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ELEMENTARY COURSE

OF

GEOMETRY.

FOR THE USE OF

SCHOOLS AND COLLEGES.

BY

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PREFACE.

The materials for the following work, like those of the author's treatise on Algebra, have been drawn from the latest and best foreign sources, and from the results of a varied experience of near twenty years as an instructor, commencing at the United States Military Academy.

The definitions of angles and parallel lines, upon which so much depends, will be found quite different from those in ordinary use; yet it is believed that no others hitherto suggested are so direct and distinct, so free from metaphysical objections, or so easily apprehended by the learner; and none, certainly, are productive of so much simplicity, generality, and brevity in the depending demonstrations.

The treatment of proportions as equalities of ratios, it is thought, will give greater clearness to the proof of those propositions in which they are used.

Great care has been taken to remedy all the little imperfections of demonstration in older treatises, and to supply some propositions which have been heretofore unaccountably omitted.

The infinitesimal system has been adopted without hesitation, and to an extent somewhat unprecedented. The usual expedients for avoiding this, result in tedious methods, involving the same principle, only under a more covert form. The idea of the infinite is certainly a simple idea, as natural to the mind as any other, and even an antecedent condition of the idea of the finite.

A peculiar feature of the work will be observed in the "Exercises," which occur at intervals, commencing immediately after the Axioms. These are intended to develop the original powers of the learner, and to bring into play his inventive faculties, the ordinary text tasking the powers of perception alone. The Exercises are so arranged as to make the progress from the easy to the more difficult so gradual that they will be found to excite a lively interest even in students of moderate capacity.

They will be especially convenient in the instruction of large classes, the members of which may all pursue the text, while the exercises upon it will afford scope for the students of greatest ability.

The appendices will be found to contain some recent and elegant improvements. They leave much to be done by the learner, it being supposed that none will be likely to attempt them except such as have some taste and talent for geometry. The previous exercises will have furnished the skill requisite to master this part of the work with facility.

The work might have been arranged in more elegant form by a rigid classification of subjects, all the theorems relating to a particular class of magnitudes being given together, after the manner of some of the latest and best French and German treatises. This arrangement, though in the main preserved, has been occasionally departed from, for the sake of rendering a knowledge of the whole subject most easy of acquisition by the student.

By omitting the fine print in the volume, the student will obtain a very short course of geometry, but one fully adequate as a preparation for the study of all the higher branches of mathematics, while the whole work contains, probably, the most complete system of purely elementary geometry to be found in any single treatise in any language.

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DEFINITIONS.

Geometry is the science of position and extension.

1. A Point is position without magnitude or dimensions. It has neither length, breadth, nor thickness.

2. A Line has one dimension only, length.

3. A Surface or Superficies has extension in two dimensions, length and breadth; but is without thickness.*

4. A Body or Solid has three dimensions, length, breadth, and depth or thickness.

5. A Right Line, or Straight Line, is one which has every where the same direction.

When the term Line is used in this work without an adjective, a Right Line is understood.

A line is designated by two letters placed upon it. Thus we say the line AB.

6. A Broken Line is one which changes its direction at intervals.

7. A Curve, or Curve Line, is one which is continually changing its direction.

8. Parallel Lines are those which have the same direction.†

* A surface may be boundless, and a line interminable.

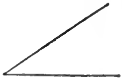
† Parallel lines are sometimes said to meet at an infinite distance; in other words, they never meet. This follows evidently from the definition.

The case where one line is the prolongation of another, or others, which seems to be embraced in this definition, is to be excluded; for these lines, in the unrestricted sense of the term, form one and the same straight line. Or, when two parallel lines coincide, they become one and the same straight line.

9. One line is Perpendicular to another when the first inclines not more toward the second on the one side than on the other.

10. An Angle is the difference of direction of two lines.*

The point where the two lines meet is called the vertex of the angle.



11. Angles are Right or Oblique.

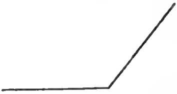
12. A Right Angle is that which is made by one line perpendicular to another.

Or, when the angles on either side of one line meeting another are equal, they are right angles.

13. Oblique Angles are either Acute or Obtuse.

14. An Acute Angle is less than a right angle.

15. An Obtuse Angle is greater than a right angle.



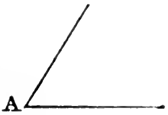
* When one line, having coincided with another, begins to move round the point at one extremity, it begins to have a different direction, and the amount of this difference depends upon the amount of the movement, which is evidently measured by the portion of the circumference described by the other extremity. The length of this portion is usually expressed in degrees, each degree being the $\frac{1}{360}$th part of the whole circumference.

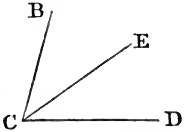
It is immaterial whether the revolving line be longer or shorter, when it has attained the same position with respect to the stationary line, or the same difference of direction from it, the portion of circumference described will contain the same number of degrees in both circumferences; each degree in the smaller circumference being smaller, since it is the 360th part of its own circumference.

Parallel lines having no difference of direction, the angle which they make with each other is zero or 0°.

Fractions of a degree are expressed usually in minutes and seconds, a minute being the 60th part of a degree, and a second the 60th part of a minute. This is called the *sexagesimal measurement of angles*, or *division of the circumference*. Another mode of division sometimes used, is of the whole circumference into four parts called *quadrants*, the quadrant into 100 parts called *grades*, or centesimal degrees, each grade into 100 centesimal minutes, and each minute into 100 centesimal seconds. This is called the centesimal division of the circumference. To convert one kind of degree into the other, it is only necessary to observe that a grade is 0·9 of a degree.

An angle is named from the letter at its vertex. Thus we say the angle A. When, however, there are two angles whose vertices are at the same point, this method would be ambiguous. It is necessary, then, to designate the angle to be pointed out by three letters, naming the one at the vertex always in the middle. Thus, the angle formed by the two lines CB and CE is called the angle BCE, or ECB; and the angle formed by the two lines CE and CD is called the angle ECD, or DCE.





Angles are susceptible of addition, subtraction, and multiplication. Thus the angle BCD = BCE + ECD.

16. Superficies are either Plane or Curved.

17. A Plane Superficies, or a Plane, is that which is straight in every direction, or with which a right line, joining any two points of it, will coincide throughout the length of the line. But if not, it is curved.

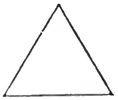
18. More accurately, a Curve Surface is one of which the section made by some plane cutting it is a curve.*

19. A Plane Figure is a portion of a plane, bounded either by right lines or curves.

20. Plane figures that are bounded by right lines are called Polygons, and have names according to the number of their sides, or of their angles; the number of sides and angles being the same. The least number of sides requisite to form a polygon is three.

21. A Polygon of three sides and three angles is called a Triangle. And it receives particular denominations from the relations of its sides and angles.

22. An Equilateral Triangle is one the three sides of which are all equal.



* A curve surface may or may not be straight in certain directions. See the cone and cylinder, toward the end of the volume.

23. An Isosceles Triangle is one which has two sides equal.

24. A Scalene Triangle is one whose three sides are all unequal.

25. A Right-angled Triangle is a triangle having one right angle.

It will be shown hereafter that no triangle can have more than one right angle, or more than one obtuse angle.

26. Other triangles are Oblique-angled, and are either obtuse or acute.

27. An Obtuse-angled Triangle has one obtuse angle.

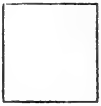
28. An Acute-angled Triangle has its three angles acute.

29. A figure of Four sides and angles is called a Quadrangle, or a Quadrilateral.

30. A Parallelogram is a quadrilateral which has both its pairs of opposite sides parallel. And it takes the following particular names, viz., Rectangle, Square, Rhombus, Rhomboid.

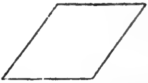
31. A Rectangle is a right-angled parallelogram.

32. A Square is an equilateral rectangle.



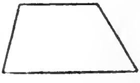
33. A Rhomboid is an oblique-angled parallelogram.

34. A Rhombus is an equilateral rhomboid.



35. A Trapezium is a quadrilateral which has not its opposite sides parallel.

36. A Trapezoid is a quadrilateral which has only one pair of opposite sides parallel.



37. A Pentagon is a polygon of five sides; a Hexagon is one of six sides; a Heptagon, one of seven; an Octagon, one of eight; a Nonagon, one of nine; a Decagon, one of ten; an Undecagon, one of eleven; and a Dodecagon, one of twelve sides.

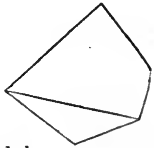
The Perimeter of a polygon is the sum of its bounding lines.

A Convex Polygon is one the perimeter of which can be intersected by a straight line in but two points.

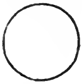
38. A Polygon is Equilateral when all its sides are equal; and it is Equiangular when all its angles are equal. A Regular Polygon is one which is both equiangular and equilateral.

39. An Equilateral Triangle is a regular polygon of three sides, and the square is one of four; the former being also called a trigon, and the latter a tetragon.

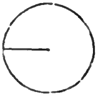
40. A Diagonal is a line joining any two angles of a polygon not adjacent.



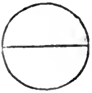
41. A Circle is a plane figure bounded by a curve line, called the Circumference, every point of which is equidistant from a certain point within, called the Center.



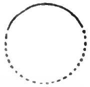
42. The Radius of a circle is a line drawn from the center to the circumference.



43. The Diameter of a circle is a line drawn through the center, and terminating both ways at the circumference.

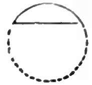


44. An Arc of a circle is any part of the circumference.*

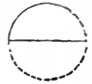


* It will be shown hereafter that the circumference of a circle may be obtained by multiplying the radius by 6·2832: in order to obtain the absolute length of any arc given in degrees and parts of a degree, or grades and parts, it is necessary to ascertain what fraction of a

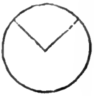
45. A Chord is a right line joining the extremities of an arc.



46. A Segment is any part of a circle bounded by an arc and its chord.

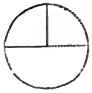


47. A Semicircle is half the circle, or a segment cut off by a diameter. A Semicircumference is half the circumference.



48. A Sector is a part of a circle which is bounded by an arc, and two radii.

Note.—A sector is a surface, as is also a segment.

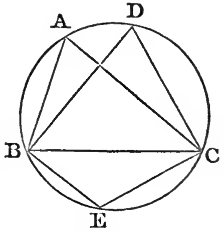


49. A Quadrant, or Quarter of a circle, is a sector having a quarter of the circumference for its arc, its two radii being perpendicular to each other. A quarter of the circumference is also called a Quadrant.

Note.—A semicircle contains 180 degrees, and a quadrant 90 degrees.

50. Concentric Circles are those which have the same center.

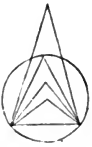
51. Circles are said to be Eccentric with respect to one another when they have not the same center. In this case, the one circumference may be, with respect to the other, Exterior, Interior, Tangent Externally, Tangent Internally, or, finally, the two circumferences may intersect.



52. An Angle in a Segment is that which is contained by two lines, drawn from any point in the arc of the segment, to the two extremities of that arc. Thus A and D are both angles in the segment BADC. They are also called *inscribed* angles, and are said to be inscribed in the segment.

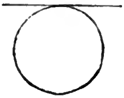
circumference the arc is, by reducing 360° to the lowest denomination in the given arc for a denominator, and the degrees, &c., of the arc to the same denomination, for a numerator, then to multiply this fraction by the product of the radius and the number 6·2832.

53. An Angle on a Segment, or an Arc, is that which is contained by two lines, drawn from any point in the opposite part of the circumference to the extremities of the arc, and containing the arc between them. Thus A and D (in the last figure) are both angles upon the arc BEC.

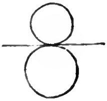


54. An *Angle at the Center* is one whose vertex is at the center of the circle. An *Eccentric Angle* is one whose vertex is not at the center. An *Angle at the Circumference* is one whose vertex is in the circumference. This last is also called an *Inscribed angle.*

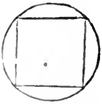
55. Similar arcs, in different circles, are those which subtend equal angles at the center.



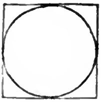
56. A right line is a Tangent to a circle, or touches it, when it has but one point in common with the circle.



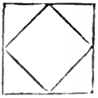
57. Two circles Touch each other when they have but one point common, or when they have a common tangent.



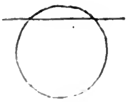
58. A right-lined figure is Inscribed in a circle, or the circle Circumscribes the figure, when all the angular points of the figure are in the circumference of the circle.



59. A right-lined figure Circumscribes a circle, or the circle is Inscribed in the figure, when all the sides of the figure touch the circumference of the circle.

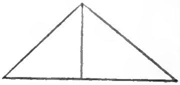


60. One right-lined figure is inscribed in another, or the latter circumscribes the former, when all the angular points of the former are placed in the sides of the latter.



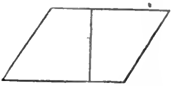
61. A Secant is a line that cuts a circle, lying partly within and partly without it.

62. The Altitude of a triangle is a perpendicular let fall from the vertex of either angle upon the opposite side, called the base.

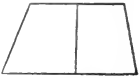


63. In a right-angled triangle, the side opposite the right angle is called the Hypothenuse; and the other two sides are called the Legs, and sometimes the Base and Perpendicular.

64. The altitude of a parallelogram or trapezoid is the perpendicular distance between the parallel sides.



The bases of a trapezoid are the parallel sides.



65. Two triangles, or other right-lined figures, are said to be mutually equilateral when all the sides of the one are equal to the corresponding sides of the other, each to each; and they are said to be mutually equiangular when the angles of the one are respectively equal to those of the other.

66. Identical polygons are such as are both mutually equilateral and equiangular, or that have all the sides and all the angles of the one respectively equal to all the sides and all the angles of the other, each to each; so that if the one figure were applied to, or laid upon the other, all the sides of the one would exactly fall upon and cover all the sides of the other; the two becoming, as it were, but one and the same figure.

67. Similar polygons are of the same shape, but not the same size; they have all the angles of the one equal to all the angles of the other, each to each, and the corresponding or homologous sides, as they are called, proportional.* The homologous sides are those

* Perhaps it will be a little plainer to say that the homologous sides in the two figures have the same ratio. Thus, if the first side in the one figure (beginning in both at the sides adjacent equal angles) be three times as great as the first side in the other, the second side in the first figure will be three times as great as the second side in the other figure, and so on.

The ratio of the corresponding sides of the polygon is called the *ratio of similitude.*

similarly situated, or those adjacent equal angles, or, in triangles, those opposite equal angles.

68. A Proposition is something which is either proposed to be done, or to be demonstrated, and is either a problem or a theorem.

69. A Problem is something proposed to be done.

70. A Theorem is a truth proposed to be demonstrated.

71. A Hypothesis is a supposition made in the enunciation of a proposition, or in the course of a demonstration.

72. A Lemma is something which is premised, or demonstrated, in order to render what follows more easy.

73. A Corollary is a consequent truth, gained immediately from some preceding truth or demonstration.

74. A Scholium is a remark or observation made upon something going before it, and may require a demonstration or may not.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. When equals are added to equals, the wholes are equal.

3. When equals are taken from equals, the remainders are equal.

4. When equals are added to unequals, the wholes are unequal.

5. When equals are taken from unequals, the remainders are unequal.

6. Things which are double the same thing, or equal things, are equal to each other.

7. Things which are halves of the same thing are equal.

8. The whole is greater than its part.

9. Every whole is equal to all its parts taken together.

10. Things which coincide, or fill the same space, are identical, or mutually equal in all their parts.

11. All right angles are equal.

12. Angles that have equal measures, or arcs, are equal.

13. A straight line is the shortest distance between two points. *Corollary.*—One side of a triangle is less than the sum of the other two.

14. But one straight line can be drawn between two points.*

EXERCISE WITH RULE AND DIVIDERS UPON THE RIGHT LINE AND ANGLE.

1. Make a line equal to the sum of two given lines. Of four.
2. Make a line equal to the difference of two given lines.
3. Make a line equal to five times one given line and six times another.
4. Find how many times one given line is contained in another.
5. Find a common measure of two given lines.†
6. Make a straight line equal in length to a broken line.
7. Make a straight line equal in length approximately to a curve.‡
8. With several given points as centers, to describe circles with given lines as radii.
9. To find a point which shall be at given distances from two given points.
10. Draw the radius of a circle as a chord of the same.
11. Make an angle double a given angle. Triple.
12. Measure the number of degrees in a given angle by means of a brass or paper circle or semicircle, divided into degrees, called a protractor.
13. Make an angle equal to the sum of several given angles.
14. Draw a line through a given point parallel to a given line.
15. Draw through given points several parallels to a given line.

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* A straight line joining two points is the direction of the one from the other. Two points are said to determine a line. Two points of a line being given, the line is given; for it is the line joining them.

† This is done in a manner analogous to the corresponding operation in Arithmetic and Algebra, by applying the smaller line to the larger as many times as it will go; and the remainder to the smaller given line, and so on.

‡ This may be done by taking such small portions of the curve as are nearly straight.

16. Draw through a given point, without a given line, a line forming with it a given angle.

17. To make an angle with two given lines for sides.

18. In how many points may 20 lines cut each other, no two of which are parallel?

19. In how many when twelve of them are parallel?

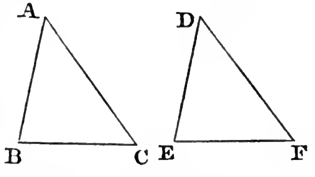
20. In how many when 4 are parallel in one direction, 5 in another, and 6 in another?

21. In how many points will 36 lines intersect, 24 of which pass through the same point?

THEOREM I.

If two triangles have two sides and the included angle in the one equal to two sides and the included angle in the other, the triangles will be identical, or equal in all respects.

In the two triangles ABC, DEF, if the side AB be equal to the side DE, the side AC equal to the side DF, and the angle A equal to the angle D, then will the two triangles be identical, or equal in all respects.



For, conceive the triangle ABC to be applied to, or placed on,* the triangle DEF, in such a manner that the point A may coincide with the point D, and the side AB with the side DE, which is equal to it.

Then, since the angle A is equal to the angle D (by hyp.),† the side AC must differ in direction from the side AB by the same amount that the side DF does from DE; hence AC must take the same direc-

* The student will do well, at first, to cut two triangles out of pasteboard or paper, and place one upon the other; or imagine the first of the above triangles to be cut out of the page and placed upon the other; or conceive the sides to be fine wires, so that the triangle can be taken off the page.

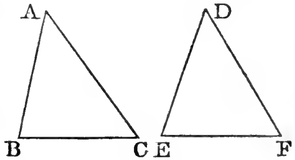
† Hyp. stands for hypothesis. This term is much used, and signifies generally that what is stated is given or supposed true at the outset.

tion as DF, and, since AC is (by hyp.) equal to DF, the point C must fall on the point F, and, by Ax. 14, the line BC must fall on EF; thus the two triangles coincide throughout, and (Ax. 10) are identical. Q. E. D.*

THEOREM II.

When two triangles have two angles and the included side in the one equal to two angles and the included side in the other, the triangles are identical, or have their other sides and angles equal.

Let the two triangles ABC, DEF have the angle A equal to the angle D, the angle B equal to the angle E, and the side AB equal to the side DE; then these two triangles will be identical.



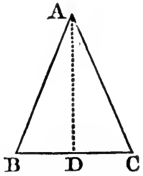
For, conceive the triangle ABC to be placed on the triangle DEF in such manner that the side AB may fall exactly on the equal side DE. Then, since the angle A is equal to the angle D (by hyp.), the side AC must fall on the side DF; and, in like manner, because the angle B is equal to the angle E, the side BC must fall on the side EF, and the two sides AC and BC coinciding respectively with the two DE and EF, they must meet in the same point, that is, the point C must fall on the point F. Thus the three sides of the triangle ABC will be exactly placed on the three sides of the triangle DEF; consequently, the two triangles are identical (ax. 10), and the other two sides AC, BC will be equal to the two DF, EF, and the remaining angle C equal to the remaining angle F. Q. E. D.

* These letters are the initials of the words "quod erat demonstrandum," signifying "which was to be demonstrated."

THEOREM III.

In an isosceles triangle, the angles at the base are equal. Or, if a triangle have two sides equal, their opposite angles will also be equal.*

If the triangle ABC have the side AB equal to the side AC, then will the angle B be equal to the angle C.



For, conceive the angle A to be bisected, or divided into two equal parts by the line AD, making the angle BAD equal to the angle CAD.

Then the two triangles BAD, CAD have two sides and the contained angle of the one equal to two sides and the contained angle of the other, viz., the side AB equal to AC, the angle BAD equal to CAD, and the side AD common; therefore these two triangles are identical, or equal in all respects (th. 1); and, consequently, the angle C equal to the angle B. Q. E. D.

Corol. 1. Hence the line which bisects the vertical angle of an isosceles triangle bisects the base, and is also perpendicular to it. (See def. 12.)†

Corol. 2. Hence, too, it appears that every equilateral triangle is also equiangular, or has all its angles equal; for an equilateral triangle is isosceles, whichever side may be taken for the base.

* The base of an isosceles triangle is the side unequal to either of the other two.

† The line AD passes through the vertex, bisects the vertical angle, bisects the base, and is perpendicular to the base. Any two of these four conditions determine the position of the line, and the other two conditions follow. Whence the following theorems in addition to cor. 1.

1. The line joining the vertex of an isosceles triangle with the middle of the base bisects the vertical angle, and is perpendicular to the base.

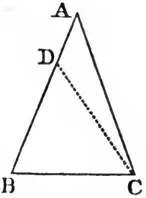
2. The perpendicular at the middle of the base passes through the vertex, and bisects the vertical angle.

3. The perpendicular from the vertex to the base bisects the base and the vertical angle.

To prove each of these independently will be an exercise.

THEOREM IV.

When a triangle has two of its angles equal, the sides opposite to them are also equal.



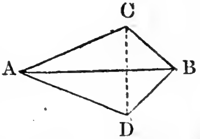
If the triangle ABC have the angle C equal to the angle B, it will also have the side AB equal to the side AC.

For if not, let BA be greater than AC, and take BD equal to AC, and join the points C and D by the line CD; then the two triangles ABC and CBD having the side BC common, the side BD of the one equal (by construction) to the side AC of the other, and the contained angle BCA of the former equal to the contained angle B of the latter, are equal (by th. 1); but the triangle CBD is evidently only a part of the triangle ABC, and a part can not be equal to the whole (ax. 8). The hypothesis must, therefore, be wrong, and AB can not be greater than AC. In a similar manner it might be proved that AC can not be greater than AB; hence AB = AC. Q. E. D.*

Corol. Hence every equiangular triangle is also equilateral.

THEOREM V.

When two triangles have all the three sides in the one equal to all the three sides in the other, the triangles are identical, or have also their three angles equal, each to each.



Let the two triangles ABC, ABD have their three sides respectively equal, viz., the side AB equal to AB, AC to AD, and BC to BD; then shall the two triangles be identical,

* The kind of demonstration employed here is called negative rea soning, because, by proving that the negative can not be true, it proves the affirmative. It is also called the reductio ad absurdum, because it proves that the denial of the proposition leads to an absurdity.

or have their angles equal, viz., those angles that are opposite to the equal sides; that is to say, the angle BAC to the angle BAD, the angle ABC to the angle ABD, and the angle C to the angle D.

For, conceive the two triangles to be joined together by their longest equal sides, and draw the line CD.

Then, in the triangle ACD, because the side AC is (by hyp.) equal to AD, the angle ACD is equal to the angle ADC (th. 3). In like manner, in the triangle BCD, the angle BCD is equal to the angle BDC, because the side BC is equal to BD. Hence, then, the angle ACD being equal to the angle ADC, and the angle BCD to the angle BDC, by equal additions the sum of the two angles ACD, BCD, is equal to the sum of the two ADC, BDC (ax. 2), that is, the whole angle ACB equal to the whole angle ADB.

Since, then, the two sides AC, CB are equal to the two sides AD, DB, each to each (by hyp.), and their contained angles ACB, ADB, also equal, the two triangles ABC, ABD are identical (th. 1), and have the other angles equal, viz., the angle BAC to the angle BAD, and the angle ABC to the angle ABD. Q. E. D.

General Scholium. From the foregoing propositions it appears that triangles will be identical when they have two sides and the included angle, two angles and the included side, or three sides of the one equal to the same in the other.*

EXERCISES.

1. To make a triangle when two sides and the angle which they form are given.

* A triangle is composed of *six elements*, three sides and three angles. It is only necessary, in general, that three of these, provided one be a side, in one triangle should be equal to the same in the other to render the triangles equal. (See cor. 7, th. 15, and see prob. 8 for an exception.)

Three given parts are also said to *determine* a triangle; by which is to be understood. that with the three given parts only one triangle can be formed, or that all which may be formed with them will be identical. This remark, like the above, does not include the case where three angles are given.

2. When two angles and the side of the triangle between them are given.

3. On a given line to construct an equilateral triangle.

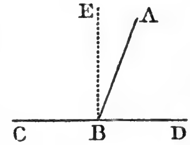
4. To construct an isosceles triangle with a given base and given side.

5. To construct a triangle with three given lines for sides.

N.B. The given sides must be subject to the condition expressed in the corollary to ax. 13.

THEOREM VI.

When one line meets another, the angles which the first line makes on the same side of the second are together equal to two right angles.



Let the line AB meet the line CD; then will the two angles ABC, ABD, taken together, be equal to two right angles.

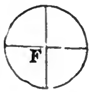
For, suppose BE drawn perpendicular to CD. Then the two angles CBA, ABD fill the same angular space with the two CBE, EBD; but the latter are right angles, hence the former are together equal to two right angles.

Corol. 1. Conversely, if the two angles ABC, ABD, on opposite sides of the line AB, make up together two right angles, then CB and BD form one continued right line CD. For no other line from the point B than BD, the continuation of CB, can form with BA an angle equal to ABD, which, added to ABC, makes two right angles, by the above theorem.

Corol. 2. All the angles which can be made at any point B, by any number of lines, on the same side of the right line CD, are, when taken all together, equal to two right angles, since they fill the same angular space.

Corol. 3. And as all the angles that can be made on the other side of the line CD are also together equal to two right angles; therefore, all the angles that can be made quite round a point B, by any number of lines, are together equal to four right angles.

Corol. 4. Hence, also, the whole circumference of a circle, being the sum of the measures of all the angles that can be made about the center F, is the measure of four right angles; a semicircle, or 180 degrees, is the measure of two right angles; and a quadrant, or 90 degrees, the measure of one right angle. (See def. 49.)



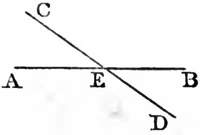
Two angles which together amount to a right angle, or 90°, are called complements of each other. Thus, 40° is the complement of 50°, and 50° of 40°.

Two angles which together equal two right angles, or 180°, as 110° and 70°, are called supplements of each other.

THEOREM VII.

When two lines intersect each other, the opposite angles are equal.

Let the two lines AB, CD intersect in the point E; then will the angle AEC be equal to the angle BED, and the angle AED be equal to the angle CEB.

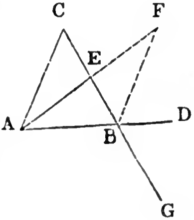


For EA is exactly the opposite direction from EB, and ED exactly the opposite direction from EC. Hence their difference of direction will be the same. Q. E. D. (See def. 10.)

THEOREM VIII.

When one side of a triangle is produced, the outward angle is greater than either of the two inward opposite angles.

Let ABC be a triangle, having the side AB produced to D; then will the outward angle CBD be greater than either of the inward opposite angles A or C.



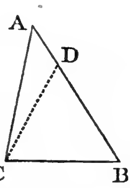
For, conceive the side BC to be bisected in the point E, and draw the line AE, producing it till EF be equal to AE; and join BF.

Then, since the two triangles AEC, BEF have the side AE = the side EF, and the side CE = the side BE (by constr.), and the included or opposite angles at E also equal (th. 7), therefore those two triangles are equal in all respects (th. 1), and have the angle C = the corresponding angle EBF. But the angle CBD is greater than the angle EBF (ax. 8); consequently, the said outward angle CBD is also greater than the angle C.

In like manner, if CB be produced to G, and AB be bisected, it may be shown that the outward angle ABG, or its equal (th. 7) CBD, is greater than the other inward angle A.

THEOREM IX.

The greater side of every triangle is opposite to the greater angle, and the greater angle opposite to the greater side.



Let ABC be a triangle, having the side AB greater than the side BC; then will the angle ACB, opposite the greater side AB, be greater than the angle A, opposite the less side CB.

For, on the greater side ABD take the part BD, equal to the less side BC, and join CD. Then, since ACD is a triangle, the outward angle BDC is greater than the inward opposite angle A (th. 8). But the angle BCD is equal to the said outward angle BDC, because the triangle BDC is isosceles, BD being equal to BC (th. 3). Consequently, the angle BCD, also, is greater than the angle A. And, since the angle BCD is only a part of ACB, much more must the whole angle ACB be greater than the angle A. Q. E. D.

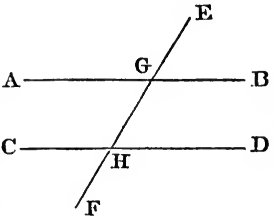
Again, conversely, in the given triangle ABC, if the angle C be greater than the angle A, then will the side AB, opposite the former, be greater than the side BC, opposite the latter.

For if AB be not greater than BC, it must be either equal to it or less than it. But it can not be equal, for then the angle C would be equal to the angle A (th. 3), which it is not, by the supposition. Neither can it be less, for then the angle C would be less than the angle A, by the former part of this theorem, which is also contrary to the supposition. The side AB, then, being neither equal to BC, nor less than it, must necessarily be greater. Q. E. D.

THEOREM X.

When a line intersects two parallel lines obliquely it will form with them four acute angles and four obtuse; the four acute will be equal to one another, and the four obtuse.

This follows from definitions 8 and 10, and th. 7.



Corol. 1. Any one of the acute and any one of the obtuse will be supplements of each other. (See th. 6.)

Corol. 2. If one of the angles be right, the whole eight will be right angles.

Scholium 1. The line cutting the two parallels is sometimes called the secant line. The two angles within the parallels, and on different sides of the secant line, are commonly called alternate internal or interior angles, or simply *alternate* angles. The two outside the parallels, and on different sides the secant, *alternate external* angles; and the two on the same side of the secant, one within and the other without the parallels, and not adjacent, are called *opposite internal and external,* or *outward and inward on the same side.*

Scholium 2. The distance of two parallel lines is the length of the line between them, drawn perpendicular to both, as FH or EG in the next diagram.

THEOREM XI.

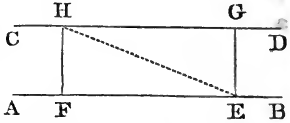
When one straight line meets two others so as to make equal angles with them, the latter are parallel; in other words, if two lines make the same angle with a third, they will be parallel to each other.

For, having the same difference of direction from the same line, they must have the same direction with one another, and are therefore (def. 8) parallel.

Note.—Two lines parallel to a third are parallel to one another. This follows from def. 8.

THEOREM XII.

Parallel lines are every where equally distant.



To prove this it will only be necessary to prove any two perpendiculars, FH and EG, drawn at random between them, equal. Join EH. Then the lines FH and EG both being at right angles to AB, will, by the last theorem, be parallel; and the two triangles EFH, EGH will have the side EH common, and the two angles adjacent this side in the one equal to the same in the other; hence (th. 2) these triangles are equal in all respects, and, therefore, the side FH of the one equal to the corresponding side EG of the other.

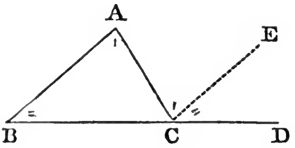
Corol. 1. Parallel lines, being every where at the same distance. however far produced, can never meet. This is sometimes expressed by saying that they meet at an infinite distance.

Corol. 2. If the extremities of two equal perpendiculars to a given line be joined, a parallel will be obtained.

THEOREM XIII.

*When one side of a triangle is produced, the outward angle is equal to both the inward opposite angles taken together.**

Let the side BC of the triangle ABC be produced to D; then will the outward angle ACD be equal to the sum of the two inward opposite angles A and B.

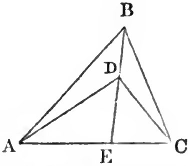


For, conceive CE to be drawn parallel to the side AB of the triangle. Then AC, meeting the two parallels BA, CE, makes the alternate angles A and ACE equal (th. 10). And BD, cutting the same two parallels BA, CE, makes the inward and outward angles on the same side, B and ECD, equal to each other (th. 10). Therefore, by equal additions, the sum of the two angles A and B is equal to the sum of the two ACE and ECD, that is, to the whole angle ACD (by ax. 2). Q. E. D.

THEOREM XIV.

If two angles stand upon the same base, the vertex of one being within that of the other, the latter will be the less angle.

In the diagram the angle ABC is less than the angle ADC; for (th. 8) ADE > ABD, and EDC > EBC ∴ by equal addition ADE + EDC > ABD + EBC, or ADC > ABC. Q. E. D.



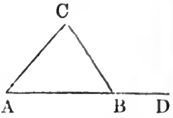
* The two opposite angles are the two neither of which is adjacent the outward angle.

One angle is said to be adjacent to another when the two have a common vertex and a common side. An angle is said to be adjacent to a line when that line is one of its sides.

N.B.—It will be found convenient, when angles have been proved equal, to mark them with the same number of dots as in the figure.

THEOREM XV.

In any triangle the sum of all the three angles is equal to two right angles.



Let ABC be any plane triangle; then the sum of the three angles A + B + C is equal to two right angles.

For, let the side AB be produced to D. Then the outward angle CBD is equal to the sum of the two inward opposite angles A + C (th. 13). To each of these equals add the inward angle B; then will the sum of the three inward angles A + B + C be equal to the sum of the two adjacent angles ABC + CBD (ax. 2). But the sum of these two last adjacent angles is equal to two right angles (th. 6). Therefore, also, the sum of the three angles of the triangle A + B + C is equal to two right angles (ax. 1). Q. E. D.

Corol. 1. If two angles in one triangle be equal to two angles in another triangle, the third angles will also be equal; for they make up two right angles in both.

Corol. 2. Two right-angled triangles will be equiangular when they have an acute angle in each equal.

Corol. 3. The sum of two angles of any triangle and the third angle are supplements of each other.

Corol. 4. If one angle in one triangle be equal to one angle in another, the sums of the remaining angles will also be equal (ax. 3).

Corol. 5. If one angle of a triangle be right, the sum of the other two will also be equal to a right angle, and each of them singly will be acute, or less than a right angle, and will be the complement of the other.

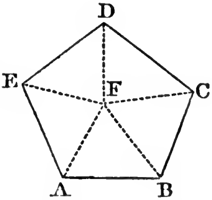
Corol. 6. The two least angles of every triangle are acute, or each less than a right angle. In other words, there can be but one right angle, or one obtuse angle in a triangle.

Corol. 7. Any two angles and a side of one triangle being equal to the same in another, the triangles will be equal (th. 2).

Corol. 8. One side and an acute angle of a right-angled triangle being equal to the same in another, the triangles are equal.

THEOREM XVI.

The sum of all the inward angles of a polygon is equal to twice as many right angles, wanting four, as the figure has sides.



Let ABCDE be any figure; then the sum of all its inward angles, A + B + C + D + E, is equal to twice as many right angles, wanting four, as the figure has sides.

For, from any point F, within it, draw lines, FA, FB, FC, &c., to all the angles, dividing the polygon into as many triangles as it has sides. Now the sum of the three angles of each of these triangles is equal to two right angles (th. 15); therefore, the sum of the angles of all the triangles is equal to twice as many right angles as the figure has sides. But the sum of all the angles about the point F, which are so many of the angles of the triangles, but no part of the inward angles of the polygon, is equal to four right angles (corol. 3, th. 6), and must be deducted out of the former sum. Hence it follows that the sum of all the inward angles of the polygon alone, A + B + C + D + E, is equal to twice as many right angles as the figure has sides, wanting the said four right angles. Q. E. D.

Corol. 1. In any quadrangle, the sum of all the four inward angles is equal to four right angles.

Corol. 2. Hence, if three of the angles be right ones, the fourth will also be a right angle.

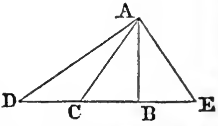
Corol. 3. And if the sum of two of the four angles be equal to two right angles, the sum of the remaining two will also be equal to two right angles.

Corol. 4. The sum of the angles of a pentagon is $5 \times 2 - 4 = 6$ right angles. Of a hexagon, $6 \times 2 - 4 = 8$ right angles.* Of a polygon of n sides $(n \times 2 - 4)$ right angles.

The rule may be thus expressed: to obtain the sum of the angles of a polygon, double the number of sides and subtract 4, the right angle being the unit of measure.†

THEOREM XVII.

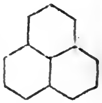
A perpendicular is the shortest line that can be drawn from a given point to an indefinite line. And, of any other lines drawn from the same point, those that are equally distant from the perpendicular are equal, and those nearest the perpendicular are less than those more remote.



If AB, AC, AD, &c., be lines drawn from the given point A, to the indefinite line DE, of which AB is perpendicular; then shall the perpendicular AB be less than AC, AE = AC if BE = BC, and AC < AD if BC < BD.

For, the angle B being a right one, the angle C of

* Each angle of a regular hexagon would be the sixth part of 8 right angles, or $\frac{8}{6}$ or $\frac{4}{3}$ or $1\frac{1}{3}$ right angles, *i. e.*, 120°, 90° being a right angle, this is $\frac{1}{3}$ of 360°, or 4 right angles; so that if three regular hexagons were placed together they would fill up the whole angular space about a point.



This would not be the case with pentagons, for $\frac{5 \times 2 - 4}{5} = \frac{6}{5} = 1\frac{1}{5}$, which, multiplied by 3, gives $3\frac{3}{5} < 4$ right angles, and, multiplied by 4, gives $4\frac{4}{5} > 4$ right angles, so that they would fit together neither by threes nor fours.

It will be found, on examination, that no other figures except squares and triangles have this same property with the hexagon. Hence these three kinds of figures alone are employed for paving blocks.

† The above applies only to convex polygons, that is, those in which the angles point outward. No convex polygon can have more than three acute angles.

the triangle ABC is acute (by cor. 6, th. 15), and therefore less than the angle B. But the less angle of a triangle is subtended by the less side (th. 9). Therefore the side AB is less than the side AC.

Again, in the right-angled triangles ABC, ABE the two sides AB, BC being respectively equal to the two AB, BE, the third sides are equal (th. 1).

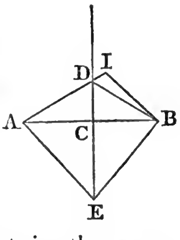
Corol. Every point, as A, of a perpendicular at the middle of a given line CE is equally distant from its extremities C and E.

Finally, the angle ACB being acute, or less than a right angle, as before, the adjacent angle ACD will be greater than a right angle, or obtuse (by th. 6); consequently, the angle D is acute (corol. 6, th. 15), and therefore is less than the angle C. And since the less side is opposite to the less angle, therefore the side AC is less than the side AD. Q. E. D.

Corol. The least distance of a given point from a line is the perpendicular. For if it were an oblique line the perpendicular would be shorter, and thus less than the least distance, which is impossible.

THEOREM XVIII.

Every point out of a perpendicular at the middle of a given line is at unequal distances from the extremities of the line.

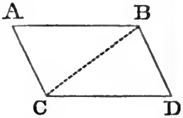


Let DC be a perpendicular at the middle of AB, and I a point out of the perpendicular, then shall $IB < IA$. For join BD; then, $BI < BD + DI$; or since $BD = AD$ (th. 17), $BI < AD + DI$, or $BI < AI$. Q. E. D.

Scholium. There can be but one perpendicular through a given point to a given line. For there can be but one line through the same point in the same direction, or having the same difference of direction from a given line.

THEOREM XIX.

The opposite sides and angles of a parallelogram are equal to each other, and the diagonal divides it into two equal triangles.



Let ABDC be a parallelogram, of which the diagonal is BC; then will its opposite sides and angles be equal to each other, and the diagonal BC will divide it into two equal parts, or triangles.

For, since the sides AB and DC are parallel, as also the sides AC and BD (def. 30), and the line BC meets them; therefore the alternate angles are equal (th. 10), namely, the angle ABC to the angle BCD, and the angle ACB to the angle CBD. Hence the two triangles, having two angles in the one equal to two angles in the other, have also their third angles equal (cor. 1, th. 15), namely, the angle A equal to the angle D, which are two of the opposite angles of the parallelogram.

Also, if to the equal angles ABC, BCD be added the equal angles CBD, ACB, the wholes will be equal (ax. 2), namely, the whole angle ABD to the whole ACD, which are the other two opposite angles of the parallelogram. Q E. D.

Again, since the two triangles are mutually equiangular, and have a side in each equal, viz., the common side BC; therefore the two triangles are identical (th. 2), or equal in all respects, namely, the side AB equal to the opposite side DC, and AC equal to the opposite side BD,* and the whole triangle ABC equal to the whole triangle BCD. Q. E. D.

Corol. 1. Hence, if one angle of a parallelogram be a right angle, all the other three will also be right

* The student will observe that in identical triangles the equal sides are opposite equal angles. Thus, in the diagram, the side BD opposite the angle C, in the lower triangle, is equal to the side AC, opposite the equal angle B in the upper.

angles, and the parallelogram a rectangle. (See cors. to th. 16.)

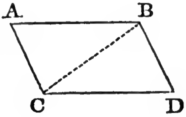
Corol. 2. Hence, also, the sum of any two adjacent angles of a parallelogram is equal to two right angles.

Corol. 3. If two parallelograms have an angle in each equal, the parallelograms are equiangular.

THEOREM XX.

Every quadrilateral whose opposite sides are equal is a parallelogram, or has its opposite sides parallel.

Let ABDC be a quadrangle having the opposite sides equal, namely, the side AC equal to BD, and AB equal to CD; then shall these equal sides be also parallel, and the figure a parallelogram.



For, let the diagonal BC be drawn. Then the triangles ABC, CBD being mutually equilateral (by hyp.), they are also mutually equiangular (th. 5), or have their corresponding angles equal; consequently, the opposite sides, having the same difference of direction in opposite ways from the same line BC, have the same direction one way, and are parallel (def. 8); viz., the side AB parallel to DC, and AC parallel to BD, and the figure is a parallelogram (def. 30). Q. E. D.

THEOREM XXI.

Those lines which join the corresponding extremities of two equal and parallel lines are themselves equal and parallel.

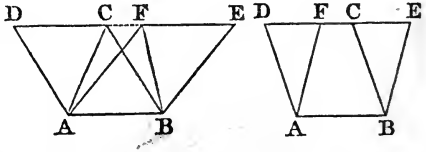
Let AB, DC be two equal and parallel lines; then will the lines AC, BD, which join their extremes, be also equal and parallel. [See the fig. above.]

For, draw the diagonal BC. Then, because AB and DC are parallel (by hyp.), the angle ABC is equal to the alternate angle DCB (th. 10). Hence, then, the two triangles having two sides and the contained angles equal, viz., the side AB equal to the side DC, and the side BC common, and the contained angle ABC equal to the contained angle DCB, they have the remaining sides and angles also respectively equal (th. 1); consequently AC is equal to BD, and also parallel to it (th. 11). Q. E. D.

General Scholium. From the foregoing theorems it appears that a quadrilateral will be a parallelogram: 1. When it has its opposite sides parallel; 2. When it has its opposite sides equal; 3. When it has two of its opposite sides equal and parallel. A quadrilateral is also a parallelogram: 4. When two sides are parallel and two opposite angles equal; 5. When the opposite angles are equal. The proof of the last two cases is left as an exercise for the learner.

THEOREM XXII.

Parallelograms, as also triangles, standing on the same base, and between the same parallels, are equal to each other.



Let ABCD, ABEF be two parallelograms, and ABC, ABF two triangles, standing on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be equal to the parallelogram ABEF, and the triangle ABC equal to the triangle ABF.

For the two triangles ADF, BCE are equiangular, having their corresponding sides in the same direction; and having the two corresponding sides AD, BC equal (th. 19), being opposite sides of a parallelogram, these two triangles are identical, or equal in all respects (th. 2). If each of these equal triangles, then, be taken from the whole space ABED, there will remain the parallelogram ABEF in the one case, equal to the parallelogram ABCD in the other (by ax. 3).

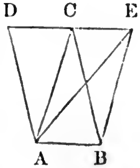
Also, the triangles ABC, ABF, on the same base AB, and between the same parallels, are equal, being the halves of the said equal parallelograms (th. 19).* Q. E. D.

Corol. 1. Parallelograms, or triangles, having the same base and altitude are equal. For the altitude is the same as the perpendicular or distance between the two parallels, which is every where equal, by theorem 12.

Corol. 2. Parallelograms, or triangles, having equal bases and altitudes are equal. For, if the one figure be applied with its base on the other, the bases will coincide, or be the same, because they are equal; and so the two figures, having the same base and altitude, are equal.

THEOREM XXIII.

If a parallelogram and a triangle stand on the same base, and between the same parallels, the parallelogram will be double the triangle, or the triangle half the parallelogram.



Let ABCD be a parallelogram, and ABE a triangle, on the same base AB, and between the same parallels AB, DE; then will the parallelogram ABCD be double the triangle ABE, or the triangle half the parallelogram.

For, draw the diagonal AC of the parallelogram, dividing it into two equal parts (th. 19). Then, because the triangles ABC, ABE on the same base, and between the same parallels, are equal (th. 22); and because the one triangle ABC is half the parallelogram ABCD (th. 19), the other equal triangle

* The triangles being given, with their vertices C and F taken at pleasure in the line DE, the lines BE and AD must be drawn parallel to the sides AF, BC of the triangles, to complete the parallelograms

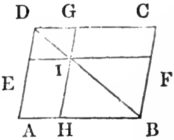
The above theorem may be proved by th. 1, and also by th. 5.

ABE is also equal to half the same parallelogram ABCD. Q. E. D.

Corol. A triangle is equal to half a parallelogram of the same base and altitude.

THEOREM XXIV.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to each other.



Let AC be a parallelogram, BD a diagonal, EIF parallel to AB and DC, and GIH parallel to AD and BC, making AI, IC complements to the parallelograms EG, HF, which are about the diagonal DB; then will the complement AI be equal to the complement IC.

For, since the diagonal DB bisects the three parallelograms AC, EG, HF (th. 19); therefore, the whole triangle DAB being equal to the whole triangle DCB, and the parts DEI, IHB respectively equal to the parts DGI, IFB, the remaining parts AI, IC must also be equal (by ax. 3). Q. E. D.

THEOREM XXV.

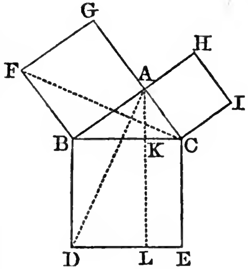
A trapezoid is equal to half a parallelogram, whose base is the sum of the two parallel sides, and its altitude the perpendicular distance between them.

Let ABCD be the trapezoid, having its two sides AB, DC parallel; and in AB produced take BE equal to DC, so that AE may be the sum of the two parallel sides; produce DC also, and let EF, GC, BH be all three parallel to AD. Then is AF a parallelogram of the same altitude with the trapezoid ABCD, having its base AE equal to the sum of the parallel sides of the trapezoid; and it is to be proved that the trapezoid ABCD is equal to half the parallelogram AF.

Now, since triangles, or parallelograms, of equal bases and altitudes, are equal (corol. 2, th. 22), the parallelogram DG is equal to the parallelogram HE, and the triangle CGB equal to the triangle CHB; consequently, the line BC bisects or equally divides the parallelogram AF, and ABCD is the half of it. Q. E. D.

THEOREM XXVI.

In any right-angled triangle, the square of the hypothenuse is equal to the sum of the squares of the other two sides.



Let ABC be a right-angled triangle, having the right angle A; then will the square of the hypothenuse BC be equal to the sum of the squares of the other two sides AC, AB. Or $BC^2 = AC^2 + AB^2$.

For, on BC describe the square BE, and on AC, AB, the squares CH, BG; then draw AL parallel to BD, and join CF, AD.

Now, because the line AB meets the two AG, AC, so as to make the sum of the two adjacent angles equal to two right angles, these two form one straight line GC (corol. 1, th. 6). And because the angle FBA is equal to the angle DBC, being each a right angle, or the angle of a square; to each of these equals add the common angle ABC, so will the whole angle or sum FBC be equal to the whole angle or sum ABD. But the line FB is equal to the line BA, being sides of the same square; and the line BD to the line BC, for the same reason; so that the two sides FB, BC, and the included angle FBC, are equal to the two sides AB, BD, and the included angle ABD, each to each; therefore the triangle FBC is equal to the triangle ABD (th. 1).

But the square BG is double the triangle FBC on the same base FB, and between the same parallels FB, GC (th. 23); in like manner, the parallelogram BL is double the triangle ABD, on the same base BD, and between the same parallels BD, AL. And since the doubles of equal things are equal (by ax. 6), therefore the square BG is equal to the parallelogram BL.

In like manner, the other square CH is proved equal to the other parallelogram EK. Consequently, the two squares BG and CH together are equal to the two parallelograms BL and EK together, or to the whole square BE; that is, the sum of the two squares on the two less sides is equal to the square on the greatest side. Q. E. D.

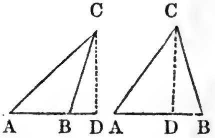
Corol. 1. Hence the square of either of the two less sides is equal to the difference of the squares of the hypothenuse and the other side (ax. 3); or equal to the rectangle contained by the sum and difference of the said hypothenuse and other side; for (Alg. 13) $a^2-b^2=(a+b)\ (a-b)$.

Corol. 2. Hence, also, if two right-angled triangles have two sides of the one equal to two corresponding sides of the other, their third sides will also be equal, and the triangles identical.

THEOREM XXVII.*

In any triangle, the difference of the squares of the two sides is equal to the difference of the squares of the segments of the base, or of the two lines, or distances, included between the extremes of the base and the perpendicular.

Let ABC be any triangle having CD perpendicular to AB; then will the difference of the squares of AC, BC be equal to the difference of the squares of AD, BD; that is, $AC^2-BC^2=AD^2-BD^2$.



* The two following theorems require the aid of the following algebraic formulas:

$$(a+b)^2=a^2+2ab+b^2=a^2+b^2+2ab$$
$$(a-b)^2=a^2-2ab+b^2=a^2+b^2-2ab$$

For, since ACD and BCD are right-angled triangles,

$$\left.\begin{aligned} AC^2 &= AD^2 + CD^2 \\ \text{and } BC^2 &= BD^2 + CD^2 \end{aligned}\right\} \text{(by th. 26);}$$

$\therefore$ By subtraction, $AC^2 - BC^2 = AD^2 - BD^2$.

Corol. The rectangle of the sum and difference of the two sides of any triangle is equal to the rectangle of the sum and difference of the distances between the perpendicular and the two extremes of the base, or equal to the rectangle of the base and the difference or sum of the segments, according as the perpendicular falls within or without the triangle.

That is, $(AC + BC) . (AC - BC) = (AD + BD) . (AD - BD)$.

Or, $(AC + BC) . (AC - BC) = AB\ (AD - BD)$ in the 2d fig.

And, $(AC + BC) . (AC - BC) = AB . (AD + BD)$ in the 1st fig.

THEOREM XXVIII.

In any obtuse-angled triangle, the square of the side subtending the obtuse angle is greater than the sum of the squares of the other two sides, by twice the rectangle of one of the sides containing the obtuse angle and the distance of the perpendicular drawn from the opposite vertex upon this side, from the obtuse angle.

Let ABC be a triangle, obtuse-angled at B, and CD perpendicular to AB; then will the square of AC be greater than the squares of AB, BC, by twice the rectangle of AB, BD. That is, $AC^2 = AB^2 + BC^2 + 2\ AB . BD$. See the 1st fig. above.

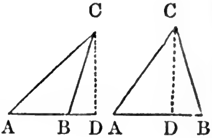
For, $AD^2 = (AB + BD)^2 = AB^2 + BD^2 + 2AB . BD$; adding CD^2 to both members of this equality, $AD^2 + CD^2 = AB^2 + BD^2 + CD^2 + 2AB . BD$ (ax. 2.)

But $AD^2 + CD^2 = AC^2$, and $BD^2 + CD^2 = BC^2$ (th. 26).

Therefore, by substitution in the last equality but one, $AC^2 = AB^2 + BC^2 + 2AB . BD$. Q. E. D.

THEOREM XXIX.

In any triangle, the square of the side subtending an acute angle is less than the squares of the other two sides, by twice the rectangle of one of the sides containing the acute angle and the distance of the perpendicular upon this side from the acute angle.



Let ABC be a triangle, having the angle A acute, and CD perpendicular to AB; then will the square of BC be less than the sum of the squares of AB, AC by twice the rectangle of AB, AD; that is, $BC^2 = AB^2 + AC^2 - 2\,AD \,.\, AB$.

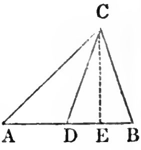
For $BD^2 = (AD \sim AB)^2 = AD^2 + AB^2 - 2AD \,.\, AB$.

And $BD^2 + DC^2 = AD^2 + DC^2 + AB^2 - 2AD \,.\, AB$ (ax. 2).

Therefore $BC^2 = AC^2 + AB^2 - 2AD \,.\, AB$ (th. 26). Q. E. D.

THEOREM XXX.

In any triangle the double of the square of a line drawn from the vertex to the middle of the base, together with double the square of the half base, is equal to the sum of the squares of the other two sides.



Let ABC be a triangle, and CD the line drawn from the vertex to the middle of the base AB, bisecting it into the two equal parts AD, DB; then will the sum of the squares of AC, CB be equal to twice the sum of the squares of CD, AD; or $AC^2 + CB^2 = 2CD^2 + 2AD^2$.

For $AC^2 = CD^2 + AD^2 + 2AD \,.\, DE$ (th. 28).

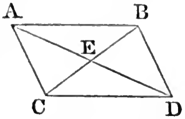
And $BC^2 = CD^2 + BD^2 - 2AD \,.\, DE$ (th. 29).

Therefore, by addition (ax. 2),

$$AC^2 + BC^2 = 2CD^2 + AD^2 + BD^2$$
$$= 2CD^2 + 2AD^2 \text{ (by hyp.)}. \quad \text{Q. E. D.}$$

THEOREM XXXI.

In any parallelogram the two diagonals bisect each other, and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.



Let ABDC be a parallelogram whose diagonals intersect each other in E; then will AE be equal to ED and BE to EC, and the sum of the squares of AD, BC will be equal to the sum of the squares of AB, BD, CD, CA; that is,

$$AE = ED, \text{ and } BE = EC,$$

and $$AD^2 + BC^2 = AB^2 + BD^2 + CD^2 + CA^2,$$

For, in the triangles AEB, DEC, the two lines AD, BC meeting the parallels AB, DC, make the angle BAE equal to the angle CDE, and the angle ABE equal to the angle DCE, and the side AB is equal to the side DC (th. 19); therefore these two triangles are identical, and have their corresponding sides equal (th. 2), viz., $AE = DE$, and $BE = EC$.

Again, since AD is bisected in E, the sum of the squares, $CA^2 + CD^2 = 2CE^2 + 2DE^2$ (th. 30).

In like manner, $BA^2 + BD^2 = 2DE^2 + 2BE^2$ or $2CE^2$.

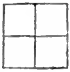
Therefore, by addition, $AB^2 + BD^2 + DC^2 + CA^2 = 4CE^2 + 4DE^2$ (ax. 2).

But because the square of a whole line is equal to 4 times the square of half the line;* that is, $BC^2 = 4BE^2$, and $AD^2 = 4DE^2$;

Therefore $AB^2 + BD^2 + DC^2 + CA^2 = BC^2 + AD^2$ (ax. 1). Q. E. D.

Cor. If $AB = AC$, or the parallelogram be a rhombus, then the triangles AEB, AEC will be mutually equilateral, and, consequently (th. 5), the angle BEA of the one will be equal to the angle AEC of the other. Hence (def. 12) the diagonals of a rhombus intersect at right angles.

* This may be seen from the accompanying diagram, or, algebraically, from considering that the square of $\frac{1}{2}a$ is $\frac{1}{4}a^2$.



EXERCISES.

1. To construct an isosceles triangle with a given base and given vertical angle.

2. Prove that every point of the bisectrix of a given angle is equally distant from the sides.

3. Two angles of a triangle being given, to find the third.

4. To construct an isosceles triangle so that the vertex shall fall at a given point, and the base fall in a given line.

5. An isosceles triangle so that the base shall be a given line and the vertical angle a right angle.

6. With two angles and a side opposite one of them, to construct a triangle.

7. To construct a triangle when the base, the angle opposite, and the sum of this and one of the other two angles are given.

8. The same, except the difference instead of the sum given.

9. To construct a quadrilateral when the four sides and one angle are given.

10. When three of the sides and the two angles included between them are given.

11. When two sides and the included angle and two other angles.

12. To construct a parallelogram with two adjacent sides and the diagonal given.

13. To construct a parallelogram with given base, altitude, and diagonal.

14. With two adjacent sides and the altitude.

15. To make a hexagon equal in all respects to a given irregular hexagon.

16. To construct a triangle with the angles at the base and the altitude given.

17. With the vertical angle, one of its sides and the altitude given.

18. With the base, altitude, and one of the angles at the base given.

19. To construct a trapezoid when three sides and the angle contained between two of them are given.

20. A line and two points without it being given, to find a point in the line equidistant from the two given points.*

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* *Geometric Analysis.*—The best method for *discovering* the solution of problems is what is termed the *analytic*. This consists in supposing the problem solved, making the diagram accordingly, and, then, by examination of the required and given parts of the diagram in their relations to one another, considering what known theorems of geometry connect them together. This is a sort of going back from the result sought by a chain of relations—depending upon known

21. The data as above to draw two lines from the two given points, meeting in the given line, and making equal angles with it.

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theorems to what is given (or may be obtained), and is the natural process of discovery or invention.

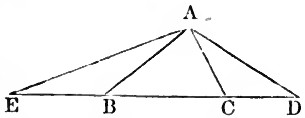
The required result having been thus obtained by *analysis*, or *resolution*, the demonstration of its correctness is made by *synthesis*, or *composition*; the order in which is the reverse of the former, and carries us forward from the data, by means of the truths on which the result depends, to the result itself. Analysis is, then, the method of *discovery*, synthesis of *demonstration* after the discovery is made. The one has for its object to find unknown truths, the other to prove known ones. Analysis and synthesis are both of them applicable to theorems as well as problems. In submitting a *problem* to analysis, its solution, in the first instance, is assumed; and from this assumption a series of consequences are drawn, until at length something is found which can be done upon established principles. In the *synthesis, or solution*, beginning with the construction indicated by the final result of the analysis, the process ends with the performance of what was required by the problem, and is the first step of the analysis.

When a *theorem* is submitted to analysis, the thing to be determined is whether the statement expressed by it be true or not. In the analysis this statement is, in the first instance, assumed to be true, and a series of consequences deduced from it, until some result is obtained, which either is an established or admitted truth, or contradicts an established or admitted truth. If the former, the theorem may be proved by retracing the steps of the investigation, commencing with the final result and concluding with the proposed theorem. But if the final result contradict an established truth, the proposed theorem must be false, since it leads to a false conclusion.

We give a specimen of the analytic investigation of a problem below. Of synthesis the student has already had specimens in all the preceding theorems, and will find others in the problems which follow the remaining theorems of plane geometry.

Specimen of the Analysis of a Problem.

Given two angles, and the sum of the three sides of a triangle, to construct it.



Suppose it done, and that ABC is the triangle sought. Produce BC till CD = CA and BE = BA; join EA, DA; then the triangles ACD and ABE being isosceles, the angle ABC = twice the angle AEB, and the angle ACB = twice the angle ADC.

Hence the following construction: at the extremities of a line ED, equal to the given sum of the three sides, draw lines making, with this, angles each equal to half one of the given angles, and from the point A, where they meet, draw lines making angles with AE and AD, respectively equal to the angles E and D; ABC will be the triangle required.

The demonstration synthetically would be as follows:

The angle EAB being = the angle E, the triangle is isosceles, and

22. The same when the two given points are on opposite sides of the line.

23. When every side of a polygon is produced out, prove that the sum of the outward angles is equal to four right angles.

24. Show that in an isosceles triangle the square of the line drawn from the vertex to any point of the base, together with the rectangle of the segments of the base, is equal to the square of one of the equal sides of the triangle.

25. Prove that the square of a line is equal to the square of its projection on another line added to the square of the difference of the perpendiculars which determine this projection.

26. Prove that the sum of the squares on two lines, together with twice their rectangle, is equal to the square on their sum.

27. That the square on the difference of two lines is equal to the sum of their squares minus twice their rectangle.

28. To construct a quadrangle when three sides, one angle, and the sum of two other angles are given.

29. When three angles and two opposite sides.

30. Prove that two parallelograms are equal when they have two sides and the included angle equal.

31. Prove that the greater diagonal of a parallelogram is opposite the greater angle.

32. Prove that two rhombi are equal when a side and angle of the one are equal to the same in the other.

33. That if the diagonals of a quadrilateral bisect each other at right angles, the figure will be a rhombus.

34. That the diagonals of a rectangle are equal; and the converse.

35. Prove that the line joining the middle points of the inclined sides of a trapezoid is parallel to the bases, and that it is equal to half the sum of the bases.

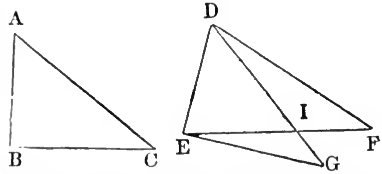
36. Prove that two convex polygons are identical: 1°. When they have the same vertices. 2°. When one side in each equal, and the distances of the corresponding vertices from its extremities equal. 3°. When composed of the same number of equal triangles, similarly placed. 4°. When they have all their sides equal and all their angles but two. 5°. When all their sides but one and all their angles but one.

37. Prove that there can be but one perpendicular from a given point to a given line.

AB = BE; for a similar reason AC = CD; hence the sum of the three sides of the triangle ABC = the given sum ED. Again, the angle ABC = 2AEB, and ACB = 2ADC ∴ ABC and ACB are equal the given angles. Q. E. D.

THEOREM XXXII.

If two triangles have two sides of the one equal to two sides of the other, but the included angles unequal, the third sides will be unequal, and the greater will be in that triangle which has the greater included angle.



Let ABC, DEF be two triangles in which AB = DE, AC = DF, BAC < EDF. Then will EF > BC.

For at the point D make the angle EDG equal to the angle BAC; take DG equal to AC, and join GE. Then will the triangle DEG equal the triangle ABC (th. 1) and EG = BC.

But EG < EI + IG, and DF < DI + IF (ax. 13).

By addition of these inequalities, EG + DF < EI + IF + DI + IG; or, EG + DF < EF + DG.

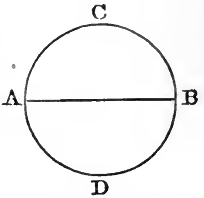
Taking away the equals DF and DG from the members of the last inequality, there remains EG < EF. But EG = BC ∴ BC < EF. Q. E. D.

If the point G fall within instead of without the triangle DEF, we should have DG + GE < DF + EF (th. 14); and, taking away the equals DG and DF, there remains EG < EF. If the point G fall on EF, the theorem is evident.

The converse of this proposition is also true, viz., that if two sides of one triangle be equal to two sides of another, and the third sides unequal, the angle opposite the smaller third side will be less than the one opposite the larger. For if the angle were greater, by the above proposition the third side must be greater; and if it were equal, it must, by (th. 1), be equal. But the third side is neither greater nor equal; therefore the angle opposite, being neither greater nor equal than the angle of the other triangle, must be less.

THEOREM XXXIII.

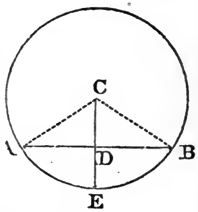
Every diameter bisects a circle and its circumference.



Let ACBD be a circle, AB a diameter. Conceive the part ADB to be turned over and applied to the part ACB, it will coincide with it exactly, otherwise there would be points in the one portion of the circumference or the other unequally distant from the center; but this is contrary to the definition (def. 41); hence the two parts are equal (ax. 10).

THEOREM XXXIV.

If a line drawn through the center of a circle bisect a chord, it will be perpendicular to the chord; or, if it be perpendicular to the chord, it will bisect both the chord and the arc of the chord.



Let AB be any chord in a circle, and CD a line drawn from the center C to the chord. Then, if the chord be bisected in the point D, CD will be perpendicular to AB.

Draw the two radii CA, CB. Then the two triangles ACD, BCD, having CA equal to CB, being radii of the same circle (def. 41), and CD common, also AD equal to DB (by hyp.); they have all the three sides of the one equal to all the three sides of the other, and so have their angles also equal (th. 5). Hence, then, the angle ADC being equal to the angle BDC, these angles are right angles, and the line CD is perpendicular to AB (def. 12).

Again, if CD be perpendicular to AB, then will the chord AB be bisected at the point D, or have AD equal to DB; and the arc AEB bisected in the point E, or have AE equal EB.

For, the two triangles ACD, BCD being right-angled at D, and having two sides of the one equal to the same in the other, viz., AC = CB and CD common, are equal (th. 26, cor. 2), ∴ AD = BD.

Also, since the angle ACE is equal to the angle BCE, the arc AE, which measures the former, is equal to the arc BE, which measures the latter, since equal angles must have equal measures.

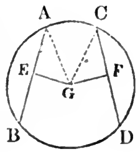
Scholium. Two conditions determine a line such as that it shall pass through two given points, or that it shall pass through one point and be perpendicular to a given line, or pass through a point and make a given angle with a given line.

The line CE in the last diagram fulfills four conditions. It passes through the center C, through the point D, the middle of the chord, through the point E, the middle of the arc, and, finally, is perpendicular to the chord AB. Either two of these involves the other two. Thus, if a line pass through the middle of the chord and be perpendicular to it, it will pass through the middle of the arc and the center of the circle; if it pass through the middle of the arc and center of the circle, it will pass through the middle of the chord and be perpendicular to it; if it pass through the middle of the arc and chord, it will be perpendicular to the latter, and pass through the center of the circle, &c.

THEOREM XXXV.

Any chords in a circle which are equally distant from the center are equal to each other; or, if they be equal to each other, they will be equally distant from the center.

Let AB, CD be any two chords at equal distances from the center G; then will these two chords AB, CD be equal to each other.



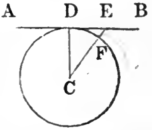
Draw the two radii GA, GC, and the two perpendiculars GE, GF, which are the equal distances of AB, CD from G (th. 17).* Then the two right-angled triangles, GAE, GCF, having the side GA equal the side GC (being radii), and the side GE equal the side GF, are identical (cor. 2, th. 26), and have the line AE equal to the line CF. But AB is the double of AE (th. 34), and CD is the double of CF; therefore AB is equal to CD (by ax. 6). Q. E. D.

Again, if the chord AB be equal to the chord CD, then will their distances from the center, GE, GF, also be equal to each other.

For, since in the right-angled triangles AEG, CFG, AE the half of AB is equal to CF, the half of CD, and the radii GA, GC are equal, therefore the third sides are equal (cor. 2, th. 26), or the distance GE is equal the distance GF. Q. E. D.

THEOREM XXXVI.

A line perpendicular to a radius at its extremity is a tangent to the circle.



Let the line ADB be perpendicular to the radius CD of a circle; then is any other point, except D, as E of the line AB, without the circle. For CE, an oblique line, is greater than the perpendicular CD (th. 17), or greater than the radius. Hence, the line AB having but one point, D, in common with the circle, is a tangent (def. 56).

THEOREM XXXVII.

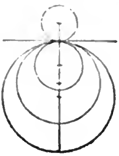
When a line is a tangent to a circle, a radius drawn to the point of contact is perpendicular to the tangent.

For if oblique, a line shorter can be drawn perpendicular to the tangent, and the tangent must then pass within the circle, which is contrary to definition.

Corol. 1. Hence, conversely, a line drawn perpendicular to a tangent, at the point of contact, passes through the center of the circle; for there can be but one perpendicular to a given line through a given point.

* By the distance of a point from a line is understood the shortest distance.

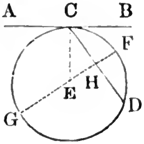
Corol. 2. If any number of circles touch each other at the same point, their centers must be in the same line perpendicular to their common tangent; for the perpendicular to the tangent at the common point must pass through the center of each.



THEOREM XXXVIII.

The angle formed by a tangent and chord is measured by half the arc of that chord.

Let AB be a tangent to a circle, and CD a chord drawn from the point of contact C; then is the angle BCD measured by half the arc CFD, and the angle ACD measured by half the arc CGD.



Draw the radius EC to the point of contact, and the radius EF perpendicular to the chord at H.

Then the radius EF, being perpendicular to the chord CD, bisects the arc CFD (th. 34). Therefore CF is half the arc CFD.

But the angle CEF is equal to the angle BCD, because the sides of the two angles are respectively perpendicular to each other, and consequently have the same difference of direction. Moreover, the angle CEF is measured by the arc CF (def. 10, note), which is the half of CFD; therefore the equal angle BCD must also have the same measure, namely, half the arc CFD of the chord CD.

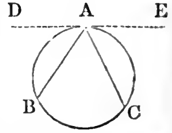
Again, GEF being a diameter, CG is the supplement of CF, and is equal to GD, the supplement of FD. ∴ CG is half the arc CGD. Now, since the line CE, meeting FG, makes the sum of the two angles at E equal to two right angles (th. 6), and the line CD makes with AB the sum of the two angles at C equal to two right angles; if from these two equal sums there be taken away the parts or angles CEH and BCH, which have been proved equal, there remains the angle CEG, equal to the angle ACH. But the former of these, CEG, being an angle at the center, is measured by the arc CG (def. 10, note); consequently the equal angle ACD must also have the same measure CG, which is half the arc CGD of the chord CD. Q. E. D.

Corol. 1. The sum of two right angles is measured by half the circumference. For the two angles BCD, ACD, which make up two right angles, are measured by the arcs CF, CG, which make up half the circumference, FG being a diameter.

Corol. 2. Hence, also, one right angle must have for its measure a quarter of the circumference, or 90 degrees.

THEOREM XXXIX.

An angle at the circumference of a circle is measured by half the arc that subtends it.



Let BAC be an angle of the circumference; it has for its measure half the arc which subtends it.

For, suppose the tangent DE to pass through the point of contact A; then, the angle DAC being measured by half the arc ABC, and the angle DAB by half the arc AB (th. 38), it follows, by equal subtraction, that the difference, or angle BAC, must be measured by half the arc BC, upon which it stands. Q. E. D.

Corol. 1. All angles in the same segment of a circle, or standing on the same arc, are equal to each other.

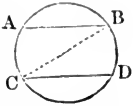
Corol. 2. An angle at the center of a circle is double the angle at the circumference, when both stand on the same arc.

Corol. 3. An angle in a semicircle is a right angle.

THEOREM XL.

Any two parallel chords intercept equal arcs.

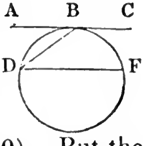
Let the two chords AB, CD be parallel; then will the arcs AC, BD be equal; or AC = BD.



Draw the line BC. Then, because the lines AB, CD are parallel, the alternate angles B and C are equal (th. 10). But the angle at the circumference B is measured by half the arc AC (th. 39); and the other equal angle at the circumference C is measured by half the arc BD; therefore the halves of the arcs AC, BD, and consequently the arcs themselves, are also equal. Q. E. D.

THEOREM XLI.

When a tangent and chord are parallel to each other, they intercept equal arcs.

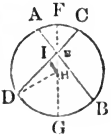


Let the tangent ABC be parallel to the chord DF; then are the arcs BD, BF equal; that is, BD = BF.

Draw the chord BD. Then, because the lines AB, DF are parallel, the alternate angles D and B are equal (th. 10). But the angle B, formed by a tangent and chord, is measured by half the arc BD (th. 38); and the other angle at the circumference D is measured by half the arc BF (th. 39); therefore the arcs BD, BF are equal. Q. E. D.

THEOREM XLII.

When two lines, meeting a circle each in two points, cut one another, either within or without it, the rectangle of the parts of the one is equal to the rectangle of the parts of the other, the parts of each being measured from the point of meeting to the two intersections with the circumference.



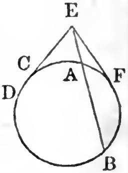
Let the two lines AB, CD meet each other in E; then the rectangle of AE, EB will be equal to the rectangle of CE, ED. Or, AE . EB = CE . ED.

For through the point E draw the diame-

ter FG; also, from the center H draw the radius DH, and draw HI perpendicular to CD.

Then, since DEH is a triangle, and the perpendicular HI bisects the chord CD (th. 34), the line CE is equal to the difference of the segments DI, EI, the sum of them being DE. Also, because H is the center of the circle, and the radii DH, FH, GH are all equal, the line EG is equal to the sum of the sides DH, HE; and EF is equal to their difference.

But the rectangle of the sum and difference of the two sides of a triangle is equal to the rectangle of the sum and difference of the segments of the base (cor., th. 27); therefore the rectangle of FE, EG is equal to the rectangle of CE, ED. In like manner, it is proved that the same rectangle of FE, EG is equal to the rectangle of AE, EB. Consequently, the rectangle of AE, EB is also equal to the rectangle of CE, ED (ax. 1). Q. E. D.



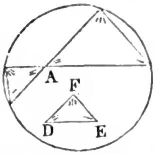
Corol. 1. When one of the lines in the second case, as DE, by revolving about the point E, comes into the position of the tangent EC or ED, the two points C and D running into one; then the rectangle of CE, ED becomes the square of CE, because CE and DE are then equal. Consequently, the rectangle of the parts of the secant, AE . EB, is equal to the square of the tangent, CE^2.

Corol. 2. Hence both the tangents EC, EF, drawn from the same point E, are equal; since the square of each is equal to the same rectangle or quantity AE . EB.

THEOREM XLIII.

In equiangular triangles, the rectangles of the corresponding or like sides, taken alternately, are equal.

Let ABC, DEF be two equiangular triangles, having the angle A = the angle D, the angle B =



the angle E, and the angle C = the angle F; also, the like sides AB, DE, and AC, DF, being those opposite the equal angles; then will the rectangle of AB, DF be equal to the rectangle of AC, DE.

In BA, produced, take AG equal to DF; and through the three points B, C, G, conceive a circle BCGH to be described, meeting CA, produced, at H, and join GH.

Then the angle G is equal to the angle C on the same arc BH, and the angle H equal to the angle B on the same arc CG (th. 39); also, the opposite angles at A are equal (th. 7): therefore the triangle AGH is equiangular to the triangle ACB, and consequently to the triangle DFE also. But the two like sides AG, DF are also equal, by construction; consequently, the two triangles AGH, DFE are identical (th. 2), and have the two sides AG, AH equal to the two DF, DE, each to each.

But the rectangle GA . AB is equal to the rectangle HA . AC (th. 42); consequently, the rectangle DF . AB is equal to the rectangle DE . AC. Q. E. D.

EXERCISES.

1. Find the length of an arc of $20^\circ\ 45'$ to a radius of 10.

2. Through two given points, to draw a circumference of given radius.

3. Divide an arc into 2, 4, 8, 16 . . . . equal parts.

4. Prove that every other chord is less than the diameter.

5. That parallel tangents include semicircumferences between their points of contact.

6. Draw a tangent to a given circle parallel to a given line.

7. To describe a circle of given radius tangent to a given line at a given point.

8. To describe a circle of given radius touching the two sides of a given angle.

9. To describe a circumference which shall be embraced between two parallels, and pass through a given point.

10. To place a chord of given length and direction in a given circle.

11. Prove that the chords of equal arcs are equal, and the converse.

12. To find in one side of a triangle the center of a circle which shall touch the other two sides.

13. To find the radius of a circle when a chord and perpendicular from the centre to the chord are given.

14. With given radii to describe two circumferences which shall intersect in a given point, and have their centers in a given line.

15. With given radii to describe two circles which shall touch each other either externally or internally.

16. Three circles with equal given radii touching each other externally.

17. The same with unequal radii.

18. Through a given point on a circumference, and another given point without, to describe a circle touching the given circumference.

19. The same when, instead of the point upon the circumference, the radius of the required circle is given.

20. To describe a circle of given radius touching two given circles.

21. To construct a right-angled triangle with the hypothenuse and one of the perpendicular sides given.

22. In a given circle to inscribe a right angle, one side of which is given.

23. In a given circle to construct an inscribed triangle of given altitude and vertical angle.

24. Also, a quadrangle, when one side and two angles not adjacent this side are given. (See Exercise 31, below.)

25. To find the center of a circle in which two given lines meeting in a point shall be a tangent and chord.

26. In a given circle to inscribe a triangle equiangular to a given triangle.

27. Show how to circumscribe a square about a given circle, and how to inscribe a circle in a given square.

28. That a straight line touching a circle can have with it but one point of contact.

29. To inscribe in an equilateral triangle three equal circles touching each other, and the sides of the triangle.

30. Prove that an eccentric angle is measured by half the sum of the opposite arcs subtending it, if the vertex be within the circle; and by half the difference of the arcs if it be without.

31. That the opposite angles of an inscribed quadrilateral are supplements.

32. That if one of the sides of an inscribed quadrilateral be produced out, the outward angle will be equal to the inward opposite angle.

33. That the sums of the opposite sides are equal.

34. That a regular polygon may be circumscribed with a circle.

35. That a circle may be inscribed in any regular polygon.

36. If one circle touch another externally or internally, any straight line drawn through the point of contact will cut off similar segments.*

37. Prove that only one tangent can be drawn to a circle at a given point on the circumference.

38. That of two chords the greater is nearer the center of the circle.

Numerical Problems.

1. In a triangle suppose two of the sides to be 8.76 and 5.26, and the perpendicular from the vertex in which they meet 4.38; required the third side.

Suppose the two segments of the required side to be represented by x and y.

$$x = \sqrt{(8.76)^2 - (4.38)^2},\ y = \sqrt{(5.26)^2 - (4.38)^2},$$

or,

$$x = \sqrt{(8.76 + 4.38)(8.76 - 4.38)},\ y = \sqrt{(5.26 + 4.38)(5.26 - 4.38)}.$$

$$\begin{array}{rl} \log.\ 13.14 = & 1.1185954 \\ \log.\ \ 4.38 = & \underline{0.6414741} \\ & 2)\underline{1.7600695} \\ x = 7.586,\ \log. = & 0.8800347 \end{array}$$

By a similar method, $y = 2.9126$

$$\therefore x + y = 10.4990$$

for the value of the third side if the perpendicular falls within the triangle; and $x \sim y = 4.67$

for the value if the perpendicular falls without.

2. Given in a triangle the base 88; one of the sides 128.49; and the perpendicular upon the base from the vertex opposite 96.45, to find the third side. *Ans.* 96.50.

3. Two chords cut each other in a circle; the segments of the one are 13 and 25; the segments of the other are in the ratio of 4 to 7; required the length of the latter chord. *Ans.* 37.47.

4. To find the absolute length of an arc of $45^\circ\ 20'$ in a circle whose radius is 5.4, supposing the ratio of the circumference to the diameter of a circle to be 3.1416. *Ans.* 4.2726.

5. The side of a square being given 0.25, to find the side of an equilateral triangle equal to the square. *Ans.* 0.37994.

6. Given the area of a circle 33.1830, to find its radius. *Ans.* 3.25.

7. Find the chord of the sum of two arcs, the chords of the arcs being given 10 and 12, and the radius 16.

8. Find the chord of half an arc, the chord of the whole arc being 12 and 16.

* Similar segments are those which correspond to similar arcs.

OF RATIOS AND PROPORTIONS.

DEFINITIONS.

DEF. 75. RATIO is the proportion or relation which one magnitude bears to another magnitude of the same kind with respect to quantity.

Note.—The measure or quantity of a ratio is conceived by considering what part or parts the leading quantity, called the Antecedent, is of the other, called the Consequent;* or what part or parts the number expressing the quantity of the former is of the number denoting, in like manner, the latter. So the ratio of a quantity, expressed by the number 2 to a like quantity expressed by the number 6, is denoted by 2 divided by 6, or $\frac{2}{6}$ or $\frac{1}{3}$; the number 2 being 3 times contained in 6, or the third part of it. In like manner, the ratio of the quantity, 3 to 6, is measured by $\frac{3}{6}$ or $\frac{1}{2}$; the ratio of 4 to 6 is $\frac{4}{6}$ or $\frac{2}{3}$; that of 6 to 4 is $\frac{6}{4}$ or $\frac{3}{2}$, &c. The ratio of two lines is the ratio of the number of times which each contains the common measure of the two lines. When the terms of a ratio are equal, it is called a ratio of Equality. When unequal, a ratio of Inequality.

76. Proportion is an equality of ratios. Thus,

77. Three quantities are said to be proportional when the ratio of the first to the second is equal to the ratio of the second to the third. As of the three quantities, $A=2$, $B=4$, $C=8$, where $\frac{2}{4}=\frac{4}{8}=\frac{1}{2}$, both the same ratio.

78. Four quantities are said to be proportional when the ratio of the first to the second is the same as the ratio of the third to the fourth. As of the four, A (4), B (2), C (10), D (5), where $\frac{4}{2}=\frac{10}{5}=2$, both the same ratio.

* The antecedent and consequent are called the terms of a ratio.

Note.—To denote that four quantities, A, B, C, D, are proportional, they are usually stated or placed thus, A : B : : C : D ; and read thus, A is to B as C is to D. The two dots : must be understood as representing the sign of division ; the four dots : : the sign of equality. The same proportion or equality of ratios may be written thus, $\frac{A}{B} = \frac{C}{D}$, or A : B = C : D. When three quantities are proportional, the middle one is repeated, and they are written thus, A : B : : B : C.

79. Of three proportional quantities, the middle one is said to be a Mean Proportional between the other two ; and the last a Third Proportional to the first and second.

80. Of four proportional quantities, the last is said to be a Fourth Proportional to the other three, taken in order.

81. Quantities are said to be Continually Proportional, or in Continued Proportion, when the ratio is the same between every two adjacent terms, viz., when the first is to the second as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on, all in the same common ratio.

As in the quantities 1, 2, 4, 8, 16, &c., where the common ratio is equal to 2.

82. Of any number of quantities, A, B, C, D, the ratio of the first A, to the last D, is said to be compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

83. Inverse ratio is, where the antecedent is made the consequent, and the consequent the antecedent. Thus, if 1 : 2 : : 3 : 6 ; then inversely, or by inversion, 2 : 1 : : 6 : 3.

84. Alternate proportion is where antecedent is compared with antecedent, and consequent with consequent. As, if 1 : 2 : : 3 : 6 ; then, by alternation or permutation, it will be 1 : 3 : : 2 : 6.

85. Compound ratio is, where the sum of the antecedent and consequent is compared either with the consequent or with the antecedent. Thus, if 1 : 2 : :

$3 : 6$; then, by composition, $1 + 2 : 1 :: 3 + 6 : 3$, and $1 + 2 : 2 :: 3 + 6 : 6$.

86. Divided ratio is, when the difference of the antecedent and consequent is compared either with the antecedent or with the consequent. Thus, if $1 : 2 :: 3 : 6$; then, by division, $2-1 : 1 :: 6-3 : 3$, and $2-1 : 2 :: 6-3 : 6$.

Note.—The term Division here means subtracting, or parting; being used in the sense opposed to compounding, or adding, in def. 85.

THEOREM XLIV.

Equimultiples of any two quantities have the same ratio as the quantity themselves.

Let A and B be any two quantities, and mA, mB any equimultiples of them, m being any number whatever; then will mA and mB have the same ratio as A and B, or $A : B :: mA : mB$.

For $$\frac{mB}{mA} = \frac{B}{A}. \quad \text{Q. E. D.}$$

Corol. Hence like parts of quantities have the same ratio as the wholes; because the wholes are equimultiples of the like parts, or A and B are like parts of mA and mB.

Corol. 2. If $\frac{B}{A}$ represent the first ratio of a proportion, or equality of ratios, $\frac{mB}{mA}$ must be the form of the second ratio, m being a quantity entire or fractional, rational or irrational. In the following theorems the form of a proportion will always be assumed in accordance with this corollary.

THEOREM XLV.

If four quantities of the same kind be proportionals, they will be in proportion by alternation or permutation, or the antecedents will have the same ratio as the consequents.

Let $A : B :: mA : mB$; then will $A : mA :: B : mB$.

For $\frac{mA}{A} = \frac{m}{1}$, and $\frac{mB}{B} = \frac{m}{1}$, both the same ratio.

THEOREM XLVI.

If four quantities be proportional, they will be in proportion by inversion or inversely.

Let $A : B :: mA : mB$; then will $B : A :: mB : mA$.

For $$\frac{mA}{mB} = \frac{A}{B}.$$

Otherwise. Let $A : B :: C : D$; then shall $B : A :: D : C$.

For let $\frac{A}{B} = \frac{C}{D} = r$; then $A = Br$, and $C = Dr$: therefore $B = \frac{A}{r}$ and $D = \frac{C}{r}$. Hence $\frac{B}{A} = \frac{1}{r}$, and $\frac{D}{C} = \frac{1}{r}$. Whence it is evident that $\frac{B}{A} = \frac{D}{C}$ (ax. 1), or $B : A :: D : C$.

In a similar manner may most of the other theorems of proportion be demonstrated.

THEOREM XLVII.

If four quantities be proportional, they will be in proportion by composition and division.

Let $A : B :: mA : mB$;

Then will $B \pm A : A :: mB \pm mA : mA$,

and $B \pm A : B :: mB \pm mA : mB$.

For $\frac{mA}{mB \pm mA} = \frac{A}{B \pm A}$; and $\frac{mB}{mB \pm mA} = \frac{B}{B \pm A}$.

Corol. 1. If $A : B :: mA : mB$, then $B + A : B - A :: mB + mA : mB - mA$.

Corol. 2. It appears from hence that the sum of the greatest and least of four proportional quantities of the same kind exceeds the sum of the other two. For since $A : A + B :: mA :: mA + mB$, where A is the least, and $mA + mB$ the greatest; then $\overline{m + 1} . A$

$+ m\text{B}$, the sum of the greatest and least, evidently exceeds $\overline{m+1}\,.\,\text{A}+\text{B}$, the sum of the two other quantities, since $m\text{B} > \text{B}$.

THEOREM XLVIII.

If of four proportional quantities there be taken any equimultiples whatever of the two antecedents, and any equimultiples whatever of the two consequents, the quantities resulting will still be proportional.

Let $\text{A} : \text{B} :: m\text{A} : m\text{B}$; also, let $p\text{A}$ and $pm\text{A}$ be any equimultiples of the two antecedents, and $q\text{B}$ and $qm\text{B}$ any equimultiples of the two consequents; then will - - - - $p\text{A} : q\text{B} :: pm\text{A} : qm\text{B}$.

For $$\frac{qm\text{B}}{pm\text{A}} = \frac{q\text{B}}{p\text{A}}.$$

THEOREM XLIX.

If there be four proportional quantities, and the two consequents be either augmented or diminished by quantities that have the same ratio as the respective antecedents, the results and the antecedents will still be proportionals.

Let $\text{A} : \text{B} :: m\text{A} : m\text{B}$, and $n\text{A}$ and $nm\text{A}$ any two quantities having the same ratio as the two antecedents; then will $\text{A} : \text{B} \pm n\text{A} :: m\text{A} : m\text{B} \pm nm\text{A}$.

For $$\frac{m\text{B} \pm nm\text{A}}{m\text{A}} = \frac{\text{B} \pm n\text{A}}{\text{A}}.$$

THEOREM L.

If any number of quantities be proportional, then any one of the antecedents will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.

Let $\text{A} : \text{B} :: m\text{A} : m\text{B} :: n\text{A} : n\text{B}$, &c.; then will $\text{A} : \text{B} :: \text{A} + m\text{A} + n\text{A} : \text{B} + m\text{B} + n\text{B}$, &c.

For $$\frac{\text{B} + m\text{B} + n\text{B}}{\text{A} + m\text{A} + n\text{A}} = \frac{(1+m+n)\,\text{B}}{(1+m+n)\,\text{A}} = \frac{\text{B}}{\text{A}}.$$

THEOREM LI.

If a whole magnitude be to a whole as a part taken from the first is to a part taken from the other, then the remainder will be to the remainder as the whole to the whole.

Let $$A : B :: \frac{m}{n} A : \frac{m}{n} B;$$

then will $$A : B :: A - \frac{m}{n} A : B - \frac{m}{n} B.$$

For $$\frac{B - \frac{m}{n}B}{A - \frac{m}{n}A} = \frac{B}{A}.$$

THEOREM LII.

If any quantities be proportional, their squares or cubes, or any like powers or roots of them, will also be proportional.

Let $A : B :: mA : mB$; then will $A^n : B^n :: m^nA^n : m^nB^n$.

For $$\frac{m^nB^n}{m^nA^n} = \frac{B^n}{A^n}.$$

THEOREM LIII.

If there be two sets of proportionals, then the products or rectangles of the corresponding terms will also be proportional.

Let $$A : B :: mA : mB,$$
and $$C : D :: nC : nD;$$
then will $$AC : BD :: mnAC : mnBD.$$

For $$\frac{mnBD}{mnAC} = \frac{BD}{AC}.$$

THEOREM LIV.

If four quantities be proportional, the rectangle or product of the two extremes will be equal to the rectangle or product of the two means. And the converse.

Let $A : B :: mA : mB$;
then is $A \times mB = B \times mA = mAB$, as is evident.

THEOREM LV.

If three quantities be continued proportionals, the rectangle or product of the two extremes will be equal to the square of the mean. And the converse.

Let A, mA, m^2A be three proportionals,
or $A : mA :: mA : m^2A$;
then is $A \times m^2A = m^2A^2$, as is evident.

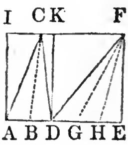
THEOREM LVI.

If any number of quantities be continued proportionals, the ratio of the first to the third will be duplicate, or the square of the ratio of the first and second; and the ratio of the first and fourth will be triplicate, or the cube of that of the first and second, and so on.

Let A, mA, m^2A, m^3A, &c., be proportionals;
then is $\frac{A}{mA} = \frac{1}{m}$; but $\frac{A}{m^2A} = \frac{1}{m^2}$; and $\frac{A}{m^3A} = \frac{1}{m^3}$, &c.

THEOREM LVII.

Triangles, and also parallelograms, having equal altitudes, are to each other as their bases.



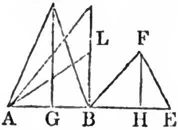
Let the two triangles ADC, DEF have the same altitude, or be between the same parallels AE, IF; then is the surface of the triangle ADC to the surface of the triangle DEF as the base AD is to the base DE. Or AD : DE :: the triangle ADC : the triangle DEF.

For, let the base AD be to the base DE as any one number m (which we have taken 2 in the diagram), to any other number n (which we have taken 3 in the diagram, though the reasoning would be the same for any other numbers); and divide the respective bases into those parts, AB, BD, DG, GH, HE, all equal to one another; and from the points of division draw the lines BC, GF, HF to the vertices C and F. Then will these lines divide the triangles ADC, DEF into the same number of parts as their bases, each equal to the triangle ABC, because those triangular parts have equal bases and altitudes (cor. 2, th. 22); namely, the triangle ABC equal to each of the triangles BDC, DFG, GFH, HFE. So that the triangle ADC is to the triangle DFE as the number of parts m (2) of the former to the number n (3) of the latter, that is, as the base AD to the base DE. (See end of note to def. 75.)

The parallelograms ADKI, DEFK being doubles of the triangles ACD, DFE, are in the same ratio, viz., that of the base AD to the base DE. Q. E. D

THEOREM LVIII.

Triangles, and also parallelograms, having equal bases, are to each other as their altitudes.



Let ABC, BEF be two triangles having the equal bases AB, BE, and whose altitudes are the perpendiculars CG, FH; then will the triangle ABC : the triangle BEF :: CG : FH.

For, let BK be perpendicular to AB, and equal to CG; in which let there be taken BL = FH; drawing AK and AL.

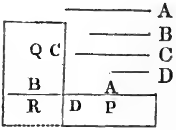
Then, triangles of equal bases and heights being equal (cor. 2, th. 22), the triangle ABK is = ABC, and the triangle ABL = BEF. The two triangles ABK, ABL may, therefore, be compared, instead of the two given triangles, which are respectively equal to them; and having the same altitude AB, they will be as their bases (th. 57), namely, the triangle ABK : the triangle ABL : : BK = CG : BL = FH.

Therefore, the triangle ABC : triangle BEF : : CG : FH.

And since parallelograms are the doubles of these triangles, having the same bases and altitudes, they will likewise have to each other the same ratio as their altitudes. Q. E. D.

THEOREM LIX.

If four lines be proportional, the rectangle of the extremes will be equal to the rectangle of the means; and, conversely, if the rectangle of the extremes of four lines be equal to the rectangle of the means, the four lines will be proportional.



Let the four lines A, B, C, D be proportionals, or A : B : : C : D; then will the rectangle of A and D be equal to the rectangle of B and C; or the rectangle A . D = B . C.

For, let the four lines be placed with their four extremities meeting in a common point, forming at that point four right angles; and draw lines parallel to them to complete the rectangles P, Q, R, where P is the rectangle of A and D, Q the rectangle of B and C, and R the rectangle of B and D.

Then the rectangles P and R, being between the same parallels, are to each other as their bases A and B (th. 57); and the rectangles Q and R, being between the same parallels, are to each other as their bases C and D. But the ratio of A to B is the same as the ratio of C to D, by hypothesis; therefore, the ratio of P to R is the same as the ratio of Q to R, and, consequently, the rectangles P and Q are equal. Q. E. D.

Again, if the rectangle of A and D be equal to the rectangle of B and C, these lines will be proportional, or A : B : : C : D.

For, the rectangles being placed the same as before; then, because parallelograms between the same parallels are to one another as their bases, the rectangle P : R : : A : B, and Q : R : : C : D. But as P and Q are equal by supposition, they have the same ratio to R, that is, the ratio of A to B is equal to the ratio of C to D, or A : B : : C : D. Q. E. D.

Corol. 1. When the two means, namely, the second and third terms, are equal, their rectangle becomes a square of the second term, which supplies the place of both the second and the third. And hence it follows that, when three lines are proportionals, the rectangle of the two extremes is equal to the square of the mean; and, conversely, if the rectangle of the extremes be equal to the square of the mean, the three lines are proportionals.

Corol. 2. If the sides about the equal angles of parallelograms or triangles be reciprocally proportional,* the parallelograms or triangles will be equal; and, conversely, if the parallelograms or triangles be equal, their sides about the equal angles will be reciprocally proportional. It is only necessary to suppose P and Q parallelograms to prove this. (See also th. 19.)

THEOREM LX.

Rectangles are to each other as the products of their bases by their altitudes.

For, in the last figure, let the two rectangles P and Q be unequal, and be placed as before. Then (th. 57),

$$P : R :: A : B;$$
$$R : Q :: D : C.$$

Multiplying the two proportions, and striking out the common factor R from the two terms of the first ratio of the resulting proportion, we have

$$P : Q :: A \times D : B \times C. \quad \text{Q. E. D.}$$

Scholium. The area or space of a rectangle may,

* *i. e.*, the first side of the first is to the first of the second as the second of the second is to the second of the first. By multiplying the extremes and means, this will make the rectangle of the two sides of the one figure equal to the rectangle of the two sides of the other.

then, be represented or expressed by the product of its length and breadth multiplied together. And, in geometry, the rectangle of two lines signifies the same thing as their product. (Compare ths. 59, 54.) Also, a square is similar to, or represented by, its side multiplied by itself, or written with an exponent 2.

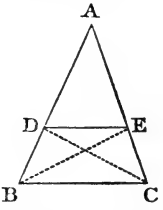
Corol. 1. Since, by th. 22, corol. 2, rhomboids are equivalent to rectangles having the same base and altitude, it follows that the areas of all parallelograms will be expressed by the product of the base by the altitude, and of triangles which are the halves of parallelograms of the same base and altitude, by the product of the base by half the altitude, or the altitude by half the base, or half the product of the base by the altitude.

Corol. 2. Parallelograms or triangles having equal bases will be to each other as their altitudes; those having equal altitudes will be to each other as their bases; and those having neither equal will be as the products of their bases by their altitudes.

Corol. 3. Parallelograms or triangles having an angle in each equal, are in proportion to each other as the rectangles of the sides which are about these equal angles. This may be proved from the last diagram, supposing P and Q to be parallelograms.

THEOREM LXI.

If a line be drawn in a triangle parallel to one of its sides, it will cut the other two sides proportionally.



Let DE be parallel to the side BC of the triangle ABC; then will AD : DB : : AE : EC.

For, draw BE and CD. Then the triangles DBE, DCE are equal to each other, because they have the same base DE, and are between the same parallels DE, BC (th. 22). But the two triangles ADE, BDE, on the bases AD, DB, have the same altitude, viz., the perpendicular from their common vertex E to the line of their bases BA; and the two triangles ADE, CDE, on the bases AE, EC, have also a common altitude; and because triangles of the same altitude are to each other as their bases, therefore

the triangle ADE : BDE : : AD : DB,
and triangle ADE : CDE : : AE : EC.

But BDE is = CDE; hence the first ratio is the same in these two proportions, and the second ratios must be equal; therefore AD : DB : : AE : EC. Q. E. D.

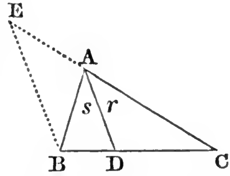
Corol. 1. Also, the whole lines AB, AC are proportional to their corresponding proportional segments (th. 47),

viz., AB : AC : : AD : AE,
and AB : AC : : BD : CE.

Corol. 2. The converse of the above proposition is also true, viz., that a line which divides the two sides of a triangle proportionally must be parallel to the base. For any other line through D than the parallel DE, meeting AC in some other point than E, must divide AC into two parts, having a different ratio from AE to EC, and, consequently, different from the ratio AD : DB.

THEOREM LXII.

A line which bisects any angle of a triangle divides the opposite side into two segments, which are proportional to the other two adjacent sides.



Let the angle BAC, of the triangle ABC, be bisected by the line AD; then will the segment BD be to the segment DC as the side AB is to the side AC.

For, let BE be drawn parallel to AD, meeting CA produced at E. Then, because the line BA meets the two parallels AD, BE, it makes the angle ABE equal to the alternate angle *s*. (th. 10), and therefore also equal to the angle r, which is (by hyp.) equal to s. Again, because the line CE cuts the two parallels AD, BE, it makes the angle E equal to the angle r* on the same side (th. 10). Hence, in the triangle ABE, the angles B and E, being each equal to half the bisected angle of the triangle, are equal to each other, and, consequently, their opposite sides AB, AE are also equal (th. 4).

But now, in the triangle CBE, the line AD, being parallel to the side BE, cuts the other two sides, CB, CE, proportionally (th. 61), making CD to DB, as is CA to AE, or to its equal AB. Q. E. D.

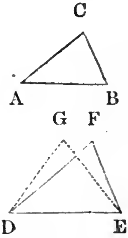
THEOREM LXIII.

Equiangular triangles are similar, or have their like sides proportional.

For, by th. 43, the rectangles of the corresponding sides taken alternately are equal, and by the second part of th. 59, these corresponding or like sides† are, in consequence, directly proportional.

THEOREM LXIV.

Triangles which have their sides proportional are also equiangular.



In the two triangles ABC, DEF, if AB : DE : : AC : DF : : BC : EF, the two triangles will have their corresponding angles equal.

For, if the triangle ABC be not equiangular with the triangle DEF, suppose some other triangle, as DEG, constructed upon the side DE, to be equiangular with ABC. But this is impossible; for if the two triangles ABC, DEG were equiangular, their sides would be proportional (th. 63), viz.,

* The use of small letters to designate angles may be adopted in other propositions.

† The corresponding sides are called *homologous.*

$$AB : DE :: AC : DG,$$

but, by hypothesis,

$$AB : DE :: AC : DF;$$
$$\therefore DG = DF.$$

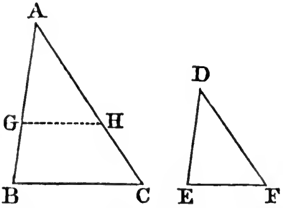
In the same manner, it may be proved that

$$EG = EF;$$

$\therefore$ (th. 5), Δ* DEF is identical with Δ DEG, which is absurd, the angles being different.

THEOREM LXV.

Triangles which have an angle in the one equal to an angle in the other, and the sides about these angles proportional, are equiangular.



Let ABC, DEF be two triangles, having the angle A = the angle D, and the sides AB, AC proportional to the sides DE, DF; then will the triangle ABC be equiangular with the triangle DEF.

For, make AG = DE, and AH = DF, and join GH.

Then the two triangles DEF, AGH, having two sides equal, and the contained angles A and D equal, are identical and equiangular (th. 1), having the angles G and H equal to the angles E and F. But, since the sides AG, AH are proportional to the sides AB, AC, the line GH is parallel to BC (th. 61, corol. 2); hence the angles B and C are equal to the angles G and H respectively (th. 10), and, consequently, to their equals E and F. Q. E. D.

General Scholium.—Triangles will be similar, 1°. When they have their angles equal, or two of their angles equal (th. 15, corol. 1); 2°. When they have their homologous sides proportional;† 3°. When

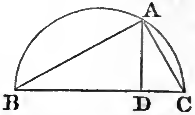
* This sign (Δ) stands for the word triangle.

† Triangles are the only polygons in which one part of the definition (def. 67) of similar figures involves the other as a necessary consequence. Thus a square and a rectangle are equiangular quadri-

they have an angle in each equal, and the sides about the equal angles proportional; 4°. When they have their sides respectively parallel or perpendicular, or in any way equally inclined.

THEOREM LXVI.

In a right-angled triangle, a perpendicular from the right angle is a mean proportional between the segments of the hypothenuse, and each of the sides about the right angle is a mean proportional between the hypothenuse and the adjacent segment.



Let ABC be a right-angled triangle, and AD a perpendicular from the vertex of the right angle A to the hypothenuse CB; then will

AD be a mean proportional between BD and DC;
AB a mean proportional between BC and BD;
AC a mean proportional between BC and DC.

For, the two right-angled triangles ABD, ABC, having the angle B common, are equiangular (cor. 2, th. 15). For a similar reason, the two triangles ABC, ADC are equiangular.

Hence, then, all the three triangles, viz., the whole triangle and the two partial triangles ABC, ABD, ADC, being equiangular, will have their like sides proportional (th. 63),

viz.,* BD : AD : : AD : DC;
and BC : AC : : AC : DC;
and BC : AB : : AB : BD. Q. E. D.

laterals, but their homologous sides are not proportional, the adjacent ones of the square having a ratio of equality, those of the rectangle a ratio of inequality. Again, the sides of a square and rhombus are proportional, having in both the ratio of equality, but the angles are not equal, those of the square being right, those of the rhombus oblique.

* The student will be aided by saying BD, the long perpendicular side of the left triangle, is to AD, the long perpendicular side of the right triangle, as AD, the short perpendicular side of the former, is to DC, the short perpendicular side of the latter. Again, BC, the hypothenuse of the whole triangle, is to AB, the hypothenuse of the left partial triangle, &c.

Corol. 1. Because the angle in a semicircle is a right angle (corol. 3, th. 39), it follows that if, from any point A in the periphery of the semicircle, a perpendicular be drawn to the diameter BC, and the two chords CA, AB be drawn to the extremities of the diameter; then are AD, AB, AC the mean proportionals as in this theorem, or (by th. 55) $AD^2 = CD . BD$; $AB^2 = BC . BD$; and $AC^2 = CB . CD$.

Corol. 2. Hence $AB^2 : AC^2 :: CD : BD$.

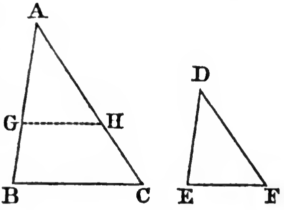
Corol. 3. Hence we have another demonstration of th. 26.

For, since $AB^2 = BC . BD$, and $AC^2 = BC . CD$;
By addition, $AB^2 + AC^2 = BC (BD + CD) = BC^2$.

THEOREM LXVII.

Similar triangles are to each other as the squares of their like sides.

Let ABC, DEF be two similar triangles, AB and DE being two like sides; then will the triangle ABC be to the triangle DEF as the square of AB is to the square of DE, or as AB^2 to DE^2.



For, the triangles being similar, they have their like sides proportional (def. 67);

therefore $AB : DE :: AC : DF$;
and $AB : DE :: AB : DE$, an identity of ratios;
therefore $AB^2 : DE^2 :: AB . AC : DE . DF$ (th. 53).

But the triangles are to each other as the rectangles of the like pairs of their sides (cor. 3, th. 60); or

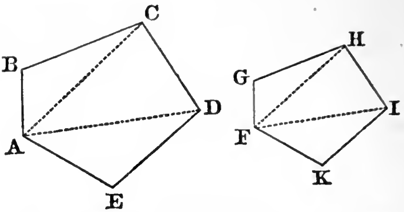
$$\triangle ABC : \triangle DEF :: AB . AC : DE . DF;$$

therefore $\triangle ABC : \triangle DEF :: AB^2 : DE^2$. Q. E. D.

THEOREM LXVIII.

The perimeters of all similar figures are to each other as their homologous sides, and the surfaces as the squares of their homologous sides.

Let ABCDE, FGHIK be any two similar figures, their like sides being AB, FG, and BC, GH, and so on in the same order; then



will the perimeter of the figure ABCDE be to the perimeter of the figure FGHIK as AB to FG, and the surface as the square of AB to the square of FG, or as AB^2 to FG^2.

For (by def. 67) AB : BC : CD, &c. : : FG : GH : HI, &c. And (by th. 50) AB + BC + CD, &c.; or the perimeter of the first polygon is to FG + GH + HI, &c.; or the perimeter of the second polygon as AB : FG.

Again, draw AC, AD, FH, FI, dividing the figures into an equal number of triangles by lines from two equal angles A and F.

The two figures being similar (by hyp.), they are equiangular, and have their like sides proportional (def. 67).

Then, since the angle B is = the angle G, and the sides AB, BC proportional to the sides FG, GH, the triangles ABC, FGH are equiangular (th. 65). If, from the equal angles BCD, GHI there be taken the equal angles ACB, GHF, there will remain the equals ACD, FHI; and since, from the similarity of the triangles ABC, FGH, and of the whole polygons, AC and FH, as well as CD and HI, have the same ratio that BC and GH have, they must have the same ratio as one another; hence the triangles ACD, FHI, having an equal angle contained by proportional sides are (th. 65) similar.

In the same manner, ADE may be proved similar to FIK. Hence each triangle of the one figure is equiangular with each corresponding triangle of the other.

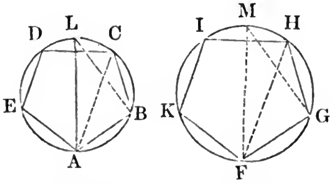
But equiangular triangles are similar (th. 63), and are proportional to the squares of their like sides (th. 67).

Therefore the $\triangle$ ABC : $\triangle$ FGH : : $AB^2 : FG^2$;
and $\triangle$ ACD : $\triangle$ FHI : : $DC^2 : HI^2$;
and $\triangle$ ADE : $\triangle$ FIK : : $DE^2 : IK^2$.

But as the two polygons are similar, their like sides are proportional, and, consequently, their squares also proportional; so that all the ratios AB^2 to FG^2, and DC^2 to HI^2, and DE^2 to IK^2, are equal among themselves, and, consequently, the corresponding triangles also, ABC to FGH, and ACD to FHI, and ADE to FIK, have all the same ratio, viz., that of AB^2 to FG^2; and hence the sum of the antecedents, or the figure ABCDE, have to the sum of the consequents, or the figure FGHIK, still the same ratio, viz., that of AB^2 to FG^2 (th. 50). Q. E. D.

THEOREM LXIX.

Similar figures inscribed in circles have their like sides, and also their whole perimeters, in the same ratio as the diameters of the circles in which they are inscribed.



Let ABCDE, FGHIK be two similar figures, inscribed in the circles whose diameters are AL and FM; then will each side AB, BC, &c., of the one figure be to the like side FG, GH, &c., of the other figure, or the whole perimeter AB + BC +, &c., of the one figure, to the whole perimeter FG + GH +, &c., of the other figure, as the diameter AL to the diameter FM.

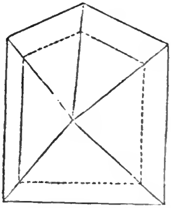
For, draw the two corresponding diagonals, AC, FH, as also the lines BL, GM. Then, since the polygons are similar, they are equiangular, and their like sides have the same ratio (def. 67); therefore the two triangles ABC, FGH have the angle B = the angle G, and the two sides AB, BC proportional to the two sides FG, GH; consequently, these two triangles are equiangular (th. 65), and have the angle ACB = FHG. But the angle ACB = ALB, standing on the same arc AB; and the angle FHG = FMG, standing on the same arc FG; therefore the angle ALB = FMG (ax. 1). And since the angle ABL = FGM, being both right angles, because in a semicircle; therefore the two triangles ABL, FGM, having two angles equal, are equiangular; and, consequently, their like sides are proportional (th. 63); hence AB : FG : : the diameter AL : the diameter FM.

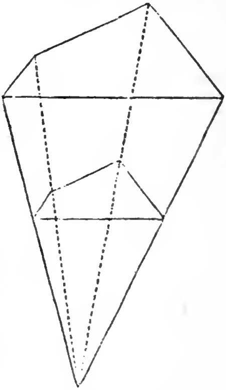
In like manner, each side BC, CD, &c., has to each side GH, HI, &c., the same ratio of AL to FM; and, consequently, the sums of them are still in the same ratio, viz., AB + BC + CD, &c. : FG + GH + HI, &c., : : the diam. AL : the diam. FM (th. 50). Q. E. D.*

THEOREM LXX.

Similar figures inscribed in circles are to each other as the squares of the diameters of those circles.

Let (see last fig.) ABCDE, FGHIK be two similar figures, inscribed in the circles whose diameters





* Similar figures admit of a better definition than that given at definition 67, and which can now be understood.

They are those which can be so placed that lines drawn through the angular points of the one from some point O, within or without, shall also pass through the angular points of the other; and the distances from the angular points of the two figures to the point O shall be proportional.

This definition admits of being enlarged. Similar geometrical *magnitudes* are those which admit of lines being drawn from a point through the corresponding points of both, the distances of which from the radiant point are proportional (See Appendices II. and V.)

are AL and FM; then the surface of the polygon ABCDE will be to the surface of the polygon FGHIK as AL^2 to FM^2.

For the figures, being similar, are to each other as the squares of their like sides, AB^2 to FG^2 (th. 68). But by the last theorem, the sides AB, FG are as the diameters AL, FM; and, therefore, the squares of the sides AB^2 to FG^2 as the squares of the diameters AL^2 to FM^2 (th. 52). Consequently, the polygons ABCDE, FGHIK are also to each other as the squares of the diameters AL^2 to FM^2 (ax. 1). Q. E. D.

THEOREM LXXI.

The circumferences of all circles are to each other as their diameters.

Let D, d denote the diameters of two circles, and C, c their circumferences; then will $D : d :: C : c$, or $D : C :: d : c$.

For (by th. 69) similar polygons inscribed in circles have their perimeters in the same ratio as the diameters of those circles.

Now, as this property belongs to all polygons, whatever may be the number of the sides, conceive the number of the sides to be indefinitely great, and the length of each infinitely small. But the perimeter of the polygon of an indefinite number of sides becomes the same thing as the circumference of the circle. Hence it appears that the circumferences of circles, being the same as the perimeters of such polygons, are to each other in the same ratio as the diameters of the circles. Q. E. D.

Corol. 1. Since the radius is half the diameter, circumferences are also as their radii.

Corol. 2. Similar arcs being the same parts of their respective circumferences, are to each other as their radii.

Corol. 3. The ratio of the arc which subtends an angle to its radius, being an arc which subtends the same angle in the circle whose radius is unity, it follows that angles at the centers of different circles are to each other as the ratio of the arcs which subtend them to their radii.

THEOREM LXXII.

The areas or spaces of circles are to each other as the squares of their diameters, or of their radii.

Let A, a denote the areas or spaces of two circles, and D, d their diameters; then $A : a :: D^2 : d^2$.

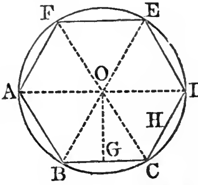
For (by th. 70) similar polygons inscribed in circles are to each other as the squares of the diameters of the circles.

Hence, conceiving the number of the sides of the polygons to be increased more and more, or the length of the sides to become less and less, the polygon approaches nearer and nearer to the circle, till at length, by an infinite approach, they coincide, and become, in effect, equal; and then it follows that the spaces of the circles, which are the same as of the polygons, will be to each other as the squares of the diameters of the circles. Q. E. D.

Corol. The spaces of circles are also to each other as the squares of the circumferences; since the circumferences are in the same ratio as the diameters (by th. 71).

THEOREM LXXIII.

The area of any circle is equal to the rectangle of half its circumference and half its diameter.



Conceive a regular polygon to be inscribed in a circle, and radii drawn to all the angular points, dividing it into as many equal triangles as the polygon has sides, one of which is OBC, of which the altitude is the perpendicular OG,*

* The line OG is called the *apophthegm* of the polygon.

from the center to the base BC. The other triangles will have the same altitude (th. 35).

Then the triangle OBC is equal (th. 60, cor. 1) to the rectangle of the half base BC and the altitude OG; consequently, the whole polygon, or all the triangles added together which compose it, is equal to the rectangle of the common altitude OG, and the halves of all the sides, or the half perimeter of the polygon.

Now, conceive the number of sides of the polygon to be indefinitely increased; then will its perimeter coincide with the circumference of the circle, and the altitude OG will become equal to the radius, and the whole polygon equal to the circle. Consequently, the space of the circle, or of the polygon in that state, is equal to the rectangle of the radius and half the circumference. Q. E. D.

Scholium. It will be shown that the ratio of the circumference of a circle to its diameter may be expressed approximately by the mixed decimal 3.1415926, a number which in all mathematical books it is customary to represent by the Greek letter π.*

If now d denote the diameter of a circle, r the radius, c the circumference, and a the area, we shall have the following formulæ derived from the foregoing theorems:

$$c = \pi d \quad \ldots (1)$$
$$c = 2\pi r \ldots (2)$$
$$a = \pi r^2 \quad \ldots (3)$$

(1) is obtained by multiplying d by the ratio of c and d;
(2) " " substituting $2r$ for d in (1);
(3) " " multiplying (2) by $\frac{1}{2}r$, in accordance with th. 73.

EXERCISES.

1. Prove that no two lines in a circle bisect each other except two diameters.

2. Prove that lines drawn from the vertex of a triangle divide the base and a parallel to the base proportionally.

* For the investigation of the ratio of the circumference of a circle to its diameter, see Mensuration, at the last part of this volume.

3. Prove that if a line bisect the external angle of a triangle, the distances of the point in which it meets the side opposite from the extremities of that side produced, will have the same ratio as the other two sides of the triangle.

4. Through a given point, situated between the sides of an angle, to draw a line terminating at the sides of the angle, and in such a manner as to be equally divided at the point.

5. Prove that the hypothenuse of a right-angled triangle is to either segment formed by a perpendicular upon the hypothenuse from the opposite vertex, as the square on the hypothenuse is to the square on the side adjacent the segment.

6. Construct a quadrangle similar to a given quadrangle, the sides of the latter having to the former the ratio of 2 to 3.

7. Divide a line into parts proportional to three given lines.

8. To draw a line parallel to the base of a given triangle in such a manner as to halve the triangle.

9. If from a point in the circumference of a circle two chords be drawn to the extremities of any diameter and a perpendicular; supposing the diameter to be 20, and the ratio of the segment into which the perpendicular divides it 2 : 3, what are the lengths of the chords?

10. To divide a given line into two parts, such that they shall have the ratio of two given squares.

11. To find a line which shall be to a given line as $\sqrt{8} : \sqrt{10}$.

12. To construct a triangle, having given the base, the vertical angle, and the ratio of the two sides which contain it. (See Pr. 21, p. 84)

13. The same with the same ratio, the altitude and one of the angles at the base given.

14. The same with the altitude, the ratio of the two segments of the base, and the vertical angle.

15. The same when the base, the vertical angle, and the sum of the squares of the two sides are given.

16. The same when the base, the altitude, and the sum of the squares of the two sides.

17. The same with the difference of the squares.

18. To inscribe in a given triangle another triangle of given angles in such a manner that one of its sides may be parallel to one of the sides of the given.

19. Also, when the required triangle is required to be similar to the given, and one of its vertices at a given point in one of the sides of the given triangle.

20. The same similar to another instead of the given triangle.

21. To describe a circle passing through a given point, and touching the two sides of a given angle.

22. Through two points touching a given line.

23. Touching a line and circle, and passing through a given point.

24. To construct a square when the difference between its diagonal and side are given.

25. To find a point in a given line from which, if lines be drawn to two given points without, they shall have a given ratio.

26. Prove that if two circles touch each other, the secants through the point of contact and terminating in the two circumferences are divided proportionally at that point.

27. Prove that the two common tangents and the line joining the centers of two circles meet in the same point.

28. Draw a tangent to two circles of different centers and radii.

29. Prove that if a line be drawn through the points of intersection of two circles, tangents drawn from any point of this line will be equal.

30. That two parallelograms are similar when they have an angle in each equal contained by proportional sides.

31. Prove that the ratio of the diagonal to the side of a square is that of $\sqrt{2}$ to 1.

32. To construct a polygon similar to a given one, and bearing to it a given ratio.

33. To construct a polygon similar to one given polygon and equal to another.

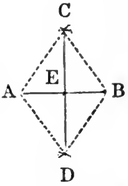
34. Construct a rectangle equal to a given square, having the sum of its sides equal to a given line. The same, except the difference of the sides equal to a given line.

35. To find a point such that the sum of the squares of its distances from two given points shall be equal to a given square.

PROBLEMS.

PROBLEM I.*

To bisect a given line AB.

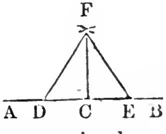


From the two centers A and B, with any equal radii greater than AE, describe arcs of circles, intersecting each other in C and D; and draw the line CD, which will bisect the given line AB in the point E.

For C and D both belong to the perpendicular at the middle of AB (th. 17, corol. 1); and as but one line can be drawn through two points, CD must be this bisecting perpendicular.

PROBLEM II.

At a given point C, *in a line* AB, *to erect a perpendicular.*



From the given point C, with any radius, cut off any equal parts CD, CE of the given line; and from the two centers D and E, with any one radius, describe arcs intersecting in F; then join CF, which will be perpendicular, as required.

For the points F and C both belong to the perpendicular at the middle of DE, and determine it. (See note to Axioms.)†

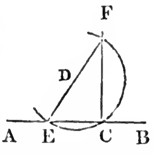
* Many of the following problems are for the benefit of those who omit the exercises.

† The most convenient way of drawing perpendiculars through points within or without lines is by means of a rule and triangle made of wood, metal, or any other hard substance.

The triangle is right-angled, and its hypothenuse being placed against the rule, with one of its perpendicular sides coinciding with the given line, the triangle is moved up or down obliquely, sliding along the rule, till the other perpendicular side passes through the given point; a line drawn along this latter side will be the perpendicular required.

OTHERWISE.

When the point is near the end of the line.



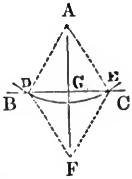
Analysis.* Suppose the perpendicular CF drawn; FCA will then be a right angle. But we know that a right angle is inscribed in a semicircle. Hence the following construction.

From any point D, assumed above the line, as a center, through the given point C describe a circle, cutting the given line at E; and through E and the center D draw the diameter EDF; then join CF, which will be the perpendicular required.

Synthesis. For the angle at C, being an angle in a semicircle, is a right angle, and therefore the line CF is a perpendicular (by def. 12).

PROBLEM III.

From a given point A, *to let fall a perpendicular on a given line* BC.



From the given point A as the center, with any convenient radius, describe an arc, cutting the given line at the two points D and E; and from the two centers D, E, with any radius, describe two arcs, intersecting at F; then draw AGF, which will be perpendicular to BC, as required.

With the rule and triangle, parallel lines to a given line may be drawn through given points in an obvious manner.

Another method is by means of what is called a T rule, from its resemblance to this letter, the cross-piece being thicker than the other, so as to project below the edge of a rectangular drawing board, upon which the paper is pasted. The lines drawn with this will be always parallel to the edges of the board, unless there be a movement of one arm of the rule, so that it may be placed at any angle with the other.

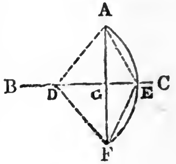
* The student may be exercised in giving the analysis of some others of the problems.

For the points A and F are both equally distant from the points D and E; hence AF is perpendicular at the middle of DE.

OTHERWISE.

When the point is nearly the opposite end of the line.

From any point D, in the given line BC, as a center, describe the arc of a circle through the given point A, cutting BC in E; and from the center E, with the radius EA, describe another arc, cutting the former in F; then draw AGF, which will be perpendicular to BC, as required.

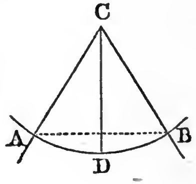


For the chords AE, EF being equal, their arcs are equal, and the line DE, drawn through the center D and middle point E of the arc AEF, is perpendicular to the chord AF of that arc. (See th. 34.)

PROBLEM IV.

To bisect a given angle.

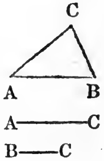
Let ACB be the given angle. With C as a center, describe an arc, cutting the sides of the given angle in A and B. Draw the chord AB, and from C the perpendicular CD to this chord, which (th. 34) will bisect the given angle.



PROBLEM V.

To make a triangle with three given lines AB, AC, BC.

With the center A, and distance AC, describe an arc. With the center B, and distance BC, describe another arc, cutting the former in C. Draw AC, BC, and ABC will be the triangle required.

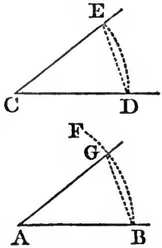


For the radii, or sides of the triangle, AC, BC are equal to the given lines AC, BC by construction.

Note. If any two of the lines are not together greater than the third, the construction is impossible.

PROBLEM VI.

At a given point A, *in a line* AB, *to make an angle equal to a given angle* C.

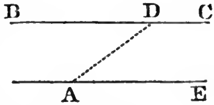


From the centers A and C, with any one radius, describe the arcs DE, BF. Then, with radius DE, and center B, describe an arc, cutting BF in G. Through G draw the line AG, and it will form the angle required.

Let the equal lines or radii, DE, BG, be drawn. Then the two triangles CDE, ABG, being mutually equilateral, are mutually equiangular (th. 5), and have the angle at A equal to the angle at C.*

PROBLEM VII.

Through a given point A, *to draw a line parallel to a given line* BC.



From the given point A draw a line AD to any point in the given line BC. Then draw the line AE, making the angle at A equal to the angle at D (by prob. 6); so shall AE be parallel to BC, as required.

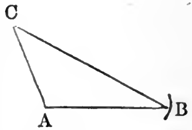
* Angles are made most conveniently with a *protracter,* which is commonly a semicircle of metal, horn, or paper, divided into degrees and parts of a degree. An angle equal to a given angle is *protracted* by placing the diameter of the instrument upon one side of the given angle, the center being at the vertex, and then the other side of the given angle will pass through the number of the degrees which it contains. If, then, the protracter be taken up and placed with its diameter upon the given line, and center at the given point, and a point marked on the paper at the same degree or division of the circumference, and this point joined with the given point, the required angle will be formed.

For, the angle D being equal to the alternate angle A, the lines BC, AE are parallel, by th. 10.

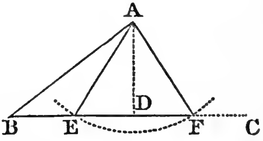
PROBLEM VIII.

Given two sides of a triangle and the angle opposite one of them to construct the triangle.

There will be two cases:



1. Where the given triangle is right or obtuse. Draw two lines, AB, AC, making with each other the given angle. Take AC, equal to one of the given sides, and with C as a center and the given side opposite the given angle as a radius, cut the indefinite side AB in B. Join CB, and ABC will be the triangle constructed with the given angle and sides.



2. If the given angle be acute, and the side opposite be less than the other given side, there will be two solutions. The triangle ABF or ABE, either of them being constructed with the given angle B and side AE or AF. This is called a doubtful or ambiguous solution.

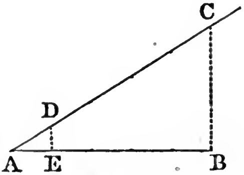
If AE or AF be just long enough to reach BC, the two solutions coalesce, and the resulting triangle ADB is right-angled at D. If AF is too short to reach BC, the solution is impossible.

N.B.—With the exception of this case, it may be said, as a general scholium, that a triangle is determined when any three parts* are given, one of which is a side. Or, two triangles are equal when any three parts, one of which is a side in the one, are equal to the same in the other.

* The parts of a triangle are the three sides and the three angles—six in all.

PROBLEM IX.

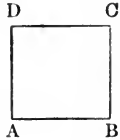
To divide a line AB *into any proposed number of equal parts.*



Draw any other line AC, forming any angle with the given line AB; on which set off any line AD as many times as there are to be parts in AB ending at C. Join BC, parallel to which draw DE, then AE will apply exactly the required number of times to AB. For those parallel lines divide both the sides AB, AC proportionally, by th. 61.

PROBLEM X.

To make a square on a given line AB.

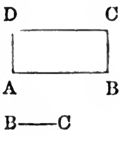


Raise AD, BC, each perpendicular and equal to AB, and join DC; so shall ABCD be the square sought.

For all the three sides AB, AD, BC are equal, by the construction, and DC is equal and parallel to AB (by th. 21); so that all the four sides are equal, and the opposite ones are parallel. Again, the angle A or B of the parallelogram, being a right angle, the angles are all right ones (cor. 1, th. 19). Hence, then, the figure, having all its sides equal and all its angles right, is a square (def. 32).

PROBLEM XI.

To make a rectangle or a parallelogram of a given length and breadth, AB, BC.



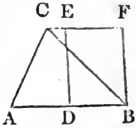
Erect AD, BC perpendicular to AB, and each equal to BC; then join DC, and it is done.

The demonstration is the same as in the last problem.

And in the same manner is described any oblique parallelogram, only drawing AD and BC to make the given oblique angle with AB, instead of perpendicular to it.

PROBLEM XII.

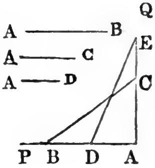
To make a rectangle equal to a given triangle ABC.



Bisect the base AB in D; then raise DE and BF perpendicular to AB, and meeting CF parallel to AB at E and F; so shall DF be the rectangle equal to the given triangle ABC (by cor. of th. 23).

PROBLEM XIII.

To make a square equal to the sum of two or more given squares.

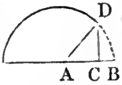


Let AB and AC be the sides of two given squares. Draw two indefinite lines, AP, AQ, at right angles to each other, in which place the sides AB, AC of the given squares; join BC: then a square described on BC will be equal to the sum of the two squares described on AB and AC (th. 26).

In the same manner, a square may be made equal to the sum of three or more given squares. For, if AB, AC, AD be taken as the sides of the given squares, then, making AE = BC, AD = AD, and drawing DE, it is evident that the square on DE will be equal to the sum of the three squares on AB, AC, AD. And so on for more squares.

PROBLEM XIV.

To make a square equal to the difference of two given squares.

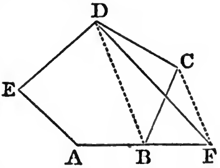


Let AB and AC, taken in the same straight line, be equal to the sides of the two given squares. From the center A, with the distance AB, describe a circle, and make CD perpendicular to AB, meeting the circumference in D: so shall a square described on CD be equal to $AD^2 - AC^2$, or $AB^2 - AC^2$, as required (cor. 1, th. 26).

PROBLEM XV.

To make a triangle equal to a given polygon ABCDE.

Draw DB and CF parallel to it, meeting AB produced at F; then draw DF; so shall the polygon DFAE be equal to the given polygon ABCDE.



For the triangle DFB = DCB (th. 22); therefore, by adding DBAE to the equals, the sums are equal (ax. 2), that is, DBAE + DBF = DBAE + DCB, or the quadrilateral DFAE = to the pentagon ABCDE.

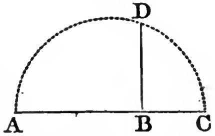
In a similar manner the number of sides of a polygon may be repeatedly reduced, by one each time, till the polygon is changed into an equivalent triangle.

PROBLEM XVI.

To make a square equal to a given rectangle.

Let AB, BC be equal to the adjacent sides of the rectangle.

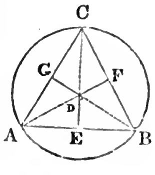
Produce one side AB till BC be equal to the other side. On AC as a diameter describe a circle meeting the perpendicular BD at D; then will BD be the side of the square equal to the given rectangle, as appears by cor. 1, th. 66.



PROBLEM XVII.

To describe a circle about a given triangle ABC.

Bisect any two sides with two of the perpendiculars FD, ED, or GD, and the point D, in which they intersect, will be the center.



For every point of the line FD must be equally distant from the points B and C (th. 17, corol. 1), and every point of the line ED must be equally distant from A and B; hence the point D, common to these two lines, must be at equal distances from the three points A, B, and C, and the center of a circle passing through them.

Scholium. There is but one such circle. For its center could not be out of the line FA (th. 18), nor out of EC, and they intersect in but one point D.*

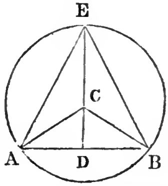
Note. The problem is the same, in effect, when it is required

To describe the circumference of a circle through three given points A, B, C, *or to find the center of a given circle or arc.*

Draw chords BA, BC, and bisect these chords perpendicularly by lines meeting in D, which will be the center. (See last diagram.)

PROBLEM XVIII.

An isosceles triangle ABC *being given, to describe another on the same base* AB, *whose vertical angle shall be only half the vertical angle* C.



From C as a center, with the distance CA, describe the circle ABE. Bisect AB in D, join DC, and produce to the circumference at E; join EA and EB, and ABE shall be the isosceles triangle required.

For every point of the perpendicular DE is equally distant from A and B (th. 17, corol. 1); hence the side EA must be equal to the side EB of the triangle AEB, which is, therefore, isosceles, and the angle ACB at the center must be double of the angle AEB at the circumference, for they both stand on the same segment AB.

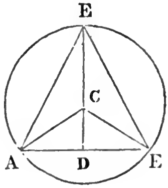
* If the given triangle be acute angled, the center of the circle will be within it; and if the triangle be equilateral, as in the diagram, the center of the circle will be the center of the triangle, and the perpendiculars at the middle of the sides will pass through the vertices of the opposite angles.

If the triangle be obtuse angled, the center of the circle will fall without; if right angled, the center will fall upon the hypothenuse.

PROBLEM XIX.

Given an isosceles triangle AEB, *to erect another on the same base* AB, *which shall have double the vertical angle* E.

Describe a circle about the triangle AEB, find its center C, and join CA, CB, and ACB is the triangle required.

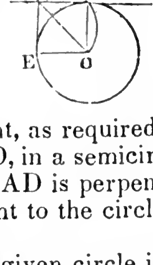


The angle C at the center is double of the angle E at the circumference, and the triangle ACB is isosceles; for the sides CA, CB, being radii of the same circle, are equal.

PROBLEM XX.

To draw a tangent to a circle, through a given point A.

1. When the given point A is in the circumference of the circle, join A and the center O; perpendicular to which draw BAC, and it will be the tangent, by th. 36.



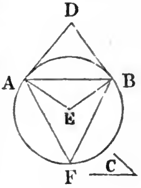
2. When the given point A is out of the circle, draw AO to the center O; on which, as a diameter, describe a semicircle, cutting the given circumference in D; through which draw BADC, which will be the tangent, as required.

For join DO. Then the angle ADO, in a semicircle, is a right angle, and, consequently, AD is perpendicular to the radius DO, or is a tangent to the circle (th. 36).

Scholium. The circle ADO cuts the given circle in two points; and there will be two tangents, AD and AE, to the given circle from the same point A, without. These tangents are equal in length, and the line joining the point without and the center bisects the angle which the tangents make with each other; for the right-angled triangles ADO, AEO, having the side OA common, and the side OD = OE being radii of the same circle, the triangles are equal ∴ AD = AE and angle DAO = angle EAO.

PROBLEM XXI.

On a given line AB *to describe a segment of a circle capable of containing a given angle.*



At the ends of the given line make angles DAB, DBA, each equal to the given angle C. Then draw AE, BE perpendicular to AD, BD; and with the center E, and radius EA or EB, describe a circle; so shall AFB be the segment required, as any angle F made in it will be equal to the given angle C.

For the two lines AD, BD, being perpendicular to the radii EA, EB (by construction), are tangents to the circle (th. 36); and the angle A or B, which is equal to the given angle C by construction, is equal to the angle F, being all three measured by half the arc AB (th. 38 and 39).*

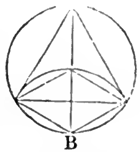
Scholium. One of the lines AD, BD may be omitted, and a perpendicular drawn at the middle of AB to meet the other at the point E.

* This problem is particularly useful in the survey of harbors. Three points on the shore are chosen, which, being connected by lines, form a triangle; then from a boat, where a sounding is to be made, the angles subtended by two of the sides of this triangle are measured with a sextant.

To transfer this to a map, there must first be made upon the paper the triangle whose sides unite the three points upon the shore. Then upon one of the sides of this triangle, by the above problem, make a segment capable of containing one of the observed angles, and upon the other a segment capable of containing the other observed angle; the point in which the arcs of these two segments intersect will be the point on the map corresponding to that where the sounding was made, and there the depth in fathoms or feet may be written down.

PROBLEM XXII.

To inscribe an equilateral triangle in a given circle.

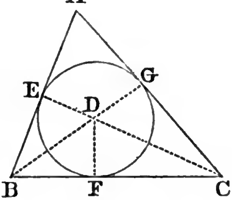


Through the center C draw any diameter AB. From the point B as a center, with the radius BC of the given circle, describe an arc DCE. Join AD, AE, DE, and ADE is the equilateral triangle sought.

Join DB, DC, EB, EC. Then DCB is an equilateral triangle, having each side equal to the radius of the given circle. In like manner, BCE is an equilateral triangle. But the angle ADE is equal to the angle ABE or CBE, standing on the same arc AE; also, the angle AED is equal to the angle CBD, on the same arc AD; hence the triangle DAE has two of its angles, ADE, AED, equal to the angles of an equilateral triangle, and therefore the third angle at A is also equal to the same; so that the triangle is equiangular, and therefore equilateral.

PROBLEM XXIII.

To inscribe a circle in a given triangle ABC.

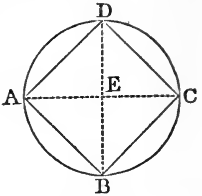


Bisect any two angles C and B with the two lines CD, BD. From the intersection D, which will be the center of the circle, draw the perpendiculars DE, DF, DG, and they will be the radii of the circle required.

For, since the sides CB, CA, are to be tangents, the line CD, bisecting the angle which they form, must pass through the center. (Prob. 20, *schol.*) For a similar reason, BD must pass through the center. Hence it is at the intersection D of these two lines.

PROBLEM XXIV.

To inscribe a square in a given circle.

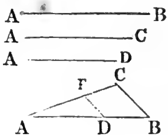


Draw two diameters AC, BD, crossing at right angles in the center E. Then join the four extremities A, B, C, D with right lines, and these will form the inscribed square ABCD.

For the four right-angled triangles AEB, BEC, CED, DEA are identical, because they have the sides EA, EB, EC, ED all equal, being radii of the circle, and the four included angles at E all equal, being right angles, by the construction. Therefore, all their third sides, AB, BC, CD, DA, are equal to one another, and the figure ABCD is equilateral. Also, all its four angles, A, B, C, D, are right ones, being angles in a semicircle. Consequently, the figure is a square.

PROBLEM XXV.

To find a fourth proportional to three given lines, AB, AC, AD.



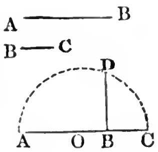
Place two of the given lines AB, AC, or their equals, to make any angle at A; and on AB set off, or place, the other line AD, or its equal. Join BC, and parallel to it draw DE; so shall AE be the fourth proportional, as required.

For, because of the parallels BC, DE, the two sides AB, AC are cut proportionally (th. 61); so that AB : AC : : AD : AE.

PROBLEM XXVI.

To find a mean proportional between two lines AB, BC.

Place AB, BC, joined in one straight line AC; on which, as a diameter, describe the semicircle ADC; to meet which erect the perpendicular BD, and it will be the mean proportional sought between AB and BC (by cor. 1, th. 66).



PROBLEM XXVII.

To make a square equal to a given triangle.

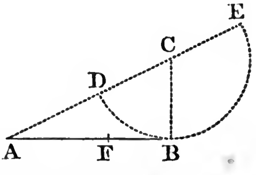
Find a mean proportional between the base and half the altitude, or between the altitude and half the base of the triangle, and it will be the side of the square required.

Corol. To find a square equal to a given polygon, first find a triangle equal to the given polygon by Prob. 15, and then a square equal to the given triangle. This is called quadrating the polygon.*

PROBLEM XXVIII.

To divide a given line in extreme and mean ratio.

Let AB be the given line to be divided in extreme and mean ratio, that is, so that the whole line may be to the greater part as the greater is to the less part.



Draw BC perpendicular to AB, and equal to half AB. Join AC; and with center C and distance CB, describe the circle BD; then with center A and distance AD, describe the arc DF; so shall AB be divided in F in extreme and mean ratio, or so that AB : AF : : AF : FB.

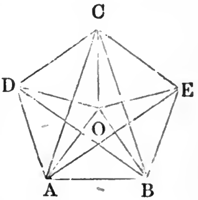
For produce AC to the circumference at E. Then, ADE being a secant, and AB a tangent, because B is a right angle; therefore the rectangle AE·AD is equal to AB^2 or AB·AB (cor. 1, th. 42); taking the first as

* The quadrature of the circle has occupied ingenious minds in a fruitless undertaking from a remote antiquity. The impossibility of this problem may be very satisfactorily proved.

means, and second as extremes of a proportion (th. 54), we have AB : AE or AD + DE : : AD : AB. But AF is equal to AD, by construction, and AB = 2 BC = DE; therefore, AB : AF + AB : : AF : AB or AF + FB; and, by division (th. 47), AB : AF : : AF : FB.

PROBLEM XXIX.

To describe a regular pentagon on a given line AB.



On AB erect the isosceles triangle ACB, having each of the angles at the base double of its vertical angle;* on AB again construct another isosceles triangle whose vertical angle AOB is double of A CB, and about the vertex O place the isosceles triangles AOD, DOC, COE, and EOB, each = AOB; these triangles will compose a regular pentagon.

For the angle AOB, being the double of ACB, which is the fifth part of two right angles, must be equal to the fifth part of four right angles; and, consequently, five angles, each of them equal to AOB, will adapt themselves about the point O. But the bases of those central triangles, and which form the sides of the pentagon, are all equal; and the angles at their bases being likewise equal, they are equal in the collective pairs which constitute the internal angles of the figure. It is, therefore, a regular pentagon.

PROBLEM XXX.

To describe a hexagon upon a given line AB.

From A and B as centers, with AB as radius, describe arcs intersecting in O (fig. to the next problem). From O as a center, with the same radius, describe a circle ABCDEF. Within this circle set off from B

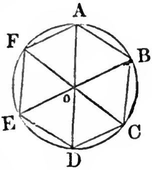
* In the last diagram, AB being the given base, AE will be the side of the isosceles triangle. (See Prob. 33.)

the chords BC, CD, DE, EF, FA in succession, each equal to AB: they will, together with AB, form the hexagon required.

The demonstration is analogous to that of the following problem.

PROBLEM XXXI.

To inscribe a regular hexagon in a circle.



Apply the radius AO of the given circle as a chord, AB, BC, CD, &c., quite round the circumference, and it will complete the regular hexagon AB CDEF.

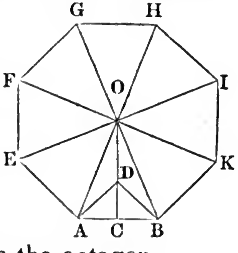
For, draw the radii AO, BO, CO, DO, EO, FO, completing six equal triangles; of which any one, as ABO, being equilateral (by constr.), its three angles are all equal (cor. 2, th. 3), and any one of them, as AOB, is one third of the whole, or of two right angles (th. 15), or one sixth of four right angles. But the whole circumference is the measure of four right angles (cor. 4, th. 6). Therefore the arc AB is one sixth of the circumference of the circle, and, consequently, its chord AB one side of an equilateral hexagon inscribed in the circle.

Cor. 1. The chord of 60° is equal to the radius of the circumscribing circle.

Cor. 2. To inscribe an equilateral triangle, join the alternate vertices of the inscribed hexagon.

PROBLEM XXXII.

On a given line AB *to construct a regular octagon.*

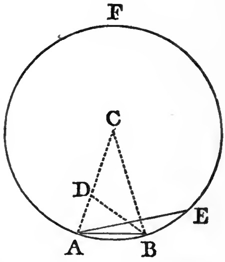


Bisect AB by the perpendicular CD, which make = CA or CB; join DA and DB; produce CD, making DO = DA or DB, draw AO and BO, thus forming an angle equal to the half of AD B (Prob. 18), and about the vertex O repeat the equal triangles AOB, AOE, EOF, FOG, GOH, HOI, IOK, and KOB to compose the octagon.

For CA, CD, and CB being all equal by construction, the angle ADB is contained in a semicircle, and is, therefore, a right angle. Consequently, AOB is equal to the half of a right angle, and eight such angles will adapt themselves about the point O. Whence the figure BAEFGHIK, having eight equal sides and equal angles, each angle, as ABK, being the double of ABO, is a regular octagon.

PROBLEM XXXIII.

To inscribe a regular decagon in a circle.



Divide the radius into extreme and mean ratio, and the greater segment will apply to the circumference ten times. For let D be the point of division on the radius. Then, since we have, by construction,

$$CA : AB :: AB : AD,$$

the two triangles CAB, DAB have the angle A common, and the sides about the common angle proportional; they are, therefore (th. 65), similar. But CAB is isosceles, CA and CB being radii; therefore, ADB is isosceles or $AB = BD$; and since, by construction, $AB = CD$ $\therefore$ $BD = DC$, and the triangle DCB is isosceles, and hence the angle $C = CBD$. But the exterior angle ADB of the triangle DCB is equal to the sum of the two interior and opposite (th. 13); or, since these are equal, the angle ADB, or its equal $DAB = CBA$, is equal to double the angle C. Hence the triangle CAB is such that the angles A and B at the base are each double the angle C at the vertex, or together are four times the angle C; or all three of the angles of the triangle CAB are together equal to five times the angle C. Hence the angle $C =$ one fifth of two right angles $=$ one tenth of four right angles, and the arc AB, therefore, which measures the angle C, is one tenth the circumference.

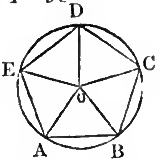
Corol. 1. By joining the alternate vertices of the decagon, a pentagon may be inscribed.

Corol. 2. A pentedecagon may be inscribed by first finding the arc of a decagon, then the arc of a hexagon, and the difference between them will be the arc of the figure required.

For $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$.

PROBLEM XXXIV.

To describe a circle about a regular polygon.



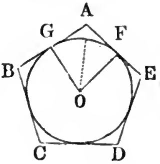
Bisect any two of the angles C and D with the lines CO, DO; then their intersection O will be the center of the circumscribing circle; and OC or OD will be the radius.

For, draw OB, OA, OE, &c., to the angular points of the given polygon. Then the triangle OCD is isosceles, having the angles at C and D equal, being the halves of the equal angles of the polygon BCD, CDE; therefore, their opposite sides CO, DO are equal (th. 4). But the two triangles OCD, OCB, having the two sides OC, CD equal to the two OC, CB, and the included angles OCD, OCB also equal, will be identical (th. 1), and have their third sides BO, OD equal.

AO may be proved equal to BO in the same manner that BO was proved equal to CO, and so on; and thus the point O be shown to be equidistant from all the vertices of the polygon.

PROBLEM XXXV.

To inscribe a circle in a regular polygon.



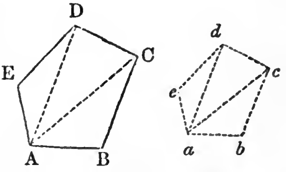
Bisect any two sides of the polygon by the perpendiculars GO, FO, and their intersection O will be the center of the inscribed circle, and OG or OF will be the radius.

For if a circle be circumscribed about the polygon, the perpendiculars

GO, FO, at the middle of the chords, will meet in its center O (schol. to th. 34), and the distances OG, OF, &c., of these chords are equal (th. 35).

PROBLEM XXXVI.

On a given line to construct a rectilinear figure similar to a given rectilinear figure.



Let *abcde* be the given rectilinear figure, and AB the side of the proposed similar figure that is similarly posited with *ab*.

Place AB in the prolongation of *ab*, or parallel to it. Draw AC, AD, AE, &c. parallel to *ac*, *ad*, *ae*, respectively. Draw BC parallel to *bc*, meeting AC in C; CD parallel to *cd*, and meeting AD in D; DE parallel to *de*, and meeting AE in E; and so on till the figure is completed. Then ABCDE will be similar to *abcde*, from the nature of parallel lines and similar figures (th. 68).

Otherwise, divide the given figure up into triangles as in the diagram; then upon *ab* make the triangle *abc* equiangular with the triangle ABC, and upon *ac*, *acd*, equiangular with ACD, and so on till the figure is completed. (See the demonstr. of th. 68.)

GENERAL NOTE UPON THE METHOD OF SOLUTION OF PROBLEMS.

In every problem of Plane Geometry, it is necessary to trace upon a plane, in accordance with given conditions, one or more right lines or curves, one or more angles, one or more points.

The problem can be solved by the aid of the rule and compass, if the entire system of lines to be traced, and the lines of construction, are reduced to a system of right lines and circumferences of circles.

A line is determined when two of its points are known; a circumference when its center and a point of it, or when three of its points; and an angle when the two sides, or the vertex and another point in each of the sides. The tracing, then, of a system of lines, circles, angles, and points, and, consequently, the solution of a problem of Geometry, when this problem is resolvable by the aid of the rule and compass, can be reduced to the determination of a certain number of unknown points.

We may call that a *simple problem* which is reduced to the determination of a single unknown point, and that a *compound problem* which requires the determination of several points. For a compound problem, the nature of the solution may vary not only with the number and nature of the points proposed to be determined, but also with the order of their determination; and it is easy to perceive from hence how the same problem of Geometry admits of different solutions more or less elegant. But as the different unknown points must be determined one after another, it is clear that, to resolve a compound problem, it is only necessary to resolve successively a number of simple problems.

It remains to consider how a simple problem is to be solved.

In every simple problem the unknown point is generally determined by two conditions. By virtue of one of these conditions alone the unknown point is not completely determined; it will only be subjected to the necessity of coinciding with one of the points situated upon a certain right line or curve corresponding to this condition. But if we have regard to the conditions united, the unknown point must be situated, at the same time, upon the two lines corresponding to the two conditions, and can, therefore, only be at one of the points common to these two lines. Then, if the two lines do not meet, the proposed geometrical problem is impossible, or incapable of solution. It will admit of a single solution, if the two lines meet in a single point; it will admit of several distinct solutions, if the two lines intersect in several points. Thus a simple determinate problem may be considered as resulting from the combination of two other simple problems, but indeterminate, each of which consists in finding a point which fulfills a single condition, or, rather, the geometric locus of all the points (infinite in number) which fulfill the given condition. If this condition is reduced to that of the unknown point being found upon a certain line, the geometric locus sought will evidently be this line itself. It may be added, that very often the geometric locus corresponding to a given condition will comprehend the system of a number of right lines or curves. Thus, for instance, if the unknown point is required to be at a given distance from a given right line, the geometric locus sought will be the system of two parallel lines drawn at the given distance from this line.

Let it be observed, moreover, that a simple problem, determinate or indeterminate, will be resolvable by the rule and dividers, if each of the geometric loci which serve to resolve it is reduced to a system of right lines or circumferences.

To illustrate the above, the solutions of some simple and indeterminate problems will now be indicated.

1° *Prob.* To find a point which shall be situated upon a given line.

Solution. The geometric locus which resolves this problem is the line itself.

2° *Prob.* To find a point which shall be situate upon the circumference of a given circle.

Solution. The geometric locus which resolves this problem is the circumference of the given circle itself.

3° *Prob.* To find a point which shall be at a given distance from a given point.

Solution. The geometric locus which resolves this problem is the circumference of a circle described with the given point as a center, and with a radius equal to the given distance.

4° *Prob.* To find a point which shall be situated at a given distance from a given line.

Solution. The geometric locus which resolves this problem is the system of two lines drawn parallel to the given line, and separated from it by the given distance.

5° *Prob.* To find a point which shall be at a given distance from the circumference of a given circle.

Solution. The geometric locus which resolves this problem is the system of two circumferences of circles which are concentric with the given circle, and have radii equal to its radius, increased or diminished by the given distance.

6° *Prob.* To find a point which shall be situated at equal distances from two given points.

Solution. The geometric locus which resolves this problem is the perpendicular erected at the middle of the line which joins the two given points.

7° *Prob.* To find a point which shall be situated at equal distances from two given parallel lines.

Solution. The geometric locus which resolves this problem is a third line parallel to the two others, and which divides their mutual distance into two equal parts.

8° *Prob.* To find a point which shall be at equal distances from two lines which intersect.

Solution. The geometric locus which resolves this problem is the system of two new lines which bisect the angles comprehended between the given lines.

9° *Prob.* To find a point situated at equal distances from the circumferences of two given concentric circles.

Solution. The geometric locus which resolves this problem is a third circumference concentric to the other two, and which divides their mutual distance into equal parts.

10° *Prob.* To find a point from which lines drawn to the extremities of a line given in length and position, form, with each other, a right angle.

Solution. The geometric locus which resolves this problem is the circumference of a circle which has the given line for a diameter.

11° *Prob.* To find a point from which lines drawn to the extremities of a given line form, with each other, an obtuse or acute angle.

Solution. The geometric locus which resolves this problem is the system of two segments of a circle described on the given line as a chord, and capable of containing the given angle.

12° *Prob.* To find a point the distances of which, from two given points, shall have a given ratio.

Solution. The geometric locus which resolves this problem is the circumference of a circle, one diameter of which has, for extremities, the two points which fulfill the prescribed condition upon the line drawn through the two given points.

13° *Prob.* To find a point the distances of which, from two given lines, shall be in a given ratio.

Solution. The geometric locus which resolves this problem is the system of two new lines which divide the angles comprehended between the given lines into parts, the trigonometric sines of which have the given ratio.*

14° *Prob.* To find a point the distances of which, from two given points, are the sides of squares, the difference of which is equal to a given square.

Solution. The geometric locus which resolves this problem is the perpendicular erected upon the line which joins the two given points at the point of this line which fulfills the given condition.

15° *Prob.* To find a point the distances of which, from two given points, are sides of squares, the sum of which is equal to a given square.

Solution. The geometric locus which resolves this problem is the circumference of a circle, one diameter of which has, for extremities, the two points which fulfill the prescribed condition, upon the line joining the two given points.

16° *Prob.* To find a point such that the oblique line drawn from this point to a given line, under a given angle, shall have a given length.

Solution. The geometric locus which resolves this problem is a system of two lines drawn parallel to the given line through the extremities of a secant line, which, having its middle point upon the given line, cuts it at the given angle, and has a length double the given length.

17° *Prob.* To find a point such that the secant, drawn from this point to the circumference of a given circle and parallel to a given line, shall be of given length.

Solution. The geometric locus which resolves this problem is the system of two new circumferences, the radii of which are equal to that of the given circumference, and the centers of which are the extremities of a line which, having its middle point at the center of the given circle, is parallel to the given line, and of a length equal to double the given length.

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* This solution, of course, requires a knowledge of the first principles of Trigonometry.

18° *Prob.* A point and a line being given, to find a second point which shall be the middle of a secant drawn from the given point to the given line.

Solution. The geometric locus which resolves this problem is a new line drawn parallel to the given line, and which divides into equal parts the distance of the given point from this line.

19° *Prob.* A point and the circumference of a circle being given, to find a second point which shall be the middle of a secant drawn from this point to the circumference.

Solution. The geometric locus which resolves this problem is a new circumference of a circle which has for its radius the half of the radius of the given circumference, and for its center the middle of the distance of the given point, from the center of the given circle.

20° *Prob.* To find a point the distance of which, from a given point, has its middle upon a given line.

Solution. The geometric locus which resolves this problem is a new line drawn parallel to the given line, at a distance equal to that which separates this line from the given point.

21° *Prob.* To find a point the distance of which, from a given point, has its middle upon the circumference of a given circle.

Solution. The geometric locus which resolves this problem is a new circumference which has for its radius the double of the radius of the given circumference, and for its center the extremity of a line, the half of which is the distance of the given point from the center of the given circle.

22° *Prob.* Two points being given, symmetrically placed on opposite sides of a given axis, to find a third point such that the line drawn from this third point to the first shall meet the given axis at equal distances from the second and third points.

Solution. The geometric locus which resolves this problem is a line drawn parallel to the given axis, at a distance equal to that which separates it from the given point.

23° *Prob.* A circle being given and a chord, to find a point such that the line drawn from this point to one of the extremities of a chord shall meet the circumference of the circle at equal distances from this point and from the other extremity.

Solution. The geometric locus which resolves this problem is the system of two new circumferences of circles which have for a common chord the given chord, and for centers the extremities of the diameter perpendicular to this chord in the given circle.

24° *Prob.* Two lines perpendicular to each other being given, to find a point which shall be at the middle of a secant of given length comprehended between these two lines.

Solution. The geometric locus which resolves this problem is a circumference of a circle which has for its center the point common to the two lines, and for its radius half the given length.

25° *Prob.* To find in a given circle a point which shall be the middle of a chord of given length.

Solution. The geometric locus which resolves this problem is a circumference which has for its center the center of the given circle, and for its radius the distance from this center to any one of the chords drawn so as to be of the given length.

26° *Prob.* To find out of a given circle a point which must be the extremity of a tangent of given length.

Solution. The geometric locus which resolves this problem is a circumference of a circle which has for its center the center of the given circle, and for its radius the distance from this center to the extremity of any one whatever of the tangents, drawn in such a manner as to be of the given length.

27° *Prob*. To find out of a given circle the point of meeting of two tangents, drawn through the extremities of a chord which contains a given point.

Solution. The geometric locus which resolves this problem is the polar line corresponding to the given point. (See Appendix II.)

The solutions above given are easily deduced from well-known theorems of Geometry. A great number of problems, both simple and indeterminate, could be pointed out, the solutions of which would reduce themselves, in a similar manner, to systems of right lines and circumferences of circles. Let it be observed, moreover, that from the solutions of n problems of this kind, in each of which the unknown point is subjected to a single condition, we can deduce immediately the solutions of $\frac{n(n+1)}{2}$ simple and determinate problems, in each of which the unknown point is subjected to two conditions. For, to obtain a simple and determinate problem, it is sufficient to combine two conditions corresponding to two simple but indeterminate problems, or even two conditions alike and corresponding to a single indeterminate problem. But the number of combinations of n quantities, two and two, is (see Alg., art. 203),

$$\frac{n(n-1)}{2};$$

and, adding to this number that of the quantities themselves, the result is,

$$\frac{n(n-1)}{2}+n=\frac{n(n+1)}{2}.$$

This result increases very rapidly with n. Thus, if $n=27$, $\frac{n(n+1)}{2}=378$; that is, the solution of the 27 indeterminate problems enunciated above furnishes already the means of resolving 378 simple and determinate problems.

In order that the principles just brought to view may be the better apprehended, they will now be applied to the solution of some determinate problems.

Suppose, first, that it is required to draw a tangent to a circle through a point without. The question may be reduced to seeking the unknown point of contact of the tangent with the circle. The two conditions which this point must satisfy are, 1°. That it shall be upon the circumference of the given circle. 2°. That lines drawn from this point to the given point and the center of the circle should make a right angle with each other. Then the question to be resolved will be a determinate problem, resulting from the combinations of the indeterminate problems, 2 and 10.

The combined solutions of 2 and 10 furnish, in fact, the solutions heretofore given (Prob. 20, p. 83).

Suppose, secondly, that it is required to circumscribe a circle about a given triangle. The question can be reduced to seeking for the center of the circle. But the two conditions which this center must satisfy will be those of being not only at equal distances from the first and second vertex of the given triangle, but also at equal distances from the first and third. Then the question to resolve will be a determinate problem resulting from the combination of two indeterminate problems identical with each other and with problem 6. In fact, the solution of problem 6, twice repeated, will furnish two geometric loci, reduced to two right lines, which cut each other in a single point, and thus will be obtained the known solution of the problem proposed.

Suppose, next, that the question is how to draw a circle tangent to the three sides of a given triangle.

The question can be reduced to finding the center of the given circle. But the two conditions which this center must satisfy will be not only to be at equal distances from the first and second side of the given triangle, but also at equal distances from the first and third sides. Then the question to be resolved will be a determinate problem, resulting from the combination of two indeterminate problems identical with one another, and with problem 8. In fact, the solution of problem 8 twice will furnish two geometric loci, which, reduced each to the system of two right lines, will cut each other in four points, and thus four solutions will be obtained of the proposed problem.

Suppose, finally, that the question is to inscribe between a chord of a circle and its circumference a line equal and parallel to a given line. The question can be reduced to seeking either one of the two points which will form the extremities of this line, and, consequently, to a determinate problem resulting from the combination of two indeterminate problems, to wit, problems 1 and 17, or problems 2 and 16. In fact, by the aid of this combination, the question proposed is resolved without difficulty. And one of the extremities of the line sought will be found determined either by the meeting of the circumference of the given circle with a new line, or by the meeting of the given chord with a new circumference.

It is seen here how the solution obtained may be modified when the order comes to be inverted in which the unknown points are determined.

The construction of the geometric locus which corresponds to a simple and indeterminate problem may itself require the resolution of one or more determinate problems. It should be observed, upon this subject, that in the case where the problem is resolvable by the rule and compass, the geometric locus should reduce to a system of right lines and circles. Then, since each line or each circumference finds itself completely determined when there are known two or three points of it, the construction of the geometric locus, corresponding to a simple and indeterminate problem, can always be deduced from the construction of a certain number of points suitable to verify the condition which ought to be fulfilled, in virtue of the enunciation of the problem, by the unknown point.

Thus, for example, to resolve problem 6; that is to say, to find a point which shall be situated at equal distances from two given points, and, consequently, to construct the geometric locus which shall contain every point suitable to fulfill this condition. We begin by seeking such a point, for example, one the distance of which from the given points is sufficiently great. But the solution of this last problem deduces itself immediately from problem 3, which, twice repeated, will furnish at once two points that fulfill the proposed condition; consequently, two points which suffice to determine the geometric locus required.

Thus, again, to resolve problem 15; that is to say, to find a point the distances of which, from two points given, shall furnish squares the sum of which shall be equal to a given square, and, consequently, to construct the geometric locus of every point suitable to fulfill this condition. We can commence by seeking such a point; for example, that which shall be situated at equal distances from the two given points, and, consequently, separated from each of them by a distance equal to half the diagonal of the given square. But the solution of this last problem deduces itself immediately from the solution of problem 3, and, twice repeated, will furnish at once, also, two points which will fulfill the proposed condition. Moreover, these two points are precisely the extremities of a diameter of the circle, the circumference of which represents the geometric locus required.

The above note is from a recent article by the celebrated Cauchy. Although designed by him as an introduction to a new method of resolving determinate geometric problems by means of the Indeterminate Analysis (for an exhibition of which, see Comptes Rendus de L'Acadamie des Sciences, No. 17, 21 Avril, 1843, p. 867, and No. 19, 15 Mai, 1843, p. 1039), yet it is calculated to afford important aid to the solution of problems by the processes of ordinary geometry.

M. Cauchy acknowledges that it is but the development of some principles, the memory of which he has preserved, which were contained in the course of lectures given by Dinet, at the Lycée Napoleon, some forty years ago.

MISCELLANEOUS EXERCISES IN PLANE GEOMETRY.

1. Prove that all regular polygons of the same number of sides are similar figures.

2. That if a line join the middle points of two sides of a triangle, it will be parallel to the third side and equal to its half.

3. To describe a circle about a given square.

4. To divide a right angle into three equal parts.

5. To circumscribe about a given circle a triangle one side of which is given.

6. Find the length of the circumference of a circle in seconds of a degree.

7. Find the length of the radius in seconds of a degree.

8. Find the length of $1''$ to radius 1.

9. Prove that a straight line can meet a circumference in but two points.

10. From a given point without a circle to draw a secant such that the part within the circle shall be equal to a given line.

11. To draw to a circle a tangent of given length, and terminating at a given line, which cuts the circumference.

12. An inscribed polygon being given, to circumscribe another similar.

13. Prove that of two convex lines, broken or curved, terminating at the same points, the enveloped is less than the enveloping line.

14. Prove that the difference between the sum of the two perpendicular sides of a right-angled triangle and the hypothenuse is equal to the diameter of the inscribed circle.

15. To trisect a given finite straight line.

16. Prove that if, from the extremities of the diameter of a semicircle, perpendiculars be let fall on any line cutting the semicircle, the parts intercepted between those perpendiculars and the circumference are equal.

17. If, on each side of any point in a circle, any number of equal arcs be taken, and the extremities of each pair joined, the sum of the chords so drawn will be equal to the last chord produced to meet a line drawn from the given point through the extremity of the first arc.

18. That if two circles touch each other, and also touch a straight line, the part of the line between the points of contact is a mean proportional between the diameters of the circles.

19. From two given points in the circumference of a given circle to draw two lines to a point in the same circumference, which shall

cut a line given in position, so that the part of it intercepted by them may be equal to a given line.

20. Prove that if, from any point within an equilateral triangle, perpendiculars be drawn to the sides, they are together equal to a perpendicular drawn from any of the angles to the opposite side.

21. That if the three sides of a triangle be bisected, the perpendiculars drawn to the sides, at the three points of bisection, will meet in the same point.

22. If from the three vertices of a triangle lines be drawn to the points of bisection of the opposite sides, these lines intersect each other in the same point.

23. The three straight lines which bisect the three angles of a triangle meet in the same point.

24. If from the angles of a triangle perpendiculars be drawn to the opposite sides, they will intersect in the same point.

25. If any two chords be drawn in a circle, to intersect at right angles, the sum of the squares of the four segments is equal to the square of the diameter of the circle.

26. In a given triangle to inscribe a rectangle whose sides shall have a given ratio.

27. Prove that the two sides of a triangle are together greater than the double of the straight line which joins the vertex and the bisection of the base.

28. That if, in the sides of a square, at equal distances from the four angles, four other points be taken, one in each side, the figure contained by the straight lines which join them shall also be a square.

29. That if the sides of an equilateral and equiangular pentagon be produced to meet, the angles formed by these lines are together equal to two right angles.

30. That if the sides of an equilateral and equiangular hexagon be produced to meet, the angles formed by these lines are together equal to four right angles.

31. If squares be described on the three sides of a right-angled triangle, and the extremities of the adjacent sides of any two squares be joined, the triangles so formed are equal, though not identical, to the given triangle, and to one another.

32. If the squares be described on the hypothenuse and sides of a right-angled triangle, and the extremities of the sides of the former square, and those of the adjacent sides of the others, be joined, the sum of the squares of the lines joining them will be equal to five times the square of the hypothenuse.

33. To bisect a triangle by a line drawn parallel to one of its sides.

34. To divide a circle into any number of concentric equal annuli.

35. To inscribe a square in a given semicircle.

36. Prove that if, on one side of an equilateral triangle, as a diameter, a semicircle be described, and from the opposite angle two straight lines be drawn to trisect that side, these lines produced will trisect the semi-circumference.

37. Draw straight lines across the angles of a given square, so as to form an equilateral and equiangular octagon.

38. Prove that the square of the side of an equilateral triangle, inscribed in a circle, is equal to three times the square of the radius.

39. To draw straight lines from the extremities of a chord to a point in the circumference of the circle, so that their sum shall be equal to a given line. N.B. The given line must evidently be limited.

40. In a given triangle to inscribe a rectangle of a given area.

41. Given the perimeter of a right-angled triangle, and the perpendicular from the right angle upon the hypothenuse, to construct the triangle.

42. In an isosceles triangle to inscribe three circles touching each other, and each touching two of the three sides of the triangle.

43. To construct a trapezoid when four sides are given.

44. The same when three sides and the sum of the angles at the base are given.

45. When the two sides not paralel, the altitude and an angle are given.

46. When the difference of the parallel sides, the diagonal, a third side, and an angle.

47. Prove that the line drawn to the middle of the hypothenuse from the vertex of the right angle in a right-angled triangle is equal to half the hypothenuse.

48. Prove that if the four angles of a parallelogram be bisected, and the points in which two of the bisecting lines adjacent one side meet be joined with that in which the two bisecting lines adjacent the opposite side meet; 1°, that the joining line will be parallel to the other two sides; 2°, that it will be equal to the difference of two adjacent sides.

49. That in any quadrilateral the lines joining the middle points of the opposite sides, and the line joining the middle points of the diagonals, meet in the same point, and all three bisect one another.

50. That the rectangle of the two sides of any triangle is equal to the rectangle of the perpendicular upon the third side from the vertex opposite, and the diameter of the circle circumscribing the triangle.

51. Prove that the rectangle of the diagonals of an inscribed quadrilateral is equal to the sum of the rectangles of the opposite sides.

52. Prove that the square of the line bisecting the vertical angle of a triangle, together with the rectangle of the two segments of the base, is equal to the rectangle of the other two sides of the triangle.

53. To find two lines that shall have the same ratio as two given rectangles.

54. Draw a transverse line to two circles such that the parts comprehended within the circumferences shall be equal to a given line.

55. To inscribe in a circle (radius not given) a triangle of given base, vertical angle, and altitude.

56. In a given circle to place six others, so that each shall touch two others, and the given.

57. To construct a figure similar to two given similar figures, and equal to their sum or difference.

APPENDIX I.

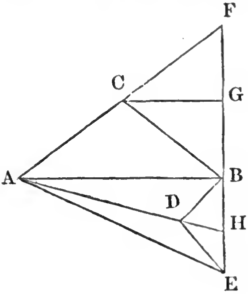
ISOPERIMETRY.

Def. 1. A *maximum* is the greatest quantity among those of the same kind; a *minimum* the least.*

Def. 2. Isoperimetrical figures are those which have equal perimeters.

THEOREM I.

Among all triangles of the same perimeter, that is, a MAXIMUM *in which the undetermined sides are equal.*



Let ABC, ABD be the two triangles. With C as center, and radius CA, describe a circle cutting AC produced in F; ABF, inscribed in a semicircle, will be a right angle; produce FB, making $DE = DB$, and draw the perpendiculars DH, CG. It will be easy to show of the oblique lines that $AE < AF \therefore BE < BF \therefore BH < BG$; these last two being the altitudes of the given triangles which have a common base.

THEOREM II.

Of all isoperimetric polygons the maximum has its sides equal.

By drawing a diagonal in the polygon so as to cut off two sides forming a triangle; if these two sides be not equal, they may be replaced by two others which are equal, and which, by the last theorem, will inclose a greater triangle. This process may be repeated with all the sides of the polygon.

THEOREM III.

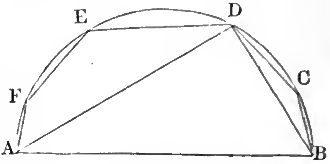
Of all triangles formed with two given sides, that is, a maximum in which the two given sides make a right angle.

For with the same base it will have the greater altitude.

* These definitions are suited to our present purpose.

THEOREM IV.

Of all polygons formed with sides all given except one side, that is, a maximum, of which the given sides are inscribed in a semicircle, of which the side not given is the diameter.

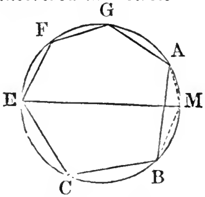


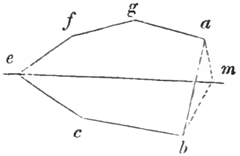
Let ABCDEF be the maximum polygon formed with sides all given except AB. Join AD, BD; then ADB must be a right angle: otherwise, preserving the parts BCD and ADEF the same, the triangle ADB might be increased (th. 3); the point D must, therefore, be in the semicircumference described on AB as a diameter. In the same manner it may be proved that the points E, F, C, &c., must be in the semicircumference. Q. E. D.

Schol. There is but one way of forming the polygon; for if, after having found a circle which satisfies the requisition, a larger circle be supposed, the chords which are the sides of the polygon correspond to smaller angles at the center, and the sum of these will be less than two right angles.

THEOREM V.

Of all polygons formed with given sides, that is, a maximum which can be inscribed in a circle.



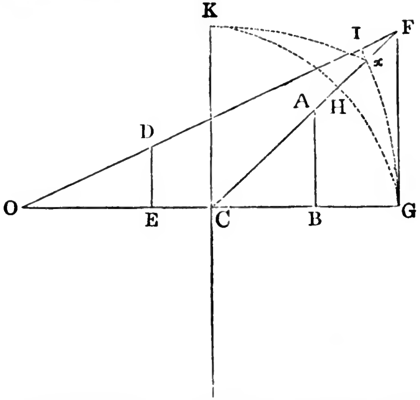


Let ABCEFG be an inscribed polygon, *abcefg* one which is not capable of being inscribed, and of equal sides with the former; draw the diameter EM; join AM, MB; upon *ab* make the triangle *abm* = ABM, and join *em*. By th. 4, EFGAM > *efgam* and ECBM > *ecbm* ∴ by addition, EFGAMBC minus AMB > *efgambc* minus *amb*. Q. E. D.

Corollary from the two last propositions. The *regular* polygon is the maximum among all isoperimetric polygons of the same number of sides.

THEOREM VI.

Of all regular isoperimetric polygons, that is the greatest which has the greatest number of sides.



Let DE be the half side of one of the polygons, O its center, OE its apothegm. AB, C, CB the same for the other; DOE, ACB will be the half angles at the centers of the polygons; and as these angles are not equal, the lines CA, OD, prolonged, will meet at the point F; from this point draw FG perpendicular to OC; with O and C as centers, describe arcs GI, GH.

Now, DE is to the perimeter of the first polygon as O is to four right angles, and AB : perim. 2d polyg. : : C : 4 r. angs. ∴ DE : AB : : O : C, and ∴ (th. 71, corol. 3), DE : AB : : $\frac{GI}{OG} : \frac{GH}{CG}$. Multiplying the antecedents by OG, and the consequents by CG,

$$DE \times OG : AB \times CG :: GI : GH.$$

But the similar triangles ODE, OFG give

$$OE : OG :: DE : FG \therefore OE \times FG = DE \times OG \text{ (th. 54).}$$

In the same manner it may be shown that

$$AB \times CG = CB \times FG$$

Therefore, by substitution,

$$OE \times FG : CB \times FG : GI : GH.$$

If, then, it can be shown that the arc GI > arc GH, it will follow that the apothegm OE is greater than the apothegm CB.

Make the figures $CKx = CGx$, $CKH = CGH$.

Then $KxG > KHG$ (see exercise 13 of the miscellaneous exercises).

$$\therefore Gx = \tfrac{1}{2}\, KxG > GH = \tfrac{1}{2} KHG.$$

Much more $\qquad GI > GH.$ Q. E. D.*

Corollary from the preceding Propositions.—The *circle is the greatest of all figures of the same perimeter;* for it may be regarded as a regular polygon of an infinite number of sides.

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* It has been seen in the note to corol. 4, th. 16, that equilateral triangles, squares, and regular hexagons are the only figures which will, in juxtaposition, leave no intervening space. It appears, also, from the present proposition, that the space inclosed in the regular hexagon is greater than that inclosed in the square or triangle of the same perimeter. Some writers on natural theology call attention to the fact that the cells of the bee-hive being made in the form of regular hexagons, thus affording the greatest space with a given amount of the material employed in their construction, indicate an instinct working in accordance with the most recondite principles of geometry.

APPENDIX II.

CENTERS OF SYMMETRY.

Def. 1. When the vertices of two polygons, or of the same polygon, are two and two upon lines meeting in a point interior, and at equal distances from this point, the point is called the *center of symmetry.*

Theorem 1. Prove that all lines drawn to opposite parts of the figure through the center of symmetry are equally divided at this point. 2. That the opposite sides of the figure or figures are equal, parallel, and arranged in a reverse order. 3. The converse.

Def. 2. Two points are said be situated symmetrically with respect to a line when this line is perpendicular to that which joins the two points, and divides it into two equal parts.

OF AXES OF SYMMETRY.

Two polygons, or portions of one polygon, are said to be symmetrical with respect to a line when their corresponding vertices are symmetrical. The line in such a case is called an *axis of symmetry.*

Def. 3. An isosceles trapezoid is one whose inclined sides are equal.

Theorem. Prove that the line joining the middle points of the parallel bases of such a figure is an axis of symmetry.

Theorem 2. Prove that the isosceles trapezoid may be inscribed in a circle.

General Theorem. Prove that every figure which has two axes of symmetry perpendicular to each other has a center of symmetry at their intersection.

Schol. Show that these axes divide the figure into four equal parts.

OF DIAMETERS.

Theorem 3. When the vertices of two polygons or of a same polygon are two and two upon lines parallel, and equally divided by a MEDIAN line, this median line bisects, also, every other line parallel to the former, and terminating at the sides of the figure or figures.

CENTER OF MEAN DISTANCES.

Def. 4. If the middle points of the consecutive sides of a polygon be joined, a new polygon will be formed of less perimeter and area, evidently, than the perimeter and area of the first. Proceeding in the same manner with the second, a third is obtained, still smaller, and so on. These operations being continued indefinitely, the result will be at length a polygon, infinitely small, which may be regarded as a point. This point is called the *center of mean distances.* It has a remarkable property. *The distance of this point from any given line is equal to the quotient of the sum of the distances of all the vertices of the polygon from this given line, divided by the number of vertices.* The student may prove this by proving the sum of the distances of the vertices of the second polygon equal to that of the first, and so on, till the polygons are reduced to a point, the center of mean distances.

Cor. From this will follow a construction for determining the center of mean distances, viz., determine its distance from two given lines by dividing the sum of the distances by the number of vertices of the polygon, and, drawing parallels to these two lines at the distances thus determined, these parallels will, by their intersection, determine the point required.

Def. 5. Polygons are said to be *inversely* similar when one is similar to a polygon symmetric with the other.

CENTERS OF SIMILITUDE.

Def. 6. *The center of similitude* is a point placed in such a manner with reference to two polygons, *directly* similar, as that, if a line be drawn through it to two homologous vertices of the polygons, the direction of these vertices from the point shall be the same, and the lines proportional to the homologous sides.

The distances from this point to the homologous vertices are called *radii of similitude.*

If, from a point taken at pleasure, lines be drawn to the vertices of a polygon, and upon these lines or their prolongations parts be taken proportional to them, the points thus obtained will determine a new polygon similar to the given polygon, and the arbitrary point will be the center of similitude of the two polygons.

The center of similitude may be either *external* or *internal* to the two polygons. (See diagram of note to th. 69 for an external center.)

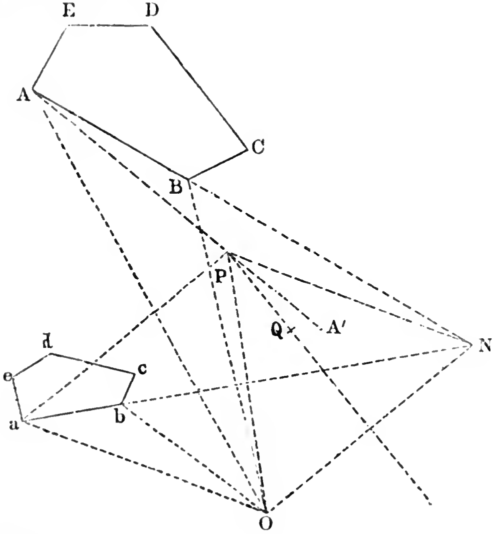
Two similar polygons which have their sides respectively parallel, and directed the same or contrary ways, have in the first case an *external,* and in the second case an *internal* center of similitude. In the latter case it is between homologous vertices.

Theorem. Prove that when three similar polygons have their sides respectively parallel, their three centers of similitudes are upon the same line.

THEOREM.

Two polygons (directly) *similar, situated in any manner upon a plane, have always a common homologous point.**

By this is to be understood that there exists in the plane of the two polygons a point such that, if it be joined with the vertices of the two polygons, the homologous lines of junction will have the ratio of similitude of the two polygons,† and that the angles formed by these lines are equal each to each.



Let ABCDE, *abcde* be the two polygons, and N the point in which the two sides AB, *ab* meet. Find P, the center of a circle passing through A, *a* and N, or equally distant from these three points, and Q a point equally distant from B, *b* and N. Join PQ. Then find the point O symmetric to N, with reference to the line PA. O will be the point required.

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* That is, a point from which homologous lines, drawn to the vertices of the two polygons, will have the ratio of similitude of the polygons, and form with each other equal angles.

This theorem is due to M. Chasles, and is demonstrated in the Bulletin des Sciences Mathématique of Férussac for 1830.

† See note to def. 67.

For APN, APO are isosceles triangles, and give the angle PAN $=$ PNA, and angle PAO $=$ POA $\therefore$ NPA$'$ $=$ 2PAN, OPA$'$ $=$ 2PAO; or, by addition, NPO $=$ 2NAO, *i. e.*, NPO $=$ 2BAO.

Similarly, the two triangles aPN, aPO give

$$NPO = 2Na O, \text{ or } NPO = 2baO \therefore BAO = baO.$$

Reasoning upon the four points Q, B, b, N in the same manner as upon the four points P, A, a, N, it may be proved that the angles ABO and abO are equal. Thus the triangles OAB, Oab are similar, and give

$$OA : Oa :: OB : Ob :: AB : ab.$$

The same would result from any number of triangles.

Scholium. The point O is the only *common homologous point* of the two polygons. Every line passing through it is called a *common homologous line,* and conversely.

The *center of similitude* of two polygons is their *common homologous point.*

There is another mode of construction which seems more natural than that just given.

Determine the circumference of a circle, every point of which shall be at distances from two homologous vertices A, a in the ratio of similitude of the two polygons (see Prob. 12 of General Note, p. 95); *repeat* the same construction for two other vertices B, b; one of the points in which the two circumferences intersect will be the point sought.

CENTERS OF SIMILITUDE IN CIRCLES.

THEOREM.

Two circles, as well as two regular polygons of an even number of sides, have two centers of similitude, the one internal and the other external.

When the two circles are exterior to each other, prove that the points in which their common tangents meet are centers of similitude. This point may be found by dividing the line, joining the centers in the ratio of the radii.

When the circles touch each other externally, prove that the point of contact is an internal center of similitude; and that if they touch each other internally, the point of contact is an external center of similitude.

There exist several other remarkable particular cases.

For two concentric circles, the centers of similitude unite in the common center.

For two equal circles, the internal center of similitude is at the middle of the line joining their centers; the external is at an infinite distance.

When one of the circles degenerates into a right line, 1°. The centers of similitude are, at the extremities of a diameter of the other circle, perpendicular to the line. 2°. If one of the circles reduces to a point, that point is itself the center of similitude, both internal and external.

Scholium. To be proved. When three circles are situated upon the same plane which gives six centers of similitude; 1°. *The three external* centers of similitude; 2°. *One external and two internal—are upon a same line,* which gives *four* lines, passing through six points combined, *three and three.*

RADICAL AXIS AND RADICAL CENTER.

Definitions. A radical axis of two circles is the locus of points from each of which equal tangents can be drawn to the two circles.

Construction. Divide the line joining the centers of the two circles in such a manner that the difference of the squares of the two parts is equal to the difference of the squares of the radii, and the perpendicular to this line at the point of division will be a radical axis.

Prove that to find the point on the line joining the centers it is only necessary to lay off from the middle of this line, on the side toward the smaller circle, a distance equal to half a third proportional to the distance between the centers and the square root of the difference of the squares of the radii. (See Prob. 14, p. 80.)

Each particular case, however, presents a more simple construction. 1°. If the circles be exterior, or in any position for which there exists a common tangent, as the middle point of the portion of this tangent comprehended between the two points of contact belongs to the radical axis, we draw through this point a perpendicular to the line joining the centers of the circles, and thus have this axis. 2°. When the circles touch either exteriorly or interiorly, the common point of the two circumferences belongs necessarily to the radical axis, and thus leads to its determination, as before. It is then the common tangent to the circles at this point. 3°. When the circles cut each other, the common chord produced both ways is the radical axis.

Concentric circles have no radical axis. When the two circles are equal, the radical axis is the perpendicular at the middle of the line joining the centers.

If one of the circles be reduced to a point, the radical axis is obtained by joining the *middles of the tangents drawn from this point to the other circle.*

If one of the circles degenerates into a right line, the radical axis is the line itself.

Radical Center.—*Three circles* situated in the same plane (the centers of which are not in the same line) give, by their combination two and two, *three* radical axes; and these *three axes* cut each other *in the same point.*

For the two first cutting each other, and being respectively perpendicular to two lines which cut, their point of intersection is such that there can be drawn from this point to the three circumferences equal tangents; consequently, it belongs to the third radical axis.

This point is called the *radical center* of the three circles.

From this definition, and from what has been shown above, it follows, that if three circumferences intersect, the three chords which unite their points of intersection meet in the same point.

When this *point* of intersection is exterior to the three circles, the *six tangents* from this point are equal.

Theorem. Prove *that if, from any point of a radical axis of two circles, a secant be drawn meeting the circumferences in* FOUR *points, these four points will be in the circumference of a third circle.*

This may be proved by aid of the theorem that the rectangle of a secant and its external segment is constant (th. 42), together with the construction of a radical axis.

Theorem. If *through one of two centers of similitude* (external or internal) *of two circles two secants to these circles be drawn,* 1°. *The eight points of intersection combined, four and four, in a suitable manner, form* FOUR *groups, situated respectively upon as many new circumferences;* 2°. *These four circumferences have for a* COMMON RADICAL CENTER *that center of similitude which served to determine these circumferences.*

Note. The above theory will be found of great use in the solution of all problems involving the contact of circles.

CONJUGATE POINTS, POLES, AND POLAR LINES.

Conjugate points are two points situated the one within, the other without, a circle, in such a manner that the distances of every point in the circumference from these two points are in a constant ratio. The circle is called the regulating circle.

The *point within* the circle being given, to determine its *conjugate,* erect at the given point a perpendicular to the line joining the given point and the center, and at the point where this perpendicular meets the circumference draw a tangent which will meet the line joining the given point and center produced in the point required. Prove this.

A *chord of contact* is a line joining the points of contact of two tangents drawn from the same point.

THEOREM. The chords of contact of all tangents which meet in one and the same line will meet in the same point, and the conjugate of this point is the foot of a perpendicular let fall from the center of the circle upon the line in which the tangents meet.

The point in which all the chords of contact meet is called a pole, and the line in which the tangents all meet, a polar line.

GEOMETRY OF PLANES.*

DEFINITIONS.

1. The angle formed by two lines not in the same plane is the angle formed by one of them with a line drawn through any point of it parallel to the other.

2. A plane is a surface in which, if any two points be taken, the straight line which joins these points will be wholly in that surface.

3. A straight line is said to be perpendicular to a plane when it is perpendicular to all the straight lines in the plane which pass through the point in which it meets the plane.

This point is called the foot of the perpendicular.

4. The inclination of a straight line to a plane is the acute angle contained by the straight line, and another straight line drawn from the point in which the first meets the plane, to the point in which a perpendicular to the plane, drawn from any point in the first line, meets the plane.

5. A straight line is said to be parallel to a plane when it can not meet the plane, to whatever distance both be produced.

6. It will be proved in Prop. 2, that the common intersection of two planes is a straight line; this being premised,

The angle contained by two planes, which cut one another, is measured by the angle contained by two straight lines drawn, one in each of the planes, perpendicular to their common intersection at the same point.

This angle may be acute, right, or obtuse.

* Students intending to pursue that subject, may here with advantage take up Plane Trigonometry before going on with the Geometry of Planes and Solids.

If it be a right angle, the planes are said to be perpendicular to each other.

The angle formed by two planes is called *diedral.*

7. Two planes are parallel to each other when they can not meet, to whatever distance both be produced.

8. A plane is ordinarily represented, in a diagram, by a parallelogram, and called by the two letters at the opposite (diagonally) angle. This plane, which must be conceived to be indefinitely extended, divides space into two indefinite portions called *regions.*

Two or more planes are represented in their relative position not accurately, but by a sort of perspective.

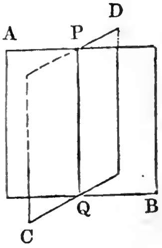
PROP. I.

A straight line can not be partly in a plane and partly out of it.

For, by def. (1), when a straight line has two points common with a plane, it lies wholly in that plane.

PROP. II.

If two planes cut each other, their common intersection is a straight line.



Let the two planes AB, CD cut each other, and let P, Q be two points in their common section.

Join P, Q;

Then, since the points P, Q are in the same plane AB, the straight line PQ which joins them must lie wholly in that plane (def. 2).

For a similar reason, PQ must lie wholly in the plane CD.

∴ The straight line PQ is common to the two planes, and is ∴ their common intersection.

Note. In this and the following diagrams concealed lines are drawn dotted.

PROP. III.

Any number of planes may be drawn through the same straight line.

For let a plane, drawn through a straight line, be conceived to revolve round the straight line as an axis. Then the different positions assumed by the revolving plane will be those of different planes drawn through the straight line

PROP. IV.

One plane, and one plane only, can be drawn,

1°. *Through a straight line, and a point not situated in the given line.*

2°. *Through three points which are not in the same straight line.*

3°. *Through two straight lines which intersect each other.*

4°. *Through two parallel straight lines.*

1. For if a plane be drawn through the given line, and be conceived to revolve round it as an axis, it must in the course of a complete revolution pass through the given point, and so assume the position enounced in 1°.

Also, one plane only can answer these conditions, for if we suppose a second plane passing through the same straight line and point, it must have at least two intersections with the first, which is evidently impossible.

2. Join two of the points; this case is then reduced to the last.

3. Take a point in each of the lines which is not the point of intersection; join these two points; the case is now the same as the two former.

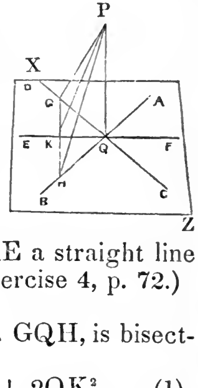
4. Parallel straight lines are in the same plane, and, by the first case, one plane only can be drawn through either of them, and a point assumed in the other.

Cor. Hence the position of a plane is determined by,

1. *A straight line, and a point not in the given straight line.*
2. *A triangle, or three points not in the same straight line.*
3. *Two straight lines which intersect each other.*
4. *Two parallel straight lines.*

PROP. V.

If a straight line be perpendicular to two other straight lines which intersect at its foot in a plane, it will be perpendicular to every other straight line drawn through its foot in the same plane, and will therefore be perpendicular to the plane.



Let XZ be a plane, and let the straight line PQ be perpendicular to the two straight lines AB, CD which intersect in Q in the plane XZ.

We shall prove that PQ will be perpendicular to any other straight line EF, drawn through Q in the plane XZ.

Draw through any point K in QE a straight line GH, such that $GK = KH$. (See exercise 4, p. 72.)

Join P, G; P, K; P, H;

Then, since GH, the base of the $\triangle$ GQH, is bisected in K;

$\therefore$ (by th. 30), $GQ^2 + HQ^2 = 2GK^2 + 2QK^2 \ldots (1)$

Similarly, since GH, the base of $\triangle$ GPH, is bisected in K;

$$\therefore GP^2 + HP^2 = 2GK^2 + 2PK^2.$$

But the triangles PQG, PQH are right-angled at Q; $\therefore$ the last expression becomes

$PQ^2 + GQ^2 + PQ^2 + HQ^2 = 2GK^2 + 2PK^2 \ldots (2)$

Taking (1) from (2), there remains

$$2PQ^2 = 2PK^2 - 2QK^2,$$

$\therefore$ dividing by 2, and transposing,

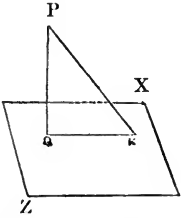
$$PQ^2 + QK^2 = PK^2.$$

Hence the triangle PQK is right-angled at Q, for in the right-angled triangle alone the sum of the squares of two of the sides is equal to the square of the third (see theorems 26, 28, 29.)

In like manner, it may be proved that PQ is at right angles to every other straight line passing through Q in the plane XZ.

PROP. VI.

A perpendicular is the shortest line which can be drawn to a plane from a point without.



Let PQ be perpendicular to the plane XZ;

From P draw any other straight line PK to the plane XZ;

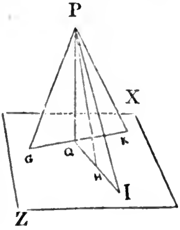
Then PQ < PK.

In the plane XZ draw the straight line QK, joining the points Q, K.

Then, since the line PQ is perpendicular to the plane XZ, it is perpendicular to QK, a line of the plane; and ∴ PQ is less than PK. (Geom. Theor., 17.)

PROP. VII.

Oblique lines equally distant from the perpendicular are equal, and, if two oblique lines be unequally distant from the perpendicular, the more distant is the larger.



That is, if QG, QH, QK . . . . . . are all equal, then PG, PH, PK . . . . . . are all equal; and if QI be greater than QG, then PI is greater than PG. For the three right-angled triangles PQG, PQH, PQK having two sides in each equal, the third sides are equal (th. 26, corol. 2); and since PH < PI (th. 17), ∴ PG < PI.*

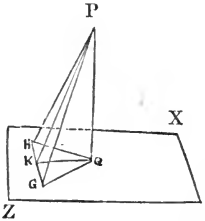
* This Proposition affords a method of finding the foot of a perpendicular to a given plane from a point without. With a straight

Cor. A perpendicular measures the distance of any point from a plane. The distance of one point from another is measured by the straight line joining them, because this is the shortest line which can be drawn from one point to another. So, also, the distance from a point to a line is measured by a perpendicular, because this line is the shortest that can be drawn from the point to the line. In like manner, the distance from a point to a plane must be measured by a perpendicular drawn from that point to the plane, because this is the shortest line that can be drawn from the point to the plane.

PROP. VIII.

If, from a given point without a plane, a perpendicular be let fall to the plane, and from its foot a perpendicular be drawn to a line of the plane, and the point of intersection be joined with the point without, the last line will be perpendicular to the line of the plane.

Let PQ be a perpendicular on the plane XZ, and GH a straight line in that plane; if from Q, the foot of the perpendicular, QK be drawn perpendicular to GH, and P, K be joined; then PK will be perpendicular to GH.



Take KG = KH; join P, G; P, H; Q, G; Q, H;

Because KG = KH, and KQ is common to the triangles GQK, HQK, and the angle GKQ = angle HKQ, each being a right angle,

∴ QG = QH,

∴ PG = PH (last Prop.),

Hence the two triangles GKP, HKP have the two sides GK, KP equal to the two sides HK, KP, and

rod, one end of which is fixed at the given point, touch the given plane in three points not in the same right line; find the center of the circle passing through these three points, and it will be the foot of the perpendicular required.

the remaining side GP, equal to the remaining side HP.

∴ Angle GKP = angle HKP, and ∴ each of them is a right angle (def. 12).

Cor. GH is perpendicular to the plane PQK, for GH is perpendicular to each of the two straight lines KP, KQ (Prop. 5).

REMARK.—The two straight lines PQ, GH present an example of two straight lines which do not meet, because they are not situated in the same plane.

The shortest distance between these two lines is the straight line QK, which is perpendicular to each of them.

For, join any two other points, as P, G;

$$\left.\begin{array}{ll}\text{Then} & PG > PK \\ \text{And} & KP > KQ\end{array}\right\} \text{last Prop.}$$
$$\therefore \quad PG > KQ.$$

The two lines PQ, GH, although not situated in the same plane, are considered to form a right angle with each other. For PQ, and a straight line drawn through any point in PQ parallel to GH, would form a right angle.

In like manner, PG and QK, which represent any two straight lines not situated in the same plane, are considered to form with each other the same angle which PG would make with any parallel to QK, drawn through a point in PG.

PROP. IX.

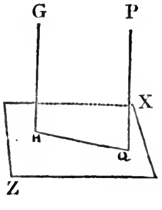
If two straight lines be perpendicular to the same plane, they will be parallel to each other.

Let each of the straight lines PQ, GH be perpendicular to the plane XZ.

Then PQ will be parallel to GH.

In the plane XZ draw the straight line QH, joining the points Q, H.

Then, since PQ, GH are perpendicular to the plane XZ, they are perpendicular to the straight line QH in



that plane; and, since PQ, GH are both perpendicular to the same line QH, they have the same direction and are parallel to each other.

PROP. X.

Conversely, if two straight lines be parallel, and if one of them be perpendicular to any plane, the other will also be perpendicular to the same plane.

For let GH, PQ be the two lines. Draw through P a perpendicular to the plane; this, by the last Prop., will be parallel to GH, and must, therefore, be identical with QP, since through a given point but one parallel can be drawn to a given line.

Cor. Two straight lines parallel to a third are parallel to each other.

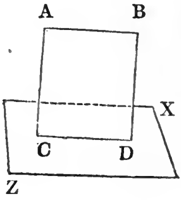
For, conceive a plane perpendicular to any one of them, then the other two being parallel to the first, will be perpendicular to the same plane; hence, by the last Prop., they will be parallel to each other.

The three straight lines are not supposed to be in the same plane.

This corollary follows, also, from our definition of parallel lines (def. 8, p. 1).

PROP. XI.

If a straight line, without a given plane, be parallel to a straight line in the plane, it will be parallel to the plane.



Let AB, lying without the plane XZ, be parallel to CD, lying in the plane.

Then AB is parallel to the plane XZ.

Through the parallels AB, CD pass the plane ABCD, and suppose it, as well as the lines AB and CD, to extend indefinitely.

If the line AB can meet the plane XZ, it must meet it in some point of the line CD, which is the common intersection of the two planes, for the line AB can not get out of the plane AD (def. 2).

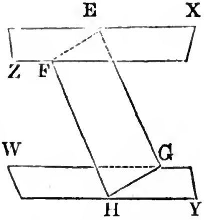
But AB can not meet CD, because AB is parallel to CD.

Hence AB can not meet the plane XZ, *i. e.*, AB is parallel to the plane XZ (def. 5).

PROP. XII.

The sections made by a plane cutting two parallel planes are parallel.

Let FE, GH be the sections made by the plane GF which cuts the parallel planes XZ, WY;



Then FE will be parallel to GH.

For if the lines FE, GH, which are situated in the same plane, be not parallel, they will meet if produced. Therefore, the planes XZ, WY, in which these lines lie, will meet if produced, and ∴ can not be parallel, which is contrary to the hypothesis.

∴ FE is parallel to GH.

PROP. XIII.

Parallel straight lines included between two parallel planes are equal.

Let (see last fig.) the parallels EG, FH be cut by the parallel planes XZ, WY, in the points G, H, E, F.

Then EG = FH,

Through the parallels EG, FH, draw the plane EFGH, intersecting the parallel planes in GH, FE.

Then GH is parallel to FE, by last Prop.

And GE is parallel to HF (by hyp.);

∴ GHFE is a parallelogram; and, therefore,

EG = FH.

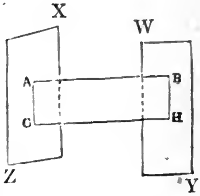
Cor. Two parallel planes are every where equidistant. For the perpendiculars which measure their distance, being parallels, are every where equal.

PROP. XIV.

If two planes be parallel to each other, a straight line which is perpendicular to one of the planes will be perpendicular to the other also.

Let the two planes XZ, WY be parallel, and let the straight line AB be perpendicular to the plane XZ;

Then will AB be perpendicular to WY.



For, from any point H in the plane WY, draw HG perpendicular to the plane XZ, and draw AG, BH.

Then, since BA, HG are both perpendicular to XZ, they are (Prop. 9) parallel to each other.

And, since the planes XZ, WY are parallel, the parallels BA, HG are (by the last Prop.) equal.

Hence (th. 21) AG is parallel to BH; and AB, being perpendicular to AG, is perpendicular to its parallel BH also.

In like manner, it may be proved that AB is perpendicular to any other line which can be drawn from B in the plane WY.

∴ AB is perpendicular to the plane WY.

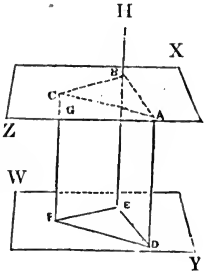
PROP. XV.

Conversely, if two planes be perpendicular to the same straight line, they will be parallel to each other.

For if they could meet, from any common point of the two planes draw two lines, one in each plane, to the extremities of the line to which they are both perpendicular; we should thus have two perpendiculars from the same point to the same line, which is impossible.

PROP. XVI.

If two straight lines which form an angle be parallel to two other straight lines which form an angle in the *same direction, although not in the same plane with the former, the two angles will be equal, and their planes will be parallel.*



Let the two straight lines AB, BC, in the plane XZ, be parallel to the two DE, EF, in the plane WY;

Then angle ABC = angle DEF.

For, make BA = ED, BC = EF; join A, C; D, F; A, D; B, E; C, F;

Then the straight lines AD, BE, which join the equal and parallel straight lines AB, DE, are themselves equal and parallel.

For the same reason, CF, BE are equal and parallel.

∴ AD, CF are equal and parallel (cor., Prop. 10), and ∴ AC, DF are also equal and parallel (th. 21).

Hence the two triangles ABC, DEF, having all their sides equal, each to each, have their angles also equal.

∴ angle ABC = angle DEF.

Again, the plane XZ is parallel to the plane WY.

For, if not, let a plane drawn through A, parallel to DEF, meet the straight lines FC, EB in G and H.

Then DA = EH = FG (Prop. 13).

But DA = EB = FC

∴ EH = EB, FG = FC,

which is absurd.

Cor. 1. If two parallel planes XZ, WY are met by two other planes ADEB, CFEB, the angles ABC, DEF, formed by the intersection of the parallel planes, will be equal.

For the section AB is parallel to the section DE (Prop. 12).

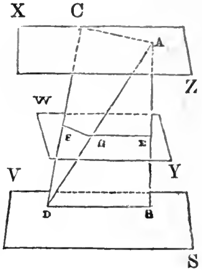
So, also, the section BC is parallel to the section EF.

∴ angle ABC = angle DEF.

Cor. 2. If three straight lines AD, BE, CF, not situated in the same plane, be equal and parallel, the triangles ABC, DEG, formed by joining the extremities of these straight lines, will be equal, and their planes will be parallel.

PROP. XVII.

If two straight lines be cut by parallel planes, they will be cut in the same ratio.



Let the straight lines AB, CD be cut by the parallel planes XZ, WY, VS, in the points A, E, B; C, F, D;

Then AE : EB : : CF : FD.

Join A, C; B, D; A, D; and let AD meet the plane WY in G; join E, G; G, F;

Then the intersections EG, BD of the parallel planes WY, VS, with the plane ED, are parallel (Prop. 12).

∴ AE : EB : : AG : GD (th. 61).

Again the intersection AC, GF of the parallel planes XZ, YW, with the plane CG, are parallel.

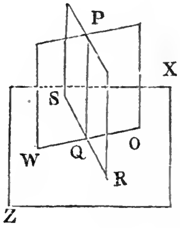
∴ AG : GD : : CF : FD.

∴ substituting the second ratio for the first of this proportion in the previous proportion, we have

AE : EB : : CF : FD.

PROP. XVIII.

If a straight line be perpendicular to a plane, every plane which passes through it will be at right angles to that plane.



Let the straight line PQ be perpendicular to the plane XZ.

Through PQ draw any plane PO, intersecting XZ in the line OQW.

Then the plane PO is perpendicular to the plane XZ.

For, draw RS. in the plane XZ, perpendicular to WQO.

Then, since the straight line PQ is

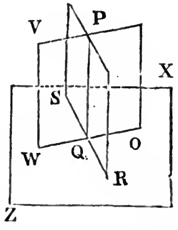
perpendicular to the plane XZ, it is perpendicular to the two straight lines RS, OW, which pass through its foot in that plane.

But the angle PQR is contained between PQ, QR, which are perpendiculars at the same point to OW, the common intersection of the planes XZ, PO; this angle, therefore, measures the angle of the two planes (def. 6); hence, since this angle is a right angle, the two planes are perpendicular to each other.

Cor. If three straight lines, such as PQ, RS, OW, be perpendicular to each other, each will be perpendicular to the plane of the other two, and the three planes will be perpendicular to one another.

PROP. XIX.

If two planes be perpendicular to each other, a straight line drawn in one of the planes perpendicular to their common section will be perpendicular to the other plane.



Let the plane VO be perpendicular to the plane XZ, and let OW be their common section.

In the plane VO draw PQ perpendicular to OW;

Then PQ is perpendicular to the plane XZ.

For, from the point Q, draw QR in the plane XZ, perpendicular to OW.

Then, since the two planes are perpendicular, the angle PQR is a right angle (def. 6).

∴ The straight line PQ is perpendicular to the straight lines QR, QO, which intersect at its foot in the plane XZ.

∴ PQ is perpendicular to the plane XZ (Prop. 5).

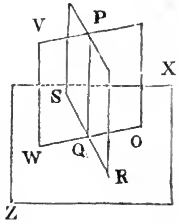
Cor. If the plane VO be perpendicular to the plane XZ, and if from any point in OW, their common intersection, we erect a perpendicular to the plane XZ, that straight line will lie in the plane VO.

For if not, then we may draw from the same point a straight line in the plane VO, perpendicular to OW, and this line, by the Prop., will be perpendicular to the plane XZ.

Thus we should have two straight lines drawn from the same point in the plane XZ, each of them perpendicular to this plane, which is impossible.

PROP. XX.

If two planes which cut each other be each of them perpendicular to a third plane, their common section will be perpendicular to the same plane.



Let the two planes VO, TW, whose common section is PQ, be both perpendicular to the plane XZ.

Then PQ is perpendicular to the plane XZ.

For, from the point Q, erect a perpendicular to the plane XZ.

Then, by cor. to last Prop., this straight line must be situated at once in the planes VO and TW, and is ∴ their common section.

EXERCISES.

1. Prove that but one plane can be passed through a given point perpendicular to a given line.

2. Prove that but one perpendicular can be drawn from a given point to a given plane.

3. That when a plane is perpendicular at the middle of a given line, every point of the plane is equally distant from the extremities of the line, and that every point out of the plane is unequally distant.

4. That through a given line in a plane only one plane perpendicular to the given plane can be passed.

5. That through a line parallel to a given plane but one plane can be passed perpendicular to the given plane.

6. That if two planes which intersect contain two lines parallel to each other, the intersection of the planes will be parallel to the lines.

7. That if a line be parallel to a plane, every other plane passed

through this line and meeting the former, will intersect it in a second line parallel to the first.

8. That when a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

9. That a line parallel to a plane is every where equally distant from that plane. The same of two parallel planes.

10. That two lines are always either in one and the same plane or two parallel planes.

Note.—These planes, the system of which is unique for each system of two lines not situated in the same plane, are called *the parallel planes* of these lines.

11. Show that but one plane can be drawn through a given point parallel to a given plane.

12. Prove that two planes parallel to a third are parallel to each other.

13. Draw a perpendicular to two lines not in the same plane.

14. Prove that if two lines are parallel in space, and planes be passed through them perpendicular to a third plane, the two planes will be parallel.

15. That if a line be parallel to one of two perpendicular planes, and a plane be passed through the line perpendicular to the other plane, it will be parallel to the first plane.

16. To place a perpendicular to a given plane at a given point of the plane.

17. To place a plane perpendicular to a given plane, and intersecting it in a given line.

18. To place a plane parallel to a given plane.

19. To place a line under a given angle to a given plane.

20. To place a plane under a given angle to a given plane, and intersecting it in a given line.

21. To place a plane perpendicular to two give planes.

POLYHEDRAL ANGLES.

DEFINITION.

A *polyhedral angle*, improperly called a *solid angle*, is the angular space contained between several planes which meet in the same point. This point is called the vertex.

Three planes at least are required to form a polyhedral angle.

A polyhedral angle is called a *trihedral, tetrahedral*, &c., *angle*, according as it is formed by *three, four*, &c., plane angles.

A polyhedral angle is named from the letter at its vertex.

A polyhedral angle is called *regular* when all its plane angles are equal and all its diedral angles equal.

A trihedral is called *birectangular, trirectangular*, when two or three of its diedral angles are right angles.

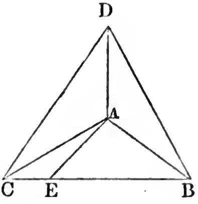
When two of the diedral angles are equal, it is called *isohedral.*

PROP. I.

If a polyhedral angle be contained by three plane angles, the sum of any two of these angles will be greater than the third.

It is unnecessary to demonstrate this proposition, except in the case where the plane angle, which is compared with the two others, is greater than either of them.

Let A be a polyhedral angle contained by the three plane angles BAC, CAD, DAB, and let BAC be the greatest of these angles.



Then CAD + DAB > BAC.

For, in the plane BAC, draw the straight line AE, making the angle BAE = angle BAD.

Make, also, AE = AD, and through E draw any straight line BEC, cutting AB, AC in the points B, C; join D, B; D, C;

Then, because AD = AE, and AB is common to the two triangles DAB, BAE, and the angle DAB = angle BAE, by construction,

$$\therefore BD = BE.$$

But, in the triangle BDC,

$$BD + DC > BE + EC \text{ (ax. 13, cor.)},$$
$$\therefore DC > EC.$$

Again, ∵* AD = AE, and AC is common to the two triangles DAC, EAC, but the base DC > base EC,

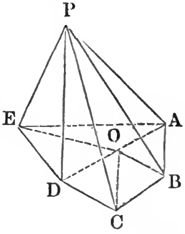
$$\therefore \text{angle } DAC > \text{angle } EAC \text{ (th. 32)}.$$

But $\quad$ angle DAB = angle BAE;

$\therefore$ angle CAD + angle DAB > angle BAE + angle EAC > angle BAC.

PROP. II.

The sum of the plane angles which form a polyhedral angle is always less than four right angles.



Let P be a polyhedral angle contained by any number of plane angles APB, BPC, CPD, DPE, EPA.

Let the polyhedral angle P be cut by any plane ABCDE.

Take any point O in this plane; join A, O; B, O; C, O; D, O; E, O.

Then, since the sum of all the angles of every triangle is always equal to two right angles, the sum of all the angles of the triangles APB, BPC, ... about the point P, will be equal to the sum of all the angles of the equal number of triangles AOB, BOC, ... about the point O.

Again, by the last Prop., angle ABC < angle ABP + angle CBP; in like manner, angle BCD < angle BCP + DCP, and so for all the angles of the polygon ABCDE.

* This sign ∵ signifies "because."

Hence the sum of the angles at the bases of the triangles whose vertex is O, is less than the sum of the angles at the bases of the triangles whose vertex is P.

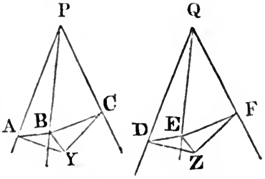
∴ The sum of the angles about the point O must be greater than the sum of the angles about the point P.

But the sum of the angles about the point O is four right angles.

∴ The sum of the angles about the point P is less than four right angles.

PROP. III.

If two trihedral angles be formed by three plane angles which are equal, each to each, the planes in which these angles lie will be equally inclined to each other.



Let P, Q be two trihedral angles;

Let angle APC = angle DQF, angle APB = angle DQE, and angle BPC = angle EQF.

Then the inclination of the planes APC, APB will be equal to the inclination of the planes DQF, DQE.

Take any point B in the intersection of the planes APB, CPB.

From B draw BY perpendicular to the plane APC, meeting the plane in Y.

From Y draw YA, YC, perpendiculars on PA, PC; join A, B; B, C.

Again, take QE = PB; from E draw EZ perpendicular to the plane DQF, meeting the plane in Z; from Z draw ZD, ZF, perpendiculars on QD, QF; join D, E; E, F.

BA is perpendicular to PA (Geom. of Planes, Prop. 8), and the triangle PAB is right-angled at A, and the triangle QDE is right-angled at D.

Also, the angle APB = angle DQE, by hyp.

Moreover, the side PB = side QE (by construction); ∴ the two triangles APB, DQE are identical (cor. 8, th. 15).

∴ PA = QD, and AB = DE.

In like manner, we can prove that

PC = QF, and BC = EF.

Let now the angle APC be placed upon the equal angle DQF, then the point A will fall upon the point D, and the point C on the point F, because PA = QD, and PC = QF.

At the same time, AY, which is perpendicular to PA, will fall upon DZ, which is perpendicular to QD; and, in like manner, CY will fall upon FZ.

Hence the point Y will fall on the point Z, and we shall have

AY = DZ, and CY = FZ.

But the triangles AYB, DZE are right-angled in Y and Z, the hypothenuse AB = hypothenuse DE, and the side AY = side DZ; hence these two triangles are equal (th. 26, cor. 2).

∴ angle YAB = angle ZDE.

The angle YAB is the inclination of the planes APC, APB (Geom. of Planes, def. 6); and

The angle ZDE is the inclination of the planes DQF, DQE.

∴ These planes are equally inclined to each other.

In the same manner, we prove the angle YCB = angle ZFE, and, consequently, the inclination of the planes APC, BPC is equal to the inclination of the planes DQF, EQF.

We must, however, observe that the angle A, of the right-angled triangle YAB, is not, properly speaking, the inclination of the two planes APC, APB, except when the perpendicular BY falls upon the same side of PA as PC does; if it fall upon the other side, then the angle between the two planes will be obtuse, and, added to the angle A of the triangle YAB, will make up two right angles. But, in this case, the angle between the two planes DQF, DQE will also be obtuse, and, added to the angle D of the triangle ZDE, will make up two right angles.

Since, then, the angle A will always be equal to the angle D, we infer that the inclination of the two planes APC, APB will always be equal to the inclination of the two planes DQF, DQE. In the first case, the inclination of the plane is the angle A or D; in the second case, it is the supplement of those angles.

Scholium. If two trihedral angles have the three plane angles of the one equal to the three plane angles of the other, each to each, and, at the same time, the corresponding angles *arranged in the same manner* in the two trihedral angles, then these two trihedral angles will be equal; and if placed one upon the other, they will coincide. In fact, we have already seen that the quadrilateral PAYC will coincide with the quadrilateral QDZF. Thus the point Y falls upon the point Z, and, in consequence of the equality of the triangles AYB, DZE, the straight line YB, perpendicular to the plane APC, is equal to the straight line ZE, perpendicular to the plane DQE; moreover, these perpendiculars lie in the same direction; hence the point B will fall upon the point E, the straight line PB on the straight line QE (their extremities already coinciding), and the two trihedral angles will entirely coincide with each other.

This coincidence, however, can not take place except we suppose the equal plane angles to be *arranged in the same manner* in the two trihedral angles; for if the equal plane angles be *arranged in an inverse order*, or, which comes to the same thing, if the perpendiculars YB, ZE, instead of being situated both on the same side of the planes APC, DQF, were situated on opposite sides of these planes, then it would be impossible to make the two trihedral angles coincide with each other. It would not, however, be less true, according to the above theorem, that the planes, in which the equal angles lie, would be equally inclined to each other; so that the two trihedral angles would be equal in all their constituent parts, without admitting of superposition. This species of equality is termed *symmetry*.

Thus the two trihedral angles in question, which have the three plane angles of the one equal to the three plane angles of the other, each to each, but *arranged in an inverse order*, are termed *angles equal by symmetry*, or, simply, *symmetrical angles*.

The same observation applies to polyhedral angles formed by more than three plane angles. Thus, a polyhedral angle formed by the plane angles A, B, C, D, E, and another polyhedral angle formed by the same angles in an inverse order, A, E, D, C, B, may be such that the planes in which the equal angles are situated are equally inclined to each other. These two polyhedral angles, which would in this case be equal, although not admitting of superposition, would be termed *polyhedral angles equal by symmetry*, or *symmetrical polyhedral angles*.

In plane figures there is no species of equality to which this designation can belong, for all those cases to which the term might seem to apply are cases of *absolute equality*, or *equality of coincidence*. The reason of this is, that in a plane figure one may take the upper part for the under, and vice versâ. This, however, does not hold in solids, in which the third dimension may be taken in two different directions.

This term *symmetrical* is of very extensive application. Two magnitudes are said to be symmetrical with respect to a plane when the corresponding points are on opposite sides of the plane in the same perpendicular to it, and at equal distances from it.

Thus the two halves of the human body are symmetrical with respect to what anatomists call the median plane. (See Appendix V.)

A plane figure may be said to be symmetrical with respect to a median *line* when points on one side of the median line are at equal perpendicular distances from it with opposite points on the other side (see Appendix II., Def. 2).

EXERCISES.

1. To make a trihedral angle with three given plane angles.

2. Prove that in a trihedral angle the sum of the diedral angles is greater than two and less than six right angles.

3. That two trihedral angles are equal when they have two plane angles and the included diedral angle equal [disposed in the same order].

4. Also, when they have one plane angle and two adjacent diedral angles.

5. Also, when they have three diedral angles equal.

6. Show that if from a point within a trihedral angle perpendiculars be drawn to each of the planes which compose it, a new trihedral will be formed whose plane angles will be supplements of the diedral angles of the first, and vice versâ.

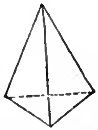
7. Prove that the three planes bisecting the diedral angles of a trihedral angle meet in the same line.

SOLID GEOMETRY.

DEFINITIONS.

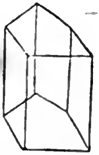
1. A POLYHEDRON is a solid bounded by planes. The intersection of any two of the planes is called a side or edge of the polyhedron. Each bounding plane will be a polygon, and is called a face of the polyhedron.

2. Similar polyhedrons are such as have all their solid angles equal, each to each, and are contained by the same number of similar planes.*



3. A Pyramid is a solid figure contained by triangular planes meeting in one point, called the Vertex, and terminating in the sides of a polygon, called the Base of the pyramid.

A Regular Pyramid is one the base of which is a regular polygon, and the vertex in a perpendicular to the base at its center.



4. A Prism is a solid figure contained by plane figures, of which two that are opposite are equal, similar, and parallel to each other, called bases; and the others are parallelograms. The latter are together called the Lateral Surface of the prism.

A Right Prism is one in which the parallelograms are perpendicular to the bases.

Pyramids and prisms are called Triangular, Quadrangular, Pentagonal, &c., according as their base is a triangle, quadrilateral, pentagon, &c.

5. A Sphere is a solid figure described by the revolution of a semicircle about its diameter, which remains unmoved. The moving semicircle is called the *generatrix*.

6. The Axis of a sphere is the fixed right line about which the semicircle revolves.

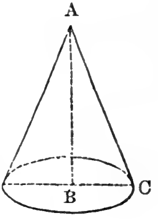
* For a more comprehensive definition of similar solids, see Appendix V.

7. The Center of a sphere is the same with that of the semicircle.

8. The Diameter of a sphere is any right line which passes through the center, and is terminated both ways by the superficies of the sphere. The axis is a diameter.

9. A right Cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.

Thus the side AC, revolving round AB, one of the sides which contains the right angle and remains fixed, generates a cone.



If the fixed side be equal to the other side containing the right angle, the cone is called a right-angled cone; if it be less than the other side, an obtuse-angled; and if greater, an acute-angled cone.

10. The Axis of a cone is the fixed right line about which the triangle revolves.

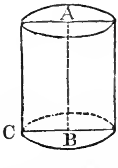
In the figure above, AB is the axis.

The moving side of the triangle is called the *generatrix* of the cone, and any one of the positions of the generatrix is called an *element* of the cone. The *length* of the element is called the *apophthegm* of the cone.

11. The Base of a cone is the circle described by that side containing the right angle which revolves.

12. A Cylinder is a solid figure described by the revolution of a right-angled parallelogram about one of its sides which remains fixed.

Thus, the revolution of the parallelogram AC about its side AB, which remains fixed, generates a cylinder.*



13. The axis of a cylinder is the fixed right line about which the parallelogram revolves.

* The solids above defined are properly *right* cones and cylinders. These may also be oblique.

The Altitude of a pyramid or cone is the perpendicular let fall from the vertex to the plane of the base, produced if necessary.

The altitude of a prism or cylinder is the perpendicular distance between the parallel bases.

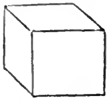
The altitude of a cone or cylinder is identical with the axis.

14. The bases of a cylinder are the circles described by the two revolving opposite sides of the parallelogram.*

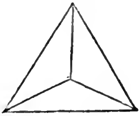
15. Similar cones and cylinders are those which have their axes and the diameters of their bases proportionals.

16. A regular polyhedron is one, all whose solid angles are equal, and whose faces are equal polygons.

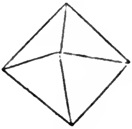
17. A Cube is a regular solid figure contained by six equal squares.



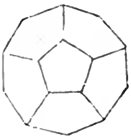
18. A regular Tetrahedron is a solid figure contained by four equal and equilateral triangles.



19. A regular Octahedron is a solid figure contained by eight equal and equilateral triangles.

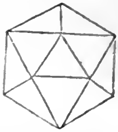


20. A regular Dodecahedron is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.



* Every section of a cylinder made by a plane perpendicular to the axis is a circle, and every section through the axis is a rectangle double the generating rectangle.

The moving side of the parallelogram is called the *generatrix* of the cylinder, and any one of its positions is called an *element*.



21. An Icosahedron is a solid figure contained by twenty equal and equilateral triangles.

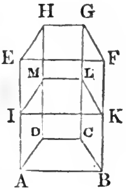
These five, it will be shown hereafter, are the only regular polyhedrons which can be formed.

22. A Parallelopipedon is a solid figure contained by six parallelograms, the planes of every opposite two whereof are parallel. The parallelopipedon is a prism with parallelograms for bases.

PROPOSITIONS.

PROP. I.

If a prism be cut by a plane parallel to its base, the section will be equal and like to the base.



Let AG be any prism, and IL a plane parallel to the base AC; then will the plane IL be equal, and like to the base AC, or the two planes will have all their sides and all their angles equal.

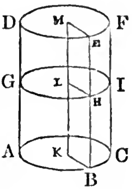
For the two planes AC, IL, being parallel, by hypothesis, and two parallel planes, cut by a third plane, having parallel sections; therefore, IK is parallel to AB, KL to BC, LM to CD, and IM to AD. But AI and BK are parallels, by def. 4, last page but two; consequently, AK is a parallelogram; and the opposite sides, AB, IK, are equal. In like manner, it is shown that KL is = BC and LM = CD, and IM = AD, or the two planes AC, IL are mutually equilateral. But these two planes, having their corresponding sides parallel, have the angles contained by them also equal (Geom. of Planes, Prop. 16); namely, the angle A = the angle I, the angle B = the angle K, the angle C = the angle

L, and the angle D = the angle M. So that the two planes AC, IL have all their corresponding sides and angles equal, or are equal and like. Q. E. D.

PROP. II.

If a cylinder be cut by a plane parallel to its base, the section will be a circle equal to the base.

Let AF be a cylinder, and GHI any section parallel to the base ABC; then will GHI be a circle equal to ABC.



For, let the plane KE pass through the axis of the cylinder MK, and meet the section GHI in the line LH.

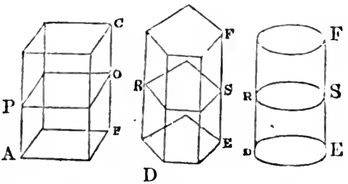
Then, since KL, BH are parallel (def. 12, Sol. Geom.); and the plane KH meeting the two parallel planes ABC, GHI, makes the two sections KB, LH parallel (Prop. 12, Geom. of Planes); the figure KLHB is, therefore, a parallelogram, and, consequently, has the opposite sides LH, KB equal, where KB is a radius of the circular base.

In like manner, it may be shown that any other line drawn from the point L to the circumference of the section GHI, is equal to the radius of the base; consequently, GHI is a circle, and equal to ABC. Q. E. D.

PROP. III.

All prisms, and a cylinder, of equal bases and altitudes, are equal to each other.

Let AC, DF be two prisms and a cylinder, upon equal bases AB, DE, and having equal altitudes; then will the solids AC, DF be equal.



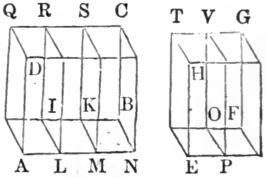
For, let PQ, RS be any two sections parallel to the bases, and equidistant from them. Then, by the last two propositions, the section PQ is equal to the base AB, and the section RS equal to the base DE. But the bases AB, DE are equal by the hypothesis; therefore the sections PQ, RS are also equal. And in like manner it may be shown that any other corresponding sections are equal to one another.

Since, then, every section in the prism AC is equal to its corresponding section in the prism, or cylinder RS, the prisms and cylinder themselves, which are composed of those sections (which will be the same in number,* the altitudes being equal), must also be equal. Q. E. D.

Corol. Every prism, or cylinder, is equal to a rectangular parallelopipedon, of an equal base and altitude.

PROP. IV.

Rectangular parallelopipedons, of equal altitudes, have to each other the same proportion as their bases.



Let AC, EG be two rectangular parallelopipedons, having the equal altitudes AD, EH; then will AC be to EG as the base AB is to the base EF.

For, let the proportion of the base AB to the base EF be that of any one number m (3) to any other number n (2). And conceive AB to be divided into m equal parts, or rectangles, AI, LK, MB (by dividing AN into that number of equal parts, and drawing IL, KM parallel to BN). And let EF be divided, in like manner, into n equal parts, or rectangles EO, PF: all of these parts of both bases being mutually equal among themselves. And through the lines of division let the plane sections LR, MS, PV pass parallel to AQ, ET.

* The number in each will be infinite, but these infinities will evidently be equal.

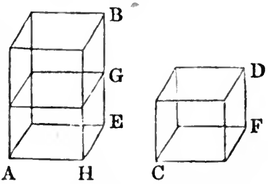
Then the parallelopipedons AR, LS, MC, EV, PG are all equal, having equal bases and heights. Therefore, the solid AC is to the solid EG as the number of parts in AC to the number of equal parts in EG, or as the number of parts in AB to the number of equal parts in EF; that is, as the base AB to the base EF. Q. E. D.

Note. If the bases be incommensurable, the divisions must be infinitely small.

Corol. From this proposition, and the corollary to the last, it appears that all prisms and cylinders of equal altitudes are to each other as their bases; every prism and cylinder being equal to a rectangular parallelopipedon of an equal base and height.

PROP. V.

Rectangular parallelopipedons of equal bases are in proportion to each other as their altitudes.



Let AB, CD be two rectangular parallelopipedons standing on the equal bases AE, CF; then will AB be to CD as the altitude EB is to the altitude DF.

For, let AG be a rectangular parallopipedon on the base AE, and its altitude EG equal to the altitude FD of the solid CD.

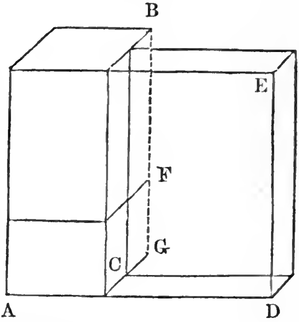
Then AG and CD are equal, being prisms of equal bases and altitudes. But if HB, HG be considered as bases, the solids AB, AG of equal altitude AH, will be to each other as those bases HB, HG. But these bases HB, HG being parallelograms of equal altitude HE, are to each other as their bases EB, EG; and, therefore, the two prisms AB, AG are to each other as the lines EB, EG. But AG is equal CD, and EG equal FD; consequently, the prisms AB, CD are to each other as their altitudes EB, FD; that is, AB : CD : : EB : FD. Q. E. D.

Corol. From this proposition and the corollary to

Prop. 3, it appears that all prisms and cylinders of equal bases are to one another as their altitudes.

PROP. VI.

Rectangular parallelopipedons are to each other as the products of their bases by their altitudes.



The parallelopipedon AF is to the parallelopipedon CE as the base AG × the altitude GF, is to the base CD × altitude DE.

For the parallelopipedons AB and CE, having the same altitude, are to each other as their bases AG and CD; and the parallelopipedons AF and AB, having the same base, are to each other as their altitudes GF, GB; or,

$$AB : CE :: AG : CD,$$
$$AF : AB :: GF : GB.$$

Multiplying these two proportions together and striking out AB from the two terms of the first resulting ratio, we have

$$AF : CE :: AG \times GF : CD \times GB.*$$

* As rectangular parallelopipeds are always to each other as the products of their bases by their altitudes, this may be taken as the measure of parallelopipeds; and as every parallelopiped is equal to a prism or cylinder of the same base and altitude (Prop. 3), it follows that the measure of any prism or cylinder is the product of its base by its altitude.

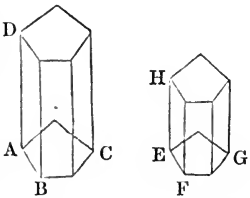
If the convex surface of a cylinder be developed, it opens out into a parallelogram, of which the circumference of the cylinder's base is the base, and the altitude of which is that of the cylinder; and as this parallelogram is measured by the product of its base by its altitude, we have for the measure of the convex surface of a cylinder the product of the circumference of its base by its altitude.

A plane determined by an element of a cylinder, and the tangent line to the base at the point where the element meets it, is a tangent plane to the cylinder.

The contact is along the whole length of the element, which is called the element of contact.

PROP. VII.

Similar prisms and cylinders are to each other as the cubes of their altitudes, or of any other like linear dimensions.

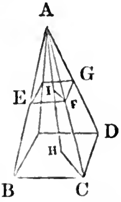


Let ABCD, EFGH be two similar prisms; then will the prism CD be to the prism GH as AB^3 to EF^3, or as AD^3 to EH^3.

For the solids are to each other as the products of their bases and altitudes (by the note to the last Prop.), that is, as AC . AD to EG . EH. But the bases being similar planes, are to each other as the squares of their like sides, that is, AC to EG as AB^2 to EF^2; therefore, substituting the ratio or fraction $AB^2 : EF^2$ for AC : EG, we have the solid CD to the solid GH as $AB^2 . AD$ to $EF^2 . EH$. But BD and FH being similar planes, have their like sides proportional, that is, AB : EF : : AD : EH, or $AB^2 : EF^2 :: AD^2 : EH^2$; multiply this by the identical proportion AD : EH : : AD : EH, and we have $AB^2 . AD : EF^2 . EH :: AD^3 : EH^3$; and, consequently, the solid CD : solid GH : : $AD^3 : EH^3$, or $AB^3 : EF^3$. Q. E. D.

PROP. VIII.

In a pyramid, a section parallel to the base is similar to the base, and these two planes will be to each other as the squares of their distances from the vertex.



Let ABCD be a pyramid, and EFG a section parallel to the base BCD, also AIH a line perpendicular to the two planes at H and I; then will BD, EG be two similar planes, and the plane BD will be to the plane EG as AH^2 to AI^2.

For, join CH, FI. Then, because a plane cutting two parallel planes makes parallel sections, therefore the plane ABC, meeting the two parallel planes BD, EG, makes the sections BC, EF parallel; in like manner, the plane ACD makes the sections CD, FG parallel. Again, because two pair of parallel lines make equal angles, the two EF, FG, which are parallel to BC, CD, make the angle EFG equal the angle BCD. And, in like manner, it is shown that each angle in the plane EG is equal to each angle in the plane BD, and, consequently, those two planes are equiangular.

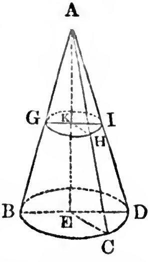
Again, the three lines AB, AC, AD, making with the parallels BC, EF, and CD, FG, equal angles; and the angles at A being common, the two triangles ABC, AEF are equiangular, as also the two triangles ACD, AFG, and have, therefore, their like sides proportional, namely, $AC : AF :: BC : EF :: CD : FG$. And, in like manner, it may be shown that all the lines in the plane EG are proportional to all the corresponding ones in the base BD. Hence these two planes, having their angles equal and their sides proportional, are similar.

But similar planes being to each other as the squares of their like sides, the plane $BD : EG :: BC^2 : EF^2$: or $:: AC^2 : AF^2$, by what is shown above. But the two triangles AHC, AIF, having the angles H and I right ones, and the angle A common, are equiangular, and have, therefore, their like sides proportional, namely, $AC : AF :: AH : AI$, or $AC^2 : AF^2 :: AH^2 : AI^2$. Consequently, the two planes BD, EG, which are as the former squares AC^2, AF^2, will be also as the latter squares AH^2, AI^2, that is, $BD : EG :: AH^2 : AI^2$.

PROP. IX.

In a right cone a section parallel to the base is a circle, and this section is to the base as the squares of their distances from the vertex.

Let ABCD be a right cone, and GHI a section parallel to the base BCD; then will GHI be a circle,



and BCD, GHI will be to each other as the squares of their distances from the vertex.

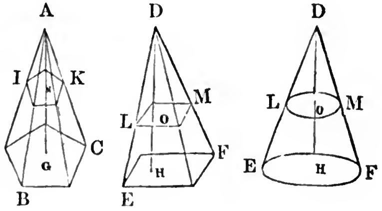
For, let the planes ACE, ADE pass through the axis of the cone AKE, meeting the section in the three points H, I, K.

Then, since the section GHI is parallel to the base BCD, and the planes CK, DK meet them, HK is parallel to CE, and IK to DE. And from similar triangles, shown to be such (as in the last Prop.), KH : EC :: AK : AE :: KI : ED. But EC is equal to ED, being radii of the same circle; therefore, KI is also equal to KH. And the same may be shown of any other lines drawn from the point K to the circumference of the section GHI, which is, therefore, a circle.

Again, since AK : AE :: KI : ED, hence AK^2 : AE^2 :: KI^2 : ED^2; but KI^2 : ED^2 :: circle GHI : circle BCD (th. 72); therefore, AK^2 : AE^2 :: circle GHI : circle BCD. Q. E. D.

PROP. X.

All pyramids and right cones of equal bases and altitudes are equal to one another.



Let ABC, DEF be any pyramids and a cone, of equal bases BC, EF, and equal altitudes AG, DH; then will the pyramids and cone ABC and DEF be equal.

For, parallel to the bases, and at equal distances, AN, DO, from the vertices, suppose the planes IK, LM to be drawn.

Then, by Prop. 8 and 9,

$$DO^2 : DH^2 :: LM : EF,$$

and $$AN^2 : AG^2 :: IK : BC.$$

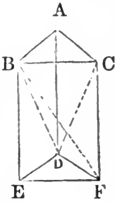
But, since AN^2, AG^2 are equal to DO^2, DH^2; therefore IK : BC : : LM : EF. But BC is equal to EF, by hypothesis; therefore IK is also equal to LM.

In the same manner, it is shown that any other sections, at equal distance from the vertex, are equal to each other.

Since, then, every section in the cone is equal to the corresponding section in the pyramids, and the heights are equal, the solids ABC, DEF, composed of those sections, must be equal also. Q. E. D.

PROP. XI.

Every triangular prism may be divided into three equal triangular pyramids of the same base and altitude with the prism.



Let ABCDEF be a prism.

In the planes of the three sides of the prism, draw the diagonals BF, BD, CD. Then the two planes BDF, BCD divide the whole prism into the three pyramids BDEF, DABC, DBCF; which are proved to be all equal to one another as follows:

Since the opposite ends of the prism are equal to each other, the pyramid whose base is ABC and vertex D, is equal to the pyramid whose base is DEF and vertex B (Prop. 10), being pyramids of equal base and altitude.

But the latter pyramid, whose base is DEF and vertex B, may be considered as having BEF for its base and D for its vertex, and this is equal to the third pyramid, whose base is BCF and vertex D, being pyramids of the same altitude (since they have the same vertex and their bases are in the same plane) and equal bases BEF, BCF (th. 19).

Consequently, all the three pyramids which compose the prism are equal to each other, and each pyramid is the third part of the prism, or the prism is triple of the pyramid. Q. E. D.

Corol. 1. Any triangular pyramid is the third part of a triangular prism of the same base and altitude (this follows from the last Prop., 10).

Corol. 2. Every pyramid, whatever its figure may be, is the third part of a prism of the same base and altitude. This follows from Props. 3 and 10, or may be proved by dividing the given prism into triangular prisms, and the given pyramid into triangular pyramids, all having a common altitude.

Corol. 3. Any right cone is the third part of a cylinder, or of a prism, of equal base and altitude; since it has been proved that a cylinder is equal to a prism, and a cone equal to a pyramid, of equal base and altitude.

Corol. 4. The measure of a pyramid or cone will be the product of its base by the third of its altitude. (See note to Prop. 6.)

Scholium. Whatever has been demonstrated of the proportionality of prisms or cylinders holds equally true of pyramids or cones, the former being always triple the latter when they have the same base and altitude; viz., that similar pyramids or cones are as the cubes of their like linear sides, or diameters, or altitudes, &c.

The tangent plane to a cone is analogous to that of a cylinder. The contact is a right-lined element. Every tangent plane to a cone passes through the vertex.

The surface of a cone developes into the sector of a circle, and is measured by the circumference of the base multiplied by half the apophthegm.

A pyramid is inscribed in a cone when the base of the pyramid is inscribed in that of the cone, and they have the same vertex.

A cone is a pyramid of an infinite number of triangular faces.

EXERCISES.

1. Prove that the four diagonals of a parallelopipedon meet in the same point.

2. That the square of each diagonal of a rectangular parallelopipedon is equal to the sum of the squares of its three edges.

3. Construct a parallelopipedon upon three lines perpendicular to each other as edges.

4. Prove that the two lines joining the points of the opposite faces of a parallelopipedon, in which the diagonals of those faces intersect, bisect each other at the point where the diagonals of the solid meet.

5. Prove that two polyhedrons which have the same vertices are identical.

6. That two prisms are equal when they have three faces forming a polyhedral angle of the one equal to the same in the other, and arranged in the same order.

7. That two right prisms are equal when they have equal bases and altitudes.

8. That two tetrahedrons are equal, 1°. When they have a diedral angle equal, comprehended between equal faces, arranged in the same manner in both; 2°. When they have one trihedral angle, comprehended by three equal faces in each, and arranged in the same order. 3°. *Corol.* When they have their edges all equal and arranged in the same order. 4°. When they have two faces and two adjacent diedrals equal.

9. That two pyramids are equal when they have their bases and two other faces forming a trihedral angle, with the base equal in each.

10. Polyhedrons may be divided into tetrahedrons by planes passing diagonally through the edges.

11. Polyhedrons are equal when composed of the same number of equal tetrahedrons.

12. Prove that polyhedrons are equal when their faces and diedral angles are equal, and disposed in the same order.

13. Prove that similar polyhedrons are composed of the same number of similar tetrahedrons.

14. That polyhedrons are similar when their faces are all equal, each to each, and equally inclined.

15. That polyhedrons are similar when they have a face in each similar, and their homologous vertices out of this face are determined by tetrahedrons having a triangular face in the homologous face.*

16. That two pyramids are similar when they have their edges parallel.

17. That two regular polyhedrons of the same kind are similar.

18. That a plane passed through two edges of a parallelopiped, diagonally opposed to each other, divides the parallelopiped into two symmetrical triangular prisms equal in volume.

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* This is the definition of similar polyhedrons given by Legendre.

19. That two prisms of the same base are proportional to their altitudes.

20. That two regular pyramids are equal when their base and an edge of the one are equal to the same in the other.

21. That similar pyramids are to each other as the cubes of their homologous sides.

22. That similar polyhedrons are as the cubes of their homologous sides.

23. That the surfaces of similar polyhedrons are as the squares of their homologous sides.

24. Cut a pyramid by a plane, parallel to the base, in such a way that the section shall be to the base in the ratio of two given lines.

25. Also, so that the convex surface of the superior portion shall be one third that of the whole pyramid.

26. Show how to construct a pyramid when the base and two adjacent triangular faces are given.

27. A prism with the same data.

28. A parallelopipedon with given base, and edge meeting it.

29. With given edges to construct the faces of a tetrahedron.

30. With three edges forming a trihedral angle, and their angles, to construct the faces of a triangular prism.

31. The same for a pentagonal prism, three faces, forming a trihedral angle, being given.

32. Show how to construct a cylinder similar to a given cylinder, and whose base shall be to that of the given in the ratio of two given lines.

33. Show in what the frustum of a cone develops, and what is the measure of its surface.

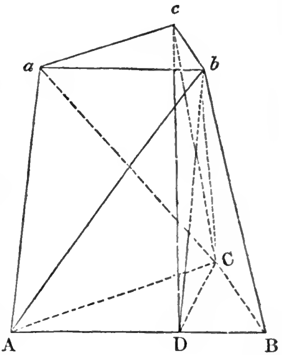
34. Prove that every plane parallel to the axes of a cylinder cuts its surface in two lines parallel to the axis.

35. That if a circle and a line tangent to it revolve about a common axis passing through the center of the circle, the curve of contact of the cone generated by the line, and the sphere generated by the circle, will be a circle whose plane is perpendicular to the axis.

36. That similar cones and cylinders are proportional, their surfaces to the squares, and their volumes to the cubes of their altitudes, elements, diameters of bases, or any homologous lines.

PROP. XII.

The frustum of a pyramid is composed of three pyramids having for a common altitude the altitude of the frustum, and for bases the upper and lower base of the frustum and a mean proportional between them.



Let a plane be passed through the three points a, C, b; it cuts off a triangular pyramid having acb for a base, and C in the plane of the lower base of the frustum for a vertex. This is evidently the first pyramid of the enunciation. Again: suppose a plane be passed through the three points A, C, b; it cuts off a pyramid having ABC for base and b for vertex, the second pyramid of the enunciation. There remains the pyramid CAba, which is equal to the pyramid CADb (bD being drawn parallel to aA), having the same vertex C and an equal base, since the diagonal Ab bisects the parallelogram ADba. This last pyramid being considered as having its vertex at b and its base ADC, has the altitude of the frustum, and it remains to show that the triangle ADC, which is its base, is a mean proportional between the triangles ABC and abc. For this purpose, let us observe that

$$\triangle ABC : \triangle ADC : AB : AD \quad . \quad . \quad . \quad . \quad . \quad (1)$$

because they have a common vertex, and, therefore, are to each other as their bases, which are in the same straight line. Also, that

$$\triangle ADC : \triangle abc : AC : ac \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

because the angle A = the angle a (corol. 1, Prop. 16, Geom. of Planes), and (th. 60, cor. 3) $\triangle$ ADC : abc : : AD $\times$ AC : $ab \times ac$, and AD = ab. Also, that

$$AB : ab :: AC : ac \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since the triangles ABC, *abc* are similar (Prop. 8, ante).

Substituting now the first ratio of (3) for its equivalent, the second ratio of (2), and then the first ratio of (2) for its equivalent (since $AD = ab$), the second ratio of (1), we have

$$\triangle ABC : \triangle ADC :: \triangle ADC : \triangle abc. \quad \text{Q. E. D.}$$

Corol. The same proposition is true of the frustum of any pyramid or of a cone (Prop. 9 and 10) which is equivalent to three cones having the upper base, the lower base and a mean proportional between the two for bases, and for a common altitude the altitude of the frustum. In symbols r and r', being the radii of the bases, and h the altitude of the frustum, its volume would be expressed by (th. 73, schol.)

$$\pi(r^2 + r'^2 + rr')h.$$

PROP. XIII.

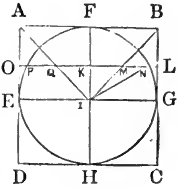
If a sphere be cut by a plane, the section will be a circle.

Because the radii of the sphere are all equal, each of them being equal to the radius of the describing semicircle, it is evident that if the section pass through the center of the sphere, then the distance from the center to every point in the periphery of that section will be equal to the radius of the sphere, and the section will, therefore, be a circle of the same radius as the sphere. But if the plane do not pass through the center, draw a perpendicular to it from the center, and draw any number of radii of the sphere to the intersection of its surface with the plane; then these radii are evidently the hypothenuses of a corresponding number of right-angled triangles, which have the perpendicular from the center on the plane of the section, as a common side; consequently, their other sides are all equal, and, therefore, the section of the sphere by the plane is a circle, whose center is the point in which the perpendicular cuts the plane.

Scholium. All the sections through the center are equal to one another, and are greater than any other section which does not pass through the center. Sections through the center are called *great circles*, and the other sections *small* or *less circles.*

PROP. XIV.

Every sphere is two thirds of its circumscribing cylinder.



Let ABCD be a section of the cylinder, and EFGH a section of the sphere through the center I, and join AI, BI. Let FIH be parallel to AD or BC, and EIG and KL parallel to AB or DC, the base of the cylindric section; the latter line KL meeting BI in M, and the circular section of the sphere in N.

Then, if the whole plane HFBC be conceived to revolve about the line HF as an axis, the square FG will describe a cylinder AG, and the quadrant IFG will describe a hemisphere EFG, and the triangle IFB will describe a cone IAB. Also, in the rotation, the three lines, or parts, KL, KN, KM, as radii, will describe corresponding circular sections of these solids, viz., KL a section of the cylinder, KN a section of the sphere, and KM a section of the cone.

Now, FB being equal to FI or IG, and KM parallel to FB, then, by similar triangles, $IK = KM$ (Geom. Theor., 63), and IKN is a right-angled triangle; hence IN^2 is equal to $IK^2 + KN^2$ (theor. 26). But KL is equal to the radius IG or IN, and $KM = IK$; therefore KL^2 is equal to $KM^2 + KN^2$, or the square of the longest radius of the above-mentioned circular sections is equal to the sum of the squares of the two others. Now circles are to each other as the squares of their diameters, or of their radii, therefore the circle described by KL is equal to both the circles described by KM and KN; or the section of the cylinder is equal to both the corresponding sections of the sphere and cone. And as this is always the case in every parallel position of KL, it follows that the cylinder EB, which is composed of all the former sections, is equal to the hemisphere EFG and cone IAB, which are composed of all the latter sections, the number of the sections being the same, because the three solids have the same altitude.

But the cone IAB is a third part of the cylinder EB (Prop. 11, cor. 3); consequently, the hemisphere EFG is equal to the remaining two thirds, or the whole sphere EFGH is equal to two thirds of the whole cylinder ABCD.

Corol. 1. A cone, hemisphere, and cylinder of the same base and altitude are to each other as the numbers 1, 2, 3.*

Corol. 2. All spheres are to each other as the cubes of their diameters, all these being like parts of their circumscribing cylinders.

Corol. 3. From the foregoing demonstration it appears that the spherical zone or frustum EGNP is equal to the difference between the cylinder EGLO and the cone IMQ, all of the same common height IK. And that the spherical segment PFN is equal to the difference between the cylinder ABLO and the conic frustum AQMB, all of the same common altitude FK.

Scholium. By the scholium to Prop. 11, we have

$$\text{cone } AIB : \text{cone } QIM :: IF^3 : IK^3 :: FH^3 : (FH-2FK)^3$$

$$\therefore \text{cone } AIB : \text{frust. } ABMQ :: FH^3 : FH^3-(FH-2FK)^3$$

$$:: \dagger FH^3 : 6FH^2.FK-12FH.FK^2+8FK^3;$$

but cone $AIB =$ one third of the cylinder ABGE; hence

$$\text{cyl. } AG : \text{frust. } ABMQ :: 3FH^3 : 6FH^2.FK-12FH.FK^2 + 8FK^3.$$

$$\text{Now cyl. } AL : \text{cyl. } AG :: FK : FI.$$

Multiplying the last two proportions, and striking out the common factors from the ratios, observing, also, that $FI = \frac{1}{2}FH$, we have

* The surfaces of the sphere and circumscribing cylinder are in the same ratio as their solidities. For the demonstration, see Mensuration.

† Raising the binomial to the third power.

cyl. AL : frust. ABMQ : : $6FH^2 : 6FH^2 - 12FH.FK + 8FK^2$

$\therefore$ (dividendo, and by corol. 3 of this Prop., 14),

cyl. AL : segment PFN : : $6FH^2 : 12FH.FK - 8FK^2$

$: : \frac{3}{2}FH^2 : FK(3FH - 2FK)$.

But cylinder AL = circular base whose diameter is AB or FH, multiplied by the height FK; hence cylinder AL = circle EFGH $\times$ FK.

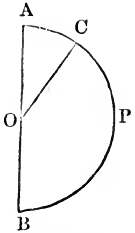
$$\therefore \text{segment PFN} = \frac{2}{3} \cdot \frac{\text{circle EFGH}}{FH^2}(3FH - 2FK)FK^2.$$

SPHERICAL GEOMETRY.

DEFINITIONS.

1. A SPHERE is a solid terminated by a curve surface, and is such that all the points of the surface are equally distant from an interior point, which is called the *center* of the sphere.

We may conceive a sphere to be generated by the revolution of a semicircle APB about its diameter AB; for the surface described by the motion of the curve APB will have all its points equally distant from the center O.



The sector of a circle AOC at the same time generates a *spherical sector.*

2. The *radius* of a sphere is a straight line drawn from the center to any point on the surface.

The *diameter* or *axis* of a sphere is a straight line drawn through the center, and terminated both ways by the surface.

It appears from Def. 1 that all the radii of the same sphere are equal, and that all the diameters are equal, and each double of the radius.

3. It will be demonstrated (Prop. 1) that every section of a sphere made by a plane is a circle; this being assumed,

A *great circle* of a sphere is the section made by a plane passing through the center of the sphere.

A *small circle* of a sphere is the section made by a plane which does not pass through the center of the sphere.

4. The *pole* of a circle of a sphere is a point on the surface of the sphere equally distant from all the points in the circumference of that circle.

It will be seen (Prop. 2) that all circles, whether great or small, have two poles.

5. A *spherical triangle* is the portion of the surface of a sphere included by the arcs of three great circles.

6. These arcs are called the *sides* of the triangle, and each is supposed to be less than half of the circumference.

7. The *angles* of a spherical triangle are the angles contained between the planes in which the sides lie. Or the angle formed by any two arcs of great circles is the angle formed by the planes of the great circles of which the arcs are a part.

8. A *spherical polygon* is the portion of the surface of a sphere bounded by several arcs of great circles.

9. A plane is said to be a tangent to a sphere when it contains only one point in common with the surface of the sphere.

10. A *zone* is the portion of the surface and a spherical segment, the portion of the volume of a sphere between two parallel planes, or cut off by one plane.

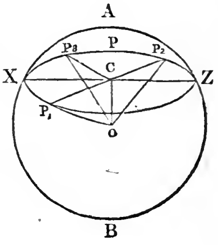
The circles in which the planes intersect the sphere are called bases of the zone or segment.

11. A lune is the portion of the surface of a sphere comprehended between two great semicircles.

12. A spherical wedge or ungula is the solid bounded by a lune and the planes of its two circles.

PROP. I.

Every section of a sphere made by a plane is a circle.



Let AZBX be a sphere whose center is O.

Let XPZ be a section made by the plane XZ.

From O draw OC perpendicular to the plane XZ.

In XPZ take any points P_1, P_2, P_3, ...

Join CP_1; CP_2; CP_3; . . . . . . also, OP_1; OP_2; OP_3; . . . .

Then, since OC is perpendicular to the plane XZ, it will be perpendicular to all straight lines passing through its foot in that plane. (Def. 3, Geometry of Planes.)

Hence the angles OCP_1, OCP_2, OCP_3 . . . . . . are right angles

$$OP_1^2 = CP_1^2 + OC^2;$$
$$OP_2^2 = CP_2^2 + OC^2;$$
$$OP_3^2 = CP_3^2 + OC^2.$$

But, since P_1, P_2, P_3 . . . . . . are all points upon the surface of the sphere, $\because$ by def. 1, $OP_1 = OP_2 = OP_3 =$ . . . . . .

$$\therefore CP_1 = CP_2 = CP_3 \ldots\ldots$$

Hence XPZ is a circle whose center is C, and every other section of a sphere made by a plane may, in like manner, be proved to be a circle.

Cor. 1. If the plane pass through the center of the sphere, then $OC = 0$, and the radius of the circle will be equal to the radius of the sphere.

Cor. 2. Hence all great circles are equal to one another, since the radius of each is equal to the radius of the sphere.

Cor. 3. Hence, also, two great circles and their circumferences always bisect each other; for, since both pass through the center, their common intersection passes through the center, and is a diameter of the sphere and of each of the two circles.

Cor. 4. The center of a small circle and that of the sphere are in a straight line, which is perpendicular to the plane of the small circle.

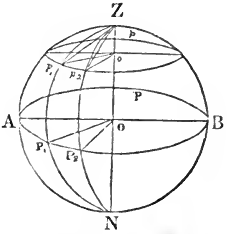
Cor. 5. We can always draw one, and only one, great circle through any two points on the surface of a sphere; for the two given points and the center of the sphere give three points, which determine the position of a plane.

If, however, the two given points are the extremities of a diameter, then these two points and the center of the sphere are in the same straight line, and an infinite number of great circles may be drawn through the two points. (Prop. 3, Geom. of Planes.)

Distances on the surface of a sphere are measured by the arcs of great circles. The reason for this is, that the shortest line which can be drawn upon the surface of a sphere, between any two points, is the arc of a great circle joining them, which will be proved hereafter.

PROP. II.

If a diameter be drawn perpendicular to the plane of a great circle, the extremities of the diameter will be the poles of that circle, and of all the small circles whose planes are parallel to it.



Let APB be a great circle of the sphere whose center is O.

Draw ZN, a diameter perpendicular to the plane of the circle APB.

Then Z and N, the extremities of this diameter, are the poles of the great circle APB, and of all the small circles, such as *apb*, whose planes are parallel to that of APB.

Take any points P_1, P_2, ... in the circumference of APB.

Through each of these points respectively, and the points Z and N, describe great circles, ZP_1N, ZP_2N.

Join OP_1, OP_2, ...

Then, since ZO is perpendicular to the plane of APB, it is perpendicular to all the straight lines OP_1, OP_2, ... drawn through its foot in that plane.

Hence all the angles ZOP_1, ZOP_2, ... are right angles, and ∴ the arcs ZP_1, ZP_2, ... are quadrants.

Thus it appears that the points Z and N are at a quadrant's distance, and ∴ equally distant from all the points in the circumference of APB, and are ∴ the poles of that great circle.

Again; since ZO is perpendicular to the plane APB, it is also perpendicular to the parallel plane *apb* (Geometry of Planes, Prop. 14).

Hence the oblique lines Zp_1, Zp_2, ... drawn to p_1, p_2, in the circumference of *apb*, will be equal to each other. (Prop. 7, Geometry of Planes.)

∴ The chords Zp_1, Zp_2, ... being equal, the arcs Zp_1, Zp_2, ... which they subtend, will also be equal.

∴ The point Z is the pole of the circle *apb;* and the point N is also a pole, the arcs Np_1, &c., being supplements of the arcs Zp_1, &c.

Def. The diameter of the sphere perpendicular to the plane of a circle is called the *axis* of that circle.

Cor. 1. Every arc P_1Z, drawn from a point in the circumference of a great circle to its pole, is a quadrant, and this arc P_1Z makes a right angle with the arc AP_1B. For, the straight line ZO being perpendicular to the plane APB, every plane which passes through this straight line will be perpendicular to the plane APB (Prop. 18, Geometry of Planes); hence the angle between these planes is a right angle, or, by def. 7, the angle of the arcs AP_1 and ZP_1 is a right angle.

Cor. 2. In order to find the pole of a given arc AP_1 of a great circle, take P_1Z, perpendicular to AP_1,* and equal to a quadrant, the point Z will be a pole of the arc AP_1; or, from the points A and P_1 draw two arcs AZ and P_1Z perpendicular to AP_1, the point Z in which they meet is a pole of AP_1.

Cor. 3. Reciprocally, if the distance of the point Z from each of the points A and P_1 is equal to a quadrant, then the point Z is the pole of AP_1, and each of the angles ZAP_1, ZP_1A is a right angle.

For, let O be the center of the sphere; draw the radii OA, OP_1, OZ;

Then, since the angles AOZ, P_1OZ are right angles, the straight line OZ is perpendicular to the straight lines OA, OP_1, and is ∴ perpendicular to their plane; hence, by the above prop., the point Z is the pole of

* A perpendicular arc to AP_1 at P_1 is described by means of its pole, which will be in AP_1B, at a quadrant's distance from P_1.

AP_1, and $\therefore$ (corol. 1), the angles ZAP_1, ZP_1A are right angles.

Cor. 4. Great circles, such as ZA, ZP_1, whose planes are at right angles to the plane of another great circle, as APB, are called its *secondaries;* and it appears from the foregoing corollaries, that,

1. The planes of all secondaries pass through the axis, and their circumferences through the poles of their primary; and that the poles of any great circle may always be determined by the intersection of any two of its secondaries.

2. The arcs of all secondaries intercepted between the primary and its poles are $= 90°$.

3. A secondary bisects all circles parallel to its primary, the axis of the latter passing through all their centers.

Cor. 5.* Let the radius of the sphere $= R$, radius of small circle parallel to it $= r$. Distance of two circles, or $Oo = \delta$.

Join Op_1, and let the arc P_1p_1, in degrees and fractions of a degree, be expressed by ϕ. Then will $\delta = \sin. \phi$ and $r = \cos. \phi$ to the radius R, and we have the equations

$$R^2 = r^2 + \delta^2;$$
$$r = R \cos. \phi;$$
$$\delta = R \sin. \phi;$$

in which $\cos. \phi$ and $\sin. \phi$ express these trigonometrical lines to radius 1; the usual radius of the tables.

Cor. 6. Two secondaries intercept similar arcs of circles parallel to their primary, and these arcs are to each other as the cosines of the arcs of the secondaries between the parallels and the primary.

For the arcs of the parallels subtend at their respective centers, angles equal to the inclinations of the planes of the secondaries, and these arcs will, therefore (def. 55), be similar. Again: let p_1p_2 in the diagram be one of these arcs, and imagine another,

* The two following corollaries require a knowledge of the first principles of Trigonometry.

q_1q_2, between this and P_1P_2; then if r_1, r_2 be the radii of the two small parallels p_1p_2, q_1q_2, the rest of notation as before, we shall have

$$\frac{\text{arc } p_1p_2}{\text{arc } q_1q_2}=\frac{\text{whole circumference of 1st}}{\text{whole circumference of 2d}};$$

$$=\frac{r_1}{r_2}\text{ (th. 71, cor. 1)};$$

$$=\frac{\text{R cos. }\phi}{\text{R cos. }\phi'};$$

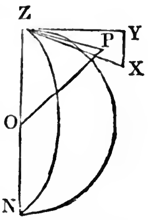
$$=\frac{\text{cos. }\phi}{\text{cos. }\phi'}.$$

If the second arc q_1q_2 becomes P_1P_2, $\phi'=0$ and cos. $\phi'=1$.

$\therefore \dfrac{\text{arc } P_1P_2}{\text{arc } p_1p_2}=\dfrac{1}{\text{cos. }\phi}$, or $\dfrac{\text{arc } p_1p_2}{\text{arc } P_1P_2}=\text{cos. }\phi$, or arc $p_1p_2=$ cos. ϕ arc P_1P_2.*

PROP. III.

Every plane perpendicular to a radius at its extremity is a tangent to the sphere in that point.



Let ZXY be a plane perpendicular to the radius OZ.

Then ZXY touches the sphere in Z.

Take any point P in the plane; join ZP; OP;

Then (Prop. 6, Geom. of Planes) OP $>$ OZ.

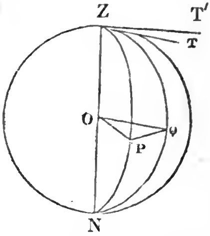
Hence the point P is without the sphere; and, in like manner, it may be shown that every point in XYZ, except Z, is without the sphere.

Therefore the plane XYZ is a tangent to the sphere.

* These formulas are of frequent use in Astronomy, serving to express the relation between the distance moved on a parallel of declination and in right ascension of a star, and various other useful relations of a similar kind.

PROP. IV.

The angle formed by two arcs of great circles is equal to the angle contained by the tangents drawn to these arcs at their point of intersection, and is measured by the arc described from their point of intersection as a pole, and intercepted between the arcs containing the angle.



Let ZPN, ZQN, arcs of great circles, intersect in Z.

Draw ZT, ZT′, tangents to the arcs at the point Z.

With Z as pole, describe the arc PQ.

Take O, the center of the sphere, and join OP, OQ.

Then the spherical angle PZQ is equal to the angle TZT′, and is measured by the arc PQ.

For the tangent ZT, drawn in the plane ZPN, is perpendicular to radius OZ; and the tangent ZT′, drawn in the plane ZQN, is perpendicular to radius OZ; hence the angle TZT′ is equal to the angle contained by these two planes (def. 6, Geom. of Planes), that is, to the spherical angle PZQ.

Again; since the arcs ZP, ZQ are each of them equal to a quadrant;

∴ Each of the angles ZOP, ZOQ is a right angle, or OP and OQ are perpendicular to ZO.

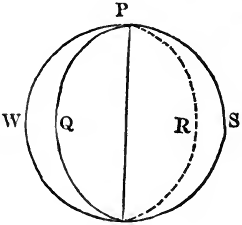
∴ The angle QOP is the angle contained by the planes ZPN, ZQN.

∴ The arc PQ, which measures the angle POQ, measures the angle between the planes, that is, the spherical angle PZQ.

Cor. 1. The angle under two great circles is measured by the distance between their poles. For the axes (def. in Prop. 2) of the great circles drawn through their poles being perpendicular to the planes of the circles, will be perpendicular to all lines of these planes, consequently, to the lines which measure the angles of the planes, and ∴ (see th. 65, Gen. Sch., 4°) the angles under these axes will be equal to the angle between the circles; but the angle under the axes is obviously measured by the arc which joins their extremities, that is, by the distance between their poles.

Cor. 2. The angle under two great circles is measured by the arc of a common secondary intercepted between them.

Cor. 3. Vertical spherical angles, such as QPW, RPS, are equal, for each of them is the angle formed by the planes QPR, WPS.

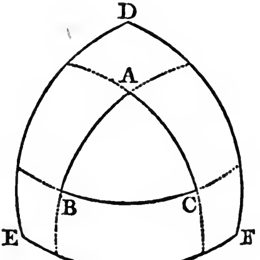


Also, when two arcs cut each other, the two adjacent angles QPW, QPS, when taken together, are always equal to two right angles.

PROP. V.

If from the angular points of a spherical triangle considered as poles, three arcs be described forming another triangle, then, reciprocally, the angular points of this last triangle will be the poles of the sides opposite to them in the first.

Let ABC be a spherical triangle.



From the points A, B, C, considered as poles, describe the arcs EF, DF, DE, forming the spherical triangle DEF.

Then D will be the pole of BC, E of AC, and F of AB.

For, since B is the pole of DF, the distance from B to D is a quadrant.

And, since C is the pole of DE, the distance from C to D is a quadrant.

Thus, it appears that the point D is distant by a quadrant from the points B and C.

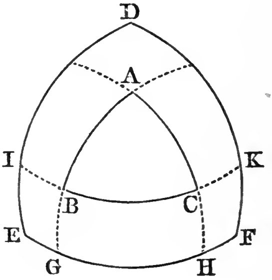
∴ (cor. 1, 2, Prop. 2) D is the pole of the arc BC.

Similarly, it may be shown that E is the pole of AC, and F the pole of AB.

Note. D having been shown to be the pole of the arc passing through the points B and C, it must be of the arc BC, because but one arc of a great circle can be made to pass through the two points B and C (cor. 5, Prop. 1).

PROP. VI.

The same things being given as in the last proposition, each angle in either of the triangles will be measured by the supplement of the side opposite to it in the other triangle.



Produce BC to I and K, AB to G, and AC to H.

Then, since A is the pole of EF, the angle A is measured by the arc GH at a quadrant's distance from A (Prop. 4).

But, because F is the pole of AG, the arc FG is a quadrant.

And, because E is the pole of AH, the arc EH is a quadrant.

$$\therefore\ EH + GF = 180°,$$

or

$$EF + GH = 180°;$$

$$\therefore\ GH = 180° - EF.$$

In a similar manner, it may be proved that the angle B is measured by 180° — DF, and the angle C by 180° — DE.

Again; since D is the pole of BC, the angle D is measured by IK.

But, because B is the pole of DK, the arc BK is a quadrant.

And, because C is the pole of DI, the arc CI is a quadrant.

$$\therefore IC + BK = 180°,$$

or

$$IK + BC = 180°;$$

$$\therefore IK = 180° - BC.$$

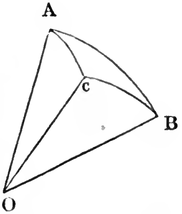
But IK is the measure of the angle D (Prop. 4).

In the same manner, it may be proved that the angle E is measured by 180° — AC, and the angle F by 180° — AB.

These triangles ABC, DEF are, from their properties, usually called *Polar triangles*, or *Supplemental triangles*.

PROP. VII.

In any spherical triangle any one side is less than the sum of the other two.



Let ABC be a spherical triangle, O the center of the sphere. Draw the radii OA, OB, OC.

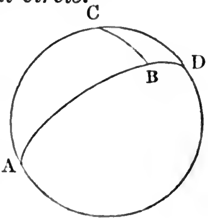
Then the three plane angles AOB, AOC, BOC form a trihedral angle at the point O, and these three angles are measured by the arcs AB, AC, BC.

But each of the plane angles which form the trihedral angle is less than the sum of the two others (Prop. 1, Polyhedral Angles).

Hence each of the arcs AB, AC, BC, which measures these angles, is less than the sum of the other two.

PROP. VIII.

The sum of the three sides of a spherical triangle is less than the circumference of a great circle.



Let ABC be any spherical triangle.

Produce the sides AB, AC to meet in D.

Then, since two great circles always bisect each other (Prop. 1, cor. 3), the arcs ABD, ACD are semicircles.

Now, in the triangle BCD,

$$BC < BD + DC, \text{ by last Prop.}$$

$$\therefore AB + AC + BC < AB + BD + AC + CD,$$

$$< ABD + ACD,$$

$$< \text{circumference of great circle.}$$

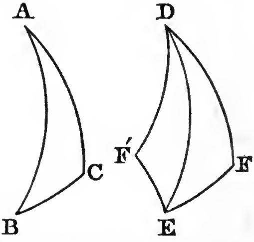
Note. In elementary geometry the only spherical triangles considered are those in which each side is less than a semicircumference, and each angle less than two right angles.

Should a spherical triangle be taken without these restrictions, it will be found that the residue of the surface of the sphere will be a triangle having portions of the same circumferences as boundaries with the given triangle, and falling within the restrictions; when all the parts of this latter triangle are known, the parts of the other may be derived from them by subtracting the known angles and the known sides from 180 or 360 degrees.

Triangles not limited by the restrictions above mentioned, therefore, being dependent upon those which are thus limited, the first class may be rejected, and our attention, as it has been in the preceding theorems, confined to the second.

PROP. IX.

Two spherical triangles are either identical or symmetrical, 1°. *When they have two sides and the included angle of the one equal to the same in the other;* 2°. *When they have a side and two adjacent angles;* 3°. *When they have three sides respectively equal.*

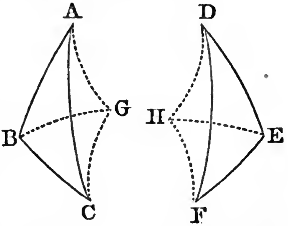


These follow from the corresponding theorems in trihedral angles (Prop. 3, and exer. 3, 4, 5), but may be proved by superposition of the given triangles, the one upon the other, or its symmetrical triangle, as at th. 1, 2, &c., in Plane Geometry.

The preceding diagram exhibits symmetrical triangles, viz., EDF and EDF′, or ABC and EDF′. The one could not be superposed upon the other, for, on turning it over so as to bring the equal parts opposite to each other, the convexities of the two surfaces would be turned toward each other, and could touch in but one point.

PROP. X.

Symmetrical triangles are nevertheless *equal in surface*, which may be proved as follows:



Let ABC, DEF be two symmetrical triangles, in which AB = DE, AC = DF, BC = EF.

Let G be the pole of the small circle passing through the three points A, B, C, and H the pole of the small circle passing through the three D, E, F. Join G with A, B, C, and H with D, E, F by arcs of great circles. The triangles AGB, BGC, AGC, HDE, HFE, HDF are all isosceles, and the corresponding ones in the two diagrams admit of superposition, because, in turning them over to bring the convexities of their surfaces the same way, equal sides are not turned away from each other, and this arises from the triangles being isosceles. The three triangles of the one diagram being respectively identical, therefore, with the three of the other, we have AGB + BGC — AGC = DHE + EHF — DHF, or A BC = DEF. Q. E. D.

PROP. XI.

An isosceles spherical triangle has its two angles equal, and conversely.

PROP. XII.

In any spherical triangle, the greater side is opposite the greater angle, and conversely.

These may be proved precisely as in plane triangles.

PROP. XIII.

Two spherical triangles (unlike two plane triangles in this respect) *are equal when the three angles of the one are equal to the three angles of the other, each to each.*

For the polar triangles of the two given triangles will have equal sides (Prop. 6), and, consequently, equal angles (Prop. 9). Hence the given triangles will have equal sides.

NOTE.

The equal triangles in question in the preceding theorems need not be supposed on the same sphere, if their sides and angles are given in degrees and fractions of a degree. Indeed, there would be much advantage gained by discarding spherical triangles from geometry except for purposes of mensuration on the surface of the sphere, and using trihedral angles in their place, especially in the application to Astronomy, which, as a science of observation, depends entirely on angular measurements.

PROP. XIV.

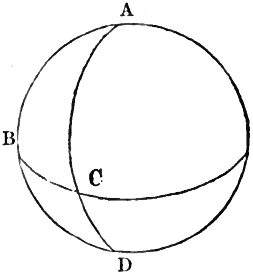
The sum of the angles of a spherical triangle is greater than two and less than six right angles.

For each angle is less than two (note to Prop. 8), hence the sum of the three is less than six right angles. Again, each angle being measured by a semicircumference, minus the side opposite in the polar triangle, the sum of the three angles will be three semicircumferences, minus the sum of the three sides of the polar triangle, but the latter is less than a circumference (Prop. 8); hence the measure of the sum of the three angles will be greater than one semicircumference or two right angles.

Note. In a *birectangular* spherical triangle, *two* of the sides are *quadrants;* and in a *trirectangular* triangle *all three* of the sides are *quadrants*. This latter triangle is sometimes taken as the *unit of measure* on the surface of the sphere. As there are four such triangles in each hemisphere, the whole surface of the sphere would be expressed by the number 8.

PROP. XV.

The surface of a lune is to the whole surface of the sphere as the angle of the lune is to four right angles, or as the arc which measures the angle of the lune is to a circumference.



It is evident, from a mere inspection of the diagram, that the lune ABDC is the same aliquot part of the whole surface of the sphere that the arc BC is of a whole circumference, or that the angle BAC, measured by this arc, is of four right angles. The demonstration may be made more full by dividing the triangle ABC into a number of equal triangles, having their common vertex at A and their bases equal portions of BC,* and dividing, also, the hemisphere into triangles of the same size, and thus showing that the ratio of the triangle ABC to the hemisphere is the same as the ratio of BC to the whole circumference, because both are in the ratio of the same two numbers, viz., the number of triangles in ABC to the number in the hemisphere, or the number of bases in each.

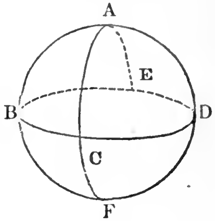
Schol. The angle of the lune is to four as twice this angle is to eight. Hence, if the whole sphere be expressed by 8, the lune will be expressed by 2A.

* These triangles will be equal because their sides are equal.

PROP. XVI.

The two opposite spherical triangles on a hemisphere are together equal to a lune having the same angle.

Let the two triangles ABC, ADE be on the same hemisphere, having their common vertex at A. Then will their sum be equal to the lune ABFC.

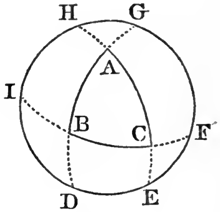


For the triangle BFC may be proved equilateral with the triangle ADE, and, therefore, of the same surface. Q. E. D.

PROP. XVII.

The measure of a spherical triangle is the excess of the sum of its angles above two right angles.

Let ABC be a spherical triangle; its measure will be $A + B + C - 2$.*



For (by the last two propositions),

$$\triangle DAE + \triangle GAH = 2A;$$
$$\triangle FBG + \triangle IBD = 2B;$$
$$\triangle HCI + \triangle ECF = 2C.$$

If we add the first members together, we obtain evidently the whole hemisphere, which is expressed by four, together with twice the triangle ABC.

$$\therefore 4 + 2 \triangle ABC = 2A + 2B + 2C;$$
$$\therefore \quad ABC = A + B + C - 2.$$

Q. E. D.

Corol. 1. The spherical triangle is equivalent to a lune whose angle is half the above expression.

Corol. 2. Two spherical triangles are of equal surface when the sum of their angles is the same, and *vice versâ.*

* A, B, and C must be here understood as expressed not in degrees, &c., but in fractions of a right angle.

EXERCISES.

1. Prove that every spherical triangle may be inscribed in a circle.

2. Through a given point on the arc of a great circle to draw an arc of a great circle perpendicular to the former.

3. The same through a point without the given arc.

4. Prove that the rectangles of the parts of all lines passing through the same point within a sphere, and terminating at the surface, are equal.

5. Prove that circles whose planes are equidistant from the center of the sphere are equal.

6. Prove that every plane passing through the point of contact of a tangent plane to a sphere cuts this plane in a line tangent to the circle cut from the sphere.

7. That the *line of centers* of two spheres which cut each other is perpendicular to the plane of the circle of section of the two spheres.

8. Prove that their intersection is a circle.

9. Show how to construct a spherical triangle with any three parts given.

10. Prove that the sum of all the sides of a spherical polygon is less than the circumference of a great circle.

11. Make a sphere pass through four given points, or prove that every tetrahedron may be circumscribed by a sphere.

12. Also, inscribed.

13. Prove that the measure of the surface of a spherical polygon is equal to the excess of the sum of its angles over as many times two right angles as the figure has sides less two.

14. Make a great circle tangent* to a small circle on the surface of a sphere.

15. Change a spherical quadrangle into an equivalent spherical triangle.

16. Upon the base of a spherical triangle to construct an isosceles spherical triangle of equal surface.

17. To construct on the base of a given spherical triangle another of equal surface, 1°, having a given base angle; 2°, having a given side.

18. Prove that the sums of the opposite angles of a spherical quadrilateral inscribed in a circle of the sphere are equal.†

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* One circle is said to be tangent to another on the surface of a sphere when the two circles have a common tangent line at a common point.

† This is done by connecting the four vertices of the quadrilateral

19. Prove that if two spherical triangles, having a common base, be inscribed in the same circle of a sphere, the difference between the sum of the base angles and the vertical angle will be equal in the two triangles.

Corol. Spherical triangles having the same base, and the sums of their base angles equal, and also their vertical angles equal, have their vertices lying in the same circumference on the sphere.

20. Prove that if the base of a spherical triangle be prolonged to become a complete circumference, and the other two sides prolonged beyond the vertex till they meet this; then, if through the points of meeting and the vertex a small circle of the sphere be made to pass, every triangle having its vertex in this, and its base the same with the given triangle, will have an equal surface.

Corol. If one of the other sides of the triangle falls in the prolongation of the base, and the vertex coincides with one of the above-mentioned points of meeting, the small circle passing through the three points vanishes or reduces to a point, viz., the point in which these three points coalesce; the triangle then degenerates into a lune, which is still, however, equal to the given triangle in surface.

21. Upon the base of a given spherical triangle to construct another of equal surface of which the vertex shall lie in a given great circumference.

22. To change a spherical triangle into another of equal surface with a given side and given angle adjacent.

23. To construct a spherical triangle with two given sides and of surface equal to a given triangle.

24. Prove that if P denote the number of polyhedral angles of a polyhedron, F the number of its faces, and E the number of its edges,

$$P + F = E + 2.$$

25. Also, that the sum of the plane angles of a polyhedron is equal to $P - 2$ times four right angles.

26. To construct the length of the radius of a sphere when confined to its exterior.

27. To describe the circumference of a great circle through two given points.

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with the pole of the circle in which it is inscribed, thus forming four isosceles spherical triangles.

APPENDIX III.

PROVE that if $\frac{m}{n}$ expresses the ratio of an arc to a quadrant, $\frac{m}{n} \cdot \frac{\pi}{2}$ will express the ratio of the arc to the radius.

Knowing the ratio of an arc to the radius, show how to find the number of degrees which it contains.

Two angles subtended by arcs of different radii are to each other as the ratios of the arcs to their respective radii (th. 71, corol. 3). In symbols, if V and V′ be two angles, A and A′ the arcs subtending them, described with the radii R and R′,

$$V : V' :: \frac{A}{R} : \frac{A'}{R'}.$$

Taking the right angle as the unit of angles, supposing for a moment V′ to be this unit, A′ the unit of arc, and R′ the unit of length, the above proportion becomes

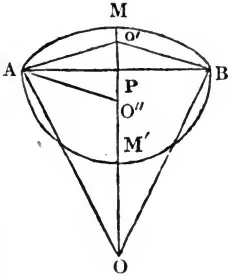
$$V : 1 :: \frac{A}{R} : 1 \ \therefore \ V = \frac{A}{R};$$

that is, an angle at the center has for its measure the quotient of the arc which subtends it, divided by the radius. It must, however, be understood that the quantities V, A, and R are referred to their respective units.

THEOREM.

Of two arcs, each less than a semicircumference, subtended by the same chord, the shortest is that whose center is furthest from the middle of the chord.

Let AB be the common chord, AMB, AM′B the two arcs, O the center of the former, O′ of the latter. Then, if $OP > O'P$, $AMB < AM'B$.



For (by th. 17) $OA > O'A$; and, if the arc AM′B be turned over round AB as a hinge, it will evidently contain the arc AMB within it;* and it may be easily

* This may be seen more distinctly by observing that an indefinitely small portion of the arc of a circle may be regarded as a

proved, that of two lines, the one enveloping the other, and terminating at the same points, the enveloped line is the least. This may be shown, supposing them to be polygonal lines at first, by repeated application of the principle that a straight line is the shortest distance between two points, and then supposing the straight portions of the polygonal lines to become infinitely small, or the polygonal lines to become curves.

Prove that every small circle of a sphere has a less radius than the sphere.

THEOREM.

The arc of a great circle comprehended between two given points on the surface of a sphere is less than any arc of any small circle comprehended between the same two points.

This follows from the last theorems.

THEOREM.

The shortest path from one point to another on the surface of a sphere is the arc of a great circle.

To prove this, let it be observed that the sphere is perfectly round in all directions, so that every section of it made by a plane is a circle. This being premised, suppose an irregular line upon its surface between the two given points; this may be considered either an arc of a small circle, or made up of small portions of such circles. In the first case, it has already been proved that the arc of a great circle between the points is shorter than this. In the second case, arcs of great circles between the extremities of the portions are less than these portions, and, by the repetition of the principle that one side of a spherical triangle is less than the sum of the other two, it may be shown that the arc of a great circle between the two given points is less than the polygonal combination of arcs of great circles between the same points, so that in both cases the theorem is demonstrated.

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straight line, which, prolonged both ways, becomes a tangent; the tangent, therefore, shows the direction of the curve at the point of contact. If, now, after the arc AM′B is turned over, it be observed that the direction of this arc at the point A is perpendicular to AO″, while the direction of AMB is perpendicular to AO, it is evident that the latter arc will run within the former.

By joining the point O″ with any point of the inverted arc AM′B, and the point in which this line intersects the arc AMB with the point O, it may be shown that the arc AM′B is every where diverging in direction from the arc AMB, except at M, M′.

APPENDIX IV.

ISOPERIMETRY ON THE SPHERE.

1. Prove that of all spherical triangles formed with two given sides, the greatest is that in which the angle formed by the given sides is equal to the sum of the other two angles.

2. That of all spherical triangles formed with one side, and the perimeter given, the greatest is that in which the undetermined sides are equal.

3. That of all isoperimetrical spherical polygons, the greatest is an equilateral polygon.

4. That of all spherical polygons formed with given sides, and one side taken at pleasure, the greatest is that which can be inscribed in a circle, of which the chord of the undetermined side is the diameter.

5. The greatest of spherical polygons formed with given sides is that which can be inscribed in a circle of the sphere.

6. The greatest of spherical polygons having the same perimeter and same number of sides is that in which the sides and angles are equal.

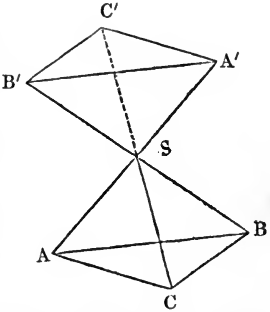
Note. All the above apply, also, to polyhedral angles, of which the spherical triangles are the measures.

APPENDIX V.

SYMMETRY IN SPACE.

THERE are two kinds of *symmetry* for polyhedrons, symmetry of *form* and symmetry of *position*.

To give an idea of these two kinds of symmetry, let us consider, first, a tetrahedron SABC, and upon its edges, prolonged above the vertex S, take distances $SA' = SA$, $SB' = SB$, $SC' = SC$, and draw $A'B'$, $A'C'$, $B'C'$; the parts of the two tetrahedrons (edges, faces, diedral angles) are evidently equal each to each, but disposed in an inverse order. They are called *symmetric*.



The second tetrahedron may be detached from the first, and is still symmetric, whatever may be their relative position.

Two *polyhedrons* are said to be symmetric [and that independent of their position in space] when they *can be decomposed into the same number of tetrahedrons symmetric each to each, and disposed in an inverse order*.

Whence it follows that, 1°. A polyhedron can have but one symmetric with it. 2°. Two symmetric polyhedrons have their edges, faces, diedral and polyhedral angles equal each to each.

SYMMETRY OF POSITION.

This exists in three ways: 1°. With reference to a *point*, which is called a *center of symmetry;* 2°. With reference to a *line*, called an *axis of symmetry;* 3°. With reference to a plane, called the *plane of symmetry*. We shall treat, first, of

SYMMETRY RELATIVE TO AN AXIS.

Definition. Two points are *symmetrical* with respect to a line when the line which joins them is perpendicular to the first, and divided by it into two equal parts.

A polyhedron is symmetric, or *two* polyhedrons are symmetric with reference to a line, when this line passes through the middle point of all the lines [other than the edges or diagonals of the faces] which join the vertices of the polyhedron, two and two, and is perpendicular to them.

THEOREM 1. *Two figures which are symmetric with reference to a line are identical.*

This may be proved by revolving the perpendiculars about the axis; the vertices will all describe similar arcs.

Corollaries. In a polyhedron *symmetric* with reference to an axis, 1°. *Every line meeting the axis at right angles, and terminating at the surface, is equally divided by the axis.* 2°. *Every plane through the axis cuts the polyhedron into two equal parts.* 3°. *Every plane perpendicular to the axis determines a symmetric section with reference to the point of intersection of this plane with the axis,* and this point is *the center of symmetry* of *the section.*

Schol. 1. The most simple of polyhedrons symmetrical with reference to an axis is the *right prism,* the base of which is *symmetric with reference to a point.*

When the base of the right prism is a rectangle it has for axes of symmetry the three lines which join the centers of the opposite faces.

If, moreover, the base is a square, there exist two other axes of symmetry which join the middle of the opposite edges.

When the base of the right prism is a *rhombus,* there are *three* axes of symmetry, *one* joining the centers of the two bases, and *two* others joining the middle points of the opposite edges.

Schol. 2. The axis of a *regular pyramid* is also an axis of *symmetry* when the number of lateral faces is even.

Schol. 3. Symmetry, with reference to an axis, is, properly speaking, merely symmetry of position, since, by the preceding theorem, the figures are equal and capable of superposition. But the same is not the case with symmetry with reference to a point, or symmetry with reference to a plane, which are at the same time symmetry of form and position. For this reason we have commenced with symmetry referred to a line.

SYMMETRY WITH REFERENCE TO A POINT OR PLANE.

Definitions. Two points are said to be symmetrical with reference to a *point* when the latter divides into two equal parts the line joining the two former; and, with reference to a *plane,* when this plane is perpendicular to the line which joins the two points and bisects it.

THEOREM 2. *If three points are in a right line, their symmetric* *points with reference to a point or plane are in a right line.* The student will easily prove this.

Corollaries. 1°. *Two lines of determinate length, and symmetric with respect to a point, are equal and parallel.* 2°. *Two triangles symmetric with respect to a point are equal and their planes parallel.* 3°. *Two lines of determinate length, and symmetric with reference to a plane, are equal, make equal angles with this plane,* and, being prolonged, *meet it at the same point, unless they are parallel.*

THEOREM 3. *If four points are in the same plane, their symmetric points, with reference to a point, are also in a same plane.*

Schol. When the four points are in different planes their symmetrics are also, and then the two systems of points determine two tetrahedrons, whose angles, diedral and trihedral, are symmetric, and, consequently, the tetrahedrons themselves *symmetric.*

THEOREM 4. *When two polyhedrons have their vertices, two and two, symmetric with reference to a point or a plane* [in which case the polyhedrons are said to be symmetric], 1°. These polyhedrons *have their faces equal each to each, their diedral angles equal each to each, and their polyhedral angles symmetric.* 2°. These polyhedrons *are symmetric in form.*

The first part of this theorem results from the last corollaries, and the second by observing that the two polyhedrons are composed of the same number of tetrahedrons symmetric, two and two, and inversely disposed.

THEOREM 5. *When the vertices of a polyhedron are situated symmetrically with reference to a point,* 1°. *This polyhedron has necessarily an even number of edges, equal and parallel two and two; and it is the same with the faces;* 2°. *The plane angles and diedral angles are also equal each to each; the polyhedral angles are symmetric in pairs;* 3°. *Every line passing through the center of symmetry and terminating at the surface is divided at this point into two equal parts;* 4°. *Finally, every plane passing through the center divides the polyhedron symmetrically.*

This follows from the last corollaries and scholium.

Schol. 1. The most simple of polyhedrons with reference to a point is the parallelopipedon. It has for a *center of symmetry* its *center of figure.* As every diagonal plane passes through its center of figure, such a plane divides the parallelopipedon symmetrically.

Schol. 2. After the parallelopipedon, the most simple are prisms having for bases polygons symmetric with reference to a point. The center of symmetry is the middle of the line which joins the centers of the two bases.

General Scholium upon symmetry with reference to a point and a plane compared with absolute symmetry.

It follows, from theorems four and five, that two polyhedrons symmetric with reference to a point or to a plane are at the same time absolutely symmetric.

Reciprocally, *two polyhedrons symmetric to each other* (absolutely) *can always be placed symmetrically with reference to a point* in space, or with reference to *a plane*, this point or plane being a common vertex or face of the two polyhedrons.*

THEOREM 6. *Two symmetric polyhedrons are equivalent.*

It is only necessary, after the first definition, to demonstrate this for tetrahedrons. These may be shown to have the same base and height (see th. 5, 3°, of this App.), and are, consequently, equal.

The two following propositions may be easily established:

1°. When there exist in a polyhedron *two planes of symmetry perpendicular to each other*, their common intersection is *an axis of symmetry*; 2°. And if there exist *three*, the point common to these three planes is a *center of symmetry*.

OF DIAMETRAL PLANES.

When a plane passes through a polyhedron in such a manner that a system of *parallel lines* terminating at the surface are equally divided by the plane, it is called a *diametral plane*. N.B.—The parallels are not necessarily perpendicular to the plane.

THEOREM 7. *When the vertices of a polyhedron, or of two polyhedrons, are situated in pairs upon parallel lines, and a certain plane passes through the middle points of these lines,* 1°. *Each couple of homologous edges produced will meet at a point of the plane, unless they are parallel*; 2°. *Each couple of homologous planes determined by three vertices of the one polyhedron and three corresponding of the other, intersect each other in a line of the first-mentioned plane* (unless they are parallel); 3°. *Every line parallel to any of the lines joining the homologous vertices, and terminating on either side the plane at the polyhedral surface, is equally divided by this plane, which is, consequently, a diametral plane.*

N.B.—When the lines joining the homologous vertices are equal and parallel, the figures determined by the vertices are equal and their planes parallel.

* An object and its reflected image present a familiar example of two figures symmetric to each other.

The human body is a figure composed of two parts symmetric, with reference to what is called a *median* plane.

Schol. 1. The preceding theorem comprehends, as a particular case, figures symmetrical with reference to a plane.

Schol. 2. Every triangular prism, *right* or *oblique*, has *four* diametral planes, one of which is the plane half way between the bases parallel to them; and the three others are the planes passing through the lateral edges and through the diameters of the bases.

CENTER OF MEAN DISTANCES.

The point which has been named center of mean distances in a polygon in a previous appendix (II., def. 4), has a property with reference to a plane which we have shown it to have with reference to a line.

THEOREM 8. *The perpendicular let fall from the center of mean distances upon a plane drawn at pleasure in space, is equal to the quotient of the algebraic sum of the perpendiculars let fall from the different vertices upon this plane divided by the number of vertices.* This sum is *zero* when the plane passes through the center of mean distances, and *vice versâ.*

The demonstration will be similar to that in the corresponding one in a previous appendix (II., def. 4, et seq.).

It is to be observed, that the vertices of which the point in question is the center of mean distances need not be in the same plane as they were supposed to be in the previous appendix.

Schol. To determine the center of mean distances for any number of points not in the same plane, draw three planes at pleasure which cut each other (suppose, for the sake of simplicity, at right angles). Let fall, from the different vertices upon each of these planes, perpendiculars; find afterward for each plane the algebraic sum of its perpendiculars, and divide this sum by the number of vertices. Finally, at distances equal to the three quotients, draw planes parallel to the first three, and their common intersection will be the point sought.

When four points are not in the same plane, these points, combined three and three, determine a *tetrahedron.* This being observed:

THEOREM 9. *In every tetrahedron the lines which join the middle points of the edges not adjacent, all meet in a point which is the center of mean distances of the four vertices.*

Schol. This point is also found in three planes parallel to the faces, and at a distance equal to one quarter the distance of the opposite vertex from each face.

THEOREM 10. *The four lines joining the vertices with the centers of mean distances of the opposite faces, meet in a point which is the cen-*

ter of mean distances of the vertices. This point is one quarter the distance from the center of mean distances in each face to the opposite vertex.

OF CENTERS OF SIMILITUDE.

THEOREM 11. *If all the vertices of a polyhedron be joined with a point in space by lines, and upon these lines, or three prolongations, portions be taken proportional to the lines themselves, the vertices of a new polyhedron will be thus obtained, which is directly or inversely similar to the first.*

This point is called a *center of similitude,* EXTERNAL in the first case, INTERNAL in the second.

The proof of the above is in all respects similar to that in a previous appendix (App. II.).

CENTERS OF SIMILITUDE OF SPHERES.

If lines be drawn tangent to two circles meeting each other, one pair *internally* and the other pair *externally;* and if these circles and tangents be set in revolution about the line joining the centers, the circles will generate spheres, and the tangents, cones enveloping the spheres, and the points of contact will generate circles which will be the curves of contact of the cones and spheres; the planes of these circles of contact will be perpendicular to the axis. The vertices of these cones, at the points in which the tangents intersect, are called *centers of similitude* of the two spheres, the one *internal,* the other *external.*

Prove that every plane tangent to one of these conic surfaces is tangent to the two spheres.

And, conversely, that every plane tangent to the two spheres is tangent to one of the conic surfaces.

Two spheres in space would have an infinite number of common tangent planes. One of these would be determined by another condition, as, that it should pass through a given point, or be parallel to a given line, or tangent to a third sphere, &c.; and there would be two planes which would fulfill the required condition in the first two cases; in the last there might be *four* systems of *two* planes tangent to the *three* spheres, to wit: two planes comprehending the three spheres between them, and *six* placed two and two between one of the spheres and the two others.

This second case gives rise to a remarkable theorem analogous to one in a previous appendix (App. II.), for three circumferences of a circle.

Theorem 12. *The six centers of similitude of three spheres exterior to one another are situated* three and three upon a same line, to wit, the *three external* centers of similitude, then *one* of the *external* and *two internal*, giving in all *four* lines.

For, first, let us consider the two tangent planes which embrace the three spheres between them. These planes being tangent to the three cones which envelop the spheres, must both pass through the vertices of these cones, and, consequently, their intersection must. The other tangent planes will, in a similar manner, serve to demonstrate the other part of the theorem.

This theorem serves to prove the correctness of the theorem for the case of three circumferences (App. II.), because the centers of similitude of these circles are the same as the centers of similitude of three spheres, of which these circles are great circles.

It is thus that sometimes propositions in Plane Geometry may be demonstrated in a more simple manner by the aid of truths relating to geometry in space.

REGULAR POLYHEDRONS.

A regular polyhedron is one in which the faces are equal regular polygons, and the diedral angles equal. From this definition it will follow that the polyhedral angles will also be equal.

THEOREM.

There can be but five regular polyhedrons.

This follows from Prop. 2, of Polyhedral Angles, that a polyhedral angle can not be formed unless the sum of the plane angles which form it is less than four right angles.

If we take equilateral triangles, each angle of which is two thirds of a right angle, to form a polyhedral angle, we may combine these in threes, fours, and fives, but not more, because $6 \times \frac{2}{3} = \frac{12}{3} = 4$ right angles.

If we take squares, each angle of which is one right angle, to form a polyhedral angle, we can combine them in threes alone, for $4 \times 1 = 4$ right angles.

If regular pentagons, each angle of which is $\frac{5 \times 2 - 4}{5} = 1\frac{1}{5}$, they can be combined but in threes.

If hexagons, each angle of which is $1\frac{1}{3}$, they can not be combined even in threes to form a polyhedral angle, and three is the least number of planes that can be employed for this purpose.

It is evident that still less can regular polygons of a greater number of sides be employed.

There can, therefore, be formed but three regular polyhedrons of triangular faces, but one of square faces, and but one of pentagonal faces, in all five, which is the greatest number that can possibly exist.

Schol. It remains to be shown that five regular polyhedrons can be formed.

CONSTRUCTION OF REGULAR POLYHEDRONS.

1°. TO CONSTRUCT A REGULAR TETRAHEDRON.

Take an equilateral triangle; erect at the center of its inscribed circle a perpendicular to its plane; with one of its vertices as a center, and a radius equal in length to one of its edges, cut this perpendicular in a point; join this point with the vertices of the triangle, and the regular tetrahedron will be formed.

2°. TO CONSTRUCT A REGULAR HEXAHEDRON OR CUBE.

We leave this to the student, being too easy to require explanation.

3°. A REGULAR OCTAHEDRON.

Upon a line equal to one of the sides of the equilateral triangle, which is to be a face, construct a square; erect at the center of this square a perpendicular to its plane, and take upon this perpendicular, on each side of the plane, a distance equal to one half the diagonal of the square; joining the points thus determined with the vertices of the square, the polyhedron required is formed.

N.B.—The center of the square is a *center of symmetry*. It is also the *center of figure*.

4°. A REGULAR ICOSAHEDRON.

Construct first a pentagon upon the side of the given equilateral triangle; at the center of this figure erect a perpendicular to its plane; with a radius equal to the side of the triangle, cut this perpendicular in a point; this point being joined with the vertices of the pentagon, will furnish five equilateral triangles formed about it; form now a second pentahedral angle, with one of the angles of the pentagon as a vertex, and two of its faces will be the same as those of the first pentahedral angle formed; with a third vertex of the same triangle, to which the other two already employed belonged, form a third pentahedral angle; for this purpose two new faces will be required. There will thus be united *ten* triangles, forming a sort of polyhedral cap, such that the angles at the border are formed by alternately two and three triangles. This polygonal line, which terminates the surface, has its sides equal, but its vertices not in the same plane. If now a second polyhedral cap be constructed equal to the first, its diedral angles will have the same value as those in the other. Then, without breaking the continuity, we can unite the *double angles* of the border of the first cap with the *triple angles* of the border of the second, and *vice versâ;* whence will result a figure of twenty equal faces equally inclined.

5°. A REGULAR DODECAHEDRON.

Suppose that with three *regular pentagons* a *trihedral* be formed, which is possible (see last th.). The three diedral angles of this trihedral angle are equal. Now with new pentagons, equal to the preceding, can be formed in the same manner, successively, at the vertices of one of these pentagons, other trihedral angles, all of the same magnitude. There will result *six* regular pentagons, composing a polygonal cap, such that the angles of the border are formed alternately of *one* and of *two* plane angles.

[The same remark as above applies to this border.]

If a second cap be imagined, equal to the first, they can be united, border to border, so that the single angles of the one accord to the double angles of the other; and thus will be formed a figure of twelve faces, equal, and equally inclined to one another.

Schol. 1. To construct a regular polyhedron mechanically, taking one of the faces as a base of construction, upon a sheet of pasteboard make the development of all the faces, then fold these different faces upon their edges in a suitable manner.

Schol. 2. All the regular polyhedrons except the tetrahedron have a *center of symmetry* which is identical with the *center of figure.*

All have, also, *planes of symmetry.* These are, in general, planes perpendicular upon the middle of the edges, or upon the middle points of lines joining opposite vertices, taken two and two, or else planes passing through the opposite edges, two and two.

Schol. 3. *The regular tetrahedron* has 4 vertices, 4 faces, and 6 edges.

The cube	8	"	6	"	12	"
The octahedron	6	"	8	"	12	"
The dodecahedron	20	"	12	"	30	"
The icosahedron	12	"	20	"	30	"

General Scholium upon Polyhedrons. These expressions, which can be easily verified upon the figures, are contained in the enunciation of a theorem by the celebrated Euler, and which is translated by the formula

$$V + F = E + 2;$$

V designating the number of vertices, F the number of faces, and E the number of edges. This formula has been previously given.

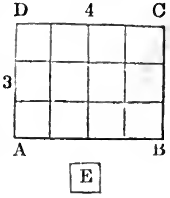
There are a great many theorems more or less important upon polyhedrons as well as upon polygons, similar to those in a previous appendix, for which, see a memoir of M. Poinsot, in the Journal de l'Ecole Polytechnique, 10^e cahier, t. iv., p. 6, et seq. Also, a memoir of M. Cauchy, in the same journal, 16^e cahier, t. ix., p. 77. Also, Annales de Mathematiques of M. Gergonne, particularly tome xv., page 157.

MENSURATION OF PLANES.

THE area of any plane figure is the measure of the space contained within its extremes or bounds, without any regard to thickness.

This area, or the content of the plane figure, is estimated by the number of little squares that may be contained in it; the side of each of those little measuring squares being an inch, a foot, a yard, or any other fixed quantity. And hence the area or content is said to be so many square inches, or square feet, or square yards, &c. In other words, the area of a surface is the *numerical ratio of this surface to its unit.*

Thus, if the figure to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit proposed; then, as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain, which in the present case is 12.



PROBLEM I.

To find the area of any parallelogram, whether it be a square, a rectangle, a rhombus, or a rhomboid.

Multiply the length by the perpendicular breadth or height, and the product will be the area.*

* The truth of this rule is proved in the Geometry, Theor. 60, Schol.

The same is otherwise proved thus: Let the foregoing rectangle be the figure proposed; and let the length and breadth be divided into equal parts, each equal to the linear measuring unit, being here four for the length and three for the breadth; and let the opposite points of division be connected by right lines. Then it is evident that these lines divide the rectangle into a number of little squares, each equal to the square measuring unit E; and further, that the number of these

EXAMPLES.

Ex. 1. To find the area of a parallelogram whose length is 12·25, and height 8·5.

$$\begin{array}{r l} 12{\cdot}25 & \text{length.} \\ 8{\cdot}5 & \text{breadth.} \\ \hline 6125 & \\ 9800 & \\ \hline 104{\cdot}125 & \text{area.} \end{array}$$

Ex. 2. To find the area of a square whose side is 35·25 chains. *Ans.* 124 acres, 1 rood, 1 perch.

Ex. 3. To find the area of a rectangular board whose length is $12\frac{1}{2}$ feet, and breadth 9 inches.

Ans. $9\frac{3}{8}$ feet.

Ex. 4. To find the content of a piece of land in form of a rhombus, its length being 6·20 chains, and perpendicular height 5·45.

Ans. 3 acres, 1 rood, 20 perches.

Ex. 5. To find the number of square yards of painting in a rhomboid whose length is 37 feet, and breadth 5 feet 3 inches. *Ans.* $21\frac{7}{12}$ square yards.

PROBLEM II.

To find the area of a triangle.

Rule I. Multiply the base by the perpendicular height, and half the product will be the area.* Or, multiply the one of these dimensions by half the other.

little squares, or the area of the figure, is equal to the number of linear measuring units in the length, which is the same as the number of square units in a horizontal row, repeated as often as there are linear measuring units in the breadth or height, which is the same as the number of horizontal rows, that is here 4×3 or 12.

And it is proved (Geometry, theor. 22) that a rectangle is equal to any oblique parallelogram of equal length and perpendicular breadth. Therefore the rule is general for all parallelograms whatever.

* The truth of this rule is evident, because any triangle is the half of a parallelogram of equal base and altitude, by Geometry, Theor. 23.

EXAMPLES.

Ex. 1. To find the area of a triangle whose base is 625, and perpendicular height 520 links?*

Here $625 \times 260 = 162500$ square links,

or equal 1 acre, 2 roods, 20 perches, the answer.

Ex. 2. How many square yards contains the triangle, whose base is 40, and perpendicular 30 feet?

Ans. $66\frac{2}{3}$ square yards.

Ex. 3. To find the number of square yards in a triangle whose base is 49 feet, and height $25\frac{1}{4}$ feet.

Ans. $68\frac{5}{7}\frac{3}{2}$, or 68·7361.

Ex. 4. To find the area of a triangle whose base is 18 feet 4 inches, and height 11 feet 10 inches.

Ans. 108 feet, $5\frac{2}{3}$ inches.

Rule II. When two sides and their contained angle are given: Multiply the two given sides together, and take half their product: Then say, as radius is to the sine of the given angle, so is that half product to the area of the triangle.

Or, multiply that half product by the natural sine of the said angle.†

Ex. 1. What is the area of a triangle whose two sides are 30 and 40, and their contained angle 28° 57′ 18″?

Here $\frac{1}{2} \times 40 \times 30 = 600$,

Therefore, ·4841226 nat. sin. 28° 57′ 18″

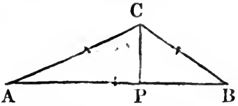
$$\begin{array}{r} \cdot 4841226 \\ 600 \\ \hline 290\cdot 47356 \end{array}$$

290·47356, the answer.

Ex. 2. How many square yards contains the tri-

* 100 links make a chain, 10,000 square links a square chain, and 10 square chains an acre.

† The following demonstration requires an acquaintance with Trigonometry. For, let AB, AC be the two given sides, including the given angle A. Now $\frac{1}{2}$AB $\times$ CP is the area, by the first rule, CP being perpendicular. But, by Trigonometry, CP = sine angle A $\times$ AC, taking radius = 1. Therefore, the area $\frac{1}{2}$AB $\times$ CP is $= \frac{1}{2}$AB $\times$ AC $\times$ sin. angle A, to radius 1; or,



as radius : sin. angle A : : $\frac{1}{2}$AB $\times$ AC : the area.

angle, of which one angle is 45°, and its containing sides 25 and $21\frac{1}{4}$ feet? *Ans.* 20·86947.

RULE III. When the three sides are given: Add all the three sides together, and take half that sum. Next, subtract each side severally from the said half sum, obtaining three remainders. Lastly, multiply the said half sum and those three remainders all together, and extract the square root of the last product for the area of the triangle.*

Ex. 1. To find the area of the triangle whose three sides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	25, first rem.	15, second rem.	5, third rem.
2)90			
45, half sum.			

Then $45 \times 25 \times 15 \times 5 = 84375$.

The root of which is 290·4737, the area.

Ex. 2. How many square yards of plastering are in a triangle whose sides are 30, 40, 50?

Ans. $66\frac{2}{3}$.

Ex. 3. How many acres, &c., contains the triangle whose sides are 2569, 4900, 5025 links?

Ans. 61 acres, 1 rood, 39 perches.

PROBLEM. III.

To find the area of a trapezoid.

Add together the two parallel sides; then multiply

* For, let a, b, c denote the sides opposite respectively to A, B, C, the angles of the triangle A B C (see last figure); then, by theor. 29, Geom., we have $BC^2 = AB^2 + AC^2 - 2AB.AP$, or $a^2 = b^2 + c^2 - 2c.AP \therefore AP = \frac{b^2 + c^2 - a^2}{2c}$; hence we have

$$CP^2 = b^2 - \frac{(b^2+c^2-a^2)^2}{4c^2} = \frac{4b^2c^2-(b^2+c^2-a^2)^2}{4c^2} = \frac{(2bc+b^2+c^2-a^2)\cdot(2bc-b^2-c^2+a^2)}{4c^2}$$

$$\therefore 4c^2.CP^2 = \left\{(b+c)^2-a^2\right\}\cdot\left\{a^2-(c-b)^2\right\} = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$

$$\therefore \frac{1}{2}AB.CP = \frac{1}{2}c.CP = \sqrt{\left\{\frac{a+b+c}{2}\cdot\frac{-a+b+c}{2}\cdot\frac{a-b+c}{2}\cdot\frac{a+b-c}{2}\right\}} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c) =$ half the sum of the three sides.

their sum by the perpendicular breadth or distance between them; and half the product will be the area, by Geometry, theorem 25.

Ex. 1. In a trapezoid the parallel sides are 750 and 1225, and the perpendicular distance between them 1540 links: to find the area.

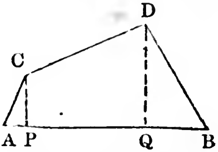
1225
750

$1975 \times 770 = 152075$ sq. links = 15 acres, 33 perches.

Ex. 2. How many square feet are contained in the plank whose length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the less end 11 inches? *Ans.* $13\frac{13}{24}$ feet.

Ex. 3. In measuring along one side AB of a quadrangular field, that side and the two perpendiculars let fall on it from the two opposite corners, measured as below: required the content.

AP = 110 links.
AQ = 745 "
AB = 1110 "
CP = 352 "
DQ = 595 "



Ans. 4 acres, 1 rood, 5·792 perches.

PROBLEM IV.

To find the area of any trapezium.

Divide the trapezium into two triangles by a diagonal: then find the areas of these triangles, and add them together.

Note. If two perpendiculars be let fall on the diagonal, from the other two opposite angles, the sum of these perpendiculars being multiplied by the diagonal, half the product will be the area of the trapezium.

Ex. 1. To find the area of the trapezium whose diagonal is 42, and the two perpendiculars on it 16 and 18.

Here $16 + 18 = 34$; its half is 17.
Then $42 \times 17 = 714$, the area.

Ex. 2. How many square yards of paving are in the trapezium whose diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet?

Ans. $222\frac{1}{12}$ yards.

Ex. 3. In the quadrangular field ABCD, on account of obstructions, there could only be taken the following measures, viz.: the two sides BC 265, and AD 220 yards, the diagonal AC 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, AE 100, and CF 70 yards. Required the area in acres when 4840 square yards make an acre.

Ans. 17 acres, 2 roods, 21 perches.

PROBLEM V.

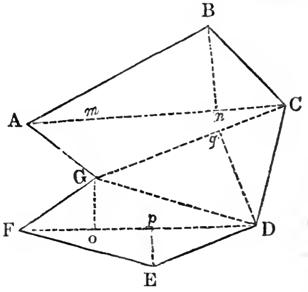
To find the area of an irregular polygon.

Draw diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of all these separately, and add them together for the content of the whole polygon.

Ex. To find the content of the irregular figure ABCDEFGA, in which are given the following diagonals and perpendiculars, namely:

AC 55
FD 52
GC 44
Gm 13
Bn 18
Go 12
Ep 8
Dq 23

Ans. 1878·5.



PROBLEM VI.

To find the area of a regular polygon.

Rule I. Multiply the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its center on one of its sides, and take half the product for the area.*

* The demonstration of this is given in th. 73.

Ex. 1. To find the area of the regular pentagon, each side being 25 feet, and the perpendicular from the center on each side 17·2047737.

Here $25 \times 5 = 125$, is the perimeter.

And $17{\cdot}2047737 \times 125 = 2150{\cdot}5967125$.

Its half, 1075·298356, is the area sought.

Rule II. Square the side of the polygon; then multiply that square by the area or multiplier set against its name in the following table, and the product will be the area.*

No. of Sides.	Names.	Areas or Multipliers.
3	Trigon, or triangle	0·4330127
4	Tetragon, or square	1·0000000
5	Pentagon	1·7204774
6	Hexagon	2·5980762
7	Heptagon	3·6339124
8	Octagon	4·8284271
9	Nonagon	6·1818242
10	Decagon	7·6942088
11	Undecagon	9·3656399
12	Dodecagon	11·1961524

Ex. Taking here the same example as before, namely, a pentagon, whose side is 25 feet.

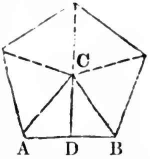
Then, 25^2 being $= 625$,

And the tabular area 1·7204774;

Therefore, $1{\cdot}7204774 \times 625 = 1075{\cdot}298375$, as before.

* This rule is founded on the property that regular polygons of the same number of sides, being similar figures, are as the squares of their sides. Now the multipliers in the table are the areas of the respective polygons to the side 1. Whence the rule is manifest.

Note. The areas in the table, to each side 1, may be computed in the following manner, with the aid of plane trigonometry: From the center C of the polygon draw lines to every angle, dividing the whole figure into as many equal triangles as the polygon has sides; and let ABC be one of those triangles, the perpendicular of which is CD. Divide 360 degrees by the number of sides in the polygon, the quotient gives the angle at the center ACB. The half of this gives the angle ACD; and this taken from 90°, leaves the angle CAD. Then, as radius is to AD, so is tangent angle CAD to the perpendicular CD. This, multiplied by AD, gives the area of the triangle ABC; which, being multiplied by the number of the triangles, or of the sides of the polygon, gives its whole area, as in the table.



Ex. 2. To find the area of the trigon, or equilateral triangle, whose side is 20. *Ans.* 173·20508.

Ex. 3. To find the area of a hexagon whose side is 20. *Ans.* 1039·23048.

Ex. 4. To find the area of an octagon whose side is 20. *Ans.* 1931·37084.

Ex. 5. To find the area of a decagon whose side is 20. *Ans.* 3077·68352.

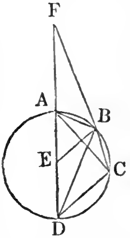
PROBLEM VII.

To find the diameter and circumference of any circle, the one from the other.

This may be done nearly by either of the two following proportions, viz. :

As 7 is to 22, so is the diameter to the circumference.

Or, as 1 is to 3·1416, so is the diameter to the circumference.*



* For, let ABCD be any circle whose center is E, and let AB, BC be any two equal arcs. Draw the several chords as in the figure, and join BE ; also, draw the diameter DA, which produce to F,* till BF be equal to the chord BD.

Then the two isosceles triangles DEB, DBF are equiangular, because they have the angle at D common; consequently, DE : DB : : DB : DF. But the two triangles AFB, DCB are identical, or equal in all respects, because they have the angle F = the angle BDC, being each equal the angle ADB (see th. 39, cor. 1); also, the exterior angle FAB of the quadrangle ABCD is equal the opposite interior angle at C (exercise 32, p. 48); and the two triangles have, also, the side BF = the side BD; therefore, the side AF is also equal the side DC. Hence the proportion above, viz., DE : DB : : DB : DF = DA + AF, becomes DE : DB : : DB : 2DE + DC. Then, by taking the rectangles of the extremes and means, it is $DB^2 = 2DE^2 + DE \cdot DC$.

Now if the radius DE be taken = 1, this expression becomes $DB^2 = 2 + DC$; and hence $DB = \sqrt{2 + DC}$. That is, if the measure of the supplemental† chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of the circle, let the arc AC be taken equal to one sixth of the circumference,

* The point F may be found by describing an arc with B as center, and radius = BD.

† The supplemental chord is the chord of the supplement.

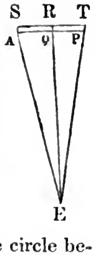
Ex. 1. To find the circumference of the circle whose diameter is 20.

and be successively bisected by the above theorem: thus, the chord AC of one sixth of the circumference is the side of the inscribed regular hexagon, and is, therefore, equal the radius AE or 1; hence, in the right-angled triangle ACD, we shall have $DC = \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1{\cdot}7320508076$, the supplemental chord of one sixth of the periphery.

Then, by the foregoing theorem, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th, the 48th, the 96th, &c., parts of the periphery; thus,

$$\left.\begin{array}{l}\sqrt{3{\cdot}7320508076} = 1{\cdot}9318516525\\ \sqrt{3{\cdot}9318516525} = 1{\cdot}9828897227\\ \sqrt{3{\cdot}9828897227} = 1{\cdot}9957178465\\ \sqrt{3{\cdot}9957178465} = 1{\cdot}9989291743\\ \sqrt{3{\cdot}9989291743} = 1{\cdot}9997322757\\ \sqrt{3{\cdot}9997322757} = 1{\cdot}9999330678\\ \sqrt{3{\cdot}9999330678} = 1{\cdot}9999832669\\ \sqrt{3{\cdot}9999832669} = \ldots\ldots\ldots\end{array}\right\}\begin{array}{c}\text{for the}\\ \text{supplemental}\\ \text{chord of}\end{array}\left\{\begin{array}{c}\frac{1}{12}\\ \frac{1}{24}\\ \frac{1}{48}\\ \frac{1}{96}\\ \frac{1}{192}\\ \frac{1}{384}\\ \frac{1}{768}\\ \frac{1}{1536}\end{array}\right\}\begin{array}{c}\text{of the}\\ \text{periphery.}\end{array}$$

Since, then, it is found that 3·9999832669 is the square of the supplemental chord of the 1536th part of the periphery, let this number be taken from 4, the square of the diameter, and the remainder 0·0000167331 will be the square of the chord of the said 1536th part of the periphery, and, consequently, the root $\sqrt{0{\cdot}0000167331} = 0{\cdot}0040906112$ is the length of that chord; this number, then, being multiplied by 1536, gives 6·2831788 for the perimeter of a regular polygon of 1536 sides inscribed in the circle; which, as the sides of the polygon nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.



But now, to show how near this determination is to the truth, let $AQP = 0{\cdot}0040906112$ represent one side of such a regular polygon of 1536 sides, and SRT a side of another similar polygon described about the circle; and from the center E let the perpendicular EQR be drawn, bisecting AP and ST in Q and R. Then, since AQ is $= \frac{1}{2}AP = 0{\cdot}0020453056$, and $EA = 1$, therefore $EQ^2 = EA^2 - AQ^2 = {\cdot}9999958167$, and, consequently, its root gives $EQ = {\cdot}9999979084$; then, because of the parallels AP, ST, we have the proportion EQ : ER :: AP : ST :: the whole inscribed perimeter : the circumscribed one; that is, as ·9999979084 : 1 :: 6·2831788 : 6·2831920, the perimeter of the circumscribed polygon. But the circumference of the circle being greater than the perimeter of the inner polygon, and less than that of the outer, it must, consequently, be greater than 6·2831788,
but less than 6·2831920,
and must, therefore, be nearly equal half their sum or a mean between them, or 6·2831854, which, in fact, is true to the last figure, which should be a 3 instead of the 4.

By the first rule, as $7:22::20:62\frac{6}{7}$, the answer.

Ex. 2. If the circumference of the earth be 25,000 miles, what is its diameter?

By the 2d rule, as $3.1416:1::25000:7957\frac{3}{4}$, nearly the diameter.

Corol. To find the circumference from the radius, multiply the latter by 6·2832.

PROBLEM VIII.

To find the length of any arc of a circle.

Multiply the degrees in the given arc by the radius of the circle, and the product, again, by the decimal ·01745, for the length of the arc.*

Ex. 1. To find the length of an arc of 30 degrees, the radius being 9 feet. *Ans.* 4·7115.

Ex. 2. To find the length of an arc of $12^\circ\ 10'$, or $12^\circ\frac{1}{6}$, the radius being 10 feet. *Ans.* 2·1231.

PROBLEM IX.

To find the area of a circle.

Rule I.† Multiply half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take $\frac{1}{4}$ of the product.

Hence the circumference being 6·2831854 when the diameter is 2, it will be the half of that, or 3·1415927, when the diameter is 1 (th. 71), to which the ratio in the rule, viz., 1 to 3·1416, is very near. Also, the other ratio in the rule, 7 to 22, or 1 to $3\frac{1}{7} = 3{\cdot}1428$, &c., is another near approximation.

* It having been found, in the demonstration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumference is 6·2831854, which consists of 360 degrees; therefore, as $360^\circ : 6{\cdot}2831854 :: 1^\circ : {\cdot}01745$, &c., the length of the arc of 1 degree. Hence the number ·01745, multiplied by any number of degrees, will give the length of the arc of those degrees. And, because the circumferences and arcs are as the diameters, or radii of the circles; therefore, as the radius 1 is to any other radius r, so is the length of the arc above mentioned to $r \times {\cdot}01745 \times$ degrees in the arc, which is the length of that arc, as in the rule.

† This first rule is proved in the Geometry, theor. 73.

And the second rule is derived from formula (3), schol., of the same theorem, π being 3·1416. Rule 3 is derived from the same formula, observing that $\pi r^2 = \frac{1}{4}\pi d^2$, and $\frac{1}{4}\pi = {\cdot}7854$.

RULE II. Square the radius, and multiply that square by 3·1416.

RULE III. Square the diameter, and multiply that square by the decimal ·7854, for the area.

Ex. 1. To find the area of a circle whose diameter is 10, and its circumference 31·416.

By Rule 1.

$$\begin{array}{r} 31{\cdot}416 \\ 10 \\ \hline 4)\overline{314{\cdot}16} \\ 78{\cdot}54, \text{ the area.} \end{array}$$

By Rule 3.

$$\begin{array}{r} {\cdot}7854 \\ 100 = 10^2 \\ \hline 78{\cdot}54 \end{array}$$

Ex. 2. To find the area of a circle whose diameter is 7, and circumference 22. *Ans.* $38\frac{1}{2}$.

Ex. 3. How many square yards are in a circle whose diameter is $3\frac{1}{2}$ feet? *Ans.* 1·069.

Ex. 4. Find the area of a circle whose radius is 10. *Ans.* 314·16.

PROBLEM X.

To find the area of a circular ring or space included between two concentric circles.

Take the difference between the areas of the two circles, as found by the last problem. Or, since circles are as the squares of their diameters, subtract the square of the less diameter from the square of the greater, and multiply their difference by ·7854. Or, lastly, multiply the sum of the diameters by the difference of the same, and that product by ·7854;* which is still the same thing, because the product of the sum and difference of any two quantities is equal to the difference of their squares.

Ex. 1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here $10 + 6 = 16$, the sum; and $10 - 6 = 4$, the difference.

Therefore $\cdot 7854 \times 16 \times 4 = \cdot 7854 \times 64 = 50{\cdot}2656$, the area.

* In this last method logarithms may be advantageously applied.

Ex. 2. What is the area of the ring, the diameters of whose bounding circles are 10 and 20? *Ans.* 235·62.

PROBLEM XI.

To find the area of the sector of a circle.

Rule I. Multiply the radius, or half the diameter, by half the arc of the sector, for the area. Or, multiply the whole diameter by the whole arc of the sector, and take one quarter of the product. The reason for which is, that the sector bears the same proportion to the whole circle that its arc does to the whole circumference.

Rule II. As 360 is to the degrees in the arc of the sector, so is the area of the whole circle to the area of the sector.

This is evident, because the sector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1. To find the area of a circular sector whose arc contains 18 degrees, the diameter being 3 feet.

1. By the 1st Rule.

First, $3{\cdot}1416 \times 3 = 9{\cdot}4248$, the circumference.

And $360 : 18 :: 9{\cdot}4248 : {\cdot}47124$, the length of the arc.

Then ${\cdot}47124 \times 3 \div 4 = {\cdot}11781 \times 3 = {\cdot}35343$, the area.

2. By the 2d Rule.

First, ${\cdot}7854 \times 3^2 = 7{\cdot}0686$, the area of the whole circle.

Then, as $360 : 18 :: 7{\cdot}0686 : {\cdot}35343$, the area of the sector.

Ex. 2. To find the area of a sector whose radius is 10, and arc 20. *Ans.* 100.

Ex. 3. Required the area of a sector whose radius is 25, and its arc containing 147° 29′.

Ans. 804·4017.

PROBLEM XII.

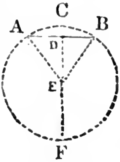
To find the area of a segment of a circle.

Rule I. Find the area of the sector having the same arc with the segment, by the last problem.

Find, also, the area of the triangle formed by the chord of the segment and the two radii of the sector.

Then take the sum of these two for the answer, when the segment is greater than a semicircle: or take their difference for the answer, when it is less than a semicircle; as is evident by inspection.

Ex. 1. To find the area of the segment ACBDA, its chord AB being 12, and the radius AE or CE 10.



*First, AD $\div$ AE = sin. angle D = sin. $36° \; 52'\frac{1}{5} = 36{\cdot}87$ degrees, the degrees in the angle AEC or arc AC. Their double, 73·74, are the degrees in the whole arc ACB.

Now $\cdot 7854 \times 400 = 314{\cdot}16$, the area of the whole circle.

Therefore $360° : 73{\cdot}74 :: 314{\cdot}16 : 64{\cdot}3504$, area of the whole sector ACBE.

Again, $\sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 =$ DE.

Therefore, $AD \times DE = 6 \times 8 = 48$, the area of the triangle AEB.

Hence sector ACBE — triangle AEB = 16·3504, area of seg. ACBDA.

Rule II. Divide the height of the segment by the diameter, and find the quotient in the column of heights in the following tablet: Take out the corresponding area in the next column on the right hand, and multiply it by the square of the circle's diameter, for the area of the segment.†

Note. When the quotient is not found exactly in the

* This requires a knowledge of plane trigonometry.

† The truth of this rule depends on the principle of similar plane figures, which have the ratio of their like lines (as the height and radius of a segment) equal, and are to one another as the square of their like linear dimensions. The segments in the table are those of a circle whose diameter is 1; and the first column contains the quotients of corresponding heights, or versed sines, divided by the diameter, which are the same for similar segments of all diameters. Thus, then, the area of the similar segment, taken from the table, and multiplied by the square of the diameter, gives the area of the segment corresponding to this diameter.

table, proportion may be made between the next less and greater area, in the same manner as is done for logarithms or any other table.

TABLE OF THE AREAS OF CIRCULAR SEGMENTS.

Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.	Height.	Area of the segment.
·01	·00133	·11	·04701	·21	·11990	·31	·20738	·41	·30319
·02	·00375	·12	·05339	·22	·12811	·32	·21667	·42	·31304
·03	·00687	·13	·06000	·23	·13646	·33	·22603	·43	·32293
·04	·01054	·14	·06683	·24	·14494	·34	·23547	·44	·33284
·05	·01468	·15	·07387	·25	·15354	·35	·24498	·45	·34278
·06	·01924	·16	·08111	·26	·16226	·36	·25455	·46	·35274
·07	·02417	·17	·08853	·27	·17109	·37	·26418	·47	·36272
·08	·02944	·18	·09613	·28	·18002	·38	·27386	·48	·37270
·09	·03502	·19	·10390	·29	·18905	·39	·28359	·49	·38270
·10	·04088	·20	·11182	·30	·19817	·40	·29337	·50	·39270

Ex. 2. Taking the same example as before, in which are given the chord AB 12, and the radius 10, or diameter 20.

And having found, as above, $DE = 8$; then $CE - DE = CD = 10 - 8 = 2$. Hence, by the rule, $CD \div CF = 2 \div 20 = \cdot 1$, the tabular height. This being found in the first column of the table, the corresponding tabular area is ·04088. Then $\cdot 04088 \times 20^2 = \cdot 04088 \times 400 = 16\cdot 352$, the area, nearly the same as before.

Ex. 3. What is the area of the segment whose height is 18, and diameter of the circle 50?

Ans. 636·375.

Ex. 4. Required the area of the segment whose chord is 16, the diameter being 20.

Ans. 44·7292.

PROBLEM XIII.

To measure long irregular figures.

Take or measure the breadth in several places at equal distances; then add all these breadths together, and divide the sum by the number of them for the mean breadth, which multiply by the length for the area.*

Note 1. Take half the sum of the extreme breadths for one of the said breadths.

Note 2. If the perpendiculars or breadths be not at equal distances, compute all the parts separately as so many trapezoids, and add them all together for the whole area.

Or else add all the perpendicular breadths together, and divide their sum by the number of them for the mean breadth, to multiply by the length; which will give the whole area not far from the truth.

Ex. 1. The breadths of an irregular figure, at five equidistant places, being 8·2, 7·4, 9·2, 10·2, 8·6; and the whole length 39: required the area.

First, $(8{\cdot}2 + 8{\cdot}6) \div 2 = 8{\cdot}4$, the mean of the two extremes.

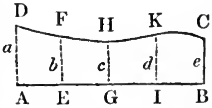
Then $8{\cdot}4 + 7.4 + 9{\cdot}2 + 10{\cdot}2 = 35{\cdot}2$, sum of breadths.

And $35{\cdot}2 \div 4 = 8{\cdot}8$, the mean breadth.

Hence $8{\cdot}8, \times 39 = 343{\cdot}2$, the answer.

Ex. 2. The length of an irregular figure being 84, and the breadths at six equidistant places, 17·4, 20·6, 14·2, 16·5, 20·1, 24·4; what is the area?

Ans. 1550·64.



* This rule is made out as follows: Let ABCD be the irregular piece, having the several breadths AD, EF, GH, IK, BC at the equal distances AE, EG, GI, IB. Let the several breadths in order be denoted by the corresponding letters a, b, c, d, e, and the whole length AB by l; then compute the areas of the parts into which the figure is divided by the perpendiculars, as so many trapezoids by Problem 3, and add them all together. Thus the sum of the parts is,

$$\frac{a+b}{2} \times \mathrm{AE} + \frac{b+c}{2} \times \mathrm{EG} + \frac{c+d}{2} \times \mathrm{GI} + \frac{d+e}{2} \times \mathrm{IB};$$

$$= \frac{a+b}{2} \times \tfrac{1}{4}l + \frac{b+c}{2} \times \tfrac{1}{4}l + \frac{c+d}{2} \times \tfrac{1}{4}l + \frac{d+e}{2} \times \tfrac{1}{4}l;$$

$$= (\tfrac{1}{2}a + b + c + d + \tfrac{1}{2}e) \times \tfrac{1}{4}l = (m + b + c + d)\,\tfrac{1}{4}l,$$

which is the whole area, agreeing with the rule; m being the arithmetic mean between the extremes and 4 the number of the parts. And the same for any other number of parts.

MENSURATION OF SOLIDS.

By the Mensuration of Solids are determined the spaces included by contiguous surfaces; and the sum of the measures of these including surfaces is the whole Surface or Superficies of the body.

The measure of a solid is called its solidity, capacity, or content. A better term is *volume.*

Solids are measured by cubes, whose sides are inches, or feet, or yards, &c. And hence the volume of a body is said to be so many cubic inches, feet, yards, &c., as will fill its capacity or space, or another of equal magnitude.

The least ordinary solid measure, or measure of volume, is the cubic inch, other cubes being taken from it according to the proportion in the following table:

Table of Cubic or Solid Measures.

1728 cubic inches make . .	1 cubic foot.
27 cubic feet make . .	1 cubic yard.
$166\frac{3}{8}$ cubic yards make . .	1 cubic pole.
64000 cubic poles make . .	1 cubic furlong.
512 cubic furlongs make .	1 cubic mile.

PROBLEM I.

To find the superficies of a prism.

Multiply the perimeter of one end of the prism by the altitude of one of the parallelograms, and the product will be the lateral surface. To which add, also, the area of the two ends of the prism, when required.*

* And the rule is evidently the same for the surface of a cylinder, which may be regarded as a prism of an infinite number of lateral faces.

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1. To find the surface of a cube, the length of each side being 20 feet. *Ans.* 2400 feet.

Ex. 2. To find the whole surface of a triangular prism whose length is 20 feet and each side of its end or base 18 inches. *Ans.* 91·948 feet.

Ex. 3. To find the convex surface of a round prism, or cylinder, whose length is 20 feet and diameter of its base is 2 feet. *Ans.* 125·664.

Ex. 4. What must be paid for lining a rectangular cistern with lead at 2*d.* a pound weight, the thickness of the lead being such as to weigh 7 lbs. for each square foot of surface; the inside dimensions of the cistern being as follows, viz., the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches? *Ans.* £2 3*s.* $10\frac{1}{2}$*d.*

To find the superficies of an irregular polyhedron.

Find the superficies of each of its bounding polygonal faces, and add the results.

To find the superficies of a regular polyhedron.

Find the area of one of its faces by Prob. VI., and multiply this by the number of faces.

PROBLEM II.

To find the surface of a regular pyramid or cone.

Multiply the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the convex surface or the sum of the areas of all the triangles which form it. To which add the area of the end or base, if requisite.

Note. The slant height of a regular pyramid is the perpendicular from the vertex to the middle of one of the sides of the base.

Ex. 1. What is the upright surface of a triangular pyramid, the slant height being 20 feet, and each side of the base 3 feet? *Ans.* 90 feet.

Ex. 2. Required the convex surface of a cone, or circular pyramid, the slant height being 50 feet, and the diameter of its base $8\frac{1}{2}$ feet. *Ans.* 667·59.

PROBLEM III.

To find the surface of the frustum of a regular pyramid or cone, being the lower part, when the top is cut off by a plane parallel to the base.

Rule I. Add together the perimeters of the two ends, and multiply their sum by the slant height, taking half the product for the answer. Because the lateral faces of the frustum of a pyramid are trapezoids, having their opposite sides parallel, and the frustum of a cone is the frustum of a pyramid of an infinite number of lateral faces.

Rule II. Multiply the perimeter of the section midway between the two bases by the slant height. This depends upon the fact that the perimeter of the middle section is half the sum of the perimeters of the bases, as may be easily shown.

Ex. 1. How many square feet are in the surface of the frustum of a square pyramid whose slant height is 10 feet; also, each side of the base or greater end being 3 feet 4 inches, and each side of the less end 2 feet 2 inches? *Ans.* 110 feet.

Ex. 2. To find the convex surface of the frustum of a cone, the slant height of the frustum being $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4. *Ans.* 90 feet.

PROBLEM IV.

To find the volume of any prism or cylinder.

Find the area of the base, or end, whatever the figure of it may be; and multiply it by the altitude* of the prism, or cylinder, for the volume.

* The altitude of a prism is the perpendicular distance between its parallel bases. The cylinder, as well as the prism, may be oblique. Prop. 3 of Solid Geom., upon which, with the note to Prop. 6 of the same, the demonstration of this depends, may evidently be extended to an oblique cylinder.

Ex. 1. Find the solid content of a cube whose side is 24 inches. *Ans.* 13824.

Ex. 2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thickness 2 feet 6 inches? *Ans.* $21\frac{1}{9}$.

Ex. 3. How many gallons of water will the cistern contain whose dimensions are the same as in the last example, when 277·274 cubic inches are contained in one gallon? *Ans.* 131·566.

Ex. 4. Required the solidity of a triangular prism whose length is 10 feet, and the three sides of its triangular end or base are 3, 4, 5 feet. *Ans.* 60.

Ex. 5. Required the content of a round pillar, or cylinder, whose length is 20 feet, and circumference 5 feet 6 inches. *Ans.* 48·1459.

PROBLEM V.

To find the volume of any pyramid or cone.

Find the area of the base, and multiply that area by the perpendicular height; then take one third of the product for the volume. (See Prop. XI., Solid Geom., and corollaries.)

Ex. 1. Required the solidity of the square pyramid, each side of its base being 30, and its perpendicular height 25. *Ans.* 7500.

Ex. 2. To find the content of a triangular pyramid whose perpendicular height is 30, and each side of the base 3. *Ans.* 38·97117.

Ex. 3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three sides of its base 5, 6, 7. *Ans.* 71·0352.

Ex. 4. What is the content of a pentagonal pyramid, its height being 12 feet, and each side of its base 2 feet? *Ans.* 27·5276.

Ex. 5. What is the content of the hexagonal pyramid whose height is 6·4, and each side of its base 6 inches? *Ans.* 1·38564 feet.

Ex. 6. Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its base 9 feet. *Ans.* 22·56093.

PROBLEM VI.

To find the volume of the frustum of a cone or pyramid.

RULE I. Add into one sum the areas of the two ends, and the mean proportional between them, or the square root of the product, and one third of that sum will be a mean area; which, being multiplied by the perpendicular height or length of the frustum, will give its content.

RULE II. For the cone. Add together the squares of the radii of the two bases and their product, and multiply the sum by 3·1416, and the product by one third of the altitude. (See Prop. XII., Solid Geom., and corol.)

Ex. 1. To find the number of solid feet in a piece of timber whose bases are squares, each side of the greater end being 15 inches, and each side of the less end 6 inches; also, the length or perpendicular altitude 24 feet? *Ans.* $19\frac{1}{2}$.

Ex. 2. Required the content of a pentagonal frustum whose height is 5 feet, each side of the base 18 inches, and each side of the top or less end 6 inches. *Ans.* 9·31925 feet.

Ex. 3. To find the content of a conic frustum, the altitude being 18, the greatest diameter 8, and the least diameter 4. *Ans.* 527·7888.

Ex. 4. What is the volume of the frustum of a cone, the altitude being 25; also, the circumference at the greater end being 20, and at the less end 10? *Ans.* 464·216.

Ex. 5. If a cask, which is two equal conic frustums joined together at the bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches, how many gallons of wine will it hold? *Ans.* 79·0613.

PROBLEM VII.

To find the surface of a sphere, or any segment.

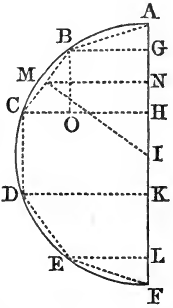
RULE I. Multiply the circumference of the sphere by its diameter, and the product will be the whole surface of it.*

RULE II. Multiply the square of the diameter by 3·1416, and the product will be the surface.

Note. For the surface of a segment, multiply the circumference of a great circle of the sphere by the altitude of the segment.

Ex. 1. Required the convex superficies of a sphere whose diameter is 7, and circumference 22.

Ans. 154.



* For if a regular semi-polygon be revolved about a diameter of the figure, each of the trapezoids, as BGHC, will describe the frustum of a cone, the convex surface of which will be measured by the circumference of MN, described by the middle point of its inclined side, multiplied by the slant height BC (Prob. 3, Rule 2). But by the similarity of the triangles IMN and BCO, whose sides are respectively perpendicular,

BC : BO : : IM : MN : circum. IM : circum. MN (Geom., th. 71).

$\therefore$ BC $\times$ circum. MN $=$ BO $\times$ circum. IM.

In the same manner, the convex surface of the frustum described by the revolution of the trapezoid HCDK may be shown to be measured by HK $\times$ circum. IM. Of that described by the revolution of DKLE by KL $\times$ circum. IM. And, by addition, the surface described by the portion of the perimeter BCDE is measured by GL $\times$ circum. IM. The same result will be obtained when the number of sides of the semi-polygon is infinite and it becomes a semicircle, generating a sphere by its revolution; and the portion BCDE generating a zone, of which GL is the altitude. The circum. IM in this case becomes the circumference of a great circle of the sphere. When the whole semi-polygon or semicircle revolves, the altitude becomes the diameter AF, and the surface is measured by the circumference of a great circle multiplied by its diameter. This is equal to four times the area of a great circle (see th. 73, Geom.).

Corol. The convex surface of a cylinder circumscribing a sphere is measured by the rectangle of the circumference of the base by the altitude, which, being equal to the diameter of the sphere, and the base of the cylinder equal a great circle, it follows that the measure of the surface of the sphere is equal to that of the convex surface of the cylinder. If now we add the two bases of the cylinder, since the surface of the sphere is equal to four great circles, we shall have the surface of the cylinder equal six great circles, so that the surfaces of the sphere and circumscribed cylinder are as 4 to 6, or as 2 to 3. Rule 2 follows obviously from Rule 1.

Ex. 2. Required the superficies of a globe whose diameter is 24 inches. *Ans.* 1809·5616.

Ex. 3. Required the area of the whole surface of the earth, its diameter being $7957\frac{3}{4}$ miles, and its circumference 25000 miles.

Ans. 198943750 sq. miles.

Ex. 4. The axis of a sphere being 42 inches, what is the convex superficies of the segment whose height is 9 inches? *Ans.* 1187·5248 inches.

Ex. 5. Required the convex surface of a spherical zone whose breadth or height is 2 feet, and cut from a sphere of $12\frac{1}{2}$ feet diameter. *Ans.* 78·54 feet.

PROBLEM VIII.

To find the surface of a lune.

Multiply the arc which measures the angle of the lune by the diameter of the sphere. For the lune is to the whole surface of the sphere as its arc is to a circumference.

Cor. 1. The measure of a spherical wedge, or *ungula*, is for a similar reason the product of the lune which serves for its base, multiplied by one third the radius of the sphere (see next Prob.).

Cor. 2. The measure of a *spherical triangle* is the arc of a great circle subtending *half* the excess of the sum of its angles over two right angles, multiplied by the diameter of the sphere. This depends on the above and Prop. XVII., cor. 1, Spher. Geom.

The measure of the surface of a *spherical polygon* is the arc of a great circle subtending *half* the excess of the sum of its angles over as many times two right angles as the figure has sides, wanting two, multiplied by the diameter of the sphere.*

Or in symbols, s denoting the sum of the angles of the polygon in fractions of a right angle, n the number of its sides,

$$[s-2\,(n-2)] \div 8,$$

* This may be easily proved by dividing the polygon into triangles.

will express the fraction which the polygon is of the whole surface of the sphere.

Ex. 1. Required the surface of the lune whose arc is 8 and diameter 10. *Ans.* $8 \times 10 = 80$.

Ex. 2. Required the measure of the lune whose angle is 30° 20′, and diameter 12.

Ans. $12 \times 3{\cdot}1416 \times 12 \times \frac{30° \, 21'}{360°} = 38{\cdot}139$.*

Ex. 3. Required the area of a spherical triangle, of which the three angles are 30°, 100°, and 80°, the diameter of the sphere being 40.

Ans. $\frac{15°}{360°} \times 3{\cdot}1416 \times 40 \times 40$.

Ex. 4. Required the area of a spherical pentagon, the angles of which are 60°, 110°, 150°, 160°, and 100°, the diameter of the sphere being 50.

Ans. $\{\frac{1}{2}[60+110+150+160+100-(5-2)\,180°] \div 360\} \times 3{\cdot}1416 \times 50 \times 50$.

Ex. 5. What fraction of the whole surface of a sphere is a spherical heptagon, the angles of which are in fractions of a right angle $1\frac{1}{2}$, $1\frac{7}{8}$, $1\frac{3}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{3}{4}$, $1\frac{1}{4}$.

Ans. $\frac{11\frac{3}{8}-10}{8} = 1\frac{3}{8} \div 8 = \frac{11}{64}$.

PROBLEM IX.

To find the volume of a sphere or globe.

Rule I. Multiply the surface by the diameter, and take one sixth of the product for the content.

Rule II. Multiply the cube of the diameter by the decimal ·5236 for the content.

Ex. 1. To find the content of a sphere whose axis is 12. *Ans.* 904·7808.

Ex. 2. To find the solid content of the globe of the earth, supposing its circumference to be 25,000 miles.

Ans. 263,857,437,760 miles.

* This answer is of the same denomination as the diameter, except that it is square units instead of linear. Logarithms may here be conveniently applied, using the arithmetical complement of the logarithm of the divisor 360° reduced to minutes.

PROBLEM X.

To find the volume of a spherical sector.

Multiply the area of the zone which serves for its base by one third of the radius of the sphere.*

Ex. 1. Required the volume of a spherical sector, the altitude of the zone which serves for a base being 12, and the diameter of the sphere being 30.

Ans. $12 \times 30 \times 3{\cdot}1416 \times 5$.

Ex. 2. Required the volume of a spherical sector, a great section of the zone base being an arc of 40°, and the diameter of the sphere being 100.

PROBLEM XI.

To find the volume of a spherical segment.

Rule I. From three times the diameter of the sphere take double the height of the segment; then multiply the remainder by the square of the height, and the product by the decimal ·5236 for the content. (See Schol. to Prop. XIV., Sol. Geom.)

Rule II. To three times the square of the radius of the segment's base add the square of its height; then multiply the sum by the height, and the product by ·5236, for the content.

Rule III. When the segment has two bases, multiply the half sum of the parallel bases by the altitude, and add the volume of the sphere of which this altitude is the diameter.

* The spherical sector may be supposed to be made up of an infinite number of indefinitely small cones, each having an evanescent portion of the surface of the zone base for a base, and the radius of the sphere for an altitude. The sum of these will be measured by the sum of their bases, or the zone multiplied by one third their com mon altitude, or the radius of the sphere.

When the zone becomes the whole surface of the sphere the sector becomes the whole solid sphere. Note that one third the radius is one sixth the diameter.

For, Rule 2, observe that $\pi d^2 =$ surface of sphere (Prob. 7), and $\frac{1}{6}\pi = {\cdot}5236$. The rule is similar for a spherical pyramid having a spherical polygon for a base and the center of the sphere for a vertex.

Ex. 1. To find the content of a spherical segment of two feet in height cut from a sphere of 8 feet in diameter. *Ans.* 41·888.

Ex. 2. What is the solidity of the segment of a sphere, its height being 9, and the diameter of its base 20? *Ans.* 1795·4244.

EXERCISES IN MENSURATION.*

1. Transform a given parallelogram into another of double the altitude which shall have a given angle.

2. To transform a triangle into another of the same base and given vertical angle.

3. To construct a triangle of given base, vertical angle, and area.

4. The same, except the altitude instead of the base given.

5. To construct a triangle similar to a given triangle, and equal to a given square.

6. A triangle with given angles at the base, and equal to a given rectangle.

7. The same, when the base, vertical angle, and rectangle of the other two sides are given.

8. The same, when the base, the altitude, and the product of the two sides.

9. The same, when the altitude, the area, and the ratio of one of the sides to the base.

10. The same, when the ratio of the base and altitude, the vertical angle and the area.

11. Make a regular hexagon equivalent to a given polygon.

12. To construct a figure similar to a given figure, and its area having to that of the given figure a given ratio.

13. A quadrilateral capable of being inscribed, in which two adjacent angles, the angle which its diagonals make with each other and its area, are given.

14. A quadrilateral that may be inscribed, in which three angles and the area are given.

15. A circle equal to the sum of several circles.

16. A square in a given semicircle.

17. A circle equal to the ring between two circles.

18. A quadrant equal to a given semicircle.

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* Many of these will conveniently admit the application of logarithms.

19. A sextant equal to a given quadrant.

20. To determine the side of an equilateral triangle, the area of which is 73·45.

21. Also, of a regular hexagon, the area of which is 168.

22. The side of a regular pentagon is 21·7. What is that of another half as large?

23. To find the radius of a semicircle equal to a triangle whose base is 14, and altitude 9.

24. What is the diameter of a circle equal to a trapezoid, of which the base is 17·4, the opposite side 12·7, and the altitude 10·08?

25. To find the content of a regular octagon when the radius of its inscribed circle is equal to 12.

26. Of a regular decagon when the radius of the inscribed circle is equal to 17·2.

27. How large is the angle at the center of a circular sector, the area of which is equal to that of an equilateral triangle whose side is 14, the radius of the sector being 8?

28. To determine the side of a square which shall be equal to a sector whose arc is 18°, and radius 7·5.

29. To determine the diameter of a circle which shall be equal to the segment of a circle whose radius is 120, and arc 135°.

30. To determine the radii of the inscribed and circumscribed circles of a triangle whose sides are 10, 12, and 14.

31. To find the convex surface of a regular hexagonal prism, the longest diameter of which is $2r$, and height h.

32. Of a regular hexagonal pyramid with the same data.

33. Of a zone of one base, the radius of which is r, and altitude h.

34. Of a spherical sector, the chord of which $=c$, and rad. sphere $=r$.

35. To find the volume of a solid generated by the revolution of a sector of a circle about a line through the center, and exterior to the sector.

36. Find the volume of the solid generated by the revolution of any triangle about one of its sides.

37. Find the volume of a solid generated by the revolution of the segment of a circle about a line passing through the center of the circle, and exterior to the segment.

38. Prove that the surfaces of two spheres are as the squares, and the volumes as the cubes of their radii.

39. Find the volume left of a cylinder after a spherical segment having one base equal to that of the cylinder, and the same altitude with the cylinder, has been abstracted.

40. The altitude and surface of a regular hexagonal prism, of

which the greatest diameter is 18, and the volume of which is equal to that of a regular triangular pyramid, of which the base side is equal to 8, and altitude 20.

41. In a quadrangular and hexagonal prism each side of the bases is 7, the height 13. What is the ratio of their volumes and surfaces?

42. To find the altitude of a regular quadrangular pyramid, the side of whose base is 28·7, and volume equal to that of a rectangular parallelopipedon whose edges are 13, 17, and 23.

43. The ratio of two homologous edges of two similar polyhedrons is 5 : 7. To find the ratio of their surfaces and volumes.

44. The ratio of the volumes of two similar polyhedrons is 14 : 29. To find that of their homologous edges.

45. What is the ratio of the surfaces of two regular pyramids, the one triangular, the other quadrangular, if the base in both is 24, and the altitude 7?

46. A regular tetrahedron, the edge of which is 15, has the third of its altitude cut off by a plane parallel to the base; required the volume of the frustum left. Also, the surface.

47. About a sphere of 16 inches radius a polyhedron is circumscribed, containing 20,800 cubic inches. What is the area of the surface of the latter?

48. A cylinder and cone have their radii 14 and 8, their altitudes 6 and 9. What is the ratio of their volumes and surfaces?

49. Find the radius of a sphere equal to a cube, the diagonal of which is 17·22.

50. Also, of a sphere equal to a regular octahedron, the diagonal of which is 31·5.

51. Find the radius of an inscribed sphere in a regular tetrahedron, the edge of which is a.

52. The same for an octahedron.

53. Find the ratio of the surfaces of a regular tetrahedron and inscribed sphere.

54. The same for an octahedron and sphere.

55. What is the ratio of a hemisphere to a cone of the same base and altitude?

56. Find the ratio of the solids generated by a triangle and rectangle revolving about a common base, and the altitude of the former being double that of the latter.

57. To find the volume of a spherical segment when the radius is 5·86, and the arc of a great section 162° 14′.

58. Find the base of a square pyramid which shall contain a cubic yard, and the altitude of which shall be 1 foot.

59. The sides of the base of a tetrahedron are 12, 15, 17, its altitude 9. Required its volume. *Ans.* 263·248.

60. A regular tetrahedron contains 19·683 cubic yards. Required its edges and surface. *Ans.* Edge 5·50705, surface 52·5289.

61. Required the volume of a frustum of a regular triangular pyramid, the larger base of which has 0·9 for its side, and the smaller base 0·4, and of which the lateral edge is 0·5. *Ans.* 0·078371.

62. Given the volume of a sphere equal to 1843·086278 to find its radius. *Ans.* 7·61.

63. Given the edge of a cube 0·36. Required the volume of the circumscribed sphere. *Ans.* 0·126937.

64. Find the area of a spherical triangle, the angles of which are respectively 85 grades, 17′, 103^{gr}, 35′, 67^{gr}, 49′, the radius of the sphere being 1·54. *Ans.* 2·0865.

65. There is a crucible in the form of a conic frustum, the bottom of which is 0·03 in diameter, the top 0·06, and the altitude 0·08; this crucible contains a quantity of melted metal, the surface of which is 0·05 in diameter: it is required to make a sphere of it. What is the diameter of the proper mold? *Ans.* 0·507414.

66. Given the side or apophthegm of a cone 25·15, and its height 17·3, to find its convex surface and volume.

Ans. A = 1442·32, V = 6037·01.

67. Find the quantity of glass in a lens, of which the diameters of the surfaces are 0·03, and the thickness of the lens 0·004.*

Ans. 0·000001422094.

68. Supposing the earth to be perfectly spherical, and a quarter of the meridian to be expressed by 10,000,000; find the expression for its radius, the area of its surface, its volume and weight, supposing the mean density of the earth to be 5·6604.†

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* This solid is a double segment of a sphere.

† This number is the result of the experiments of Sir Francis Bailey, given in the xivth vol. of the Memoirs of the Royal Ast. Soc. of Lond., 1844.