Harriet E. Glazier

Oct 27. 1916

ELEMENTARY ALGEBRA

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One of the greatest mathematicians of all time. As a young man he invented the Binomial Theorem which is now studied in second year courses in algebra. Newton wrote a *Universal Arithmetic*, which was really a book on algebra. He was the first to use fractional and negative exponents as we write them. He first used them in a letter dated June 13, 1676.

ELEMENTARY ALGEBRA

First Year Course

BY

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New York

THE MACMILLAN COMPANY

1916

Set up and electrotyped. Published June, 1915. Reprinted June, 1916.

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

In this book algebra is presented to beginners in a simpler, clearer, and more practical form than is usually found in school texts.

The treatment is simplified by the omission of certain redundant terms and notations, by maintaining an intimate connection between algebra and arithmetic, and by starting with definite assumptions of the laws of signs in subtraction and multiplication rather than with complicated and unsatisfactory proofs. The part of algebra that deals with the mechanical manipulation of algebraic expressions is reduced in amount as much as is consistent with the acquirement of accuracy. Simple fractions and easy radicals are introduced early; the discussion of fractions with binomial and trinomial denominators, and consideration of the more difficult parts of radicals, are postponed to a time near the close of the course. Ratio and proportion are brought into closer touch with the equation and with graphs.

Clearness has been sought by careful definition, copious illustration, and by the use of language which recalls the axiomatic processes involved. Such phrases as "clearing of fractions" are objectionable because of the danger of their being applied by the pupil, not to the equation alone, but to algebraic expressions in general. "Canceling," as now used, is ambiguous, for it sometimes involves subtraction, at other times division.

The treatment is rendered more practical by a careful selection of problems. Mechanical problems involving the theory of the lever, specific mass, specific heat, the motion of falling bodies, and technical terms in electricity are omitted at first, and even near the end they are used only sparingly. Experience has shown that first year pupils in the high school cannot cope successfully with the abstract concepts of mechanics. These topics belong more properly to the course in physics usually given in the third or fourth year. In this text stress is laid on practical applications to problems arising in business. In such applications the pupil is merely continuing on a somewhat higher plane the subjects first approached in arithmetic. A distinguishing feature of this text is the practical application of graphs. The ordinary procedure is to use the graph merely in presenting to the eye the behavior of two variables in an equation. In this text it serves, in addition, for the determination by inspection of results decidedly practical in character.

The authors are indebted for valuable criticisms and suggestions to several teachers, but more particularly to Principal E. L. Brown of the North Side High School in Denver.

FLORIAN CAJORI.
LETITIA R. ODELL.

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$4 \div 2$

$4 \div 2$

xyi   4e

ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION

1. Algebra, like arithmetic, deals largely with the solution of problems involving numbers. It is an extension of arithmetic which enables one to solve more complicated problems. It extends the field of arithmetic by means of three devices: the use of letters as well as Hindu-Arabic numerals in the study of numbers; the introduction of new kinds of numbers to be used in conjunction with those of arithmetic; and the development of simple methods of operation with those numbers.

The signs of operation used in arithmetic are used also in algebra.

Thus addition is indicated by $+$, subtraction by $-$, multiplication by $\times$, division by $\div$.

The sign of multiplication ($\times$) is often replaced in algebra by a dot ($\cdot$), which is written above the line to distinguish it from a decimal point. Thus $3 \cdot 4$ means 3×4.

The product of two numbers, one or both of which are represented by letters, is usually indicated by writing the numbers one after the other without any sign between them. Thus ab means $a \times b$ or $a \cdot b$, $4c$ means $4 \times c$ or $4 \cdot c$.

When only one of two numbers is represented by a letter, it is customary to write the letter last. Thus we write $4c$, but not $c4$.

In arithmetic, and also in algebra, division is frequently expressed in the form of a fraction. Thus $\frac{4}{5} = 4 \div 5$, $\frac{a}{b} = a \div b$.

$=$ is the sign of equality.

2. Any combination of numbers, letters, and symbols of operation, which represents a number, is called an *algebraic expression.*

Thus, $a + b$, $\frac{a-b}{2c}$, $4ab \div c$ are algebraic expressions.

3. In arithmetic it is customary to use abbreviations, such as ft. for "foot," in. for "inch," A. for "acres," and so on. In algebra the practice of using abbreviations is carried much further. If we write i for "interest," p for "principal," r for "rate," and t for "time," then the statement

"The interest is equal to the product of the principal, the rate, and the time"

can be expressed briefly by the equation

$$i = prt.$$

By the use of such a system of shorthand much time and labor is saved. The same letter may be used for different things in different problems.

The letter m may be used for "miles" in one problem and "minutes" in another. The letter x is used extensively to designate a number which is unknown at the outset, but which is to be determined by the solution of the problem in hand. In one case x may mean the number of "dollars," in another case the number of "pounds," in a third case the number of "years." The meaning of a letter must be made plain in each problem. To avoid confusion, no letter should stand for two different things in the same problem.

If i means "inches" and f means "feet," then

$48\,i = 4\,f$ means "48 inches are equal to 4 feet."

$15f + 4i$ means "15 feet and 4 inches." If n stands for any number, then $2n + 50$ means "2 times any number, increased by 50," or "twice any number, plus 50." If n is taken equal to 3, then $2n + 50 = 2 \cdot 3 + 50 = 56$. If $n = 25$, then $2n + 50 = 2 \cdot 25 + 50 = 100$.

ORAL EXERCISES

4. Express in algebraic symbols a number which is

1. 5 more than x.
2. 5 less than x.
3. .8 more than three times x.
4. 10 less than five times c.
5. 20 less x.
6. 1.5 less five times x.

7. 6 less than y.

8. 1.2 more than y.

9. .18 less than ten times z.

10. 5 more than $\frac{1}{4}$ of y.

11. 1 less than twice y.

Supplying for each letter a given number, express in words exercises 12–25:

12. $16g$.

13. $g+1$.

14. $2x+31$.

15. $2x-5$.

16. $7x-13$.

17. $13-5x$.

18. $\frac{1}{5}y-3$.

19. $\frac{1}{4}y+7$.

20. $3y+5$.

21. $7-4y$.

22. $4y-\frac{1}{3}y-5$.

23. $15-\frac{2}{3}y$.

24. $17x+5x=22x$.

25. $12x-3x=9x$.

26. If $i=$ interest, $p=$ principal, $t=$ time, $r=$ rate, express in words $i=prt$, $p=\frac{i}{rt}$, $r=\frac{i}{pt}$, $t=\frac{i}{pr}$.

Find the values of the following expressions if $a=2$, $b=3$, $c=5$, $d=19$:

27. $8a+2b$.

28. $3b+c$.

29. $6c-b$.

30. $9c+4d$.

31. $a+b+c+d$.

32. $a+2b+3c-d$.

33. $d-5a+7b-c$.

34. $\frac{1}{4}+a$.

35. $59-5c+10b$.

36. $2d+199c-25a-19$.

If $m=15$, $n=12$, $p=20$, $q=0$, what are the values of the following algebraical expressions?

37. $m+n+p+q$.

38. $q+5p-6m$.

39. $12n+2p+5m-q$.

40. $6p+12+2n-3m$.

(When $q=0$, then $12q$ is equal to 12 times 0; this gives the product 0.)

41. $2m+n+12q$.

42. $3p+5q+m$.

43. $5n+m-7q$.

44. $8q+5p-2m$.

EQUATIONS

5. An *equation* expresses an equality. In other words, an equation is a statement that two expressions stand for the same number. Thus $x + 3 = 3x - 1$ is an equation. Equations are used in the solution of problems. Some number, which at the outset is unknown, is represented by a letter; an equation is formed, which enables one to ascertain the value of that unknown number.

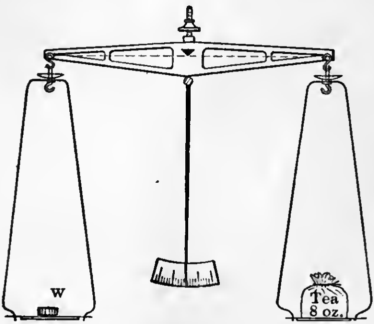


Fig. 1

An equation is like a balance which is in equilibrium when the weights placed in one scale pan are together equal to the weights placed in the other scale pan.

The equilibrium of a balance is not disturbed so long as like changes in the weights are made simultaneously on both sides.

So in equations, we may *add* the same number to both sides, or *subtract* the same from both sides, or we may *multiply* or *divide* both sides by the same number (except division by zero); the equality is maintained during all these changes.

On the other hand, the equality is destroyed if more is added to or subtracted from one side than the other, or if one side is multiplied or divided by a larger number than is the other side.

6. Illustrative Problem. If 13 is added to twice a certain number, the sum is 41. Find the number.

Arithmetical Solution. Here we do not use equations. We subtract 13 from 41 and obtain 28. Then we divide 28 by 2 and obtain 14, which is the required number. That is, we begin with 41, and go back to the required number by *subtraction* and *division*, which are respectively the *inverse* of the operations of *addition* and *multiplication*, named in the problem.

Algebraical Solution. Here we use equations. Let some letter stand for the unknown number and then perform upon it the *direct* operations named in the problem and obtain an equation. This *direct* way is usually easier than the *inverse*. The solution is as follows:

Let x be the unknown number. Then $2x$ is twice that number, and $2x + 13$ is the sum obtained by *adding* 13 to twice that number.

But this sum is equal to 41, as is stated in the problem. That is,

$$2x + 13 = 41.$$

We want to find x. Remembering that an equality is like a balance, that the equality is not destroyed if 13 is subtracted from both sides, we obtain

$$2x + 13 - 13 = 41 - 13, \text{ or}$$
$$2x = 28.$$

The equality remains true if both sides are divided by 2. Hence

$$\frac{2x}{2} = \frac{28}{2}, \text{ or}$$
$$x = 14.$$

We have now *solved* the equation $2x + 13 = 41$ and have found the previously unknown number x to be equal to 14. We see that the *solution* of the equation consists in making such changes in the equation that *x finally stands alone on one side of the equation.*

7. Second Illustrative Problem. If from $\frac{1}{3}$ of a certain number 17 is subtracted, the number resulting is 35. Find the number.

Let x be the required number. Then

$\frac{x}{3}$ is $\frac{1}{3}$ of that number, and

$\frac{x}{3} - 17$ is $\frac{1}{3}$ of that number, less 17.

But this difference is equal to 35, as is stated in the problem. Hence

$\frac{x}{3} - 17 = 35$. To solve this, add 17 to both sides. We get

$\frac{x}{3} = 52$. Multiplying both sides by 3, we obtain

$x = 156$, which is the answer.

This answer is *correct*, for $\frac{1}{3}$ of $156 = 52$, and $52 - 17 = 35$, as stipulated in the problem.

PROBLEMS

8. 1. If 23 is added to three times a certain number, the sum is 74. Find the number.

2. A wagon loaded with 12 sacks of flour weighs 1876 lb.; the empty wagon weighs 700 lb. Find the weight of 1 sack.

3. A train travels from Baltimore to New York, a distance of 185 mi., in 4 hr. Find the average velocity of the train.

Work by arithmetic; then by algebra.

In the latter case, let v be the velocity in miles per hour; form an equation which expresses the relation:

$$\text{velocity} \times \text{time} = \text{distance}.$$

4. The distance from New York to St. Louis is 1058 mi. What is the velocity of a train which travels this distance in 23 hr.?

5. How long will it take a train to travel the distance of 960 mi. from New York to Chicago at an average speed of 48 mi. an hour?

Let t = the no. of hours ("time").

6. A lot sold for \$2475. What was the frontage, if it sold at \$75 a front foot?

7. The sum of two numbers is 31; the larger exceeds the smaller by 7. Find the two numbers.

Let the smaller number	$= x.$
Then the larger number	$= x + 7.$
The sum of the two numbers	$= x + x + 7.$
But the sum of the two numbers	$= 31.$
Hence	$x + x + 7 = 31.$
Since $x + x = 2x$,	$2x + 7 = 31.$
Subtract 7 from both sides,	$2x = 24.$
Divide both sides by 2,	$x = 12$, the smaller number.
Whence	$x + 7 = 19$, the larger number.

8. The sum of two numbers is 75; the larger exceeds the smaller by 15. Find the two numbers.

9. Find two numbers whose difference is 34 and whose sum is 126.

10. A father earns \$ 24 a week more than his son. Together they earn \$ 36 a week. What are the weekly wages of each?

11. Divide \$ 96 between father and son so that the son gets $\frac{1}{7}$ of what the father gets.

12. Divide \$124 between two brothers so that one receives \$36 more than the other.

13. I propose to a boy the following puzzle: "Think of a number, multiply it by 10, add 30, subtract 20." He gives the result as 60. Find the number.

14. To find the weight of a golf ball a man puts 10 golf balls into the left scale pan of a balance and a 1-lb. weight into the right; he finds that too much, but the balance is restored if he puts 1 oz. into the left scale pan. What was the weight of a golf ball?

15. A path half a mile long is to have a curb on both sides. The curbstones used are 40 in. long, and 200 of them have already been supplied. How many more are wanted?

16. The cost of housekeeping for a family of n persons is estimated at $7.5 + 2.5\,n$ dollars per week. How many persons are there in a family whose weekly expenses are \$ 25?

17. A father leaves \$ 14,000 to be divided among his three children, so that the eldest child receives \$ 1000 more than the second, and twice as much as the third. What is the share of each?

18. Divide \$ 24,000 among A, B, and C, so that A's share may be three times that of B, and C may have $\frac{1}{2}$ of what A and B have together.

19. A piece of silver is found to weigh a certain number of ounces in a pair of scales; on taking out a weight of 3 oz. from one scale and placing it in the scale containing the silver, the contents of one scale is twice as heavy as the contents of the other. Determine the weight of the silver.

20. Find the number whose excess over 2 is three times the excess of its half over 4.

21. If the length of a circle is 3.1416 times its diameter, find the radius of a circle (correct to two decimal places), when the length of the circle is 7.56 ft.

To determine the nearest figure in the second decimal place, it is necessary to compute the figure in the third decimal place. If the decimal figures are, for example, .546, write .55; if the decimal figures are .543, write .54; if the decimal figures are .545, write either .54 or .55.

22. If 6 be added to 7 times a certain number, the result is equal to 144. Find the number, correct to two decimal places.

23. The sum of two numbers is 11.3; 8 times the greater exceeds 11 times the smaller number by 2.5. Find the numbers, correct to three decimal places.

FACTOR, COEFFICIENT, TERM, EXPONENT

9. Each of the quantities which multiplied together form a product is called a *factor* of the product.

Thus, each of the numbers 3, 5, x, y, is a factor of the product $15\,xy$. Also, since $15\,x$ times y gives the product $15\,xy$, $15\,x$ is called a factor of $15\,xy$.

In general, the product of any two of the simple factors, 3, 5, x, y, is also called a factor of $15\,xy$; also, the product of any three of the factors 3, 5, x, y, is called a factor of $15\,xy$.

10. In $4\,c$, 4 is called the *coefficient* of c. Similarly,

In $16\,a$, 16 is called the coefficient of a.

In $16\,ab$, 16 is called the coefficient of ab.

Here the word *coefficient* is applied to the factor which is expressed in the Hindu-Arabic numerals. This is the most common use of the word *coefficient*, but in a broader sense, *either of two factors which are multiplied together to form a product* is called a coefficient of the other factor.

For instance, $16\,ab$ is the product of two factors $16\,a$ and b; hence $16\,a$ is the coefficient of b, and b is the coefficient of $16\,a$.

If no numerical coefficient is expressed, 1 is understood; x is the same as $1x$. Notice also that $0x$, or $0 \times x$, is equal to 0.

11. An algebraic expression may consist of parts which are separated by the $+$ or $-$ signs; these parts with the signs immediately preceding them are called *terms*.

Thus, the expression $3a - 4b + 5c$ is separated by the $+$ or $-$ signs into three parts; it has the terms $+3a$, $-4b$, $+5c$.

12. An algebraic expression of one term is called a *monomial*, of two terms a *binomial*, of three terms a *trinomial*, and of several terms a *polynomial*.

13. The product of two equal factors $a \cdot a$ is called the *square* of a.

$a \cdot a$ is usually written a^2 and is read "a square" or "the second power of a."

The product of three equal factors $a \cdot a \cdot a$ is called the *cube* of a.

$a \cdot a \cdot a$ is usually written a^3, and is read "a cube" or "the third power of a."

The product of four equal factors $a \cdot a \cdot a \cdot a$ is called "a fourth" or "the fourth power of a."

In a^2, a^3, a^4, a^5, a is called the *base*, the 2, 3, 4, 5, are called *exponents*.

14. An *exponent* is a number or letter placed a little above and to the right of another number or letter, called the *base*.

When the exponent is a positive integer, it indicates *how many times the base is taken as a factor*.

Avoid saying that a^2 means a multiplied by itself 2 times. This is not true. a^2 means $a \cdot a$, where the base a is multiplied by itself only once.

Why is it incorrect to say that a^3 means a multiplied by itself 3 times?

When no exponent is expressed, the exponent is regarded as 1. Thus, a means the same as a^1, 7 the same as 7^1.

Later we shall extend the definition of an exponent, in order to give meaning to such expressions as $x^{\frac{1}{2}}$, or $c^{\frac{2}{3}}$.

Care must be taken to distinguish between exponents and coefficients. Notice that e^4 means $e \times e \times e \times e$, but $4e$ means $e + e + e + e$. If $e = 3$, then $e^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$, while $4e = 3 + 3 + 3 + 3 = 12$.

15. If the factors of a number are all equal, any one of them is called a *root* of the number.

Nine has two equal factors, 3 and 3. We call 3 the *square root* of 9. Similarly, 27 has three equal factors 3, 3, and 3; 3 is called the *cube root* of 27. Again 256 has four equal factors, each being 4; hence 4 is called the *fourth root* of 256.

Roots are indicated as follows: $\sqrt[2]{9}=3$, $\sqrt[3]{27}=3$, $\sqrt[4]{256}=4$. The figure of the radical sign shows what root of the number is to be taken. This figure is called the *index* of the root. If no figure is expressed, square root is understood.

EXERCISES

16. Express in words the following:

1. a^5.	6. $41\,m$.	11. $\sqrt[3]{125}-\sqrt[3]{64}$.
2. b^3.	7. $3\,y^2$.	12. 19^3-5^4.
3. a^5+b^3.	8. $5\,k^4+c$.	13. $a+s^2+g^5-8\,p^7$.
4. e^2-4.	9. $6\,h-9\,x$.	14. x^2+y^2+z-mn.
5. 8^2-29.	10. $\sqrt{36}-\sqrt{25}$.	15. $h^3-u^5+k^2-\sqrt{3\,x}$.

16. Compute the value of the expression in exercises 1–15, if $a=2$, $b=4$, $c=50$, $e=30$, $g=1$, $h=5$, $k=11$, $m=6$, $n=10$, $p=1$, $s=12$, $u=0$, $x=3$, $y=9$, $z=29$.

17. Find the value of $4\,c^4$ for each of the following values if $c=1, 2, 5, 6, 4, 3, 10$.

18. When $s=10$, compute the values of $s^2-2\,s$, $s^3-3\,s$, $s^4-4\,s$, $s^5-5\,s$.

19. When $t=\frac{1}{2}$, compute the values of t^2, t^3, $5\,t^2+6\,t^3$, $3\,t-t^3$, $4\,t-t^4$.

20. When $b=0$, find the values of b^2, b^3.

PARENTHESES

17. When terms are to be grouped together, parentheses are used.

Thus, $(x+y)^2$ means that $x+y$, considered as a single number, is to be squared. That is, $(x+y)^2=(x+y)(x+y)$.

Again, $5a-(b-2c+d)$ means that the entire expression $b-2c+d$ is to be subtracted from $5a$.

When several parentheses are used in the same expression, confusion may be avoided by using different forms. All these forms go by the general name of "parentheses," but they are designated by special names when it is desirable to distinguish between them. Thus [ ] is called a "bracket," { } is called a "brace," —— is called a "vinculum." But ( ) is always called a "parenthesis."

That the sum of a and b is to be multiplied by c may be indicated in four different ways as follows: $(a+b)c$, $[a+b]c$, $\{a+b\}c$, $\overline{a+b}\cdot c$.

EXERCISES

18. Read and tell the meaning of expressions 1–12:

1. $20(5-3)$.
2. $10(x-2y)$.
3. $(x+y)(x-y)$.
4. $(4x+3)(x-y)$.
5. $3x(x+y)$.
6. $3x+(x+y)$.
7. $(c+d)-5$.
8. $(2-s)+[10-4]$.
9. $\frac{3}{x}-\left(\frac{2}{5}-\frac{1}{3}\right)$.
10. $\overline{b+c}\cdot d$.
11. $\overline{e+f}-\{g+h\}$.
12. $[a+b-(c+d)]-e$.

13. Show that $a^2-b^2=(a-b)(a+b)$, when $a=12$ and $b=10$.
14. Show that $\{a+b\}^2=a^2+b^2+2ab$, when $a=9$ and $b=5$.
15. Show that $(a-b)^2=a^2+b^2-2ab$, when $a=12$, $b=8$.

19. Expressions like $19+(7+3)$ may be worked out in two ways:

(1) First simplify inside the parenthesis and then add.

(2) Add 7 to 19 and then add 3 to the result. That is,

$$\begin{aligned} 19+(7+3) &= 19+10=29, \\ &= 19+7+3=26+3=29. \end{aligned}$$

In the same way there are two ways of working $19+(7-3)$ or $19-(7+3)$ or $9\times(7+3)$ or $9\times(7-3)$, as appears from the following:

$$
\begin{aligned}
&(1)\quad 19+(7-3)=19+4=23,\\
&(2)\quad\qquad\qquad\;\; =19+7-3=23.\\
&(1)\quad 19-(7+3)=19-10=9,\\
&(2)\quad\qquad\qquad\;\; =19-7-3=9.\\
&(1)\quad 9\times(7+3)=9\times 10=90,\\
&(2)\quad\qquad\qquad\;\; =9\times 7+9\times 3=90.\\
&(1)\quad 9\times(7-3)=9\times 4=36,\\
&(2)\quad\qquad\qquad\;\; =9\times 7-9\times 3=36.
\end{aligned}
$$

EXERCISES

20. Perform each of the following exercises in two ways:

1. $8+[6+3]$.
2. $\{10+5\}+122$.
3. $30-[4+8]$.
4. $16+(11-6)$.
5. $15+[5+6-2]$.
6. $5\times(5+5)$.
7. $[3+9+5-3]\times 5$.
8. $25\times(1+2+3-6)$.
9. $30(10-4+3-1)$.

21. In expressions like $9-4+3$ or $5\cdot 6-2\cdot 10+2\cdot 3$ or $3+3\cdot 6\div 3\cdot 2$ it is understood that the multiplications and divisions (if there are any) are performed first in their order from left to right; the additions and subtractions are carried out afterwards in their order from left to right.

Notice that, in a term like $3\cdot 5\div 3\cdot 2$, it is understood that the divisor is 3, not $3\cdot 2$. If we want $3\cdot 2$ to be the divisor, we must inclose it in a parenthesis and write the expression thus, $3\cdot 5\div(3\cdot 2)$. Then $3\cdot 2$ will be regarded as a single number. Bearing these things in mind, we see that

$$5-4+3=1+3=4.$$
$$5\cdot 6-2\cdot 10+2\cdot 3=30-20+6=16.$$
$$3+3\cdot 6\div 3\cdot 2=3+18\div 3\cdot 2=3+6\cdot 2=15.$$
$$3+3\cdot 6\div(3\cdot 2)=3+18\div 6=3+3=6.$$

ORAL EXERCISES

22. Simplify the following:

1. $10 - 5 + 3 - 8$.
2. $10 - (5 + 3) + 8$.
3. $10 \times 2 - 9 \div 3 + 5 \times 2$.
4. $12 \cdot 4 \div 2 \cdot 4 - 3$.
5. $12 \cdot 4 \div (2 \cdot 4) - 3$.
6. $12 \cdot 4 \div 2(4 - 3)$.
7. $10 - [12 \div 3 - 2]$.
8. $10 - 12 \div 3 - 2$.

Of the following results, which are wrong?

9. $50 - 10 + 8 = 50 - 18 = 32$.

10. $20 + 24 \div 3 \cdot 4 - 2 = 20 + 24 \div 12 - 2 = 20 + 2 - 2 = 20$.

11. $15 + 36 \div (3 + 9) + 5 = 15 + 36 \div 3 + 9 + 5 = 41$.

EVALUATION OF ALGEBRAIC EXPRESSIONS

23. When $a = 5$, $b = 3$, $c = 2$, $d = 6$, and $e = 4$, find the values of:

1. $\dfrac{1}{b} + \dfrac{1}{c}$.
2. $abc + d^2 - e^3$.
3. $5a^3 - (2ab + d)$.
4. $\dfrac{d - b}{e}$.
5. $\dfrac{10a + 8b - e}{a + c + e}$.
6. $\dfrac{a^2 + b^3 - 3b}{c + d^2 - a^2 + 7}$.
7. $2\{a + e - d + b^4\} - 4b$.
8. $(a + b)(c \times d - 5)$.
9. $b\sqrt{36} - c\sqrt{16} + d\sqrt{144}$.
10. $\sqrt{a + b + c + d + e - 4}$.

11. Does $x^2 + 5x - 36 = 0$, when $x = 4$? When $x = 5$?

12. Does $x^3 - x^2 + x - 10 = 18$, when $x = 5$? When $x = 3$?

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

24. The thermometer in your classroom indicates a temperature of, perhaps, "67° above zero." In the northern states the winter temperature out of doors sometimes drops to "15° below zero" or even lower. A shorter way of expressing this is as follows:

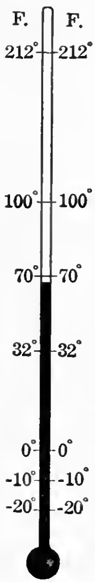


FIG. 2

For "67° above zero" write "+ 67°."
For "15° below zero" write "– 15°."

A + indicates that the temperature is "above zero"; a – indicates that it is "below zero."

But such *plus* and *minus* numbers are convenient in other ways. A man who takes in and pays out money may, for brevity, mark the sums taken in by prefixing the + sign, and the sums paid out by prefixing the – sign. It is customary also to indicate the amount a man owns by + and the amount a man owes by –.

In the same way we may write 1916 A.D. as +1916 and 500 B.C. as – 500. These illustrations make it plain how plus and minus numbers, or *positive* and *negative* numbers, as they are more usually called, may be used in ordinary affairs of life. As we proceed further we shall see that the use of such numbers makes the solution of many problems much easier and shorter. Such numbers can always be used when there are pairs of quantities which are the exact opposites of each other, as are the quantities in the above illustrations.

ORAL EXERCISES

25. State the answers by prefixing + or − to the numbers:

1. The temperature at noon is $+40°$ and falls $25°$ by night. State the temperature at night. Give the answer if it falls $50°$ (that is, if it falls $40°$ and then $10°$ more).

2. A boy purchases a book which costs \$1.75 and pays \$.50; the balance he has charged. How may he indicate the amount charged?

3. His father gives him two dollars and tells him to pay the balance on that book. How much has the boy left?

26. One of the most common modes of representing positive and negative numbers to the eye is by distances along a straight line, as is done in the thermometer. It does not matter in what direction the line is drawn. Usually it is most convenient to draw the line horizontally.

Some point 0 is taken as the starting point (corresponding to the *zero* in the thermometer). Distances to the right of 0 are usually indicated by the *positive numbers*, and distances to the left of 0 by the *negative numbers.*

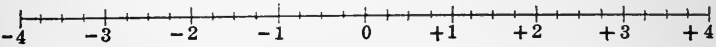


FIG. 3

In the figure, the distance from 0 to $+1$ is taken as unit distance or 1 space. The point marked $+2$ is 2 spaces to the right of 0; the point marked -2 is 2 spaces to the left of 0.

In the same way, a number $+2\frac{1}{4}$ is indicated by a point $2\frac{1}{4}$ spaces to the right of 0; $-2\frac{1}{4}$ is indicated by a point $2\frac{1}{4}$ spaces to the left of 0.

If to $+1$ we desire to *add* $+2$, we start at the point marked $+1$, and go 2 units to the *right*, giving $+3$ as the answer.

If from $+1$ we desire to *subtract* $+2$, we start at $+1$ and go 2 units to the *left*, giving -1 as the answer.

ADDITION

ORAL EXERCISES

27. 1. Locate on the line (Fig. 3) the following numbers:

$$+4,\ -3,\ +1\tfrac{1}{2},\ -2\tfrac{1}{2},\ 3\tfrac{1}{4},\ \tfrac{1}{4}.$$

2. To -1 add 3.

We start at the point -1, and add 3 by going 3 units to the *right*, giving $+2$ as the answer.

3. From $+1$ subtract 4.

4. Add 4 and -3, -4 and -3, $+1\frac{1}{2}$ and 2, $+4\frac{1}{3}$ and -2.

5. Add $+5\frac{1}{2}$ and -2, $+4\frac{1}{2}$ and $1\frac{1}{4}$, $+6$ and -5, 15 and -25.

28. The student will have noticed that each of the symbols $+$ and $-$ is used in algebra in two senses. Thus $+$ is used sometimes to indicate addition and at other times it is used to show that the number to which it is prefixed is positive. Similarly, $-$ sometimes signifies *subtraction* and at other times is used to represent a *negative number.* This double use of these symbols may at first seem confusing, but we soon learn how to interpret algebraic expressions in which they are used.

To express in algebraic symbols the addition of $+5$ and -4 we inclose the numbers in parentheses and write thus:

$$(+5)+(-4).$$

The $+$ between the two parentheses means *addition;* the $+$ in $(+5)$ and the $-$ in (-4) indicate whether the number is *positive* or *negative.*

29. By the *absolute value* of a number is meant its value without regard to the sign before it. Thus the absolute value of both $+5$ and -5 is 5.

Proceeding as in the examples given in § 27, verify the following:

$$(+7)+(+5)=+12.$$
$$(+7)+(-5)=+2.$$
$$(-7)+(+5)=-2.$$
$$(-7)+(-5)=-12.$$

We see that these answers can be obtained by the following rules which we shall find very useful in practice:

The sum of two numbers having the same sign is found by adding their absolute values and writing their common sign before the result.

The sum of two numbers having opposite signs is found by subtracting the less absolute value from the greater and writing before the result the sign of the number having the greater absolute value.

EXERCISES

30. Work exercises 1–9 by these rules and then verify the answers by using the straight line as in the previous exercises.

1. $(+3)+(-6)$.
2. $(-3)+(+6)$.
3. $(-3)+(-6)$.
4. $(-7)+(+17)$.
5. $(-17)+(-15)$.
6. $(+14)+(-10)$.
7. $(+5\frac{1}{2})+(4\frac{1}{4})$.
8. $(+5\frac{1}{2})+(-4)$.
9. $(-12)+(2\frac{1}{4})$.

10. Explain each of the above answers when the positive and negative numbers represent *assets* and *debts;* also when they represent temperatures above and below zero.

11. $(+6)+(+10)+(+6)+(-9)=?$
12. $(+50)+(-30)+(+40)+(-10)=?$
13. $(-9)+(+20)+(-3)+(+20)=?$
14. $(+100)+(+900)+(-500)=?$

SUBTRACTION

31. If the temperature is $+10°$ in the morning and $+40°$ at noon, we can find the rise in temperature by subtracting $+10°$ from $+40°$. The rise is $(+40°)-(+10°)$ or $+30°$. If the temperature in the morning is $0°$, the rise is $(+40°)-(0°)$ or $+40°$. If the morning temperature is $-5°$, then the rise must be still greater; namely, $+45°$.

We have then $(+40°)-(-5°)=+45°$.

An easy way to carry out this subtraction is to change the sign of $-5°$ to $+5$ and then to add $+5$ to $40°$.

We assume the following general rule:

To subtract a number, change its sign and add.

This rule carries the operation of subtraction back to that of addition. Notice that we have made no pretense of actually proving this rule to be true in all cases. We take for granted that it is. In fact, assuming this rule amounts to a definition of subtraction in algebra. To the question, what is meant by "subtracting a number," the rule gives the answer:

"To subtract a number is to change its sign and add."

In subtracting a positive number from a larger positive number, as 7 from 12, it is easier to follow at once the familiar process used in arithmetic. But the algebraic rule just given can be applied to this case also. For we have

$$(+12)-(+7)=(+12)+(-7)=+5.$$

ORAL EXERCISES

32. 1. Verify the following subtractions:

$$\begin{array}{r} +7 \\ -10 \\ \hline +17 \end{array} \qquad \begin{array}{r} +22 \\ +8 \\ \hline +14 \end{array} \qquad \begin{array}{r} -7 \\ -10 \\ \hline +3 \end{array} \qquad \begin{array}{r} -7 \\ +8 \\ \hline -15 \end{array} \qquad \begin{array}{r} -20 \\ -5 \\ \hline -15 \end{array}$$

2. Check each answer in Ex. 1, by adding the remainder to the subtrahend. What should the sum be equal to in each case?

3. Explain by the thermometer (or by *assets* and *debts*, or by a straight line with a starting point 0) how it is possible to subtract 5 from a smaller number 3. Does the introduction of negative numbers make subtraction always possible in algebra?

Perform the following subtractions:

4. $\begin{array}{r} +20 \\ +30 \\ \hline \end{array}$ 5. $\begin{array}{r} +15 \\ -30 \\ \hline \end{array}$ 6. $\begin{array}{r} -25 \\ -40 \\ \hline \end{array}$ 7. $\begin{array}{r} -60 \\ +10 \\ \hline \end{array}$ 8. $\begin{array}{r} +30 \\ -50 \\ \hline \end{array}$

9. $8-(+12)$.
10. $5-(-40)$.
11. $15-(+60)$.
12. $(-30)-(-10)$.
13. $0-(+10)$.
14. $0-(-10)$.
15. $(-20)-(+20)$.
16. $50-(-30)$.

17. $100-[+30+10]$.

18. $(-5)-\{20+40\}$.

19. $0-(+55+15)$.

20. $(+20+30)-(+45+25)$.

Simplify the following:

21. $5+(+10)-(+60)$.

22. $6-[+9]-[-8]-[+7]$.

23. $(-30)-(-10)-(-40)$.

24. $(-30)-(+10)-(+40)$.

25. $85+\{-90\}-\{+90\}$.

26. $85-\{+90\}+\{-90\}$.

MULTIPLICATION INVOLVING NEGATIVE NUMBERS

33. In arithmetic we define 3×4 as meaning $4+4+4$; that is, 4 is to be taken as many times as there are units in 3. In the multiplication of fractions, say, $\frac{1}{2}\times\frac{2}{5}$, the above definition becomes inapplicable, for the reason that we cannot take $\frac{2}{5}$ a fractional number of times. We are therefore driven to a different definition of multiplication; we define the product as the result obtained by multiplying the numerators together for a new numerator and multiplying the denominators together for a new denominator. Thus, $\dfrac{1}{2}\times\dfrac{2}{5}=\dfrac{1\times 2}{2\times 5}$.

In algebra we are now studying a new type of numbers; namely, negative numbers. When we multiply one number by another, and one or both numbers are negative, the question arises: is the product a *positive* number or a *negative* number? We give the answer to the question in the following *definition:*

The product of two numbers having like signs is a positive number and the product of two numbers having unlike signs is a negative number.

In multiplying together two numbers, first find the product of their absolute values and then write before it the proper sign.

Thus we have, according to this definition,

$$(+3)\times(+4)=+12. \qquad (+3)\times(-4)=-12.$$
$$(-3)\times(+4)=-12. \qquad (-3)\times(-4)=+12.$$

Illustration. If a man *owes* \$10 to each of two persons, he *owes* them \$20 all together. Denote the \$10 *owed* by $-$ \$10.

Then evidently, $(+2)(-\$10) = -\$20$.

Second Illustration. Denote a *profit* of \$10 by + \$10.

Denote a *loss* of \$10 by $-$ \$10.

Denote *receive* (add) by +.

Denote *cancel* (subtract) by $-$. Then

a. To *receive* 5 *profits* of \$10 each is to be worth \$50 *more*. That is, $(+5)(+\$10) = +\$50$.

b. To *receive* 5 *losses* of \$10 each is to be worth \$50 *less*. That is, $(+5)(-\$10) = -\$50$.

c. To *cancel* 5 *profits* of \$10 each is to be worth \$50 *less*. That is, $(-5)(+\$10) = -\$50$.

d. To *cancel* 5 *losses* of \$10 each is to be worth \$50 *more*. That is, $(-5)(-\$10) = +\$50$.

ORAL EXERCISES

34. State the following products:

1. $\begin{array}{r}+5\\ \underline{-11}\end{array}$	2. $\begin{array}{r}-12\\ \underline{-12}\end{array}$	3. $\begin{array}{r}-70\\ \underline{+5}\end{array}$	4. $\begin{array}{r}+33\\ \underline{-11}\end{array}$

5. $(+5) \times (-5)$.
6. $(-5) \times (+5)$.
7. $0 \times (-5)$.
8. $\frac{3}{4} \times \frac{5}{7}$.
9. $(-\frac{2}{3}) \times (+\frac{1}{5})$.
10. $(-.5) \times (+.61)$.
11. $[+2\frac{1}{2}] \times [-2\frac{1}{2}]$.
12. $\{-1.2\}\{-1.2\}$.
13. $(+5)(-1.6)$.

PROBLEMS

35. 1. A man saved \$175 a month for 4 months; then, during a sickness of 3 months, he lost \$200 each month. How much did he have at the end of the 7 months?

2. During 12 months a merchant expended \$200 each month, and took in \$215 monthly. What was his profit during the year?

3. A boy received from his father on July 4, \$1.25; he earned one dime, spent 85¢ for firecrackers, 25¢ for a lunch, and paid four 5¢ car fares. How much did he have at the end of the day?

4. A road from town A to town B leads over rolling country; 5 times the road passes over hillocks rising each 37 ft. and 6 times it descends from the hillocks 34 ft. How much higher or lower is town B than town A?

5. The water in a reservoir rises 3 in., then falls 6 in., rises again 7 in., falls 9 in., and finally rises 13 in. How much higher or lower is it than at first?

6. An explorer 200 mi. south of the north pole travels north 47 mi., then 31 mi. south, and again 10 mi. farther south, then 86 mi. north. How far north from the starting point is he finally?

7. The boiling point of water is $+212°$. How much above or below the boiling point are the following temperatures: $+250°$, $+200°$, $+100°$, $+50°$, $-10°$?

8. Colorado Springs is 6000 ft. above sea level. How much higher or lower are the following elevations above sea level: 14,000 ft., 5000 ft., 400 ft., 100 ft., -100 ft.? How do you interpret -100 ft.?

9. The midday temperatures for one week were $+39°$, $+48°$, $+60°$, $+57°$, $+39°$, $+55°$, $+60°$. Find the average of these temperatures.

The average of a series of numbers is found by dividing their sum by the number of them. In this case there are 7 numbers. Hence you find their average by dividing their sum by 7.

10. The midnight temperatures for that same week were as follows: $+10°$, $-3°$, $-12°$, $+8°$, $+12°$, $-20°$, $-5°$. Find the average of these temperatures.

11. A merchant's monthly profits for five consecutive months were $+\$400$, $+\$200$, $-\$100$, $-\$300$, $+\$500$. Find the average monthly profits.

12. The latitude of Key West, Fla., is 24° 33′ and of Chicago is 41° 50′. What is the latitude of a place halfway between the two?

13. The latitude of the Cape of Good Hope is $-34° 21'$ (the $-$ indicating here *south* latitude); the latitude of Athens in Greece is $37° 58'$. Find the latitude of a place halfway between them.

14. At the seashore the rise and fall of the tides are measured from a certain arbitrarily chosen level. A tide which falls below that level is called a "minus" tide. If one day the tide rises to $6' 3''$ and then falls to $-1' 2''$, what is the average water level for that day?

15. The Greek philosopher, Plato, died -347. How many years ago was that?

16. The elevation above sea level of the top of Mont Blanc is $16{,}050'$, and of the lowest part of the Atlantic Ocean is $-27{,}800'$. What is the difference in elevation between the two?

17. A man is rowing upstream. In still water his rate of rowing is 6 mi. an hour. The rate of the stream is 2 mi. an hour. How long will it take him to reach a place 12 mi. upstream?

18. If a boat is steaming southward on a river at the rate of 13 mi. an hour, while a man on the deck is walking toward the stern at the rate of 3 mi. an hour, what is the man's actual motion with respect to the shore?

19. A balloon capable of exerting an upward pull of 395 lb. is attached to a car weighing 146 lb. What is the net upward or downward pull?

DIVISION

36. From the rule of signs for multiplication we can ascertain what the rule of signs should be for division.

In arithmetic, we test the correctness of a division by multiplying the quotient by the divisor; their product should be the dividend. That is,

$$\textit{Divisor} \times \textit{Quotient} = \textit{Dividend}.$$

We see that $15 \div 3 = 5$, because $3 \times 5 = 15$.

Similarly with negative numbers:

$(+15) \div (-3) = -5$, because $(-3) \times (-5) = +15$.
$(-15) \div (+3) = -5$, because $(+3) \times (-5) = -15$.
$(-15) \div (-3) = +5$, because $(-3) \times (+5) = -15$.

From these examples we see that the rule of signs for division is the same as for multiplication; namely,

The quotient of two numbers having like signs is a positive number.

The quotient of two numbers having unlike signs is a negative number.

From the above it appears that division is the *reverse* of multiplication. In multiplication, we are given two factors, to find the product; in division, we are given the product and one of the factors, to find the other factor.

ORAL EXERCISES

37. Carry out the indicated divisions and check the answers:

1. $(+36) \div (+12)$. 4. $(-42) \div (-7)$. 7. $(-51) \div (+17)$.
2. $(-64) \div (+16)$. 5. $(7.5) \div (1.5)$. 8. $(+5.1) \div (-1.7)$.
3. $(-\frac{3}{4}) \div (-\frac{3}{2})$. 6. $(-1.44) \div (+1.2)$. 9. $(+\frac{1}{5}) \div (+.5)$.

Perform the indicated operations:

10. $(+45) \div (-9) - (-6)$.
11. $(+10) - (-7) - (+3)$.
12. $(-4) \div (-2) - (+5) + (+3)$.
13. $(+9) \div (-3) \times (+12) \times (-1)$.
14. $(-18) \cdot (+3) \div (-6) - (-4)$.
15. $(+50) \times (-\frac{1}{5}) \div (-5 + 7)$.
16. $(-2x) - (+3x) + (-5x)$.
17. $(-8) \cdot (-5) \div (-10)$.
18. $(-16) \div (+2) \times (-9) \div (-8)$.
19. $(-24) \div (-24) \times (-24) - (+24)$.

PROBLEMS

38. Use positive and negative numbers in the solutions of the following problems:

1. A man has a contract to dig a well 56 ft. deep. If he digs on an average 4 ft. a day, how many days will it take him to dig the well?

2. If the foundations of a wall are 5 ft. below the surface and the wall is 41 ft. high above the surface, how many feet are there from the foundations to the top of the wall? How many cubic feet of brick in it if it is 20 ft. long and 2 ft. thick?

A SIMPLIFIED NOTATION

39. We have seen that the expression

$$(+8)+(-5)$$

means "to positive 8 add negative 5." The + between the two parentheses means *addition;* the + and − in $(+8)$ and (-5) serve the purpose of showing the quality of the numbers; namely, that the 8 is positive and the 5 is negative.

We know by our rules for addition and subtraction that

$$(+8)+(-5)=8-5=3 \quad \text{and}$$
$$(+8)-(+5)=8-5=3.$$

This shows that we can simplify the above notation by writing,

$$8-5$$

and that $8-5$ may have two interpretations. It may signify either

$$(+8)-(+5) \quad \text{or} \quad (+8)+(-5).$$

In the first interpretation the − sign means an *operation;* in the second interpretation the − sign means a *quality.*

The same argument shows that

$$(+3)-(-4)=3+4=7 \quad \text{and}$$
$$(+3)+(+4)=3+4=7.$$

We can therefore simplify expressions as in the following examples:

$$(+9)-(-2)+(+6)-(+7)=9+2+6-7.$$
$$(-6)+(+9)-(-8)-(+3)=-6+9+8-3.$$

The sign + may be omitted in expressions involving multiplications and divisions like the following:

For $(+27)\cdot(+9)$ we may write $27\cdot 9$.
For $(-27)\div(+9)$ we may write $(-27)\div 9$.
For $(+15)\div(+5)$ we may write $15\div 5$.

EXERCISES

40. Simplify the following expressions:

1. $(-5)+(+9)-(-12)-(5)$.
2. $(+4)(+8)+(-6)-(+12)$.
3. $(-24)-(+30)+(+36)\div(+7)$.
4. $(-35)\cdot(10)-(+8)-(-7)$.
5. $(+18)\div(+6)-(+5)+(+5)$.
6. $(24+12)-(-42)+(+30)\div(-28)$.
7. $(4+3-2)-(1+6)+(-1+8)-[-8-2+3]$.
8. $3(3+1)-4(5+8)+(2-7)-(5+6-7)+9(8-5)$.
9. $10\{10-6+1\}+[7+3-12]\cdot 4+5-15+(3+4)5$.

SIMILAR TERMS

41. Terms which have the same literal factors are called *similar*.

Thus, $12a$ and $-5a$ are similar; so are $-13xy^2$ and $25xy^2$.

On the other hand, terms which do not contain the same literal factors are called *dissimilar*.

$12a$ and $15b$ are dissimilar terms; so are $-13x^2y$ and $23xy^2$.

If f signifies "feet," we know from arithmetic that

$$10f+5f=15f.$$

We find the sum by *adding the numerical coefficients.* This mode of procedure is general. If f means "forks" or if f stands for any abstract number, as 12, the same process of addition holds.

On the other hand, if the terms are dissimilar, as $5f$ and $10i$, then their sum is not $15f$ nor is it $15i$; all we can do in such a case is to *indicate* the addition by writing

$$10f + 5i.$$

If one boy has 5 ducks + 6 hens + 3 rabbits, and another boy has 3 ducks + 7 hens + 8 rabbits, then the two together have 8 ducks + 13 hens + 11 rabbits. Abbreviating, we write

$$(5d + 6h + 3r) + (3d + 7h + 8r) = 8d + 13h + 11r.$$

This result is true, no matter what d, h, r may mean.

Similar remarks apply to the *subtraction* of similar and dissimilar terms.

EXERCISES

42. Find the sums in exercises 1–11:

1. $3a + 4a + 7a - 5a + 6a.$
2. $(+5c) - (+3c) - 5c.$
3. $3x + 3y + 4z + 9y + z + 2x.$
4. $-3x + 6x + 8x - 6x.$
5. $(+7x) + (3x + 4y) + 6y + 4x.$
6. $(20c - 10d + 30e) + (4d - 5c + 9e).$
7. $(3a + 5b - 6c) + (10b + 4e - 2a) + 6c - 9a + 20b.$
8. $5(a + b) + 3(a + b) + 7(a + b) - (a + b).$
9. $-5a^2b^2 + 4ab^2 - 2a^2b^2 - 6ab^2 + 7ab^2c + 0ab^2c.$
10. $+3fg, +4gh, -5gh, -7gh, +6fg.$
11. $2(a + b) + c, 3a, 7b, 9c.$
12. From $-8ax$ take $-5ax$, then add to the result $-6ax$.

13. Add $9\,mn$ to $+4\,mn$; from the result take $10\,mn$.

14. Subtract $+6\,kl$ from $-24\,kl$, then add $+12\,kl$.

15. Subtract $3\,bc$ from $-5\,ac$, then add $-9\,cd$.

16. Simplify: $40\,st - 5\,st + 20\,st + ty - 10\,ty$.

17. From $6\,(a+x)$ take $7\,x$, then again $-4\,a$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

43. When polynomials are to be added together, or when one polynomial is to be subtracted from another, it is convenient to write similar terms in the same column and to add or subtract the terms in each column, proceeding from left to right.

Add $2\,a + 4\,b - 5\,c$, $-2\,a + 4\,b + 7\,c$, $-8\,a - 5\,b + 9\,c$.

Solution.
$$\begin{array}{rrr} 2\,a & +\,4\,b & -\ \ 5\,c \\ -\,2\,a & +\,4\,b & +\ \ 7\,c \\ -\,8\,a & -\,5\,b & +\ \ 8\,c \\ \hline -\,8\,a & +\,3\,b & +\,10\,c, \end{array}$$
the required sum.

From $9\,x + 8\,y - 5\,z$ take $6\,x - 7\,y + 4\,z$.

Solution.
$$\begin{array}{rrr} 9\,x & +\ \ 8\,y & -\,5\,z \\ 6\,x & -\ \ 7\,y & +\,4\,z \\ \hline 3\,x & +\,15\,y & -\,9\,z, \end{array}$$
the required difference.

EXERCISES

44. Add the following polynomials:

1. $2\,ac^2 + 3\,b^2c - c^3$, $6ac - b^2c + 9\,ac^2$, $ac + b^2c - 5\,c^3$.
2. $a^2 + 8\,b^2 + 4\,c^2$, $2\,a^2 - 5\,b^2 - 8\,c^2$, $6\,b^2 + 7\,c^2$.
3. $6\,x - 7\,y + 4\,xy$, $-4\,x + 5\,y + 10\,xy$, $20\,x - 9\,xy + y$.
4. $3\,x + y - 8\,z + 3\,w$, $9\,x - 8\,y - 6\,z + 4\,w - 3\,v$.
5. $a + b$, $a + c$, $b + c$, $2\,a - 3\,b + 4\,c$.
6. $a + b + c - d$, $2\,b + 3\,c + 7\,d$, $4\,a + 5\,b - 6\,d$.
7. $1\,a + 6\,b + 3\,c - 10\,d + f$, $a - 5\,b + 8\,c + 2\,f$, $2\,b + 3\,c + 4\,d$.
8. $a^3b + 2\,ab^3 - 6ab$, $3\,a^3b - 5\,ab + 9\,ab^3$.

In exercises 9–11, subtract the second polynomial from the first and check your answer by adding the remainder to the subtrahend:

9. $a + b + c + d + e$, $2a + 3b - 4c - 3e$.

10. $10x + 10y - 10z + 10w$, $5w + 3x - 8y + 3z - v$.

11. $a + b - x + y + 3z$, $a + b + x - y - 3z + 5x$.

12. To $a + c - d$ add $4a + 5c + 6d$ and from the sum take $3a + 6c + 9d$.

13. From $2m + 4n + 9p$ take $9m - 8n + 7p$ and to the result add $4m + 4n + 4p$.

14. From $27f + 24g + 23h$ take $19f - 15h + 15g$ and from the result take $20f + 3g - 9h$.

15. Subtract $2b + 4c + 7d + 6e$ from $5b - 4d + 6c + 9e$ and check.

16. From $3a + r + s$ take the sum of $10r + 30s$ and $5r - 7s + 20a$.

17. From the sum of $a + 3b$ and $2a - 6b$ take the sum of $-3a + 24b$ and $a + b + c$.

18. From $x + y + z + w$ take $a - b - c - d$, then add to the result $3x + 4w - 7c + 3d$.

19. From $4(a + b)$ take $6(a + b + 5c)$.

20. Add $2(a + 4b + 3c)$, $3(-2a + 3b + 7c)$, $4(b + c)$.

21. Add $4a^4 + 4a + a^4b$, $0a^4 - 3a + 5a^4b$, $-6a^4b$, $-9a$.

Evaluate the following monomials and polynomials, when $a = -2$, $b = -3$, $c = -1$, $d = 10$:

22. ab, abc, bc, $2ab$, $-2ab$, $-3bc$, $3bc$, cd.

23. $a + b + c + d$, $a - b + c + 2d$, $-5a - 3b + ac$.

24. a^2, b^2, c^2, d^2, a^2b^2.

25. $abcd$, a^2bcd, ab^2cd, abc^3d, $abcd^2$.

26. a^3, b^3, c^3, d^3, a^3b^3, a^3c^3, c^3d^3.

27. a^2c^3, a^2d^3, bd^3, cd^3, c^2d^3.

CHECKING BY SUBSTITUTION OF NUMBERS

EXERCISES

45. 1. Add $3x^2-4y$, $-x^2+7y$, $5x^2-128y$.

$$\begin{array}{lrcrr} & 3x^2 - & 4y = & 3 - & 8 \\ & -x^2 + & 7y = & -1 + & 14 \\ & 5x^2 - & 128y = & 5 - & 256 \\ \hline \text{Sum} & 7x^2 - & 125y & 7 - & 250 \\ & 7x^2 - & 125y = & -243. & \end{array}$$

In the polynomials added and in their sum, let each letter equal some simple number and compare as indicated below. If a mistake has been made in addition, this test almost always reveals it.

In each of the three binomials and also in their sum, let $x=1$, $y=2$. We obtain the numbers on the right. Adding the two columns of numbers, the sums are $7-250$. This is the very same result as that obtained by letting $x=1$ and $y=2$ in the sum $7x^2-125y$. This checks the addition.

Add the following and check by letting $x=2$, $y=1$:

2. $5x^2-6x+4y$, $x^2+7x-3y$, $-3x^2-5x-2y$.

3. $-x^2-7x-2y$, $10x^2+8x-4y$, $-6x^2+4x-10y$.

4. $10x^2-x-y$, $11x^2-8x-9y$, $-20x^2+7x-5y$.

In exercises 5–7, subtract the second polynomial from the first and check by letting $a=2$, $b=3$:

5. $10a^2+a-b$, $5a^2-4a+b$.

6. $25a^2-10a+2b$, $20a^2-11a+3b$.

7. $13a^2+14a+15b$, $20a^2-15a+b$.

8. $10a^3+2b^2$, $7a^3-3b^2$.

9. $12a^3-5b^3$, $7a^3-7b^3$.

10. $7a^3-24b^4$, $8a^3-20b^4$.

REMOVAL OF PARENTHESES

46. In an example like $2a+(a+b+c-d)$, the polynomial within the parenthesis is to be added to what precedes. This may be accomplished by adding each term separately. We may at first merely indicate this separate addition by rewriting the expression with the parenthesis omitted; thus,

$$2a+(a+b+c-d)=2a+a+b+c-d.$$

Thereupon like terms are combined. In this instance we obtain, $3a + b + c - d$.

In an example like $2a - (a + b + c - d)$, the polynomial within the parenthesis is to be subtracted from what precedes. This may be accomplished by subtracting each term separately. We may at first merely indicate this subtraction by rewriting the expression, but changing the sign of each term as it is taken out of the parenthesis; thus,

$$2a - (a + b + c - d) = 2a - a - b - c + d.$$

Thereupon like terms are combined. In this instance we obtain, $a - b - c + d$.

We have then the following rules for the removal of parentheses:

A parenthesis preceded by the + sign may be removed without changing the signs of the terms that were within the parenthesis.

A parenthesis preceded by the − sign may be removed if we change the sign of each term that was within the parenthesis.

When an expression contains a parenthesis within another parenthesis, as $-(5a - [4b - 5c])$, remove one parenthesis first, then the other. Thus,

$$-(5a - [4b - 5c]) = -(5a - 4b + 5c) = -5a + 4b - 5c.$$

It is usually found easier to remove the inner parentheses first.

The student should form the habit of guarding carefully against error in the application of these simple rules.

EXERCISES

47. Remove the parentheses and simplify:

1. $a + (3a + 5 - 4c)$.
2. $b - (b + 2c + 3d)$.
3. $(a + b + 3c) + (a + b) - (a - b)$.
4. $(a + 4b - 5c) - (3a + 4b + 2c) + (a + b)$.
5. $[10x - 2n + 4q] - \{10x + 2n + 4q\}$.

6. $-(-3d+4e-5f)-(3e+7f-6g)$.
7. $\{-91+8m-7n\}+[-71+6n+4m]-(3r+5t-9y)$.
8. $[2x+(2y+z)+w]-\{5y-(2x-z)\}+(10x+9y)$.
9. $[\{a+b\}-\{2a-3b\}-9a+6b]-(4a+5b)$.
10. $-(-p+q)-(4q+6p+3r)-(-5p+r-5q)$.
11. $6c-(4d+[e-3f])$.
12. $4a+\{-2b-(c-4d)+e\}$.
13. $(-2a+\{4b-5c\}-\{2b-3c\}+6a)$.
14. $5b-(4c-a)+(-3b+c)-(5a-\{b+c\}+3c)$.
15. $-(m-n)+(3m+4n)-\{-m+(3m-n)\}$.
16. $\{-(m+4n-p)-(p-2n)+(4p-3m)-7m\}$.

INSERTION OF PARENTHESES

48. This process is the inverse of the preceding. Since in removing a parenthesis with the $+$ sign before it the signs of the terms taken out are not changed, it follows that when the parenthesis is restored, the signs of the terms inserted in the parenthesis are not changed, provided the sign before the parenthesis is $+$.

Since in removing a parenthesis with the $-$ sign before it the sign of every term taken out is changed, it follows that when the parenthesis is inserted, the sign of every term in the parenthesis must be changed, provided the sign before the parenthesis is $-$.

Accordingly $\quad a+b-c+d=a+(b-c+d)$, and

$$=a-(-b+c-d).$$

EXERCISES

49. Inclose in parentheses all the terms except the ones involving the letters x and y; in the odd-numbered exercises use the $+$ sign before the parentheses; in the even-numbered exercises use the $-$ sign.

1. $x+a+b-c+d+y-m+n-s$.
2. $2x+3a-4b+5c-y-6k+8l+7f-4g$.

3. $3a+2w-3e+x-4r+5t+y+5t-6u+7i-x+a+b.$
4. $-9m+8i-7u+6y+5t-4r+3e+x+2w+3a.$
5. $3y-c+3a-2x+4b-2d.$
6. $-5b-x+3y-4c+5d.$
7. $-5m+6n+7x-3p-4y.$
8. $3y-3c+2b-5a+5x.$

MAGIC SQUARES

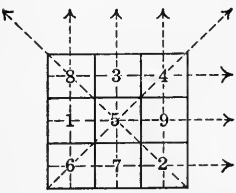


FIG. 4

1. Figure 4 is a magic square. Add the numbers along each of the three horizontal dotted lines, each of the three vertical lines, and each of the two diagonal lines. What is there peculiar about the sums?

2. This magic square (Fig. 5) has algebraic expressions arranged along the eight lines. Find the sum of the three algebraical expressions along each of these lines. What is there peculiar about the sums?

$3b-2a$	$8b-7a$	b
$2b-a$	$4b-3a$	$6b-5a$
$7b-6a$	a	$5b-4a$

FIG. 5

3. Take $a=1$, $b=2$, and compute the resulting numerical magic square. Select other integral values for a and b, and construct other magic squares.

4. If $a=2$ and $b=1$, then a magic square is obtained, most of whose numbers are negative. Is the sum of the numbers still the same for each of the 8 different lines?

5. Find the sum of the expressions in each of the 10 lines in the larger magic square (Fig. 6).

a	$14b-13a$	$13b-12a$	$3b-2a$
$11b-10a$	$5b-4a$	$6b-5a$	$8b-7a$
$7b-6a$	$9b-8a$	$10b-9a$	$4b-3a$
$12b-11a$	$2b-a$	b	$15b-14a$

FIG. 6

6. Answer questions 2 and 3, given above, for this larger magic square.

ORAL EXERCISES

50. Represent a number:

1. Greater than n by 2.
2. Greater than x by 5.
3. Less than y by 3.
4. Less than r by 6.
5. Less than $7n$ by a.
6. 5 times x less 2.
7. 10 greater than $a + b$.
8. x greater than 9.
9. a greater than $n - x$.
10. y less than 2 times x.

What is the other part if

11. One part of 15 is 6?
12. One part of n is 2?
13. One part of x is y?
14. One part of y is x?
15. One part of 24 is a?
16. One part of $a + b$ is 10?

17. The difference between two numbers is 16; the greater is 20. What is the smaller?

18. The difference between two numbers is a; the greater is x. What is the smaller?

19. The smaller of two numbers is y; their difference is 3. What is the greater?

20. The sum of two numbers is 27; the smaller is n. What is the greater?

21. The sum of two numbers is a; the greater is y. What is the smaller?

22. By how much does 28 exceed 15?

23. By how much does x exceed 11?

24. Express the excess of 52 over 45.

25. Express the excess of x over y.

26. Express the excess of $a + b$ over $a - b$.

27. If one book costs \$2, what will 7 books cost?

28. If one pencil costs 5¢, what will b pencils cost?

29. If one coat cost x dollars, what will 3 coats cost?

30. If 7 hats cost \$$17\frac{1}{2}$, what will 1 hat cost?

31. If 2 oranges cost d cents, what will 1 orange cost?

32. If x apples cost 15¢, what will 1 apple cost?

33. What is 5% of \$200? 4% of d dollars?

34. What is the interest on \$250 at 6%?

35. What is the interest on \$300 at r%?

36. What is the interest on p dollars at r%?

37. If a man is 53 years old now, how can you represent his age x years hence?

38. If a man's age now is n years, how can you represent his age 20 years ago?

39. If a man is twice as old as his son, and his son is b years old, how old is the father?

40. The side of a square is 6 in. What is its perimeter?

41. The side of a square is x in. What is its perimeter?

42. The length of a rectangle is twice its width. Express its length and its perimeter, if w represents its width.

43. The length of a rectangle is 4 ft. more than twice its width. Express its length and its perimeter if x represents its width.

44. If I have x dimes, how can I express the number of cents I have?

45. If n represents a certain number, how can I represent the next greater number?

46. If n represents a certain even number, how can I represent the next even number?

47. Express two consecutive odd numbers.

SOLUTION OF EQUATIONS

51. We have seen that an equation is like a balance (§ 5); the equality or balance is not disturbed as long as the same changes are made simultaneously on both sides.

Thus, in solving	$9x - 10 = 4x$.
Adding 10 to both sides,	$9x = 4x + 10$.
Subtracting $4x$ from both sides,	$5x = 10$.
Dividing both sides by 5,	$x = 2$, the answer.

It will be noticed that adding the 10 to both sides has the same effect as simply moving the 10 to the other side of the equation and *changing its sign.*

In the same way, subtracting $4x$ from both sides had the same effect as taking $4x$ over to the left-hand side and *changing its sign.*

This operation is called *transposition. It consists in moving a term from one side of the equation to the other and changing the sign of the term.*

Thus, in solving	$4 - 6x = 2x - 36$.
Transposing 4 yields	$-6x = 2x - 36 - 4$.
Or	$-6x = 2x - 40$.
Transposing $2x$ gives	$-8x = -40$.
Dividing both sides by -8,	$x = 5$.

Check: Substitute 5 for x in the original equation. The result is

$$4 - 6 \cdot 5 = 2 \cdot 5 - 36, \text{ or } \quad 4 - 30 = 10 - 36, \text{ or}$$
$$-26 = -26$$

Since this is correct, we know that 5 is the correct value for x, provided that no mistake was made in checking.

The method of solution which we have explained applies equally well when the coefficients of the equation are numbers represented by letters.

Consider, for example,	$mx + b - c = d - mx$.
Transpose $-mx$, $+b$, $-c$,	$mx + mx = d - b + c$.
Combine $mx + mx$,	$2mx = d - b + c$.

Divide both sides of the equation by $2m$, $x = \dfrac{d - b + c}{2m}$, the answer.

The answer obtained in the solution of an equation is called a *root* of that equation.

A *root* of an equation is a quantity which, when substituted for the unknown in the equation, "satisfies" the equation by reducing both members to identical numbers.

EXERCISES

52. Solve the following equations, and check each answer:

1. $6x + 5 = 3x + 17.$
2. $13x - 5 = 11x + 5.$
3. $4 - 5x = -6x + 2.$
4. $4x - 30 = 30 - 6x.$
5. $3y + 8 = 5y - 5.$
6. $6y + 45 = 7y + 50.$
7. $40 + 4 = 90 - 6z.$
8. $1 + x - 6x + 30 = x + 35 - 3x.$
9. $\frac{1}{2}x + 5 = 6.$
10. $\frac{1}{2}x - 10 = 2x + 20.$
11. $50y + 20y + 40 = 40 - 3y + 70 - 7y.$
12. $\frac{1}{2}z + 10 + \frac{1}{2}z - 35 = \frac{3}{2}z + 50 - 45.$
13. $2x + b = a.$
14. $2x + b = x + a - c.$
15. $ax + b = c - ax.$
16. $5x = a + b - x.$
17. $\frac{1}{2}y - a = -b - \frac{1}{2}y.$
18. $ax + 4 = 5.$
19. $cx - 3c = 10c.$
20. $az + 2b = 3c.$

53. In a balance the equilibrium is not disturbed if we substitute on one side, say, a 10-pound weight in place of a 4-pound weight and a 6-pound weight. So, in an equation, it is permissible to leave one side unaltered and to make any change on the other side we wish, provided this change does not alter the *value* of that other side.

Only in an operation which changes the value of one side of a balance or equation we must be careful that exactly the same change in value is made on the other side. For example:

$$2(4x + 5) - 3x = 4 - 3(5x - 9) + 13x.$$
$$8x + 10 - 3x = 4 - 15x + 27 + 13x.$$
$$5x + 10 = -2x + 31.$$
$$7x = 21.$$
$$x = 3.$$

EXERCISES

54. Solve:

1. $2(5x+6)+20=4x+5(x+2)$.
2. $\frac{1}{2}(6x+8)-3x+10=6(x+4)-21$.
3. $10z-20+25=\frac{1}{5}(10z+20)-60$.
4. $\frac{1}{4}(x+2)=x-20$.
5. $\frac{1}{2}x-\frac{1}{3}x=10$.
6. $3(x-5)-2(x+2)=5(x-7)$.

PROBLEMS

55. 1. The length of a rectangle exceeds its width by 10 ft. Its perimeter is 52 ft. Find its length and width.

2. The length of a rectangle is 70 in. less than its perimeter. Find its length, when its width is 20 in.

3. A lot is 3 times as long as it is wide. If the perimeter is 480 ft., what are its dimensions?

4. A garden in the form of a rectangle is 70 yd. wide. The perimeter is 4 times the length. Find the length.

5. A grocer desires to mix two grades of coffee costing 35¢ and 45¢ a pound, so as to obtain a mixture weighing 100 lb. and worth 38¢ a pound. How many pounds of each kind must he take?

6. How many pounds of 50¢ coffee must be mixed with 60 lb. of 40¢ coffee to get a mixture worth 43¢ a pound?

7. Two grades of coffee costing 35¢ and 30¢ are to be mixed so as to obtain a mixture weighing 100 lb. and selling for 50¢ a pound at a profit of 50% on the cost. How many pounds of each kind are needed?

8. The sum of two numbers is 206; the larger exceeds the smaller by 46. Find the numbers.

9. What two numbers have 216 for their sum and 300 for their difference?

10. The difference between two numbers is 25; their sum is 0. Find the numbers.

11. If the difference between two numbers is 24, what must each number be in order that their sum shall be 16?

12. Two bicyclists start in the same direction, one going at the rate of 10 mi. an hour and the other at the rate of 12 mi. an hour. How long before the second will be 1 mi. in advance of the first?

Let x = the no. of hr. required.

Then $10x$ = the no. of mi. the first goes in x hr.

$12x$ = the no. of mi. the second goes in x hr.

In x hr. the second has gone 1 mi. farther than the first. Hence if we add 1 mi. to the distance the first has traveled, we get $12x$, the distance traveled by the second. That is,

$$10x + 1 = 12x.$$

Solving this, we get $x = \frac{1}{2}$.

13. Two boys start to run in the same direction. One runs at the rate of 300 yd. a minute; the other at a rate of 340 yd. a minute. How long before the second will be half a mile in advance of the first?

14. If the two boys in problem 13 were to start at the same moment and run in opposite directions on a two-mile circular running track, in how many minutes would they meet?

15. A boy has \$1.35 in his pocket, all in dimes and nickels. How many coins of each kind has he, if their total number is 19?

16. How many dimes and quarters, 16 coins in all, are necessary to amount together to \$2.65?

17. A newsboy has collected \$1.99, the amount consisting of nickels and pennies. How many has he of each, the total number of coins being 103?

18. I have \$7.45 in 19 coins, which are quarters, dimes, and 5 one-dollar pieces. Tell the number of coins of each kind.

19. The average of the highest and the lowest temperature of a winter's day is $17\frac{1}{2}°$; the difference between them is 55°. Find the extreme temperatures.

20. If the lowest temperature one day was $-8°$, and the average for that day was 20°, find the maximum temperature.

21. A certain man is worth \$3000 more than when he started out in business; one third of the sum of what he has now and what he did have at first is \$1915. How much does he possess now?

22. The sum of two investments is \$2700; the first yields annually at 5% \$30 less than does the second investment at 6%. Find each investment.

23. A man divides \$5000 into two investments, of which one at 6% brings annually \$40 more than the other at 7%. Into what sums was the \$5000 divided?

24. John can write twice as fast on a typewriter as James can write by hand. How many words can each write per minute, if together they write 162 words per minute?

25. The sum of two numbers is 81; their difference is 11. Find the two numbers.

26. The sum of two numbers is $4a+b$; their difference is $a+b$. Find the numbers.

Let the first number $= x.$
Then the second number $= 4a + b - x.$
The first number subtracted from the second $= 4a + b - x - x$
$= 4a + b - 2x.$
By the conditions of the problem, this difference $= a + b.$
Hence we have the equation $4a + b - 2x = a + b.$
Solve this equation.

27. The sum of two numbers is a; their difference is b. Find the two numbers.

28. The difference between two numbers is c; their sum is 0. Find the two numbers.

29. How many dimes and quarters will make a sum of $10a + 25b$ cents, if the number of coins is $a + b$?

30. If the lowest temperature one day was $-t°$, and the average temperature was $T°$, what was the maximum temperature?

31. A merchant owns now d dollars more than when he started out in business; one third of the sum of what he owns now and what he did own at the beginning is D dollars. How much does he own now?

32. A father earns 4 times as much as his son. Together they earn a dollars per day. How much does each earn?

33. The perimeter of a rectangle is p; its length exceeds its width by q. Find the length and breadth.

34. A rectangle is 3 times as long as it is wide. Find its dimensions, if the perimeter is P.

35. Find two consecutive integers whose sum is 25.

If the smaller integer is x, then the next higher integer is $x + 1$.

36. Find three consecutive integers whose sum is 75.

37. What three consecutive integers produce the sum s?

38. Find two consecutive even integers whose sum is 102.

If x is any integer, then we are certain that $2x$ is an even number, for the reason that any integer, when doubled, gives an even number.

Letting $2x$ stand for the smaller even integer, what must be added to this to give the next higher even integer?

LEONHARD EULER

One of the greatest mathematicians of the eighteenth century. In 1770 he published an algebra which was translated from German into French, English, Italian, and Latin, and was widely used. Euler was a Swiss, but spent most of his time in Russia and Germany.

CHAPTER III

MULTIPLICATION AND DIVISION

MULTIPLICATION OF MONOMIALS

56. By the definition of an exponent (§ 14), we know that a^2 means $a \cdot a$, b^3 means $b \cdot b \cdot b$; hence

$a^2 \times a^3$ means $a \cdot a \times a \cdot a \cdot a$ or a^5. Similarly,

$c^3 \times c^4 = c^7$, and $a^3b^2c^4 \times a^2b^3c^5 = a^5b^5c^9$.

Hence *in the multiplication of terms containing like letters, the exponent of any letter in the product is the sum of the exponents of that letter in the factors.*

The product of $12\,abc$ and $5\,ab^2c$ is $60\,a^2b^3c^2$; that is, the coefficients are multiplied together. Bearing in mind the rule of signs in multiplication, the following results are seen to hold:

$$+3\,xy \times (+5\,x^2yz) = +15\,x^3y^2z.$$
$$-4\,xy^2 \times (+6\,x^2yz^2) = -24\,x^3y^3z^2.$$
$$+5\,xyz^2 \times (-6\,x^2y^2z) = -30\,x^3y^3z^3.$$
$$-6\,xyz \times (-7\,xyz) = +42\,x^2y^2z^2.$$

ORAL EXERCISES

57. Perform the following indicated multiplications:

1. $+3\,x(+5\,xy)$.

2. $+5\,xz \cdot (-5\,xy^4z)$.

3. $-7\,a^3b^2c$ times $-8\,b^4c^3d^2$.

4. $+x^1y^2z^3$ times 60.

5. $(8\,m^1n^4o^3) \cdot (-7\,m^6n^4o^2)$.

6. $-12\,h^8 \times -7\,h^8jk$.

7. $+50\,df^3 \times -9\,f^2gh$.

8. $+20$ times $+30\,e^3$.

9. $-30\,acf \times -10\,ab^2cdf^3$.

10. $(-2\,xy) \cdot (+4\,x^2y) \cdot (-1\,xyz)$.

11. $(-2\,a)(-3\,a^2)(-2\,ab)$.

12. $(+2\,ab)(-2\,a^2b)(-3\,ab^3)$.

13. If two, four, or six negative terms are multiplied together, what is the sign of the product? Why?

14. If three, five, or seven negative terms are multiplied together, what is the sign of the product? Why?

15. If among the terms to be multiplied together a certain number of terms is positive and the number of negative terms is even, what is the sign of the product? Does the number of positive terms require special attention?

16. If the number of negative terms is odd, what is the sign of the product?

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

58. It is seen that $4(2+3)=4\cdot 5=20,$
and that $4\cdot 2+4\cdot 3=20.$
Hence $4(2+3)=4\cdot 2+4\cdot 3.$
In general, $a(b+c)=ab+ac,$
and $a(b+c-d)=ab+ac-ad.$

Hence, in multiplication of a polynomial by a monomial, we have the rule:

Multiply each term of the polynomial by the monomial, and write down in succession the products with their respective signs.

Observe that the following results hold:

$$+2\,ab^2(a^2+b-c^3)=+2\,a^3b^2+2\,ab^3-2\,ab^2c^3.$$
$$-3\,a^2c[-a+2\,bc-4\,c]=+3\,a^3c-6\,a^2bc^2+12\,a^2c^2.$$

EXERCISES

59. 1. Multiply $2\,x+ey+6\,z-5\,w$ by $3\,xy$.

2. Multiply $a-2\,b+ec-4\,d-e$ by $10\,abc$.

3. Find the product of $9\,xy^2z$ and $xz+4\,y^3z-5\,xyz^2-2\,x^2$.

Simplify

4. $(3\,m+5\,n+4\,o-5\,p)(-2\,n^3o^2)$.

5. $-7(5-2\,a+2\,b-3\,c+4\,e)$.

6. $(-1)(-2\,m+5\,n^2+6\,o-5\,p)(-2)$.

7. $(a - b^2 - 2c^3 - 3d^3)(-4abcd)(-1)$.

8. $(-2ax^3)\cdot(9b + 3x - 8d + uf - g)$.

9. $(-3)(+2)(-2a + 3b + 4c - 5d)$.

10. $(-5)(-1)(-2x + qy - 3z - 0)$.

11. $12(1 + 2 - 3 + 4 - 5 + 6 - 7 + 8)$.

12. What are the two ways of working the last exercise? Which of those two ways is the shorter? Why?

MULTIPLICATION OF POLYNOMIALS

60. It is readily seen that $(3 + 4)(2 + 6) = 56$. It might be worked also in this manner:

$$(3+4)(2+6) = 3(2+6) + 4(2+6) = 3\cdot 2 + 3\cdot 6 + 4\cdot 2 + 4\cdot 6 = 56.$$

This second mode of procedure is longer in this particular example. It is given here because it illustrates the process which *must* be followed when the terms inside the parentheses are not like terms. For instance, it can be used in a multiplication like the following:

$$\begin{aligned}(x + 2y)(3x - 4y) &= x(3x - 4y) + 2y(3x - 4y)\\ &= x\cdot 3x - x\cdot 4y + 2y\cdot 3x - 2y\cdot 4y\\ &= 3x^2 - 4xy + 6xy - 8y^2 = 3x^2 + 2xy - 8y^2.\end{aligned}$$

We see from this that *we can find the product of two polynomials by multiplying one polynomial by each term of the other and then adding the partial products.*

In written exercises the work may be arranged as follows:

$$\begin{array}{r} 3x - 4y \phantom{{} - 8y^2} \\ \underline{x + 2y \phantom{{} - 8y^2}} \\ 3x^2 - 4xy \phantom{{} - 8y^2} \\ \underline{6xy - 8y^2} \\ 3x^2 + 2xy - 8y^2 \end{array}$$

Frequently a polynomial has some letter raised to different powers in its terms. Thus, in the polynomial $x^4 + 2x^3 - x^2 + 4x - 5$, all the terms contain x, except the last term;

moreover, the term x^4, which is the highest power of x in the polynomial, is written first. Then follow the terms containing the lower powers of x, arranged in descending order. This polynomial is said to be *arranged* according to the *descending* powers of x.

When written in reverse order, thus, $-5+4x-x^2+2x^3+x^4$, we say that the polynomial is *arranged* according to the *ascending* powers of x. When polynomials of this kind are used, it is convenient to arrange them first according to the descending or the ascending powers of some letter. If in the multiplication of two polynomials both are arranged according to the ascending powers of some letter, or both according to the descending powers, the partial products can be written down and added with much greater ease.

Example. Find the product of $-3x^2+4x-5+2x^3$ and $2x+x^2-3$.

Arranging both polynomials according to the descending powers of x, the multiplication is as follows:

$$\begin{array}{rrrrrr}
2x^3 & -3x^2 & +4x & -5 & & \\
x^2 & +2x & -3 & & & \\
\hline
2x^5 & -3x^4 & +4x^3 & -5x^2 & & \\
 & +4x^4 & -6x^3 & +8x^2 & -10x & \\
 & & -6x^3 & +9x^2 & -12x & +15 \\
\hline
2x^5 & +x^4 & -8x^3 & +12x^2 & -22x & +15
\end{array}$$

One method of checking a multiplication of this kind is to arrange both polynomials according to the ascending powers of the letters and then to find their product, thus:

$$\begin{array}{rrrrrr}
-5 & +4x & -3x^2 & +2x^3 & & \\
-3 & +2x & +x^2 & & & \\
\hline
+15 & -12x & +9x^2 & -6x^3 & & \\
 & -10x & +8x^2 & -6x^3 & +4x^4 & \\
 & & -5x^2 & +4x^3 & -3x^4 & +2x^5 \\
\hline
+15 & -22x & +12x^2 & -8x^3 & +x^4 & +2x^5
\end{array}$$

The two products are the same, except that one is arranged according to the ascending powers of x, and the other according to the descending powers. The order in which the terms of a polynomial are written does not affect its value, hence we say that the two products are the same. The second multiplication indicates that the first multiplication was correct.

A second method of checking multiplication is by assigning some particular small number as the value of x. Let $x = 2$, then

$$\begin{array}{lr}
2x^3 - 3x^2 + 4x - 5 & = 7 \\
x^2 + 2x - 3 & = 5 \\
\hline
2x^5 - 3x^4 + 4x^3 - 5x^2 & \\
\quad + 4x^4 - 6x^3 + 8x^2 - 10x & \\
\qquad - 6x^3 + 9x^2 - 12x + 15 & \\
\hline
2x^5 + x^4 - 8x^3 + 12x^2 - 22x + 15 & \overline{35}
\end{array}$$

Let $x = 2$, $64 + 16 - 64 + 48 - 44 + 15 = 35$.

Hence we are reasonably certain that the multiplication is correct.

EXERCISES

61. Multiply together the following polynomials, and check:

1. $x + 1$ and $x + 2$.
2. $2x + 1$ and $x + 3$.
3. $2x - 1$ and $x - 3$.
4. $5x^2 + x + 2$ and $x + 4$.
5. $6x^2 - 2x - 2$ and $-x + 3$.
6. $y^3 - 2y^2 + 4$ and $y - 5$.
7. $y^4 - 1$ and $y^2 - y - 1$.
8. $z^2 - 2z + 1$ and $z + 1$.
9. $y - 5$ and $y^3 - 2y^2 + 3y - 1$.
10. $2z + 6$ and $-z^3 + 2z^2 - 10$.
11. $3x + y$ and $2x^4 - 3x^2 + 5$.
12. $x^2 + x$ and $5x^2 + 4x + 2$.
13. $z^2 - 7$ and $6z^3 - 1$.
14. $x^2 - x + \frac{1}{2}$ and $2x + 4$.
15. $x^2 - x + 3$ and $x - \frac{1}{3}$.
16. $z^4 + z^2 + 1$ and $z^2 - 1$.

Arrange the polynomials according to the descending powers of the letter and multiply. Verify by arranging the polynomials according to the ascending powers, and multiply:

17. $x - 1$ and $3x^2 - 4 + 2x$.
18. $6 + z^2 - 5z$ and $2 - z^2 + 3z$.
19. $-1 + 3y^3 + 2y$ and $6 + 5y - y^2$.

20. $x^2 + x^3 - 2$ and $-5 + 2x^2 - 3x + 8x^3$.

21. $x^3 + 2x^4 - 3x^2 - 4$ and $-3 + 2x + x^2$.

Perform the indicated multiplications:

22. $(3x + y + z)(5x - 2y - z + 5)$.

23. $[a + b - c][2a - 2b + 3c]$.

24. $\{x^2 + 2xy + x\}\{x^2 - xy + x + y\}$.

25. $(x^2 + y^2 + z^2)(x + y + z)$.

26. $(a - e + 3f - 2g)^2$.

27. $(a + b - c + d + e)^2$.

28. $(a + b)(a - b)(a^2 + 2ab - b^2)$.

29. $(x - 2y)^3$.

30. $(x - y + z)^3$.

31. $(a - b - c)^3$.

HARDER EQUATIONS

62. 1. Solve $(x + 2)(x - 7) - (x + 5)^2 = 0$.

Solution. $(x^2 - 5x - 14) - (x^2 + 10x + 25) = 0.$

Remove parentheses, $x^2 - 5x - 14 - x^2 - 10x - 25 = 0.$

Transpose, $x^2 - 5x - x^2 - 10x = 24 + 14.$

Combine, $-15x = 39.$

Divide both sides by -15, $x = -\frac{39}{15} = -2\frac{3}{5}.$

Check:

$$(-\tfrac{13}{5} + 2)(-\tfrac{13}{5} - 7) - (-\tfrac{13}{5} + 5)^2 = 0.$$
$$(-\tfrac{3}{5})(-\tfrac{48}{5}) - (\tfrac{12}{5})^2 = 0.$$
$$+\tfrac{144}{25} - \tfrac{144}{25} = 0.$$
$$0 = 0.$$

Solve:

2. $(x + 1)(x - 1) - (x + 2)(x - 3) = 0$.

3. $x^2 - (x + 4)(x + 1) = 0$.

4. $(y + 2)(y - 3) + (y - 5)(y - 1) - 2y^2 = 0$.

5. $(2y + 1)^2 - 4(y + 1)^2 = 0$.

6. $[z + 1][z - 2] - [z - 3][z + 4] = 0$.

7. $x^2 - 2x + 5 - (x + 2)(x - 3) = 0$.

8. $(y + 3)(y - 5) + (y + 2)(y + 1) - (y + 2)(y - 3) - (y - 3)(y + 4) = 0$.

9. $(x+1)(x-1)-(x+2)(x-2)+x+10=0.$

10. $(3-x)(2-x)-(5+x)(1-x)-2x^2=6.$

11. $0=(1-z)(z-2)+(z+4)(z-2)+8.$

12. $3=5+x^2-\{6+x\}\{x+1\}+5x.$

13. $(x-5)^2-(x+2)^2=5(x+3).$

14. $6(x^2+1)-(5x+6)(x+1)=(x+2)(x-2).$

15. $(x^2+x+1)(x-5)-(x^2-x-7)(x-3)=5(x+7).$

16. $x+10=-(x+5)(x-5)-(4+x)(5-x).$

17. $(y+3)(6y+7)=(2y-3)(3y-4)+6.$

18. $2z-5=(z-5)(z+4)-(2z+1)(z-1)+z^2.$

19. $5+(x+1)(x-1)(x^2+1)=(x^2+1)(x^2-1)+x.$

20. $(x^2+x+2)(x-1)=(x^2-x-1)(x+1).$

63. Figure 7 shows that a rectangle 4 in. long and 3 in. wide has an area of 12 sq. in. Similar relations hold when the foot, yard, or some other unit of length is taken. These considerations suggest the following *definitions*:

in. × in. = sq. in.
ft. × ft. = sq. ft.
yd. × yd. × yd. = cu. yd., etc.

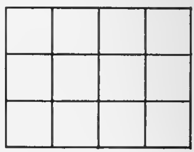


FIG. 7

ORAL EXERCISES

64. Find the areas of rectangles whose dimensions are:

1. 7 ft. and 13 ft.
2. 6 in. and $9\frac{1}{3}$ in.
3. $3\frac{1}{2}$ ft. and $4\frac{1}{2}$ ft.
4. 5 yd. and x yd.
5. b ft. and 9 ft.
6. ab yd. and c yd.
7. $3x$ mi. and $4y$ mi.
8. $2ab$ in. and $7abc$ in.
9. x^2 yd. and xz yd.

Find the volumes of rectangular solids whose dimensions are:

10. 3 in., 4 in., 5 in.
11. 5 ft., 6 ft., n ft.
12. 2 yd., b yd., c yd.
13. l in., w in., h in.
14. ab yd., abc yd., and a^2b yd.
15. $4x$ in., $5y$ in., and $2z$ in.
16. $5y$ ft., $2y^2$ ft., and $2yz$ ft.
17. x rd., y rd., and xyz rd.

Find the area of a square whose edges are:

18. $6s$ ft.
19. $4.5a$ in.
20. $4a^2$ in.
21. mn^2 ft.
22. $(a+b)$ yd.
23. $(x+2y)$ in.

Find the volume of a cube whose edges are:

24. $4a$.
25. $2b^2$.
26. $(c+d)$ ft.

27. If the box shown in the adjoining figure has $l = 4$ in., $w = 3$ in., $h = 2$ in.,

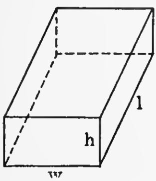


Fig. 8

(*a*) What is the area of the bottom?

(*b*) What is the area of the bottom and top taken together?

(*c*) What is the area of one end of the box? Of both ends?

(*d*) What is the total surface of the box?

(*e*) What is the volume of the box?

28. Answer the questions in Ex. 27 when $l = 1$, $w = 2$, and h is any number.

29. Answer the questions in Ex. 27 when $l = 10$ and w and h are any numbers.

30. Answer the questions in Ex. 27 when l, w, h are any numbers.

Find the area of a rectangle whose dimensions are as follows:

31. $9, (a+b)$.
32. $(x-6), 5$.
33. $(a+b), (a-b)$.
34. $(3-x), (4+x)$.
35. $(a+2b), (a-2b)$.
36. $[5a-2b], [2a+3b]$.

37. Find the numerical value of the area in Ex. 36 when $a = 10$, $b = 5$.

DIVISION BY MONOMIALS

65. Just as in arithmetic, $3 \div 4$ may be written $\frac{3}{4}$;

so in algebra, $n \div d$ may be written $\frac{n}{d}$.

In algebra, as in arithmetic, $\frac{3}{4}$ may be interpreted in two ways:

(1) As an *indicated division*.

(2) As a regular fraction, showing that 3 of the 4 equal parts of a unit are taken.

The same is true of $\frac{n}{d}$. It means "n divided by d," or "a unit is divided into d equal parts and n such parts are taken." We can adopt either interpretation.

We have learned in arithmetic that the value of an expression (quotient or fraction) is not changed, when the dividend and divisor in a division (or the numerator and denominator of a fraction) are both multiplied or both divided by the *same* number.

That is, $\frac{4}{8} = \frac{1}{2} = \frac{3}{6}$.

This principle is used in simplifying the dividend and divisor, or in reducing a fraction to its lowest terms.

Thus, dividing by $5a$ both terms of the fraction $-\frac{15\,abc}{5\,ad}$, we obtain its equal, $-\frac{3\,bc}{d}$.

The rule of signs must be followed carefully. In division, as in multiplication, "like signs give plus, unlike signs give minus."

We know (§ 13) that $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, and $2^3 = 2 \cdot 2 \cdot 2$.

Hence $$\frac{2^5}{2^3} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2} = 2^2 \text{ or } 2^{5-3}.$$

Similarly, $$\frac{a^5}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a^2 \text{ or } a^{5-3}.$$

This gives the following very important rule:

The exponent of any letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

It is thus seen that in division we *subtract* the exponents of any letter, while in multiplication (§ 56) we add its exponents.

In general, in multiplication, $a^m \cdot a^n = a^{m+n}$;

in division, $\dfrac{a^m}{a^n} = a^{m-n}$.

ORAL EXERCISES

66. Perform the indicated divisions:

1. $\dfrac{12\,abc}{4\,ab}$, $\dfrac{24\,x^2y}{6\,xy}$, $\dfrac{36\,aby^3}{12\,aby}$, $\dfrac{8\,cdx^2}{12\,dx}$.

2. $\dfrac{-a^2}{-a}$, $\dfrac{-c^3}{c}$, $\dfrac{+z^4}{z^2}$, $\dfrac{-3m^3}{+3m}$, $\dfrac{+5\,x^5}{-x^3}$.

3. $\dfrac{a^5b}{ab}$, $\dfrac{-ab^4c}{-2\,ab^3}$, $\dfrac{-4\,x^3y^4}{+4\,x^3y^2}$, $\dfrac{+9\,m^2n^5}{-3\,m^2n^2}$.

67. EXERCISES

1. $-124\,a^4b^8c^5 \div 12\,a^1b^3c^7$.
2. $-25\,x^6y^7z^9 \div 15\,x^2y^7z^5$.
3. $236\,m^6n^2o^5 \div 24\,m^3n^5o^8$.
4. $+50\,bc^4d^2e^6 \div (-60\,b^3c^5d^6e^3)$.

Reduce the following fractions to the lowest terms:

5. $\dfrac{-28\,p^5q^4r^9}{+56\,p^4q^6r^7s^2}$. 6. $\dfrac{+72\,a^2t^4c^{12}}{-44\,a^9t^8c^4d^7}$. 7. $\dfrac{-64\,l^{10}m^{10}n^{10}}{-196\,l^2m^6n^{10}o^5}$.

Let A be the area of a rectangle, L its length, W its width. Find the second dimension in the following rectangles:

8. $A = 56\,a^2$ sq. ft., $L = 8\,a$ ft.
9. $A = 136\,x^2y^2$ sq. in., $L = 12\,xy$ in.
10. $A = 365\,m^3n^9o^6$, $W = 25\,m^3n^5o^2$.
11. $A = 256\,bc^8d^3$, $W = 72\,b^3c^5d^2$.
12. $A = 1.25\,x^4y^8$, $L = .05\,x^2y^6$.

MULTIPLICATION OF FRACTIONS

68. Figure 9 shows that $\frac{1}{2}$ of $\frac{3}{4}$ of an inch is $\frac{3}{8}$ of an inch; this recalls to mind the arithmetical rule for the multiplication of fractions: *Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

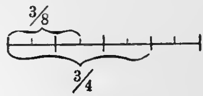


FIG. 9

The process is the same in algebra.

It is easiest, as a rule, at first merely to *indicate* the multiplication, then to divide both numerator and denominator by every factor common to them.

$$\text{Thus,}\quad \frac{25\,a^4b^3}{c^4}\text{ times }\frac{8\,c^3d^4}{75\,a^2b}=\frac{\overset{a^2b^2}{\cancel{25}\,\cancel{a^4}\cancel{b^3}}\cdot 8\,\cancel{c^3}d^4}{\underset{c}{\cancel{c^4}}\cdot \underset{3}{\cancel{75}}\,\cancel{a^2}\cancel{b}}=\frac{8\,a^2b^2d^4}{3\,c}.$$

EXERCISES

69. Find the product of the following fractions:

1. $\dfrac{3\,ab}{4\,cd},\ \dfrac{12\,cd}{15\,a}.$

2. $\dfrac{16\,zw^2}{15\,at},\ \dfrac{85\,a^2t^5}{64\,zw}.$

3. $\dfrac{1}{8\,x^3y^2z},\ \dfrac{64\,xy^3z^3}{abc}.$

4. $\dfrac{-2\,ab}{6\,xy},\ \dfrac{-3\,x^2y}{8\,a^2b^2}.$

5. $\dfrac{+7\,t^4}{-24}\times\dfrac{8\,t^5}{-14}.$

6. $\dfrac{-24\,a^2bc^4}{-36\,m^4n^2}\times\dfrac{-2\,m^3n^2}{14\,a^2c^2}.$

Since in division "unlike signs give minus," it is evident that the following equalities hold:

$$\frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b}.$$

That is, the $-$ sign may be in front of the fraction, or before a factor of the numerator, or before a factor of the denominator, and there is no change in the value of the fraction.

7. $\dfrac{-25\,a^2c^4}{8\,d^2e^5}\times\dfrac{+2\,d^2e^3}{5\,a^4c^2}.$

8. $\dfrac{27\,k^2l^5m}{12\,s^3t^4}\times\dfrac{3\,r^4s^2t^5}{-9\,k^8l^4m^2}.$

9. $\dfrac{-x^6y^6z^3}{-a^3b^2c^5}\times\dfrac{-ab^2c^9}{-x^6yz}.$

10. $\dfrac{8\,f^3g^2h^4}{-15\,t^2yw^2}\times\dfrac{5\,t^5y^3w}{+4\,f^2g^2h^4}.$

If a fraction $\frac{a}{b}$ is to be multiplied by an integer c, it is easiest to write for c its equal $\frac{c}{1}$, and then to multiply $\frac{a}{b}$ by $\frac{c}{1}$, by the rule for the multiplication of fractions.

11. $24\,a^2c^3e^3$ times $\dfrac{-4\,ab^2cd^3}{6\,e^2fg}$.

12. $\dfrac{30\,h^4i^2k}{21\,x^3y^2z^5}$ times $3\,x^2z^4w^2$.

13. $32\,b^3c^1d^6$ times $\dfrac{2\,ert^5}{b^2c^5d^3}$.

14. $\dfrac{20\,h^8j^6k^4}{-3\,m^2n^4o^2}$ times $-5\,hm^2n^3$.

DIVISION OF FRACTIONS

70. In algebra, as in arithmetic, a fraction is divided by another "by inverting the divisor and multiplying."

This rule may be proved as follows: Let the quotient of $\frac{a}{b} \div \frac{c}{d}$ be x. The dividend is $\frac{a}{b}$; the divisor is $\frac{c}{d}$. Since

divisor $\times$ quotient $=$ dividend, we have,

$$\frac{c}{d} \times x = \frac{a}{b}, \text{ or}$$

$$\frac{cx}{d} = \frac{a}{b}.$$ Multiplying both sides of the equation by bd gives $$\frac{bcdx}{d} = \frac{abd}{b},$$ or, reducing the fractions to the lowest terms, $$bcx = ad.$$ Dividing both sides by the coefficient of x, we get $$x = \frac{ad}{bc}.$$

This value of x is the required quotient. One sees at once that a quick way of getting this answer is to *invert the divisor and multiply.*

ORAL EXERCISES

71. Perform orally the following divisions:

1. $\frac{3}{4} \div \frac{1}{2}$.
2. $\frac{4}{5} \div \frac{2}{3}$.
3. $\frac{9}{10} \div \frac{3}{5}$.
4. $1\frac{3}{4} \div 2\frac{1}{4}$.
5. $3\frac{1}{5} \div 2\frac{2}{5}$.
6. $\dfrac{b}{c} \div \dfrac{d}{e}$.
7. $\dfrac{mn}{pq} \div \dfrac{pq}{m^3}$.
8. $\dfrac{4\,a^2}{b} \div \dfrac{8\,ab}{c}$.

In divisions like $a \div \frac{b}{c}$ or $\frac{b}{c} \div a$, it is easiest to write $\frac{a}{1}$ in place of a; then proceed as in the previous exercises.

9. $c \div \frac{a}{b}$.
10. $\frac{x}{y} \div z$.
11. $5 \div \frac{m}{n}$.
12. $\frac{1}{2} \div \frac{a}{b}$.
13. $\frac{bc}{de} \div \frac{3}{4}$.
14. $\frac{2a}{3b} \div \frac{2}{3}$.

72. WRITTEN EXERCISES

1. $\frac{14as^2d^3}{25x^2y} \div \frac{28a^2s^2d}{125xy^2}$.
2. $\frac{256e^5r^4t^5}{36ab^4c^3} \div \frac{128e^3r^2t^2y}{144a^3b^2c^2d}$.
3. $\frac{72m^2n^3o^9}{-91k^9l^{10}} \div \frac{288mn^3o^4p}{98k^5l^6}$.
4. $\frac{-135d^2f^2g^3}{84x^2y^4z} \div \frac{155df^3gh^4}{91x^4y^6z^2}$.

ORAL PROBLEMS

73. 1. How many 25¢ pieces make \$4?

2. How many 25¢ pieces make \$$d$?

3. A man "steps off" the length of a playground and finds it to be 125 paces. How many feet is this, if his pace is 3 ft.?

4. If a man's pace is 3 ft., what distance do p such paces measure?

5. A man finds the width of a lot to be a lengths of a certain rod which is b ft. long. How wide is the lot?

6. The water in a canal flows at the rate of $\frac{a}{b}$ mi. an hour. In what time will it flow $\frac{h}{k}$ mi?

7. An aviator flies at the rate of 95 mi. an hour. How long will it take him to go $\frac{r}{s}$ of a mile?

8. A girl cuts off two-thirds of a ribbon $2\frac{1}{2}$ yd. long. How many feet of ribbon does she cut off?

9. A woman gets the $\frac{1}{a}$ part of a supply of coffee weighing $\frac{b}{c}$ pounds. How many pounds does she receive?

DIVISION OF POLYNOMIALS BY MONOMIALS

74. This is easily done by *dividing each term of the polynomial by the monomial and connecting these partial results by the proper signs.*

Thus, $(15\,x^2y^2 - 6\,xy^3 + 12\,xy) \div (3\,xy)$

$$= \frac{15\,x^2y^2}{3\,xy} - \frac{6\,xy^3}{3\,xy} + \frac{12\,xy}{3\,xy} = 5\,xy - 2\,y^2 + 4, \text{ the quotient.}$$

Again, $(a^2b^3 - 3\,a^2b^4 - 5\,a^3b^3) \div 5\,a^2b^2$

$$= \frac{a^2b^3}{5\,a^2b^2} - \frac{3\,a^2b^4}{5\,a^2b^2} - \frac{5\,a^3b^3}{5\,a^2b^2} = \frac{b}{5} - \frac{3\,b^2}{5} - ab, \text{ the answer.}$$

The correctness of the answer can be tested by multiplying the quotient by the divisor. The resulting product should be equal to the dividend.

EXERCISES

75. Perform the indicated divisions and check your results:

1. $\dfrac{8\,x^2 - 6\,xy}{2\,x}$.

2. $\dfrac{12\,a^3b^2 + 6\,a^2b^2}{3\,a^2b^2}$.

3. $\dfrac{9\,mk^4 - 12\,m^2k^4}{3\,mk^4}$.

4. $\dfrac{15\,b^3c^2 - 20\,b^2cd}{15\,b^2c}$.

5. $\dfrac{7\,hl^3 + 14\,h^2l^4}{7\,hl^3}$.

6. $\dfrac{24 - 12\,xz^3}{12}$.

7. $\dfrac{6\,x^5y^2 - 4\,x^2y^5 + 8\,x^2y^2}{4\,x^2y^2}$.

8. $\dfrac{36\,r^4s^3 + 48\,r^3s^3 - 24\,r^2s^3}{12\,r^3s^3}$.

9. $\dfrac{8\,x^2z^2 - 10\,xyz^2 - 8\,x^2y^2z^2}{6\,x^2y^2z^2}$.

10. $\dfrac{3\,w^2z^3 + 5\,w^2z^7 + 7\,w^3z^3}{4\,w^2z^3}$.

11. $\dfrac{-5\,abc - 6\,a^2bc^2 + 8\,a^2bc}{-abc}$.

12. $\dfrac{c^2d^3e - 4\,cd^2e^2 + c^2d^3e^3}{-cd^3e}$.

13. $\dfrac{20\,x^4y - 24\,x^5y^2 + 12\,x^4y^6}{-12\,x^4y}$.

14. $\dfrac{18\,ab^2c^3 - 12\,a^2b^4c^3 + 24\,a^2b^5c^6}{-12\,ab^2c^3}$.

DIVISION OF POLYNOMIALS BY BINOMIALS

76. An inspection of the following multiplication will suggest some of the steps to be taken in the inverse operation — Division.

$$
\begin{array}{r}
4x^2 + 3x + 5 \\
x - 2 \\
\hline
4x^3 + 3x^2 + 5x \\
- 8x^2 - 6x - 10 \\
\hline
4x^3 - 5x^2 - x - 10
\end{array}
$$

If we are required to divide $4x^3 - 5x^2 - x - 10$ by $x - 2$, then the quotient which must be the other factor, $4x^2 + 3x + 5$, can be found as follows:

$$
\begin{array}{lr|l}
\text{Dividend} & 4x^3 - 5x^2 - x - 10 & x - 2 \quad \text{Divisor} \\
\cline{3-3}
\text{1st partial product} & \underline{4x^3 - 8x^2} & 4x^2 + 3x + 5 \quad \text{Quotient} \\
\text{1st remainder} & + 3x^2 - x - 10 & \\
\text{2d partial product} & \underline{+ 3x^2 - 6x } & \\
\text{2d remainder} & + 5x - 10 & \\
\text{3d partial product} & \underline{+ 5x - 10} & \\
\text{Last remainder} & 0 &
\end{array}
$$

Notice that both the given dividend and divisor are arranged according to the descending powers of the leading letter x. The ascending powers would have answered equally well. It is a very great convenience to have the dividend and divisor arranged either both according to the ascending or both according to the descending powers of a leading letter. If they are not so arranged, then they should be, before the division is begun.

In the division above, notice the following:

(1) *The first term in the quotient* $4x^2$ *is obtained by dividing the first term in the dividend* $4x^3$ *by the first term in the divisor* x.

(2) *The first partial product is the product of the first term in the quotient* $4x^2$ *and the divisor* $x - 2$.

(3) *The first remainder* $3x^2 - x - 10$ *is obtained by subtracting this partial product from the dividend.*

The $- 10$ is not needed in the second step of the division and may therefore be omitted until the third step.

The remaining steps of the division, should there be any, are repetitions of steps above.

77. Explain:

(1) How is the term $+3x$ of the quotient found?

(2) How is the second remainder found?

(3) How are the $+5$ in the quotient and the third partial product found?

(4) How can you check your division?

In many cases the division does not come out exact and there will be a final remainder. In exercises 1–23 which follow, the divisions will come out exact.

Divide $9-9x+8x^2-7x^3+2x^4$ by $3-2x$.

$$\begin{array}{l|l}
9-9x+8x^2-7x^3+2x^4 & 3-2x \\ \cline{2-2}
\underline{9-6x} & 3-x+2x^2-x^3 \\
\quad -3x+8x^2 & \\
\quad \underline{-3x+2x^2} & \\
\qquad\quad +6x^2-7x^3 & \\
\qquad\quad \underline{+6x^2-4x^3} & \\
\qquad\qquad\quad -3x^3+2x^4 & \\
\qquad\qquad\quad \underline{-3x^3+2x^4} &
\end{array}$$

EXERCISES

78. Divide:

1. x^2+2x+1 by $x+1$.
2. y^2+4y+4 by $y+2$.
3. x^2-x-6 by $x-3$.
4. $z^2-7z+10$ by $z-5$.
5. x^3-2x^2+2x-1 by $x-1$.
6. w^3+1 by $w+1$.
7. c^3-1 by $c-1$.
8. x^4-16 by $x-2$.
9. $6x^4+x^3-15x^2+38x-30$ by $x-1$.
10. $4+28x+29x^2-30x^3+25x^4$ by $5x+2$.
11. $y^5+2y^4-3y^3-y^2-2y+3$ by $y+3$.
12. a^2-b^2 by $a-b$.
13. x^2-y^2 by $x+y$.
14. $x^2+2xy+y^2$ by $x+y$.
15. $x^2-2xy+y^2$ by $x-y$.

16. $m^3 + n^3$ by $m + n$.

17. $m^3 - n^3$ by $m - n$.

18. $x^5 - y^5$ by $x - y$.

19. $a^6 - b^6$ by $a^2 - b^2$.

20. $c^6 - d^6$ by $c^3 - d^3$.

21. $a^6 - b^6$ by $a^3 + b^3$.

22. $x^6 - y^6$ by $x - y$.

23. $x^5 + y^5$ by $x + y$.

DIVISIONS INVOLVING REMAINDERS

79. $11 \div 3$ gives the quotient 3 and the remainder 2; the division is not exact; the answer is 3 and $\frac{2}{3}$. We see that

$$\text{dividend} \div \text{divisor} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Divide $x^5 + 3x^3 - 2x^2 + x - 2$ by $x^3 - 2$.

$$\begin{array}{rrrr|l}
x^5 + 3x^3 & - 2x^2 & + x - 2 & & x^3 - 2 \\
x^5 & - 2x^2 & & & \overline{x^2 + 3} \\
\hline
+ 3x^3 & & + x - 2 & & \\
+ 3x^3 & & - 6 & & \\
\hline
& & + x + 4, & \text{the remainder.} &
\end{array}$$

Hence the quotient is $\quad x^2 + 3 + \dfrac{x + 4}{x^3 - 2}.$

Check: dividend = quotient × divisor + remainder.

EXERCISES

80. Divide:

1. $x^3 + 2x^2 + 7x - 5$ by $x - 9$.
2. $y^4 - 4y^3 + 5y^2 - 3y + 12$ by $y^2 + 4$.
3. $z^5 - 16z^3 - 6z^2 + 3z + 5$ by $z^2 - 4z$.
4. $w^8 + 1$ by $w^2 + 1$.
5. $a^4 + 3a^2 + 9a - 10$ by $a^3 - 7$.
6. $m^5 - 5m^4 - 8m^3 + 4m + 8$ by $m^3 + 1$.
7. $r^{16} + 1$ by $r^4 + 1$.
8. $s^{16} - 1$ by $s^5 - 2$.

CHAPTER IV

PROPORTION

81. A *proportion* expresses the equality of two common fractions.

Thus, $\frac{12}{18}$ and $\frac{2}{3}$ are equal to each other; hence, $\frac{12}{18} = \frac{2}{3}$ is a proportion.

A common fraction may be said to indicate the division of the number in the numerator by the number in the denominator. When so considered, the fraction is frequently called a *ratio*. The ratio $\frac{a}{b}$ is often written $a : b$.

Since a proportion is a special type of equation, it may be treated like an equation. If one of the terms of either fraction in the proportion is unknown, the rules for the solution of equations enable us to find the value of the unknown term.

1. Solve the proportion $\dfrac{x}{12} = \dfrac{3}{5}$.

Multiplying both sides of the equation by numbers which will remove the denominators, that is, multiplying both sides by 12 and by 5, we get

$$\frac{5 \cdot \cancel{12} \cdot x}{\cancel{12}} = \frac{\cancel{5} \cdot 12 \cdot 3}{\cancel{5}}.$$

Or $$5x = 36.$$

Divide both sides by 5, $\quad x = \frac{36}{5} = 7\frac{1}{5}$, the answer.

2. Solve the proportion $\dfrac{a-b}{c} = \dfrac{x}{d}$.

Multiplying both sides by c, and also by d,

$$\frac{\cancel{c}d(a-b)}{\cancel{c}} = \frac{c\cancel{d}x}{\cancel{d}}.$$

Or $$d(a-b) = cx.$$

Change sides, for convenience, $\quad cx = d(a-b)$.

Divide both sides by c, $\quad x = \dfrac{d(a-b)}{c}$, the answer.

EXERCISES

82. Solve the following proportions:

1. $\frac{x}{6}=\frac{23}{3}$.
2. $\frac{x}{5}=\frac{19}{3}$.
3. $\frac{8}{9}=\frac{x}{3}$.
4. $\frac{10}{7}=\frac{y}{-4}$.
5. $\frac{7}{8}=\frac{x}{-8}$.
6. $\frac{10.5}{6}=-\frac{x}{8}$.
7. $\frac{a}{5}=\frac{z}{15}$.
8. $\frac{x}{12}=\frac{mn}{4}$.
9. $\frac{-c}{16}=\frac{y}{-32}$.
10. $\frac{5}{x}=\frac{10}{25}$.
11. $\frac{x}{c-d}=\frac{3}{4}$.
12. $\frac{x}{a}=\frac{m}{n}$.
13. $\frac{t}{20}=\frac{x}{m}$.
14. $\frac{r}{-6}=\frac{y}{s}$.
15. $x:5=19:3$.
16. $-10:7=y:4$.
17. $10.5:6=-x:8$.
18. $x:12.5=-mn:4$.
19. $5.5:x=10.7:25.4$.
20. $-x:a=m:n$.
21. $-r:-6.8=y:s$.

PROBLEMS IN PROPORTION

83. 1. If 5 acres of land are worth \$875, what are 7 acres of land of the same quality worth?

Here the number of acres is said to be in the "same ratio" as or "directly proportional" to the cost. That is,

$$\frac{5 \text{ acres}}{7 \text{ acres}}=\frac{\$875}{\$x}.$$

Since $\frac{5 \text{ acres}}{7 \text{ acres}}=\frac{5}{7}$ and $\frac{\$875}{\$x}=\frac{875}{x}$, the original proportion can be written in the simplified form,

$$\frac{5}{7}=\frac{875}{x}.$$

We get $x=1225$. Hence, the answer is \$1225.

2. If 11 men can build a wall in 80 days, how long would it take 14 men to do the same work?

Here the *larger* the number of men, the *less* the time for doing a given piece of work; the number of men is *not* in the "same ratio" as the times needed to complete the work, but in the "inverse ratio." That is, the number of men is "inversely proportional" to the time required to finish the work. We have, therefore,

$$\frac{11 \text{ men}}{13 \text{ men}} = \frac{x \text{ days}}{80 \text{ days}}, \text{ or more simply,}$$

$$\frac{11}{13} = \frac{x}{80}.$$

Solve this proportion.

3. If, in a broad jump, a boy can jump 15 ft. when he takes a running start of 20 ft., how far can he jump with a running start of 400 ft.?

Is the distance a boy can jump either "directly" or "inversely" proportional to the running start?

Can this example be worked by proportion?

In working problems we must consider the following:

(1) Can the problem be worked by proportion; that is, do the quantities named in the problem vary in either direct or in inverse ratio?

(2) If they do, determine whether the variation is in "direct ratio" or in "inverse ratio."

It should be observed that, while a variation may not be strictly in "direct" or in "inverse ratio," the variation may be so nearly in that ratio that the method of proportion yields answers that are sufficiently accurate for practical purposes.

For example, it does not necessarily follow that 4 men can do a piece of work in half the time that 2 men can do the work. The 4 men may not be equally efficient or 4 men may be in each other's way, so that 4 men could not work as efficiently as 2 men can. But in many such cases the differences are so slight as to be negligible.

4. Find two numbers in the ratio of 4 to 5 whose sum is 207.

Let the two numbers be $4x$ and $5x$.

5. Find two numbers in the ratio of 3 to 8 whose sum is -275.

6. Find two numbers in the ratio 2 to -9 whose sum is -217.

7. Find two numbers in the ratio of $-2:5$, such that the first number minus the second yields -133.

8. Two sums of money bear interests which are in the ratio of 6 to 7. Find each interest, if the two interest sums together amount to 650 dollars.

9. Two sums of money, at the same time and rate, bear interests which are in the ratio of 3 to 4. What is the larger sum, if the smaller is \$3360?

10. Two equal sums of money, both at the same rate, bear interests which are to each other as 10 is to 13. If the time for the lesser interest is 5 years and 5 months, what is the time for the other?

11. What is the height of a flagstaff which casts upon level ground a shadow of 84 ft., if at the same moment a vertical rod, 10 ft. long, casts a shadow 7 ft. long?

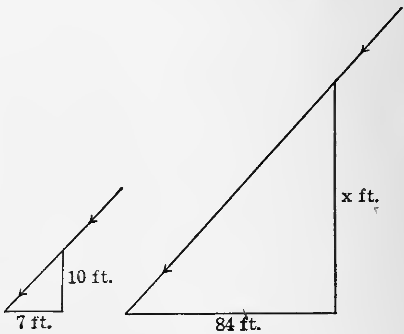


FIG. 10

As shown by Fig. 10, two triangles are formed which have their sides parallel, respectively. Such triangles have the same shape, but not the same size. It is shown in geometry that such triangles have their corresponding sides proportional. In this case the horizontal sides are to each other as the vertical sides and as the slanting sides. In our problem the slanting sides are not used in forming the proportion. We form the proportion thus:

$$\frac{\text{Horiz. side of small tri.}}{\text{Horiz. side of large tri.}} = \frac{\text{Vert. side of small tri.}}{\text{Vert. side of large tri.}}.$$

$$\frac{7}{84} = \frac{10}{x},$$

x being the required height of the flagstaff.

12. A pole for wireless telegraphy casts a shadow of 105 ft. upon level ground at the same time that a vertical rod, 12 ft. long, casts a shadow 7 ft. long. Compute the length of the pole.

13. By the method just explained determine the height of the flagpole of your school, or of some similar high object.

14. At the moment when the sun sets behind a mountain peak, a vertical rod 12 ft. high casts a shadow of 27 ft. on level ground. If the mountain is known to tower 7000 ft. above the plain, what is the distance in a horizontal line to the center of the mountain?

15. Concrete is made by mixing cement, sand, gravel, and water. A $1:2:4$ concrete is made up of one volume of cement, twice that volume of sand, and 4 times that volume of stone. The volume of the mixed cement is less than the sum of the volumes of the parts, because of the open spaces between the stones which must be filled in making the concrete. One cubic foot of concrete contains:

for $1:2:4$ concrete: .22 cu. ft. cement, .44 cu. ft. sand, .88 cu. ft. gravel.

for $1:2\frac{1}{2}:5$ concrete: .19 cu. ft. cement, .475 cu. ft. sand, .95 cu. ft. gravel.

In the building of a silo, 1125 cu. ft. of concrete is needed, $\frac{4}{5}$ of which must be $1:2:4$ concrete, the rest $1:2\frac{1}{2}:5$ concrete. How many cubic feet of cement, of sand, and of gravel are needed for making the $1:2:4$ cement?

Let the required number of cubic feet of cement, sand, and gravel be, respectively, $22\,x$, $44\,x$, and $88\,x$.

16. In Ex. 15, how many cubic feet of cement, of sand, and of gravel are needed for making the $1:2\frac{1}{2}:5$ cement?

17. A contributes \$ 2000 to a speculation, B \$ 3500, C \$ 1500. The total profits are \$ 725. What amount of the gain ought each to receive?

18. Find a number such that its excess over 8 bears the same ratio to its excess over 9 as the number itself does to its excess over 2.

19. Two pounds of tea cost as much as 3 lb. of coffee, and as much as 5 lb. of butter. The cost of 1 lb. of tea, 1 lb. of coffee, and $\frac{1}{2}$ lb. of butter is \$ 1.40. Find the cost per pound of each article.

GRAPHIC REPRESENTATION OF TEMPERATURE

84. If we desire information on the changes of temperature for any period of time, we may take readings of a thermometer at certain intervals of time. We may take hourly readings of the outdoor temperature during the afternoon and evening of a winter day and obtain data, in degrees Fahrenheit, perhaps as follows:

Noon, 20°.	7 P.M., 15°.
1 P.M., 21°.	8 P.M., 9°.
2 P.M., 22°.	9 P.M., 5°.
3 P.M., 21.5°.	10 P.M., 0°.
4 P.M., 20°.	11 P.M., −2.°
5 P.M., 17°.	12 P.M., − 3°.
6 P.M., 16°.	

These readings convey an idea of the changes of temperature during the 12 hours following midday. We see that the maximum temperature was at 2 P.M., the minimum at midnight. The temperature dropped most rapidly between 7 and 8 P.M.

These relations can be seen much more easily, if the changes of temperature are shown *graphically*. Draw a horizontal line, as in Fig. 11, and let successive equal intervals from o along the line ox indicate the successive hours after midday. We mark in this way the hours 1, 2, 3, etc.

At each of these points we draw a line perpendicular to ox to represent the temperature at that particular time. If the temperature is +, we measure the distance from ox *upward*, but if the temperature is −, we measure the distance from ox *downward*. If 1° is represented by the length of one of the divisions up or down, then the midday temperature is represented by 20 spaces from *ox up*, while $-3°$ at midnight is measured from *ox* by 3 spaces *down*. If we connect the ends

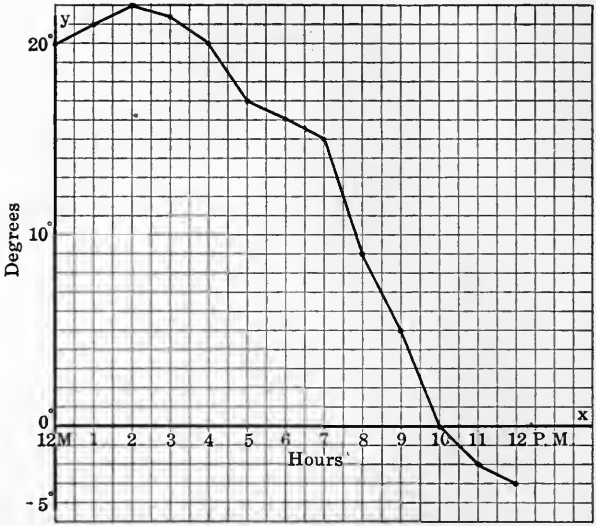


FIG. 11

of the lines thus drawn, we obtain a broken line which shows to the eye the changes in temperature. This broken line is the *graphic representation* of *temperature.* It is called a *graph* or a diagram. Drawing such a graph or diagram is called plotting it; locating a point is called *plotting* the point.

How can you tell from the graph the time when the temperature was highest? The time when it was lowest? The time when the temperature changed least? The time when the temperature changed most rapidly? What was the drop in temperature between 2 P.M. and 9 P.M.? Between 7 P.M. and 10 P.M.? Between 8 P.M. and 12 P.M.? Between 1 P.M. and 11 P.M.?

Care should be taken, in plotting, to use a scale not too small and yet small enough so that all the statistics can be used.

EXERCISES

85. 1. Plot the curve of temperatures for the next 12 hours the observed temperatures being as follows:

12 P.M., − 3°.	7 A.M., − 4°.
1 A.M., − 4°.	8 A.M., 0°.
2 A.M., − 4°.	9 A.M., + 3°.
3 A.M., − 5°.	10 A.M. + 7°.
4 A.M., − 5°.	11 A.M. + 10°.
5 A.M., − 6°.	12 Noon, + 15°.
6 A.M., − 6°.	

2. Find the rise in temperature between 5 A.M. and 11 A.M.

3. At what time was the rise in temperature most rapid?

4. What was the difference in temperature between 12 P.M. and 12 noon?

5. Plot the curve of mean temperatures for 1913, Denver, Col.

Jan. 30°.	July 72°.
Feb. 22°.	Aug. 73°.
Mch. 38°.	Sept. 59°.
Apr. 49°.	Oct. 46°.
May 58°.	Nov. 44°.
June 67°	Dec. 23°.

6. Plot the rise and fall of the tide on the coast of Southern California, Aug. 2, 1913.

At 3 : 20 A.M., − 1.2 ft.
9 : 50 A.M., − 4.2 ft.
2 : 50 P.M., 1.7 ft.
9 : 00 P.M., 7 ft.

F

FLUCTUATIONS IN THE PRICE OF COAL

ORAL EXERCISES

86. Figure 12 shows the changes in the price of anthracite coal in the United States from 1860 to 1914.

The years are shown on a horizontal line, the cost on a ver-

Fig. 12

tical line. For instance, in 1890 the price per ton is seen to be $ 3.90; in 1910, $ 4.80.

1. What effect had the Civil War upon the price of coal?

2. In what year was the price $ 4.50?

3. Name the highest price shown.

4. Name the lowest price shown.

5. How many tons of coal could be bought in 1893 for $ 57.20?

6. What was the difference in the price per ton in 1874 and 1861?

7. What was the difference in the price of 100 tons in 1866 and 1899?

DRAWING EXERCISE

87. 1. Draw a diagram showing the fluctuations in the price of iron per ton during the years 1870 to 1910, the data being as follows:

1870,	$ 33.	1885,	$ 18.	1900,	$ 20.
1872,	$ 44.	1887,	$ 21.	1902,	$ 16.
1874,	$ 30.	1889,	$ 18.	1903,	$ 21.
1877;	$ 19.	1890,	$ 18.50	1904,	$ 19.
1878,	$ 18.	1894,	$ 13.	1905,	$ 13.
1880,	$ 28.	1895,	$ 13.	1906,	$ 20.
1881,	$ 25.	1898,	$ 12.	1907,	$ 23.
1882,	$ 26.	1899,	$ 19.	1910,	$ 17.

2. Make a table of varying quantities in your own experience and draw the graph. Select the number of pupils in your school, the amount of money you have spent in successive weeks or months, the amount of rainfall in successive months, the number of immigrants coming to America in different years, or some similar data which you are able to obtain.

88. From these graphs we can see, at a glance, the variation in one thing as another changes. For instance in Fig. 12 there is a change in the *price of coal per ton* as the *time* increases.

These quantities are called *variables*; the value of one variable depends upon the value of the other.

89. This idea can be carried over to the relation between two unknown quantities expressed by an equation.

For instance, in $x + y = 7$:

$$\begin{array}{llll} \text{if } x = 3, & \text{then } & 3 + y = 7, & \text{hence } y = 4; \\ \text{if } x = 2, & \text{then } & 2 + y = 7, & \text{hence } y = 5; \\ \text{if } x = 1, & \text{then } & 1 + y = 7, & \text{hence } y = 6; \\ \text{if } x = 0, & \text{then } & 0 + y = 7, & \text{hence } y = 7; \\ \text{if } x = -1, & \text{then } & -1 + y = 7, & \text{hence } y = 8; \\ \text{if } x = -4, & \text{then } & -4 + y = 7, & \text{hence } y = 9. \end{array}$$

As x increases, y decreases; and as x decreases, y increases.

Thus x and y are two *variables*, the value of y depending upon what the value of x may be, and the value of x depending upon the value of y; thus x is a *function* of y and y is a *function* of x.

90. Let us make a table, assuming ten values of x and computing the corresponding values for y in the equation $x - y = 5$. The result is the following table:

$x - y = 5$

x	y	Point
6	1	*A*
5	0	*B*
4	-1	*C*
3	-2	*D*
2	-3	*E*
$1\frac{1}{2}$	$-3\frac{1}{2}$	*F*
1	-4	*G*
0	-5	*H*
-1	-6	*I*
-2	-7	*J*

We count off values of x on the horizontal line OX (called the *x-axis*) and the corresponding

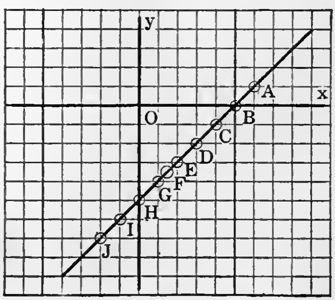


FIG. 13

values of y on the vertical line OY (called the *y-axis*), counting to the *right* of the *origin* O for + values of x and to the *left* for − values of x; *up* for + values of y, and *down* for − values of y.

This becomes plainer, if we plot each of the points in Fig. 13. The first point, $x = 6$ and $y = 1$, is the point A, 6 spaces to the right and 1 space up. The second point, $x = 5$, $y = 0$, is the point B, 5 spaces to the right, on the x-axis. The third point, $x = 4$, $y = -1$, is the point C, 4 spaces to the right and 1 space down, and so on. Points represented by fractional values of x and y are located in the same way. Thus $x = 1\frac{1}{2}$, $y = 3\frac{1}{2}$, locate the point F.

The values of x and y which satisfy the equation $x - y = 5$ are represented by points which form a straight line. Furthermore, every point in the line corresponds to values of x and of y which satisfy the equation.

Thus the graph of a *linear* equation in *two* unknowns is a *straight* line.

EXERCISES

91. Tabulate five pairs of values of x and y which satisfy each of the following equations. Draw a separate pair of axes for each equation and plot the points.

1. $2x + 3y = 6$.
2. $3x - 4y = 12$.
3. $x - y = 2$.
4. $x + y = 0$.
5. $x = 2y$.
6. $y = 2x$.

92. In the preceding exercises the student has observed that a linear equation in two unknowns produces a graph which is a straight line. As two points determine the position of any straight line, it is *necessary* and *sufficient* to find only two points on the graph of a linear equation in order to fix the position of the line. The two points most easily found are $x = 0$, $y = ?$ and $x = ?$, $y = 0$.

If $x = 0$ and $y = 0$ satisfy the equation, it will be necessary to let x have some value other than 0 and to find the corresponding value of y.

For instance, in $2x - 5y = 0$, if $x = 0$, $y = 0$, and if $y = 0$, $x = 0$. But if $x = 5$, $y = 2$.

Plot (0, 0), and (5, 2), and draw the line, as in Fig. 14.

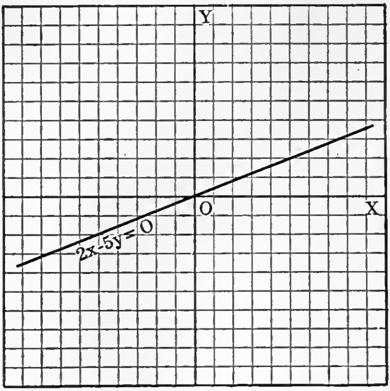


FIG. 14

EXERCISES

93. Represent the following equations graphically:

1. $x + y = 3$.
2. $x - y = 5$.
3. $2x + y = 4$.
4. $x - 2y = 6$.
5. $2x - 4y = 10$.
6. $x + 2y = 0$.
7. $x - 3y = 0$.
8. $3x - y = 0$.
9. $5x = 4y$.

PRACTICAL APPLICATIONS OF GRAPHS[1]

94. One of the simplest applications of graphs is in the reduction of denominate numbers from one unit to another without the labor of numerical computation. Thus, Fig. 15 enables us by inspection to reduce miles to Kilometers, or Kilometers to miles. The following example shows how the graph may be drawn.

95. If 10 miles are equal to 16.1 Kilometers, how many Kilometers are equal to x miles?

Let y be the required number of Kilometers.

Then by proportion, $\dfrac{x \text{ mi.}}{10 \text{ mi.}} = \dfrac{y \text{ Km.}}{16.1 \text{ Km.}}$.

Omitting the names of units, $\dfrac{x}{10} = \dfrac{y}{16.1}$.

Multiply both sides by 16.1, $1.61x = y$.

Or $y = 1.61x$.

[1] May be postponed for the present.

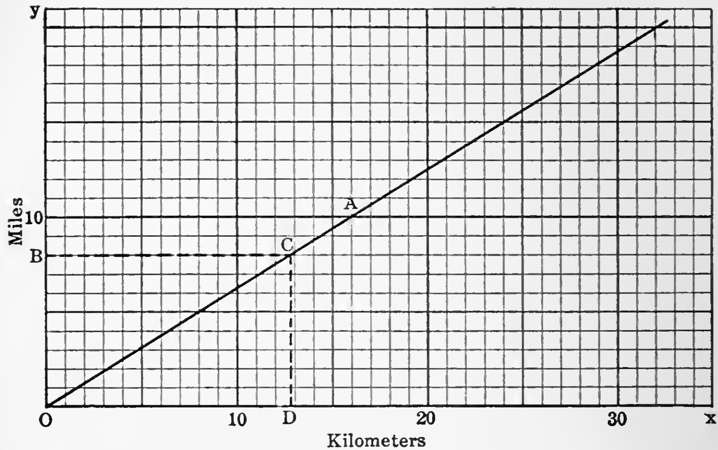


Fig. 15

The graph of this equation is a straight line.

To draw this graph, carefully locate two points on it and then draw a straight line through the two points. We obtain

$$y = 1.61\,x$$

x	y	Point
0	0	O
10	16.1	A

In the graph, Fig. 15, Kilometers are measured along the x-axis, miles are measured along the y-axis.

To determine how many Kilometers are equal to 8 miles, we find the point B on the y-axis which indicates 8 miles, then proceed to the right (in a direction parallel to the x-axis) to the point C on the graph, then proceed downward to the point D on the x-axis. The number of Kilometers equivalent to 8 miles is seen to be approximately 13.

Results that are absolutely accurate cannot be obtained for three reasons: *First*, the relation "10 miles = 16.1 Kilometers" is correct only to *tenths* of Kilometers; *secondly*, even if this relation were accurate, the graph could not be drawn with absolute accuracy; *third*, to estimate the fraction of a Kilometer at the point *C*, without some slight error, is hardly possible. Of necessity all measurements are only approximate.

ORAL EXERCISES

96. Estimate, to the first decimal, the number of Kilometers in

1. 7 mi.	**4.** 13 mi.	**7.** 17 mi.
2. 9 mi.	**5.** 14 mi.	**8.** 18 mi.
3. 11 mi.	**6.** 16 mi.	**9.** 20 mi.

Estimate, to the first decimal, the number of miles in

10. 8 Km.	**13.** 17 Km.	**16.** 23 Km.
11. 12 Km.	**14.** 19 Km.	**17.** 27 Km.
12. 13 Km.	**15.** 21 Km.	**18.** 32 Km.

CONSTRUCTION AND USE OF GRAPHS

97. 1. 10 Kilograms are approximately equal to 22 pounds. Draw a graph for reducing at sight, Kilograms to pounds, and pounds to Kilograms. Change at sight, 6 lb., 7 lb., 9 lb. to Kilograms; change also 21 Kg., 17 Kg., 13 Kg. to pounds.

2. $5\frac{1}{2}$ yards $= 1$ rod. Construct a graph for converting yards to rods and rods to yards. At sight change to yards, 2.2 rd., 1.7 rd., 1.3 rd., .8 rd.

Let 1 in. along the x-axis stand for 1 rd., and $\frac{1}{8}$ in. along the y-axis stand for 1 yd. This gives a graph which is convenient for reducing a small number of rods to yards.

3. Draw a graph for finding the lengths of circles when the diameters are given, and *vice versa*.

Take the length of a circle equal to $\frac{22}{7}$ times its diameter.

4. Draw a graph for converting temperatures on the Fahrenheit scale to temperatures on the Centigrade scale, and *vice versa*.

We know that a temperature of 0° C. is the same as one of 32° F.; also that a temperature of 20° C. is the same as one of 68° F. From these data we locate the points A and B, in Fig. 16, which determine the line. This graph is applicable to temperatures below zero, or *negative* temperatures.

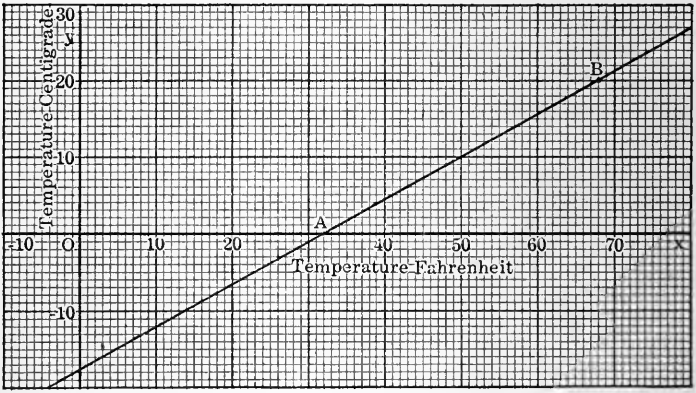


FIG. 16

5. In Fig. 16 change the following to temperatures on the Fahrenheit scale: 30° C., 25° C., 17° C., 0° C., – 10° C., – 20° C.

6. In Fig. 16 change the following to temperatures on the Centigrade scale: 80° F., 65° F., 52° F., 32° F., 15° F., 8° F., 0° F., – 3° F.

7. A wholesale dealer's profit when he sells to a retail merchant is 20%. Draw a graph for ascertaining the cost price when the selling price is known, and *vice versa.*

When there is a profit of 20%, the selling price is to the cost, as 120 is to 100.

Let x be the selling price and y the corresponding cost.

Then, by proportion, $$\frac{x}{120} = \frac{y}{100}.$$

Multiply both sides by 100, $$\tfrac{5}{6}x = y.$$

Or $$y = \tfrac{5}{6}x.$$

The graph of this equation is a straight line.

From this equation the following data are obtained :

x	y	Point
0	0	O
30	25	A

Through the points O and A, in Fig. 17, draw a straight line, which is the graph required. It shows the relation between the cost and the selling price.

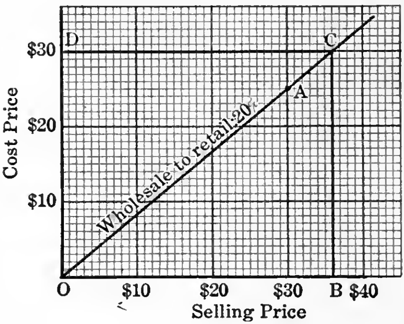


Fig. 17

To determine the cost of an article that sells for $36, take the point B 36 divisions from O on the x-axis, then OB represents the selling price. From B pass vertically to C on the "wholesale to retail" graph, thence horizontally to D. Then OD, or $30, is the cost to the wholesale dealer.

8. In Fig. 17, find the cost of articles sold to the retail dealer for $10, $14, $17, $24, $28, $32.

9. In Fig. 17, find the selling price to the retail dealer of articles which cost $5, $15, $20, $23, $25.

10. A wholesale dealer makes a profit of 15 % when he sells to a retail dealer. The retail dealer makes a profit of 65 % when he sells to a consumer. Draw a graph for finding at sight the cost to the wholesale dealer of articles whose selling price to the consumer is known.

When there is a profit of 15%, the selling price is to the cost as 115 is to 100.

Let x be the selling price to the retail dealer and y the cost to the wholesale dealer.

Then, by proportion, $\frac{x}{115} = \frac{y}{100}$.

Multiply both sides of the equation by 100, $\frac{100}{115}x = y$.

Or $$y = \frac{20}{23}x.$$

When $x = 0$, $y = 0$, hence the graph passes through O.

When $x = 46$, $y = 40$, hence the graph passes through A.

Draw the line OA.

The retail dealer sells at a profit of 65 %. That is, his selling price to the consumer is to his buying price as 165 is to 100.

Let x be the selling price to the consumer and y the cost to the retail dealer.

Then, $$\frac{x}{165} = \frac{y}{100}.$$

Or $$y = \frac{20}{33}x.$$

When $x = 0$, $y = 0$, hence the graph passes through O in Fig. 18.

When $x = 66$, $y = 40$, hence the graph passes through B.

Draw the line OB.

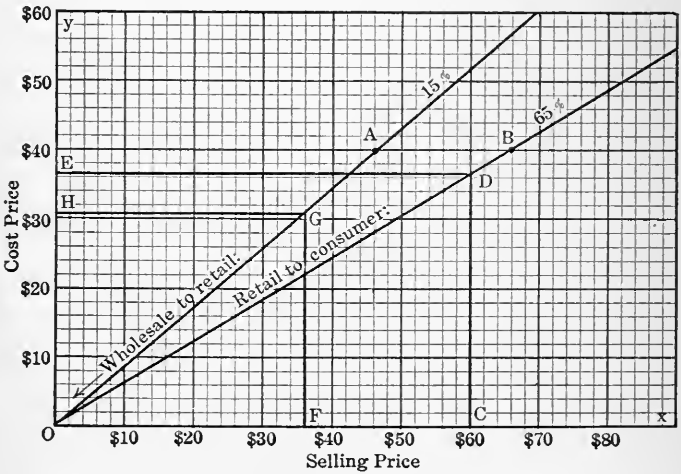


Fig. 18

To determine the cost to the wholesale dealer of an article for which the consumer pays $60, two steps are necessary.

The first step is to find its cost to the retail dealer. Take the point C 60 divisions on the x-axis, then OC represents the price paid by the consumer. From C pass vertically to D on the "retail to consumers" graph, 3p.20

thence horizontally to E. Then OE, or about \$52.20, is the cost to the retail dealer.

The second step is to find the cost to the wholesale dealer. Take the distance OF equal to OE; from F pass vertically to G on the "wholesale to retail" graph, thence horizontally to H. Clearly OH is the cost to the wholesale dealer, about \$$31\frac{1}{2}$. This is approximately the required answer.

11. What is the cost to the wholesale dealer of articles which are sold to the consumer for \$20? \$30? \$50? \$75?

12. What does the consumer pay for articles which cost to the wholesale dealer \$50? \$40? \$15?

$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$\frac{1}{2} + \frac{1}{2} = 1$ $\quad \frac{2}{3} \div \frac{1}{3}$

$\frac{1}{4} \div$

$\frac{1}{2} \div \frac{1}{4}$

$\frac{1}{2} \cdot \frac{4}{1} \quad \frac{4}{2} = 2$

$\frac{1}{2} \div \frac{1}{4}$

$\frac{1}{2} \times 4$

Cost $= \frac{190}{x}$

$x = 190 \div$ cost

CHAPTER V

EQUATIONS INVOLVING FRACTIONAL COEFFICIENTS

98. Solve $\frac{1}{3}x + \frac{1}{2}x = 15$.

First Solution. $\qquad \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$.

Hence we have $\qquad \frac{5}{6}x = 15$.

Divide both sides by $\frac{5}{6}$, $\qquad x = \frac{15 \cdot 6}{5} = 18$. *Ans.*

Second Solution. $\qquad \frac{1}{3}x + \frac{1}{2}x = 15$.

Multiply both sides by 6, $\quad 2x + 3x = 90$.

$$5x = 90.$$

$$x = 18. \quad \textit{Ans.}$$

Check:

$$\tfrac{1}{3}(18) + \tfrac{1}{2}(18) = 15.$$

$$6 + 9 = 15.$$

$$15 = 15.$$

In the second solution we *multiply both sides of the equation by some number which will remove the denominators.* It is usually easiest to select the *least* number into which *each* denominator will go without a remainder. In the example above, that number is 6.

EXERCISES

99. Solve and check:

1. $\dfrac{2x}{3} = 8.$

2. $\frac{3}{5}y + \frac{2}{5}y = 1.$

3. $\dfrac{2v}{3} = \dfrac{5}{6} - \dfrac{2v}{3}.$

4. $\dfrac{n}{8} + 2n - \dfrac{5n}{6} = \dfrac{3n}{4} + 13.$

5. $2r + \dfrac{3r}{7} = \dfrac{3r+2}{4} + 23.$

6. $\dfrac{7x}{12} - \dfrac{1}{4} = 2x - \dfrac{5}{3}.$

7. $\dfrac{r-8}{2} = 6 - \dfrac{r}{3}.$

8. $\dfrac{2y+3}{5} - \dfrac{y-3}{3} = 2.$

9. $\dfrac{y-1}{2} - \dfrac{y-2}{3} = \dfrac{2}{3} - \dfrac{y-3}{4}.$

10. $\dfrac{t-3}{7} + \dfrac{t+5}{3} - \dfrac{t+2}{6} = 4.$

11. $\frac{5}{4}(6 - x) = 5.$

12. $\dfrac{4s+2}{11} = \dfrac{1}{5}(s+5).$

13. $\frac{4}{5}(t+1)=2.$

14. $\frac{1}{2}(s+4)-\frac{1}{6}s-8=0.$

15. $.5x=3.$

16. $.2x=48-.04x.$

17. $.2w+3=3.8+.04w.$

18. $\frac{7}{8}(z+2)+\frac{2}{3}(z-12)=\frac{1}{5}(z+9).$

19. $\frac{1}{2}(n-1)-\frac{1}{3}(n-2)=\frac{2}{3}-\frac{1}{4}(n-3).$

20. $\frac{1}{3}(y+1)-\frac{1}{5}(y+4)+\frac{1}{4}(y+3)=16.$

PROBLEMS

100. 1. One half a certain number plus one fourth that number increased by one third the number equals 26. Find the number.

2. The sum of two numbers is 29. One half the first plus one third the second is 12. What are the numbers?

3. The difference between two numbers is 6. One ninth of the first plus one third of the second is 6. Find the numbers.

4. Separate 48 into two parts so that $\frac{1}{5}$ the larger minus $\frac{1}{3}$ the smaller equals $\frac{1}{6}$ the number itself.

5. One fourth of a certain number increased by $1\frac{1}{4}$, and the result diminished by the quotient obtained by dividing the sum of twice the number and 4 by 9, equals 1. Find the number.

6. The width of a rectangle is $\frac{4}{5}$ of its length and its perimeter is 54 in. Find its dimensions.

7. The length of a rectangle is 12 in. more than the width. The sum of the length and width is twice their difference. Find the area.

8. The width of a rectangle is 5 ft. less than the length. If the length be decreased 2 ft. and the width increased 3 ft., the area will be increased 11 sq. ft. Find the area.

9. A certain square has the same area as a rectangle whose dimensions are 3 ft. longer and 2 ft. shorter than those of the square. What is the area of each?

10. Find two consecutive integers such that $\frac{1}{5}$ the first plus $\frac{1}{2}$ the second equals 7 less than the first.

11. Find three consecutive even integers such that $\frac{1}{3}$ of the first and $\frac{1}{2}$ of the second, plus $\frac{1}{4}$ of the third, equals the third.

12. What three consecutive integers have their sum equal to 48? How does the sum compare with the second integer? Is this true of any three consecutive integers?

13. The sum of $\frac{1}{6}$ of a certain even integer, $\frac{1}{5}$ of the next odd one, and $\frac{1}{2}$ of the next even one equals the even integer just before the one first mentioned. Find the series of integers.

14. A is 15 years old; B is 25 years old. In how many years will A be $\frac{3}{4}$ as old as B?

15. A is twice as old as B; five years ago he was $2\frac{1}{4}$ times as old. How old is each now?

16. A man's age 6 years ago was $\frac{1}{2}$ of what his age will be 30 years hence. Find his age now.

17. A father is 3 times as old as his son; in 5 years the father will be $2\frac{1}{2}$ times as old as the son. What is the age of each?

18. A man who is 35 years old has a son 10 years old. In how many years will the boy be $\frac{1}{2}$ as old as his father?

19. A man gave to one son $\frac{1}{3}$ of his money, to a second $\frac{1}{5}$ of his money. He had left \$10 less than the number of dollars he gave to both. How much had he at first?

20. I spent \$1.75 for 1¢ and 2¢ stamps, buying 25 more of the former than of the latter. How many of each kind did I buy?

21. I took a trip of 105 miles, partly by trolley, partly by train. If I went $\frac{1}{4}$ as far again by train as by trolley, how far did I go by each?

22. A collection of 5¢ pieces and quarters amounts to $1.60. There are 2 more 5¢ pieces than quarters. How many are there of each?

23. I have three more half dollars than quarters. The value of the half dollars exceeds the value of the quarters by $4.50. How much money have I?

24. Thirty coins — dimes, nickels, and quarters — amount to $5.30. There are $\frac{1}{8}$ as many nickels as quarters. How many are there of each?

25. Three boys have 150 marbles. If the third has $\frac{4}{5}$ as many as the second, and the second $\frac{5}{6}$ as many as the first, how many marbles has each boy?

26. The same number is subtracted from 60 and from 45. One third of the first remainder equals $\frac{1}{2}$ of the second. What is the number?

27. In going a certain distance, a train traveling at the rate of 40 miles an hour takes $2\frac{1}{2}$ hours longer than one traveling 50 miles an hour. What is the distance?

Use $d = rt$.

28. I have 13 hours at my disposal. How far may I go at the rate of 3 miles an hour, so as to return home in time, coming back at the rate of $3\frac{1}{2}$ miles an hour?

29. A man sold a home for $4944, gaining 3% of the cost. How much did the home cost?

Hint. $x + .03x = 4944.$

30. An agent deducted 5% commission for the sale of property, remitting $6222.50. Find the selling price.

31. At 6% interest for a certain time, $350 amounted to $444.50. What was the length of time?

Use $i = prt$.

32. The interest of $365 for 2 years 3 months at a certain rate was $32.85. What was the rate?

33. The amount of a certain principal at 6 % in $5\frac{1}{2}$ years is \$7714. What is the principal?

34. A man invests a part of \$1500 at 5 % and the remainder at 4 %. The 5 % investment yields \$39 more a year than the 4 %. How much has he invested at 4 %?

35. A part of \$3680 is invested at $5\frac{1}{2}$ % and the remainder at 6 %. The total interest in 2 years amounted to \$411.60. How many dollars are there in each investment?

36. If the valuation of a certain property is \$17,000, and the annual tax thereon is \$161.50, find the tax rate in mills on a dollar.

Let x mills = the tax on a dollar. Then $\frac{17000x}{1000} = 161.50$.

Find the tax rate in mills on a dollar:

	Valuation	Tax		Valuation	Tax
37.	\$6,560	\$55.76	**40.**	\$125,500	\$909.875
38.	\$29,800	\$186.25	**41.**	\$375,400	\$2909.35
39.	\$90,700	\$589.55	**42.**	\$875,500	\$6566.25

43. I wish to get \$7280 from a bank. For what sum must I make out a 60-day note, to obtain this sum, bank discount being at the rate of 6 % per annum?

Bank discount is computed on the *face* of the note.

Let $\$x$ = the required sum. The bank discount for 60 da. is 1 % of $\$x$, or $\$.01x$. Subtracting $\$.01x$ from $\$x$, leaves $\$.99x$, the proceeds. Hence, $.99x = 7280$.

44. A man needs \$2975. For what sum must he make out a 60-day note to obtain the \$2975 as proceeds, if the bank discounts the note at 6 % per annum?

45. Mr. Murray makes out a 90-day note, which is discounted by the bank at 2 % for the 90 days. What must be the face of the note, to yield \$9700 as proceeds?

46. John Allen issued a note to a bank which secured for him, after it was discounted at 3 % for the term of 4 months, the sum of \$29,750. Find the face of the note.

The federal income tax in the United States requires that every unmarried person shall pay an annual tax of 1 % on his net income in excess of \$ 3000, and that married persons living together shall pay an annual tax of 1 % on their joint income in excess of \$ 4000. When the net income exceeds \$ 20,000, an *additional tax* thereon is levied as follows:

1 % on part of income over \$ 20,000 and not above \$ 50,000.

2 % on part of income over \$ 50,000 and not above \$ 75,000, etc.

Thus, an unmarried man pays on an income of \$ 70,000 the following tax: 1 % on \$ 67,000 + 1 % on \$ 30,000 + 2 % on \$ 20,000.

47. An unmarried person whose net annual income is below \$ 20,000 pays an income tax of \$ 145.70. Find his income.

48. A man and wife have a joint income that is below \$ 20,000; their income tax is \$ 135.45. Find their income.

49. A man and wife pay an income tax of \$ 200. Does their annual income exceed \$ 20,000? Find their income.

Let \$ x = income. Then $.01(x - 4000) + .01(x - 20{,}000) = 200$.

50. The annual income tax of an unmarried man is \$ 450. Is his income above \$ 20,000? Above \$ 50,000? Find it.

51. The joint income tax of a man and his wife is \$ 1360. Is the annual income above \$ 50,000? Above \$ 75,000? Compute their income.

52. What is the annual net income of an unmarried man whose income tax is \$ 850?

53. A boy on a farm receives $2\frac{1}{2}$¢ a dozen for delivering 98 dozen eggs and agrees to have 27¢ deducted from his pay for every dozen eggs that he breaks and does not deliver. He is paid \$ 1.55. How many dozen eggs did he break?

54. Two persons traveling toward each other set out at the same time from towns 198 miles apart. One person walks 10 miles a day, the other 12 miles. In what time will they meet, and how many miles will each have walked?

55. At an election $\frac{1}{5}$ of a constituency abstained from voting, and $\frac{2}{3}$ of the constituency voted for the successful candidate.

At a subsequent election $\frac{2}{5}$ abstained from voting, and $\frac{1}{2}$ voted for the successful candidate. The minority in the latter election was 120 less than in the former. Find the number of voters in the constituency.

56. A rectangular tank is 10 ft. 6 in. long and 7 ft. 3 in. wide, and contains 400 gallons of water. Find the depth of the water, correct to one tenth of an inch, having given that a gallon contains 231 cu. in.

57. A train is 5 minutes late when it performs its journey at the rate of $29\frac{1}{2}$ miles an hour, and is 2 minutes late when it travels at 30 miles an hour. What is the number of miles in the journey?

58. There are two lawns in one garden. One is square; the second is 3 yd. narrower and 4 yd. longer than the first, but of equal area. Find the area of each.

CHAPTER VI

SPECIAL PRODUCTS

101. I. $(a+b)^2$.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2\,ab+b^2 \end{array}$$

II. $(a-b)^2$.

$$\begin{array}{r} a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2\,ab+b^2 \end{array}$$

$$\therefore\ (a+b)^2 = a^2 + 2\,ab + b^2.$$
$$(a-b)^2 = a^2 - 2\,ab + b^2.$$

RULE. *The square of the* $\left\{\begin{matrix}\textit{sum}\\ \textit{difference}\end{matrix}\right\}$ *of two numbers is equal to the square of the first,* $\left\{\begin{matrix}\textit{plus}\\ \textit{minus}\end{matrix}\right\}$ *twice the product of the first and the second, plus the square of the second.*

ORAL EXERCISES

102. Find, by inspection, the values of:

1. $(m+n)^2$.
2. $(c-d)^2$.
3. $(a-2\,b)^2$.
4. $(3\,x+y)^2$.
5. $(4+a)^2$.
6. $(b-7)^2$.
7. $(x^2-1)^2$.
8. $(2\,a^2-3\,b)^2$.

9. $(4c + 3d^3)^2$.
10. $(7ab - 2)^2$.
11. $(9x^2 - 2y^2)^2$.
12. $(5a^3 + 4x^2)^2$.
13. $(1 + z)^2$.
14. $(\frac{1}{3}a - 3)^2$.
15. $(\frac{2}{5}c - \frac{4}{7}d)^2$.
16. $(x^a + y^b)^2$.
17. $(3a^m - 2b^n)^2$.
18. $(7a - b)^2$.
19. $(5d - 3e)^2$.
20. $(m + 12n)^2$.
21. $(9x - 3y)^2$.
22. $(2a - 7b)^2$.

23. What must be added to $a^2 + 10a$ to make it the square of $a + 5$?

24. What must be added to $x^2 - xy + y^2$ to make it the square of $x - y$?

25. What must be added to $36 - 12a^2$ to make it a perfect square?

In the following, supply terms necessary to form perfect squares:

26. $c^2 + (\ ) + d^2$.
27. $m^2 + (\ ) + 4n^2$.
28. $x^2 + (\ ) + 9$.
29. $t^2 - (\ ) + 16$.
30. $4a^2 - (\ ) + 1$.
31. $9x^2 + (\ ) + 36$.
32. $16b^2 - (\ ) + 49$.
33. $25g^4 - 10g^2 + (\ )$.
34. $64y^2 + 32y + (\ )$.
35. $121m^6 - 22m^3 + (\ )$.
36. $9a^{10} + 12a^5b^2 + (\ )$.
37. $(\ ) - 6b^2 + 9$.
38. $(\ ) + 10x^4y^3 + 25y^6$.
39. $(\ ) + 80hk^2 + 64k^4$.
40. $(\ ) - 24s^5 + 144$.
41. $(\ ) - 182m^2 + 169$.

Find the binomials whose squares are:

42. $x^2 + 6x + 9$.
43. $4a^2 - 4ab + b^2$.
44. $x^2 + 22xy + 121y^2$.
45. $1 - 14y + 49y^2$.
46. $m^2 + 40m + 400$.
47. $n^2 - 4n + 4$.
48. $16y^2 - 8y + 1$.
49. $25c^4 + 160c^3 + 256c^2$.
50. $9a^2 - 30axy^2 + 25x^2y^4$.
51. $36b^2 + 60a^2bc^3 + 25a^4c^6$.
52. $64c^8 - 16c^4d^5 + d^{10}$.
53. $4x^2y^2 + 28xy + 49$.

103. III. $(a+b)(a-b)$.

$$\begin{array}{l} \quad a+b \\ \quad \underline{a-b} \\ a^2+ab \\ \quad \underline{\quad -ab-b^2} \\ a^2 \qquad\quad -b^2 \end{array}$$

$$\therefore \boldsymbol{(a+b)(a-b)=a^2-b^2}.$$

Rule. *The product of the sum of two numbers and their difference is equal to the square of the first number minus the square of the second.*

ORAL EXERCISES

104. Find, by inspection, the values of:

1. $(k+y)(k-y)$.
2. $(1-3a)(1+3a)$.
3. $(2h+7)(2h-7)$.
4. $(5-b)(b+5)$.
5. $(10-3a)(3a+10)$.
6. $(3n+7)(3n-7)$.
7. $(3x-a)(3x+a)$.
8. $(5ab+3)(5ab-3)$.
9. $(2n+7)(7-2n)$.
10. $(6b-1)(6b+1)$.
11. $(a^2-9b^2)(a^2+9b^2)$.
12. $(13x^5+y)(13x^5-y)$.
13. $(4ab-c)(4ab+c)$.
14. $[(a+b)+c][(a+b)-c]$.
15. $(x+b+2)(x+b-2)$.
16. $[(m-n)-3][(m-n)+3]$.
17. $[t+(s+u)][t-(s+u)]$.
18. $[7+x+y][7-x-y]$.
19. $[5+(a-b)][5-(a-b)]$.
20. $(2+c-d)(2-c+d)$.

Find two binomials whose product is:

21. x^6-4y^6.
22. $16-a^4$.
23. $4x^2-16$.
24. $9a^2-64b^4$.
25. $25m^4-49n^6$.
26. a^4-9.
27. $9x^2y^2-25$.
28. $121c^2d^2-144x^4$.
29. $225b^6-1$.
30. $1-81a^{10}$.
31. $100b^2-36$.
32. $y^{2a}-225$.
33. $\frac{25}{36}-a^6$
34. $16m^2-25n^2$.

35. $(x-y)^2-z^2$.

36. $(c+d)^2-e^2$.

37. $(m-n)^2-64$.

38. $a^2-(b+c)^2$.

39. $t^2-(s-u)^2$.

40. $81-(a-b)^2$.

In each of these exercises (21–40), the two binomials found are called *factors* of the given expression. Thus, in Ex. 21, x^3+2y^3 and x^3-2y^3 are said to be *factors* of x^6-4y^6.

105. IV. $(x+a)(x+b)$.

$$\begin{array}{r} x+a \\ x+b \\ \hline x^2+ax \qquad\qquad \\ +bx+ab \\ \hline x^2+(a+b)x+ab \end{array}$$

$$\therefore (x+a)(x+b)=x^2+(a+b)x+ab.$$

Rule. *The product of two binomials having a common term is equal to the square of the common term, and the sum of the unlike terms times the common term, plus the product of the unlike terms.*

ORAL EXERCISES

106. Find, by inspection, the values of:

1. $(x+3)(x+4)$.
2. $(x-3)(x-4)$.
3. $(x+3)(x-4)$.
4. $(x-3)(x+4)$.
5. $(y+2)(y-7)$.
6. $(a-9)(a-3)$.
7. $(b-5)(b+6)$.
8. $(ax+1)(ax-2)$.
9. $(v-a)(v-2a)$.
10. $(n+b)(n+4b)$.
11. $(a+4c)(a-c)$.
12. $(2t+3)(2t-5)$.
13. $(x-16)(x+20)$.
14. $(3k-1)(3k-19)$.
15. $(m+17)(m-2)$.
16. $(c-x)(c+3x)$.
17. $(s-8)(s-3)$.
18. $(7h-15)(7h+2)$.
19. $(4g-a^2)(4g+5a^2)$.
20. $(5c^2-9)(5c^2-4)$.

Find two binomials (two factors) whose product is:

21. $a^2 + 7a + 12$.

22. $x^2 - 9a + 20$.

23. $b^2 + 2b - 35$.

24. $a^2 + 11a - 12$.

25. $c^2 - 20c + 51$.

26. $m^2 - 25m + 100$.

27. $t^2 - 25t + 66$.

28. $4r^2 + 22r + 30$.

29. $25s^2 - 30s + 8$.

30. $9d^2 - 60d + 19$.

31. $100m^2 + 10m - 12$.

32. $x^2y^2 + 12xy - 64$.

33. $16s^4 + 16s^2 - 5$.

34. $15 - 8m + m^2$.

35. $8 - 42a + 49a^2$.

36. $m^2 - 33m + 90$.

37. $121t^2 - 77t - 18$.

38. $9b^2 - 9b - 40$.

39. $100x^2 - 40x - 12$.

40. $f^2 - 13f - 30$.

CHAPTER VII

FACTORING

107. In the preceding chapter we found the products of certain binomials. We performed also the inverse operation: given the product, to find the binomials.

For instance, starting with the product $x^2 - 4y^2$, we found two binomials, $x + 2y$ and $x - 2y$, whose product is $x^2 - 4y^2$.

This inverse operation is called *factoring;* $x + 2y$ and $x - 2y$ are called *factors* of $x^2 - 4y^2$.

In this chapter we enter upon a fuller treatment of factoring.

Factoring an expression is the process of finding two or more expressions which, multiplied together, produce the given expression.

In factoring it is frequently necessary to find roots of monomials.

108. The *square root* of a monomial is one of the two equal factors whose product is the monomial.

Since $5 \cdot 5 = +25$, and $(-5)(-5) = +25$, it follows that there are two square roots of 25; namely, $+5$ and -5.

Similarly there are two square roots of a^2; namely, $+a$ and $-a$.

The two square roots of 25 are usually written in the form ± 5; similarly the two square roots of a^2 are written $\pm a$.

The positive square root is called the *principal square root.*

We write $\sqrt{25} = 5$. When $\sqrt{}$ has no sign before it, or has the $+$ sign before it, the principal root is always understood. When we write $-\sqrt{}$ we mean the negative root.

Thus, $\sqrt{25}$ stands for $+5$, $\sqrt{25}$ does *not* stand for -5.
$-\sqrt{25}$ stands for -5, $-\sqrt{25}$ does *not* stand for $+5$.
$\pm\sqrt{25} = \pm 5$.

Since $x^3 \cdot x^3 = x^6$, and $(-x^3)(-x^3) = x^6$, we have $\sqrt{x^6} = +x^3$, $-\sqrt{x^6} = -x^3$, $\pm\sqrt{x^6} = \pm x^3$.

Similarly, $\sqrt{16\,x^6} = +4\,x^3$, $-\sqrt{16\,x^6} = -4\,x^3$, $\pm\sqrt{16\,x^6} = \pm 4\,x^3$.

Also, $\pm\sqrt{4\,a^6b^4} = \pm 2\,a^3b^2$, $\pm\sqrt{36\,a^8b^{10}} = \pm 6\,a^4b^5$.

How is the exponent of a letter in the square root found?

RULE. *The two square roots of a monomial are found by writing ± the square root of the numerical coefficient, times the letters of the monomial, each with an exponent that is half of its exponent in the given monomial.*

Similar results hold for the fourth or sixth roots, or higher *even* roots of monomials.

Thus, $\sqrt[4]{16} = +2$, $-\sqrt[4]{16} = -2$, $\pm\sqrt[4]{16} = \pm 2$, $\sqrt[4]{81\,x^8} = +3\,x^2$, $-\sqrt[4]{81\,x^8} = -3\,x^2$, $\pm\sqrt[4]{81\,x^8} = \pm 3\,x^2$, $\pm\sqrt[6]{x^6} = \pm x$, $\pm\sqrt[6]{64\,a^{12}} = \pm 2\,a^2$.

109. The *cube root* of a monomial is one of the three equal factors whose product is the monomial.

Thus, $\sqrt[3]{8} = 2$, because $2 \cdot 2 \cdot 2 = 8$.

Notice that $\sqrt[3]{8}$ is *not* -2, for $(-2)(-2)(-2)$ is *not* 8.

$\sqrt[3]{-8} = -2$, because $(-2)(-2)(-2) = -8$.

Observe that a negative number can be the *cube* root of a negative number, but a negative number *cannot* be the *square* root of a negative number.

That is, -4 is $\sqrt[3]{-64}$, because $(-4)(-4)(-4) = -64$, but -8 is not $\sqrt{-64}$, because $(-8)(-8)$ is *not* -64.

Nor indeed can $\sqrt{-64}$ be $+8$, because $8 \cdot 8$ is *not* -64.

In other words, neither a positive number nor a negative number can be the square root of a *negative* number.

From $\sqrt[3]{8} = 2$ and $\sqrt[3]{-8} = -2$ we see that the cube root of a monomial has the same sign as the monomial.

Since $(3a^2)(3\,a^2)(3\,a^2) = 27\,a^6$, we have $\sqrt[3]{27\,a^6} = 3\,a^2$.

110. The square root of 2 cannot be exactly expressed by the Hindu-Arabic numerals. One can only *approximate* its value by extracting the square root to three or four, or more, decimal places. The radical $\sqrt{2}$, and other radicals of the same kind, like $\sqrt{3}$, $\sqrt[3]{5}$, etc., whose values cannot be found exactly, but only approximately, represent numbers called *irrational* numbers.

ORAL EXERCISES

111. Give the values of:

1. $\sqrt{121}$.	5. $\pm\sqrt{16x^2y^4}$.	9. $-\sqrt{\frac{1}{4}a^2b^2}$.
2. $-\sqrt{121}$.	6. $-\sqrt{25y^8z^6}$.	10. $\pm\sqrt{4a^2x^{10}y^4}$.
3. $\sqrt{9m^2}$.	7. $+\sqrt{100a^2b^2c^2}$.	11. $-\sqrt{36r^2s^4t^8}$.
4. $-\sqrt{9m^2}$.	8. $\sqrt{x^6y^2z^4}$.	12. $\sqrt{49m^6n^4o^2p^2}$.

Give both square roots of:

13. $144a^{12}$.	15. $225x^{12}y^4z^2$.	17. $400a^{16}b^{24}c^{20}$.
14. $169a^6b^8$.	16. $100r^2s^{16}$.	18. $900p^{30}q^{20}r^{10}$.

Give the values of:

19. $\sqrt[3]{125}$.	23. $\sqrt[5]{32}$.	27. $\sqrt{a^{24}b^{48}c^{72}}$.
20. $\sqrt[3]{-125}$.	24. $\sqrt[4]{a^4b^8c^{12}}$.	28. $\sqrt[3]{a^{24}b^{48}c^{72}}$.
21. $\sqrt[4]{16a^4b^8}$.	25. $\sqrt[3]{-216a^9}$.	29. $\sqrt[4]{a^{24}b^{48}c^{72}}$.
22. $\sqrt[6]{64x^6}$.	26. $\sqrt[3]{8x^{24}y^{21}z^{18}}$.	30. $\sqrt[6]{a^{24}b^{48}c^{72}}$.

31. Simplify $\sqrt{32}$.

32 is equal to $16 \cdot 2$, where 16 is a perfect square.

Hence $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$. We take the square root of 16, which is 4, and write the 4 as a factor before the $\sqrt{}$. The factor 2 is not a perfect square and is kept under the radical sign.

32. Simplify $\sqrt{50a^2b^5}$.

Here $50a^2b^5 = 25a^2b^4$ times $2b$. Since $25a^2b^4$ is a perfect square, we have $\sqrt{50a^2b^5} = 5ab^2\sqrt{2b}$.

Simplify the following:

1. $\sqrt{200\,x^2y^2}$.
2. $\sqrt{512}$.
3. $\sqrt{18\,a^2b^3}$.
4. $\sqrt{27\,a^2b^2c^4}$.
5. $\sqrt{12\,a^2b^2c^5}$.
6. $\sqrt{300\,m^2np^2}$.
7. $\sqrt{500\,yz^2w^3}$.
8. $\sqrt{128\,ab^2c^3d^4}$.

112. An *integral number* or *expression* is one that is free from fractions.

For example, 125 and $7\,a + 4\,b^2$ are integral numbers or expressions.

A *prime factor* of an integral number is an integral factor which is exactly divisible only by *itself* and *one*.

Thus 3 and 7 are prime factors of 63.

A *prime factor* of an integral algebraic expression is an integral factor which cannot itself be resolved into rational factors other than *itself* and *one*.

Since $(a^2 + 2\,b^2)(a^2 - 2\,b^2) = a^4 - 4\,b^4$, we know that $a^2 + 2\,b^2$ and $a^2 - 2\,b^2$ are factors. Moreover, these are *prime* factors. By multiplying we see that $(a + \sqrt{2}\,b)(a - \sqrt{2}\,b) = a^2 - 2\,b^2$. But $(a + \sqrt{2}\,b)$ and $(a - \sqrt{2}\,b)$ are not *prime* factors, because they contain the irrational $\sqrt{2}$.

In this chapter we shall find the prime factors of expressions; we shall not search for factors involving irrationals.

Factors of monomials are easily found by inspection. Thus the monomial $30\,a^2x^3y = 2 \cdot 3 \cdot 5\,a^2x^3y = 2 \cdot 3 \cdot 5 \cdot a \cdot a \cdot x \cdot x \cdot x \cdot y$.

If a polynomial contains a monomial factor, it may be found easily by inspection. In factoring a polynomial, proceed as follows:

(1) Divide the polynomial by the highest monomial factor common to *all* the *terms*.

(2) When possible, factor the quotient thus obtained.

Thus, in $32\,a^3b^3 + 28\,a^2b^2 - 12\,ab$, we see that each term contains the factor $4\,ab$. Divide the polynomial by $4\,ab$; the quotient is $8\,a^2b^2 + 7\,ab - 3$, which (as we shall see later) cannot itself be resolved into factors free from inexact roots.

Therefore, $32\,a^3b^3 + 28\,a^2b^2 - 12\,ab = 4\,ab(8\,a^2b^2 + 7\,ab - 3)$.

113. To check an example in factoring either

(I) Multiply the factors together, or

(II) Substitute numbers for the letters and simplify.

In $32\,a^3b^3 + 28\,a^2b^2 - 12\,ab = 4\,ab(8a^2b^2 + 7\,ab - 3)$, let $a = b = 1$. Then $32\,a^3b^3 + 28\,a^2b^2 - 12\,ab = 48$; $4\,ab = 4$, $8\,a^2b^2 + 7\,ab - 3 = 12$, and $4 \cdot 12 = 48$.*

114. Since factoring is the inverse of multiplication, certain type forms can always be factored.

I. Type form: $ay + by - cy$.

$$ay + by - cy = y(a + b - c),$$

wherein y stands for the highest monomial factor common to *all* the terms.

Hence $15\,b^3x - 10\,b^3y - 5\,b^3z = 5\,b^3(3\,x - 2\,y - z)$.

EXERCISES

115. Factor:

1. $4\,a - 8$.
2. $x^2 - x^3$.
3. $3\,ab - 6\,ac$.
4. $2\,x^2y - 6\,x^2z$.
5. $a^2b + a^2c - a^2d$.
6. $2\,m^2n - 6\,m^2n^2 + 8\,m^2n^3$.
7. $15\,a^4 - 10\,a^3 - 25\,a^2$.
8. $16\,c^3 + 8\,c^2d - 64\,c^4d^2$.
9. $x - x^2 + x^3 + x^4$.
10. $12\,a^6 - 24\,a^7 + 6\,a^5$.
11. $x^2 + 2\,xy - x - xz$.
12. $18\,d^5 - 81\,d^7 + 27\,d^2 - 9\,d^3$.
13. $17\,a^3y^4 - 34\,a^4y^3 + 68\,a^3$.

* If an error exists, checking by the substitution of particular numbers for the letters usually reveals the error, but not always. There is the possibility that some particular numbers will be hit upon which fail to reveal the error. For instance, $a^2 - b^2 \neq (a^2 - b)\,(a + b)$ (where $\neq$ means "is not equal to"), yet $a = 1$, $b = 2$ yields $1 - 4 = (1 - 2) \cdot 3$, or $-3 = -3$.

Again, $a = 2$, $b = 2$ yields $0 = 2 \cdot 0$, or $0 = 0$. Again $a = 0$, $b = 10$ yields $-100 = -100$. If the substitution yields an inequality, the factoring is certainly wrong; if the substitution yields an equality, the factoring is very probably correct, but the test is not infallible in this case.

14. $3x^2yz + 3xy^2z + 3xyz^2$.

15. $44m^3n^2 - 55m^3n + 66m^3r$.

16. $a^3b^3c^2 - a^3b^2c^3 + a^2b^3c^3$.

17. $125c^2d^2 + 100c^2d^3 - 150c^3d^2$.

18. $15b^3gh^2 + 17bg^2h^2 - b^2g^2h - 6bgh$.

19. $a^nb - a^nc - a^nd$.

20. $2x^a - 6bx^a + 4cx^a$.

21. $3a^nb^nc - 9a^nb^ncd + 6a^nb^nce$.

22. $cy^{2n} - dy^{2n} + gy^{2n}$.

23. $5a^2b^n + 10a^3b^n - 15a^4b^n$.

24. $8x^ay^b - 4y^b - 2x^ay^bz^c$.

116. II. Type form: $ax + ay + bx + by$.

Group the first two terms together and the last two terms together. Then factor each group. Notice that the factoring should reveal the *same* binomial or polynomial factor in all the groups.

We obtain

$$(ax + ay) + (bx + by) = a(x + y) + b(x + y).$$

Divide by $x + y$ and we obtain the quotient $a + b$.

Taking the divisor $x + y$ as one factor, and the quotient $a + b$ as the other factor, we have

$$ax + ay + bx + by = (x + y)(a + b).$$

Factor $2a - 6b - ad + 3bd$.

Grouping, $(2a - 6b) - (ad - 3bd)$.

Factoring each group, $2(a - 3b) - d(a - 3b)$.

The factor $a - 3b$ is seen to occur in both groups.

Dividing by $a - 3b$, we obtain $2 - d$.

Taking the divisor $a - 3b$ as one factor, and the quotient $2 - d$ as another factor, we have

$$2a - 6b - ad + 3bd = (a - 3b)(2 - d).$$

The groups may contain two or more terms, but all the groups must have the same polynomial factor. In grouping, the terms may be taken in any order which promises to reveal a common polynomial factor.

For example, $ab - c - bc + a$ may be written $ab - bc + a - c$. Then $ab - bc + a - c = b(a - c) + (a - c) = (a - c)(b + 1)$.

EXERCISES

117. Factor:

1. $a(x - y) + b(x - y)$.
2. $a(x + 2y) - b(x + 2y)$.
3. $x(c - a) + y(c - a)$.
4. $3(c - d) - z(c - d)$.
5. $a(3c + d) - b(3c + d) + c(3c + d)$.
6. $-2(a + x) + a(a + x) - e(a + x)$.
7. $b(a - c) - c(c - a)$.
8. $r(x - y) + s(x - y) + t(y - x)$.
9. $ax - ay - bx + by$.
10. $3x - 3y - dx + dy$.
11. $ac + ad - bc - bd$.
12. $xy - bx - ay + ab$.
13. $cd + 3c - 2d - 6$.
14. $6rs + 4r - 9s - 6$.
15. $y^3 + y^2 + y + 1$.
16. $a^2 - a - b + ab$.
17. $6bx - 12by - 50cy + 25cx$.
18. $3b - 5by^3 - 6by + 10by^4$.
19. $49ar - 14at + 21br - 6bt$.
20. $1 + a + b + ab$.
21. $ax + ay + az + bx + by + bz$.
22. $ax - a - cx + c - bx + b$.
23. $ax - ay + 2bx - 2by + cx - cy$.
24. $2a - 2b + 2c + ax - bx + cx + ay - by + cy$.

118. III. Type form: $\boldsymbol{a^2 + 2ab + b^2}$.

$$a^2 \pm 2ab + b^2 = (a \pm b)^2.$$

1. Factor $25x^4 - 60x^2y^3 + 36y^6$.

Here $\quad a^2 = 25x^4,\ b^2 = 36y^6.$

Hence $\quad a = 5x^2, \quad b = 6y^3,$

and $\quad -2ab = -2 \cdot 5x^2 \cdot 6y^3 = -60x^2y^3.$

This product $-60x^2y^3$ is the same as the middle term of the trinomial which we are factoring.

Hence that trinomial is factored by type form III, and we obtain —

$$25x^4 - 60x^2y^3 + 36y^6 = (5x^2 - 6y^3)^2.$$

2. Factor $4x^2 + 2x + 1$.

Here $a^2 = 4x^2,\ b^2 = 1.$

We obtain $a = 2x, \quad b = 1, \ +2ab = 4x.$

The product $4x$ is *not* the same as the middle term of the given trinomial.

Hence this trinomial cannot be factored by type form III.

The type form here considered enables one to factor any trinomial which is a perfect square.

EXERCISES

119. Factor and check by multiplying the factors together:

1. $m^2 + 2mn + n^2$.
2. $a^4 - 4a^2b + 4b^2$.
3. $a^4 - 6a^2b^2 + 9b^4$.
4. $x^2 + 24x + 144$.
5. $b^6 + 16 + 8b^3$.
6. $d^2 + 16d + 64$.
7. $\frac{1}{4} + y^2 - y$.
8. $\frac{1}{9}x^2 + \frac{1}{6}x + \frac{1}{16}$.
9. $36a^2 - 12ab^3 + b^6$.
10. $121e^8 - 22e^4f + f^2$.
11. $9y^2 + 6y + 1$.
12. $100 + 20h + h^2$.
13. $9w^2 - 18w + 9$.
14. $1 - 26b + 169b^2$.
15. $25 + 40a + 16a^2$.
16. $9c^2 + 54cd^3e^3 + 81d^6e^6$.
17. $16n^2 - 56np + 49p^2$.
18. $225t^2 - 120ty + 16y^2$.
19. $4m^2 + 64mn + 256n^2$.
20. $225s^2 - 60st^5 + 4t^{10}$.
21. $(a + b)^2 - 6(a + b) + 9$.
22. $(x - y)^6 + 8(x - y)^3 + 16$.
23. $36(r - s)^2 - 60(r - s) + 25$.
24. $9(a + b)^2 - 6(a + b)(c + d) + (c + d)^2$.
25. $4 - 12(a - x) + 9(a - x)^2$.
26. $x^2 + 2xy - 2xz + y^2 - 2yz + z^2$.
27. $c^2 + 2cd + 2ce + d^2 + 2de + e^2$.
28. $a^2 - 2ab + 4a + b^2 - 4b + 4$.
29. $4m^2 + 4mn + 4m + n^2 + 2n + 1$.
30. $g^2 + 6gh + 2g + 9h^2 + 6h + 1$.

120. IV. Type form: $a^2 - b^2$.

$$a^2 - b^2 = (a + b)(a - b).$$

Factor $81 x^6y^2 - 100 z^4$.

Here $a^2 = 81 x^6y^2,\ b^2 = 100 z^4$.

We obtain $a = \sqrt{81 x^6y^2} = 9 x^3y,\ b = \sqrt{100 z^4} = 10 z^2$.

Hence $81 x^6y^2 - 100 z^4 = (9 x^3y + 10 z^2)(9 x^3y - 10 z^2)$.

We see that every binomial which is the difference of two perfect squares is composed of two binomial factors.

RULE. (1) *Find the square root of each perfect square.*

(2) *Take the sum of the square roots as one factor, and*

(3) *The first square root minus the second square root as the other factor.*

EXERCISES

121. Factor:

1. $x^2 - y^2$.
2. $a^2 - 9$.
3. $1 - b^2$.
4. $4 - m^4$.
5. $y^6 - 25$.
6. $64 c^2 - z^2$.
7. $25 c^6 - 49 d^4$.
8. $81 - 121 x^{12}$.
9. $9 r^2 - 16 t^2$.
10. $100 p^2q^2 - 81 w^2$.
11. $36 x^{10} - 49 y^{12}$.
12. $144 w^2 - 25 x^4$.
13. $h^2 - 169$.
14. $4 a^2 - 1$.
15. $16 a^2b^2 - 25 c^2$.
16. $225 q^2 - 144 z^2$.
17. $100 u^2 - 256 v^4$.
18. $9 x^2y^2 - 121$.
19. $49 c^6 - 64 t^8$.
20. $289 a^2b^2c^2 - 25 d^2$.
21. $(x + y)^2 - 1$.
22. $(a + 3 b)^2 - c^2$.
23. $(x - 2 y)^2 - z^2$.
24. $(m + 6 n)^2 - 4 r^2$.
25. $(7 d - r)^2 - 25 s^2$.
26. $a^2 + 2 ab + b^2 - 4 c^2$.
27. $x^2 - 2 xy + y^2 - z^2$.
28. $c^2 - 6 cd + 9 d^2 - e^2$.
29. $b^2 - 18 b + 81 - 49 s^2$.
30. $d^2 - 12 d + 36 - 25 t^2$.
31. $1 - (a + b)^2$.
32. $16 - (x - y)^2$.
33. $9 - (m - 3 n)^2$.
34. $k^2 - (h + 10)^2$.
35. $a^2 - b^2 - 2 bc - c^2$
$= a^2 - (b^2 + 2 bc + c^2)$
$= a^2 - (b + c)^2$.
36. $t^2 - m^2 + 2 mn - n^2$.

37. $49 - c^2 - 6\,cd - 9\,d^2$.

38. $81\,p^2 - 36\,q^2 + 12\,q - 1$.

39. $100\,s^2 - 16\,h^2 - 16\,h - 4$.

40. $(a+b)^2-(c+d)^2=[(a+b)+(c+d)][(a+b)-(c+d)]$
$=(a+b+c+d)(a+b-c-d)$.

41. $(x+2\,y)^2-(m-n)^2$.

42. $9(c-d)^2-4(x-y)^2$.

43. $64(h+l)^2-49(r-t)^2$.

44. $144(m-x)^2-121(n+y)^2$.

45. $36(y-z)^2-196(w+x)^2$.

46. $c^2+2\,cd+d^2-m^2+2\,mr-r^2$.

122. V. Type form: $\boldsymbol{x^2+gx+h}$.

$$x^2+gx+h=(x+a)(x+b),$$

wherein $g=a+b$ and $h=ab$.

1. Factor $x^2-5\,x+6$.

Here $g=-5,\ h=6.\ \therefore a+b=-5,\ ab=6$.

By trial, find two numbers whose sum is -5, and whose product is 6. These numbers are -3 and -2.

Hence $x^2-5\,x+6=(x-2)(x-3)$.

2. Factor $x^2-2\,x-35$.

Here $g=2,\quad h=-35$.

That is, $a+b=2,\ ab=-35$.

By trial, find two numbers whose sum is $+2$ and whose product is -35.

These numbers are $+7$ and -5.

Hence $x^2+2\,x-35=(x+7)(x-5)$.

3. Factor $x^2+5\,x-7$.

Here $g=5,\qquad h=-7$.

$a+b=5,\qquad ab=-7$.

We cannot find two integers whose sum is 5 and whose product is -7. Hence this trinomial cannot be factored by this type form.

ORAL EXERCISES

123. Factor:

1. $x^2+5\,x+6$.
2. x^2-x-6.
3. x^2+x-6.
4. $x^2+9\,x+20$.

5. $x^2 - x - 20$.
6. $x^2 - 9x + 20$.
7. $x^2 + x - 20$.
8. $x^2 - x - 12$.
9. $x^2 + 9x + 14$.
10. $x^2 - 3x - 18$.
11. $x^2 + 14x + 33$.
12. $x^2 - 16x - 36$.
13. $x^2 + 14x + 45$.
14. $x^2 + 14x + 24$.
15. $x^2 - x - 72$.
16. $x^2 - 16x + 28$.
17. $a^2 - 3a - 10$.
18. $m^2 + 7m + 12$.
19. $a^3 + 3a + 2$.
20. $a^2 - a - 2$.
21. $d^2 + 7d + 10$.
22. $d^2 - 14d - 15$.
23. $f^2 - 15f + 26$.
24. $g^2 + 9g + 8$.
25. $h^2 + 10h - 39$.
26. $h^2 - 11h - 60$.
27. $k^2 - 12k + 32$.
28. $k^2 - 3k - 54$.
29. $k^2 + 27k - 90$.
30. $l^2 - 25l + 100$.
31. $m^2 + 9m - 112$.
32. $r^2s^2 + 12rs + 20$.
33. $a^2b^2 + 16ab + 55$.
34. $x^2y^2 - xy - 380$.
35. $c^2d^2 - 12cd - 13$.
36. $(x+y)^2 - 7(x+y) - 18$.

124. VI. Type form: $px^2 + qx + r$.

$$px^2 + qx + r = (ax + b)(cx + d),$$

wherein $p = ac$, $q = bc + ad$, $r = bd$.

1. Factor $6x^2 - x - 12$.

Here $p = 6$, $q = -1$, $r = -12$.

If possible, we must find numbers a, b, c, and d, such that $ac = 6$, $bd = -12$, $bc + ad = -1$.

Here, $ac = 6$ suggests the values $a = 3$, $c = 2$, or $a = 1$, $c = 6$, etc., and $bd = -12$ suggests the values $b = 4$, $d = -3$, or $b = -6$, $d = 2$, etc.

Try the various sets of values of a and c, whose product is 6, with each pair of values of b and d, whose product is -12, until a combination is found which satisfies $bc + ad = -1$; that is, a combination which gives $-x$ as the middle term of the product.

Try the sets of values, $a = 3$, $c = 2$, $b = 4$, $d = -3$.

$$+8\,x$$

Examine $$(3\,x+4)(2\,x-3).$$

$$-9\,x$$

We see that the sum of the two cross products $+8\,x$ and $-9\,x$ is $-x$, which is the middle term of $6\,x^2-x-12$.

Hence $$6\,x^2-x-12=(3\,x+4)(2\,x-3).$$

2. Factor $8\,x^2-34\,x+35$.

Try $a=8$, $c=1$, $b=-7$, $d=-5$.

$$-7\,x$$

Examine $$(8\,x-7)(x-5).$$

$$-40\,x$$

The sum of the cross products $-40\,x$ and $-7\,x$ is $-47\,x$.
But the middle term in $8\,x^2-34\,x+35$ is not $-47\,x$.
Hence $(8\,x-7)(x-5)$ are *not* factors of $8\,x^2-34\,x+35$.
Try $a=4$, $c=2$, $b=-7$, $d=-5$.

$$-14\,x$$

Examine $$(4\,x-7)(2\,x-5).$$

$$-20\,x$$

The sum of the cross products $-20\,x$ and $-14\,x$ is $-34\,x$.
$-34\,x$ is the middle term of $8\,x^2-34\,x+35$.

Hence $$8\,x^2-34\,x+35=(4\,x-7)(2\,x-5).$$

EXERCISES

125. Factor:

1. $6\,x^2+7\,x+2$.
2. $6\,x^2+13\,x+5$.
3. $12\,x^2+7\,x-5$.
4. $7\,x^2+11\,x+4$.
5. $3\,x^2+7\,x+2$.
6. $20\,x^2+33\,x+7$.
7. $2\,a^2-27\,a+13$.
8. $8\,m^2-31\,m+30$.
9. $10\,b^2-37\,b+30$.
10. $6\,y^2-31\,y+35$.
11. $6\,t^2-19\,t+10$.
12. $3\,s^2+13\,s-10$.
13. $24\,s^2+s-3$.
14. $30\,y^2+19\,y-5$.
15. $8\,m^2-22\,m-21$.
16. $3\,m^2-m-2$.

17. $15\,m^2+14\,m-8$.
18. $18\,a^2+9\,a-2$.
19. $6\,c^2+35\,c-6$.
20. $6\,d^2-13\,d-5$.
21. $11\,g^2-21\,g-2$.
22. $2\,g^2+gh-15\,h^2$.
23. $3\,a^2+11\,ab-4\,b^2$.
24. $3\,a^2-7\,ab-6\,b^2$.

126. VII. Type forms: $a^3 \pm b^3$.

$$(a^3+b^3)=(a+b)(a^2-ab+b^2).$$
$$(a^3-b^3)=(a-b)(a^2+ab+b^2).$$

Factor x^3+27.

$$a^3=x^3,\ b^3=27;\ a=x,\ b=3.$$
$$\therefore x^3+27=(x+3)(x^2-3\,x+9).$$

Again, factor $125\,m^3-8$.

Here $a^3=125\,m^3,\ b^3=8;\ a=5\,m,\ b=2.$

$$\therefore 125\,m^3-8=(5\,m-2)(25\,m^2+10\,m+4).$$

Factor:

1. x^3+y^3.
2. x^3-y^3.
3. m^3+1.
4. m^3-1.
5. a^3b^3+8.
6. c^3d^3-27.
7. $64\,a^3-1$.
8. $125\,b^3+216$.
9. $8\,g^3-27\,h^3$.
10. x^3y^3+343.
11. x^6+y^6.
12. $8\,a^6-b^6$.
13. $1-27\,a^9$.
14. $m^{12}+n^{12}$.
15. $27\,c^{15}+1$.
16. $8\,a^{3n}-1$.
17. $b^{9n}+c^{6n}$.
18. x^9+y^9.
19. $64\,a^3-125\,b^3$.
20. $216-27\,x^{18}$.

The foregoing are the simpler type forms of factoring in algebra; more complicated ones are given in more advanced texts on algebra.

127. Skill in factoring depends upon an easy recognition of the type forms and a facility in applying correctly the method of each one.

Some suggestions will be helpful.

I. See if the terms contain a common monomial factor. If so, divide the expression by it, and keep it as one factor; the quotient will be the other factor.

II. Determine to which type form the quotient thus obtained belongs and factor accordingly.

III. Inspect each factor and, if possible, resolve it into new factors.

TYPE FORMS AND FACTORS

128. I. $ay + by - cy = y(a + b - c)$.

II. $ax + ay + bx + by = a(x+y) + b(x+y) = (x+y)(a+b)$.

III. $a^2 \pm 2\,ab + b^2 = (a \pm b)^2$.

IV. $$\begin{cases} a^2 - b^2 = (a + b)(a - b). \\ a^2 + 2\,ab + b^2 - c^2 = (a+b)^2 - c^2 = (a+b+c)(a+b-c). \\ a^2 + 2\,ab + b^2 - c^2 - 2\,cd - d^2 = (a + b)^2 - (c + d)^2 \\ \qquad = (a + b + c + d)(a + b - c - d). \end{cases}$$

V. $x^2 + gx + h = (x + a)(x + b)$; $g = a + b$, $h = ab$.

VI. $px^2 + qx + r = (ax + b)(cx + d)$; $p = ac$, $q = bc + ad$, $r = bd$.

VII. $$\begin{cases} a^3 + b^3 = (a + b)(a^2 - ab + b^2). \\ a^3 - b^3 = (a - b)(a^2 + ab + b^2). \end{cases}$$

REVIEW EXAMPLES

129. Factor:

1. $a^3 + 2\,a^2 + a$.
2. $2\,x^4 + 12\,x^2 + 18$.
3. $9\,y^4 - 24\,y^6 + 16\,y^8$.
4. $b^3 - b$.
5. $x^4 - 16$.
6. $a^2 + 6\,a - 7$.
7. $a^2 + 16\,a + 15$.
8. $ax - 2\,x - ay + 2\,y$.
9. $a^2m - b^2m - a^2n + b^2n$.
10. $c^3 + c^2 - c - 1$.
11. $x^6 - 6\,x^3 - 7$.
12. $k^4 - 17\,k^2 + 72$.

13. $64 - x^6$.

14. $h^3 - 7h^2 - 2h + 14$.

15. $81h^4 - k^4$.

16. $x^2 - 2xy + y^2 - z^2$.

17. $3m^4 - 8m^2 - 35$.

18. $y^3 - 64$.

19. $L^2 + L + \frac{1}{4}$.

20. $x^3 + 2x^2 - 4x - 8$.

21. $(a+b)(a^2+b^2) + (a+b)(2ab)$.

22. $2a^3 - 14a^2 + 70 - 10a$.

23. $(a+b)^2 + 2(a+b) + 1$.

24. $a^2 + b^2 - 2ab - 9$.

25. $c^2 - a^2 + 2ab - b^2$.

26. $1 + x^{12}$.

27. $r^2 + 3r - 154$.

28. $x^2 + 11x - 210$.

29. $a^{12} - b^{12}$.

30. $2s^2 - 5s + 2$.

31. $7a^2 + 175 - 70a$.

32. $18 - 6x - 12x^2$.

33. $17w^2 + 25w - 18$.

34. $9x^{2a} - 25$.

35. $a^2 - ac + 2ab - 2bc$.

36. $c^{16} - 216c^7$.

37. $a^2 - 4b^2 + 4bc - c^2$.

38. $a^2 + m^2 - n^2 - 2am$.

39. $3x^2 - 3ax + 3x - 3a$.

40. $7c^2 + 20c + 13$.

41. $e^3 + 512f^3$.

42. $6y^2 - 56z^2 + 41yz$.

43. $a^2 - 12ab + 36b^2 - t^2$.

44. $1 - 9x^2 - 30xy - 25y^2$.

45. $9c^4 - 16c^2d^2 + 7d^4$.

46. $9m^4 - 13m^2n^2 + 4n^4$.

47. $a^2 - b^2 - a - b$.

48. $a^3 + b^3 + a^2 - b^2$.

49. $4a^2 - 4ab + b^2 - 9c^2 - 24cd - 16d^2$.

50. $(x+y)^2 - 5(x+y) + 6$.

51. $a^5xy - ab^4xy$.

52. $4x^{5n} - x^n$.

53. $49a^{2x} - 84a^xb^y + 36b^{2y}$.

54. $25x^6 - 61x^3y^3 + 36y^6$.

55. $x^6 + 729y^6$.

56. $196r^6(y-z)^2 - 225s^2$.

57. $4x^2 + y^2 + 1 + 4xy + 4x + 2y$.

58. $21c - 5d + 3cd - 2de - 14e - 35$.

59. $(x^2-1)^2 - (y^2-1)^2$.

60. $49y^2 - 70yz + 25z^2$.

CHAPTER VIII

EQUATIONS SOLVED BY FACTORING

130. The degree of an equation in one unknown is the same as the *highest power* of the unknown in the equation.

An equation of the *first* degree is a *linear* equation; one of the *second* degree is a *quadratic* equation; one of the *third* degree a *cubic* equation; one of the *fourth* degree a *quartic* equation; etc.

$ax + b = 0$ is a linear equation.

$ax^2 + bx + c = 0$ is a quadratic equation.

$ax^3 + bx^2 + cx + d = 0$ is a cubic equation.

$ax^4 + bx^2 + c = 0$ is a quartic equation.

An equation of second, third, fourth, or higher degree may sometimes be solved by the methods of factoring which we have learned.

The solution of such equations by the method of factoring depends upon the principle that if the product of two or more factors equals zero, at least *one* of the factors must equal zero.

131. 1. Solve $x^2 - 7x = -12$.

Transposing -12, *so that the right-hand member of the equation may be* 0,

$$x^2 - 7x + 12 = 0.$$

Factoring, $$(x-4)(x-3) = 0.$$

Since the *product* of the factors $(x-4)(x-3)$ is equal to zero, one or the other of the factors must be equal to zero.

When $$x - 4 = 0,$$

we obtain the root $$x = 4.$$

When $x - 3 = 0$,

we obtain the other root, $x = 3$.

Hence the given quadratic equation has the two roots, 3 and 4.

Check: If $x = 4$, $x^2 - 7x = -12$ becomes

$$16 - 28 = -12.$$
$$-12 = -12.$$

If $x = 3$, $x^2 - 7x = -12$ becomes

$$9 - 21 = -12.$$
$$-12 = -12.$$

2. Solve $x^3 - x = 2x^2 - 2$.

Transpose, $x^3 - x - 2x^2 + 2 = 0$.

Factor, $x(x^2 - 1) - 2(x^2 - 1) = 0$.

$$(x^2 - 1)(x - 2) = 0.$$
$$(x + 1)(x - 1)(x - 2) = 0.$$

If $x + 1 = 0$, $x = -1$.

If $x - 1 = 0$, $x = 1$.

If $x - 2 = 0$, $x = 2$. *Ans.* $-1, 1, 2$.

Check: If $x = -1$, $-1 + 1 = 2 - 2,\ 0 = 0$.

If $x = 1$, $1 - 1 = 2 - 2,\ 0 = 0$.

If $x = 2$, $8 - 2 = 8 - 2,\ 6 = 6$.

Rule. *Transpose all the terms to the left side so that the right side may be zero., Factor the left side. Make each factor containing the unknown equal to zero and solve each resulting equation.*

If there is a factor that does not contain the unknown, divide both members of the equation by it. Thus, $7x^2 + 21x - 63 = 0$ has the factor 7. Dividing both sides of the equation by it, the simplified equation is $x^2 + 3x - 9 = 0$.

132. Observe that a *second*-degree equation will produce *two* factors, a *third*-degree equation *three* factors, and so on. There will be as many roots as linear factors containing the unknown, and hence an equation has as many roots as the highest power of the unknown in it.

Some of the roots may be alike, as in

$$x^2 - 4x = -4.$$
$$x^2 - 4x + 4 = 0.$$
$$(x - 2)(x - 2) = 0.$$
$$x - 2 = 0,\ x = 2.$$
$$x - 2 = 0,\ x = 2. \quad \textit{Ans. } 2,\ 2.$$

This is called a *double* root.

In solving an equation care must be taken not to discard a factor which contains the unknown quantity; by so doing, a root will be lost.

Solve $x^2 - 2x = 0$.

Divide both sides by x, $x - 2 = 0$; $x = 2$, *one* root.

But by factoring, $x(x - 2) = 0$.

Whence $x = 0$; $x = 2$, *two* roots.

EXERCISES

133. Solve and check:

1. $x^2 - 25 = 0$.
2. $x^2 = 36$.
3. $x^2 = 4x$.
4. $x^2 = ax$.
5. $x^2 - 7x = -10$.
6. $x^2 - 3x = 10$.
7. $2x^2 - 14x = 0$.
8. $5x^2 = 15x$.
9. $x^2 + x = 72$.
10. $x^2 - 2ax = -a^2$.
11. $2x^2 = 3x$.
12. $bx^2 - ax = 0$.
13. $cx^2 = a^2c$.
14. $2x^2 + 6 = 7x$.
15. $12x^2 + x = 6$.
16. $25x^2 - 70x = -49$.
17. $25x^2 - 25x + 6 = 0$.
18. $x^2 - ax - bx + ab = 0$.
19. $x^2 - ax - bx = 0$.
20. $6x^2 - 3ax - 2cx = -ac$.
21. $x^3 = 4x$.
22. $x^3 + 8 = 2x^2 + 4x$.
23. $x^3 - 3x^2 + 2x = 0$.
24. $x^3 + 20x = 9x^2$.

PROBLEMS

134. Solve the following problems and see which roots of the equations satisfy the conditions of the problem:

1. What number is it whose square plus the number itself equals 42?

2. The product of a certain number decreased by 2 and the same number decreased by 3 equals 20. Find the number.

3. If to the square of a certain number, the number itself and 5 be added, the result will be 115. Find the number.

4. If a certain number be subtracted from 19 and 21, in turn, the product of the remainders will be 80. What is the number?

5. The difference between two numbers is 3 and the sum of their squares is 89. What are the numbers?

6. A rectangular field is 5 rd. longer than it is wide. If each dimension be increased by 1 rd., its area would be 50 sq. rd. What is its present area?

Let x rd. = the width.

Then $(x+5)$ rd. = the length.

$(x+1)$ rd. = the width increased by 1 rd.

$(x+6)$ rd. = the length increased by 1 rd.

We obtain the equation $(x+1)(x+6)=50$.

$$x^2+7x+6=50.$$

$$x^2+7x-44=0.$$

$$(x-4)(x+11)=0.$$

The roots of the equation are $+4$ and -11.

We discard the root -11 as being inapplicable to our problem, on the ground that we do not speak of a rectangle having a *negative* length and a *negative* breadth; the problem involves the condition that only the positive roots shall be used.

7. If the length of a certain square be increased 3 ft. and the width decreased 2 ft., the area would not be changed. What are the dimensions of the square?

8. By how many inches must the length, 38 in., and the width, 22 in., of a rectangle be increased, in order to increase the area by 189 sq. in.?

9. How may \$84 be divided among a number of persons so that each person shall receive \$5 more than the number of persons?

10. Is the difference of the squares of two consecutive numbers even or odd? Prove your answer.

11. The sum of the squares of two consecutive numbers is 85. What are the numbers?

12. The sum of the squares of two consecutive odd numbers is 202. What are the numbers?

13. A schoolroom contains 48 seats, and the number in a row is 4 less than twice the number of rows. How many seats in a row?

14. If a cubical box has its dimensions increased by 1, 2, 3 inches respectively, the capacity of the box has been increased 93 cu. in. What are the dimensions?

15. A rectangular yard has a fence around it measuring 100 ft.; the area of the yard is 600 sq. ft. How does its width compare with its length?

16. The surface of a cube contains 384 sq. in. Find its dimensions.

The *altitude* of a triangle is its height, or the length of a perpendicular line from its vertex to its base.

The *area* of a triangle equals one half its altitude multiplied by its base. Area $= \frac{1}{2}ab$.

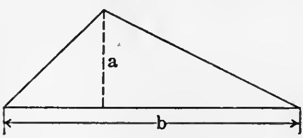


Fig. 19

17. If the area of a triangle is 36 sq. in. and the base is 12 in., find the altitude.

18. The base of a triangle is 3 times the altitude and the area is 24 sq. ft. Find the base.

19. The area of a triangle is 35 sq. ft. and the base is 3 ft. more than the altitude. Find the altitude and the base.

The hypotenuse of a right triangle is its longest side. The square on the hypotenuse is equal to the sum of the squares on the legs.

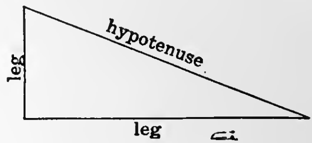


FIG. 20

20. One leg of a right triangle is 3 ft. longer than the other, and the area is 54 sq. ft. Find each leg and the hypotenuse.

21. The sides of a certain right triangle are in the ratio 3 : 4 : 5. If the perimeter is 72 in., what is its area?

22. The diagonal of a square is 50 in. What is its perimeter?

23. The diagonal of a rectangle is 640 in. If the length of the rectangle is three times its width, what is its area?

24. The area of a rectangle is 48 sq. ft. If its length is 2 ft. more than the width, what will its diagonal be?

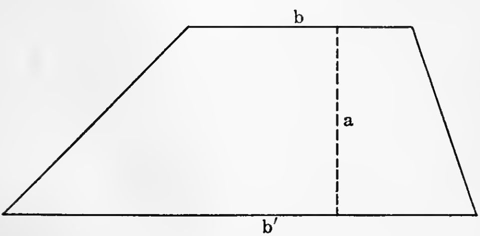


FIG. 21

The area of a trapezoid is equal to one half the altitude times the sum of the bases.

$$\text{Area} = \frac{a(b + b')}{2}.$$

25. The altitude of a trapezoid is 12 in., one base is 3 in. shorter than the other; the area is 162 sq. in. How long is the other base?

26. One base of a trapezoid is 2 ft. longer than the other and the altitude is 1 ft. longer than the longer base; the area is 120 sq. ft. Find the bases and the altitude.

The length of a circle (circumference), $C = 2\pi R$ (wherein π (*pī*) is a Greek letter standing for 3.1416, or $\frac{22}{7}$, approximately).

The area included by a circle $A = \pi R^2$.

27. If a circle is 88 in. long, what is its radius?

28. Over how much ground may a horse graze if tied to a stake by a 30-foot rope?

FRANÇOIS VIETA

Vieta's Notation	*Modern Notation*
$1\,C - 8\,Q + 16\,N$ aequ. 40;	$x^3 - 8\,x^2 + 16\,x = 40.$

C stands for *cubus* or "cube" of the unknown.
Q stands for *quadratus* or "square" of the unknown.
N stands for *numerus* or the unknown itself.
aequ. stands for *aequalis* or *equal*.

CHAPTER IX

MULTIPLICATION AND DIVISION OF POLYNOMIALS

REVIEW OF MULTIPLICATION

135. Arrange both polynomials that are to be multiplied together, according to the descending or ascending power of some letter. Multiply *each* term of one polynomial by *each* term of the other, and add the partial products.

Thus, find the product of:

1. $2a^2 + 3a - 5$ and $3a^2 - 2a + 1$.
2. $3a^3 - a^2 + 4a - 2$ and $a^2 + a - 3$.
3. $3x^2 - x + 4$ and $2x^2 + 3x - 5$.
4. $x^2 + xy + y^2$ and $x - y$.
5. $2x^2 - 5xy + 2y^2$ and $2x - 3y$.
6. $x^4 - x^2 + 1$ and $x^4 + x^2 + 1$.
7. $1 - a - 2a^2 + a^4$ and $1 - 2a + a^2$.
8. $c^6 - 2c^3 + 1$ and $c^2 - 2c + 1$.
9. $x - y$ and $x^3y + xy^3 + y^4 + x^2y^2 + x^4$.
10. $1 + 5a^3 - 6a^4$ and $3a^2 + 1 - a$.

Simplify:

11. $(m^4 - 9m^2 + 12m - 4)(m^2 + 3m - 2)$.
12. $(3r^3 - 2r + 7)(5r^4 + 3r^2 - 1)$.
13. $(a^3 + a^2b + ab^2 + b^3)(a - b)(a^4 + b^4)$.
14. $(x^2 + y^2 + z^2 - xy - xz - yz)(x + y + z)$.
15. $(x^2 + 3x - 2)(4x^2 - 5x + 1)(2x^2 - x - 3)$.

DIVISION OF POLYNOMIALS

136. We learned (§ 76) to divide a polynomial by a binomial. This process may be extended to cases in which the divisor is itself a polynomial.

(1) Divide $6x^4-5x^3-7x^2+8x-2$ by $2x^2-3x+1$.

$$\begin{array}{rrrrr|l}
6x^4 - 5x^3 & - 7x^2 & + 8x & - 2 & & 2x^2-3x+1 \\ \cline{6-6}
\underline{6x^4 - 9x^3} & \underline{+ 3x^2} & & & & 3x^2+2x-2 \\
4x^3 & - 10x^2 & + 8x & - 2 & & \\
\underline{4x^3} & \underline{- 6x^2} & \underline{+ 2x} & & & \\
 & - 4x^2 & + 6x & - 2 & & \\
 & \underline{- 4x^2} & \underline{+ 6x} & \underline{- 2} & &
\end{array}$$

There is no remainder; the division is exact. The quotient is $3x^2+2x-2$.

(2) Divide $16+5a^4-3a^3-32a+36a^2$ by a^2+4-4a.

$$\begin{array}{rrrrr|l}
16 & - 32a & + 36a^2 & - 3a^3 & + 5a^4 & 4-4a+a^2 \\ \cline{6-6}
\underline{16} & \underline{- 16a} & \underline{+ 4a^2} & & & 4-4a+4a^2 \\
 & - 16a & + 32a^2 & - 3a^3 & + 5a^4 & \\
 & \underline{- 16a} & \underline{+ 16a^2} & \underline{- 4a^3} & & \\
 & & 16a^2 & + a^3 & + 5a^4 & \\
 & & \underline{16a^2} & \underline{- 16a^3} & \underline{+ 4a^4} & \\
 & & & 17a^3 & + a^4 &
\end{array}$$

There is a remainder, and the division is "not exact." The quotient is

$$4-4a+4a^2+\frac{17a^3+a^4}{4-4a+a^2}.$$

137. Emphasis must be placed upon the necessity of *arranging both polynomials, and each remainder, according to the ascending or the descending power of some letter.*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the entire divisor by the first term of the quotient.

Subtract the product from the dividend.

Treat the remainder as a new dividend and proceed as before.

To check an example in division:

I. Multiply the quotient by the divisor, add the remainder to this product. The result should be the dividend; or

II. Substitute particular numbers for the letters and divide the resulting values.

To illustrate the second method of checking: in our first division, let $x = 2$; then

$$6x^4 - 5x^3 - 7x^2 + 8x - 2 = 42.$$
$$2x^2 - 3x + 1 = 3.$$
$$3x^2 + 2x - 2 = 14.$$
$$\text{Then, } 42 \div 3 = 14.$$

Hence the division is correct. In this mode of checking, care must be taken to select numbers for the letters which will not reduce the divisor to zero.

EXERCISES

138. Divide and check:

1. $x^4 - 9x^3 + 19x^2 - 25x + 6$ by $x^2 - 2x + 3$.
2. $6a^4 - a^3b - 10a^2b^2 + 31ab^3 - 20b^4$ by $3a^2 + 4ab - 5b^2$.
3. $8a^4 + 8a^3b - 28a^2b^2 + 53ab^3 - 21b^4$ by $4a^2 - 6ab + 7b^2$.
4. $x^5 - x^4 + 4x^3 - 3x^2 + 5x - 6$ by $x^3 + x - 2$.
5. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
6. $10a^5 + 11a^4 - 39a^3 + 42a^2 - 17a + 5$ by $2a^3 + 3a^2 - 7a + 5$.
7. $r^8 + r^7 - r^6 - r^5 + r^3 + r^2 - 1$ by $r^4 - r^2 + 1$.
8. $a^4 + a^2b^2 + b^4$ by $a^2 + b^2 - ab$.
9. $x^5 - y^5$ by $x - y$.
10. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

Simplify:

11. $(x^2 + 2x + 2)(x^2 - 4x - 5) \div (x - 5)$.
12. $(x + y)^2(x^2 - xy + y^2)$.
13. $[(a^3 + b^3) \div (a^2 - ab + b^2)][a^2 + ab + b^2]$.
14. $(a - b)^3(a + b)^2 \div (a^2 - b^2)$.
15. $[(x + y)^3 - (x - y)^3] \div (3x^2 + y^2)$.

I

CHAPTER X

SQUARE ROOT

139. The **square root** of an expression is one of the *two equal* factors of the expression.

Since $$(a+b)^2 = a^2 + 2ab + b^2,$$
$$\sqrt{a^2 + 2ab + b^2} = a + b.$$

Thus it is a simple matter, by inspection, to extract the square root of a trinomial, which is seen to be of the form $a^2 + 2ab + b^2$.

Or we may find the square root in this way:

$$\begin{array}{lr|l}
 & & a^2 + 2ab + b^2 \quad \underline{|a+b} \\
 & & a^2 \\
\hline
\text{Trial divisor,} & 2a & 2ab + b^2 \\
\text{Complete divisor,} & 2a + b & \underline{2ab + b^2}
\end{array}$$

Thus the square root is $a + b$.

The study of this simple case enables one to devise a rule which is applicable to more complicated cases. The procedure is as follows:

I. Extract the square root of the first term and subtract its square from the polynomial, leaving $2ab + b^2$.

II. We see in $2ab$ the factor b which we know is the second term of the root. In cases when b is not known, we see that b may be obtained by dividing $2ab$ by $2a$. We call $2a$ the trial divisor. Since a is the part of the root already found, we see that the *trial divisor is double the root already found.* After b has been found, add it to the trial divisor and we have $2a + b$, the complete divisor.

III. Multiply $2a + b$ by b, and subtract.

This process may now be extended to finding the square root of any polynomial.

140. The extraction of the square root of $49x^4 - 42x^3 + 79x^2 - 30x + 25$ is as follows:

	$49x^4-42x^3+79x^2-30x+25$	$\underline{\mid 7x^2-3x+5}$
	$\underline{49x^4}$	
1st trial divisor, $2(7x^2)=14x^2$	$-42x^3+79x^2-30x+25$,	1st remainder
1st complete divisor, $14x^2-3x$	$\underline{-42x^3+9x^2}$	
2d trial divisor, $2(7x^2-3x)=14x^2-6x$	$+70x^2-30x+25$,	2d remainder
2d complete divisor, $14x^2-6x+5$	$\underline{+70x^2-30x+25}$	
	0	3d remainder

The process is as follows:

(1) The square root of $49x^4$ is $7x^2$. Here $a = 7x^2$; $2a = 14x^2$, the first trial divisor.

(2) The first term in the first remainder is $-42x^3$. Take $-42x^3 = 2ab$.

(3) To find b, divide $-42x^3$ by $14x^2$. We obtain $b = -3x$.

(4) Multiply the first complete divisor by $-3x$ and subtract. The second remainder is what is left after subtracting $(7x^2 - 3x)^2$ from the given polynomial.

(5) Proceed with the second remainder as you did with the first. Let $7x^2 - 3x$ be the new value of a.

(6) Divide the first term of the second remainder by the first term of the second trial divisor; you obtain $+5$.

(7) Multiply the second complete divisor by $+5$, and subtract. The third remainder is what is left over after subtracting $(7x^2 - 3x + 5)^2$ from the given polynomial. As this remainder is zero, $(7x^2 - 3x + 5)^2$ is equal to that polynomial.

Hence $7x^2 - 3x + 5$ is the required square root.

EXERCISES

141. Find the square root of:

1. $a^4 - 4a^3 + 6a^2 - 4a + 1$.
2. $x^4 - 6x^3 + 13x^2 - 12x + 4$.
3. $c^6 + c^4 - 26c^3 + 6c^5 + 8c + 10c^2 + 1$.
4. $4x^8 - 12x^6 - 15x^4 + 36x^2 + 36$.

5. $25x^6 - 20x^5y + 34x^4y^2 - 32x^3y^3 + 17x^2y^4 - 12xy^5 + 4y^6.$

6. $9c^4d^2 - 18c^3d^3 - 3c^2d^4 + 12cd^5 + 4d^6.$

7. $m^4n^4 + 16r^4 + m^2n^2r^2 - 6m^3n^3r + 24mnr^3.$

8. $x^4 - \frac{2x^3}{3} + \frac{x^2}{9}.$

9. $\frac{a^2}{4} - \frac{ab}{3} + a + \frac{b^2}{9} - \frac{2b}{3} + 1.$

10. $4x^2 - 20x - \frac{5}{x} + \frac{1}{4x^2} + 27.$

SQUARE ROOT OF ARITHMETICAL NUMBERS

142. $1^2 = 1$, $2^2 = 4$, $\cdots$ $9^2 = 81$, $10^2 = 100$, $90^2 = 8100$, $100^2 = 10{,}000$, $1000^2 = 1{,}000{,}000$.

This is sufficient to show that a perfect square consisting of *two* digits has *one* digit in the root; one consisting of *three* or *four* digits has *two* digits in the root; one consisting of *five* or *six* digits has *three* digits in the root; and so on. In other words, the square of a number has twice as many digits, or one less than twice as many, as the number itself.

Hence, to find the square root we must separate the number into *periods* of *two* digits each, *towards the left and right from the decimal point.* The period farthest to the left may have one or two digits, whereas the one farthest to the right must have two, a cipher being annexed, if necessary, to complete it.

The square root has therefore one digit corresponding to every period in its square. Separating the square number into periods enables one to find one digit in the root at a time.

This is seen more clearly if we consider that the square of any number of tens, say 4 tens (40), ends in two ciphers (1600); hence the two digits on the right are not needed to find the tens' digit (4), and are set aside until the units' digit is to be found.

Likewise, the square of any number of hundreds, say 4 hundreds (400), ends in four ciphers (160,000), and all of these may be set aside until the hundreds' digit is found and they are needed for the tens' and units' digits.

After pointing off the number into periods, the method is similar to the one used for polynomials. The procedure is based on the formula $a^2 + 2ab + b^2 = (a + b)^2$.

When dividing by the trial divisor, *exclude, for brevity, the right-hand digit in the remainder.*

The reason for this is seen in the example which follows; the trial divisor 2 is equal to 20 of the next lower units. Now $12 \div 2$ gives the same digit in the root as $127 \div 20$.

Find the square root of 2272254.76.

	$\dot{2}2\dot{7}2\dot{2}5\dot{4}.\dot{7}6$	1507.4	
	1		
1st trial divisor = 2(1) = 2	127		1st remainder
1st complete divisor = 25	125		
2d trial divisor = 2(15) = 30	22254		2d remainder
3d trial divisor = 2(150) = 300			
3d complete divisor = 3007	21049		
4th trial divisor = 2(1507) = 3014	120576		3d remainder
4th complete divisor = 30144	120576		

Ans. 1507.4.

When the 2d trial divisor (30) will not go into 22, a cipher is placed in the root; the next trial divisor becomes 2(150) or 300 and the next period is brought down. Then 2225 is divided by 300.

143. The square root of a fraction may be found by taking the square root of the numerator and of the denominator separately, or by reducing the fraction to an equivalent decimal and taking the square root of the decimal. Unless the denominator of the fraction is a perfect square, the second method of procedure is far better.

EXERCISES

144. Find the square root of:

1. 3249.	3. 51529.	5. 248004.
2. 6084.	4. 93636.	6. 118.1569.

7. 1010025.

8. 10.4976.

9. .00564001.

10. 64.1601.

11. .01522756.

12. 1459.24.

13. 420.496036.

14. 7753607.3209.

Find, to *three* decimal places, the square root of:

15. 2. 16. 3. 17. 5. 18. $\frac{2}{3}$. 19. $\frac{4}{5}$. 20. $\frac{8}{9}$.

CHAPTER XI

GRAPHS OF SIMULTANEOUS EQUATIONS

145. The plotting of linear equations of the form $ax + by = c$ was explained in § 90.

In the equation $x + 0y = 2$, y may have any value; but x must equal 2. Consequently such points as (2, 0), (2, -3), (2, 2), (2, 5), (2, -1), (2, -2) would lie on the graph. Thus the graph must be a line parallel to the y-axis, two units to the right.

EXERCISES

146. Represent equations 1–8 graphically:

1. $x = 4$.	**3.** $x = -7$.	**5.** $x = 0$.	**7.** $y = \pm 2$.
2. $y = -1$.	**4.** $y = 3$.	**6.** $y = 0$.	**8.** $x = \pm 3$.

9. Is the point (3, 4) on the graph of the equation $2x - y = 7$? Can you tell without making a graph of the equation?

10. Can you tell without making a graph of an equation whether the graph goes through the origin or not?

11. Which of the equations 1–8 above go through the origin?

12. What would you note about the graphs of the equations $2x - 3y = 6$ and $2x - 3y = 12$?

13. Do $x + y = 7$ and $2x + 2y = 14$ stand for two distinct graphs or only one? Why?

14. Can you tell by looking at a linear equation whether its graph is parallel to the x-axis, or to the y-axis, or parallel to neither of them? How?

15. Why are the equations in Ex. 12 called "inconsistent"? Will the same finite values of x and y satisfy both of them?

16. Will the same values of x and y satisfy both equations in Ex. 13?

TWO EQUATIONS IN X AND Y

147. The solution of a pair of linear equations may be obtained by drawing their graphs, referred to the same pair of axes.

For instance:

$$x + 2y = -4, \quad (1)$$
$$3x - y = 9. \quad (2)$$

In (1), if $x = 0$, $y = -2$.
$x = -4$, $y = 0$.

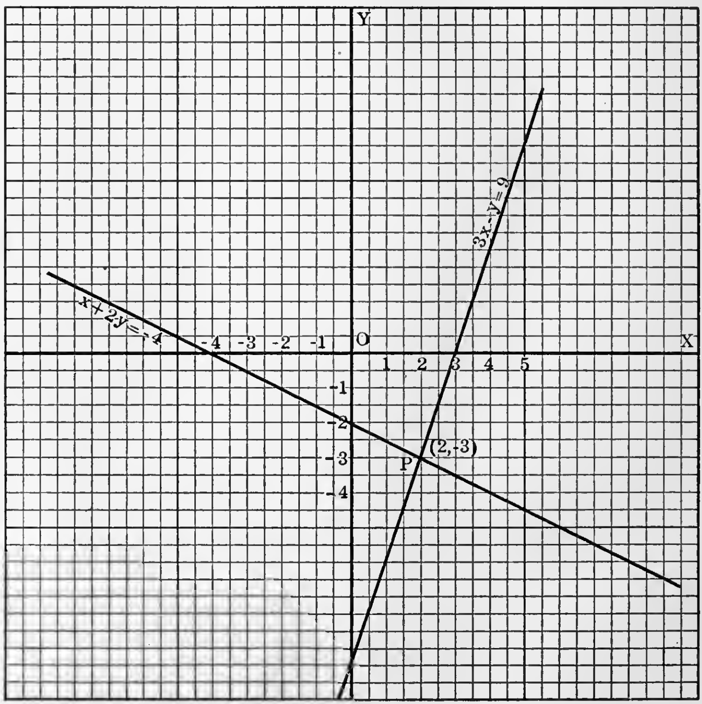


Fig. 22

Plot these points and draw the graph.

In (2), if $x = 0$, $y = -9$.

$x = 3$, $y = 0$.

Plot these points and draw the graph.

The graphs intersect at point P, whose coördinates are 2, -3. Since two straight lines can intersect in only one point, there can be only one pair of values of x and y which will satisfy two linear equations.

Test these coördinates by substituting them in both equations.

Let $x = 2$, $y = -3$; $x + 2y = -4$ becomes $-4 = -4$; $3x - y = 9$ becomes $9 = 9$.

Hence $x = 2$, $y = -3$ satisfy both equations.

What would you infer if the graphs were parallel?

What would you infer if the graphs were identical?

EXERCISES

148. Plot the following pairs of equations, each pair on the same set of axes; determine the coördinates of the intersection, and test the result in each pair of equations:

1. $x + y = 3$,
$x + 4y = 9$.

2. $x - y = 4$,
$2x + y = 14$.

3. $x - y = 5$,
$x + y = 9$.

4. $x + 2y + 3 = 0$,
$2x - 3y - 8 = 0$.

5. $3x - 4y = -18$,
$4x + 3y = 1$.

6. $2x - y = 3$,
$4x - 2y = 6$.

7. $x + y = 7$,
$x = y$.

8. $x - 2y = 1$,
$x = 2\frac{1}{2}y$.

9. $x - y = 1$,
$y = -2$.

10. $3x - 2y = 0$,
$2x + 3y = 0$.

11. $x = -4$,
$y = 3$.

12. $4x - 5y = 20$,
$4x - 5y = 40$.

13. What is the equation of the x-axis?

14. What is the equation of the y-axis?

PRACTICAL APPLICATIONS OF GRAPHS*

149. Draw a graph for changing prices expressed in dollars-per-yard to prices expressed in francs-per-meter.

1 yard = .915 meter, 1 dollar = 5.18 francs.

Hence \$1 per yard = 5.18 fr. per .915 meter.

If .915 meter cost 5.18 fr., then 1 meter costs 5.18 fr. ÷ .915 = 5.66+fr.

Hence 1 dollar-per-yard = 5.66 fr. per meter, nearly.

Let x = francs-per-meter,

and y = dollars-per-yard.

Then $x = 5.66\,y$.

When $y = 0$, then $x = 0$; these values determine the point O in Fig. 23.

When $y = 5$, then $x = 28.3$; these values determine the point A.

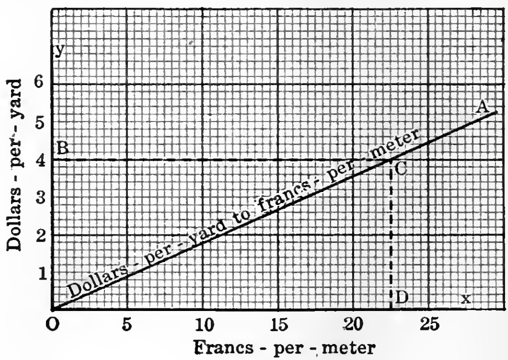


FIG. 23

The line OA is the required graph. To reduce 4 dollars-per-yard to francs-per-meter, pass from B to C, and from C to D. The answer is 22.6 francs-per-meter, nearly.

EXERCISES

150. By inspection, change the following to francs-per-meter:

1. \$2 per yard.
2. \$3 per yard.
3. \$6 per yard.
4. \$$5\frac{1}{2}$ per yard.
5. \$$4\frac{1}{2}$ per yard.
6. \$$3\frac{1}{2}$ per yard.

* May be postponed for the present.

Change the following to dollars-per-yard:

7. 5 fr. per meter.
8. 10 fr. per meter.
9. 15 fr. per meter.
10. 20 fr. per meter.
11. 25 fr. per meter.
12. 27 fr. per meter.

13. Given 1 mile = 1.61 Kilometers, and 1 dollar = 5.18 francs, construct a graph for changing dollars-per-mile to francs-per-Kilometer.

14. Draw a graph for changing miles-per-hour to feet-per-minute.

Hint. 1 mile-per-hour = 5280 feet per 60 minutes = 5280 ÷ 60 feet-per-minute.

15. Draw a graph for changing rods-per-minute to feet-per-second.

16. A freight train starts from Denver at noon, running at the rate of 20 miles an hour. Four hours later a passenger train starts in the same direction at the rate of 40 miles per hour. Draw a graph showing the distances of the trains from Denver during the first ten hours.

Let $x =$ the number of hours after 12 M, and $y =$ number of miles traveled.

(*a*) For the freight train, $y = 20x$.

When $x = 0$, then $y = 0$; these values determine the point O in Fig. 24.

When $x = 10$, then $y = 200$; these values determine the point A.

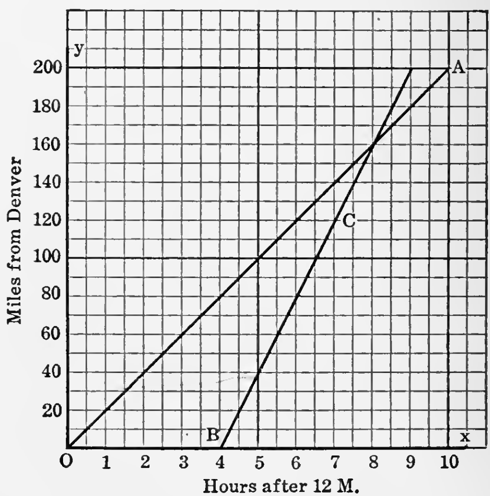


FIG. 24

The line OA shows the distances traveled in different times.

(*b*) For the passenger train, $y = 40(x - 4)$.

When $x = 4$, then $y = 0$; these values determine the point B.

When $x = 7$, then $y = 120$; these values determine the point C.

The line BC shows distances traveled by the passenger train.

(*c*) How far from Denver is each train after $4\frac{1}{2}$ hr.? 5 hr.? $8\frac{1}{2}$ hr.?

(*d*) How far from Denver does the passenger train overtake the freight train?

(*e*) At what time does it overtake the freight train?

17. A starts off on a bicycle at 7 miles an hour. Two hours later B rides in the same direction at 10 miles an hour. Draw a graph and tell from it how far apart the two men are after 3 hr., 5 hr., $6\frac{1}{2}$ hr. Where and when does the second man overtake the first?

18. A train leaves Denver for Colorado Springs at 8 A.M., traveling 30 miles an hour. At 8:30 A.M. a second train leaves Colorado Springs for Denver, traveling 40 miles an hour. The two stations are 80 miles apart. Draw a graph and tell from it how far apart the trains are at 9 o'clock, 9:30 o'clock, 10 o'clock. Where and when do the trains meet?

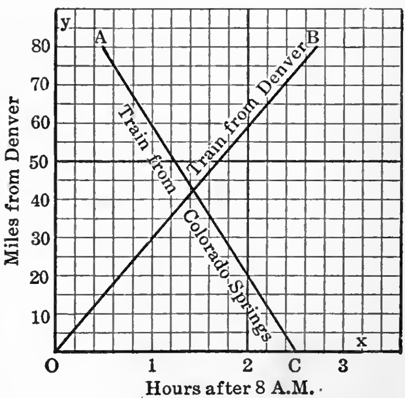


FIG. 25

Let x = number of hours after 8 A.M., and y = number of miles from Denver.

(*a*) The point O marks the time and place where the first train starts. Explain.

(*b*) The point A marks the time and place where the second train starts. Explain.

(*c*) Explain how the line OB is drawn.

(*d*) Explain how the line AC is drawn.

19. A man cycles from A to B at 8 miles an hour, and returns at 10 miles an hour. If he takes 4 hr. to go there and back, find the distance from A to B. Draw a graph showing how far he was from B after 1 hr., 2 hr., 3 hr. When was he at B?

20. A boy begins work with a weekly wage of \$9 and receives an increase of 25¢ every week. Another boy starts with a weekly wage of only \$6, but receives an increase of 50¢ every week. Draw a graph which shows the wage of each at the beginning of every week, for 20 weeks. When will their wages be the same?

Let x weeks = the time,
and y dollars = the wage.

(*a*) After x weeks the wage of the first boy is $y = 9 + \frac{x}{4}$.

When $x=0$, $y=9$; these fix the point A.

When $x = 20$, $y = 14$; these fix the point B.

The line AB shows the wage of the first boy for every week.

(*b*) After x weeks the wage of the second boy is $y = 6 + \frac{x}{2}$.

When $x=0$, $y=6$; these fix point C.

When $x = 20$, $y = 16$; these fix D.

The line CD shows the wage of the second boy for every week.

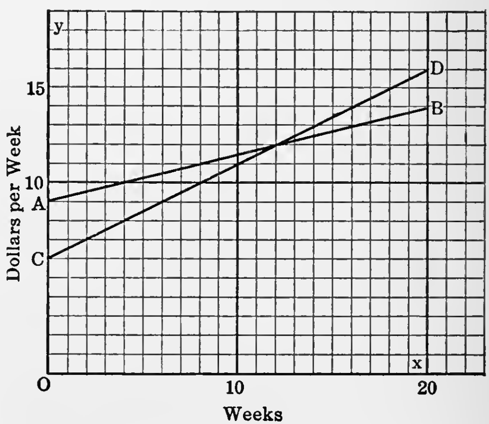


FIG. 26

(*c*) During which week are the wages the same?

(*d*) State the difference in wage after 7, 10, 13, and 19 weeks.

21. Fred has \$7 in the bank and adds to this 75¢ a day. David has \$5 in the bank and adds to this \$1.25 a day. Draw a graph showing the amounts for the first 7 days.

CHAPTER XII

SIMULTANEOUS LINEAR EQUATIONS

151. In a *linear* equation in one unknown, that unknown can have just one value. Thus $2x = 8$, $x = 4$.

In a *linear* equation in two unknowns, one unknown may vary at pleasure; the other unknown varies also, but it is dependent upon the values of the first. In $2x - y = 4$, if $x = 0$, $y = -4$; if $x = 2$, $y = 0$; if $x = 3$, $y = 2$, etc.

But if besides $2x - y = 4$, we have a second relation between the variables, wholly independent of the first relation, as $2x + y = 12$, then we have what is called a *system* of linear equations in which x and y are subjected to two conditions which restrict x and y to a single value each.

The solution of equations by graphs is not always easily accomplished and may not be accurate.

To solve the system without graphs we must reduce the two equations in two unknowns to one equation in one unknown. This process is called *elimination.*

The elimination may be accomplished in either of two ways: I. *addition,* or *subtraction,* or II. *substitution.*

SOLUTION BY ADDITION OR SUBTRACTION

152. I. Solve

$$x + y = 7. \quad (1)$$

$$2x - 3y = 4. \quad (2)$$

Multiply (1) by 2, *to make the coefficients of x alike in both equations.*

$$2x + 2y = 14. \quad (3)$$

$$2x - 3y = 4 \quad (2)$$

Subtract,

$$5y = 10$$

$$y = 2.$$

Substitute 2 for y in (1),

$$x + 2 = 7.$$

$$x = 5. \quad \textit{Ans. } 5, 2.$$

Check: Substituting 5 for x and 2 for y in (1), and also in (2), we obtain:

$$5+2=7,\ 7=7.$$
$$10-6=4,\ 4=4.$$

II. Solve $\qquad 2x-3y=4. \qquad (1)$

$\qquad\qquad 7x+4y=-15. \qquad (2)$

Multiply (1) by 4, $\quad 8x-12y=16 \qquad (3)$

Multiply (2) by 3, $\quad 21x+12y=-45 \qquad (4)$

Add, $\quad 29x=-29$

$\qquad x=-1.$

Substitute -1 for x in (1), $-2-3y=4.$

$\qquad -3y=6.$

$\qquad y=-2.$ *Ans.* $-1,\ -2.$

Check: Substituting in (1), $-2+6=4;\quad 4=4.$

Substituting in (2), $\quad -7-8=-15;\ -15=-15.$

Notice that the coefficients of x or of y in the two equations may be made alike in absolute value by multiplying both sides of one equation or of each equation by some appropriate number. Then either add or subtract to eliminate one letter. By substitution, find the value of the other letter.

EXERCISES

153. Solve and check:

1. $2x-3y=-1,$
   $5x+2y=45.$
2. $7x+2y=1,$
   $4x-y=7.$
3. $x=2y,$
   $3x+7y=130.$
4. $6x-3y=3,$
   $x+y=3.$
5. $3x-2y=1,$
   $4x+3y=2\frac{3}{4}.$
6. $5a+7b=14,$
   $3a-4b=-8.$
7. $3t-10s=32,$
   $6t-20s=64.$
8. $2m=3n,$
   $9m-17n=-7.$
9. $3a+6b=7,$
   $6a+12b=15.$
10. $10x+3y=159,$
    $3x+10y=166.$
11. $3x-5y=23,$
    $7x+y=-35.$
12. $2x+5y=49,$
    $3x-2y=-50.$

SOLUTION BY SUBSTITUTION

154. Solve

$$2x - 3y = 27. \qquad (1)$$
$$5x + 2y = 1. \qquad (2)$$

From (1),

$$2x = 27 + 3y.$$
$$x = \frac{27 + 3y}{2}. \qquad (3)$$

Substitute in (2), $\dfrac{135 + 15y}{2} + 2y = 1. \qquad (4)$

Multiply both sides by 2,

$$135 + 15y + 4y = 2. \qquad (5)$$

Whence

$$19y = -133.$$
$$y = -7.$$

Substitute in (3), $x = \dfrac{27 - 21}{2} = 3.$ *Ans.* 3, − 7.

Check: In (1), $6 + 21 = 27;\ 27 = 27.$
In (2), $15 - 14 = 1;\ \ 1 = 1.$

Notice that the value of x was found in terms of y in *one* equation and substituted in the *other;* the resulting equation was solved, and the value of the other letter found by substitution.

Either letter may be eliminated, its value in terms of the other letter being found in either equation and substituted in the other equation.

EXERCISES

155. Solve, by substitution, and check. Which two equations are inconsistent (§ 146, Ex. 15)? Which are not independent?

1. $4x + 5y = 10,$
   $7x + 3y = 6.$

2. $3x + 2y = 60,$
   $2x + 3y = 60.$

3. $4x - 6y = -96,$
   $10x + 3y = 120.$

4. $3x + 5y = 50,$
   $x - 7y = 8.$

5. $x + 21y = 2,$
   $27y + 2x = 19.$

6. $6p + 5q = 2,$
   $p + 3q = 9.$

7. $8m + 5n = -1,$
   $4m = 10n + 7.$

8. $k = 5l + 3,$
   $l = 2k - 24.$

9. $4r + 3s = 5,$
$4s - 3r = 2\frac{1}{2}.$

10. $a - 2b = 2,$
$2b - 6a = 3.$

11. $x + y = 7,$
$2x + 2y = 14.$

12. $3x - 4y = 5,$
$6x - 8y = 7.$

EQUATIONS CONTAINING FRACTIONS

156. Solve $$\frac{m}{3} + \frac{n}{4} = 4. \quad (1)$$
$$\frac{m}{6} - \frac{n}{2} = -3. \quad (2)$$

Multiply (1) by 3, $m + \frac{3n}{4} = 12.$ (3)

Multiply (2) by 6, $m - 3n = -18.$ (4)

Subtract (4) from 3, $3n + \frac{3n}{4} = 30.$ (5)

Multiply (5) by 4, $12n + 3n = 120.$
$15n = 120.$
$n = 8.$

Substitute 8 for n in (4), $m - 24 = -18.$
$m = 6.$ *Ans.* $m = 6,\ n = 8.$

Check this answer by substitution in (1) and (2).

EXERCISES

157. Solve, and check:

1. $\frac{a}{3} + \frac{b}{2} = \frac{4}{3},$
$\frac{a}{2} + \frac{b}{3} = \frac{7}{6}.$

2. $\frac{a}{8} - \frac{b}{9} = 0,$
$\frac{a}{3} - \frac{b}{4} = \frac{5}{12}.$

3. $x - \frac{y}{2} = \frac{1}{2},$
$2y - \frac{11x}{3} = \frac{7}{3}.$

4. $\frac{x}{3} - \frac{y}{4} = \frac{1}{12},$
$\frac{x}{2} - \frac{3y}{16} = \frac{1}{2}.$

5. $\frac{m}{7} + \frac{n}{6} = 3,$
$\frac{3m}{2} - n = 15.$

6. $\frac{3x}{5} + \frac{4y}{5} + 1 = 0,$
$\frac{5x}{3} + \frac{4y}{5} + 1 = 0.$

7. $\dfrac{r+3}{2}+5s=9,$
$\dfrac{s+9}{10}-\dfrac{r-2}{3}=0.$

8. $\dfrac{u+2}{9}-\dfrac{v-3}{2}=-2,$
$\dfrac{v+7}{11}-\dfrac{u-4}{10}=-1.$

9. $\dfrac{x+y+2}{x-y+2}=3,$
$\dfrac{x-y-2}{x+y-2}=\dfrac{1}{5}.$

10. $\dfrac{a+2}{3}-\dfrac{b-5}{3}=0,$
$\dfrac{2a-7}{3}-\dfrac{13-b}{6}=10.$

11. $\dfrac{5+3x}{7}-\dfrac{5y-2}{4}=-2,$
$6x+8y=108.$

12. $\dfrac{a+b}{b-a}=-\dfrac{5}{9},$
$3a-\dfrac{3b+11}{7}=20\tfrac{2}{7}.$

13. $x=2y,$
$\dfrac{x+y}{5}+\dfrac{x-y}{5}=\dfrac{3x-2y}{3}+\dfrac{3y-x}{2}-\dfrac{31}{10}.$

14. $\dfrac{x}{y}=\dfrac{2}{3},$
$\dfrac{5x+6y}{10}-\dfrac{4x-3y}{3}=y-\dfrac{4}{3}.$

LITERAL SIMULTANEOUS EQUATIONS

158. Solve

$$ax+y=b, \quad (1)$$
$$cx-dy=e. \quad (2)$$

Multiply (1) by d, $\quad adx+dy=bd, \quad (3)$
$\quad cx-dy=e. \quad (2)$

Add (2) and (3), $\quad (ad+c)x=bd+e. \quad (4)$

Divide both sides of (4) by $(ad+c)$,

$$x=\frac{bd+e}{ad+c}.$$

By substituting this value of x in (1) or in (2) we can find y.

But it is easier, in this case, to find y by the same method which gave x. Accordingly,

Multiply (1) by c, $\quad acx+cy=bc. \quad (5)$

Multiply (2) by a, $\quad acx-ady=ae. \quad (6)$

Subtract (6) from (5), $(c + ad)y = bc - ae.$

$$y = \frac{bc - ae}{c + ad}.$$

EXERCISES

159. Solve:

1. $x + y = a,$
$x - y = b.$

2. $ax + by = c,$
$x - y = d.$

3. $cx - dy = m,$
$dx + cy = n.$

4. $x = ay,$
$bx + dy = bcd.$

5. $mx + ny = 1,$
$cx - dy = 1.$

6. $3x - 2y = m + n,$
$2x - 3y = m - n.$

7. $x - y = 1,$
$(a + b)x - (a - b)y = 0.$

8. $(a - b)x - y = 0,$
$x + (a - b)y = 0.$

9. $r + ds = 3,$
$dr - s = 3d.$

10. $m + an = 0,$
$bm + n = 1.$

11. $2bx - 3ay = c,$
$3bx + 2ay = d.$

12. $\frac{x}{a} + \frac{y}{b} = c,$
$\frac{x}{a} - \frac{y}{b} = d.$

13. $\frac{y}{a} + \frac{x}{b} = 1,$
$\frac{y}{b} + \frac{x}{a} = 1.$

14. $\frac{r}{a} + \frac{s}{b} = m,$
$\frac{r}{b} + \frac{s}{a} = n.$

15. Solve for R: $C = 2\pi R.$

16. Solve for a and b: $A = \frac{ab}{2}.$

17. Solve for a and h: $A = \frac{1}{2}(a + b)h.$

18. Solve for r and t: $d = rt.$

19. Solve for M: $F = Ma.$

20. Solve for a, l, and n: $S = \frac{n}{2}(a + l)$.

21. Solve for r and t: $A = P + Prt$.

22. Solve for d and n: $l = a + (n - 1)d$.

23. Solve for F: $C = \frac{5}{9}(F - 32)$.

24. Solve for s and a: $\frac{s}{C} = \frac{a}{360}$.

160. Solve

$$\frac{2}{x} + \frac{3}{y} = 20, \qquad (1)$$

$$\frac{3}{x} - \frac{2}{y} = 17. \qquad (2)$$

If we multiply both sides of each equation by xy, the resulting equations are much more difficult to solve. Hence *do not multiply* by xy, but eliminate the fraction containing either x or y.

Multiply (1) by 3, and (2) by 2,

$$\frac{6}{x} + \frac{9}{y} = 60 \qquad (3)$$

$$\frac{6}{x} - \frac{4}{y} = 34 \qquad (4)$$

Subtract,

$$\frac{13}{y} = 26$$

$$y = \tfrac{1}{2}.$$

In (1) substitute $\frac{1}{2}$ for y, $\frac{2}{x} + 6 = 20,$

$$\frac{2}{x} = 14,$$

$$2 = 14x,$$

$$x = \tfrac{1}{7}. \qquad \textit{Ans. } \tfrac{1}{7}, \tfrac{1}{2}.$$

Check:

$14 + 6 = 20.$ (1)
$20 = 20.$

$21 - 4 = 17.$ (2)
$17 = 17.$

EXERCISES

161. Solve and check:

1. $\frac{5}{x} + \frac{6}{y} = -27,$
$\frac{15}{x} - \frac{3}{y} = -39.$

2. $\frac{5}{x} - \frac{4}{y} = \frac{1}{2},$
$\frac{7}{x} + \frac{3}{y} = 5.$

3. $\frac{2}{x}+\frac{3}{y}=-5,$
$\frac{8}{x}-\frac{5}{y}=31.$

4. $\frac{3}{2x}-\frac{5}{3y}=1\frac{1}{4},$
$\frac{5}{3x}+\frac{7}{2y}=4\frac{3}{5}.$

5. $\frac{1}{x}+\frac{1}{y}=a,$
$\frac{1}{x}-\frac{1}{y}=b.$

6. $\frac{a}{x}+\frac{1}{y}=b,$
$\frac{c}{x}-\frac{1}{y}=d.$

7. $\frac{2}{x}-\frac{m}{y}=a,$
$\frac{3}{x}+\frac{n}{y}=b.$

8. $\frac{1}{ax}-\frac{1}{by}=-1,$
$\frac{1}{bx}-\frac{1}{ay}=-1.$

9. $\frac{6}{ax}-\frac{5}{by}=1,$
$\frac{2}{ax}-\frac{3}{by}=-3.$

10. $\frac{m}{nx}+\frac{n}{my}=m+n,$
$\frac{n}{x}+\frac{m}{y}=m^2+n^2.$

THREE SIMULTANEOUS LINEAR EQUATIONS

162. If the system consists of *three* independent equations containing *three* unknowns, one of the unknowns must be eliminated between *two* pairs of the equations. The resulting equations may then be solved and the third unknown found by substitution.

Solve
$$x+y-z=0, \quad (1)$$
$$2x-y+3z=9, \quad (2)$$
$$3x+2y+z=10. \quad (3)$$

Eliminate y between (1) and (2),

$$x+y-z=0 \quad (1)$$
$$2x-y+3z=9 \quad (2)$$

Add, $$3x+2z=9 \quad (4)$$

Eliminate y between (2) and (3), by multiplying (2) by 2,

$$4x-2y+6z=18 \quad (5)$$
$$3x+2y+z=10 \quad (3)$$

Add, $$7x+7z=28 \quad (6)$$

Divide by 7, $x + z = 4.$ (7)

Eliminate z between (4) and (7), by multiplying (7) by 2,

$$\begin{array}{rcll} 2x + 2z &=& 8 & (8) \\ 3x + 2z &=& 9 & (4) \\ \hline -x &=& -1. & \end{array}$$

Subtract,

$$x = 1.$$

In (7) substitute 1 for x, $1 + z = 4$; $z = 3$.

In (1) substitute 1 for x, and 3 for z, $1 + y - 3 = 0$; $y = 2$.

Ans. 1, 2, 3.

Be sure that after eliminating one unknown between two pairs of the equations, you have two equations containing the *same* two unknowns.

Should there be more than three unknowns, reduce the system to a system containing one less equation and one less unknown. Continue this process until you have one equation with one unknown.

EXERCISES

163. Solve and check:

1. $2x - 3y + 4z = 20,$
   $3x + y - z = 8,$
   $5x - y + 3z = 28.$

2. $x + y + z = 2,$
   $x - y + z = 16,$
   $x + y - z = -2.$

3. $2x - 5y - 4z = -4,$
   $x + y + z = 0,$
   $4x - 3y - 2z = -4.$

4. $4a - 3b + 3c = 20,$
   $-7a + 2b + 6c = 5,$
   $8a - b + 2c = 45.$

5. $2a - 3b + 4c = 2,$
   $3a - 3b - 15 = 0,$
   $7a - 4c - 31 = 0.$

6. $4m - 5n = 22,$
   $6m - 5p = -12,$
   $10n + 3p = -2.$

7. $2r - 3s = 5,$
   $3r + t = 10,$
   $s - 2t = 5.$

8. $3a + 5b = 95,$
   $a - 2c = -9,$
   $7c - 4b = 44.$

9. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 6,$
   $\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = 2,$
   $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0.$

10. $\frac{1}{x} + \frac{1}{y} = -1,$
    $\frac{1}{x} + \frac{1}{z} = -2,$
    $\frac{1}{y} + \frac{1}{z} = 3.$

11. $\frac{2}{a}-\frac{3}{b}+\frac{4}{c}=12,$
$\frac{5}{a}+\frac{6}{b}+\frac{7}{c}=-28,$
$\frac{1}{a}-\frac{1}{b}+\frac{8}{c}=-43.$

12. $a+b+c+d=1.1,$
$a-b-c+d=-.3,$
$a+2b+3c-4d=.8,$
$a+b-4c+3d=-.8.$

PROBLEMS

164. 1. The sum of two numbers is 78; their difference is 4. Find the numbers.

2. The sum of two numbers is 76; their difference is 26. Find the numbers.

3. If the difference of two numbers is 12 and their sum is 45, what are the numbers?

4. There are two numbers whose sum is 25. If the larger be subtracted from the smaller, the remainder is -6. Find the numbers.

5. Twice a certain number plus another number is 35. The difference between the two numbers is 1. What are the numbers?

6. The sum of two numbers is 72. If the second be subtracted from twice the first, the remainder is 3. Find the numbers.

7. The difference between two numbers is 10; their sum is -98. What are the numbers?

8. The sum of two numbers is 26; their difference is 2. Find them.

9. A father earns \$45 more a month than his son. Together they earn \$125 a month. What is the monthly salary of each?

10. A real estate dealer purchases two houses for \$11,250; one house is worth \$250 more than the other. Find the value of each.

11. A mother is 21 years older than her daughter; their combined ages are 71 years. Ascertain the age of each.

12. Five pounds of sugar and three dozen of eggs cost $1.20; seven pounds of sugar and five dozen of eggs cost $1.92. What will one dozen of eggs and one pound of sugar cost?

13. I bought 10 yards of silk and 5 yards of cambric for $14. If I had bought 2 yards less of silk and 3 yards more of cambric, my bill would have been $1.60 less. What is the price of each per yard?

14. Find two numbers such that the greater exceeds the less by 11, and one fourth the less plus one third the greater is 13.

15. Find the fraction in which if 1 be added to the numerator its value is 1, but if 6 be added to the denominator the value is $\frac{1}{2}$.

16. The sum of three numbers is 29; twice the first plus 3 equals the second; twice the third minus 1 equals the second. Find the numbers.

17. A man hired 3 boys for a day for $4.25. The second boy received half again as much as the first, and the third received 25¢ more than the second. How much did each receive?

18. Find two numbers such that three times the greater exceeds twice the less by 61, and twice the greater exceeds 3 times the less by 4.

19. A and B have equal sums of money. A spent half of his and B earns $10; then B has 3 times as much as A. How much had each at first?

20. If A gives B $5, A will have $\frac{1}{4}$ as much as B; but if B gives A $25, B will have the same amount as A. How much has each?

21. A can do a piece of work in 3 days, B can do the same work in 4 days. In what time will both together do the work?

22. A and B together do some work in 4 days, A and C together in 5 days, B and C in 10 days. How long would it take each alone?

23. A company of boys bought a boat. If there had been 5 boys more, each would have paid \$1 less; but if there had been 3 boys fewer, each would have paid \$1 more. How many boys were there? What did each pay? What did the boat cost?

24. A two-digit number has the sum of its digits equal to 9. If 9 be added to the number, the digits will be interchanged. Find the number.

Let $t =$ the digit in tens' place, $u =$ the digit in units' place. Then $10\,t + u =$ the number.

25. The tens' digit is twice the units' digit. If 36 be subtracted from the number, the digits will be reversed. What is the number?

26. The sum of the digits of a three-digit number is 11. The units' digit is twice the tens'. If the digits are reversed, the number will be 396 more than it is. Find the number.

27. Three cities are located at the vertices of a triangle. The distance from A to B by way of C is 150 miles; from C to A by way of B is 140 miles; from B to C by way of A is 110 miles. How far apart are the cities?

28. If a rectangle were 1 inch longer and 2 inches narrower, it would contain 20 square inches less; if it were 3 inches wider, it would become a square. Find its dimensions.

29. A train maintains a uniform speed for a certain distance. If the speed had been 10 miles an hour less, the time would have been $2\frac{1}{2}$ hours more; if the speed had been 10 miles an hour more, the time would have been $1\frac{2}{3}$ hours less. Find time, rate, distance.

30. A man invests a certain sum of money at 4% and a second sum at $3\frac{1}{2}$%. His income from the two investments is \$445. Had the first sum been invested at 5%, his yearly income would have been \$50 more. How much had he in each investment?

31. B's yearly income is $\frac{3}{4}$ of A's, and A spends \$500 per annum more than B. At the end of 3 years A has saved \$4500 and B \$2250. What is the yearly income of each?

32. A person had \$1750 invested so as to bring in an annual income of \$77. Part was bearing 4% interest, the rest 5%. How much was invested at 4%?

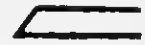
33. A merchant mixes two qualities of tea in the ratio of two parts of the cheaper to one part of the dearer, and, by selling the mixture at 60¢ per pound, he gains a profit of 25%; on mixing them in the ratio of three parts of the cheaper to two parts of the dearer, and selling the mixture at 65¢ per pound, he gains the same rate of profit. Find the prices at which he bought.

THE BEGINNINGS OF ALGEBRA

165. The earliest traces of algebra are found in a very old Egyptian papyrus, written by Ahmes about 1700 or 2000 B.C. This papyrus is a copy of a still older document. The Ahmes papyrus was found in 1868 and is the most important source of our knowledge of ancient Egyptian mathematics. It contains interesting matters on fractions, geometry, algebra, and even the rudiments of trigonometry. It gives solutions of linear equations. It might be expected that the symbols used 3000 years ago would be as different from ours as their language and speech were different from ours. To give an idea of the writing in the Ahmes papyrus, we reproduce one equation and place beneath it the English translation and the equation written in modern symbols.

Heap, its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, it makes : 37

That is, $x\left(\frac{2}{3} + \frac{1}{2} + \frac{1}{7} + 1\right) = 37.$

Notice particularly that  means $\frac{2}{3}$,  means $\frac{1}{2}$,  is the symbol to indicate that 7 is the denominator of a unit fraction,  means 10.

No further progress was made in algebra until the time of the Greek Diophantus (about 300 B.C.), who was very skillful in solving problems. He had a symbol of his own for the unknown x and for x^2. He had also a symbol to express subtraction and equality. Juxtaposition, which with us represents multiplication, meant addition with him. A most momentous impulse to the development of algebra was given by the Hindus, after the fifth century A.D. They displayed wonderful skill in developing arithmetic and algebra. The Hindu and Greek knowledge of algebra was later transmitted to the Arabs who, in turn, became the teachers of European nations at the close of the dark ages. The early European writers on algebra used hardly any symbols; everything except arithmetical numbers was fully written out in words. In the sixteenth century the modern symbols began to be invented. These symbols constitute a sort of shorthand method of indicating algebraic relations and processes, whereby the power and usefulness of algebra in the solution of problems is very greatly increased. How important they are can be realized when the attempt is made to solve difficult problems without their aid. The sixteenth-century symbols are more clumsy and less convenient than the modern. The great French algebraist, Vieta, wrote in 1595 the polynomial

$$2-16x^2+20x^4-8x^6+x^8, \text{ thus: } 2-16\,Q+20\,QQ-8\,CC+QCC.$$

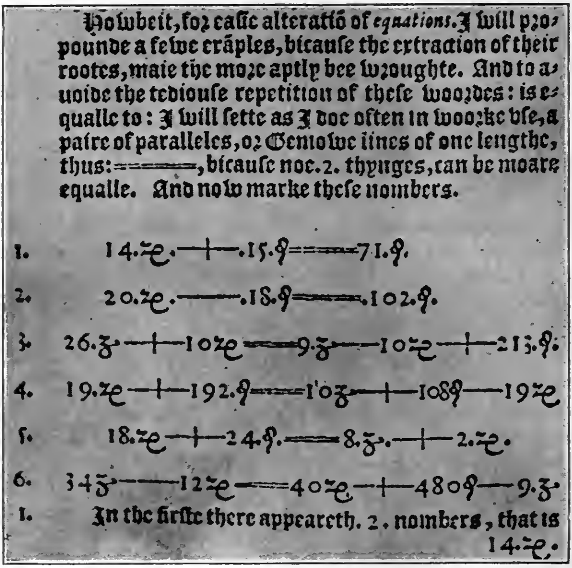
Here Q (quadratus) stands for x^2, C (cubus) stands for x^3. The Portuguese mathematician, Pedro Nuñez, in 1567, wrote

$\sqrt[3]{\sqrt{756}+27}-\sqrt[3]{\sqrt{756}-27}$ thus:

R. V. cu. R. 756 · p̃ 27 · m̃ · R. V. cu. R. 756 · m̃ · 27.

Here R. V. cu. means *radix vniversalis cubica* or general cube root, p̃ means *plus*, m̃ means *minus*. The notations of Vieta and Descartes will be illustrated by other examples which we place under their portraits opposite pages 110 and 140. The notation of Descartes was nearly the same as the modern notation.

The sign of equality (=) was first used for this purpose by an English writer on arithmetic and algebra, Robert Recorde. The following photograph of part of a page in his algebra, entitled the *Whetstone of Witte*, published in London in 1557. shows the place in that book where the sign = is first introduced.



Howbeit, for easie alteratiō of *equations*. I will propounde a fewe exāples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to avoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: ======, bicause noe. 2. thynges, can be moare equalle. And now marke these nombers.

1. 14.ze.—+—.15.ϑ ====== 71.ϑ.

2. 20.ze.——.18.ϑ ====== .102.ϑ.

3. 26.ʒ.—+—10ze ====== 9.ʒ——10ze—+—213.ϑ.

4. 19.ze—+—192.ϑ ====== 10ʒ—+—108ϑ——19ze

5. 18.ze—+—24.ϑ. ====== 8.ʒ.—+—2.ze.

6. 34ʒ——12ze ====== 40ze—+—480ϑ——9.ʒ.

1. In the firste there appeareth. 2. nombers, that is 14.ze.

From Recorde's *Whetstone of Witte*, 1557.

FIG. 27

RENÉ DESCARTES

Descartes' Notation	*Modern Notation*
$x^3 * + px + q \propto 0$;	$x^3 + px + q = 0.$

Descartes used $\propto$ as the sign of equality. The * after x^3 signified that the next lower power of x was missing from the expression.

CHAPTER XIII

QUADRATIC EQUATIONS

166. The equation $$x^2 + ax + b = 0$$
is called a *complete quadratic equation.* It is a *quadratic* equation because the highest power of the unknown x is the *second;* it is *complete* because it contains also a term involving the unknown x to the first power and a term b (often called the *absolute term*) which is free from x.

Quadratic equations of the forms
$$x^2 = b,$$
$$x^2 + ax = 0,$$
are called *incomplete quadratic equations,* because either the term involving the unknown x to the first power or the absolute term b is absent.

INCOMPLETE QUADRATIC EQUATIONS

167. Equations of the form $x^2 + ax = 0$ can be solved by the method of "completing the square," to be explained later. Much easier is the solution by the method of factoring, treated in a previous chapter.

Factoring, we obtain $x(x + a) = 0.$

Making the first factor equal to zero, $x = 0.$

Making the second factor equal to zero, $x + a = 0.$
$$x = - a.$$

Very simple is the solution of $x^2 = b$. For greater convenience, write it $x^2 = c^2.$

Extracting the square root of both sides, $x = \pm c.$

Since the square root of both sides has been extracted, it might be claimed that the sign $\pm$ should be written on *both* sides, giving
$$\pm x = \pm c.$$

But this result is the same as when we write $x = \pm c$.
For the equation $\pm x = \pm c$ means here

(1) $+x = +c$. (3) $+x = -c$.
(2) $-x = -c$. (4) $-x = +c$.

Of these four sets, the first two are the same, and the last two are the same. Hence, $x = \pm c$ gives all the values of x.

EXERCISES

168. 1. Solve $\frac{9x^2}{2} - 3 = \frac{x^2}{2} + 5$.

Transposing and combining,

$$4x^2 = 8.$$

Hence $$x^2 = 2,$$
and $$x = \pm\sqrt{2}.$$

Check: $$\frac{9 \cdot 2}{2} - 3 = \frac{2}{2} + 5, \text{ or } 6 = 6.$$

Solve the following:

2. $4x^2 - \frac{5}{3} = \frac{10}{3} - 5x^2$.
3. $\frac{x}{2} + \frac{1}{x} = \frac{3}{x}$.
4. $\frac{1}{x^2} + \frac{5}{x^2} = 2$.
5. $4y^2 = 3y^2 + \frac{1}{4}$.
6. $7x^2 - 63 = 7$.
7. $11z^2 - 43 = 5z^2 - 7$.
8. $\frac{2y^2 - 8}{5} = 2$.
9. $\frac{3x^2 + 15}{13} = 3$.
10. $\frac{x^2 + 2}{3} = \frac{x^2 - 2}{2}$.
11. $\frac{x^2 - 3}{4} + \frac{3x^2 - 4}{2} = 13$.
12. $2x^2 + 5.78x = 0$.
13. $7x^2 = 10.5x$.
14. $3x - \frac{4}{3x} = 0$.
15. $\frac{5.5}{x} - 6.4x = 0$.
16. $.4x - \frac{.06}{x} = 0$.
17. $.01x^2 - .08x = 0$.

COMPLETE QUADRATIC EQUATIONS

169. In § 131 we solved complete quadratic equations by the method of factoring. That method is the best when the factors can be found by inspection. But there are equations, like $x^2+3x+1=0$, which cannot be solved by any method of factoring thus far studied. Let us consider another method of solving complete quadratic equations.

Of what expression is the trinomial $a^2+2ab+b^2$ a perfect square? Of what is $x^2+2bx+b^2$ a perfect square? What must be added to x^2+2bx, to make the resulting trinomial a perfect square? How can b^2 be found from x^2+2bx? The process of finding this third term, b^2, when the two terms x^2+2bx are given is called *completing the square.*

RULE. *To complete the square of x^2+2bx, add to this the square of half the coefficient of x.*

ORAL EXERCISES

170. Complete the square of the following:

1. x^2+2x.	3. x^2-4x.	5. z^2+3z.	7. y^2+16y.
2. x^2+6x.	4. y^2-8y.	6. x^2+5x.	8. z^2+11z.

SOLUTION OF QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

171. 1. Solve the equation $x^2+6x-7=0$.

(1) Transpose the term -7 which does not involve x or x^2. $x^2+6x-7=0.$

(2) Complete the square in the left member, by adding 3^2 to both sides. $x^2+6x+9=16.$

(3) Take the square root of both sides. $x+3=\pm 4.$

(4) Transpose the 3 to the right side. $x=\pm 4-3.$

Simplify. $x=+1$ or -7.

Checking in the original equation, $x^2+6x-7=0$.

The value $x=1$ gives:	The value $x=-7$ gives:
$1+6(1)-7=0.$	$(-7)^2+6(-7)-7=0.$
$1+6-7=0.$	$49-42-7=0.$
Hence 1 is a root.	-7 is a root.

The reason for substituting the values of x in the original equation, rather than in some equation derived from it, is evident. An error may have been committed in getting the derived equation, but this derived equation may be correctly solved. Substituting the values of x in the derived equation would not reveal the error.

2. Solve $9x^2 - 5 = -4x$.

(1) Transpose $-4x$ to the left side and -5 to the right,

$$9x^2 + 4x = 5.$$

(2) Divide both sides by 9, the coefficient of x^2,

$$x^2 + \tfrac{4}{9}x = \tfrac{5}{9}.$$

(3) Complete the square by adding $(\tfrac{2}{9})^2$ to both sides,

$$x^2 + \tfrac{4}{9}x + (\tfrac{2}{9})^2 = \tfrac{5}{9} + \tfrac{4}{81} = \tfrac{49}{81}.$$

(4) Find the square root of both sides, $\quad x + \tfrac{2}{9} = \pm\tfrac{7}{9}.$

(5) Transposing and combining, $\quad x = \pm\tfrac{7}{9} - \tfrac{2}{9}.$

$$x = \tfrac{7}{9} - \tfrac{2}{9} = \tfrac{5}{9}.$$

$$x = -\tfrac{7}{9} - \tfrac{2}{9} = -1.$$

Checking in the original equation, $9x^2 - 5 = -4x$.

The value $x = -1$ gives:	The value $x = \tfrac{5}{9}$ gives:
$9(-1)^2 - 5 = -4(-1).$	$9(\tfrac{5}{9})^2 - 5 = -4(\tfrac{5}{9}).$
$9 - 5 = 4.$	$\tfrac{25}{9} - 5 = -\tfrac{20}{9}.$
$4 = 4.$	$-\tfrac{20}{9} = -\tfrac{20}{9}.$
Hence -1 is a root.	Hence $\tfrac{5}{9}$ is a root.

172. To solve complete quadratic equations:

(1) *Transpose the terms in x^2 and in x to the left side, and the terms free of x to the right.*

(2) *Divide both sides by the coefficient of x^2.*

(3) *Complete the square by adding to each side the square of half the coefficient of x.*

(4) *When possible, simplify the right side by combining its terms.*

(5) *Extract the square root of both sides, and use both signs $\pm$ on the right.*

(6) *Find the two values of x.*

EXERCISES

173. Solve the following quadratic equations by completing the square, and check:

1. $x^2 + 6x - 16 = 0.$
2. $x^2 + 10x = 24.$
3. $4x^2 + 12x = 27.$
4. $25x^2 + 20x = 21.$
5. $36x^2 + 24x = 45.$
6. $9y^2 = 6y + 63.$

7. Solve $9x^2 + 36x - 2 = 0$.

(1) Transpose -2, $\qquad 9x^2 + 36x = 2.$

(2) Divide both sides by 9, $\qquad x^2 + 4x = \frac{2}{9}.$

(3) Complete the square by adding 4 to both sides,

$$x^2 + 4x + 4 = \frac{38}{9}.$$

(4) Find the square root of both sides,

$$x + 2 = \pm\frac{\sqrt{38}}{3}.$$

(5) Solve for x, $\qquad x = -2 \pm \frac{\sqrt{38}}{3}.$

The value of the radical $\sqrt{38}$ cannot be computed accurately. Taking $\sqrt{38} = 6.16^+$, we obtain the following *approximate* values for x:

$$x = \frac{+6.16}{3} - 2 = 2.05 - 2 = .05.$$

$$x = -2.05 - 2 = -4.05.$$

Except when the approximate values are needed in some computation, it is customary to retain the radical in the expression for x; thus,

$$x = -2 \pm \frac{\sqrt{38}}{3}.$$

Solve the following:

8. $4y^2 + 20y = 15.$
9. $9y^2 - 12y = 10.$
10. $49x^2 - 28x = 16.$
11. $10z - 1 = z^2.$
12. $81x^2 = 13 - 54x.$
13. $21 - 200x = 100x^2.$
14. $100y^2 = 25 + 50y.$
15. $121z^2 - 22z = 8.$
16. $x^2 + \frac{3}{2}x = 1.$
17. $4x^2 - 6x = 0.$
18. $17y^2 + 34y = 0.$
19. $2z^2 - 5z = -1.$
20. $4x^2 - 20x + 5 = 0.$
21. $10 - 7x = 25x^2.$
22. $16x^2 - 5x = 4.$
23. $3x + 5 = 9x^2.$
24. $3x = 10 - 16x^2.$
25. $36x^2 + 24x = 5.$
26. $y^2 + 7y = 13.$
27. $13 = x^2 + 14x.$
28. $21 - 5x - 2x^2 = 0.$
29. $3x^2 + 8x = 2.$
30. $5y^2 + 9y - 1 = 0.$
31. $4x - 1 = x^2.$

32. $2x + 5 = 3x^2$.

33. $x^2 - x + \frac{2}{9} = 0$.

34. $16 = 11x + 3x^2$.

35. $15x = 18 - 2x^2$.

36. $4x^2 = 6x + 1$.

37. $4x^2 - 5 = 9x$.

38. $x^2 + 2x + \frac{24}{25} = 0$.

39. $4x^2 + 4x - 15 = 0$.

40. $9x^2 - 12x - 5 = 0$.

41. $x^2 - 3x = 5$.

42. $3x^2 - 5 = 5x - x^2 + 2$.

43. $6x^2 - 7x - 5 = 3x + 5$.

44. Solve $x^2 - 2x - 31 = 0$.

We have $x^2 - 2x + 1 = 32$.

$$x - 1 = \pm\sqrt{32}.$$

$$x = 1 \pm \sqrt{32} = 1 \pm 4\sqrt{2}.$$

The radical $\sqrt{32}$ can be simplified as follows (§ 111): As $32 = 2.16$, and 16 is a perfect square, we have $\sqrt{32} = \sqrt{16.2} = 4\sqrt{2}$. Proceeding in the same way, show that $\sqrt{50} = 5\sqrt{2}$, $\sqrt{8x^2} = 2x\sqrt{2}$, $\sqrt{32a^3} = 4a\sqrt{2a}$.

45. Simplify the following radicals:

$$\sqrt{18},\ \sqrt{8},\ \sqrt{12a^2},\ \sqrt{50a^2b^2},\ \sqrt{300x^5}.$$

Solve the following:

46. $x^2 - 4x - 4 = 0$.

47. $x^2 - 6x - 9 = 10$.

48. $4x^2 - 12x = 23$.

49. $9x^2 - 12x - 46 = 0$.

50. $16x^2 - 40x = 175$.

51. $x^2 - 6x - 11 = 0$.

52. Solve $\dfrac{1}{x-1} = \dfrac{2x}{x+1}$.

When x appears in the equation in the denominators of fractions, multiply both sides of the equation by a number which will remove the denominators. In this case multiply by $(x-1)(x+1)$. We obtain

$$\frac{(x-1)(x+1)}{x-1} = \frac{2x(x-1)(x+1)}{x+1}.$$

Dividing both terms of each fraction by the factor common to those terms, we get

$$\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \frac{2x(x-1)\cancel{(x+1)}}{\cancel{x+1}}$$

Or
$$x+1=2x(x-1).$$
$$2x^2-3x=1.$$
$$x^2-\tfrac{3}{2}x=\tfrac{1}{2}.$$
$$x^2-\tfrac{3}{2}x+\tfrac{9}{16}=\tfrac{1}{2}+\tfrac{9}{16}=\tfrac{17}{16}.$$
$$x-\frac{3}{4}=\pm\frac{\sqrt{17}}{4}.$$
$$x=\frac{3}{4}\pm\frac{\sqrt{17}}{4}.$$

Solve the following:

53. $\dfrac{5}{x-1}=\dfrac{x}{2}.$

54. $\dfrac{1}{2x-1}=x.$

55. $\dfrac{x+1}{x-1}=\dfrac{1}{2}x.$

56. $\dfrac{20}{x-1}-\dfrac{20}{x}=1.$

57. $y=\dfrac{2}{3}y+\dfrac{1}{y}.$

58. $\dfrac{5}{x-4}+\dfrac{5}{x}=6.$

59. $\dfrac{1}{x+1}+\dfrac{1}{x-4}=\dfrac{3}{10}.$

60. $\dfrac{2}{x+1}=\dfrac{x}{x-1}.$

61. $\dfrac{y}{y+5}=\dfrac{2}{y+2}.$

62. $\dfrac{3}{z-5}=\dfrac{z}{z-4}.$

PROBLEMS

174. 1. The sum of two numbers is 10, their product is 24. Find the numbers.

Let $x=$ one of the numbers, then
$10-x=$ the other number.
$x(10-x)=$ their product, which must be equal to 24.

Hence
$$x(10-x)=24.$$
$$x^2-10x=-24.$$
$$x^2-10x+25=1.$$
$$x=\pm 1+5.$$
$$x=6 \text{ or } 4.$$
$$10-x=4 \text{ or } 6.$$

Hence the two numbers are 6 and 4.

2. The product of two numbers is 143; their sum is 24. Find the numbers.

3. The area of a rectangle is 112 sq. in.; the sum of its length and breadth is 23 in. Find its length and breadth.

4. Find the length and breadth of a rectangle whose area is 4 sq. in. and whose perimeter is $8\frac{2}{15}$ in.

5. The perimeter of a window frame is 18 ft.; the area of the window is $19\frac{1}{4}$ sq. ft. Find the dimensions of the window.

6. If a rectangular table is 7 ft. longer than wide, and it contains $36\frac{3}{4}$ sq. ft., what are its dimensions?

Are both roots of the quadratic equation to which the problem leads applicable to the problem?

7. A tinner wants to make a square box 2 inches deep, with a capacity of 72 cubic inches. From each corner of a square sheet of tin a 2-inch square is cut; the four rectangular strips thus made are turned up. What must be the dimensions of the sheet of tin?

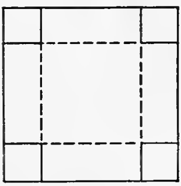


Fig. 28

8. The longer side of a rectangular box exceeds the shorter side 2 in. The box is 3 in. deep and has a capacity of 105 cu. in. If made out of a single piece of cardboard, what were the dimensions of the cardboard, supposing that there is no other waste than the 3-inch squares cut from each corner?

9. A flower bed is $12' \times 15'$. How wide a walk must surround the bed, to increase the total area by 160 sq. ft.?

10. The area of a rectangle exceeds that of a square by 24 sq. in. If the side of the square is equal to half a shorter side of the rectangle, and the longer side of the rectangle exceeds the shorter side by 3 in., find the dimensions of the rectangle.

11. In a right triangle the hypotenuse is 10 ft. longer than the longer leg, and the shorter leg is 10 ft. shorter than the longer leg. Find all three sides.

12. Two legs of a right triangle are equal. The hypotenuse exceeds the length of a leg by 3 in. Find the three sides.

13. The product of two consecutive integers is 3540. Find them.

14. The longer leg of a right triangle exceeds the shorter by 5 in. The area of the triangle is 88 sq. in. Find the length of each leg.

15. Find a number such that, if 20 is subtracted from it, and the remainder is multiplied by the number itself, the product is 189.

16. A school board has \$ 22 to spend on geographies. If the bookseller reduces the price of each book 5 ¢, 4 more books can be purchased than at the original price. How many books can be bought at the reduced rate?

17. Had a man's daily wage been \$ 1 less, he would have been obliged to work 7 days longer to earn \$ 140. How many days did the man work?

Do both roots apply to the problem? Why?

18. A picture $8'' \times 12''$ is placed in a frame of uniform width. If the area of the frame is equal to that of the picture, how wide is the frame? Draw a figure.

19. In a trapezoid, AB exceeds CD by 4 in., and CD exceeds the altitude CE by 1 in. The area of the trapezoid, computed by taking the product of the altitude and one half the sum of the parallel sides, is 28 sq. in. Find the lengths of the parallel sides and of the altitude.

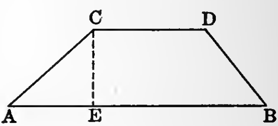


FIG. 29

20. The difference between the parallel sides of a trapezoid is 20 ft.; the altitude is equal to the shorter of the parallel sides; the area is 119 sq. ft. Find the lengths of the parallel sides.

21. Find two consecutive integers whose product is 30,800.

22. Find two consecutive even numbers whose product is 1088.

Hint. Let x be an integer; then $2x$ must be an *even* integer. How much must be added to $2x$ to get the next higher *even* integer?

23. Find two consecutive odd numbers whose product is 783.

Hint. If $2x$ is an even integer, what must be added to $2x$ to get the next higher odd integer?

24. The edges of a rectangular box (with cover) are in the ratio $2:3:6$. If the entire surface is 648 sq. ft., find the lengths of the edges.

Hint. Let the number of feet in the edges be $2x$, $3x$, $6x$, respectively.

SOLUTION OF THE GENERAL QUADRATIC EQUATION

175. The equation $ax + bx + c = 0$ is called the *general* quadratic equation, in which the coefficients a, b, c may be any numbers.

Let us solve it by completing the square.

(1) Divide both sides by a and transpose $\frac{c}{a}$ to the right side.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

(2) Complete the square by adding $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

(3) Perform the indicated subtraction on the right side.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

(4) Find the square root of both sides.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

(5) Transpose $\frac{b}{2a}$ to the right side.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

(6) Another way of writing the answer.

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}. \quad \ldots \text{I.}$$

SOLUTION OF QUADRATIC EQUATIONS BY FORMULA

EXERCISES

176. 1. Solve $3x^2 + 5x + 2 = 0$.

Here $a = 3$, $b = 5$, $c = 2$. Substitute these numbers in the formula I above. We obtain

$$x = \frac{-5 \pm \sqrt{25 - 24}}{6} = \frac{-5 \pm 1}{6} = -\frac{2}{3} \text{ or } -1.$$

Check: (1)

$$3(-\tfrac{2}{3})^2 + 5(-\tfrac{2}{3}) + 2 = 0.$$
$$\tfrac{4}{3} - \tfrac{10}{3} + 2 = 0.$$
$$0 = 0.$$

(2)

$$3(-1)^2 + 5(-1) + 2 = 0.$$
$$3 - 5 + 2 = 0.$$
$$0 = 0.$$

Solve:

2. $5x^2 - 7x + 2 = 0$.

3. $7x^2 + 10x - 15 = 0$.

4. $3x^2 - x - 10 = 0$.

5. $5x^2 - 6x - 3 = 0$.

6. $x^2 - 10x + 3 = 0$.

7. $\frac{3}{2}x^2 - x - 2 = 0$.

8. $\frac{1}{3}x^2 - 12x = 10$.

9. $.1x^2 + x - .3 = 0$.

10. $.3x^2 + .2x - .5 = 0$.

11. $.1x^2 - 3x + \frac{1}{2} = 0$.

12. $x^2 - 101x + 7 = 0$.

MISCELLANEOUS QUADRATIC EQUATIONS

EXERCISES

177. Solve by the method of factoring, whenever the trinomial can be factored at sight. Otherwise solve by the formula or by completing the square.

1. $x^2 - 7x - 18 = 0$

2. $x^2 - 6x - 8 = 0$.

3. $x^2 + 32 = -18x$.

4. $2x^2 + 5x + 2 = 0$.

5. $.2x^2 + x - .5 = 0$.

6. $1.5x^2 + 5x + 1.2 = 0$.

7. $x^2 + 24 = 10x$.

8. $\dfrac{x}{x + 60} = \dfrac{7}{3x - 5}$.

9. $x^2 + 3x = 18$.

10. $x^2 + 7x = 0$.

11. $40 = x^2 + 3x$.

12. $x^2 - 5x + 6 = 0.$

13. $\frac{x+12}{x} + \frac{x}{x+12} = \frac{26}{5}.$

14. $.1x^2 - .1x = 10.$

15. $x(x-3) = (x-1)(x+1)$

16. $x^2 - 8x + 15 = 0.$

17. $4x^2 - 7x - 2 = 0.$

18. $(x-2)(x-3) = (x-5)(x+1).$

19. $(x - \frac{3}{2})(x+2) = 0.$

20. $x(x+10) = (x-1)(x-2).$

21. $6x^2 - 19x + 10 = 0.$

22. $x(x^2 - x - 1) = (x-1)(x^2+1).$

23. $7x^2 - 3x = 160.$

24. $\frac{1}{x} - 10x = 3.$

25. $\frac{1}{x+1} + \frac{1}{x-1} = 1.$

26. $\frac{1}{x^2-1} + \frac{1}{x} = 0.$

27. $\frac{1}{x^2-1} + \frac{2}{x^2+1} = 0.$

28. $\frac{x}{x-1} + \frac{1}{x} = \frac{1}{3}.$

29. $\frac{x-1}{x} + \frac{1}{x-1} = 9.$

30. $\frac{1}{x} + x = 5.$

IMAGINARY ROOTS

178. Solve $x^2 + 2x + 5 = 0.$

Complete the square, $\quad x^2 + 2x + 1 = -4.$

Extract the square root, $\quad x + 1 = \pm\sqrt{-4}.$

$$x = -1 \pm \sqrt{-4}.$$

We cannot express $\sqrt{-4}$ as a number of the type studied thus far:

$\sqrt{-4}$ is not $+2$, because $(+2)(+2) = +4.$

$\sqrt{-4}$ is not -2, because $(-2)(-2) = +4.$

$\sqrt{-4}$ is a *new* type of number; it is called an *imaginary* number.

Since $\qquad -4 = (+4)(-1)$, we may write

$$-\sqrt{4} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1}.$$

The roots of the quadratic equation may be written

$$x = -1 \pm 2\sqrt{-1}.$$

We call $\sqrt{-1}$ the *imaginary unit.* The geometrical meaning of imaginary numbers will be explained in the advanced course. Whenever a quadratic equation yields roots involving the *square root of a negative number,* it is evident that some condition is called for which cannot be satisfied by the ordinary positive and negative numbers.

For example, find two numbers whose sum is 2 and whose product is 3.

Let $\qquad x =$ one of the numbers.

Then $\qquad 2 - x =$ the other number,

and $\qquad x(2-x) = 3.$

Solving,
$$x^2 - 2x = -3,$$
$$x^2 - 2x + 1 = -2,$$
$$x - 1 = \pm\sqrt{-2},$$
$$x = 1 \pm \sqrt{-2}.$$

The solution involves $\sqrt{-2}$; it shows that no two numbers of the kind thus far studied can satisfy the conditions. Only these new numbers, these *imaginary* numbers, satisfy the conditions.

EXERCISES

Solve:

1. $x^2 + x + 1 = 0.$
2. $x^2 + 2x + 6 = 0.$
3. $2x - 5 = x^2.$
4. $2x^2 + 3x + 4 = 0.$

GRAPHIC SOLUTION OF QUADRATIC EQUATIONS

179. 1. Solve graphically $\dfrac{x^2}{4} + 2x - 1 = 0.$

We draw the graph of $y = \dfrac{x^2}{4} + 2x - 1$. Assume different values for x and compute $2x$, $\dfrac{x^2}{4}$, and y, as shown in the following table:

x	$2x$	$\frac{x^2}{4}$	-1	y
2	4	1	-1	4
1	2	$\frac{1}{4}$	-1	$1\frac{1}{4}$
0	0	0	-1	-1
-1	-2	$\frac{1}{4}$	-1	$-2\frac{3}{4}$
-2	-4	1	-1	-4
-3	-6	$2\frac{1}{4}$	-1	$-4\frac{3}{4}$
-4	-8	4	-1	-5
-5	-10	$6\frac{1}{4}$	-1	$-4\frac{3}{4}$
-6	-12	9	-1	-4
-7	-14	$12\frac{1}{4}$	-1	$-2\frac{3}{4}$
-8	-16	16	-1	-1
-9	-18	$20\frac{1}{4}$	-1	$1\frac{1}{4}$
-10	-20	25	-1	4

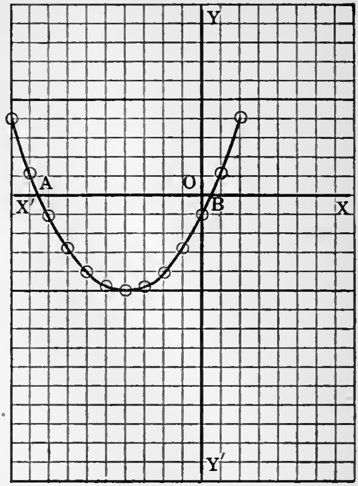


Fig. 30

When $y = 0$, we have $\frac{x^2}{4} + 2x - 1 = 0$. Hence values of x in the graph which correspond to $y = 0$ furnish roots of the equation $\frac{x^2}{4} + 2x - 1 = 0$. We see that $y = 0$ at the points A and B. At A, $x = -8\frac{1}{2}$, nearly; at B, $x = \frac{1}{2}$, nearly. Hence $-8\frac{1}{2}$ and $+\frac{1}{2}$ are approximations to the roots of the quadratic equation.

Solve graphically:

2. $\frac{x^2}{5} - 2x + 1 = 0$.

3. $x^2 - 5x + 2 = 0$.

TWO SIMULTANEOUS EQUATIONS: ONE LINEAR, THE OTHER QUADRATIC

180. I. Solve the two equations
$$y = x^2,$$
$$y = 2x + 3.$$

To find the values of x and y that will satisfy both these equations, eliminate either x or y. The elimination of y is easier. Writing in the first equation $2x + 3$ in place of y we obtain

$$2x + 3 = x^2.$$
$$x^2 - 2x + 1 = 4.$$
$$x - 1 = \pm 2.$$
$$x = 3 \text{ or } -1.$$

From the equation $y = 2x + 3$, we see that, when

$$x = 3, \quad y = 9,$$

and when

$$x = -1, \quad y = 1.$$

The same values of y are obtained with even greater ease from the equation

$$y = x^2.$$

The answers are $\begin{cases} x = 3 \\ y = 9 \end{cases}$ and $\begin{cases} x = -1 \\ y = 1. \end{cases}$

It is important to study the geometrical meaning of problems like this.

In the equation

$$y = 2x + 3,$$

give various values to x and compute the corresponding values of y. We obtain, say,

$$x = 2,\ 1,\ 0,\ -1,\ -2,\ -3.$$
$$y = 7,\ 5,\ 3,\ \ 1,\ -1,\ -3.$$

Plotting the pairs of values of x and y, as is done in Fig. 31, we see that all of them lie on a straight line.

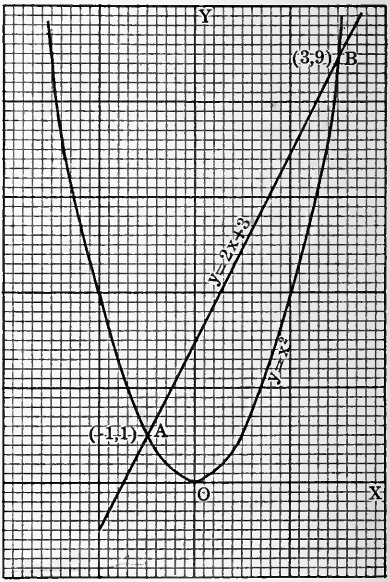


FIG. 31

In the equation $y = x^2$, give different values to x and compute the corresponding values of y; we obtain pairs of values, as follows:

$$x = 3,\quad 2,\quad 1,\quad 0,\quad -1,\quad -2,\quad -3.$$
$$y = 9,\quad 4,\quad 1,\quad 0,\quad 1,\quad 4,\quad 9.$$

These points lie on the curve shown in Fig. 31. This curve is called a *parabola.*

We see that the straight line cuts the parabola in the two points A and B, having the coördinates $x = -1$ and $y = 1$, $x = 3$ and $y = 9$. *These points of intersection represent the values of x and y which satisfy both of the two given equations.*

II. Solve $x = 2y^2$.
$x + 2y = 4$.

Here x is easier to eliminate. In the second equation write $2y^2$ for x. We obtain $2y^2 + 2y = 4$.

(1) Divide both sides by 2, $y^2 + y = 2$.
(2) Complete the square, $y^2 + y + \frac{1}{4} = \frac{9}{4}$.
(3) Take the square root of both sides, $y + \frac{1}{2} = \pm \frac{3}{2}$.
(4) Transposing, $y = \pm \frac{3}{2} - \frac{1}{2} = 1$ or -2.
(5) From $x + 2y = 4$, we get $x = 2$ when $y = 1$, and $x = 8$ when $y = -2$.

The answers are $\begin{cases} x = 2 \\ y = 1 \end{cases}$ and $\begin{cases} x = 8 \\ y = -2. \end{cases}$

Solve the following:

1. $y^2 = 4x$,
$x + 2y = 5$.

2. $3y^2 = 4x$,
$2x + 4y = 14$.

3. $x^2 = 5y$,
$5y - 60 = 4x$.

4. $x^2 = 10y$,
$2y + 7x = 90$.

5. $3x = 7y^2$,
$x - y = \frac{4}{3}$.

6. $2x + 3y = 0$,
$x^2 - 4y = 0$.

7. $y^2 + 3x = 0$,
$6x + y = -15$.

8. $y^2 + 4x = 1$,
$4x - 5y = -23$.

9. $y^2 - 2x + 7 = 0$,
$x - y = 5$.

10. $x^2 + 3y + 5 = 0$,
$y + 2x = 0$.

PROBLEMS

181. 1. The sum of two numbers is 18; the square of one number is equal to 3 times the other. Find the two numbers.

Let $x =$ one number,
and $y =$ the other number.

Then $x + y = 18,$

and $x^2 = 3y.$

Solving, we get from the first equation,

$$y = 18 - x.$$

Eliminate y, $x^2 = 3(18 - x) = 54 - 3x.$

Transpose, $x^2 + 3x = 54.$

Complete the square, $x^2 + 3x + (\frac{3}{2})^2 = 54 + \frac{9}{4} = \frac{225}{4}.$

Extract the square root, $x + \frac{3}{2} = \pm \frac{15}{2}.$

$$x = \pm \frac{15}{2} - \frac{3}{2} = 6 \text{ or } -9.$$

$$y = 18 - x = 12 \text{ or } 27.$$

The answers are $\begin{cases} x = 6 \\ y = 12 \end{cases}$ and $\begin{cases} x = -9 \\ y = 27. \end{cases}$

This problem could have been solved by the use of only one unknown quantity.

Using both x and y solve the following:

2. Find two numbers such that 10 times the square of one is equal to 3 times the other, and their sum is 33.

3. Find two numbers whose difference is $\frac{1}{3}$, the square of the larger being equal to $\frac{4}{3}$ times the smaller.

4. Five times one number is equal to 3 times another. The square of the smaller is equal to 18 times the other. Find the numbers.

EXERCISES

182. 1. Solve $xy = 6.$

$$x - y = 5.$$

The second equation gives $x = 5 + y.$

Substitute in the first, $(5 + y)y = 6.$

Or $y^2 + 5y - 6 = 0.$

Factor, $(y + 6)(y - 1) = 0.$

$$y = -6, +1.$$

Substitute in $x = 5 + y$, $x = -1$, when $y = -6.$

$$x = 6, \text{ when } y = 1.$$

The answers are $\begin{cases} x = -1 \\ y = -6 \end{cases}$ and $\begin{cases} x = 6 \\ y = 1. \end{cases}$

Let us plot the equation $x - y = 5$; compute values as follows:

$y = 2,\quad 1,\quad 0,\quad -1,\quad -2,\quad -6.$

$x = 7,\quad 6,\quad 5,\quad +4,\quad +3,\quad -1.$

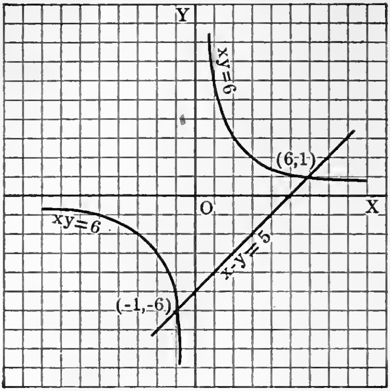


FIG. 32

The graph is a straight line, as shown in Fig. 32.

Plot next $xy = 6$. We have

$y = 6, \quad 3, \quad 2, \quad 6, \quad -1, \quad -2, \quad -3, \quad -6.$

$x = 1, \quad 2, \quad 3, \quad 1, \quad -6, \quad -3, \quad -2, \quad -1.$

The pairs of points which are here computed lie on a curve composed of two branches, called *hyperbola.*

The straight line and the hyperbola intersect in two points,

$x = 6$ and $y = 1$,

$x = -1$ and $y = -6$.

These pairs of values are the answers to the simultaneous equations.

Solve the following:

2. $x + y = 9$,
   $xy = -36$.

3. $x + y = 23$,
   $xy = 132$.

4. $x + y = \frac{5}{6}$,
   $xy = \frac{1}{6}$.

5. $xy = 18.75$,
   $x - y = 11$.

6. $xy + y = 40$,
   $x - y = 2$.

7. $xy + x = -36$,
   $xy = -30$.

8. $xy + x + y = 0$,
   $x + y = \frac{1}{6}$.

9. $xy + x = 9$,
   $x + y = 5$.

10. Solve $x^2 + y^2 = 25$.
    $x + y = 7$.

(1) The second equation gives $x = 7 - y$.

(2) Write $(7 - y)$ in place of x in the first equation.

$$(7 - y)^2 + y^2 = 25.$$
$$49 - 14y + y^2 + y^2 = 25.$$
$$2y^2 - 14y = -24.$$
$$y^2 - 7y = -12.$$

$$y^2 - 7y + (\tfrac{7}{2})^2 = -12 + \tfrac{49}{4} = \tfrac{1}{4}.$$
$$y - \tfrac{7}{2} = \pm \tfrac{1}{2}.$$
$$y = \pm \tfrac{1}{2} + \tfrac{7}{2} = 4 \text{ or } 3.$$
$$x = 7 - y = 3 \text{ or } 4.$$

The answers are $\begin{cases} x = 3 \\ y = 4 \end{cases}$ and $\begin{cases} x = 4 \\ y = 3. \end{cases}$

The graph of $x + y = 7$ can be drawn from the following values:

$$x = 3,\quad 1,\quad 0,\quad -1,\quad -2.$$
$$y = 4,\quad 6,\quad 7,\quad 8,\quad 9.$$

These pairs of values are points lying on the line in Fig. 33.

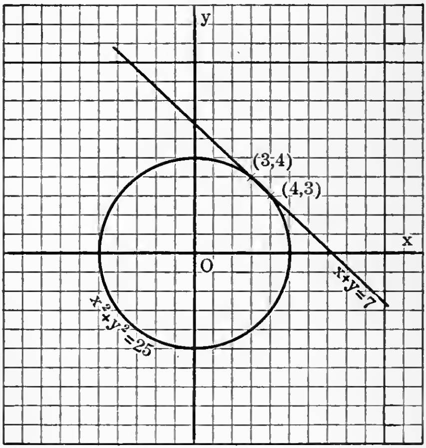


Fig. 33

To find the graph of $x^2 + y^2 = 25$, assume values of x and compute corresponding values of y as follows:

$x=5,$	$4,$	$3,$	$2,$	$1,$	$0,$	$-1,$	$-2,$	$-3,$	$-4,$	$-5.$
$y=0,$	$\pm 3,$	$\pm 4,$	$\pm 4.6,$	$\pm 4.9,$	$\pm 5,$	$\pm 4.9,$	$\pm 4.6,$	$\pm 4,$	$\pm 3,$	$0.$

Notice that for every value of x (except $x = 5$ and -5) there are two corresponding values of y. The pairs of points all lie on a circle.

11. $x^2 + y^2 = 13,$
$x - y = 1.$

12. $x^2 + y^2 = 41,$
$2x - y = 3.$

13. $x^2 + y^2 = 29,$
$x - 2y = 1.$

14. $3x + 2y = 37,$
$x^2 + y^2 = 113.$

15. $x^2 - y^2 = 0,$
$x + 2y = 15.$

16. $x^2 - 2y^2 = -17,$
$x + y = 4.$

17. $x^2 + 2y^2 = 36,$
$x + y = 6.$

18. $2x^2 - y^2 = 14,$
$x - y = -5.$

19. $x^2 + y^2 = 221,$
$2x - 3y = -13.$

20. $x^2 + y^2 = 2,$
$5x - y = 4.$

21. $4x + y = 50,$
$x^2 + y^2 = 200.$

22. $7x + y = 17,$
$x^2 + y^2 = 13.$

23. $3x^2 - y^2 = 242,$
$x - y = 0.$

24. $2x^2 + 3y^2 = 5,$
$x + y = 0.$

25. $3x^2 - 2y^2 = 43,$
$x - y = 1.$

26. $2y^2 - 4x^2 = 82,$
$3x + 5y = 41.$

HISTORY OF QUADRATIC EQUATIONS

183. While the earliest known solution of equations of the first degree is found in the Ahmes papyrus, about 2000 B.C., the earliest known solutions of quadratic equations occur in Euclid's *Elements* of geometry, about 300 B.C., — about 1700 years later. Euclid solved quadratic equations by geometric construction. In proposition 11, of Book II, of the *Elements*, Euclid solved by drawing lines the problem: To divide a given straight line into two parts, so that the rectangle contained by the whole line and one part of it may be equal to the square on the other part. In algebra, this problem demands the solution of the quadratic equation $a(a - x) = x^2$, where x is the length of the required line. About two centuries after Euclid, a Greek writer, Heron, gives a root of a quadratic equation, showing that he knew how to solve the equation algebraically, but he nowhere tells how he did it. Equally non-committal is Diophantus, the celebrated Greek writer on arithmetic and algebra, of the fourth century A.D. Diophantus gives correct answers to several quadratic equations. Part of his book is lost; the parts that are extant nowhere explain his mode of solution. Diophantus does not give more than one root of a quadratic equation, even if both are positive, nor does he ever recognize a negative root. He nowhere uses negative numbers. In that respect his algebra was like our school arithmetics.

Marked advance in algebra was made by the Hindus. In the fifth and sixth centuries they solved quadratics by a process which can be explained easiest by using our modern notation. They wrote the roots of $ax^2 + bx = c$; thus: $x = \dfrac{\sqrt{ac + \left(\frac{b}{2}\right)^2} - \frac{b}{2}}{a}$. Somewhat later Çridhara, another Hindu mathematician, uses the form of $x = \dfrac{\sqrt{4\,ac + b^2} - b}{2\,a}$. To obtain the first form of x one begins by multiplying both sides of $ax^2 + bx = c$ by a. To obtain the second form of x, one begins by multiplying both sides of the equation by $4\,a$. This last process is now called the "Hindu method" of solution. It has the advantage of excluding fractions under the radical sign.

The Hindu mathematician Bhaskara (twelfth century A.D.) recognized that a quadratic equation has two roots, when both of them were positive. He gives $x = 50$ or -5 as the roots of $x^2 - 45\,x = 250$, and remarks: "But the second value is in this case not to be taken, for it is inadequate; people do not approve of negative roots." Thus negative roots were seen, but not admitted. The recognition of negative roots came much later. The Italian mathematician, Cardan (1539), and the Belgian mathematician, Simon Stevin (1585), admitted negative roots. Soon after, Albert Girard (1629) took the still more advanced position of admitting the possibility of imaginary roots, thereby recognizing the theorem that every quadratic equation has two roots or, more generally, that every equation of the nth degree has n roots. But opposition to the acceptance of imaginary roots continued for over a century after Girard. Not until the beginning of the nineteenth century did all opposition to them vanish. This opposition to imaginary roots grew mainly out of the fact that they could not be explained geometrically. A positive root could be pictured to the eye by the length of a line drawn in a given direction; a negative root could be interpreted by the length of a line drawn in the *opposite* direction. But how could imaginary roots be consistently represented by lines? Such geometric representation is, however, possible. How this can be done was first shown by the Danish engineer, C. Wessel, in 1797. Most influential in securing the general recognition of imaginary numbers was the great German mathematician, Gauss.

CHAPTER XIV

FRACTIONS

184. We have seen in previous exercises that fractions in algebra are subject to the same general rules as fractions in arithmetic. In algebra, as in arithmetic, the value of a fraction is not changed, *if both of its terms are multiplied by the same factor or if both of its terms are divided by the same factor.*

Thus, $$\frac{2}{3}=\frac{4}{6},\quad \frac{a}{b}=\frac{ac}{bc},\quad \frac{ax+bx}{ax-bx}=\frac{a+b}{a-b}.$$

The most common mistake made by beginners is to say that $\frac{x^2-5}{x^2+6}$ is equal to $-\frac{5}{6}$. But this answer cannot be obtained by multiplying or dividing both terms of the given fraction by any number.

EXERCISES

185. 1. Is $\dfrac{x^2-5}{x^2+6}$ equal to $\dfrac{-5}{6}$ when $x=1$? When $x=2$?

2. Show by an example that *subtracting* a number from the numerator and denominator changes the value of a fraction.

3. Which of the following equalities are correct?

$$\frac{3+1}{3-2}=\frac{1}{-2},\quad \frac{(a+b)(a-b)}{(a+2\,b)(a-b)}=\frac{a+b}{a+2\,b},\quad \frac{a+a^2}{b-a^2}=\frac{a}{b}.$$

4. Reduce to the lowest terms, $\dfrac{12\,x^5y^5-48\,x^3y^7}{16\,x^5y^5-32\,x^4y^6}$.

Factor numerator and denominator, then divide both terms by their common factors. We get

$$\frac{12\,x^3y^5(x^2-4\,y^2)}{16\,x^4y^5(x-2\,y)}=\frac{\overset{3}{\cancel{12\,x^3y^5}}(x+2\,y)\cancel{(x-2\,y)}}{\underset{4\,x}{\cancel{16\,x^4y^5(x-2\,y)}}}=\frac{3(x+2\,y)}{4\,x}.$$

Reduce to the lowest terms:

5. $\frac{156}{1068}$.

6. $\dfrac{mx}{my}$.

7. $\dfrac{mx}{m^2y}$.

8. $\dfrac{a(x+y)}{a(x+y)}$.

9. $\dfrac{xc^2+yc^2}{xc^2-yc^2}$.

10. $\dfrac{x(x+1)^2}{2x^2(x-1)(x+1)}$.

11. $\dfrac{3x+3y}{4x+4y}$.

12. $\dfrac{a^2b+ab^2}{abx}$.

13. $\dfrac{x+x^2}{y+xy}$.

14. $\dfrac{(x-y)^2}{y-x}$.

15. $\dfrac{x^2-y^2}{y-x}$.

16. $\dfrac{x^2-y^2}{(x+y)^2}$.

17. $\dfrac{12x^2y(x+2)(x+4)}{9xy^2(x+4)^2}$.

18. $\dfrac{(y-x)(y^2+x^2)}{x^2-y^2}$.

19. $\dfrac{6(a-b)(b-c)}{(3b-3a)(4b-4c)}$.

20. $\dfrac{a^2-2a-3}{a-3}$.

21. $\dfrac{a^2+ab-6b^2}{5(a-2b)}$.

22. $\dfrac{2a^2b(a^3-b^3)}{6ab(a^2+ab+b^2)}$.

23. $\dfrac{m^2-1}{(1-m)(m^2+4m+3)}$.

24. $\dfrac{(2y-1)^2}{2y^2-5y+2}$.

25. $\dfrac{n^3-n^2-12n}{4n^2-16n}$.

26. $\dfrac{ax-ay}{(y-x)^2}$.

27. $\dfrac{5a-10b}{2b-a}$.

28. $\dfrac{mx+my}{nx+ny}$.

29. $\dfrac{5a+5b}{15a^2b+15ab}$.

30. $\dfrac{y^2-5y+4}{y^3-y}$.

31. $\dfrac{4x^2-4xy+y^2}{ay-2ax}$.

32. $\dfrac{(3y-x)^2}{2x-6y}$.

MULTIPLICATION AND DIVISION OF FRACTIONS

186. In algebra we have the same rules of operation as in arithmetic; namely,

1. *In finding the product, multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

2. *In finding the quotient, invert the divisor and then proceed as in the multiplication of a fraction by a fraction.*

Thus, $$\tfrac{2}{3} \div \tfrac{5}{7} = \tfrac{2}{3} \times \tfrac{7}{5} = \tfrac{14}{15}.$$

The correctness of these rules is shown in arithmetic. However, it is easy to establish the truth of them by means of the equation.

For instance, the second rule may be proved as follows:

Let $$\frac{a}{b} = x, \qquad \frac{c}{d} = y,$$

then $$a = bx, \qquad c = dy,$$

and $$ad = bdx, \quad bc = bdy.$$

Dividing equals by equals,

$$\frac{ad}{bc} = \frac{bdx}{bdy} = \frac{x}{y}.$$

But $$\frac{x}{y} = \frac{a}{b} \div \frac{c}{d}.$$

Hence $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

This proves the rule for the division of one fraction by another.

Notice that the rules for the multiplication and division of fractions are still applicable when in place of one of the fractions we have an integral expression; we need only write the integral expression with the denominator 1.

Thus, $$\frac{a}{b} \cdot c = \frac{a}{b} \cdot \frac{c}{1} = \frac{ac}{b}.$$

$$\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}.$$

Perform the indicated operations and simplify by reducing the resulting fraction to its lowest terms:

1. $\frac{120}{255} \cdot \frac{155}{160}$.

Here
$$\frac{\overset{3}{\cancel{120}} \cdot \overset{31}{\cancel{155}}}{\underset{\underset{17}{\cancel{51}}}{\cancel{255}} \cdot \underset{4}{\cancel{160}}} = \frac{31}{4 \cdot 17} = \frac{31}{68}.$$

2. $\dfrac{x^2 - y^2}{x^2 - 4y^2} \div \dfrac{x + y}{x - 2y}$.

Inverting the divisor and multiplying, we obtain

$$\frac{(x^2 - y^2)(x - 2y)}{(x^2 - 4y^2)(x + y)}.$$

It is usually best *not* to carry out the indicated multiplications in the numerator and denominator. On the contrary, it is usually better, if possible, to factor still further, with the purpose of exposing to view all the factors that are common to the numerator and denominator.

Factoring and then dividing both terms by their common factors, we obtain

$$\frac{\cancel{(x+y)}(x - y)\cancel{(x - 2y)}}{(x + 2y)\cancel{(x - 2y)}\cancel{(x + y)}} = \frac{x - y}{x + 2y}.$$

3. $\dfrac{a - b}{a^2 + ab} \div \dfrac{a^2 - ab}{a^2 - b^2}$.

4. $\dfrac{4a^3}{b^2} \div \dfrac{2a^2}{b}$.

5. $\dfrac{6x^5}{14y^3} \cdot \dfrac{35y^2}{9x^3}$.

6. $\dfrac{3x}{x - y} \cdot \dfrac{x^2 - y^2}{9}$.

7. $\dfrac{1 - y^2}{3} \cdot \dfrac{6x}{1 + y}$.

8. $\dfrac{x^2 - y^2}{x} \cdot \dfrac{1}{y + x} \cdot \dfrac{x}{x - y}$.

9. $\dfrac{a^2 - 1}{3} \div \dfrac{a + 1}{a - 1}$.

10. $\dfrac{15m - 30}{2m} \cdot \dfrac{3m}{5m - 10}$.

11. $\dfrac{y^2 - x^2}{x} \cdot \dfrac{y^2 + x^2}{xy - x^2}$.

12. $\dfrac{7x - 14y}{5} \div \dfrac{7(x - 2y)}{4}$.

13. $\dfrac{x^3 + y^3}{xy} \div \dfrac{x + y}{x^2}$.

14. $\dfrac{x^3 - y^3}{5} \div \dfrac{x^2 + xy + y^2}{20}$.

15. $\dfrac{6(x^3 - y^3)}{5} \div (x^2 + xy + y^2)$.

16. $(x^2 + y^2) \div \dfrac{x^2y + y^3}{3}$.

17. $\dfrac{3\,xy}{1+8\,x^3} \div \dfrac{6\,y^2}{1+2\,x}$.

18. $\dfrac{6\,x^3}{1-27\,y^3} \cdot \dfrac{1-3\,y}{2\,y^3}$.

19. $(x^2-xy+y^2) \cdot \dfrac{12\,xy}{x^3+y^3}$.

20. $\dfrac{3\,x+1}{x^3+y^3} \cdot (x+y)$.

21. $\dfrac{2\,x^2+5\,x+3}{x-1} \cdot \dfrac{2\,x^2+x-3}{x+1}$.

22. $\dfrac{2\,x^2+x-6}{ax+x^2} \cdot \dfrac{a+x}{2\,x^2+7\,x-15}$.

23. $\dfrac{4\,y^2+6\,y+2}{3\,x^2y+6\,x^2} \div \dfrac{2\,xy+2\,x}{6\,y^2+15\,y+6}$.

24. $\dfrac{x^2-y^2}{x+1} \cdot \dfrac{x^3+1}{x^2+x+1} \cdot \dfrac{x^3-1}{x-y}$.

COMPLEX FRACTIONS

187. A complex fraction can usually be simplified most easily by multiplying both its numerator and denominator by the lowest common denominator of the fractions occurring in the numerator and denominator of the complex fraction.

1. Simplify $\dfrac{\dfrac{a}{x}+\dfrac{b}{y}}{\dfrac{a}{x}-\dfrac{b}{y}}$.

The l. c. d. of the minor fractions $\dfrac{a}{x}$ and $\dfrac{b}{y}$ is xy. Multiply both the numerator and the denominator of the complex fraction by xy. We obtain

$$\frac{\dfrac{a}{x}+\dfrac{b}{y}}{\dfrac{a}{x}-\dfrac{b}{y}}=\frac{ay+bx}{ay-bx}.$$

2. Simplify $\dfrac{\dfrac{4}{a}+\dfrac{5}{b}-\dfrac{6}{c}}{\dfrac{1}{a}-\dfrac{3}{b}-\dfrac{4}{c}}$.

Multiply both terms of the complex fraction by abc.

Then, $$\frac{\frac{4}{a}+\frac{5}{b}-\frac{6}{c}}{\frac{1}{a}-\frac{3}{b}-\frac{4}{c}}=\frac{4\,bc+5\,ac-6\,ab}{bc-3\,ac-4\,ab}.$$

3. Simplify $\dfrac{\frac{25}{49}}{\frac{35}{42}}$.

In this case the process explained in arithmetic is easier:
Invert the divisor and multiply.

$$\frac{\frac{25}{49}}{\frac{35}{42}}=\frac{\overset{5}{\cancel{25}}}{\underset{7}{\cancel{49}}}\cdot\frac{\overset{6}{\cancel{42}}}{\underset{7}{\cancel{35}}}=\frac{30}{49}.$$

Simplify:

4. $\dfrac{\frac{2}{5x}-\frac{1}{7y}}{\frac{1}{5x}+\frac{1}{7y}}$.

5. $\dfrac{\frac{m}{n}-\frac{n}{m}}{\frac{n}{m}+\frac{m}{n}}$.

6. $\dfrac{1+\frac{1}{m}}{2-\frac{1}{m}}$.

7. $\dfrac{\frac{3}{4m}-\frac{1}{3n}}{4+\frac{1}{mn}}$.

8. $\dfrac{1+\frac{1}{2m+1}}{1-\frac{1}{2m+1}}$.

9. $\dfrac{\frac{3}{5}+\frac{5}{6}}{\frac{27}{60}-\frac{7}{20}}$.

10. $\dfrac{\frac{22}{63}}{\frac{55}{72}}$.

11. $\dfrac{1+\frac{a}{b}}{1-\frac{a^2}{b^2}}$.

12. $\dfrac{x-1-\frac{2}{x+2}}{x-2-\frac{3}{x+2}}$.

13. $\dfrac{\frac{1}{1-x}+\frac{1}{1+y}}{\frac{1}{1-x}-\frac{1}{1+y}}$.

THE HIGHEST COMMON FACTOR

188. Only expressions which are free of fractions and of radicals will be considered here.

The *highest common factor* (h. c. f.) of two or more expressions is the product of such prime factors as occur in *each* expression, every factor being taken the least number of times it occurs in any one expression.

This definition indicates the process of finding the h. c. f.

1. Find the h. c. f. of $12\,a^2b^3c$, $24\,a^3bc^2$, $36\,a^4b^2c^3$.

Resolving into prime factors, we obtain

$$12\,a^2b^3c = 2\cdot 2\cdot 3\cdot a\cdot a\cdot b\cdot b\cdot b\cdot c.$$
$$24\,a^3bc^2 = 2\cdot 2\cdot 2\cdot 3\cdot a\cdot a\cdot a\cdot b\cdot c\cdot c.$$
$$36\,a^4b^2c^3 = 2\cdot 2\cdot 3\cdot 3\cdot a\cdot a\cdot a\cdot a\cdot b\cdot b\cdot c\cdot c\cdot c.$$

The h. c. f. $= 2\quad 2\cdot 3\cdot a\cdot a\cdot b\cdot c = 12\,a^2bc.$

2. Find the h. c. f. of $a^2(a-2)^2$, $ab(a-2)^3$, $3\,a(a-2)^4$.

We may write

$$a^2(a-2)^2 = a\cdot a(a-2)(a-2).$$
$$ab(a-2)^3 = a\cdot b(a-2)(a-2)(a-2).$$
$$3\,a(a-2)^4 = 3\cdot a(a-2)(a-2)(a-2)(a-2).$$
$$\therefore \text{ the h. c. f.} = a(a-2)(a-2) = a(a-2)^2.$$

EXERCISES

189. Find the h. c. f. of:

1. a^2b, a^2c, a^2d.
2. axy, bxy, cxy.
3. $10\,a^2$, $15\,a^3$, $25\,a^4$.
4. $6\,a^2bc$, $12\,a^2b^2c^2$, $18\,a^2b^3c^2$.
5. a^2-ab, a^2-ac, a^2+ad.
6. $mx-my$, $nx-ny$, $px-py$.
7. $2\,x^2+2\,y^2$, ax^2+ay^2.
8. $2\,a+2\,b$, a^2-b^2.
9. $x-y$, x^2-y^2.
10. x^2-1, $x^2-2\,x+1$.
11. $2\,a^2+4\,a+2$, $4\,a^2-4$.
12. $3\,ab+3\,b^2$, $a^2+2\,ab+b^2$.
13. $x^2-2\,xy+y^2$, x^3-y^3.
14. x^2-4, x^2+x-6.
15. x^3+y^3, x^2-y^2, $x^2+2\,xy+y^2$.
16. m^2-9, $m^2+5\,m+6$, m^2-m-12.
17. a^2-1, a^3-1, $ab-b-2\,a+2$.

18. $3a^6 - 48a^2$, $3a^4 + 13a^3 + 14a^2$.

19. $(x+y)^4(x-y)^2(a+b)$, $(x+y)^2(x-y)^3(a-b)$.

20. $r^2 - 6rs + 8s^2$, $r^2 + 2rs - 8s^2$, $r^2 - 5rs + 6s^2$.

LOWEST COMMON MULTIPLE

190. The *lowest common multiple* (l. c. m.) of two or more expressions is the product of all the prime factors that occur in the expressions, every factor being taken the *greatest* number of times it occurs in any one expression.

This definition indicates the process of finding the l. c. m.

1. Find the l. c. m. of $3a^2$, $6a^3$, $5ab^2$.

$$3a^2 = 3 \cdot a \cdot a;\; 6a^3 = 2 \cdot 3 \cdot a \cdot a \cdot a;\; 5ab^2 = 5 \cdot a \cdot b \cdot b$$

Hence the l. c. m. $= 2 \cdot 3 \cdot 5 \cdot a \cdot a \cdot a \cdot b \cdot b = 30a^3b^2$.

2. Find the l. c. m. of $a^2 + 2ab + b^2$, $a^2 - b^2$, $2a^2 + 2ab$.

$$a^2 + 2ab + b^2 = (a+b)(a+b).$$
$$a^2 - b^2 = (a+b)(a-b).$$
$$2a^2 + 2ab = 2a(a+b).$$

Hence the l. c. m. $= 2a(a+b)^2(a-b)$.

EXERCISES

191. Find the l. c. m. of:

1. x, y, z.
2. a^3b, ab^3.
3. b^2cd, bc^2d, bcd^2.
4. $5c$, $7d$, $10e$.
5. a^2, $a^2 + a$.
6. $a^2 + a$, $a^2 - a$.
7. $c^2 + cd$, $cd + d^2$.
8. $a^2 + ab$, $ab - b^2$.
9. $a^2 - 4$, $a^2 + a - 6$.
10. $x^2 - y^2$, $x^2 + 2xy + y^2$, $(x-y)^3$.
11. $a^2 - b^2$, $a^3 - b^3$, $a^2 - 2ab + b^2$.
12. $m + n$, $m^2 + n^2$, $m^2 + 2mn + n^2$.
13. $3m - 3n$, $m^3 - n^3$, $m^2 + mn + n^2$.
14. $x^2 + 6x + 9$, $x^2 + 5x + 6$.

15. $a^2 - 6a + 8,\ a^2 - a - 12.$

16. $1 - x^2,\ 1 - x^3,\ 1 + x.$

17. $xyz,\ x^2y - xy^2,\ x^3y - xy^3.$

18. $x^2 - 9,\ x + 3,\ x^2 + 10x + 21.$

19. $a^2 - 1,\ a^3 + a^2 + a + 1,\ a^3 - a^2 + a - 1.$

20. $x^2 - 4xy + 4y^2,\ x^2 + 2xy - 8y^2,\ x^2 - xy - 2y^2.$

192. Find the h. c. f. and l. c. m. of:

1. $24, 36.$
2. $96, 144.$
3. $49, 98, 147.$
4. $252, 216, 180.$
5. $8xy^3,\ 4x^5yz,\ 10x^2y^2z^2.$
6. $25,\ 75cd^4,\ 100ac^3d.$
7. $3a,\ a^2 - ab.$
8. $4ax,\ 4a^3x^2 - 4ax^3.$
9. $x^2 - 9,\ x^2 + 5x + 6.$
10. $ax + ab,\ bx + b^2.$
11. $x^2 - y^2,\ x^2 + 2xy + y^2,\ x^3 + y^3.$
12. $27 - a^3,\ (3 - a)^2,\ 4(3 - a).$
13. $a^2 - 2ab,\ a^2 - 6ab + 8b^2,\ a^2 + ab - 6b^2.$
14. $c^2 - cd - 2d^2,\ cm + dm - cn - dn.$

ADDITION AND SUBTRACTION OF FRACTIONS

193. The process is the same as in arithmetic. If the fractions do not all have the same denominator, they must be reduced to fractions having a common denominator. The *lowest common denominator* (l. c. d.) is obtained by finding the *lowest common multiple* (l. c. m.) of the given denominators.

1. Add $\dfrac{x}{2ab^3} - \dfrac{x^2}{3a^2b} + \dfrac{x^3}{4a^2b^2}.$

The l. c. d. of $2ab^3$, $3a^2b$, $4a^2b^2$ is $12a^2b^3$.

$$12a^2b^3 \div 2ab^3 = 6a, \qquad \frac{x}{2ab^3}\cdot\frac{6a}{6a} = \frac{6ax}{12a^2b^3}.$$

$$12a^2b^3 \div 3a^2b = 4b^2, \qquad -\frac{x^2}{3a^2b}\cdot\frac{4b^2}{4b^2} = -\frac{4b^2x^2}{12a^2b^3}.$$

$$12a^2b^3 \div 4a^2b^2 = 3b, \qquad \frac{x^3}{4a^2b^2}\cdot\frac{3b}{3b} = \frac{3bx^3}{12a^2b^3}.$$

We obtain $\frac{x}{2ab^3}-\frac{x^2}{3a^2b}+\frac{x^3}{4a^2b^2}=\frac{6ax-4b^2x^2+3bx^3}{12a^2b^3}$.

In practice, much of this work can be done mentally and need not be written down.

2. $\frac{2x}{(x-y)^2(x+y)}+\frac{6y}{(x-y)(x+y)^2}$.

The l. c. d. $=(x-y)^2(x+y)^2$.

$(x-y)^2(x+y)^2$ divided by $(x-y)^2(x+y)=x+y$.

$(x-y)^2(x+y)^2$ divided by $(x-y)(x+y)^2=x-y$.

Hence $\frac{2x}{(x-y)^2(x+y)}+\frac{6y}{(x-y)(x+y)^2}=\frac{2x(x+y)+6y(x-y)}{(x-y)^2(x+y)^2}$

$$=\frac{2x^2+8xy-6y^2}{(x-y)^2(x+y)^2}.$$

194. EXERCISES

1. Add $\frac{5}{6}$, $\frac{1}{8}$, $\frac{7}{12}$.

2. Add $\frac{x}{5}$, $\frac{y}{15}$, $\frac{2x}{25}$.

3. Add $\frac{3}{x^2y}$, $\frac{5}{3xy^2}$, $\frac{4}{15x^2y^2}$.

4. Add $\frac{a^2b}{5}$, $-\frac{ab^2}{25}$, $\frac{3a^2b^2}{50}$.

Change to a single fraction and reduce that to its lowest terms:

5. $\frac{x+y}{5}-\frac{x-y}{10}$.

6. $\frac{2}{x+y}-\frac{1}{x-y}$.

7. $\frac{2}{(x-y)^2}+\frac{1}{x-y}$.

8. $\frac{x}{x+y}-\frac{1}{(x+y)^2}$.

9. $\frac{x}{x^2-y^2}-\frac{1}{x+y}$.

10. $\frac{y}{x(x+y)}+\frac{1}{x+y}$.

11. $\frac{x-y}{2x+2y}+\frac{x+y}{3x-3y}$.

12. $\frac{1}{x+y}+\frac{1}{x-y}$.

13. $\frac{1}{x-y}-\frac{1}{x+y}$.

14. $\frac{m}{x(m-x)}+\frac{1}{m-x}$.

15. $\frac{x-y}{x^2+xy}-\frac{x}{x^2-y^2}$.

16. $\frac{4}{6x+15}+\frac{5}{8x+20}$.

17. $\frac{y-1}{15y-5}-\frac{y-2}{18y-6}$.

18. $\frac{x+2}{5x+5}+\frac{x-2}{7x+7}$.

19. $\dfrac{30x}{9x^2-1}+\dfrac{4}{3x-1}-\dfrac{5}{3x+1}$.

20. $\dfrac{1}{(z-x)(x-y)}+\dfrac{1}{(x-y)(y-z)}+\dfrac{1}{(y-z)(z-x)}$.

21. $\dfrac{1}{(a+b)(a+c)}+\dfrac{1}{(a+b)(b+c)}+\dfrac{1}{(b+c)(a+c)}$.

EQUATIONS CONTAINING FRACTIONS

195. In solving equations containing fractions it is usually best to multiply both sides of the equation by an expression which removes all fractions.

1. Solve $\dfrac{x}{6}+\dfrac{x-2}{3}=\dfrac{7}{3}$.

Multiplying both sides by 6, $x+2x-4=14.$

$$3x=18.$$
$$x=6.$$

Check: Substituting 6 for x in the original equation,

$$\tfrac{6}{6}+\tfrac{4}{3}=\tfrac{7}{3}.$$
$$\tfrac{7}{3}=\tfrac{7}{3}.$$

2. Solve $\dfrac{1}{x}+\dfrac{1}{x+1}=\dfrac{2}{x-1}$.

The l. c. d. $=x(x+1)(x-1)$. It is better to retain the l. c. d. in the factored form.

Multiplying both sides by $x(x+1)(x-1)$, we obtain

$$(x+1)(x-1)+x(x-1)=2x(x+1).$$
$$x^2-1+x^2-x=2x^2+2x.$$
$$-3x=1.$$
$$x=-\tfrac{1}{3}.$$

Check:

$$-3+\frac{1}{\frac{2}{3}}=\frac{2}{-\frac{4}{3}}.$$
$$-3+\tfrac{3}{2}=-\tfrac{3}{2}.$$
$$-\tfrac{3}{2}=-\tfrac{3}{2}.$$

3. Solve $\dfrac{3}{x^2+x}+\dfrac{4}{x+1}=\dfrac{11}{6x-6}$.

The l. c. d. $=6x(x+1)(x-1)$.

Multiply both sides by the l. c. d.

$$18x - 18 + 24x^2 - 24x = 11x^2 + 11x.$$
$$13x^2 - 17x - 18 = 0.$$
$$(x-2)(13x+9) = 0.$$
$$x = 2 \text{ or } -\tfrac{9}{13}.$$

EXERCISES

196. Solve the following:

1. $\frac{1}{x} = \frac{7}{8}$.

2. $\frac{2}{3} = \frac{5}{2x}$.

3. $\frac{2x-1}{2} + \frac{x-2}{3} = \frac{2}{3}$.

4. $\frac{x+1}{3} + \frac{x}{4} = \frac{13}{4}$.

5. $\frac{2}{3}x + 5x = 7$.

6. $\frac{6-x}{3-x} - \frac{4}{3-x} = \frac{4}{3}$.

7. $\frac{0.6}{x} = \frac{7}{12}$.

8. $\frac{3x}{x+2} = \frac{1}{x-2}$.

9. $\frac{13x}{x+4} = \frac{14x}{x+5}$.

10. $\frac{.001}{x+2} = \frac{.025}{24}$.

11. $\frac{11.5x+3}{4} = \frac{1.75}{6}$.

12. $\frac{3x-1}{x+7} = \frac{1}{5}$.

13. $2x + \frac{1}{3}(4+x) = 3\frac{2}{3}$.

14. $\frac{x-1}{2x-3} = \frac{3x-4}{6x-5}$.

15. $\frac{1}{3}x - .5x = \frac{5}{6}$.

16. $\frac{2x^2}{2x^2+1} = \frac{x}{x-1}$.

17. $\frac{2}{x-1} - \frac{1}{2-x} = \frac{6}{x}$.

18. $\frac{1}{x-1} + \frac{2}{x-2} = \frac{5}{6}$.

19. $\frac{2}{x-1} + \frac{3}{x-2} = 0$.

20. $y + 2 = \frac{25}{y+2}$.

21. $\frac{x-1}{x+1} + \frac{10}{x-1} = 0$.

22. $\frac{x+1}{x-1} = \frac{3}{x-3}$.

23. $\frac{x-5}{4} = \frac{4}{5-x} + \frac{3x-1}{4}$.

24. $\frac{x^2+1}{x-1} = 2x+1$.

25. $x + 12 = -\frac{36}{x}$.

26. $\frac{1}{x-4} = -\frac{x}{4}$.

27. $\dfrac{1}{x-1}+\dfrac{2}{x-2}=\dfrac{29}{90}.$

28. $\dfrac{x+1}{x-1}+\dfrac{3-x}{3+x}=\dfrac{8}{x}.$

29. $\dfrac{3}{z+1}-\dfrac{2}{z+2}=\dfrac{1}{z+3}.$

30. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x+3.$

PROBLEMS

197. 1. By what number must 72 be divided, that the quotient may exceed the divisor by 6?

2. What number, when divided by 13, yields a quotient that is less than the number by 168?

3. A is 8 years old, B is 33 years old. When will A be $\frac{1}{2}$ as old as B?

4. A is 36 years old, B is 48 years old. How many years ago was A $\frac{1}{3}$ as old as B?

5. What fraction, equal to $\frac{1}{3}$, becomes equal to $\frac{3}{4}$ when 15 is added to its numerator and denominator?

Hint. Let $\dfrac{x}{3x}$ be the fraction.

6. What must be the dimensions of a rectangle containing 192 sq. in., in order that the perimeter be 56 in.?

7. A rectangle whose area is $\frac{1}{6}$ sq. ft. has a perimeter of $1\frac{2}{3}$ ft. Find its dimensions.

8. Divide 84 into two parts, such that the fraction formed by these parts is $\frac{5}{7}$.

9. A number exceeds the sum of its one third, one fourth, and one fifth by 26. Find the number.

10. The rate of one train running between two towns exceeds the rate of another by 10 miles an hour. The difference in time is 1 hour. Find the rate of each train, if the towns are 200 miles apart.

Negative answers for the rates may be rejected as not permissible in our problem.

11. An oil tank can be filled by one pump in 6 hours, or by another pump in 12 hours. How long will it take to fill the tank when both pumps are working?

Let $x =$ time in hours, when both pumps are working.

Then $\frac{1}{x} =$ part of tank filled by both pumps in 1 hr.

$\frac{1}{6} =$ part of tank filled by the first pump in 1 hr.

$\frac{1}{12} =$ part of tank filled by the second pump in 1 hr

Hence $$\frac{1}{6} + \frac{1}{12} = \frac{1}{x}\cdot \text{ Solve.}$$

12. How long will it take two pumps to fill an oil tank, when one pump alone can fill it in 6 hr., the other pump alone can fill it in 8 hr.?

13. Two pumps working at the same time can fill an oil tank in 5 hr. One pump working alone can fill it in 8 hr. How long will it take the second pump alone to fill it?

14. In a debating society a motion was carried 4 to 3; on a reconsideration, 4 affirmative votes changed over to the negative, and the motion was lost 10 to 11. Find the number that voted in the affirmative on the first ballot.

Hint. Let the affirmative and negative vote on the first ballot be $4x$ and $3x$, respectively.

15. Find the world's production of rubber in a year, when South America produced $\frac{1}{3}$, Africa $\frac{1}{5}$ of the total amount, and the rest of the world produced 2800 tons.

16. A man spends $\frac{1}{4}$ of his income in board, $\frac{1}{10}$ in clothes, $\frac{1}{8}$ in sundries, and has \$630 left. What is his income?

17. A man placed at interest \$6000 at 5%, and 6 months later \$5500 at 6%. When will the interest on the two sums be equal?

18. Two ball teams in a league have the following record: A won 50, lost 24, B won 56, lost 24 games. The two teams play a final series of 10 games together. How many must A win in order that the ratio of games won to games lost be equal for both teams?

19. The height of a bin is 1 foot less than the width and 4 feet less than the length. Find the capacity of the bin if the height is equal to half the length.

HISTORY OF FRACTIONS

198. Fractions occur in some of the earliest historic records, but ancient peoples experienced great difficulty in computing with fractions. At the present time fractions may have any number as numerator and any number as denominator. Not so in olden times. The Babylonians, as early as 2000 B.C., perhaps even earlier, used fractions with the denominator 60. In measuring time and angles, they used subdivisions on the scale of 60. The subdivisions of the hour into 60 minutes and the minute into 60 seconds, as also of a degree (angular measure) into 60 minutes and a minute into 60 seconds, are of Babylonian origin. Every time we measure angles and every time we consult our watches we are using units of measure invented by astronomers near the banks of the Ganges over 3000 years ago! When the Babylonians took 60 as the denominator of all fractions, there was no need of actually writing the denominator. They wrote simply the numerator. To distinguish it from a whole number, they placed it a little to the right of the ordinary position for a word or number.

A fractional system involving the same denominator was in vogue also among the Romans. They usually took 12 as the denominator; with them 12 was also the number of subdivisions of weights and measures; the coin called *as* was subdivided into 12 *unciæ*, the foot into 12 inches.

While the Babylonians and Romans preferred the use of fractions with the same denominator and different numerators, the Egyptians and Greeks had, as a rule, fractions with constant numerators and different denominators. In the Ahmes papyrus, an old Egyptian mathematical treatise found about fifty years ago, much attention is given fractions. All fractions in that papyrus, except the fraction $\frac{2}{3}$, have the numerator unity. In this case there was no need of writing the numerators; a fraction was designated by writing the denominator and then placing over it a dot or another simple mark. Two ninths was indicated in the Ahmes papyrus as $\frac{1}{6}\ \frac{1}{18}$, the sum of $\frac{1}{6}$ and $\frac{1}{18}$ being $\frac{2}{9}$.

The more modern point of view in computation with fractions appeared among the Hindus after the fifth century of our era and, somewhat later, among certain Arabic scholars, and among European writers of the renaissance. The Hindus wrote the numerator over the denominator, but did not separate the two by a fractional line. Among the first to use this line was the Italian mathematician Leonard of Pisa (thirteenth century). At the beginning of the sixteenth century the fractional line had come into general use.

Even as late as the fifteenth century certain European writers of prominence experienced difficulty in explaining the multiplication of fractions. If to "multiply" means to "increase," how can the product of two proper fractions be smaller than either of the two fractions? The earliest algebraist to make extensive use of letters as the representatives of numbers was the Frenchman, Vieta (1540–1603). The use of fractions involving letters was introduced since his day.

The most conspicuous figure in the invention of decimal fractions is the Belgian mathematician, Simon Stevin. He explained them in a booklet, *La Disme*, published in 1585. He did not use the decimal point. He had a clumsy notation; he wrote the fraction 5.912 thus $\overset{0}{5}\ \overset{1}{9}\ \overset{2}{1}\ \overset{3}{2}$ or thus 5 ⓪ 9 ① 1 ② 2 ③. The small digit above, or inside a circle, indicated the decimal place of the digit affected. Among the first, if not the first, to simplify the notation of decimals by the use of the decimal point, or comma, was John Napier, of Scotland, in publications of 1616 and 1617.

Thus it is seen that the modern fraction and the modern methods of computing with fractions are the result of many stages of evolution, reaching over a period of not less than 4000 years, and that Asiatic, African, and European nations shared in effecting this evolution.

CHAPTER XV

RADICALS AND GENERAL EXPONENTS

199. All numbers of elementary algebra can be placed in one of two groups, — *real numbers* or *imaginary numbers.*

$\sqrt{2}$, $\frac{2}{3}$, -7, 4.675 are real numbers.

$\sqrt{-3}$, $-2\sqrt{-1}$, $\frac{2}{3}\sqrt{-1}$ are imaginary numbers.

Imaginary numbers or expressions are numbers which involve the square roots of negative numbers. *Real numbers* are rational or irrational. *Rational numbers* are positive or negative integers, and numbers which are the quotients of such integers, such as 9, -5, $\frac{7}{8}$, $.125$.

Irrational numbers or *irrationals* are numbers that are real but not rational, such as $\sqrt{2}$, $\sqrt[3]{9}$, $\sqrt[4]{5}$.

A *radical* is a root of any algebraic expression, indicated by the use of a radical sign, such as $\sqrt{9}$, $\sqrt[3]{6}$, $\sqrt{x^2-x-6}$.

From this definition it is evident that a radical may be a rational or an irrational number.

The integer of the rational sign is called its *index.* Thus the index of the radical sign $\sqrt[4]{}$ is 4. If no index appears, the index 2 is understood.

A number by which a radical is multiplied is called its *coefficient.*

Thus, in $7\sqrt{5}$, 7 is the coefficient of the radical $\sqrt{5}$.

200. The value of an irrational number, like $\sqrt{2}$, cannot be expressed exactly in Hindu-Arabic numerals, but we may represent it accurately by means of lines. For instance, the diagonal of a square whose side is 1 is equal to $\sqrt{2}$. We cannot find $\sqrt{2}$ exactly, but we can draw the square and its diagonal.

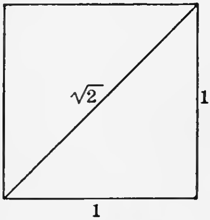


FIG. 34

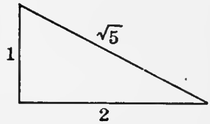


FIG. 35

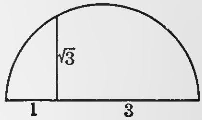


FIG. 36

Likewise, in the right triangle whose legs are 1 and 2, the hypotenuse is $\sqrt{1^2+2^2}=\sqrt{5}$.

In this semicircle whose diameter is 4, the perpendicular erected at the distance 1 from one end equals $\sqrt{3}$, as is shown in geometry.

FRACTIONS AS EXPONENTS

201. All the exponents used thus far have been positive integers. We defined a^5 as $a \cdot a \cdot a \cdot a \cdot a$. See § 14. That is, the exponent 5 indicates that a is taken as a factor five times.

What is the meaning of $a^{\frac{1}{2}}$? It would be absurd to say that $a^{\frac{1}{2}}$ signifies a taken as a factor one half times. A number can be taken as a factor only a whole number of times; for example, three times or ten times.

For the purpose of attaching a meaning to $a^{\frac{1}{2}}$, we stipulate that we shall be able to multiply $a^{\frac{1}{2}}$ by $a^{\frac{1}{2}}$ according to the same rule by which a^2 is multiplied by a^2. The product of a^2 and a^2 is found by *adding the exponents*. That is, $a^2 \cdot a^2 = a^4$.

If we add the exponents in the multiplication of $a^{\frac{1}{2}}$ by $a^{\frac{1}{2}}$, we obtain $a^{\frac{1}{2}+\frac{1}{2}}$ or a^1. That is,

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a.$$

It is seen that $a^{\frac{1}{2}}$ is *one of the two equal factors of a*, or the square root of a.

Hence $a^{\frac{1}{2}}$ *is another way* of writing the square root of a.* That is, $a^{\frac{1}{2}} = \sqrt{a}$.

In the same way, since

$$a^{\frac{3}{2}} \cdot a^{\frac{3}{2}} = a^{\frac{3}{2} + \frac{3}{2}} = a^3,$$

$a^{\frac{3}{2}}$ is another way of writing the square root of a^3.

That is, $$a^{\frac{3}{2}} = \sqrt{a^3}.$$

Similarly, since

$$a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{2}{3} + \frac{2}{3} + \frac{2}{3}} = a^2,$$

it follows that $$a^{\frac{2}{3}} = \sqrt[3]{a^2},$$

and $$x^{\frac{5}{4}} = \sqrt[4]{x^5}, \text{ etc.}$$

Thus *the numerator of a fractional exponent indicates the power of the base and the denominator indicates the root.**

202. In finding the value of $8^{\frac{2}{3}}$, we may take the cube root of 8, which is 2, and square 2, getting 4 as a result.

Or we may square 8, which gives us 64, and then take the cube root of 64, which is 4.

That is, $$8^{\frac{2}{3}} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2.$$

All irrational numbers which are expressed by the use of radical signs can be expressed by the use of fractional exponents. In fact, the simplification of expressions can be effected more easily by the latter, as will appear later.

Thus, $$\sqrt[4]{16\,a^6b} = 16^{\frac{1}{4}}a^{\frac{6}{4}}b^{\frac{1}{4}} = 2\,a^{\frac{3}{2}}b^{\frac{1}{4}}.$$

* It can be shown that $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$ has n different root values, but in elementary algebra only one of these values is usually considered, the so-called *principal value*. If a is a *positive* number, then the principal value of $\sqrt[n]{a}$ is its positive root; if a is *negative* and the index n is *odd*, there is no positive root, and the negative root is taken as the principal root. If a is negative and the index n is *even*, then all the roots are imaginary.

ORAL EXERCISES

203. Express with fractional exponents and simplify:

1. $\sqrt[3]{a^2b}$.
2. $\sqrt{xy^2}$.
3. $\sqrt{3\,mn}$.
4. $\sqrt[4]{2\,x^3y^5}$.
5. $\sqrt{5\,xy^3z^2}$.
6. $a\sqrt[5]{ax^3}$.
7. $2\sqrt{2\,x^5}$.
8. $\sqrt[a]{x^b} \cdot \sqrt[a]{x^{2b}}$.
9. $24\,a^3\sqrt[3]{ay^4}$.
10. $7\sqrt[4]{16\,c^2d^3}$.
11. $4\sqrt[3]{a^2}$.
12. $-2\sqrt[5]{xy^3z^4}$.
13. $6\sqrt[7]{a^5b^3}$.
14. $\sqrt{m} \cdot \sqrt[3]{m} \cdot \sqrt[4]{m}$.
15. $\frac{2}{3}\sqrt[4]{w^3} \cdot \frac{5}{4}\sqrt{w}$.
16. $3\sqrt{\frac{a}{b}}$.
17. $-7\sqrt[4]{\frac{x}{y^2}}$.
18. $5\sqrt[n]{c^m}$.
19. $\sqrt{a+b}$.
20. $3\sqrt[3]{(a-b)^2}$.

EXERCISES

204. Write with radical signs:

1. $a^{\frac{3}{4}}$.
2. $b^{\frac{5}{2}}$.
3. $x^{\frac{2}{3}}y^{\frac{4}{3}}$.
4. $(7\,a)^{\frac{1}{2}}$.
5. $2\,(5\,x)^{\frac{1}{3}}$.
6. $2\,a^{\frac{2}{3}}$.
7. $(2\,a)^{\frac{2}{3}}$.
8. $2\,x^{\frac{1}{2}}y^{\frac{1}{3}}$.
9. $6\,a^{\frac{5}{6}}y^{\frac{1}{3}}$.
10. $7\,(3\,m)^{\frac{4}{5}}$.
11. $a^{\frac{1}{2}}b^{\frac{1}{2}}$.
12. $3\,x^{\frac{5}{4}}$.
13. $(x+y)^{\frac{3}{4}}$.
14. $4\,(c-d)^{\frac{2}{3}}$.
15. $5\,a^{\frac{2}{3}}(x-y)^{\frac{1}{2}}$.
16. $ab^{\frac{m}{n}}$.

MEANING OF A ZERO EXPONENT

205. Thus far, no meaning has been assigned to a°. Can we not discover a meaning? By the law of exponents in division (*i.e.* subtracting the exponent in the divisor from that in the dividend) we obtain $a^3 \div a^3 = a^0$. But when a is not zero, $a^3 \div a^3 = \dfrac{a^3}{a^3} = 1. \quad \therefore a^0 = 1.$

Therefore *any number except* 0 *with a zero exponent is equal to* 1.

MEANING OF A NEGATIVE EXPONENT

206. Reasoning as in § 205, we obtain

$$a^5 \div a^8 = a^{-3}, \text{ also } a^5 \div a^8 = \frac{a^5}{a^8} = \frac{1}{a^3}. \quad \therefore a^{-3} = \frac{1}{a^3}.$$

Therefore, *any quantity with a negative exponent is equal to* 1 *divided by that quantity with the exponent positive.*

Further, $$\frac{a^{-2}b^3}{a^{-3}b^{-4}c} = \frac{\dfrac{1}{a^2} \cdot b^3}{\dfrac{1}{a^3} \cdot \dfrac{1}{b^4} \cdot c} = \frac{\dfrac{b^3}{a^2}}{\dfrac{c}{a^3b^4}} = \frac{a^3b^4 \cdot b^3}{a^2c} = \frac{ab^7}{c}.$$

Therefore, *a factor may be moved from the numerator to the denominator of a fraction provided the sign of its exponent be changed.*

Care must be taken to transfer only *factors;* in $\dfrac{a^{-2}+1}{b}$, a^{-2} is not a factor of the numerator. $\dfrac{a^{-2}+1}{b}$ is *not* equal to $\dfrac{1+1}{a^2b}$. Since $a^{-2} = \dfrac{1}{a^2}$, it follows that $\dfrac{a^{-2}+1}{b} = \dfrac{\dfrac{1}{a^2}+1}{b} = \dfrac{1+a^2}{a^2b}$.

EXERCISES

207. Simplify:

1. $36^{-\frac{1}{2}}$.
2. $64^{\frac{1}{2}}$.
3. $125^{\frac{1}{3}}$.
4. $7^0 \cdot 16^{-\frac{1}{4}}$.
5. $9^{\frac{3}{2}}$.
6. $(-27)^{\frac{2}{3}}$.
7. $(\frac{1}{25})^{-\frac{1}{2}}$.
8. $(\frac{1}{64})^{\frac{1}{3}}$.
9. $36^{\frac{1}{2}} \cdot 343^{\frac{1}{3}}$.

10. $(-8)^{-\frac{2}{3}} \div (\frac{1}{225})^{\frac{1}{2}}$.

11. $9^{\frac{1}{2}} \div (\frac{1}{9})^{\frac{1}{2}}$.

12. $(\frac{4}{9})^{\frac{1}{2}} \cdot 2^0$.

13. $(-2)^{-2} \cdot (-3)$.

14. $(\frac{25}{49})^{-\frac{1}{2}} \cdot (x-y)^0$.

15. $\dfrac{1}{2^{-1}} \cdot \dfrac{1}{3}$.

16. $\dfrac{3^{-1}}{4^{-1}}$.

17. $\left(\dfrac{4}{5}\right)^0 \cdot \dfrac{1}{5^{-2}}$.

18. $\dfrac{8^0}{7}$.

19. $\dfrac{9}{3^0}$.

20. $\dfrac{5^0}{6^0}$.

21. $2^{-1}b$.

22. $\dfrac{3x^{-1}y^2}{2^{-2}x^{-1}y^{-3}}$.

23. $\dfrac{6x}{y^{-3}z}$.

24. $\dfrac{4^{-2}a^{-2}}{5^{-1}b^{-3}}$.

25. $a^{-1}+b^{-1}$.

26. $\dfrac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}$.

REDUCTION OF RADICALS

208. A radical is in its simplest form when the following three conditions are all satisfied:

(1) There are no fractions under the radical sign.

$\sqrt{\frac{1}{2}}$ is not in its simplest form.

(2) The number under the radical sign contains no factor raised to a power equal to or exceeding the index of the radical sign.

$\sqrt[3]{a^4}$ is not in its simplest form, because the exponent 4 exceeds the index 3 of the radical sign.

(3) In a radical of the form $\sqrt[m]{a^n}$, m and n must have no common factor other than 1.

$\sqrt[4]{a^2}$ is not in the simplest form, because 2 and 4 have the common factor 2.

Simplify $\sqrt[4]{64\,a^8b^2}$. We obtain

$$\sqrt[4]{64\,a^8b^2} = 2^{\frac{6}{4}} \cdot a^{\frac{8}{4}} \cdot b^{\frac{2}{4}} = 2^{\frac{3}{2}}\,a^2b^{\frac{1}{2}} = 2 \cdot 2^{\frac{1}{2}} \cdot a^2b^{\frac{1}{2}} = 2\,a^2\sqrt{2\,b}.$$

ORAL EXERCISES

209. Simplify:

1. $\sqrt[4]{4}$.
2. $\sqrt[6]{9}$.
3. $\sqrt[4]{25\,a^2}$.
4. $\sqrt[8]{81}$.
5. $\sqrt[6]{64\,x^6}$.
6. $\sqrt[4]{49\,r^2}$.
7. $\sqrt[8]{16\,x^4y^4}$.
8. $\sqrt[4]{\dfrac{25\,m^2}{49\,n^2}}$.
9. $\sqrt[6]{\dfrac{4\,x^2}{9\,y^2}}$.
10. $\sqrt[6]{\dfrac{27\,r^3}{64\,s^3}}$.
11. $\sqrt[4]{\dfrac{16\,a^2b^2}{(a-b)^2}}$.
12. $\sqrt[10]{\dfrac{32\,c^5d^{10}}{(x+y)^5}}$.
13. $\sqrt[5]{32\,x^{10}y^{15}}$.
14. $\sqrt[6]{729\,x^3y^{12}}$.
15. $\sqrt[5]{ab^{10}c}$.
16. $\sqrt[7]{\dfrac{x^{14}y^{21}}{z^7}}$.

210. 1. Simplify $\sqrt{98}$.

$\sqrt{98}=\sqrt{49\cdot 2}$, wherein 49 is the *largest* factor of 98 which is a *perfect square.* $\sqrt{49\cdot 2}=7\sqrt{2}$. $\therefore\ \sqrt{98}=7\sqrt{2}$.

Again $5\sqrt[3]{54}=5\sqrt[3]{27\cdot 2}$, wherein 27 is the *largest* factor of 54 which is a perfect cube.

$$5\sqrt[3]{27\cdot 2}=5\cdot 3\sqrt[3]{2}=15\sqrt[3]{2}.$$

2. Simplify $\sqrt{\frac{2}{3}}$.

Multiply both numerator and denominator by some number which will make the denominator a perfect square; multiplying by 3, $\sqrt{\frac{2}{3}}=\sqrt{\frac{6}{9}}=\sqrt{\frac{1}{9}\cdot 6}$, wherein 1 is the *largest* factor of 6 which is a square, and 9 is used for its denominator. Since $\sqrt{\frac{1}{9}}=\frac{1}{3}$, $\sqrt{\frac{1}{9}\cdot 6}=\frac{1}{3}\sqrt{6}$. $\therefore\ \sqrt{\frac{2}{3}}=\frac{1}{3}\sqrt{6}$.
Again $7\sqrt[3]{\frac{16}{25}}=7\sqrt[3]{\frac{16}{25}\cdot\frac{5}{5}}=7\sqrt[3]{\frac{8}{125}\cdot 10}=\frac{14}{5}\sqrt[3]{10}$.

EXERCISES

211. Simplify:

1. $\sqrt{8}$.
2. $\sqrt{12}$.
3. $\sqrt{20}$.
4. $\sqrt{24}$.
5. $\sqrt{28}$.
6. $\sqrt{40}$.

7. $3\sqrt{44}$.
8. $4\sqrt{18}$.
9. $-6\sqrt{27}$.
10. $5\sqrt{48}$.
11. $2\sqrt{56}$.
12. $7\sqrt{121}$.
13. $-8\sqrt{160}$.
14. $2\sqrt{75}$.
15. $3\sqrt{125}$.
16. $-9\sqrt{300}$.
17. $3\sqrt{162}$.
18. $2\sqrt{320}$.
19. $-\frac{2}{7}\sqrt{147}$.
20. $\frac{5}{9}\sqrt{108}$.
21. $\sqrt[3]{16}$.
22. $2\sqrt[3]{54}$.
23. $5\sqrt[3]{32}$.
24. $7\sqrt[3]{56}$.
25. $3\sqrt[3]{88}$.
26. $\frac{3}{5}\sqrt[3]{250}$.
27. $10\sqrt[3]{10000}$
28. $\sqrt[4]{32}$.
29. $3\sqrt[4]{48}$.
30. $7\sqrt[4]{80}$.
31. $\sqrt[5]{224}$.
32. $\sqrt{x^3}$.
33. $\sqrt{c^5}$.
34. $\sqrt{2y^7}$.
35. $a\sqrt{c^3}$.
36. $b\sqrt{x^2y}$.
37. $m\sqrt{m^3n}$.
38. $5\sqrt{\frac{1}{3}}$.
39. $\sqrt{\frac{4}{5}}$.
40. $\sqrt{\frac{7}{8}}$.
41. $\sqrt{\frac{5}{6}}$.
42. $\sqrt{\frac{1}{12}}$.
43. $\sqrt{\frac{1}{18}}$.
44. $\frac{2}{3}\sqrt{\frac{1}{20}}$.
45. $\sqrt[3]{\frac{2}{3}}$.
46. $\sqrt[3]{\frac{7}{8}}$.
47. $\sqrt[3]{\frac{2}{27}}$.
48. $3\sqrt[3]{\frac{2}{9}}$.
49. $\sqrt[4]{\frac{7}{8}}$.
50. $\sqrt[4]{\frac{5}{9}}$.
51. $3\sqrt[4]{\frac{1}{16}}$.
52. $\sqrt{\frac{1}{x}}$.
53. $\sqrt{\frac{a}{b}}$.
54. $a\sqrt{\frac{1}{a^3}}$.
55. $\sqrt{\frac{1}{x+y}}$.
56. $\frac{x^2}{y}\sqrt[4]{\frac{y}{x^3}}$.
57. $(x-y)\sqrt{\frac{x+y}{x-y}}$.
58. $mn\sqrt{\frac{m^3}{m-n}}$.
59. $\frac{1}{a+b}\sqrt{\frac{(a+b)^2}{a-b}}$.
60. $(x^2-y^2)\sqrt{\frac{1}{x-y}}$.

ADDITION AND SUBTRACTION OF RADICALS

212. *Similar radicals* are radicals having the same radical sign and the same number under the radical sign.

$4\sqrt[3]{a}$, $-2\sqrt[3]{a}$, $b\sqrt[3]{a}$ have the same radical sign $\sqrt[3]{\ }$, and the same number a under the radical sign. Hence they are similar.

Add $2\sqrt{8}$, $3\sqrt{10}$, $5\sqrt{50}$, $-7\sqrt{40}$.

We obtain $2\sqrt{8}+3\sqrt{10}+5\sqrt{50}-7\sqrt{40}$

$$=4\sqrt{2}+3\sqrt{10}+25\sqrt{2}-14\sqrt{10}=29\sqrt{2}-11\sqrt{10}.$$

EXERCISES

213. Simplify:

1. $\sqrt{3}-2\sqrt{3}$.
2. $5\sqrt{2}+3\sqrt{2}-\sqrt{2}$.
3. $8\sqrt{11}-5\sqrt{11}$.
4. $9\sqrt{a}-2\sqrt{a}$.
5. $a\sqrt{b}+b\sqrt{b}$.
6. $2\sqrt{2}-3\sqrt{\frac{1}{2}}$.
7. $5\sqrt{3}+9\sqrt{\frac{1}{3}}$.
8. $\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{6}}$.
9. $\sqrt{\frac{3}{5}}-\sqrt{\frac{5}{3}}$.
10. $\sqrt{\dfrac{a}{b}}+\sqrt{\dfrac{b}{a}}$.
11. $\sqrt{20}+\sqrt{45}-\sqrt{5}$.
12. $2\sqrt{2}-\sqrt{18}+3\sqrt{50}$.
13. $5\sqrt{28}+3\sqrt{63}-\sqrt{112}$.
14. $\sqrt{\frac{1}{2}}-2\sqrt{2}+\sqrt{\frac{9}{2}}$.
15. $\sqrt{a^2b}+\sqrt{b^3}-\sqrt{9\,b}$.
16. $2\sqrt{a^3b}-\sqrt{abc^2}+\sqrt{4\,ab}$.
17. $\sqrt{\dfrac{1}{x}}+\sqrt{\dfrac{1}{x^3}}+\sqrt{\dfrac{1}{x^5}}$.
18. $\sqrt{(a+b)^3}-\sqrt{a^2-b^2}$.
19. $\sqrt{mn}+\sqrt{\dfrac{m}{n}}-\sqrt{\dfrac{1}{mn}}$.
20. $\sqrt{(x^2-y^2)(x-y)}-\sqrt{4\,x^3y^2+4\,x^2y^3}$.
21. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{128}$.
22. $\sqrt[3]{5}-\sqrt[3]{7}+\sqrt[3]{\frac{1}{25}}$.
23. $\sqrt[3]{8\,x-16}+\sqrt[3]{27\,x-54}$.
24. $\sqrt[4]{(a-3)^2}+\sqrt{9\,a-27}$.

MULTIPLICATION OF RADICALS HAVING THE SAME INDEX

214. Multiply $5\sqrt{a}$ by $7\sqrt{b}$ and we have

$$5\cdot 7\cdot \sqrt{a}\cdot \sqrt{b}=35\sqrt{ab}.$$

Multiply $-3\sqrt{2}$ by $4\sqrt{10}$ and we have

$$-3\cdot 4\cdot \sqrt{2}\cdot \sqrt{10}=-12\sqrt{20}=-24\sqrt{5}.$$

RULE. *Multiply the rational factors and the radical factors, and simplify.*

It will be observed that the multiplication of two radicals, $\sqrt{x}$ and $\sqrt{y}$, rests on the identity $\sqrt{xy}=\sqrt{x}\cdot\sqrt{y}$.

ORAL EXERCISES

215. Find the indicated products:

1. $\sqrt{3}\cdot\sqrt{27}$.
2. $\sqrt{x}\cdot\sqrt{x^5}$.
3. $\sqrt{3}\cdot\sqrt{5}$.
4. $\sqrt{3}\cdot\sqrt{6}$.
5. $\sqrt{2a}\cdot\sqrt{3a}$.
6. $\sqrt{6x}\cdot\sqrt{10x^3}$.
7. $\sqrt{abc}\cdot -\sqrt{ac}$.
8. $\sqrt[3]{ab^2c^2}\cdot\sqrt[3]{a^2bc}$.
9. $\sqrt[3]{2x^2y}\cdot\sqrt[3]{4xy}$.
10. $\sqrt[4]{ab^2c^3}\cdot\sqrt[4]{a^3b^2c^2}$.
11. $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{2}{3}}$.
12. $\sqrt{\frac{5}{6}}\cdot\sqrt{1\frac{1}{5}}$.
13. $\sqrt{\frac{7}{8}}\cdot\sqrt{\frac{12}{35}}$.
14. $\sqrt{\frac{x}{y}}\cdot\sqrt{\frac{y}{z}}$.
15. $\sqrt[n]{a^2}\cdot\sqrt[n]{a^{n-2}b}$.
16. $\sqrt[n]{x^2y}\cdot\sqrt[n]{xy^{n-1}}$.
17. $\sqrt[3]{2}\cdot\sqrt[3]{3}$.
18. $\sqrt[4]{\frac{3}{4}}\cdot\sqrt[4]{\frac{4}{3}}$.
19. $\sqrt[6]{\frac{3}{5}}\cdot\sqrt[6]{\frac{5}{4}}$.
20. $5\sqrt[3]{2}\cdot 2\sqrt[3]{5}$.

216. Multiply $\sqrt{2}+\sqrt{3}-\sqrt{5}$ by $\sqrt{2}-\sqrt{3}+\sqrt{5}$.

$$
\begin{array}{llll}
\sqrt{2}+\sqrt{3}-\sqrt{5} & & & \\
\underline{\sqrt{2}-\sqrt{3}+\sqrt{5}} & & & \\
2+\sqrt{6} & -\sqrt{10} & & \\
-3-\sqrt{6} & & & +\sqrt{15} \\
\underline{-5\phantom{-\sqrt{6}}} & \underline{+\sqrt{10}} & & \underline{+\sqrt{15}} \\
-6 & & & +2\sqrt{15}
\end{array}
$$

RULE. *Multiply each term of one polynomial by each term of the other, simplify, and combine.*

EXERCISES

217. Multiply:

1. $\sqrt{6}-\sqrt{7}-2\sqrt{2}$ by $\sqrt{3}$.
2. $\sqrt{7}+\sqrt{3}-\sqrt{11}$ by $\sqrt{5}$.
3. $3\sqrt{2}-2\sqrt{3}+\sqrt{5}$ by $\sqrt{6}$.

4. $\sqrt{5}-4\sqrt{2}+\sqrt{10}$ by $\sqrt{10}$.

5. $\sqrt{3}+7\sqrt{6}-5\sqrt{12}$ by $\sqrt{6}$.

6. $3+\sqrt{2}$ by $7-\sqrt{3}$.

7. $8-\sqrt{6}$ by $\sqrt{5}-4$.

8. $3\sqrt{2}+4\sqrt{3}$ by $2\sqrt{2}-5\sqrt{3}$.

9. $2\sqrt{7}+7\sqrt{2}$ by $2\sqrt{7}-7\sqrt{2}$.

10. $5\sqrt{3}+3\sqrt{5}$ by $5\sqrt{3}-3\sqrt{5}$.

11. $\sqrt{6}-\sqrt{7}+\sqrt{8}$ by $\sqrt{8}+\sqrt{7}-\sqrt{6}$.

12. $2\sqrt{5}-3\sqrt{6}-4\sqrt{7}$ by $\sqrt{10}-\sqrt{5}-\sqrt{2}$.

13. $\sqrt{\frac{1}{5}}+\sqrt{\frac{1}{6}}-\sqrt{\frac{1}{7}}$ by $\sqrt{5}+\sqrt{6}-\sqrt{7}$.

14. $x\sqrt{y}+\sqrt{xy}-y\sqrt{x}$ by $\sqrt{x}+\sqrt{y}$.

15. $\sqrt[3]{2}-\sqrt[3]{3}+\sqrt[3]{4}$ by $\sqrt[3]{4}+\sqrt[3]{9}-\sqrt[3]{2}$.

DIVISION OF RADICALS HAVING THE SAME INDEX

218. 1. Divide $2\sqrt{2}$ by $\sqrt{3}$.

(*a*) $2\sqrt{2}\div\sqrt{3}=2\sqrt{\frac{2}{3}}=\frac{2}{3}\sqrt{6}$, or

(*b*) $2\sqrt{2}\div\sqrt{3}=\dfrac{2\sqrt{2}}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{2\sqrt{6}}{3}=\dfrac{2}{3}\sqrt{6}$.

Unless the division of the quantities under the radical signs gives a whole number, the second method is preferable.

In (*b*) we express the division in the form of a fraction, then multiply both numerator and denominator by some number which will make the denominator rational.

2. Divide 3 by $2+\sqrt{3}$.

$$3\div(2+\sqrt{3})=\frac{3}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{6-3\sqrt{3}}{4-3}=6-3\sqrt{3}.$$

Again, $(3\sqrt{2}+4\sqrt{3})\div(3\sqrt{2}-4\sqrt{3})$

$$=\frac{3\sqrt{2}+4\sqrt{3}}{3\sqrt{2}-4\sqrt{3}}\cdot\frac{3\sqrt{2}+4\sqrt{3}}{3\sqrt{2}+4\sqrt{3}}=\frac{18+24\sqrt{6}+48}{18-48}=\frac{66+24\sqrt{6}}{-30}=-\frac{33+12\sqrt{6}}{15}.$$

If the denominator is of the form $\sqrt{a} \pm \sqrt{b}$, multiply both terms of the fraction by $\sqrt{a} \mp \sqrt{b}$. This is called *rationalizing* the denominator.

To find the numerical value of $2\sqrt{2} \div \sqrt{3}$ in its present form requires the extracting of two square roots and a long division. It is much less work to rationalize the denominator, then in the resulting expression, $\frac{2}{3}\sqrt{6}$, find $\sqrt{6}$, and take $\frac{2}{3}$ of it.

EXERCISES

219. Divide:

1. $\sqrt{2}$ by $\sqrt{5}$.
2. 1 by $\sqrt{3}$.
3. 24 by $\sqrt{6}$.
4. 2 by $\sqrt[3]{2}$.
5. 36 by $\sqrt{48}$.
6. 6 by $2-\sqrt{3}$.
7. a by $\sqrt{a+b}$.
8. $\sqrt{x+y}$ by $\sqrt{x-y}$.
9. 5 by $\sqrt{5}+1$.
10. $3\sqrt{7}$ by $\sqrt{7}-4$.
11. $\sqrt{2}+\sqrt{3}$ by $\sqrt{5}+\sqrt{6}$.
12. $\sqrt{3}+\sqrt{5}$ by $\sqrt{2}-\sqrt{6}$.
13. $\sqrt{10}-\sqrt{2}$ by $\sqrt{3}-\sqrt{5}$.
14. $\sqrt{15}-\sqrt{3}$ by $\sqrt{5}+\sqrt{6}$.
15. $\sqrt{11}-\sqrt{12}$ by $\sqrt{11}+\sqrt{12}$.
16. $2\sqrt{2}+3\sqrt{3}$ by $2\sqrt{2}-3\sqrt{2}$.
17. $2\sqrt{3}-3\sqrt{2}$ by $2\sqrt{3}+3\sqrt{2}$.
18. $a\sqrt{b}+b\sqrt{a}$ by $a\sqrt{b}-b\sqrt{a}$.
19. $a-\sqrt{a+b}$ by $\sqrt{a+b}$.
20. $\sqrt{x+y}-\sqrt{x-y}$ by $\sqrt{x+y}+\sqrt{x-y}$.

RADICAL EQUATIONS

220. 1. Solve $\sqrt{x-2}=5$.

Square both sides, $\quad x-2=25$; $x=27$. *Ans.* 27.

Check: $\quad \sqrt{27-2}=5$; $5=5$.

2. Solve $\sqrt{x+5}+\sqrt{x-1}=3$.

Transpose, $\sqrt{x+5}=3-\sqrt{x-1}$.

Square both sides, $x+5=9-6\sqrt{x-1}+x-1$.

Transpose and combine, $6\sqrt{x-1}=3$.

$\sqrt{x-1}=\frac{1}{2}$.

Square both sides, $x-1=\frac{1}{4}$; $x=\frac{5}{4}$. *Ans.* $\frac{5}{4}$.

Check: $\sqrt{\frac{5}{4}+5}+\sqrt{\frac{5}{4}-1}=3$.

$\sqrt{\frac{25}{4}}+\sqrt{\frac{1}{4}}=3$.

$\frac{5}{2}+\frac{1}{2}=3$.

$3=3$.

221. Rule. *If the equation contains but* **one** *radical, transpose so that it will stand alone.*

If the equation contains **two** *radicals, transpose so that there is* **one radical on each side** *and so that the rational terms are combined on one side.*

Raise both sides to a power corresponding to the degree of the radicals. Solve the resulting equation.

In squaring both sides of an equation, a value of x may be obtained which will not satisfy the original equation. Such a value is called an *extraneous root*. For instance, squaring both sides of the equation

$$\sqrt{x+5}=1-\sqrt{x},$$

there results $x+5=1-2\sqrt{x}+x$.

Transpose and simplify, $2=-\sqrt{x}$.

Square both sides, $4=x$.

Substitute in $\sqrt{x+5}=1-\sqrt{x}$, $3=1-2$, which is absurd.

Hence $x=4$ is not a root of the original equation, but an extraneous root. In fact, there is no value of x that will satisfy $\sqrt{x+5}=1-\sqrt{x}$. In other words, this equation has no roots.

Only by substitution in the original equation involving radicals can we tell whether a value obtained for x is a root of that equation or whether it is an extraneous root.

EXERCISES

222. Solve and check:

1. $\sqrt{x-5}=2.$
2. $3\sqrt{x-2}=1.$
3. $\sqrt{x+5}=2\sqrt{x+2}.$
4. $3\sqrt{x-3}=\sqrt{72}.$
5. $x-3=-\sqrt{x^2-6}.$
6. $x=5-\sqrt{x^2-x-6}.$
7. $\sqrt[3]{4x-16}=2.$
8. $8-\sqrt{x-1}=5+\sqrt{x+2}.$
9. $x=13-\sqrt{x^2-13}.$
10. $3-\sqrt{x}=\sqrt{x+3}.$
11. $\sqrt{x^2-7x+6}=\sqrt{x^2-x-12}.$
12. $\dfrac{\sqrt{x}+1}{\sqrt{x}-2}=\dfrac{\sqrt{x}-3}{\sqrt{x}+4}.$
13. $\sqrt{x+17}=x-3.$
14. $\sqrt{-x-5}-x=7.$
15. $\sqrt{x+3}=\dfrac{6}{\sqrt{x+3}}.$
16. $\sqrt{x-2}-\dfrac{5}{\sqrt{x-2}}=0.$

CHAPTER XVI

REVIEW OF THE ESSENTIALS OF ALGEBRA

DEFINITIONS

223. Define the following:

Coefficient	Monomial	Linear equation
Exponent	Binomial	Quadratic equation
Term	Polynomial	Root of an equation
Factoring		

EXERCISES ON PARENTHESES

224. Remove the parentheses and simplify:

1. $(5-6)-(7-8)+(5-6-7)-(-5-4+6)$.
2. $11-(3-[4-5])+[4-(3-5)+(4-1)]$.
3. $4(3-2)+4(3\div 2)-4\div(4-2)$.
4. $(4a+b)-(8b-c)-(a-7b)$.
5. $4x^2-7y^2-[7x^2-6x+8y^2]-[5x-3y]$.
6. $3x^2-5xy-\{2xy-4x^2-9y^2\}-\{7x^2-5xy\}$.
7. $a-[b+(c+2d)-(-c-d)]-[2a-3b]$.
8. $-(a-\overline{c+d})+(\overline{2a-3b}+4c)-(4a-[5b-6c])$.

Group all terms with a single letter in a parenthesis with + before it, and all terms with two different letters in a parenthesis with − before it:

9. $a-b+4c-ab+2ac-4bc+a^2b$.
10. $-x+5y-4z+xy-3xz+7yz$.
11. $2x^3-5y^2+3z^2-2xz+2yz+4xy$.
12. $cde-4cd-3de+f-d+2c-3e$.

EXERCISES IN ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION

225. Simplify:

1. $2x + (x-1)^2$.
2. $4t + (2t+1)^2$.
3. $4t + (2t-1)^2$.
4. $4y - (2y+1)^2$.
5. $(x-y)^2 + 4xy$.
6. $2m(m-4) - m^2$.
7. $3(x+y) - 2(x-y)$.
8. $(x+y)^2 + (x-y)^2$.
9. $(a-b)^2 + (a-b)^2$.
10. $a(a-2b) + (a+b)(a-b)$.

Find the values of the following at sight:

11. $4^3, 3^4, 4^1, 1^4, 2^0, 8^0, 25^{\frac{1}{2}}$.
12. $a^2a^3, a^4a, 4c^4c^3, x^0x^5, y^5y^5$.
13. $\frac{a^4}{a^9}, \frac{b^7}{b^{10}}, \frac{4c^3}{5c^5}, x^8 \div x^5, x^6 \div x^8$.
14. $\frac{4a^2b^3}{6a^3b}, \frac{12a^3b^4}{4ab^2}, \frac{15x^5y^5}{25x^5y}, \frac{75x^2yz}{15x^3y^2z}$.

Simplify:

15. $(a+b)^2 + (a-b)^2$.
16. $(a-2b)^2 + (a+b)^2$.
17. $(a+b-2c)^2$.
18. $(a+2b)^2 - (a+b)(a-b)$.
19. $(2a-b)^2 + (a-b)^2$.
20. $(a-b)^2 - (a+2b)(a-2b)$.

Multiply orally:

21. $(x+3)(x+9)$.
22. $(y+7)(y+4)$.
23. $(7-z)(5-z)$.
24. $(a-6)(a+8)$.
25. $(b-5)(b-4)$.
26. $(2-z)(3-z)$.
27. $(x^2+7)(x^2+5)$.
28. $(ay+1)(ay+4)$.

Perform the indicated divisions:

29. $(y^3 + 2y^2 - y - 2) \div (y+1)$.
30. $(y^3 + 21y + 342) \div (y+6)$.
31. $(4x^5 - 5x^4 + 20x^3 + 50x^2 + 2x - 12) \div (x^3 - 2x^2 + 6x + 5)$.

EXERCISES IN FACTORING

226. Factor:

1. $10\,xy^3 - 15\,x^2y^3$.
2. $(2\,x + y)\,a + (2\,x + y)\,b$.
3. $5\,x(a + b) - 3\,x(a + b)$.
4. $x^6 - 5\,x^5 - 3\,x^4 + 3\,x^3$.
5. $x^6 - 5\,x^5 + x^4 - 5\,x^3$.
6. $(a - b)\,a^2 - (a - b)\,b^2$.
7. $(x^2 - 1)\,m + (x^2 - 1)\,n$.
8. $3\,x^2(x - 1) - (1 - x)$.
9. $3\,m(a^2 + b^2) - 2\,n(b^2 + a^2)$.
10. $(m - n)^2 - (m - n)$.
11. $ab - ax + by - xy$.
12. $a^3 - a^2b + ab^2 - b^3$.
13. $y^2 + 10\,y + 25$.
14. $x^2 + 8\,x + 16$.
15. $a^2b^2 + 12\,ab + 36$.
16. $a^2 + b^2 + c^2 - 2\,ab - 2\,ac + 2\,bc$.
17. $m^2 - 49$.
18. $64 - x^2$.
19. $81\,x^4 - 25\,y^6$.
20. $(a - b)^2 - c^2$.
21. $x^2 - (5\,x + y)^2$.
22. $a^2 + 2\,ab + b^2 - c^2$.
23. $x^2 - 5\,x + 4$.
24. $x^2 + 5\,x + 4$.
25. $y^2 + 6\,y - 16$.
26. $y^2 - 6\,y - 16$.
27. $x^2 - 7\,x + 10$.
28. $x^2 + 7\,x + 10$.
29. $x^2 + 3\,x - 10$.
30. $x^2 - 3\,x - 10$.
31. $a^2 + 7\,a - 18$.
32. $b^2 - 12\,b - 45$.
33. $c^2 + 14\,c - 32$.
34. $x^2 + 10\,x - 75$.
35. $x^2 - 10\,x - 75$.
36. $x^2 - 7\,xy - 8\,y^2$.
37. $x^2 + 15\,x^2y + 14\,x^2y^2$.
38. $70\,a^2 + 17\,a + 1$.
39. $2\,x^2 + 9\,x + 9$.
40. $2 - y - 6\,y^2$.
41. $6 - m - 15\,m^2$.
42. $a^3 + 8\,b^3$.
43. $a^3 - 8\,b^3$.
44. $8\,x^3 - 27\,y^3$.
45. $8\,x^3 + 27\,y^3$.
46. $x^4 - 1$.

EXERCISES ON FRACTIONS

227. 1. Show that $\frac{+a}{+b}=\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}=+\frac{a}{b}.$

2. Show that $\frac{+a}{-b}=\frac{-a}{+b}=-\frac{+a}{+b}=-\frac{a}{b}.$

3. $\frac{1}{x-y}=\frac{?}{y-x}$, $\frac{1}{(x-y)(z-x)}=\frac{?}{(x-y)(x-z)}.$

4. $\frac{1}{(x-y)^2}=\frac{?}{(y-x)^2}$, $\frac{1}{(x-y)(x-z)}=\frac{?}{(y-x)(z-x)}.$

Reduce each fraction to its lowest terms:

5. $\frac{6(x+y)}{3(x+y)}.$

6. $\frac{5(x+y)^3}{20(x+y)^2}.$

7. $\frac{a-b}{2a-2b}.$

8. $\frac{a^2+ab}{a^2-ab}.$

9. $\frac{3-2c}{9-4c^2}.$

10. $\frac{bx+ax}{na^2-nb^2}.$

11. $\frac{5x^2+5xy}{x^2-y^2}.$

12. $\frac{x^2+x-6}{(x-2)^3}.$

Multiply:

13. $\frac{8x}{y^3}$ by $\frac{5yz^3}{16x^2}.$

14. $\frac{15x^2y^3}{17z^2}$ by $\frac{34z^3}{5xy^4}.$

15. $\frac{x-y}{2x}$ by $\frac{10x}{y^2-x^2}.$

16. $\frac{xy^2-y^3}{x^2+xy}$ by $\frac{x^3-xy^2}{5y^2}.$

Divide:

17. $\frac{x-3}{x+1}$ by $\frac{x^3-27}{x^2+2x+1}.$

18. $\frac{m-n}{c-n}$ by $\frac{c+n}{m+n}.$

19. $\frac{6m^3}{n^3-m^3}$ by $\frac{3m}{m-n}.$

20. (a^2-x^2) by $\frac{a+x}{a}.$

21. $\frac{a^4-4b^2}{a+b}$ by $(a^2-2b).$

22. $\frac{x^2-5x+6}{x^2-2x-35}$ by $\frac{x-3}{x+5}.$

Perform the indicated additions and subtractions of fractions:

23. $\frac{1}{a^3} - \frac{a^2 - 1}{a^5}$.

24. $\frac{x}{2y} + \frac{5x}{3y} - \frac{3x}{8y}$.

25. $\frac{1}{(x-y)^3} - \frac{1}{(y-x)^2}$.

26. $\frac{2}{(x-1)^3} - \frac{1}{(1-x)^3}$.

27. $\frac{2}{x-y} + \frac{3}{x+y}$.

28. $\frac{3}{a-3} - \frac{4}{a-4}$.

Reduce each expression to an equivalent fraction:

29. $3x + \frac{1-5x}{4}$.

30. $2y + \frac{5x-3y}{4}$.

31. $3a - \frac{2a^2}{a+b}$.

32. $m - \frac{3m-4}{3-m}$.

33. $x^2 + xy + y^2 + \frac{y^3}{x-y}$.

34. $x^2 - xy + y^2 - \frac{y^3}{x+y}$.

EXERCISES IN LINEAR EQUATIONS

228. Solve:

1. $8x + 19 = 11 - 5x + 13$.

2. $\frac{1}{2}y + 7 = -\frac{1}{2}y + 8$.

3. $\frac{1}{5}x = 12 - x$.

4. $\frac{5}{3}x - 2 - \frac{3}{5}x = 4x$.

5. $(16x + 5)(9x + 31) = (4x + 12)(36x + 10)$.

6. $x(x+1) + x(x+2) = (x+3)(2x+4)$.

7. $mx = nx + 3$.

8. $ay + by = c$.

9. $2ay - 2by = 16 + 2a$.

10 $ax + bx - cx = d$.

11. $mx - nx - x = m + n$.

12. $(x+a)^2 = (x-b)^2$.

13. $(x-4)^2 - (x-5)^2 = 4x + 5$.

14. $\frac{x+4}{x-5} = 5$.

15. $5 - \frac{2}{x} = 3$.

16. $\frac{x-1}{x+2} = \frac{2}{5}$.

17. $\frac{x-1}{x-4} = \frac{x-3}{x-2}$.

18. $\dfrac{x-1}{2x-1}=\dfrac{x-3}{2x-2}$.

19. $\dfrac{y}{y+1}=\dfrac{3y}{y+2}-2$.

20. $\dfrac{x-1}{x+1}+\dfrac{1}{x}=1$.

21. $\dfrac{x}{x-1}=\dfrac{1}{x+1}+1$.

22. $\dfrac{3x-5}{x-4}=-\dfrac{2}{3}$.

23. $\dfrac{x^2+4}{2x}=\dfrac{x+1}{2}$.

24. $\dfrac{y^2+y+3}{2y+1}=\dfrac{y+1}{2}$.

25. $\dfrac{2x+5}{2x+1}=\dfrac{x+5}{x+7}$.

26. $\begin{cases} x+y=35, \\ x-y=18. \end{cases}$

27. $\begin{cases} 2x+y=74, \\ 2x-y=36. \end{cases}$

28. $\begin{cases} 3x+2y=16, \\ 4x+5y=33. \end{cases}$

29. $\begin{cases} y=3x-5, \\ 4x-2y-5=0. \end{cases}$

30. $\begin{cases} \dfrac{x}{3}-\dfrac{y}{5}=1, \\ x-y=1. \end{cases}$

31. $\begin{cases} -3x+8y=5, \\ x+y=2. \end{cases}$

32. $\begin{cases} 3x+cy=3c^2, \\ 3x-cy=c^2. \end{cases}$

EXERCISES IN QUADRATIC EQUATIONS

229. Solve:

1. $x^2-6x+5=0$.
2. $x^2-5x+5=0$.
3. $x^2-5x=-4$.
4. $(3x-2)(x-1)=14$.
5. $(3x-2)(x-1)=200$.
6. $(2x-1)^2=-8x$.
7. $x^2-2ax=b^2$.
8. $x^2-(a-1)x+a=0$.
9. $x^2+mx+n=0$.
10. $px^2+qx+r=0$.
11. $\dfrac{4}{x-1}=\dfrac{3x}{x^2-1}+2$.
12. $15x+\dfrac{2}{x}=11$.
13. $x+\dfrac{1}{x}=5+\dfrac{1}{5}$.
14. $\dfrac{3}{1+x}+\dfrac{3}{1-x}=8$.
15. $\dfrac{x-1}{c}=\dfrac{d}{x+1}$.
16. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=7$.
17. $\begin{cases} 3x=2y, \\ xy=54. \end{cases}$

18. $\begin{cases} 2x = 3y, \\ -3y^2 = 24 - 2x^2. \end{cases}$

19. $\begin{cases} x^2 + y^2 = 17, \\ 3x - y = -1. \end{cases}$

20. $\begin{cases} y^2 = 4x, \\ x + y = 15. \end{cases}$

21. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5, \\ \dfrac{1}{x} - \dfrac{1}{y} = 1. \end{cases}$

22. $\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} = 4, \\ \dfrac{1}{x} - \dfrac{2}{y} = 2. \end{cases}$

23. $\begin{cases} x^2 + y^2 = 25, \\ x^2 - y^2 = 7. \end{cases}$

24. $\begin{cases} x^2 = 16y, \\ x = 2y. \end{cases}$

PROBLEMS

230. 1. What number added to the numerator and denominator of $\frac{9}{19}$ will give a fraction equal to $\frac{3}{4}$?

2. A train runs 300 miles in a certain time. If the train were to run 10 miles an hour faster, it would take it 1 hour less to travel the same distance. Find the time and rate.

3. The length of the rim of one of the hind wheels of a carriage exceeds the length of the rim of one of the front wheels by 3 ft. The front wheel makes the same number of revolutions in running 200 ft. that the hind wheel makes in running 260 ft. Find the length of the rim of each wheel.

4. A farmer found that the supply of feed for his cows would last only 20 weeks. He therefore sold 50 cows, and his supply lasted 30 weeks. How many cows had he at first?

5. Two trains start together and run in opposite directions, one at the rate of 47 miles an hour, the other at the rate of 34 miles an hour. After how many hours will they be 1053 miles apart?

6. The area of a rectangle is 1850 sq. in. Its length exceeds its width by 13 in. How long is each side?

7. A farmer expected to receive \$5.52 for his eggs. He broke 6 eggs, but by charging 2 ¢ a dozen more for the remaining eggs, he received the desired amount. How many dozen eggs had he originally?

8. Weighed on a defective balance a body appears to weigh P pounds when the true weight is $a + bP$ pounds, where a and b are numbers that are the same for all weights. If the balance shows 10 pounds for a real weight of 11.2 pounds, and 30 pounds for a real weight of 33.2 pounds, find a and b. What will the balance show for a real weight of 40 pounds?

9. According to tradition, the following problem was assigned by Euclid of Alexandria to his pupils, about 3 centuries B.C.: A mule and a donkey were going to market laden with wheat. The mule said to the donkey, "If you were to give me 1 measure, I would carry twice as much as you, if I were to give you 1 measure, our burdens would be equal." What was the burden of each?

10. A room shaped as in Fig. 37 has a floor area of 320 sq. ft. If the lengths marked a are all alike, and b is 22 ft., find a.

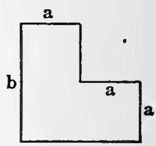


FIG. 37

11. To pass from one corner of a rectangular park to the opposite corner I must go 700 feet, if I go around the sides; if I walk diagonally across, I save 200 feet. What are the dimensions of the park?

12. It takes a cyclist 18 minutes longer to go a distance of 27 miles from A to B than it takes to return by the same route. If he rides up hill 6 miles an hour and down hill 15 miles an hour, how many miles of his trip going is up-hill travel, and how many miles is down hill?

13. A dealer sells bicycles at a 20 % profit. A rival dealer obtains the same kind of bicycles \$3 cheaper and sells them \$3 cheaper, thereby realizing a profit of $21\frac{7}{8}$ %. What price does the first dealer pay for his bicycles?

14. If speculum metal contains 67 % of copper and 33 % of tin (by weight) and gunmetal contains 90 % of copper and 10 % of tin, how many pounds of gunmetal should be melted with 300 pounds of speculum metal, to obtain an alloy in which there is 5 times as much copper as tin?

15. Water in freezing expands 10 % of its volume. How much water when frozen will fill a 5-gallon freezer?

16. In a price list, the cost of sewer pipe, per foot of length, is given by the formula $c = .4\,d^2 + 14$, where d is the diameter of the pipe in inches and c is the cost in cents. What will 400 ft. of 15 in. pipe cost?

PROBLEMS ON THE LEVER

231. **1.** What force must be applied by the hand to balance or raise a known weight $W = 748$ pounds?

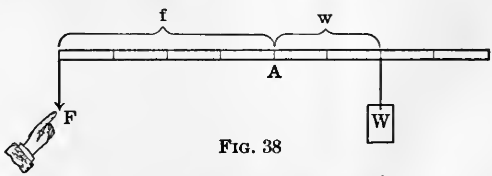


FIG. 38

By careful measurement it has been ascertained that *the force multiplied by the force arm of the lever is equal to the weight multiplied by the weight arm.* Notice that the weight arm is measured from the point of application of the weight to the prop or fulcrum, A, and the force arm is measured from the point of application of the force to the prop. In this case $f = 4$, $w = 2$.

Let F = the required force in pounds, then

$$4\,F = 2 \times 748.$$

$$F = 374 \text{ pounds, the answer.}$$

2. How large a weight can be raised with a force of 500 lb., if the force arm is 7.5 ft. and the weight arm .5 ft.? Draw a figure.

3. The force arm is 16 times greater than the weight arm. What force will balance a weight of 732 pounds?

4. The force arm is 2 ft. longer than the weight arm. How long is the weight arm, if a force of 12 pounds balances a weight of 35 pounds?

5. A lever is 75 in. long. Where must the prop be placed in order that a force of 2 pounds at one end may move 5 pounds at the other end?

6. Two men carry a load of 212 pounds on a pole between them. If the load is 3.5 ft. from one man and 5.4 ft. from the other, how many pounds does each man carry?

7. A man weighing 170 pounds has a crowbar 6 ft. long. What pressure can he exert toward moving a rock, if the prop is 3 in. from the lower end of the crowbar?

8. On an untrue balance a weight of 13 oz. appears to weigh 13.2 oz. If the beam is 6 in. long, and the error is due to a displacement of the fulcrum, how much longer is one arm than the other?

9. A team is hitched to a doubletree 4 ft. long. At what point must the doubletree be attached to the plow, so that one horse will pull $1\frac{1}{5}$ times as much as the other?

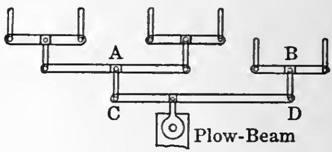


FIG. 39

10. Each of two horses hitched to the doubletree *A* in Fig. 39 pulls $\frac{4}{5}$ as much as a horse at *B*. At what point should a plow beam be attached to the doubletree evener *CD*, which is 5 ft. long?

PROBLEMS ON FALLING BODIES

232. Bodies falling from a state of rest obey laws expressed by the following formulas:

$$v = 32.2\,t,$$
$$s = 16.1\,t^2,$$
$$v = \sqrt{64.4\,s},$$

where v means the velocity, t the time in seconds, s the space or distance (in feet) through which the body falls.

1. Express each of the three formulas in words.

2. Eliminate t from the first two formulas, and thereby derive the third formula.

3. Eliminate s from the last two formulas, and thereby derive the first.

4. Eliminate v from the first and third, and thereby derive the second.

5. A stone dropped into a mine shaft is heard to strike bottom after 5 seconds. Neglecting the time it takes sound to travel, estimate the depth of the shaft.

6. With what velocity will a body strike the bottom of a mining shaft 1000 ft. deep?

7. A stone is heard to strike the bottom of a mine shaft after ten seconds, sound traveling, in this instance, at a rate of 1126 ft. a second. What is the depth of the shaft?

INDEX

Numbers refer to sections

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