5. $-1\frac{1}{3}, 2\frac{1}{3}$.	11. 5, -2, -1.	17. $2 + \sqrt{-1}, 2 - \sqrt{-1}$.
6. $-2\frac{2}{3}, -2\frac{3}{4}$.	12. $-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}$.	18. $-2 + \sqrt{-3}, -2 - \sqrt{-3}$.

350. Some Important Laws of Physics. A force that will give to a mass of one pound a velocity of one foot in one second is called a **poundal**; a force that will give to a mass of one gram a velocity of one centimeter in one second is called a **dyne.**

The *work* done in raising one pound a distance of one foot against gravity is called a **foot-pound**, and is chosen as the *unit of work.*

The power to do 33,000 foot-pounds of work a minute is called a **horse power.**

The **momentum** of a body is equal to the *mass* multiplied by the *velocity.*

1. **Falling Bodies.** (1) *The velocity acquired is directly proportional to the time of falling.*

(2) *The space described varies as the square of the time.*

(3) *The space described varies as the square of the velocity.*

Let $g =$ the constant acceleration each second, $v =$ the velocity, $t =$ the time in seconds, and $s =$ the space fallen.

Then (1) $v = gt$; (2) $s = \frac{1}{2}gt^2$; (3) $v^2 = 2gs$.

At the latitude of New York $g = 32.16$ feet, or 9.80 meters.

2. **Motion down an Inclined Plane.** The formulas of Law 1 apply if for g is substituted g multiplied by the number found by dividing the height of the plane by its length (hypotenuse).

3. **Projectiles.** *Gravity always impresses upon a body, free to move, a downward velocity of* g *units each second, whether the body starts from a state of rest or is moving already with any velocity in any direction.*

Let $u =$ the initial velocity and g, v, t, s be used as in Law 1.

(i) If the body is thrown vertically downwards,

$$(1)\ v = u + gt;\quad (2)\ s = ut + \tfrac{1}{2}gt^2.$$

(ii) If the body is thrown vertically upwards,

$$(1)\ v = u - gt;\quad (2)\ s = ut - \tfrac{1}{2}gt^2;\quad (3)\ s = \frac{u^2}{2g}.$$

4. **The Pendulum.** *The time of vibration varies as the square root of the length divided by the acceleration of gravity.*

Let t = the time of vibration in seconds, l = the length of the pendulum, g = the acceleration of gravity, and $\pi = 3.1416$.

Then $$t = \pi\sqrt{\frac{l}{g}}.$$

5. **Effect of a Constant Force.** *If a body is acted upon by a constant force so that no rotation takes place, the acceleration is equal to the moving force divided by the mass moved.*

Let a = the acceleration (or retardation), v = the velocity acquired (or destroyed), t = the time in seconds, and s = the space described.

Then (1) $v = at$; (2) $s = \frac{1}{2}vt = \frac{1}{2}at^2$; (3) $v^2 = 2as$.

6. **Force in Circular Motion.** *The centripetal force varies as the mass multiplied by the square of the velocity, and divided by the radius of the circle.*

Let m = the mass of the revolving body, F = the centripetal force in poundals or dynes, v = the velocity, r = the radius of the circle, and a = the acceleration.

Then (1) $F = \dfrac{mv^2}{r}$; (2) $F = \dfrac{4\pi^2 mr}{t^2}$.

NOTE. It takes g poundals to make a pound and g dynes to make a gram.

7. **Universal Gravitation.** *Every particle of matter in the universe attracts every other particle with a force that varies as the product of their masses divided by the square of the distance between them.*

Let m and m' = the attracting masses, d = their distance apart, F = the force of attraction, and k = the value of F when m, m', and d are each unity.

Then $$F = \frac{kmm'}{d^2}.$$

8. **Kinetic Energy.** *The kinetic energy of a mass of* m *pounds moving with a velocity of* v *feet per second is* $\frac{\mathrm{mv}^2}{2\,\mathrm{g}}$ *foot-pounds. The kinetic energy of a body is equal to the work expended in giving to the body the velocity which it possesses.*

Let $F =$ the force in poundals, $m =$ the mass, $t =$ the time in seconds, $v =$ the velocity acquired, $s =$ the space described.

Then (1) $Ft = mv$; (2) $s = \frac{1}{2}vt$; (3) $Fs = \frac{1}{2}mv^2$.

9. **Kinetic Energy of Falling Bodies.** Let $m =$ the mass of the body, $h =$ the distance it has fallen, $v =$ the velocity acquired, and $g =$ the acceleration of gravity.

Then $$mh = \frac{mv^2}{2\,g}.$$

Exercise 119

1. In (2) Law 1 find t in terms of the other letters.

2. In (3) Law 1 find v in terms of the other letters.

3. In Law 2, if h denotes the height and l the length of the inclined plane, find the three formulas that correspond to (1), (2), and (3) of Law 1. Find the values of t and the value of v in terms of g, h, l, and s.

4. In Law 3 find the values of t in terms of the other letters. Find the value of u in (3).

5. In Law 5 find the value of t in terms of a and s. Find the value of v in terms of a and s.

Find in terms of the other letters:

6. The value of l and the value of g in Law 4.

7. The value of v and the value of t in Law 6.

8. The value of d in Law 7.

9. The values of v in Law 8.

10. The value of v in Law 9.

11. How long will it take a stone starting from rest to fall 1600 feet? What velocity will the stone acquire? What is the mean velocity?

NOTE. In these examples use 32 feet for g.

12. A stone is thrown vertically upwards with a velocity of 160 feet per second. How high will it rise? How many seconds will it be in the air?

13. With what initial velocity must a bullet be fired upwards that it may rise to a height of 6400 feet?

14. A stone is thrown down a shaft 164 feet deep with an initial velocity of 50 feet per second. How many seconds will it take to reach the bottom?

15. A stone is dropped into a vertical shaft, and the sound of its striking the bottom is heard after 10 seconds. If the velocity of sound is 1120 feet per second, how deep is the shaft?

16. A stone is dropped into a vertical shaft 1024 feet deep, and the sound of its striking the bottom is heard after 8.9 seconds. Find the velocity of sound.

17. When a balloon is just a mile above the ground and is rising at the rate of 22 feet per second, a stone is thrown vertically downwards with an initial velocity of 50 feet per second. How long will it take the stone to reach the ground?

NOTE. Since the balloon is *rising* and the stone is thrown *downwards* with an initial velocity of 50 feet, the value of u is $50 - 22$, or 28, feet.

Similarly, in Example 18, the value of u is $50 + 22$, or 72, feet.

18. If the stone of the last example was thrown vertically *upwards*, how long would it take to reach the ground?

19. One body is allowed to slide down a smooth inclined plane 400 feet long and 144 feet high. Another body is allowed to fall vertically through the height of the plane. Find the velocity of each body on reaching the base of the plane, and the time required for each to fall.

351. Problems involving Quadratics. Problems that involve quadratic equations apparently have two solutions, since a quadratic equation has two roots. When both roots are positive integers, they will, in general, give two solutions.

Fractional and negative roots will, in some problems, give solutions; in other problems they will not give solutions.

No difficulty will be found in selecting the result which belongs to the particular problem we are solving.

Sometimes, by a change in the statement of the problem, we may form a new problem that corresponds to the result which was inapplicable to the original problem.

Imaginary roots will, in some problems, give solutions. Their interpretation in such cases will be given in the chapter on Complex Numbers.

1. The square of the sum of two consecutive numbers exceeds the sum of their squares by 220. Find the numbers.

Let $x =$ the smaller number.

Then $x + 1 =$ the larger number,

$2x + 1 =$ the sum of the numbers,

$(2x + 1)^2 =$ the square of the sum of the numbers,

$x^2 =$ the square of the smaller number,

$(x + 1)^2 =$ the square of the larger number,

and $x^2 + (x + 1)^2 =$ the sum of the squares of the numbers.

$$\therefore (2x + 1)^2 = x^2 + (x + 1)^2 + 220.$$

Remove parentheses, $4x^2 + 4x + 1 = x^2 + x^2 + 2x + 1 + 220.$

Transpose and combine, $2x^2 + 2x = 220.$

Complete the square, $4x^2 + 4x + 1 = 441.$

Extract the square root, $2x + 1 = \pm 21.$

Transpose and combine, $2x = 20, \text{ or } -22.$

$$\therefore x = 10, \text{ or } -11.$$

$$x + 1 = 11, \text{ or } -10.$$

The positive root of x gives for the numbers, 10 and 11.

The negative root of x is not applicable to the problem, as *consecutive numbers* are commonly understood to be integers which follow one another in the common scale 1, 2, 3, 4, $\cdots$

Therefore, the numbers required are 10 and 11.

2. A laborer worked a number of days and received for his labor \$36. Had his wages been 20 cents more per day, he would have received the same amount for two days less labor. What were his daily wages, and how many days did he work?

Let $x =$ the number of days the laborer worked.

Then $\frac{36}{x} =$ the number of dollars he received per day,

$\frac{36}{x} + \frac{1}{5} =$ the number of dollars he would have received per day if he had received 20 cents more,

$x - 2 =$ the number of days he would have worked,

and $(x-2)\left(\frac{36}{x} + \frac{1}{5}\right) =$ the number of dollars he would have received.

$$\therefore (x-2)\left(\frac{36}{x} + \frac{1}{5}\right) = 36.$$

Clear of fractions, $(x-2)(180 + x) = 180x.$

Simplify, $x^2 - 2x - 360 = 0.$

Factor, $(x-20)(x+18) = 0.$

$$\therefore x = 20, \text{ or } -18.$$

$$\frac{36}{x} = 1.80, \text{ or } -2.$$

Therefore, the laborer worked 20 days at \$1.80 per day.

Suppose the problem is changed to read: A laborer worked a number of days and received for his labor \$36. Had his wages been 20 cents *less* per day, he would have received the same amount for two days *more* labor. What were his daily wages and how many days did he work?

The algebraic statement would then be

$$(x+2)\left(\frac{36}{x} - \frac{1}{5}\right) = 36,$$

which leads to the quadratic $x^2 + 2x = 360$, the solution of which gives $x = 18$, or -20, and $\frac{36}{x} = 2$, or -1.80.

Hence, the solution -18, which is not applicable in the original problem because it is negative, is here the true result, while the solution -20 is not applicable in this problem.

Exercise 120

1. The difference between two numbers is 12, and the sum of the squares of the numbers is 1130. Find the numbers.

2. Two men, A and B, can together do a piece of work in 20 days. If B requires 9 more days than A to do the whole work, how many days would it take each alone?

3. The length of a rectangular lot is 19 feet longer than the width. If the width was 40 feet greater and the length was 33 feet greater, the area of the lot would be doubled. Find the dimensions of the lot.

4. If 11 is subtracted from a certain whole number, the result is the square of an integral number; if 24 is added to the same whole number, the result is the square of the integral number one larger than before. Find the number.

5. A company of gentlemen engaged a supper and agreed to pay $80 for it. Four of the gentlemen failed to attend and each of the rest paid $1 more than he expected to pay. How many were present at the supper?

6. If the edges of a rectangular box were increased by 2 inches, 3 inches, and 4 inches respectively, the box would become a cube and its contents would be increased by 1008 cubic inches. Find the length of each edge of the box.

7. A railway train makes a run of 799 miles in a certain time. If the average speed was reduced $4\frac{1}{2}$ miles an hour, the run would take 1 hour 48 minutes longer. How long does it take the train to make the run?

8. A body of soldiers can form a hollow square 4 men deep. If the outer side of the hollow square is diminished by 36 men, the soldiers can form a solid square. Find the number of soldiers.

9. The difference between the cubes of two consecutive numbers is 1519. Find the numbers.

10. Find the momentum and the kinetic energy of a mass of 5 pounds as it strikes the ground after falling 900 feet.

11. A mass of 10 pounds falls from rest 225 feet to the ground. Find its kinetic energy on reaching the ground.

12. A ball weighing 2 pounds is fired vertically upwards with an initial velocity of 128 feet a second. Compute its kinetic energy at the end of each second of its ascent.

13. Find the effective horse power of a steam engine that raises 100 cubic feet of water per minute to a height of 132 feet. A cubic foot of water weighs $62\frac{1}{2}$ pounds.

14. Find the time of vibration of a simple pendulum 1.63 feet long at a place where $g = 32.16$ feet.

15. Find the time of vibration of a simple pendulum 60 centimeters long at a place where $g = 980$ centimeters.

16. Find the length in meters of a simple pendulum that vibrates once a second at a place where $g = 9.8$ meters.

17. Find the length in inches of a simple pendulum that vibrates once a second at New York, where $g = 32.16$ feet.

18. If an iron ball is suspended by a fine wire 550 feet 8 inches long from the top of Washington monument, what will be the time of vibration if $g = 32.16$ feet?

19. Two masses of 48 grams and 50 grams are attached by a cord passing over a pulley. Starting from rest, each mass moves 10 centimeters in 1 second. Find g in centimeters.

20. How many times a minute must a mass of 16 pounds revolve horizontally at the end of a wire 4 feet long that the tension of the wire may equal the weight of 800 pounds?

21. A weight of $1\frac{1}{2}$ tons drops 24 feet on the head of a pile and drives the pile a distance of 9 inches. Find the resistance of the ground, neglecting the weight of the pile.

22. Find the horse power of the engine that raises the weight of Example 21 back to its original position in 3 seconds.

CHAPTER XXI

SIMULTANEOUS QUADRATIC EQUATIONS

352. Pairs of simultaneous quadratic equations that involve two unknown numbers require different methods for their solution according to the forms of the equations.

In the next three sections will be shown general methods of solving different kinds of pairs of quadratic equations.

353. Case I. When One of the Equations is a Simple Equation. The general rule is to find in the simple equation the value of one of the unknown numbers in terms of the other and then substitute this value in the other equation.

Solve $$\left.\begin{aligned} x^2 - y^2 &= 16 \\ 3x - y &= 12 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Transpose (2), $y = 3x - 12.$ (3)

Substitute the value of y in (1),

$$x^2 - 9x^2 + 72x - 144 = 16.$$

Transpose and combine, $8x^2 - 72x + 160 = 0.$

Divide by 8, $x^2 - 9x + 20 = 0.$

Factor, $(x - 4)(x - 5) = 0.$

$\therefore\ x = 4, \text{ or } 5.$

Substitute the value of x in (3), $y = 0, \text{ or } 3.$

If $x = 4,\ y = 12 - 12 = 0.$

If $x = 5,\ y = 15 - 12 = 3.$

Hence, we have the *pairs* of values $\left.\begin{aligned} x &= 4 \\ y &= 0 \end{aligned}\right\}$, or $\left.\begin{aligned} x &= 5 \\ y &= 3 \end{aligned}\right\}$.

The given equations are both satisfied by *either* pair of values; but the values $x = 4$, and $y = 3$ will not satisfy the equations, nor will the values $x = 5$, and $y = 0$.

The student must be careful to join to each value of x *the corresponding value of* y.

354. Case II. When Each Equation is Homogeneous and of the Second Degree. The general rule is to substitute vx for y in each equation, find the value of x^2 in each equation in terms of v, equate these values of x^2, and solve for v.

Solve
$$\left.\begin{aligned} x^2 + xy + 2y^2 &= 44 \\ 2x^2 - xy + y^2 &= 16 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Put $y = vx$ in (1), $\quad x^2 + vx^2 + 2v^2x^2 = 44.$ (3)

Put $y = vx$ in (2), $\quad 2x^2 - vx^2 + v^2x^2 = 16.$ (4)

From (3), $$x^2 = \frac{44}{1 + v + 2v^2}.$$

From (4), $$x^2 = \frac{16}{2 - v + v^2}.$$

Equate the values of x^2, $$\frac{44}{1 + v + 2v^2} = \frac{16}{2 - v + v^2}.$$

Divide by 4 and clear of fractions,
$$22 - 11v + 11v^2 = 4 + 4v + 8v^2.$$

Transpose and combine, $\quad 3v^2 - 15v + 18 = 0.$

Divide by 3, $\quad v^2 - 5v + 6 = 0.$

Factor, $\quad (v - 2)(v - 3) = 0.$

$$\therefore v = 2, \text{ or } 3.$$

If $v = 2$, $y = vx = 2x$.
Substitute in (2),
$$2x^2 - 2x^2 + 4x^2 = 16.$$
$$4x^2 = 16.$$
$$x^2 = 4.$$
$$\therefore x = \pm 2.$$
$$y = 2x = \pm 4.$$

If $v = 3$, $y = vx = 3x$.
Substitute in (2),
$$2x^2 - 3x^2 + 9x^2 = 16.$$
$$8x^2 = 16.$$
$$x^2 = 2.$$
$$\therefore x = \pm\sqrt{2}.$$
$$y = 3x = \pm 3\sqrt{2}.$$

Therefore, $x = 2$, and $y = 4$; $x = -2$, and $y = -4$; $x = \sqrt{2}$, and $y = 3\sqrt{2}$; or $x = -\sqrt{2}$, and $y = -3\sqrt{2}$.

355. Case III. When Each Equation is Symmetrical with respect to *x* and *y*; that is, when *x* and *y* are similarly involved.

Thus, the equations $x^2 + y^2 = 8xy$, $ax^2 + bxy + ay^2 = 0$, $2xy - 3x - 3y + 1 = 0$, $x^4 - 3x^3y + 5x^2y^2 - 3xy^3 + y^4 = 0$, are symmetrical equations.

In this case the general rule is to combine the equations in such a manner as to remove the highest powers of x and y.

Solve
$$\left.\begin{aligned} x^3 + y^3 &= 9xy \\ x + y &= 6 \end{aligned}\right\} \quad \begin{aligned}(1)\\(2)\end{aligned}$$

Cube (2), $x^3 + 3x^2y + 3xy^2 + y^3 = 216.$ (3)

Subtract (1) from (3), $3x^2y + 3xy^2 = 216 - 9xy.$

Factor, $3xy(x + y) = 216 - 9xy.$ (4)

Substitute the value of $x + y$ from (2) in (4),

$18xy = 216 - 9xy.$

Transpose and combine, $27xy = 216.$

Divide by 27, $xy = 8.$ (5)

Square (2), $x^2 + 2xy + y^2 = 36.$ (6)

Multiply (5) by 4, $4xy = 32.$ (7)

Subtract (7) from (6), $x^2 - 2xy + y^2 = 4.$

Extract the square root, $x - y = \pm 2.$ (8)

Add (2) and (8), $2x = 8$, or 4.

$\therefore x = 4$, or 2.

Subtract (8) from (2), $2y = 4$, or 8.

$\therefore y = 2$, or 4.

Therefore, $x = 4$, and $y = 2$; or $x = 2$, and $y = 4$.

356. The preceding three cases are *general methods* for the solution of equations that belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may be found more easily than by the general method. A few illustrations will be given.

1. Solve
$$\left.\begin{aligned} x^2 + y^2 &= 41 \\ x + y &= 9 \end{aligned}\right\} \quad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (2), $x^2 + 2xy + y^2 = 81.$ (3)

Subtract (1) from (3), $2xy = 40.$ (4)

Subtract (4) from (1), $x^2 - 2xy + y^2 = 1.$

Extract the square root, $x - y = \pm 1.$ (5)

Add (2) and (5), $2x = 10$, or 8.

$\therefore x = 5$, or 4.

Substitute the value of x in (2), $y = 4$, or 5.

Therefore, $x = 5$, and $y = 4$; or $x = 4$, and $y = 5$.

2. Solve
$$\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{1}{2}\\ \frac{1}{x^2}+\frac{1}{y^2}&=\frac{5}{36}\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Square (1), $\frac{1}{x^2}+\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{4}.$ (3)

Subtract (2) from (3), $\frac{2}{xy}=\frac{1}{9}.$ (4)

Subtract (4) from (2), $\frac{1}{x^2}-\frac{2}{xy}+\frac{1}{y^2}=\frac{1}{36}.$

Extract the square root, $\frac{1}{x}-\frac{1}{y}=\pm\frac{1}{6}.$ (5)

Add (1) and (5), $\frac{2}{x}=\frac{2}{3}$, or $\frac{1}{3}.$

$\therefore x=3$, or 6.

Substitute the value of x in (1), $y=6$, or 3.

Therefore, $x=3$, and $y=6$; or $x=6$, and $y=3$.

3. Solve
$$\left.\begin{aligned}x^2+y^2+x+y&=18\\ xy&=\ \ 6\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Multiply (2) by 2, $2xy=12.$ (3)

Add (1) and (3), $x^2+2xy+y^2+x+y=30.$

$(x+y)^2+(x+y)=30.$

Transpose, $(x+y)^2+(x+y)-30=0.$

Factor, $[(x+y)-5][(x+y)+6]=0.$

$\therefore x+y=5$, or -6. (4)

Square (4), $x^2+2xy+y^2=25$, or 36. (5)

Multiply (2) by 4, $4xy=24.$ (6)

Subtract (6) from (5), $x^2-2xy+y^2=1$, or 12.

Extract the square root, $x-y=\pm 1$, or $\pm 2\sqrt{3}$. (7)

Add (4) and (7), $2x=6$, 4, or $-6\pm 2\sqrt{3}$.

$\therefore x=3$, 2, or $-3\pm\sqrt{3}$.

Substitute the value of x in (2), $y=2$, 3, or $-3\mp\sqrt{3}$.

Therefore, $x=3$, and $y=2$; $x=2$, and $y=3$; $x=-3+\sqrt{3}$, and $y=-3-\sqrt{3}$; or $x=-3-\sqrt{3}$, and $y=-3+\sqrt{3}$.

4. Solve
$$\left.\begin{aligned}x^3-y^3&=19\\ x-y&=\ \ 1\end{aligned}\right\}\qquad\begin{aligned}(1)\\(2)\end{aligned}$$

Divide (1) by (2), $x^2 + xy + y^2 = 19.$ (3)

Square (2), $x^2 - 2xy + y^2 = 1.$ (4)

Subtract (4) from (3), $3xy = 18.$

$\therefore xy = 6.$ (5)

Add (3) and (5), $x^2 + 2xy + y^2 = 25.$

Extract the square root, $x + y = \pm 5.$ (6)

Add (6) and (2), $2x = 6, \text{ or } -4.$

$\therefore x = 3, \text{ or } -2.$

Substitute the value of x in (2), $y = 2, \text{ or } -3.$

Therefore, $x = 3$, and $y = 2$; or $x = -2$, and $y = -3$.

5. Solve
$$\left.\begin{aligned} x^2 + xy + 2y^2 &= 44 \quad (1)\\ 2x^2 - xy + y^2 &= 16 \quad (2)\end{aligned}\right\}$$

Multiply (1) by 4, $4x^2 + 4xy + 8y^2 = 176.$ (3)

Multiply (2) by 11, $22x^2 - 11xy + 11y^2 = 176.$ (4)

Subtract (3) from (4), $18x^2 - 15xy + 3y^2 = 0.$

Divide by 3, $6x^2 - 5xy + y^2 = 0.$

Factor, $(2x - y)(3x - y) = 0.$

$\therefore 2x - y = 0, \text{ or } 3x - y = 0.$

$\therefore y = 2x, \text{ or } 3x.$ (5)

Substitute the value of y in (2),

$$2x^2 - 2x^2 + 4x^2 = 16, \text{ or } 2x^2 - 3x^2 + 9x^2 = 16.$$

$\therefore x^2 = 4, \text{ or } 2.$

$\therefore x = \pm 2, \text{ or } \pm\sqrt{2}.$

Substitute the value of x in (5), $y = \pm 4, \text{ or } \pm 3\sqrt{2}.$

Therefore, $x = 2$, and $y = 4$; $x = -2$, and $y = -4$; $x = \sqrt{2}$, and $y = 3\sqrt{2}$; or $x = -\sqrt{2}$, and $y = -3\sqrt{2}$.

Exercise 121

Solve:

1. $\left.\begin{aligned} x^2 + y^2 &= 40\\ x &= 3y\end{aligned}\right\}$

2. $\left.\begin{aligned} 3x - y &= 5\\ xy - x &= 0\end{aligned}\right\}$

3. $\left.\begin{aligned} xy &= 12\\ 2x + 3y &= 18\end{aligned}\right\}$

4. $\left.\begin{aligned} 5x^2 + y &= 3xy\\ 2x - y &= 0\end{aligned}\right\}$

5. $\left.\begin{aligned} x^2 - xy + y^2 &= 7\\ 2x - 3y &= 0\end{aligned}\right\}$

6. $\left.\begin{aligned} x^2 - xy - 2y^2 &= 7\\ x - y &= 3\end{aligned}\right\}$

7. $\left.\begin{array}{l} 6x^2 - 13xy + 6y^2 = 26 \\ x - 2y + 1 = 0 \end{array}\right\}$

8. $\left.\begin{array}{l} x + xy = 35 \\ y + xy = 32 \end{array}\right\}$

9. $\left.\begin{array}{l} 2x^2 - 3xy + 5y = 5 \\ (x-2)(y-1) = 0 \end{array}\right\}$

10. $\left.\begin{array}{l} x^3 - y^3 = 152 \\ x - y = 2 \end{array}\right\}$

11. $\left.\begin{array}{l} x^3 + y^3 = 152 \\ x + y = 8 \end{array}\right\}$

12. $\left.\begin{array}{l} x^2 + xy = 77 \\ xy + y^2 = 44 \end{array}\right\}$

13. $\left.\begin{array}{l} x^2 - xy = 45 \\ xy - y^2 = -36 \end{array}\right\}$

14. $\left.\begin{array}{l} x^2 + xy = 55 \\ y^2 + xy = 66 \end{array}\right\}$

15. $\left.\begin{array}{l} x^2 - y^2 = 40 \\ xy = 21 \end{array}\right\}$

16. $\left.\begin{array}{l} x^2 + y^2 = 250 \\ x - y = 4 \end{array}\right\}$

17. $\left.\begin{array}{l} x - y = 5 \\ xy = 36 \end{array}\right\}$

18. $\left.\begin{array}{l} x^2 - xy + y^2 = 39 \\ 2x^2 - 3xy + 2y^2 = 43 \end{array}\right\}$

19. $\left.\begin{array}{l} x + xy + y = 5 \\ x^2 + xy + y^2 = 7 \end{array}\right\}$

20. $\left.\begin{array}{l} \frac{1}{x} + \frac{1}{y} = \frac{3}{2} \\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4} \end{array}\right\}$

21. $\left.\begin{array}{l} \frac{x+1}{y+1} = 2 \\ \frac{x^2+1}{y^2+1} = 5 \end{array}\right\}$

22. $\left.\begin{array}{l} x - y = 3 \\ x^3 - y^3 = 387 \end{array}\right\}$

23. $\left.\begin{array}{l} x^2 + y^2 + x + y = 18 \\ x^2 - y^2 + x - y = 6 \end{array}\right\}$

24. $\left.\begin{array}{l} 2x^2 - 5xy + 3x - 2y = 10 \\ 5xy - 2x^2 + 7x - 8y = 10 \end{array}\right\}$

25. $\left.\begin{array}{l} 5x + y + 3 = 2xy \\ xy - 2x + y = 9 \end{array}\right\}$

26. $\left.\begin{array}{l} x^3 + y^3 = 37 \\ x^2 - xy + y^2 = 37 \end{array}\right\}$

27. $\left.\begin{array}{l} x^2 + y = 5(x - y) \\ x + y^2 = 2(x - y) \end{array}\right\}$

28. $\left.\begin{array}{l} x^2 + y^2 + x + y = 58 \\ xy + 14 = 0 \end{array}\right\}$

29. $\left.\begin{array}{l} x^2 + y^2 = xy + 103 \\ x + y = xy - 79 \end{array}\right\}$

30. $\left.\begin{array}{l} x^2 + y^2 = 57 + xy \\ x + y = xy - 41 \end{array}\right\}$

31. $\left.\begin{array}{l} x + xy + y = 11 \\ x^2 + x^2y^2 + y^2 = 49 \end{array}\right\}$

32. $\left.\begin{array}{l} x^2 + y^2 - 12 = x + y \\ xy + 8 = 2(x + y) \end{array}\right\}$

33. $\left.\begin{array}{l} x^2 - xy + y^2 = 37 \\ x^2 - y^2 = 40 \end{array}\right\}$

34. $\left.\begin{array}{l} x^4 + x^2y^2 + y^4 = 21 \\ x^2 + xy + y^2 = 7 \end{array}\right\}$

35. $\left.\begin{array}{l} x^2 - 5xy + y^2 + 5 = 0 \\ xy - x - y + 1 = 0 \end{array}\right\}$

36. $\left.\begin{array}{l} x^2 + 2xy - y^2 = 41 \\ 3x^2 + 2xy + 2y^2 = 103 \end{array}\right\}$

37. $\left.\begin{array}{l} 2x^2 - xy + 5y^2 = 10(x + y) \\ x^2 + 4xy + 3y^2 = 14(x + y) \end{array}\right\}$

38. $\left.\begin{array}{l} 8x + 8y - x^2 - xy = 10 \\ 5x + 5y - xy - y^2 = 20 \end{array}\right\}$

39. $\left.\begin{array}{l} 4x^2 - 9y^2 = 0 \\ 4x^2 + y^2 = 8(x + y) \end{array}\right\}$

40. $\left.\begin{array}{l} x^2 - 5y^2 - 3x - y + 22 = 0 \\ (x - 3)(y - 2) = y^2 - 3y + 2 \end{array}\right\}$

41. $\left.\begin{array}{l} x^2 + y^2 = xy + 37 \\ x + y = xy - 17 \end{array}\right\}$

42. $\left.\begin{array}{l} x^4 + y^4 = 97 \\ x + y = 5 \end{array}\right\}$

43. $\left.\begin{array}{l} x^4 + y^4 = 272 \\ x - y = 2 \end{array}\right\}$

44. $\left.\begin{array}{l} x^5 + y^5 = 33 \\ x + y = 3 \end{array}\right\}$

45. $\left.\begin{array}{l} x^5 - y^5 = 211 \\ x - y = 1 \end{array}\right\}$

46. $\left.\begin{array}{l} x^4 + y^4 = 162 \\ x^2 + y^2 = 3xy - 9 \end{array}\right\}$

47. $\left.\begin{array}{l} \dfrac{x + y}{xy - 1} = \dfrac{7}{11} \\ \dfrac{y - x}{xy + 1} = \dfrac{1}{13} \end{array}\right\}$

48. $\left.\begin{array}{l} \dfrac{x + x^2}{y + y^2} = \dfrac{14}{3} \\ \dfrac{y + x^2}{x + y^2} = \dfrac{13}{4} \end{array}\right\}$

49. $\left.\begin{array}{l} x^2 + y^2 = x^2y^2 + 1 \\ x + y = 2xy - 2 \end{array}\right\}$

50. $\left.\begin{array}{l} x^3 + y^3 = -147 - 3xy \\ x^2 + y^2 = xy + 39 \end{array}\right\}$

51. $\left.\begin{array}{l} x + y = 8 \\ x^4 + y^4 = 706 \end{array}\right\}$

52. $\left.\begin{array}{l} x + y = 5 \\ x^5 + y^5 = 275 \end{array}\right\}$

53. $\left.\begin{array}{l} x^3 + 3xy^2 + 171 = 0 \\ 3x^2y + y^3 + 172 = 0 \end{array}\right\}$

54. $\left.\begin{array}{l} x^4 + y^4 = 97 \\ x^2 + y^2 = 49 - x^2y^2 \end{array}\right\}$

55. $\left.\begin{array}{l} 16x^2 + 16y = 17 \\ 4x + 4y^2 = 5 \end{array}\right\}$

56. $\left.\begin{array}{l} x + xy + y = 5 \\ x^3 + x^3y^3 + y^3 = 17 \end{array}\right\}$

57. $\left.\begin{array}{l} xy = a \\ \dfrac{x}{y} = b \end{array}\right\}$

58. $\left.\begin{array}{l} x^2 + xy = a \\ y^2 + xy = b \end{array}\right\}$

59. $\left.\begin{array}{l} x^2y = a \\ xy^2 = b \end{array}\right\}$

60. $\left.\begin{array}{l} x + y = a(x^2 + y^2) \\ x - y = b(x^2 + y^2) \end{array}\right\}$

61. $\left.\begin{array}{l} x + y = a \\ x^3 + y^3 = b \end{array}\right\}$

62. $\left.\begin{array}{l} \dfrac{a - x}{b - y} + \dfrac{b - y}{a - x} = \dfrac{34}{15} \\ x - y = 3(a - b) \end{array}\right\}$

63. $\left.\begin{array}{l} x + y = a \\ \dfrac{x}{b - y} + \dfrac{b - y}{x} = \dfrac{5}{2} \end{array}\right\}$

64. $\left.\begin{array}{l} x^3 + xy^2 = a \\ y^3 + x^2y = b \end{array}\right\}$

65. $\left.\begin{array}{l} x + y^2 = ax \\ x^2 + y = by \end{array}\right\}$

66. $\left.\begin{array}{l} x + y^2 = ay^2 \\ x^2 + y = bx^2 \end{array}\right\}$

67. $\left.\begin{array}{l} x + y = a \\ x^3 + y^3 = bxy \end{array}\right\}$

68. $\left.\begin{array}{l} x^3 + y^3 = a(x + y) \\ x^4 + y^4 = b(x + y)^2 \end{array}\right\}$

69. $\left.\begin{array}{l} x^2 - y^2 = a \\ x^3 + y^3 = b(x - y) \end{array}\right\}$

70. $\left.\begin{array}{l} x + y = a \\ x^4 + y^4 = b \end{array}\right\}$

71. $\left.\begin{array}{l} (x + y)xy = a \\ x^5 + y^5 = bxy \end{array}\right\}$

72. $\left.\begin{array}{l} x^4 + x^2y^2 + y^4 = a \\ x^2 - xy + y^2 = 1 \end{array}\right\}$

73. $\left.\begin{array}{l} \dfrac{x + y}{1 + xy} = \dfrac{2a}{1 + a^2} \\ \dfrac{x - y}{1 - xy} = \dfrac{2b}{1 + b^2} \end{array}\right\}$

74. $\left.\begin{array}{l} \dfrac{x + y}{1 - xy} = \dfrac{2a}{1 - a^2} \\ \dfrac{x - y}{1 + xy} = \dfrac{2b}{1 - b^2} \end{array}\right\}$

75. $\left.\begin{array}{l} ax^2 + by^2 = cx^3 \\ cx^2 - dy^2 = ax \end{array}\right\}$

76. $\left.\begin{array}{l} x^2 + xy + y^2 = 2a \\ x^2 - xy + y^2 = 2b \end{array}\right\}$

77. $\left.\begin{array}{l} x^2 - xy + y^2 = 2a \\ x^4 - x^2y^2 + y^4 = 2b \end{array}\right\}$

78. $\left.\begin{array}{l} a(x - y) = b(x + y) \\ xy = a(x - y) \end{array}\right\}$

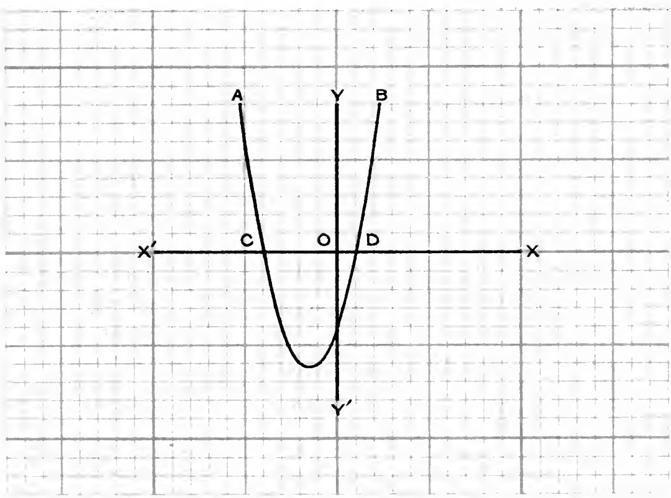
357. Graphs of Quadratic Equations. The graph of any given quadratic equation in x and y may be drawn by the use of the method shown in the solution of the following example.

Plot the graph of the function $x^2 + 3x - 4$.

Put $x^2 + 3x - 4 = y$. Then $x = \frac{1}{2}(-3 \pm \sqrt{25 + 4y})$.

If $y = -6\frac{1}{4}$, $x = -1.5$;
$y = -6$, $x = -1$ or -2;
$y = -5$, $x = -0.38$ or -2.62;
$y = -4$, $x = 0$ or -3;
$y = -3$, $x = +0.30$ or -3.30;
$y = -2$, $x = +0.56$ or -3.56;
$y = -1$, $x = +0.79$ or -3.79.

If $y = 0$, $x = +1$ or -4;
$y = +1$, $x = +1.19$ or -4.19;
$y = +2$, $x = +1.37$ or -4.37;
$y = +3$, $x = +1.54$ or -4.54;
$y = +4$, $x = +1.70$ or -4.70;
$y = +6$, $x = +2$ or -5;
$y = +8$, $x = +2.27$ or -5.27.



Plot the points found $(-1.5, -6\frac{1}{4})$, $(-1, -6)$, $(-2, -6)$, and so on. Through these points with a free hand draw the smooth curve AB. The curve AB is the graph of the function $x^2 + 3x - 4$. This graph consists of one symmetrical branch of infinite length and is called a *parabola*. For values of y less than $-6\frac{1}{4}$, the corresponding values of x are imaginary.

When $y=0$, then $x=1$ and -4, the roots of the equation $x^2-3x-4=0$.

To solve an equation in x it is necessary only to find the points in which the graph cuts the axis of x. The abscissas of these points are the roots of the given equation.

358. A more rapid method of solving a quadratic by the use of graphs is shown in the solution of the following equation.

Solve the equation $x^2 - x - 2 = 0$.

Let $x^2 = y$, and put y for x^2. Then the equation becomes $y - x - 2 = 0$.

The graph of the equation $y - x - 2 = 0$ is the straight line AB, as shown in the left-hand figure on page 307, and the graph of the equation $x^2 = y$ is the parabola CD.

The abscissas of the intersections of AB and CD are 2 and -1, which are the roots of the given equation, as may be shown by solving it.

The great advantages of this method are that the same parabola may be used in the solution of different equations, and that it takes less time to construct the graph of an equation of the first degree than of the second degree.

Exercise 122

Plot the graphs of the following functions:

1. $x^2 + 5x + 4$.
2. $x^2 + x - 2$.
3. $x^2 - 7x + 6$.
4. $x^2 - 5x + 6$.
5. $2x^2 - 7x + 5$.
6. $3x^2 + 4x - 4$.

Find, by the method of graphs, the roots of:

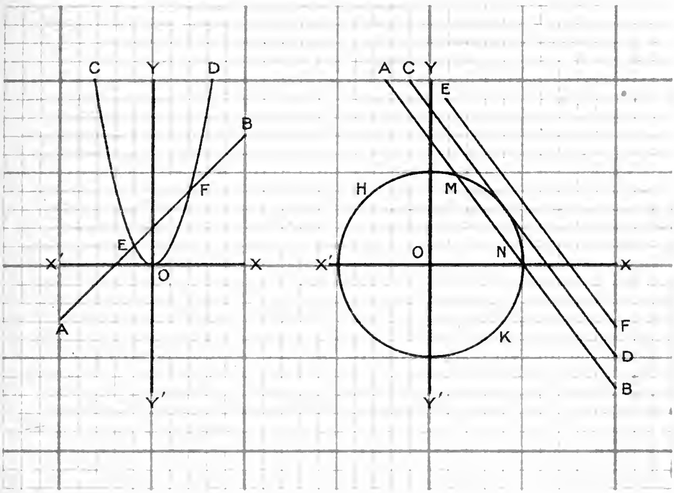
7. $x^2 - x - 6 = 0$.
8. $2x^2 - 9x + 9 = 0$.
9. $5x^2 - 20 = 0$.
10. $7x^2 + 14x - 21 = 0$.
11. $4x^2 - 12x + 9 = 0$.
12. $25x^2 + 60x + 36 = 0$.

Find approximately the roots of:

13. $3x^2 - 2x - 7 = 0$.
14. $5x^2 - 3x - 30 = 0$.
15. $5x^2 - 7x - 1 = 0$.
16. $7x^2 + 5x - 31 = 0$.

17. What is the nature of the roots of the equation $3x^2 + 2x + 4 = 0$? Construct the graph of the equation.

18. How does the graph of a quadratic equation indicate the fact that the roots of the equation are real and unequal? real and equal? imaginary?



359. Solution of Simultaneous Quadratic Equations. In general, the method of solving simultaneous linear equations (p. 206, § 226) should be followed.

Solve $\left.\begin{aligned} x^2 + \ y^2 &= 25 \\ 4x \ + 3y \ &= 20 \end{aligned}\right\}$ (1) (2)

In (1), if $x = \ 0$, $y = \pm 5$; $x = \pm 3$, $y = \pm 4$;
$x = \pm 1$, $y = \pm 4.90$; $x = \pm 4$, $y = \pm 3$;
$x = \pm 2$, $y = \pm 4.58$; $x = \pm 5$, $y = \ 0$.

If $x > +5$ or < -5, the value of y is imaginary.

Equation (1) is symmetrical; its graph also is symmetrical and is the circle HK, as shown in the right-hand figure. The graph of (2) is the straight line AB intersecting the circle at $M\,(1\frac{2}{5}, 4\frac{4}{5})$ and $N\,(5, 0)$.

Hence, the solution gives $x = 5$, $y = 0$ or $x = 1\frac{2}{5}$, $y = 4\frac{4}{5}$.

The straight line CD is the graph of $4x + 3y = 25$. (3)

The solution of (1) and (3) gives the double solution $x = 4$, $y = 3$.

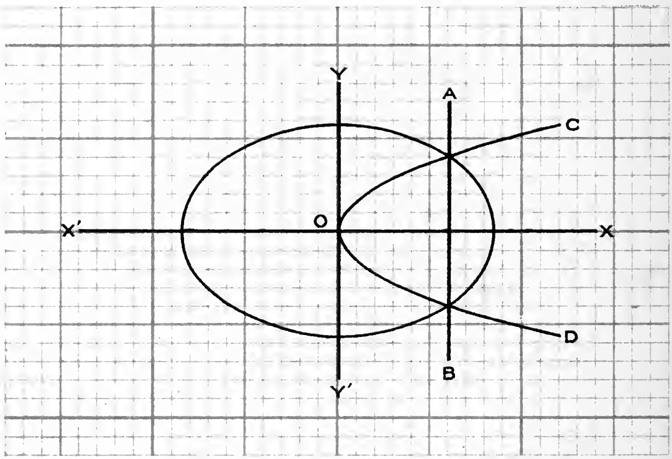
The straight line EF is the graph of $4x + 3y = 30$. (4)

The solution of (1) and (4) gives imaginary roots, since the graphs do not intersect.

EXERCISE 123

1. What does the right-hand figure on page 307 show about the relation of AB, CD, and EF? How do the coefficients of x and y in equations (2), (3), and (4) show this? Are the graphs of $ax + by + c = 0$ and $ax + by + d = 0$ parallel?

2. Write the equations of two parallel lines and construct their graphs.



3. If a straight line and a circle touch each other, how many values has x? how many has y? How many values have x and y when the line cuts the circle? What is the nature of the roots of two equations when the graphs do not intersect?

Solve exactly or approximately by the method of graphs:

4. $\left.\begin{array}{l} x^2 + y^2 - 169 = 0 \\ 3x - 2y + 9 = 0 \end{array}\right\}$

5. $\left.\begin{array}{l} x^2 + y^2 = 100 \\ 5x + y = 46 \end{array}\right\}$

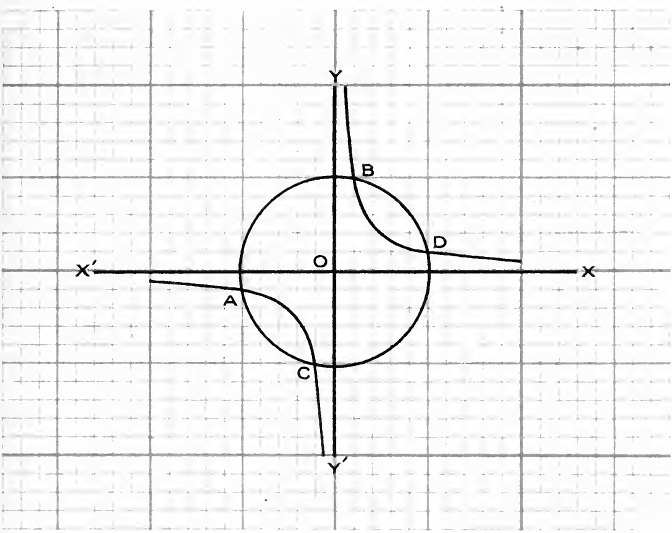
6. $\left.\begin{array}{l} x^2 + y^2 = 100 \\ 3x + 4y = 50 \end{array}\right\}$

7. $\left.\begin{array}{l} x^2 + y^2 = 100 \\ 3x + 4y = 60 \end{array}\right\}$

8. Solve the equations of Example 7 as simultaneous equations and explain why their graphs do not intersect.

9. The figure on page 308 shows the graph of the ellipse $4x^2 + 9y^2 = 288$, and the graph of the parabola $3y^2 = 8x$. What roots satisfy these equations?

10. The equation of the circle $ax^2 + ay^2 = c$ differs in what respect from the equation of the ellipse $ax^2 + by^2 = c$? What is the shape of the ellipse when a and b differ greatly in value? when a and b are nearly equal? when a and b are equal?



11. The figure on this page shows the graph of the circle $x^2 + y^2 = 26$, and the graph of the hyperbola $xy = 5$. What are the coördinates of their points of intersection? What roots satisfy the equations?

12. Solve $x^2 + y^2 = 26$ and $xy = 5$ as simultaneous quadratics and notice that the results are the answers to Example 11.

Solve by graphs:

13. $\left.\begin{array}{r} x^2 + y^2 = 80 \\ xy = 32 \end{array}\right\}$

14. $\left.\begin{array}{r} x^2 + y^2 = 34 \\ xy = 15 \end{array}\right\}$

15. $\left.\begin{array}{r} x^2 + y^2 = 74 \\ 3x^2 + y^2 = 172 \end{array}\right\}$

16. $\left.\begin{array}{r} 5x^2 + y^2 = 321 \\ 5y^2 - 196x = 0 \end{array}\right\}$

17. $\left.\begin{array}{r} x^2 + 6y^2 = 79 \\ 5x^2 - 4y^2 = 89 \end{array}\right\}$

18. $\left.\begin{array}{r} 3x^2 - 5y^2 = 43 \\ xy = 4 \end{array}\right\}$

19. Explain the meaning of the imaginary roots of the equations in Example 18.

NOTE. The equations in Example 18 are types of the two simple hyperbola equations. The difference in form is due to the difference in the position of the curves.

Determine by inspection the shape of the graph of:

20. $2x - 11y = 7$.

21. $5x^2 + 8y^2 = 6$.

22. $y^2 = 8x$.

23. $x^2 + y^2 = 18$.

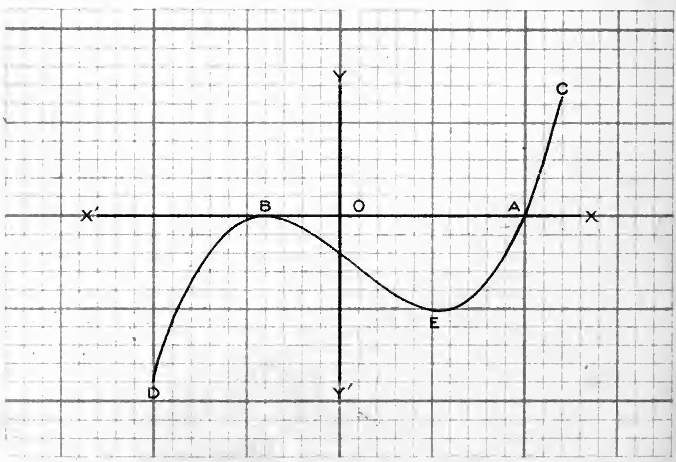
24. $x = 3y$.

25. $2x^2 = 7y$.

26. $4x^2 + 4y^2 = 27$.

27. $xy = 12$.

28. $2x^2 - 5y^2 = 12$.



360. Graphs of Higher Equations. The graphs of equations and functions of higher degree than the second may be plotted by the method already shown (p. 305, § 357).

In general, the number of real roots of an equation in x is equal to the number of times the graph cuts the axis of x. If the graph is tangent to the axis of x, there is a double root or a multiple root; if the graph does not cut or touch the axis of x, the roots are imaginary.

Plot the graph of the function $x^3 - x^2 - 16x - 20$.

Put $x^3 - x^2 - 16x - 20 = y$.

If $x = +6$,	$y = +64$;	If $x = +0.5$,	$y = -28.13$;
$x = +5.5$,	$y = +28.13$;	$x = 0$,	$y = -20$;
$x = +5$,	$y = 0$;	$x = -0.5$,	$y = -12.38$;
$x = +4.5$,	$y = -21.13$;	$x = -1$,	$y = -6$;
$x = +4$,	$y = -36$;	$x = -1.5$,	$y = -1.63$;
$x = +3.5$,	$y = -45.38$;	$x = -2$,	$y = 0$;
$x = +3$,	$y = -50$;	$x = -2.5$,	$y = -1.88$;
$x = +2.7$,	$y = -50.81$;	$x = -3$,	$y = -8$;
$x = +2.5$,	$y = -50.63$;	$x = -3.5$,	$y = -19.13$;
$x = +2$,	$y = -48$;	$x = -4$,	$y = -36$;
$x = +1.5$,	$y = -42.88$;	$x = -4.5$,	$y = -59.38$;
$x = +1$,	$y = -36$.	$x = -5$,	$y = -90$.

To make the figure compact use two spaces of the coördinate paper for one unit of x, and one space for ten units of y. The curve $CAEBD$ (p. 310) is the graph of the function $x^3 - x^2 - 16x - 20$. The graph shows that the roots of the equation $x^3 - x^2 - 16x - 20 = 0$ are 5, -2, and -2.

It is evident that for values of x greater than 6 the curve extends indefinitely above XX', and for values less than -5 indefinitely below XX'.

To determine more accurately the shape of the curve, it is often desirable to assume for x several values between two consecutive units.

Exercise 124

Find by a graph the roots of:

1. $x^3 - x^2 - 12x = 0$.
2. $2x^3 - x^2 - 26x + 40 = 0$.

Find by a graph the number of real roots of:

3. $x^3 - 8 = 0$.
4. $x^3 - 5x^2 + 8x + 14 = 0$.

Exercise 125

1. The sum of the squares of two numbers is 130 and twice the product of the numbers is 126. Find the numbers.

2. The sum of two numbers added to the sum of their squares is 686, and the difference of the numbers added to the difference of their squares is 74. Find the numbers.

3. If a number of two digits is divided by the product of the digits, the quotient is 3. If the digits are reversed and the resulting number is divided by the sum of the digits, the quotient is 7. Find the number.

4. The product of two numbers is 91 greater than ten times the first number, and 51 greater than ten times the second number. Find the numbers.

5. There are two numbers formed of the same two digits in reverse order. The sum of the numbers is 55 times the difference between the two digits, and the difference between the squares of the two numbers is 1980. Find the numbers.

6. Divide 16,120 into two parts such that the sum of their cube roots shall be 40.

7. If a number of two digits is divided by the product of the digits, the quotient is 5 and the remainder 2. If the order of the digits is reversed and the resulting number divided by the product of the digits, the quotient is 2 and the remainder 5. Find the number.

8. A walks a quarter of a mile an hour faster than B, and, in consequence, requires a quarter of an hour less time to walk 15 miles. Find the rate at which each walks.

9. Two square gardens have together a surface of 3469 square yards. A rectangular garden whose dimensions are respectively equal to the sides of the two squares contains $24\frac{1}{2}$ square yards less than half the area of the two squares together. Find the sides of the two squares.

10. The sum of the squares of two numbers is 370. If the smaller number was 1 greater and the larger number 3 greater, the sum of their squares would be 500. Find the numbers.

11. The diagonal of a rectangle is 89 inches. If each side of the rectangle was 3 inches shorter, the diagonal would be 85 inches. Find the length of the sides.

12. The diagonal of a rectangle is 65 feet. If the shorter side of the rectangle was 17 feet shorter and the longer side 7 feet longer, the length of the diagonal would be unchanged. Find the length of the sides.

13. A farmer received $245 for some apples. If he had sold 10 barrels more and had received for each barrel a quarter of a dollar more, he would have received $300. How many barrels of apples did he sell, and what was the price per barrel?

14. If the speed of a railway train was increased 5 miles an hour, $37\frac{1}{2}$ minutes would be saved in making a certain run. If the speed was decreased 5 miles an hour, 50 minutes would be lost in making the same run. Find the length of the run and the rate of the train.

15. Find the fraction that is increased by $\frac{1}{40}$ when 2 is added to the numerator and 3 to the denominator, and is diminished by $\frac{1}{10}$ when 2 is subtracted from the numerator and 3 from the denominator.

16. Find the number of two digits which is 4 less than the sum of the squares of its digits, and is 5 greater than twice the product of the digits.

17. Two rectangular fields were supposed to contain each 4 acres; but on accurate measurement the first field was found to contain 140 square yards more than 4 acres, and the second 160 square yards less than 4 acres. The second field was 10 yards longer and 10 yards narrower than the first. Find the dimensions of the fields.

18. The altitude of a trapezoid is 18 feet. Its area is equal to that of a rectangle with sides equal to the parallel bases of the trapezoid. Three times the smaller base added to the larger base is four times the altitude of the trapezoid. Find the two bases.

19. A certain sum at simple interest at a certain per cent amounts to \$22,781 in one year. If the sum was \$200 greater and the rate of interest $\frac{1}{4}$ per cent higher, the amount would be \$23,045. Find the sum and the rate of interest.

20. The product of the sum and the difference of two numbers is a, and the quotient of the sum divided by the difference is b. Find the numbers.

21. Find the specific gravity of gold and of copper, if a compound of 28 ounces of gold and 11 ounces of copper has a specific gravity of 14.4, and if the specific gravity of gold is 10.4 greater than that of copper.

22. Find the specific gravity of two substances A and B, if a compound of a ounces of the first and b ounces of the second has a specific gravity of m, and a compound of c ounces of the first and d ounces of the second has a specific gravity of n.

23. A farmer sold a certain number of sheep for \$286. He received for each sheep \$2 more than he paid for it and gained thereby on the cost of the sheep half as many per cent as each sheep cost him dollars. Find the number of sheep.

24. A and B together take a contract to set in type in 12 days a certain number of pages for \$90, and agree with each other to divide the \$90 in proportion to the amount each does. After working 8 days, however, they call in C to help them and finish the work on time. They still divide the \$90 in proportion to the amount each does, and B receives \$4 less than if he and A had finished the work alone. A sets just half the pages. How many days would it take B alone and C alone to set all the pages?

CHAPTER XXII

RATIO, PROPORTION, AND VARIATION

Ratio

361. Ratio of Numbers. The relative magnitude of two numbers is called their **ratio**, when expressed by the indicated quotient of the first by the second.

Thus, the ratio of a to b is $\frac{a}{b}$, or $a \div b$, or $a : b$; the quotient is generally written in the last form when it is intended to express a ratio.

362. Antecedents and Consequents. The first term of a ratio is called the **antecedent**, and the second term the **consequent.**

When the antecedent is *equal* to the consequent, the ratio is called a **ratio of equality**; when the antecedent is *greater* than the consequent, the ratio is called a **ratio of greater inequality**; when the antecedent is *less* than the consequent, the ratio is called a **ratio of less inequality.**

Thus, the ratio $5:5$ is a ratio of equality; the ratio $5:4$ is a ratio of greater inequality; the ratio $4:5$ is a ratio of less inequality.

363. Inverse Ratio. If the antecedent and the consequent of a given ratio are interchanged, the resulting ratio is called the **inverse** of the given ratio.

Thus, the ratio $3:6$ is the *inverse* of the ratio $6:3$.

364. Commensurable Magnitudes. If two magnitudes of the *same kind* are so related that a unit of measure can be found which is contained in each of the magnitudes an integral number of times, this unit of measure is called a **common measure** of the two magnitudes, and the two magnitudes are said to be **commensurable.**

Two magnitudes *different in kind* can have no ratio.

If two commensurable magnitudes are measured by the same unit, their **ratio is the ratio of their numerical measures.**

Thus, $\frac{1}{6}$ of a foot is a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet, being contained in the first 15 times and in the second 22 times.

Therefore, the ratio of $2\frac{1}{2}$ feet to $3\frac{2}{3}$ feet is the ratio 15 : 22.

365. Incommensurable Magnitudes. Two magnitudes of the same kind that cannot *both* be expressed in *integers* in terms of a common unit are said to be **incommensurable,** and the *exact value* of their ratio cannot be found. But by taking the unit sufficiently small, an *approximate value* can be found that shall differ from the true value of the ratio by less than any assigned value, however small.

Suppose a and b to be two incommensurable magnitudes of the *same kind.* Divide b into any integral number, n, of equal parts, and suppose one of these parts is contained in a more than m times and less than $m + 1$ times. Then $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$ and cannot differ from either of these by so much as $\frac{1}{n}$.

But, by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

Hence, the ratio of two incommensurable magnitudes, although it cannot be expressed *exactly* by numbers, may be expressed *approximately* to any desired degree of accuracy.

Thus, if b represents the length of the side of a square, and a the length of the diagonal, then $a : b = \sqrt{2}$. Now, $\sqrt{2} = 1.41421356 \cdots$, a value greater than 1.414213, but less than 1.414214. If, then, a *millionth part* of b is taken as the unit, the value of the ratio $a : b$ lies between 1.414213 and 1.414214, and therefore differs from either by less than 0.000001.

By carrying the decimal farther, a value may be found that will differ from the true value of the ratio by less than a *billionth*, a *trillionth*, or *by less than any other assigned value whatever.*

Hence, the ratio $a : b$, while it cannot be expressed by numbers *exactly*, may be expressed by numbers *to any degree of accuracy we please.*

366. Incommensurable Ratio. The ratio of two incommensurable magnitudes of the same kind is called an **incommensurable ratio.** An incommensurable ratio has *a definite fixed value* such that an approximate value can be found which will differ from this fixed value by a quantity whose absolute value shall be less than that of any assigned constant, however small.

367. Equal Incommensurable Ratios. As the treatment of Proportion in Algebra depends upon the assumption that it is possible to find fractions which will represent ratios, and as it appears that no fraction can be found to represent exactly the value of an incommensurable ratio, it is necessary to show that *two incommensurable ratios are equal if their approximate values remain equal when the unit of measure is indefinitely diminished.*

Thus, let $a : b$ and $a' : b'$ be two incommensurable ratios whose true values lie between the approximate values $\frac{m}{n}$ and $\frac{m+1}{n}$, when the unit of measure is indefinitely diminished. Then they cannot differ from each other by so much as $\frac{1}{n}$.

Let d denote the difference (if any) between $a : b$ and $a' : b'$; then

$$d < \frac{1}{n}.$$

Now the true values of $a : b$ and $a' : b'$ being fixed, their difference, d, must be fixed, that is, *d must be a constant.*

By increasing n we can make the value of $\frac{1}{n}$ less than any assigned value, however small; hence, $\frac{1}{n}$ can be made less than d if d is not zero.

Therefore, d is zero, and there is no difference between the ratios $a : b$ and $a' : b'$. Therefore, $a : b = a' : b'$.

368. *The value of a ratio is not changed if both terms are multiplied by the same number.*

For the ratio $a : b$ is represented by $\frac{a}{b}$, and the ratio $ma : mb$ by $\frac{ma}{mb}$.
Since $\frac{ma}{mb} = \frac{a}{b}$, therefore, $ma : mb = a : b$.

369. *The value of a ratio is changed if its terms are multiplied by different positive numbers; and is increased or diminished according as the multiplier of the antecedent is greater than or less than that of the consequent.*

For $ma : nb >$ or $< a : b$,

according as $\frac{ma}{nb} >$ or $< \frac{a}{b}\left(= \frac{na}{nb}\right)$,

according as $ma >$ or $< na$,

according as $m >$ or $< n$.

370. *The value of a ratio of greater inequality is diminished, and of a ratio of less inequality increased, if the same positive number is added to both its terms.*

For $a + m : b + m >$ or $< a : b$,

according as $\frac{a + m}{b + m} >$ or $< \frac{a}{b}$,

according as $ab + bm >$ or $< ab + am$,

according as $bm >$ or $< am$,

according as $b >$ or $< a$.

371. *The value of a ratio of greater inequality is increased, and of a ratio of less inequality diminished, if the same positive number is subtracted from both its terms.*

For $a - m : b - m >$ or $< a : b$,

according as $\frac{a - m}{b - m} >$ or $< \frac{a}{b}$,

according as $ab - bm >$ or $< ab - am$,

according as $am >$ or $< bm$,

according as $a >$ or $< b$.

372. Ratios are **compounded** by taking the product of the fractions that represent them.

Thus, the ratio compounded of $a : b$ and $c : d$ is $ac : bd$.

The ratio compounded of $a : b$ and $a : b$ is called the **duplicate** ratio $a^2 : b^2$.

The ratio compounded of $a : b$, $a : b$, and $a : b$ is called the **triplicate** ratio $a^3 : b^3$; and so on.

373. Ratios are **compared** by comparing the fractions that represent them.

Thus, $a : b >$ or $< c : d$,

according as $\frac{a}{b} >$ or $< \frac{c}{d}$.

EXERCISE 126

1. Compound the ratios $5:7$ and $14:45$.

2. Compound the duplicate of $4:9$ with the triplicate of $3:8$.

3. Arrange the ratios $4:7$, $2:3$, $5:6$, $9:14$, $11:21$ in order of magnitude.

Find the ratio compounded of:

4. $15:28$, $7:33$, $22:45$.

5. $17:13$, $39:44$, $77:85$.

6. $8a^2b : c$, $4ab : cd$, $c^2d : 8a^3$, $cd : 4b$.

7. $x^3 + a^3 : x^2 - 9a^2$ and $x + 3a : x + a$.

8. Find the condition that the ratio $a : b$ shall be the duplicate of the ratio $a + c : b + c$.

9. What numbers must be added to the terms of the ratio $a : b$ that it may become equal to the ratio $c : d$?

10. Find to six places of decimals the value of the ratio of the diagonal of a cube to an edge of the cube.

11. Two numbers are in the ratio $3:4$, and if 12 is added to each, the sums are in the ratio $5:6$. Find the numbers.

12. The ages of two brothers are in the ratio of $9:5$, and in $2\frac{1}{2}$ years their ages will be in the ratio of $8:5$. In how many years will their ages be in the ratio of $7:5$?

13. An alloy is composed of a pounds of copper and b pounds of tin. If there had been a pounds of tin and b pounds of copper, the volume of the alloy would have been increased by 9 per cent. The weights of equal volumes of copper and tin are in the ratio 11 to 9. Find the ratio of a to b.

Proportion

374. Proportion. An equation consisting of two equal ratios is called a **proportion.**

375. The algebraic test of a proportion is that the two fractions which represent the ratios of the quantities compared shall be equal.

Thus, the ratio $a : b$ is equal to the ratio $c : d$ if the fraction that represents the ratio $a : b$ is equal to the fraction that represents the ratio $c : d$.

376. If the ratios $a : b$ and $c : d$ form a proportion, the proportion is written

$$a : b = c : d$$

(*read the ratio of* a *to* b *is equal to the ratio of* c *to* d),

or

$$a : b :: c : d$$

(*read* a *is to* b *in the same ratio as* c *is to* d).

The four numbers a, b, c, d are called **proportionals,** and are said to be **in proportion.**

The first and last terms, a and d, are called the **extremes.**

The two middle terms, b and c, are called the **means.**

377. The **fourth proportional** to three given numbers is the fourth term of the proportion which has for its first three terms the three given numbers *taken in order.*

Thus, d is the fourth proportional to a, b, and c in the proportion

$$a : b = c : d.$$

378. The numbers $a, b, c, d, e, \cdots$ are said to be in **continued proportion** if $a : b = b : c = c : d = d : e = \cdots$

If three numbers are in continued proportion, the second is called the **mean proportional** between the other two numbers, and the third is called the **third proportional** to the other two numbers.

Thus, b is the mean proportional between a and c in the proportion $a : b = b : c$; and c is the third proportional to a and b.

379. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

For, if $a : b = c : d,$

then $\frac{a}{b} = \frac{c}{d}.$

Multiply by bd, $ad = bc.$

The equation $ad = bc$ gives

$$a = \frac{bc}{d}, \qquad b = \frac{ad}{c};$$

so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean.

If any three terms of a proportion are given, it appears that the fourth term has one value and but one value.

380. *If the product of two numbers is equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.*

For, if $ad = bc,$

divide by bd, $\frac{ad}{bd} = \frac{bc}{bd},$

or $\frac{a}{b} = \frac{c}{d}.$

$$\therefore a : b = c : d.$$

381. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **inversion**; *that is,* b : a = d : c.

For, if $a : b = c : d,$

then $\frac{a}{b} = \frac{c}{d},$

and $1 \div \frac{a}{b} = 1 \div \frac{c}{d},$

or $\frac{b}{a} = \frac{d}{c}.$

$$\therefore b : a = d : c.$$

382. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **composition**; *that is,* a + b : b = c + d : d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

$$\therefore a + b : b = c + d : d.$$

383. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **division**; *that is,* a − b : b = c − d : d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

$$\therefore a - b : b = c - d : d.$$

384. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **composition and division**; *that is,*

a + b : a − b = c + d : c − d.

For $$\frac{a+b}{b} = \frac{c+d}{d}, \quad (\S\ 382)$$

and $$\frac{a-b}{b} = \frac{c-d}{d}. \quad (\S\ 383)$$

Divide, $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore a + b : a - b = c + d : c - d.$$

385. *If four numbers,* a, b, c, d, *are in proportion, they are in proportion by* **alternation**; *that is,* a : c = b : d.

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiply by $\frac{b}{c}$, $$\frac{ab}{bc} = \frac{bc}{cd},$$

or $$\frac{a}{c} = \frac{b}{d}.$$

$$\therefore\ a : c = b : d.$$

386. *Like powers of the terms of a proportion are in proportion.*

For, if $$a : b = c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Raise both sides to the nth power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

$$\therefore\ a^n : b^n = c^n : d^n.$$

387. If $a : b = c : d$, any ratio whose terms are two polynomials in a and b, homogeneous and both of the same degree, is equal to the ratio whose terms are found from those of the preceding ratio by substituting c for a and d for b.

To prove this in any particular case, it will be found sufficient to substitute ra for b and rc for d.

388. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For, if $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h} = \cdots,$$

we may put r for each of these ratios.

Then $$\frac{a}{b} = r,\ \frac{c}{d} = r,\ \frac{e}{f} = r,\ \frac{g}{h} = r \cdots$$

$$\therefore\ a = br,\ c = dr,\ e = fr,\ g = hr, \cdots$$

$$\therefore\ a + c + e + g + \cdots = (b + d + f + h + \cdots)r.$$

$$\therefore\ \frac{a + c + e + g + \cdots}{b + d + f + h + \cdots} = r = \frac{a}{b}.$$

$$\therefore\ a + c + e + g + \cdots : b + d + f + h + \cdots = a : b.$$

In like manner, it may be shown that

$$ma + nc + pe + qg + \cdots : mb + nd + pf + qh + \cdots = a : b.$$

389. *If four numbers,* a, b, c, d, *are in* **continued proportion,** *then* $a : c = a^2 : b^2$ *and* $a : d = a^3 : b^3$.

For, if $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d},$$

then $$\frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b},$$

or $$\frac{a}{c} = \frac{a^2}{b^2}.$$

$$\therefore\ a : c = a^2 : b^2.$$

Also, $$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b},$$

or $$\frac{a}{d} = \frac{a^3}{b^3}.$$

$$\therefore\ a : d = a^3 : b^3.$$

390. *The mean proportional between two numbers is equal to the square root of their product.*

For, if $$a : b = b : c,$$

then $$b^2 = ac.$$ (p. 321, § 379)

$$\therefore\ b = \sqrt{ac}.$$

391. *The products of the corresponding terms of two or more proportions are in proportion.*

For, if $$a : b = c : d,\ e : f = g : h,\ k : l = m : n,$$

then $$\frac{a}{b} = \frac{c}{d},\ \frac{e}{f} = \frac{g}{h},\ \frac{k}{l} = \frac{m}{n}.$$

Take the product of the left members, and also of the right members of these equations,

$$\frac{aek}{bfl} = \frac{cgm}{dhn}.$$

$$\therefore\ aek : bfl = cgm : dhn.$$

392. In order that four *quantities* a, b, c, d may be in proportion, a and b must be of the same kind and c and d of the same kind; but c and d need not necessarily be of the same kind as a and b. In applying *alternation,* however, *all four quantities must be of the same kind.*

393. The laws that have been established for ratios should be remembered when ratios are expressed in fractional form.

Solve the equation $\frac{x^2 + ax - b}{x^2 - ax + b} = \frac{2x^2 + b}{2x^2 - b}$.

By composition and division (p. 322, § 384),

$$\frac{2x^2}{2ax - 2b} = \frac{4x^2}{2b}.$$

Clear of fractions and simplify, $x^2(2ax - 3b) = 0$.
Hence, the equation is satisfied when $x^2 = 0$, or $2ax - 3b = 0$.
The roots of the equation $x^2 = 0$ are 0, 0.
The root of the equation $2ax - 3b = 0$ is $\frac{3b}{2a}$.
Therefore, the roots of the given equation are 0, 0, $\frac{3b}{2a}$.

Exercise 127

Find the third proportional to:

1. 12 and 24. 2. 27 and 3. 3. $\frac{b^2}{a^2 - b^2}$ and $\frac{ab - b^2}{(a+b)^2}$.

Find the mean proportional between:

4. 64 and 81. 5. 25 and 196. 6. $1\frac{1}{2}$ and $16\frac{2}{3}$.

7. $\frac{b + c + a}{b + c - a}$ and $\frac{2bc + b^2 + c^2 - a^2}{4b^2c^2}$.

8. $\left(x - \frac{xy - y^2}{x + y}\right)\left(x - \frac{xy^2 - y^3}{x^2 + y^2}\right)$ and $\frac{x^2}{x^2 - xy + y^2}$.

Find the fourth proportional to:

9. 5, 18, and 20.
10. 9, 16, and 36.
11. 12, 17, and 21.
12. 64, 48, and 56.
13. $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$.

Find the value of x in the proportion:

14. $8 : 23 = 12 : x$.
15. $9 : 18 = x : 45$.
16. $7 : x = 24\frac{1}{2} : 52$.
17. $x : 16 = 7 : 8$.

Show that if $a:b=c:d$:

18. $3a+7b:3a-7b=3c+7d:3c-7d.$

19. $2a^2+5b^2:2a^2-5b^2=2c^2+5d^2:2c^2-5d^2.$

Show that if $a:b=b:c$:

20. $ab+b^2:b^2=bc+c^2:c^2.$

21. $a-c:c=a(a+2b)-c(2b+c):(b+c)^2.$

Find x from the proportion:

22. $6x+a:4x+b=3x-b:2x-a.$

23. $5x:5x-4=6-7x:10-7x.$

24. $2x+11:6x+47=2x+3:6x+16.$

25. $x-1:x-2=4x+5:3x+2.$

26. $x-4:x-7=2x-1:x+1.$

27. $x+a+b:x-3a+b=2x-3a+2b-c:x+b+c.$

Find x and y from the proportions:

28. $x+y-4:2x+y+1=1:2;\ 2x+y-9:x+2y+7=3:4.$

29. $x+y:x-y=a:b-c;\ x+c:y+b=a+b:a+c.$

30. Two numbers are in the ratio $5:4$. If 8 is added to the first number and 4 to the second, the difference of the squares of the new numbers is to the difference of the squares of the numbers in the ratio $35:3$. Find the numbers.

31. What number must be added to each of the numbers 2, 7, 14, 29, in order that the sums may be in proportion?

32. In a continued proportion the sum of the three proportionals is 39, and the sum of their squares is 741. What is the proportion?

33. Find two numbers such that their sum is to their difference as $5:1$, and their sum to their product as $5:4$.

34. If 3 is subtracted from the greater of two numbers and 2 is added to the smaller, the results are in the ratio 2 : 3; but if 1 is added to the greater number and 2 is subtracted from the smaller, the results are in the ratio 3 : 2. Find the numbers.

35. On a division of the house when all members present must vote, if 50 members more had voted for the motion, it would have been carried in the ratio of 5 : 3; but if 60 members more had voted against it, it would have been lost in the ratio of 4 : 3. How many members attended at the division of the house?

36. A pendulum that beats seconds at the sea level is taken to a place where the force of gravity is 0.9 that at the surface of the earth. In what time will the pendulum make one vibration? If the pendulum was attached to a clock, how much time would the clock lose in 24 hours?

VARIATION

394. Variation. One quantity is said to **vary** as another when the two quantities are so related that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other.

Thus, if it is said that the weight of water varies as its volume, the meaning is that *one* gallon of water is to *any specified number* of gallons of water as the weight of *one* gallon of water is to the weight of *the specified number* of gallons of water.

395. Function of a Variable. Two variables may be so related that when a value of one is given the corresponding value of the other can be found. In this case one variable is said to be a *function* of the other (p. 199, § 219).

Thus, if the rate at which a man walks is known, the distance he walks can be found when the time is given; the distance in this case is a *function* of the time.

396. If two variable magnitudes X and Y, not necessarily of the same kind, are so related that when X is changed in any ratio Y is changed in the same ratio, Y is said to vary as X.

Thus, the area of a triangle with a given base varies as its altitude; for, if the altitude is changed in any ratio, the area is changed in the same ratio.

If Y varies as X, this relation is written $Y \propto X$. The sign $\propto$, called the **sign of variation**, is read *varies as*.

If $Y \propto X$, and if, when X has a definitely assigned value A, Y takes the value B, then

$$B : Y = A : X, \tag{1}$$

and therefore, by the theory of proportion, B has a value definitely determined by the value of A.

Let the numerical measures of A, B, X, and Y be a, b, x, y respectively, so that

$$a : x = A : X, \text{ and } b : y = B : Y.$$

Therefore, by (1), $\quad b : y = a : x.$

$$\therefore\ b : a = y : x. \tag{2}$$

Since a and b are the numerical measures of the definitely assigned magnitudes A and B, they are themselves constant and their ratio, $b : a$, is constant. Also, x and y are the numerical measures of the variable magnitudes X and Y; hence, by (2),

When two variable magnitudes X and Y are so related that $Y \propto X$, the ratio of their numerical measures is constant.

Hence, if $y \propto x$, the ratio $y : x$ is constant; and if this constant is represented by m,

$$y : x = m : 1, \text{ or } \frac{y}{x} = m. \quad \therefore\ y = mx.$$

Again, if y', x' and y'', x'' are two sets of corresponding values of y and x, then

$$y' : x' = y'' : x'',$$

or

$$y' : y'' = x' : x''.$$

397. Inverse Variation. When x and y are so related that the ratio of y to $\frac{1}{x}$ is constant, y is said to vary *inversely* as x; this relation is written $y \propto \frac{1}{x}$.

Thus, the time required to do a certain amount of work varies inversely as the number of workmen employed; for, if the number of workmen is doubled, halved, or changed in any other ratio, the time required is halved, doubled, or changed in the inverse ratio.

In this case, $$y : \frac{1}{x} = m. \qquad (\S\ 396)$$

$$\therefore\ y = \frac{m}{x}, \text{ and } xy = m;$$

that is, the product xy is constant.

As before, $$y' : \frac{1}{x'} = y'' : \frac{1}{x''},$$

$$x'y' = x''y'', \qquad (\text{p. } 321,\ \S\ 379)$$

or $$y' : y'' = x'' : x'. \qquad (\text{p. } 321,\ \S\ 380)$$

398. If the ratio of $y : xz$ is constant, then y is said to *vary jointly* as x and z.

In this case, $$y = mxz,$$

and $$y' : y'' = x'z' : x''z''. \qquad (\S\ 396)$$

399. If the ratio $y : \frac{x}{z}$ is constant, then y varies *directly* as x and *inversely* as z.

In this case, $$y = \frac{mx}{z},$$

and $$y' : y'' = \frac{x'}{z'} : \frac{x''}{z''}.$$

400. Theorem I. If $y \propto x$, and $x \propto z$, then $y \propto z$.

For $$y = mx, \text{ and } x = nz. \qquad (\S\ 396)$$

$$\therefore\ y = mnz.$$

$$\therefore\ y \propto z. \qquad (\S\ 396)$$

401. Theorem II. If $y \propto x$, and $z \propto x$, then $(y \pm z) \propto x$.

For $$y = mx,$$
and $$z = nx. \qquad \text{(p. 328, § 396)}$$
$$\therefore\ y \pm z = (m \pm n)x.$$
$$\therefore\ (y \pm z) \propto x. \qquad \text{(p. 328, § 396)}$$

402. Theorem III. If $y \propto x$ when z is constant, and $y \propto z$ when x is constant, and if x and z are independent of each other, then $y \propto xz$ when x and z are both variable.

Let x', y', z' and x'', y'', z'' be two sets of corresponding values of the variables.

Let x change from x' to x'', while z remains constant, and let the corresponding value of y be Y.

Then, by § 396, $$y' : Y = x' : x''. \qquad (1)$$

Now, let z change from z' to z'', while x remains constant.

Then, by § 396, $$Y : y'' = z' : z''. \qquad (2)$$

From (1) and (2), $$y'Y : y''Y = x'z' : x''z'', \qquad \text{(p. 324, § 391)}$$
or $$y' : y'' = x'z' : x''z'', \qquad \text{(p. 317, § 368)}$$
or $$y' : x'z' = y'' : x''z''. \qquad \text{(p. 322, § 385)}$$

Therefore, the ratio $\frac{y}{xz}$ is constant, and $y \propto xz$.

In like manner, it may be shown that if y varies as each one of any number of independent values $x, z, u, \cdots$, when the rest are unchanged, then when they all change, $y \propto xzu \cdots$

Thus, the area of a rectangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of the base and altitude when both vary.

The volume of a rectangular solid varies as the length when the breadth and height remain constant; as the breadth when the length and height remain constant; as the height when the length and breadth remain constant; but as the product of the length, breadth, and height when all three vary.

403. Examples. 1. If z varies directly as x and inversely as y, and when $z = 12$ the corresponding values of x and y are 6 and 8 respectively, find the value of z when $x = 7$ and $y = 4$.

Here $$z = \frac{mx}{y}, \text{ or } m = \frac{yz}{x}.$$

$$\therefore m = \frac{8 \times 12}{6} = 16.$$

Substitute 16 for m, 7 for x, and 4 for y in the equation $z = \frac{mx}{y}$;

then $$z = \frac{16 \times 7}{4} = 28.$$

2. The weight of a sphere of given material varies as the volume of the sphere, and the volume varies as the cube of the diameter. If a sphere 4 inches in diameter weighs 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
V represent the volume,
and D represent the diameter.

Then $$W \propto V, \text{ and } V \propto D^3.$$

$$\therefore W \propto D^3. \qquad \text{(p. 329, § 400)}$$

Put $$W = mD^3.$$

Then, since 20 and 4 are corresponding values of W and D,

$$20 = m \times 64.$$

$$\therefore m = \tfrac{20}{64} = \tfrac{5}{16}.$$

$$\therefore W = \tfrac{5}{16} D^3.$$

Hence, when $D = 5$, $$W = \tfrac{5}{16} \times 5^3 = 39\tfrac{1}{16}.$$

Therefore, a sphere 5 inches in diameter weighs $39\frac{1}{16}$ pounds.

Exercise 128

1. If $y \propto x$, and $y = 6$ when $x = 9$, find y when $x = 15$.

2. If $y \propto \frac{1}{x}$, and $y = 32$ when $x = 6$, find y when $x = 21$.

3. If z varies jointly as x and y, and 4, 5, 6 are simultaneous values of x, y, z, find z when $x = 8$ and $y = 9$.

4. If the cube of x varies inversely as the square of y, and $x = 2$ when $y = 3$, find the equation between x and y.

5. If z varies directly as x and inversely as y, and if $x = 6$ and $y = 15$ when $z = 7$, find z when $x = 12$ and $y = 8$.

6. The volume of a rectangular solid varies jointly as the length, breadth, and height. A cube of clay 12 inches on an edge is molded into a right prism whose base is a rectangle 16 inches by 8 inches. Find the height of the prism.

7. The volume of a sphere varies as the cube of the diameter. The diameter of a sphere whose volume is 75 cubic feet 1377 cubic inches is 5 feet 3 inches. Find the diameter of a sphere whose volume is 179 cubic feet 1152 cubic inches.

8. The intensity of light varies inversely as the square of the distance from the source. How far from a lamp is a point that receives half as much light as another point 16 feet away?

9. The volume of a right cylinder varies jointly as its height and the square of its radius. One cylindrical vessel is 5 inches high and $1\frac{1}{2}$ inches in radius, and another is 6 inches high and 2 inches in radius. Find the radius of a third cylindrical vessel 7 inches high that will hold as much as the other two vessels together.

10. If a body falling freely falls 144 feet in 3 seconds and acquires a velocity of 96 feet per second, find the velocity acquired and the distance fallen in 5 seconds; in 7 seconds; in 9 seconds (see p. 289, § 350, Law 1).

11. The deflection of a horizontal beam bent by a heavy weight varies directly as the weight, directly as the cube of the length of the beam, inversely as the breadth, and inversely as the cube of the thickness. If a pine beam 5 feet long, 2 inches wide, 3 inches thick is bent 1 inch by a weight of 600 pounds, how much will a pine beam 15 feet long, 4 inches wide, 6 inches thick be bent by a weight of 2 tons?

CHAPTER XXIII

PROGRESSIONS

404. Series. A succession of numbers that proceed according to some fixed law is called a **series**; the successive numbers are called the **terms** of the series.

A series that ends at some particular term is a **finite series**; a series that continues without end is an **infinite series.**

405. The number of different forms of series is unlimited; in this chapter we shall consider only arithmetical series, geometrical series, and harmonical series.

ARITHMETICAL PROGRESSION

406. Arithmetical Progression. A series is called an **arithmetical series** or an **arithmetical progression** when each succeeding term may be obtained by adding to the preceding term a constant called the *constant difference.*

The general representative of such a series is

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \cdots,$$

in which a is the first term and d the common difference.

The series is *increasing* or *decreasing* according as d is positive or negative.

407. The *n*th Term. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d is always one less than the number of the term, so that the nth term is $a+(n-1)d$.

If the nth term is represented by l, we have

$$\mathbf{l = a + (n-1)d.} \qquad \textbf{(I)}$$

408. Sum of the Series. If l denotes the nth term, a the first term, n the number of terms, d the common difference, and s the sum of n terms, it is evident that

$$s = a + (a + d) + (a + 2d) + \cdots + (l - d) + l$$

or

$$s = l + (l - d) + (l - 2d) + \cdots + (a + d) + a$$

$$\therefore 2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l)$$
$$= n(a + l).$$

Therefore, $$\mathbf{s = \frac{n}{2}(a + l).} \qquad \textbf{(II)}$$

409. From the two formulas (I) and (II), when any *three* of the numbers a, d, l, n, s are given the other *two* may be found.

1. Find the sum of eight terms of the series $3, 7, 11, 15, \cdots$

Here $a = 3,\ d = 4,\ n = 8.$
From (I), $l = 3 + (8 - 1)4 = 3 + 28 = 31.$
Substitute in (II), $s = \frac{8}{2}(3 + 31) = 4 \times 34 = 136.$
Therefore, the sum of the series is 136.

2. The first term of an arithmetical series is 2, the last term 29, and the sum of the series 155. Find the series.

From (I), $29 = 2 + (n - 1)d.$ (1)

From (II), $155 = \frac{n}{2}(2 + 29).$ (2)

From (2), $n = 10.$
Substitute in (1), $29 = 2 + (10 - 1)d.$
$\therefore d = 3.$

Therefore, the series is 2, 5, 8, 11, 14, 17, 20, 23, 26, 29.

3. How many terms of the series $6, 11, 16, 21, \cdots$ must be taken in order that their sum may be 402?

From (I), $l = 6 + (n - 1)5.$
$\therefore l = 5n + 1.$ (1)

From (II), $402 = \frac{n}{2}(6 + l).$ (2)

Substitute in (2) the value of l found in (1),

$$402 = \frac{n}{2}(5n + 7).$$

Simplify, $5n^2 + 7n = 804.$

Complete the square, $100n^2 + (\ ) + 49 = 16{,}129.$

Extract the square root, $10n + 7 = \pm 127.$

Transpose and combine, $10n = 120, \text{ or } -134.$

$$\therefore\ n = 12, \text{ or } -13\tfrac{2}{5}.$$

We use only the positive result.

Therefore, the required number of terms is 12.

4. Find a when d, l, and s are given.

We have $$l = a + (n-1)d, \quad (1)$$

and $$s = \frac{n}{2}(a + l). \quad (2)$$

From (1), $$n = \frac{l - a + d}{d}. \quad (3)$$

From (2), $$n = \frac{2s}{a + l}. \quad (4)$$

From (3) and (4), $$\frac{l - a + d}{d} = \frac{2s}{a + l}.$$

Clear of fractions, $l^2 - a^2 + ad + ld = 2ds.$

Transpose, $a^2 - ad = l^2 + ld - 2ds.$

Complete the square, $4a^2 - (\ ) + d^2 = 4l^2 + 4ld + d^2 - 8ds.$

Extract the square root, $2a - d = \pm\sqrt{(2l + d)^2 - 8ds}.$

$$\therefore\ a = \tfrac{1}{2}\left[d \pm \sqrt{(2l + d)^2 - 8ds}\right].$$

410. The table on page 336 contains the results of all possible problems in arithmetical series in which three of the numbers a, l, d, n, s are given and the other two required.

The student should work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown numbers are represented by letters other than x, y, z.

411. The **arithmetical mean** between two numbers is the number which, when placed between them, makes with them an arithmetical series.

If a and b represent two numbers, and A their arithmetical mean, then, by the definition of an arithmetical series,

$$A - a = b - A.$$

$$\therefore\ A = \frac{a + b}{2}.$$

No.	Given	Required	Result
1	$a\ d\ n$	l	$l = a + (n-1)d$
2	$a\ d\ s$		$l = \frac{1}{2}\left[-d \pm \sqrt{8ds + (2a-d)^2}\right]$
3	$a\ n\ s$		$l = \frac{2s}{n} - a$
4	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d]$
6	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}$
7	$a\ n\ l$		$s = \frac{n}{2}(a+l)$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d]$
9	$d\ n\ l$	a	$a = l - (n-1)d$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}$
11	$d\ l\ s$		$a = \frac{1}{2}\left[d \pm \sqrt{(2l+d)^2 - 8ds}\right]$
12	$n\ l\ s$		$a = \frac{2s}{n} - l$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}$
14	$a\ n\ s$		$d = \frac{2(s-an)}{n(n-1)}$
15	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}$
16	$n\ l\ s$		$d = \frac{2(nl-s)}{n(n-1)}$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1$
18	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}$
19	$a\ l\ s$		$n = \frac{2s}{l+a}$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}$

412. Sometimes it is required to insert several arithmetical means between two numbers.

Insert five arithmetical means between 4 and 22.

Here the whole number of terms is 7; the first term is 4 and the seventh, or last, term is 22.

By (I), $$22 = 4 + 6d.$$
$$\therefore 6d = 18.$$
$$\therefore d = 3.$$

Hence, the complete series is 4, 7, 10, 13, 16, 19, 22.

Therefore, the five means required are 7, 10, 13, 16, 19.

NOTE. When the sum of a number of terms in arithmetical progression is given, it is convenient to represent three terms by $x - y$, x, $x + y$; four terms by $x - 3y$, $x - y$, $x + y$, $x + 3y$; and so on.

The sum of three numbers in arithmetical progression is 69, and twice the product of the two extremes exceeds the square of the mean by 79. Find the numbers.

Let $x - y$, x, $x + y$ represent the numbers.

Then $$x - y + x + x + y = 69, \quad (1)$$
and $$2(x - y)(x + y) = x^2 + 79. \quad (2)$$
From (1), $$3x = 69.$$
$$\therefore x = 23. \quad (3)$$
From (2), $$x^2 - 2y^2 = 79. \quad (4)$$
Substitute in (4) the value of x from (3),
$$529 - 2y^2 = 79.$$
$$\therefore y^2 = 225.$$
$$\therefore y = \pm 15.$$

Therefore, the numbers are 8, 23, 38; or 38, 23, 8.

EXERCISE 129

1. Find the eighth term of 7, 10, 13, $\cdots$
2. Find the tenth term of 2, 11, 20, $\cdots$
3. Find the fifteenth term of 4, 16, 28, $\cdots$
4. Find the twelfth term of 144, 138, 132, $\cdots$

5. Find the sixth term of $5, -1, -7, \cdots$

6. Find the fourteenth term of $12, 4, -4, \cdots$

Find the sum of:

7. Seven terms of $18, 21, 24, \cdots$

8. Ten terms of $-6, -2, 2, \cdots$

9. Eighteen terms of $3\frac{1}{8}, 3\frac{7}{8}, 4\frac{5}{8}, \cdots$

10. Twenty-four terms of $1\frac{5}{9}, 3, 4\frac{4}{9}, \cdots$

11. n terms of $3, 5\frac{2}{3}, 8\frac{1}{3}, \cdots$

12. n terms of $a, a+4b, a+8b, \cdots$

13. Given $a=4$, $l=34$, $n=11$; find d and s.

14. Given $a=120$, $n=16$, $s=960$; find d and l.

15. Given $a=-12$, $d=4$, $l=40$; find n and s.

16. Given $a=7$, $l=49$, $s=812$; find d and n.

17. Given $d=\frac{2}{3}$, $n=24$, $s=56$; find a and l.

18. Given $a=4$, $d=3$, $s=246$; find l and n.

19. Insert four arithmetical means between 8 and 23.

20. Insert six arithmetical means between 96 and 47.

21. Insert eight arithmetical means between 4 and 58.

22. Insert twelve arithmetical means between 1 and 10.

23. The first term of an arithmetical progression is 7, and the third term is 23. Find the sum of ten terms.

24. The first term of an arithmetical progression is 4, and the sum of seven terms is 175. What term is 60?

25. How many terms of the series $-9, -3, 3, \cdots$ must be taken in order that their sum may be 135?

26. The sum of three numbers in arithmetical progression is 27, and the sum of their squares is 293. Find the numbers.

27. The sum of four numbers in arithmetical progression is 42, and the sum of their squares is 686. Find the numbers.

28. How many terms of the arithmetical series 3, 7, 11, ⋯ must be taken that the sum of the first half may be to the sum of the second half in the ratio 11 : 31?

29. The sum of the squares of the extremes of four numbers in arithmetical progression is 450, and the sum of the squares of the means is 306. What are the numbers?

30. If a number composed of three digits in arithmetical progression is divided by the sum of the digits, the quotient is 63 and the remainder 6; if 792 is subtracted from the number, the order of the digits is reversed. Find the number.

31. In a potato race 100 potatoes are placed 3 feet apart in a straight line. A runner picks up one potato at a time and carries it to a basket in the line of the potatoes, and 3 feet from the first potato. How far must the contestant run?

32. In a potato race, if there are 30 potatoes 4 feet apart, and the basket is 4 feet from the first potato, how far must the contestant run?

33. A body falling freely falls 16.08 feet in the first second, and in each succeeding second 32.16 feet more than in the second immediately preceding. If a stone dropped from a stationary balloon reaches the ground in 12 seconds, how far does it fall in the last second, and how high is the balloon?

34. A stone is dropped from the top of a tower 402 feet high. In how many seconds does it reach the ground?

35. A stone is dropped from a stationary balloon 3618 feet high. In how many seconds does it reach the ground?

36. In astronomical time the hours of a day are numbered from 1 to 24. If a clock should strike the hours of an astronomical day, how many strokes would it strike in one day?

37. A body falling freely falls 4.9 meters the first second, and each succeeding second 9.8 meters more than in the second immediately preceding. How far will a body fall in 1 minute?

38. If a bullet, when fired vertically upwards, traverses 490 meters the first second, how high will it rise, and how long will it be before it reaches the earth again?

39. In order to sell a football a boy marks tickets from 1 to 25 and obtains twenty-five persons to draw each a ticket. Each person pays as many cents as the number on the ticket he draws, and the owner of the football is determined by lot. How much does the boy receive for the football?

40. In selling a gold watch a man used the same system as in Example 39, but sold 80 tickets marked from 1 to 80. How much did he receive for the watch?

41. In selling a Persian rug a dealer used the same system as in Example 39, but sold 250 tickets marked from 1 to 250. How much did he receive for the rug?

Geometrical Progression

413. Geometrical Progression. A series is called a **geometrical series** or a **geometrical progression** when each succeeding term may be obtained by multiplying the preceding term by a *constant multiplier* called the *ratio*.

The general representative of such a series is

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \cdots,$$

in which a is the first term and r the ratio.

The terms increase or decrease in numerical magnitude according as r is numerically greater than or less than unity.

414. The *n*th Term. Since the exponent of r increases by one for each succeeding term after the first, the exponent is always one less than the number of the term, so that the nth term is ar^{n-1}.

If the nth term is represented by l, we have

$$\mathbf{l = ar^{n-1}}. \qquad \textbf{(I)}$$

415. Sum of the Series. If l represents the nth term, a the first term, n the number of terms, r the common ratio, and s the sum of n terms, then

$$s = a + ar + ar^2 + \cdots + ar^{n-1}. \quad (1)$$

Multiply by r, $rs = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n$. (2)

Subtract (1) from (2), $rs - s = ar^n - a$.

Factor, $(r-1)s = a(r^n - 1)$.

Divide by $r - 1$, $$\mathbf{s = \frac{a(r^n - 1)}{r - 1}}. \quad \text{(II)}$$

Since $l = ar^{n-1}$, then $rl = ar^n$, and (II) may be written

$$\mathbf{s = \frac{rl - a}{r - 1}}. \quad \text{(III)}$$

416. From (I) and (II), or (I) and (III), when any *three* of the numbers a, r, l, n, s are given, the other *two* may be found.

1. The sum of a geometrical series is 1456, the first term 4, and the last term 972. Find the ratio and the number of terms.

From (I), $972 = 4r^{n-1}$. (1)

From (III), $$1456 = \frac{972r - 4}{r - 1}. \quad (2)$$

From (2), $r = 3$.

Substitute the value of r in (1), $972 = 4 \times 3^{n-1}$.

Divide by 4 and transpose, $3^{n-1} = 243$.

Since $243 = 3^5$, $n - 1 = 5$, and $n = 6$.

Therefore, the ratio is 3, and number of terms is 6.

2. Find l when r, n, s are given.

From (I), $$a = \frac{l}{r^{n-1}}.$$

Substitute in (III), $$s = \frac{rl - \frac{l}{r^{n-1}}}{r - 1}.$$

$$(r-1)s = \frac{(r^n - 1)}{r^{n-1}} l.$$

$$\therefore l = \frac{(r-1)r^{n-1}s}{r^n - 1}.$$

417. The table on page 344 contains the results of all possible problems in geometrical series in which three of the numbers a, r, l, n, s are given and the other two required, with the exception of those in which n is required; these last require the use of logarithms, with which the student is supposed to be not yet acquainted.

The student should work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown numbers are represented by letters other than x, y, z.

418. The **geometrical mean** between two numbers is the number which when placed between them makes with them a geometrical series.

If a and b denote two numbers, and G their geometrical mean, then, by the definition of a geometrical series,

$$\frac{G}{a} = \frac{b}{G}.$$

$$\therefore\ G = \sqrt{ab}.$$

419. Sometimes it is required to insert several geometrical means between two numbers.

Insert three geometrical means between 3 and 48.

Here the whole number of terms is 5, the first term is 3, and the fifth term is 48.

By (I),
$$48 = 3r^4.$$
$$\therefore\ r^4 = 16.$$
$$\therefore\ r = \pm 2.$$

Therefore, the series is either of the following:

3, [6, 12, 24,] 48; or 3, [−6, 12, −24,] 48.

The terms inclosed by the brackets are the terms required.

420. Infinite Geometrical Series. When r in absolute value is less than 1, the successive terms become numerically smaller and smaller; by taking n large enough we can make the nth term, ar^{n-1}, as small as we please, although we cannot make it absolutely equal to zero.

By changing the signs of the numerator and denominator, the sum of n terms, $\frac{ar^n - a}{r-1}$, may be written $\frac{a - ar^n}{1-r}$, which is equal to $\frac{a}{1-r} - \frac{ar^n}{1-r}$; this sum differs from $\frac{a}{1-r}$ by the fraction $\frac{ar^n}{1-r}$; by taking enough terms we can make ar^n, and consequently this fraction, as small as we please; the greater the number of terms taken the nearer is their sum to $\frac{a}{1-r}$. Hence, $\frac{a}{1-r}$ is called the *sum* of an infinite number of terms of the series.

1. Find the sum of the infinite series $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

Here $a = 1$, and $r = -\frac{1}{3}$.

The sum of the series is $\frac{a}{1-r} = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$.

Therefore, the sum of n terms is

$$\frac{3}{4} - \frac{1 \times (-\frac{1}{3})^n}{1+\frac{1}{3}} = \frac{3}{4} - \frac{3}{4}\left(-\frac{1}{3}\right)^n = \frac{3}{4}\left[1 - \left(-\frac{1}{3}\right)^n\right].$$

This sum evidently approaches $\frac{3}{4}$ as n is increased.

2. Find the value of the recurring decimal $0.5243243\cdots$

Consider first the part that recurs; this may be written

$$\tfrac{243}{10000} + \tfrac{243}{10000000} + \tfrac{243}{10000000000} + \cdots$$

In this infinite geometrical series, $a = \frac{243}{10000}$ and $r = \frac{1}{1000}$.

Hence, the sum is $\frac{a}{1-r} = \frac{\frac{243}{10000}}{1 - \frac{1}{1000}} = \frac{243}{9990} = \frac{9}{370}$.

Add 0.5, the part of the decimal that does not recur.

We obtain for the value of the whole decimal

$$0.5 + \tfrac{9}{370} = \tfrac{1}{2} + \tfrac{9}{370} = \tfrac{194}{370} = \tfrac{97}{185}.$$

Exercise 130

1. Find the sixth term of 4, 8, 16, $\cdots$
2. Find the ninth term of 2, 14, 98, $\cdots$
3. Find the tenth term of 128, $-$ 64, 32, $\cdots$

No.	Given	Required	Result
1	$a\ r\ n$	l	$l = ar^{n-1}$
2	$a\ r\ s$		$l = \dfrac{a + (r-1)s}{r}$
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$
4	$r\ n\ s$		$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}$
5	$a\ r\ n$	s	$s = \dfrac{a(r^n-1)}{r-1}$
6	$a\ r\ l$		$s = \dfrac{rl - a}{r-1}$
7	$a\ n\ l$		$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$
8	$r\ n\ l$		$s = \dfrac{lr^n - l}{r^n - r^{n-1}}$
9	$r\ n\ l$	a	$a = \dfrac{l}{r^{n-1}}$
10	$r\ n\ s$		$a = \dfrac{(r-1)s}{r^n - 1}$
11	$r\ l\ s$		$a = rl - (r-1)s$
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$
13	$a\ n\ l$	r	$r = \sqrt[n-1]{\dfrac{l}{a}}$
14	$a\ n\ s$		$r^n - \dfrac{s}{a}r + \dfrac{s-a}{a} = 0$
15	$a\ l\ s$		$r = \dfrac{s-a}{s-l}$
16	$n\ l\ s$		$r^n - \dfrac{s}{s-l}r^{n-1} + \dfrac{l}{s-l} = 0$

4. Find the twelfth term of 4, -3, $2\frac{1}{4}$, $\cdots$

5. Find the twenty-fifth term of 2, 6, 18, $\cdots$

6. Find the nth term of 2, $-1\frac{1}{9}$, $\frac{50}{81}$, $\cdots$

Find the sum of:

7. Nine terms of 8, 24, 72, $\cdots$
8. Twelve terms of 9, -3, 1, $\cdots$
9. Sixteen terms of 81, 27, 9, $\cdots$
10. Nineteen terms of 5, -10, 20, $\cdots$
11. n terms of $2\frac{1}{4}$, $1\frac{1}{2}$, 1, $\cdots$

Find the sum of the infinite series:

12. $6 - 3 + 1\frac{1}{2} - \cdots$
13. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$
14. $1 - \frac{2}{3} + \frac{4}{9} - \cdots$
15. $2 + \frac{4}{5} + \frac{8}{25} + \cdots$
16. $2 - \frac{2}{5} + \frac{2}{25} - \cdots$
17. $4 + 2\frac{2}{3} + 1\frac{7}{9} + \cdots$

Find the value of the recurring decimal:

18. $0.454545 \cdots$
19. $0.020303 \cdots$
20. $0.7283283 \cdots$
21. $0.11342342 \cdots$
22. $0.64389389 \cdots$
23. $0.55862862 \cdots$
24. $2.4336336 \cdots$
25. $3.731843184 \cdots$

26. Insert three geometrical means between 3 and 768.

27. Insert four geometrical means between 2 and 6250.

28. Insert five geometrical means between 243 and $\frac{1}{3}$.

29. Given $a = 8$, $r = 2$, $s = 248$; find l and n.

30. Given $a = 343$, $n = 9$, $l = \frac{1}{343}$; find r and s.

31. Given $r = 1\frac{1}{2}$, $n = 8$, $s = 1050\frac{5}{6}$; find a and l.

32. Given $n = 6$, $s = 945\frac{7}{9}$, $l = -227\frac{5}{9}$; find a and r.

33. Given $r = \frac{1}{6}$, $n = 5$, $l = 1296$; find a and s.

34. If the first term of a geometrical series is 5 and the ratio 3, what term is 1215?

35. The fourth term of a geometrical series is 160 and the ratio is 4. Find the sixth term and the eighth term.

36. Four numbers are in geometrical progression. The sum of the first and fourth is 130, and the sum of the second and third is 40. Find the numbers.

37. The sum of three numbers in arithmetical progression is 45. If 2 is added to the first number, 3 to the second, and 7 to the third, the new numbers are in geometrical progression. Find the numbers.

38. A man deposited $1000 in a savings bank that pays 3 per cent interest, compounded annually. If the depositor withdrew no money from the bank for 6 years, what was the average amount he had on deposit?

39. A man deposited $1000 in a savings bank that pays 4 per cent interest, compounded semiannually. If the depositor withdrew no money from the bank for 6 years, what was the average amount he had on deposit?

40. In using an air pump to exhaust air from a receiver, if the capacity of the piston barrel is a and that of the receiver is b, and the height of the barometer is c inches, the pressure within the receiver is reduced by one stroke of the piston to $\frac{b}{a+b}$ of c inches, by two strokes to $\left[\frac{b}{a+b}\right]^2$ of c inches, and so on. If the capacity of the receiver is 5 cubic feet, that of the piston barrel 1 cubic foot, and the original pressure of the air is 30 inches, what is the pressure in the receiver after five strokes of the piston?

41. If the capacity of the receiver of an air pump is 8 liters, and that of the piston barrel is 1 liter, and the original pressure of the air is 76 centimeters, what is the pressure within the receiver after six strokes of the piston?

42. If a rubber ball, falling freely upon a marble floor, rebounds to one third the height from which it has fallen, how great a distance will it traverse if tossed to a height of 50 feet?

43. If a rubber ball rebounds to three fifths of the height from which it has fallen, how great a distance will it traverse if tossed to a height of 30 feet?

Harmonical Progression

421. Harmonical Progression. A series is called a **harmonical series,** or a **harmonical progression,** when the reciprocals of its terms form an *arithmetical series.*

The general representative of such a series is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}.$$

422. Questions relating to harmonical series are generally best solved by writing the reciprocals of its terms, and thus forming an arithmetical series.

423. The **harmonical mean** between two numbers is the number which when placed between them makes with them a harmonical series.

If a and b denote two numbers, and H their harmonical mean, then by the definition of a harmonical series,

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}.$$

$$\therefore \frac{2}{H} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}.$$

$$\therefore H = \frac{2ab}{a+b}.$$

424. Sometimes it is required to insert several harmonical means between two numbers.

Insert four harmonical means between 6 and 24.

First find four arithmetical means between $\frac{1}{6}$ and $\frac{1}{24}$.
These are found to be $\frac{17}{120}$, $\frac{7}{60}$, $\frac{11}{120}$, and $\frac{1}{15}$.
Therefore, the harmonical means are

$$\tfrac{120}{17},\ \tfrac{60}{7},\ \tfrac{120}{11},\ \tfrac{15}{1};\ \text{or}\ 7\tfrac{1}{17},\ 8\tfrac{4}{7},\ 10\tfrac{10}{11},\ 15.$$

425. Since $A = \dfrac{a+b}{2}$, $H = \dfrac{2ab}{a+b}$, and $G = \sqrt{ab}$,

therefore, $\qquad H = \dfrac{G^2}{A}, \quad$ or $\quad G = \sqrt{AH}.$

That is, the geometrical mean between two numbers is also the geometrical mean between the arithmetical and harmonical means of the numbers, or

$$A : G = G : H.$$

Hence, G lies in numerical value between A and H.

Exercise 131

1. Insert four harmonical means between 2 and 5.

2. Insert seven harmonical means between $\frac{2}{3}$ and $\frac{2}{7}$.

3. Find the 8th term of the harmonical series 4, $4\frac{4}{5}$, 6, $\cdots$

4. Find the 6th term of the harmonical series $9\frac{3}{5}$, $6\frac{6}{7}$, $5\frac{1}{3}$, $\cdots$

5. The fourth and seventh terms of a harmonical progression are $\frac{1}{10}$ and $\frac{1}{16}$. Find the first seven terms.

6. The difference between two numbers is 12, and the harmonical mean between them is $6\frac{2}{5}$. Find the numbers.

7. The difference between the arithmetical and harmonical means between two numbers is $2\frac{1}{2}$, and one of the numbers is three times the other. Find the numbers.

8. The arithmetical mean between two numbers exceeds the geometrical mean by 68, and the geometrical mean exceeds the harmonical mean by 60. Find the numbers.

9. The sum of three numbers in harmonical progression is 39, and the square of the first is greater by 99 than the sum of the squares of the second and third. Find the numbers.

10. Show that a, b, and c are in arithmetical progression, in geometrical progression, or in harmonical progression, according as $a - b : b - c$ is equal to $a : a$, to $a : b$, or to $a : c$.

11. Show that if a, b, c are in harmonical progression, then $a - \frac{1}{2}b$, $\frac{1}{2}b$, $c - \frac{1}{2}b$ are in geometrical progression.

12. Show that if a, b, c are in harmonical series, a, $a - c$, $a - b$ are in harmonical series, as are c, $c - a$, $c - b$.

CHAPTER XXIV

VARIABLES AND LIMITS

426. Variables and Constants. A number that, under the conditions of the problem into which it enters, may take *different values* is called a **variable.**

A number that, under the conditions of the problem into which it enters, has a *fixed value* is called a **constant.**

Variables are generally represented by the last letters of the alphabet, x, y, z, etc.; constants, by the Arabic numerals, and by the first letters of the alphabet, a, b, c, etc.

427. Functions. Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case the second variable is said to be a **function** of the first.

Thus, if a man walks on a road at a uniform rate of a miles per hour, the number of miles he walks and the number of hours he walks are both variables, and the first is a function of the second. If y is the number of miles he has walked at the end of x hours, y and x are connected by the relation $y = ax$, and y is a function of x. Also, $x = \frac{y}{a}$; hence, x is also a function of y.

When one of two variables is a function of the other, the relation between them is generally expressed by an equation. If any value of the variable is assumed, the corresponding value or values of the function may be found from the given equation.

The variable of which the value is assumed is generally called the *independent* variable; and the function is called the *dependent* variable.

In the last example we may assume values of x, and find the corresponding values of y from the relation $y = ax$; or assume values of y, and find the corresponding values of x from the relation $x = \frac{y}{a}$. In the first case x is the independent variable, and y the dependent; in the second case y is the independent variable, and x the dependent.

428. Limits. As a variable changes its value, it may approach some constant; if the variable may be made to approach the constant *as near as we please*, the variable is said to *approach the constant as a limit*, and the constant is called the **limit** of the variable.

Let x represent the sum of n terms of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

Then (p. 341, § 415), $x = \dfrac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = \dfrac{1 - \dfrac{1}{2^n}}{\frac{1}{2}} = \dfrac{2^n - 1}{2^{n-1}} = 2 - \dfrac{1}{2^{n-1}}$.

Suppose n to increase; then $\dfrac{1}{2^{n-1}}$ decreases, and x approaches 2.

Since we may take as many terms of the series as we please, n may be made as large as we please; therefore, $\dfrac{1}{2^{n-1}}$ may be made as small as we please, and x may be made to approach 2 as near as we please.

If we take *any assigned* positive constant, as $\frac{1}{10000}$, we may make the difference between 2 and x less than this assigned constant; for we have only to take n so large that $\dfrac{1}{2^{n-1}}$ is less than $\dfrac{1}{10000}$; that is, that 2^{n-1} is greater than 10,000: this is accomplished by taking n as large as 15. Similarly, by taking n large enough, we may make the difference between 2 and x less than *any* assigned positive constant.

Since $2 - x$ may be made as small as we please, it follows that the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, as n is constantly increased, approaches 2 *as a limit.*

429. Test for a Limit. In order to prove that a variable approaches a constant as a limit, it is *necessary* and *sufficient* to prove that the difference in absolute value between the variable and the constant may become and remain *less than any assigned constant, however small.*

A variable may approach a constant without approaching it *as a limit.*

Thus, in the last example x approaches 3, but not as a limit; for $3 - x$ cannot be made as near to 0 as we please, since it cannot be made less than 1.

430. Infinitesimals. As a variable changes its value, it may constantly decrease in absolute value; if the variable may become and remain less in absolute value than any assigned constant *however small,* the variable is said to *decrease without limit,* or to *decrease indefinitely.* In this case the variable approaches zero as a limit.

When a variable that approaches zero as a limit is conceived to become and remain less in absolute value than any assigned constant however small, the variable is said to become *infinitesimal;* such a variable is called an *infinitesimal number,* or simply an **infinitesimal.**

431. Infinites. As a variable changes its value, it may constantly increase in absolute value; if the variable may become and remain greater in absolute value than any assigned constant *however great,* the variable is said to *increase without limit,* or to *increase indefinitely.*

When a variable is conceived to become and remain greater in absolute value than any assigned constant however great, the variable is said to become *infinite;* such a variable is called an *infinite number,* or simply an **infinite.**

432. Infinites and infinitesimals are *variables,* not constants. There is no idea of *fixed value* implied in either an infinite or an infinitesimal.

A *constant* whose absolute value may be shown to be less than the absolute value of any assigned constant however small can have no other value than zero.

433. Finites. A number that cannot become an infinite or an infinitesimal is said to be a **finite number,** or simply a **finite.**

434. Relations between Infinites and Infinitesimals.

I. *If* x *is infinitesimal and* a *is finite and not* 0, *then* ax *is infinitesimal.*

For ax can be made less in absolute value than any assigned constant, since x can be made less than any assigned constant.

II. *If* X *is infinite and* a *is finite and not* 0, *then* aX *is infinite.*

For aX can be made larger in absolute value than any assigned constant however large, since X can be made larger in absolute value than any assigned constant however large.

III. *If* x *is infinitesimal and* a *is finite and not* 0, *then* $\frac{a}{x}$ *is infinite.*

For $\frac{a}{x}$ can be made larger in absolute value than any assigned constant however large, since x can be made less in absolute value than any assigned constant however small.

IV. *If* X *is infinite and* a *is finite and not* 0, *then* $\frac{a}{X}$ *is infinitesimal.*

For $\frac{a}{X}$ can be made less in absolute value than any assigned constant however small, since X can be made larger in absolute value than any assigned constant however large.

In the above theorems a *may be a constant or a variable; the only restriction on the value of* a *is that it shall not become either infinite or zero.*

435. Abbreviated Notation. An infinite is often represented by ∞. In § 434, III and IV are sometimes written

$$\frac{a}{0} = \infty, \quad \frac{a}{\infty} = 0.$$

The expression $\frac{a}{0}$ cannot be interpreted literally since we cannot divide by 0; neither can $\frac{a}{\infty} = 0$ be interpreted literally, since we can find no number such that the quotient obtained by dividing a by that number is zero.

$\frac{a}{0} = \infty$ is simply an abbreviated way of writing: *if* $\frac{\text{a}}{\text{x}} = \text{X}$, *and* x *approaches* 0 *as a limit,* X *increases without limit.*

$\frac{a}{\infty} = 0$ is simply an abbreviated way of writing: *if* $\frac{\text{a}}{\text{X}} = \text{x}$, *and* X *increases without limit,* x *approaches* 0 *as a limit.*

The symbol $\doteq$ is used for the phrase *approaches as a limit.*

Thus, as $x \doteq a$ means and is read *as* x *approaches* a *as a limit.*

436. Approach to a Limit. When a variable approaches a limit it may approach its limit in one of three ways.

1. The variable may be always less than its limit.
2. The variable may be always greater than its limit.
3. The variable may be sometimes less and sometimes greater than its limit.

If x represents the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$, x is always less than its limit 2.

If x represents the sum of n terms of the series $3 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots$, x is always greater than its limit 2.

If x represents the sum of n terms of the series $3 - \frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots$, we have (p. 341, § 415)

$$x = \frac{3 - 3(-\frac{1}{2})^n}{1 + \frac{1}{2}} = 2 - 2(-\tfrac{1}{2})^n.$$

As n is indefinitely increased, x evidently approaches 2 as a limit.

If n is even, x is less than 2; if n is odd, x is greater than 2. Hence, if n is increased by taking each time one more term, x is alternately less than and greater than 2. If, for example,

$$\begin{array}{llllll} n = 2, & 3, & 4, & 5, & 6, & 7, \\ x = 1\frac{1}{2}, & 2\frac{1}{4}, & 1\frac{7}{8}, & 2\frac{1}{16}, & 1\frac{31}{32}, & 2\frac{1}{64}. \end{array}$$

In whatever way a variable approaches a constant, the test for a limit given in § 429 always applies.

437. Equal Variables. *If two variables are always equal, and each approaches a limit, then their limits are equal.*

Let x and y be increasing variables, a and b their limits.

Now $\qquad a = x + x'$, and $b = y + y'$, (p. 350, § 429)

where x' and y' are variables that approach 0 as a limit.

Then, since the equation $x = y$ always holds true,

$$a - b = x' - y'.$$

Since $x' - y'$ is always equal to the constant $a - b$, it follows that $x' - y'$ must be a constant.

But $x' - y'$ is less than any assigned constant, since x' and y' can each be made less than any assigned constant.

The only constant, however, which is less than any assigned constant is 0. (p. 351, § 432)

Therefore, $\qquad x' - y' = 0.$

Hence, $\qquad a - b = 0,$

or $\qquad a = b.$

Similarly if x and y are decreasing variables.

438. Limit of a Sum. *The limit of the algebraic sum of any finite number of variables is the algebraic sum of their limits.*

Case I. *When each of the limits is zero.*

Let $x, y, z, \cdots$ be variables, of which none is greater than z. Let n be the number of variables.

Then $\qquad x + y + z + \cdots = nz$ or $< nz.$

Now nz can be made less than any assigned constant (§ 434, I).

Therefore, $x + y + z + \cdots$, which is equal to or less than nz, can be made less than any assigned constant;

that is, the limit of $x + y + z + \cdots$ is 0, (p. 350, § 429)

which is the sum of the limits of $x, y, z, \cdots$

CASE II. *When none of the limits is zero.*

Let x, y, z, $\cdots$ be variables, and a, b, c, $\cdots$ their limits.

Then $a-x$, $b-y$, $c-z$, $\cdots$ are variables, the limit of each of which is 0. (p. 350, § 429)

Hence, the limit of $(a-x)+(b-y)+(c-z)+\cdots$ is 0. (Case I)

That is, the limit of $(a+b+c+\cdots)-(x+y+z+\cdots)$ is 0.

$\therefore\ a+b+c+\cdots$ is the limit of $(x+y+z+\cdots)$. (§ 429)

CASE III. *When some of the limits are zero and the others are not zero.*

The proof is left as an exercise for the student.

439. Limit of a Product. *The limit of the product of any finite number of variables is the product of their limits.*

Let x and y be variables, a and b their limits.

Put $x=a-x'$, $y=b-y'$; then x' and y' are variables that can be made less than any assigned constant. (p. 350, § 429)

Now $\quad xy=(a-x')(b-y')=ab-ay'-bx'+x'y'.$

$$\therefore\ ab-xy=ay'+bx'-x'y'.$$

As each term on the right contains x' or y', the right member can be made less than any assigned constant. (p. 354, § 438)

Hence, $ab-xy$ can be made less than any assigned constant.

Therefore, ab is the limit of xy. (p. 350, § 429)

Similarly for three or more variables.

440. Limit of a Quotient. *The limit of the quotient of two variables is the quotient of their limits, if the divisor is not zero.*

Let x and y be variables, a and b their limits.

Put $a-x=x'$, and $b-y=y'$; then x' and y' are variables with limit 0. (p. 350, § 429)

We have $\qquad x=a-x',\ y=b-y',\ \text{and}\ \dfrac{x}{y}=\dfrac{a-x'}{b-y'}.$

Now $\qquad \dfrac{a}{b}-\dfrac{x}{y}=\dfrac{a}{b}-\dfrac{a-x'}{b-y'}=\dfrac{bx'-ay'}{b(b-y')}.$

The numerator of the last expression approaches 0 as a limit, and the denominator approaches b^2 as a limit; hence, the expression approaches 0 as a limit. (p. 352, § 434, I)

Therefore, $\frac{a}{b} - \frac{x}{y}$ approaches 0 as a limit.

Therefore, $\frac{a}{b}$ is the limit of $\frac{x}{y}$. (p. 350, § 429)

441. Vanishing Fractions. When variables are involved in both numerator and denominator of a fraction it may happen that for certain values of the variables the numerator and the denominator both vanish. The fraction then assumes the indeterminate form $\frac{0}{0}$, a form without meaning; as even the interpretation of p. 352, § 435, fails, since the numerator is 0.

If, however, there is but *one* variable involved, we may obtain a value as follows: Let x be the variable, and a be the value of x for which the fraction assumes the indeterminate form. Give to x a value a little greater than a, as $a + z$; the fraction now has a definite value. Find the limit of this last value as z is indefinitely decreased.

This limit is called the **limiting value** of the fraction.

The fundamental indeterminate form is $\frac{0}{0}$, since all other indeterminate forms may be reduced to this.

Thus,
$$\frac{\infty}{\infty} = \frac{a}{0} \div \frac{b}{0} = \frac{a}{0} \times \frac{0}{b} = \frac{0}{0};$$
$$0 \times \infty = 0 \times \frac{a}{0} = \frac{0 \times a}{0} = \frac{0}{0};$$
$$\infty - \infty = \frac{a}{0} - \frac{b}{0} = \frac{0 \times a - 0 \times b}{0} = \frac{0}{0}.$$

1. Find the limiting value of $\frac{x^2 - a^2}{x - a}$ as $x \doteq a$.

When x has the value a the fraction assumes the form $\frac{0}{0}$.
Put $x = a + z$; the fraction becomes

$$\frac{(a+z)^2-a^2}{(a+z)-a}=\frac{2az+z^2}{z}.$$

Since z is not 0, we divide by z and obtain $2a+z$.
As z is indefinitely decreased, $2a+z$ approaches $2a$ as a limit.
Therefore, $2a$ is the limiting value of the fraction as $x \doteq a$.

2. Find the limiting value of $\dfrac{2x^3-4x+5}{3x^3+2x^2-1}$ as $x \doteq \infty$.

Divide each term of the numerator and denominator by x^3.

Then $$\frac{2x^3-4x+5}{3x^3+2x^2-1}=\frac{2-\dfrac{4}{x^2}+\dfrac{5}{x^3}}{3+\dfrac{2}{x}-\dfrac{1}{x^3}}.$$

As x increases indefinitely, $\dfrac{4}{x^2}, \dfrac{5}{x^3}, \dfrac{2}{x}, \dfrac{1}{x^3}$, each approaches 0 as a limit (p. 352, § 434, IV) and the fraction approaches $\frac{2}{3}$ as a limit.

EXERCISE 132

Find the limiting value of:

1. $\dfrac{x^2-5x+4}{x^2-6x+5}$ as $x \doteq 1$.
2. $\dfrac{27-x^3}{3-x}$ as $x \doteq 3$.
3. $\dfrac{\sqrt{x+1}-\sqrt{2}}{x-1}$ as $x \doteq 1$.
4. $\dfrac{\sqrt{a}-\sqrt{x}}{\sqrt{a-x}}$ as $x \doteq a$.
5. $\dfrac{x^2-3ax+2a^2}{x^2+4ax-5a^2}$ as $x \doteq a$.
6. $\dfrac{3x^3-4x^2-1}{2x^3+2x-5}$ as $x \doteq \infty$.
7. $\dfrac{\sqrt[4]{5x-1}-\sqrt{2}}{x-1}$ as $x \doteq 1$.
8. $\dfrac{y^3-2y^2-y+2}{y^2-5y+6}$ as $y \doteq 2$.
9. $\dfrac{x^2+y^2+2xy-4}{x+y-2}$ as $x \doteq 1$ and $y \doteq 1$.
10. $\dfrac{2xz-x^2+y^2-z^2}{2xy-x^2-y^2+z^2}$ as $z \doteq x-y$.
11. $\dfrac{4x^3+3x^2+2x+1}{5x^3-3x^2+7x+3}$ as $x \doteq \infty$.

CHAPTER XXV

SERIES

442. Convergent Series. In the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

we find, by p. 350, § 428, that the sum of the first n terms, as n increases without limit, approaches 2 as its limit. Such a series is called a **convergent series.** That is,

An infinite series is a convergent series if the sum of the terms, as the number of terms is indefinitely increased, approaches some fixed finite value as a limit.

443. Divergent Series. In the infinite series

$$1 + 2 + 4 + 8 + \cdots$$

the sum of the first n terms, as n increases without limit, does not approach a fixed finite value as its limit, but increases without limit. Such a series is called a **divergent series.** That is,

An infinite series is a divergent series, if the sum of the terms, as the number of terms is indefinitely increased, increases numerically without limit.

444. Oscillating Series. In the infinite series

$$1 - 1 + 1 - 1 + \cdots$$

the sum of the first n terms, as n increases without limit, is either 1 or 0, according as n is an odd number or an even number. Such a series is called an **oscillating series.** That is,

An infinite series is an oscillating series, if the sum of the terms, as the number of terms is indefinitely increased, is repeatedly in turn one of a fixed number of finite values.

EXERCISE 133

Determine the kind of series:

1. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$
2. $1 - 2 + 4 - 8 + 16 - \cdots$
3. $1 - 2 - 1 + 2 + 1 - 2 - 1 + 2 + \cdots$
4. Show that the harmonic series

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \cdots \text{ is divergent.}$$

HINT. Write the series thus: $\frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots$ Then show that the value of each group is greater than $\frac{1}{2}$ and that the sum of the series increases without limit.

5. Show that the series

$$\frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \cdots \text{ is convergent.}$$

HINT. Show that each term after the second is less than the corresponding term of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$; therefore, the sum of any number of terms is less than 2.

445. Series Resulting from Division. By performing the indicated division we obtain from $\dfrac{1}{1 - x}$ the infinite series

$$1 + x + x^2 + x^3 + \cdots$$

For different values of x this series assumes different forms.

Thus, for the value $x = \frac{1}{3}$, the series becomes

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \cdots,$$

which is convergent, since it approaches the fixed value $\frac{3}{2}$ as the number of terms is indefinitely increased.

For the value $x = 3$, the series becomes

$$1 + 3 + 9 + 27 + \cdots,$$

which is plainly divergent, since each term is greater than the sum of all the preceding terms.

For the value $x = -1$, the series becomes

$$1 - 1 + 1 - 1 + \cdots,$$

which is oscillating.

If for x, in the fraction $\frac{1}{1-x}$, these same values of x are substituted, we obtain the values $\frac{3}{2}$, $-\frac{1}{2}$, and $\frac{1}{2}$ respectively.

By comparing values of the fraction with corresponding values of the series it is seen that in general the value of the fraction is not equal to the corresponding value of the series. Only in case the series is convergent can the value of the fraction be said to equal the value which the series is found to approach.

In all that follows, the application of infinite series is restricted to such series as are convergent, or to such values of the variables as make the series convergent.

446. *If two infinite series, arranged by powers of* x, *are equal for all values of* x *that make both series convergent, the corresponding coefficients are equal each to each.*

For, if

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots,$$

by transposition

$$A - A' = (B' - B)x + (C' - C)x^2 + (D' - D)x^3 + \cdots$$

Since this is true for all values of x, it is true when x approaches 0 as a limit. Hence, by taking x sufficiently small, the right member of this equation can be made *less* than any assigned value whatever, and therefore can be made less than $A - A'$, if $A - A'$ has any value whatever.

Hence, $A - A'$ cannot have any value.

Therefore, $\quad A - A' = 0, \quad$ or $\quad A = A'.$

Therefore,

$$Bx + Cx^2 + Dx^3 + \cdots = B'x + C'x^2 + D'x^3 + \cdots,$$

or $$(B - B')x = (C' - C)x^2 + (D' - D)x^3 + \cdots$$

Divide by x,

$$B - B' = (C' - C)x + (D' - D)x^2 + \cdots$$

By the same proof as for $A - A'$,

$$B - B' = 0, \quad \text{or} \quad B = B'.$$

In like manner, $C = C'$, $D = D'$, and so on.

Hence, the equation

$$A + Bx + Cx^2 + Dx^3 + \cdots = A' + B'x + C'x^2 + D'x^3 + \cdots,$$

if true for all finite values of x, is an **identical equation**; that is, *the coefficients of like powers of* x *are equal.*

It is easily shown that the theorem is true also when one of the series is finite or both series are finite.

447. Series Resulting from Undetermined Coefficients. We shall illustrate this method by two examples.

1. Expand $\dfrac{2 + 3x}{1 + x + x^2}$ in ascending powers of x.

Assume $$\frac{2 + 3x}{1 + x + x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$

To determine the coefficients in the right member of this equation, clear of fractions and collect the coefficients of like terms in the right member. Then

$$2 + 3x = A + (B + A)x + (C + B + A)x^2 + (D + C + B)x^3 + \cdots$$

Now, if we regard the left member as an infinite series with coefficients of all terms after the second as 0, by p. 360, § 446,

$$A = 2,\ B + A = 3,\ C + B + A = 0,\ D + C + B = 0, \text{ and so on;}$$

whence, $$B = 1,\ C = -3,\ D = 2,\ E = 1, \text{ and so on.}$$

$$\therefore \frac{2 + 3x}{1 + x + x^2} = 2 + x - 3x^2 + 2x^3 + x^4 - \cdots$$

This series is, of course, equal to the fraction for only such values of x as make the series convergent.

By dividing the numerator of the given fraction by the denominator, it is found that the quotient thus obtained is the same as the series resulting from the above process.

NOTE. In employing the method of undetermined coefficients the form of the given expression determines what powers of the variable x must be assumed. It is *necessary* that the assumed equation, when simplified, shall have in the right member every power of x that is found in the left member.

If any powers of x occur in the *right* member that are not in the *left* member, the coefficients of these powers in the right member will vanish, so that in this case the method still applies; but if any powers of x occur in the *left* member that are not in the *right* member, then the coefficients of these powers of x must be put equal to 0 in equating the coefficients of like powers of x; and this leads to absurd results. Thus, if it were assumed that

$$\frac{2+3x}{1+x+x^2} = Ax + Bx^2 + Cx^3 + \cdots,$$

there would be in the simplified equation no term on the right corresponding to 2 on the left; so that, in equating the coefficients of like powers of x, 2, which is $2x^0$, would be put equal to $0x^0$; that is, $2 = 0$, an absurdity.

2. Expand $\sqrt{a-x}$ by the use of undetermined coefficients.

Assume $\sqrt{a-x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$

Square, $$a - x = A^2 + 2ABx + \left.\begin{array}{r} B^2 \\ +2AC \end{array}\right| x^2 + \left.\begin{array}{r} 2AD \\ +2BC \end{array}\right| x^3 + \left.\begin{array}{r} C^2 \\ +2AE \\ +2BD \end{array}\right| x^4 + \cdots$$

Equate the coefficients of like powers of x.

$$A^2 = a; \qquad \therefore A = \sqrt{a}.$$

$$2AB = -1; \qquad \therefore B = -\frac{1}{2\sqrt{a}} = -\frac{\sqrt{a}}{2a}.$$

$$B^2 + 2AC = 0; \qquad \therefore C = -\frac{1}{4a \cdot 2\sqrt{a}} = -\frac{\sqrt{a}}{8a^2}.$$

$$2AD + 2BC = 0; \qquad \therefore D = -\frac{1}{8a^2 \cdot 2\sqrt{a}} = -\frac{\sqrt{a}}{16a^3}.$$

$$C^2 + 2AE + 2BD = 0; \qquad \therefore E = -\frac{5}{64a^3 \cdot 2\sqrt{a}} = -\frac{5\sqrt{a}}{128a^4}.$$

$$\therefore \sqrt{a-x} = \sqrt{a} - \frac{\sqrt{a}}{2a}x - \frac{\sqrt{a}}{8a^2}x^2 - \frac{\sqrt{a}}{16a^3}x^3 - \frac{5\sqrt{a}}{128a^4}x^4 - \cdots$$

$$= \sqrt{a}\left(1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3} - \frac{5x^4}{128a^4} - \cdots\right).$$

This series is the same series as may be obtained by the extraction of the square root of $a - x$.

Here, again, the series is equal to the expression from which it is derived only for such values of x as make the series convergent.

Exercise 134

Expand to four terms:

1. $\dfrac{1}{1+2x}$.
3. $\dfrac{1-x}{1+x+x^2}$.
5. $\dfrac{3-x}{1+x+x^2}$.
2. $\dfrac{1}{3-5x}$.
4. $\dfrac{5-x}{1+x-x^2}$.
6. $\dfrac{4-5x}{1-2x+3x^2}$.

Expand to five terms:

7. $\dfrac{5}{3+x}$.
9. $\dfrac{7-3x}{1+5x-x^2}$.
11. $\dfrac{1-8x}{1-x-6x^2}$.
8. $\dfrac{3-x}{4+x}$.
10. $\dfrac{x^2+x-1}{x(x+2)}$.
12. $\dfrac{x^2+x+1}{(x+1)(x^2+1)}$.

Expand to four terms:

13. $\sqrt{1+2a}$.
15. $\sqrt{4-3a}$.
17. $\sqrt{1-x+x^2}$.
14. $\sqrt{1-2a}$.
16. $\sqrt{1+x}$.
18. $\sqrt[3]{1-x+x^2}$.

448. Partial Fractions. To resolve a fraction into *partial fractions* is to express it as the sum of a number of fractions the denominators of which are the factors of the denominator of the given fraction. This process is the reverse of the process of *adding* fractions that have different denominators.

Resolution into partial fractions may be easily accomplished by the use of undetermined coefficients and the theorem of p. 360, § 446.

In decomposing a given fraction into its simplest partial fractions it is important to determine what form the assumed fractions must have.

Since the given fraction is the *sum* of the required partial fractions, each assumed denominator must be a factor of the given denominator, and since the required partial fractions are to be in their simplest form, incapable of further decomposition, the numerator of each required fraction must be assumed with reference to this condition.

1. *If all the linear factors are real and different.*

In this case we take each factor of the given denominator as the denominator of one of the partial fractions.

Thus, $$\frac{3x^2-5x-7}{x(x-1)(x+2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}.$$

2. *If all the linear factors are real and equal.*

In this case we take as denominators every power of the repeated factor from the given power down to the first.

Thus, $$\frac{x^2-7x-1}{(x+2)^3}=\frac{A}{(x+2)^3}+\frac{B}{(x+2)^2}+\frac{C}{x+2}.$$

3. *If all the linear factors are real and some equal.*

In this case we combine the methods of the first two cases.

Thus, $$\frac{3x^3-2x^2+47x-101}{(x+1)(x-2)^3}=\frac{A}{x+1}+\frac{B}{(x-2)^3}+\frac{C}{(x-2)^2}+\frac{D}{x-2}.$$

4. *If all or some of the linear factors are imaginary.*

The imaginary factors occur in pairs of conjugate imaginaries, so that the product of each pair is a real quadratic factor.

Thus, if a quadratic factor of the given denominator is x^2+2x+5, it is equal to $\left[(x+1)+2\sqrt{-1}\right]\left[(x+1)-2\sqrt{-1}\right]$.

In this case we assume a fraction of the form $\dfrac{Ax+B}{x^2+ax+b}$ for each quadratic factor in the given denominator.

Thus,
$$\frac{5x^4-3x^3+9x^2+5x-17}{(x+2)(x^2+3x-7)(x^2-5x+11)}=\frac{A}{x+2}+\frac{Bx+C}{x^2+3x-7}+\frac{Dx+E}{x^2-5x+11}.$$

In general, for the assumed fractions only fractions the degree of whose numerator is lower than the degree of the denominator need be considered.

1. Resolve $\frac{3x-7}{x^2-5x+6}$ into partial fractions.

Since $x^2-5x+6=(x-2)(x-3)$, the denominators of the partial fractions will be $x-2$ and $x-3$.

Assume $$\frac{3x-7}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}.$$

Clear of fractions, $$3x-7=A(x-3)+B(x-2),$$

or $$3x-7=(A+B)x-(3A+2B).$$

$$\therefore\ A+B=3, \text{ and } 3A+2B=7. \quad \text{(p. 360, § 446)}$$

From these two relations of A and B we obtain

$$A=1, \text{ and } B=2.$$

Therefore, $$\frac{3x-7}{x^2-5x+6}=\frac{1}{x-2}+\frac{2}{x-3}.$$

This result may be verified by actual simplification.

2. Resolve $\frac{2x^2-7x+1}{x^3-1}$ into partial fractions.

Since $x^3-1=(x-1)(x^2+x+1)$, the denominators of the partial fractions will be $x-1$ and x^2+x+1.

Assume $$\frac{2x^2-7x+1}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.$$

Clear of fractions, $$2x^2-7x+1=Ax^2+Ax+A+Bx^2-Bx+Cx-C,$$

or $$2x^2-7x+1=(A+B)x^2+(A-B+C)x+(A-C).$$

$$\therefore\ A+B=2,\ A-B+C=-7, \text{ and } A-C=1. \quad \text{(p. 360, § 446)}$$

From these three relations of A, B, and C we obtain

$$A=-\tfrac{4}{3},\ B=\tfrac{10}{3}, \text{ and } C=-\tfrac{7}{3}.$$

Therefore, $$\frac{2x^2-7x+1}{x^3-1}=\frac{-\frac{4}{3}}{x-1}+\frac{\frac{10}{3}x-\frac{7}{3}}{x^2+x+1},$$

or $$\frac{2x^2-7x+1}{x^3-1}=\frac{10x-7}{3(x^2+x+1)}-\frac{4}{3(x-1)}.$$

The result may be verified by actual simplification.

3. Resolve $\dfrac{x^3+4x^2-8x-20}{x^2(x+2)^2}$ into partial fractions.

The denominators of the partial fractions are x, x^2, $x+2$, $(x+2)^2$.

Assume $$\frac{x^3+4x^2-8x-20}{x^2(x+2)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}+\frac{D}{(x+2)^2}.$$

$$\therefore\ x^3+4x^2-8x-20=Ax^3+4Ax^2+4Ax+Bx^2+4Bx+4B+Cx^3+2Cx^2+Dx^2,$$

or $$x^3+4x^2-8x-20=(A+C)x^3+(4A+B+2C+D)x^2+(4A+4B)x+4B.$$

$\therefore\ A+C=1,\ 4A+B+2C+D=4,\ 4A+4B=-8,\ 4B=-20;$

whence $A=3,\ B=-5,\ C=-2,\ D=1.$

$$\therefore\ \frac{x^3+4x^2-8x-20}{x^2(x+2)^2}=\frac{3}{x}-\frac{5}{x^2}-\frac{2}{x+2}+\frac{1}{(x+2)^2}.$$

Exercise 135

Resolve into partial fractions:

1. $\dfrac{2x+1}{x^2+x-30}$.

2. $\dfrac{2x}{x^2-1}$.

3. $\dfrac{9ac}{a^3-27c^3}$.

4. $\dfrac{2a^3}{1+a^2+a^4}$.

5. $\dfrac{2}{x^3+3x^2+2x}$.

6. $\dfrac{2-d}{d(d^2-1)}$.

7. $\dfrac{x^2+2x+6}{x^3-1}$.

8. $\dfrac{4x-3x^2-8}{x^3+1}$.

9. $\dfrac{7x-19}{x^2-6x+9}$.

10. $\dfrac{7x^2-44x+32}{x^3-8x^2+16x}$.

11. $\dfrac{5x^3-11x^2+8x-6}{x^4-2x^3+4x-4}$.

12. $\dfrac{4x^2}{1-x^4}$.

13. $\dfrac{5x^3-13x^2-4x+35}{x^2(x^2-5x+7)}$.

14. $\dfrac{x^3+1}{x(x-1)^3}$.

15. $\dfrac{3x^3-7x^2-35x+6}{x^4-2x^3-3x^2+4x+4}$.

16. $\dfrac{3x^3-21x^2-6x+51}{x^4-6x^3+3x^2+8x-6}$.

CHAPTER XXVI

LOGARITHMS

449. We have already learned that

$$a^m \times a^n = a^{m+n}, \qquad \text{(p. 43, § 83)}$$
$$a^m \div a^n = a^{m-n}, \qquad \text{(p. 236, § 282)}$$
$$(a^m)^n = a^{mn}. \qquad \text{(p. 236, § 283)}$$

If M is put equal to a^m and N equal to a^n, then

$$M \times N = a^{m+n},$$
$$M \div N = a^{m-n},$$
$$M^n = a^{mn}.$$

Hence, if all numbers are expressed as powers of the same number, called the **base**,

A product of two or more factors may be obtained by adding the exponents of the factors.

A quotient of one number by another may be obtained by subtracting the exponent of the divisor from the exponent of the dividend.

A power of a number may be obtained by multiplying the exponent of the number by the index of the required power.

For example, if $384 = 10^{2.5843}$, $625 = 10^{2.7959}$, $24 = 10^{1.3802}$,

then $\qquad 384 \times 625 = 10^{2.5843} \times 10^{2.7959} = 10^{2.5843 + 2.7959} = 10^{5.3802}.$

Now $\qquad 10^{5.3802} = 10^4 \times 10^{1.3802} = 10{,}000 \times 24 = 240{,}000.$

Therefore, $\qquad 384 \times 625 = 240{,}000.$

The number 2.5843, which is the exponent of 10, is called the **logarithm** *of* 384 *to the base* 10. Also the number 2.7959 is the logarithm of 625 to the base 10, and the number 1.3802 is the logarithm of 24 to the base 10.

When exponents are thought of as logarithms, it is customary to express the relation

$$384 = 10^{2.5843}$$

in the following terms:

$$\log_{10} 384 = 2.5843.$$

This is read *the logarithm of* 384 *to the base* 10 *is* 2.5843.

In general, if $N = a^n$, then $\log_a N = n$.

This last statement expressed in ordinary language means that if numbers are expressed as powers of the same base a, the *exponents* are the *logarithms* of the respective numbers.

When 10 is used as the base, it is customary in writing numbers to omit the 10.

Thus, the logarithm of 384 to the base 10 is written $\log 384 = 2.5843$.

450. Common or Briggs logarithms. When numbers are thus referred to the *base ten*, the corresponding logarithms are said to form a system of **common logarithms** or **Briggs logarithms.**

451. From § 449 it is evident that:

The logarithm of a product is equal to the sum of the logarithms of the factors.

The logarithm of a quotient is equal to the logarithm of the dividend less that of the divisor.

The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Exercise 136

1. What is the logarithm of 24 if $24 = 10^{1.3802}$?
2. What is the logarithm of $\frac{1}{10}$?
3. What is the logarithm of $10^8 \times 10^4$?
4. What is the logarithm of 8 to the base 2?

5. May 1 be taken as the base of a system of logarithms? Why?

6. What is the logarithm of -27 to the base -3?

7. May a negative number be used as the base of a complete system of logarithms? Why?

8. Write the common logarithm of each of the following:

$$1000,\ 0.001,\ 10^8,\ \sqrt{10},\ \sqrt[3]{10^2},\ 10^{\frac{2}{3}} \div 10^{\frac{3}{4}}.$$

9. If the logarithm of 4.94 is 0.6937, what is the logarithm of 494? of 0.494?

10. If the logarithm of 2 is 0.3010, find the logarithm of 2^6; of $\sqrt{2}$; of $\sqrt[3]{2}$.

452. The logarithms of 1, 10, 100, etc., and of 0.1, 0.01, 0.001, etc., are integral numbers. The logarithms of all other numbers are either fractions or mixed numbers.

$$\begin{aligned}
\text{Since } 1 &= 10^0, & &\text{the logarithm of } 1 = 0;\\
10 &= 10^1, & &\text{the logarithm of } 10 = 1;\\
100 &= 10^2, & &\text{the logarithm of } 100 = 2;\\
1000 &= 10^3, & &\text{the logarithm of } 1000 = 3;\\
0.1 &= \tfrac{1}{10} = 10^{-1}, & &\text{the logarithm of } 0.1 = -1;\\
0.01 &= \tfrac{1}{100} = 10^{-2}, & &\text{the logarithm of } 0.01 = -2;\\
0.001 &= \tfrac{1}{1000} = 10^{-3}, & &\text{the logarithm of } 0.001 = -3.
\end{aligned}$$

Hence, the common logarithm of every number between

1 and 10	is	$0 +$ a fraction;
10 and 100	is	$1 +$ a fraction;
100 and 1000	is	$2 +$ a fraction;
1 and 0.1	is	$-1 +$ a fraction;
0.1 and 0.01	is	$-2 +$ a fraction;
0.01 and 0.001	is	$-3 +$ a fraction.

453. If the logarithm is less than 1, the logarithm is negative (§ 452), but is written in such a form that the fractional part is always positive.

454. Characteristic and Mantissa. Every logarithm, therefore, consists of two parts: a positive or negative integral number, which is called the **characteristic**, and a *positive* decimal fraction, which is called the **mantissa.**

Thus, the logarithm of 3326 is 3.5219, in which the positive integral number 3 is the characteristic and the positive fraction .5219 is the mantissa; the logarithm of 0.0439 is $\bar{2}.6425$, in which the negative integral number -2 is the characteristic and the positive fraction .6425 is the mantissa. The negative sign above the integral number 2 in $\bar{2}.6425$ is so placed to show that only the 2 is negative; whereas -2.6425 means that the entire number is negative.

455. If the logarithm has a negative characteristic, it is customary to change its form by adding 10, or a multiple of 10, to the characteristic, and then indicating the subtraction of the same number from the result.

Thus, the logarithm $\bar{2}.6425$ is changed to $8.6425-10$ by adding 10 to the characteristic and writing -10 after the result; the logarithm $\overline{12}.6425$ is changed to $8.6425-20$ by adding 20 to the characteristic and writing -20 after the result.

456. Characteristic. From p. 369, § 452, it is evident that the *characteristic* of the logarithm of a given number may be written at once.

For, if the given number is greater than 1, the characteristic of the logarithm is one unit less than the number of integral figures in the number;

If the given number is less than 1, the characteristic of the logarithm is negative and is one unit more than the number of zeros between the decimal point and the first significant figure of the number.

Thus, the characteristic of log 2.8 is 0; the characteristic of log 4562.76 is 3; the characteristic of log 0.97 is -1, or $9-10$; the characteristic of log 0.045 is -2, or $8-10$.

457. Mantissa. The *mantissa* of the common logarithm of any integral number, decimal fraction, or mixed number depends only upon the sequence of the digits of the number, and is unchanged so long as the *sequence of the digits* remains the same.

For, changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its common logarithm, therefore, is increased or diminished by the *exponent* of that power of 10; and since this exponent is *integral*, the *mantissa*, or decimal part of the logarithm, is unaffected.

Thus,

$$271{,}940 = 10^{5.4345}, \qquad 2.7194 = 10^{0.4345},$$
$$27{,}194 = 10^{4.4345}, \qquad 0.27194 = 10^{9.4345-10},$$
$$2719.4 = 10^{3.4345}, \qquad 0.027194 = 10^{8.4345-10},$$
$$271.94 = 10^{2.4345}, \qquad 0.0027194 = 10^{7.4345-10},$$
$$27.194 = 10^{1.4345}, \qquad 0.00027194 = 10^{6.4345-10}.$$

The great advantage of using the number *ten* as the base of a system of logarithms consists in the fact that the *mantissa* of a given number depends only on the *sequence of the digits*, and the *characteristic* depends only on the *position of the decimal point.*

The mantissa of the logarithm may be found by means of a Table of Logarithms.

458. Table of Four-Place Logarithms. A table of *four-place* common logarithms is given on pages 372 and 373. This table contains the common logarithms of all numbers under 1000, *the decimal point and characteristic being omitted.* The logarithms of numbers composed of the single digits 1, 2, 3, ⋯ are found at 10, 20, 30, ⋯

In working with a four-place table, the numbers corresponding to the logarithms, that is, the **antilogarithms**, as they are called, may be carried to *four significant digits* without reasonable chance of error.

Tables that contain logarithms to more places than four can be procured, but this four-place table will serve for many practical uses, and will enable the student to use tables of five-place, seven-place, and ten-place logarithms in work that requires greater accuracy.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

459. The mantissa of a logarithm can, in general, be expressed only approximately, and in a four-place table all figures that follow the fourth are rejected. *Whenever the fifth figure is* 5 *or more the fourth figure is increased by* 1.

Thus, if the mantissa of a logarithm written to seven places is 5328372, it is written in a four-place table 5328; if the mantissa is 5328732, it is written in a four-place table 5329.

460. To Find the Logarithm of a Given Number from the Table.

I. If the given number has *two* significant figures, the number is in the column headed **N** (pp. 372, 373), and the *mantissa* of its logarithm in the column headed **0**, and in the same line with the number. To this mantissa the proper characteristic should be prefixed.

Thus,	$\log 25 = 1.3979$,	$\log 2500 = 3.3979$;
	$\log 37 = 1.5682$,	$\log 3.7 = 0.5682$;
	$\log 72 = 1.8573$,	$\log 0.072 = 8.8573 - 10$.

II. If the given number has *three* significant figures, the first two significant figures are in the column headed **N**, and the third figure is found at the top of the page in the line containing the figures **0**, **1**, **2**, **3**, etc. The mantissa is found in the column headed by the third figure, and in the same line with the first two significant figures.

Thus,	$\log 453 = 2.6561$,	$\log 4.53 = 0.6561$;
	$\log 768 = 2.8854$,	$\log 76{,}800 = 4.8854$;
	$\log 935 = 2.9708$,	$\log 0.935 = 9.9708 - 10$.

III. If the given number has *four or more* significant figures, a process called **interpolation** is required.

Interpolation is based on the *assumption* that between two consecutive mantissas of the table the change in the mantissa is proportional to the change in the number. This assumption is not strictly correct, for the numbers form an arithmetical series, and their logarithms form a geometrical series. However, this method yields remarkably close approximations.

1. Find the logarithm of 3424.

The required mantissa is the same as the mantissa for 342.4; therefore, it will be found by adding to the mantissa for 342 four tenths of the difference between the mantissas for 342 and 343.

The mantissa for 342 is 5340; the mantissa for 343 is 5353.

The difference between the mantissas for 342 and 343 is 13.

Hence, the mantissa for 342.4 is $5340 + (0.4 \text{ of } 13) = 5340 + 5.2$, or 5345 to four figures.

The characteristic of the logarithm for 3424 is 3.

Therefore, the logarithm of 3424 is 3.5345.

2. Find the logarithm of 0.015764.

The required mantissa is the same as the mantissa for 157.64; therefore, it will be found by adding to the mantissa for 157 sixty-four hundredths of the difference between the mantissas for 157 and 158.

The mantissa for 157 is 1959; the mantissa for 158 is 1987.

The difference between the mantissas for 157 and 158 is 28.

Hence, the mantissa for 157.64 is $1959 + (0.64 \text{ of } 28) = 1959 + 17.92$, or 1977 to four figures.

The characteristic of the logarithm for 0.015764 is $8 - 10$.

Therefore, the logarithm of 0.015764 is $8.1977 - 10$.

Note. When the fraction of a unit in the part to be added to the mantissa for three figures is less than 0.5, it is to be neglected; when it is 0.5 or more, it should be taken as one unit.

461. To Find the Antilogarithm of a Given Logarithm. If the given mantissa can be found in the table, the first two figures of the required antilogarithm are in the same line with the mantissa in the column headed **N**, and the third figure of the antilogarithm is the figure in heavy type at the top of the column containing the mantissa.

The position of the decimal point is determined by the characteristic. For, if the characteristic is *positive*, the number of figures in the integral part of the corresponding number is *one more* than the number of units in the characteristic; if the characteristic is *negative*, the number of zeros between the decimal point and the first significant figure of the corresponding number is *one less* than the number of units in the characteristic (see p. 370, § 456).

1. Find the antilogarithm of 0.9201.

The number corresponding to the mantissa 9201 is 832. The characteristic is 0; hence, the required antilogarithm is 8.32.

2. Find the antilogarithm of 4.0969.

The number corresponding to the mantissa 0969 is 125. The characteristic is 4; hence, the required antilogarithm is 12,500.

3. Find the antilogarithm of 8.5065 − 10.

The number corresponding to the mantissa 5065 is 321. The characteristic is −2; hence, the required antilogarithm is 0.0321.

If the given mantissa cannot be found in the table, find in the table the two adjacent mantissas between which the given mantissa lies, and the three figures corresponding to the smaller of these two mantissas are the first three significant figures of the required antilogarithm. To obtain the fourth significant figure of the antilogarithm it is necessary to use interpolation.

4. Find the antilogarithm of 1.8895.

The mantissa 8895 cannot be found in the table, but the two adjacent mantissas between which it lies are 8893 and 8899. The difference between these adjacent mantissas is 6, and the difference between 8893 and the given mantissa 8895 is 2. Therefore, two sixths of the difference between the numbers corresponding to the mantissas 8893 and 8899 must be added to the number corresponding to the mantissa 8893.

The number corresponding to the mantissa 8893 is 7750.

The number corresponding to the mantissa 8899 is 7760.

The difference between these numbers is 10, and

$$7750 + \tfrac{2}{6} \text{ of } 10 = 7753.$$

The characteristic is 1; hence, the required antilogarithm is 77.53.

NOTE. In finding antilogarithms it should be remembered that an antilogarithm can be carried safely without danger of error to only four significant digits (p. 371, § 458).

5. Find the antilogarithm of $9.0940-10$.

The mantissa 0940 cannot be found in the table, but the two adjacent mantissas between which it lies are 0934 and 0969. The difference between these adjacent mantissas is 35, and the difference between 0934 and the given mantissa 0940 is 6. Therefore, six thirty-fifths of the difference between the numbers corresponding to the mantissas 0934 and 0969 must be added to the number corresponding to the mantissa 0934.

The number corresponding to the mantissa 0934 is 1240.

The number corresponding to the mantissa 0969 is 1250.

The difference between these numbers is 10, and

$$1240+\tfrac{6}{35} \text{ of } 10=1242.$$

The characteristic is $9-10$; hence, the required antilogarithm is 0.1242.

462. Cologarithms. The logarithm of the reciprocal of a given number is called the **cologarithm** of the number.

Thus,
$$\begin{aligned}\text{cologarithm } N &= \log\frac{1}{N}\\ &= \log 1-\log N\\ &= 0-\log N\\ &= -\log N\\ &= (10-\log N)-10.\end{aligned}$$

Hence, the cologarithm of a given number is equal to the logarithm of the number with the minus sign prefixed, and the negative sign affects *the entire logarithm.*

463. To avoid a negative mantissa in the cologarithm, it is customary to use for $-\log N$ its equivalent $(10-\log N)-10$.

The cologarithm is abbreviated *colog*, and is most easily found by beginning with the characteristic of the logarithm and subtracting each figure from 9, down to the last significant figure, and subtracting that figure from 10.

1. Find the cologarithm of 70.

$$\begin{aligned}\log 1 &= 10. \qquad -10\\ \log 70 &= \underline{1.8451\qquad\quad}\\ \text{colog } 70 &= 8.1549-10\end{aligned}$$

In practice, to find colog 70, we subtract, mentally, 1 from 9, 8 from 9, 4 from 9, 5 from 9, 1 from 10, and write the resulting figure at each step.

2. Find the cologarithm of 0.00673.

$$\begin{aligned} \log 1 &= 10. \qquad\quad -10 \\ \log 0.00673 &= \underline{\ \ 7.8280 - 10} \\ \text{colog}\, 0.00673 &= \ \ 2.1720 \end{aligned}$$

In practice, we subtract 7 from 9, 8 from 9, 2 from 9, 8 from 10.

464. From the two examples of § 463 it may be seen that if the logarithm of a given number does not have -10 affixed to it the cologarithm of the number does, and if the logarithm of a given number has -10 affixed to it the cologarithm of the number does not.

465. From § 462 the logarithm of a quotient may be found by *adding* the *logarithm* of the dividend to the *cologarithm* of the divisor.

Thus,
$$\begin{aligned} \log \frac{5}{0.002} &= \log 5 + \text{colog}\, 0.002 \\ &= 0.6990 + 2.6990 \\ &= 3.3980. \end{aligned}$$

Also,
$$\begin{aligned} \log \frac{0.07}{2^3} &= 8.8451 - 10 + 9.0970 - 10 \\ &= 17.9421 - 20 \\ &= 7.9421 - 10. \end{aligned}$$

Here
$$\log 2^3 = 3 \log 2 = 3 \times 0.3010 = 0.9030.$$

466. In finding the logarithm of a root if the characteristic of the logarithm of the power is *negative*, there should be added to the logarithm equal positive and negative numbers such that the resulting negative number, when divided by the index of the root, gives a quotient of -10.

Thus,
$$\log 0.002^{\frac{1}{3}} = \tfrac{1}{3}\,(7.3010 - 10).$$

The expression $\frac{1}{3}\,(7.3010 - 10)$ is put in the form $\frac{1}{3}\,(27.3010 - 30)$, which is $9.1003 - 10$, since the addition of 20 to the 7, and of -20 to the -10, produces no change in the *value* of the logarithm.

That is,
$$\begin{aligned} \log 0.002^{\frac{1}{3}} &= \tfrac{1}{3}\,(27.3010 - 30) \\ &= 9.1003 - 10. \end{aligned}$$

Again,
$$\begin{aligned} \log 0.0002^{\frac{1}{5}} &= \tfrac{1}{5}\,(6.3010 - 10) \\ &= \tfrac{1}{5}\,(46.3010 - 50) \\ &= 9.2602 - 10. \end{aligned}$$

Exercise 137

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find the logarithms of the following numbers by resolving the numbers into factors and taking the sum of the logarithms of the factors:

1. 6.
2. 15.
3. 21.
4. 49.
5. 60.
6. 7.5.
7. 0.021.
8. 0.35.
9. 0.0035.
10. 12.5.
11. 420.
12. 0.004.
13. 0.63.
14. 105.
15. 0.0105.
16. 0.05.
17. 1.05.
18. 0.056.
19. 8×9.
20. 25×32.

21. Find $\log 5$, knowing that $\log 2 = 0.3010$ and $\log 10 = 1$.

Find the logarithms of the following quotients:

22. $\frac{2}{5}$.
23. $\frac{3}{7}$.
24. $\frac{7}{5}$.
25. $\frac{3}{5}$.
26. $\frac{7}{0.5}$.
27. $\frac{0.05}{3}$.
28. $\frac{0.0007}{0.2}$.
29. $\frac{0.003}{7}$.
30. $\frac{420}{125}$.
31. $\frac{0.02}{0.005}$.
32. $\frac{27}{10}$.
33. $\frac{100}{0.5}$.
34. $\frac{84}{25}$.
35. $\frac{0.05}{0.7}$.
36. $\frac{6 \times 7}{5 \times 5}$.
37. $\frac{7 \times 9}{8 \times 5}$.
38. $\frac{0.7 \times 0.9}{0.8 \times 0.5}$.
39. $\frac{0.07 \times 1.5}{0.002}$.
40. $\frac{0.1 \times 0.7 \times 0.2}{5 \times 0.03}$.
41. $\frac{0.02 \times 0.005}{0.07 \times 0.03}$.

Find the logarithms of the following powers and roots:

42. 2^3.
43. 7^4.
44. 5^5.
45. $2^{\frac{1}{4}}$.
46. $3^{\frac{1}{3}}$.
47. $\sqrt{7}$.
48. $3^{\frac{2}{5}}$.
49. $3^{\frac{9}{11}}$.
50. $\sqrt[3]{35}$.
51. $\sqrt{4.2}$.
52. $2.1^{\frac{1}{3}}$.
53. $7^{\frac{11}{7}}$.
54. 10.5^3.
55. $1.2^{\frac{1}{2}}$.
56. $6^3 \times 5^2$.
57. $7^{\frac{3}{5}} \times 3^{\frac{2}{9}}$.
58. $1.8^{\frac{1}{3}} \times 2.7^3$.
59. $3^{\frac{1}{2}} \times 4^{\frac{1}{3}} \times 5^{\frac{1}{4}}$.
60. $0.02^{\frac{1}{3}}$.
61. $1.05^{\frac{2}{5}} \times 0.03^{\frac{1}{2}} \times 0.02^3$.

Find from the table the common logarithm of:

62. 57.
63. 109.
64. 857.
65. 9901.
66. 5406.
67. 4552.
68. 5433.
69. 90,871.
70. 10,007.
71. 10,070.
72. 56.27.
73. 12.16.
74. 0.8770.
75. 0.0567.
76. 1.006.
77. 8.778.
78. 73.896.
79. 0.07069.
80. 0.008974.
81. $\pi = 3.1416$.

Find the antilogarithms of the following common logarithms:

82. 2.8142.
83. 3.6064.
84. 5.2695.
85. 1.7750.
86. 3.8941.
87. 2.1562.
88. 1.2982.
89. $\bar{1}.7903$.
90. $\bar{3}.6233$.
91. $\bar{4}.1547$.
92. 0.4382.
93. 4.2488 − 10.
94. 8.6330 − 10.
95. 2.5310 − 10.
96. 9.9486 − 10.
97. 1.9730.
98. 9.8800 − 10.
99. 0.2788.
100. 8.0060 − 10.
101. 7.0216 − 10.
102. 8.6582 − 10.

Find cologarithms corresponding to the following logarithms:

103. 0.6990.
104. 2.4843.
105. $\bar{1}.9274$.
106. 0.3010.
107. 9.7404 − 10.
108. 2.8779 − 10.
109. 9.6542 − 10.
110. 7.1673 − 10.
111. 3.7559.
112. 11.4263.
113. 5.0414 − 10.
114. 3.8260 − 10.

Find from the table the cologarithms of the following numbers:

115. 363.
116. 792.
117. 8652.
118. 178,000.
119. 4820.
120. 8.104.
121. 14.26.
122. 0.0243.
123. 43.256.
124. 15.7643.
125. 0.02634.
126. 2,716,000.
127. 0.0003827.
128. 0.045826.
129. 0.002835.

467. Computation by Logarithms.

1. Find the product of $26.45 \times 0.02687 \times 3.194$.

$$\begin{aligned} \log 26.45 &= 1.4224 \\ \log 0.02687 &= 8.4293 - 10 \\ \log 3.194 &= \underline{0.5043} \\ & 0.3560 \quad = \log 2.270. \end{aligned}$$

Therefore, the required product is 2.270.

By regular multiplication the product is 2.270012531.

2. Find the product of $5.674 \times (-0.8624) \times 0.3252$.

$$\begin{aligned} \log 5.674 &= 0.7539 \\ \log 0.8624 &= 9.9357 - 10\,n \\ \log 0.3252 &= \underline{9.5122 - 10} \\ & 0.2018\,n \quad = \log 1.591. \end{aligned}$$

When any factor is *negative* find its logarithm without regard to the sign; write n after the logarithm that corresponds to a negative number. If the number of logarithms marked with n is *odd*, the product is *negative*; if *even*, the product is *positive*.

Therefore, the required product is -1.591.

By regular multiplication the product is -1.59128737152.

3. Find the quotient of $53.25 \div 163.7$.

$$\begin{aligned} \log 53.25 &= 1.7263 \\ \text{colog } 163.7 &= \underline{7.7860 - 10} \\ & 9.5123 - 10 = \log 0.3253. \end{aligned}$$

Therefore, the required quotient is 0.3253.

By regular division the quotient is $0.3252901649 \cdots$

4. Find the quotient of $\dfrac{33.47 \times 0.04316 \times 264.6}{87.62 \times 29.68 \times 0.9585}$.

$$\begin{aligned} \log 33.47 &= 1.5246 \\ \log 0.04316 &= 8.6351 - 10 \\ \log 264.6 &= 2.4226 \\ \text{colog } 87.62 &= 8.0574 - 10 \\ \text{colog } 29.68 &= 8.5275 - 10 \\ \text{colog } 0.9585 &= \underline{0.0184} \\ & 9.1856 - 10 = \log 0.1533. \end{aligned}$$

Therefore, the required quotient is 0.1533.

By regular division the quotient is $0.153344331 \cdots$

5. Find the fourth power of 0.0872.

$$\begin{array}{r} \log 0.0872 = 8.9405 - 10 \\ 4 \\ \hline 5.7620 - 10 = \log 0.00005781. \end{array}$$

Therefore, the fourth power of 0.0872 is 0.00005781.

6. Find the fourth root of 0.00862.

$$\begin{array}{rl} \log 0.00862 = & 7.9355 - 10 \\ & 30. \qquad - 30 \\ & 4\overline{)37.9355 - 40} \\ & 9.4839 - 10 = \log 0.3047. \end{array}$$

Therefore, the fourth root of 0.00862 is 0.3047.

7. Find the value of $\sqrt[5]{\dfrac{3.1416 \times 4771.2 \times 2.718^{\frac{1}{3}}}{30.13^4 \times 0.4343^{\frac{1}{2}} \times 69.89^4}}$.

$$\begin{array}{rll} \log 3.1416 = 0.4971 & = & 0.4971 \\ \log 4771.2 = 3.6786 & = & 3.6786 \\ \frac{1}{3}\log 2.718 = \frac{1}{3}(0.4343) & = & 0.1448 \\ 4 \text{ colog } 30.13 = 4(8.5210 - 10) & = & 4.0840 - 10 \\ \frac{1}{2} \text{ colog } 0.4343 = \frac{1}{2}(0.3622) & = & 0.1811 \\ 4 \text{ colog } 69.89 = 4(8.1556 - 10) & = & 2.6224 - 10 \\ \hline & & 11.2080 - 20 \\ & & 30. \qquad - 30 \\ & & 5\overline{)41.2080 - 50} \\ & & 8.2416 - 10 = \log 0.01744. \end{array}$$

Therefore, the required value is 0.01744.

468. An **exponential equation**, that is, an equation in which the exponent involves the unknown number, is easily solved by logarithms.

Find the value of x in $81^x = 10$.

$$81^x = 10.$$

$$\therefore \log(81^x) = \log 10.$$

$$\therefore x \log 81 = \log 10.$$

$$\therefore x = \frac{\log 10}{\log 81} = \frac{1.0000}{1.9085} = 0.5240.$$

Exercise 138

Find by logarithms the value of:

1. 562.3×0.7854.
2. 22.87×0.05826.
3. 0.31416×282.4.
4. 5985×4268.
5. 820.25×2.6354.
6. 0.00857×0.00693.
7. -46.82×94.37.
8. $5280 \times (-63.42)$.
9. $-62{,}762 \times (-0.0046)$.
10. $28.46 \div 7.423$.
11. $26{,}250 \div 832.6$.
12. $0.04287 \div 0.56875$.
13. $0.003625 \div 0.000987$.
14. $(38.42 \times 63.95) \div 428.3$.
15. $263 \times (-512) \div 1728$.
16. $5.8404 \div 0.003764$.
17. $48.792 \div 148.79$.
18. $31.58 \div 0.0007854$.
19. $\dfrac{862 \times 978 \times 562}{718 \times 1255 \times 243}$.
20. $\dfrac{52.634 \times (-36.875)}{-152.6 \times (-87.65)}$.
21. $\dfrac{84.296 \times 48.75}{7.862 \times (-6.827)}$.
22. $\dfrac{87.57 \times 495 \times 0.823}{3.1416 \times 0.045 \times 8662}$.
23. $\dfrac{0.007854 \times 0.09863}{363 \times 98.47 \times 106.8}$.
24. $\dfrac{89.76 \times 98.54 \times 26.63}{0.005862 \times 0.8271}$.
25. 3.1416^{12}.
26. 862.7^{5}.
27. 47.58^{3}.
28. $81.48^{\frac{1}{2}}$.
29. $16{,}327^{\frac{1}{3}}$.
30. $26.87^{\frac{1}{5}}$.
31. $\sqrt[5]{0.00452}$.
32. $\sqrt[4]{0.8751}$.
33. $802.53^{\frac{1}{4}}$.
34. 2.851^{7}.
35. $\left(\frac{5}{4}\frac{1}{3}\right)^{\frac{1}{20}}$.
36. $0.00893^{-\frac{1}{7}}$.
37. 0.5137^{5}.
38. 0.2675^{7}.
39. 0.1818^{9}.
40. $0.02852^{\frac{1}{5}}$.
41. $0.07225^{\frac{1}{6}}$.
42. $0.03687^{\frac{1}{7}}$.
43. $\dfrac{\sqrt[3]{0.0047}}{\sqrt[4]{0.00872}}$.
44. $\dfrac{4 \times (0.5326)^{\frac{3}{5}}}{(-41.752)^{\frac{2}{3}}}$.
45. $\dfrac{(-0.04563)^{\frac{4}{5}}}{(-0.6578)^{\frac{4}{7}}}$.
46. $\dfrac{5086 \times 0.0008769^{3}}{9802 \times 0.001984^{4}}$.
47. $\dfrac{8094 \times \sqrt[5]{0.031}}{5408 \times \sqrt[6]{0.017}}$.
48. $\dfrac{109}{716}\sqrt[5]{\dfrac{7628}{9317}}$.

49. $\sqrt[4]{\dfrac{348.7^2 \times 2.685^{\frac{1}{3}} \times 3.082}{2.687^{\frac{1}{5}} \times 0.08216^{\frac{1}{4}} \times 8000}}$.

50. $\sqrt[5]{\dfrac{0.002452^{\frac{1}{4}} \times 86.47^3 \times 128.72}{5280 \times 0.07115^2 \times 62.47}}$.

51. $\sqrt[6]{\dfrac{4.382^{\frac{3}{5}} \times 36.15^4 \times 0.00819^{\frac{2}{3}}}{128600 \times 42.9^{\frac{1}{5}} \times 72.653^{\frac{1}{4}}}}$.

52. $\sqrt[7]{\dfrac{3742^2 \times 0.006543^3 \times 22.63^4}{52.78^{\frac{1}{4}} \times 0.0008157^{\frac{1}{3}} \times 4618^{\frac{2}{3}}}}$.

Find x from the equation :

53. $8^x = 35$.

54. $12^x = 7$.

55. $1.5^{x+3} = 32$.

56. $2.66^{x-5} = 12$.

57. $32.5^{\frac{1}{x}} = 8$.

58. $4.02^{\frac{1}{x-4}} = 2.37$.

59. Express in the logarithmic form the following:

$$\frac{a^2}{4}\sqrt{3};\ \pi r^2;\ \sqrt{s(s-a)(s-b)(s-c)};\ \frac{c}{2\pi};\ \frac{4}{3}\pi r^3.$$

60. The indicated area of a triangle is $\sqrt{153 \times 52 \times 51 \times 50}$. Find the value by means of logarithms.

61. The formula for the number of terms in a geometrical series, in terms of l, a, and r, is $r^{n-1} = \frac{l}{a}\cdot$ Put this equation in logarithmic form and solve for n.

62. Find the logarithm of 3 to the base 2.

HINT. $2^x = 3$.

63. Find the logarithm of 100 to the base 12.

64. Find the logarithm of 10 to the base 12.

65. Find the logarithm of 0.1 to the base 12.

66. Solve for n the equation $l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.

67. Find $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4}$, $\sqrt[5]{5}$, $\sqrt[6]{6}$, $\sqrt[7]{7}$, $\sqrt[8]{8}$, $\sqrt[9]{9}$, $\sqrt[10]{10}$.

68. Show that $\log\sqrt{a^2 - b^2} = \frac{1}{2}[\log(a+b) + \log(a-b)]$.

Compound Interest, Annuities, and Bonds

469. Compound Interest. Interest is *compounded* when it is added to the principal and becomes a part of the principal at specified intervals.

Compound interest is compounded annually, semiannually, quarterly, or monthly, according to agreement. Compound interest is compounded annually unless otherwise stated.

470. In **interest problems** four elements are considered: *principal*, *rate*, *time*, and *interest* or *amount*. If three of the elements are known, the fourth may be found.

471. Let r stand for the interest on \$1 for 1 conversion (compounding), t for the time in years between two successive conversions, n for the number of conversions, P for the principal, and A for the amount.

Then A after 1 conversion $= P(1+r)$,
after 2 conversions $= P(1+r)^2$,
after 3 conversions $= P(1+r)^3$,
after n conversions $= P(1+r)^n$.

$$\therefore\ \mathbf{A = P(1+r)^n}.$$

If R is put for $1+r$, this formula becomes

$$\mathbf{A = PR^n}.$$

In the case of simple interest, $n = 1$.

If there is a *broken* period whose time in years is t', t' being less than t, the rate of increase for t' is by commercial usage taken to be $1 + rt'$.

Find the amount of \$2650 for 10 years at 4 per cent compound interest.

$$A = PR^n = 2650 \times 1.04^{10}.$$

$$\begin{aligned} \log 2650 &= 3.4232 \\ \log 1.04^{10} &= \underline{0.1700} \\ &\ 3.5932 = \log 3919. \end{aligned}$$

Therefore, the required amount is \$3919.

472. Annuities. A sum of money that is payable yearly, or in parts at fixed periods in the year, is called an **annuity**.

To find the amount of an unpaid annuity when the interest, time, and rate per cent are given.

Let S denote the amount of the annuity.

The sum due at the *end* of the

$$\begin{aligned} \text{first year} &= S, \\ \text{second year} &= S + SR, \\ \text{third year} &= S + SR + SR^2, \\ n\text{th year} &= S + SR + SR^2 + \cdots + SR^{n-1}. \end{aligned}$$

That is, the amount $A = S + SR + SR^2 + \cdots + SR^{n-1}$.

$$\therefore AR = SR + SR^2 + SR^3 + \cdots + SR^n.$$

$$\therefore AR - A = SR^n - S.$$

$$\therefore A = \frac{S(R^n - 1)}{R - 1},$$

or

$$\mathbf{A} = \frac{\mathbf{S(R^n - 1)}}{\mathbf{r}}.$$

An annuity of \$1200 was unpaid for 6 years. What was the amount due if interest is reckoned at 4 per cent?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$1200(1.04^6 - 1)}{0.04} = \$7948 \text{ (by four-place logs).}$$

473. *To find the present worth of an annuity when the time it is to continue and the rate per cent are given.*

Let P denote the present worth. Then the amount of P for n years is equal to A, the amount of the annuity for n years.

But the amount of P for n years

$$= P(1 + r)^n = PR^n, \quad \text{(p. 385, § 471)}$$

and

$$A = \frac{S(R^n - 1)}{R - 1}. \quad \text{(§ 472)}$$

$$\therefore PR^n = \frac{S(R^n - 1)}{R - 1}.$$

$$\therefore \mathbf{P} = \frac{\mathbf{S}}{\mathbf{R^n}} \times \frac{\mathbf{R^n - 1}}{\mathbf{R - 1}}.$$

Find the present worth of an annual pension of \$1600 for 8 years at 4 per cent interest.

$$P = \frac{S}{R^n} \times \frac{R^n - 1}{R - 1} = \frac{\$1600}{1.04^8} \times \frac{1.04^8 - 1}{1.04 - 1} = \$10,760 \text{ (by logs).}$$

474. *To find the annuity when the present worth, the time, and the rate per cent are given.*

Let P denote the present worth and S denote the amount of the annuity.

Then $$P = \frac{S(R^n - 1)}{R^n(R - 1)}. \qquad (\S\ 473)$$

$$\therefore S = \frac{PR^n(R-1)}{R^n - 1}.$$

$$\therefore \mathbf{S = Pr \times \frac{R^n}{R^n - 1}}.$$

What annuity for 5 years will \$4440 yield when interest is reckoned at 4 per cent?

$$S = Pr \times \frac{R^n}{R^n - 1} = \$4440 \times 0.04 \times \frac{1.04^5}{1.04^5 - 1} = \$1000 \text{ (by logs).}$$

475. Sinking Funds. If the sum set apart at the end of each year to be put at compound interest is represented by S,

The amount of the sinking fund at the end of the

first year $= S$,
second year $= S + SR$,
third year $= S + SR + SR^2$,
nth year $= S + SR + SR^2 + \cdots + SR^{n-1}$.

That is, $$\mathbf{A = \frac{S(R^n - 1)}{r}}. \qquad (\S\ 472)$$

1. If \$10,000 is set apart each year, and put at 4 per cent compound interest for 10 years, what will be the amount?

$$A = \frac{S(R^n - 1)}{r} = \frac{\$10,000(1.04^{10} - 1)}{0.04}.$$

By four-place logarithms the amount is \$119,700.

2. A town owes \$50,000. What sum must be set apart annually, as a sinking fund, to cancel the debt in 10 years, provided money is worth 4 per cent?

$$S = \frac{Ar}{R^n - 1} = \frac{\$50{,}000 \times 0.04}{1.04^{10} - 1} = \$4176 \text{ (by four-place logs).}$$

The amount of tax required yearly is \$2000 for the interest on the debt and \$4176 for the sinking fund, making \$6176 in all.

476. Bonds. If P denotes the price of a bond or mortgage that has n years to run, and bears r per cent interest, S the face of the bond, and q the current rate of interest, what interest on his investment will a purchaser of such a bond receive?

Let x denote the rate of interest on the investment.

Then $P(1+x)^n$ is the value of the purchase money at the end of n years.

The annual interest on the bond is Sr.

At the end of n years the purchaser is paid the face of the bond in full; that is, after n years he is paid S.

Hence, $Sr(1+q)^{n-1} + Sr(1+q)^{n-2} + \cdots + Sr + S$ is the amount received on the bond if the interest received from the bond is put immediately at compound interest at q per cent.

But $Sr(1+q)^{n-1} + Sr(1+q)^{n-2} + \cdots + Sr$ is a geometrical progression in which the first term is Sr, the ratio $1+q$, and the number of terms n.

$$\therefore Sr(1+q)^{n-1} + Sr(1+q)^{n-2} + \cdots + Sr + S$$

$$= S + \frac{Sr[(1+q)^n - 1]}{q}. \quad \text{(p. 341, § 415)}$$

$$\therefore P(1+x)^n = S + \frac{Sr[(1+q)^n - 1]}{q}.$$

$$\therefore 1 + x = \left[\frac{S}{P} + \frac{Sr[(1+q)^n - 1]}{Pq}\right]^{\frac{1}{n}}.$$

$$\therefore \mathbf{1 + x = \left[\frac{Sq + Sr(1+q)^n - Sr}{Pq}\right]^{\frac{1}{n}}}.$$

1. What interest will a purchaser receive on his investment if he buys at 114 a 4 per cent bond that has 26 years to run, money being worth $3\frac{1}{2}$ per cent?

Here $S = 100,\ P = 114,\ q = 0.035,\ r = 0.04,\ n = 26.$

$$\therefore\ 1 + x = \left(\frac{3.5 + 4 \times 1.035^{26} - 4}{114 \times 0.035}\right)^{\frac{1}{26}} = 1.033 \text{ (by logs).}$$

Therefore, the purchaser will receive $3\frac{1}{3}$ per cent for his money.

2. At what price must 7 per cent bonds, running 12 years, with the interest payable semiannually, be bought in order that the purchaser may receive on his investment 5 per cent, interest semiannually, which is the current rate of interest?

$$P = \frac{Sq + Sr(1+q)^n - Sr}{q(1+x)^n}.$$

In this case $S = 100$; and, as the interest is semiannual,

$$q = 0.025,\ r = 0.035,\ n = 24,\ x = 0.025.$$

Hence, $$P = \frac{2.5 + 3.5(1.025)^{24} - 3.5}{0.025(1.025)^{24}}.$$

By logarithms, $P = 117.8.$

Therefore, the purchase price must be 117.8.

Exercise 139

1. Find the amount of $6326 in 25 years at 4 per cent compound interest, the interest being compounded annually.

2. Find the amount of $6326 in 25 years at 4 per cent compound interest, the interest being compounded semiannually.

3. Find the amount of $142.50 in 3 years 8 months at 5 per cent compound interest.

4. What sum will amount to $5000 in 22 years if put at compound interest at $3\frac{1}{2}$ per cent?

5. In how many years will $2500 amount to $10,000 at 4 per cent compound interest?

6. At what rate per cent will $1500 amount in 15 years to $2717.05 at compound interest, compounded semiannually?

7. If a corporation sets aside each year for 20 years $5000 as a sinking fund and puts it at compound interest at $3\frac{1}{2}$ per cent, what will be the amount of the sinking fund?

8. A city has a funded debt of $500,000 due in 18 years. What sum must be set apart each year, as a sinking fund, to cancel the debt, provided money is worth 4 per cent?

9. Find the present worth of an annuity of $1200 for 10 years at $3\frac{1}{2}$ per cent interest.

10. A man buys an annuity of $900 for 15 years. If money is worth $4\frac{1}{2}$ per cent, what should he pay for the annuity?

11. An annuity of $500 was unpaid for 10 years. What was the amount due, if interest is reckoned at $3\frac{3}{4}$ per cent?

12. A man 59 years old is expected to live just 15 years. He has $20,000 with which to buy a life annuity. If money is worth $4\frac{1}{4}$ per cent, how large an annuity may he expect?

13. A man 35 years old is expected to live just 31 years. He has $18,625 with which to buy a life annuity. If money is worth $3\frac{1}{4}$ per cent, how large an annuity may he expect?

14. What interest on his investment will a purchaser receive if he buys at 108 Boston & Maine $4\frac{1}{2}$ per cent bonds due in 39 years with semiannual coupons, if money is worth 4 per cent?

15. What interest on his investment will a purchaser receive if he buys at 103 West End 4 per cent bonds due in 12 years with semiannual coupons, if money is worth $3\frac{1}{2}$ per cent?

16. At what price must Atchison, Topeka & Santa Fe 4 per cent bonds due in 90 years with semiannual coupons be bought in order to net the purchaser $4\frac{1}{4}$ per cent on his investment, if money is worth $3\frac{3}{4}$ per cent?

17. At what price must Old Colony 4 per cent bonds due in 33 years with semiannual coupons be bought in order to net the purchaser $3\frac{1}{2}$ per cent on his investment, if money is worth $4\frac{1}{2}$ per cent?

CHAPTER XXVII

PERMUTATIONS AND COMBINATIONS

477. If three paths, A, B, and C, lead to the top of a mountain, there is obviously a choice of three different ways of ascending the mountain; and when the top of the mountain has been reached, there is again a choice of three different ways of descending.

If a traveler ascends by path A, he may descend by path A or by path B or by path C, thus giving three different ways of making the round trip to the summit and return. So, also, if he ascends by path B, or by path C.

Therefore, there are in all 3×3, or 9, different ways of making the trip to the summit of the mountain and return.

If, however, the same path is not to be used in descending as in ascending, then for each of the ways of ascending there are *two* different ways of descending; that is, 3×2, or 6, different ways of doing both.

These examples illustrate the following:

478. Fundamental Principle of Choice. *If one thing can be done in* **a** *different ways and, when it has been done in any one of these* **a** *ways, another thing can be done in* **b** *different ways, then both things together can be done in* **a** $\times$ **b** *different ways.*

For, corresponding to the *first* way of doing the first thing there are b different ways of doing the second thing; corresponding to the *second* way of doing the first thing there are b different ways of doing the second thing; and so on for *each* of the a ways of doing the first thing. Hence, there are $a \times b$ different ways of doing both things together.

The Fundamental Principle may be extended so as to be perfectly general, thus:

479. *If one thing can be done in* a *different ways and then a second thing can be done in* b *different ways, then a third thing in* c *different ways, and so on to an* r*th thing which can be done in* n *different ways, the number of different ways all the things together can be done is* $a \times b \times c \times \cdots \times n$.

The truth of this principle follows from the successive application of the Fundamental Principle. Evidently the first two things together can be done in $a \times b$ different ways. Then for each of the different $a \times b$ ways the first two things together can be done there are c different ways the third thing can be done. Hence, the first three things together can be done in $a \times b \times c$ different ways; and so on for any number of things.

Exercise 140

1. In how many ways may a girl and a boy be chosen from a school that consists of 18 girls and 24 boys?

2. On a shelf are a set of 6 English books, a set of 8 French books, and a set of 10 German books. In how many ways can three books be chosen so that there shall be one from each set?

3. A boy has 5 different routes to his school building. In how many ways may he go and return by a different route?

4. In how many ways can a vowel and a consonant be chosen from the alphabet of 6 vowels and 20 consonants?

5. After a particular vowel and a particular consonant have been chosen, in how many ways can a two-lettered word be made? In how many different ways can a two-lettered word be made, if it is to consist of a vowel and a consonant?

480. Combinations and Permutations. The last two examples of Exercise 140 show the difference between a *selection*, or *combination,* of different things and an *arrangement,* or *permutation,* of the same things.

Thus, the vowel *a* and the consonant *c* together form a *combination* of two letters, while *ac* and *ca* form two different *permutations* of this combination.

Again, *a, b,* and *c* is a *combination* of three letters from the alphabet. This combination, then, admits of six different arrangements, or *permutations;* thus,

abc, *acb,* *bac,* *bca,* *cab,* *cba.*

A **selection**, or **combination**, of any number of elements or things means a group of that number of elements or things put together without regard to their order of sequence.

An **arrangement**, or **permutation**, of any number of elements or things means a group of that number of elements or things put together with reference to their order of sequence.

481. In how many ways can the letters of the word *Cambridge* be arranged, taken all at a time?

There are 9 letters, all different. Hence, the first place can be filled in 9 ways. When the first place has been filled in any one of the 9 ways, the second place can be filled in 8 ways. Then the third place can be filled in 7 ways, and so on to the last place, which can be filled in 1 way. Therefore, the total number of ways in which the nine letters can be arranged is

$$9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362{,}880. \quad (\S\ 479)$$

This example is an illustration of the following principle:

482. *The number of arrangements, or permutations, of* n *different elements or things taken all at a time is*

$$n(n-1)(n-2) \times \cdots \times 3 \times 2 \times 1.$$

For the first place can be filled in n ways, then the second place in $n-1$ ways, then the third place in $n-2$ ways, and so on, to the last place, which can be filled in only 1 way.

Hence (p. 392, § 479), the whole number of permutations is the continued product of these numbers; that is,

$$n(n-1)(n-2)\times\cdots\times 3\times 2\times 1.$$

It is customary to write the continued product

$$n(n-1)(n-2)\times\cdots\times 3\times 2\times 1 \text{ as } n! \text{ or } \underline{|n}.$$

$n!$ or $\underline{|n}$ is read *factorial* n.

483. If, instead of taking all the letters of the word *Cambridge* to make a permutation, there are taken *four* letters each time, the whole number of permutations is $9\times 8\times 7\times 6$.

For the first place can be filled in 9 ways, then the second place in 8 ways, then the third place in 7 ways, and then the fourth place in 6 ways.

This illustrates the following modification of § 482:

484. *The number of arrangements, or permutations, of* n *different elements or things taken* r *at a time is*

$$\text{n}(\text{n}-1)(\text{n}-2)\cdots(\text{n}-\text{r}+1).$$

For the first place can be filled in n ways, then the second in $n-1$ ways, then the third in $n-2$ ways, and so on, to the rth place, which can be filled in $n-(r-1)$, or $n-r+1$, ways.

485. The number $n(n-1)(n-2)\cdots(n-r+1)$

$$=\frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\times\cdots\times 3\times 2\times 1}{(n-r)(n-r-1)\times\cdots\times 3\times 2\times 1}$$

$$=\frac{n!}{(n-r)!}.$$

The Principles of § 482 and § 484 may be conveniently expressed in symbols thus:

$$P_{n,n}=n! \quad (1); \qquad P_{n,r}=\frac{n!}{(n-r)!} \quad (2).$$

If *repetitions* are allowed, the right members of (1) and (2) become respectively n^n and n^r.

For each place can then be filled in n ways.

Exercise 141

1. Find the value of $5!$; of $\frac{10!}{6!}$; of $\frac{12!}{3!} \times \frac{8!}{7!} \times \frac{15!}{16!}$.

2. Find the value of $P_{9,5}$; of $P_{20,1}$; of $P_{6,6}$; of $(P_{4,4})^2$.

3. Find the value of each in Example 2 when repetitions are allowed.

4. In how many ways can a class of ten pupils be arranged at a blackboard?

5. How many changes can be rung with a peal of 6 bells?

6. How many changes can be rung with a peal of 7 bells, a particular one always being first?

7. If $P_{8,r} = 1680$, find the value of r.

8. If $P_{n,n} = 362{,}880$, find the value of n.

9. If $P_{n,2} = 110$, find the value of n.

10. In how many ways can the members of a baseball nine be arranged in the field, if the same two must always pitch and catch?

11. How many numbers of four figures each can be made with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, repetitions not being allowed? How many if repetitions are allowed?

12. In how many ways can eight guests be seated along one side of a table?

13. In how many ways can eight guests be seated at a round table relative to the eight places at the table? relative to each other?

14. In how many ways can the letters of the word *combine* be arranged, if *b* is always to be the middle letter?

15. How many different signals can be given with 6 differently colored flags, if 3 are displayed on a staff each time? How many, if any number of flags are displayed each time?

486. In how many ways can three letters be selected from the letters in the word *Cambridge?*

We have already seen (p. 394, § 484) that the number of arrangements of three letters from the letters in the word *Cambridge* is $9 \times 8 \times 7$, or 504.

These 504 arrangements might be obtained by first forming all the possible *selections of three letters* and then *arranging* the three letters in each selection in as many ways as possible.

Now the three letters of each selection can be arranged in 3!, or 6, ways.

Hence (p. 391, § 478), the number of selections multiplied by 6 is equal to the number of arrangements, or 504.

Therefore, the number of selections is $504 \div 6$, or 84.

This example illustrates the following Principle of Combinations:

487. Principle of Combinations. *The number of different selections, or combinations, of* n *different elements, taken* r *at a time, is the number of arrangements of* n *elements, taken* r *at a time, divided by* r!, *or* $\frac{n!}{(n-r)!\,r!}$.

For, if $C_{n,r}$ stands for the number of combinations of n different things, taken r at a time, then

$$P_{n,r} = C_{n,r} \times r!;$$

since for each combination of r things there are $r!$ permutations.

Therefore, $C_{n,r} = \frac{P_{n,r}}{r!} = \frac{n!}{(n-r)!} \div r! = \frac{n!}{(n-r)!\,r!}$.

488. Corollary. *The number of combinations of* n *elements, taken* r *at a time, is the same as the number of combinations of* n *elements, taken* n − r *at a time.*

This becomes evident if for r we substitute $n - r$ in the formula; also from the fact that for each selection of r elements there is left a selection of $n - r$ elements.

EXERCISE 142

1. Find the value of $C_{9,5}$; of $C_{20,1}$; of $C_{6,6}$.

2. Show that $C_{n,r} = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$.

3. Show that the number of combinations of 9 things, taken 5 at a time, is the same as the number of combinations taken 4 at a time.

4. What is the number of combinations of 100 things taken 98 at a time?

5. In an examination paper there are 10 questions, from which 8 are to be selected by the pupil. In how many ways can he do this?

6. How many ways are there of electing five members of a board of education from eight candidates?

7. If $C_{6,r} = 20$, find the value of r.

8. If $24\,C_{n,r} = P_{n,r}$, find the value of r.

9. There are 9 coins of different denominations on the table. How many different sums of money can be made up by taking 4 coins each time?

10. Expand $(x+y)^5$ and show that it is equal to

$$x^5 + C_{5,1}x^4y + C_{5,2}x^3y^2 + C_{5,3}x^2y^3 + C_{5,4}xy^4 + C_{5,5}y^5.$$

11. In how many ways can a committee of three be chosen from a council of eight members?

12. If one of the committee of Example 11 is to be chairman, one secretary, and one treasurer, in how many ways may the committee be organized?

13. If $C_{16,r} = C_{16,r+2}$, find the value of r.

14. Show that $C_{n,1} = P_{n,1}$, but that $C_{n,r} \neq P_{n,r}$ unless $r = 1$.

15. Make up a problem of your own in combinations or permutations and then solve it.

CHAPTER XXVIII

BINOMIAL THEOREM

489. Binomial Theorem; Positive Integral Exponent. By successive multiplications we obtain the following identities:

$$(a + b)^2 \equiv a^2 + 2\,ab + b^2;$$
$$(a + b)^3 \equiv a^3 + 3\,a^2b + 3\,ab^2 + b^3;$$
$$(a + b)^4 \equiv a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4.$$

The expressions on the right may be written in a form better adapted to show the law of their formation:

$$(a + b)^2 \equiv a^2 + 2\,ab + \frac{2 \cdot 1}{1 \cdot 2} b^2;$$
$$(a + b)^3 \equiv a^3 + 3\,a^2b + \frac{3 \cdot 2}{1 \cdot 2} ab^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} b^3;$$
$$(a + b)^4 \equiv a^4 + 4\,a^3b + \frac{4 \cdot 3}{1 \cdot 2} a^2b^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} ab^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} b^4.$$

490. Let n represent the exponent of $(a + b)$ in any one of these identities; then, in the expressions on the right, we observe that the following laws hold true:

1. The number of terms is $n + 1$.

2. The first term is a^n, and the exponent of a decreases by one in each succeeding term. The first power of b occurs in the second term, the second power in the third term, and the exponent of b increases by one in each succeeding term.

The sum of the exponents of a and b in any term is n.

3. The coefficients of the terms taken in order are 1, n, $\frac{n(n-1)}{1 \cdot 2}$, $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$, and so on.

491. The Coefficient of Any Term. The number of factors in the numerator of the coefficient of any term is the same as the number in the denominator. The number of factors in each numerator and denominator is the same as the exponent of b in that term, and is one less than the number of the term.

492. Proof of the Theorem. We know that the laws of § 490 hold for the *fourth power* (§ 489). Suppose, for the moment, that they hold for the kth power, k being any positive integer. Then

$$(a+b)^k \equiv a^k + ka^{k-1}b + \frac{k(k-1)}{1\cdot 2}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}a^{k-3}b^3 + \cdots \quad (1)$$

Multiply both members of (1) by $a+b$; the result is

$$(a+b)^{k+1} \equiv a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots \quad (2)$$

In the right member of (1) for k put $k+1$; this gives

$$a^{k+1} + (k+1)a^k b + \frac{(k+1)(k+1-1)}{1\cdot 2}a^{k-1}b^2 + \frac{(k+1)(k+1-1)(k+1-2)}{1\cdot 2\cdot 3}a^{k-2}b^3 + \cdots$$

This last expression, simplified, is identical with the right member of (2), and this by (2) is identical with $(a+b)^{k+1}$.

Hence, (1) holds when for k we put $k+1$; that is, if the laws of § 490 hold for the kth power, then they hold for the $(k+1)$th power.

But the laws hold for the *fourth* power (§ 489); therefore, they hold for the *fifth* power.

Holding for the *fifth* power, they hold for the *sixth* power; and so on for *any positive integral* power.

Therefore, they hold for the nth power if n is a positive integer; and we have

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots \qquad \text{(A)}$$

493. This formula is known as the **binomial theorem.**

The expression on the right is known as the **expansion** of $(a+b)^n$; this expansion is a *finite series* when n is a positive integer. That the series is finite may be seen as follows:

In writing the successive coefficients we shall finally arrive at a coefficient that contains the factor $n-n$; the term vanishes. The coefficient of each succeeding term likewise contains the factor $n-n$, and, therefore, all these terms vanish.

494. If a and b are interchanged, the identity (A) is written

$$(a+b)^n \equiv (b+a)^n \equiv b^n + nb^{n-1}a + \frac{n(n-1)}{1\cdot 2}b^{n-2}a^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}b^{n-3}a^3 + \cdots$$

This last expansion is the expansion of (A) written in reverse order. Comparing the two expansions, we see that the coefficient of the last term is the same as the coefficient of the first term; the coefficient of the last term but one is the same as the coefficient of the first term but one; and so on.

In general, the coefficient of the rth term from the end is the same as the coefficient of the rth term from the beginning.

In writing an expansion by the binomial theorem, after arriving at the middle term, we can shorten the work by observing that the remaining coefficients are those already found, written in reverse order.

495. If b is negative, the terms that involve *even* powers of b are *positive;* and the terms that involve *odd* powers of b are *negative.* Hence,

$$(a-b)^n \equiv a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots \quad \text{(B)}$$

If we put 1 for a and x for b in (A),

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots \quad \text{(C)}$$

If we put 1 for a and x for b in (B),

$$(1-x)^n \equiv 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots \quad \text{(D)}$$

496. Examples. 1. Expand $(1+2a)^6$.

In (C) put $2a$ for x and 6 for n. The result is

$$(1+2a)^6 \equiv 1 + 6(2a) + \frac{6\cdot 5}{1\cdot 2}(2a)^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}(2a)^3 + \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}(2a)^4 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5}(2a)^5 + \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(2a)^6$$

$$\equiv 1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6.$$

2. Expand to four terms $\left(\frac{2}{x} - \frac{3x^2}{4}\right)^7$.

In (B) put $\frac{2}{x}$ for a and $\frac{3x^2}{4}$ for b,

$$\left(\frac{2}{x}-\frac{3x^2}{4}\right)^7 \equiv \left(\frac{2}{x}\right)^7 - 7\left(\frac{2}{x}\right)^6\left(\frac{3x^2}{4}\right) + 21\left(\frac{2}{x}\right)^5\left(\frac{3x^2}{4}\right)^2 - 35\left(\frac{2}{x}\right)^4\left(\frac{3x^2}{4}\right)^3 + \cdots$$

$$\equiv \frac{128}{x^7} - \frac{336}{x^4} + \frac{378}{x} - \frac{945x^2}{4} + \cdots$$

497. Any Required Term. From (A) it is evident (§§ 491, 492) that the $(r+1)$th term in the expansion of $(a+b)^n$ is

$$\frac{n(n-1)(n-2)\cdots \text{to } r \text{ factors}}{1\cdot 2\cdot 3\cdots r}a^{n-r}b^r.$$

The $(r+1)$th term in the expansion of $(a-b)^n$ is the same as the above if r is even, and the *negative* of the above if r is *odd*.

Find the eighth term of $\left(4-\frac{x^2}{2}\right)^{10}$.

Here $\quad a=4,\ b=\frac{x^2}{2},\ n=10,\ r=7.$

The eighth term is $-\frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(4)^3\left(\frac{x^2}{2}\right)^7$, or $-60x^{14}$.

498. The Greatest Coefficient. The coefficient of the $(r+1)$th term and the coefficients of the terms immediately preceding and following are as follows:

rth term, $\quad \frac{n(n-1)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}$;

$(r+1)$th term, $\quad \frac{n(n-1)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r}$;

$(r+2)$th term, $\quad \frac{n(n-1)\cdots(n-r+2)(n-r+1)(n-r)}{1\cdot 2\cdot 3\cdots(r-1)r(r+1)}$.

The coefficient of the rth term may be found by multiplying that of the $(r+1)$th by $\frac{r}{n-r+1}$; the coefficient of the $(r+2)$th term, by multiplying that of the $(r+1)$th by $\frac{n-r}{r+1}$. If the coefficient of the $(r+1)$th is *numerically the greatest,*

$$\frac{r}{n-r+1}<1, \qquad \text{and} \qquad 1>\frac{n-r}{r+1}.$$

Therefore, $\quad r<n-r+1,$ and $r+1>n-r.$

Therefore, $\quad r<\frac{n+1}{2},\quad$ and $\quad r>\frac{n-1}{2}.$

If n is even, $r = \frac{n}{2}$, and $r + 1 = \frac{n+2}{2}$; then the coefficient of the $\frac{n+2}{2}$th term is the greatest coefficient.

If n is odd, $r = \frac{n+1}{2}$ and $\frac{n-1}{2}$, and $r + 1 = \frac{n+3}{2}$ and $\frac{n+1}{2}$; then the coefficients of the $\frac{n+3}{2}$th and $\frac{n+1}{2}$th terms are alike and are the two greatest coefficients.

499. A polynomial of more than two terms, if put in the form of a binomial, may be expanded by the binomial theorem.

Expand $(a^2 - 4a + 3)^3$.

$$(a^2 - 4a + 3)^3 = [a^2 - (4a - 3)]^3$$
$$= (a^2)^3 - 3(a^2)^2(4a - 3) + 3(a^2)(4a - 3)^2 - (4a - 3)^3$$
$$= a^6 - 12a^5 + 9a^4 + 48a^4 - 72a^3 + 27a^2 - 64a^3 + 144a^2 - 108a + 27$$
$$= a^6 - 12a^5 + 57a^4 - 136a^3 + 171a^2 - 108a + 27.$$

Exercise 143

Expand:

1. $(a - 3b)^5$.
2. $(2x - 5y)^6$.
3. $\left(\frac{1}{x} - \frac{x^2}{2}\right)^4$.
4. $(3 - 2x^2)^7$.
5. $(1 + 2a)^8$.
6. $\left(\frac{2x^2}{y} + \frac{y^2}{3x}\right)^5$.
7. $(2x + 3y)^9$.
8. $(4x + y)^{10}$.
9. $\left(\frac{3x}{2a} - \frac{y}{4b}\right)^6$.
10. $\left(\sqrt{\frac{x}{2}} + \sqrt[3]{\frac{y}{3}}\right)^7$.
11. $\left(\sqrt[3]{\frac{x^2}{2}} - \frac{1}{2\sqrt{x}}\right)^8$.
12. $(1 - 2x - 3x^2)^4$.
13. $\left(\sqrt{-1} + \frac{\sqrt[3]{x}}{2\sqrt{-1}}\right)^5$.
14. $\left(\frac{a\sqrt{-1}}{3} + \frac{1}{b\sqrt{-1}}\right)^6$.
15. $(a^3 - 2a^2b + 3ab^2 - 4b^3)^3$.
16. Find the third term of $(2x - \frac{1}{2}y)^8$.
17. Find the sixteenth term of $(3x - 2y)^{18}$.
18. Find the thirty-second term of $(3x - \frac{1}{3}y)^{37}$.
19. Find the middle term of $(3a - 2b)^{12}$.

20. Find the fifth term of $\left(x + \frac{1}{2x}\right)^9$.

21. Find the eighth term of $\left(x - \frac{1}{4x^2}\right)^{11}$.

22. Find the $(r+1)$th term of $\left(\frac{3}{4x} - \sqrt[3]{\frac{x^2}{2}}\right)^{15}$.

23. Find the two middle terms of $\left(\frac{a}{\sqrt{x}} - \frac{b}{\sqrt[3]{y}}\right)^{21}$.

Expand:

24. $(a^{\frac{1}{2}} + b^{\frac{1}{3}})^5$.

25. $(x^{-2} - y^{-3})^6$.

26. $(m^{-\frac{1}{2}} - n^2)^5$.

27. $(a^{-3} + b^{-\frac{2}{3}})^5$.

28. $(3a^{-2} + 2b^{-3})^7$.

29. $(x^{-\frac{3}{4}} - y^{-\frac{2}{3}})^6$.

30. $(2x^{-\frac{1}{2}} + 3y^{-\frac{2}{3}})^8$.

31. $(2x^{-2}y^{\frac{1}{2}} - y\sqrt{x})^6$.

32. $(3x\sqrt{-1} + 2y\sqrt{-1})^5$.

33. $(\frac{1}{2}a\sqrt{-1} - \frac{1}{3}b\sqrt{-1})^8$.

500. Binomial Theorem, Any Exponent. We have seen (p. 401, § 495) that when n is a positive integer we have

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

This formula holds true when n is fractional or negative, provided x is so taken as to render the series of the expansion convergent (see Wentworth's *College Algebra*, p. 321).

The series obtained is an infinite series unless n is a positive integer (p. 400, § 493).

501. If $x < a$,

$$(a+x)^n = a^n\left(1 + \frac{x}{a}\right)^n$$

$$= a^n\left[1 + n\frac{x}{a} + \frac{n(n-1)}{1 \cdot 2}\frac{x^2}{a^2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\frac{x^3}{a^3} + \cdots\right]$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}x^3 + \cdots;$$

if $x > a$,

$$(a+x)^n = (x+a)^n = x^n\left(1+\frac{a}{x}\right)^n$$

$$= x^n\left[1 + n\frac{a}{x} + \frac{n(n-1)}{1\cdot 2}\frac{a^2}{x^2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{a^3}{x^3} + \cdots\right]$$

$$= x^n + nax^{n-1} + \frac{n(n-1)}{1\cdot 2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^3x^{n-3} + \cdots$$

502. If x is negative, the formula of § 500 becomes

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

1. Expand to four terms $(a+b)^{-2}$.

$$(a+b)^{-2} = a^{-2} + (-2)a^{-3}b + \frac{(-2)(-2-1)}{1\cdot 2}a^{-4}b^2$$
$$+ \frac{(-2)(-2-1)(-2-2)}{1\cdot 2\cdot 3}a^{-5}b^3 + \cdots$$
$$= a^{-2} - 2a^{-3}b + 3a^{-4}b^2 - 4a^{-5}b^3 + \cdots$$

2. Expand to four terms $(1+x)^{\frac{1}{2}}$.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots$$

3. Expand to four terms $(a-b)^{-\frac{1}{3}}$.

$$(a-b)^{-\frac{1}{3}} = a^{-\frac{1}{3}} - \left(-\frac{1}{3}\right)a^{-\frac{4}{3}}b + \frac{(-\frac{1}{3})(-\frac{1}{3}-1)}{1\cdot 2}a^{-\frac{7}{3}}b^2$$
$$- \frac{(-\frac{1}{3})(-\frac{1}{3}-1)(-\frac{1}{3}-2)}{1\cdot 2\cdot 3}a^{-\frac{10}{3}}b^3 + \cdots$$
$$= a^{-\frac{1}{3}} + \tfrac{1}{3}a^{-\frac{4}{3}}b + \tfrac{2}{9}a^{-\frac{7}{3}}b^2 + \tfrac{14}{81}a^{-\frac{10}{3}}b^3 + \cdots$$

4. Find the sixth term of $\left(x - \dfrac{2}{3\sqrt[3]{x}}\right)^{-\frac{1}{2}}$.

Here $\qquad a = x,\ b = \dfrac{2}{3\sqrt[3]{x}} = \dfrac{2}{3x^{\frac{1}{3}}},\ n = -\dfrac{1}{2},\ r = 5.$

The sixth term is $-\dfrac{-\frac{1}{2}\cdot -\frac{3}{2}\cdot -\frac{5}{2}\cdot -\frac{7}{2}\cdot -\frac{9}{2}}{1\cdot 2\cdot 3\cdot 4\cdot 5}(x)^{-\frac{11}{2}}\left(\dfrac{2}{3x^{\frac{1}{3}}}\right)^5 = \dfrac{7}{216x^{\frac{43}{6}}}.$

It is often convenient to extract a root of an arithmetical number by means of an expansion.

5. Extract to eight decimal places the fifth root of 247.

$$247 = 243 + 4 = 243\left(1 + \tfrac{4}{243}\right) = 3^5\left(1 + \tfrac{4}{243}\right).$$

$$\therefore \sqrt[5]{247} = 3\left(1 + \tfrac{4}{243}\right)^{\frac{1}{5}}$$

$$= 3\left[1 + \frac{1}{5}\left(\frac{4}{243}\right) + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)}{1\cdot 2}\left(\frac{4}{243}\right)^2 + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)}{1\cdot 2\cdot 3}\left(\frac{4}{243}\right)^3 + \frac{\frac{1}{5}\left(\frac{1}{5}-1\right)\left(\frac{1}{5}-2\right)\left(\frac{1}{5}-3\right)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{4}{243}\right)^4 + \cdots\right]$$

$$= 3(1 + 0.003292181 - 0.000021677 + 0.000000214 - 0.000000002 + \cdots)$$

$$= 3(1.003270716)$$

$$= 3.00981215.$$

Exercise 144

Expand to four terms:

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{-2}$.
3. $(1+x)^{-\frac{1}{2}}$.
4. $(1-x)^{-3}$.
5. $(1-x)^{-\frac{3}{2}}$.
6. $(1-3x)^{\frac{1}{3}}$.
7. $(a-bx)^{-\frac{2}{3}}$.
8. $(2a-3b)^{\frac{2}{3}}$.
9. $(3x^2+2y)^{-\frac{3}{2}}$.
10. $(a^2-4x^2)^{\frac{3}{4}}$.
11. $\sqrt[4]{3-2x}$.
12. $\sqrt[5]{1+4a}$.
13. $\sqrt[6]{3-5b}$.
14. $\sqrt[4]{5a^2+7b^3}$.
15. $(2a-5b)^{-4}$.
16. $(3x+4y)^{-3}$.
17. $(2a^2-3b^3)^{\frac{3}{4}}$.
18. $(2a^{-2}-3b^{-3})^{-\frac{3}{4}}$.
19. $\left(\dfrac{1}{2a}+\dfrac{3}{b}\right)^{\frac{1}{3}}$.
20. $\left(\dfrac{a}{3b}+\dfrac{2m^{\frac{1}{3}}}{x}\right)^{\frac{1}{4}}$.
21. $\left(\dfrac{3a}{b}-\dfrac{2a^2}{x}\right)^{-\frac{2}{5}}$.
22. $\left(\dfrac{4x}{3y}-\dfrac{3a}{4b}\right)^{-\frac{3}{4}}$.
23. $\left(\dfrac{2a^2}{3b^3}+\dfrac{4x^3}{y^4}\right)^{-\frac{4}{7}}$.
24. Find the sixth term of $(1-3x)^{\frac{1}{3}}$.
25. Find the tenth term of $(2a-3b)^{-\frac{3}{4}}$.
26. Find the eighth term of $(a^2+4x^{-2})^{-\frac{5}{2}}$.
27. Expand $(3a^2-a-2)^{\frac{2}{5}}$ to four terms.
28. Find the sixth root of 66 to six decimal places.
29. Find the fifth root of 239 to seven decimal places.

CHAPTER XXIX

MISCELLANEOUS EXAMPLES

Exercise 145

Reduce to lowest terms:

1. $\dfrac{2x^4+3x^3+5x^2+9x-3}{3x^4-2x^3+10x^2-6x+3}$.

2. $\dfrac{x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5}{x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5}$.

3. $\dfrac{x^4-x^3y-x^2y^2+xy^3}{x^5-x^4y-xy^4+y^5}$.

4. $\dfrac{(x+y)^5-x^5-y^5}{(x+y)^4+x^4+y^4}$.

Simplify:

5. $\dfrac{1+a}{1-a}+\dfrac{4a}{1+a^2}+\dfrac{8a}{1+a^4}-\dfrac{1-a}{1+a}$.

6. $\dfrac{a}{a-2}+\dfrac{a-9}{a-7}-\dfrac{a+1}{a-1}-\dfrac{a-8}{a-6}$.

7. $\dfrac{a}{a+b}+\dfrac{b}{a-b}-\dfrac{ab}{ab-b^2}+\dfrac{ab}{a^2+ab}$.

8. $\dfrac{1}{2x+2}-\dfrac{4}{x+2}+\dfrac{9}{2(x+3)}-\dfrac{x-1}{(x+2)(x+3)}$.

9. $\dfrac{a^3+a^2b+ab^2+b^3}{a^3-a^2b-ab^2+b^3}\times\dfrac{(a+b)^2-3ab}{(a-b)^2+3ab}\times\dfrac{(a-b)^3-a^3+b^3}{(a+b)^3-a^3-b^3}$.

10. $\dfrac{a^2+ab+ac+bc}{ax-ay-x^2+xy}\times\dfrac{a^2-ax+ay-xy}{a^2+ac+ax+cx}\div\dfrac{a^2-a(y-b)-by}{x^2-x(y-a)-ay}$.

11. $\dfrac{ax-ay+bx-by+a+b}{ax-ay-bx+by+a-b}\times\dfrac{a^2-b^2-ac+bc}{ab+ac+b^2-c^2}$.

12. $\dfrac{a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5}{a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5}\div\dfrac{a^2-2ab+b^2}{a^2+2ab+b^2}$.

Solve:

13. $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{4x+6}{7}.$

14. $\frac{132x+1}{3x+1}+\frac{8x+5}{x-1}=52.$

15. $\frac{3}{x-7}+\frac{1}{x-9}=\frac{4}{x-8}.$

16. $\frac{ax-2a}{ax-2b}=\frac{ax-2b}{ax+2a}.$

17. $\frac{a-x}{bc}+\frac{b-x}{ac}+\frac{c-x}{ab}=0.$

18. $\frac{ax^2-bx+c}{mx^2-nx+p}=\frac{a}{m}.$

19. $\frac{b-x}{a+x}+\frac{c-x}{a-x}=\frac{a(c-2x)}{a^2-x^2}.$

20. $\frac{x-a}{b+c}+\frac{x-b}{a+c}+\frac{x-c}{b+a}=3.$

21. $\frac{2x+a}{3(x-a)}+\frac{3x-a}{2(x+a)}=2\frac{1}{6}.$

22. $\frac{a}{c}+\frac{cx}{ax-b}=\frac{c}{a}+\frac{ax}{cx-b}.$

23. The sum of \$2.37 is divided among A, B, and C. If B is given 20 per cent more than A and 25 per cent more than C, how much does each receive?

24. A man bought two pairs of shoes, and one pair cost 50 cents more than the other pair. If he had paid 50 cents less for each pair, he would have paid four fifths of what he did pay. How much did each pair cost?

25. A sum of money is divided between A and B. A receives \$72 more than B, and A receives seven twelfths of the whole. What is the sum divided?

26. A farmer bought some pigs at \$2.50 each. He sold 40 of them at \$2.60 each and the rest at \$3.00 each. If his total gain was 10 per cent, how many pigs did he buy?

27. A boat that travels $10\frac{1}{2}$ miles an hour downstream requires three times as long to travel a certain distance up the river as down the river. Find the rate of the current.

28. From the contents of a keg a man draws 1 gallon less than two fifths of the contents, and from the remainder he draws 1 gallon less than three fourths of what is left. He finds that the keg then contains 1 gallon more than one sixth of the original contents. Find the original contents.

Solve:

29. $\left.\begin{aligned} x + 4y &= 37 \\ 2x + 5y &= 53 \end{aligned}\right\}$

30. $\left.\begin{aligned} 8x - 15y &= -30 \\ 2x + 3y &= 15 \end{aligned}\right\}$

31. $\left.\begin{aligned} 5x + 7y &= 17 \\ 7x - 5y &= 9 \end{aligned}\right\}$

32. $\left.\begin{aligned} 10x + 7y &= -4 \\ 6x + 5y &= -2 \end{aligned}\right\}$

33. $\left.\begin{aligned} \tfrac{2}{3}x + \tfrac{3}{5}y &= 17 \\ \tfrac{3}{4}x + \tfrac{2}{3}y &= 19 \end{aligned}\right\}$

34. $\left.\begin{aligned} ax + by &= c \\ mx + ny &= p \end{aligned}\right\}$

35. $\left.\begin{aligned} 3x - y &= 2(a + b)^2 \\ 3y - x &= 2(a - b)^2 \end{aligned}\right\}$

36. $\left.\begin{aligned} 5x - 2y &= 3(a + 7c) \\ 5y - 2x &= 3(a + 7b) \end{aligned}\right\}$

37. $\left.\begin{aligned} ax + bc &= by + ac \\ x + y &= c \end{aligned}\right\}$

38. $\left.\begin{aligned} (a + b)x + (a + c)y &= a + b \\ (a + c)x + (a + b)y &= a + c \end{aligned}\right\}$

39. A market woman spent \$1.30 for apples, some at a cent each, and the remainder at three for two cents. She sold the apples for \$2.40, thereby gaining half a cent on each apple. How many at each price did she buy?

40. A power boat can travel 15 miles an hour downstream. It is found that the boat can cover a certain distance in 45 minutes in still water and the same distance in 1 hour against the current. Find the distance, the rate of the boat in still water, and the rate of the current.

41. There are two numbers whose sum is 3; and the quotient of the first by the second is also 3. Find the numbers.

42. A man has a certain sum of money invested at a certain per cent. If he had \$1000 more invested at a per cent $\frac{1}{2}$ less, the interest would be \$35 less; but if he had \$500 less invested at a per cent $\frac{1}{2}$ higher, the interest would be \$50 more. Find the capital and the rate per cent.

43. A and B working together can build a wall in 12 days. If A works 2 days and B works 3 days, they will build a fifth of the wall. How long will it take each alone?

Solve:

44. $x^2 - 17x + 60 = 0.$

45. $\frac{6}{5}x^2 + 10 = 7x.$

46. $20x^2 + 159 = 136x.$

47. $\frac{7}{2x-3} + \frac{5}{x-1} = 12.$

48. $x + 5\sqrt{37 - x} = 43.$

49. $\sqrt{x+1} + \sqrt{x-4} = \sqrt{2x+9}.$

50. $3x - 4\sqrt{x-7} = 2(x+2).$

51. $a^2 - x^2 = (a - x)(b + c - x).$

52. $\frac{2a+b}{a+x} - \frac{2a-b}{a-x} = \frac{2a}{b}.$

53. $\sqrt{a-x} + \sqrt{x-b} = \sqrt{a-b}.$

54. A man drives a certain distance in 6 hours. If he had driven three quarters of it at a rate $1\frac{1}{2}$ miles an hour faster, he could have driven the remainder at a rate 3 miles an hour slower and finished in the same time. Find the distance and the rate of driving.

55. Two friends, A and B, are 221 miles apart and set out on bicycles to meet. A travels 10 miles an hour and B travels 6 miles less per hour than the number of hours each travels. How many miles does each travel before they meet?

56. The sum of a certain number and its square root is 72. Find the number.

57. A man divides 120 nuts among 9 children, giving half to the boys and half to the girls. If each girl receives 3 more nuts than each boy, how many boys and how many girls are there?

58. The denominator of a certain fraction is greater than the numerator by 3. If 2 is added to each term, the new fraction is $\frac{2}{21}$ greater than the given fraction. Find the fraction.

59. A tourist who has finished a journey of 540 miles would have required 2 days more if he had gone each day 3 miles less. How many days was he on his journey, and how many miles did he travel each day?

60. A body of troops was marching in solid column with 14 more men in depth than in front. When the front was increased by 828 men, there were 5 full lines. Find the number of men.

Solve:

61. $\left.\begin{aligned} 5x^2 + 2y^2 &= 22 \\ 3x^2 - 5y^2 &= 7 \end{aligned}\right\}$

62. $\left.\begin{aligned} x + xy &= 45 \\ y + xy &= 48 \end{aligned}\right\}$

63. $\left.\begin{aligned} x^2 + y^2 &= 130 \\ \frac{x+y}{x-y} &= 8 \end{aligned}\right\}$

64. $\left.\begin{aligned} x^2 + y^2 &= axy \\ x + y &= bxy \end{aligned}\right\}$

65. $\left.\begin{aligned} x &= a(x^2 + y^2) \\ y &= b(x^2 + y^2) \end{aligned}\right\}$

66. $\left.\begin{aligned} b^2x^2 &= a^2y^2 \\ a - x &= b - y \end{aligned}\right\}$

67. $\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{3} \\ x^2 + y^2 &= 160 \end{aligned}\right\}$

68. $\left.\begin{aligned} x(y+1) &= 10(y-1) \\ 2y(x+1) &= 9(x-1) \end{aligned}\right\}$

69. If the sum of the squares of two numbers is divided by the first number, the quotient is 14 and the remainder 4; if divided by the second number, the quotient is 10 and the remainder 4. Find the numbers.

70. The diagonal of a rectangle is 85 feet. If each dimension is increased by 2 feet, the area is increased by 230 square feet. Find the dimensions of the rectangle.

71. The sum of two numbers is 24 and the sum of the fourth powers of the numbers is 85,922. Find the numbers.

72. A garrison is supplied with bread for 11 days. If there were 200 men more, each man would receive each day $\frac{1}{4}$ of a pound less bread; if the garrison was 300 men smaller, each man would receive each day $\frac{1}{2}$ of a pound more bread. How large was the garrison and how large was the daily ration of bread?

73. The fore wheel of a carriage turns 132 times more than a hind wheel while the carriage goes 1 mile. If the circumference of each was 2 feet greater, the fore wheel would turn only 88 times more. Find the circumference of each wheel.

74. The third digit of a number is the sum of the other two digits. The product of the first and third digits exceeds the square of the second by 5. If 396 is added to the number, the order of the digits is reversed. Find the number.

75. An express train travels a miles in b hours with one stop of 5 minutes. An accommodation train covers the same distance in c hours, making 15 stops of average length of 2 minutes. How many miles will each train travel in one hour, if each moves at a uniform rate between stations? Compare the rates of the two trains.

76. Find two numbers in the ratio $1\frac{1}{2} : 2\frac{2}{3}$ such that when each is increased by 15 they shall be in the ratio $1\frac{2}{3} : 2\frac{1}{2}$.

77. Two companies of a regiment went into a battle in strength as 9 to 11. After the battle the relative strength was as 5 to 8. Of the men in the two companies 35 per cent were disabled, and 30 of these belonged to the second company. Find the strength of each company at first.

78. Show that the product of the least and the greatest of any four consecutive numbers is less by 2 than the product of the two intermediate numbers.

79. Show that if 1 is added to the product of any four consecutive numbers, the sum is a perfect square.

80. If $\dfrac{x^3 + ax^2 - bx + c}{x^3 - ax^2 + bx + c} = \dfrac{x^2 + ax - b}{x^2 - ax + b}$, show that $x = \dfrac{b}{a}$.

81. $x^4 + x^3 - 16x^2 - 4x + 48$ may be resolved into two factors of the form $x^2 + mx + 6$ and $x^2 + nx + 8$. Find the factors.

82. A man who has deposited \$500 at compound interest each year at 5 per cent finds that at the age of 54 he owns \$33,250. At what age did he begin to deposit his money?

83. December 31, 1895, a man deposited \$1200 in a savings bank that pays 3 per cent compound interest. If he deposited \$300 on the last day of December of each year, what will be the amount of his deposit December 31, 1905?

84. A man borrows \$10,000 at 4 per cent interest and pays each year $12\frac{1}{2}$ per cent of that sum. In how many years will the debt be paid?

503. Laws of Concurrent Forces. *Concurrent* forces are forces whose lines of action pass through the same point.

1. *If two equal concurrent forces act on a rigid body in opposite directions, they balance each other.*

2. *If two concurrent forces acting on a rigid body are balanced, they must be equal and opposite in direction.*

3. *If two concurrent forces are represented by two straight lines drawn from any point, and a parallelogram is constructed upon these lines as sides, the resultant of the forces is represented by the diagonal through that point.*

4. *The resultant of any two of three balanced forces is equal to and opposite to the third force.*

Exercise 146

1. Forces of 5 pounds and 12 pounds act on a point. Find their resultant if the forces act in the same direction; if they act in opposite directions; if they act at right angles.

2. Two forces of 25 pounds act on a point at right angles. Find the resultant, and the angle it forms with each force.

3. The resultant of two concurrent forces acting at right angles is 10 pounds. One of the forces is 8 pounds. Find the other force.

4. Resolve a force of 75 pounds into two perpendicular components that make equal angles with the given force.

5. Resolve a force of 40 pounds into two perpendicular components one of which is three times as great as the other.

6. What force parallel to a smooth inclined plane 18 feet long and 12 feet high will support on the plane a body weighing 75 pounds?

7. The base of a smooth inclined plane is 80 inches and the height is 39 inches. What force acting parallel to the plane will support on it a weight of 178 pounds?

8. Resolve a force of 250 pounds acting northeast into two forces, one acting north and the other east.

9. Two forces in the ratio of 3 to 5 act at right angles upon a point and produce a resultant force of 136 pounds. Find the value of the forces.

10. A body moves east with a constant velocity of 25 feet per second and south with a constant velocity of 30 feet per second. Find the actual rate of motion in a straight line.

11. A man rows a boat at right angles to the course of a river three times as fast as the river flows, and reaches the opposite bank half a mile below the starting point. What is the width of the river?

12. A boy sits on his sled on the side of a hill that rises 1 foot in every 6 feet. If the weight of the boy and sled is 100 pounds, what force acting parallel to the ground will keep the sled from sliding down the hill?

13. A stone weighing 150 pounds rests on an inclined plane 8 feet long and 3 feet high. What must be the force of friction that will keep the stone from sliding down the plane?

14. Find the tension on the rope if a mass of 150 pounds is suspended from the middle of a rope 16 feet long fastened to each end of a horizontal beam 12 feet long.

15. Three concurrent balanced forces act, one toward the north, one toward the east, and one toward the southwest. If the third force is 25 pounds, find the other two forces.

16. A ladder 24 feet long weighs 36 pounds and rests with one end against a wall and the other end on the ground 6 feet from the wall. The center of gravity of the ladder is $11\frac{1}{2}$ feet from the base. Find the horizontal pressure against the wall.

17. If 200 pounds is supported in front of a wall by a horizontal rod and a brace making an angle of 45° with the rod at its end, find the pull of the rod and the push of the brace.

504. Specific Gravity. The *specific gravity* of a substance is the number that expresses the ratio of the weight of a given volume of the substance to the weight of an equal volume of water.

$$\textit{Specific gravity} = \frac{\textit{weight of the body}}{\textit{weight of an equal bulk of water}}.$$

NOTE. If metric units are used, 1 cubic centimeter of water at 4° C. weighs 1 gram, and the density of a substance is numerically equal to the specific gravity. If English units are used, 1 cubic foot of water weighs 62.4 pounds, and the density in pounds per cubic foot is equal to 62.4 times the specific gravity.

Examples. 1. A piece of copper weighs 89 grams in air and 79 grams in water. Find the specific gravity of copper.

The weight of an equal bulk of water is $89 - 79$, or 10, grams.

Therefore, the specific gravity of copper is $89 \div 10 = 8.9$.

2. A piece of wood weighs 40 grams in air and a lead sinker weighs 50 grams in water. The wood and the sinker tied together weigh only 30 grams in water. Find the specific gravity of the wood.

The buoyant force on the wood supports the wood and takes away $50 - 30$, or 20, grams from the weight of the sinker. Hence, the buoyant force upon the wood, or the weight of an equal bulk of water, is $40 + 20$, or 60, grams.

Therefore, the specific gravity of the wood is $\frac{40}{60}$, or $\frac{2}{3}$.

3. An empty bottle weighs 80 grams. Filled with water the bottle weighs 200 grams, and filled with chloroform it weighs 260 grams. Find the specific gravity of chloroform.

The weights of equal volumes of water and chloroform are evidently $200 - 80$, or 120, grams; and $260 - 80$, or 180, grams.

Therefore, the specific gravity of chloroform is $180 \div 120 = 1.5$.

EXERCISE 147

1. Find the volume of 1 pound of cast iron, specific gravity 7.2. What does the iron weigh under water?

2. A block of wood, placed in a vessel full of water, floats 0.6 submerged. Sixty cubic centimeters of water runs out. Find the weight, volume, and specific gravity of the wood.

3. A piece of wood weighs 150 grams in air, and a piece of lead weighs 40 grams in water. They together weigh 30 grams in water. Find the specific gravity of the wood.

4. A stone weighs 20.4 grams in air, 14.4 grams in water, and 14.94 grams in linseed oil. Find the specific gravity of the stone and of the oil, and the volume of the stone.

5. A block of wood weighing 42 grams in air floats in water with 0.8 of its volume submerged. How many grams must be placed upon the block of wood to submerge it?

6. A block of wood floats in kerosene, specific gravity 0.79, with three fourths of its volume submerged. How much of the wood will be submerged in sea water, specific gravity 1.026?

7. How many grams of lead, specific gravity 11.4, must be fastened to a cubic centimeter of cork, specific gravity 0.24, that it may just float in sulphuric acid, specific gravity 1.84?

8. A block of lead weighs 236.5 grams. Find the specific gravity of a liquid in which the lead weighs 206.3 grams.

9. The specific gravity of pure milk is 1.03. A sample has been adulterated with water and has a specific gravity of 1.0275. What per cent of the sample is water?

10. A bar of gold contains some copper, and weighs 352.6 grams in air and 332.3 grams in water. Find the number of cubic centimeters of gold and of copper in the bar. The specific gravity of gold is 19.3, that of copper 8.9.

11. A piece of lead weighing 45 grams and a piece of copper are fastened to the ends of a string passing over a pulley, and are in equilibrium when immersed in water. Find the weight of the copper if the specific gravity of lead is 11.4, and that of copper is 8.9.

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