ELEMENTARY DYNAMICS

OF

PARTICLES AND SOLIDS

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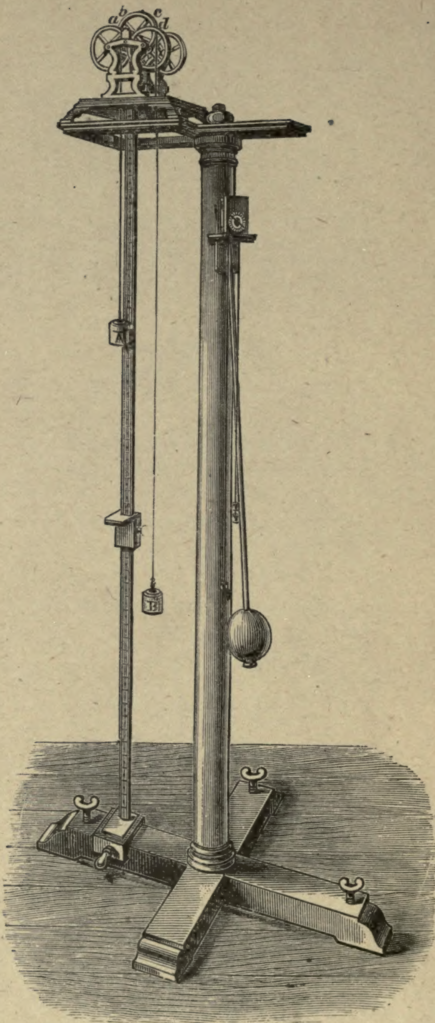
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OF

PARTICLES AND SOLIDS

BY

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PREFACE

THE following treatise is based on courses of lectures delivered in Firth College, Sheffield. It is intended as an introduction to the principles of dynamics for the use of students with no knowledge of mathematics beyond the elements of algebra and pure geometry. It will thus be found useful, not only in colleges and schools, but also to that large class of mechanical engineers to whom a knowledge of dynamics is valuable, but whose acquaintance with mathematics is slight. The wants of this latter class have been kept in view throughout, although developments properly to be found in technical treatises are not introduced. The chapters on the motion of rigid bodies will, it is hoped, be especially useful to them. This part will also be of value to ordinary students, who will thus be introduced to the simple principles of rigid dynamics freed from the intricacies of the differential and integral calculus, which usually accompany them. For engineers, the knowledge of the properties of moments of inertia and their values for simple bodies are as important as that of their centres of gravity, and it is hoped that the methods of Chapter XIX will be found as simple as those employed in finding centres of gravity.

Although a knowledge of trigonometry has not been assumed in the text, it has been occasionally introduced in some of the worked-out examples, and examples have been added which require trigonometry for their solution. The book is thus rendered useful to a larger circle of students.

The chief points of novelty in the presentment of the subject are the following: (1) no separation has been made between Statics and Kinetics—but they have been considered together, the former merely as a special case of the latter; (2) the way in which the idea of Mass and its measure is introduced, and the discussion of Momentum before that of Force, depart from the order usually followed. The author believes, however, that this is the only logical way of treating the subject, based, as it must be ultimately, on experimental laws. This gives to the student a vivid realisation of the essential property of matter and inertia at the very commencement of the subject. He would strongly recommend the student himself to perform or see the experiments described.

The author owes a great debt to Miss PERRIN, late scholar of Girton College, and Mr. G. M. HICKS, late scholar of Clare College, Cambridge, for their kindness in going through the proof-sheets, in working the answers to the examples, and for most valuable help and suggestions as the book was passing through the press.

W. M. HICKS.

December 1889.

PREFACE TO THE FOURTH EDITION

Advantage has been taken of the call for a new edition to add a chapter on motion under central forces. This has been done at the request of teachers, so as to render the book useful to students preparing for examinations in which this subject is included. It is placed in an appendix with some additional matter which appeared desirable.

The author has again to thank various correspondents who have made suggestions and corrections. He has especially to thank his friend and colleague Professor Leahy for valuable advice and help.

March 1897.

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INTRODUCTION

PHYSICAL science is concerned with the relations of the fundamental things, time, space, and matter. Each of these is said to be *sui generis*—neither can be explained nor defined in terms of the others. Although, however, it is not possible to say what they are in themselves, it is possible to investigate the relations which subsist between them. That science which treats of space by itself is called geometry. That which treats of the relations of matter to space and time is called dynamics. Kinematics deals with questions connected with space and time, such as velocity and change of position with time. Dynamics, again, is subdivided into kinetics, which deals with matter in motion, and statics, which deals with the conditions of rest. Before entering on the consideration of these latter subjects it will be necessary first to learn about motion, apart from the thing which moves; consequently we shall be obliged to make some study of kinematics as a preliminary to the larger subject of dynamics.

The object of all physical science is to reduce phenomena to measurement. In the case of any of the fundamental things, time, space, or matter, all we can do is to say that a certain time, space, or matter is so many times another portion of the same thing. A similar statement holds good with respect to any other kind of quantity, although, as we shall see later, it is possible to express them in terms of the fundamental ones. The expression of the magnitude of any physical quantity is therefore composed of two factors—one giving the portion with which it is compared, and the other the number of times that the quantity in question contains it. The former is called the unit, the latter the measure. Thus a certain length may be 3 feet, a certain time 3 hours. Here the measure is 3, the units are "feet" and "hours." The chief fundamental units in use are the following:—

Unit of Time.—The unit of time in use throughout the world depends on the average time which the earth takes to make one turn on its axis, relatively to the sun, or, as it is called, the mean solar day. This is subdivided into 24 hours, an hour into 60 minutes, and a minute into 60 seconds.

Unit of Space.—Different nations use different units of length. The principal British unit is called the *yard.* It is the distance between the centres of two transverse lines on a bronze bar kept in the office of the Exchequer in London, the distance being measured at a temperature of 62° F. The yard is subdivided into 3 feet, and each foot into 12 inches. The unit used chiefly on the continent and in scientific measurements is called the *meter.* It was introduced in 1795 by the French Republic, and was intended to be one ten-millionth of the length of a meridian from the earth's pole to the equator. It is now defined as the distance, measured at the temperature of melting ice, between the ends of a platinum rod kept in the archives at Paris. The system of units based on this is called the metrical system. The meter is subdivided into smaller units, one-tenth, one-hundredth, and one-thousandth of the meter, and named by placing the words deci, centi, milli respectively before meter. On the contrary, when larger units are needed, deca placed before meter means 10 meters, hecto 100, and kilo 1000. It is thus possible to change the measure of a length from one unit to another merely by changing the decimal point. Thus 1102·167 meters = 1·102167 kilometers, 110216·7 centimeters, or 1102167 millimeters. This is one advantage of the metrical system of units, but a greater advantage still is the way in which they are correlated with the unit of mass. The relations between the metrical and British units are given in Tables I., II. below.

An area requires a unit of its own kind by which to be measured. It is the area of a square of which a side is the unit of length. But for certain purposes special units are employed—thus, for measuring land an acre is equal to 4840 square yards, a rood is equal to a quarter of an acre, and a pole contains $30\frac{1}{4}$ square yards. The metrical unit of area is 100 square meters, and is called an are. For land measures the hectare or square hectometer is employed.

Again, the unit proper to measure a volume is the volume of a cube whose side is the unit of length. Other units are also employed—for instance, 1 gallon is equal to 277·274 cubic inches, and contains 10 lbs. of pure water at 62° F. This is subdivided into 4 quarts and 8 pints. The unit of volume corresponding to the decimeter unit of length, that is a cubic decimeter, is called a liter.

Units of Matter.—The quantity of matter in a body is called its mass. The British unit of mass is called the pound. It is a quantity of matter equal to that contained in a piece of platinum which is preserved in the office of the Exchequer in London.

The system of units employed on the continent and in scientific measurements is based on the gram. It was intended to represent the quantity of matter in 1 cubic centimeter of pure water at 4° C. It is now defined as the one-thousandth of the quantity of matter in a piece of platinum preserved in the archives at Paris.

The multiples and subdivisions are made as in the case of the meter, and are called the kilogram, hectogram, decagram, decigram, centigram, milligram.

The system of units depending on the centimeter, the gram, and the second is often called the C.G.S. system.

It will be useful for reference to collect here the following tables and numbers :—

I. Measures of Space.

A. Length—

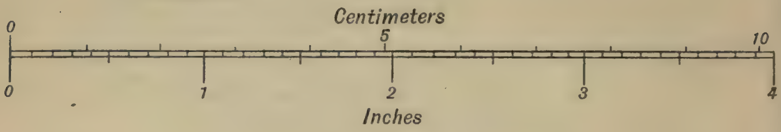


Table I.

1 centimeter	=	·3937079 inch.
1 meter	=	39·37079 inches.
,,	=	3·2809 feet.
1 kilometer	=	1093·6 yards.
,,	=	·6213 mile.

Table II.

1 inch	=	2·539954 centimeters.
1 foot	=	30·479449 centimeters.
1 yard	=	·91438347 meter.
1 mile	=	1·60932 kilometer.

The following are roughly approximate numbers:—

Distance from pole to equator	= 10,000,000 meters.
1 decimeter	= 4 inches.
8 kilometers	= 5 miles.
Diameter of 1 halfpenny	= 1 inch.

B. Area—

Table III.

1 sq. centimeter	=	·155006 sq. inch.
1 sq. meter	=	10·7643 sq. feet.
1 sq. hectometer or 1 hectare	=	2·47114 acres.
1 sq. kilometer	=	·38611 sq. mile.

Table IV.

1 sq. inch	=	6·45137 sq. centimeters.
1 sq. foot	=	928·997 sq. centimeters.
1 sq. yard	=	·836097 sq. meter.
1 acre	=	·404672 hectare.
1 sq. mile	=	2·58989 sq. kilometers.

C. VOLUME—

TABLE V.

1 cubic centimeter	= ·0610271 cubic inch.
1 liter or 1 cubic decimeter	= 61·0271 cubic inches. = 1·76172 pint.
1 cubic meter	= 35·3166 cubic feet.

TABLE VI.

1 cubic inch	= 16·3866 cubic centimeters.
1 cubic foot	= 28·3153 liters.
1 cubic yard	= ·764513 cubic meter.
1 pint	= ·567627 liter.
1 gallon	= 4·54102 liters.

II. MEASURES OF MASS.

TABLE VII.

1 centigram	= ·154323 grain.
1 gram	= 15·4323 grains.
,,	= ·0353739 oz.
1 kilogram	= 2·20462 lbs.

TABLE VIII.

1 grain	= ·064799 gram.
1 oz.	= 28·3496 grams.
1 lb.	= ·453593 kilogram.
1 ton	= 1·01605 tonne = 1016·05 kilos.

---

1 gram = mass of 1 cubic centimeter of pure water at 4° C.
1 kilogram = ,, 1 liter of pure water at 4° C.
1 gallon = 277·274 cubic inches.
The gallon contains 10 lbs. of pure water at 62° F.
1 cubic foot of water contains about 1000 oz.
The pint contains 20 fluid oz.
The mass of 1 sovereign = 123·274 grains.
,, ,, 48 pennies = 1 lb.
,, ,, 3 pennies = 1 oz.
Acceleration of gravity at London = 32·182 feet per second per second
= 980·889 centimeters per second per second.
1 dyne = $\frac{1}{981}$ weight of gram = weight of 1 milligram about.
1 poundal = weight of $\frac{1}{2}$ oz. about.

Two masses of 1 gram each, distant 1 centimeter attract each other with a force $= 6{\cdot}58 \times 10^{-8}$ dynes.

Diameter of earth, equatorial, 7926 miles.

" polar, 7900 miles.

Sound travels in air at about 1100 feet per second.

Light travels at the rate of 186,000 miles per second, or 300,000 kilometers per second.

The heat necessary to raise 1 lb. of water 1° F. if expended in work would raise 1 lb. 779 feet high ; *or*

1 lb. of water raised 1° C. would raise 1 lb. 1402 feet high ; *or*

1 kilogram of water raised 1° C. would raise 1 kilogram 427·4 meters high.

Areas and Volumes.

Parallelogram .	area	= one side × distance from opposite side.
Triangle . .	area	$= \frac{1}{2}$ base × altitude.
Parallelopiped .	volume	= area of one face × distance from opposite face.
Pyramid or cone	volume	$= \frac{1}{3}$ area of base × height.
Circle, rad. $= r$.	length	$= 2\pi r$.
" "	area	$= \pi r^2$.
Sphere, rad. $= r$	surface	$= 4\pi r^2$.
" "	volume	$= \frac{4}{3}\pi r^3$.
Zone on sphere between parallel planes .	surface	$= 2\pi r$ × distance between planes.

Where $\pi = 3{\cdot}14159 \ldots = \frac{22}{7}$ nearly.

PART I

RECTILINEAR MOTION OF A PARTICLE

CHAPTER I

MOTION IN A STRAIGHT LINE

1. WHEN a point is changing its position in space it takes a certain time in which to do it. The shorter the time the more quickly it is said to move. In this chapter we shall consider the simplest case—viz. where the point is moving along a straight line. If it always moves over the same distance in the same time it is said to have a constant velocity. If it does not, the velocity is variable. The magnitude of a velocity is called its speed. *It is measured when constant by the space passed over in the unit of time. When not constant, the measure at any time is the space which would be passed over in a unit of time if the velocity were throughout the same as at the instant in question.* This way of measuring is quite familiar; thus, if we say that a man walks at the rate of 3 miles an hour, or that the speed of a train is 20 miles per hour, no one will suppose us to mean that in the next hour the man will walk 3 miles or the train will travel 20 miles, but only that they would do so if they went at the *same* speed for an hour.

Velocity is a quantity which is different in kind from each of the fundamental conceptions of space, time, or matter. It will therefore require a unit of its own kind to measure it; but this unit can be expressed in terms of two of the fundamental ones. Suppose, for instance, the units of space and time are 1 foot and 1 second, then we may say that a certain velocity is (say) 10 feet per second, in other words it is ten times a velocity of 1 foot per second. Here the measure is 10, the unit is a velocity of 1 foot per second. And so in general *the unit of velocity is the velocity of a point which passes over the unit of space in the unit of time.* When we say that the velocity of a body is denoted by v, we mean that it is v times the unit, or, which is the same thing, a velocity of v units of space passed over in a unit of time.

It is easy to find a formula connecting the space passed over in a given time by a given velocity, when the velocity is constant. Thus, let the velocity be denoted by v, and the time by t (*i.e.* t units of time). Then in one unit of time v units of space are passed over, therefore in t units of time, t times as much, or vt units of space will be passed over. That is, if s denotes the number of units of space,

$$s = vt \qquad (1).$$

It is often necessary to pass from one system of units to another. Thus, for example, suppose we want to express a velocity of 1800 miles per day in feet per second.

In one day 1800 miles or $1800 \times 1760 \times 3$ feet are passed over,

$$\therefore \text{ in one hour } \frac{1800 \times 1760 \times 3}{24} \text{ feet,}$$

$$\text{or in one second } \frac{1800 \times 1760 \times 3}{24 \times 60 \times 60}$$

$$= 110 \text{ feet;}$$

in other words, the velocity is 110 feet per second.

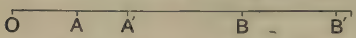
2. We have defined a velocity to be the space passed over in a unit of time. Let us consider for a moment how this distance is to be determined. We cannot do this without referring the positions of the point at the beginning and at the end of the interval to some other point which we suppose fixed. This is evident, for when it is in its final position we cannot tell where its initial one was without referring to surrounding objects. We call the point of reference a fixed point, but we cannot say whether any point is absolutely fixed in space—in fact,

such a statement could have no meaning. To illustrate this, take the case of a train travelling along a line due east, with constant velocity. If at the end of an hour's time it was 30 miles distant from the starting-point, we should say its velocity was 30 miles per hour. But suppose a person looking at it from a point outside the earth, fixed with reference to the sun, this person would see that the train was carried in the hour through an enormously greater distance than 30 miles. Or, to take another case of a passenger walking along the deck of a steamboat from stern to bow. To a person in the ship looking at him he might seem to walk at the rate of 3 miles per hour, whereas to a person in the sea he might appear to be moving along at 23 miles per hour, owing to the motion of the ship (20 miles per hour). Thus we see that all velocity is essentially relative to something else, and when we speak of a fixed point we only mean that we refer the positions of all other points to that, and consider their motions *relatively* to it.

Now, if every point of a system of bodies be displaced through the same distance, their relative positions remain unaltered, and therefore any relative motions they may have will remain unchanged if they receive the same displacement every second. In other words, we may impress any the same velocity on every point without altering their relative motion. This remark enables us to find the relative velocities of two points when their motion is given with reference to some third point. Thus, let the points A, B be moving along the same straight line with velocities u, v relative to some fixed point. Since the relative motion is unaffected if we impress the same velocity on both A and B, let us impress on both a velocity *equal* and *opposite* to that of A. Then the velocity of A becomes zero—that is, it becomes the "fixed point," and the velocity of B becomes $v - u$. This way of treating the question is of great value when we come to treat of more complicated motions than are considered in this chapter.

We can, however, look at it from another point of view. The relative velocity of two points *moving along the same* *straight line* is the rate at which the distance between them increases.

Let O be the fixed point, A, B the initial positions of A, B and A′, B′ their positions after unit of time.



Then $AA' = u$ units of length, $BB' = v$ units of length, and the change of distance in the unit of time is

$$A'B' - AB = BB' - AA' = v - u \text{ units of length.}$$

The following questions illustrate the foregoing result :—

EXAMPLE I. *Two trains on the same line are* 21 *miles apart ; the foremost is going* 13 *miles an hour, and the hindmost* 20. *When will they collide ?*

As above, the rate at which the distance increases is $13 - 20 = -7$, or the distance between decreases at the rate of 7 miles per hour, and when the collision takes place this distance has decreased from 21 miles to 0. Therefore, if t be the number of hours,

$$21 = 7t \text{ or } t = 3 \text{ hours.}$$

If they had been moving towards one another, the relative velocity would have been $-13 - 20 = -33$, and the time would have been $\frac{21}{33}$ hour $= 38\frac{2}{11}$ minutes.

EXAMPLE II. *A column of soldiers* 315 *yards long is marching at* $3\frac{1}{2}$ *miles per hour past an onlooker walking at* 2 *miles per hour in the same direction. How long will the column take to pass him ?*

Here the velocity of the head of the column relative to the onlooker is $1\frac{1}{2}$ mile per hour $= 2640$ yards per hour. The time taken to pass is the time the head of the column takes to increase its distance from 0 to 315 yards,

Therefore the time is $\frac{315}{2640}$ hour $= \frac{21}{176}$ hour $= 7\frac{7}{44}$ minutes.

3. The formula (1) gives the space passed over in any time when the velocity remains constant, but fails in other cases. When the velocity is altering, the point is said to be accelerated. One very important case is where the velocity is increasing at a constant rate while the point is moving. We shall then say that the point has constant acceleration.* It is usual to say that it has uniform acceleration, but we shall find it best to use the word constant for regularity with reference to time and uniform for regularity

* For a case of varying acceleration see example 50 at the end of this chapter.

with reference to space. This distinction between the words will be held to throughout the book.

Acceleration is measured in an analogous way to velocity. *When constant it is measured by the increase of velocity in a unit of time, and when variable by the increase of velocity which would take place in a unit of time if the acceleration remained the same as at the instant in question.*

Here, again, acceleration is a new *kind* of quantity, and we require a new kind of unit to measure it. This unit is an acceleration in which the unit of velocity is added on per unit of time. Thus, for instance, if the foot and second are the respective units of space and time, an acceleration 10 would mean one in which 10 units of velocity were added on in a second—*or* in which a velocity of 10 feet per second was added on every second. It is clear here that the unit of time enters twice. We shall consider this point further in Chapter V; here we will only show by an example how to change from one set of units to another. Thus let it be required to express an acceleration of 2400 yard minute units in foot second units. We proceed as follows—

In one minute a velocity of 2400 yards per minute is added on, therefore in one second one-sixtieth of this will be added on, or 40 yards per minute.

But a velocity of 40 yards per minute is 120 feet per 60 seconds, that is 2 feet per second. Hence

In one second a velocity of 2 feet per second is added on, *or* the acceleration is 2 feet per second per second—that is, the measure in the new unit is 2.

When the acceleration is constant it is important to be able to know what the velocity is after any time, and also what the space passed over amounts to. Let the velocity at the beginning of the time be denoted by u, and the acceleration by a. After a time denoted by t let the velocity become v. Then, since in a unit of time a velocity a is added on, in t units of time a velocity at will be added on. In other words, the velocity which at the beginning was u has increased by at, or in symbols

$$v = u + at \qquad (2).$$

Thus a train going down an incline at an acceleration of $\frac{3}{2}$ mile per hour per hour, and starting at 20 miles per hour, will have, after 30 minutes (or $\frac{1}{2}$ hour), a velocity $= 20 + \frac{3}{2} \times \frac{1}{2} = 20\frac{3}{4}$ miles per hour. Again, as we shall see later, any body falling freely is constantly accelerated at the rate of about 32 feet per second per second. Hence a stone dropped from rest will after 5 seconds have a velocity of $32 \times 5 = 160$ feet per second.

If the point be retarded instead of accelerated, then the velocity in time t has decreased by at, and

$$v = u - at.$$

If we regard a retardation as a negative acceleration the formula (2) will therefore be true in all cases.

4. It remains now to obtain a formula which shall give us the *space* described in any time. As before, let u be the initial velocity, a the acceleration, t the time, and s the space passed over. Let V be the velocity at the middle of the time, *i.e.* at a time $\frac{1}{2}t$ from the beginning. Then by formula (2)

$$V = u + \tfrac{1}{2}at.$$

Now notice that since the velocity of the point is increased constantly at the same rate, at any interval (τ) before the middle the velocity will be just as much less than V, as it is greater than V at the same interval τ after. In other words, the rate at which space is being passed over at the former instant is as much less than V as it is greater than V at the latter, and therefore the total effect is the *same* as if the point is moving at both instants with velocity V. As this is the case, whatever the interval τ may be, the whole space described will be the same as if there is a constant velocity V throughout. That is

$$s = Vt.$$

But we have just seen that

$$V = u + \tfrac{1}{2}at,$$

$$\therefore\ s = (u + \tfrac{1}{2}at)t = ut + \tfrac{1}{2}at^2 \qquad (3).$$

As an example of the use of this formula, suppose a stone thrown downwards with a velocity of 12 feet per second, and let it be required to find the space described in 5 seconds. Here, as above, $a=32$, and

$$s=12\times 5+\tfrac{1}{2}\times 32\times 25=460 \text{ feet.}$$

If it had been projected upwards, it would be retarded, and the distance upwards would be

$$s=12\times 5-\tfrac{1}{2}\times 32\times 25=-340 \text{ feet,}$$

the negative sign showing that it is 340 feet *below* the point of projection. If the time had been $\frac{1}{2}$ second the distance in the second case would have been $12\times\frac{1}{2}-\frac{1}{2}\times 32\times\frac{1}{4}=+2$ feet, that is 2 feet *above*.

Equation (3) will also enable us to find in what time the point will have passed over a given distance. For now s is known, and t is required to be found. The equation, regarded as one to find t, is a quadratic, and therefore has two roots. In other words, the point will have been in a given position at two different times. This can be illustrated by the case of a stone thrown up in the air. It will pass through each position twice; once on the way up, and once on the way down. Thus, suppose a stone projected up with a velocity of 40 feet per second, and it is required to know at what times it will be at a height of 16 feet,

The equation is (remembering that if we measure space upwards $a=-32$)

$$16=40t-\tfrac{1}{2}\times 32t^2=40t-16t^2;$$

i.e.

$$t^2-\frac{5}{2}t=-1,$$

$$t^2-\frac{5}{2}t+\left(\frac{5}{4}\right)^2=\frac{25-16}{16}=\frac{9}{16},$$

$$t-\frac{5}{4}=\pm\frac{3}{4};$$

$$t=\frac{5+3}{4}=2 \text{ or } =\frac{5-3}{4}=\frac{1}{2}.$$

We learn, therefore, that it was 16 feet high half a second after starting (on way up), and again two seconds after starting (on way down).

Suppose the stone had been projected downwards instead of upwards, and we wished to know when it was 16 feet below, then, measuring the space downwards,

$$16 = 40t + 16t^2;$$

$$\left(t + \frac{5}{4}\right)^2 = \frac{25}{16} + 1 = \frac{41}{16},$$

$$t = -\frac{5}{4} \pm \frac{\sqrt{41}}{4} = \frac{-5 \pm 6{\cdot}403}{4} = {\cdot}35 \text{ or } -2{\cdot}85.$$

The value ·35 has a definite meaning, viz. it will be at the point named ·35 seconds after starting. But what does – 2·85 mean? To determine this we notice that the motion is precisely the same whether the stone was initially projected down, or whether it was just then falling freely through the starting-point. In this latter case it would have had a previous motion, and therefore at some time previously it would have been moving upwards and at the point named. The answer above then means, that if it had been moving freely before it would have been at the given point 2·85 seconds before the instant at which we supposed it to start. If the equation to find t should have imaginary roots, that would mean that the velocity of projection was not large enough to bring it to the position given.

Formula (2) gives a relation between the velocity and time, whilst (3) gives the space in terms of the time. Between them, therefore, it is possible to find the velocity after the point has moved over a given distance, and this we now proceed to do. Squaring both sides of the equation $v = u + at$, there results

$$v^2 = (u + at)^2 = u^2 + 2uat + a^2t^2,$$
$$= u^2 + 2a(ut + \tfrac{1}{2}at^2).$$

But (3) $$s = ut + \tfrac{1}{2}at^2,$$
$$\therefore v^2 = u^2 + 2as \qquad (4).$$

If the point is retarded instead of being accelerated $-a$ must be put instead of a, as in the former cases. This formula serves to answer the question how far a retarded body will move before it comes to rest. For instance, a stone is projected up with a velocity of 20 feet per second, how far will it go? The point to notice is that at the highest point the velocity is zero,

$$\therefore\ 0^2 = 20^2 - 2 \times 32 \times s,$$

or
$$64s = 400,$$

$$s = \frac{25}{4} = 6\tfrac{1}{4} \text{ feet.}$$

If the question had been, how long before it reaches its greatest height, it would be best to use the formula $v = u + at$,

thus
$$0 = 20 - 32t,$$

whence
$$t = \frac{20}{32} = \frac{5}{8} \text{ second.}$$

5. In the same way as it was shown that there is no such thing as an absolute velocity, it can be seen that acceleration must be also relative. The relative acceleration of two points *moving along the same line* is the rate at which rate of increasing distance between them is altered. In the same way as for velocity, it is easy to see that the relative accelerations of a series of points will be unaltered if we impress on each any the same acceleration. Thus let A, B move along the same straight line with velocities u, u' at any instant and accelerations a, a'. If we impress on both a velocity $(-u)$, A will be for the instant at rest, and the velocity of B will be $u' - u$. If at the same time we also impress on both an acceleration equal and opposite to a, A will *remain* at rest, and B will have an acceleration $a' - a$. Hence the relative acceleration of B with respect to A is $a' - a$.

The same result may also be deduced by formula (3) from the figure on p. 12. For, if A′, B′ be the positions of A, B after time t,

$$AA' = ut + \tfrac{1}{2}at^2, \quad BB' = u't + \tfrac{1}{2}a't^2,$$

and
$$A'B' - AB = BB' - AA' = (u' - u)t + \tfrac{1}{2}(a' - a)t^2.$$

That is, comparing with formula (3), the initial relative velocity is $u' - u$, and relative acceleration is $a' - a$.

EXAMPLES—I.

1. A river is flowing at the rate of 3 miles per hour. How long will it take a log floating with it to pass over 100 feet?

2. Express 45 miles per hour in feet per second and in yards per minute.

3. Express 2 feet per second in centimeters per minute.

4. Supposing the earth to describe a circle relative to the sun of 92,000,000 miles radius, and that the year is 365 days 6 hours, find the velocity of the centre of the earth in miles per second, and also when the units are 1 day and 1000 miles.

5. Light travels at the rate of 186,000 miles per second; express this in kilometers per minute.

6. The distance of the sun from the earth is 91,000,000 miles. How long does it take light to travel thence to us? (See previous question.)

7. The earth rotates on its axis in 23 hours 56 minutes, and its equatorial diameter is 7925 miles. Find the velocity relative to the earth's centre of a point (1) on the equator, (2) in latitude 60°.

8. Two trains whose lengths respectively are 130 and 110 feet, moving in opposite directions on parallel rails, are observed to be 4 seconds in completely passing each other, the velocity of the longer train being double that of the other. Find how many miles per hour each train is moving.

9. Two trains pass one another moving in opposite directions on parallel lines of rail, with velocities of 45 and 60 miles per hour. The length of one is 420 feet and of the other 350 feet. How long will they be in passing one another?

10. If in the previous question they had been going in the same direction with the slow one in front, how long would they take to pass? Also, how far would each go before they are just clear?

11. Two boats, each 30 feet long, are rowed at 8 and 7 miles per hour respectively, the latter being 80 feet ahead of the former. Find how long before it is bumped; also the time before the former draws level with it, and the extra time necessary to pass it.

12. In the former question determine how far the first boat has gone (1) when it bumps, and (2) when it is clear.

13. Two men a mile apart are walking towards one another with a velocity of 264 and 176 feet per minute. When and where do they meet?

14. In a storm the thunder was heard 12 seconds after the flash was seen. How far off was the point of discharge? (Velocity of sound 1100 feet per second.)

15. A train moving uniformly describes 88 yards in 3 seconds. Find its velocity in miles per hour. In what time will it travel 600 miles with a stoppage of 5 minutes after every 100 miles?

16. Supposing the circumference of the earth at the equator to be 25,000 miles, and the time of the earth's rotation to be 24 hours, find the velocity relative to the earth's centre, in miles per hour, of a cannon-ball at the equator when it is fired with a velocity of 1650 feet per second (1) in the direction of the earth's rotation, (2) in the opposite direction.

17. If the minute-hand of a clock be 6 inches long, what is the linear velocity of its extremity?

18. Compare the velocities of the extremities of the hour, minute, and second hands of a watch, their lengths being ·48, ·8, and ·24 inches respectively.

19. The velocity of the extremity of the minute-hand of a clock is 48 times the velocity of the extremity of the hour-hand, which is 3 inches long. Find the length of the minute-hand.

20. From a train moving with velocity V, a carriage on a road parallel to the line at a distance d from it, is observed to move so as to appear always in a line with a more distant object, whose least distance from the railway is D. Find the velocity of the carriage.

21. A sportsman covers a bird flying in a straight line at the rate of 6 miles an hour, moving his gun round his shoulder as a fixed point. If when the bird is nearest to him it be 20 yards distant, find the angular velocity of the gun at that instant.

22. Two points describe the same circle in such a manner that the line joining them always passes through a fixed point. Show that at any moment their speeds are proportional to their distances from the fixed point.

23. A stone falls with an acceleration of 32 feet per second per second. Express this in yard minute, and also in centimeter second units.

24. What is the measure of the acceleration of a falling body when an inch and an hour are units of space and time?

25. Compare the following accelerations—(a) one in which a velocity of 20 feet per second is added on every minute, (b) one in which 20 feet per minute is added on every second.

26. Determine what would have to be the value of the acceleration of gravity, in order that a body starting from rest should fall 10,000 feet in 10 seconds.

27. A point moves from rest with an acceleration of 20 yards per minute per minute. What distance will it pass over (1) in the fifth minute, (2) in the next 12 seconds?

28. A point moves along a straight line under an acceleration of 10 centimeters per second per second. If the initial velocity be 7 centimeters per second, what is the velocity after it has passed over 12 centimeters?

29. A point moving from rest under constant acceleration passes over 36 yards in 3 seconds. What is the acceleration? Also what is its velocity after the 3 seconds?

30. A point moving from rest under constant acceleration has a velocity after 4 seconds of 18 feet per second. What is the acceleration? Also how far will it have moved in 16 seconds?

31. How long does it take a stone to fall 625 feet from rest?

32. A train which has constant acceleration starts from rest, and at the end of 3 seconds has a velocity with which it would travel through 1 mile in five minutes. Find the acceleration.

33. A point starting from rest passes over 121 feet in the sixth second. What is the acceleration?

34. A point passed over 126 feet during the fourth second and 246 feet during the eighth second. What was its initial velocity and its acceleration?

35. A body is projected upwards with a certain velocity, and it is found that when in its ascent it is at a point 960 feet from the ground it takes 4 seconds to return to the same point again. Find the velocity of projection and the whole height ascended.

36. A train reduces speed from 45 miles an hour to 15 miles an hour in 800 yards. How much farther will it go before stopping?

37. A bullet is projected upwards with a velocity of 1000 feet per second. Find the height to which it will rise.

38. In the previous question another bullet is projected vertically upwards with a velocity of 920 feet per second, 5 seconds after the first. Where and when will they meet?

If the second had been projected 2 seconds after the first, would they have met?

39. A tower is 288 feet high; at the same instant one body is dropped from the top of the tower and another projected vertically upwards from the bottom, and they meet half way. Find the initial velocity of the projected body and its velocity when it meets the descending body.

40. A stone is thrown up with a velocity of 320 feet per second, and 4 seconds afterwards another with a velocity of 190 feet per second. Will they ever meet?

41. A stone is let fall, and 5 seconds afterwards another is projected down with a velocity of 200 feet per second. When and where will it overtake the former?

42. A train goes from one station to another 5 miles off in 8 minutes, first moving with constant acceleration and then with an equal retardation. Find its greatest speed.

43. Two trains pass through stations 100 miles apart towards one another with velocities of 10 and 15 miles per hour; they are respectively accelerated at 1760 and 3520 yards per hour per hour. When and where will they meet, and what will be their velocities then?

44. A body projected vertically downwards describes 720 feet in t seconds and 2240 feet in $2t$ seconds. Find t and the velocity of projection.

45. A train stops at a station by constantly slackening speed; it begins to do so one mile from the station, when it is going at 30 miles per hour. What is its retardation?

46. A stone is let fall down a well and the splash is heard 5·94 seconds later. What is the depth of the well? (See question 14.)

47. A train going 60 miles per hour overtakes one going at 20 miles per hour, but they see each other just in time to prevent a collision, the one slackening and the other increasing speed at the rate of 5 feet per second per second. How far off were they when they sighted each other and how long before they just not collide?

48. From a balloon which is ascending with a velocity of 32 feet per second, a stone is let fall and reaches the ground in 17 seconds. How high was the balloon when the stone was dropped?

49. If a, b, c be the spaces described in the pth, qth, and rth seconds by a point moving with constant acceleration, prove that

$$a(q-r)+b(r-p)+c(p-q)=0.$$

50. i. O, B, A are three points arranged in this order on a straight line. A particle is projected from A towards B with velocity u and reaches B with velocity v, being acted on during the motion by a force which produces constant acceleration $\frac{\mu}{\text{OB}\,.\,\text{OA}}$ towards O. Prove that

$$\tfrac{1}{2}(v^2-u^2)=\mu\left(\frac{1}{\text{OB}}-\frac{1}{\text{OA}}\right).$$

ii. O, A_n, A_{n-1}, . . ., A_2, A_1, A are points arranged in this order on a straight line. A particle is projected from A towards A_n with velocity u and reaches A_n with velocity v, being acted on during the motion by a force of the following nature: whilst the particle moves from A to A_1 it produces constant acceleration $\frac{\mu}{\text{OA}\,.\,\text{OA}_1}$, whilst from A_1 to A_2 it produces constant acceleration $\frac{\mu}{\text{OA}_1\,.\,\text{OA}_2}$; from A_2 to A_3 $\frac{\mu}{\text{OA}_2\,.\,\text{OA}_3}$; and so on. Prove that

$$\tfrac{1}{2}(v^2-u^2)=\mu\left(\frac{1}{\text{OA}_n}-\frac{1}{\text{OA}}\right).$$

iii. Show how, by keeping the points A_n, A fixed in position, and making n indefinitely great, to prove the formula

$$\tfrac{1}{2}v^2-\frac{\mu}{r}=\text{C},$$

where v is the velocity, $\frac{\mu}{r^2}$ the acceleration at the distance r from O, and C is a constant.

CHAPTER II

MASS—MOMENTUM—COLLISION

6. In this chapter we enter on the consideration of matter in its relations to motion. The quantity of matter in a given body is called its *mass.* To measure this mass we must refer it to another portion of matter which we take for unit mass, and say how many times it contains this unit; and this leads us to the question how this ratio is to be determined.

Now, in general, in order to determine the ratio between two quantities of the same kind, the first step is to get some criterion which shall determine when they are equal. For instance, suppose we have a given thing A (whether mass or any other quantity) and a criterion by which to determine if it is equal to another thing B of the same kind. It will then be possible, by applying the criterion of equality, to make any number of things of the same kind each equal to A. By combining two of them we can get one whose magnitude is twice that of A; by combining three one whose magnitude is three times that of A, and so on. In a similar way we can obtain two equal ones which together equal A—in other words, we can get one whose magnitude is one-half that of A, and so on. Having obtained these larger and smaller units we can, by combining them, build up a quantity whose magnitude (according to the criterion) is equal to that of B which we wished to measure. To illustrate this, consider the case of measurement of length, the unit A being one foot—*i.e.* the length of a certain rod. Here the "criterion of equality" is, that when laid alongside with one end coinciding, the other coincides also.

Applying this criterion we can make a series of rods each equal in length to one foot. So also, by making twelve equal pieces which together equal the one foot, we get other units equal to one-twelfth of a foot. If then we wish to measure the length of anything, say the side of a table, we first lay along it end to end a series of our foot rods, until one more would be too long; then a series of the smaller, until one more would be too long, and so on. We have then built up a length equal to that whose measure is required, and the ratio of this composite one to the unit—one foot—is known by the process of its formation. In practice this is simplified, but the rationale is as here given, and is the same whatever the kind of quantity we may wish to measure.

The first point then to be determined is the criterion of equality of two masses, and this criterion must depend on the properties of matter. For a knowledge of these properties recourse must be had to experiment.

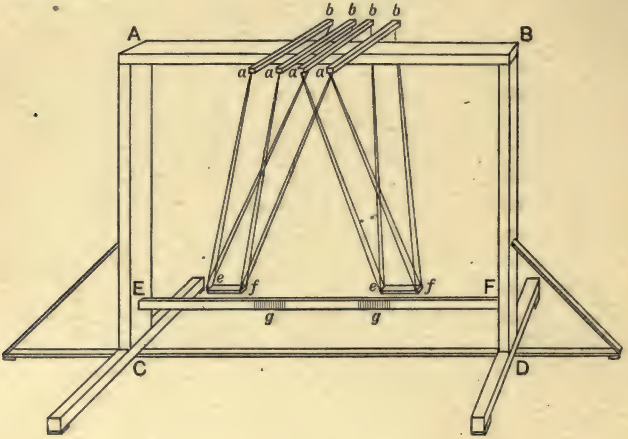
In general, when two bodies impinge they fly apart after the collision, but if by any means this is prevented, such as by a catch or sticky cement, the single body composed of these two will in general still have a motion different from that of either before impact. Now suppose two particles to impinge on one another in opposite directions in the same straight line with equal speeds, and to stick to one another. If they do not come to rest, clearly the masses are not equal, for everything else is the same for both. If they do come to rest we shall say that they are equal. This will give the necessary criterion for equal masses; it may be defined as follows—

Two masses are equal, if when they are made to impinge on one another in opposite directions with equal speeds, and stick together, they come to rest. It is supposed here that no rotations are set up.

A simpler practical method of determining equality of masses will be *deduced* later based on another property of matter. The above is, however, the simplest and most fundamental conception of equal masses.

7. The criterion supposes that the actual value of the

relative velocity of the masses has no influence. Before therefore applying it, it will be necessary to see by experiment whether two masses which, according to the criterion, are equal with one velocity, are also equal with another velocity. To test this and also to employ our definition in the measurement of actual masses it is necessary to have some instrument by which it is possible to give determinate velocities to the bodies to be compared, and to measure their velocities after impact. Such an instrument may be called a ballistic balance.* One form of construction is as follows—



ABCD is a framework across the top of which the bars *ab* can be adjusted at right angles to AB. From the extremities of these bars two carriers *ef* are supported by sets of wires, which cause them to move always parallel to themselves in a definite path, so as to strike one another perpendicularly. The ends of one carrier are fitted with two springs, which can clasp the other and so prevent them

* The arrangement shown in the text is satisfactory when it has to be movable. If, however, the apparatus is to be used always in the same room, the arrangement explained in the Appendix is preferable.

from rebounding. Each carrier also has a pointer which moves in front of two scales *g*, *g*. On the bar EF, to which the scales are fixed, are fitted movable catches, which hold back the carriers to any point of the scale, and which are capable of being set free at the same moment. If a carrier be pulled aside and then let go it will arrive at the lowest point with a velocity which depends only on the distance from which it falls. This velocity is marked on the scale, and must be determined for each instrument. For instance, when the carriers are 109 centimeters below their point of support, it will be found that if the deflections are not very large a centimeter scale will serve, each centimeter corresponding to a velocity of 3 centimeters per second. It will also be found that the carriers will arrive at their lowest position at the same moment, if they start simultaneously, and that this will be the case whether they start from equal distances or not.

With this instrument it is easy to show that the criterion of the equality of masses is independent of the actual velocities of impact. For it will be found that the carriers will come to rest after striking with any velocity, if they do so with some given velocity.

The complete expression of the mass of a body necessitates that it should be compared with the mass of another body, which is called the unit. In Great Britain this is the pound, on the continent generally the gram. The relations between these are stated in the Introduction (pp. 3-5).

8. The degree of concentration of matter in a body is called its *density*. It is measured in one of the two following ways—

A. By the mass contained in the unit of volume, such as a cubic foot or a cubic centimeter.

B. By the ratio of the mass of any volume of the body to the mass of an *equal* volume of some substance which is taken as a standard of comparison. The substance usually taken as the standard is water.

We may illustrate the two ways by taking a case where the units of space and mass are 1 foot and 1 lb., and the density 5. Then according to

A. One cubic foot of the body contains 5 lbs. of matter.

B. One cubic foot contains five times as much matter as 1 cubic foot of water. Now 1 cubic foot of water contains about 1000 oz., or $62\frac{1}{2}$ lbs. Hence 1 cubic foot of the body in question contains 5000 oz., or $312\frac{1}{2}$ lbs.

It is clear then that in general the numerical value of the density of a body will be very different according as it is stated by methods A or B. But now consider the same case when the units of mass and length are the gram and centimeter.

Then by A. Mass of 1 cubic centimeter of the body is 5 grams.

by B. Mass of 1 cubic centimeter of the body is five times the mass of 1 cubic centimeter of water, *i.e.* 5 grams.

Here the measures are the same, and they are so *because the unit of mass has been taken to be the mass of the unit of volume of the standard.* Whenever this is the case, the two measures agree. This is one advantage in using the centimeter-gram or the decimeter-kilogram system of units.

The densities of bodies are always tabulated by method B, and are then sometimes called specific gravities. The standard substance is generally water. In the following table the density-ratios or specific gravities of several common substances are given. The exact values in any case will depend on the temperature, the extent of hammering, or other process to which the bodies have been subjected—

Air at 0° C.*	·0012759	Tin	7·4
Alcohol at 0° C.	·791	Iron	7·7
Turpentine at 0° C.	·870	Copper	8·8
Ice	·92	Silver	10·5
Sea water at 0° C.	1·026	Lead	11·4
Crown glass	2·5	Mercury at 0° C.	13·596
Flint glass	3·0	Gold	19·3
Aluminium	2·6	Platinum	21·5
Zinc	7·0		

* When the pressure on any area is the same as that of a column of mercury 760 millimeters high.

9. We cannot determine from any *à priori* considerations what the properties of matter may be. Any further steps in our knowledge then must be based on experiment. The question that first presents itself is, "How do the velocities of bodies which strike each other depend on their velocities before impact and on their masses?" We must try to obtain from experiment some general statement of what takes place. These general statements are often called laws. To do this we make use of the ballistic balance, and make a long series of determinations of the resulting velocities where different masses impinge with all kinds of velocities. First, we shall simplify matters by examining what happens when the masses are prevented from rebounding after collision, and when consequently both take the same velocity. Let v denote this common velocity after impact, u_1, u_2 the velocities of the right and left-hand carriers respectively, and m_1, m_2 the masses, including that of the carriers.

Then it will be found that, if the measurements are accurate, the value of v will be given by the formula

$$(m_1 + m_2)v = m_1u_1 + m_2u_2,$$

the values of the velocities being taken to be positive when they are in the same direction. If u_2 be *opposite* to the direction of u_1 and v be considered as positive in the direction of u_1, we must put

$$(m_1 + m_2)v = m_1u_1 - m_2u_2.$$

If this formula gives v a negative value, it means that it is opposite to the direction of u_1 and in the same direction as u_2.

The student is strongly advised to himself carry out a set of actual observations with the ballistic balance to convince himself of the truth of this law.

By means of this formula we can determine the change of velocity when the two bodies do not rebound after impact. *It is particularly to be noticed that it expresses the results of experiment*, and has not been deduced from any *à priori* considerations.

A similar equation is found also to hold good even when the colliding bodies are free to rebound after striking.

In this case the rebounding bodies will have different velocities. In experimenting for this case observations of the two distances which the carriers reach after the impact will be required, and therefore two observers will be needed. In other respects the experiment will be carried out as in the previous cases, the clips having been removed. It will be found that, if v_1, v_2 denote the velocities afterwards,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

10. If we multiply together the numbers expressing the measures of a mass and the velocity with which it is moving we get another number, the magnitude of which depends on the units of mass, space, and time employed. The quantity of which this product is the measure is called the *momentum* of the moving mass, and sometimes the *quantity of motion*. If several masses are moving, then the momentum of the whole is the sum of the momenta of each separately. The result of the foregoing experiments can therefore be shortly summed up by the statement that the whole momentum of the bodies remains unaltered by impact.

The equation

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$$

may also be written

$$m_1v_1 - m_1u_1 = m_2u_2 - m_2v_2.$$

In words, the momentum gained by m_1 = the momentum lost by m_2. We are thus led to regard *the momentum as a physical quantity which is capable of being transmitted in whole or in part without loss from one body to another.*

This capacity of matter to possess momentum is called *inertia.*

Momentum then, being a real physical quantity, requires a unit of its own kind by which it may be measured. This unit is the momentum possessed by a unit of mass moving with unit of velocity. Its magnitude therefore depends on all three of the fundamental units. A unit of mass moving with v units of velocity will then possess v units of momentum, and if a mass m times as big, *or* of mass m, is moving with this velocity it will possess mv units of momentum. Thus then, when it is said that the momentum of a body is mv, what is meant is, that its momentum is mv times that momentum which unit mass moving with unit velocity possesses.

11. If we confine our attention to a single body only, what happens at a collision is the sudden change of momentum in it. The cause of the change may be the impact of a big mass moving with a small velocity or a small mass with a big velocity, or a moderate mass with a moderate velocity in an infinite number of possible arrangements.

In order, however, to avoid having to take account of the actual circumstances of a collision, it is sometimes useful to suppose an intermediate effect and cause, and to say that the change of momentum is produced by a blow. The magnitude of the blow is called the *impulse*, or sometimes shortly the "blow." *It is measured by the change of momentum produced.*

It has been seen that in any collision the gain of momentum by one body is equal to the loss of momentum by the other—*or* the changes of momentum produced are equal and opposite. Expressed in this other language, the impulses on the two masses are equal and opposite. This is found to be the case in all kinds of changes of momentum; and is a case of Newton's third law of motion that action and reaction are equal and opposite.

In considering then changes of motion of any body due to impacts, we may leave out of consideration the acting bodies and merely suppose it acted on by a series of impulses.

12. We will now apply the principles so far developed in the present chapter to the solution of a few examples.

EXAMPLE I. *Two inelastic balls moving in the same direction with velocities of* 10 *and* 8 *feet per second impinge. The masses being* 4 *and* 5 *lbs. respectively, what is the common velocity after impact?*

Note.—Bodies are said to be inelastic when they do not rebound after impact.

Since the bodies are inelastic, they have the same velocity after impact. Denote this by v.

By the experimental law that the momentum is unchanged by the impact

$$(4+5)v = \text{mom. after impact,}$$
$$= \text{mom. before impact,}$$
$$= 4 \times 10 + 5 \times 8;$$

i.e. $9v = 80,$

or $v = \frac{80}{9} = 8\frac{8}{9}$ feet per second.

If they had been moving in opposite directions

$$9v = 4 \times 10 - 5 \times 8 = 0,$$
$$v = 0,$$

or they would have come to rest.

EXAMPLE II. *A bullet whose mass is* 1 *oz. is fired with a velocity of* 1210 *feet per second into a mass of* 1 *cwt. of wood at rest. What is the velocity with which the wood begins to move?*

Since the momentum is unchanged, and the bullet and wood move on together afterwards, we have

Momentum of wood and bullet after = momentum before.

Using ounce, foot, second units, this gives, since 1 cwt. = 1792 oz.,

$$(1+1792)v = 1 \times 1210 + 1792 \times 0,$$
$$1793v = 1210,$$
$$v = \frac{1210}{1793} = \frac{110}{163} \text{ feet per second,}$$

or $v = 8\frac{16}{163}$ inches per second.

This is the principle of the ballistic pendulum, an instrument to measure the velocity of rifle bullets.

EXAMPLE III. *An* 80-*ton gun on a smooth horizontal plane fires horizontally a shot of* $\frac{1}{2}$ *cwt. with a velocity of* 1800 *feet per second. If the mass of the powder be negligible, find the velocity of recoil.*

Here the momentum before the blow by the powder is zero. Hence the momentum afterwards is also zero, or the momentum of the gun is equal and opposite to that of the shot. Let v denote the velocity of the gun; its mass is 1600 cwts. Hence

$$1600v = -\tfrac{1}{2} \cdot 1800,$$
$$\therefore v = -\tfrac{9}{16} \text{ foot per second.}$$
$$= \tfrac{9}{16} \text{ foot per second opposite to that of the shot.}$$

13. Up to this point we have only dealt completely with the case where after impact the two bodies move on together—that is, with bodies which are named inelastic. Now in general, if two bodies strike one another they rebound, and after the impact move with different velocities. To complete the study of the subject of collision it will be necessary then to consider this more general case. In any

given case we know the circumstances before impact—that is, the magnitudes of the two masses and of the two velocities. The unknown quantities are two, viz. the two velocities after impact. The principle of the constancy of the whole momentum gives one equation only between the above quantities. In order then to completely determine the velocities after impact another equation is needed, and this must be found from experimental observations. These observations were first made by Newton with an apparatus on the same principle as that described in § 7. He found that *when two bodies collide, the velocity of separation after impact is in a constant ratio to the velocity of approach before impact.* This ratio depends on the nature of the colliding bodies. It is sometimes called the coefficient of restitution, or of rebound. We shall always denote it by the letter e. The law may be expressed in other words, and perhaps more suitably for application thus: *The relative velocity after impact is* $-e$ *times the relative velocity before.* This law is not rigorously true in all cases, for the value of e will depend to some extent on the *form* of the colliding bodies, and on the velocities. It is, however, in nearly all cases very approximately true. This law, combined with the constancy of momentum, enables us completely to determine the effect of a given collision. Let m_1, m_2 be the two masses, and u_1, u_2 their respective velocities before and v_1, v_2 after impact. Then, since the momentum is unaltered,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2.$$

Since the relative velocities after and before are in the ratio $-e$,

$$v_1 - v_2 = -e(u_1 - u_2).$$

If one of the bodies is a fixed plane, $v_2 = 0$, $u_2 = 0$, and $v_1 = -eu_1$. From these two equations v_1, v_2 can be found. To do this, multiply the latter throughout by m_2, add this to the first, then

$$m_1v_1 + m_2v_1 = m_1u_1 - m_2eu_1 + m_2(1 + e)u_2,$$
$$(m_1 + m_2)v_1 = (m_1 + m_2)u_1 - m_2(1 + e)u_1 + m_2(1 + e)u_2,$$
$$v_1 = u_1 - (1 + e)\frac{m_2}{m_1 + m_2}(u_1 - u_2);$$

and similarly

$$v_2 = u_2 - (1 + e)\frac{m_1}{m_1 + m_2}(u_2 - u_1).$$

These equations completely give the motion. The student is, however, advised not to attempt to remember them, but in all cases to start from the two experimental laws of collision. Thus, take as an example the following—

Two balls of masses 3 *and* 7 *lbs. are moving in the same direction with velocities of* 20 *and* 15 *feet per second, so that the former overtakes the latter; the coefficient of rebound is* ·3. *What are the velocities afterwards?*

The two principles stated above give

$$3v_1 + 7v_2 = 3 \times 20 + 7 \times 15 = 165$$
$$v_1 - v_2 = -\cdot 3(20 - 15) = -1\cdot 5.$$

Multiplying the second by 7, and adding to the first,

$$3v_1 + 7v_1 = 165 - 1\cdot 5 \times 7,$$
$$10v_1 = 165 - 10\cdot 5 = 154\cdot 5,$$
$$v_1 = 15\cdot 45 \text{ feet per second.}$$

Also $v_2 = v_1 + 1\cdot 5 = 15\cdot 45 + 1\cdot 5 = 16\cdot 95$ feet per second.

If they had been moving towards one another, the velocity of the 7 lbs. would be denoted by -15, and the equations would have been

$$3v_1 + 7v_2 = 3 \times 20 - 7 \times 15 = -45,$$
$$v_1 - v_2 = -\cdot 3(20 + 15) = -10\cdot 5\,;$$

whence

$$10v_1 = -45 - 73\cdot 5 = -118\cdot 5,$$
$$v_1 = -11\cdot 85,$$
$$v_2 = -11\cdot 85 + 10\cdot 5 = -1\cdot 35.$$

That is, the velocity of the 3 lbs. is *reversed* and becomes 11·85 feet per second, whereas the 7 lbs. moves in the same direction as before, but with the diminished velocity of 1·35 feet per second.

The student can easily verify the truth of Newton's law that the velocity of separation is in a constant ratio to the velocity of approach by means of the apparatus already described. In order, however, to avoid rotations set up by the irregular shape of the colliding bodies, it will be necessary to make them of symmetrical form about the line of impact, such as spheres for instance. These can be suspended by two strings at a single point of the sphere, or by four strings at two points, as in the arrangement described in § 7. By employing balls of glass, steel, ivory, or other materials, it is then easy to find, as before, the velocities before and after impact and to verify the truth of the law.

14. Let us consider a little more fully what happens during a collision. The collision commences as soon as the surfaces of the bodies come in contact. Then the substance around the point of contact gets compressed, the velocities altering during this compression. At the instant of greatest compression they will have no *relative* motion, or the velocities of the two will be the same. Up to this point then the bodies behave in the same way as inelastic bodies, and their common velocity at the moment of greatest compression can be found in the same way as if inelastic. From this instant of greatest compression the surfaces begin to regain their form with different velocities, so that a relative velocity is developed, and they tend to fly apart. At the instant when the surfaces separate, the collision is at an end, and the bodies separate with the velocity they had at that instant. Denote now the impulses of the blows during the first and second periods by I_1, I_2. Then the whole blow of the impact is $I = I_1 + I_2$.

The effect of I_1 is to reduce the initial velocities u_1, u_2 to a common velocity, which we will denote by u. As it is measured by the change of momenta produced,

$$I_1 = m_1(u_1 - u) = m_2(u - u_2),$$

whence
$$u = \frac{m_1u_1 + m_2u_2}{m_1 + m_2},$$

and
$$I_1 = \frac{m_1m_2}{m_1 + m_2}(u_1 - u_2).$$

The effect of I_2 is to alter the velocity of m_1 from u to v_1, and of m_2 from u to v_2. Hence, being measured by the change of momentum,

$$I_2 = m_1(u - v_1) = m_2(v_2 - u),$$

and
$$I_2 = \frac{m_1m_2}{m_1 + m_2}(v_2 - v_1).$$

From this

$$\frac{I_2}{I_1} = \frac{v_2 - v_1}{u_1 - u_2} = e.$$

We learn then that the blows during the two portions of a collision are always in a constant ratio for the same bodies. The ratio is in fact the coefficient of rebound.

EXAMPLES—II.

1. Find the mass of a cube of iron of 1 yard side.

2. Find the masses of 1 cubic decimeter of platinum, gold, silver, and aluminium.

3. Find the mass of the earth in tons, having given, mean density of earth $=5\cdot6$, mean radius $=4000$ miles.

4. Find the size of a plate of iron 1 inch thick and weighing 1 ton.

5. Find the difference in the masses of 100 cubic decimeters of water and of ice.

6. Find the difference in the masses of 1 cubic yard of water and 1 cubic yard of ice.

7. What is the volume of (1) 1 cwt. of copper, (2) 4 kilograms of lead?

8. The mass of 20 liters of a certain fluid is 16 kilograms. What is its density?

9. The mass of 10 cubic feet of a certain substance is 2 tons. What is its density?

10. Find the density of an alloy of 2 parts by weight of copper to 3 of zinc, supposing no alteration of volume to take place.

11. The density of a solution of zinc sulphate is 1·2. How much pure water must be mixed with 1 gallon of it, that the density of the mixture may be the same as that of sea water?

12. The density of a certain substance, when water is the standard, is 2, and that of another substance, when mercury at 0° C. is the standard, is 1·6. Compare their densities.

[*In questions* 13-21 *the bodies are inelastic.*]

13. Two spherical masses of 3 and 5 tons impinge directly on one another with velocities of 4 and 5·5 feet per second respectively. Find their final velocity (1) when they are moving in the same, (2) when in opposite directions.

14. Two particles of 3 lbs. and 14 oz. are moving in opposite directions, and impinge on one another; the first has a velocity of $3\frac{1}{2}$, and the latter of 9 feet per second. In what direction will they move after impact?

15. A particle whose mass is 16 lbs. is moving with a velocity of 25 miles per hour, and impinges on another moving in the opposite direction. The two come to rest. If the mass of the latter were 28 lbs., what was its velocity?

16. If in the former question the velocity of the second particle were 66 feet per second, what was its mass?

17. A mass of 1 cwt. strikes an object of very large mass with a velocity of 20 feet per minute. What is the measure of the blow when the fundamental units are foot, pound, second?

18. Three balls of equal size, and masses of 2, 3, 5 oz., lie on a straight line at rest. If the first have communicated to it a velocity of 15 centimeters per second towards the others, what will the ultimate velocity of the balls be?

19. If in the preceding question the balls had velocities of 15, 12, and 10 centimeters per second in the same direction, what would their ultimate velocity be?

If the third had a velocity of 13·5 centimeters per second, show that the others would never overtake it.

20. A shot of 600 lbs. is fired from a 10-ton gun with a velocity of 1000 feet per second. If the mass of the powder be neglected, find the velocity of recoil.

21. An 1800-lb. shot moving with a velocity of 2000 feet per second impinges on a plate of 10 tons, passes through it, and goes on with a velocity of 400 feet per second. If the plate be free to move, find its velocity.

22. Two elastic masses of 3 lbs. and 30 oz. impinge on one another with velocities of 3 and 5 feet per second. Can it be possible for their velocities afterwards to be (1) 1 inch and 1 foot per second, (2) 5 feet and 3 feet per second?

23. Two spheres of glass of 5 and 8 oz. impinge on one another with velocities of 10 and 4 feet per second (1) in the same, (2) in opposite directions; the coefficient of rebound is $\frac{15}{16}$. Determine the motion after impact.

24. Two balls ($e=\frac{5}{6}$) impinge directly on one another; their masses are as 2 to 1, and their respective velocities before impact as 1 to 2, and in opposite directions. Show that each ball will move back after impact with five-sixths of its original velocity.

25. Two equal bodies impinge on one another with velocities of 60 and 45 miles per hour in opposite directions, and their velocities after impact are 7 and 8 miles per hour in the same direction. What was the coefficient of rebound?

26. Two masses of 4 lbs. and 8 lbs. impinge on one another in opposite directions with velocities of 5 and 2 feet per second; the former flies back with a velocity of 3 feet per second. Find the velocity of the other and the coefficient of rebound.

27. An elastic sphere (m) impinges on another elastic sphere (m_1) at rest. If, after the impact, (m) remains at rest and (m_1) moves on with

one-eighth of the velocity with which it is struck, find the coefficient of rebound and the ratio of the radii of the two spheres, supposed of the same material.

28. Two masses impinge on one another and interchange their velocities. Prove that e must be unity and the masses equal.

29. A ball at rest on a smooth horizontal plane at the distance of one yard from a wall, is impinged on directly by another equal ball moving towards the wall with the velocity of one yard in a minute. If the coefficient of rebound between the balls and wall be ·5, prove that the balls will impinge a second time after 2 minutes 24 seconds, the radii of the balls being of inconsiderable length.

30. A ball A impinges directly upon an exactly equal and similar ball B lying upon a smooth horizontal plane. If e be the coefficient of rebound, prove that after the impact B's velocity will be to A's velocity as $1+e : 1-e$.

31. If, after the impact described in the foregoing question, B moves on and impinges directly upon a perfectly elastic vertical wall from which it rebounds and meets A a second time, prove that after the second impact B will be reduced to rest.

32. Two equal marbles A, B lie in a horizontal circular groove at opposite ends of a diameter; A is projected along the groove, and after a time t impinges on B. Show that a second impact will take place after a further interval $2t/e$, e being the coefficient of rebound.

33. A ball A of mass p impinges directly upon another ball B of mass q, which is at rest; after the impact B impinges directly upon a third ball C of mass r, which is also at rest. If C has imparted to it the same velocity as A had at first, and all the balls are perfectly elastic, show that $(p+q)(q+r)=4pq$.

34. Two equal balls A, B are lying very nearly in contact on a smooth horizontal table; a third equal ball impinges directly on A, the three centres being in one straight line. Prove that if $e>3-2\sqrt{2}$, the final velocity of B will bear to the initial velocity of the striking ball the ratio $(1+e)^2 : 4$.

35. A smooth circular ring, mass M radius a, rests on a smooth horizontal table; a small spherical mass m is projected from the centre of the circle with velocity v. Prove that the whole time that elapses until the second impact is

$$\frac{a}{v} \cdot \frac{2+e}{e},$$

where e is the coefficient of rebound. Find the ultimate velocity of the ring and sphere after an infinite time.

36. Three imperfectly elastic balls of different masses and materials are capable of moving in a straight line and are originally at rest and

not in contact; one of the extreme balls is then projected with a velocity V against the middle ball, which strikes the other extreme ball and communicates to it a momentum M. Show that if the third ball had been projected in the opposite direction with the same velocity V, it would have communicated the same momentum M to the first ball.

Show also, that P, Q, R being the masses and e_1, e_2 the coefficients of restitution, if

$$PR(1+e_1e_2) > Q(P+Q+R)(e_1+e_2),$$

the middle ball will after another collision with the first strike the third again and impart to it a momentum which is the same whichever of the extreme balls be originally projected.

CHAPTER III

FORCE

15. In the preceding chapter the circumstances attending the sudden change of velocity in a mass have been investigated, and our experiments have led us, amongst other things, to the conception of a new kind of physical quantity which has been called momentum. The idea of impulse has also been introduced as the cause of a sudden change of momentum of a body. Now, clearly a given change of momentum may be gradually caused by a gradual change of the velocity, as well as suddenly by an impulse. Or it may be supposed to be generated by a series of smaller impulses acting at any intervals after one another. The smaller these impulses are the more of them will be required in order to produce a given change. If the successive impulses act more quickly, yet so that the sum of those acting in a given time remains finite, we approach nearer and nearer to the case where the momentum is gradually changed. The change of momentum in a given time will be equal to the sum of all the small impulses in this time. That is, the change of momentum per second = whole impulse per second.

Now an impulse per second is another *kind* of thing from an impulse, the time being introduced differently in the two cases. It therefore has a name of its own, and is called a *force*.

If two equal blows be given to a body in opposite directions, the body behaves precisely as if no blow acted on it at all. Similarly, if two equal forces acted in opposite directions on a body, no change of momentum would be produced. In this case each would *tend* to produce a change of momentum, and would *actually* produce it if the other were absent. We can then give the following definition of force, and statement of how it is measured—

Def. *When a gradual change of momentum is either produced or tends to be produced in a body, that body is acted on by force.*

Force is measured by the rate at which the momentum changes, or the rate at which it would change if all other forces were removed.

Whether there is an actual entity corresponding to force there is no evidence to show. Our sensation of an effort required to make a body move with our hands is a subjective one, and has nothing to do with us at present. The student will do well to banish any *à priori* ideas about force founded on his own muscular feelings, as they will only lead him astray. It will, however, be of great assistance to *regard* force as something having a real existence, and as measured by the effect it produces.

16. Momentum depends on two factors, the mass and the velocity; and it may change, therefore, either by a change in the mass or in the velocity.

The mass of a body, however, in general remains the same, and any change therefore is due to a change of velocity. Let the mass of a body be denoted by m. Then the rate of change of momentum, *i.e.* of $mv = m \times$ rate of change of velocity,

$$= m \times \text{acceleration}.$$

Hence, if we denote the force by F and the acceleration produced by it a,

$$F = ma.$$

Two accelerations along a straight line are equivalent to a single one whose magnitude is the sum of the two. Hence it follows that if forces act along the same line on a particle, the effects produced are the same as if a single force equal to the sum of the others acted on it. This statement will still hold when some of the forces are opposed to the other, if they be regarded as negative, and the algebraical sum be taken.

17. Suppose the constant force F to act for a time t. The acceleration being a, the change of velocity in that time is at, and the change of momentum = mat. But $ma = \text{F}$, therefore the change of momentum in the time $= \text{F}t$. Thus the product of a force and time gives us a momentum. In other words, this product and momentum are physical quantities of the same kind. It is sometimes called the "time integral of the force" or the "impulse of the force" during the time in question.

18. We have seen that if a change of momentum is produced in any body an equal and opposite change must be produced in some other bodies. This will apply also to forces—which are merely impulses per time. This statement is embodied in Newton's third law of motion, viz. action and reaction are equal and opposite.

It is usual to give, in connection with the present subject, Newton's three statements of the laws of motion. They are as follows:—

I. *A body at rest will remain at rest, or if in motion will move with a uniform velocity in a straight line unless acted on by some external force.*

As we know nothing about force, this is really a definition of it. The definition given above is equivalent to it, but worded so as to bring out more definitely the fact that it is merely a definition.

II. *Change of momentum is proportional to the impressed force and takes place in the direction of the force.*

Here again the "change of momentum"—or, as is meant, the rate of change of momentum—is a definite thing we can measure, whereas the force, apart from the change produced, is unknown. The "law" is then merely a statement of how force is to be measured.

III. *Action and reaction are equal and opposite.*

This statement is based on experiment. We have seen in the previous chapter how it is verified for impulses. Regarding force as impulse per second, the statement is also seen to be true for forces. It is, however, easily shown directly, as for instance where two bodies are joined by a stretched elastic string, and are allowed to approach. It can then be observed that the rates of increase of momentum of the two—that is, the forces—are equal and opposite. This law is of extreme importance.

19. Where new physical quantities are introduced, units of their own kind are required to measure them. Now the equation

$$F = ma$$

means that, if the *numbers* of units of mass and of acceleration be substituted in ma, the result gives the *number* of units of force in the force. If $m = 1$ and $a = 1$, then $F = 1$ unit of force. In other words, the unit of force produces unit acceleration in unit mass. We shall define it then as follows—

Def. *The unit of force is such that, if it acts on unit of mass, it will produce unit of acceleration;*

or,

The unit of force is such that, if it acts on unit of mass for unit of time, it will produce unit of velocity.

On account of their importance, definite names have been given to the units of force in the two cases where the fundamental units are the pound, foot, second, and the gram, centimeter, second, or C.G.S. units respectively. We shall define them anew as follows—

1. *The* POUNDAL *is that force which, acting on a mass of one pound for one second, gives it a velocity of one foot per second.*

2. *The* DYNE *is that force which, acting on a mass of one gram for one second, gives it a velocity of one centimeter per second.*

A system of two equal and opposite forces is called a *stress.* Clearly, if a stress act on a perfectly rigid body, there will be no effect produced by which the presence of the stress will be indicated. When, however, the components of the stress act on different bodies, or different portions of a non-rigid body, effects are produced. Thus in the first case the bodies move towards or from one another with equal changes of momentum. In the second case the different portions of the body actually move, but are again brought to rest by the calling into play of other stresses. The body is deformed, and the deformation is called a *strain.*

When the two components of a stress act *from* one another, as at any point of a stretched bar, it is called a *tension* :—if the bar is cut at any part, the two sides, where it is cut, move apart. When the components act towards one another, as at any point of a compressed bar, it is sometimes though not correctly * called a *pressure* :—if the bar is cut at any part, the two sides remain in contact and *tend* to move towards one another.

EXAMPLE I. *A certain force acting on* 5 *lbs. for* 3 *seconds gives it a velocity of* 4 *feet per second. How long would it take to move a mass of* 8 *lbs. through* 15 *feet?*

Here the first thing is to find the magnitude of the force, next the acceleration it produces in the 8 lbs.

It generates in 5 lbs. a velocity of 4 feet per second in 3 seconds, that is, a velocity of $\frac{4}{3}$ feet per second in 1 second—in other words, the acceleration is $\frac{4}{3}$.

Hence the force $=5\times\frac{4}{3}=\frac{20}{3}$ poundals.

Next, let a be the acceleration it produces in 8 lbs. Then

$$\text{Force}=8a \text{ poundals},$$

$$\therefore\ 8a=\frac{20}{3},$$

or

$$a=\frac{5}{6}.$$

The acceleration it will produce in the 8 lbs. is then $\frac{5}{6}$ foot per second per second.

Let t be the time required to move it through 15 feet.

Then by the formula $s=\frac{1}{2}at^2$

$$15=\frac{1}{2}\times\frac{5}{6}t^2,$$

$$t^2=36,$$

whence

$$t=6 \text{ seconds}.$$

EXAMPLE II. *Two masses of* 6 *and* 11 *lbs. are connected by an inelastic string of no mass; they are acted on by forces of* 48 *and* 14 *poundals in opposite directions. Determine the motion.*

Since the string is inelastic, they will move on together. Let the

* Pressure is properly the intensity of the stress over a surface, as for instance 10 poundals per square foot.

acceleration be a. Then the whole force acting on them is $48-14=34$ poundals, whereas the mass moved is $6+11=17$ lbs. Hence

$$17a=34,$$
$$a=2 \text{ feet per second per second.}$$

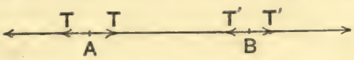
EXAMPLE III. *If the fundamental units are the ton, yard, and minute, express the unit of force in poundals.*

The unit force produces an acceleration of 1 yard per minute per minute in a mass of 1 ton.

Now 1 yard per minute per minute $=3$ feet per 60 seconds in 60 seconds $=\frac{1}{20}$ foot per second in 60 seconds $=\frac{1}{20\times 60}$ foot per second per second, also 1 ton $=2240$ lbs. Therefore

$$\text{The unit of force}=2240\times\frac{1}{20\times 60} \text{ poundals,}$$
$$=\frac{112}{60}=\frac{28}{15}=1\frac{13}{15} \text{ poundals.}$$

20. When equal and opposite forces are applied at the ends of a string or bar, the tension or pressure is the same at every point. For consider the two points A, B of a stretched string, and suppose the tension at the two points to be T, T′.



Then, confining our attention to what happens to the portion AB, the whole force on it is $T-T'$. But the string is at rest; hence there is no force, *i.e.* $T-T'=0$ or $T=T'$. If, however, the string is in motion and is accelerated, this is no longer the case. In fact, if m be the mass of the string between A and B, and a the acceleration, then

$$T-T'=ma.$$

In many cases the mass of a string is so slight compared with the other masses treated that it may be regarded as nothing. In this case again $T=T'$, and the tension may be regarded as the same throughout.

As an example, suppose a force equal to 10 poundals to act at one end of a rope 100 feet long, whose mass per foot is 8 oz., and let us find the tension at a distance 20 feet from the free end.

The first thing is to find the acceleration. Now the mass $=100\times 8$ oz. $=50$ lbs. and the force is 10 poundals Hence acceleration $=\frac{10}{50}=\frac{1}{5}$ foot per second per second.

Let T be the tension at a point distant 20 feet from the free end. Now T is the force which drags along the 20 feet of rope with acceleration $\frac{1}{5}$. The mass $= 20 \times 8$ oz. $= 10$ lbs.

Hence $$T = 10 \times \tfrac{1}{5} = 2 \text{ poundals.}$$

21. *Gravitation.*—So far we have concerned ourselves with one only of the fundamental properties of matter, viz. inertia. There is another, to which the name of gravitation is given. It shows itself in a tendency for any two masses to move towards one another—in other words, a force of attraction exists between every portion of matter and every other portion. Here, however, it will be necessary to enter fully into one case only of it, viz. the force between the earth and any body at its surface. If any body be free to move at the surface of the earth, it always falls towards the earth; whence, according to the first law of motion, a force must act on it. This force is called the *weight* of the body.

A falling body is found to obey the following laws—

(1) It falls with a constant acceleration.

(2) This acceleration is independent of the mass of the falling body, but depends on the locality.

We shall see immediately in §§ 22, 23 how the truth of these statements can be shown experimentally, and other more exact methods will be given later. At present let us see what deductions can be drawn from them.

From (1), since the mass and the acceleration remain constant, the force acting is constant. In other words, the weight of a body is the same at different heights, and is independent of the velocity of the body. This is not found to be true for very great differences of heights, nor is it rigorously true even for small heights. The deviation from constancy, however, is so small as to be quite inappreciable except by the most delicate apparatus.

From (2) we learn that the weight of a body at a given place is proportional to its mass. For let m_1, m_2 be the masses of two bodies and a_1, a_2 their accelerations, then the forces acting on them—*i.e.* their weights—are

$$F_1 = m_1a_1, \quad F_2 = m_2a_2.$$

But, since $a_1 = a_2$ according to law (2),

$$\frac{F_1}{F_2} = \frac{m_1a_1}{m_2a_2} = \frac{m_1}{m_2},$$

or the weights are proportional to the masses.

This fact, that the mass of a body is proportional to its weight, is that on which is based the practical way of measuring masses referred to in § 6.

The acceleration of gravity is usually denoted by the letter g. It is found to vary at different parts of the earth's surface from 32·091 feet per second per second at the equator to 32·255 near the pole. The value of g at a point h units of length above the sea level and in latitude λ is, in F.P.S. units, $g = 32{\cdot}173 - {\cdot}082 \cos 2\lambda - {\cdot}000003h$; in C.G.S. units, $g = 980{\cdot}6056 - 2{\cdot}5028 \cos 2\lambda - {\cdot}000003h$.

If the place is on land at a height h, the term depending on h has to be modified for the attraction of the extra land between the point and sea level.

The following table, based on Everett's *Units and Physical Constants*, gives the value of g at sea level in a few localities—

	Latitude.	Value of g	
		in F.P.S. units.	in C.G.S. units.
The equator .	0° 0′	32·091	978·10
Paris . . .	48° 50′	32·183	980·94
London . .	51° 29′	32·191	981·17
Berlin . . .	52° 30′	32·194	981·25
Edinburgh . .	55° 57′	32·203	981·54
The pole . .	90° 0′	32·255	983·11

Unless we wish to be very exact, we shall take the numerical value of g to be 32 and 981 in the two systems of measurement respectively.

Denote the weight of a body by W and its mass by m. Then the force W acting on a mass m produces an acceleration g. Therefore

$$W = mg \text{ units of force.}$$

This will enable us to get some idea of the magnitude of a poundal or of a dyne in terms of forces with which we are familiar. Thus—

Weight of 1 lb. $= 1 \times 32$ poundals;
whence 1 poundal $= \frac{1}{32}$ weight of 1 lb.,
$= \frac{16}{32} = \frac{1}{2}$ weight of 1 oz.,
= weight of $\frac{1}{2}$ oz.

So also

Weight of 1 gram = 981 dynes;
or 1 dyne $= \frac{1}{981}$ weight of 1 gram,
= 1·02 weight 1 milligram,
= weight of 1 milligram nearly.

The student must keep clearly distinguished in his mind the difference between the mass of a body and its weight. The mass is the quantity of matter in it; the weight is the force with which the earth pulls it. The mass is constant, the weight depends on the locality.

EXAMPLE. *A force which can just support* 2 *cwts. acts for* 1 *minute on* 3 *tons. Find the velocity it produces in it.*

The force = weight of 224 lbs.,
$= 224 \times 32$ poundals.

Let a be the acceleration it produces in 3 tons or 6720 lbs. Then

$$\text{Force} = 6720a.$$

$$\therefore\ 6720a = 224 \times 32,$$

$$a = \frac{224 \times 32}{6720} = \frac{16}{15} \text{ foot per second per second.}$$

$\therefore$ velocity in 1 minute $= 60a = \frac{16}{15} \times 60 = 64$ feet per second.

The student should not, however, get into the habit of always expressing the magnitudes in F.P.S. or C.G.S. units. Thus in the preceding example it would be better to take the cwt. as unit of mass. Then the

Force = weight of 2 cwts.,
$= 2g$ units.

Also $3 \text{ tons} = 60 \text{ cwts.}$

Then Force $= 60a$ units ;

$\therefore\ 60a = 2g,$

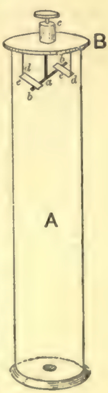
$a = \frac{g}{30} = \frac{16}{15}$ foot per second per second.

22. The facts that the acceleration of a falling body is constant, and that it is independent of the mass of the body and of its velocity can only be known as the results of observation. It is easy to see the approximate truth of the first statement by letting a stone fall from, say, a first-floor and second-floor window to the ground and noting the times it takes in each case to reach the ground. It will be found that the heights are to one another in the same ratio as the square of the times of fall. This shows that the acceleration is constant, for when the acceleration is constant

$$s = \tfrac{1}{2}at^2,$$

or, the spaces are proportional to the squares of the times. It will be seen below, in finding the actual magnitude of the acceleration, how the statement can be verified with greater exactness.

The following experiment will show the truth of the second statement, viz. that the acceleration is independent of the mass. A cylindrical glass receiver A is fitted with a cover B, to which it is ground accurately true and made air-tight with a little grease. Through this cover passes a rod working in an air-tight collar *c* and having arms *bab* inside ; *d*, *d* are small pillars on the cover to which are hinged small plates *e*, *e* so arranged that their other ends can rest one on each arm *b*, whilst they will slip off the arms at the same instant when *bab* is rotated. On these plates are placed different bodies, as for instance a piece of cork, lead, and a feather. The air is then exhausted from the interior. If now the rod be turned, the bodies will fall at the same instant, and it will be observed that they reach the bottom at the same instant also. That is, the accelerations with which they move are the same. It is necessary



to make the experiment in a vacuum, for otherwise the effect of gravity is masked by others due to the presence of the air.

The following examples will serve to illustrate the principles developed in this chapter, and will also lead up to a method of determining the value of the acceleration of gravity by experiment.

EXAMPLE I. *A body of mass* m *lies on a smooth horizontal table; to it is connected one end of an inextensible string which passes over a pulley at the edge of the table and supports a mass* m′; *the system will move. Find the acceleration and the tension of the string.*

It is to be noticed that the effect of the pulley is merely to change the direction of the string, and not to alter its tension. Also, since the table is smooth, there is no force tending to stop the mass m when it is once in motion. The only force then on m is the tension of the string. The string has no mass, and therefore requires no force to set it in motion. Consequently the tension is the same at every point. (See § 20.)

Let T denote the tension of the string and a the acceleration of either mass, it being the same for each, since the string is inextensible.

We shall treat the question in two ways.

Method 1. Consider the circumstances of each mass separately.

The only force on the mass m, moving it on the table, is the tension T; the acceleration produced is a. Hence

$$\mathrm{T}=ma,$$

or

$$a=\frac{\mathrm{T}}{m}.$$

The forces on m' are the tension T upwards and its weight $m'g$ downwards. These are equivalent to one force $m'g-\mathrm{T}$ downwards. This acts on a mass m' and produces acceleration a. Therefore

$$a=\frac{m'g-\mathrm{T}}{m'}=g-\frac{\mathrm{T}}{m'}.$$

But we have already seen that

$$a=\frac{\mathrm{T}}{m},$$

$$\therefore\ \frac{\mathrm{T}}{m}=g-\frac{\mathrm{T}}{m'};$$

or

$$m'\mathrm{T}+m\mathrm{T}=mm'g,$$

$$\mathrm{T}=\frac{mm'}{m+m'}g.$$

That is, the tension is $\frac{mm'}{m+m'}g$ units of force, or is equal to the *weight* of a mass $\frac{mm'}{m+m'}$.

Also
$$a=\frac{\mathrm{T}}{m}=\frac{m'}{m+m'}g.$$

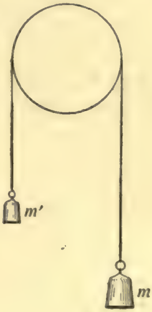
Method 2. In this method regard the two bodies as one, which is legitimate since they move as such. The only force acting on the composite body and which produces motion is the weight of the hanging portion, *i.e.* $m'g$. The mass moved is the sum of the two, *i.e.* $m+m'$. Hence the acceleration is

$$a=\frac{\text{force}}{\text{mass}}=\frac{m'g}{m+m'}.$$

To find the tension we must consider the motion of one, as in the former method

$$\mathrm{T}=ma=\frac{mm'}{m+m'}g.$$

EXAMPLE II. *Two masses* m, m' *are connected by a string whose mass can be neglected, and which passes over a massless pulley. Determine the acceleration and the tension of the string.*



Suppose m to be the larger, then the force on m downwards is $mg-\mathrm{T}$ as in the former case,

$$\therefore\ a=\frac{mg-\mathrm{T}}{m}=g-\frac{\mathrm{T}}{m}.$$

So also force on m' is upwards and $=\mathrm{T}-m'g$,

$$\therefore\ a'=\frac{\mathrm{T}-m'g}{m'}=\frac{\mathrm{T}}{m'}-g.$$

Now $a=a'$,

$$\therefore\ g-\frac{\mathrm{T}}{m}=\frac{\mathrm{T}}{m'}-g,$$

$$\mathrm{T}\left(\frac{1}{m}+\frac{1}{m'}\right)=2g;$$

$$\therefore\ \mathrm{T}=\frac{2mm'}{m+m'}g;$$

also
$$a=g-\frac{\mathrm{T}}{m}=\left(1-\frac{2m'}{m+m'}\right)g,$$
$$=\frac{m-m'}{m+m'}g.$$

If $m=m'$, $a=0$. That is, there is no acceleration, and they will be at rest, *or will move with constant velocity.*

It will be noted that a is the acceleration of m *down*, and a' of m' *up*. It might be thought, therefore, with regard to what has been said as to direction, that the true equation should be $a = -a'$. But this would not be correct. In fact the presence of the pulley alters the direction in space, but does not do so in reference to the motion along the string, which is what we have to deal with here. The accelerations a and a' are both in the direction of the string's motion, and therefore $a = a'$.

By taking m and m' nearly equal, $m - m'$ becomes very small, in which case the acceleration a will also be small, and therefore easily measurable. This being done, the value of g can be deduced, viz.—

$$g = \frac{m + m'}{m - m'} a.$$

This was the method employed by Atwood (see Art. 23).

Example III. *A length of* 20 *feet of a heavy chain is suspended by a wire, whose mass may be neglected, over a pulley; the other end of the wire is fastened to a mass of* 56 *lbs.; the mass per foot of the chain is* $2\frac{1}{2}$ *lbs. Find* (1) *the acceleration,* (2) *the tension of the wire,* (3) *the tension of the chain at a point* 6 *feet from its lower end.*

(1) The mass of the chain is $20 \times 2\frac{1}{2} = 50$ lbs. Therefore the whole mass moved is $56 + 50 = 106$ lbs. The force moving it is the weight of $(56 - 50)$ lbs. $= 6g$ poundals.

$$\therefore \text{Acceleration is } \frac{6g}{106} = \frac{3}{53} g.$$

(2) Also force on chain alone $= (\text{T} - 50g)$ poundals, where T is the tension of the wire.

$$\therefore \text{Acceleration} = \frac{\text{T} - 50g}{50}.$$

$$\therefore \frac{\text{T}}{50} - g = \frac{3}{53} g,$$

$$\frac{\text{T}}{50} = g + \frac{3}{53} g = \frac{56}{53} g,$$

$$\text{T} = \frac{2800}{53} g \text{ poundals},$$

$$= \text{weight of } \frac{2800}{53} \text{ lbs.}$$

(3) Let T′ be the tension of the chain at a point 6 feet from the lower end. Then the function of T′ is to take part in accelerating the 6 feet of chain below it.

The mass of this portion $= 6 \times 2\frac{1}{2} = 15$ lbs. The force on it is $\text{T}' - 15g$ upwards.

$$\therefore \text{Acceleration} = \frac{\text{T}' - 15g}{15} = \frac{\text{T}'}{15} - g.$$

But the acceleration is that found before, viz. $\frac{3}{53}g$.

$$\therefore \frac{T'}{15} - g = \frac{3}{53}g,$$

$$T' = 15 \times \frac{56}{53}g = \frac{840}{53}g \text{ poundals,}$$

$$= \text{weight of } \frac{840}{53} \text{ lbs.}$$

EXAMPLE IV. *One of the masses in Example II suddenly strikes another* M, *and the two go on together. Determine the circumstances of the whole motion.*

The circumstances will differ according to which of the two moving ones has the collision. If it is the less (say m'), it is moving up at the moment, the collision makes it suddenly go more slowly, there is an impulsive jerk on the string, and m is also made to go more slowly.

After collision the system will go on with another constant acceleration. If m' and M are together less than m, the acceleration is in the same direction as before, and the only effect is that they are accelerated less. If m' and M are together greater than m, the acceleration is in the opposite direction, and therefore they will go more and more slowly, come to rest, and then begin to move in the opposite direction. If $m' + M = m$, there will be no acceleration after the impact, and the system will go on with constant velocity. This is all clear without any calculations. To find the actual changes we must have recourse to calculation.

Before the collision—

$$\text{Acceleration} = \frac{m - m'}{m + m'}g,$$

$$\text{Tension} = \frac{2mm'}{m + m'}g.$$

At the collision—

Let the common velocity just before impact be u and afterwards be v. The impact takes place in so short a time that gravity during the collision has no effect. It is therefore a case of simple impact, in which case the momentum after impact = momentum before,

i.e. $$(m + m' + M)v = (m + m')u$$

$$v = \frac{m + m'}{m + m' + M}u,$$

and

$$\text{Change of velocity} = u - v = u - \frac{m + m'}{m + m' + M}u = \frac{M}{m + m' + M}u.$$

After the collision—

$$\text{Acceleration} = \frac{m - (m' + M)}{m + m' + M} g,$$

$$\text{Tension} = \frac{2m(m' + M)}{m + m' + M} g,$$

for the m' and M are equivalent to one $= m' + M$.

Here if $m' + M < m$, acceleration is $+$, *i.e.* in the same direction as before; if $m' + M > m$, acceleration is $-$, *i.e.* in the opposite direction.

On the second supposition, that the collision takes place with the greater mass, the result is more complicated. The mass m is going downwards, and by the impact is made to go more slowly, consequently the string now becomes *slack*. As the tension of the string is now zero, m' is retarded by its weight at the rate g. M and m are accelerated at the same rate, until the string again becomes tight. Then another jerk takes place, and the two suddenly take up a common velocity and proceed with a greater acceleration than the original one.

Let us consider each of these steps in detail.

Before the collision—

$$\text{Acceleration} = \frac{m - m'}{m + m'} g,$$

$$\text{Tension} = \frac{2mm'}{m + m'} g.$$

At the collision—

Suppose the common velocity just before to be u. Then m' is unaffected by the collision, and its velocity is unchanged and $= u$. The string becomes slack, and the mass $m + M$ goes on with altered velocity. If this be v,

$$(m + M)v = mu,$$

$$v = \frac{m}{m + M} u.$$

While the string is slack—

Afterwards m' moves freely under a retardation g, and with initial velocity u. Also $m + M$ moves freely with acceleration g and initial velocity v. The string gets loose, and after, say a time t, gets tight again. This will happen when the space gone up by m' is equal to the space gone down by $m + M$, for they are then again at the same distance from one another.

Now space gone up by m' $= ut - \frac{1}{2}gt^2$,

" " down by $m + M = vt + \frac{1}{2}gt^2$;

$$\therefore ut - \tfrac{1}{2}gt^2 = vt + \tfrac{1}{2}gt^2,$$

$$u - v = gt.$$

But
$$v=\frac{m}{m+M}u;$$
$$\therefore gt=\frac{M}{m+M}u,$$
$$t=\frac{M}{m+M}\frac{u}{g}.$$

Also at this time the
$$\text{velocity of } m'=u-gt=v,$$
$$\text{velocity of } m+M=v+gt=u,$$
or, the masses have interchanged their velocities.

When the string becomes tight—

The velocity of m' before $=v=\frac{m}{m+M}u$. The velocity of $m+M$ is u. Let V be the common velocity afterwards. Then
$$(m'+m+M)V=m'v+(m+M)u,$$
$$=\left(\frac{mm'}{m+M}+m+M\right)u;$$
$$\therefore V=\frac{mm'+(m+M)^2}{(M+m)(m'+m+M)}u.$$

Final state—
$$\text{Acceleration}=\frac{m+M-m'}{m+M+m'}g,$$
$$\text{Tension}=\frac{2m'(m+M)}{m+M+m'}g.$$

23. *Experimental determination of the value of* g.—The acceleration of a body falling freely is so great that it is not at all an easy matter to observe its magnitude. Moreover, when moving quickly, the resistance of the air is so large as to introduce great errors. The earliest attempts were based on methods which diminished the acceleration in a known ratio; this diminished acceleration was then observed, and the acceleration of free fall deduced from it. Thus, if the method produced an acceleration $\frac{1}{20}$ that of gravity, and this diminished acceleration was observed to be 1·6 feet per second per second; the value of g would be $20 \times 1\cdot6 = 32$. Apart from the fact that apparatus to diminish the acceleration introduces complications due to friction and other causes, the method is a bad one for another reason, which perhaps may best be illustrated by an example. In all physical measurements a certain small error of observation is always liable to

occur. Suppose now in the above observation an error so small as ·01 were made, so that instead of 1·6 the true observation should have been 1·61, then the calculated value of g would have been $1·61 \times 20 = 32·2$, showing a large error in the value of g of ·2, twenty times that of the original one.

In order to diminish the acceleration, Galileo employed an inclined plane, allowing a body to slide down it under the action of gravity. By the principles developed in Chapter VII, it is possible to calculate the ratio of this diminished acceleration to that of free fall when the inclination is known, and thence to deduce a value of g. It is difficult, however, to avoid errors due to friction.

In 1784 Atwood published a new method for finding the value of g. Although it is not susceptible of giving very accurate values, yet his machine, as it is called, is very interesting, as it may be used to exemplify in a striking manner the principles developed in this chapter. The frontispiece is reduced from the plate in his book *On the Rectilinear Motion and Rotation of Bodies.* The essential part is a pulley over which passes a light string carrying masses at its two ends. In order to reduce the friction as much as possible, the axle of the pulley rests on two pairs of other wheels called friction pulleys, which diminish the friction. The pendulum shown is for the purpose of measuring the time. The masses A and B are equal. Then A is loaded with a small mass. This causes A to descend and B to ascend with an acceleration, as we have already seen,

$$= \frac{M}{2m + M} g,$$

where m is the mass of A or of B, and M of the additional mass placed on A. The pendulum beats seconds, or at least some known interval of time. The masses are then placed so that A is at the top of the graduations on the graduated rod, and they are set free at one tick of the pendulum. A then falls until it strikes against the small shelf. The position of this shelf is so arranged that A strikes it just as the pendulum makes the second tick. This can be done with great accuracy after several trials, for the ear can easily decide whether the sounds of the tick and the blow occur simultaneously. It only remains then to measure the distance to the shelf, which gives the distance fallen by A in one second, and from this we deduce at once its acceleration from the formula

$$s = \tfrac{1}{2} \cdot a \cdot 1^2 = \tfrac{1}{2}a.$$

Thus, for instance, if $m = 62$ oz. and $M = 4$ oz., we know that

$$a = \frac{M}{2m + M}g = \frac{4}{124 + 4}g = \frac{1}{32}g;$$

also by observation from the machine it will be found that the shelf must be placed at a distance 6 inches below the top, whence

$$a = 2s = 2 \times \tfrac{1}{2} = 1 \text{ foot per second per second,}$$

$$\therefore \frac{g}{32} = 1,$$

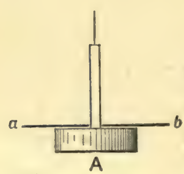
$$g = 32 \text{ feet per second per second.}$$

By taking the interval more than one second, the position of the shelf will be farther down, and the measurements may be made more accurately.

Modifications of Atwood's machine have been made, and various contrivances to give more accurate observations, into which we need not enter, especially as there is a much more accurate way of determining the value of g given in §§ 162, 203.

The application of Atwood's machine, as here described, can, however, never give accurate, or even nearly accurate, values of g. For not only have corrections to be applied to take account of the friction of the apparatus, the resistance of the air, and the weight of the string; but a much larger correction has to be applied owing to the mass of the pulleys over which the string passes. The weight of the pulleys of course produces no effect in moving the system, but as the pulleys have to be moved, and their angular motion is accelerated, as well as the motion of the suspended masses, part of the moving force is used up in this acceleration of the pulleys. This is always comparable with the whole effect. We shall see how this is to be taken account of in Part III. (See § 200.)

If the student can have access to an Atwood's machine it will, however, be of very great value to him to practise with it, not only in finding the value of g, but in illustrating the laws deduced in the preceding examples—for instance, in proving that the spaces described are proportional to the squares of the times, that the velocity generated is proportional to the time of fall, or to the square root of the distance through which it has fallen. The velocity can be measured in the following manner: The additional weight M is made of the form ab in the adjoining figure, and can be slipped over the mass A as in the second figure. In its fall A comes to a ring through which A itself can pass, but which catches off ab. After this has taken place the two moving masses are equal, and the acceleration is therefore zero. Consequently the bodies move on with the same velocity that they had at that instant. This velocity can then be measured either by observing the time which A takes to move from the ring to a lower shelf, or the space through which A travels in a given time. From this we deduce what the velocity of A was when ab was taken off by means of the formula



$$s = vt.$$

By making observations for different intervals and different distances of fall, it will be found that the value of g comes out always the same, which proves the statement in § 21 that the acceleration of gravity is constant.

24. *Pressure due to continuous impact.*—When a large number of particles impinge continuously against an obstacle, the same effect is produced as if the obstacle was acted on by a force. Thus, for instance, a water jet or

sand blast directed against a plate exerts a force on the plate due to the fact that the momentum of the water or sand is being continuously destroyed. We shall illustrate the treatment of this class of problems by considering the pressure produced on a floor by a heavy chain falling on it, lying on the floor as it falls. To fix our ideas, suppose the chain held up at one end with the other hanging vertically down, and that then it is let fall. It falls on the floor, coiling up as it does so. Suppose that the velocity with which the chain is falling in is v feet per second, which for the present we shall suppose is kept constant. Also let m lbs. denote the mass of a foot of the chain. Then in one second a length $v \times 1 = v$ feet falls in, and its mass is therefore $m \times v = mv$. Its momentum before falling in was mass $\times$ velocity $= mv \times v = mv^2$, and this is destroyed every second. In other words, momentum is being destroyed at the rate mv^2, but the force produced is the rate of change of momentum. Hence the pressure produced by the falling chain

$$= mv^2 \text{ poundals.}$$

This is the pressure when the velocity is v, and does not depend on whether the velocity was different before or after. If then the velocity be changing, at the instant when the velocity is v the pressure is still mv^2. Hence after falling t seconds the velocity is gt, and the pressure is mg^2t^2.

EXAMPLE. *A uniform chain of* 1 *lb. to the yard has its lower end just touching the ground, and is let fall. Find the pressure on the ground after* 5 *seconds.*

The mass of one foot is $\frac{1}{3}$ lb. and the velocity is $5g$, hence the pressure due to the fall is $25g^2/3$ poundals. Also a length $g/2 \times 5^2 = 25g/2$ has already fallen in, and it presses on the ground with its weight. Its mass $= 1/3 \times 25g/2$, and its weight $25g^2/6$ poundals. Therefore whole pressure

$$= \frac{25}{3}g^2 + \frac{25}{6}g^2 = \frac{25}{2}g^2 \text{ poundals,}$$

$$= \text{weight of } \frac{25g}{2} \text{ lbs.,}$$

$$= \text{weight of } 400 \text{ lbs.,}$$

whilst the real weight of the portion on the floor is only that of $133\frac{1}{3}$ lbs.

EXAMPLES—III.

1. A mass of 7 lbs. is suspended from a fixed point by a uniform string which weighs 18 oz. Find the tension of the string at its middle point and at its extremities.

2. If the resistance of the air is supposed to be always four-fifths of the weight of a body, find how high a body will go if shot vertically upwards with velocity 900 feet per second, and prove that the body will reach the point of projection again after 62·5 seconds, taking $g=32$ [F.P.S.]

3. A force of 5 lbs. weight acts on a mass of 48 lbs. Find the acceleration produced.

4. A certain force acting on 1 ton for 5 minutes gives it a velocity of 6 yards per minute. Express it in poundals.

5. A certain force acting on 1 kilogram for an hour gives it a velocity of 100 kilometers per minute. Express it in dynes.

6. A ball whose mass is 3 lbs. is falling at the rate of 100 feet per second. What force, in addition to its weight, expressed in lbs. weight, will stop it (1) in 2 seconds, (2) in 2 feet?

7. How many dynes are there in one poundal? (See table on p. 4.)

8. A force acting on 7 lbs. gives it an acceleration of 96 feet per second. What acceleration would it produce in 1 cwt., and how long would it take to move it through 8 feet 4 inches?

9. A force acting on 1 milligram for 1 second gives it a velocity of 100 meters per minute. What velocity would it give to a kilogram in 1 day?

10. A train is moving on a horizontal rail at the rate of 15 miles an hour. If the steam be suddenly turned off, how far will it run before it stops, the resistances being taken at 8 lbs. weight per ton?

11. Prove that an engine capable of exerting a uniform pull of 3 tons weight can take a train of 120 tons on the level from rest at one station to stop at the next station 2 miles off in 3 minutes 38 seconds, the speed being kept uniform when it has reached 45 miles per hour and the brakes bringing the train to rest in 368½ yards. (Neglect passive resistances.)

12. A certain force acting on a mass of 10 lbs. for 5 seconds produces in it a velocity of 100 feet per second. Compare the force with the weight of 1 lb. and find the acceleration it would produce if it acted on a ton.

13. A certain force can just support a mass of 8 tons. How far would it move a mass of 16 tons in 1 minute if no other force acted on it?

14. If the force in the preceding question were applied to lift a mass of 6 tons against gravity, how far would it raise it in 15 seconds?

15. A man of 12 st. weight is riding on a trolly of 1 ton; he pulls on a fixed rope with a force equal to the weight of 56 lbs. Find the acceleration of the trolly. Also find the acceleration if he gets off and pulls the trolly with the same force.

Find the force with which his feet push along the ground.

16. A bullet with an initial velocity of 1500 feet per second strikes a target 1200 yards distant with a velocity of 900 feet per second, the range of the bullet being assumed to be horizontal. Compare the mean resistance of the air with the weight of the bullet.

17. A ball of elasticity $=\frac{1}{2}$ falls from a height of 64 feet upon a horizontal plane. Find the height to which it will rise at the first rebound and the time at which the rebounding will cease.

18. A heavy elastic ball drops from the ceiling of a room and after twice rebounding from the floor just reaches a point half the height of the room. Show that the coefficient of rebound $=1/\sqrt[4]{2}$.

19. Two particles are projected at the same instant with the same velocity v, one vertically upwards and the other vertically downwards, from a point at a height h above a perfectly elastic horizontal plane. Prove that they will meet again at the same point if $v^2=\frac{2}{3}gh$.

20. If a given impulse acting on a stone causes it to rise to a height h, how high will a stone of half the weight of the former rise under the action of the same impulse?

21. A ball is projected vertically upwards with a velocity of 160 feet per second; when it has reached its greatest height it is met in direct impact by an equal ball which has fallen through 64 feet. Find the times from the instant of impact to those in which the balls reach the ground, the coefficient of rebound between them being $\frac{1}{2}$.

22. Given that a quadrant of the earth's surface is 10^9 centimeters, and that the mean density of the earth is 5·67, prove that the C.G.S. unit of force will be the attraction of two particles, each of 3928 grams, at a distance of one centimeter, the acceleration of gravity at the earth's surface being 981 centimeters per second per second.

23. If the attraction of gravitation between two unit masses at the unit distance from one another be taken as the unit force, express the unit mass in lbs., when the units of space and time are a foot and a second respectively: gravity at the earth's surface being regarded as due solely to the attraction of the earth considered as a sphere of radius 21,000,000 feet and of uniform density equal to $5\frac{2}{3}$ of the density of water.

[*Note.*—A sphere attracts as if the whole mass is condensed at its centre. The attraction can then be found by the law in § 21.]

24. Find the attraction of two pound-weights, a foot apart, in terms of the weight of a lb. (Rough approximations to the numerical results will suffice.)

25. Find the difference in the weight of one ton at London and at Edinburgh.

26. If the time of a body's fall from a certain height at one place on the earth's surface be m seconds less than that at another place, and the velocity acquired in the fall be a feet per second greater, prove that a/m is the geometric mean of the acceleration of gravity at the two places.

27. A pile is driven a feet vertically into the ground by n blows of a steam-hammer fastened to the head of the pile. Prove that if p is the mean pressure of the steam in lbs. per square inch, d the diameter of the piston in inches, e the length of the stroke in feet, w the weight in lbs. of the moving parts of the hammer, and W the weight of the pile and the fixed parts of the steam-hammer attached to it, then the mean resistance of the ground in lbs. weight is

$$\mathrm{W} + w + \frac{nw}{\mathrm{W} + w}\left(w + \tfrac{1}{4}\pi d^2 p\right)\frac{e}{a}.$$

28. Find the ratio of the units of force in the two cases where the fundamental units are (1) ton, mile, hour, and (2) stone, yard, minute.

29. Two masses of 8 and 3 kilograms are connected by a string, and forces of 1600 and 500 dynes act on them respectively in opposite directions. Find the acceleration and the tension of the string.

30. Two masses of 18 and 12 grams are connected over a small pulley. Find the acceleration and the tension of the string.

31. Two masses of 5 lbs. and 4 lbs. are connected by an inextensible string which hangs over a smooth pulley. Find the acceleration of each and the tension of the string. After the greater has descended 4 feet the string breaks, how far will each move in the next second?

32. A uniform chain whose mass is 28 lbs. has one end attached to a mass of one cwt. and the other end is pulled with a force equal to the weight of 56 lbs. Find the acceleration and the tension of the chain at its middle point.

33. If in the preceding question the mass and chain lay on a smooth horizontal table and the end were connected by a massless rope over a pulley at the edge of the table with a mass of 56 lbs. hanging freely, find the acceleration and the tension at the middle of the chain.

34. Two weights, each equal to 8 oz., are in equilibrium over a pulley, and $\frac{1}{10}$ oz. is then added to one of them. Determine how long it will be in descending 10 feet and what velocity it will acquire in so doing.

35. Two equal scale pans are suspended over a pulley, the mass of

each being 6 oz. A mass of 8 oz. is then placed in one of them. How long will it take to fall through 10 feet and what will its velocity then be?

36. In the preceding question find the pressure between the scale pan and the 8-oz. mass.

37. A man descends in a lift with an acceleration $\frac{1}{10}$ that of gravity. What is the ratio of his pressure on the lift to his weight?

38. If a weight W be connected by a weightless string hanging over a smooth pulley with a scale pan containing two weights, each equal to W, lying one upon the other, find the pressures during free motion between these weights—the weight of the pan being neglected.

39. A bucket containing a cwt. of coal is being drawn up from a coal-pit, so that the pressure of the coal on the bottom of the bucket is equal to that of 126 lbs. Find the acceleration of the bucket.

40. Masses of 4 and 6 lbs. hang over a pulley; the 4-lb. mass is projected downwards with a velocity of 128 feet per second. How long is it before its velocity is zero?

41. Inelastic masses of 15 and 17 lbs. hang over a pulley, and are at the same height of 16 feet above a rigid plane. Describe generally the subsequent history and discuss in detail what happens up to the second impact.

42. Discuss the same question as the preceding in the case where the coefficient of rebound between the masses and the plane is $\frac{1}{2}$.

43. (*a*) Two masses of 8 and 4 oz. hang over a massless pulley. After moving for 2 seconds, the 4 oz. catches up another mass of 2 oz. previously at rest. Determine the subsequent motion.

Also if the mass caught up had been 6 oz.

(*b*) If in the previous case the larger mass had caught up the 2 oz., determine the subsequent motion.

(*c*) Determine the motion in the example 43*a*, if the 2 oz. had been moving with a velocity of 10 feet per second upwards at the moment when it was caught up by the 4 oz. mass.

44. Two balls, unequal in weight and connected by a string, hang over a pulley; they are allowed to move from rest so that one of them encounters a fixed horizontal plane and rebounds. Show that if the modulus of elasticity be $\frac{1}{2}$, the string will become tight again just at the time when the heavier ball ceases to rebound, and that ball will have oscillated through spaces amounting cumulatively to two-thirds of the free fall required to produce its original velocity.

45. Two masses m, m' are suspended freely over a pulley; the pulley itself is moved upwards with acceleration a. Find the acceleration of each mass and the tension of the string.

46. If in the preceding question the pulley itself be suspended over another fixed pulley by a string attached to a mass M, find the acceleration of each mass and the tension of each string.

47. A mass of 20 lbs. rests on a smooth horizontal table. A string tied to it passes over a pulley A on the table under another B, and is fixed at C so that the strings between are vertical; from B is suspended a mass of 10 lbs. Find the acceleration of each mass and the tension of the string.

If the mass of 20 lbs. hung freely over A, find the acceleration and the tension.

48. For one of the weights in Atwood's machine a pulley is substituted, round which passes a string connecting two masses P, Q which hang freely. Show that, if the ratio of P to Q lie between 3 and $\frac{1}{3}$, certain values of the other weight may be found which will keep either P or Q stationary, and that these values are to one another respectively as $3P - Q$ to $3Q - P$.

49. In case a fine string passing over a smooth pulley carries two small smooth pulleys, and these in turn carry strings with weights 1 lb. 2 lbs. and 1 lb. 3 lbs. tied to their ends respectively, show that the accelerations of the movable pulleys are each equal to $g/17$.

Find also the accelerations of the weights and the tensions of the strings.

50. Two equal buckets are connected by a string without weight passing over a smooth pulley, and over one of the buckets a heavy chain is held by its upper end with its lower end just above the base of the bucket. If the upper end be let go, prove that the equilibrium may be maintained by pouring water gently and uniformly into the other bucket, provided the weight of water which can be poured in is three times the weight of the chain.

* * * * *

51. On what mass, expressed in grams, must a force equal to the weight of a gram act, so as to add to it a velocity of 1 centimeter per second every second?

52. Masses of 2 lbs. and 8 lbs. hang in equilibrium over a wheel and axle, whose mass may be neglected. A mass of 1 lb. falling vertically strikes the 2 lbs. with a velocity of 14 feet per second. The masses are inelastic. Show that immediately after impact the 2 lbs. moves with a velocity of 4 feet per second. Also determine the acceleration and the tensions after the impact.

53. Masses of 2 and 3 lbs. hang in equilibrium over a wheel and axle. The masses are interchanged. Determine the motion ensuing.

CHAPTER IV

WORK—ENERGY—POWER

In the present chapter we propose to touch shortly on certain matters which are of extreme importance in physical science, leaving a fuller discussion until the student has obtained some acquaintance with the properties of motion in a plane.

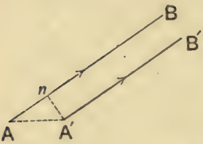
25. When a force acts on a body it makes it move, and the farther it moves it the greater the velocity it gives the body. The force in moving the body is said to *do work* on it, and we *define* work and its measurement as follows—

Def. *When a force acting at a fixed point of a body moves its point of application it is said to do work. The work is measured by the product of the measure of the force into the measure of the displacement estimated parallel to the force.*

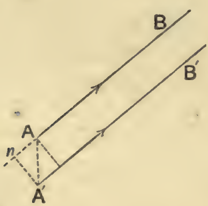
A remark or two on the wording of the definition is necessary. A force may be supposed to act at any point of its line of action: thus in the case of a sphere suspended by a string, the tension may be supposed to be applied at its upper end, or at its point of attachment, or at the centre of the sphere. When, in the definition, a displacement of the point of application of a force is spoken of, no displacement of this kind is meant. For instance, suppose a string attached to the lower end of a rod at B and passing through a clip at A. The point of application of the force is at B. If, however, the string be cut between

A and B and at the same time the clip be closed at A, the point of application is transferred to A. But this is no displacement in the sense of the definition. The force would not be said to do work. If, however, it lifted the rod bodily through a height BA, then it would be said to have done an amount of work measured by the product of the force by the distance AB.

Next, as to the force of the words "estimated parallel to the force." Let the force act at A in the direction AB, and suppose the point of application to move to A′ so that now the force acts along A′B′.



Draw A′n perpendicular to AB. Then AA′ is the actual displacement, but An is the displacement "estimated parallel to the force." The work done will then be measured by F × An.



If, however, the displacement had been as in the second figure, the displacement An would be in the opposite direction to the force, and work would be said to be done against the force; *or* the work done by the force would be measured by – F × An.

In the present part we are only treating of motion in a straight line, and consequently in all cases the actual displacement will be that required. The reason for the complete definition will be seen later.

26. In Chapter I. it was shown that if a point move through a space s with an acceleration a and change its velocity from u to v, then

$$v^2 = u^2 + 2as.$$

In the last chapter it was seen that if a force F acted on a mass m, the acceleration (a) produced was given by

$$a = \frac{F}{m}.$$

If then a force acts on a mass m, and moves it through a distance s, the change in velocity will be given by

$$v^2 = u^2 + 2\frac{F}{m}s,$$

or

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = Fs.$$

Now the expression on the right hand is, according to the above definition, the work done by the force, and the result is that the quantity $\frac{1}{2}mu^2$ is changed to $\frac{1}{2}mv^2$. The quantity $\frac{1}{2}mv^2$ is called the kinetic energy of the particle. It is measured by half the product of the measure of the mass into the square of the measure of the velocity with which the mass is moving. We can then state the result of the above equation in words as follows—

When a force acts on a particle, the change in its kinetic energy is equal to the work done by the force.

This raises the magnitudes, kinetic energy and work, from being mere definitions to the rank of quantities which have a real physical meaning. It may be compared with the analogous result obtained in the last chapter

$$mv - mu = Ft,$$

or the change in the momentum is equal to $F \times t$, the "impulse of the force."

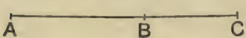
Def. *The unit of work is the work done by the unit force acting through the unit of distance.*

No special name has been given to the unit work when the units are the foot, pound, second. It is the work done by a poundal in moving through a foot, and is called shortly a foot-poundal. In the C.G.S. unit the unit of work is that done by a dyne when its point of application is moved through a centimeter. It is called an erg.

In engineering and for practical purposes generally it is more convenient to measure forces in terms of the *weight* of unit mass, such as the weight of a pound or of a gram. In these cases the practical unit of work will be the work done in moving the weight of unit mass through unit distance, or, which is the same thing, the work necessary to lift unit mass vertically through unit distance. In the British units this is called the foot-pound. In countries using the metrical system it is the kilogrammeter, or the work done in lifting one kilogram through one meter.

In electrical measurements the C.G.S. system of units is always employed. For practical purposes, however, the erg is too small a unit, and a larger one is used called a joule. A joule is $10,000,000 = 10^7$ ergs.

27. Consider a particle moving along a straight line under the action of a force which may or may not be constant, provided it is always the same at the same point. When it has moved from A to B, the force has done a certain amount of work, which will depend only on the path AB, and the kinetic energy will be increased. If it be moved back again to A, it will lose kinetic energy, and the loss will be exactly equal to the work formerly done by the force in moving from A to B. Thus, if a mass move from A to C and then back to B, the work done on the whole is equal to that done if it had moved at once from A



to B. In other words, the change of kinetic energy between A and B depends only on the positions of A and B and not on the previous history of how the mass moved from A to B. This theorem is true for the forces of nature, even if the particle pursued any path whatever in going from A to B. So far, however, we have only proved it true if the path lies along ABC. The complete proof is given later, § 102. The mass has a given energy at the point A. If it moves to B or C it alters by a certain amount, greater or less, depending on the position of B and C. Now notice two facts: (1) when it is at A it has a certain amount of kinetic energy, and (2) it is also in a position from which it can gain more by moving away from it. This power of getting kinetic energy, which depends on its position, is called potential energy. The potential energy at any point is measured by the energy which the mass can gain by moving from that point to some fixed position (say C). Clearly the sum of these two energies must be constant, for this statement simply means that the kinetic energy which the mass actually possesses, together with that which it can obtain by moving to a fixed point C, is constant. This statement is expressed shortly by saying that the whole energy, or simply the energy, is constant.

When a body is moving so that its kinetic energy is decreasing, we have seen that the work done is against the forces acting—that is, work is done *by* the moving body. Since action always has its equal reaction, this work is capable of being applied to other bodies, and so of producing useful effects.

Thus a body in motion is enabled to do work in virtue of its motion, the whole amount available being measured by its kinetic energy. The study of physical science shows that apparently other things besides matter in motion are capable of doing work. The following definition of energy will include all forms.

Def. *Anything that is capable of doing work is called energy.* The more advanced our knowledge of nature becomes, however, the more we are led to believe that all forms of energy are ultimately referable to motion of matter.

The above principles find their widest applications in more complex motions than the rectilinear ones considered in this first part. An example or two will, however, show their use.

EXAMPLE I. *A* 13-*ton gun recoils on being fired with a velocity of* 10 *feet per second, and is brought to rest by a uniform friction equal to the weight of* $4\frac{1}{16}$ *tons. How far does it recoil ?*

Take as units the ton, foot, and second.

Let x feet be the distance. The force overcome is $4\frac{1}{16}\times g$ units.

$$\therefore \text{Work done} = \tfrac{65}{16}g \times x.$$

The kinetic energy at starting is $\frac{1}{2}\times 13 \times 10^2 = 650$.

Now, since the kinetic energy at the end is zero, it is all used up in doing this work.

$$\therefore \tfrac{65}{16}gx = 650,$$

$$x = \frac{16\times 650}{65\times 32} = 5 \text{ feet.}$$

EXAMPLE II. *A train of* 100 *tons running on a level line is kept going by the locomotive at a uniform pace of* 50 *miles per hour ; the steam is suddenly shut off, and the train comes to rest after it has travelled*

2 miles farther. What was the force applied by the locomotive to the train, supposing the resistance of the rail and air to be constant?

Here the force on the whole train at first is compounded of the pull forward and the resistance backwards. Since the velocity is constant, the whole force is zero. Therefore the pull on the train just equals the resistance.

Afterwards the resistance destroys the whole kinetic energy, and does an amount of work measured by the product of the resistance into the distance. For units use the ton, mile, hour. Let R be the resistance. Then the distance $=2$, and

$$\text{Work} = 2 \times \text{R},$$
$$\text{Energy destroyed} = \tfrac{1}{2} \times 100 \times (50)^2,$$
$$\therefore 2\text{R} = 125{,}000,$$
$$\text{R} = 62{,}500 \text{ units of force.}$$

But this is equal to the pull of the engine.

From a practical point of view this does not give us much idea of the magnitude of the force. If, however, g denote the acceleration of gravity in the above units,

$$\text{R} = \text{the weight of } \frac{62{,}500}{g} \text{ tons.}$$

Now g is the acceleration of 32 feet per second per second, measured in miles per hour per hour. We must, therefore, determine the measure of g in these units. Now

$$32 \text{ feet} = \frac{32}{1760 \times 3} \text{ mile.}$$

Hence g is an acceleration of $\frac{32}{1760 \times 3}$ mile per second added on in 1 second, that is,

an acceleration of $\frac{32 \times 60 \times 60 \times 60 \times 60}{1760 \times 3}$ miles per hour added on in 1 hour

$$= \frac{864{,}000}{11} \text{ miles per hour per hour;}$$

$$\therefore \text{R} = \text{weight of } \frac{11 \times 62{,}500}{864{,}000} = \text{weight of } \frac{1375}{1728} \text{ tons.}$$

28. When a blow acts on a body, its velocity, and therefore its kinetic energy, are altered, and work is done on it. In fact, the work is done by a very big force acting through a very short space. Suppose this force to be F, the space s, and the time of impact, supposed very small, to be t. Then

$$\text{Work done} = \text{F}s = \text{F}t \times \frac{s}{t}.$$

Now Ft is the change of momentum produced by the force. But this change is measured by the impulse, which we will denote by I.

Also s/t is the average velocity of the body during the time t. If u be the velocity before and v the velocity afterwards, this average velocity will be $\frac{1}{2}(v+u)$. Hence

$$\text{Work done} = \text{I}\,.\,\frac{v+u}{2} = \frac{1}{2}\text{I}(v+u).$$

Now look at the question from the other side, viz. at the changes produced. The mass being m, the change in the kinetic energy is $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}m(v^2-u^2)$.

Also $$\text{I} = mv - mu.$$

Now

$$\begin{aligned}\text{Change in kinetic energy} &= \tfrac{1}{2}m(v^2-u^2),\\ &= \tfrac{1}{2}m(v-u)(v+u),\\ &= \tfrac{1}{2}\text{I}(v+u).\end{aligned}$$

In other words,

$$\text{Change in kinetic energy} = \text{work done by blow.}$$

29. When an impact takes place the whole momentum remains unchanged. Is this the case with the kinetic energy or not? This question we now proceed to consider. First, take the simpler case where the impinging bodies are inelastic.

Let their masses be denoted by m, m', velocities before impact by u, u', common velocity after impact by v. Then, since the momentum is unaltered,

$$mu + m'u' = (m+m')v.$$

The kinetic energy before impact is

$$\text{E}_1 = \tfrac{1}{2}mu^2 + \tfrac{1}{2}m'u'^2,$$

afterwards it is

$$\begin{aligned}\text{E}_2 &= \tfrac{1}{2}(m+m')v^2,\\ &= \frac{1}{2}(m+m')\,.\left(\frac{mu+m'u'}{m+m'}\right)^2,\\ &= \frac{1}{2}\frac{(mu+m'u')^2}{m+m'}.\end{aligned}$$

Hence

$$2(E_1 - E_2)$$
$$= mu^2 + m'u'^2 - \frac{(mu + m'u')^2}{m + m'},$$
$$= \frac{m(m + m')u^2 + m'(m + m')u'^2 - (m^2u^2 + 2mm'uu' + m'^2u'^2)}{m + m'},$$
$$= \frac{mm'(u^2 - 2uu' + u'^2)}{m + m'}.$$
$$E_1 - E_2 = \frac{1}{2}\frac{mm'(u - u')^2}{m + m'}.$$

Now $(u - u')^2$ being a square number, is always positive whatever the velocities u, u' may be. Hence $E_1 - E_2$ is always positive, or E_1 is greater than E_2. There is therefore always a *loss* of energy, and the loss is given by the value of $E_1 - E_2$.

This result might have been arrived at more shortly by employing the expression obtained above for the work done by an impulse.

Let I denote the impulse, then

$$I = m(u - v),$$

also
$$I = m'(v - u').$$

Multiply the first by m', the second by m, and add. Then

$$(m + m')I = mm'(u - u').$$

Also

The loss of energy of one $= \frac{1}{2}I(v + u)$,

The gain of energy of the other $= \frac{1}{2}I(v + u')$;

∴ Whole loss by the collision $= \frac{1}{2}I(v + u) - \frac{1}{2}I(v + u')$,

$$= \frac{1}{2}I(u - u'),$$
$$= \frac{1}{2}\frac{mm'}{m + m'}(u - u')^2,$$

the same result as before.

30. In the more general case where the bodies are elastic, let the velocities after impact be v, v', the other quantities remaining as before. Also let I denote the whole impact and I_1 the impact up to the instant when the bodies

have a common velocity. The magnitude of this has just been found, viz.

$$I_1 = \frac{mm'}{m+m'}(u-u').$$

But it was shown in § 14 that the impulses during the two parts of the collision are always in the ratio $1 : e$. Therefore

$$I = I_1 + eI_1 = (1+e)\frac{mm'}{m+m'}(u-u').$$

Now the loss of energy by the first body $= \frac{1}{2}I(v+u)$,
gain ,, ,, second $= \frac{1}{2}I(v'+u')$;

$$\therefore \text{ Whole loss} = \tfrac{1}{2}I(v - v' + u - u').$$

But one of the laws of collision is that the relative velocities are reversed and in the constant ratio $e : 1$, that is

$$v - v' = -e(u-u').$$

Therefore, substituting for $v - v'$,

$$\begin{aligned}\text{Whole loss} &= \frac{1}{2}I(1-e)(u-u'),\\ &= \frac{1}{2}(1+e)\frac{mm'}{m+m'}(u-u')(1-e)(u-u'),\\ &= \frac{1}{2}(1-e^2)\frac{mm'}{m+m'}(u-u')^2.\end{aligned}$$

Now e is always less than unity. Hence, as before, the right-hand side is positive, and there is always a loss of kinetic energy—at least of the kinetic energy of visible motion. This energy, apparently lost, reappears in the form of heat, sound, and vibrations of the colliding bodies.

In the limiting case of perfectly elastic bodies $e = 1$, and there is no loss of energy.

30 *a*. Questions as to work done by the stoppage of bodies in motion, such as the driving of piles, nails, or penetration of shot, well serve to illustrate the ideas developed in this chapter. Piles are driven into the ground by repeated blows of a heavy mass falling from a height. In some cases an apparatus for raising the weight is fixed to the top of the pile, and then the height of fall is always the same. In other cases the mass falls

from a fixed staging, and the height of fall increases as the pile is driven in. In an improved arrangement the blow is produced by a small steam hammer, whose energy is produced both by a fall and the expansion of steam. In all cases kinetic energy is produced, a part of which is used in doing the required work. The work done is always equal to the kinetic energy of the blow, but as was shown in the previous article, a portion of the energy is wasted in producing heat, noise, etc. It is necessary, therefore, to know what proportion of the original energy is utilised for penetration.

After the blow, the pile originally at rest starts with a certain velocity, and is brought to rest after penetrating a short distance by the resistance of the ground. The work done by this resistance is equal to the kinetic energy of the pile and mass just after the blow, together with the work done by gravity on them in moving through the distance penetrated.

The following example will illustrate the method—

A pile weighing 4 *cwt. is driven* 1 *foot into the ground by* 10 *blows of a mass of* 5 *cwt. falling* 15 *feet. What is the resistance of the ground?*

The momentum of the 5 cwt. before striking is equal to the momentum of the $5+4=9$ cwt. after. Hence the common velocity after$=\frac{5}{9}$ of that before.

$$\therefore \text{ Velocity after}=\frac{5}{9}\times\sqrt{(2g\times 15)},$$

$$\therefore \text{ Kinetic energy after}=\frac{1}{2}\cdot 9\cdot\frac{25}{81}\cdot 30g=\frac{125}{3}\text{ foot-cwts.}$$

Also distance moved$=\frac{1}{10}$ foot. Therefore, if resistance of ground be the weight of R cwt.

$$\frac{R}{10}=\frac{125}{3}+9\times\frac{1}{10},$$

$$\text{whence } R=425\tfrac{2}{3},$$

or resistance = weight of $425\frac{2}{3}$ cwts., or about $21\frac{1}{4}$ tons.

31. Steam engines or other agents which give a continuous supply of work or energy differ from one another in

the rate at which they supply this energy. The rate at which any agent supplies work is called its *power*, and is measured by the quantity of work it does in a given unit of time. In the foot, pound, second units the unit of power is a power which will do one foot-poundal per second, *i.e.* move against a poundal through one foot in one second. In the C.G.S. units the unit power is that which will do one erg per second.

For engineering purposes the unit of work is the foot-pound. A power which can do 33,000 foot-pounds of work per minute is called a *horse-power*. This is the unit in which the power of prime movers is measured. For electrical purposes—in which units based on the C.G.S. system are used—the unit of power employed is a joule per second, that is $10{,}000{,}000 = 10^7$ ergs per second. It is called a watt. A horse-power contains about 746 watts.

EXAMPLE. *What must be the horse-power exerted by an engine which pumps up* 4257 *tons of water a day from a depth of* 100 *feet?*

The work done in a day is the raising of 4257 tons or 4257×2240 lbs. 100 feet high against its weight.

Hence the work is $4257 \times 224{,}000$ foot-pounds.

This is done in 1 day.

Therefore the work in 1 minute $= \dfrac{4257 \times 224{,}000}{60 \times 24}$ foot-pounds,

$= 473 \times 1400$ foot-pounds.

But 1 horse-power $= 33{,}000$ foot-pounds per minute.

$$\therefore \text{ The horse-power} = \frac{473 \times 1400}{33{,}000} = \frac{301}{15} = 20\frac{1}{15}.$$

EXAMPLES—IV.

[*Further examples on this subject are given at the end of Chapter XI.*]

1. A cannon-ball whose mass is 60 lbs. falls through a vertical height of 400 feet. What is its energy? With what velocity must such a ball be projected from a cannon to have initially an equal energy?

2. How much energy is there in a body weighing 1 oz. and moving at the rate of 30 miles an hour?

3. Find the energy per second of a waterfall whose ledge is 30 yards high and ¼ mile broad, where the mass of water is 20 feet deep and has

a velocity of $7\frac{1}{2}$ miles an hour when it arrives at the fall. The weight of water is 1024 oz. per cubic foot.

4. A half-ton shot is discharged from an 81-ton gun with a velocity of 1620 feet per second. What will be the velocity with which the gun will begin to recoil if the mass of the powder be neglected? Will the gun or the shot be able to do more work before coming to rest, and in what proportion?

5. Prove that if a hammer weighing 2 lbs. striking a round nail one-tenth of an inch in diameter and weighing 1 oz., with a velocity of 50 feet per second, drives the nail 1 inch into a plank of wood, then a bullet half an inch in diameter, and weighing 1 oz., striking with a velocity of 1500 feet per second, will penetrate about 1·16 inches of the wood, supposing the resistance uniform and proportional to the sectional area of the hole.

6. From a point 28 feet above the ground a mass of 3 lbs. is projected upwards with a velocity of 20 feet per second. What is its kinetic energy, how far will it rise, and what will be its kinetic energy when it reaches the ground?

7. How many ergs are there in a foot-poundal and how many in a foot-pound? ($g=32{\cdot}2$ F.S. units.)

8. How many watts are there in a horse-power?

9. A man of 10 st. goes to the top of a house 45 feet high. What work does he do? If he does it in 1 minute, what power does he exert?

10. If the unit of energy be that required to raise 1 lb. through 1 foot (without gain of velocity), find the number of units of kinetic energy in a mass of 1 oz. moving 10 feet per second.

11. A man of 12 st. weight climbs up a mine shaft 800 feet deep by a ladder. What work does he do? If he exerts $\frac{1}{5}$ horse-power, how long will he be?

12. A train of 100 tons is pulled by a locomotive on a level at the constant speed of 60 miles per hour; the resistance is 18 lbs. weight per ton. Find the minimum horse-power of the locomotive.

13. Find the horse-power of a locomotive which moves a train of mass 50 tons at the rate of 30 miles an hour along a level railroad, the resistance from friction and the air being 16 lbs. weight per ton.

14. The mass of a complete train is 80 tons, and the resistance to its motion on a level 20 lbs. weight per ton; the horse-power of the locomotive is 256. What is the highest speed at which it can pull the train?

15. An engine is required to raise in 3 minutes a weight of 13 cwts. from a pit whose depth is 840 feet. Find the horse-power of the engine.

16. Determine the resistance to the motion of a steamer when 8000

effective horse-power is required to drive it at $17\frac{1}{4}$ knots (of 6080 feet an hour).

17. Determine the horse-power transmitted by a belt moving with a velocity of 600 feet a minute, passing round two pulleys, supposing the difference of tension of the two parts is the weight of 1650 lbs.

18. Supposing that the band of a friction break extends over the upper half of the fly-wheel of radius r feet, and that the band is kept tight by means of a weight of W lbs. hung at one end, and a spring-balance at the other end, prove that if the spring-balance registers a tension of W′ pounds when the engine is making n revolutions a minute, the horse-power of the engine is $2\pi nr(W - W')/33,000$.

19. The weights of an eight-day clock are together 11 lbs., and when the clock is wound up they are raised a yard. How many such clocks could an engine of one horse-power drive?

20. If n equal masses are placed in contact in a line on a smooth table, each being connected with the next by an inelastic string of length a, and another equal mass is attached to the foremost of the n masses by a string which passes over a pulley at the edge of the table, show that of the *vis viva* generated until the last mass is set in motion the fraction $\frac{n}{2(n+1)}$ is lost, supposing that none of the n masses leave the table before the last is set in motion.

21. How do the values of a horse-power and a watt depend on the locality? Find the difference between a horse-power at the equator and in London.

22. An inelastic pile of $\frac{1}{2}$ ton is driven 12 feet into the ground by 30 blows of a hammer of two tons falling 30 feet. Prove that it would require 120 tons in addition to the hammer to be superimposed on the pile to drive it down very slowly.

CHAPTER V

UNITS

32. As we have seen, the measure of any physical quantity consists of two factors, the unit and the measure or ratio of the quantity measured to the unit. The measure simply states how many of the units must be joined together in order to form a quantity equal to the one in question. Clearly, if the standard taken as the unit be changed, so will the measure be also. Thus a length of 10 yards when expressed by means of foot units becomes 30 feet, because, a foot being only one-third the size of a yard, it will require three times as many of them to make up the length as it would when a yard was the unit. So in general we are led to the important law that if the unit is increased or decreased in any ratio, the measure is decreased or increased in the same ratio—or, as it is stated shortly, *the measure of a quantity varies inversely as the magnitude of the unit.* The truth of this is easy to see; for since the measure states how many of the units must be taken to make a quantity equal to the given one, if the unit be twice as big it will need only half as many to make the same quantity as before—or if half as big it will need twice as many for the same purpose. And the same reasoning holds for any other proportion. This rule enables us at once to find the new measure when the ratio of the new unit to the old is given.

33. There are three kinds of things *sui generis*, and which, as such, require units, none of which have any

relation to the others—viz. matter, space, and time. The units used in measuring these quantities are called fundamental. In the preceding chapters examples of several other kinds of units have occurred; but they have all reference to two or more of these fundamental units. It is not, however, necessary in measuring them that recourse be had to the fundamental units. Thus, for instance, in measuring a velocity, we might take as the unit of velocity the velocity of the earth round the sun, or the velocity of sound, and say that a certain velocity is so many times a sound-speed. Or again, in measuring a force, we may take, as is very generally done, the force exerted on a certain piece of matter by the earth as the unit, and say that a certain force is so many times this weight. Thus for practical purposes people often speak of a force of so many pounds, speaking shortly for a force equal to the weight of so many pounds. These secondary or derivative units, however, are generally expressed in terms of the fundamental ones, and when such is the case they are called *absolute* units. Thus, instead of measuring a velocity by comparing it with a sound-speed, it is measured by comparing it with a velocity in which say a foot is passed over in a second; or, instead of referring a force to the weight of a pound, it is referred to the poundal, or the force which in one second will make a pound move with a velocity of one foot per second.

In absolute measurement then it is clear that the magnitude of a unit will depend on the magnitude of the fundamental units. The way in which a particular unit depends on them is called the *dimensions* of the unit. When the dimensions of a unit are known we can at once determine how the unit is changed if the fundamental units are changed; and from the law that the measure is inversely as the unit, we can immediately write down the new measure. The determination of the dimensions of the different derivative units is therefore a matter of importance.

The dimension of a unit with reference to either of the fundamental units tells us the kind of proportion in which that unit is altered when the fundamental unit is altered. Thus, suppose that the unit of mass be made x times as big, and it is found that the derivative unit becomes thereby x^3 times as big. The unit is then said to be of 3 dimensions with reference to mass, and we should know that if the unit of mass be doubled, the unit in question will be increased 2^3 or 8 times; or if the unit be changed from ounces to pounds, the derivative unit will be increased 16^3 or become 4096 times as big.

Take the case of any unit which depends on all three fundamental units. Suppose the unit of mass becomes M times as big as before, the unit of space L times, and the unit of time T times. Suppose also that in consequence of the increase of the unit mass, the derivative unit becomes M^x times as large; on account of the increase of the length unit, L^y as large; and on account of the increase of the time unit, T^z as large. Then this fact is represented in the following way,

$$\text{Dimensions of unit} = [M^x L^y T^z],$$

and it is said that the dimensions of the unit are x in mass, y in space, and z in time.

34. The easiest way, perhaps, to find the dimensions of a unit in any particular case is to suppose one of the fundamental units, say the space, doubled, and then to examine how many times the unit is increased. This number, expressed in the form 2^x, will then give the dimensions. This is the method we shall employ in the following determinations. We will take the units in the order in which they have been introduced in the preceding pages.

I. *Velocity*.—Velocity depends on two fundamental units only, those of space and time. The unit velocity is one in which the unit of space is passed over in a unit of time. If the unit of space be doubled, the new unit is such that twice the space is passed over in the same time—that is, the unit is doubled, or is altered in the same proportion as the unit of space. Hence it is *one* dimension in space.

If, on the contrary, the unit of time be doubled, the same space is passed over in twice the time, and therefore the new unit is only one-half the old. Hence the unit of velocity varies *inversely* as the unit of time. The final result then is

$$[\text{Velocity}] = \left[\frac{L}{T}\right] = [LT^{-1}].$$

EXAMPLE. *Express a velocity of* 120 *yards per minute in miles per hour.*

Here the unit of space is increased 1760 times, and the unit of time 60 times, or L = 1760, T = 60,

$$\therefore \text{New unit} = \frac{1760}{60} = \frac{88}{3} \text{ old unit.}$$

But the measure is inversely as the unit,

$$\therefore \text{New measure} = 120 \times \tfrac{3}{88} = \tfrac{45}{11} = 4\tfrac{1}{11} \text{ miles per hour.}$$

II. *Acceleration.*—This unit also depends only on space and time. It is an acceleration in which unit of velocity is added on in unit of time. Let now the unit of space be doubled; by the preceding result the unit of velocity is also doubled. Therefore the new unit is one in which twice the velocity is added on in the same time as before—in other words, the new unit is double the old. Doubling the unit of space doubles the unit of acceleration. Hence the unit is *one* dimension in space.

Next, double the time. We have seen that the new unit of velocity is one-half the original. Hence the new unit of acceleration is one in which *half* the original velocity is added on in twice as long a time. It is therefore one-quarter the magnitude of the old one. Doubling the unit of time therefore diminishes the unit of acceleration to $1/4 = 1/2^2$ its original value, or so far as the time is concerned

$$[\text{Acceleration}] = \left[\frac{L}{T^2}\right].$$

The final result, therefore, is

$$[\text{Acceleration}] = \left[\frac{L}{T^2}\right] = [LT^{-2}]$$

III. *Density.*—Two ways of measuring the density of a body were given in Chapter II. One way was to state the ratio of the mass of a body to that of an equal volume of some standard substance. In this method density is merely a ratio between two quantities of the same kind, and therefore is a pure number, independent of any units. In the other method the density is measured by stating the mass contained in a unit of volume, and is, therefore, a quantity which depends on the units of mass and space, or

$$[\text{Density}] = [M^x L^y].$$

This unity of density is one in which unit volume contains unit of mass. If the mass be doubled, the new unit of density will contain twice the mass, and will therefore also be doubled. Hence $x = 1$.

If the unit of space be doubled, the unit of volume is increased to eight times its former volume. Hence in the new unit density the same mass occupies a volume eight times larger than the old one. The density is therefore only $\frac{1}{8}$ its former value. and since $\frac{1}{8} = 1/2^3 = 2^{-3}$, we see that $y = -3$, or

$$[\text{Density}] = \left[\frac{M}{L^3}\right] = [ML^{-3}].$$

IV. *Momentum, impulse.*—Momentum and impulse are quantities of the same kind, and therefore are of the same dimensions. They involve mass and velocity, and therefore all three of the fundamental units.

If the unit of mass is doubled, so also is the unit of momentum, which is the momentum of unit mass moving with unit velocity. These two units are therefore so far directly proportional.

Also, if the unit of velocity is doubled, the new unit of momentum is that of the same mass moving with double the velocity, and is therefore itself doubled also. Hence

$$[\text{Momentum}] = [MV] = \left[\frac{ML}{T}\right] = [MLT^{-1}].$$

V. *Force.*—The unit of force is that which will give unit acceleration to unit mass. If the mass be doubled, the new unit of force must give the same acceleration to double the mass, and must therefore itself be twice as big as before. In other words, so far as mass is concerned, it is proportional to the unit of mass.

Again, if the unit acceleration is doubled, so also must be the force. Hence

$$[\text{Force}] = [\text{M . Acceleration}] = \left[\frac{\text{ML}}{\text{T}^2}\right] = [\text{MLT}^{-2}].$$

VI. *Work, energy.*—These are quantities of the same kind $= [\text{M}^x\text{L}^y\text{T}^z]$. The unit of work is the work done in moving unit force through unit distance.

If the mass be doubled, so is unit force, and therefore unit work,

$$\therefore x = 1.$$

If the length be doubled, so is unit force, and the new unit work is that done when twice the force moves through twice the distance, and is therefore four times $= 2^2$ as big as before,

$$\therefore y = 2.$$

If the time be doubled, the unit force is $\frac{1}{4}$ the original one, and therefore also the unit of work, or

$$z = -2.$$

$$\therefore [\text{Work}] = \left[\frac{\text{ML}^2}{\text{T}^2}\right] = [\text{ML}^2\text{T}^{-2}].$$

VII. *Power.*—The unit power is unit work per unit time. It will, therefore, depend on mass and space in the same way as unit work. If the unit of time be doubled, the new unit of work is $\frac{1}{4}$ its original value, and the new unit power does one-quarter the work in twice the time, or is only $\frac{1}{8} = 1/2^3$ its original value.

$$\therefore z = -3\,;$$

or

$$[\text{Power}] = \left[\frac{\text{ML}^2}{\text{T}^3}\right] = [\text{ML}^2\text{T}^{-3}].$$

These results are useful for reference, and are here collected—

$$\text{Velocity} = \left[\frac{L}{T}\right],$$

$$\text{Acceleration} = \left[\frac{L}{T^2}\right],$$

$$\text{Density} = \left[\frac{M}{L^3}\right],$$

$$\left.\begin{matrix}\text{Momentum,}\\ \text{Impulse}\end{matrix}\right\} = \left[\frac{ML}{T}\right],$$

$$\text{Force} = \left[\frac{ML}{T^2}\right],$$

$$\left.\begin{matrix}\text{Work,}\\ \text{Energy}\end{matrix}\right\} = \left[\frac{ML^2}{T^2}\right],$$

$$\text{Power} = \left[\frac{ML^2}{T^3}\right].$$

One example of the use of these formulæ has been given. Another, more complicated, is to express a power of 25 foot-pounds per minute in the C.G.S. system.

Firstly, the first number is not an absolute measure.

It must therefore be made absolute by expressing the weight of a pound in units of force. To do this we have to multiply by g (foot minute units).

Now in foot second units $g=32$.

$$\therefore \text{ In foot minute, new unit acceleration} = \frac{1}{60^2} \times \text{the old} = \frac{1}{3600},$$

$$\therefore \textit{Measure} \text{ in new units is } 32 \div \frac{1}{3600} = 32 \times 3600 = 115{,}200,$$

or, absolute measure of the power $= 25 \times 115{,}200 = 2{,}880{,}000$ ft. lb. min. units.

The question then is to find the measure of a power $= 2{,}880{,}000$ ft. lb. min. units. in C.G.S. units.

$$\text{Now } 1 \text{ cm.} = \cdot 0328 \text{ ft.},$$

$$1 \text{ gr.} = \cdot 0022 \text{ lbs.},$$

$$1 \text{ sec.} = \tfrac{1}{60} \text{ min.},$$

and by VII above $\text{Power} = \left[\frac{ML^2}{T^3}\right];$

$$\therefore \text{ New unit of power} = \frac{\cdot 0022 \times (\cdot 0328)^2}{(\frac{1}{60})^3} \text{ the old,}$$

$$= \cdot 0022 \times (\cdot 0328)^2 \times (60)^3,$$

$$= \cdot 511{,}239.$$

Hence new measure of the power in question (being inversely as unit) $= \frac{2,880,000}{\cdot 5112} =$ about 5,610,000 ergs per second.

35. These tables of dimensions are useful also for another purpose; they enable us to check any result we may have arrived at, and often to discover if the result is wrong. For instance, to take a simple case, suppose a result came out that 5 lbs. were equal to 3 feet. It would be clear at once that the result had no meaning. Again, suppose the equation

$$3ms + 2ft - 10v = 0$$

had been arrived at, where the letters denote the usual quantities. It is not evident at sight whether this would be a possible equation or not. But it is clear that if it represents an equation between real quantities, that equation must be independent of the units in which they happen to be measured and must still hold if the units are altered in any manner.

Let us then test the equation by the results obtained above. Firstly by the space. By I and II, v, f are each of one dimension in space, and therefore, if the unit of space be increased L times, each term is multiplied by $1/L$, and the only effect is to multiply the whole equation by $1/L$; in other words, the equation still subsists. So far then we find no objection to it. Next let us test by the mass. If the mass be increased M times, the first term in the equation being of dimension + 1 in mass, is multiplied by $1/M$, but the others not containing mass remain unaltered; in other words, we get a new equation by altering the unit of mass. The equation is therefore just as unmeaning as that 5 lbs. = 3 feet. So also if we increase the unit of time T times, the first term remains unaltered, the second is multiplied by $(T \times 1/T^2)^{-1} = T$, and the last by T; again showing that the equation is really nonsense.

All the above reasoning may be done at a glance by putting the units in evidence by their dimensions. Thus the equation would become

$$3ms[\mathrm{ML}] + 2ft\left[\frac{\mathrm{L}}{\mathrm{T}^2} \times \mathrm{T}\right] - 10v\left[\frac{\mathrm{L}}{\mathrm{T}}\right] = 0,$$

or
$$3ms[\mathrm{ML}] + 2ft\left[\frac{\mathrm{L}}{\mathrm{T}}\right] - 10v\left[\frac{\mathrm{L}}{\mathrm{T}}\right] = 0.$$

It is then clear that ML and $\frac{\mathrm{L}}{\mathrm{T}}$ are different kinds of things, and therefore cannot be added or subtracted from one another.

Such a process of testing whether a result is wrong is called counting the dimensions. The following examples will exemplify the method.

Determine whether any of the following equations are impossible or not,

(1) $10\mathrm{F}vst + 8mv^2s - 3mg^3t^4 = 0,$

(2) $v^3t - 4mfs + 3\mathrm{F} = 0,$

(3) $6m^2\mathrm{V} + 2g^3\mathrm{F}s^2\rho t^8 - 3\mathrm{F}^2st^4 = 0,$

where V is a volume.

Putting in evidence the dimensions of the units only, the terms are—

In (1)
$$\frac{\mathrm{ML}}{\mathrm{T}^2} \times \frac{\mathrm{L}}{\mathrm{T}} \times \mathrm{L} \times \mathrm{T}, \quad \mathrm{M} \times \left(\frac{\mathrm{L}}{\mathrm{T}}\right)^2 \times \mathrm{L}, \quad \mathrm{M}\left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^3 \mathrm{T}^4,$$

or
$$\frac{\mathrm{ML}^3}{\mathrm{T}^2}, \quad \frac{\mathrm{ML}^3}{\mathrm{T}^2}, \quad \frac{\mathrm{ML}^3}{\mathrm{T}^2}.$$

Each term is therefore a quantity of the same kind and the equation is possible.

In (2)
$$\left(\frac{\mathrm{L}}{\mathrm{T}}\right)^3 \mathrm{T}, \quad \mathrm{M}\frac{\mathrm{L}}{\mathrm{T}^2}\mathrm{L}, \quad \frac{\mathrm{ML}}{\mathrm{T}^2},$$

or
$$\frac{\mathrm{L}^3}{\mathrm{T}^2}, \quad \frac{\mathrm{ML}^2}{\mathrm{T}^2}, \quad \frac{\mathrm{ML}}{\mathrm{T}^2},$$

or each term refers to different kinds of physical quantities, and the equation is therefore nonsense.

In (3)
$$\mathrm{M}^2 \times \mathrm{L}^3, \quad \left(\frac{\mathrm{L}}{\mathrm{T}^2}\right)^3 \times \frac{\mathrm{ML}}{\mathrm{T}^2} \times \mathrm{L}^2 \times \frac{\mathrm{M}}{\mathrm{L}^3} \times \mathrm{T}^8, \quad \left(\frac{\mathrm{ML}}{\mathrm{T}^2}\right)^2 \times \mathrm{L} \times \mathrm{T}^4,$$

or
$$\mathrm{M}^2\mathrm{L}^3, \quad \mathrm{M}^2\mathrm{L}^3, \quad \mathrm{M}^2\mathrm{L}^3.$$

Each term is of the same kind, and the equation is therefore possible.

EXAMPLES—V.

1. Find the ratio of the units of the following quantities when the fundamental units are changed from the foot, pound, second system to the yard, ton, hour—

(α) Acceleration,
(β) Force,
(γ) Work.

2. Find the ratio of the measures of the *same* acceleration, force, and work when the fundamental units are (1) the foot, pound, second, and (2) the yard, ton, hour.

3. Find the ratio of the units of the following quantities when the fundamental units are changed from the yard, cwt., minute system to the foot, pound, second—

(α) Momentum,
(β) Force,
(γ) Power.

4. Find the ratio of the poundal to the dyne.

5. Express an acceleration of 314 feet per second per second in terms of yards per hour per hour.

6. Express 18 poundals in units, depending on the yard, pound, minute.

7. How many watts are there in one horse-power? ($g=32{\cdot}2$.)

8. Determine the unit of time in order that, the foot being the unit of length, g may be expressed by unity.

9. If the acceleration of a falling body be the unit of acceleration, and if a velocity of a yard per minute be the unit of velocity, find the units of space and time.

10. The acceleration of a body falling in vacuum and the velocity it acquires in 1 second are taken as the units of acceleration and velocity respectively. What are the units of space and time?

11. If the acceleration due to gravity be taken as the unit acceleration and the velocity generated in one minute as the unit of velocity, find the unit of length.

12. A particle describes a foot from rest and acquires a velocity denoted by b in d seconds, with a uniform acceleration denoted by c. Find the units of length and time.

13. Determine the unit of mass that the absolute unit of energy may be the foot-pound, the second and foot being units of time and length.

14. What are the dimensions of energy per volume of a liquid?

15. Show that mvg is the same kind of quantity as power.

16. Have the following expressions any meaning—

(α) $smv + 3Ft$,

(β) $3va + mg$,

(γ) $8m^2vs + 2F^2t^3 - 3mFst = 0$,

(δ) Work $= \dfrac{3mv}{t} - 2Fvt$?

where the letters have the meanings applied in this book.

17. Two particles, whose masses are m_1, m_2, and distances from a fixed point are r_1, r_2, are moving with velocities u_1, u_2 under accelerations f_1, f_2. A person deduces from some investigations the following equations—

$$(m_1+m_2)(r_1^2-r_2^2)(u_1^2-u_2^2) = \sqrt{\left\{(m_1r_1^2u_1+m_2r_2^2u_2)(m_1r_1^2f_1^2+m_2r_2^2f_2^2)\left(\frac{r_2^2}{u_1}-\frac{r_1^2}{u_2}\right)\right\}}.$$

Energy of system $= f_1\sqrt{m_1m_2(r_1^2-r_2^2)} + f_2\sqrt{(m_1+m_2)(r_1+r_2)}$. Show that he has made an error somewhere in his calculations.

18. Show that the expression mv^2 in Art. 24, Chap. iii., is of the dimensions of a force.

PART II

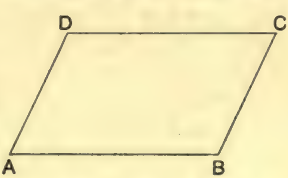
FORCES IN ONE PLANE

CHAPTER VI

COMPOSITION OF VELOCITIES AND ACCELERATIONS

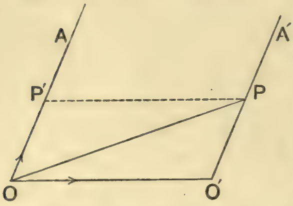
HITHERTO we have confined our attention to the simplest kind of motion—that of translation along a straight line, in which therefore the direction remains unchanged. We now proceed to consider the more complex motions in which change of direction occurs, confining our attention to cases where all the motions take place in the same plane.

36. *Resultant of two velocities.*—If a board ABCD be moving in the direction AB with a velocity u, and a ball *on* it move over it parallel to AB with a velocity u' relative to the board, we have seen that the actual velocity of the ball relative to the "fixed point" is $u + u'$. But suppose that the ball, instead of moving in direction AB, had moved in direction AD; what would its actual velocity then have been? This is the problem we first proceed to discuss.



The ball is in reality moving with two simultaneous velocities u and u', and is therefore changing its position in space in a certain manner. If looked at from outside the board, it would appear to be moving with a velocity different from either u or u'. This velocity is called the *resultant* of the other two. It produces by itself the same rate of change of position as the others do together. It clearly depends both on the magnitudes of u and u' and also on their directions.

In order to fix our ideas, we will suppose the simultaneous velocities to be produced in the following manner. Let a ring slide on a straight rod OA with velocity u' relative to the rod, while at the same time the rod moves parallel to itself with velocity u along OO′. At the beginning of the time let the rod be OA and the ring at O. After t units of time let the rod have moved to O′A′ and the ring have slipped along the rod to P. Then OO′ is the space passed over by the rod in time t, and is therefore measured by ut. Also O′P is the space passed over by the ring along the rod in the same time t, and is therefore measured by $u't$. Hence



$$\frac{O'P}{OO'} = \frac{u't}{ut} = \frac{u'}{u},$$

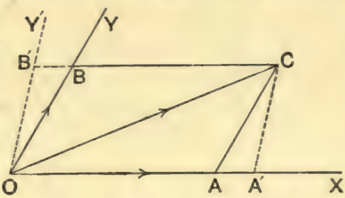
or P moves so that at any time O′P/OO′ is constant, and therefore P must describe a straight line. It follows therefore that OP/OO′ is constant. Now OO′ is described at a constant rate u; therefore also OP is described at a constant rate—that is, the resultant velocity of u and u' is a constant velocity. To find its value, let the time t be taken to be unity, then $OO' = u$, $O'P = u'$, and OP will then represent the displacement in unit of time—that is, the resultant velocity in magnitude and direction.

If a line PP′ be drawn through P parallel to OO′, OP′ is equal to O′P, and OP′PO′ is a parallelogram whose sides are equal to u, u'. The theorem, called the parallelogram of velocities, may then be thus stated—

If two velocities be represented by two straight lines meeting at a point, and the parallelogram be described having these lines as adjacent sides, then the resultant velocity is represented by that diagonal which passes through the point.

Note.—The lines represent both the magnitude and direction of the velocities—that is, they have the same direction and contain as many units of length as the velocities contain units of velocity.

37. Let two velocities be represented by OA, OB. Complete the parallelogram OACB. We have just seen that OC represents a velocity which produces the same effect as OA, OB combined. Now clearly we may regard the question in the opposite light, and look on OA, OB as velocities which combined produce the same effect as OC. Regarded in this light, OA, OB are said to be *components* of OC.



Two given velocities have only one definite resultant, but a given velocity may be replaced by pairs of components in an infinite number of ways. If, however, the directions of these components be given, their magnitudes are determinate. For instance, let us see how to decompose OC into two components whose directions lie along OX, OY. Through C draw CA parallel to OY and CB parallel to OX. Then by the parallelogram of velocities OC is the resultant of OA and OB, or OA, OB are the components of OC in the given directions.

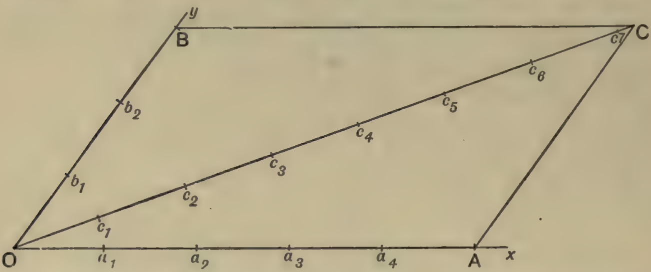
Suppose now the direction OY were changed to OY′, then the components would be OA′, OB′ as in the figure, so that not only is the component along OY′ altered, but also that along OX.

Hence it is necessary in finding the component along OX to know the direction in which the other is acting. OA is called "the component of OC along OX, when the other component acts along OY." In the case, however, where OY is perpendicular to OX, it is usual to call OA simply the component of OC along OX.

38. It is important to be able to determine the resultant of two velocities, when their speeds and directions are given. This can always be done approximately by a

graphical construction. Thus suppose we require the resultant of two velocities of 5 and 3 feet per second at an angle of 57° 32′.

Draw two lines OX, OY making the required angle with one another. Along OB mark off lengths Ob_1, b_1b_2, b_2B equal to one another, and along OX mark off five each



equal to these, and let A be the last. Then OB, OA will represent the velocities 3, 5. Complete the parallelogram OACB and join OC. Mark off on OC points $c_1, c_2, \ldots, c_7$ at distances equal to Ob_1. The last (in this case) falling inside OC is c_7. Then the measure of the resultant is

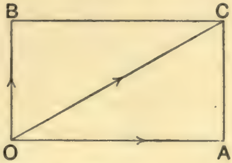
$$7 + \frac{Cc_7}{Oc_1}.$$

The angle AOC, or direction of the resultant, can then easily be determined roughly by a suitable scale. In the present case it will be found to be 20° 57′.

Clearly, by drawing with care on a sufficiently large scale, it is possible to obtain a very fair degree of accuracy by the graphical method. Frequently, however, we require to obtain the result either quite accurate or with a known amount of accuracy. In this case recourse must be had to methods of calculation. In general this requires the use of trigonometrical tables. In many cases, however, especially as to magnitude, we can obtain the result by a little easy geometry. It will be useful to consider some of these geometrical results first.

39. I. *Two velocities* u, u′ *perpendicular to one another.*

The parallelogram now becomes a rectangle. OAC is a right angle, and, by Eucl. I. 47,



$$OC^2 = OA^2 + AC^2,$$
$$= OA^2 + OB^2,$$

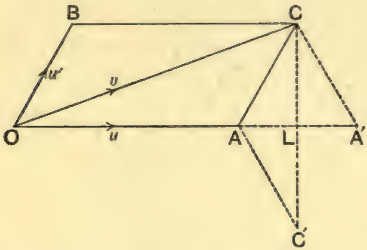
or $$v^2 = u^2 + u'^2.$$

II. *Two velocities* u, u′ *at an angle of* 60°.

Let OA, OB denote u, u' and let AOB = 60°.

From C draw CL perpendicular to OA produced. Then

$$CAL = 60°, \quad ACL = 30°.$$



Draw CA′, making A′CL = 30°. Then ACA′ = 60°, and the triangle ACA′ is equilateral.

Hence, CL being the perpendicular from C,

$$AL = \tfrac{1}{2}AA' = \tfrac{1}{2}AC = \tfrac{1}{2}u',$$

also $$CL^2 = AC^2 - AL^2 = u'^2 - \tfrac{1}{4}u'^2 = \tfrac{3}{4}u'^2.$$

Now $$OL = OA + AL = u + \tfrac{1}{2}u',$$

$$\therefore OC^2 = OL^2 + CL^2;$$

or $$v^2 = (u + \tfrac{1}{2}u')^2 + \tfrac{3}{4}u'^2,$$

$$v^2 = u^2 + u'^2 + uu'.$$

III. *Two velocities* u, u′ *at an angle of* 45°.

Making a similar construction,

$$CAL = 45°,$$
$$\therefore ACL = 45°,$$
$$\therefore AL = CL,$$

and $$AL^2 + CL^2 = AC^2;$$

or
$$2AL^2 = u'^2,$$
$$AL = \frac{u'}{\sqrt{2}} = CL;$$
$$\therefore OL = OA + AL,$$
$$= u + \frac{u'}{\sqrt{2}};$$
and
$$OC^2 = OL^2 + CL^2;$$
$$\therefore v^2 = \left(u + \frac{u'}{\sqrt{2}}\right)^2 + \frac{u'^2}{2},$$
$$= u^2 + u'^2 + uu'\sqrt{2}.$$

IV. *Two velocities* u, u′ *at an angle of* 30°.

Here CAL = 30°, ACL = 60°.

Draw CL perpendicular to OA and produce it to C′ so that LC′ = CL. Join AC′. Then C′AL = CAL = 30°.

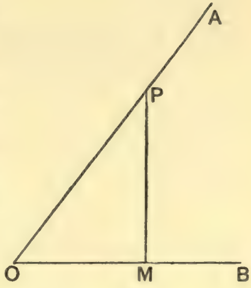
∴ CAC′ = 60° and CAC′ is an equilateral triangle.
$$\therefore CL = \tfrac{1}{2}CC' = \tfrac{1}{2}AC = \tfrac{1}{2}u',$$
$$AL^2 = \tfrac{3}{4}AC^2;$$
or
$$AL = \frac{\sqrt{3}}{2} \cdot u';$$
$$\therefore OL = OA + AL,$$
$$= u + \frac{\sqrt{3}}{2}u',$$
$$OC^2 = OL^2 + CL^2,$$
$$v^2 = \left(u + \frac{\sqrt{3}}{2}u'\right)^2 + \frac{1}{4}u'^2,$$
$$= u^2 + u'^2 + uu'\sqrt{3}.$$

The student may prove as an exercise that when the angles are
$$120°,\ v^2 = u^2 + u'^2 - uu',$$
$$135°,\ v^2 = u^2 + u'^2 - uu'\sqrt{2},$$
$$150,\ v^2 = u^2 + u'^2 - uu'\sqrt{3}.$$

V. *In the general case, if* AL *be known,* the resultant can be calculated by geometry. For, by Eucl. II 13,
$$OC^2 = OA^2 + AC^2 + 2 \cdot OA \cdot AL,$$
or
$$v^2 = u^2 + u'^2 + 2uu' \cdot \frac{AL}{AC}.$$

Now the ratio AL/AC depends only on the angle CAL (or BOA), and not on the lengths of OA or OB. The value of this ratio for different angles is given in books of trigonometrical tables. When the angle between the velocities is known, all that is necessary is to turn to the tables and find the value of the corresponding ratio, substitute its value, and the further determination of v is a mere question of arithmetic.

40. The properties of this ratio, and of other ratios of an angle, are considered in trigonometry. In this book we assume no knowledge of trigonometry, but it will be convenient to know the names of certain ratios and regard their values for different angles as given by the tables.



Let then AOB be any angle. In OA (or OB) take any point P and draw PM perpendicular to OB. Then the ratios of the sides OP, PM, OM are independent of the position of P.

The ratio $\frac{PM}{OP}$ is called the sine of AOB, and is written

$$\frac{PM}{OP} = \sin AOB.$$

So also $\frac{OM}{OP}$ is the cosine of AOB $= \cos AOB$,

and $\frac{PM}{OM}$ is the tangent of AOB $= \tan AOB$.

As stated, these ratios depend only on the angle AOB. Their values are tabulated in trigonometrical tables for all angles differing successively by 1 minute from 0° to 90°.

If we denote the angle by A, the above may be written

$$PM = OP \sin A,$$
$$OM = OP \cos A,$$
$$PM = OM \tan A.$$

So that if the angle and OP be known, the values of PM and OM are at once found by looking out in the tables the values of sin A and cos A.

The formula of the last article may now be written in the new notation: for

$$\frac{AL}{AC} = \cos CAL = \cos AOB,$$

or if we denote the angle between the velocities by θ,

$$AL/AC = \cos\theta,$$

and $$v^2 = u^2 + u'^2 + 2uu' \cos\theta.$$

This is a most important formula.

41. The formula $v^2 = u^2 + u'^2 + 2uu' \cos\theta$ gives the magnitude of the resultant velocity. It remains to obtain a formula to give its direction.

The direction is known if we can determine the angle COA, or COB. Now, by the definition above,

$$\tan COL = \frac{CL}{OL}.$$

But $$\frac{CL}{AC} = \sin CAL,$$

$$\therefore CL = u' \sin\theta,$$

so $$OL = OA + AL = u + u' \cos\theta.$$

Hence $$\tan COL = \frac{u' \sin\theta}{u + u' \cos\theta}.$$

This enables us to find the value of tan COL. Then look up in the tables what angle has its tangent equal to this, and we know at once the value of the angle COA.

A numerical example will render the above reasoning clearer. We will take the case whose graphical solution was given in § 38, viz. to find the resultant of 3 and 5 feet per second at an angle of 57° 32′.

The tables give cos 57° 32′ = ·53681, sin 57° 32′ = ·84370. Hence

$$v^2 = 3^2 + 5^2 + 2 \times 3 \times 5 \times {\cdot}53681,$$
$$v^2 = 9 + 25 + 16{\cdot}1043,$$
$$= 50{\cdot}1043;$$
$$\therefore v = 7{\cdot}078 \text{ feet per second.}$$

Next, to find the direction—

$$\tan COA = \frac{3 \sin 57^\circ\ 32'}{5 + 3 \cos 57^\circ\ 32'},$$
$$= \frac{3 \times \cdot 84370}{5 + 3 \times \cdot 53681},$$
$$= \frac{2 \cdot 5311}{6 \cdot 61043},$$
$$= \cdot 38289.$$

Now in the tables it will be found that

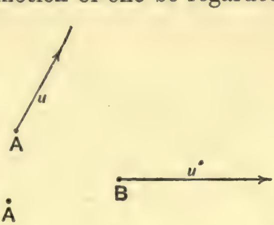
$$\tan 20^\circ\ 57' = \cdot 38286,$$
$$\tan 20^\circ\ 58' = \cdot 38319.$$

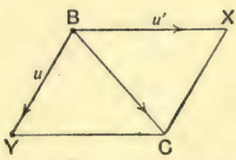
Hence the angle COA lies between 20° 57′ and 20° 58′. If we are satisfied to get the angle correct to 1′, COA = 20° 57′. If we desire a closer approximation, it is easy to obtain one by the theory of "proportional parts," but the consideration of this would lead us too far.

The final result then is that the resultant velocity is one of 7·078 feet per second, making an angle of 20° 57′ with the direction of the 5 feet per second component.

Note.—For graphical purposes it is much better to know the tangent of an angle than the angle itself, as to draw the latter requires a special scale or protractor, whereas the tangent can be drawn at once by the aid of a pair of compasses.

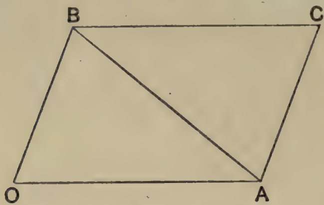
42. *Relative velocity.*—Suppose two points moving with different velocities u, u'. If the motion of one be regarded from the position of the other, it will *appear* to move with a velocity different from either u or u'. This is called the relative velocity of the two points. To find it in any case, notice that the relative motion is unaffected if we give any the same velocity to *both* points. Apply then to both a velocity equal and opposite to that of A. The consequence is that A becomes "at rest," whilst B has the velocity $-u$, in addition to its own—





represented in the second figure by BX, BY. These are equivalent to a resultant BC. In other words B, as seen from A, will appear to move with a velocity represented in magnitude and direction by BC. This then is the relative velocity.

Note.—It is to be especially noted that the relative velocity is not the rate at which the distance between A and B increases. This rate depends on the positions of A and B as well as their velocities.

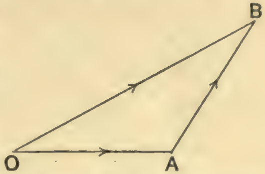


Clearly YB may be taken to represent the velocity of A and YC that of B, and we get thus the following construction to give the relative velocity in any case. If OA, OB represent the simultaneous velocities of *two* points, their relative velocity is AB—or if the parallelogram be described having those lines for adjacent sides, the *relative* velocity is that diagonal which does not pass through O. On the contrary, we have seen that if they represent the simultaneous velocities of *one* point, then their *resultant* is that diagonal which does pass through O.

43. *Acceleration.*—Acceleration has been defined in Chapter I. as rate of change of velocity. As a velocity depends on both direction and speed, acceleration will involve both change in direction as well as change in speed. The question of change in speed alone has been already considered; it will now be requisite to take up the general problem where both conditions vary.

As an example of change of velocity without change of speed, we may take the following. Suppose a point moving with a velocity represented by OA (the student should draw the figure himself), and that its velocity is changed by the addition of another, OB, equal to it in magnitude, but at an angle of 120° to it. Then the new velocity is OC, the diagonal of the parallelogram OBCA, equal to OA in magnitude, but in a different direction. The *change of velocity* is represented by AC or OB.

We have already seen that if OA denote any velocity, and AB another velocity added to it, then the resulting velocity is OB—or conversely, if OA be changed to OB, the change is represented by AB. Suppose now that in the unit of time the velocity OA changes to OB, then AB is the change of velocity per unit time, or in other words represents the acceleration.



Now just as a point may be considered to have two velocities simultaneously, so also we may consider each of these velocities as altering,—that is, consider the point as having simultaneous accelerations in different directions. These will have a single resultant which produces the same effect as the two combined. To find this resultant, we notice that the accelerations are velocities added on per unit time. These are equivalent to a single resultant velocity, found by the parallelogram law, and this resultant will represent the whole change of velocity per unit of time, in other words the resultant acceleration. We see then that accelerations are compounded in precisely the same manner as velocities, and the statement of the law in § 36 will exactly hold good for accelerations when the word "velocity" is replaced by "acceleration."

All the results obtained with reference to components and resultants of velocities and to relative velocity hold also with respect to accelerations.

The methods for finding the resultant of several velocities or accelerations are precisely analogous to those for finding the resultant of forces given in the next chapter, and will be there considered (§§ 45, 46).

EXAMPLES—VI.

[*Further examples in resolving and compounding will be found in the next chapter.*]

1. A particle moves in a straight line along a horizontal smooth plane with a velocity of 3 feet per second; after 2 seconds a velocity of 8 feet per second is imparted to it in a direction at right angles to its original motion. Find the distance of the particle from its starting-point after it has been in motion for 4 seconds.

2. The velocity of a ship in a straight course on an even keel is $8\frac{1}{8}$ miles an hour; a ball is bowled across the deck, perpendicular to the ship's length, with a uniform velocity of 3 yards in a second. Describe the true path of the ball in space, and show that it will pass over 45 feet in 3 seconds nearly.

3. A particle is moving with a velocity of 3 feet per minute along the diagonal of a square, which is itself moving with a velocity of 4 feet per minute parallel to an edge. Find the actual velocity of the particle.

4. A particle is moving with a velocity of 12 yards per minute along the side of an equilateral triangle, which is itself moving at 2 feet per second parallel to the base. Find the actual velocity of the particle.

5. A particle revolves uniformly in a vertical circle with a velocity (v); find the vertical and horizontal velocity at any point. If the time of one revolution is 8 seconds, and the radius of the circle 12 inches, and the particle start from the highest point, find the horizontal and vertical velocity after it has revolved for 1 second.

6. A body descends uniformly down an inclined plane 1 mile in length in 1 hour and 20 minutes. If the plane rises 1 foot vertical for 100 feet in length, find the vertical velocity of the body in feet per second.

7. Rain is falling vertically, and it is observed that the splashes made by the drops on the window of a moving railway carriage are inclined to the vertical. Explain this, and point out in which direction the splashes are inclined.

8. If in the previous question the train was travelling at 60 miles per hour and the inclination of the splashes to the vertical was 30°, what was the velocity of the falling drops?

9. A horseman at full gallop fires at a stationary animal. Show that he must aim behind the animal.

10. If in the previous question the animal is also running in a parallel direction, show that the horseman must aim in front or behind according as his speed is less or greater than that of the animal.

11. A steamboat is going north at 15 miles per hour while an east wind is blowing at 5 miles per hour. Find the angle the direction of the smoke appears to make with the ship's keel.

12. Knowing the direction of the true wind and the velocity and direction of the apparent wind on a ship, as shown by the direction of the vane on the mast, determine the velocity of the ship, supposing there is no lee-way.

13. In a ship sailing at 16 miles per hour it is observed that the direction of the wind is apparently 30° to the line of keel and from the

bows; its velocity is apparently 4 miles per hour. What is its true direction and magnitude?

14. If two points move in two straight lines with uniform acceleration, the path of either relative to the other will be rectilinear, if at any one instant their velocities be to one another respectively as their accelerations.

15. Two particles are started simultaneously from the points A and B, 5 feet apart, one from A towards B with a velocity which would cause it to reach B in 3 seconds and the other at right angles to the former and with three-fourths of its velocity. Find their relative velocity in magnitude and direction, the shortest distance between them, and the time at which they are nearest to one another.

16. A circular ring moves uniformly in a straight line in its own plane, and a point on the ring moves uniformly round the ring. Find the actual velocity of the point when the line joining it to the centre makes angles of (1) 90°, (2) 45°, (3) 0° with the direction of motion.

17. Two planets are moving in concentric circles, radii a, b, in the same direction with velocities u, v. Determine the angle between them when their relative velocity is along the line joining them. Describe generally the appearance of the motion of each planet as seen from the other.

CHAPTER VII

FORCES ACTING AT ONE POINT

44. When a body is changing its state of rest or motion, the rate at which the change of momentum is taking place is called the force acting on it. It may be that more than one cause is tending to make a particle move, and in such a case what is observed is only the effect due to the whole combined. By observing then the motion of any particle it is not possible to determine all the causes tending to make it move, but on the other hand, if we have given the causes and the effects which each would produce separately (in other words their forces), it must be possible to determine what the combined effect will be. This is the problem to which we address ourselves in this chapter: and we begin with the simplest case, viz. that of two forces acting at a point.

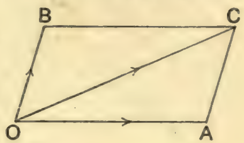
If m denote the mass of a particle and v its velocity, the momentum is measured by mv. The momentum can change, therefore, either by alteration of the mass or by alteration of the velocity. In most cases the latter takes place alone. The former would take place if matter was being created; the same result is in effect produced in certain cases, such as the fall of a raindrop through a cloud, where the drop continually increases by condensation from the mist.

Now notice, that momentum is quite analogous to velocity in that it has magnitude and direction. It will therefore obey the same laws as velocity with regard to composition—*i.e.* there is a theorem analogous to the parallelogram of velocities which may be called the parallelogram of momenta. And in precisely the same manner as the parallelogram of accelerations (*i.e.* of rate of change of velocity) was deduced from that of velocity, so a theorem of the parallelogram of forces (*i.e.* of rate of change of momentum) may be deduced from that of momentum. On account of the importance of this theorem, it is here stated in its proper form.

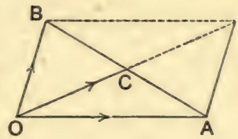
If two forces acting at a point be represented in magnitude and direction by two sides of a parallelogram, their resultant is represented by that diagonal of the parallelogram which passes through the point.

45. The constructions given for velocities also hold good in the case of forces. We thus have the following graphical methods of finding the resultant of two forces represented by two lines OA, OB—

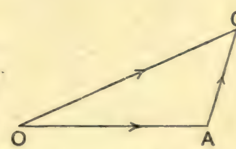
(1) Complete the parallelogram AB. Join OC. Then OC is the resultant.



(2) Join AB and bisect it in C. Join OC. Then the resultant is 2OC.



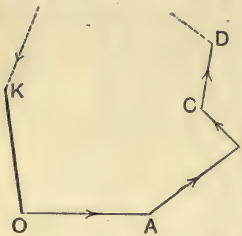
(3) From A draw AC to represent the second force, and join OC. Then the resultant is OC.



Conversely OC is equivalent to any two OA, AC.

The third method gives the means of finding graphically the resultant of any number of forces acting at a point. We shall suppose the forces are given graphically by straight lines.

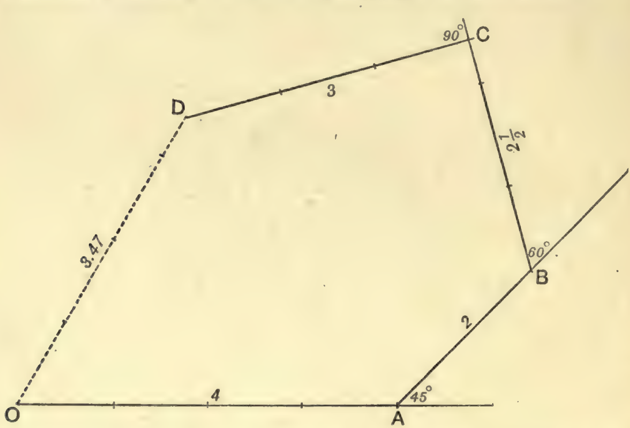
Take any point O, and from it draw the line OA to represent one of the forces in magnitude and direction. From A draw AB equal and parallel to another force; from B, BC equal and parallel to another; and so on until all the forces have been represented: and let the end of the last line be denoted by K. Then the resultant of all the forces is represented in magnitude and direction by the line OK.



For, by what has gone before, the resultant of the first two forces is represented by OB. They may, therefore, be replaced by the single force OB. Again, this and the third force may be replaced by a single force OC, and so on, until at last the whole will be replaced by the single force OK.

This method requires only the use of a scale and an instrument to measure angles, and is of wide application.

Example. *Find by the graphical method the resultant of* 4, 2, $2\frac{1}{2}$, 3 *poundals at angles of* 45°, 60°, 90° *successively with one another.*



Constructing the figure as described,* OA = 4 inches. AB is drawn making 45° with this and 2 inches measured on it, BC is then drawn making 60° with AB and $2\frac{1}{2}$ inches measured off; finally CD is drawn at right angles to BC and 3 inches marked off ending at D. Then,

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* The figure in the text is reduced from a larger one, constructed as described.

applying the ruled scale to OD, it will be found to be about 3·47 inches in length.

Hence the resultant force is 3·47 poundals, making an angle DOA with the first force.

46. The preceding paragraphs give the means of obtaining directly by graphical methods the resultant of any number of forces acting at a point. We now proceed to deduce a method by which it may be calculated in any given case to any desired degree of accuracy.

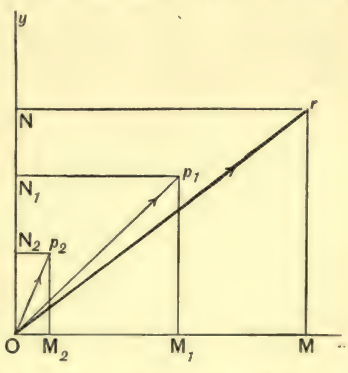
When there are only two forces P, Q, and the angle between them A is given, we can at once apply the formula already obtained for velocities. Thus, if R denote the resultant,

$$R^2 = P^2 + Q^2 + 2PQ \cos A,$$

and the direction is given by a formula similar to that in § 41.

When there are more than two forces, it is best to proceed in a different manner. Take any two straight lines Ox, Oy at right angles to one another through the point of application. Then, since the angles between the forces are given, so are their inclinations to these lines.

Let the forces be denoted by P_1, P_2, P_3, . . . and the angles they make with Ox by α_1, α_2, α_3, . . . Draw from O lines Op_1, Op_2, . . . to represent the forces. Draw p_1M_1, p_1N_1 perpendicular to Ox, Oy respectively, and similarly with the other forces. Then we know that Op_1 is equivalent to two forces OM_1, ON_1, and may therefore be replaced by them. Similarly Op_2 may be replaced by OM_2 and ON_2, and so on with all the original forces P_1, P_2, . . .



We now have a system of forces precisely equivalent to the original one, viz. a series all acting along Ox equivalent to a single one, and another series along Oy also equivalent to a single one. Denote these by X, Y. Then

$$X = OM_1 + OM_2 + OM_3 + \ldots,$$
$$Y = ON_1 + ON_2 + ON_3 + \ldots,$$

where, if M lies to the left of O, or N below O, the corresponding lines must be subtracted.

The result now is that all the original forces $P_1, P_2, \ldots$ may be replaced by the two X, Y at right angles to one another. Let them be represented by OM, ON, and complete the parallelogram OMrN. Then the resultant is

$$R = Or,$$

whence $$R^2 = OM^2 + ON^2 = X^2 + Y^2.$$

Often the angles $\alpha_1, \alpha_2, \ldots$ are such that the values of OM, etc. can easily be determined by geometrical methods, *e.g.* in the case where they are 0°, 30°, 45°, 60°, 90°, etc. In general, however, this is not the case, and recourse must be had to trigonometrical tables. Referring to the definitions given in § 40,

$$\frac{OM_1}{Op_1} = \cos \alpha_1, \quad \frac{ON_1}{Op_1} = \frac{p_1M_1}{Op_1} = \sin \alpha_1,$$

whence, since $Op_1 = P_1$,

$$OM_1 = P_1 \cos \alpha_1, \quad ON_1 = P_1 \sin \alpha_1,$$

and so with the others. Hence

$$X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \ldots = \Sigma(P \cos \alpha) \text{ say},$$
$$Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \ldots = \Sigma(P \sin \alpha).$$

Also, to find the angle rOM = A (say), which gives the direction of the resultant,

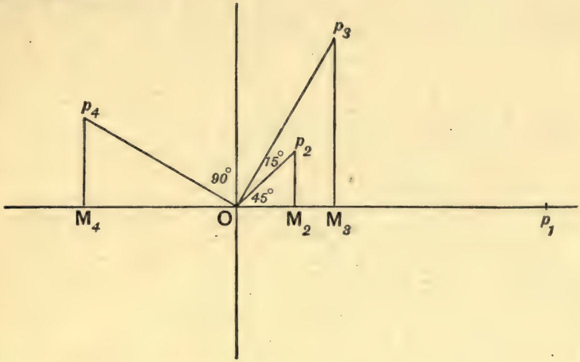
$$\tan A = \frac{rM}{OM} = \frac{ON}{OM} = \frac{Y}{X}.$$

Substituting the values of X and Y obtained above, we get the value of tan A, and A may then be found by reference to the tables. The student will perhaps best understand the method by the following examples :—

EXAMPLE I. *Four forces equal to the weight of* 8, 2, 5, $13\sqrt{3} - 18$

lbs. act so that the angles between successive forces are 45°, 15°, *and* 90°. *Determine their resultant.*

The direction of the line of reference Ox is at our disposal. We will then, in order to simplify the calculation, choose it to be along the direction of the first force. The diagram will then be as in the adjoining figure.



Here $\quad OM_1 = Op_1 = 8, \quad ON_1 = 0;$

$p_2OM_2 = 45°, \quad \therefore OM_2 = \dfrac{Op_2}{\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}, \quad ON_2 = p_2M_2 = \sqrt{2};$

$p_3OM_3 = 60°, \quad \therefore OM_3 = \dfrac{1}{2}Op_3 = \dfrac{5}{2}, \quad ON_3 = p_3M_3 = \dfrac{\sqrt{3}}{2} \cdot Op_3 = \dfrac{5\sqrt{3}}{2};$

$p_4OM_4 = 30°, \quad \therefore OM_4 = -\dfrac{\sqrt{3}}{2}Op_4 = -\dfrac{\sqrt{3}}{2}(13\sqrt{3} - 18), \quad ON_4 = p_4M_4$

$$= \frac{13\sqrt{3} - 18}{2};$$

$$\therefore X = 8 + \sqrt{2} + \frac{5}{2} - \frac{39}{2} + 9\sqrt{3} = \sqrt{2} + 9\sqrt{3} - 9,$$

$$Y = 0 + \sqrt{2} + \frac{5\sqrt{3}}{2} + \frac{13\sqrt{3} - 18}{2} = \sqrt{2} + 9\sqrt{3} - 9.$$

If the approximate values of $\sqrt{2}$ and $\sqrt{3}$ be substituted, there results

$$X = 8 \cdot 0026,$$
$$Y = 8 \cdot 0026,$$

whence $\quad R^2 = (8 \cdot 0026)^2 + (8 \cdot 0026)^2,$

$\quad = 128 \cdot 08321352,$

and $\quad R = 11 \cdot 3173,$

also $\quad \tan A = \dfrac{Y}{X} = 1,$

whence the angle is 45°. The resultant, therefore, is equal to the weight of 11·3173 lbs., and makes an angle of 45° with the force 8.

EXAMPLE II. *Find resultant of* 3, 2·5, 6 *poundals, making angles of* 22° 30′, 73° *on one side, and* 114° *on the other with a straight line* Ox.

The latter makes an angle of 180° − 114° = 66° with Ox produced backwards (say Ox').

This is a case where recourse must be had to the tables. They give the following values—

$$\cos 22°\,30' = \cdot 92388, \quad \sin 22°\,30' = \cdot 38268,$$
$$\cos 73° = \cdot 29237, \quad \sin 73° = \cdot 95630,$$
$$\cos 66° = \cdot 40674, \quad \sin 66° = \cdot 91354.$$

Hence
$$X = 3\cos 22°\,30' + 2\cdot 5\cos 73° - 6\cos 66°,$$
$$= 2\cdot 77164 + \cdot 73092 - 2\cdot 44044,$$
$$= 1\cdot 06212.$$
$$Y = 3\sin 22°\,30' + 2\cdot 5\sin 73° - 6\sin 66°,$$
$$= 1\cdot 14804 + 2\cdot 39075 - 5\cdot 48124,$$
$$= -1\cdot 94245.$$

That is, Y acts along Oy' instead of Oy.

Hence
$$R^2 = (1\cdot 06212)^2 + (1\cdot 94245)^2,$$
$$= 1\cdot 1281 + 3\cdot 7729,$$
$$= 4\cdot 9010,$$
$$R = 2\cdot 213 \text{ poundals.}$$

and
$$\tan A = \frac{1\cdot 94245}{1\cdot 06212} = 1\cdot 8288.$$

Now
$$\tan 61°\,19' = 1\cdot 8278,$$
$$\tan 61°\,20' = 1\cdot 8290,$$

whence, neglecting seconds,

$$A = 61°\,19',$$

or the direction makes an angle 180° − 61° 19′ = 128° 41′ with the positive direction of Ox.

EXAMPLES—VII (a).

1. What are the rectangular components of 10 poundals along lines making two equal angles with it?

2. What are the components of 10 poundals along lines making 30° with it on each side?

3. Find the resultant of the following forces—

(1) 4 and 5 lbs. weight at 30°,
(2) 2 and 6 lbs. weight at 60°,
(3) 4 lbs. weight and 4 poundals at 45°,
(4) 6 and 7 poundals at 135°.

4. Find the resultant of 20 and 35 lbs. weight at an angle of 120°.

5. Forces equal to the weights of 6, 7, and 8 lbs. act at a point in directions making angles of 120° with each other. Find their resultant.

6. Find graphically the magnitude and direction of

(1) 3 and 5 lbs. weight at 35°,

(2) 3, 6, 2, 8 lbs. weight,

making angles of 15°, 120°, 60° with each other in succession.

7. Calculate by the help of the trigonometrical tables the magnitude and direction of the resultants of

(1) 3 and 5 lbs. weight at 35°,

(2) 3, 6, 2, 8 lbs. weight,

making angles of 15°, 120°, 60° with each other in succession.

8. Show that if the angle between two forces is increased, their resultant is diminished.

9. A cricket-ball, mass m, being supposed to move with uniform velocity in a horizontal plane, occupies t seconds in traversing the distance, 22 yards, from the bowler to the batsman. In what direction must it be hit so as to go off with equal velocity in a direction at right angles to that in which it first moves? Give a numerical measure of the blow, supposing $m=5\frac{1}{2}$ oz., $t=5\frac{1}{2}$.

10. The greatest resultant that two forces can have is P, and the least resultant they can have is Q. Find what their resultant is when they act at right angles.

11. Prove that the resultant of two forces P and P + Q acting at an angle of 120° is equal in magnitude to the resultant of two forces Q and P + Q acting at the same angle.

12. The horizontal and vertical components of a certain force are equal to the weights of 5 lbs. and 12 lbs. respectively. What is the magnitude of the force?

13. Supposing this force to act for 10 seconds on a mass of 8 lbs., which is also exposed to the action of gravity and is initially at rest, what velocity will be communicated to the mass, the vertical component of the force acting upward?

14. The sum of two forces is 36 lbs. weight, and the resultant, which is at right angles to the smaller of the two, is 24 lbs. weight. Find the magnitude of the forces.

15. Forces of 12 lbs. and 6 lbs. weight act at a point in a plane; the angle between them is 120°. Determine the direction of a force which exactly counterbalances these.

16. The magnitudes of two forces are as 3 : 5 and the direction of their resultant is at right angles to that of the smaller force. Compare the magnitudes of the larger force and the resultant.

17. If the resultant R of the two forces P and Q inclined to each

other at any given angle make the angle θ with P, prove that the resultant of the forces (P + R) and Q at the same angle will make the angle $\theta/2$ with P + R.

18. The resultant of two forces which are inclined to each other at an angle of 60° divides the angle in the ratio 3 : 1. What is the magnitude of the smaller force, if that of the greater be 20 lbs. weight?

19. Forces of 1, 2, 4, 6, 8 act from the centre of a regular pentagon to the corners. Find the resultant. Given

$$\cos 72° = \cdot 30901, \quad \sin 72° = \cdot 95105,$$
$$\cos 36° = \cdot 80901, \quad \sin 36° = \cdot 58778.$$

20. The sides of a quadrilateral taken in order are 1, 2, 9, 7 inches respectively; forces at a point of 2, 4, 8, 14 lbs. weight respectively act parallel to them, the same way round. What is the resultant?

21. ABDC is a parallelogram, and AB is bisected in E. Prove that the resultant of the forces represented by AD, AC is double the resultant of those represented by AE, AC.

22. Four forces are represented by the sides AB, BC, CD, AD of the rectangle ABCD. Find their resultant and its line of action.

23. Three forces are completely represented by the lines joining the angular points of a triangle with the middle points of the opposite sides. Show that they are in equilibrium.

24. ABC is a triangle, D, E are points in AB and AC respectively, BE, CD cut in O. Indicate the direction of the resultant of forces represented by CD, BE.

25. The side BC of an equilateral triangle ABC is bisected in D, and forces are represented in direction and magnitude by BA, BD. Find the magnitude of their resultant if the force along BD be equal to the weight of 1 lb.

26. The side BC of an equilateral triangle ABC is bisected at D and a point O is taken in AD so that OD is equal to twice OA; two forces act along OB and OC, each equal to the weight of $\sqrt{7}$ lbs. Find the magnitude and direction of their resultant.

27. The sides OB, BA, AO of the triangle OAB are respectively 4, 5, and 6 inches long. Find the magnitude of a force which, together with a force of 4 lbs. weight acting along OB, will be equivalent to a force of 12 lbs. weight acting along OA.

28. ABC is a triangle, D, E are the middle points of AB, AC. Show that forces acting at a point represented in magnitude and direction by DB, BC, CE are equivalent to forces represented by DA, AE.

29. Four forces, P, Q, R, and S, no two of which are parallel, act in a plane; the resultant of P and Q meets that of R and S in A, the resultant of P and R meets that of Q and S in B, and the resultant of P and S meets that of Q and R in C. Prove that A, B, and C are in the same straight line.

30. Centres of force repelling directly as the distance are arranged symmetrically round a circle, and one of them is destroyed. Find the resultant force upon a particle placed at any point in the plane.

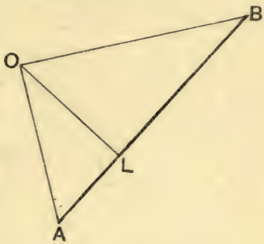
31. It is required to decompose a force whose magnitude and line of action are given into two equal forces passing through two given points. Give a geometrical construction for solving the problem (1) when the two points are on the same side of the line, (2) on opposite sides.

47. The following definition and theorem are of very great importance.

Def. *The product of the measures of a force and its distance from a point is called the moment of the force about the point.*

Note.—The "dimensions" of a force are $[ML/T^2]$, § 34 V. Hence the dimensions of a moment are $[ML^2/T^2]$, or the same as those of energy.

Let O be the point, and AB represent the force. Join OA, OB and draw OL perpendicular to AB. Then the moment is represented by the product OL × AB, *i.e.* by twice the area of the triangle OAB. Thus, just as a force may be represented geometrically by a straight line, so may its moment about a point be represented by twice the area of the triangle formed by joining the point to the extremities of the line representing the force.



It will be convenient to regard the moment of a force as of different signs, according as the point is on different sides of the force, *or* according as the direction of the force looked at from the point tends in the direction of the hands of a watch, or opposite. With this convention, the following important theorem is true, viz.—

The sum of the moments of any number of forces about a point is equal to the moment of their resultant about the same point.

We first proceed to prove the theorem in the case of two forces acting at a point.

Let the forces act along the lines Oa, Ob and their resultant along Oc. Let P be the point about which the moments are taken. Through P draw BPC parallel to Oa, and through C draw CA parallel to Ob. Then OA, OB may be taken to represent the forces and OC their resultant. Join PO, PA. Then we require to show that

(1) If P is placed as in the first figure,

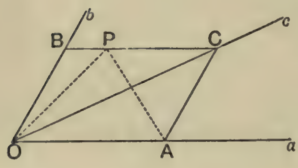
$$2 \triangle \text{OPA} - 2 \triangle \text{OBP} = 2 \triangle \text{OPC}.$$

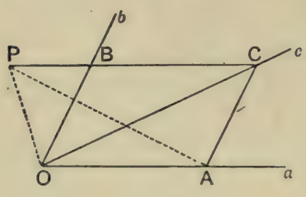
(2) If P is placed as in the second figure,

$$2 \triangle \text{OPA} + 2 \triangle \text{OPB} = 2 \triangle \text{OPC}.$$

Now in (1)

$$\begin{aligned} 2 \triangle \text{OPA} - 2 \triangle \text{OBP} &= 2 \triangle \text{OCA} - 2 \triangle \text{OBP}, \\ &= 2 \triangle \text{OCB} - 2 \triangle \text{OBP}, \\ &= 2 \triangle \text{OPC}; \end{aligned}$$





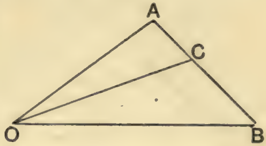
and in (2)

$$\begin{aligned} 2 \triangle \text{OPA} + 2 \triangle \text{OPB} &= 2 \triangle \text{OCA} + 2 \triangle \text{OBP}, \\ &= 2 \triangle \text{OCB} + 2 \triangle \text{OBP}, \\ &= 2 \triangle \text{OPC}. \end{aligned}$$

The theorem is therefore true for two forces. It is then easy to see that it is true in general for any number of forces P_1, P_2, P_3, . . . For the moments of P_1, P_2 are equal to that of their resultant. Hence the moments of P_1, P_2, P_3 are equal to those of the two forces P_3 and the resultant of P_1, P_2, that is of the resultant of the three—and so on.

48. The preceding result enables us to prove a theorem which is often of great use. It is this. Let forces act along the lines OA, OB, and of magnitudes $l \,.\, \text{OA}$, $m \,.\, \text{OB}$ respectively. Then the resultant acts along OC, where $\text{AC} : \text{BC} = m : l$, and is represented in magnitude by $(l + m)\text{OC}$. For if the resultant R acts along OC, the

moments of the forces round C must vanish. Now, since the forces are l . OA and m . OB, the moments are $l \times$ moment of OA $- m \times$ moment of OB,



$\therefore l . \triangle OAC - m . \triangle OBC = 0.$

But $\triangle$ OAC, $\triangle$ OBC are on bases AC, CB, and of the same altitude,

$$\therefore \frac{\triangle OAC}{\triangle OBC} = \frac{AC}{BC},$$

$$\therefore l . AC = m . BC,$$

which proves the first part. To find the value of R, take moments about A. Then, if $R = x . OC$,

$$m . \text{ moment of } OB = x . \text{ moment of } OC,$$

or

$$m . \triangle OAB = x . \triangle OAC,$$

$$m . AB = x . AC.$$

But

$$AB = AC + BC = \frac{l+m}{m} . AC,$$

$$\therefore (l+m)AC = x . AC,$$

$$x = l + m,$$

$$R = (l+m)OC,$$

which proves the second part.

STATICS OF FORCES ACTING AT A POINT.

49. In the succeeding part of this chapter we devote our attention to the conditions of the equilibrium of a particle acted on by any forces in one plane. In other words, we proceed to consider the science of statics in so far as it has reference to a single particle.

As the particle does not change its motion, no resultant force can act on it. This then is the simple condition, viz. the resultant of all the forces must be zero. We can from this deduce easily the two following sets of conditions, either of which is necessary and sufficient.

(1) *The resolved parts of all the forces in any two directions at right angles must vanish.*

This is necessary, for as there is no resultant there can be no components—it is also sufficient, for the resolved parts are components of the force, and if these two components vanish, the diagonal of the parallelogram formed by them must also be zero—*i.e.* there can be no resultant force.

If the resolved parts in any two directions vanish, so must the resolved parts in any directions at right angles. Hence the limiting condition "at right angles" may be omitted.

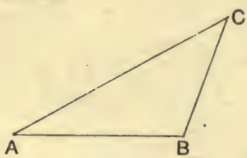
(2) *The moments of the forces about any two points not in the same straight line with the particle must vanish.*

For the said moments are equal to the moment of their resultant. Now this moment can vanish, either because the resultant itself vanishes, or because the resultant passes through the point.

If, however, the particle and the two points do not lie in one straight line, it is clear that the resultant cannot pass through both points, and therefore, if the moment vanishes for both, it must be because the resultant force itself vanishes. That is, the particle is at rest. This also is clearly necessary and sufficient.

50. The following theorems, easily deduced from the foregoing principles, are often of great use.

(1) *The triangle of forces.*—If three forces *acting on a particle* be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.



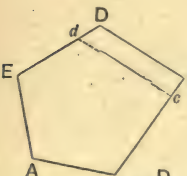
For let the forces be represented in magnitude and direction (though not line of action) by AB, BC, CA. Then we know that the resultant of AB, BC is AC (§ 45). Hence the resultant of the three is that of AC and CA, that is zero—or the particle will be at rest.

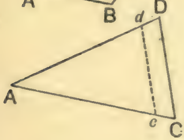
Note.—The converse theorems of this are also true, viz. if three forces acting on a particle be in equilibrium, and a triangle be formed (α) whose sides are parallel to them they will also be proportional to them; or (β) if the sides are proportional to them they shall make the same angles with one another as the corresponding forces do.

(2) *The polygon of forces.*—If any number of forces acting on a particle can be represented by the sides of a closed polygon taken in order, they will be in equilibrium.

For let the polygon be OABC . . . K. By § 45 the resultant is found by joining O to the extremity of the sides representing the forces. But here this extremity is at O itself. Therefore the resultant is zero.

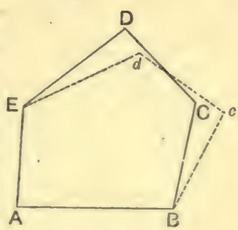
Note.—The converse theorems of this are not necessarily true, viz. if a polygon be drawn with its sides parallel to the forces they will not necessarily be proportional to them. For suppose the polygon ABCDE to have its sides parallel and proportional to the forces. Draw *cd* parallel to CD, then AB*cd*E has also its sides parallel to the forces, but clearly its sides are not proportional to those of ABCDE, *i.e.* to the forces.





In the case of a triangle, however, the sides of A*cd* are proportional to those of ACD.

So also with the other converse, viz. if the sides are proportional to the forces they will not necessarily contain angles equal to those between the forces. For suppose this the case with the polygon ABCDE. Form another polygon by displacing EDCB to E*dc*B, keeping the lengths unaltered. Then AB*cd*E is a polygon with its sides proportional to the forces, but the angles



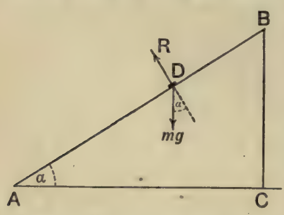
are clearly not the same as in ABCDE—that is, as between the forces.

This illustration fails in the case of the triangle, as it ought, for when three sides are given there is only one triangle possible.

The following examples will not only serve to illustrate the principles developed in this chapter, but are also important in themselves :—

EXAMPLE I. *A particle is placed on a smooth inclined plane. Determine the motion.*

A smooth surface is one which offers no resistance to a particle moving over it. The reaction between the particle and the surface is therefore always perpendicular to the surface at the point where the particle is.



Let m be the mass of the particle, and its weight therefore mg. Let BA be the inclined plane making an angle BAC $(=\alpha)$ with the horizontal, and suppose the particle at D. The first thing we have to do is to find the forces on it. In this case there are only two, viz. its weight mg acting vertically and the resistance of the plane, which we will call R, and which, as we have seen, is perpendicular to AB.

Resolve the forces in two directions, one along AB and the other perpendicular to it.

Now the weight can be decomposed into $mg \cos \alpha$ perpendicular to AB and $mg \sin \alpha$ along BA. Hence

$$\text{Force down BA} = mg \sin \alpha.$$

This acts on the mass m.

$$\therefore \text{Acceleration down BA} = \frac{mg \sin \alpha}{m} = g \sin \alpha.$$

Also Force perpendicular to AB is

$$R - mg \cos \alpha.$$

But there is no acceleration perpendicular to AB (else the particle would go through the plane),

$$\therefore \text{the force} = 0,$$

$$\therefore R - mg \cos \alpha = 0,$$

or

$$R = mg \cos \alpha = W \cos \alpha,$$

where W is the weight.

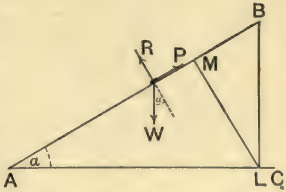
Hence the final result is that the particle moves down the plane with acceleration $g \sin \alpha$.

If BC be perpendicular to AC, this may be expressed,

$$\text{Acceleration} = g\frac{\text{BC}}{\text{AB}} = \frac{\text{height of plane}}{\text{length of plane}}g.$$

EXAMPLE II. *A mass of weight* W *is placed on an inclined plane, and is kept at rest by a force acting up the plane. Determine the magnitude of this force and the pressure on the plane.*

Let AB be the plane, AC the horizontal, and BL perpendicular to AC. Let P denote the force up the plane, R the reaction. Then the forces acting on the particle are as represented in the figure. We shall treat the question in two ways.



(*a*) Draw LM perpendicular to AB. Then in the triangle LMB the sides LM, MB, BL are parallel to the forces R, P, W respectively. Therefore, by the converse of the triangle of forces, they are proportional to them.

$$\therefore \frac{\text{P}}{\text{BM}} = \frac{\text{R}}{\text{LM}} = \frac{\text{W}}{\text{BL}}.$$

But the triangle LMB is similar to ALB, and therefore has its sides proportional to those of the triangle ALB respectively—

$$\therefore \frac{\text{P}}{\text{BL}} = \frac{\text{R}}{\text{AL}} = \frac{\text{W}}{\text{AB}};$$

or

$$\text{P} = \text{W}\,.\,\frac{\text{BL}}{\text{AB}} = \text{W}\,.\,\frac{\text{height of plane}}{\text{length of plane}},$$

$$\text{R} = \text{W}.\frac{\text{AL}}{\text{AB}} = \text{W}\,.\,\frac{\text{base of plane}}{\text{length of plane}}.$$

E.g. *a smooth board* 1 *yard long has one end raised* 9 *inches. Find the force necessary to keep* 1 *ton weight from slipping down.*

Here $\text{P} = \frac{9}{36} \times 1$ ton weight = weight of 560 lbs.

(β) In the second method let the inclination of the plane to the horizontal be given, say α. Then W makes an angle α with the perpendicular to AB.

The condition of equilibrium in § 49 is that the resolved parts in any two directions must vanish. Resolve then along AB and perpendicular to it. Then

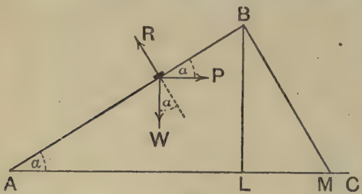
$$\text{P} - \text{W}\sin\alpha = 0,$$
$$\text{R} - \text{W}\cos\alpha = 0;$$

whence

$$\text{P} = \text{W}\sin\alpha,$$
$$\text{R} = \text{W}\cos\alpha.$$

Note.—By resolving along AB, R does not enter into the equation, and by resolving perpendicular to AB, P does not enter. Hence the equations will give P and R at once. The advantage of this is obvious. In general, in applying the conditions of equilibrium, we should if possible resolve perpendicular to a force whose magnitude is not wanted.

Example III. *A particle of weight* W *is placed on a smooth inclined plane, and is kept there by a horizontal force. Determine its magnitude and the pressure on the plane.*



Denote the force by P, the pressure by R. Draw BL perpendicular to AC and BM to AB.

Then in the triangle BLM the sides BL, LM, MB are parallel to the forces W, P, R. Therefore

$$\frac{W}{BL}=\frac{P}{LM}=\frac{R}{MB}.$$

But BLM is equiangular to ALB, and therefore the sides of the two triangles are proportional. Hence

$$\frac{W}{AL}=\frac{P}{BL}=\frac{R}{AB};$$

or

$$P=W.\frac{BL}{AL}=W\frac{\text{height of plane}}{\text{base of plane}},$$

$$R=W.\frac{AB}{AL}=W\frac{\text{length of plane}}{\text{base of plane}}.$$

If we proceed by the second method, we shall have, resolving along the plane and vertically,

$$P\cos\alpha-W\sin\alpha=0,$$
$$R\cos\alpha-W=0;$$

whence

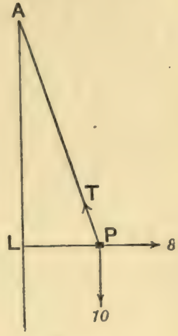
$$P=W\frac{\sin\alpha}{\cos\alpha}=W\tan\alpha,$$

$$R=\frac{W}{\cos\alpha}.$$

Example IV. *A mass of* 10 *lbs. is suspended from a point by a string* 4 *feet long; it is acted on by a horizontal force equal to the weight of* 8 *lbs. How far is it displaced and what is the tension of the string?*

Take the weight of 1 lb. for the unit of force.

Let A be the point of suspension and P the position of rest of the mass. Draw PL, AL horizontal and vertical. Then the forces 8, 10, T are parallel to LP, AL, PA, and are therefore proportional to them. Therefore



$$\frac{8}{PL}=\frac{10}{AL}=\frac{T}{AP}=\frac{T}{4};$$

also $$PL^2+AL^2=AP^2=16.$$

But $$\frac{AL}{PL}=\frac{10}{8}=\frac{5}{4}.$$

$$\therefore AL=\frac{5}{4}PL,$$

and $$PL^2+\frac{25}{16}PL^2=16,$$

$$41PL^2=16\times 16,$$

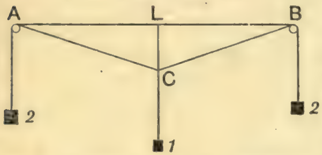
$$PL=\frac{16}{\sqrt{41}}\text{ feet,}$$

which gives the displacement.

Also $$\frac{T}{4}=\frac{8}{PL}=\frac{8\sqrt{41}}{16},$$

$$\therefore T=2\sqrt{41}\text{ lbs. weight.}$$

EXAMPLE V. *Two masses of* 2 *lbs. each are connected by a string and are suspended over two smooth pegs in a horizontal line and* 1 *foot apart; a mass of* 1 *lb. is then hooked on to the string between the pegs. What is the position of rest?*



The particles will hang as in the figure. AB=1 foot.

The tension along CB and CA must be equal to the weight of 2 lbs. Hence at C three forces act, viz. 2 lbs. weight along CB, 2 lbs. weight along CA, and 1 lb. weight vertical. CA and CB must be equally inclined to the vertical. Hence, resolving vertically,

$$2\cos BCL+2\cos ACL-1=0.$$

But $$BCL=ACL,$$

$$\therefore 4\cos ACL=1,$$

or $$4\cdot\frac{CL}{AC}=1,$$

$$AC=4CL.$$

Now $$AL=\tfrac{1}{2}\text{ foot}=6\text{ inches,}$$

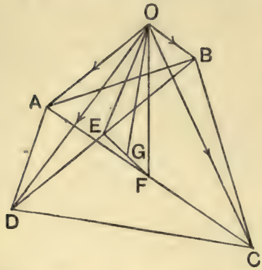
and $$AC^2=AL^2+CL^2,$$

$$\therefore 16CL^2=36+CL^2,$$

$$15CL^2=36,$$

$$CL=\frac{6}{\sqrt{15}}=\frac{2}{5}\sqrt{15}\text{ inches.}$$

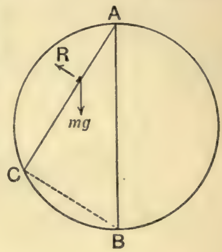
Example VI. ABCD *is a quadrilateral and* O *any point; forces are represented by* OA, OB, OC, OD. *Prove that their resultant is* 4.OG, *where* G *is the mid point of the line joining the mid points of the diagonals.*



Join the diagonals BD, AC, and let E, F be their mid points. Join EF and bisect it in G.

Then the resultant of	OA, OC = 2OF,
,, ,,	OB, OD = 2OE,
and resultant of	OE, OF = 2OG,
∴ ,, ,,	2OE, 2OF = 4OG.

51. The time occupied by a particle in falling from the highest point of a vertical circle down any chord is the same for all chords. To prove this, let AB be the vertical diameter of the circle and AC any chord through A. The particle is acted on by its weight mg vertically and the reaction R, which is normal to the chord. The force effective in moving the particle is therefore the resolved part of the weight along AC. Since ACB is a right angle, this resolved part is AC/AB of the weight. The acceleration is therefore



$$\frac{\mathrm{AC}}{\mathrm{AB}} \cdot g.$$

Hence, if t be the time down AC,

$$\mathrm{AC} = \frac{1}{2} \cdot \frac{\mathrm{AC}}{\mathrm{AB}} g t^2,$$

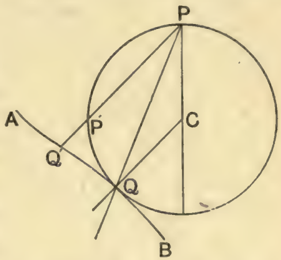
whence $$t = \sqrt{\frac{2\mathrm{AB}}{g}},$$

or the time down AC is equal to that down AB, and is therefore the same for all chords through A.

The same theorem is also clearly true for chords ending in the lowest point B.

This curious theorem is of importance, since it enables us to find the straight line of quickest descent from a fixed point to a curve, and also from one curve to another.

Thus let AB be any curve in a vertical plane and P a point in the same plane. It is required to find a straight line from P to the curve down which a particle will slide more quickly than down any other straight line. Through P draw a circle having its highest point at P and touching the curve AB. Let it touch it in Q. Then PQ is the line required. For, if not, let PQ′ be a line down which the time is shorter. This will cut the circle at some point P′ between P and Q′. Now by the foregoing theorem the time down PP′ is the same as that down PQ. But clearly it takes longer to slide down PQ′ than PP′. Hence the time down PQ′ is longer than down PQ—so that PQ *is* the line of quickest descent.



The construction may be modified thus. Let C be the centre of the circle. Join CP, CQ. Then CP is vertical, and CQ is normal to the curve at Q. Also the angle CPQ is equal to the angle CQP. Hence PQ must be in such a direction that it bisects the angle at Q between the vertical and the normal to the curve at Q.

EXAMPLES—VII (b).

1. Equal forces act at the centre of a regular pentagon along the lines drawn from the centre to the angles of the pentagon. Prove that the resultant of any two of these adjacent forces is equal and opposite to the resultant of the other three.

2. Five forces acting on a body keep it in equilibrium; the resultant of three of these forces is known. Find the resultant of the other two.

3. Three forces cannot be made to balance if the sum of two of them is less than the third.

4. A weight of 10 lbs. is supported by two forces, one of which acts horizontally and the other at an angle of 30° with the horizon. Find the magnitude of these forces.

5. A mass of 12 lbs. is hanging by a string from a fixed point; it is acted on by a horizontal force equal to the weight of 9 lbs. Find the tension of the string.

Also the horizontal distance through which it is displaced, the length of the string being 5 feet.

6. What is the inclination to the horizon of the steepest plane on which a force of 5 lbs. weight will support a weight of 10 lbs.?

7. On an inclined plane, whose height is 6 feet and length 18 feet, a mass of 12 lbs. is supported by a string which, passing over a pulley at the top, supports a mass m hanging freely. Determine the mass m that the 12 lbs. may be in equilibrium.

8. A number of loaded trucks, each containing 1 ton, on one part of a tramway whose inclination to the horizon is α, supports an equal number of empty trucks on another part whose inclination is β. Find the weight of a truck.

9. A man pushes a garden roller of weight w up a plank 6 feet long and resting with one end on a step 1 foot high; he holds the handle horizontally. Find the force necessary to push it, and the pressure on the plank.

10. Prove that if R, R′ be the pressures on a plane when a given weight W is supported by a force parallel to the base and a force parallel to the plane respectively, $RR' = W^2$.

11. Equal weights are attached to the ends of a string, one of which rests on a plane inclined at 45° to the horizon, and the other hangs vertically over the summit of the plane and rests on the ground beneath. Find the pressure of the latter on the ground.

12. A string is fastened to two points A, B in a horizontal line, and supports a mass of 8 lbs. at its middle point. Determine the tension of the string, when the length of the string is 1 foot and the depth of its middle point below AB is 2 inches.

13. If ABCD is any quadrilateral figure, and O be the point of intersection of the two straight lines bisecting the opposite sides of the quadrilateral, then the forces represented in magnitude and direction by OA, OB, OC, OD will be in equilibrium at O.

14. ABCD is a quadrilateral with a point fixed; forces act along AB, BC, AD, DC proportional to those lines, and there is equilibrium; also if forces be represented by BA, AD, BC, CD there is equilibrium. Find the position of the fixed point, and prove that the line joining it to the intersection of the diagonals is bisected by the line which bisects two opposite sides of the quadrilateral.

15. ABC is a triangle, AE, BF, CD lines drawn from the angles to the points of bisection of the opposite sides. Show that the forces represented by AE, BF, and CD are in equilibrium.

16. If four straight lines lie in a plane, and no three of them meet in a point nor are parallel, prove that it is always possible to arrange forces acting along them so as to be in equilibrium.

If the four straight lines are the sides of a quadrilateral inscribed in a circle, prove that the force acting along each side is proportional to the opposite side of the quadrilateral.

17. On two inclined planes of equal height two weights are respectively supported by means of a string passing over the common vertex; the length of one plane is double its height, and the length of the other is double its base. Show that the pressure on one plane is three times the pressure on the other.

18. A continuous string, without weight, length l, hung over two smooth pegs in the same horizontal line, distant a apart, hangs in two loops, on each of which is placed a small, smooth, heavy ring, one of weight W, the other of weight W′. Find an equation to determine the tension of the string.

19. A string is fastened to two points A, B and has equal particles (m) attached at C, D trisecting the string; CD is horizontal and AC, BD inclined at angles of 45° to the horizon. Find the tensions in AC, CD.

20. Four forces acting in the sides of the quadrilateral ABCD are in equilibrium. If the straight lines representing the forces in the sides AB, BC, CD, DA are to the lengths of the respective sides as $p:q:r:s$, prove by considering the moments of the forces about the angular points that $pr=qs$.

EXAMPLES—VII (c).

1. Prove that the time of falling down a plane from rest is the same as the time of moving over the same distance with a uniform velocity equal to half that acquired in falling down the plane.

2. A heavy particle slides from rest down a smooth inclined plane 15 feet long and 12 feet high. What velocity will it possess when it reaches the bottom and how many seconds will be occupied in the descent? How long would it have taken to fall vertically through a height of 12 feet?

3. A heavy body slides down a smooth plane inclined 30° to the horizon. Through how many feet will it fall in the fourth second of its motion?

4. A body begins to slide down a smooth inclined plane from the top, and at the same instant another body is projected upwards from the foot of the plane with such a velocity that the bodies meet in the middle of the plane. Find the velocity of projection and determine the velocities of each body when they meet.

5. Find the horse-power of an engine which is taking a train of 120 tons down an incline of 1 in 224 at 50 miles per hour, supposing a resistance of 35 lbs. a ton on the level at this speed.

6. Find the horse-power of an engine which is drawing a train of 200 tons up an incline of 1 in 250 at 20 miles an hour; the resistance on the level being 8 lbs. a ton.

7. Supposing a tricycle and rider, weighing together 200 lbs., to run uniformly at 8 miles per hour down an incline of 1 in 100 against the resistance of the air and of the road, without working the pedals; prove that to go up an incline of 1 in 200 at the same speed the rider must be working at the rate of ·064 horse-power, and that the mean pressure on each pedal will then be about 8·06 lbs. weight, supposing the cranks 5 inches long and making 100 revolutions a minute.

8. A train runs from rest for a mile down a plane whose descent is 1 foot vertical for 100 feet in length. If the resistances are 8 lbs. per ton, how far will the train be carried along the horizontal level at the foot of the incline?

9. If a train ascends a gradient of 1 in 40 by its own momentum for a distance of 1 mile, the resistance from friction etc. being 10 lbs. weight per ton, find its initial velocity.

10. A train of mass 200 tons is running at 40 miles an hour down an incline of 1 in 120. Find the resistance necessary to stop the train in half a mile.

11. A train runs from rest down an incline of 1 in 100 for a distance of 1 mile (no engine attached); it then runs up an equal gradient with its acquired velocity for 500 yards before stopping. Find the total resistance in pounds per ton which has been opposing its motion.

12. If particles start from rest from a given point to run down a number of smooth inclined planes, show that at the end of t seconds they will all be at the same distance from a point $gt^2/4$ feet below that from which they started.

13. P hangs vertically and is 9 lbs. Q is 6 lbs on a plane whose inclination is 30°. Show that P will draw Q up the whole length of the plane in half the time that Q hanging vertically would draw P up the plane.

14. A weight of W lbs. is drawn from rest up a smooth inclined plane of height h and length l by means of a string passing over a pulley at the top of the plane, and supporting a weight of w lbs. hanging freely. Prove that in order that W may just reach the top of the plane, w must be detached after it has descended a distance $\frac{W+w}{w} \cdot \frac{hl}{h+l}$.

15. Two equal masses of 8 lbs. are connected by a string; one lies on an inclined plane and the other hangs freely over the top. If the inclination of the plane be 30° and the string be just on the point of breaking, find the greatest weight which the string would support if it were suspended from a fixed point vertically.

16. If two particles P and Q start simultaneously from A, one sliding down the plane AB at the angle α to the horizon and the other falling freely, prove that their relative vertical acceleration is ($g \cos^2 \alpha$).

17. Three weights (w) are fastened to a string whose length (l) is equal to that of an inclined plane; one weight is attached to each end, and the other weight to the middle of the string; when one weight hangs over the top of the plane the weights are in equilibrium. If the second weight also is just made to hang vertically, find the velocity with which the third weight reaches the top of the plane.

18. A smooth wedge in the form of a right-angled triangle ABC has its hypotenuse AB horizontal; the angle ABC is 30°; equal masses lie on the faces and are connected by a string passing over the top. Determine the acceleration, the tension of the string, and the pressure produced by the string at the top.

19. A man stands on the upper end of a long rough plank of length a and mass M, which lies along a smooth straight groove on an inclined plane, and has its upper end supported by a cord. The cord is cut, and at the same instant the man starts off and runs with very short steps down the plank at such a rate that the plank does not move. Prove that the velocity of the man at the lower end of the plank is

$$\sqrt{\left\{2ga\cos\alpha\frac{m+M}{m}\right\}},$$

where m is the mass of the man and α the inclination of the groove to the vertical.

20. A wedge with angle 60° is placed upon a smooth table, and a weight of 20 lbs. on the slant face is supported by a string lying on that face passing through a smooth ring at the top and supporting a weight W hanging vertically. Find the magnitude of W. Find also the force necessary to keep the wedge at rest (1) when the ring is not attached to the wedge, (2) when it is so attached.

21. A particle is fastened on an inclined plane whose height is 6 inches and base 18 inches, and which rests on a horizontal board; it is set free to slide down the plane. Find the change in the pressure on the board.

With what acceleration must the board be moved upwards that the pressure on it may be the same? In this case compare the time of the particle's fall down the plane with that in the former case.

22. If two vertical circles touch each other a their lowest point, and any straight line be drawn from that point to cut the inner and to meet the outer circle, show that the time of a heavy particle falling from rest along the part of the line (considered as an inclined plane) intercepted between the circles is constant.

23. A straight line without a circle and in the same plane with it is

parallel to its vertical diameter. Find the straight line of quickest descent from the given line to the circle, and determine the angle which the line so drawn makes with the tangent at the lowest point of the circle.

24. Show that the line of quickest descent from one curve to another bisects at each end the angles between the vertical and the normal to the curve there.

25. Show how to find the time of quickest descent from a vertical curve to a point below it.

26. Find the direction of the line of quickest descent between two parallel lines in a vertical plane.

27. Find the direction of the line of quickest descent from a point to a circle.

28. Find the line of longest descent from a point to a circle.

29. Find (1) the line of quickest and (2) the line of longest descent from one circle to another, both lying in a vertical plane.

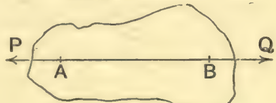
CHAPTER VIII

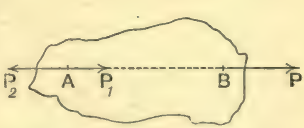
FORCES ON A RIGID BODY—PARALLEL FORCES

52. When we are dealing with the motion, or with the conditions of equilibrium of particles, all the forces under consideration pass through one point. This is no longer the case when we have to deal with bodies of finite size, and our methods will therefore have to be modified accordingly. The modification depends on the principle that we may suppose a force to *act* at any point in its line of action, which is in rigid connection with the body. In practice, however, it is best to suppose that all points are rigidly connected, and consider afterwards the actual circumstances of each case.

The truth of the principle can be rendered evident by the following considerations.

Let A, B be two points in a rigid body and suppose two equal and opposite forces acting at the points A, B, the one in the line BA and the other in the line AB. Then it is evident that no change of motion can take place—in other words, the state of rest or motion is unaltered if we suppose the two forces altogether removed.



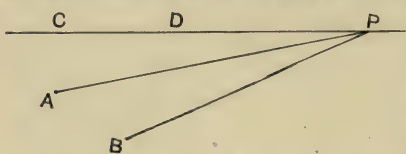


Now let a force P act at a point B of a rigid body, and let A be any other point in its line of action, rigidly connected with B. At A apply two opposite forces (P_1, P_2) each equal to P in magnitude and acting, P_1 along AB and P_2 along BA. These, being in equilibrium by themselves, will not affect the motion or state of rest. But by the preceding considerations we may take away P and P_2, and there remains P_1 acting at A, instead of P at B, whilst the state of rest or motion of the body is unaltered. In other words, we may suppose P to have its point of application at any point in its line of action.

53. If a system of forces in one plane acts on a body, their directions will in general intersect. We can then by the foregoing principle suppose two to act at their point of intersection, and replace them by their resultant. We can then take the resultant of this and another of the forces, and so on, until at last we have the single resultant of the whole. This will be possible by foregoing principles, provided each two of the forces under consideration do actually intersect; but should they be parallel these principles are no longer directly applicable. Before, therefore, proceeding to the general case it will be necessary to consider the special case of parallel forces acting on a rigid body.

Two Parallel Forces.

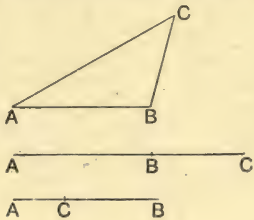
54. Suppose A, B to be two fixed points and CD a fixed straight line. Let P be any point on the line CD; join PA, PB. Now suppose P to move farther and farther to an infinite distance along CD, then PA, PB will become more and more parallel to CD, and by making P go far enough we may make the angles APC, BPC smaller than any given quantity, however small. That is, ultimately the angles become nothing and CD, PA, PB become parallel.



So, *vice versa*, all parallel lines may be supposed to intersect in the same point at an infinite distance away.

This consideration at once brings parallel forces within the category of forces acting at a point, and hence the results already obtained are applicable to these forces also.

(1) To find the *magnitude* of the resultant of two parallel forces we notice that if AB, BC be equal and parallel to two forces P, Q, their resultant (R) is AC. If now P, Q are parallel and in the same direction, the angle ABC becomes two right angles and



$$AC = AB + BC,$$

or $$R = P + Q.$$

If, on the contrary, they are in opposite directions, the angle ABC becomes zero, and

$$AC = AB - BC,$$

or $$R = P - Q.$$

(2) The *direction* of R is clearly parallel to that of P or Q, for R goes through their intersection, which is at an infinite distance.

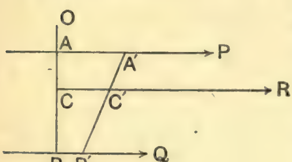


FIG. 1.

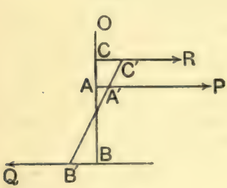


FIG. 2.

(3) The *position* of R is easily found by applying the theorem of moments proved in the last chapter.

Take any point O and draw OAB perpendicular to the directions of the forces, meeting them in A and B, and let the resultant act at C. Then the moments of the forces are equal to the moment of the resultant, whence

(in Fig. 1) $P \,.\, OA \pm Q \,.\, OB = R \,.\, OC = (P + Q)OC,$

(in Fig. 2) $P \,.\, OA \mp Q \,.\, OB = R \,.\, OC = (P - Q)OC,$

the + or − sign being taken according as O is outside A, B, or between them.

In the first case C is between A, B; in the second case outside, on the side of the bigger force.

The point O may have any position; suppose it at C, then $OC = 0$, and

$$P \,.\, CA - Q \,.\, CB = 0,$$

or

$$\frac{AC}{BC} = \frac{Q}{P}.$$

i.e. C divides AB inversely as the forces, *internally* if the forces are in the same direction, *externally* if in opposite directions.

By supposing the point O at B we get

$$P \,.\, AB = R \,.\, BC,$$

or

$$BC = \frac{P}{R}AB = \frac{P}{P \pm Q}AB,$$

a form which is sometimes useful.

Draw the line $A'C'B'$ inclined to the forces in Figs. 1, 2. Then we may suppose the forces to act at A', B', C', and we know by geometry that $A'B'$ is divided at C' in the same way as AB is at C. In other words,

$$\frac{A'C'}{B'C'} = \frac{Q}{P}.$$

Now notice that the position of C' on $A'B'$ depends on the magnitude of the forces alone, and not on their common direction. Hence if P, Q act at A', B' in any parallel directions, their resultant will always act at C' in the like direction. C' is called the centre of the parallel forces P, Q acting at A', B'. *It remains the same when* P, Q *are turned through any the same angle about* A, B.

55. The following proof is often given of this important theorem—

Suppose the forces to be P, Q acting at the points A, B. Apply at A any force S along BA and at B an equal force S along AB. These will balance one another and will not modify the state of motion of the rigid body to which they are applied. Replace P and S, acting at A, and Q and S acting at B, by their resultants. Then, if the forces

P, Q act in the like direction as in Fig. 1, it is clear that the two resultants will intersect at some point O. If, however, they act in unlike directions, as in Fig. 2, they will only intersect if P and Q are unequal. For it is clear that if they be equal, then the resultants of P and S and of Q and S will be parallel, whilst if Q is not equal to P the resultants will cease to be parallel, and will intersect somewhere, say O, as in Fig. 2.

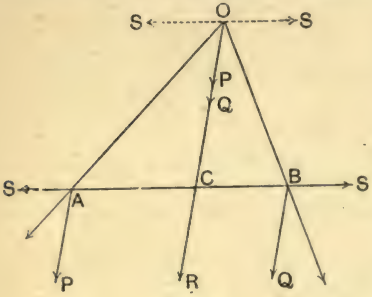


FIG. 1.

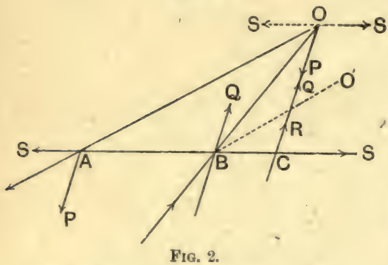


FIG. 2.

If then the case of equal forces be excepted for the present, the resultants in both cases of like and unlike forces will meet at a point O. Through O draw OC parallel to P or Q and meeting the line AB in the point C. When the forces are like, as in Fig. 1, C will clearly fall between A and B. When, however, they are unlike, as in Fig. 2, C will fall outside AB *on the side of the larger force.* To prove this, suppose Q the larger force. If it were equal to P, the resultant of it and S would act along BO′ parallel to AO; but as Q is greater than P, this resultant must lie somewhere in the angle QBO′, and therefore will intersect AO to the right of BQ—in other words, OC will fall on the side of Q away from P.

We have then now two forces at A and B whose directions intersect at a point O. Transfer their points of application to O. Each can be decomposed into two forces equal and parallel to their former components, viz. the one into P along OC and S parallel to BA, and the other into Q along OC (Fig. 1) or CO (Fig. 2) and S parallel to AB. The two S forces balance one another and may be removed. We are then left with

(Fig. 1, like forces) $P+Q$ along OC,
(Fig. 2, unlike forces) $Q-P$ along CO.

It only remains to determine the position of C. Now notice that in the case of the force along OA it is decomposed into two parallel to OC, CA. The forces are therefore proportional to these sides—that is,

$$R : P : S = OA : OC : AC,$$

whence
$$\frac{OC}{AC} = \frac{P}{S}.$$

Similarly
$$\frac{OC}{BC} = \frac{Q}{S}.$$

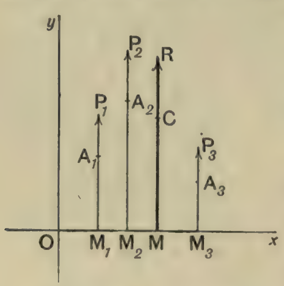
Therefore, dividing the first of these by the second,

$$\frac{BC}{AC} = \frac{P}{Q},$$

or AB is divided at C inversely as the forces, internally when like, and externally when unlike. Here also it is to be noticed that the position of C depends only on the magnitudes of P and Q and not on their directions.

With this method no proof is given that the moment of two parallel forces about any point is equal to the moment of their resultant. If this method be adopted, therefore, it will be necessary to prove this further proposition. This is left as an exercise to the student.

56. Having now obtained the means of finding the resultant of two parallel forces, we can proceed to find that of any number. Suppose then we are given any number of parallel forces P_1, P_2, P_3, . . . acting at the points A_1, A_2, A_3, . . . Then P_1, P_2 may be replaced by another parallel force R_1 acting at a definite point B_1. So R_1 and P_3, acting at B_1 and A_3, by another R_2 at a definite point B_2, and so on; until all the forces are replaced by a single force acting at some definite point C; which point, as we have just seen, depends only on the magnitudes of the forces and their points of application, and not at all on their common direction. This point is called the centre of parallel forces of the system. We proceed to obtain formulæ by which its position may be determined.



We shall suppose the positions of A_1, A_2, . . . determined by their distances from two straight lines, Ox, Oy

perpendicular to one another, and we shall denote their distances from Oy respectively by x_1, x_2, . . ., and from Ox by y_1, y_2, . . ., also the corresponding distances of C by x, y. Now we know that the position of C is independent of the common direction of the forces. Suppose then that they are all parallel to Oy, and draw the forces cutting Ox in M_1, M_2, . . ., and let the resultant cut it in M.

Then the resultant R is

$$R = P_1 + P_2 + P_3 + \ldots,$$

and the moment of the forces round O is the same as the moment of their resultant. Hence

$$R \,.\, OM = P_1 \,.\, OM_1 + P_2 \,.\, OM_2 + \ldots;$$

or

$$x = \frac{P_1 \,.\, x_1 + P_2 \,.\, x_2 + P_3 \,.\, x_3 + \ldots}{R},$$

$$= \frac{P_1x_1 + P_2x_2 + P_3x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}.$$

Similarly we may suppose the forces to be all parallel to Ox, and by again taking the moments in this case about O, we get, as before,

$$y = \frac{P_1y_1 + P_2y_2 + \ldots}{P_1 + P_2 + \ldots};$$

x, y are therefore now known, and the position of C is determined.

These results are often written

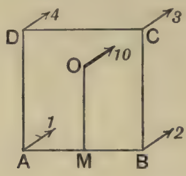
$$x = \frac{\Sigma Px}{\Sigma P}, \quad y = \frac{\Sigma Py}{\Sigma P}.$$

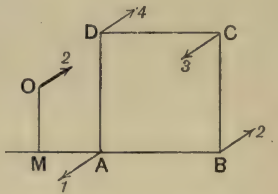
ΣPx standing for summation of all terms P_1x_1, P_2x_2, etc.

Note.—If any force act in the opposite direction to the others, we must put $-P$ instead of P, for its moment is altered in direction, and it diminishes the resultant. Also if a point A lies to the left of Oy, its value of x must be considered negative, for here again the corresponding moment acts in the opposite direction. Similarly, if it lie below Ox, its value of y must be negative. With these conditions, the formulæ above are universally true.

A few examples will illustrate the use of the above formulæ.

EXAMPLE I. *Parallel forces* 1, 2, 3, 4 *act at the angles of a square. Find their centre of parallel forces.*





Let ABCD be the square. We may take the lines of reference Ox, Oy, at our discretion. Choose them then to be AB, AD. Then

$$x=\frac{1\times 0+2\times \mathrm{AB}+3\times \mathrm{AB}+4\times 0}{1+2+3+4},$$
$$=\tfrac{5}{10}\mathrm{AB}=\tfrac{1}{2}\mathrm{AB};$$
$$y=\frac{1\times 0+2\times 0+3\times \mathrm{AD}+4\times \mathrm{AD}}{1+2+3+4},$$
$$=\tfrac{7}{10}\,.\,\mathrm{AB};$$

and $$\mathrm{R}=1+2+3+4=10.$$

Hence the resultant is 10 and it acts at O, where

$$\mathrm{AM}=\tfrac{1}{2}\mathrm{AB},\quad \mathrm{OM}=\tfrac{7}{10}\mathrm{AB}.$$

EXAMPLE II. *The same system with* 1, 3 *in the opposite direction.*

Here $$\mathrm{R}=-1+2-3+4=2.$$

$$x=\frac{-1\times 0+2\,.\,\mathrm{AB}-3\,.\,\mathrm{AB}+4\,.\,0}{2}=-\tfrac{1}{2}\mathrm{AB},$$
$$y=\frac{-1\,.\,0+2\,.\,0-3\,.\,\mathrm{AD}+4\,.\,\mathrm{AD}}{2}=\tfrac{1}{2}\mathrm{AD},$$

and the resultant acts at O as in the second figure, where AM is in the opposite direction to AB and is $\tfrac{1}{2}$AB, and OM $=\tfrac{1}{2}$AD.

EXAMPLE III. *Forces equal to the weights of* 3, 2, 6, 5, 7 *lbs. act at the five angles* A, B, C, D, E *of a regular hexagon* ABCDEF *inscribed in a circle. Find the resultant and its point of application.*

The resultant is given by

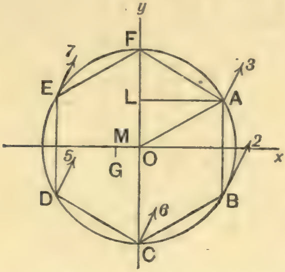
$$\mathrm{R}=3+2+6+5+7=23 \text{ lbs. weight.}$$

To find its point of application, take O at the centre of the circle, FOC for the line Oy, and a perpendicular to it for Ox.

The distances of A, B, D, E from FOC and from Ox must be found.

This is easily done when we notice that FOA is an equilateral triangle, and therefore AL bisects OF. Hence

$$\mathrm{OL} = \tfrac{1}{2}\mathrm{OF} = \tfrac{1}{2}r, \text{ where radius} = r,$$
$$\mathrm{AL}^2 = \mathrm{OA}^2 - \mathrm{OL}^2 = r^2 - \tfrac{1}{4}r^2 = \tfrac{3}{4}r^2;$$
$$\therefore \mathrm{AL} = \frac{\sqrt{3}}{2}r.$$



The *magnitudes* of the distances of A, B, D, E from Oy are therefore all equal to $\frac{\sqrt{3}}{2}r$, and from O$x = \frac{1}{2}r$. Hence

$$x = \frac{3 \times \frac{\sqrt{3}}{2}r + 2 \times \frac{\sqrt{3}}{2}r + 6 \times 0 + 5\left(-\frac{\sqrt{3}}{2}\right)r + 7\left(-\frac{\sqrt{3}}{2}\right)r}{23},$$
$$= -7\frac{\sqrt{3}}{46}r = -\cdot 263\ldots r;$$
$$y = \frac{3 \times \frac{1}{2}r + 2(-\frac{1}{2}r) + 6(-r) + 5(-\frac{1}{2}r) + 7(\frac{1}{2}r)}{23},$$
$$= -\tfrac{9}{46}r = -\cdot 1956\ldots r.$$

That is, the resultant is the weight of 23 lbs. acting at G as in the figure, where GM = ·0652 . . . r and OM = ·263 . . . r.

57. *Couples.*—We have seen that in the case of two unlike parallel forces P, Q the resultant is P – Q, and that it acts at a point C, whose distances from the points of application A, B of the components are given by

$$\mathrm{AC} = \frac{\mathrm{Q}}{\mathrm{P} - \mathrm{Q}}\mathrm{AB}.$$

If now the forces are nearly equal, the resultant P – Q is very small and AC becomes very large. The resultant therefore acts at a very great distance. If the forces are exactly equal the resultant is zero, but this zero force acts at an infinitely great distance. Its moment, however, about any point remains finite. Such a system is called a *couple.*

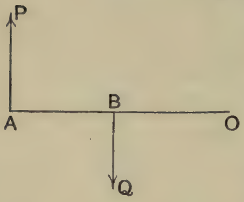
Def. *A couple is a system of two equal unlike parallel forces not in the same straight line.*

We proceed to demonstrate certain properties of couples.

I. *The sum of the moments of the forces composing a couple about any point in their plane is the same.*

For in the figure, if O be any point, the moments are

$$P.OA - Q.OB = P(OA - OB) = P.AB,$$



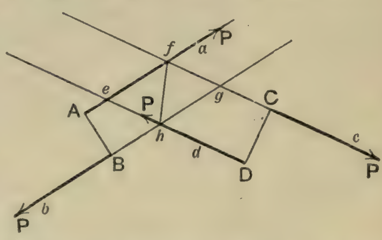
which is independent of the position of O.

The distance between the two forces of a couple is called its *arm.*

The product of one force into the arm is called the *torque or moment of the couple.* It is to be taken as positive or negative according as the forces tend to turn the body one way round or the other. The theorem just proved may then be stated thus: *the sum of the moments of the forces of a couple about any point in its plane is equal to the moment of the couple.*

58. II. *A couple may be considered as acting anywhere in its plane.*

Let *a*AB*b* represent a couple in one position, and *c*CD*d* the same transferred to any other position in the plane, AB, CD representing arms of the couple.



Then if these couples are equivalent, one of them reversed will be in equilibrium with the other. Suppose then the second reversed, and P to act along *c*C, *d*D. Let the directions of the forces meet in *e*, *f*, *g*, *h*. This forms a rhombus, or parallelogram with all its sides equal.

We may suppose the forces P along A*a* and P along

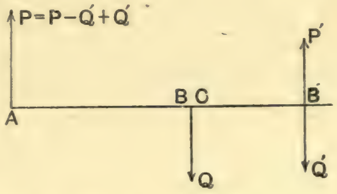
cC to act at f; and P along Bb and dD at h. Then P along ef and P along gf have a resultant R along hf. Also P along gh and P along eh have an equal resultant R along hf. These will balance one another.

Hence the two original couples are equivalent to one another.

59. III. *A couple may be replaced by another whose moment is the same.*

Let AB be the arm of the couple and the forces P, Q. Let B′ be any other point in the line AB, and at B′ apply two equal and opposite forces P′, Q′ perpendicular to AB′ so that

$$P' . AB' = P . AB.$$



The equal forces at B′ produce no effect by themselves. Now the force P at A can be replaced by two others at A, viz. $P - Q'$ and Q′. Also $P - Q'$ at A and P′ at B′ have a resultant $R = P - Q' + P' = P$ (since $P' = Q'$) acting at a point which divides AB′ in the point C, where

$$P' . AB' = R . AC = P . AC.$$

But $$P' . AB' = P . AB,$$

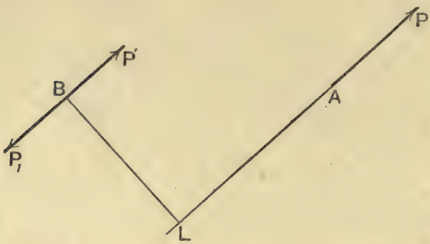
therefore C and B coincide—that is, the resultant of $P - Q'$ at A and P′ at B′ is P at B, and therefore annuls Q at B. We are left with Q′ at A and Q′ at B′, that is a couple whose moment is $Q' . AB' = P . AB$, the same as the original one.

By means of the former proposition it follows at once that the original couple can be replaced by any other in the same plane whose moment is the same.

Cor. It is clear that we can replace any couple by another in which *either* the force, *or* the arm is at our disposal.

60. IV. *A force acting at any point can be replaced by an equal force acting at any other point and a couple.*

Let the force P act at A, and let B be any other point. Draw BL perpendicular to the direction of P and apply at B two equal unlike forces P′, P_1 each equal to P. The system is unaltered. But these three forces may be regarded as a single force P′ at B together with the couple P, P_1 whose moment is P . BL.



61. V. *The resultant of two couples in a plane is another whose moment is the sum of their moments.*

For replace the two couples by others having the same arm AB and forces P, Q. Then we have a single couple whose force is P + Q and arm AB.

$$\text{Its moment is } (P+Q)AB = P \,.\, AB + Q \,.\, AB,$$
$$= \text{moment of first} + \text{moment of second.}$$

62. It is now possible for us to determine the resultant of any system of forces in a plane acting on a rigid body. For take any fixed point O in the plane, and let the forces be P_1, P_2, . . . acting at A_1, A_2, . . . Then P_1 acting at A_1 is, by IV above, equivalent to P_1 acting at O together with a couple; similarly with the others. Consequently the forces are equivalent to

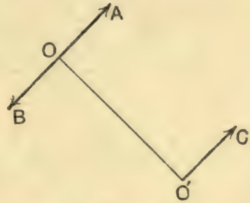
(1) A system of forces acting at a point equal and parallel to the original forces, and

(2) A system of couples in the plane of the forces.

These may be combined into a single force at the point and a single couple. Hence we get the theorem that *any number of coplanar forces acting on a rigid body are equivalent to a single force acting at* ANY *point, and a couple.*

Note.—The single force is independent of the position of the point, for it is the resultant of the original forces supposed acting at a point. The magnitude of the couple, however, does depend on the position of the point O.

In the case where there is both a force and a couple the system can be reduced to a single force alone, with, however, a definite line of action. For suppose the resultant force is R acting at O, represented by OA, alter the arm and forces of the couple (by III) so that the force is R. Then turn it (by II) until one of its forces acts at O and opposite to the force OA. Suppose the couple is then represented by OB, O′C, OO′ being the arm. Then OA, OB annul and we are left with O′C only, *i.e.* with a single force R acting along a definite line O′C.



If in the above system there is no resultant force R, then the couple remains.

In general then *any system of coplanar forces acting on a rigid body reduces to* EITHER *a single force* OR *a single couple.*

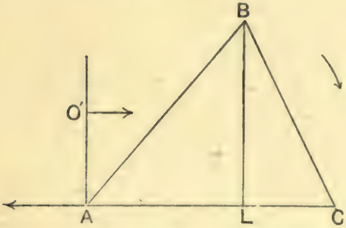
If both vanish, we get of course a special case of this theorem.

Note.—This has only been proved for forces in one plane. It is not true in general when the forces act in any direction whatever.

63. In the actual calculation of the resultant in any special case, it will generally be found most convenient to replace the original forces by their components parallel to two orthogonal lines through O before transferring them to O. The following examples will serve to illustrate the previous articles.

EXAMPLE I. *Forces act along the sides of a triangular lamina in order proportional to the sides. Find the resultant.*

Let ABC be the lamina. Take A for the point of reference. Then, by the foregoing, the system is equivalent to



(1) The resultant of the forces supposed acting at A, this by the triangle of forces is zero;

(2) A couple equal to the sum of the moments of the forces about A—*i.e.* of the force along BC alone. Hence, if the force is $k \cdot BC$, the moment of the couple is $k \times 2$ area of ABC.

EXAMPLE II. *Same case with the forces proportional to* AB, BC, AC.

Take A for point of reference. Then if BC acted at A, the resultant of AB, BC would be proportional to AC. Therefore the resultant force is proportional to 2AC and therefore equal to $2k \,.\, \mathrm{AC}$. The couple as before is $2k\triangle$, where $\triangle$ is the area of ABC, in the direction represented by the arrow. If this couple be arranged so that the force in it is $2k \,.\, \mathrm{AC}$, the arm must be

$$\frac{2k\triangle}{2k \,.\, \mathrm{AC}} = \frac{\triangle}{\mathrm{AC}}$$

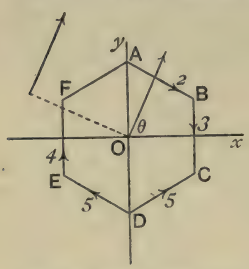
and it must be placed as in the figure, where

$$\mathrm{O'A} = \frac{\triangle}{\mathrm{AC}} = \tfrac{1}{2}\mathrm{BL}.$$

Hence the system reduces to a single force $2k \,.\, \mathrm{AC}$ parallel to AC and at a distance from it equal to $\frac{1}{2}$BL.

The foregoing exemplifies the general method. In this special case the result may be obtained more easily thus. The forces AB, BC have a resultant equal and parallel to AC acting at B, whilst the force AC itself may be supposed acting at L. Thus the system is P = AC at B and P = AC at L. Whence the resultant is 2AC at the middle point of BL.

EXAMPLE III. *Forces* of 2, 3, 5, 5, 4, 1 *poundals along the sides* AB, BC, DC, DE, EF, FA *of a regular hexagon.*

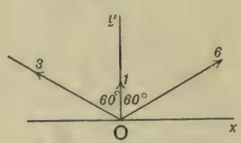


Take O the centre of the circle round the hexagon for the point of reference. Then the system of forces is equivalent to

(1) A force at O, viz. the resultant of the forces supposed all acting at O;

(2) A couple whose moment is the sum of the moments of the forces about O.

(1) The forces at O are as in the left-hand figure.



These reduce to the system in the right-hand figure. In this take Oy along the force 1, Ox perpendicular to this.

Then along Ox

$$X = 6\cos 30° - 3\cos 30° = 6\frac{\sqrt{3}}{2} - 3\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2},$$

along Oy

$$Y = 6 \times \frac{1}{2} + 1 + 3 \times \frac{1}{2} = \frac{11}{2};$$

$$\therefore R^2 = X^2 + Y^2 = \frac{27 + 121}{4} = \frac{148}{4} = 37,$$

whence $R = \sqrt{37} = 6{\cdot}082\ldots$ poundals,

and it acts in a direction making an angle θ with Ox, where

$$\tan\theta = \frac{Y}{X} = \frac{11}{3\sqrt{3}} = \frac{11\sqrt{3}}{9} = 2{\cdot}117\ldots,$$

whence, by tables of tangents, $\theta = 64° \, 43'$.

(2) The torque is equal to the moment of the forces about O. Let p denote the perpendicular distance of O from a side. Then

$$\text{Moment of couple} = 2p + 3p - 5p + 5p + 4p + 1 \times p = 10p.$$

The forces therefore reduce to a single force $6{\cdot}082\ldots$ poundals acting at O and inclined at 64° 43′ to Ox, and a couple whose moment is $10p$ (units).

But this can (by § 62) be reduced to a single force $6{\cdot}082\ldots$ alone, acting along some definite line. To find this, we have first to determine the arm of the couple when the force is $6{\cdot}082$. Now

$$(6{\cdot}082\ldots) \times \text{arm} = 10p,$$

or

$$\text{Arm} = \frac{10}{6{\cdot}082}p = 1{\cdot}644p.$$

Thus, finally, the system of forces reduces to a single force of $6{\cdot}082\ldots$ poundals acting along a line inclined 64° 43′ to Ox at a distance from O equal to $1{\cdot}644$ times the distance of a side from O. It is represented by the thick line in the figure.

Conditions of Equilibrium of a Rigid Body.

64. We are not yet in a position to determine the motion of a rigid body in general, but it is possible with the knowledge already gained to determine the conditions in order that the body may be at rest.

We have seen that any system of forces in one plane acting on a rigid body may be reduced to either a single resultant force, *or else* a single couple. The condition of equilibrium is therefore that these must be zero; and the only question now is—how best to apply this condition. In general it will be found that one of the two following statements of the conditions will be the most convenient to use—

A. The resolved parts of the forces in any two directions must vanish and their moment about any point must vanish.

B. The moments of the forces about any three points not in the same straight line must vanish.

It is easy to see the truth of these statements. Thus in the case of A—if the resolved forces in two directions vanish, there is no resultant force; if the moment about any point vanishes, it must be because the resultant force (if any) goes through the point, or because there is no couple. But the first condition has shown that there is no resultant force, therefore there is also no couple; and there being no force and no couple, the body must be in equilibrium.

So also in the case of B, the resultant cannot be a single force, for if so it would have to pass through three points *not in a straight line*, which is impossible. Hence there is no force, and further, since the moment about a point vanishes and there is no resultant force, there is also no couple. Therefore, as in A, the body must be in equilibrium—and *vice versa*, if the body is in equilibrium, conditions A and B must be satisfied.

65. The following theorem is often useful in finding the conditions of equilibrium—

If three coplanar forces keep a body in equilibrium, they must either be parallel or meet in a point.

For they are either parallel or not. If they are parallel, by a suitable arrangement of magnitude and position they may annul one another. If they are not all parallel, two of them at least must intersect. Let them intersect in A. Replace them by their resultant, also acting at A. Then there are two forces keeping a body in equilibrium, but this cannot be unless they are in the same line of action—*i.e.* unless the third force also goes through A. In other words, unless the three all go through one point.

For an application of this theorem see Example III below.

66. We cannot illustrate the foregoing principles by many examples of practical interest until we have investigated the properties of the centre of gravity in the next chapter. The following will, however, serve to exemplify the methods developed in this chapter.

EXAMPLE I. *A triangular lamina is acted on by three forces represented respectively by lines through each angle bisecting the opposite side. Prove that the lamina is in equilibrium.*

Let ABC be the triangle and DEF the middle points.

Take moments about A.

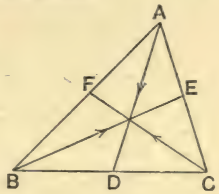
Then moments $= 2\triangle\,AFC - 2\triangle\,BEA$.

But E and F bisect AC, AB.

$\therefore$ moments $= \triangle\,ABC - \triangle\,BCA = 0$.

Similarly the moments vanish about B and C—that is, about three points not in a straight line.

Therefore the body is in equilibrium.



EXAMPLE II. *A weightless rod rests on two props* A, B, 3 *feet apart, and a* 10-*lb. weight is placed at* C, 6 *inches from one of them. Find the pressures on the props.*

Let R be the pressure on the nearer prop and R′ on the other.

For the sake of illustration, we treat this question in two different ways.

A. Notice that the forces R, R′ just counterbalance the 10-lb. weight. Hence the 10-lb. weight must be equal and opposite to the resultant of R and R′. Hence

$$R + R' = 10,$$
$$\frac{R}{R'} = \frac{BC}{AC} = \frac{2\frac{1}{2}}{\frac{1}{2}} = 5,$$
$$\therefore R = 5R',$$
$$\therefore 6R' = 10;$$

whence

$$R' = \tfrac{5}{3}, \quad R = \tfrac{25}{3}.$$

B. If the system be in equilibrium the moments about any point must vanish. Take then the moments about A. Then

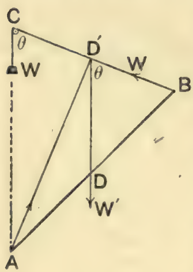
$$10 . AC = R' . AB,$$
$$3R' = 10 \times \tfrac{1}{2} = 5,$$
$$R' = \tfrac{5}{3}.$$

So also take moments about B. Then

$$10 . BC = R . AB,$$
$$3R = 10 \times \tfrac{5}{2} = 25,$$
$$R = \tfrac{25}{3}.$$

EXAMPLE III. *A rod* AB *without weight is movable about a hinge at* A, *and its end* B *is attached to a weight* W *by means of a string which passes over a pulley* C *vertically above* A, *and so that* AC = AB ; *a weight* W′ *is then suspended from the middle point of* AB. *Find the position of equilibrium.*

Let AB be the rod. It is acted on by the tension W along BC, the weight W′ vertically through D the mid point of AB, and some reaction at A—that is, by three forces.



These are not parallel, therefore they must pass through one point. Now we know the direction of W′ and W at D and B, let them intersect at D′. Then the reaction at A must act along AD′.

Because DD′ is parallel to AC and D is the mid point of AB, D′ is also the mid point of BC. Whence AD′ is perpendicular to BC, since ABC is an isosceles triangle.

Now the resolved parts of the forces in any direction must vanish. Resolve then along BC. We have, remembering that the reaction is along AD′, and therefore perpendicular to BC,

$$W - W' \cos\theta = 0,$$

θ being the inclination of BC to the vertical. Hence

$$\cos\theta = \frac{W}{W'}.$$

This gives the position of equilibrium when W, W′ are known. *E.g.*, suppose

$$W' = 2W,$$
$$\cos\theta = \tfrac{1}{2},$$

and

$$\theta = 60^\circ.$$

If we desire to find the magnitude of the reaction at A, resolve along AD′, and call the reaction R. Then

$$R - W' \sin\theta = 0,$$
$$R = W' \sin\theta,$$
$$R = W'\sqrt{1 - \cos^2\theta},$$
$$= W'\sqrt{1 - \frac{W^2}{W'^2}},$$
$$= \sqrt{W'^2 - W^2}.$$

In this case, however, the result may be more easily arrived at by noticing that the forces are parallel to the sides CA, AD′, D′C.

Whence $$\frac{W'}{CA}=\frac{W}{CD'}=\frac{R}{AD'},$$

also AD′C is a right angle,

$$\therefore AC^2=AD'^2+CD'^2.$$

Whence $$W'^2=R^2+W^2,$$

or $$R^2=W'^2-W^2,$$

also $$\cos\theta=\frac{CD'}{AC}=\frac{W}{W'}.$$

EXAMPLES—VIII.

[*In the following examples, suppose the weight of a uniform straight rod to act at its middle point.*]

1. Two forces of 10 and 15 lbs. weight act at points 30 inches apart. Find their resultant when they are in (1) like, (2) unlike directions.

2. The resultant of two unlike parallel forces of 10 lbs. and 18 lbs. weight acts in a line at a distance of 12 feet from the line of action of the less force. What is the distance between the lines of action of the two forces?

3. A lever ABC, with a fulcrum B, one-third of its length from A is divided into equal parts at D, E, F; at C, D, F forces of 12, 8, 6 lbs. weight respectively act vertically downwards, and at E a force of 16 lbs. weight acts vertically upwards. What force applied at A will cause equilibrium?

4. The resultant of two parallel forces P, Q at A, B acts at C when like, and at D when unlike. Prove that if these resultants act at C, D, then A, B will be the points at which their resultant will act in the two cases of like and unlike directions.

5. If a straight rod ABC is supported in a horizontal position by being placed under a peg at A and over a peg at B, find the reactions of the pegs due to hanging a weight W at C.

6. A lever 30 inches in length has weights 3 lbs. and 15 lbs. fastened to its ends, and balances about a point 9 inches from one end. What is the weight of the lever?

7. A uniform lever is 18 inches long, and each inch in length weighs one ounce. Find the place of the fulcrum when a weight of 27 oz. at one end of the lever balances a weight of 9 oz. at the other end. If the smaller weight be doubled, how must the position of the fulcrum be shifted to preserve the equilibrium?

8. The horizontal roadway of a bridge is 30 feet long and weighs 6 tons, and it rests on similar supports on its ends. What pressure is borne by each of the supports when a carriage weighing 2 tons is one-third of the way across the bridge?

9. A horizontal rod without weight, 6 feet long, rests on two supports at its extremities; a weight of 6 cwts. is suspended from the rod at a distance of $2\frac{1}{2}$ feet from one end. Find the reaction at each point of support. If one support could only bear a pressure of 1 cwt., what is the greatest distance from the other support at which the weight could be suspended?

10. Two equal uniform beams AB, BC are freely jointed at B, and A is fixed to a hinge at a point in a wall, about which AB can revolve in a vertical plane. At what point in BC must you apply a vertical force to keep the two beams in one horizontal line? Find the value of the force.

11. A heavy uniform beam whose mass is 40 lbs. is suspended in a horizontal position by two vertical strings, each of which can support 35 lbs. How far from the centre of the beam must a weight of 20 lbs. be placed so that one of the strings may just break?

12. A heavy beam rests horizontally over a fixed peg, weights P, Q, R being successively hung from one end. If P, Q, R be in arithmetical progression, the distances of the peg from that end of the beam are in harmonical progression.

13. Find the centre of parallel forces of forces 1, 2, 3 acting at the angles of an equilateral triangle.

14. The sides of a square are divided each in four equal parts, and parallel forces alternately like and unlike act at the angles and the points of division; their magnitudes in order are 2, 2, 3, 4, 4, 3, 2, 1, 5, 6, 7, 8, 8, 7, 6, 5. Find the resultant and the centre of parallel forces. Also when the first is 1 instead of 2.

15. A square weightless board has weights of 1, 2, 3, 4 lbs. respectively hanging from its four corners, and is suspended by a string. If the board is to remain horizontal, at what point of it must the string be fastened and what will be the tension of the string?

16. Prove that the difference of the moments of any force P about two points in the same plane with each other and the force, is equal to the moment about either of the points of a force parallel and equal to P applied at the other point.

17. If any number of forces represented by the sides of a regular hexagon taken in order act along the sides to turn the hexagon round an axis perpendicular to its plane, show that the moment of the forces is the same through whatever point within the hexagon the axis passes. Is this true if the hexagon is not regular?

18. The sides of a regular polygon taken in order represent forces

acting in the plane of the polygon. Show that the sum of their moments will be the same round any point within the figure. Find the forces of the couple, having one side of the polygon for an arm, that will keep the system in equilibrium.

19. Two equal rods AB, BC are firmly joined together at B at right angles to one another. If they were suspended from A so as to be capable of turning freely about that point, in what position would they hang? Could you get them to hang with one side vertical by attaching a heavy weight at B?

20. The arms of a bent lever are at right angles to one another, and their lengths are in the ratio of 5 : 1. The longer arm is inclined 45° to the horizon, and carries at its extremity a weight of 10 lbs. The end of the shorter arm presses against a smooth horizontal plane. Draw a figure showing the forces in action and find the pressure between the shorter arm and the plane.

21. A picture of given weight hanging vertically against a smooth wall is supported by a string passing over a smooth peg driven into the wall; the ends of the string are fastened to two points in the upper rim of the frame, which are equidistant from the centre of the rim, and the angle at the peg is 60°. Compare the tension in this case with what it will be when the string is shortened to two-thirds of its length.

22. A heavy uniform rod 15 inches in length is suspended from a fixed point by two strings fastened to its ends, the lengths of the strings being respectively 9 inches and 12 inches. If θ be the angle at which the rod is inclined to the vertical, prove that $25 \sin\theta = 24$.

23. A triangular lamina lying upon a smooth circular table has its angular points attached to strings passing over the edge of the table, and supporting weights proportional to the opposite sides of the triangle. Find the direction of the strings with reference to the triangle in the case of equilibrium.

24. ABCD is a rectangle; AB, BC, adjacent sides, are 3 and 4 feet. Along AB, BC, CD, taken in order, forces of 30, 40, 30 lbs. weight act respectively. Find their resultant.

25. ABCD is a quadrilateral, and two points P, Q are taken in AD, BC, such that AP : PD : : CQ : QB. From P, Q straight lines PP′, QQ′ are drawn equal to, parallel to, and in the same directions as BC and DA respectively. Show that forces represented by AB, CD, PP′, QQ′ are in equilibrium.

26. One end of a string is fastened to a fixed point and the other to the angle A of a weightless triangular lamina ABC. If the lamina be acted on by forces, represented by AB, BC, CA, find the weight which must be suspended from the angular point B in order to maintain the lamina in equilibrium with the side BC vertical.

CHAPTER IX

FORCES ON A RIGID BODY—CENTRE OF GRAVITY

67. Every part of a body is attracted downwards to the earth with a force proportional to its mass. When the body is not very extended, all these forces are appreciably parallel. For instance, the angle between the directions of gravity at two points a mile apart is only about one minute or a sixtieth of a degree. So that for bodies with which we have to deal practically, we may regard the direction of gravity at their different points as accurately the same.

In the case of parallel forces, we have seen in the previous chapter that there is a point, "the centre of parallel forces," through which their resultant may always be supposed to act, no matter how the forces are turned about their points of application. This will be the case, therefore, with the gravitation of bodies when the gravitation is everywhere in the same direction. In other words, we may suppose the weight to act at a definite point of every body. This point is called the "centre of gravity" of the body.

In general it is not the case that when two bodies are attracting one another the resultant attraction always passes through fixed points in the bodies. But when this is the case such point is called a centre of gravity. Thus in the case where one of two attracting bodies is a sphere the resultant attraction always passes through its centre. Here then its centre would be called a centre of gravity.

In this chapter we confine our attention to the case where gravitation is supposed to act in parallel lines, in which, as we have seen, there always exists a centre of gravity.

68. *Centre of gravity of any number of particles in a plane.* Refer the positions of the particles to two perpendicular lines Ox, Oy and let their distances from them be given by (x_1, y_1), (x_2, y_2), . . ., and the position of the centre of gravity by x, y. Also let the masses be m_1, m_2, . . . Then we have a series of parallel forces m_1g, m_2g, . . . acting at the given points, and since the centre of gravity is the centre of parallel forces of the system,

$$x = \frac{m_1gx_1 + m_2gx_2 + \ . \ . \ .}{m_1g + m_2g + \ . \ . \ .},$$

$$= \frac{m_1x_1 + m_2x_2 + \ . \ . \ .}{m_1 + m_2 + \ . \ .}.$$

Similarly

$$y = \frac{m_1y_1 + m_2y_2 + \ . \ . \ .}{m_1 + m_2 + \ . \ . \ .}.$$

The point determined by x, y depends only on the masses and their distribution, and is independent of the intensity of gravitation. It is also called, for reasons given in § 182, the centre of inertia.

69. Now any continuous solid body may be supposed to be made up of a very large number of very small parts. Hence we could use the formula just obtained to find the centre of gravity of a continuous solid, if we had the means of obtaining the value of the sum of the very large number of terms $m_1x_1 + m_2x_2 + \ . \ . \ .$ and $m_1 + m_2 + \ . \ . \ .$ To do this would require a more advanced knowledge of mathematics than is supposed in this book. It is possible, however, in several important cases to determine the centre of gravity by other methods. Thus the following considerations will often be sufficient to completely determine it.

A. If we know that the centre of gravity lies in each of two lines, or in each of three planes, it must lie at their point of intersection.

B. If a straight line or plane divides a lamina or a solid into two *equal* and *similar* portions, it is clear that the centre of gravity must lie in that straight line or plane.

C. If the centres of gravity of the two portions of a body be known we can find the centre of gravity of the whole thus. Let m_1, m_2 be the masses of the two portions and A, B the positions of their centres of gravity. Then we may suppose the body replaced by m_1 at A and m_2 at B. These will have their centre of gravity on AB at some point G where

$$\frac{AG}{BG} = \frac{m_2}{m_1},$$

and the position of G, the centre of gravity of the whole body, is determined.

The same method can also be applied to find the centre of gravity of a part when that of the whole and the other part is known.

The centre of gravity of a plane lamina of uniform thickness clearly lies in the plane half way between the faces, and its position merely depends on the shape of the boundary. We may disregard the thickness and confine our attention to the boundary. In this connection it is usual to speak of the "centre of gravity of an area." When this phrase is used it must be understood to refer to a thin lamina of *uniform* thickness with the same boundary. Similarly we may speak of the centre of gravity of a curve, meaning thereby the centre of gravity of a thin *uniform* wire of the same shape.

The rest of this chapter will be devoted to the discovery of the position of the centre of gravity for important simple cases.

70. Methods A and B enable us at once to state the position of the centre of gravity of the following bodies.

(1) *A straight line*, at its middle point.

(2) *A parallelogram*, at the intersection of lines bisecting the opposite sides, *therefore* at the intersection of the diagonals.

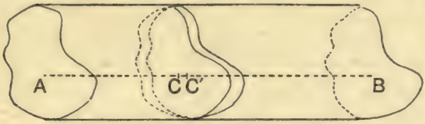
(3) *A circular disc* (*or circumference*), at its centre.

(4) *A parallelopiped*, at the intersection of planes half way between opposite faces, *therefore* at the intersections of diagonals.

(5) *A sphere* (*or spherical surface*), at its centre.

(6) *A circular cylinder with plane ends perpendicular to its axis*, at the middle point of the axis.

(7) *A right prism or cylinder of any cross section.*

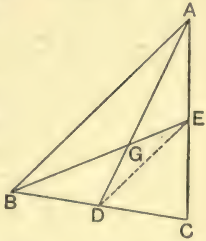


Here the centre of gravity bisects the line joining the centres of gravity of the two ends.

For let A, B be the centres of gravity of the two ends.

Draw any plane parallel to the ends and cutting AB in C. It will cut the cylinder in a curve precisely the same as the ends, and C will be its centre of gravity. If now the cylinder be divided into a large number of thin laminæ, the centres of gravity of all will lie on this line AB—that is, the centre of gravity of the whole cylinder lies somewhere on it. Also a plane through the middle point of AB parallel to the ends clearly divides the cylinder into two equal and similar parts. Hence the centre of gravity lies at the intersection of AB and this plane—that is, at the middle point of AB.

71. *The centre of gravity of three equal particles at the angles of a triangle* ABC.—Let the three particles (m) be at A, B, C. Bisect BC at D. Then m at B and m at C are equivalent to $2m$ at D.



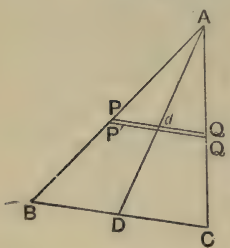
Join AD. Then m at A and $2m$ at D are equivalent to $3m$ at G, where

$$\frac{AG}{DG} = \frac{2m}{m} = 2,$$

or $$DG = \tfrac{1}{3}DA, \quad AG = \tfrac{2}{3}AD.$$

Cor. In the same way it might have been shown that the centre of gravity lay on BE, where E is the mid point of AC. Hence it lies on their intersection G. We therefore get incidentally the theorem that the lines joining the angular points of a triangle to the mid points of the opposite sides meet in one point which trisects each of them. This theorem is important for the next case to be considered.

72. *Triangular lamina.*—Let ABC be the triangle. Bisect BC in D and draw any line PQ parallel to BC, cutting the sides in PQ and AD in *d*. Then AD also bisects PQ in *d*. If, therefore, we take any narrow strip PQ′, its centre of gravity will lie on AD. Now the triangle may be cut up into an infinitely large number of strips parallel to BC, the centre of gravity of each lying on AD. Hence the centre of gravity of the whole triangle lies on AD. Similarly it may be shown to lie on BE (figure of last article); and therefore at the intersection of AD, BE. That is, the centre of gravity is the same as that of three equal particles at the angular points: therefore on AD at G, where DG = $\frac{1}{3}$DA.



The proof of the theorem may be completed without reference to the case of the three equal particles in the following way (see figure, § 71).

The centre of gravity lies at the intersection of BE and AD. Join DE. Then CE = AE and BD = DC. Therefore DE is parallel to AB.

Hence, by Eucl. VI 2,

$$\frac{AB}{DE} = \frac{AC}{CE} = 2.$$

Also, since the sides of BGA are cut by DE parallel to AB,

$$\frac{BG}{GE} = \frac{AB}{DE} = 2,$$

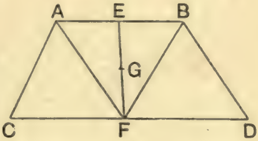
$$\therefore BG = 2GE,$$

or

$$GE = \tfrac{1}{3}BE.$$

Cor. The centre of gravity is also the same as that of three equal particles placed at the mid points of the sides. For it is the same as that of three equal ones (say P) at the angular points. Now $\frac{1}{2}$P at B and $\frac{1}{2}$P at C are equivalent to P at D. So also $\frac{1}{2}$P at C and $\frac{1}{2}$P at D are equivalent to P at E, and $\frac{1}{2}$P at A and B equivalent to P at F, which proves the statement.

73. *Centre of gravity of a trapezoid.*—Let ABCD be the trapezoid, E, F the mid points of AB and CD. Join EF. Denote AB by a, CD by b, and EF by c. Join AF, BF.



The trapezoid is now divided into three triangular parts CAF, AFB, FBD, and has therefore the same centre of gravity as the masses (or areas) of these parts supposed collected at their respective centres of gravity.

The triangles ACF, BFD on equal bases are equal. Let A denote the area of either, A′ the area of AFB. Then since their altitudes are the same, A and A′ are proportional to their bases. Therefore,

$$\frac{A}{A'} = \frac{\frac{1}{2}b}{a} = \frac{b}{2a}.$$

Now the centre of gravity of a triangle is the same as that of three equal particles (each one-third the mass of the triangle) at the angular points. Replace each triangle by these particles.

Thus we have

$$\tfrac{1}{3}A' \text{ at A, B, F,}$$
$$\tfrac{1}{3}A \text{ at C, A, F, F, B, D}$$

—that is, $\frac{1}{3}(A + A')$ at A and B, $\frac{1}{3}A' + \frac{2}{3}A$ at F, and $\frac{1}{3}A$ at C, D.

Those at A, B are equivalent to $\frac{2}{3}(A + A')$ at E, and those at C, D, F to $\frac{1}{3}A' + \frac{2}{3}A + \frac{2}{3}A = \frac{1}{3}A' + \frac{4}{3}A$ at F, and the whole $= A' + 2A$.

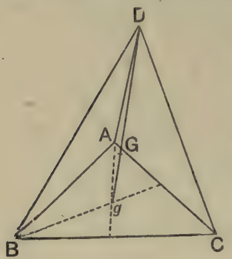
Hence the centre of gravity is at G, where

$$EG(A' + 2A) = \tfrac{1}{3}EF(A' + 4A),$$

or since $A' : 2A = a : b$;

or
$$EG(a+b) = c\left(\frac{a+2b}{3}\right),$$
$$EG = \frac{c}{3} \cdot \frac{a+2b}{a+b}.$$

74. *Four equal particles at the angles of a triangular pyramid.*—Let m denote the mass of each particle, and let ABCD be the pyramid. Let g be the centre of gravity of the base ABC. It is also the centre of gravity of the three equal particles at A, B, C, which may therefore be replaced by $3m$ at g.

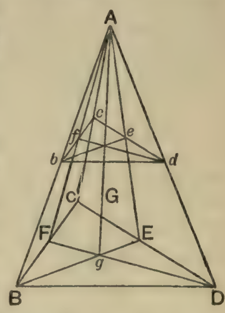


Join Dg. Then the centre of gravity of the particles is that of m at D and $3m$ at g.

Therefore it is at G where
$$\frac{DG}{Gg} = \frac{3m}{m} = 3,$$
whence $$Gg = \tfrac{1}{4}Dg \text{ and } DG = \tfrac{3}{4}Dg.$$

That is, the centre of gravity is on the line joining any angular point to the centre of gravity of the opposite face, and at a distance three-fourths of this line from the angle.

Cor. As there is only one centre of gravity, it follows that the lines joining each angle to the centre of gravity of the opposite face all pass through the same point.



75. *Triangular pyramid.*—Let ABCD be the pyramid.

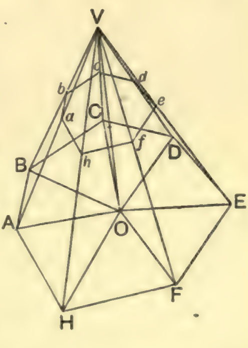
Draw a plane parallel to the base BCD and cutting the pyramid. It will cut it in a triangle bcd whose sides are parallel to BCD—viz. bc parallel to BC, and so on.

Bisect DC in E and draw a plane through the edge AB and E. It will cut the faces in BE, AE, and the triangle bcd in be. Now AE being drawn in the triangle ACD from A to the middle point of CD, will bisect *cd* which is parallel to CD. Therefore *e* is the middle point of *cd*, and therefore the centre of gravity of the triangle *bcd* is in *be*—that is, in the intersection of the plane of *bcd* with the plane through AB and E. Similarly it is in its intersection with the plane through AD and F, the mid point of BC. Therefore it lies in the intersection of these two planes.

The same is true for all triangular laminæ parallel to BCD. Therefore the centre of gravity of the whole pyramid lies in their line of intersection. This line is that joining A to g, the intersection of BE and DF, that is the line joining A to the centre of gravity of the opposite face. Similarly it may be shown to lie in the line joining any other angular point to the centre of gravity of the opposite face. It is therefore, by the corollary to the former case, the same as the centre of gravity of four equal particles at the angles. Therefore it is at G on Ag where $AG = \frac{3}{4}Ag$.

Note.—The height of G above the base is one-fourth the altitude of A.

76. *Centre of gravity of a pyramid on any base, and of a cone.*—Let the base be any polygon ABCDEF . . ., and let V be the vertex. Take any point O in the base and join O to all the angles. We then divide the pyramid into a number of smaller ones, with the same vertex V, and on triangular bases OAB, OBC, . . .



The centre of gravity of each of these is at a distance from the base equal to one-fourth the altitude of V. Therefore the centre of gravity of the whole solid must be at a distance from the base equal to one-fourth the altitude of V.

Again, take O the centre of gravity of the base and join VO.

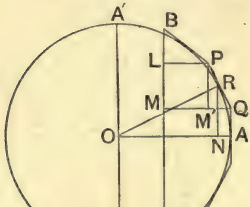
Take any plane section *abcdef* parallel to the base. It will be similar to it, and will be cut by VO in its centre of gravity. Hence the centre of gravity of every lamina parallel to the base lies in VO, and therefore so does that of the whole figure.

Hence, combining these two results, the centre of gravity of the pyramid is on the line joining the vertex to the centre of gravity of the base, and at a point one-quarter of this line from the latter point.

We can extend this result to the case where the base is a plane curve instead of a polygon. For in the above proof it does not matter how many sides the polygon has. The result is still true if we suppose the number of sides infinitely large and their lengths infinitely small. But this ultimately includes the case of a curve. Hence the centre of gravity of a cone may be determined in the same way as that of a pyramid.

77. *Surface of a pyramid and of a cone.*—Using the figure of the preceding case, we have now to do with a series of triangles VAB, VBC, etc. instead of pyramids VABO, etc. The centres of gravity of all these are at a distance from the base one-third the altitude of V. The rest of the proof proceeds in the same way. Hence the centre of gravity of the surface of a pyramid or cone is on the line joining the vertex to the centre of gravity of the perimeter of the base, and at a point distant one-third of this line from the latter point.

78. *Portion of a regular polygon and arc of a circle.*—Suppose



a wire bent into a portion of a regular polygon. Let O be the centre of the circle inscribed in the polygon. Further, let a be the radius of this circle, m the mass of one side, and n the number of parts into which the wire is bent. Then the whole mass of the wire is nm. Let OA be the radius which divides the wire symmetrically. A is either at an angle, or the mid point of the side, according as n is even or odd. The centre of gravity lies in OA. The mass of each side can be supposed to be concentrated at its mid point, where it touches the inscribed circle. The problem is then to find the centre of gravity of particles (m) at these points. If it be at G,

$$\text{OG} = \frac{\Sigma(m\,.\,\text{ON})}{nm} = \frac{\Sigma(\text{ON})}{n}.$$

If P, Q be the extremities of any side, draw PL, QM perpendicular to BC, the line joining the extremities of the wire. Also draw PM′ perpendicular to QM. Then BC is perpendicular to OA, and the sides of the triangle ORN are respectively perpendicular to the sides of the triangle PQM′, and therefore they contain equal angles and are similar. Hence

$$\frac{\text{ON}}{\text{OR}} = \frac{\text{PM}'}{\text{PQ}},$$

$$\text{ON} = \frac{\text{OR}}{\text{PQ}}\,.\,\text{PM}';$$

$$\therefore \text{OG} = \frac{\Sigma\left(\frac{\text{OR}}{\text{PQ}}\,.\,\text{PM}'\right)}{n},$$

$$= \frac{\text{OR}}{n\,.\,\text{PQ}}\Sigma(\text{PM}').$$

Now $n\,.\,\text{PQ}$ = length of wire = l (say), and OR = a,

$$\therefore \text{OG} = \frac{a}{l}\Sigma(\text{LM}),$$

$$= \frac{a}{l}\,.\,\text{BC}.$$

This gives the position of the centre of gravity in terms of the length of wire, the radius of the inscribed circle, and the chord joining the extremities. *It does not depend at all on the number of sides.* Hence the result is true if the number of sides in the inscribed polygon be infinitely large. But then it becomes indistinguishable from a circular arc. Hence the centre of gravity of a circular arc is on the radius bisecting the arc, at a distance from the centre

$$=\frac{\text{chord}}{\text{length}}\times\text{radius}.$$

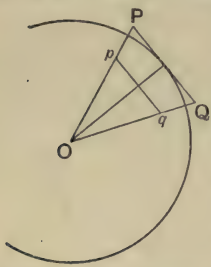
EXAMPLE I. *For a semicircular wire,*

$$OG=\frac{2a}{\pi a}.a=\frac{2a}{\pi}.$$

EXAMPLE II. *For a quadrantal wire,*

$$OG=\frac{a\sqrt{2}}{\frac{1}{2}\pi a}.a=\frac{2a\sqrt{2}}{\pi}.$$

79. *Lamina in the form of a circular sector.*—Going back to a boundary formed of a regular polygon circumscribing



the circle, consider the portion of the sector OPQ. The centre of gravity of this triangle is clearly the same as that of a uniform wire *pq*, where $Op=\frac{2}{3}OP$, $Oq=\frac{2}{3}OQ$. Hence the centre of gravity of the whole polygonal sector is the same as that of a similar polygonal wire circumscribing a circle of two-thirds the radius of the former.

This is the same for the circular sector. Its centre of gravity will therefore be the same as that of a corresponding circular arc of two-thirds the radius—that is,

$$OG=\frac{2}{3}.\frac{\text{chord}}{\text{length of arc}}\times\text{radius}.$$

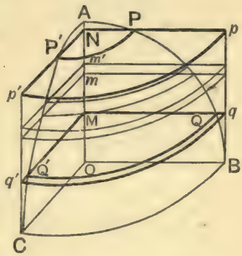
EXAMPLE. *Semicircle,*

$$OG=\frac{4a}{3\pi}.$$

80. *Surface of a sphere cut off between two parallel planes.*—Let O be the centre and PNP′, QMQ′ the bounding planes. Draw OMNA, the radius perpendicular to them.

Describe round the sphere a cylinder whose axis is OA.

Let the planes cut the cylinder in the circles pp', qq'. Then it is known that the area of the surface cut off on the sphere between the planes is equal to that cut off on the cylinder. Consider now the thin ribbons cut off these by two planes close together; they will both have their centres of gravity on OA, and will each be equivalent to their respective masses, distributed along mm'. But since these masses are equal, they are equivalent altogether. The same is therefore true for larger parts built up of these smaller ribbons. Hence the centre of gravity of the zone cut off the sphere is the same as that of the zone cut off the cylinder. This latter clearly bisects the distance between the planes. Hence so does the centre of gravity of the spherical zone.



EXAMPLE. *Hemispherical shell,*

$$OG = \tfrac{1}{2} \, . \, OA.$$

81. *Sector of a sphere.*—The volume of the sector may be divided into a larger number of very small cones, whose vertices are at the centre, and bases on the surface. The centres of gravity of these all lie on a spherical surface of three-fourths the radius. The centre of gravity of this latter surface is determined by the preceding case.

EXAMPLE. *Solid hemisphere, radius* a.

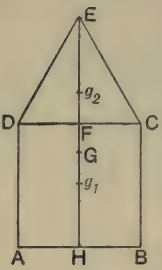
The centre of gravity is that of a hemispherical surface of radius $\tfrac{3}{4}a$. Therefore, by the preceding article,

$$OG = \tfrac{1}{2} \, . \, \tfrac{3}{4}a = \tfrac{3}{8}a.$$

82. The foregoing results must be remembered by the student. The following examples will serve to illustrate methods of finding the centre of gravity in more complicated cases :—

EXAMPLE I. *A square with an equilateral triangle on one side.*

Let the side of the square be a. Then



$$\text{Area of square} = a^2,$$

$$\text{Area of triangle} = \frac{1}{2}\text{DC} \,.\, \text{FE} = \frac{a^2\sqrt{3}}{4},$$

$$\text{and the whole} = a^2\left(1 + \frac{\sqrt{3}}{4}\right).$$

Draw EFH parallel to DA or CB, it clearly passes through g_1, g_2, the centres of gravity of the square and triangle respectively.

Replace the square by a mass a^2 at g_1 and the triangle by $a^2\sqrt{3}/4$ at g_2. Then, taking moments about F,

$$\text{FG}\left(1 + \frac{\sqrt{3}}{4}\right)a^2 = \text{F}g_1 \times a^2 - \text{F}g_2 \,.\, \frac{a^2\sqrt{3}}{4},$$

also
$$\text{F}g_1 = \frac{1}{2}a, \quad \text{F}g_2 = \frac{1}{3}\text{EF} = \frac{1}{3} \times \frac{a\sqrt{3}}{2};$$

$$\therefore \text{FG}\left(1 + \frac{\sqrt{3}}{4}\right) = \frac{1}{2}a - \frac{a\sqrt{3}}{6} \times \frac{\sqrt{3}}{4},$$

$$\text{FG}\frac{4+\sqrt{3}}{4} = \left(\frac{1}{2} - \frac{1}{8}\right)a = \frac{3}{8}a;$$

$$\therefore \text{FG} = \frac{3}{2(4+\sqrt{3})}a.$$

EXAMPLE II. *A circular disc of* 1 *foot radius has a circular hole of radius* 3 *inches cut out of it, the centre of the hole being at a distance of* 2 *inches from the centre of the disc. Find the centre of gravity.*

[The student should draw the figures for this and the next.]

Let C_1, C_2 be the centres of the circles and G the centre of gravity required. Then the whole disc is made up of two parts, viz. the figure condensed at G and the part cut out at C_2.

The area of the whole $= \pi \times 12^2 = 144\pi$,

" hole $= \pi \times 3^2 = 9\pi$,

" plate $= 144\pi - 9\pi = 135\pi$.

Hence, since C_1 is the centre of gravity of 135π at G and 9π at C_2,

$$\frac{C_1G}{C_1C_2} = \frac{9\pi}{135\pi} = \frac{1}{15},$$

$$\therefore\ C_1G = \tfrac{1}{15}C_1C_2 = \tfrac{2}{15} \text{ inch.}$$

EXAMPLE III. *A wire is bent into the shape of a triangle. Find the centre of gravity.*

Let ABC be the triangle. Bisect the sides in D. E. F and join DE, EF, FD. Calling the sides a, b, c, their masses are proportional to a, b, c, and may be supposed collected, a at D, b at E, c at F. We then have to find the centre of gravity of three particles a, b, c at the points D, E, F.

Draw DL perpendicular to FE, then, using the formulæ in § 68, the distance from FE

$$=\frac{a.\mathrm{DL}+b\times 0+c.0}{a+b+c}=\frac{a}{a+b+c}\mathrm{DL}.$$

Let Δ denote the area of DEF. Then

$$\Delta=\tfrac{1}{2}.\,\mathrm{FE}.\,\mathrm{DL}.$$

But since E, F bisect AC, AB,

$$\mathrm{EF}=\tfrac{1}{2}\mathrm{BC}=\tfrac{1}{2}a;$$

$$\therefore\ \Delta=\tfrac{1}{4}a.\,\mathrm{DL},$$

and $$a.\,\mathrm{DL}=4\Delta.$$

Hence the distance of the centre of gravity from FE is

$$\frac{4\Delta}{a+b+c}.$$

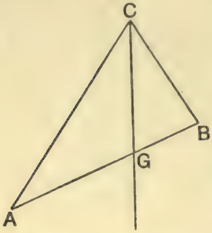
Similarly the distances from the other sides FD, DE are the same quantity; therefore it is at the same distance from each of the sides, and consequently at the centre of the circle inscribed in the triangle DEF.

83. When a heavy body is suspended from a point and is in equilibrium, its centre of gravity will lie either vertically below or vertically above the point of suspension. For the body is at rest under two forces, viz. the weight of the body supposed acting vertically through its centre of gravity and the reaction at the point of suspension. But two forces cannot balance one another unless they are in the same straight line, viz. in this case the vertical through the centre of gravity. Hence the point of suspension and the centre of gravity must be in the same vertical.

This consideration is of use in determining positions of equilibrium. Take for instance the following problem—

A rod 3 *feet long has its centre of gravity* 1 *foot from one end, and is suspended over a smooth peg by a string* 6 *feet long attached to its ends. Find the position of equilibrium.*

Notice first that at the peg the forces are the reaction and the two tensions. These are in equilibrium amongst themselves, and therefore the reaction is equal and opposite to the resultant of the tensions. But the tensions are equal, and the resultant must therefore bisect the angle between them. Hence if in the figure AB be the rod and C the peg, the reaction must bisect the angle ACB. Further, it must pass through G. That is, the vertical CG must bisect ACB. Therefore (Eucl. VI 3)



$$AC : CB = AG : GB = 2 : 1,$$

also $$AC + CB = \text{length of string} = 6\,;$$

$$\therefore\ AC = 4,$$
$$CB = 2.$$

Thus when it is placed over the peg the string will slip round until the peg divides it into portions 4 feet and 2 feet long. Also CG must be vertical. Hence CGB is the inclination of the rod to the vertical. This angle can thus be determined graphically or determined by easy trigonometry.

84. The theorem in the last article may also be applied to determine experimentally the position of the centre of gravity of a plane lamina.

Hang the lamina freely from any point A in it, and by means of a plumb line hanging from A draw on the lamina the vertical line A*a*. Then the centre of gravity must lie on A*a*. Now suspend it from another point B, and in the same way draw a vertical B*b* through it; this will intersect A*a* in some point G. Then G is the position of the centre of gravity, for the centre of gravity lies both on A*a* and B*b*, and therefore at their intersection.

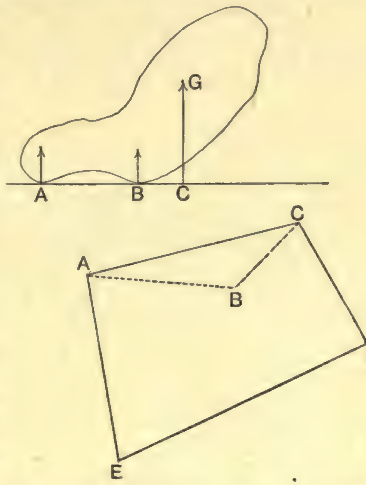
A similar method would also apply to any solids, if we could get inside to draw the lines.

The student is recommended to determine experimentally in this way the position of the centre of gravity of a triangular lamina, or other of the examples given. The lamina may be cut out of a piece of board, tin plate, or thick cardboard.

85. When a heavy body is resting on any surface, it is in equilibrium under its weight acting through its centre of gravity and the reactions at the points of contact where it is resting. These reactions will so adjust themselves that their resultant passes through the centre of gravity of the body. If it is impossible for them so to adjust themselves, then the body cannot rest in that position. For instance,

if the body is as in the figure, it is impossible for the reactions at A and B to have a resultant passing through C outside of them.

Now in the general case of a body resting on a plane—suppose that A, B, C, D, E, . . . be the points of contact. Join the outside points so that the outside figure is a convex polygon (*i.e.* has no re-entrant angles like ABC). Then it is clear that by properly choosing the reactions at A, B, C, D, E, . . . their resultant may be made to pass through any point within ACDE, but not outside. If then the vertical through the centre of gravity of the body cuts the plane in P, the weight can be counterbalanced if P lies within ACDE, but not if it lies outside. Consequently the body will rest or topple over according as the vertical through the centre of gravity falls within or without the convex polygon formed by joining the points of contact of the body with the plane.



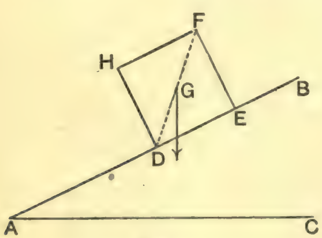


FIG. 1.

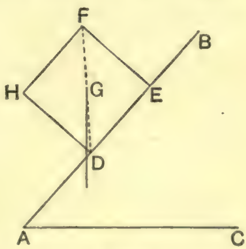


FIG. 2.

EXAMPLE. *A cube is laid on an inclined plane, and is prevented from slipping by a small peg at the lower edge. Find the inclination of the plane when the cube is just on the point of toppling over.*

If the vertical through G falls as in Fig. 1, the block will remain at rest, if as in Fig. 2, it will topple over. The limiting point between the two cases is when GD is vertical. But if GD is vertical, GDE is the inclination of DE or the plane to the vertical.

Now DEFH is a square, therefore GDE = 45°. Hence the inclination of the plane to the vertical is 45°.

EXAMPLES—IX.

1. At points in a straight line which divide it into parts, bearing to each other the ratios of 7 : 5 : 3 : 1, are placed weights which are in the ratio 1 : 3 : 5. Determine the centre of gravity of the system, the line being 5 feet 4 inches in length.

2. Find the centre of gravity of six heavy particles situated on a straight line, the successive particles weighing 1, 4, 9, 16, 25, 36 grains respectively ; the distance between the first and second particles being one inch, and between the others being respectively 3, 5, 7 and 9 inches.

3. A uniform rod of 5 lbs. is weighted with masses of 1 and 2 lbs. at the ends. Find the point about which it will balance.

4. Find the centre of gravity of weights w, $2w$, $3w$ placed at the angular points of a triangle, and determine the ratios in which the lines drawn from the angular points through the centre of gravity to the opposite sides are divided at that point.

5. Two equal heavy particles are placed in the plane of a triangle. Where must a third of equal weight be placed in order that the centre of gravity of the whole may be the same as that of the triangle?

6. Prove that, if weights 1, 2, 3, 4, 5, 6 are situated at the angles of a regular hexagon, the distance of their centre of gravity from the centre of the circumscribing circle is two-sevenths of the radius of that circle.

7. Find the centre of gravity of equal particles at all the angles but one of a regular polygon of n sides.

8. Masses are placed at four points A, B, C, D lying in one plane, respectively proportional in the areas of the triangles BCD, CDA, DAB, ABC. Find their centre of gravity.

9. A series of triangles on the same base have their vertices on a given line. Prove that their centres of gravity lie in another line parallel to the first.

10. ABC is a triangle right-angled at A, AB and AC are 12 inches and 15 inches respectively; weights of 2 oz., 3 oz., 4 oz. are placed at A, C, B respectively. Find the distances of their centre of gravity from B and C.

11. If the triangle ABC weigh 6 oz., what weight must be placed at A so that the centre of gravity of the whole may bisect the line joining A to the middle point of BC?

12. A wire is bent into the shape of a triangle. Find its centre of gravity.

13. If the sides in the above be 7, 8, 9 feet and the mass be 48 lbs., find the pressures on three props supporting it at the angular points.

14. Two cylinders of the same material 2 and 3 feet long and 1 and 4 inches diameter are joined end to end with their axes in the same straight line. Find the centre of gravity.

15. A figure is formed of two isosceles triangles on same base and on opposite sides, and of vertical angles 90° and 60°. Find the centre of gravity.

16. From an isosceles triangle another on the same base is cut away. Determine its height that the centre of gravity of the remainder may be at the vertex of the second triangle.

17. A straight line parallel to a side of a triangle is drawn so as to cut off one-ninth of the area of the triangle. Find the centre of gravity of the remainder.

18. A uniform equilateral plate is suspended by a string attached to a point in one of its sides, which it divides in the ratio of 2 : 1. Find the inclination of this side to the vertical.

19. A rod AB has a hinge at A, and is kept in a position making 60° with the vertical by a horizontal string at the other end. Find the tension of the string and the reaction at the hinge.

20. Find the centre of gravity of the laminæ represented in the figures i-iv.

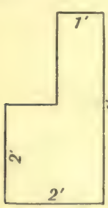


FIG. i.

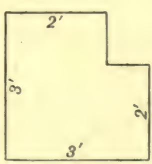


FIG. ii.

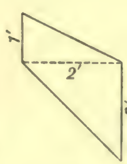


FIG. iii.

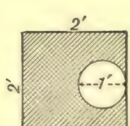
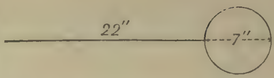


FIG. iv.

21. A wire is bent into the form 

Find its centre of gravity ($\pi = \frac{22}{7}$).

22. One corner of a square sheet of paper, whose side is 1 foot, is folded down so as to coincide with the centre of the square. Find the distance of the centre of gravity of the paper from the centre.

23. A tricycle weighing 5 st. 4 lbs. has a small wheel symmetrically placed 3 feet behind two large wheels, which are 3 feet apart. If the centre of gravity of the machine be 9 inches behind the front wheels, and that of the rider, who weighs 9 st., be 3 inches behind, find the pressures on the ground of the different wheels.

24. Two uniform balls of the same material, whose diameters are 6 inches and 12 inches respectively, in contact with one another, are firmly united at their point of contact. Find the centre of gravity of the body thus formed, it being known that the volumes of spheres are proportional to the cubes of their radii.

25. The faces of a pyramid are all equilateral triangles. Show that the position of the centre of gravity for the four faces considered as plane areas will be the same as it is for the solid pyramid.

26. If ABCD be a tetrahedron, and if the plane CDE passing through the edge CD cuts AB in E, prove that the line joining the centres of gravity of the tetrahedrons ABCD and AECD is parallel to AB.

27. The uniform quadrilateral ABCD has the sides AB, AD and the diagonal AC all equal, and the angles BAC and CAD 30° and 60° respectively. If a weight equal to two-thirds of the triangle ABC be attached to the point B, and the whole rest suspended from A, prove that the diagonal AC will be vertical.

28. A hemisphere is joined to a cylinder on the same base and of the same material. Prove that the equilibrium will be stable when the hemisphere is resting on a horizontal plane if the radius of the base is greater than $\sqrt{2}$ times the height of the cylinder.

29. From a right cone the top is cut off by a plane parallel to the base half way between the vertex and the base. Find the centre of gravity of the remainder.

30. From a right cone is cut out another with the same vertex and axis. Find the centre of gravity of the remainder.

Find the position of the centre of gravity when the part cut away becomes almost equal to the original cone. Why does the result not agree with that obtained for the centre of gravity of the surface of a cone?

31. From a uniform right cone whose vertical angle is 60° is cut out the largest possible sphere. Find the centre of gravity of the remaining portion of the cone.

32. Find the centre of gravity of the surface of a tetrahedron.

33. Given the base and height of a triangle, construct it so that it will just stand on a horizontal plane.

34. A cone whose height is equal to the diameter of its base rests on a rough inclined plane. Determine the greatest inclination that the cone may not topple over.

35. A body consists of two parts, and one of them is moved into any other position. Show that the line joining the two positions of the centre of gravity of the whole body is parallel and bears a fixed ratio to the line joining the two positions of the centre of gravity of the part moved, and apply this theorem to find the position of the centre of gravity of a circular arc.

36. An arc of a circle consists of homogeneous matter attracting with a force which varies as the distance. Find the resultant attraction on any particle.

37. Three forces PA, PB, PC diverge from the point P; and three others AQ, BQ, CQ converge to the point Q. Show that the resultant of the six is represented in magnitude and direction by 3PQ, and that it passes through the centre of gravity of the triangle ABC.

38. A set of particles attract with a force proportional to their masses and to the distance. Show that the resultant attraction at any point is proportional to the whole mass and the distance from the centre of gravity.

39. In the side CD of a uniform square plate ABCD a point E is taken and the triangle ADE is cut off. Find the length of DE so that the plate ABCE may just be able to stand with its side CE on a horizontal plane, the side of the square being a inches long.

40. A particle P descends from the highest point down the chord which is the side of a regular hexagon inscribed in a circle, and Q down the vertical diameter. If $P = 2Q$, show that their common centre of gravity will descend along the chord which is the side of an equilateral triangle inscribed in the circle, assuming that the path of the centre of gravity is a straight line.

41. Four weights are placed at four given points in space, the sum of two of the weights is given, and also the sum of the other two. Prove that their centre of gravity lies on a fixed plane.

42. A uniform lamina in the form of a right-angled triangle, such that one of the sides containing the right angle is three times the other,

is suspended by a string attached to the right angle. Prove that in the position of equilibrium the hypotenuse makes with the vertical an angle θ where $\sin\theta=\frac{3}{5}$.

43. If three heavy particles be placed in the angles A, B, C of a triangle, the weights of each being proportional to the opposite sides of the triangle a, b, c, prove that the distance of the centre of gravity of the particles from A is equal to

$$\frac{2bc}{a+b+c}\cos\frac{A}{2}.$$

44. The perpendiculars from the angles A, B, C meet the sides of a triangle in PQR. Prove that the centre of gravity of six particles proportional respectively to $\sin^2 A$, $\sin^2 B$, $\sin^2 C$, $\cos^2 A$, $\cos^2 B$, $\cos^2 C$ placed at A, B, C, P, Q, R coincides with that of the triangle PQR.

45. A body is composed of a cylinder with one conical end. Determine the ratio of lengths of cylinder to cone in order that it may just rest on a horizontal table (1) with the cylinder in contact; (2) with the cone in contact. Consider in each case the question of stability for all kinds of displacement.

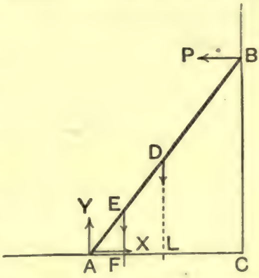
CHAPTER X

MISCELLANEOUS THEOREMS AND EXAMPLES

In this chapter the methods and results derived in the last two chapters will be illustrated by application to a few special questions. The student is advised before reading each solution to attempt to solve the problem first by himself.

86. *A ladder is placed against a smooth wall; the bottom of the ladder is* 6 *feet from the wall, and the top* 8 *feet from the ground; the mass of the ladder is* 12 *lbs., and a man of* 10 *st. is on the ladder* 2 *feet from the bottom. Find the pressure of the ladder on the wall and the reaction on the ground.*

In attacking any problem the student should always draw, in the first place, a diagram of all the forces. In this case, if AB be the ladder and BC the wall, the force at B on the ladder is a horizontal pressure, for it is perpendicular to the smooth wall. Call it P.



At A the ladder is acted on by an unknown force in an unknown direction. We shall, therefore, suppose this force given by its two components horizontally and vertically. Call them X towards the wall and Y vertically upwards.

Also the ladder is acted on by its weight, which we may suppose collected at its centre of gravity—that is, by a vertical force of 12 lbs. weight at its middle point D, and also by the weight of the man, or 140 lbs. weight vertically at E, where AE = 2 feet. The forces will, therefore, be as in the figure.

Take the weight of 1 lb. for the unit of force. For the sake of illustration we will solve this question by applying each of the conditions of § 64.

(1) The resolved forces horizontally and vertically, and the moments about some point must each vanish. Hence,

resolving horizontally, $P-X=0$ (i.)

,, vertically, $Y-140-12=0$ (ii.)

Moments about A,

$$P\times BC-12\times AL-140\times AF=0.$$

Now $BC=8, \quad AC=6. \quad \therefore AB=10$, and $AL=3$,

and $$\frac{AF}{AC}=\frac{AE}{AB}=\frac{2}{10},$$

$$\therefore AF=\frac{1}{5}\times 6=\frac{6}{5}.$$

Hence the last equation is

$$8P-36-140\times\frac{6}{5}=0 \qquad \text{(iii.)}$$

From (iii.) $8P=36+168=204$,

$\therefore P=25\frac{1}{2}$ lbs. weight.

From (ii.) $Y=152$ lbs. weight.

From (i.) $X=P=25\frac{1}{2}$ lbs. weight.

If the whole reaction of the ground be R, and if it make an angle θ with the vertical,

$$R^2=X^2+Y^2=152^2+(25\cdot5)^2,$$

whence $R=154\cdot12$,

and $$\tan\theta=\frac{X}{Y}=\frac{25\cdot5}{152}=\cdot16776,$$

whence, from the tables, $\theta=9°\ 31'$ approximately.

(2) If we had proceeded with the other set of conditions, the moments about three points not in a straight line must vanish. Take the three points to be A, B, C.

Moments about A, as before,

$$8P-36-\tfrac{6}{5}\times140=0,$$

or $P=25\frac{1}{2}$.

Moments about B,

$$X\times8-Y\times6+140\times FC+12\times CL=0,$$

or $$8X-6Y+\tfrac{24}{5}\times140+12\times3=0,$$

$$6Y-8X=36+672=708.$$

Moments about C,

$$P\times8+X\times0-Y\times6+140\times\tfrac{24}{5}+12\times3=0,$$

or $6Y-8P=708$.

Hence $6Y=8P+708=204+708=912$,

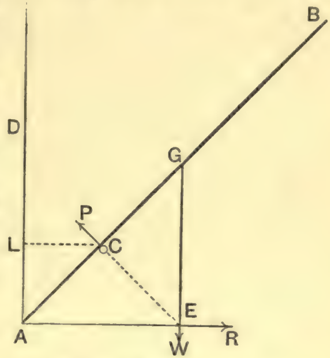
$$Y=152.$$

Also, subtracting the last equation from the second,

$$X = P = 25\tfrac{1}{2},$$

and we then proceed as before.

87. *A horizontal bar is fixed parallel to a smooth wall, and at a distance from it of* 3 *feet; a uniform heavy rod of* 8 *lbs. and* 18 *feet long is then laid across it at right angles, with one end pressing against the wall so as to rest in equilibrium. Determine its position and the pressures on the wall and on the bar.*



Let AB be the rod in its position of equilibrium, C the bar, and AD the wall. Draw the forces on the rod. These will be

(1) The weight W vertically through G the mid point of the rod.

(2) The pressure P of the bar, perpendicular to AB.

(3) The pressure R of the wall, perpendicular to it.

These are three forces in equilibrium. Hence, since they are not parallel, they must (§ 65) meet in a point. Let E denote this point. Draw CL perpendicular to the wall. Then

$$\text{Angle } LCA = CAE,$$

also $$CLA = ACE = AEG, \text{ being right angles.}$$

Hence the triangles ACL, EAC, GAE are similar,

$$\therefore \frac{AC}{CL} = \frac{AE}{AC} = \frac{AG}{AE}.$$

Let each of these ratios be represented by y.

Then $$y^3 = \frac{AC}{CL} \times \frac{AE}{AC} \times \frac{AG}{AE} = \frac{AG}{CL} = \frac{9}{3} = 3.$$

Hence $$y = \sqrt[3]{3},$$

and $$\frac{AC}{CL} = \sqrt[3]{3};$$

whence $$AC = CL\sqrt[3]{3} = 3\sqrt[3]{3},$$
$$= 4{\cdot}326 \text{ feet.}$$

This determines the position of rest of the rod. If the inclination is desired, we have

$$\sin CAL = \frac{CL}{AC} = \frac{1}{\sqrt[3]{3}} = {\cdot}6933,$$

whence, from the tables, it will be found that

$$\text{CAL} = 43° \ 53' \text{ approximately.}$$

To find the magnitude P, take moments about A. Then

$$P \times AC - W \times AE = 0.$$

But
$$\frac{AE}{AC} = y = \sqrt[3]{3},$$

$$\therefore P . AC - W \sqrt[3]{3} AC = 0,$$

or
$$P = W \sqrt[3]{3}.$$

The easiest way to find R is to notice that P must be equal in magnitude to the resultant of W and R, which are perpendicular.

$$\therefore P^2 = W^2 + R^2,$$

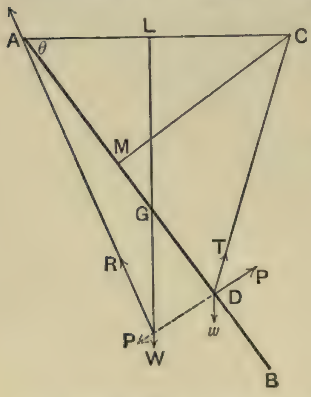
or
$$R^2 = W^2 \sqrt[3]{9} - W^2;$$

$$\therefore R = W \sqrt{(\sqrt[3]{9} - 1)}.$$

Hence
$$P = 1{\cdot}442W,$$
$$R = 1{\cdot}039W.$$

88. *A heavy rod* AB *is suspended by a hinge at* A; *a heavy ring slides on the rod and is fastened by a string to a point* C *on the same horizontal as* A, *so that* AC *is equal in length to the string. Determine the position of equilibrium, the tension of the string, and the reactions at the hinge and ring.*

This differs from foregoing questions in that two bodies have to be considered.



Let W denote the weight of the rod and $2l$ its length; also let w denote the weight of the ring and a the length of the string.

First draw the forces on the rod. These are

(1) W vertically through G, the mid point of the rod. AG $= l$.

(2) P, the pressure of the ring on the rod, acting perpendicular to the rod.

(3) The reaction R at the hinge, which by § 65 goes through the intersection of P and W.

Next draw the forces on the ring. They are

(1) The weight w vertically down.

(2) The tension T along the string.

(3) The pressure P of the rod on the ring, equal and opposite to that of the ring on the rod.

Denote the angle CAD by θ. This also is equal to CDA, since CA = CD. The angle between the rod and W or w is $90° - \theta$.

Consider first the conditions of equilibrium of the ring.

The resolved parts of the forces along the rod must vanish.

Hence $$\mathrm{T}\cos \mathrm{CDA} = w \cos w\mathrm{DB},$$

or $$\mathrm{T}\cos\theta = w\sin\theta;$$

$$\therefore \mathrm{T} = w\tan\theta \qquad \text{(i.)}$$

Again the resolved parts of the forces perpendicular to the rod must vanish. Hence

$$\mathrm{P} + w\cos\theta = \mathrm{T}\sin \mathrm{CDA},$$

or $$\mathrm{P} + w\cos\theta = \mathrm{T}\sin\theta \qquad \text{(ii.)}$$

Next consider the conditions of equilibrium of the rod.

The moments of the forces about A must vanish. Hence

$$\mathrm{P}\times\mathrm{AD} = \mathrm{W}\,.\,\mathrm{AL},$$

$$\mathrm{P}\,.\,2\mathrm{AM} = \mathrm{W}\,.\,\mathrm{AL},$$

$$2\mathrm{P}\,.\,\mathrm{AC}\,.\,\cos\theta = \mathrm{W}\,.\,\mathrm{AG}\,.\,\cos\theta;$$

$$\therefore \mathrm{P} = \frac{\mathrm{AG}}{2\,\mathrm{AC}}\,.\,\mathrm{W} = \frac{l}{2a}\mathrm{W} \qquad \text{(iii.)}$$

These equations will be sufficient to determine P, T, and the position of the rod.

From (i.) and (ii.)

$$\mathrm{P} = w\frac{\sin^2\theta}{\cos\theta} - w\cos\theta.$$

From (iii.)

$$\mathrm{P} = \frac{l}{2a}\mathrm{W}.$$

Hence $$w\frac{\sin^2\theta - \cos^2\theta}{\cos\theta} = \frac{l}{2a}\,.\,\mathrm{W},$$

or, since $$\sin^2\theta = 1 - \cos^2\theta,$$

$$1 - 2\cos^2\theta = \frac{l}{2a}\,.\,\frac{\mathrm{W}}{w}\,.\,\cos\theta.$$

This is a quadratic to determine $\cos\theta$, viz.

$$\cos\theta = \frac{\pm\sqrt{l^2\mathrm{W}^2 + 32a^2w^2} - l\mathrm{W}}{8aw}.$$

Now $\cos\theta$ is here a positive quantity; hence the value for the problem we are considering is

$$\cos\theta = \frac{\sqrt{l^2\mathrm{W}^2 + 32a^2w^2} - l\mathrm{W}}{8aw}.$$

This will, from the tables, give θ when l, a, W, w are known. For instance, if $l = a$ and $\mathrm{W} = 2w$,

$$\cos\theta = \tfrac{1}{2},$$

whence $$\theta = 60°,$$

or the ring rests at the middle of the rod.

The angle θ being known, the values of T and P are at once determined, viz.

$$\left.\begin{aligned} T &= w \tan\theta \\ P &= \frac{l}{2a}\,.\,W \end{aligned}\right\}.$$

[In the particular case above, where $\theta = 60$, $T = w\sqrt{3} = 1{\cdot}732\,.\,w$, and $P = \frac{1}{2}W = w$.]

If it is necessary to determine R, it is best done by considering it as equal and opposite to the resultant of P and W, which are inclined at an angle $180^\circ - \theta$. Hence

$$R^2 = P^2 + W^2 + 2PW\cos(180^\circ - \theta),$$

$$= W^2\left\{\frac{l^2}{4a^2} + 1 - \frac{l}{a}\cos\theta\right\},$$

and

$$R = W\left\{\frac{l^2}{4a^2} + 1 - \frac{l}{a}\cos\theta\right\}^{\frac{1}{2}}.$$

Students often find a difficulty in questions like the foregoing, where several bodies are in mutual equilibrium, *e.g.* two spheres resting inside a bowl, or a system of jointed rods. In all problems of this kind there are two sets of forces to deal with, viz. the forces acting on some of the bodies from outside, and those between one of the bodies and another; *i.e.* external forces and internal forces. The latter must be in equilibrium amongst themselves, for they consist of a number of stresses, *i.e.* to any force acting on one body must correspond the equal and opposite reaction on the adjoining body, so that these two together are in equilibrium by themselves. As this is true for all the stresses, the system of internal forces must be in equilibrium. As the whole system of forces is in equilibrium, and a part of them (the internal forces) are also in equilibrium, so must be the remainder—or the external forces. Applying the conditions of equilibrium (Art. 64) to these, we get a number of equations which may or may not be sufficient to solve the question. If not, it will be necessary to consider the internal forces as well. In doing this, each body of the system should be considered by itself as kept in equilibrium by the forces actually exerted on it, some of which may be external. In such cases the student should first indicate on a diagram all the forces by arrows, remembering that where two bodies touch, the forces on each are equal and opposite. Take, for instance, the case of Example 23 at the end of this chapter.

Consider each hemisphere and weight separately.

First, the weight is acted on by three forces, viz. its weight, the pressure of the sphere on it outwards from the centre (say P), and the tension of the string (say T). The length of the string being given or assumed, both T and P can be obtained in terms of the weight.

Next consider the right-hand hemisphere. The forces are (1) a pressure P inwards to the centre produced by the weight: this is now known from the former work; (2) the weight of the hemisphere through its centre of gravity; (3) the reaction of the ground (say R); (4) a pressure somewhere between it and the other hemisphere (say Q) acting to the right at some point L; and (5) the effect of the string over the top. This last is clearly the same as if the string were fastened to the top edge, and pulled with its tension T horizontally to the left. Applying the conditions of equilibrium to these we get three equations to find R, Q, and the unknown distance of L above the point where the system rests on the ground.

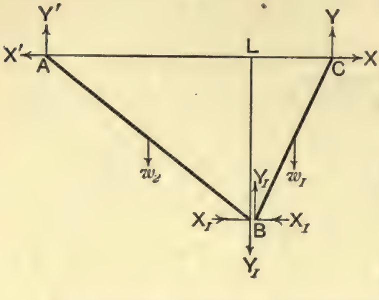
In all cases the student will find no difficulty if he carries out the rule to draw first the forces acting on each body separately, and apply the conditions of equilibrium to each. In general, however, the process may be shortened by the application of a little consideration and common sense.

In dealing with the stresses at hinges or joints, similar methods apply. The direction of a stress can often be settled at a glance from symmetry, or the fact that three forces in equilibrium must be parallel, or meet in a point.

The treatment of stresses at joints may be illustrated by the following example.

Two rods, AB, BC, are hinged at B, and jointed by smooth pins to points A and C in the same horizontal line. Determine the stresses at A, B, C.

Resolve the stresses at A, B, C horizontally and vertically, and indicate them as in the figure. Here X, Y, X′, Y′, and w_1, w_2, are external forces. Since they are in equilibrium by themselves $X = X'$, $Y + Y' = w_1 + w_2$, and Y, Y′ may be completely determined by taking moments about A. Afterwards X_1, Y_1, X can be found by considering the equilibrium of AB by itself. We will, however, work it out in full by considering each separately.



Taking equilibrium of BC only,

$$X_1 = X \quad . (1),$$

$$Y + Y_1 = w_1 \quad . (2),$$

$$X_1 . BL + Y_1 . CL = \tfrac{1}{2} w_1 . CL \ (3).$$

So equilibrium of AB gives

$$X' = X_1 = X \quad . (4),$$

$$Y' - Y_1 = w_2 \quad . (5),$$

$$X_1 . BL - Y_1 . AL = \tfrac{1}{2} w_2 . AL \ (6),$$

(3) − (6) gives

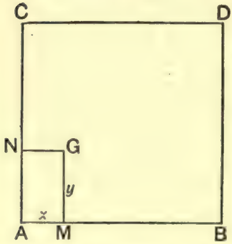
$$Y_1 . AC = \tfrac{1}{2}(w_1 . CL - w_2 . AL),$$

whence the other forces can easily be found.

In general, questions connected with jointed systems of rods are best treated by graphical methods (see Art. 91), or by the method of Art. 104.

89. In certain cases the ordinary conditions of equilibrium are insufficient to determine all the forces called into play. Such is the case, for instance, when a rigid body rests on a plane surface, touching it at more than three points. If there are three points of contact, the pressure at each of these points is quite definite and can be found by applying the ordinary conditions of equilibrium. If there are more than three points of contact, the pressures at these points will depend on other chance circumstances and may vary between given limits. To illustrate this we may take the following problem—

A square table rests on four equal legs at the corners on a horizontal plane, and a weight is placed at a given point on it. Determine the pressures on the legs.



Let G be the centre of gravity of the weight and table together, and W their combined weight. Also let the position of G be given by its distances x, y from AC and AB.

Let P, Q, R, S denote the pressures at A, B, C, D.

It is clear at once that there is a certain amount of ambiguity. For, supposing G to lie on the A side of BC and the C side of AD, it is clear that the table would still rest if we cut away either the leg at D or the leg at B. That is, the forces might be P, Q, R, 0, *or* P, 0, R, S. However, applying the conditions of equilibrium, we notice that all the forces are parallel. Hence

$$P+Q+R+S=W.$$

Taking moments round AB,

$$(R+S)a=W\,.\,y,$$

and taking moments round AC,

$$(Q+S)a=W\,.\,x.$$

These are all the equations deducible from the conditions of equilibrium. That is, three equations with four unknown quantities to find. The question is then indefinite, and in fact the pressures in any actual case will either depend on accident (such as the slightest inequality in the legs), or can be made to take to a certain extent arbitrary values. For instance, a pressure may be applied to D by the finger so that the pressure S becomes zero. If the finger be taken away, S will remain zero and the equations to determine P, Q, R become

$$P+Q+R=W,$$
$$Ra=Wy,$$
$$Qa=Wx;$$

whence

$$\left.\begin{aligned} P&=W\left(1-\frac{x}{a}-\frac{y}{a}\right)\\ Q&=\frac{Wx}{a}\\ R&=\frac{Wy}{a}\\ S&=0 \end{aligned}\right\},$$

or we may arrange that S has a given value S′

when

$$R=\frac{Wy}{a}-S',$$

$$Q=\frac{Wx}{a}-S',$$

$$P=W\left(1-\frac{x}{a}-\frac{y}{a}\right)+S',$$

provided

$$S'<\frac{Wx}{a}<\frac{Wy}{a}.$$

If the legs are in every respect equal, the actual pressures would then be determined thus. The legs have a certain amount of elasticity, the amount of "give" of each being proportional to the pressure to which it is subjected. Now this being so, suppose the legs give slightly, and let α, β, γ, δ be the small displacements of A, B, C, D. Then there must be a relation between these if we suppose the table itself rigid. For in this case the displacement of the middle of the table will be either $\frac{\alpha+\delta}{2}$ or $\frac{\beta+\gamma}{2}$. Hence

$$\alpha+\delta=\beta+\gamma.$$

But α, β, γ, δ are proportional to P, Q, R, S. Hence

$$P+S=R+Q.$$

This gives us a fourth equation, which with the others enables us to determine P, Q, R, S definitely. Solving them we get

$$P=\frac{W}{2}\left(\frac{3}{2}-\frac{x+y}{a}\right),$$
$$Q=\frac{W}{2}\left(\frac{1}{2}+\frac{x-y}{a}\right),$$
$$R=\frac{W}{2}\left(\frac{1}{2}+\frac{y-x}{a}\right),$$
$$S=\frac{W}{2}\left(\frac{x+y}{a}-\frac{1}{2}\right).$$

Another example illustrating indeterminateness of solution will be found in § 133 (3).

90. Cases often arise in which elastic strings come into operation. In these the tension of the string depends on the amount it is stretched. The law connecting the two is simply that the tension of the string is proportional to its extension, the extension being measured by the ratio of the increase of length to the original length. Thus, if l, l' be the natural and the stretched length, $l'-l$ is the increment, and the extension is $(l'-l)/l$. The above law then states that

$$T=\lambda\frac{l'-l}{l},$$

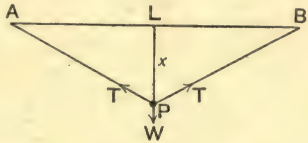
where λ is a constant, depending only on the material and section of the string. This law goes by the name of Hook's law. It is however not to be regarded as exact for large extensions. The law may be represented thus,

$$l'=l\left(1+\frac{T}{\lambda}\right).$$

If now $T=\lambda$, $l'=2l$. In other words, λ is the tension necessary to double the length of the string, provided the law held to such an extent.

This is only a particular case of elastic behaviour of matter. We do not consider this most interesting question further here, but must refer to special treatises—an example will, however, illustrate the application of Hook's law to questions involving elastic strings.

An elastic string without weight is joined to two points A, B *in a horizontal line so that* AB *is equal to the natural length of the string; a ring of weight* W *is then slipped on. Determine the position of equilibrium.*



The ring will pull down the string and will hang in a symmetrical position, as at P in the figure. Let $AB=2a$, $PL=x$. Further, let the tension of the string be T.

Then, since the ring is in equilibrium,

$$W=2T\cos APL,$$
$$=2T\cdot\frac{x}{AP}.$$

Moreover, by Hook's law,

$$T=\lambda\cdot\frac{AP-AL}{AL}=\lambda\cdot\frac{AP-a}{a}.$$

Hence $$W=\frac{2\lambda x}{a}\cdot\frac{AP-a}{AP},$$

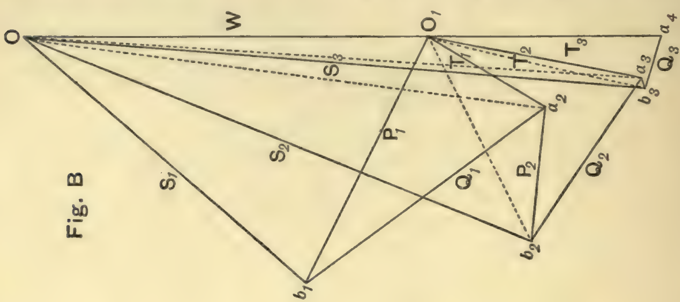
and $$AP^2=AL^2+PL^2,$$
$$=a^2+x^2;$$
$$\therefore\ \frac{W}{2\lambda}=x\left(\frac{1}{a}-\frac{1}{\sqrt{a^2+x^2}}\right),$$

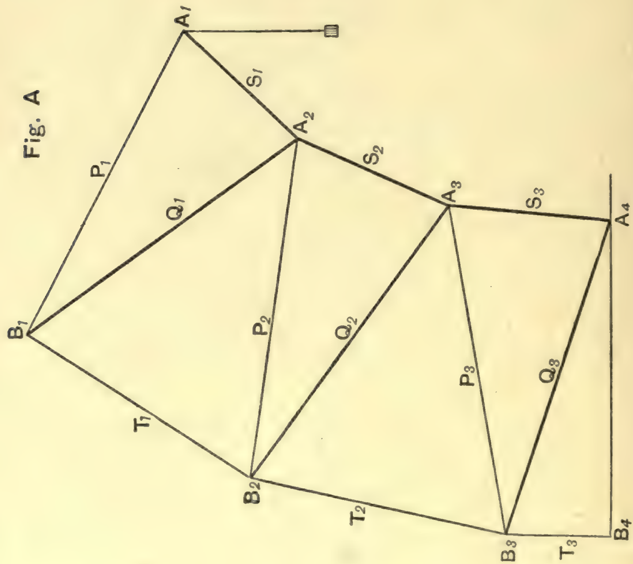
or, if we use trigonometrical expressions and call $APL=\theta$,

$$W=2T\cos\theta,$$
$$T=\lambda\frac{\frac{a}{\sin\theta}-a}{a}=\lambda\left(\frac{1}{\sin\theta}-1\right),$$
$$W=2\lambda\cos\theta\left(\frac{1}{\sin\theta}-1\right).$$

91. *Graphical methods.*—In many cases, especially in problems connected with the stresses arising in frameworks, the most expeditious way is to employ graphical methods. The following example illustrates the process. A careful

study of it and of the next article will enable the student to find the stresses called into play in any arrangement of





framework in which the ties and struts are so arranged as to make the forces definite. [See note in Appendix.]

EXAMPLE. *A bent crane, as represented in Fig. A, supports a mass of 30 tons. Find the tensions and stresses in the various parts, neglecting the weights of the bars.*

In the figure the thin lines represent ties (in which the stress is a tension) and the thick lines struts (in which the stress is a thrust).

The magnitudes of the parts are as follows—

$$A_4B_4 = 3', \quad B_3B_4 = 1',$$
$$A_1A_2 = A_2A_3 = A_3A_4 = 1'\ 6'',$$
$$B_1B_2 = B_2B_3 = 2'\ 6'',$$

and the cross pieces are all equal.

First draw a plan of the crane to scale.

Let the letters T, P, Q, S, etc. in the figure denote the stresses acting along the bars.

Now notice that A_1 is in equilibrium under the action of a known force W downwards and two unknown P_1, S_1 in two known directions. Hence P_1, S_1 can be found by the triangle of forces. So B_1 is in equilibrium under P_1 known, and T_1, Q_1 in known directions. Therefore T_1, Q_1 can be found. So also at A_2, S_1 and Q_1 are known, therefore their resultant. Hence P_2, S_2 can be found, and so on. We proceed then to find these by the "stress diagram."

Draw OO_1 (Fig. B) vertical to represent the weight W. Then (considering the point A_1) draw Ob_1 parallel to A_1A_2 and O_1b_1 to A_1B_1. Then, by the triangle of forces, Ob_1, O_1b_1 represent S_1 and P_1.

Again (considering B_1), draw b_1a_2 parallel to B_1A_2 and O_1a_2 to B_1B_2. Then these will represent Q_1, T_1.

Now at A_2 the forces S_1, Q_1 are represented by Ob_1, b_1a_2. Hence they have a resultant Oa_2, and A_2 is in equilibrium under Oa_2, P_2, S_2. Draw then Ob_2 parallel to A_2A_3 and a_2b_2 to A_2B_2. They will represent S_2, P_2.

At B_2 the resultant of T_1 and P_2 is O_1b_2. Draw O_1a_3 parallel to B_2B_3 and b_2a_3 to B_2A_3, they will represent T_2, Q_2, and Oa_3 will represent the resultant of S_2 and Q_2 $(Ob_2 + b_2a_3)$ at A_3, and so on. Whence, measuring the lengths of the different lines, we obtain the magnitudes of the various stresses. The figure B is reduced from a drawing in which 1 inch represented 10 tons. The measurements in that gave the following values—

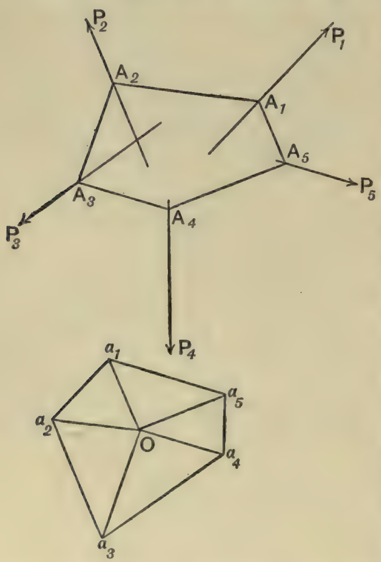
Tons.	Tons.	Tons.	Tons.
$S_1 = 28{\cdot}2$,	$T_1 = 13$,	$P_1 = 27$,	$Q_1 = 22{\cdot}5$,
$S_2 = 41$,	$T_2 = 16{\cdot}6$,	$P_2 = 10$,	$Q_2 = 14{\cdot}8$,
$S_3 = 47$,	$T_3 = 18$,	$P_3 = 1$,	$Q_3 = 4{\cdot}2$.

With a figure on a large scale and with care in drawing, great accuracy may be obtained.

If the weights of the bars are to be considered, we may regard the weight of any bar as distributed into two parts acting at the ends of a bar. This then reduces the problem to the case of a framework whose weight is neglected, but which is acted on by known forces at the various joints. To find the actual stress of any bar, the stress as determined by the above method must be combined with the weight of the bar and with two vertical upward forces at the ends, each equal to half its weight. Thus in the case here considered, if w_1, w_2, w_3 be the weights of the bars A_1A_2, B_1B_2, B_1A_1, etc. respectively, we shall have the following forces applied vertically at the various points—

at A_1	$\frac{1}{2}(w_1 + w_3)$	at A_2 and A_3	$w_1 + w_3$,
at B_1	$\frac{1}{2}(w_2 + w_3)$	at B_2, B_3	$w_2 + w_3$.

92. When a set of forces act at a point, we have learnt how to determine their resultant graphically by the polygon of forces. When the forces act in one plane on a rigid body, we know that the resultant reduces in general to a single resultant force acting along a definite line (in special cases to a couple). The *magnitude* and *direction* of this resultant force can be obtained by the polygon of forces, but as yet we have developed no method of finding graphically *the line of action*, or—which is the same thing, since the direction is known—one point in its line of action. This we now proceed to do.



Let P_1, . . . denote a system of forces in equilibrium when acting on a rigid body. Take any points A_1, . . . on them and join them so as to form a polygon A_1A_2 . . . Regard this polygon as a jointed system of bars. If they be in equilibrium the stresses in the bars can be found graphically as follows—

Take any point O. Draw Oa_1, Oa_2 parallel to A_5A_1, A_1A_2, and a_1a_2 parallel to P_1. Then, if we take a_1a_2 to represent P_1, the stresses along A_5A_1, A_1A_2 will be represented on the same scale by Oa_1, Oa_2. This follows from the triangle of forces, since A_1 is in equilibrium.

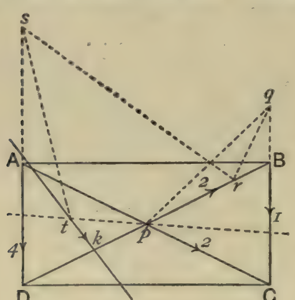
Next draw Oa_3 parallel to A_2A_3 and a_2a_3 parallel to P_2. Then again, since Oa_2 represents the stress in A_1A_2, Oa_3 will represent the stress in A_2A_3, and a_2a_3 will represent P_2. Proceed in the same way with all the sides of the polygon. Now notice that $a_1a_2a_3a_4$ . . . is the force polygon of P_1, . . . Since these are in equilibrium, the force polygon must be closed; hence the last side in the series a_1a_2 . . . will have one end falling on a_1. Call O the pole. We may then state our result thus: the second figure is such that the sides represent the applied forces, and the lines joining the pole to the angular points represent the stresses in the bars.

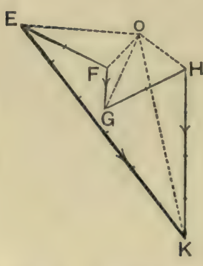
Conversely, take any pole O, and draw lines from it to the angular points of the force polygon a_1 . . . Take any point A_1 on P_1 and draw A_1A_2 parallel to Oa_2, A_2A_3 parallel to Oa_3, and so on, the last one cutting P_5 in A_5 (say). Then we shall get a *closed* polygon. For P_1 can be replaced by a_1O, Oa_2 along A_1A_5, A_1A_2; P_2 by a_2O, Oa_3 along A_2A_1, A_2A_3; P_3 by a_3O, Oa_4, etc., and so on. All of these cut one another out, with the exception of the first, a_1O along A_1A_5, and the last, Oa_1 along $A_5'A_1'$. Since P_1, . . . are in equilibrium, so must be also these two; and since they are parallel, they must be in the same straight line, or A_5' must coincide with A_5 and the polygon be closed.

Regarded from this point of view, the polygon $A_1 \ldots A_5$ is called a polar polygon or funicular with reference to the point O. It is clear that any number of polar polygons may be drawn to one pole, homologous sides being parallel.

The preceding theorem enables us now to obtain a graphical construction for the line of action of the resultant when the forces are not in equilibrium. For consider all the forces but one, say P_5. Then P_5 reversed is equal in all respects to the resultant of the others. The force polygon is unclosed, and the resultant is represented in magnitude and direction by a_1a_5.

Take now any pole O, draw Oa_1, Oa_2, and starting with say Oa_1, proceed as if to draw the polar polygon. That is, take any point A_1 on P_1, draw a line through it (A_5A_1) parallel to Oa_1. Through A_1 draw A_1A_2 parallel to Oa_2, through A_2 draw A_2A_3 parallel to Oa_3, and so on, the last one (drawn from A_4) will intersect the first drawn A_5A_1 in some point A_5. But we have just seen that this must lie on P_5, *i.e.* the line of action of the resultant. Hence A_5 is a point on the line of action. By choosing another position for A_1 and proceeding as before we get another point on the resultant. The two points then give its line of action. As, however, the magnitude and direction are known from the force polygon, it is sufficient only to determine one point.





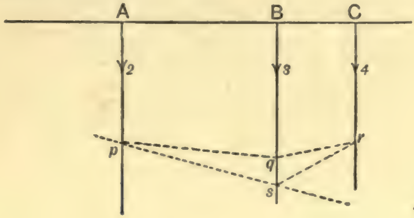
The following example will illustrate the method—

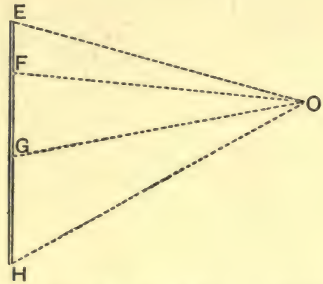
A rectangle ABCD *is acted on by the following forces,* 2 *lbs. weight along each of the diagonals* AC, DB; 1 *lb. weight along* BC; *and* 4 *lbs. weight along* AD. *Find their resultant.*

ABCD is the rectangle, EFGHK the force polygon. Hence the resultant is equal and parallel to EK (by measurement about 5·9 lbs. weight). Take any point O for pole and join OE, OF, etc.

Taking the intersection of diagonals for starting-point, *pqrst* is the polar polygon. The first (*pt*) and last (*st*) sides intersect in *t*. Hence the resultant passes through *t* and is equal and parallel to EK—is, in fact, 5·9 lbs. weight along *tk*.

When the applied forces are weights, they are all parallel. In this case the force polygon becomes a vertical straight line. As an example, take weights of 2, 3, 4 lbs. at distances of 2 and 1 feet. Find their resultant.





The force polygon is the vertical line EFGH. Therefore the resultant is EH, which closes the polygon, $=2+3+4=9$. O is a pole. Starting from p, $pqrs$ is the polar polygon, s being the intersection of the first and last side. The resultant therefore passes through s. As is seen, it falls on the force 3, as it should, since the resultant of 2 and 4 passes along the 3.

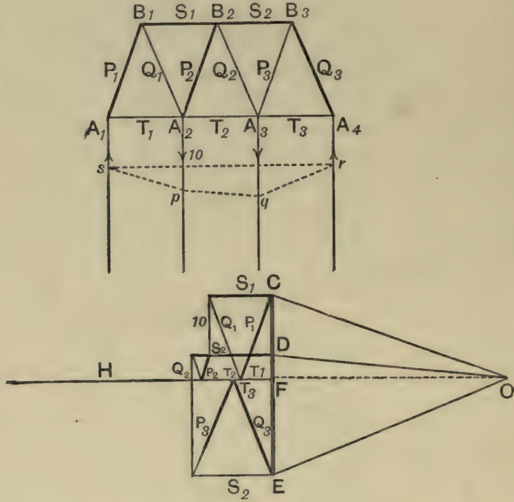
We give another example to illustrate the treatment of loads applied to the various points of a framework. For more complete information the student is referred to special treatises on the subject.

Take the case of a simple Warren girder as drawn in the figure, A_1, A_4 resting on supports and sustaining loads of 10 and 20 tons at A_2, A_3.

First, to find graphically the reactions at the ends.

Draw CDE vertically, CD to represent 10 tons and DE to represent

20. The reactions together are EC, as they form a system in equilibrium with the 10 and 20. Take any pole O, join OC, OD, OE. Starting from any point p on the force of 10 tons weight, draw $spqr$ with sides parallel to OC, OD, OE. Join rs. Then $pqrs$ must form the complete



polar polygon, and if OF be drawn parallel to rs, the lines EF, FC will complete the force polygon and will represent the reactions at the ends A_4 and A_1.

To find the stresses in the framework, describe the stress diagram as follows—

Start from A_1. Draw FH horizontal, and through C draw a line parallel to A_1B_1. The sides are the reactions at A_1, P_1, and T_1. The stress diagram is drawn in the figure, which the student should carefully study and reconstruct on a larger scale, so as to get numerical results of fair accuracy.

EXAMPLES—X.

1. Two equal uniform beams, connected at a common extremity by a smooth joint, are placed in a vertical plane, their other extremities, which rest on a smooth horizontal plane, being connected by a light rope. Find the tension of the rope and the reaction at the joint.

2. The transverse section of the timber roof of a church is given by the two equal sides of an isosceles triangle whose vertical angle is 2α where $\tan\alpha = \sqrt{12}$. Show that the total thrust outwards on each wall is W, and that it makes an angle of 30° with the vertical, where W is the weight of the roof.

3. A sphere of given weight rests between two smooth planes, one vertical and the other inclined at a given angle to the vertical. Find the pressures on the planes.

4. A vertical cylindrical cup, radius $2a$ height $3a$, rests upon a horizontal table; a rod is placed within it with its lower end at the circumference of the base; the rod rests upon the opposite point of the upper rim and projects over. If the weight of the rod be equal to that of the cylinder, how long must the rod be that it may just cause the cylinder to topple over?

5. Prove that if a pair of compasses is resting across a smooth horizontal cylinder of radius c, the frictional couple at the joint preventing the legs of the compasses from opening is

$$W\left(c\,\frac{\cos\alpha}{\sin^2\alpha} - a\sin\alpha\right),$$

where W is the weight of one leg, 2α the angle between the legs, and a the distance of the centre of gravity of one leg from the joint.

6. A uniform rod 4 inches in length is placed with one end inside a smooth hemispherical bowl of which the axis is vertical and the radius $\sqrt{3}$ inches long. Show that one-fourth of the rod will project over the rim of the bowl.

7. A uniform rod has its lower end fixed to a hinge and its other end attached to a string which is tied to a point vertically above the hinge. Show that the direction of the action at the hinge bisects the string.

8. A uniform beam rests with one end against the junction of the horizontal ground and a vertical wall; it is supported by a string fastened to the other end of the beam and to a staple in the vertical wall. Find the tension of the string and show that it will be half the weight of the beam if the length of the string be equal to the height of the staple above the ground.

9. A uniform rod of given length is to be supported in a given inclined position with its upper end resting against a smooth vertical wall by means of a string attached to the lower end of the rod and to a point of the wall. Find by a geometrical construction the point of the wall to which the string must be attached.

10. A circular disc BCD of radius a and weight W is supported by a smooth band of inappreciable weight and thickness, which surrounds the disc along the arc BCD and is fastened at its extremities to the

point A in a vertical wall, the portion AD touching the wall and the plane of the disc being at right angles to the wall. If the length of AD be b, prove that the tension of the string is $W(a^2+b^2)/2b^2$ and find the pressure at D.

11. A uniform rod of weight W is suspended horizontally from two nails in a wall by two vertical strings, each of length l, attached to its ends; a smooth weightless wedge of vertical angle 30° is pressed down with a vertical force W/2 between the wall and rod (so as not to touch the strings), its lower edge being kept horizontal, and one face touching the wall. Find the distance through which the rod is thrust from the wall.

12. A thin board in the form of an equilateral triangle and weighing 1 lb. has one-quarter of its base resting on the end of a horizontal table, and is kept from falling over by a string attached to its vertex and to a point on the table in the same vertical plane as the triangle. If the length of the string be double the height of the vertex of the triangle above the base, find the limits between which its tension must lie.

13. A square figure ABCD is formed by four equal rods jointed together, and the system is suspended from the joint A and kept in the form of a square by a string connecting A and C. Prove that the tension of the string is half the weight of the four rods and find the direction and magnitude of the action at either of the joints B or D.

14. Four weightless rods are freely jointed together forming a quadrilateral ABCD. If two equal and opposite forces are applied at two points, one on the rod AB, one on the opposite rod CD, determine the conditions that they shall balance one another.

If instead of forces two equal and opposite *couples* be applied to the rods AB, CD, what is necessary for equilibrium?

15. Three equal strings of no sensible weight are knotted together to form the equilateral triangle ABC, and a weight W is suspended from A. If the triangle and weight be supported, with BC horizontal, by means of two strings at B and C, each at the angle of 135° to BC, prove that the tension in BC is $\frac{W}{6}(3-\sqrt{3})$.

16. One of six equal heavy rods jointed at their extremities is held horizontal; the hexagon is kept regular by a horizontal rod connecting opposite corners. What is its length and the strain along it? If instead the hexagon be kept regular by a vertical string connecting the middle points of the horizontal sides, what would be the length and tension of the string?

17. Determine the tensions of the threads of a rectangular piece of network, hung from a horizontal bar, due to suspending a series of

equal weights in a horizontal line at the lowest points of the net, supposing the meshes are equal regular hexagons of which a pair of sides are vertical.

18. Four heavy rods, equal in all respects, are freely jointed together at their extremities so as to form the rhombus ABCD. If this rhombus be suspended by two strings attached to the middle points of AB and AD, each string being inclined at the angle θ to the vertical, prove that in the position of equilibrium the angles of the rhombus will be 2θ and $\pi - 2\theta$.

19. A rhombus of freely jointed rods (lengths a) is hung up by two equal strings of length l which are attached to the middle points of two adjacent sides, their other ends being fastened to two points in a horizontal line at a distance c apart. Show that the angles of the rhombus are 2θ, $\pi - 2\theta$ where $\sin\theta = c/(a+2l)$.

20. A uniform smooth plate in the form of a right-angled isosceles triangle, whose hypotenuse is c, rests with its plane vertical and its two equal edges in contact with two smooth pegs in the same horizontal line and at a distance $c/6\cos\theta$ apart. Prove that the plate can rest in a position where the hypotenuse will make an angle θ with the horizon.

21. A uniform beam AB of given length and weight has its extremity A resting in a horizontal groove AC and its extremity B in a vertical groove BC, and is kept in equilibrium by a string DC fixed at a given point D on the beam. Find the tension of the string, and the limits, as to the length and point of attachment of the string, under which equilibrium is possible.

22. Two equal hemispherical bowls are placed on a horizontal plane and are kept with their rims everywhere in contact in a vertical plane by a string which passes over them, and hangs vertically supporting two equal masses w. If W be the weight of either hemisphere, prove that if the bowls are just not falling apart, $w = \frac{1}{2}$W. Find the value of w also in case the bodies are solid hemispheres.

23. Two equal hemispherical bowls are placed on a horizontal plane, and are kept with their rims everywhere in contact in a vertical plane by a string connecting two equal weights which rest on the bowls. Find the length of the string and a limit to the magnitude of the weight.

24. Three equal cylinders, each of weight W, are bound together with their axes parallel by a string whose tension is T; they are then placed so that two rest on a horizontal plane. Find the pressures between the cylinders.

25. Two cylinders, each of weight W, are placed in contact on a smooth horizontal plane, and a third equal cylinder is placed upon

them; a rod of weight w bent into the two sides of an isosceles triangle of vertical angle 2α is then placed upon them. Prove that it is lifted if $w < \mathrm{W} \tan \alpha/\sqrt{3}$ when $\alpha < 30°$, or if $w < \mathrm{W} \tan^2 \alpha$ when $\alpha > 30°$.

26. Three spheres, each of weight W, are placed in contact on a smooth horizontal table, and a fourth equal sphere is placed upon them, and then a cone of semivertical angle α is placed over the pile of spheres. Prove that the cone will be lifted if its weight is less than $\mathrm{W} \tan \alpha/\sqrt{2}$ when $\tan \alpha < 1/\sqrt{2}$, but when $\tan \alpha > 1/\sqrt{2}$ its weight must be $< \mathrm{W} \tan^2\alpha$.

27. Three equal heavy spheres hang in contact from a fixed point by strings of equal length. Find the weight of a sphere of given radius which, when placed upon the other three, will just cause them to separate.

28. A uniform regular tetrahedron has three corners in contact with the interior of a fixed smooth hemispherical bowl of such magnitude that the completed sphere would circumscribe the tetrahedron. Prove that every position is one of equilibrium. If P, Q, R be the pressures on the bowl and W the weight of the tetrahedron, prove that

$$3(\mathrm{P}^2 + \mathrm{Q}^2 + \mathrm{R}^2) - 2(\mathrm{QR} + \mathrm{RP} + \mathrm{PQ}) = 3\mathrm{W}^2.$$

29. A hollow cylinder composed of thin material open at both ends, of radius a and height $4a$, is placed on a smooth horizontal plane; inside it are placed two smooth spheres of radius r one above the other, $2r$ being $> a$ and $< 2a$. If W be the weight of the cylinder and w the weight of one of the spheres, show that the cylinder will just stand upright without toppling over if $\frac{\mathrm{W}}{2w} = \frac{a-r}{a}$.

The following examples are to be solved by graphical methods. Where dimensions are not given the student must draw a figure, and then work to scale to this figure.

29*a*. Solve Ex. 1, 3, 4 above by graphical constructions.

29*b*. A heavy beam, resting on two smooth inclined planes, is in equilibrium in a given position. Determine its centre of gravity.

29*c*. The rod AB, whose centre of gravity is at G, is supported in a given position, with one end A in contact with a smooth vertical wall AD, by a string attached to the given point C of the rod, and also to a point D in the wall. Determine the position of the point D and the tension of the string, the weight of the rod being 60 oz.

29*d*. A rod ACB, weighing 25 oz., rests upon a smooth peg C, and its end A is attached to a fixed point O, in the same horizontal line with C, by means of a string OA. Find the position of the centre of gravity of the rod, the tension of the string, and the pressure between the rod and peg.

29*e*. A, D are fixed points; AB, BC, CD are tight strings supporting two weights resting in the given position. If the weight at B is one pound, find to two places of decimals the weight at C and the tensions of the three strings.

29*f*. ADB, BEC, DE are three rods arranged as a gibbet (AB vertical). A weight of 150 lbs. is suspended from C. Find the reactions at B and E, supposing the rods have no weight.

29*g*. Two equal weightless rods, AB and AC, are hinged together at A, and placed in a vertical plane with B and C on a smooth horizontal plane. If B and C be connected by a string, and a weight of 30 lbs. be suspended from a point D in AC, find the tension of the string BC.

30. Four equal rods are jointed to form a square, and are placed with one diagonal vertical; the other two angles are joined by a string; a mass of 2 lbs. is placed on the top. Find the stresses in the bars and strings (1) when the weights of the bars are neglected, (2) when each weighs 1 lb.

31. A regular hexagon of equal jointed rods has each pair of opposite angles joined by a string; it is placed on a horizontal plane with one of these vertical; a weight W is placed on the top. Find the tension of the strings and the stresses in the rods (without weight).

32. Show that if in the preceding question the rods be supposed to have weight, the system will collapse. Point out how this takes place.

33. The same hexagon is held with the one side (top) horizontal; two opposite angles are joined by a horizontal strut, and a weight 2W is placed on the middle of the lower. Find the stresses when the weight of each rod is W, and of the horizontal strut 2W.

34. Find all the stresses in the derrick crane (Fig. i.), supposing the framework without weight. Also, supposing the weights proportional to the lengths and that the upright weighs 28 lbs.

35. Find the stresses in the bent crane in § 91, supposing the weights of the connecting pieces all equal to 2 cwts.

36. Find all the stresses in the roofs (Figs. ii. iii. iv.)

37. Find all the stresses in the bridge (Fig. v.) and the cantilever (Fig. vi.)

38. Forces 3, 4, 5 act along the sides AB, BC, CD of a square. Find the magnitude and position of their resultant.

39. Forces 1, 2, 3 act along the sides of an equilateral triangle in order. Find completely the resultant.

40. A rectangle ABCD, with AB 4 ft. and BC $2\frac{1}{2}$ ft. long, is acted on by 5 lbs. in AB, 3 lbs. in BC, 4 lbs. in CA, and 6 lbs. in BD. Find the resultant completely.

41. Three like parallel forces of 5, 7, 9 lbs. act in lines, whose distances apart are 3 ft. and 4 ft. Find their resultant.

42. Like parallel forces 1, 2, 3 act at the angles of an equilateral triangle. Determine the line of action of their resultant.

The foregoing will serve as typical examples for graphical treatment. The student is advised to obtain facility in the method by setting himself similar examples to work out.

The following figures are merely diagrammatic; they must be drawn to scale first by the student.

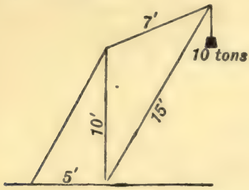


FIG. i.

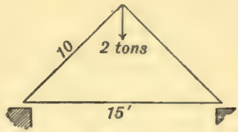


FIG. ii.

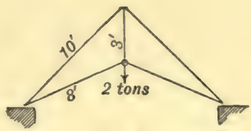


FIG. iii.

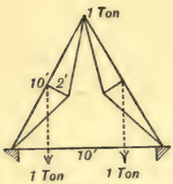


FIG. iv.

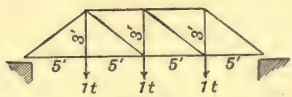


FIG. v.

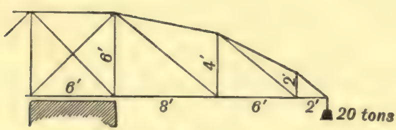


FIG. vi.

CHAPTER XI

ENERGY

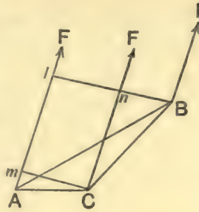
93. THE conception of work and energy has been introduced in Chapter IV, and has been there applied to cases of motion along a straight line in the direction of the force. It is, however, only when the conception is applied to the more general kinds of motion of matter that its importance becomes manifest. In the present chapter we propose to consider the subject under this extended application.

It may be well at the outset to recall the definition of work given in § 25, viz. *when a force acting at a fixed point of a body moves its point of application it is said to do work. The work is measured by the product of the measure of the force and the measure of the displacement estimated parallel to the force.*

This definition as it stands can clearly only apply to a uniform force. If, however, we first learn what we can about this, the extension to non-uniform forces will be easy.

If the particle on which a force acts be moved from one position A to another B, it can be supposed done in an infinite number of ways—either in a straight line from A to B, or a broken path, or a curved path. The question arises, does the work done depend on the path? We proceed to show that, whatever the path between the points, the work done is the same.

Let the force F act at A and its point of application be moved to B, either in one operation or in the two successive ones A to C and C to B. Draw Bl, Cm perpendicular to the force AF. Then in the figure CF being parallel to AF, Bn is perpendicular to CF, and Cn = lm.

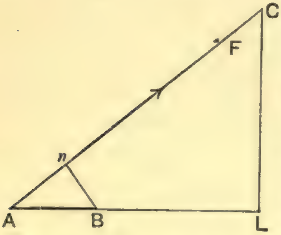


Now in the first case the work done is F × Al.

In the second case the work done in the two steps

= F × Am + F × Cn = F × Am + F × lm
= F × Al,

whence they are the same in both cases; and so in general for any series of steps AC, CD, . . . Also, since a curve can be considered as the limit of a large number of very small lines, the theorem is seen to be true also for a curved path. It is to be noted that the theorem has been proved for a uniform force only.



94. The work done by a force may also be defined as the product of the whole displacement into the resolved part of the force along the displacement, the displacement being taken positive when in the same direction as the resolved force.

For let AC denote the force, AB the displacement. Draw Bn perpendicular to AC. Then

$$\text{Work done} = \text{AC} \times \text{A}n.$$

Draw CL perpendicular to AB. Then AL is the component of the force along the displacement. Also the triangles ABn, ACL are equiangular to one another, and hence are similar, *i.e.*

$$\frac{\text{AB}}{\text{A}n} = \frac{\text{AC}}{\text{AL}},$$

$$\therefore \text{AL} \times \text{AB} = \text{AC} \times \text{A}n,$$

or resolved part of force along AB × displacement
= work done

95. We can now prove a most important proposition, viz. the work done by any system of forces acting on a particle is equal to that done by their resultant. For let AB denote the displacement. Resolve each force into two components, one along AB, the other perpendicular to it, and let X_1, X_2, . . . denote their values along AB. Then the works done by the forces are, by the previous theorem, $AB \times X_1$, $AB \times X_2$, . . .

$$\begin{aligned} \text{Hence their sum} &= AB(X_1 + X_2 + \ldots), \\ &= AB \times \text{resolved part of resultant along } AB, \\ &= \text{work done by the resultant.} \end{aligned}$$

96. It follows at once from the foregoing theorem that if *a particle* be in equilibrium under the action of any forces, and if it receive any small displacement, the work done by the forces must on the whole be zero, and *vice versa*, if the work done by the forces for *any* possible small displacement be zero, the particle will be in equilibrium. For in any case, if R denote the resultant and x any displacement estimated parallel to the resultant, the work done by the forces is Rx, and since x is not zero, if the work is zero it must be because $R = 0$, *i.e.* the particle must be at rest.

The reason for stating the above for small displacements is that for large displacements the forces will in general alter during the displacement, and the result would not then be true.

To illustrate the above, consider the case of a heavy particle resting on an inclined plane, and acted on by a force up the plane.

Let F denote the force up the plane, R the reaction, and W the weight.

(1) To find F, give the particle a displacement along the plane to D

Then Work done by $R = R \times 0 = 0$,

,, ,, $F = F \cdot DD'$,

,, ,, $W = -W \cdot D'n$.

Hence $F \cdot DD' - W \cdot D'n = 0$,

or $$F = W \cdot \frac{D'n}{DD'} = W \cdot \frac{BC}{AB} = W \cdot \frac{\text{height}}{\text{length}}.$$

(2) To find R, give the particle a displacement DD′ (Fig. 2) perpendicular to the plane. Then

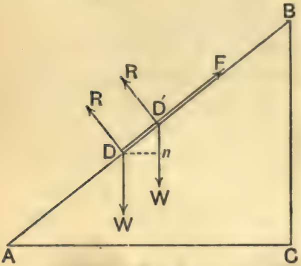


FIG. 1.

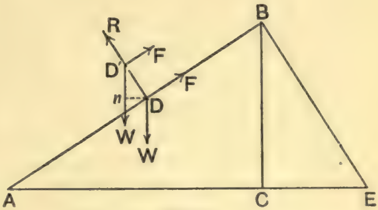


FIG. 2.

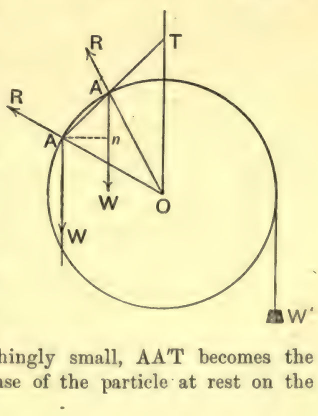
$$\text{Work by } R = R \, . \, DD',$$
$$\;\;,, \quad F = F \times 0,$$
$$\;\;,, \quad W = -W \, . \, D'n.$$

Hence $$R \, . \, DD' - W \, . \, D'n = 0,$$

or $$R = W\frac{D'n}{DD'} = W \, . \, \frac{BC}{BE} = W \, . \, \frac{AC}{AB}.$$

In this example the restriction as to smallness of displacement is not necessary. It is, however, necessary in the following.

A particle W *rests on a smooth cylinder and is kept in equilibrium by a string fastened to another particle which passes over the cylinder and hangs freely. Determine the position of equilibrium.*



Let the position of rest be at A, give the particle a small displacement along the cylinder to A′. Then, since the string is inextensible, W′ will move through a space AA′, and the whole work is

$$-W \, . \, A'n + W' \, . \, AA' = 0,$$

or $$\frac{W'}{W} = \frac{A'n}{AA'} = \cos AA'n = \cos ATO.$$

Now this equation is true, whatever the magnitude of AA′ may be, if the particle rests on the chord AA′. If, however, AA′ be vanishingly small, AA′T becomes the tangent at A and we have the case of the particle at rest on the cylinder.

Hence, if the tangent at A make an angle θ with the vertical,

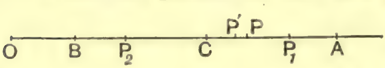
$$\cos\theta = \frac{W'}{W}.$$

97. *Work by non-uniform force.*—In general, as a particle changes its position the force acting on it changes continuously—that is, the smaller the change in position the smaller the alteration in the force. If a particle changes its position in any way, we may consider it as made up of a series of smaller changes, and the whole work done will be the sum of the works done in these smaller parts. Now if during any small displacement we regard the force as constant and equal to its value at the beginning of the displacement, the work done is measured by the definition already given, and the whole work can be determined by adding the various parts together. But the actual work done will not be exactly this, but will differ from it by a small quantity depending on the size of the parts into which the whole displacement is broken up. This difference will be smaller the smaller (and therefore the more numerous) the component parts are taken to be, and can be made as small as we please by dividing the whole displacement into a sufficiently large number of component parts.*

The foregoing considerations will show how the amount of the work done in any displacement in which the force changes is to be measured. The following example will illustrate it, and is also of importance in itself.

A particle moves in a straight line under the action of a force directed to a point in the line and proportional to the distance of the particle from the point. Find the work done while moving from one position to another.

When the particle is at P let the force be denoted by $k \cdot OP$. Let P′ be a point near P. Then in moving from P to



P′ the force will change from $k \cdot OP$ to $k \cdot OP'$, and the work done will lie between $k \cdot OP \cdot PP'$ and $k \cdot OP' \cdot PP'$.

* The argument supposes that the force is nowhere infinitely large.

If, therefore, we take the work to be $k \cdot OP \cdot PP'$, the error will be less than the difference of the two—that is, $k \cdot (PP')^2$.

Now suppose we take n *equal* divisions of AB, the error in taking the force through each division the same as at its beginning will be less than the sum of the $k \cdot (PP')^2$ or $n \cdot k \cdot (PP')^2$.

But $n \cdot PP' = AB$, hence the error will be less than $k \cdot AB \cdot PP'$. If then n be made indefinitely large or PP′ indefinitely small, the error may be made vanishingly small, and our result therefore correct. We will now proceed to find the sum of the component works by a method analogous to that by which the space in any time under constant acceleration was found in § 4.

Let C be the point midway between A and B; P_1, P_2 points at equal distances on either side of C.

Then the forces at P_1 and P_2 are

$$k \cdot OP_1 = k \cdot (OC + CP_1),$$
$$k \cdot OP_2 = k \cdot (OC - CP_2),$$

or the force at P_1 is just as much greater than that at C as that at P_2 is less. Hence the work during any small displacement at P_1 is just as much greater than that through an equal displacement at C as that at P_2 is less. The work of both is therefore the same as if both forces were equal to that at C—that is, $k \cdot OC$. And similarly for every pair of points. The work done is therefore the same as if the force were $k \cdot OC$ throughout. But in this case the work would be $k \cdot OC \times AB$. Let $OA = r_1$, $OB = r_2$.

Then $$OC = \frac{r_1 + r_2}{2}, \quad AB = r_1 - r_2.$$

Hence $$\text{Work done} = k\frac{r_1 + r_2}{2} \times (r_1 - r_2) = \tfrac{1}{2}k(r_1^2 - r_2^2).$$

This theorem enables us to find the work done in stretching an elastic string.

98. It is clear that, since the theorems proved in §§ 94-6 are true for small displacements as well as large, they will also hold universally; since, if the forces vary during any

displacement, they may be considered as built up of smaller parts, throughout which the forces may be regarded as constant. [It will not, however, follow in general that the work done is independent of the path (see § 93)]. Suppose a particle acted on by any forces to move from a position A to another B. In general the velocity will be different at the two points, and the kinetic energy of the particle will be altered. Now, however the particle is moving, we can treat the components of its velocity and acceleration in two given directions independently of one another, and consider each as due to the corresponding components of the forces. In other words, the component motions will be the same as those of two similar particles moving, one under the one set of component forces and the other under the other set. But if a particle move along a straight line under a constant force, it was shown in § 25 that the change in kinetic energy was equal to the work done by the force. The considerations above show that this is still the case even when the force is variable. Hence if u_1, v_1 are the components of the velocity at A, and u_2, v_2 the components at B,

$$\tfrac{1}{2}m(u_2^2 - u_1^2) = \text{work done by components of the forces in the direction of } u,$$

$$\tfrac{1}{2}m(v_2^2 - v_1^2) = \text{work done by components of the forces in the direction of } v.$$

Hence $\tfrac{1}{2}m(u_2^2 - u_1^2) + \tfrac{1}{2}m(v_2^2 - v_1^2) =$ work done by the forces between A and B.

But if V_1, V_2 are the actual velocities of the particle at A and B,

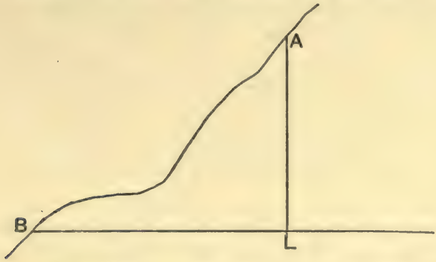
$$V_1^2 = u_1^2 + v_1^2, \quad V_2^2 = u_2^2 + v_2^2,$$

and $\tfrac{1}{2}mV_2^2 - \tfrac{1}{2}mV_1^2 =$ work done by forces,

or, change of kinetic energy = work done by forces.

99. The theorem of the last article is of extreme importance. We will illustrate its use by applying it to determine the velocity of a particle when it slides down a smooth curve under the action of gravity. The only forces are the weight of the particle and the reaction of the curve, which, since the curve is smooth, is perpendicular to

it. At every point, therefore, the particle is moving at right angles to this reaction, and consequently the reaction does no work. Hence the change in kinetic energy is



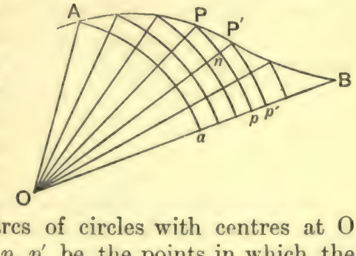
equal to the work done by gravity. But the weight is a uniform force, and the work done is consequently (§ 93) independent of the path. If, therefore, h denote the vertical height of A above B, and u be the velocity at A, the velocity v at B is given by

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = mg \, . \, \mathrm{AL} = mgh,$$

or

$$v^2 = u^2 + 2gh.$$

100. In the foregoing example the change of energy and velocity is independent of the path between A and B. This is not necessarily the case for any system of forces. But it is so in the case of all forces in nature. It is easy to prove that this must be the case for a system where the force is always directed to a fixed point, and depends only on the distance from this point. Let O denote the fixed point and AB any path along which the particle moves. Divide the path into a large number of small parts of which PP′ is one. Join each point of division to O by straight lines OP, OP′, etc.,



and through each draw arcs of circles with centres at O and cutting OB, and let p, p' be the points in which the circles through P, P′ cut OB. Then we have seen that since

PP′ is small, we may treat the force at all points of PP′ as constant, and therefore the work done along PP′ is equal to that along Pn, nP′.

But since the force is everywhere directed to O, it is everywhere perpendicular to Pn. Hence the work along Pn is zero.

Also, since the magnitude of the force depends only on the distance from O, the force at any part of P′n is the same as at the corresponding points of pp' (equal to P′n). Hence the work done through P′n is the same as would be done along $p'p$.

Therefore the work along PP′ = work along pp', and so for every corresponding portion. Hence the work done along the path AB is the same as that done along aB. So for any other path terminating at A, B the work is the same. We have, therefore, the following extremely important theorem—

If a particle move under the action of a force which is (1) *everywhere directed to a fixed point, and* (2) *depends only on the distance from that point, the work done between any two points, and therefore by* § 98 *the change of kinetic energy, is independent of the path between the points.*

Such a system of forces, where the work done between two points depends only on the position of the points, and not on the path by which a particle passes from one to the other, is called a *conservative system of forces.*

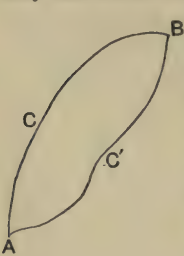
Gravitation, or the attraction between bodies, forms a conservative system. For it consists of a very large number of systems of forces directed each to a fixed point, and inversely proportional to the square of the distance from the point. Each system is therefore, as we have seen, a conservative one. Consequently the whole system is conservative.

101. If a particle move under a conservative system of forces, it is clear that if its kinetic energy be given in one position, its kinetic energy in any other position is definite and depends only on this position. Further, it is clear that, the particle being in any position, it is possible to let it have some other kinetic energy by allowing it to move

to another position. In other words, when a particle is at any point, it possesses a certain amount of kinetic energy, and is in the position to gain more. The energy that it can gain by moving from any point P to some fixed point A is called its potential energy at P. From the definition, the kinetic energy at A = kinetic energy at P + potential energy at P. Hence in all cases the whole energy remains constant. That is, in any conservative system of forces acting by themselves the sum of the kinetic and potential energies is constant. Now, as we shall see presently, all the systems of forces in nature are conservative. Hence in all physical processes the total energy remains constant. *This theorem is known as the principle of the conservation of energy.*

102. All systems of forces reducible to sets of systems which act towards points and depend only on the distances from these points are conservative forces. The same is the case for forces between particles, directed in the lines joining the particles and depending only on the distances of the particles, provided the forces between any two particles are equal and opposite. For in any small interval of time let two particles move through small distances whose components along the line joining the particles are d_1, d_2, and let the equal and opposite force between them be F. Then the work done is $Fd_1 - Fd_2 = F(d_1 - d_2)$, that is the force × relative displacement. It is therefore the same as if one had been at rest and the other moving with its motion relatively to it. But then this reduces to the case of a force to a *fixed* point, and the work depends only on the distances of the particles. This therefore forms a conservative system. Now it is probable that all natural forces are reducible to systems of forces acting between particles and depending only on their distances. If this be so, it follows at once that all forces in nature are conservative. It is impossible to devise experiments to determine the question of the nature of the forces directly in all cases. It is, however, a matter of experience that it is impossible to get work indefinitely out of any arrangement of bodies without putting energy con-

tinually into it. In other words, "perpetual motion" is impossible. So far as mechanical forces transmitted by machines are concerned, this is absolutely the case, as the foregoing principles will show (see Chapter XII). Until, however, we actually see the nature of the forces called into play by heat, electricity, or chemical action, it is impossible to prove this by demonstration. We can only say that in no known case has it been found possible to draw energy indefinitely out of bodies. From this statement it may also be deduced that all physical forces are conservative.



For suppose a particle to move from A to B along any path ACB, and to gain a certain amount of energy e, and suppose that if it moved along another path AC′B it would gain a different amount of energy e'. Then, if it had been moved from B to A along BC′A a quantity of work e' would have been lost. If then the particle were moved along ACBC′A, it would (if $e > e'$) gain energy $= e - e'$. But it would then be in the same position as before, and the same operation could be repeated over and over again, each operation yielding a quantity of work measured by $e - e'$. Thus perpetual motion would be possible. As it is not, e' must be equal to e, *or* the work between A and B must be independent of the path.

103. Other things than bodies in visible motion can be made to do work, and so must possess energy. Thus heat, light, a system of electrified bodies, the chemical action of one substance on another such as sulphuric acid on zinc, can be made to do work. In general then we need a wider definition of energy than that given above. *Anything that can be made to do work is said to possess energy, and the quantity of energy is measured by the quantity of work (foot-pounds, ergs, etc.) that can be got out of it.* Thus, for instance, heat can be transformed into work, and is therefore a form of energy. It is not the place here to go more fully into this question, which involves a knowledge of the properties of heat, electricity, chemistry, etc. But, as an illustration, we may

state that experiments by Joule and others have shown that a quantity of heat necessary to raise 1 lb. of water from 0° F. to 1° F. is capable of producing 779 foot-pounds of work.*

104. A rigid body may be regarded as a system of particles at invariable distances from one another and kept in equilibrium by forces between them. If now such a body be displaced into any other position, the work done by these internal forces will be zero. For the displacements of any two particles along the line joining them are equal, while the forces on them are equal and opposite. Hence the work done by one is equal to the work done on the other—that is, on the whole, no work is done by them. Now suppose any system of forces to act on a rigid body. In general the body will be made to move, and its kinetic energy will be altered. This more general case will be considered in Chapter XIX. Here it will be sufficient to consider the case where the body is kept at rest. In this case give the body any small displacement, either of translation or rotation. Each particle gets a small displacement, and since it is in equilibrium, the whole work done on it is zero. But of this the work done by the internal reactions is, as we have seen, also zero. Therefore the work done by the applied forces amongst themselves vanishes. In general these applied forces consist of two kinds: (1) forces applied to definite points of the surface of the body, and (2) forces, like gravity, which act on every particle of it. For practical applications, therefore, it will be necessary to know how to determine the work done by the latter kind of forces.

105. *The work done by gravity on a rigid body moving in any way from one position to another is the same as if the whole body were supposed concentrated at its centre of gravity and treated as a particle.*

For consider the body as made up of particles m_1, m_2, . . . in positions whose heights above a fixed horizontal plane are a_1, a_2, . . . In consequence of the displacement

* Heat to raise 1 lb. of water from 0° C. to 1° C. = 1402 foot-pounds.
,, ,, 1 kilogram ,, ,, = 427 kilogrammeters.

of the body, suppose m_1 to be raised through a height x_1, m_2 through x_2, and so on.

$$\text{The work done} = m_1gx_1 + m_2gx_2 + \ . \ . \ .,$$
$$= g(m_1x_1 + m_2x_2 + \ . \ . \ .).$$

Further, let $\bar{a}$ denote the height of the centre of gravity initially and $\bar{a} + \bar{x}$ the height after displacement. Then

$$(m_1 + m_2 + \ . \ . \ .)\bar{a} = m_1a_1 + \ . \ . \ .,$$
$$(m_1 + m_2 + \ . \ . \ .)(\bar{a} + \bar{x}) = m_1(a_1 + x_1) + \ . \ . \ . \ ;$$
$$\therefore (m_1 + m_2 + \ . \ . \ .)\bar{x} = m_1x_1 + m_2x_2 + \ . \ . \ . \ .$$

Hence $\qquad$ The work done $= g(m_1 + m_2 + \ . \ . \ .)\bar{x}$;

or, if W be the weight of the body,

$$W = m_1g + m_2g + \ . \ . \ .,$$

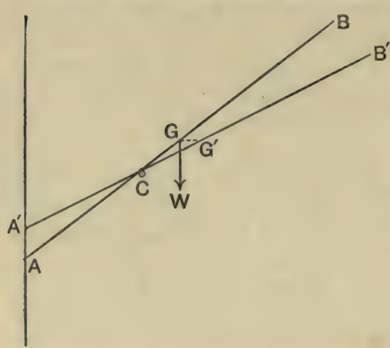
and $\qquad$ Work done $= W\bar{x}$,

which proves the proposition.

Note.—The condition that the body should be rigid has not been introduced in this proof. Hence the theorem is true for a system of bodies as well as for a rigid body.

106. The following examples will serve to illustrate the application of the foregoing principles to the equilibrium of bodies; others will be given in the chapter dealing with machines.

The first is the same as has been treated already in § 87 by a different method, viz. *a horizontal bar is fixed parallel to a smooth wall; a uniform heavy rod is laid across it at right angles with one end pressing against the wall, so as to rest in equilibrium. Determine the position of equilibrium.*

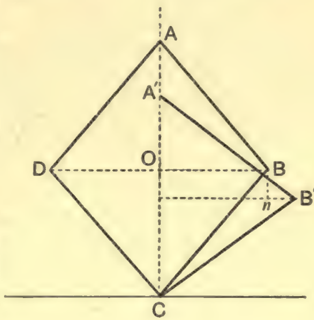


Give the rod a small displacement by pushing the end A up in contact with the wall. Then the whole work done is zero. But the work done by the pressure on the wall is zero, because the displacement is perpendicular to it. Also the work done by the reaction on the bar is zero, for the same reason. Hence the work done by the only other forces, viz. gravity, must be zero.

But this is the same as if the whole weight were collected at G. Hence, if this does no work, there must be no vertical displacement, or the rod must be in such a position that if the end A be pushed up, the centre of gravity G must begin to move horizontally.

EXAMPLE II. *Four equal rods are jointed to form a rhombus* ABCD; *the opposite angles* B, D *are joined by a string and the system is placed with* AC *vertical resting on a horizontal plane. Determine the tension of the string.*

Let T denote the tension, and suppose the string replaced by forces T on B along BD and T on D along DB.



Give the system a small displacement by pressing A down to A′, so that the rhombus now becomes A′B′CD′.

Since it is in equilibrium, the whole work done is zero. But the work done by the reactions at the hinges is zero, for the reactions at a hinge on each rod are equal and opposite and no work is done on the whole. The other forces remaining are the forces T and gravity. Hence

Work done by gravity = work done against T.

Now work by gravity is the same as if the weight of the rods were concentrated at their centre of gravity.

In the displacement let x denote the horizontal displacement of B and y the vertical displacement. Then, remembering that the centre of gravity is in BD and that consequently y denotes its vertical displacement,

$$4W \,.\, y = Tx + Tx,$$

where W = weight of one rod.

Hence $$T = 2\frac{y}{x} \,.\, W.$$

Now B moves at right angles to CB. Therefore CB′B is a right angle,

$$\therefore \frac{y}{x} = \frac{Bn}{B'n} = \tan BB'n = \cot n\,B'C = \cot \tfrac{1}{2}\,ABC.$$

Hence

$$T=2W\cot\tfrac{1}{2}ABC,$$

which gives T in terms of W and the angle between the rods.

If the length of the rods be given $(=a)$ and of the string $(=l)$, then

$$AB=a,\quad OB=\tfrac{1}{2}l,$$

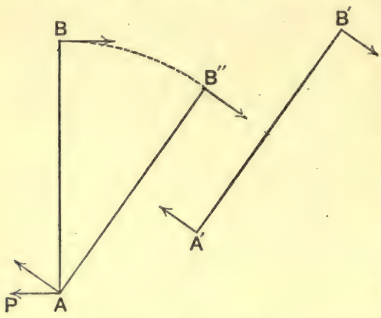
$$OA^2=a^2-\frac{1}{4}l^2=\frac{4a^2-l^2}{4},$$

and

$$T=\frac{2l}{\sqrt{(4a^2-l^2)}}W.$$

107. *Work by a couple.*—If a body on which a couple is acting receive a displacement consisting of a translation merely, the work done by the couple is zero. For each force of the couple receives the same displacement. The works done by them, therefore, are equal and opposite, and no work on the whole is done.

This is, however, not the case if the body receive a rotation as well as a displacement. Let A, B be the points of application of the forces. We may consider the forces as acting perpendicularly to AB.



By the displacement, suppose A is transferred to A′ and B to B′. Since the body is rigid, A′B′ = AB. We require the work done by the couple.

Displace the body by a translatory motion so that A′ is brought back to its former position A, and the position of B is now B″. *No work is done by this operation.* Therefore the work required is the same as that in rotating the body round A through the angle BAB″. But here the force P at A does no work, and the work is that done by P in moving from B to B″. We may regard B as moving to B″ along the arc of a circle whose centre is at A, while P always acts perpendicularly to AB, *or* along the arc at each point. Hence the work done by P at B is

$$\text{Work} = \text{P} \times \text{arc BB}'',$$
$$= \text{P} \,.\, \text{AB} \times \frac{\text{arc BB}''}{\text{AB}}.$$

Now the ratio of the arc of a circle to its radius does not depend on the radius, but on the angle subtended at the centre. This ratio is known as the circular measure of the angle. Denoting by θ the circular measure of the angle through which the body has been rotated,

$$\theta = \frac{\text{arc BB}''}{\text{AB}},$$

and $$\text{Work} = \text{P} \,.\, \text{AB} \,.\, \theta,$$
$$= \text{L} \,.\, \theta,$$

if L denote the moment of the couple.

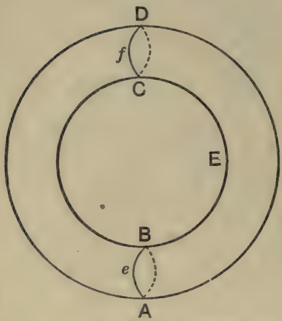
108. Considerations of energy often enable us easily to distinguish whether the equilibrium of a body is stable or unstable. A body is said to be in stable equilibrium when, if it receive any very small displacement from the position of equilibrium, it tends to move back again. If, on the contrary, it tends to move farther away, it is said to be in unstable equilibrium. When the body keeps the position to which it is displaced, it is in neutral equilibrium. In certain cases the equilibrium is stable for some displacements and unstable for others; in this case the equilibrium, as a whole, is unstable. The following examples will illustrate the distinction.

(1) *A body suspended from a point.*—Here there are two positions of equilibrium, one with the centre of gravity below the point of suspension, the other above. In the former the equilibrium is stable, in the second unstable. If the centre of gravity be at the point of suspension, the equilibrium is neutral.

(2) *A particle resting on a hollow vertical circular tubular ring.*—Here there are four positions of equilibrium—

(*a*) Inside at the lowest point A (stable).

(*b*) Outside at B (stable for displacements along BE, unstable for those along B*e*, ∴ equilibrium is unstable).



(*c*) Inside at C (unstable along CE, stable along C*f*, ∴ equilibrium is unstable).

(*d*) Outside at D (unstable for all displacements).

It is evident that it is always easy practically to put a body into one of its positions of stable equilibrium, but not into an unstable one. Witness for example the difficulty of making an egg stand on end, whilst it is always easy to put it on its side.

When a body is in equilibrium, it is so in general under a system of external forces together with a set of constraints which do no work as the body is moved about. Now suppose that the body is placed at rest in such a position that the potential energy of the acting forces is a minimum. That is, for every position of the body near this the potential energy is greater. Suppose now the body in this position to have a velocity given to it in any direction. Then, remembering that the sum of the potential and kinetic energies is always constant, and that in this case the body must move into positions where the potential energy is greater, it follows that the kinetic energy must be less. In other words, the forces called into play by the displacement tend to stop the motion and to bring back the body to its former position. We learn then that when the potential energy is a minimum there is stable equilibrium. It is of course evident that such a position *is* one of equilibrium, for, if at rest, the body must remain at rest, as it is impossible to get kinetic energy to move away with.

There is also equilibrium when the potential energy is a maximum—that is, when the potential energy for all positions near that in question is everywhere less than in the position itself. This cannot be proved in the same way as in the previous case. But suppose all the forces reversed, then clearly to a position of maximum energy for the original forces will correspond one of minimum energy in the reversed system. But this is, as we have already seen, one position of equilibrium,—it will, therefore, also be in equilibrium if the forces are all reversed again to their former state. Hence a position of maximum energy is one of equilibrium. But it is one of *unstable* equilibrium, for if it is displaced it moves to positions of less potential energy, and therefore the kinetic energy increases and the body tends to move farther away.

As illustrating these results, we notice that in the case of the suspended body the potential energy depends on the height of the centre of gravity. Hence it is stable when it is as low as possible (*i.e.* under the point of suspension), and unstable when as high as possible (*i.e.* above the point of suspension). Compare also the case of the egg. Further examples are given in § 155.

EXAMPLES—XI.

1. A ball weighing 12 lbs. leaves the mouth of a cannon horizontally with a velocity of 1000 feet per second; the gun and carriage, together weighing 12 cwts., slide upon a smooth plane whose inclination to the horizon is 30°. Find the space through which the gun and carriage will be driven up the plane by the recoil.

2. A rope 500 feet long, and weighing 2 lbs. a foot, is wound on a roller. What is the difference of its potential energy in this position and in its position when 200 feet of the rope have rolled out, neglecting friction and the weight of the roller and supposing that no part of the rope touches the ground?

3. Find the horse-power of an engine which will travel at 25 miles per hour up an incline of 1 in 100, the weight of the engine and load being 50 tons and the resistance 10 lbs. per ton.

4. Find the horse-power exerted by an engine that is drawing a train of 120 tons up an incline of 1 in 300 at 30 miles an hour; resistances 8 lbs. a ton.

5. Prove that the pull of a locomotive engine is pd^2l/D lbs. for a mean effective pressure of p lbs. on the square inch, where d denotes the diameter of each of the two cylinders, l the length of stroke, and D the diameter of the driving-wheels in inches.

6. A weight of 8 cwts. is raised through a height of 60 fathoms by means of a rope weighing 1 lb. to the foot. Find the work expended.

7. Find the horse-power of an engine that would empty a cylindrical shaft full of water in 32 hours if the diameter of the shaft be 8 feet and its depth 600 feet, the weight of a cubic foot of water being 62·5 lbs.

8. A tower is to be built of brickwork, and the base is a rectangle 22 feet by 9 and the height is 66 feet, the walls being 2 feet thick. Find the number of units of work expended on raising the bricks from the ground and the number of hours in which an engine of 3 horse-power would raise them, a cubic foot of brickwork weighing 1 cwt.

9. A man sculling does E foot-pounds of work, usefully applied, at each stroke. If the total resistance of the water when the boat is moving uniformly n miles an hour be R lbs., find the number of strokes he must take per minute to maintain the speed.

10. A particle moves in a vertical circular tube and starts from rest at the highest point. Find the velocity (1) when it is at the lowest point, (2) when the line to it from the centre makes an angle of 45°, and (3) of 135° to the vertical.

11. A ball, mass m, is just disturbed from the top of a smooth circular tube in a vertical plane; it falls and impinges on a ball, mass $2m$, at the bottom; the coefficient of rebound is $\frac{1}{2}$. Find the height to which each ball will rise in the tube after a second impact.

12. A heavy weight is suspended from a point by a string 8 feet long; it is pulled aside until the string makes an angle of 60° with the vertical and let go. What is the velocity when it reaches the lowest point?

13. A particle moves under the action of a constant force, equal to that of gravity, to a fixed point; it is projected in any direction with a velocity of 10 feet per second from a point distant 2 feet from the fixed point; after a certain time it is 3 feet from the point. Find its velocity. (See § 100.)

14. A heavy string of length l lies on a smooth horizontal table with its length perpendicular to an edge; a short piece just hangs over, and the whole is free to move. Find the velocity of the string as it just leaves the table.

15. A particle, mass M, is fastened to one end of a string which passes through a hole in a horizontal plane and is attached to another of mass m hanging vertically. Determine the velocity when m has fallen through 1 foot.

16. In the previous question M is projected with velocity u at right angles to the line joining it to the hole when the distance from the hole is a; it is again moving at right angles to the string when the distance is r. Find the velocity at that instant.

17. A particle is acted on by a force to a fixed point $= \mu \times$ distance from the point; it is at rest at a distance a. Find the velocity when it is at a distance r. (See § 97.)

18. In the previous question the particle is projected with velocity u (1) towards the point, (2) perpendicular to the distance a. Find the velocity when the particle is at a distance r.

19. Four equal rods are jointed to form a square ABCD; the opposite angles A, C are joined by a weightless rod and the whole rests with AC vertical on a horizontal plane. Find the thrust on the rod AC, the weight of the rods being given.

CHAPTER XII

MACHINES

109. A MACHINE is an instrument for the transmission of mechanical work and for the transformation of a simple motion into a more complicated system of motions, as for instance in a pump or a lace machine. The theory of the relations between the motions of different parts of a machine is called the kinematics of machinery. Into this very interesting and technical subject we do not enter here; we confine our attention to the forces called into play at different parts of a machine. If now we analyse any machine into its parts, we shall always find that it is built up of a series of simpler elements, and that these elements can usually be arranged under about five or six different heads. These elements are sometimes spoken of as the mechanical powers. But, as a power is a rate of doing work, this name is clearly wrong. We shall simply refer to them as the elementary machines. They are—

(1) The inclined plane.
(2) The lever.
(3) The wheel and axle.
(4) The pulley.
(5) The screw.

Besides these, there are others which it would be difficult to classify under one of the foregoing—such for instance as cams, link work, etc.

An instrument for the transformation of energy into that of mechanical work is called a prime mover. Thus the steam-engine transforms heat, the gas-engine the potential chemical energy of coal-gas and oxygen, the electro-motor electricity, turbines and water-wheels the potential energy due to gravity of water, wind-mills the kinetic energy of the air into mechanical work.

110. In all actual machines the case is more or less complicated by the presence of friction. In the present chapter we shall suppose, however, that all the parts are perfectly smooth, and leave to the next chapter the consideration of the modifications produced by friction.

When a certain amount of work is given in at one point of a machine an equal amount must be given out at other points. For if the forces at those other points be reversed they will form a system in equilibrium with the first, and therefore the work done *by* the first set will be just equal to that done *on* the other. When there is friction, a certain portion of this work will be expended in overcoming the friction, and is therefore lost *so far as useful effect is concerned.* The useful work given out will, therefore, always be less than the work put in. However, in the cases considered in the present chapter, where there is no friction, the work given out will be equal to that put in. Now the work in any case is produced by a certain force acting through a certain distance. In general, if the point of a machine where the force is applied be moved through a certain distance, the point where the transmitted force is exerted will move through a different distance, and since the work is the same at the two places, the forces must be different. Now for most purposes we require to change a small force into a large force, and a machine enables us to do this, though it does not enable us to increase or diminish the work done in any time.

The ratio of the force exerted by the machine to the applied force is called its mechanical advantage. Thus, if f be the applied force and F the force exerted by the machine, F/f is the mechanical advantage. When F is less than f this ratio is sometimes called the mechanical disadvantage. Suppose now two machines coupled together so that the point of exertion of the first is coupled to the point of application of the second. Let f be the applied force to the first and F′ the force it exerts. Then F′ is the applied force to the second. Let F be the force the second exerts. Also let A_1, A_2 be the mechanical advantages of the first and second and A that of the two combined. Then

$$A_1 = \frac{F'}{f}, \quad A_2 = \frac{F}{F'};$$

$$\therefore A_1 . A_2 = \frac{F'}{f} \times \frac{F}{F'} = \frac{F}{f} = A,$$

or the mechanical advantage of the two coupled is the product of the mechanical advantages of the two separately. The same is evidently true also for any number of machines—the mechanical advantage of any number arranged one after the other is equal to the product of the mechanical advantages of each separately. If then we determine the mechanical advantages of the different elementary machines, it will be an easy matter to calculate that of any complicated machine by considering the elements of which it is composed. This will be the object of a large part of the present chapter.

In any actual machine, without friction, it is an easy matter to determine the mechanical advantage by experiment. For, push the point of application through a distance a, and measure the distance through which the point of exertion moves, say b. Then, when there is no friction, the work done at the two points is the same. Hence

$$fa = Fb,$$

$$\therefore A = \frac{F}{f} = \frac{a}{b}.$$

The method of energy is in general the best to determine the mechanical advantage in all cases. For the sake of practice, however, we shall in what follows use this method as well as those developed in the earlier chapters.

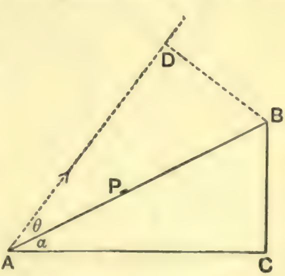
111. In actual machines a certain amount of work being spent in friction, the useful work is less than that put in. The ratio of the *useful* work transmitted to the

work put in is called the *efficiency* of the machine. The word is chiefly employed with reference to prime movers, and in this case gives the ratio of the useful work produced to the work which should have been produced if there had been no loss owing to friction, conduction of heat, electrical or other actions where work is spent uselessly. In machines without friction, for simply transmitting work, the efficiency is unity. In all other cases it is less than unity.

Properly speaking, there is no such thing as the efficiency of a prime mover or a machine. The efficiency in general depends on the power transmitted or produced, and not only on the particular machine. Thus in the case of a given water-wheel, with the water entering at the top and carried in buckets to the bottom, the *efficiency* will be greatest when it is moving slowest, or the power is least. For with a given quantity of water carried over, the whole work done is equal to the work done by the wheel + the kinetic energy given to the water. The slower, therefore, the wheel works the less kinetic energy is given to the water, and the more is therefore left to be given out by the wheel. Although, therefore, it would be incorrect to speak of the efficiency of the machine itself, it is quite correct to speak of the efficiency at which it is working.

112. *Inclined plane.*—This has been already treated of as an example in the resolution of forces in § 50.

CASE I. *Applied force along the plane.*—Let AB be the plane and BC the direction of the force to be overcome. (In the case of the weight of a body, BC is vertical.) Draw AC perpendicular to BC. Consider the work to bring the point P from A to B.



The work applied $= f \times AB$.
The work overcome $= F \times BC$.

These are equal,

$$\therefore f \,.\, AB = F \,.\, BC,$$

and $$A = \frac{F}{f} = \frac{AB}{BC}.$$

Thus, in the case of a truck hauled up an inclined plane by a rope,

$$A = \frac{\text{length of track}}{\text{height raised}}.$$

CASE II. *Applied force in any direction.*—Suppose f to act in direction AD. Draw BD perpendicular to AD. Then, as before,

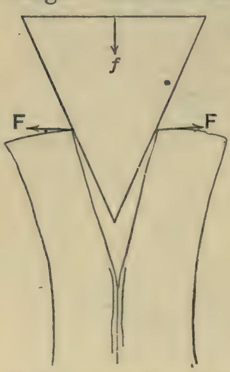
$$f.\,AD = F.\,BC,$$

or $$A = \frac{AD}{BC} = \frac{\frac{AD}{AB}}{\frac{BC}{AB}} = \frac{\cos\theta}{\sin\alpha};$$

if, for instance, f be perpendicular to F,

$$A = \frac{AC}{BC} = \frac{\text{base of plane}}{\text{height of plane}} = \cot\alpha.$$

As an example of the inclined plane, we may take a wedge. Here the thin edge is placed in a chink between two bodies M, M′, the wedge is driven down, and the bodies forced apart. Let a denote the breadth of the wedge and b its length. Then, when the wedge has been driven down, the bodies are separated by a distance $BC = a$, whilst the wedge itself has moved through a distance $= b$. Hence



$$f.\,b = F.\,a,$$

$$\therefore\ A = \frac{b}{a} = \frac{\text{length of wedge}}{\text{breadth of wedge}}.$$

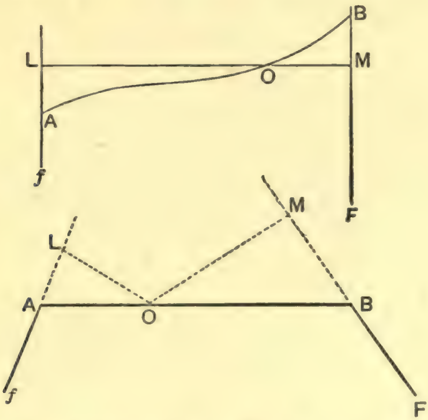
113. *The lever.*—Levers are sometimes classified according to the relative positions of the fulcrum and the points of application and exertion. Thus, in a pair of scissors or pincers, a crowbar or a pump-

handle, the fulcrum comes between the other two points; in the human arm or a sheep-shears the applied force is between; and in a pair of nutcrackers or an oar the exerted force is between the others. In all cases, however, the formula for the mechanical advantage is the same.

CASE I. *A straight lever, with parallel forces.*—Here the reaction at the fulcrum is parallel to the others, and is equal and opposite to their resultant—that is, the fulcrum is situated at the point through which the resultant of F and f acts. Hence

$$\frac{F}{f} = \frac{\text{distance of applied force from fulcrum}}{\text{distance of exerted force from fulcrum}}.$$

When, therefore, the applied force is farther from the fulcrum than the exerted force there is mechanical advantage, when otherwise there is disadvantage. Thus, with fulcrum between, there may be advantage or disadvantage; for example, in cutting with a pair of scissors, there is advantage when the cut is begun, but disadvantage toward the end of the cutting. This explains why it is



easiest to cut a hard substance when the scissors are opened wide and the object put in close to the pin. With the point of application between, there is always disadvantage; with the point of exertion between, there is always advantage.

The same formula holds for a bent lever and parallel forces. Thus in the upper figure, O being the fulcrum,

$$\frac{F}{f} = \frac{OL}{OM}.$$

CASE II. *Forces not parallel.*—Draw OL, OM perpendicular to the forces. Then, since the forces just keep the lever in equilibrium, the moments about O must vanish. Hence

$$F \,.\, OM - f \,.\, OL = 0$$

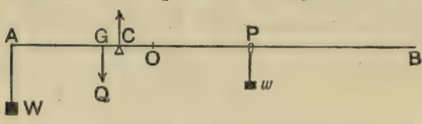
or

$$\frac{F}{f} = \frac{OL}{OM}.$$

As important applications of the lever, the steelyards and balance will require a more detailed consideration.

114. *The common steelyard.*—This consists essentially of a lever, with the fulcrum between the points of application and exertion. The body to be weighed is suspended from the shorter arm, and a constant movable weight is adjusted on the other and longer one until the beam is horizontal. The long arm is graduated so that the graduation at which the movable weight rests gives the required weight of the body. This would be a simple matter if we could neglect the weight of the steelyard itself—the weight would, in fact, be proportional to the distance of the movable weight from the fulcrum. We must then investigate how to graduate the arm when the weight of the yard is taken into consideration.

Let AB represent the yard. Let C be the fulcrum, A the point of application of the body to be weighed, and BC the long arm on which the



movable weight w moves. Let P be the point where the movable weight rests when the yard is horizontal. We require to know the number to be placed at P to give the weight of W—in other words, we require to know how to graduate the arm BC. Let Q denote the weight of the steelyard without the movable weight and the hook by which the yard is suspended from C. There will be two cases to consider, according as the centre of gravity of the yard is on one side or the other of the fulcrum.

CASE I. *The centre of gravity* G *on the shorter arm.*—Take moments about C. Then

$$W \,.\, AC + Q \,.\, GC - w \,.\, CP = 0.$$

Now take a point O on the side opposite to G, so that

$$Q \,.\, GC = w \,.\, OC;$$

then
$$W \,.\, AC = w(CP - OC),$$
$$= w \,.\, OP;$$

hence
$$OP = \frac{W}{w} \,.\, AC.$$

CASE II. G *on the longer arm.*—Here, proceeding as before, the equation of moments about C is

$$W \,.\, AC - Q \,.\, GC - w \,.\, CP = 0.$$

O must now be chosen in the shorter arm, so that

$$Q \,.\, GC = w \,.\, OC,$$

then
$$W \,.\, AC = w \,.\, OP,$$

or
$$OP = \frac{W}{w} \,.\, AC.$$

If then we mark off points P_1, P_2, P_3, . . . along CB at distances AC, 2AC, 3AC, . . . from O, these graduations will represent points for which the body to be weighed is w, $2w$, $3w$, . . . For instance, suppose w to be 1 lb. weight. Then, if the movable weight has to be placed at P_2, it shows that the weight of the body is 2 lbs.

These graduations would be too large for practice, and smaller ones are therefore necessary. For instance, in the case above, $w = 1$ lb. $= 16$ oz. If then W be expressed in ounces,

$$OP = W \,.\, \frac{AC}{16} \text{ oz.}$$

Therefore, when W is 1, 2, 3, . . ., n oz., the graduations must make

$$OP_1 = \frac{AC}{16}, \quad OP_2 = \frac{2AC}{16}, \quad OP_3 = \frac{3AC}{16}, \quad . . ., \quad OP_n = \frac{nAC}{16}.$$

Note.—It is clear that all the graduations are of the same length. For the distance between any consecutive ones $= \frac{AC}{16}$.

115. *The Danish steelyard.*—This steelyard consists of a bar with a heavy boss at one end. The body to be weighed is suspended from the other end, and the bar itself is then moved over the fulcrum until it rests in equilibrium. The graduation at which it rests then gives the weight of the body.

Let w be the weight of the yard, G its centre of gravity, W the weight of the body, P the position of the fulcrum when there is equilibrium.

Then the reaction at P $= W + w$.

Taking moments about G,

$$W \times BG - (W + w)GP = 0,$$

$$\therefore GP = \frac{W}{W + w} . GB.$$

If P_n be the graduation when $W = nw$, then

$$GP_n = \frac{n}{n+1} GB.$$

The graduations must therefore be at P_1, P_2, P_3, . . ., where

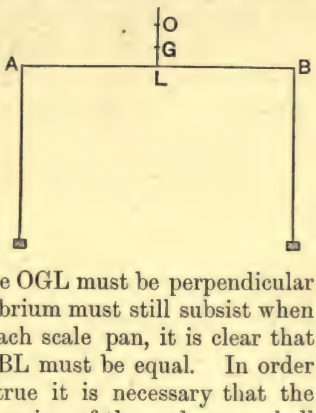
$$GP_1 = \frac{1}{2} GB, \quad GP_2 = \frac{2}{3} GB, \quad GP_3 = \frac{3}{4} GB, \quad . . .,$$

$$GP_n = \frac{n}{n+1} GB.$$

Note.—It is clear in this case that the spaces between the graduations are of unequal length. This is a great disadvantage when heavy bodies have to be weighed, for the intervals between successive graduations are then so small that it is difficult to determine them with accuracy.

116. *The balance.*—The most useful instrument for measuring masses is the balance, consisting essentially of a straight uniform lever with the fulcrum at the middle point. Like the steelyards, the balance does not directly compare masses, but only their weights. As, however, all experience hitherto has pointed to the conclusion that the weights of bodies are proportional to their masses, if the weights are equal so also will their masses be.

The figure is intended to represent diagrammatically a balance. AB is the line joining the points of suspension of the scale pans, O the point from which the beam swings, and G the centre of gravity of the beam and of the scale pans *supposed collected at their respective points of suspension.* Let OG meet AB in L. Then, when no weights are in the scale pans, OG will be vertical. But then AB must be horizontal. Therefore OGL must be perpendicular to AB. Also, since the equilibrium must still subsist when equal masses are placed in each scale pan, it is clear that the arms of the balance AL, BL must be equal. In order then that a balance may be true it is necessary that the line joining the points of suspension of the scale pans shall be (1) perpendicular to the line joining the point of support and the centre of gravity, and (2) that the arms shall be equal.

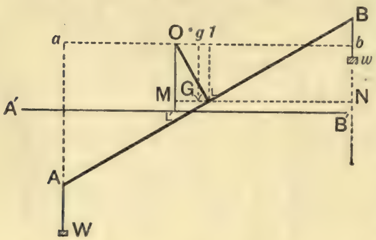


These, however, are not the only requisites for a good balance. Not only must the arm be horizontal when the masses in the pans are equal, but in order to weigh accurately it must be easy to determine when the masses are not exactly equal—in other words, if one mass be ever so slightly larger than the other, the arm must turn through an appreciable angle. The balance must in fact be *sensitive.*

Where time is an object, it is also necessary that the balance should quickly take up its position of equilibrium. In other words, when displaced from its position of rest the forces called into play should quickly bring it back again. The balance must in fact be *stable.*

It is important, therefore, to determine the conditions which must be satisfied that a balance may be (1) true, (2) sensitive, (3) stable. The conditions for the first have been just determined. To determine those for the two others it will be necessary to consider the forces called into play when the scales are loaded unequally and the beam is displaced through any angle.

In the figure let the thick lines OL, AB represent the displaced position of the balance, OL′, A′B′ its position of rest with equal weights.



Let W, w be the weights in the pans at A, B; Q the weight of the beam and scale pans. Also let

$$OG = h, \quad OL = k.$$

Then the moment of the forces tending to turn the beam round O

$$= w \,.\, Ob + Q \,.\, Og - W \,.\, Oa.$$

Now, since AL = LB, therefore $al = lb$,

whence $$Oa = la - Ol = lb - Ol,$$

also $$\frac{Og}{Ol} = \frac{OG}{OL} = \frac{h}{k}.$$

Hence the moment of the forces

$$= w(Ol + lb) + Q\frac{h}{k} \,.\, Ol - W(lb - Ol),$$

$$= Ol\left(w + W + Q\frac{h}{k}\right) + lb(w - W).$$

Now, to get the greatest stability possible, the forces tending to bring back the beam when displaced a given distance Ol, and when loaded with equal weights, must be as great as possible—that is, putting $W = w$,

$$Ol\left(2W + Q\frac{h}{k}\right)$$

must be as great as possible for a given displacement, whatever W may be.

Now the displacement LOL′ being given, Ol will be proportional to k. Hence the condition is that $k(2W + Qh/k) = 2Wk + Qh$ must be great. This may be attained by making h and k very large—that is, the point of suspension at a considerable distance from AB.

The investigation for sensitiveness becomes much simplified by the use of trigonometrical functions. We notice that if the angle LOL′ be denoted by θ,

$$Ol = LM = OL \sin \theta = k \sin \theta,$$
$$lb = LN = BL \cos BLN = a \cos \theta,$$

if a denote the length of the arm.

Putting in these values the forces tending to bring the beam back,

$$= \{(w + W)k + Qh\} \sin \theta - (W - w)a \cos \theta.$$

Now, if a balance be sensitive, its displacement will be considerable, even when the two weights are very nearly equal. That is, for a given small value of $W - w$, θ must be large. But when at rest the moment of the forces about O vanish. Hence θ is given by

$$\{(w + W)k + Qh\} \sin \theta = (W - w)a \cos \theta,$$

or $$\frac{\sin \theta}{\cos \theta} = \tan \theta = \frac{W - w}{(w + W)k + Qh}a.$$

Now for a given value of $W - w$, $\tan \theta$ must be large;

hence $$\frac{a}{(w + W)k + Qh} \text{ must be large ;}$$

that is, since $W = w$ nearly,

$$\frac{2Wk + Qh}{a} \text{ must be small.}$$

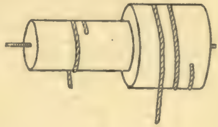
It will be noticed that this is to some extent incompatible with the condition for stability, viz. that $2Wk + Qh$ must be large. If, however, while making h, k large for stability we also make a, or the length of the arm, very long, we shall get both stability and sensitiveness. But this simply means, the larger the balance is made the better it will be. This is, however, only apparently the case, for if the arm be made longer what is called its moment of inertia (§ 175) will be larger, and the consequence will be, as will be shown in Chapter XXI, that it will move still more slowly. In fact the conditions of stability cannot be fully worked out without considering the dynamics of the balance as a moving body. The best way is, for ordinary purposes, to sacrifice great sensitiveness or accuracy (in weighing a pound of sugar it is not necessary to be accurate to the tenth of a grain) and to make balances in which OL is not extremely small. On the other hand, in cases where extreme accuracy is necessary and the time expended in the operation of weighing is a secondary consideration, as in chemical balances, everything is sacrificed to sensitiveness.

This is not the place to enter into complete details of accurate balances and of the operations necessary to secure accurate weighings. There are, however, methods whereby correct results may be obtained, even when the balance (through inequality of arms or otherwise) is inaccurate. One method, used also with the chemical balance, is to place the body in one scale and counterpoise it with small shot and sand and observe the position of the beam. Then take out the body and replace it by weights until the beam returns to its former position. It is then clear that, whether the balance is inaccurate or not, the weight of the body is equal to that of the weights put in its place.

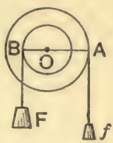
When the arms are unequal, the correct weight can still be found by weighing the body alternately in each scale pan. If w, w' be the apparent weights in the two cases, the correct weight will be $\sqrt{ww'}$. The student can prove this as an example.

117. *The wheel and axle.*—This consists essentially of two cylinders of different diameters on the same axis,

round which they can turn together. Ropes are wound in opposite directions round each of them so that, as the machine turns, one rope is wound on and the other off. A modification is when the large cylinder and rope are replaced by a handle, as in a windlass.



The second figure represents diagrammatically an end view of the machine. O is the axis; A, B the points where the ropes leave the surface.



If F and f be just in equilibrium, it is clear that they balance about O as in the case of the lever. Hence

$$\frac{F}{f}=\frac{OA}{OB}=\frac{\text{radius of wheel}}{\text{radius of axle}}.$$

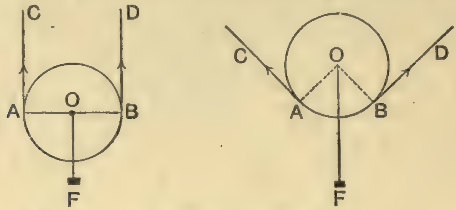
If the thickness of the rope is large, and at all comparable with the size of the wheel or axle, it is necessary to take account of it. Now in a rope we may regard the tension it transmits as spread uniformly across it, and therefore the whole tension as acting through the centre of its section. This will increase the effective diameter of the wheel or axle by half the thickness of the rope. Whence the mechanical advantage is

$$\frac{F}{f}=\frac{\text{radius of wheel}+\frac{1}{2}\text{ thickness of wheel rope}}{\text{radius of axle}+\frac{1}{2}\text{ thickness of axle rope}}.$$

Moreover it is clear that if as the wheel revolves one rope gets coiled on itself as the other uncoils, the effective radii of the wheel and of the axle alter. For instance, if the wheel is uncoiling and raising a weight, the effective radius of the wheel is diminishing whilst that of the axle is increasing. In consequence of this the ratio of the numerator to the denominator of the above fraction diminishes, or the mechanical advantage gets less. An example of varying advantage is to be seen in the fusee-wheel of a watch. When the spring is wound tight up the mechanical advantage is small, whilst as the force applied by the string gets less the mechanical advantage increases.

118. *Pulleys.*—A pulley is essentially a small wheel by which the direction of a tension is altered without affecting its magnitude. We consider first a single pulley, and then different arrangements of several pulleys by which mechanical advantage is gained.

The rope on passing round the pulley has the same tension on both sides, and the *effect on the pulley* is the same



as if two forces equal to the tension were applied at the points A, B where the rope leaves the pulley. If one end of the rope be fastened at C and the other pulled with a force f, the effect on the pulley is the same as if two forces f, f acted at A, B along AC, BD. These may be made to counterbalance a force F applied to the pin on which the pulley works.

If the two portions of the rope are parallel, it follows that

$$F = 2f,$$

or the mechanical advantage of a pulley with parallel ropes is 2.

If the two portions are not parallel. Let 2θ be the angle between them. Then each makes an angle θ with F, and resolving parallel to F,

$$F = 2f \cos \theta,$$

or the mechanical advantage is $2 \cos \theta$.

These results have been obtained on the supposition that the pulleys have no weight. Usually the error so introduced will only be a small fraction of the forces involved. In many cases, however, it will be necessary to take account of the weight of the pulleys.

If the pulley hangs vertically, the effect is to add its weight to F. Hence

$$F + w = 2f \cos \theta,$$

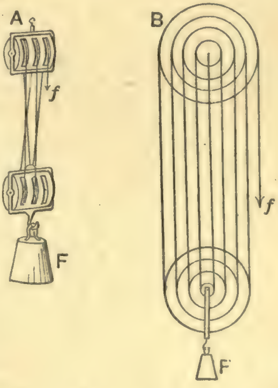
and mechanical advantage $= 2 \cos \theta - \dfrac{w}{f}$.

It is to be noticed that in this case the mechanical advantage depends partly on the applied force.

Pulleys may be combined with one another in an infinite number of ways. Three of the most common are considered in the next three articles.

119. *The same rope goes round all the pulleys.*—The pulleys are combined in two blocks running loose on two pins, and the same rope goes round all the pulleys as in the figure (A). Sometimes, however, the pulleys on each block are made of one piece, but in this case they must be of suitable diameters, as in the figure (B), or the ropes will have to slip over the surface of the pulleys and work will be lost in the friction. One end of the rope is attached to either of the blocks.

The force F applied to the lower block is counterbalanced by the tensions of the ropes passing round the pulleys in it. The tension of each rope is the same as that applied at the end, viz. f. If the other end of the rope is attached to the upper block, there will be an even number of ropes pulling up on the lower, viz. two for each pulley, but if it be fixed to the lower, then there will be an odd number. In any case, however, if there be n ropes at the lower block, the upward force is nf (since tension of each $= f$).



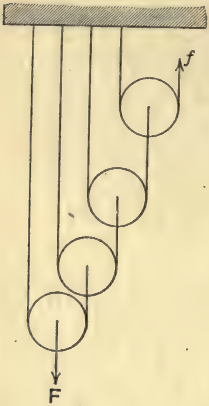
$$\therefore\ F = nf,$$

or $$A = n.$$

If the weight of the lower block is taken account of,

$$F = nf - w.$$

120. *System in which each rope has one end fixed, passes round a pulley, and is fastened to the preceding one.*—The method of arrangement is as shown in the figure. Clearly this is a case where the exerted force of one pulley is the applied force on the next. In other words, we have a series of machines each worked by the one before and working on the one in front. Hence, by § 110, the mechanical advantage is the product of those of each. In this case the mechanical advantage of each is 2 by § 118. Therefore, if there are n pulleys,



$$A = 2 \times 2 \times 2 \times \ldots \text{ (n factors)},$$
$$= 2^n;$$

or $F = 2^n f$.

This may also easily be arrived at directly, for the tension of each rope is twice that of the rope before. Hence, if t_n denote the force on the nth pulley,

$$F = t_n = 2t_{n-1} = 2^2 t_{n-2} = 2^3 t_{n-3} = \ldots = 2^n t_0 = 2^n f.$$

If the weights of the pulleys are considered, we must investigate the equilibrium of each separately.

Let $w_1, w_2, \ldots, w_n$ be the weights of the 1st, 2nd, . . ., nth pulley. Then

$$\left.\begin{aligned} F - 2t_{n-1} + w_n &= 0 \\ t_{n-1} - 2t_{n-2} + w_{n-1} &= 0 \\ t_{n-2} - 2t_{n-3} + w_{n-2} &= 0 \\ \ldots \ldots \ldots &= 0 \\ t_1 - 2f + w_1 &= 0 \end{aligned}\right\}.$$

These are n equations. Multiply the 2nd by 2, the 3rd by 2^2, the 4th by 2^3, . . ., the last (or nth) by 2^{n-1}. Then

$$\left.\begin{aligned} F - 2t_{n-1} + w_n &= 0 \\ 2t_{n-1} - 2^2t_{n-2} + 2w_{n-1} &= 0 \\ 2^2t_{n-2} - 2^3t_{n-3} + 2^2w_{n-2} &= 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ &= 0 \\ 2^{n-1}t_1 - 2^nf + 2^{n-1}w_1 &= 0 \end{aligned}\right\}.$$

If all these equations be added together, the first term of each will be cancelled by the second term of the previous equation, and

$$F - 2^nf + w_n + 2w_{n-1} + 2^2w_{n-2} + \ .\ .\ .\ + 2^{n-1}w_1 = 0,$$

or $$F = 2^nf - (w_n + 2w_{n-1} + 2^2w_{n-2} + \ .\ .\ .\ + 2^{n-1}w_1).$$

If the weights of the pulleys are all equal,

$$F = 2^nf - (1 + 2 + \ .\ .\ .\ + 2^{n-1})w,$$

let now $$S = 1 + 2 + \ .\ .\ .\ .\ .\ + 2^{n-1},$$

then $$2S = \quad 2 + 4 + \ .\ .\ .\ + 2^{n-1} + 2^n;$$

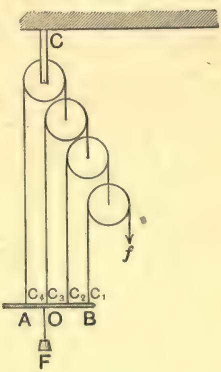
or, subtracting the 1st from the 2nd,

$$S = 2S - S = 2^n - 1.$$

Hence $$F = 2^nf - (2^n - 1)w.$$

In this system of pulleys, therefore, the mechanical advantage is diminished by the weights of the pulleys.

121. *System in which each rope is attached to the point of exertion.*—The end of each rope is attached to a bar, on which the force F acts. It is clear, from a comparison of the figures, that this is essentially the previous system inverted. Now the force acting on the pulley at C is the force F of the previous case. Call it F′, then



$$F' = 2^nf.$$

But regarding the system as one system acted on by the three forces F, F′, f,

$$F + f = F' = 2^nf,$$
$$\therefore F = (2^n - 1)f,$$

or $$A = 2^n - 1.$$

If the pulleys be of the same size, the ends of the ropes will be fixed on the bar AB at equal intervals. Their tensions will have a resultant (= F) passing through some definite point of the bar, and it is this point at which F must be applied. In order to find this point we shall require to calculate each tension separately. The method will best be exemplified by considering the case of three free pulleys, or four ropes.

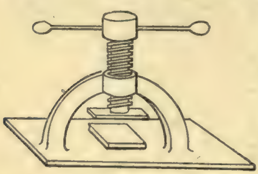
$$\begin{aligned} \text{The tension at } C_1 &= t_1 = f, \\ C_2 &= t_2 = 2f, \\ C_3 &= t_3 = 2^2 f, \\ C_4 &= t_4 = 2^3 f. \end{aligned}$$

To find their centre of parallel forces O, take moments about C_1. Let the interval between the ropes be a (= radius of a pulley). Then

$$\begin{aligned} BO &= \frac{f \times 0 + 2f \,.\, a + 2^2 f \,.\, 2a + 2^3 f \,.\, 3a}{f + 2f + 2^2 f + 2^3 f}, \\ &= \frac{2 + 8 + 24}{1 + 2 + 4 + 8} a = \frac{34}{15} a = \frac{34}{45} \,.\, 3a = \frac{34}{45} AB. \end{aligned}$$

In this system the weights of the pulleys are an advantage. The formula giving F in terms of f and the weights is left as an example for the student.

122. *The screw.*—In a screw the force is generally applied at the end of an arm, at right angles to the screw.

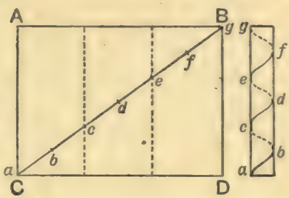


The force is exerted in one of two ways, either by the screw bolt working in a fixed nut so that the screw moves forward, or the screw working in a fixed collar with the nut free to move. In the latter case, as the screw turns the nut moves forwards or backwards.

The distance between two screw threads is called the *pitch* of the screw.

The thread forms a helix on a cylinder. The curve may be supposed generated by wrapping a piece of paper, on which a straight line has been drawn, round the cylinder.

Thus let ABCD be a piece of tracing or transparent paper on which the line BC has been drawn. Wrap it round a cylinder with AC parallel to the axis. Suppose, for example, it just goes three times round. Then, the paper being transparent, the line BC will be seen to form a helix. The screw thread then clearly makes a uniform angle with the axis. The angle BCD is the angle of the screw.



The angle of the thread may vary from square to any acute angle. The shape, however, will not affect the mechanical advantage except in so far as it modifies the friction. Its shape is determined by other considerations, such as ease of manufacture, or the stresses it has to bear and the material of which it is made.

To find the mechanical advantage, give one complete turn to the screw. Then f moves through the circumference of the circle traced out by the end of the arm on which it acts. The screw itself advances a distance equal to the distance between the threads, *i.e.* its pitch. This then is the distance through which F acts. Hence, since the work done by f = work done on F,

$$f \times \text{circumference by arm} = F \times \text{pitch},$$

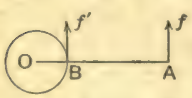
or
$$\text{Mechanical advantage} = \frac{F}{f}$$

$$= \frac{\text{circumference of circle by arm}}{\text{pitch}}.$$

If l be the length of the arm,

$$A = \frac{F}{f} = 2\pi \frac{l}{\text{pitch}}$$

It is instructive to see how the same result may be arrived at by another method.



In the figure let the circle OB represent the end of the cylinder on which the screw is made, and let OA be the arm. Then f acting at A is equivalent to f' acting at B, where

$$f \,.\, OA = f' \,.\, OB.$$

or, if r denote the radius of this cylinder,

$$f' = \frac{l}{r} \cdot f.$$

Now we have seen that the threads may be formed by wrapping an inclined plane round the cylinder. The problem is therefore the same as raising a weight F up an inclined plane by a horizontal force f'. Hence

$$\frac{F}{f'} = \frac{\text{base of plane}}{\text{height of plane}}.$$

But the base of the plane just wraps round the cylinder, and therefore measures its circumference, $= 2\pi r$; and the height of the plane = pitch of the screw,

$$\therefore \frac{F}{f'} = \frac{2\pi r}{\text{pitch}},$$

also

$$\frac{f'}{f} = \frac{l}{r}.$$

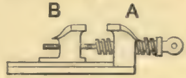
Hence, multiplying the two,

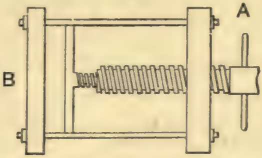
$$A = \frac{F}{f} = \frac{2\pi r}{\text{pitch}} \cdot \frac{l}{r} = \frac{2\pi l}{\text{pitch}} = \frac{\text{circumference by arm}}{\text{pitch}}.$$

By diminishing the pitch sufficiently it is therefore possible to increase the mechanical advantage to any extent. A limit, however, is set to this by the strength of the material, which would give way if the threads were made too small. The same effect is produced indirectly in another way by the differential screw.

123. *Differential screw.*—The differential screw consists of two screws of different pitch on the same axis. The figures illustrate two forms of the arrangement. In the first the two screws form one piece. The nut at A is fixed, while B is movable. As the screw is worked out of A it moves to the right, and would pull B after it at the same rate, were it not that the small screw works through B, also to the right, and so tends to leave B behind. As, however, it does not screw out of B so quickly as into A, B is still drawn towards A, though not so quickly as otherwise.

The second figure exemplifies a modification known as Hunter's screw. Here the two screws are separate, the smaller one working inside the larger one. Thus, as A is screwed in to the left it partly screws on to the small screw B and partly pushes it bodily to the left. The mechanical advantage is easily found, as in the case of the simple screw. Let a, b be the pitches of the two screws. Then on making one complete turn the large screw advances a distance a, while the small screw (or the nut in the first figure) moves b *into* it. Hence, on the whole, the distance moved through by $B = a - b$.





Since the work done is equal to the work exerted,

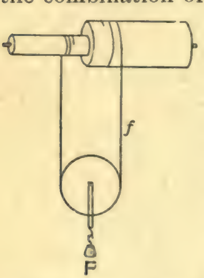
$$F(a-b) = f \times \text{circumference of circle by arm},$$

or

$$A = \frac{F}{f} = \frac{\text{circumference of circle by arm}}{\text{difference of pitches}}.$$

In order then to get a great mechanical advantage we must make the pitches nearly equal, while at the same time it is possible to make each screw as strong as may be required.

124. The differential axle and pulley is an example of the combination of two elementary machines. In this the wheel and axle are of very nearly the same diameter. The two ends of the rope are fastened, one to the axle and the other to the wheel, and the portion hanging loose supports a pulley.



Give the wheel one complete turn, then a length equal to the circumference of the wheel is wound on the wheel, and a portion wound off the axle equal to the circumference of the axle. Hence the part hanging freely is shortened by their difference; that is, the parts on each side of the pulley are shortened by half this amount. Therefore the pulley is raised a distance

$$= \tfrac{1}{2} \text{ difference of circumferences of wheel and axle.}$$

If the force f is applied to pull down the string ab, it will move through a distance equal to the circumference of the wheel. Hence, if c_1, c_2 denote the circumferences,

$$F \times \tfrac{1}{2}(c_1 - c_2) = f \cdot c_1,$$

or

$$A = \frac{F}{f} = \frac{2c_1}{c_1 - c_2}.$$

Since the circumferences are proportional to the diameters, if a, b denote the diameters,

$$A = \frac{2a}{a - b}.$$

Hence, by making b very nearly equal to a, we may increase A to any desired extent without making the axle unduly weak.

125. When a prime mover is working at its maximum efficiency it is not necessarily giving out a maximum of work. This is an important point. The following example will illustrate this difference—

Consider the action of a water-wheel. In order to simplify matters and to bring out the essential points more clearly, suppose that no water is wasted; that the water is taken in from a reservoir at the top, where the water is at rest, and that it is all given up after a quarter of a revolution—that is, after the water has fallen a distance equal to the radius of the wheel.

Let a = radius of wheel in feet.
n = number of revolutions per minute.
m = number of lbs. of water the wheel would hold if all the buckets were full.

Then in one minute the water carried over could fill n times the circumference—that is, if M denote the mass carried over every minute,

$$M = nm \text{ lbs.}$$

This falls through a distance a feet.

∴ Work done by gravity (the source of the power)

$$= mna \text{ foot-pounds.}$$

Part of this is transmitted as useful work = W, whereas part is used in giving kinetic energy to the water = E. Hence

$$W + E = mna.$$

Now the velocity of the water is that at the circumference of the wheel, which is $2\pi a$ feet long. Hence the velocity of a point there is $n \times 2\pi a$ feet per minute.

$$\therefore \text{E} = \frac{1}{2} \cdot mn \times (2\pi na)^2 \text{ (foot, pound, minute units of power)},$$

$$= \frac{2\pi^2 mn^3 a^2}{g} \text{ foot-pounds per minute},$$

where $g = 32$ feet per second per second $= 32 \times 60 \times 60$ feet per minute per minute;

$$\therefore \text{E} = \frac{\pi^2 mn^3 a^2}{16 \times 3600} \text{ foot-pounds per minute};$$

$$\therefore \text{W} = mna - \frac{\pi^2 mn^3 a^2}{16 \times 3600},$$

and efficiency $= \dfrac{\text{W}}{\text{W}+\text{E}} = 1 - \dfrac{\pi^2 n^2 a}{16 \times 3600}$.

This gives the efficiency when the speed in revolutions per minute is given. It is clearly greatest when n is least—that is, when the wheel is moving most slowly. The useful power, however, is given by

$$\text{W} = mna\left(1 - \frac{\pi^2 n^2 a}{57600}\right).$$

This is very small when the speed is very slow. It is also very small for very large speeds. For instance, it is zero when $n^2 = \dfrac{57600}{\pi^2 a}$, or $n = \dfrac{240}{\pi\sqrt{a}}$. Between these two points it first begins to increase up to a certain point, and then again to decrease to zero. There is therefore a critical speed at which the wheel is giving out its maximum amount of power. To deduce the actual value of this critical speed from the above formula necessitates the use of the differential calculus. The student may, however, satisfy himself by working out cases that the power given out for a speed $n = \dfrac{80}{\pi}\sqrt{\dfrac{3}{a}}$ is greater than that of any other speeds he may try. At this speed of maximum work the efficiency $= 1 - \frac{1}{3} = \frac{2}{3}$.

In the case of a water-wheel then, such as the above, the speed of maximum efficiency is as slow as possible, the speed of maximum work is $\dfrac{80}{\pi}\sqrt{\dfrac{3}{a}}$. This does not hold true of every motor—for instance, in the case of electro-motors the efficiency is greatest when the speed is greatest. There is in this case also a critical speed at which the power emitted is greatest.

126. We conclude this chapter with two examples to illustrate the method of calculating the motion ensuing

when the applied force differs from that just necessary to support a weight through the medium of a machine.

Example I. *In a wheel and axle a mass of* 10 *lbs. hangs from the axle, and a force equal to the weight of* 6 *lbs. is applied to the wheel; the radii of the wheel and axle are* 1 *foot and* 3 *inches respectively. Find the motion.*

Let T be the tension of the rope. Then, if the wheel and axle have no mass, T and $6g$ balance.

$$\therefore\ T.3 = 6g.12,$$
$$T = 24g \text{ poundals.}$$
$$\text{Force upon the weight} = T - 10g = 24g - 10g = 14g\ ;$$
$$\therefore\ \text{Acceleration} = \frac{14g}{10} = 1{\cdot}4g.$$

It is to be noticed here that a constant *force* is applied to the wheel. If, on the contrary, a mass of 6 lbs. had been suspended from it, the motion would be different. In this case let T′ be the tension of the string round the wheel, a', a the accelerations of the 6 and 10-lb. mass respectively. Then, a' being down and a up,

$$a' = \frac{6g - T'}{6},$$
$$a = \frac{T - 10g}{10}\ ;$$

also, *if the wheel has no mass*, T and T′ balance,

$$\therefore\ T.3 = T'.12$$

or

$$T = 4T'.$$

Moreover, since the strings are inextensible, there must be a relation between a and a'. Now if the wheel is turned once round, the 6 lbs. goes down a distance equal to the circumference of the wheel $= 2\pi$ feet, whereas the 10 lbs. goes up a distance equal to the circumference of the axle $= 2\pi$. 3 inches. But their accelerations must be proportional to these distances,

$$\therefore\ \frac{a'}{a} = \frac{2\pi \times 12}{2\pi . 3} = 4,$$

or

$$a' = 4a.$$

Hence

$$\left.\begin{aligned} 4a &= g - \frac{T'}{6} \\ a &= \frac{4T'}{10} - g \end{aligned}\right\}$$

From the first $T' = 6g - 24a$,

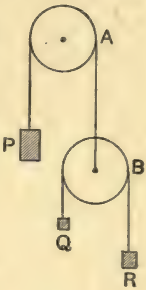
$$\therefore\ 5a = 2T' - 5g = 12g - 48a - 5g = 7g - 48a,$$
$$53a = 7g,$$

or

$$a = \tfrac{7}{53}g, \quad \text{and } a' = \tfrac{28}{53}g.$$

The mass of the wheel ought in general, however, to be taken into consideration. To do this requires the principles developed in Part III.

EXAMPLE II. *Determine the motion indicated in the annexed diagram, the pulleys being of no mass.*



Let a be the acceleration of P downwards, a', a'' the accelerations of Q, R upwards. Then B goes up with acceleration a.

$\therefore$ the acceleration of Q relative to B is $a-a'$, and of R relatively to B is $a-a''$. But, since the string QBR is inextensible, the acceleration of Q *up* relatively to B must be equal to the acceleration of R *down* relatively to B.

$$\therefore\ a-a'=-(a-a''),$$

or

$$a'+a''=2a.$$

Let T denote the tension of QBR. Then the tension of BA = 2T.

$$\therefore \text{ Force on P down} = \mathrm{P}g-2\mathrm{T},$$

$$\therefore\ a=g-\frac{2\mathrm{T}}{\mathrm{P}}.$$

So

$$a'=\frac{\mathrm{T}}{\mathrm{Q}}-g,\quad a''=\frac{\mathrm{T}}{\mathrm{R}}-g.$$

But

$$a'+a''=2a,$$

$$\therefore\ \frac{\mathrm{T}}{\mathrm{Q}}+\frac{\mathrm{T}}{\mathrm{R}}-2g=2g-\frac{4\mathrm{T}}{\mathrm{P}},$$

or

$$\mathrm{T}\left(\frac{4}{\mathrm{P}}+\frac{1}{\mathrm{Q}}+\frac{1}{\mathrm{R}}\right)=4g.$$

$$\therefore \mathrm{T}=\frac{4\mathrm{PQR}}{4\mathrm{QR}+\mathrm{PR}+\mathrm{PQ}}g,\quad a=\frac{(\mathrm{Q}+\mathrm{R})\mathrm{P}-4\mathrm{QR}}{4\mathrm{QR}+\mathrm{PR}+\mathrm{PQ}}g,$$

$$a'=-\frac{4\mathrm{QR}-3\mathrm{PR}+\mathrm{PQ}}{4\mathrm{QR}+\mathrm{PR}+\mathrm{PQ}}g,\quad a''=-\frac{4\mathrm{QR}+\mathrm{PR}-3\mathrm{PQ}}{4\mathrm{QR}+\mathrm{PR}+\mathrm{PQ}}g.$$

EXAMPLES—XII.

1. Find the advantages of the different mechanical elements by methods other than those in the text. For instance, when proved in the text by energy, deduce them by the conditions for the equilibrium of forces.

2. Two weights, each equal to 8 lbs., hanging on a straight lever at points 12 inches and 18 inches from the fulcrum, and on the same side of it, are balanced by a single vertical force acting at a point 16 inches from the fulcrum. Find the magnitude of the force.

3. Weights of 5 oz. and 7 oz. balance on a lever in which the shorter arm is 3 feet. Find the length of the lever.

4. In an 8-oar boat the oars are 10 feet long, the distance from the hand to the rowlock is 2 feet 6 inches; each man pulls with a force equal to the weight of 60 lbs. Find the force on the boat, supposing the blades of the oars not to move through the water.

5. The arms of a balance are $9\frac{1}{2}$ and $9\frac{3}{5}$ inches long respectively, and the scales balance when empty. How much would a customer gain or lose in what was weighed as a pound of tea?

6. A tradesman has a balance with unequal arms, and weighs for successive customers in alternate arms. Does he gain or lose on the average, and how much?

7. A common steelyard weighs 10 lbs.; the weight is suspended from a point 4 inches from the fulcrum, and the centre of gravity of the steelyard is 3 inches on the other side of the fulcrum; the movable weight is 12 lbs. Where should the graduation corresponding to 1 cwt. be situated?

8. A steelyard is 12 inches long, and with the scale pan weighs 1 lb., the centre of gravity of the two being 2 inches from the end to which the scale pan is attached. Find the position of the fulcrum when the movable weight is 1 lb. if the greatest weight that can be ascertained by means of the steelyard is 12 lbs.

9. In a steelyard the length of the rod is 2 feet, its weight 2 lbs., the distance of its centre of gravity from the fulcrum 1 inch towards the end of the shorter arm, the distance of the point where the weight is suspended from the fulcrum 2 inches, and the movable weight 6 oz. Find the greatest weight which can be weighed.

10. The length of the shorter arm of a common steelyard is 4 inches. If when W is increased by 2 oz. P must be moved through $\frac{1}{8}$ inch, and the division corresponding to 4 lbs. weight is 3 inches from the fulcrum, find the moment of the beam about the fulcrum.

11. A capstan has eight spokes, each 8 feet long, and three men work each at a distance of 1 foot 6 inches from one another, the outer man being at the end; the capstan is 2 feet in diameter, and each man pushes with a force of 56 lbs. weight. Find the force exerted on the anchor.

12. In the system of pulleys in which the same string goes round all the pulleys, the applied force is 2 lbs. weight; the lower block weighs 8 lbs. and contains 3 pulleys: the string is fastened to the lower block. Show that $W = 6$ lbs.

13. In that system of pulleys in which the same string goes round all the pulleys, there are 5 strings at the lower block. What pull is necessary to just raise a mass of 1 ton? Find the power exerted when the string is pulled out at a speed of 5 feet per second.

14. In a system where the same string goes round all the blocks,

one block is made of one solid piece. Find the relation between the sizes of the grooves cut in it.

15. In that system of pulleys in which each string is attached to the weight, there are three strings. Find the point of the bar to which the weight must be attached, supposing the pulleys of equal size.

16. Supposing in the previous case the diameters of the pulleys were 4, 5, 6 inches respectively, and all the strings were vertical, determine the point of attachment of the weight.

17. Find the relation between the force P and the weight W in a system of five movable pulleys in which each pulley hangs by a separate string, and the weight of each pulley is equal to P.

18. A mass of 1 ton has to be raised by a man who can only exert 34 lbs. weight. If he uses a system of pulleys in which each string has one end fixed, passes under one pulley, and is fastened to the next, how many pulleys must he use? If the weight of each pulley were 2 lbs., how many pulleys must he use?

19. There are four pulleys in which the strings are vertical, and all attached to the weight. What weight can be supported by a force of 14 lbs. weight, and what force can support a weight of 14 lbs. (1) if the weights of the pulleys themselves can be neglected, and (2) if each pulley weighs 1 lb.?

20. In a system of pulleys in which each string is attached to the weight, each pulley has the weight w, the sum of the weights of the pulleys is W′, and P and W are the applied force and weight, prove that the force $P+w$ would support the weight $W+W'$ in the same system if the pulleys had no weight.

21. In the system of pulleys where each string passes under one pulley and is attached to the next, if W be the weight supported, and $w_1, w_2, \ldots, w_n$ the weights of the movable pulleys, there will be no mechanical advantage unless

$$W-w_n+2(W-w_{n-1})+2^2(W-w_{n-2})+\ldots+2^{n-1}(W-w_1)$$

be positive.

22. In the Spanish Barton C is a fixed pulley. Two pulleys A, B hang over C by a string; another string is fastened to a fixed point D, passes under B, over A, and bears a mass P at the end; another mass Q is suspended to the pulley B. Find the relation between P and Q when there is equilibrium.

23. Find the mechanical advantage of a screw whose diameter is 6·78 inches, and the distance between whose successive threads is ·71 inch.

24. A man has to raise 10 cwts. by means of a screw; he can exert a force of 10 lbs. weight with each hand, and there is a double arm of 2 feet total length. Find the greatest pitch of screw allowable, supposing no friction.

25. A capstan, whose diameter is 20 inches, is worked by a lever which measures 5 feet from the axis of the capstan. Find in foot-pounds the amount of work done in drawing up by a rope 1 ton over 35 feet of the surface of a smooth plane whose height is three-fifths its length. The rope may be supposed to be always parallel to the surface of the plane. Find also the force applied to the end of the lever and the distance through which its point of application moves.

26. In a wheel and axle the wheel is driven by a tangent screw; the pitch of the screw is $\frac{1}{10}$ inch, and the ratio of the radii of the wheel and axle is 10 : 1. Find the mechanical advantage, the head of the screw being 1 inch diameter.

27. In the system of pulleys in which the same string goes round all the pulleys, the lower pulley bears a mass of 1 cwt. while 12 lbs. is suspended from the end of the string; there are seven strings at the lower block. Find the accelerations of the masses and the tension of the string.

28. In a wheel and axle whose radii are in the ratio 5 : 1, a mass of 1 lb. is suspended over the wheel and 10 lbs. over the axle. Find their accelerations and the tensions of the strings, supposing the wheel and axle to have no mass.

29. Prove that if W lbs. and P lbs. balance on a system of pulleys when $W=nP$, the acceleration of W downwards will be $\frac{W-nP}{W+n^2P}g$, and of P upwards will be $\frac{nW-n^2P}{W+n^2P}g$, supposing $W>nP$.

30. Fourteen horse-power is transmitted from one place to another by a rope which can only bear a strain of 56 lbs. weight. Find the least speed at which the rope can be driven.

31. A water-wheel of 10 feet diameter takes in the water from a still reservoir at its highest point, and empties its buckets after a quarter of a revolution. Compare the power it produces and its efficiency when it makes (1) 20 revolutions per minute, (2) 5 revolutions per minute.

CHAPTER XIII

FRICTION

127. HITHERTO, and especially in the last chapter, we have in the illustrations of principles treated bodies as smooth. That is, we have supposed that two bodies in contact offer no resistance to being slipped the one along the surface of the other—or, which comes to the same thing, that the mutual stress between two bodies in contact is perpendicular to their common surface. As a matter of fact, however, this is never absolutely the case. In many instances the supposition is so nearly true that no appreciable error is introduced by neglecting altogether the effects of the friction. In general, and especially in machines whose elements contain the inclined plane, axle bearings, or the screw, it is necessary to take account of the friction if any approximation to real conditions is sought.

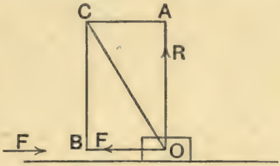
Ordinary experience teaches us that the resistance to the slipping of one body over another depends on the nature of the surfaces and the pressure with which they are forced together. Before being able to calculate the effects of friction in any case it will, therefore, be necessary to investigate the laws which it obeys. Consider now a body on the horizontal surface of another. In this case no force is exerted horizontally, and the mutual stress is perpendicular to the surfaces—no friction is called into play. If, however, a small horizontal force be applied to the body, it still does not move—friction is now called into play, and a counteracting resistance equal and opposite to the force exerted is set up. If the force be slowly increased, this counteracting resistance also increases, so as just to counterbalance it, until at last a point is reached at which the friction reaches its limit and the body gives way. The friction does not cease; it still resists the motion, but it cannot exceed a certain limit. The force called into play when the body gives way is called the *limiting friction*. The laws of limiting friction as first discovered by Coulomb may be thus stated—

(1) When the surfaces in contact remain the same, the limiting friction is directly proportional to the normal pressure, and is therefore independent of the extent of surface in contact.
(2) When in motion, the friction is independent of the relative velocities of the two surfaces.

If R denote the normal stress or reaction between the two surfaces—that is, the force with which one body is pressed on to the other—and F the limiting friction, then according to the first of these laws F/R is constant, or if we write $F/R = \mu$, μ is a constant depending only on the nature of the surfaces in contact. It does not depend only on the materials of which the bodies are made, but also on the state of polish or otherwise of the actual surfaces. It is called the coefficient of friction.

It has been found by later experimenters that these laws hold within very wide limits. They are not, however, exact for very large pressures, nor do they seem to be so for very small ones. Moreover, the coefficients of friction for the same surfaces at rest and in motion are not the same, but the former is always somewhat larger than the other—in other words, it needs a larger force to start a body on a rough plane than to keep it moving after it is once started. This statement, again, is not quite exact for exceedingly slow velocities. In fact, starting from complete rest, μ appears to decrease from its statical value to its proper value for motion—very quickly, indeed, but not by a sudden jump. The value of μ also is found to be larger when the bodies have been some time in contact.

128. In the case of a rough body, then, on a surface the reaction consists of a normal pressure R combined with a tangential force F. These are equivalent to a single reaction in some definite direction. Let the figure represent a body on a rough plane, pressed on it (by its weight or otherwise) by a pressure = R. Let a force push it to the right. This calls into play an equal force F opposite to it. Let OA, OB represent these forces. Complete the parallelogram; then the total resistance is represented by OC making an angle AOC with the normal to the surface. If the force F be gradually increased, this angle also increases up to a certain point, beyond which the friction is not sufficient to maintain equilibrium. Now



$$\tan AOC = \frac{AC}{OA} = \frac{F}{R},$$

when then the body gives way

$$\tan AOC = \mu.$$

That is, the angle AOC which the total reaction makes with the normal when the body gives way is constant for the same materials. This angle is called the *angle of friction.* If it be denoted by ϵ,

$$\tan \epsilon = \mu.$$

129. In machinery the friction would be so great that not only would a large proportion of power be lost in driving it, but the friction would soon wear away the bearings. It is enormously diminished by the use of lubricants, such as oil. In this case the oil forms a thin layer between the two surfaces so as to prevent their coming into contact, and the friction properly so called becomes extremely slight. There is, however, now another kind of resistance to be overcome, due to the viscosity of the oil. The oil sticks to the two surfaces, and the relative motion takes place in the layer of oil itself. This, however, calls

up a very much less force, and at the same time the wear of the bearings becomes almost infinitesimal. Should, however, the pressure become extremely great the layer of oil becomes very thin, and friction again comes into play.

The following table gives the values of the angle and coefficient of friction in certain cases—

	ϵ.	μ.
Wood on wood (dry)	14° to 26°	·25 to ·5.
„ „ (soaped)	2° to 11½°	·04 to ·2.
Metals on dry oak .	26½° to 31°	·5 to ·6.
„ dry elm .	11½° to 14°	·2 to ·25.
Leather on oak .	15° to 19½°	·27 to ·38.
Metals on metals .	8½° to 11½°	·15 to ·2.

130. With the above laws of friction known, the principles already developed in the foregoing chapters enable us at once to calculate the effect of friction in different cases. The following example will serve to illustrate this—

A body rests on a rough horizontal table and is attached to a weight hanging freely by a string passing over a pulley at the edge of the table. Determine (1) *the weight needed in order to just move the body, and* (2) *the motion ensuing if the weight is larger than this.*

Let W be the weight of the body, μ the coefficient of friction between it and the table. Let w be the suspended weight.

Then, if R be the pressure of the body on the plane,

$$R=W,$$

and if F be the friction called into play,

$$F=T.$$

But if *the system is at rest*, $T=w$,

$$\therefore \left.\begin{array}{r} R=W \\ F=w \end{array}\right\}.$$

If now it is just on the point of motion, F has its greatest possible value, and

$$F=\mu R,$$

$$\therefore w=\mu W.$$

This affords an easy and expeditious way to determine μ, and also to prove the truth of the first law. We have only to alter the weight w until the body just begins to move. Then observe its magnitude, and

$$\mu=\frac{w}{W}.$$

On altering the load by putting extra weights on the top of the body it will be found that w will be altered so that w/W or μ remains the same as before. Instead of hanging the weights, a spring-balance may be used to pull the body along the table, and the reading w observed when the body just gives way.

In determining the motion it has to be remembered that the coefficient of friction while moving is less than that necessary to start the body. Let this coefficient be represented by μ', and let M, m denote the masses of W and w.

Then the force tending to make the body move along the table

$$=mg-\text{F}=mg-\mu'\text{M}g,$$

and the mass moved is $m+\text{M}$;

$$\therefore \text{ Acceleration}=\frac{m-\mu'\text{M}}{m+\text{M}}g.$$

If m be the mass just necessary to start the body, and if the body be started by a slight push, then

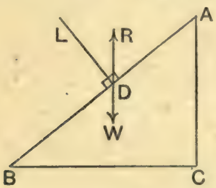
$$mg=\mu\text{M}g$$

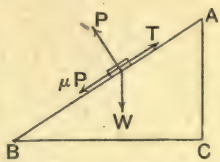
by the former part of the question, and in this case

$$a=\frac{\mu\text{M}-\text{M}'\mu}{\mu\text{M}+\text{M}}g=\frac{\mu-\mu'}{\mu+1}g.$$

In general $\mu-\mu'$ is very small, so that it would move with a constant small acceleration. If, however, the motion were stopped, the bodies would remain just at rest.

131. *Body resting on an inclined plane.*—Let us find the inclination of the plane in order that the body may just be at rest. The body is acted on by only two forces, viz. its





weight vertically downward and the reaction of the plane. Hence these must be in the same line, and therefore the reaction of the plane must be vertical. The reaction must, therefore, make with the normal to the plane the angle RDL = 90° − BDW = ABC. Therefore, as the inclination of the plane is increased, RDL increases also, until at last it becomes equal to the angle of friction, beyond which it cannot extend, and the body would begin to move. The body therefore gives way when RDL $= \epsilon$ or ABC $= \epsilon$. That is, the inclination of the plane must be the angle of friction. This is also an easy method to determine the value of μ.

If the plane had been smooth, the work done in pulling the body up the plane AB would have been equal to that done in raising it freely through the height AC. If, however, the plane be rough, the work done must be greater than this. In order to find the amount of work done, let us suppose the reaction resolved into components, a pressure P normal to the plane and a friction F along the plane. *Since the body is being pulled upwards,* F *will act downwards.* Also, since it is limiting friction, $F = \mu P$. Then, if T be the force pulling it up the plane, we have

$$P = \frac{BC}{AB} \cdot W,$$

also, resolving up the plane,

$$T = F + W\frac{AC}{AB},$$

$$= \mu P + W\frac{AC}{AB},$$

$$T = \left(\frac{\mu BC}{AB} + \frac{AC}{AB}\right)W.$$

Now the work done $= T \times AB$,

$$\therefore \text{Work} = \mu W . BC + W . AC.$$

Here W . AC is the work done in raising W freely through a height AC; also, if BC be regarded as of the same material as AB, and W be pulled along it, the friction will be μ . W and the work μW . BC. Hence we obtain this very important result—

The work done in raising a body up an inclined plane is equal to the work done in raising it through the height of the plane and dragging it along the base supposed of the same material as the plane itself.

In all cases where work is done against friction it is dissipated into heat or other energy, and is not given back again if the path be reversed. Thus here, if the body be moved down the plane, gravity returns the work formerly done against it by pulling the body through the height of the plane. But work still has to be done to drag the body back over the base.

In the rough inclined plane the mechanical advantage is

$$A = \frac{W}{T} = \frac{1}{\mu\frac{BC}{AB} + \frac{AC}{AB}},$$

$$= \frac{1}{\mu \cos \alpha + \sin \alpha}.$$

Note.—For a smooth plane $A = 1/\sin \alpha$; if the inclination of the rough plane be the angle of friction, $\mu = \tan \alpha = \sin \alpha / \cos \alpha$ and $A = 1/2 \sin \alpha$.

Hence, if a rough plane be inclined at the angle of friction, the mechanical advantage is reduced one-half by the roughness.

The above formula gives T, the force necessary to pull the weight W up the plane. If the inclination of the plane is less than the angle of friction, the body will rest on it, and it would require a force to pull it down. Let T′ be the force necessary to pull it down. In this case the friction F acts *up* the plane in opposition to T′. P is the same as before. Hence

$$T' - \mu P + W\frac{AC}{AB} = 0,$$

$$T' = W\left\{\mu \cdot \frac{BC}{AB} - \frac{AC}{AB}\right\},$$

$$= W(\mu \cos \alpha - \sin \alpha).$$

Any force then between $W(\sin \alpha + \mu \cos \alpha)$ *up* the plane and $W(\mu \cos \alpha - \sin \alpha)$ *down* the plane will be insufficient to make the body move.

132. Of the elementary machines the inclined plane and the screw are those most affected by friction. The results of the previous paragraph enable us at once to find the mechanical advantage in a rough screw.

For, give the screw one complete turn. Since the action of the screw is that of the threads sliding up an inclined plane, the work done by the screw

= work in moving freely through the pitch + work in dragging round the circumference of the screw cylinder.

Or, if F, f denote the forces exerted by the screw and applied to the arm respectively, work done by screw

= F × pitch + μF × circumference of screw.

But this is the work done by f on the screw,

= f × circumference traced out by the arm of the screw.

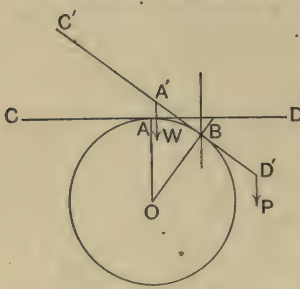
Hence, if r_1, r_2 denote the radii of the arm and screw,

$$f \cdot 2\pi r_1 = \mathrm{F} \times \text{pitch} + \mu \mathrm{F} 2\pi r_2,$$

$$\therefore \ \mathrm{A} = \frac{\mathrm{F}}{f} = \frac{2\pi r_1}{\text{pitch} + 2\pi r_2 \mu}.$$

133. The rest of this chapter will be devoted to examples more interesting for the methods of solution than for their importance in themselves.

EXAMPLE I. *A heavy uniform beam rests on the top of a rough sphere. Find the greatest weight which can be placed on one end that it may not slip off.*



Let C′A′D′ be the position when just on the point of slipping off. B the point of contact with the sphere, and A the highest point. Also let P be the greatest weight which can be placed at D′ without the beam coming off. Let a be the radius of the sphere, $2l$ the length of the beam, and W the weight of the beam.

Then the resultant of W and P must pass through B, the point of contact, and the reaction at B, being equal and opposite to this resultant, must be vertical. Therefore OB must make with the vertical the angle of friction, or

$$\mathrm{AOB} = \epsilon.$$

This gives the position.

Again $$W \,.\, A'B = P \,.\, BD',$$
and $A'B = \text{arc AB}$, since the rod rolls without slipping;
$BD' = A'D' - \text{arc AB}$.

But $$\frac{\text{arc AB}}{\text{circumference}} = \frac{\text{angle AOB}}{\text{4 right angles}},$$

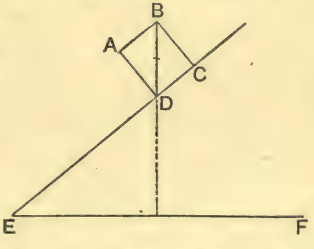
therefore, if ϵ be measured in degrees,

$$\text{arc AB} = \frac{\epsilon}{360} 2\pi a.$$

Hence
$$P = \frac{A'B}{BD'} \,.\, W,$$
$$= \frac{\text{arc AB}}{l - \text{arc AB}} \,.\, W,$$
$$= \frac{\frac{2\pi a \epsilon}{360}}{l - \frac{2\pi a \epsilon}{360}} \,.\, W,$$
$$= \frac{\epsilon}{\frac{180}{\pi a} \,.\, l - \epsilon} \,.\, W.$$

EXAMPLE II. *A rectangular block rests on a rough inclined plane. If the inclination be gradually increased, determine whether it will first begin to slide or to topple over.*

Let the figure represent the position when the diagonal BD is vertical. Then the centre of gravity is just over D, and if the plane be raised ever so little the block will topple over.



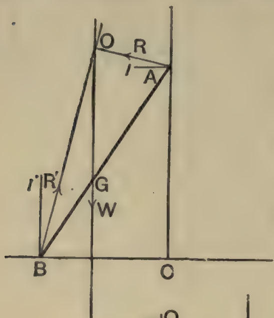
Also, if the inclination of the plane be greater than the angle of friction, it will slip.

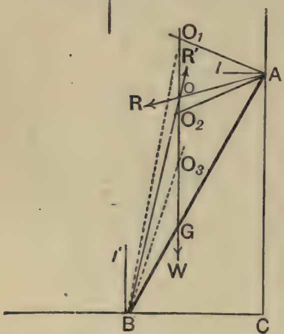
If then CEF be greater than the angle of friction, the block will have begun to slide before the inclination is so great as CEF—that is, it will slide first. If, on the contrary, CEF, or which is the same thing ADB, is less than ϵ, it will topple over first. Hence, if a denote the angle between the diagonal and the side perpendicular to the plane,

it will slide first if $a > \epsilon$,
it will topple over first if $a < \epsilon$.

EXAMPLE III. *A ladder rests against a rough vertical wall and on a rough horizontal pavement. Determine the reactions in any position and the position in which it is just on the point of slipping out.*

Let AB represent the ladder in any position, and let G be its centre of gravity, usually nearer the lower end than the top. B tends to move out, A down. Hence the total reaction at B and A will be in the directions indicated in the figure. Let them be R, R′ making angles α, α' with the horizontal Al and the vertical Bl'. Then α, α' must both be less than the angles of friction for the ladder and the wall, and the ladder and the ground respectively.





We have then the ladder acted on by three forces W, R, R′. Hence they must meet in a point (O say). This is sufficient to determine R, R′ when OAl, OBl' (or simply one of them and ABC) are given.

But there is nothing further to determine these angles beyond the fact that they must lie within certain angles.

For consider the reaction at A. Let O_1Al, O_2Al be each equal to the angle of friction $=\epsilon$.

Then the reaction R at A must lie somewhere within O_1AO_2, and must cut the vertical through G somewhere in O_1O_2.

Again, let $l'BO_3$ be equal to the angle of friction at $B=\epsilon'$ say. Then the reaction R′ at B must cut the vertical through G somewhere above O_3. Since R, R′ must meet in the vertical through G, it is clear that if O_3 be above O_1, equilibrium cannot subsist. If O_3 is just at O_1, the ladder is just on the point of slipping. If O_3 lies between O_1 and O_2, there will be equilibrium, but the reactions will be indeterminate, varying between limiting friction at A and limiting friction at B.

If O_3 be below O_2, there will be equilibrium, but the reactions will be indeterminate, varying between limiting friction at A downwards and limiting friction at A upwards. If O be given, then the reactions are perfectly determinate.

Of course in any actual case the reactions are definite. Their values are settled according to the way in which the ladder was put on and the elastic give of the material of which it is composed. The student can easily illustrate this by sitting in an easy-chair covered in plush. According to the way in which he sits down he may feel in one case that he is chiefly kept from slipping out by the friction on his trousers, and in another case chiefly by the friction on the back of his coat.

In the present case the reactions would be different if the ladder resting at B were quietly lowered on to A, from those if the ladder resting at A were quietly lowered on to B.

The less O_1O_3 is, the less ambiguity is there.

In one case, however, there is no ambiguity, viz. when the ladder is just on the point of slipping. In this case $O_1Al=\epsilon$, $O_1Bl'=\epsilon'$.

If θ be the inclination of the ladder to the vertical,

$$O_1GA=\theta,$$

also

$$BO_1G=O_1Bl'=\epsilon',$$

$$AO_1G=90^\circ-O_1Al=90^\circ-\epsilon.$$

Hence we have triangles in which

$$BG=b,\quad AG=a \text{ are known,}$$

$$GOB=\epsilon',\quad GOA=90^\circ-\epsilon,$$

and

$$OGA=\theta \text{ can be found.}$$

The angle $OBG=OGA-GOB=\theta-\epsilon'$,

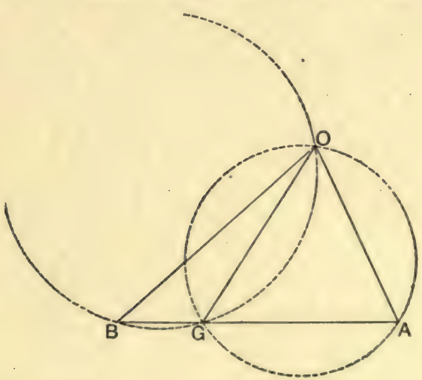
$$OAG=OGB-GOA=180^\circ-\theta-(90^\circ-\epsilon)=90^\circ+\epsilon-0.$$

Students who are acquainted with the solution of triangles will then see that

$$\frac{OG}{BG}=\frac{\sin B}{\sin BOG},\quad \frac{AG}{OG}=\frac{\sin GOA}{\sin A}.$$

Hence

$$\frac{AG}{BG}=\frac{a}{b}=\frac{\sin(\theta-\epsilon')}{\sin\epsilon'}\cdot\frac{\sin(90^\circ-\epsilon)}{\sin(90^\circ+\epsilon-\theta)}=\frac{\cos\epsilon}{\sin\epsilon'}\cdot\frac{\sin(\theta-\epsilon')}{\cos(\theta-\epsilon)},$$



whence θ may be found.

Or it may be done graphically thus: on BG describe the segment of a circle containing an angle ϵ' and on AG a segment containing an angle $90° - \epsilon$ (Eucl. III). Let them intersect at O. Join OG. Then OGA is the inclination of the ladder when it begins to slip.

An example illustrating indeterminateness of solution when bodies are treated as rigid has been given in § 89.

EXAMPLE IV. *A heavy body rests on a rough horizontal board, which is moving horizontally. Determine the acceleration that the body may remain without slipping on the board.*

Let a be the acceleration, m the mass of the body, and F the friction actually called into play. Then the force on m making it move $=$ F,

$$\therefore a = \frac{F}{m}.$$

But if it is just on the point of slipping,

$$F = \mu \times \text{weight} = \mu m g = m a\,;$$

$\therefore$ if $a = \mu g$, the body is just on the point of slipping over the board. If a be greater than this, there is relative motion; and if μ' represent the coefficient of friction when moving, the acceleration of the body is $\mu' g$, and relatively to the board is $a - \mu' g$.

EXAMPLE V. *A train is travelling on a horizontal line; it is brought to rest by the application of the brakes to some of the wheels. How far will it go before coming to rest if resistances other than those due to the brakes be neglected?*

Let m denote the mass of the whole train, v the original velocity, u the velocity after a distance s, P the pressure of a brake on a wheel, R the pressure of a wheel on the rail (both P and R measured in terms of the weight of unit mass), μ, μ' the coefficients of friction of the wheel with the brake and rail respectively, n the number of wheels braked.

Then the only work done is that by the friction. Hence

$\frac{1}{2}m(v^2 - u^2) =$ number of absolute units of work done by the friction.

Now two extreme cases will arise according as all the wheels skid along the rails, or roll on the rails and slip over the brakes. There may be intermediate cases where some wheels skid and others roll.

In the first case the work is done by the friction on the rails, *i.e.* by $\mu' R$; in the second case by μP. We shall only consider here these two cases, the intermediate ones being easily worked out in a similar way. Whichever of these cases is happening, call the friction F, so that $F = \mu' R$ in the first and μP in the second. If $\mu' R < \mu P$ the wheels will skid, if $\mu' R > \mu P$ they will roll on the rail without skidding. Hence F is to stand for the larger of $\mu' R$ or μP.

Now supposing, as is usually the case, that the brakes are applied to the rim of the wheel, the distance through which the wheel moves against F is in both cases the distance through which the train moves. Hence the work done is Fs gravitation units of work.

Therefore $\frac{1}{2}m(v^2 - u^2) = Fsg$.

It therefore comes to rest after a distance

$$s = \frac{mv^2}{2Fg},$$

where F is the greater of μ'R or μP.

EXAMPLES—XIII.

1. A force can just move a given weight up a plane of 30°, and can just prevent a weight twice as great from moving down a plane of 60°. Prove that the coefficient of friction, which is the same for both planes, $= \frac{23}{50}$ nearly.

2. Assuming that a force of 20 lbs. weight per ton of load is required to maintain the motion of a train on a level line, determine the coefficient of friction between the driving-wheels and rails when an engine of 27 tons weight can just keep in motion a train of 252 tons.

3. A weight of 10 tons is dragged in half an hour a length of 330 feet up a rough plane inclined 30° to the horizontal plane, the coefficient of friction being $1/\sqrt{3}$. Find the work expended and the horse-power of an engine by which the work could be done.

4. A weight of 30 lbs. is just supported on a rough inclined plane (coefficient of friction $\frac{3}{4}$) whose height is three-fifths of its length. Show that it will require a force of 36 lbs. weight acting parallel to the plane to be on the point of moving the weight up the plane.

5. A weight of 5 lbs can just be supported on a rough inclined plane by a weight of 2 lbs., or can just support a weight of 4 lbs. suspended by a string passing over a smooth pulley at the top of the plane. Find the coefficient of friction and the sine of the inclination of the plane to the horizon.

6. A particle of 10 lbs. mass is kept on a rough plane, inclined at 45° to the horizon, by a horizontal force. If $\mu = \frac{1}{2}$, find the least and greatest possible values of the force.

7. A weight W is just supported by friction on a plane inclined at an angle α to the horizon. Show that it cannot be moved up the plane by any horizontal force less than W tan 2α.

8. An inclined plane is partly smooth and partly rough ($\mu = \sqrt{3}/2$; a particle slips down the upper smooth part and moves on to the rough part; the inclination of the plane is 30° and the length of the smooth part is 4 feet. How far will it move before it comes to rest?

9. If the height of an inclined plane be 12 feet, the base 16 feet, find how far a body will move on the horizontal plane, supposing it to

pass from one plane to the other without loss of velocity, the coefficient for both planes being $\frac{1}{8}$.

10. A body slides down a rough inclined plane 100 feet long, the sine of whose inclination $= \cdot 6$ and coefficient of friction $= \frac{1}{2}$. Find its velocity at the bottom.

If projected up the plane with a velocity which just carries it to the top, find what height it would reach if thrown vertically upwards with that velocity.

11. The two planes AB and AC are hinged together at A, and AC is horizontal, while AB slopes downwards at the angle (α) to the horizon; two equal weights are connected by a string and placed one on AB and the other on AC, the limiting angle of friction between each weight and plane being (ϵ) and less than (α). Find through what angle AC may be slowly tilted round the hinge, always sloping towards A before motion ensues.

12. A mass rests on a rough horizontal board and is connected by a string which passes over a pulley on the board to another mass hanging vertically, and the system is just on the point of moving. Will it keep at rest if the board be lowered with an acceleration a?

13. Two weights are connected by a string and placed upon a rough inclined plane, with the string parallel to the line of greatest slope of the plane. If the coefficient of friction between the weights and the plane be different, find the angle of inclination of the plane at which they will just begin to slide down, and prove that it is intermediate between the angles of inclination for the weights taken separately.

14. Two equal heavy particles on two equally rough inclined planes of the same height, and placed back to back, are connected by a string passing over the top of the planes. Show that when the particles are on the point of moving, the limiting angle of resistance will be half the difference of the inclination of the planes.

15. A heavy string rests on two given rough inclined planes of the same material, passing over a smooth pulley at their common vertex. If the string is on the point of motion, show that the line joining its two ends is inclined to the horizon at the angle of friction.

16. A heavy slab whose under surface is rough, but the upper smooth, slides down a given inclined plane. Find the acceleration with which a small particle laid on its upper surface will move along the slab.

17. Prove that a train going 45 miles per hour will be brought to rest in about 378 yards by the brakes, supposing them to press with two-thirds of the weight on the wheels of the engine and brake-vans, which are half the weight of the train; and supposing a coefficient of friction $\cdot 18$.

18. Prove that a train going 60 miles per hour can be brought to

rest in about 313 yards by the brakes, supposing them to press on the wheels with two-thirds of the weight of the train and a coefficient of friction ·18, in addition to a passive resistance of 20 lbs. weight per ton on the level.

19. If a right cone be placed with its base on an inclined plane, friction being sufficient to prevent sliding, examine the conditions that the cone may just remain at rest on the plane.

If $1/\sqrt{3}$ be the coefficient of friction, find the angle of the cone when it is on the point both of sliding and falling over.

20. A right circular cone whose semivertical angle is α stands on a rough plane which is gradually lifted up. Show that if $4 \tan \alpha = \tan \epsilon$, where ϵ = angle of friction, it will be doubtful whether the cone will slip or upset.

21. A uniform ladder 10 feet long rests with one end against a smooth vertical wall and the other on the ground, the coefficient of friction being $\frac{1}{2}$. Find how high a man whose weight is four times that of the ladder may ascend before it begins to slip, the foot of the ladder being 6 feet from the wall.

22. A uniform ladder 70 feet long is equally inclined to a vertical wall and the horizontal ground, both rough; the weight of a man with his burden ascending the ladder is 2 cwts., and the ladder weighs 4 cwts. How far up the ladder can the man ascend before it slips, the coefficient of friction for the wall being $\frac{1}{3}$ and for the ground $\frac{1}{2}$?

23. A uniform beam rests with one end on the ground, and the other against a vertical wall. If ϵ, the angle of friction of the ground, be the same as that of the wall, show that 2ϵ is the inclination of the ladder to the vertical when it is on the point of slipping.

24. A uniform ladder rests between a vertical wall and the horizontal ground, both rough. If the coefficient of friction for the ladder and wall be $\frac{1}{3}$ and for the ladder and ground $\frac{3}{7}$, find the angle which the ladder makes with the ground when it just begins to slide.

25. A uniform ladder just rests with one end on the horizontal ground, the other leaning against a vertical wall. If θ be the angle it makes with the ground, prove that $\tan \theta = (1 - \mu\mu')/2\mu$, where μ and μ' are the coefficients of friction respectively.

26. A uniform bar is placed in a sloping position, its lower end on the ground (coefficient of friction $= \mu$), its upper in the air, the bar being supported by a smooth fixed peg against which it rests. If l = length of bar and h = height of peg from the ground, and if α be the angle the bar makes with the horizon when on the point of slipping, prove that α is to be found from the equation

$$\cos \alpha \sin^2 \alpha + \mu \sin \alpha \cos^2 \alpha = \frac{2h}{l} \mu$$

(supposing the centre of the bar below the peg).

27. A cylinder rests on a rough inclined plane with its axis horizontal, and is kept in equilibrium by a string which passing round it and over the top has one end fastened to a point in the plane, so that its length is perpendicular to the plane. Find the inclination of the plane when the cylinder is just on the point of slipping.

28. A triangle ABC formed of three uniform rods jointed together is supported by a rough peg under the middle point of AB. Prove that the least angle of friction is $\frac{1}{2}(A \sim B)$, and that the sides AC, BC are equally inclined to the vertical.

29. A hemisphere rests with its curved surface touching a vertical wall ($\mu = \frac{8}{19}$) and a horizontal plane ($\mu' = \frac{1}{8}$). Show that when the body is on the point of slipping, the inclination (θ) of the plane of its base to the horizon is given by $\sin \theta = \frac{9}{20}$.

30. Three equal spheres are placed on a horizontal table so as to touch each other; another sphere of the same radius is placed on the top. Show that if the spheres and the table are all made of the same material and the system is on the point of slipping, the coefficient of friction is $\sqrt{3} - \sqrt{2}$.

31. If the ratio of the greatest to the least forces which acting parallel to a rough inclined plane can support a weight on it, be equal to the ratio of the weight to the pressure on the plane, show that the coefficient of friction will be $\tan \alpha \tan^2 \frac{\alpha}{2}$, where α is the inclination of the plane to the horizon.

32. Two rings, each of weight w, slide upon a vertical semicircular wire with the diameter horizontal and convexity upwards; they are connected by a light string of length $2l$ (supposed less than $2a$, the diameter of the semicircle), on which is slipped a ring of weight W. Show that when the two rings that slide on the semicircle are as far apart as possible, the angle 2α subtended by them at the centre is given by the equation $(W + 2w)^2 \tan^2(\alpha + \epsilon)(l^2 - a^2 \sin^2\alpha) = W^2 a^2 \sin^2\alpha$, where $\tan \epsilon$ is the coefficient of friction.

33. Two sides of a rigid rectangular lamina are horizontal, the lower being hinged to a vertical wall and the upper connected with the wall by a string, while a rough sphere rests in equilibrium between the wall and the lamina; the coefficients of friction of the sphere with the wall and lamina being μ, μ'. If the string lengthens slightly, prove that if $\mu = \mu'$ the sphere will slide along the wall and roll along the lamina. Find the relation between μ and μ' in order that the sphere may roll along the wall and slide along the lamina.

CHAPTER XIV

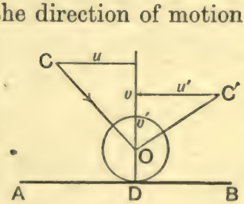
OBLIQUE IMPACT

[*This may be omitted on a first reading*]

134. When a sphere impinges directly on another—that is, so that the directions of motion of the two are in the line joining their centres—the motion ensuing is determined by the methods of Chapter II. It remains to investigate how the motion is affected when the motion before impact is not direct. The methods are not necessarily confined to spherical bodies. It is only requisite that the bodies shall be symmetrical about the line of impact, so that after impinging no rotations of the bodies shall be set up. When rotations ensue, the motion of the relative parts of the body has to be taken into account, a subject considered in the third section of this book, but too difficult to treat at any length with elementary mathematics.

The simplest case is that where a sphere impinges on a rigid plane surface.

Let AB represent the plane, CO the direction of motion before and OC′ the direction after the impact. Also let V, V′ be the velocities and a, a' the angles which their directions make with the normal DO before and after.



The velocities V, V′ can be replaced by their components parallel to the plane and perpendicular to the plane. We can then consider the alteration in each component separately.

Let u, v be the components of V parallel and perpendicular to AB, and u', v' the corresponding components of V′. Then considering first the motions parallel to AB, no effect is produced by the impact, since the blow is perpendicular to AB. Hence

$$u = u'.$$

The motion represented by v, v' is that of a direct impact on the plane. This has already been considered in Chapter II, and Newton's law, that the relative velocities after and before are in a constant ratio $-e$, gives

$$v' = -ev.$$

The motion after is therefore given by

$$\left.\begin{aligned} u' &= u \\ v' &= -ev \end{aligned}\right\}.$$

Now $$\tan\alpha = \frac{u}{v},$$

$$\tan\alpha' = \frac{u'}{v'} = \frac{u}{ev},$$

taking the positive sign, as we are now dealing with magnitudes only. Hence

$$\tan\alpha' = \frac{1}{e}\tan\alpha.$$

This gives the angle of reflection in terms of the angle of incidence and the coefficient of rebound. If the ball is perfectly elastic, $e = 1$, $\tan\alpha' = \tan\alpha$, or $\alpha' = \alpha$, and the angles of incidence and reflection are equal. Also

$$\begin{aligned} V'^2 &= u'^2 + v'^2 = u^2 + e^2v^2, \\ &= v^2\left(e^2 + \frac{u^2}{v^2}\right) = v^2(e^2 + \tan^2\alpha), \end{aligned}$$

and $$V^2 = u^2 + v^2 = v^2\left(1 + \frac{u^2}{v^2}\right) = v^2(1 + \tan^2\alpha).$$

Hence $$\frac{V'^2}{V^2} = \frac{e^2 + \tan^2\alpha}{1 + \tan^2\alpha},$$

which gives V′ in terms of V, the angle of incidence, and the coefficient of rebound.

Or thus, $u = V \sin \alpha,$

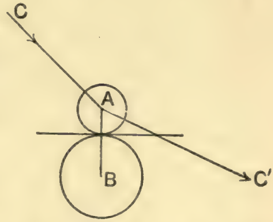
$$v = V \cos \alpha,$$

and $u'^2 + v'^2 = u^2 + e^2v^2;$

$$\therefore V'^2 = V^2 \sin^2\alpha + e^2V^2 \cos^2\alpha,$$
$$= V^2(\sin^2\alpha + e^2 \cos^2\alpha).$$

Both these formulæ are the same, since it is known from trigonometry that
$$\frac{\sin \alpha}{\cos \alpha} = \tan \alpha.$$

135. *A sphere impinges obliquely on another sphere at rest.*—Let A, B be the centres of the spheres when in contact CA, AC′ the directions of motion before and after.



Then the force on B can only be along AB. Hence after impact the second sphere will move in the direction AB. Let its velocity be V_1, and let the velocities, etc. of A be represented as in the previous case.

Then the motion perpendicular to AB is, as in the previous case, unchanged. Hence

$$u' = u \qquad \text{(i.)}$$

In the case of the motion along AB, B now moves, and the case is that of a sphere impinging directly on another at rest. Let m, m_1 be the masses of A and B. Then, by unchanged momentum,

$$m_1V_1 + mv' = mv,$$

by Newton's law, $v' - V_1 = -e(v - 0),$

whence, multiplying the second by m and subtracting,

$$(m_1 + m)V_1 = m(1 + e)v,$$

or $$V_1 = \frac{m}{m + m_1}(1 + e)v \qquad \text{(ii.)},$$

also, multiplying by m_1 and adding,

$$(m + m_1)v' = (m - m_1e)v,$$

$$v' = \frac{m - m_1e}{m + m_1}v \qquad \text{(iii.)}$$

Equations (i.), (ii.), (iii.) completely determine the motion.

A numerical case will, perhaps, illustrate the method more clearly. *A ball of* 10 *lbs. is moving with a velocity of* 15 *feet per second, and impinges on another of* 12 *lbs. making an angle of* 60° *with the line of centres. The coefficient of rebound is* ·5.

Here $$V = 15,$$
$$u = V\cos 30° = \frac{15\sqrt{3}}{2},$$
$$v = V\cos 60° = \frac{15}{2}.$$

Then $$u' = u = \frac{15\sqrt{3}}{2},$$

$$\left.\begin{aligned} v' - V_1 &= -\,\cdot 5v = -\frac{15}{4} \\ 10v' + 12V_1 &= 10v = 75 \\ 12v' - 12V_1 &= -12 \times \frac{15}{4} = -45 \end{aligned}\right\}.$$

Adding $$22v' = 30,$$
$$v' = \frac{15}{11} \text{ foot per second.}$$

Again $$10v' - 10V_1 = -10 \times \frac{15}{4} = -\frac{75}{2},$$

subtracting this from

$$10v' + 12V_1 = 75,$$
$$22V_1 = \frac{225}{2},$$
$$V_1 = \frac{225}{44} = 5\frac{5}{44} \text{ feet per second.}$$

Since v' is positive, the ball is deflected from its path, but is not reflected backwards.

The angle the direction of A makes with AB after impact is given by

$$\tan\alpha' = \frac{u'}{v'} = \frac{\frac{15\sqrt{3}}{2}}{\frac{15}{11}} = \frac{11}{2}\sqrt{3} = 9\cdot 526,$$

whence the trigonometrical tables give

$$\alpha' = 84°.$$

Also $$V'^2 = u'^2 + v'^2 = 3\left(\frac{15}{2}\right)^2 + \left(\frac{15}{11}\right)^2 = \frac{367 \times 15^2}{484} = 170\cdot 61,$$

$$\therefore V' = 13\cdot 06 \text{ feet per second.}$$

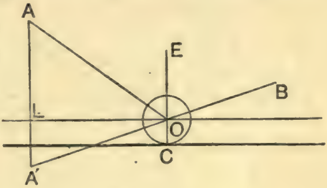
136. *Two spheres impinge obliquely.*—The method is precisely the same as the preceding. Resolve the velocities of each before and after impact perpendicular to and along the line of centres. Let them be

For A.	For B.	
u, v	u_1, v_1	before impact,
u', v'	u_1', v_1'	after impact.

Then the motion of each perpendicular to AB is unaffected, whilst the portion parallel to AB is a case of direct impact already treated.

137. In the case of a ball projected from a given point and reflected at a rigid plane, the subsequent path can easily be found as follows—

Let A be the point of projection and AO the direction. Draw LO at a distance from the plane equal to the radius of the sphere. Then the motion will be the same as if the sphere were a particle at its centre and reflected at the rigid plane LO.



Draw AL perpendicular to LO and produce it to A′, so that $A'L = e \,.\, AL$.

Join A′O and produce it to B. Then

$$\tan LAO = \frac{OL}{AL},$$

$$\tan LA'O = \frac{OL}{A'L} = \frac{OL}{e \,.\, AL} = \frac{1}{e} \tan LAO.$$

Now $\qquad LAO = AOE$, and $LA'O = BOE$.

Hence

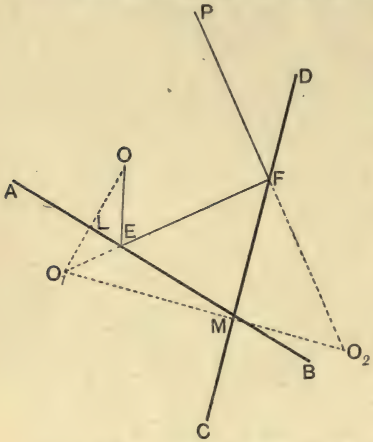
$$\tan BOE = \frac{1}{e} \tan AOE = \frac{1}{e} \tan \text{ (angle of incidence)},$$

$$\therefore BOE = \text{angle of reflection},$$

or OB is the direction of motion after reflection.

This curious result enables us to find in what direction a ball must be projected so as after impact at a given plane to pass through another point. For let A be the point of projection, B the point through which it has to pass. Find A′ as above. Join A′B, cutting LO in O. Then O must be the point of impact and AO the direction of projection.

138. The same method will serve to determine the direction of projection when it has to pass through a given point after impinging on any number of planes. Thus let AB, CD be two such planes, O the point of projection, and P the point through which it has to pass.



Draw OLO_1 perpendicular to AB and make

$$O_1L = e \cdot OL.$$

Draw O_1MO_2 perpendicular to CD and make

$$O_2M = e \cdot O_1M.$$

Join PO_2 cutting CD in F.

Join FO_1 cutting AB in E.

Join OE. Then OEFP will be the path of the ball.

For, from the foregoing, the ball after impact at AB will appear to move from O_1, and after impact from CD it will appear to move from O_2.

This construction supposes that the sphere is indefinitely small. In case the sphere is of finite size, the lines AB, CD must not represent the planes themselves, but planes parallel to them at a distance from them equal to the radius of the sphere.

EXAMPLES—XIV.

1. A sphere, mass 2 lbs., is projected with a velocity of 10 feet per second in a direction making an angle of 30° with a rigid plane. Find the subsequent motion, being given $e = \cdot 5$.

2. A sphere of given radius is projected from a given point so that its

direction after impinging on a rigid plane is perpendicular to its original direction. Show how to find by construction the direction of projection, the coefficient of rebound being given.

3. A perfectly elastic billiard-ball is struck so as to return to the same point after reflection at each of the four sides of the table. Show that it must be struck parallel to one of the diagonals of the table.

4. A smooth ball rests on the bottom of a square box; the coefficient of rebound between it and the sides of the box is $\frac{1}{2}$; it rests on a diagonal of the bottom and at one-quarter the diagonal from one corner. In what direction must it be struck so as after reflection at each side successively to return to its original position?

5. A sphere slips down a smooth inclined plane of 1 in 10 and 100 feet long; the inclined plane rests on a horizontal inelastic plane. Determine the time before the sphere reaches a point on this 20 feet from the foot of the inclined plane.

6. A sphere, mass 16 lbs., at rest is struck by another mass, 8 lbs., moving with a velocity of 20 miles per hour in a direction making an angle of 45° with the line of centres at the moment of impact; the coefficient of rebound is $\frac{1}{3}$. Determine the subsequent motion.

7. Two equal perfectly elastic spheres moving in parallel lines impinge on one another so that the line of centres makes an angle of 30° with their directions. If the velocities before impact were 29 and 16 feet per second, determine their velocities and directions afterwards.

8. If two billiard-balls were equal to each other in all respects, and were, as well as the table, perfectly smooth, prove that the direction taken by the striker's ball after hitting the object ball would be the same for all velocities of the former, and would depend only upon the point at which the latter was hit.

9. Two smooth inelastic spheres of radii 1 and 2 feet and of the same material are moving directly towards one another on a smooth inelastic plane. Determine the motion after impact.

10. Prove that the directions of the relative velocities of two perfectly elastic spheres before and after impact are equally inclined to the line of centres at the instant of impact.

11. Two balls A, B move in directions at right angles with velocities u, v. Prove that if they subsequently move at right angles to one another the coefficient of rebound $= \dfrac{A^2 u \cos \alpha + B^2 v \sin \alpha}{AB(u \cos \alpha + v \sin \alpha)}$, where α is the inclination of the direction of motion of A to the line of impact.

12. If a stream of particles of elasticity e all moving in parallel directions with velocity u impinge successively on two smooth fixed planes at right angles, prove that the average resultant of the pressures on the plane is $Mu(1+e)$, where M is the mass of the particles which strike each plane in 1 second.

CHAPTER XV

MOTION UNDER CONSTANT ACCELERATION—PROJECTILES

139. THE motion of a point moving with a constant acceleration in the direction of motion has been already investigated. When, however, the acceleration is not in the direction in which it is moving, the point will describe a curved path. The question is specially interesting, because it is the kind of motion which ensues when a body is projected in any direction under the action of gravity. In the actual case of a body thus projected, the air produces effects which in some cases may materially alter the path as deduced on the simpler hypothesis. Thus its buoyant power diminishes the effective weight of the body, the air itself has to be moved, and in addition it offers a resistance to the motion of the body through it. With slow motions, such as for instance in the case of a cricket-ball, all these effects may be neglected, but when the velocities are great, as in the case of shot, they must be taken account of. The resistance offered by the air to the motion of shot is extremely great when the velocities are great. Our results then in the case of shot must only be supposed to be a very rough first approximation. Further, in the case of light bodies, the buoyancy of the air will produce a very large disturbing effect.

We shall suppose then that the motion we consider takes place in a vacuum, but that our results may be applied without appreciable error to bodies moving in the air, provided those bodies are not too light, moving too quickly, and are not of shapes, like discs for instance, whose form produces a great resistance by the air.

140. To fix our ideas, suppose that the constant acceleration a is vertically downwards. The motions in any direction will be completely independent of those in a perpendicular direction. Let us then consider the horizontal and vertical parts independently.

Let V be the velocity of projection, α the angle of elevation.

Further, let u, v denote the horizontal and vertical components of the velocity V—that is,

$$u = V \cos \alpha, \quad v = V \sin \alpha.$$

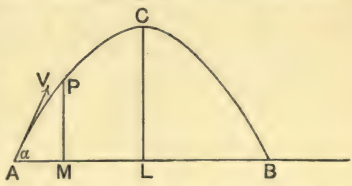
Then there is no acceleration horizontally. Hence the horizontal component of the velocity is always the same and $= u$. The vertical component is subject to acceleration a down. Hence, if v' be the vertical velocity at any time t,

$$v' = v - at,$$

and the vertical velocity at a height s is given by

$$v'^2 = v^2 - 2as.$$

Let ACB represent the path of the particle, AB the horizontal, and AV the direction of projection. It is clear that the path will be symmetrical on both sides, that in fact, if C is the highest point and CL vertical, the curve on both sides of CL will be similar.



The position of the point at any time is known if we know the lengths AM, MP at those times, *i.e.* the height above and the horizontal distance from the point of projection.

Now the horizontal velocity is constant. Hence

$$\left.\begin{aligned} AM &= ut \\ \text{also} \quad PM &= vt - \tfrac{1}{2}at^2 \end{aligned}\right\}.$$

This gives the position at any time.

The velocities are $u' = u$,

$$v' = v - at,$$

or the resultant velocity is given by

$$V'^2 = u'^2 + v'^2 = u^2 + v^2 - 2avt + a^2t^2,$$
$$= V^2 - 2aV \sin \alpha \, . \, t + a^2t^2,$$

and the direction of motion makes with the horizontal an angle θ, where

$$\tan \theta = \frac{\text{vertical velocity}}{\text{horizontal velocity}} = \frac{v'}{u'} = \frac{v - at}{u} = \tan \alpha - \frac{at}{V \cos \alpha}.$$

These formulæ give the velocity and direction after any time. When the body is at a height y,

$$v'^2 = v^2 - 2ay,$$
$$V'^2 = V^2 - 2ay.$$

The direction of motion is given by

$$\tan^2 \theta = \frac{v'^2}{u'^2} = \frac{v^2 - 2ay}{u^2} = \tan^2\alpha - \frac{2ay}{V^2 \cos^2\alpha}.$$

The path taken by a projectile belongs to a class of curves called parabolas.

141. The points of most interest are (1) the distance AB, or the *horizontal range*; (2) the time from A to B, or the *time of flight*; and (3) CL, or the greatest height.

Evidently the time of flight T is twice the time from A to C. But at C the motion is horizontal—in other words, the vertical velocity has become zero under the retardation a. Hence the vertical velocity v is destroyed in time $\frac{1}{2}$T, or

$$0 = v - a\tfrac{1}{2}T,$$
$$\therefore T = \frac{2v}{a}.$$

Next to find AB, notice that AB is described by the horizontal velocity u in time T. Hence the range R is given by

$$R = uT = \frac{2uv}{a},$$

or

$$R = \frac{2V^2 \sin \alpha \cos \alpha}{a}.$$

Thirdly, the greatest height is the height through which the vertical velocity is destroyed. Hence

$$0 = v^2 - 2ah,$$

or

$$h = \mathrm{CL} = \frac{v^2}{2g} = \frac{\mathrm{V}^2 \sin^2 a}{2g}.$$

142. With a given elevation, the range is greater the greater the velocity of projection. With a given velocity of projection, however, the range can never exceed a certain amount. The range will change with the elevation, and for a certain elevation will be a maximum. To find this, notice that

$$\mathrm{R} = \frac{2uv}{a}$$

is to be a maximum, u, v being subject to the condition

$$u^2 + v^2 = \mathrm{V}^2.$$

Now

$$(u - v)^2 = u^2 + v^2 - 2uv,$$

hence

$$2uv = \mathrm{V}^2 - (u - v)^2.$$

Here uv must be as great as possible—but, since V is given, this can only be by $(u - v)^2$ being as small as possible, *i.e.* zero. Hence the range is greatest when $u = v$. But then V bisects the angle between u, v, and must therefore make an angle of 45° with either. Hence to obtain the greatest range the elevation must be 45°. The value of this range is then

$$\mathrm{R} = \frac{2u^2}{a},$$

or, since

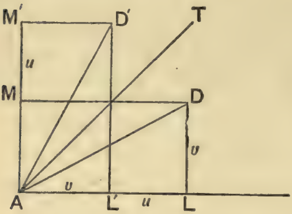
$$\mathrm{V}^2 = u^2 + u^2 = 2u^2,$$

$$\mathrm{R} = \frac{\mathrm{V}^2}{a}.$$

143. It is often a practical question to determine the elevation with a given velocity of projection in order to reach a particular object within range. Now notice that, since

$$\mathrm{R} = \frac{2uv}{a},$$

we get the same range if we interchange u, v. That is, if the elevation is such that the components are u, v, we shall get the same range as if the elevation is such as to give components v, u.



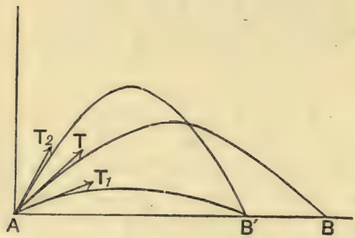
In the figure let AD, AD′ be the two velocities of projection, so that

$AL = u, \quad DL = v,$
and $AL' = v, \quad D'L' = u.$
Then

$\angle DAL = \angle D'AM',$

or the two directions are equally inclined to the horizontal and the vertical respectively.

If AT bisect LAM, it is also clear that DAT = D′AT, or the directions of projection are equally inclined to a line making 45° with the vertical.



The paths will be as in the figure, where TAB = 45° and AB is the maximum range. $T_1AT = T_2AT$, and AB′ is the range of both of them. Of course the lower one will give the quicker time of flight. If T_1, T_2 be the times of flight for the two elevations giving the same range R,

$$T_1 = \frac{2u}{a}, \quad T_2 = \frac{2v}{a},$$

$$\therefore T_1T_2 = \frac{4uv}{a^2} = \frac{2R}{a}.$$

In order to get a given range R_1 we notice

$$2uv = aR_1,$$
$$u^2 + v^2 = V^2.$$

Hence adding, subtracting,
$$\left.\begin{aligned}(u+v)^2 &= V^2 + aR_1\\ (u-v)^2 &= V^2 - aR_1\end{aligned}\right\},$$

whence
$$\left.\begin{aligned} u+v &= \sqrt{V^2+aR_1} \\ u\sim v &= \sqrt{V^2-aR_1} \end{aligned}\right\};$$
$$\therefore\ \left.\begin{aligned} 2u &= \sqrt{V^2+aR_1} \pm \sqrt{V^2-aR_1} \\ 2v &= \sqrt{V^2+aR_1} \mp \sqrt{V^2-aR_1} \end{aligned}\right\}.$$

The quicker time will be with the lower trajectory—that is, the one in which v is the less. Hence

$$2u = \sqrt{V^2+aR_1} + \sqrt{V^2-aR_1},$$
$$2v = \sqrt{V^2+aR_1} - \sqrt{V^2-aR_1}.$$

The angle of elevation may then be found from

$$\tan\theta = \frac{u}{v}.$$

But the most expeditious way is to use a property of the trigonometrical functions whereby $2\sin\alpha\cos\alpha = \sin 2\alpha$.

Then
$$R = \frac{2V^2\sin\alpha\cos\alpha}{a} = \frac{V^2}{a}\sin 2\alpha,$$
$$\therefore\ \sin 2\alpha = \frac{aR}{V^2},$$

whence 2α can be obtained at once from the tables.

144. The following example will illustrate the method:

A man firing at an elevation of 45°, *and* 800 *yards from a fort, wishes to hit the top which is* 300 *feet above the plain. Find the muzzle velocity of the shot.*

The horizontal and vertical components are both equal (u say). Then

$$AL = 2400 = ut,$$
$$BL = 300 = ut - \tfrac{1}{2}gt^2;$$
$$\therefore\ 2100 = 16t^2,$$

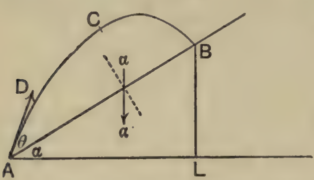
or
$$4t = 10\sqrt{21},$$
$$\therefore\ 2400 = u \times \frac{10}{4}\sqrt{21},$$
$$u = \frac{4\times 240}{\sqrt{21}} = \frac{320}{7}\sqrt{21}.$$

But
$$V^2 = u^2 + u^2 = 2u^2;$$
$$\therefore\ V = u\sqrt{2} = \frac{320}{7}\sqrt{42},$$
$$= 296{\cdot}26 \text{ feet per second.}$$

145. When a body is projected on a hillside, the range

will depend on the steepness of the hill as well as on the circumstances of projection. To find it, let AB be the inclined plane representing the hill. Then the angle BAL being known, the ratio BL/AL is known. But when the velocity of projection and the elevation are known, AL, BL can be expressed in terms of the time. The known ratio AL/BL then gives an equation to find the time to B—or the time of flight. This being solved and the time so found inserted in the expressions for AL, BL, fully determine AL, BL, and thence AB.

In all cases of inclined planes, however, it is better to proceed differently, and instead of considering the horizontal and vertical components of the motion, to treat separately the components parallel and perpendicular to the plane. In this case both components of the velocity are subject to acceleration.



Let α be the inclination of the plane.

Let θ be the elevation of the direction of projection above the plane. (DAB in the figure.)

Then the components of the velocity of projection are

$$u = \mathrm{V} \cos \theta \text{ parallel to the plane,}$$
$$v = \mathrm{V} \sin \theta \text{ perpendicular to the plane.}$$

Also, since the acceleration a makes an angle α with the normal to the plane, the components of the acceleration are

$$a \sin \alpha \text{ down the plane,}$$
$$a \cos \alpha \text{ perpendicular to the plane.}$$

Let C be the point of the path farthest from the plane. Then here the point is moving parallel to the plane, and the velocity perpendicular to the plane has been destroyed by $a \cos \alpha$. When it gets to B, this velocity has been produced again in the opposite direction, and the times taken to do this are the same. Hence the point is at C after half the time of flight.

Hence, if T denote the time of flight, v is destroyed in time $\frac{1}{2}$T, or

$$v = a\cos\alpha \,.\, \frac{T}{2},$$

whence
$$T = \frac{2u}{a\cos\alpha} = \frac{2V\sin\theta}{a\cos\alpha}.$$

Also R = AB is the space described in time T. Hence, if the projection is up the plane,

$$R = uT - \tfrac{1}{2}\,.\, a\sin\alpha\,.\, T^2,$$

$$= V\cos\theta\,.\,\frac{2V\sin\theta}{a\cos\alpha} - \tfrac{1}{2}a\sin\alpha\,.\,\frac{4V^2\sin^2\theta}{a^2\cos^2\alpha},$$

$$= \frac{2V^2\sin\theta}{a\cos^2\alpha}(\cos\theta\cos\alpha - \sin\theta\sin\alpha),$$

which gives R in terms of V, θ, α. The expression may be simplified by use of trigonometry, for

$$\cos\theta\cos\alpha - \sin\theta\sin\alpha = \cos(\theta + \alpha),$$

whence
$$R = \frac{2V^2\sin\theta\cos(\theta+\alpha)}{a\cos^2\alpha},$$

or without trigonometry the result might have been arrived at thus,

$$AL = AB\cos\alpha = R\cos\alpha.$$

But AL is the space described in time T by the horizontal velocity which remains unchanged = V cos DAL = V cos $(\theta + \alpha)$,

$$\therefore\ R\cos\alpha = V\cos(\theta+\alpha)\,.\,T,$$

$$R = \frac{V\cos(\theta+\alpha)}{\cos\alpha}\,.\,\frac{2V\sin\theta}{a\cos\alpha},$$

$$= \frac{2V^2\sin\theta\cos(\theta+\alpha)}{a\cos^2\alpha}.$$

If it is projected down the plane, AT makes an angle $(\theta - \alpha)$ with the horizontal, and the horizontal velocity = V(cos $\theta - \alpha$), whence, as before,

$$R = \frac{2V^2\sin\theta\cos(\theta-\alpha)}{a\cos^2\alpha}.$$

146. As an example, let us find the angle of projection in order that the body may strike the plane perpendicularly, and the range and time of flight in this case, when the inclination of the plane is 45° and the velocity of projection V = 1000 feet per second.

Let θ be the elevation of projection above the plane.

The velocities along and perpendicular to the plane are

$$u = V\cos\theta,$$
$$v = V\sin\theta.$$

The corresponding accelerations are

$$-g\cos 45° = -\frac{g}{\sqrt{2}},$$

and $$-g\sin 45° = -\frac{g}{\sqrt{2}}.$$

The time of flight is twice that in which $V\sin\theta$ is destroyed by $-g/\sqrt{2}$. Hence

$$V\sin\theta = \frac{g}{\sqrt{2}}\cdot\frac{T}{2},$$

or $$T = \frac{2\sqrt{2}}{g}V\sin\theta.$$

Again, when the body strikes the plane, it does so perpendicularly. Therefore the velocity there parallel to the plane vanishes. Hence

$$0 = u - \frac{g}{\sqrt{2}}\cdot T,$$

or $$T = \frac{\sqrt{2}}{g}V\cos\theta.$$

But $$T = \frac{2\sqrt{2}}{g}\cdot V\sin\theta;$$

$$\therefore 2\sin\theta = \cos\theta,$$

or $$\tan\theta = \frac{1}{2} = \cdot 5.$$

The tables give $\theta = 26°\ 33'$. This is therefore the required elevation.

Also the range will be the space through which $V\cos\theta$ is destroyed by $\frac{g}{\sqrt{2}}$. Hence

$$0 = V^2\cos^2\theta - 2\cdot\frac{g}{\sqrt{2}}\cdot R,$$

or $$R = \frac{\sqrt{2}V^2\cos^2\theta}{2g},$$

and $$T=\frac{\sqrt{2}}{g}V\cos\theta.$$

Now the tables give

$$\cos 26^\circ\ 33' = \cdot 8944.$$

Hence $$R=\frac{\sqrt{2}(1000)^2(\cdot 8944)^2}{64},$$

$$=\sqrt{2}\left(\frac{894\cdot 4}{8}\right)^2,$$

$$=\sqrt{2}(111\cdot 8)^2,$$

$$=17676\cdot 5\ldots \text{ feet.}$$

and $$T=\frac{\sqrt{2}\times 1000\times\cdot 8944}{32}\text{ seconds}$$

$$=\sqrt{2}\times\frac{111\cdot 8}{4},$$

$$=39\cdot 5\ldots \text{ seconds.}$$

EXAMPLES—XV.

1. A particle is projected with a vertical velocity 16·7 feet per second and a horizontal velocity ·81 feet per second. Prove that its distance from the point of projection after 1 second is about 1 foot.

2. A bullet is shot horizontally from a gun at the top of a high mountain with velocity u. Find the range on a horizontal plane below if it strikes at an angle of 45°, and the velocity at the point of impact. The resistance of the air is to be neglected.

3. Find the greatest range which a projectile with an initial velocity of 1600 feet per second can attain on a horizontal plane.

4. A stone is thrown from the top of a tower 32 feet high at an angle of elevation of 30°, with the velocity of $32/\sqrt{3}$ feet per second. Find its velocity and direction of motion on striking the horizontal plane through the foot of the tower.

5. Determine the angle of projection when the range is equal to the height due to the velocity of projection.

6. A man can throw a cricket-ball 50 yards vertically upwards. Find the greatest distance he can throw it on a horizontal plane.

7. A rifle has a range of 1000 yards. What would be the range under the same circumstances if fired in the moon, where the force of gravity is one-sixth that on the earth?

8. Two small elastic balls ($e=\frac{1}{2}$) are projected towards each other with equal velocities $\sqrt{ga}$ from points in the same horizontal plane distant $2a$ from each other. Prove that after impact the velocities are each $\frac{1}{2}\sqrt{5ag}$.

9. A person wishes to throw a stone so as to produce the greatest possible blow, at a point in a smooth vertical wall, at a height h from the ground; his strength is sufficient to throw the stone vertically upwards to a height $2h$. Prove that he must throw from a point distant $2h$ from the foot of the wall.

10. Determine the charge of powder required to send a 32-lb. shot to a range of 2500 yards with an elevation of 15°, supposing the initial velocity is 1600 feet a second when the charge is half the weight of the shot, and that the initial energy of the shot is proportional to the charge of powder.

11. Prove that pieces of mud thrown from the top of a cab wheel of diameter d feet, the cab moving with velocity v feet per second, will when they strike the ground be at a distance $\frac{1}{4}v\sqrt{d}$ in front of the position then occupied by the point of contact of the wheel with the ground. Prove that v must exceed $4\sqrt{d}$ feet per second for the pieces of mud to clear the wheel.

12. Two particles are simultaneously projected in opposite directions from the same horizontal line so as to describe parts of the same curve. If the mass of one be n times that of the other ($n>1$) and the coefficient of rebound be e, find their respective orbits after impact; and show from your result that if $e=\frac{1}{2}(n-1)$ one particle will fall vertically down and the velocity of the other will be reversed and increased in the ratio $n-1:1$.

13. If t be the time in which a projectile reaches a point P, and t' the time from P until it strikes the horizontal plane through the point of projection, prove that the height of P above that plane is $\frac{1}{2}gtt'$. Hence verify the expression for the greatest height.

14. A particle is projected with given velocity under the action of gravity from a point P so as to pass through a point Q. Show that if t_1, t_2 are the two possible times of flight, then $gt_1t_2=2\text{PQ}$.

15. A bird of mass M is flying horizontally at a height h with velocity V when it is struck by a bullet of mass m moving vertically with velocity v; the bullet kills the bird and remains embedded in it. Prove that the bird will fall to the ground at a distance from the point from which the bullet was projected

$$=\frac{\text{MV}}{g(\text{M}+m)^2}[mv+\sqrt{m^2v^2+2hg(\text{M}+m)^2}].$$

16. A body slides down an inclined plane of given height and impinges on an elastic horizontal plane. What must be the elevation of the plane that the range on the horizontal plane may be the greatest possible?

17. A wet open umbrella is held with the handle upright and made to rotate round that handle at the rate of 14 revolutions in 33 seconds.

If the rim of the umbrella be a circle of 1 yard diameter and its height above the ground be 4 feet, prove that the drops shaken off from the rim meet the ground in a circle of 5 feet diameter, π being $\frac{22}{7}$; the effect of the air being neglected.

18. An elastic ball projected at a given angle from a point in a horizontal plane rebounds from the plane. Find the range after the first rebound and the time of flight, the coefficient of elasticity being $\frac{1}{3}$.

19. A body is projected from a given point with velocity v. Find the direction of its projection so that it may pass through another given point, distant h horizontally and k vertically from the point of projection.

20. A shot of m lbs. is fired from a gun of M lbs. placed on a smooth horizontal plane, and elevated at an angle a. Prove that if the velocity of the shot just outside the muzzle be V, the range will be

$$2\frac{V^2}{g}\frac{\left(1+\frac{m}{M}\right)\tan a}{1+\left(1+\frac{m}{M}\right)^2\tan^2 a}.$$

21. If three bodies are projected simultaneously in the same vertical plane from the same point, prove that the area of the triangle formed by joining the three bodies at any instant of their motion will vary as the square of the time.

22. Three bodies are projected simultaneously from the same point and in the same vertical plane, one vertically upwards, another at the angle of elevation 30°, and the third horizontally. If their velocities be in the ratio of $1 : 1 : \sqrt{3}$, prove that they will always be in a straight line. How does this line move in space?

23. Free particles, projected simultaneously from points on the circumference of a vertical circle towards the highest point with the velocities which would be acquired by sliding down to those points from the highest point, all reach the circumference again in the same time; and in double that time they are all in another circle of three times the radius of the former.

24. A particle is projected up an inclined plane and making an angle of 45° with the horizontal, so as always to move in the plane. If the inclination of the plane be 60°, find its range.

25. A perfectly elastic ball is projected from a point in an inclined plane. Find the direction of projection that the ball may rebound to the point of projection.

26. A particle is projected from the foot of an inclined plane whose inclination is (β), and in a direction making an angle of 60° with the horizon. If its range on the inclined plane is equal to the distance

through which another particle would fall from rest during the time which elapses before the first particle hits the plane, find β.

27. There is a hill whose inclination is 30°. From a point on the hill one projectile is projected up the hill and the other down it with equal velocities; the angle of projection in each case is inclined to the horizon at 45°. Show that the range of one projectile is nearly $3\frac{3}{4}$ times the range of the other.

28. A smooth solid cylinder of radius r lies on a smooth horizontal plane, to which it is fastened, and an inelastic sphere of radius $2r$ moves along the plane in a direction at right angles to the axis of the cylinder. Find the condition that after collision it may pass over the cylinder without touching.

* * * * *

29. The range of a projectile on a horizontal plane was 1000 meters, and the time of flight was 10 seconds. Find the horizontal and vertical components of the initial velocity, and the greatest height reached by the projectile.

30. AB, BC are two equal planes back to back inclined at 30° to the horizon. A particle is projected up AB with a velocity due to $\frac{3}{2}$ the height of the planes. Show that it will alight at the foot of the plane BC.

CHAPTER XVI

MOTION ON CURVES

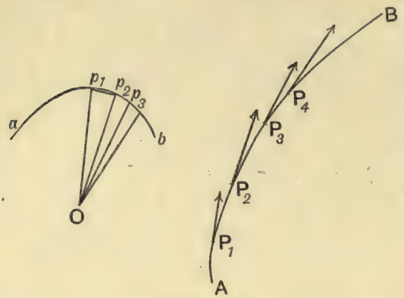
147. If a stone be whirled round at the end of a string, the string must exert a force on the stone, or if a particle be constrained to move along a given curve under any forces, the curve must in general exert a pressure on the particle. How is the amount of this tension or pressure to be determined? The answer to this depends on the acceleration of the stone or particle. When a particle is moving in any curve, it is subject to an acceleration in a definite direction at each point of the curve. In order to determine this it will be best to resolve the acceleration into two components, one along the direction of the motion at any time, the other perpendicular to this. The first component then gives the rate of change of the *magnitude* of the velocity, the second the rate at which the *direction* of the velocity changes, or the rate at which the motion of the particle is deviating from a straight line.

In studying this question the properties of a curve called the hodograph are of very great importance. The hodograph of the path of a particle may be defined as follows—

Let AB denote the path of a moving particle, and suppose that the velocity of the particle is known at every point of the path.

Take any fixed point O and draw from it lines representing the velocities at every point of the path AB. The extremities of these lines will lie on a curve, which is called the hodograph of the path.

Thus let Op_1, Op_2, . . . represent the velocities at the points P_1, P_2, . . . Then if lines similar to Op be drawn for all the points P on AB, their extremities will lie on some curve $ap_1p_2p_3b$. The curve ab is the hodograph of the path AB. The points p_1, P_1; p_2, P_2; p_3, P_3 are called corresponding points.



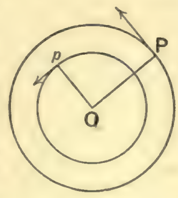
Regarding a point P as moving with the particle along the curve AB, its corresponding point p will move along the curve ab with a definite velocity at each point. We can now state the following important theorem—viz. *the acceleration of a point along the path is represented in magnitude and direction by the velocity of the corresponding point in the hodograph.*

In other words, the velocity of p represents the acceleration of P.

To prove this, let P_1, P_2 denote two points on the path and p_1, p_2 the corresponding points on the hodograph. Then the velocities at P_1, P_2 are Op_1, Op_2 respectively. Hence the total change of velocity in going from P_1 to P_2 is $Op_2 - Op_1 = p_1p_2$. In other words, the increase of velocity whilst the particle moves from P_1 to P_2 is represented by p_1p_2. If the particle takes a time t to move from P_1 to P_2, the average rate of change of velocity between these two points is therefore p_1p_2/t. In the above no supposition has been made as to the magnitude of P_1P_2, or of t. Let now P_1P_2, and therefore t, become very small. We now have p_1p_2 and t both small, but their ratio is not necessarily so; it gives in fact as before the average rate of change of velocity between P_1 and P_2. This is always the case, however near P_1 and P_2 may be. Let now P_2 move up to actual coincidence with P_1, then the limiting value of p_1p_2/t will be the rate of change of velocity at P—that is, the acceleration. But looked at as a point moving on the curve ab, p_1p_2/t denotes the velocity of p on ab. Hence the velocity on the hodograph represents the acceleration at the corresponding point of the path itself.

148. This very important theorem at once enables us to answer the question proposed above, viz. to find the acceleration when a particle is compelled to move on a given curve. Consider first the case of a particle moving uniformly round the circumference of a circle. Here the velocity is constant in magnitude, the acceleration only affects the direction.

Let the radius of the circle be denoted by a and the velocity of the particle by v. Take the centre of the circle as the fixed point for the hodograph.



When the particle is at P draw Op parallel to the tangent at P and equal to v. Then p is the point on the hodograph corresponding to P, and as P moves round its circle, p moves in another concentric one. Hence the hodograph will be a concentric circle of radius measured by v. Now, firstly, the velocity of p is along the tangent at p, *i.e.* is parallel to PO. Hence the acceleration of P is along PO. Secondly, as to its magnitude, which is represented by the velocity of p. Notice that P and p describe their circles in the same time. Hence

$$\frac{\text{velocity of } p}{\text{velocity of P}} = \frac{\text{circumference of circle } p}{\text{circumference of circle P}} = \frac{\text{O}p}{\text{OP}},$$

i.e.
$$\frac{\text{velocity of } p}{v} = \frac{v}{a},$$

or
$$\text{velocity of } p = \frac{v^2}{a}.$$

Hence the acceleration of P is directed *inwards* along the normal PO and is equal to v^2/a, or

$$\text{Acceleration of P} = \frac{v^2}{a} \text{ along PO.}$$

Now the acceleration of a point depends only on the rate of change at the moment, and does not depend on the history of the motion before or afterwards. Hence, even when the magnitude of the velocity is altering, the acceleration *along the normal* is given by the same formula. It does not, however, now represent the total acceleration. To find this, the normal acceleration as found above must be compounded with the acceleration along the tangent to the path, due to the changing speed.

Further, notice that the acceleration perpendicular to the direction of motion gives the rate at which the velocity is changing owing to the rate of change of direction. This remark points out how to find the normal acceleration when the path is any other curve than a circle. For at a point P draw a circle touching the curve of such a radius that the rate of deviation from the tangent at P is the same for the circle and the curve. If a denote the radius of this circle and v the velocity, the normal acceleration will be v^2/a. The circle is called the circle of curvature of the curve at P, and a the radius of curvature. Until, however, we have means of determining the value of the radius of curvature in particular cases we can make no use of this result. The determination of a involves a larger knowledge of mathematics than we assume in this book.

149. The following examples will illustrate the application of the foregoing results—

Example I. *A particle of mass* 3 *lbs. lies on a smooth horizontal table and is attached by a string* 2 *yards long to a fixed point in the plane; it whirls round this point at a velocity of* 20 *feet per second. Find the tension of the string.*

The acceleration is $\frac{v^2}{a}=\frac{(20)^2}{6}$ feet per second per second.

The force on the particle is the tension T of the string alone.

Hence $$\mathrm{T}=3\times\frac{400}{6}\text{ poundals}=200\text{ poundals},$$

$$=\text{weight of }\frac{200}{32}\text{ lbs.}=\text{weight of }6\tfrac{1}{4}\text{ lbs.},$$

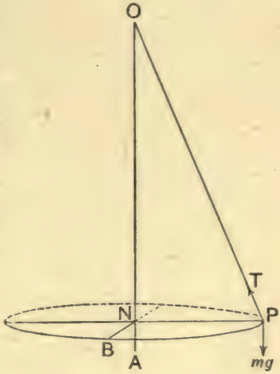
$$=\frac{25}{12}\text{. the weight of the mass itself.}$$

The student may prove that in the general case

$$T = \frac{mv^2}{a} \text{ or } T = \frac{v^2}{ag}\text{. weight of mass itself.}$$

EXAMPLE II. *Conical pendulum. A mass* m *lbs. is suspended from a point by a string of length* l *feet; the mass revolves in a horizontal circle with the string inclined to the vertical and makes* n *revolutions per second. Find the inclination of the string to the vertical and the tension of the string.*

Let O be the point of suspension, OA vertical. P describes the horizontal circle PB. Draw PN perpendicular to OA. Then PN is the radius of the circle described by the mass, also OP $= l$. The particle is acted on by two forces, viz. its weight mg downwards and the tension T along PO. Decompose this last into two components, one Y vertically upwards, the other X along PN.



Then the vertical force on m $= Y - mg$. Now the particle describes a *horizontal* circle. There is therefore no vertical motion nor vertical acceleration. Hence

$$Y - mg = 0,$$

or $$Y = mg.$$

The horizontal force on the particle is X along PN, whereas the horizontal acceleration is $\frac{v^2}{PN}$, where v denotes the velocity of the particle. Hence

$$X = m\frac{v^2}{PN}.$$

Now the particle makes n revolutions per second—that is, it passes over a distance $n \cdot 2\pi \cdot PN$ per second, or

$$v = 2\pi n \cdot PN.$$

Hence $$X = \frac{4\pi^2 mn^2 \cdot PN^2}{PN} = 4\pi^2 mn^2 \cdot PN \text{ poundals,}$$

and $$Y = mg \text{ poundals.}$$

Now X, Y are the horizontal and vertical components of T. Hence

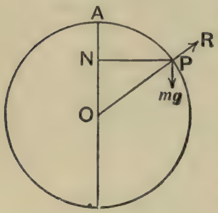
$$\frac{Y}{X} = \frac{ON}{PN}, \quad \frac{X}{T} = \frac{PN}{OP} = \frac{PN}{l};$$

or
$$T=\frac{l}{PN}\,.\,X=4\pi^2mn^2l,$$
$$=\frac{4\pi^2n^2l}{g}\,mg=\frac{4\pi^2n^2l}{g}\text{, weight of mass ;}$$
also
$$\cos POA=\frac{ON}{OP}=\frac{Y}{T}=\frac{mg}{\frac{4\pi^2n^2}{g}l\,.\,mg}=\frac{g}{4\pi^2n^2l},$$
which gives the inclination of OP to the vertical. Instead of finding POA, we may find PN. Thus
$$\frac{Y}{T}=\frac{ON}{OP},$$
or
$$ON=\frac{g}{4\pi^2n^2},$$
and
$$OP^2=ON^2+PN^2\,;$$
$$\therefore\ l^2=\frac{g^2}{(4\pi^2n^2)^2}+PN^2\,;$$
$$\therefore\ PN=l\sqrt{1-\frac{g^2}{(4\pi^2n^2l)^2}}.$$

Note that ON is the same, whatever the length of the string, provided the number of revolutions per second is the same.

If the length l is given, the inclination α, and also the fact that the particle describes a horizontal circle, the periodic time is easily found. For it makes one revolution in $\frac{1}{n}$ second. But
$$\cos\alpha=\frac{g}{4\pi^2n^2l}.$$
Hence
$$\text{Periodic time}=\frac{1}{n}=2\pi\sqrt{\frac{l\cos\alpha}{g}}.$$

EXAMPLE III. *A heavy particle is placed on the top of a smooth sphere and slips down over the surface. Where will it leave the sphere?*



Let m denote the mass of the particle, a the radius of the sphere. While the particle is in contact with the surface the sphere will exert a certain pressure on it. This pressure will just vanish when the particle leaves the sphere. In order to find where the particle leaves, we shall first find the pressure at any point, and then determine at what point this pressure vanishes.

The particle is acted on by two forces, viz. its weight mg downwards and the normal pressure (R say) outwards along OP. The resolved parts of these along PO
$$=mg\cos PON-R,\quad\text{or } mg\,.\,\frac{ON}{OP}-R.$$

This must equal the rate of change of momentum along PO. By what precedes, the acceleration along PO is v^2/a, where v is the velocity at P.

Hence
$$mg\frac{\text{ON}}{a} - \text{R} = \frac{mv^2}{a}.$$

Now v is the velocity attained in moving along the arc AP, *i.e.* in falling a vertical distance AN,

$$\therefore v^2 = 2g\,.\,\text{AN},$$

whence
$$mg\frac{\text{ON}}{a} - \text{R} = m\,.\,\frac{2g\text{AN}}{a},$$

and
$$\text{R} = mg\left(\frac{\text{ON}}{a} - \frac{2\text{AN}}{a}\right),$$
$$= \frac{mg}{a}\{\text{ON} - 2(a - \text{ON})\},$$
$$= \frac{mg}{a}(3\text{ON} - 2a),$$

or pressure at any point $\text{P} = \dfrac{3\text{ON} - 2a}{a} \times$ weight of particle.

The particle leaves the surface when this pressure becomes zero. That is, when

$$3\text{ON} = 2a,$$

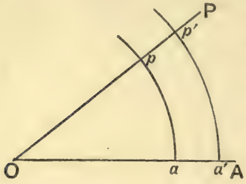
or
$$\text{ON} = \tfrac{2}{3}a = \tfrac{2}{3}\text{OA}.$$

EXAMPLE IV. *Assuming that the moon rotates round the earth as a fixed point in* 27 *days* 8 *hours, and that her distance is sixty times the earth's radius, find the acceleration of gravity at the surface of the earth. The earth's radius is to be taken as* 4000 *miles.*

If m be the mass of a body its weight at the surface of the earth is mg, whilst its weight at the distance of the moon will be $mg/(60^2)$. Its acceleration, therefore, towards the earth is $g/(60^2)$. But this is the acceleration to the earth of the moon in her orbit. Now the moon moves through a distance $2\pi r$ in 27 days 8 hours, where r is the radius of its orbit. Hence, if v denote its velocity,

$$v = \frac{2\pi r}{27\frac{1}{3} \times 24 \times 60 \times 60};$$
$$\therefore \frac{g}{60^2} = \frac{4\pi^2 r^2}{(82 \times 8 \times 60 \times 60)^2 r};$$
$$g = \frac{4\pi^2 60 \times 4000 \times 1760 \times 3}{(82 \times 8 \times 60)^2} \text{ foot second units,}$$
$$= \frac{220000}{6724} \text{ if } \pi^2 = 10,$$
$$= 32 \ldots \text{ about.}$$

150. *Angular velocity.*—Let OA be a fixed line and P a point moving in any way. The angle POA depends on the position of P, and will alter in general as P moves. The rate at which the angle POA is increasing at any moment is the *angular velocity* of P about O at that moment. It is measured when constant by the angle turned through in a unit of time, and when not constant by the angle which would be turned through in a unit of time if during that interval the angular velocity remained the same as at the beginning.



The actual numerical value of an angular velocity depends on our angular unit and unit of time. Thus we may measure the same velocity as a right angle per minute, 90° per minute, or $1\frac{1}{2}^\circ$ per second. The most convenient method, however, is to employ circular measure in which the unit is called a radian.

Draw a circle of any radius round O, and let OA, OP cut it in the points a, p. Then the arc ap will always be proportional to the angle POA so long as it is an arc of the same circle. If, however, the circle have a larger radius, ap will be larger. However, in different circles the ratios

$$\frac{ap}{Oa}, \frac{a'p'}{Oa'}$$

will be the same for the same angle. We may, therefore, measure any angle POA by drawing any circle round O and taking the ratio of the arc ap cut off between OA, OP to the radius of the circle, and say

$$\text{Angle POA} = \frac{pa}{Oa}.$$

For (1st) this ratio is independent of the size of the circle; and (2nd) it is directly proportional to the size of the angle.

Here the unit of angular measurement is that in which $ap/Oa = 1$, or the arc = radius. This unit is called a radian.

In order to express an angle given in circular measure in degrees, it is necessary to know how many degrees there are in a radian. To do this, notice that the whole circumference of a circle measures 360°.

Hence $$360° = \frac{\text{circumference of circle}}{\text{radius of circle}} \text{ . radians,}$$

$$= \frac{2\pi r}{r} = 2\pi \text{ radians.}$$

Hence $$\text{Radian} = \frac{180}{\pi} \text{ degrees,}$$

$$= 57° \text{ about.}$$

The circular measure of two right angles is π, or 3·14159.

Since the circular measure of an angle is a length divided by a length, it is independent of the units of mass, length, and time, or

$$[\text{Angular measurement}] = [M^0 L^0 T^0].$$

It is therefore simply a number.

Angular velocity is an angle per time. Hence

$$[\text{Angular velocity}] = \left[\frac{1}{T}\right].$$

Speed of rotation is sometimes given by the number of revolutions made per unit of time. If this number be n, then $4n$ right angles are described per unit of time—that is, $2n\pi$ radians per unit of time.

151. When the angular velocity of P is given and its distance from O, it is easy to express the velocity of P in the direction perpendicular to OP. For describe a circle of radius OP. Then the velocity of P in the given direction is the rate at which the arc AP is increasing.

But $$\text{Angle POA} = \frac{\text{AP}}{\text{OA}} \text{ radians.}$$

Hence rate at which the angle POA increases

$$= \frac{\text{rate at which arc AP increases}}{\text{OA}},$$

$$= \frac{\text{velocity of P}}{\text{OA}}.$$

Or, if ω denote the angular velocity of P about O and v the velocity of P in the direction perpendicular to OP,

$$\omega = \frac{v}{r},$$

or $$v = \omega r.$$

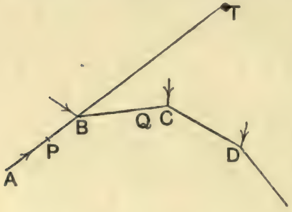
The velocity of P in the direction of OP clearly does not affect the angular velocity.

We may now express the acceleration along PO when the particle is moving in a circle round O in terms of the angular velocity. For we have seen that it is measured by v^2/r; and

$$\frac{v^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 r.$$

In general the angular velocity of a point is not constant. When it is not so it is said to be accelerated, and the acceleration is measured by the increase of angular velocity per unit of time.

152. When an impulse acts on a particle the change in its momentum is measured by the impulse. Suppose now a particle to be moving and to be acted on by a series of impulses at different intervals. The whole change of momentum will be equal in magnitude and direction to the resultant of the various impulses—*or*, which is the same thing, if we suppose its original momentum to be replaced by the corresponding impulse, then the resultant of all these impulses will be equal to the final momentum of the particle in magnitude and direction. This follows at once from the definition of impulse. But, further than this, it can be shown that if the impulses be regarded as acting at the points at which they really acted, and if the momentum of the particle at any subsequent time be replaced by its impulse reversed, acting at the point which the particle is occupying at that instant, the whole set of impulses will form a system in equilibrium.

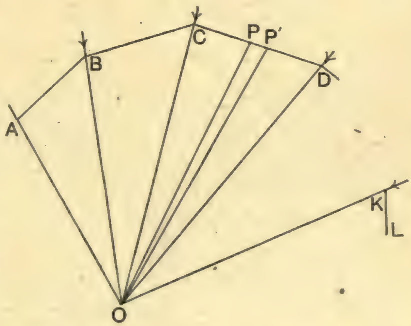


For suppose the particle to start from rest at A and to move along the path ABCD . . ., receiving given impulses I_1, I_2, I_3, . . . at A, B, C, D, . . .

After the first impulse at A it moves along the line ABT with constant momentum. This is always constant, and, as it is in direction AB, is always equivalent to the original impulse acting at A. If then it be reversed when at P, it will be a system in equilibrium with the original impulse at A. When it arrives at B, suppose another impulse to act on it, making it take the path BC. We have seen that when it reached B it had a momentum equal to the impulse at A. Replace it by this. Then the new momentum is equivalent to the resultant of I_1 and I_2 acting at B. Hence the momentum afterwards at any point Q is equal to this. But I_1, I_2 at B are equivalent to I_1 at A and I_2 at B. Hence, if the momentum at Q be reversed, it, together with I_1 at A and I_2 at B, will form a system in equilibrium. In the same way the theorem may be proved for the three impulses, and so on in general. This theorem has important applications, see next paragraph and § 171.

153. Suppose now a moving particle to be acted on by a series of impulses all directed to a fixed point O.

Let ABCD . . . KL denote the path of the particle, and let v_0 be the velocity at A, v along KL.



Then, by the preceding theorem, the following form a system in equilibrium, viz. mv_0 along AB, the impulses at B, C, D, . . ., K, and mv along LK.

Hence their moments round the point O must vanish. But all the impulses at B, C, . . ., K pass through O, and consequently their moments vanish. Therefore the moments about O of mv_0 along AB and mv along LK must vanish—that is, the moment of mv_0 about O must equal the moment of mv about the same point. But since the point K is at any point of the path, we learn that the moment about O of the momentum at any point of the path always remains the same whatever the number or magnitude of the impulses, provided they all act towards O.

Since the mass is constant, it follows that the moment of the velocity about O is always the same.

If at any time p denote the perpendicular from O on the line of motion of the particle, this theorem may be expressed thus,

$$pv = \text{constant}.$$

The above proof is quite independent of the number or magnitude of the impulses. Therefore, as in the cases before treated of, the theorem is still true when stated of a particle moving under a continuous force always directed to a fixed point.

The theorem may be stated in a different form. Referring again to the figure, let A denote the area of the figure OABC . . . PO, where P is the position of the particle at any time. In a time t, P will have moved to P′, where $PP' = vt$, and the area will have increased by OPP′. This is a triangle of base PP′ and altitude p. Hence increase of A in time $t = \frac{1}{2} . PP' . = \frac{1}{2} pvt$. That is, the increase of A per unit of time or the rate of increase of A is $\frac{1}{2}pv$. But we have seen that this is constant. Hence if a particle is describing any orbit under the action of forces directed to a fixed point, the area described about this point in any time is proportional to the time of describing it.

This theorem was first discovered by Newton, and was employed by him in conjunction with the fact discovered by Kepler, that the earth described equal areas round the sun in equal times, to deduce the law that the earth moved under the action of a force always directed towards the sun.

154. To illustrate the application of this theorem, take the following example—

A particle of mass m *is placed on a smooth horizontal table; it is attached by a string which passes through a hole in the table to a particle of mass* M, *which hangs freely; it is projected at right angles to the string with velocity* u. *Determine the velocities of the particles at any future time in terms of the distance of* m *from the hole.*

Let O denote the hole, r the distance of m from O at any time, and r_0 its initial distance. The velocity of M will be simply a vertical one. Denote it by v. The velocity of m can be decomposed into two components, one along the string and the other perpendicular to it. Since the string is inextensible, that along the string must be equal to the velocity of M. Denote the other by u. There will then be only two quantities to determine, viz. u, v, in terms of r and the initial values of r and u (say r_0, u_0). To find these, two conditions are necessary. One is given by the fact that the change of kinetic energy must be equal to the external work done. Now the only external work is that of gravity acting on M, and this has pulled M down a distance $=r_0 - r$ (for, since the string is inextensible, the vertical displacement of M is equal to the distance m has moved towards the hole). Hence

$$\text{Work done} = Mg(r_0 - r),$$

and change of kinetic energy

$$= \tfrac{1}{2}m(u^2 + v^2) + \tfrac{1}{2}Mv^2 - \tfrac{1}{2}mu_0^2,$$

$$\therefore m(u^2 - u_0^2) + (M + m)v^2 = 2Mg(r_0 - r).$$

For the second condition, notice that the force on m (which is the tension of the string) always passes through a fixed point O. Hence m moves so that the moment of its momentum round O remains unchanged. Therefore

$$mu \,.\, r = mu_0 \,.\, r_0,$$

or

$$u = \frac{r_0}{r} u_0.$$

Substituting this value of u in the previous equation,

$$mu_0^2\left(\frac{r_0^2}{r^2} - 1\right) + (M + m)v^2 = 2Mg(r_0 - r),$$

or

$$(M + m)v^2 = (r_0 - r)\left\{2Mg - m \,.\, \frac{r_0 + r}{r^2} u_0^2\right\}.$$

If M remained always at rest, the tension of the string would equal its weight, that is Mg, and in that case m would describe a circle, and

$$Mg = m\frac{u^2}{r_0}.$$

This gives the value of u_0 necessary for this purpose.

Further, the formula for v enables us to find between what limits M will oscillate, for in its two positions of rest $v = 0$. Hence

$$(r_0 - r)\left(2Mg - m\frac{r_0 + r}{r^2} u_0^2\right) = 0,$$

i.e. $r = r_0$, which is evidently the case *à priori* and the positive root of

$$2Mg - m\frac{r_0 + r}{r^2} u_0^2 = 0$$

or of $$r^2 - \frac{mu_0^2}{2Mg}r - \frac{mu_0^2}{2Mg}r_0 = 0.$$

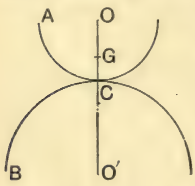
For instance, take the numerical case of $m=4$ lbs., $M=1$ lb., $r_0=2\frac{1}{4}$ feet, and $u_0=16$ feet per second.

The equation to find r is

$$r^2 - 16r - 36 = 0,$$
$$(r-18)(r+2)=0.$$

Hence m will reach a distance of 18 feet from the hole and then come back again to a distance of $2\frac{1}{4}$ feet, and so on continually. M will, therefore, oscillate up and down through a distance $18-2\frac{1}{4}=15\frac{3}{4}$ feet.

155. The knowledge we have gained in this chapter about angular acceleration enables us easily to settle an important question as to the stability of one body resting on another. If a body A rests on a rough fixed body B, the centre of gravity of A must be vertically above the point of contact C of the two bodies. The question then arises, if this condition is satisfied, will the equilibrium be stable or unstable—in other words, if A receive a small rolling displacement over B and be let go, will it tend to return to its former position or to recede farther from it? Now, as A rolls over B, its centre of gravity G will trace out a certain curve, and, by § 108, the equilibrium is stable if G is at the lowest point of this curve, and unstable if at the highest point. That is to say, if the path of G turns upwards on both sides the equilibrium is stable, whereas it is unstable if the path tends downwards. Now if the path of G tends upward as A is rolled from its position of rest, the vertical acceleration of G will be up, whereas it is downwards in the other case. The question therefore depends on whether the acceleration of G is up or down as A rolls over B.



Now suppose the surfaces near the point of contact are spherical, and let O, O′ denote the centres for A and B. Then OGO′ is a vertical straight line.

Let R, r denote the radii of the surfaces at C of the fixed and rolling bodies, viz.

$$O'C = R, \quad OC = r,$$

also let $$OG = h.$$

Now, as A rolls over B, O moves round O′ as in a circle of radius $R + r$, whereas G turns round O in a circle of radius h. Hence the acceleration of G in any direction is equal to the acceleration of O in that direction + the acceleration of G *relative* to O.

Let ω', ω denote the angular velocities of OO′ round O′ and of OG round O. Then the vertical downward acceleration of G

$$\begin{aligned} &= \text{acceleration of O along OO}' \\ &\quad - \text{acceleration of G along GO}, \\ &= \omega'^2 \, . \, OO' - \omega^2 \, . \, OG, \\ &= \omega'^2(R + r) - \omega^2 \, . \, h. \end{aligned}$$

But, as A rolls over B without slipping, the velocity of the point of contact C must be zero. That is, velocity of O + velocity of C relative to O = 0,

$$\therefore \; \omega'(r + R) - \omega r = 0.$$

Hence the equilibrium is stable or unstable

as Acceleration of G is up or down,

as $$\omega^2 h \gtrless \omega'^2(R + r),$$

as $$h \gtrless (R + r)\frac{\omega'^2}{\omega^2} \gtrless (R + r)\frac{r^2}{(R + r)^2} \gtrless \frac{r^2}{R + r}.$$

The equilibrium is then

stable, if $$OG > \frac{r^2}{R + r},$$

unstable, if $$OG < \frac{r^2}{R + r}.$$

Since $CG = r - h$, the condition may be stated thus, viz. A is stable or unstable

as $$r - h \lessgtr r - \frac{r^2}{R + r},$$

$$\lessgtr \frac{Rr}{R + r},$$

or $$\frac{1}{CG} \gtrless \frac{1}{R} + \frac{1}{r}.$$

For instance, a solid hemisphere rests with its curved surface on the top of a sphere. Find the relation between the radii that the equilibrium may be stable.

Here $OG = \frac{3}{8}r, \quad CG = \frac{5}{8}r.$

Hence the condition is $\frac{8}{5r} > \frac{1}{R} + \frac{1}{r},$

$$\frac{3}{5r} > \frac{1}{R},$$

or $R > \frac{5}{3}r.$

If the hemisphere rested with its flat surface in contact, the radius of the flat surface is infinitely great, or in the formula $r = \infty$.

Here $CG = \frac{3}{8}r,$

$$\therefore \frac{8}{3r} > \frac{1}{R} + \frac{1}{\infty} > \frac{1}{R},$$

or $R > \frac{3}{8}r.$

In case the surfaces are not spherical, the radii of curvature of the surfaces must be inserted in the formula.

When B has its concavity upwards, R must be considered negative; and when A has its concavity downwards, r must be considered negative. The student should prove this in each case as an exercise.

EXAMPLES—XVI.

1. A string 5 feet long can just sustain a weight of 20 lbs; it is fastened to a fixed point at one end and to the other to a mass of 5 lbs., which revolves round the point in a horizontal plane. Determine the greatest number of complete revolutions that can be made by the string in one minute without breaking.

2. With what number of turns per minute must a weight of 10 lbs. revolve on a horizontal table at the end of a string 15 inches long, in order to cause the same tension on the string as if a 1-lb. weight were hanging at rest held by the string vertically?

3. A weight of m lbs. is tied to a string l feet long, which has its other end tied to a fixed point on a smooth horizontal table, and makes n revolutions a second. Show that the tension of the string is $4\pi^2 lmn^2$ poundals.

4. A locomotive engine weighing 9 tons passes round a curve 600 feet in radius with a velocity of 30 miles an hour. What force tending towards the centre of the curve must be exerted by the rails so that the engine may move on this curve?

5. Find the horizontal pressure on the rails exerted by an engine of

20 tons going round a curve of 600 yards radius at 30 miles per hour.

6. A string 4 feet long can just support a weight of 9 lbs. without breaking; a weight of 8 lbs. fixed to one end of the string describes a circle uniformly round the other end, which is fixed on a smooth horizontal table. Find the greatest number of revolutions the revolving weight can make in one minute so as just not to break the string.

7. A particle rests on a rough horizontal disc capable of motion round its centre; it is distant one yard from the centre and the coefficient of friction is $\frac{2}{3}$. Determine the angular velocity of the disc when the particle just slips over the disc.

8. A particle of mass $\frac{1}{4}$ oz. rests on a horizontal disc at its edge and is attached by two strings, which are 4 feet long, to the extremities of a diameter. If the disc be made to revolve 100 times a minute about a vertical axis through its centre, find the tension of each string.

9. If the force of attraction for different distances varies inversely as the squares of the distances, and for different bodies directly as their masses, prove that if several bodies move round O in concentric circles, the squares of the times of revolution are as the cubes of the radii.

10. Two weights W and W′ are placed on a smooth table, and connected together by a string passing through a small fixed ring on the table. If they are projected with the velocities v and v' at right angles to the string, find the ratio in which the string must be divided by the ring in order that both weights may describe circles round the ring as centre.

11. If a railway carriage without flanges to its wheels moves on a circular curve, show how the effect of the centrifugal force may be counteracted by a rise of the outer rail, and find what the rise of the outer above the inner rail should be if the radius of the circle be 1320 feet, the velocity of the train 30 miles an hour, and the breadth of the track 5 feet.

12. If trains travel generally at 30 miles an hour, find how much the outer rail should be raised on a curve of half a mile radius, the gauge being 4 feet, so that there shall be no side thrust on the flange.

13. Prove that a railway carriage running round a curve of radius r will upset if the velocity is greater than $\sqrt{(gra/2h)}$, where a is the distance between the rails and h the height of the centre of gravity of the carriage above the rails.

14. A heavy particle is connected by an inextensible string 3 feet long to a fixed point, and describes a circle in a vertical plane about that point, making 600 revolutions per minute. Find the ratios of the tensions of the strings when the particle is at the highest and lowest points and when the string is horizontal, the motion of the particle being kept uniform.

15. A string AP of length a is fastened to a point A and carries a weight P. If P be projected vertically upwards from the position in which AP is horizontal, find the least velocity of projection with which it may describe a circle round A, and state what happens if the velocity be less than this. If P be projected vertically upwards when AP is below the horizontal line, and at an angle α to it, find the least value of v that P may ultimately describe a circle in this case.

16. Assuming that at the equator bodies "lose 1/289 of their weight," find the radius of the earth, assuming $g=32$ when referred to 1 foot and 1 second.

17. Prove that at the equator a shot fired westward with velocity 8333, or eastward with velocity 7407 meters per second, will, if unresisted, move horizontally round the earth in one hour and twenty minutes and one hour and a half respectively.

18. If a particle move in a smooth vertical circular tube of radius a, with velocity due to a height h above the lowest point of the circle, prove that the pressure will be proportional to the depth below a horizontal line at a height $\frac{2}{3}h+\frac{1}{3}a$ above the lowest point.

19. If a railway carriage be moving at the rate of 30 miles an hour, and the radius of one of its wheels be 2 feet, what is the angular velocity of the wheel? What also is the relative velocity of the centre and highest point of the wheel?

20. The radius of the wheel of a railway carriage is 2 feet, and it makes $3\frac{1}{2}$ revolutions per second when the train is moving at 30 miles an hour. If $\pi=\frac{22}{7}$, find the actual velocity at any instant (1) of the point in contact with the ground, (2) of either of the two points one yard above the ground.

21. If in the last question a particle weighing 1 oz. be attached to a point of the circumference of the wheel, find the magnitude of the attaching force when the particle is (1) at its highest point, (2) in the horizontal line through the centre.

22. A tube is inclined at an angle α to a vertical axis and rotates round it with angular velocity ω. Determine the position in which a particle would rest inside the tube, the latter being smooth.

Will this position be a stable or unstable one?

23. In the previous question determine what portion of the tube the particle can be at rest on, supposing that the tube is rough and the coefficient of friction is μ.

24. A particle can move in a smooth tube in the form of a circle which is rotating with angular velocity ω about a vertical diameter. Determine the points at which the particle can be at rest relatively to the tube.

25. A particle of mass m can move on a smooth horizontal plane; it is attached to one end of a weightless string which passes through a hole at a distance a in the plane, then under a pulley beneath the plane, and is fastened at the other end to a fixed point; the parts on both sides of the pulley are vertical; to the pulley is attached an equal mass m; the first particle is projected at right angles to the string. Find its velocity that the second particle may remain at rest. If the velocity be twice this, determine the distance through which the second particle will oscillate.

26. An elastic string is fixed to a point A and just reaches to a small ring in a horizontal plane without stretching; this end is pulled through the ring and fastened to a mass m whose weight would, if hanging vertically, stretch the string to double its length; the particle is then projected at right angles to the string with velocity u. Find the velocity at any time in terms of the distance from the ring. (For energy see § 95.)

27. In the previous question, find the greatest and least distances which the particle will attain from the ring.

28. A cube just rests in stable equilibrium on the top of a rough sphere of radius r. Determine its size.

29. A cone and a hemisphere on the same base and of the same uniform material are joined together. Prove that, if resting with the hemisphere in contact with a horizontal table, the equilibrium of the body will be stable or unstable according as the height of the cone is less or greater than $\sqrt{3}$ times the radius of the base.

30. A cone rests with its base on the top of a sphere of 1 foot radius. Determine the greatest height of the cone that the equilibrium may be stable.

31. A homogeneous sphere of given radius rests at the bottom of a hemispherical bowl of larger radius. If the sphere is so loaded that the height of its centre of gravity above the lowest point is $\frac{5}{4}$ of its radius, determine the radius of the hemisphere when the equilibrium is neutral.

32. Two equal spheres are fastened together at a point of their surfaces, and are placed with the line joining their centres vertical in a spherical cup with its concavity upwards. Prove that if the equilibrium be stable, the radius of the cup must be less than the diameter of either of the spheres.

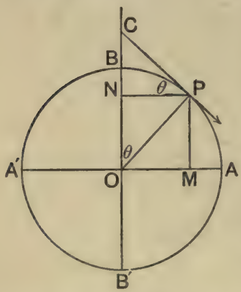
33. Find the periodic time of the pendulum in Example ii. p. 279, if the point of suspension descends with a constant acceleration $g/4$.

CHAPTER XVII

HARMONIC MOTION—SIMPLE PENDULUM

THE motion to be considered in this chapter lies at the basis of all vibratory motion, and therefore is of the very highest importance in applications of dynamics to the explanation of physical phenomena.

156. Suppose a point P to move uniformly round the circle ABA′B′. Let AOA′ be any diameter and draw PM perpendicular to AOA′. Then, as P travels with uniform velocity round the circle, M travels to and fro along the line AOA′. *The motion of* M *is called a simple harmonic motion* (S.H.M.)



The time of revolution of P, *or* of the motion of M from A to A′ and back again, is called the *period* of the motion.

The angle POB is the *phase* at M.

The distance OA is called the *amplitude.*

OM is called the *displacement* at any time.

For the sake of reference, we shall call the circle the generating circle and P the generating point.

Such a motion as the above is the up-and-down motion of the connecting rod of a locomotive. The motion of a piston, or a crank with a long piston-rod, is also very nearly a simple harmonic motion. If the crank worked in a straight slot perpendicular to the piston, the piston would have an exact S.H.M.

Let T denote the period, a the amplitude, then P goes once round the circle in time T. Hence, if u denote its velocity,

$$u\mathrm{T} = 2\pi a,$$

or

$$u = \frac{2\pi a}{\mathrm{T}}.$$

The velocity of M at any time is equal to the component of the velocity of P in a direction parallel to AOA′.

Hence
$$\text{Velocity of M} = \frac{\mathrm{PN}}{\mathrm{PC}} \text{ velocity of P},$$
$$= \frac{\mathrm{PM}}{\mathrm{OP}} \cdot u.$$

Denote the velocity of M by v, then

$$v^2 = \frac{\mathrm{PM}^2}{a^2} \cdot u^2 = \frac{\mathrm{OP}^2 - \mathrm{OM}^2}{a^2} u^2,$$
$$= (a^2 - \mathrm{OM}^2)\frac{u^2}{a^2},$$

This gives the velocity of M in terms of its distance from O. It may be expressed in terms of the phase (θ) by means of trigonometry thus,

$$v = u \cos \theta,$$

or, if ω denote the angular velocity of P and t the time from O to M,

$$\theta = \omega t, \quad u = a\omega,$$
$$v = a\omega \cos \omega t.$$

157. So also the acceleration of M is the component parallel to AOA′ of the acceleration of P. But the acceleration of P is u^2/a along PO. Hence

$$\text{Acceleration of M} = \frac{\mathrm{OM}}{\mathrm{OP}} \cdot \frac{u^2}{a} \text{ along MO},$$
$$= \frac{u^2}{a^2} \cdot \mathrm{OM}.$$

That is, the acceleration of M is *towards* O and is proportional to its distance from O. And conversely, if a particle move along a straight line with an acceleration towards a fixed point in it, and proportional to its distance from it, it moves with a simple harmonic motion.

This may also be extended to the case where a point is moving along any curve with an acceleration at every point along the curve, and proportional to its distance *measured along the curve* from a fixed point on it.

Suppose the acceleration to the point to be given by

$$\text{Acceleration} = k\ (\text{distance}).$$

Then, comparing with the foregoing,

$$k = \frac{u^2}{a^2},$$

and $$T = 2\pi\frac{a}{u} = \frac{2\pi}{\sqrt{k}}.$$

Whence we learn this important result, that if a particle move under such an acceleration as this it vibrates about the point O with a S.H.M. in a period which is independent of the amplitude. That is, from whatever distance it is started, the particle will pass through O to the other side and back again to A in the same time—in other words, it is said to be isochronous.

In the case considered of acceleration $= k$ (distance) the velocity is given by $k^2(a^2 - \text{OM}^2)$. Our results may then be stated thus, s denoting the distance from a fixed point.

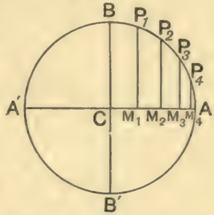
If the acceleration of a particle to a point be ks, *or* if its velocity be given by an equation of the form $v^2 = k(a^2 - s^2)$, then the period of the vibration is $2\pi/\sqrt{k}$ and the amplitude is a.

158. It is very easy to obtain a graphical representation of the motion which will render clear to the eye how the motion alters with the time.

Let ACA′ denote twice the amplitude of the motion. On it as diameter draw a circle and divide its circumference into any number of equal parts, starting from A. It will be convenient to take a multiple of 4, since then a division falls on each of B, A′, B′. Let $P_1, P_2, P_3, \ldots$ denote them.

These parts are described in equal times by the point P, which, moving along the circle, determines the harmonic motion along AOA′. If P_1M_1, P_2M_2, . . . be drawn perpendicular to AOA′, the points M will denote the position of the point moving with the S.H.M. in question, at equal intervals of time. It is clear that near A they will be closer together—that is, that M is moving very slowly, then gets quicker and quicker as it approaches C, and after passing C gets slower and slower until it comes again to rest at A′ and begins to return back over its previous course.

Fig. I.



The motion may, however, be rendered clearer to the eye by a graphical construction in which distances OM measured along a line Ox denote the times, and distances MP parallel to Oy represent the displacements at the times in question, measured up when the displacement is to the right and down when to the left.

The extremities of these lines will lie on a curve.

To illustrate the mode of drawing the curve, take the case of Fig. I., in which the circumference (or period) has been divided into twenty equal parts. We need only consider the case of the first quarter in detail, as the others are similar.

Take (Fig. II) along Ox a distance Ob to represent the period. Divide it into four equal parts at a, b', a', b. These will represent the quarter, half, and three-quarter periods, corresponding to the positions A, B′, A′ of P. Draw at a, ap_5 equal to CA, and at a' draw $a'p_{15}$ downward, and also equal to CA. Then at starting the displacement is represented by zero; at a quarter period it is ap_5; at a half period it is at p_{10} or b', *i.e.* again at zero; at three-quarters it is $a'p_{15}$, and back to its former position or zero at b at the end of the period. To find the form of the curve between, divide Oa into five equal parts by m_1, m_2, m_3, m_4 and draw m_1p_1, m_2p_2, . . . equal respectively to CM_1, CM_2, . . . Draw a continuous curve passing through $p_0p_1p_2p_3p_4$. This will re-

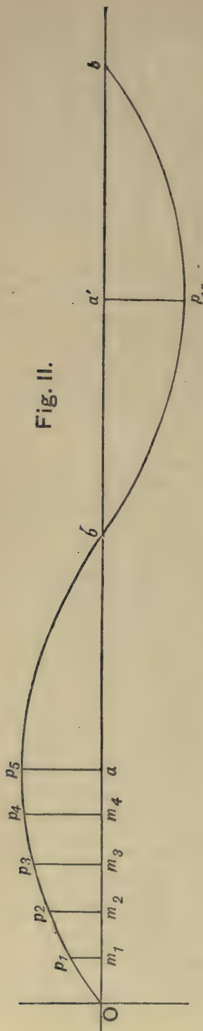


Fig. II.

present very accurately the curve required. If more accuracy is desired, it is only necessary to increase the number of divisions (P_1, P_2, . . .) or (m_1, m_2, . . .) From a to b' the curve is clearly similar to that between a and O, and so for the rest of the period, the curve being below Ox. The whole curve for one period will be as in Fig. II. As the time increases the curve repeats itself over and over again.

It is sometimes more convenient to consider the time as commencing from the instant when the point is at its extreme position, or when its velocity is zero. The only alteration to be made is that now O must be taken to be at a, instead of as in the figure.

159. Besides having a simple harmonic motion, a point may have any number of other kinds of motion or other simple harmonic motions impressed on it. The resulting motion is to be found by compounding them all according to the parallelogram law. On account of its high importance, we here consider more in detail the results of compounding various S.H.M. We may consider these as being produced by a point moving with one S.H.M. on a board, which itself moves over a second board with another S.H.M. in the same or a different direction to that of the former. Again, we may suppose this second board to have a S.H.M. of its own, and so on. We will consider first the case where the various S.H.M. are all along the same line.

In considering different S.H.M. it is to be noticed that not only may they differ in amplitude and in period, but also in phase. That is, they may not attain their greatest amplitude at the same time. In compounding, therefore, two or more S.H.M. it will be necessary to know the instants at which they first attain their greatest amplitudes —*or*, which amounts to the same thing, their phases at the commencement of the time. The phase at the commencement is called the epoch.

The composition of collinear S.H.M. is a simple matter, as we have only to add algebraically the displacements at any time to obtain the actual displacement. Thus in Fig. III the thin lines represent two S.H.M., the one having half the period and one-third the amplitude of the other and having its epoch equal to one-quarter of its period behind the second. The thick line represents the curve obtained by adding together algebraically the displacements of the components. Thus $MP = Mp + Mp'$, $QN = Nq - Nq'$.

It is clear that the curve between A and A_2 will be repeated periodically. This will always be the case if the periods of the two components are commensurate. Thus suppose for instance the periods were $5n$, $7n$. Then after a time $35n$ the first will have had seven periods and the second five

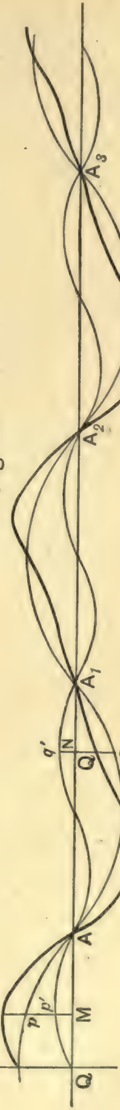


Fig. III.

periods, consequently each will be in the same condition as at starting, and the same form must therefore begin to recur. From the example given it is clear that if t_1, t_2 denote the periods, the period of the compound curve will be given by the least common multiple of t_1, t_2.

Similar results hold for any number of components. The student should himself draw the compound motion for several cases. For instance—

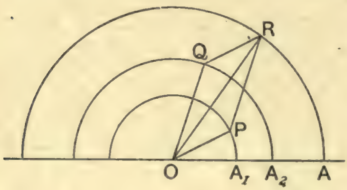
(α) Two equal S.H.M., one being half a period behind the other (the motion is always zero).

(β) Three S.H.M., amplitudes ·3, ·4, ·5 inches, periods 3, 5, 6 minutes, and (1) all with the same epoch, (2) the phase of the second at starting a half period behind the first, and the third a quarter behind.

The drawings should be made on a large scale on paper ruled into squares.

It is a fact, though not here proved, that whatever the form of any curve between two points O, A, it can be regarded as the resultant of a number of S.H. curves of suitable amplitudes and phases, and whose periods are all submultiples of OA.

When the *periods are the same* and they differ only in phase and amplitude, the resultant of two collinear S.H.M. is best found by referring them to the motions of their generating points.



In the figure let O be the centre of the generating circles, P, Q simultaneous positions of the generating points of the two component motions. Then since OP, OQ revolve at the same rate (periods being equal) the angle POQ is constant and measures the constant phase-difference of the two motions. OA_1, OA_2 are the amplitudes of the two motions. Complete the parallelogram OPRQ. Now the velocities of P, Q are proportional to OP, OQ respectively, and are perpendicular to them. Hence their resultant is proportional to OR and is perpendicular to it. Since OP, OQ are constant and POQ is constant, OR is constant and revolves at the same rate as OP or OQ. Hence R moves in a circle in the same period as the component motions.

Now since the motion of R is the resultant of those of P and Q, its component along OA is the resultant of the components of P and Q in the same direction. Hence the resultant S.H.M. is another S.H.M. of the same period, amplitude OA, and whose phase is in advance of that of P by the angle POR and behind that of Q by ROQ.

Let ϵ denote the phase-difference POQ of the two component motions, a_1, a_2 their amplitudes, and a that of their resultant. Then $a^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos \epsilon$.

Also if θ_1 be the lag of phase of R behind Q, then $\tan \theta_1 = a_1 \sin \epsilon/(a_2 + a_1 \cos \epsilon)$.

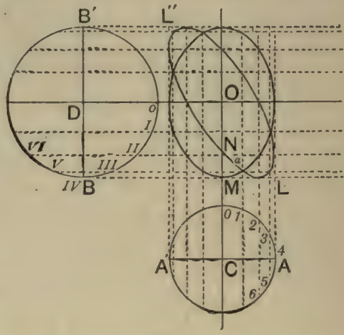
Suppose we have three S.H.M. of the same period and amplitude and with phases differing by one-third of a period. Then the resultant of the generating motion is similar to that of three equal forces inclined 120° to one another. This resultant is zero. Hence the resultant S.H.M. has zero amplitude. Any one is therefore equal and opposite to the resultant of the other two. This result is used in the transmission and transformation of electrical power by the "three-phase" method.

160. It remains to consider the case of composition of two or more S.H.M. in different directions. The method will best be understood by illustrations. We shall suppose the directions perpendicular to one another, although the method is the same when they are inclined at any angle. The curves we shall obtain will be those actually traced out by a particle moving with the two given S.H.M.

CASE I. *Two S.H.M. of equal periods but not necessarily equal amplitudes.*—Let ACA′, BDB′ be two lines representing the amplitudes of the two motions in direction and magnitude. Draw the corresponding generating circles.

Divide their circumferences into the same number ($2n$) of equal parts, and mark them, starting from the points corresponding to zero phase, the first by the numbers 0, 1, 2, 3, . . and the second by 0, I, II, III, . . .

Through the points 1, 2, 3, . . . draw lines perpendicular to ACA′ (and therefore parallel to BDB′), and through I, II, III, . . . draw lines perpendicular to BDB′. These lines will intersect to form a rectangular network (they are the dotted lines in the figure). The angles of this network we will denote by the lines meeting there. Thus the point *a* is either 1, III or 7, III or 1, V or 7, V.



The compound motion in any case will be determined by the intersection of the lines drawn through simultaneous positions of the generating points of the two S.H.M.

Different sub-cases will arise according to the differences in phase of the two component S.H.M.

Sub-case 1. *Phase the same.*—Here the generating points start from 0 and O, and successively arrive at 1, I—2, II—3, III, etc. If the points of the network represented by 1, I—2, II—3, III be joined, the resulting curve will be that traced in the compound S.H.M. It is in this case a straight line. We therefore have the very important result *that two perpendicular S.H.M. of the same period and the same phase produce a rectilinear motion, which is also harmonic, of the same period, and the square of whose amplitude is the sum of the squares of the component amplitudes.* This last result follows at once, for

$$OL^2 = OM^2 + LM^2 = AC^2 + BD^2.$$

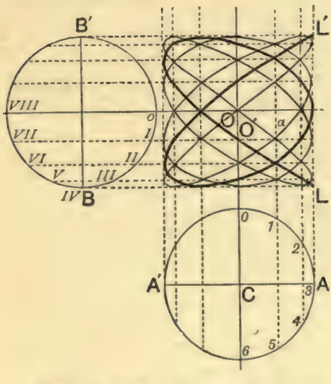
Sub-case 2. *The phase of one being $\frac{1}{8}$ ahead of the other.*—Let D be that which is $\frac{1}{8}$ ahead. Then when the generating point of C is at 0 that of D will be at II, when the first is at 1 the second is at III, and so on. Hence the points on the compound motion will be at 0, II—1, III—2, IV, etc. Join them and we get an oval curve, represented in the figure by the thin curve Na . . . It is actually an ellipse.

Sub-case 3. *Difference of phase* $\frac{1}{4}$.—Here the points start at 0, IV, and the series of points on the compound motion are 0, IV—1, V—2, VI. The result is the curve represented by the thin line Ma . . . in the figure; when the amplitudes are equal it is a circle.

Note.—When two successive points on the curve are given we can trace it without reference again to the generating circles; for the next point is the opposite angle of the succeeding rectangle, and so on.

The student can now easily find the compound motion for the cases of 3/8, and 1/2 difference of phase. When the difference is $\frac{1}{2}$, the motion becomes again rectilinear.

CASE II. *Two S.H.M. of different but commensurable periods.*—Take the case where the ratio of the periods is 3 : 4. Let C be the generating circle for the first or quicker period and D for the second. Divide the circumference of ACA′ into $3n$ (say twelve) and that of BDB′ into $4n$ (say sixteen) equal parts. Then while the generating point of C describes one of these parts, that of D will describe a corresponding part on its circle. Various sub-cases will occur, as in Case I.



Sub-case 1. *Same phase at starting.*—Here points will be 0, O—1, I—2, II, etc.

The resulting curve starts from O, and is represented by the thin line Oa . . . in the figure.

Sub-case 2. *The phase of* D $\frac{1}{16}$ *ahead at starting.*—Here the starting-point is 0I or O′, the resulting curve is shown by the thick line in the figure—it goes on to L, stops, retraces its path to O′, and on to L′, when it retraces its former path.

The student should draw the curves for other cases, especially for that where the ratio of the periods is 1 : 2. Here, when both start with their maximum amplitude, the curve is a parabola.

The Simple Pendulum.

161. A heavy particle suspended from a fixed point by a weightless string is called a *simple pendulum*. In practice a heavy body suspended by a string whose weight is very small compared with the body and whose length is great compared with any linear dimensions of the body, will behave very approximately as a simple pendulum.

If, however, a heavy body be suspended in such a way that it always moves parallel to itself, the latter condition is not necessary. The following is a method of suspending a body so that it must always move parallel to itself.

Let A, B be any two points on the body by which it is to be suspended. Attach to A, B two strings of the same length fastened to two fixed points O, O′, where O, O′ are respectively vertically above A, B, and OA = O′B. Then OO′ is equal and parallel to AB.

If now the body be pushed on one side so that A, B come to (say) A′, B′, then OA′B′O′ is a parallelogram, and A′B′ parallel to OO′ and therefore to AB. That is, the new position of the body is such that it has moved into it without any rotation. The question of the oscillations of large bodies about a fixed point will be considered in Part III.

162. *The time of oscillation of a simple pendulum.*—In the figure let O denote the point of suspension, l the length of the string, P the position of the particle at any time.

The forces on P are (1) its weight mg downwards, (2) the tension (T) of the string along PO. The first, or the weight, can be replaced by its components along PB and OP. These components are parallel to BP and OP, whilst mg is parallel to OB—that is, the forces are parallel to the sides of the triangle OBP. Hence

$$\text{Component of } mg \text{ along PB} = \frac{PB}{OB}mg = \frac{PN}{OP}mg = \frac{PN}{l}\cdot mg$$

(or $mg \sin \theta$, if $PON = \theta$).

Therefore the acceleration along the arc PA at P is $\frac{PN}{l}g$.

If now we confine our attention to small oscillations —that is, oscillations in which the angle POA is never very large—PN is very nearly equal to the arc AP. In this case the acceleration along the arc PA to A is $= \frac{g}{l} \times$ arc, *or* is proportional to the distance from A. Consequently, by § 157 of this chapter, the period is given by

$$t = 2\pi\sqrt{\frac{l}{g}}.$$

Thus, provided the string is never inclined at a large angle to the vertical, the motion is very nearly isochronous.

In using this formula, the restriction as to smallness of amplitude must not be forgotten. If the amplitude is moderately large, it will have to be taken account of, but the method by which this is to be done requires a more advanced knowledge of mathematics than we here assume.

The student should also realise the truth of the result obtained by experimenting himself. A heavy weight suspended from the ceiling of the room will serve excellently for the pendulum. By observing the time of vibration for different amplitudes it is easy to verify that the motion is isochronous, and by altering the length of the string that the time is proportional to the square root of the length. For the method of determining the time accurately, reference must be made to the various text-books on practical physics. The method depends on observing the interval between a large number of vibrations, and dividing the time observed by the number. It is to be remembered that a vibration is a complete *to and-fro* movement.

The pendulum affords the most exact method for determining the value of g, the acceleration of gravity. For it depends on the measurement of the time of vibration, an operation susceptible of extreme accuracy. When, however, very extreme accuracy is desired, various other effects have to be taken account of. Such are the buoyancy of the air, the resistance due to the viscosity of the air, the magnitude of the arc of vibration, together with other considerations dependent on the fact that the experiments are made, not with simple pendulums, but with rigid bodies.

163. It is clear, since the value of g is different at different parts of the earth, that a pendulum clock which would keep exact time at one place would not keep exact time at another unless readjusted. For instance, suppose a clock, which keeps exact time at a place where the acceleration of gravity is g, is removed to a place where it is g', and let us find how much the clock would gain or lose in an hour.

Let l denote the length of the pendulum and t, t' the times of vibration at the two places, then

$$t = 2\pi\sqrt{\frac{l}{g}},$$

$$t' = 2\pi\sqrt{\frac{l}{g'}},$$

and

$$\frac{t}{t'} = \sqrt{\frac{g'}{g}}.$$

Now at both places the hour-hand makes one turn in the same number of swings of the pendulum, and hence the times they take to make those turns will be in the same ratio as the times of the corresponding swings. Therefore

$$\frac{\text{Apparent hour at second place}}{\text{hour at first place}} = \frac{t'}{t} = \sqrt{\frac{g}{g'}}.$$

Hence

$$\frac{\text{Gain at second place per hour}}{\text{1 hour}} = \frac{t - t'}{t} = 1 - \sqrt{\frac{g}{g'}}.$$

Now as a matter of fact the differences between g and g' are very small. Let x denote the difference. Then $g' = g + x$, and

$$1 - \sqrt{\frac{g}{g'}} = 1 - \left(1 + \frac{x}{g}\right)^{-\frac{1}{2}} = 1 - \left(1 - \frac{x}{2g}\right)^{*} = \frac{x}{2g},$$

if we neglect $\left(\frac{x}{g}\right)^2$ and higher powers, which are very small;

$$\therefore \text{Gain per hour at second place} = \frac{x}{2g} \text{ hours,}$$

$$= \frac{60 \times 60x}{2g} \text{ seconds,}$$

$$= \frac{225}{4} x \text{ seconds nearly.}$$

For instance, taking the numbers given on p. 45, if the clock were taken from London to Edinburgh,

$$x = 32{\cdot}203 - 32{\cdot}191 = {\cdot}012,$$

or it would gain ·675 seconds per hour, or 16·2 seconds in the course of one day.

On the contrary, if it were taken to Paris,

$$x = 32{\cdot}183 - 32{\cdot}191 = -{\cdot}008.$$

It would gain $-\frac{225}{4} \times {\cdot}008$, or lose ·45 seconds per hour, or 10·8 per day.

The length of a simple pendulum at London when the time of vibration is one second is given by

$$l = \frac{32{\cdot}191}{4\pi^2},$$

$$= {\cdot}8154 \text{ feet,}$$

$$= 9{\cdot}7848 \text{ inches.}$$

This, however, is the length when the time of a to-and-fro or a swing-swang is one second. If the time of swing from one side to the other is to be one second, the length must be four times this, or

$$l = 39{\cdot}1392 \text{ inches.}$$

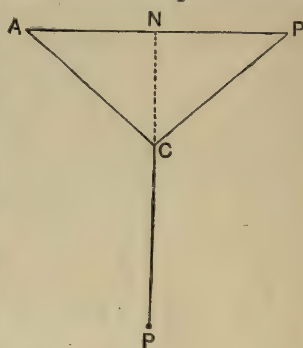
164. A pendulum can vibrate in any vertical plane through the point of suspension. In fact we may take

* By the binomial theorem. A student not acquainted with this must work out in any case the numerical value of $1 - \sqrt{\frac{g}{g'}}$.

two perpendicular vertical planes through this point, and the particle at the end of the string may simultaneously be vibrating with motions belonging to both of them. It will therefore move in a curve which is the resultant of two perpendicular S.H.M. Since the length of string is the same for both motions, the periods of the component S.H.M. are the same, the particle will therefore describe curves similar to those in Case I. of § 160. For instance, to reproduce sub-case 3, pull the particle through a distance AB and project it with a velocity at right angles to AB. It will then have the two component S.H.M., one due to AB and starting from rest at B to move along AB; the other perpendicular to AB and initially in its place of greatest velocity, or one-quarter of its period ahead of the other. According to the magnitude of velocity of projection there will result an oval (ellipse) whose longer axis is along AB, a circle, or an oval whose longer axis is perpendicular to AB.

To illustrate the general case of the composition of S.H.M. of different periods, it would be necessary for the lengths of the pendulum to be different for the two planes. This can be managed by the following arrangement.

To two points A, B in the same horizontal plane fasten a string ACB, longer than AB. At its middle point C tie another string, the other end of which is fastened to the heavy particle P. Let PC produced cut AB in N, and denote CP by l, NP by l'.



Then notice, if P be pulled aside *in* the plane ACB, it will oscillate about C as a fixed point, the strings AC, BC preventing motion of C in this plane. It will therefore vibrate with a S.H.M. in period $2\pi\sqrt{l/g}$.

If, however, P be pulled aside in a line perpendicular to the vertical plane through AB, it will swing about the point N, and will therefore move with a S.H.M. of period $2\pi\sqrt{l'/g}$.

If now P receive any other displacement it will move with these two harmonic motions, and therefore in a curve compounded of the two in a similar manner to those in Case II of § 160. If l and l' be in the ratio of 9 : 16, the periods will be in the ratio 3 : 4, and P will describe curves of the same kind as those there determined. By suitably altering the length of CP, the resultant of any kind of perpendicular S.H.M. can be obtained.

Several contrivances may be adopted to obtain traces of the motion. Thus the weight may carry a light pointer which moves over a plate of smoked glass placed horizontally under it, clearing away the sooty layer as it moves with the body. Or the weight may carry a tube through which a thin stream of sand or ink falls on a sheet of paper under it. By having a glass tube drawn out to a fine point and containing ink, it may be made to just move over a sheet of paper loosely held under—or the glass tube may be movable in a small vertical hole in the body and so adjust its position automatically as to always just rest with its point on the sheet of paper. The curves thus obtained are very beautiful. It affords an easy and extremely interesting means of verifying the results deduced *à priori* by the method of § 160.

EXAMPLES—XVII.

1. Represent graphically by a curve the nature of the motion compounded of two S.H.M. in the same direction, of equal amplitude, with periods in the ratio of 1 : 3 and such that initially the first is at rest while the second has its greatest velocity.

2. The same as in the previous question but their periods as 3 : 5.

3. A point moves in a path produced by the combination of two S.H.M. of equal amplitude in two rectangular directions, the periods of the components being as 1 : 3. Draw the paths described when the epochs are the same, and when they differ by 90°.

4. A particle oscillates in a smooth straight tube of narrow bore attracted to a centre of force outside the tube, the force varying as the distance. Find the time of oscillation and the point about which it takes place.

5. If a pendulum that oscillates seconds be lengthened by its hundredth part, find the number of oscillations it will lose in 24 hours.

6. A simple pendulum beating seconds is lengthened by one-twentieth of an inch. Find the number of seconds it will lose in 24 hours.

7. If a seconds pendulum be lengthened 1 inch, find the number of seconds it will lose in 12 hours.

8. A pendulum whose length is l makes m oscillations in 24 hours; when its length is slightly changed it makes $m+n$ oscillations in 24 hours. Show that the pendulum has been diminished in length by a part equal to $2nl/m$ nearly.

9. A pendulum oscillating seconds at one place is carried to another place at which it loses 2 minutes a day. Compare the accelerations of gravity at the two places.

10. A pendulum which would oscillate seconds at the equator would if carried to the pole gain 5 minutes a day. Show that gravity at the equator is to gravity at the pole as 144 is to 145.

11. If a seconds pendulum be carried to the top of a mountain half a mile high, how many seconds will it lose in a day if gravity vary as the inverse square of the distance from the earth's centre, which is supposed to be 4000 miles from the foot of the mountain?

12. Two pendulums oscillating at two different places lose t and τ seconds a day respectively, and if the places at which they oscillate be interchanged they lose t' and τ' seconds. Prove that

$$t+\tau=t'+\tau' \text{ nearly.}$$

13. A mass of 1 lb. attached to a stiff spring executes 10 complete vibrations per second. What will be its rate of vibration if an additional mass of 1 lb. be attached? Also find its maximum velocity in inches per second when the amplitude of oscillation is 2 inches.

14. Determine as in Art. 159 the resultant of three harmonic motions in the same line, whose amplitudes are as $1 : (\frac{1}{3})^2 : (\frac{1}{5})^2$; periods $1 : \frac{1}{3} : \frac{1}{5}$, and where at starting the first and third are in the same phase and the second 180° ahead.

[*This is nearly a zigzag line.*]

PART III

PLANE MOTION OF A RIGID BODY

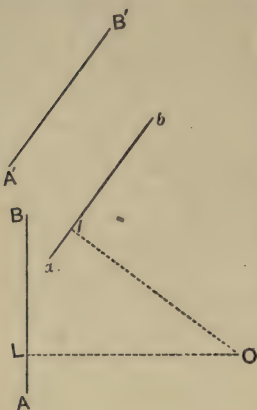
CHAPTER XVIII

KINEMATICS OF THE MOTION OF A RIGID BODY

165. WE pass now from the consideration of the motion of single particles to that of a system or systems of particles rigidly connected together, so that their relative positions to one another remain unalterable. Such a system is called a "rigid body." We shall however, as in the previous pages, confine our attention only to cases where the motion of every particle of the system is parallel to a fixed plane, and even here to the restricted case where the form of the body is symmetrical on both sides of the plane.

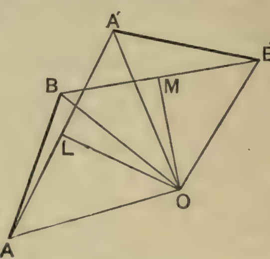
166. The position of a rigid body is determined as soon as we know the positions of three points fixed in the body, and not in the same straight line; for if two of the fixed points are given, the body may have any position found by twisting it about the line joining the points. If then another point outside this is fixed, the body itself is fixed and its position determined. If, however, the body be always constrained to move parallel to a given plane, its position will be determined as soon as two points of it in this plane are known. In considering then the change of position of a body in plane motion it is only necessary to consider the changes in position of two points fixed in the body. Each of these points is capable of any change of position in the plane, subject to the sole condition that the distance between them is invariable. We shall call them the points of reference.

A body can be removed from any position to any other in the same plane by rotating it about any fixed point in the plane and then moving it parallel to itself into the second position. For let A, B denote the points of reference of the body in the first position and A′, B′ in the second, and let O be any point whatever in the plane. First rotate the body round O until AB takes a position *ab* parallel to A′B′. Then the body may be moved parallel to the line *a*A′ or *b*B′ until *ab* coincides with A′B′, when the body arrives at the second given position. Thus the proposition is at once seen to be true.



Draw OL, O*l* perpendicular to AB, *ab*. Then LO*l* is the angle through which the body has been turned. Moreover, LO*l* is equal to the angle between AB and *ab*—that is, since *ab* is parallel to A′B′, to the angle between AB and A′B′. Hence the angle through which the body is rotated is the same wherever the point O may be taken.

167. The same result can be attained in a single operation by rotating the body around a particular point in the plane. For join AA′, BB′. Bisect them in L, M, and draw LO, MO at right angles to AA′, BB′ respectively. Let O be the point of their intersection. Then, since O is a point on LO which bisects AA′ at right angles, OA = OA′. Similarly OB = OB′. Also AB = A′B′. Therefore the three sides of the triangle OAB are equal to the sides of the triangle OA′B′. Hence the triangles are equal, and the angle AOB equal to the angle A′OB′. Adding the angle BOA′, it follows that



$$AOA' = BOB'.$$

Hence, if the body be turned round O through the angle AOA′, A will arrive at A′ and B will turn through the same angle into the position B′. That is, the body has been changed from its first position to its second by rotation about the point O.

168. In considering then the continuous motion of a rigid body, we may either take a fixed point in it and suppose the body to have simultaneously an angular velocity round this fixed point, together with a motion of translation of this point, or we may regard it as having an angular velocity about a single point, whose position, however, is not fixed in the body, but alters with the time. In either case three things will be needed to give the motion at any instant. In the first case, the angular velocity and the magnitude and direction of the motion of the fixed point. In the second case, the angular velocity and the two co-ordinates of the point round which the body moves. If this point be O, then the body is only turning about O at an instant; at a succeeding instant it is turning about another point, in general close to O. For this reason O is called the instantaneous centre of rotation.

169. We may suppose any displacement of a rigid body to be attained by two or more steps. Each of these may be produced by rotations round some two points. Hence it follows that two rotations round two points are equivalent to a resultant rotation round some other point. The amount of the rotation in each case is the angle through which the line of reference has turned. It is therefore clear that the magnitude of the resultant rotation must be the algebraical sum of the several component ones.

Regarding the displacements as taking place in a definite time by angular velocities round given points, we see that a body having two or more simultaneous angular velocities about different points moves as if under the action of an angular velocity about some other point whose magnitude is the algebraical sum of the component ones.

It remains to find this point about which the resultant angular velocity acts. Take first the case of two components. Let the body have angular velocities represented by ω_1, ω_2 about A, B. Let C be the position of the axis of the resultant angular velocity and $\Omega = \omega_1 + \omega_2$ denote its magnitude.

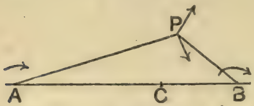


Fig. 1.

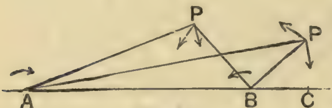


Fig. 2.

We shall determine C from the fact that its velocity due to the two angular velocities about A and B must vanish. That is, the velocity of C due to ω_1 at A must be equal and opposite to that due to ω_2 at B. It is clear, therefore, that C must lie in the line joining AB. If ω_1, ω_2 are in the same direction, C must lie between A and B, if in opposite directions, outside. For in the first case (Fig. 1) and in the second case (Fig. 2),

$$\text{Velocity of C} = \text{velocity due to } \omega_1 + \text{velocity due to } \omega_2,$$
$$= \omega_1 . AC - \omega_2 . BC = 0,$$

whence
$$\frac{AC}{BC} = \frac{\omega_2}{\omega_1}.$$

That is, the resultant of ω_1, ω_2 is found in the same way as that of two parallel forces ω_1, ω_2 at A and B.

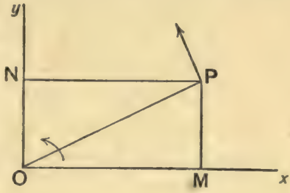
Similarly the resultant of any number of rotations may be obtained.

If ω_1, ω_2 are equal and opposite, the result is a translation $= \omega . AB$. For the velocity of any point P on AB $= \omega_1 AP + \omega_2 BP = \omega(AP + BP) = \omega . AB$, and is the same for all points.

170. Having now seen how to determine the motion of the body as a whole, it will be necessary further to express the velocities of the different particles of the body.

If O denote a fixed point in the body, we have seen that the motion of the body is determined by a velocity of translation of O and a rotation round an axis through O. The first is a translation of the body as a whole, and each particle will partake of the same.

In addition each particle will have a velocity depending on the angular velocity round O and its position in the body. Thus, if ω denote the angular velocity, it will produce in a point P a velocity $= \omega . \text{OP}$ perpendicular to OP, where OP is the perpendicular from P on the axis of rotation. The velocity of P will therefore be compounded of that of O and of $\omega . \text{OP}$ perpendicular to OP.



Let u, v denote the components of the first along the two lines Ox and Oy.

We must find the components of the velocity of P along Ox and Oy also. Now notice that ON, PN are perpendicular to Ox and Oy, whilst OP is perpendicular to the velocity of P. Hence, if the triangle PON be turned through a right angle, its sides will be parallel to the velocities. Hence

$$\text{Velocity along } \text{O}x = -\frac{\text{ON}}{\text{OP}} . \text{vel. of P} = -\frac{\text{ON}}{\text{OP}} . \omega . \text{OP}$$
$$= -\omega . \text{ON},$$

$$\text{Velocity along } \text{O}y = \frac{\text{PN}}{\text{OP}} . \text{vel. of P} = \frac{\text{OM}}{\text{OP}} . \omega . \text{OP} = \omega . \text{OM}.$$

Therefore the whole velocity of P is

$$u - \omega . \text{ON along O}x$$

and $$v + \omega . \text{OM along O}y.$$

171. If the body is altering its motion, its rate of change will be determined by the acceleration of the velocity of the point O fixed in it and the acceleration of the angular velocity round O. In this case what will be the actual acceleration of a given particle P in the body? It is compounded of the acceleration of O and the acceleration of P relative to O. But if OP is the perpendicular from P on the axis of rotation, the latter is $\omega^2 . \text{OP}$ along PO and $\dot{\omega} . \text{OP}$ perpendicular to OP—$\dot{\omega}$ being the angular acceleration. The components of $\omega^2 . \text{OP}$ along the lines Ox, Oy are $-\omega^2 . \text{OM}$, $-\omega^2 . \text{ON}$ respectively; those of $\dot{\omega} . \text{OP}$ are

$-\dot{\omega} \,.\, ON$ and $\dot{\omega} \,.\, OM$. Hence, if $\dot{u}$, $\dot{v}$ denote the acceleration of O itself, the accelerations of P are

$$\dot{u} - \omega^2 \,.\, OM - \dot{\omega} \,.\, ON,$$
$$\dot{v} - \omega^2 \,.\, ON + \dot{\omega} \,.\, OM.$$

EXAMPLES—XVIII.

1. A rigid body is moving in a plane. Prove that those points whose directions of motion at any instant pass through a fixed point lie on a circle.

2. A rigid body is moving in a plane. If O be the instantaneous centre of rotation, show how to find the position of a particle of the body whose direction of motion passes through two given points.

3. Find the instantaneous centre of rotation of a carriage wheel.

Hence find the actual velocity of a point on the wheel 30° in front of the top when the velocity of the carriage is given.

4. A fixed point in a rigid body moves round a circle with constant velocity, whilst the body itself rotates with a constant angular velocity. Find the instantaneous centre of rotation at any time.

5. A circle rolls inside another of twice the radius. Show that every point on the rim of the moving circle moves in a direction passing through the centre of the fixed one.

Hence prove that a point fixed on the moving circle oscillates along a diameter of the other.

6. A rod AB is rotating with constant angular velocity ω round A, while A moves with constant velocity u along a straight line CD. Determine the velocity and the acceleration of its centre of gravity at the instants when (1) BAC $= 0°$, (2) BAC $= 90°$.

7. If in the previous case A moves with constant velocity along the circumference of a circle of radius r, determine the velocity and acceleration of its centre of gravity when (1) AB is a tangent to the circle, (2) is perpendicular to it.

CHAPTER XIX

MOMENTUM AND ENERGY

172. We have seen that a moving particle can be brought to rest by a blow, whose impulse is measured by the momentum of the particle. If then we have any system of particles in motion, they can be brought to rest by the application of a system of blows to each particle equal and opposite to the momentum of each. Now these blows may be represented by a resultant. In the most general case (not proved here) this consists of a single blow, together with an impulsive twist round it. In the cases here considered, where the motion is always parallel to a plane, the resultant (as we have seen in § 62) may be represented by a single blow acting at any arbitrary point together with a couple—or, a single blow alone acting along a definite straight line. In some cases, however, there is no single blow, but the resultant becomes a couple only. If we reverse the directions of the impulses we arrive at a resultant which is the equivalent of the momenta of the different parts together. This is called the momentum of the motion.

Now suppose that in a system of particles the particles are connected either by rigid or elastic connections, or mutual forces, or in any way whatever. Let a blow (or couple) equal and opposite to the "momentum" of the system be applied to it. This will call into play a system of impulsive reactions between the different connections of the particles. But the change of momentum produced by these connections alone will be zero. For consider two particles A, B. The reactions between them will be equal and opposite, hence the changes of momentum produced by them will on the whole destroy one another. Consequently, whatever be the nature of the connection between the particles, the change in the momentum of the system will be measured by the external blow. Since this was equal and opposite to the original momentum, the system is reduced to such a state that its momentum is zero. It does not follow that every particle will be reduced to rest, as they may be capable of relative motion. But in the case of a rigid body such relative motion cannot take place, and consequently the body must be reduced completely to rest. We learn then this very important fact, that a rigid body moving with plane motion can always be reduced to rest by applying a suitable blow at any point rigidly connected with it, together with a suitable blow-couple.

By suitably choosing the point, the blow-couple may *in general* be made to vanish. In particular cases, however, the single blow is zero, and the blow-couple is then necessary.

In a similar way it is clear that if any blow be given to a body, the resulting motion is such that the momentum of the body is exactly equivalent to the blow. The importance then of being able to determine the momentum in any case when the motion is given by either of the methods of the preceding chapter is evident.

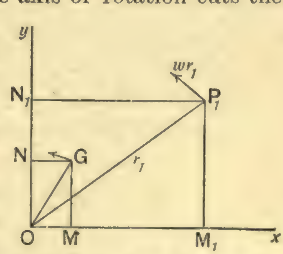
173. It will be necessary here to define more precisely the nature of the bodies, and the motions, which will be considered in the succeeding pages. In the first place, every particle of the body will be supposed to be moving parallel to a fixed plane, in which lie the centres of gravity of all the separate bodies considered. This plane we shall call the plane of motion. In the second place, the shape and nature of each body will be such that it is symmetrical both geometrically and mechanically on both sides of this plane. That is, for every point on one side there will be a corresponding point on the other, and the masses of the corresponding particles will be the same. Such bodies are, for instance, a uniform sphere spinning about any diameter and

moving in a plane perpendicular to that diameter; a prism moving round an axis parallel to one edge; or the same shaped bodies, with the densities at any point depending only on the distance from the centre of the sphere, or the distance from the plane through the centre of the prism perpendicular to its axis.

174. *Momentum of a body moving parallel to itself without rotation.*—Here the velocity of every particle is the same, and the momentum of each is therefore proportional to its mass. We have then to find the resultant of a system of parallel blows acting on particles and proportional to those particles. It is therefore the same as the sum of the momenta acting at the centre of gravity of the particles. But, since every particle has the same velocity, the magnitude is the same as that of the whole mass moving with the given velocity. Hence, if M denote the mass of the whole body and u its velocity, the momentum is Mu, and acts at the centre of gravity of the body.

Momentum of a body rotating round a fixed axis.—The momentum can be expressed (§ 172) by a single impulse acting at the point, together with an impulse-couple round it. Each of these will have to be found.

Let O be the point where the axis of rotation cuts the plane of motion, and let ω denote the angular velocity of the system of particles forming the body. Let P_1 denote one of the particles of the body, and let r_1 be its distance from the axis.



Take any two perpendicular planes through the axis of rotation, and cutting the plane of motion in Ox, Oy; and draw P_1M_1, P_1N_1 perpendicular to them. Then, by § 170, the velocity of P_1 is equivalent to

$$-\omega \,.\, ON_1 \text{ along } Ox,$$
$$\omega \,.\, OM_1 \text{ along } Oy,$$

and the momenta will be, if m_1 denote the mass of P_1,

$-m_1\omega . ON_1$ along Ox and $m_1\omega . OM_1$ along Oy. Similarly for each of the other particles. Hence the resultant will be equivalent to

$$X = -\omega\{m_1 . ON_1 + m_2 . ON_2 + \ . \ . \ .\} \text{ along } Ox$$

and

$$Y = \omega\{m_1 . OM_1 + m_2 . OM_2 + \ . \ . \ .\} \text{ along } Oy.$$

But, if G denote the centre of gravity of the particles and GM, GN are perpendiculars on Ox, Oy,

$$OM = \frac{m_1 . OM_1 + m_2 . OM_2 + \ . \ . \ .}{m_1 + m_2 + \ . \ . \ .},$$

$$ON = \frac{m_1 . ON_1 + m_2 . ON_2 + \ . \ . \ .}{m_1 + m_2 + \ . \ . \ .},$$

and $\quad m_1 + m_2 + \ . \ . \ . = \text{whole mass} = m$ say.

Hence

$$X = -\omega . m . ON,$$

$$Y = \omega . m . OM.$$

That is, the same as if the whole mass m were placed at the centre of gravity G, and revolved with it round O. For this reason the point is sometimes called the centre of inertia. Hence the single resultant impulse is $m . \omega . OG$, and acts perpendicular to OG.

To find the impulse-couple we have to take moments round the axis through O.

Now the velocity of P_1 is $\omega . OP_1$ and is perpendicular to OP_1. Hence the moment of its momentum round O $= m_1\omega . OP_1 . OP_1 = \omega . m_1 OP_1^2$. Therefore the moment of the whole momentum round O

$$= \omega\{m_1 . OP_1^2 + m_2 . OP_2^2 + \ . \ . \ .\},$$

$$= \omega\{m_1 . r_1^2 + m_2 . r_2^2 + \ . \ . \ .\}.$$

The value of $m_1 r_1^2 + m_2 . r_2^2 + \ . \ . \ .$ depends not only on the position of O but also on the shape and distribution of the particles in the body. It is called the *moment of inertia of the body about the given axis of rotation through* O. We shall denote it by I. We may then write

$$\text{Moment of impulse round } O = \omega I.$$

Let k denote such a length that

$$mk^2 = I = m_1 . r_1^2 + m_2 . r_2^2 + \ . \ . \ .,$$

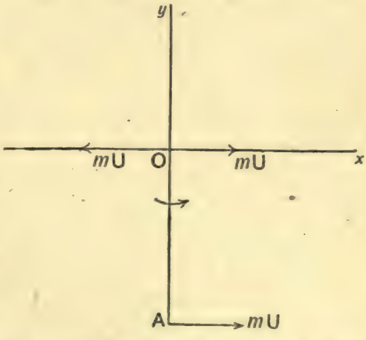
then k is called the *"radius of gyration" of the body round the axis.* No kinetical behaviour of a body can be determined until we know the value of its radius of gyration. Hence its determination is of very great importance. With this notation

$$\text{Moment of impulse round O} = mk^2\omega.$$

175. Any instantaneous motion of a body can be represented (Chapter XVIII) by a rotation round some point. Hence the preceding paragraph gives us the means of finding the resultant impulse when this instantaneous centre is known. It is, however, generally most convenient to represent the motion in the alternative way, viz. by taking a fixed point in the body and referring the motion to it. This we shall do by taking the centre of gravity for the point of reference. The angular velocity will be the same whatever the point of reference. Denote it by ω.

The velocity of any particle is composed now of two parts—one of translation equal to that of the centre of gravity, the other of rotation round the centre of gravity. These parts may be treated separately. The first is the same as if the whole mass were concentrated at the centre of gravity and moving with its velocity.

In the second part, due to rotation round G, the single impulse disappears (since OG = 0) and the impulse reduces to an impulse-couple whose moment is ωI, where I is the moment of inertia round the axis through the centre of gravity.



If U then denote the velocity of the centre of gravity, the momentum of the motion reduces to a system of a single impulse mU and an impulsive couple whose moment is ωI. This is in general the most useful form.

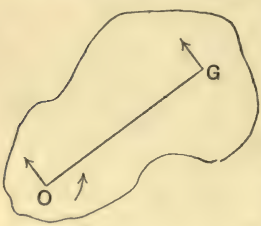
It is, however, sometimes necessary to know the single impulse which is the equivalent. To find the position of this, take Ox parallel to the velocity of translation. Then we have an impulse mU along Ox and an impulse-couple ωI round O. Take OA perpendicular to Ox and of such a length that

$$\omega I = mU \cdot OA,$$

then ωI is equivalent to mU at O along xO, and mU at A parallel to Ox. The first counterbalances the impulse mU along Ox, and there remains only the single impulse mU acting at A.

Note.—OA will be to the right of mU when ωI is positive, and to the left when ωI is negative—the positive direction of rotation being opposite to that of the hands of a watch.

176. The value of I or mk^2 will not only be different for different bodies, but will also depend on the position of the point O in the body. Let k denote the radius of gyration about an axis through the centre of gravity, and k' about a parallel axis through any other point O. Suppose the body rotating round O with angular velocity ω. Then the velocity of the centre of gravity is $\omega \cdot OG$ perpendicular to OG. Hence the momentum is equivalent to a single impulse at G



$$= m \cdot \omega \cdot OG \text{ perpendicular to OG},$$

and an impulsive couple $= m\omega k^2$.

But it is also equivalent to a single impulse at

$$O = m \cdot \omega \cdot OG,$$

and an impulsive couple $= m\omega k'^2$.

These two systems of impulses must therefore be equivalent. Hence, taking moments round G,

$$m\omega k^2 = m\omega k'^2 - m \cdot \omega \cdot OG \cdot OG,$$

$$\therefore k^2 = k'^2 - OG^2,$$

$$k'^2 = k^2 + OG^2;$$

or, if k be the radius of gyration about an axis through the centre of gravity, that about any parallel axis at a distance r is given by

$$k'^2 = k^2 + r^2,$$

and similarly

$$I' = I + mr^2.$$

This theorem enables us then to find the moment of inertia about any axis when that about a parallel axis through the centre of gravity is known. It will consequently only be necessary to find their values for different bodies for axes through their centres of gravity. This is done for certain cases in the next chapter.

177. The change in the momentum of a body is equal to the blow which acts upon it. This will always enable us to find the motion ensuing when a body is struck in a given way. *We have simply to find the change of momentum and express the condition that it is equivalent to the blow.* An example or two will make the method plain.

A system of two heavy particles (4, 8 *lbs.*) *connected by a rigid bar* 1 *foot long, whose mass is so small that it may be neglected, is struck by a blow at a point* 3 *inches from the* 4-*lb. mass and perpendicular to the bar. How will it move just subsequently?*

Let A, B be the particles. The centre of gravity is at G, where $BG = \frac{4}{12}AB = 4$ inches. Let P denote the magnitude of the blow at C.

Then after the blow the system will begin to move so that G moves with a certain velocity (u say), and it rotates round G with a certain angular velocity ω. The momentum of this motion is $(8+4)u = 12u$ acting at G, and an impulsive couple ωI round G, where

$$I = 4 \times GA^2 + 8 \times GB^2,$$
$$= 4 \times 8^2 + 8 \times 4^2 = 384 \text{ (pound inch units).}$$

These are equivalent to P at C. Hence, resolving perpendicular to AB,

$$12u = P,$$

or

$$u = \frac{P}{12}.$$

Taking moments about G,

$$384\omega = P \times CG = 5P,$$

$$\omega = \frac{5}{384}P = \frac{5 \times 12}{384}u = \frac{5}{32}u,$$

the units being inch, pound, second. Thus the motion is exactly determined so soon as the magnitude of P is known.

Further, if Q be a point on the side of G remote from C, the velocity of Q is that due to u + that due to ω

$$=u-\omega \,.\, GQ.$$

If then $GQ=\frac{u}{\omega}$, this velocity is nothing, or Q is at rest, *i.e.* just after the blow the system begins to turn round Q, where

$$GQ=\frac{u}{\omega}=\frac{32}{5}=6\tfrac{2}{5} \text{ inches.}$$

Suppose the blow had been caused by the impact of an inelastic sphere of 5 lbs. moving with a velocity of 2 feet per second, then the subsequent motion of the two will have to be found.

The sphere will behave as a particle (no rotations being set up). Let v denote its velocity after impact. Then, since 2 feet per second = 24 inches per second, change of momentum of sphere $=5(24-v)$.

This measures the blow P on the bar,

$$\therefore \left.\begin{aligned} 12u&=5(24-v) \\ 32\omega&=5u \end{aligned}\right\}.$$

Further, since the sphere is inelastic, the sphere and the bar just after impact have the same velocity at the point of contact. But the velocity of $C=u+\omega \,.\, GC$. Hence

$$v=u+5\omega.$$

Putting this in the first of the two equations,

$$12u=120-5u-25\omega,$$

or

$$17u=120-25\omega.$$

But

$$\omega=\tfrac{5}{32}u.$$

$$\therefore\ 17u+25\times\tfrac{5}{32}u=120,$$

$$544u+125u=3840,$$

$$669u=3840,$$

$$u=\tfrac{3840}{669}=5\tfrac{165}{223} \text{ inches per second,}$$

and

$$\omega=\tfrac{5}{32}u=\tfrac{200}{223} \text{ per second.}$$

In other words, AB will make one complete turn in

$$\frac{2\pi}{\omega}=\frac{223}{100}\times 3{\cdot}1416 \text{ seconds,}$$

$$=7 \text{ seconds about.}$$

As illustrating another question, *suppose the above body to be rotating round its centre of gravity and making* 2 *revolutions per second; let the point* C *suddenly strike a fixed obstacle. What will be the subsequent motion?*

In the former case the blow was supposed given. Here the subsequent velocity of the point C, viz. 0, is given. Now the blow acts through C and afterwards it *begins* to move round C with, say, angular velocity ω'. Hence the *change* of momentum, being equivalent to the blow, has a zero moment of momentum round C. That is,

$$\omega I = \omega'(I + m \cdot GC^2).$$

Here $$I = 384, \quad m = 12, \quad GC = 5;$$

$$\therefore \frac{\omega'}{\omega} = \frac{384}{384+300} = \frac{32}{57}.$$

Now the number of revolutions per second afterwards

$$= \frac{\omega'}{2\pi} = \frac{32}{57}\frac{\omega}{2\pi} = \frac{32}{57} \times 2 = \frac{64}{57},$$

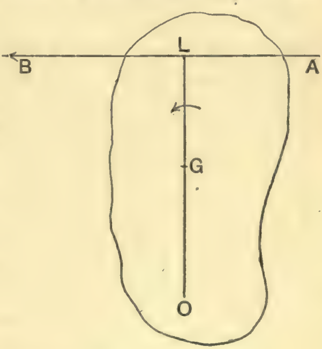
and G moves with a velocity

$$= \omega' \times GC = 5 \times \frac{32}{57}\omega,$$

$$= \frac{160}{57} \times 4\pi \text{ inches per second.}$$

178. As the question of the motion of a body just after being struck a blow is important, we here consider the general case for symmetrical bodies.

Let G be the centre of gravity of the body supposed at rest, and let the direction of the blow P be along AB at a distance $GL = h$ from G. Also let k denote the radius of gyration round an axis through G and perpendicular to the blow (which is supposed to be in the plane of symmetry through G). The velocity of G will after the blow be parallel to AB. Denote it by u and the angular velocity round G by ω. Then, by the foregoing principles,



$$mu = P,$$

$$mk^2\omega = P \cdot GL = Ph.$$

Hence $$\omega = \frac{uh}{k^2}.$$

Let now O be a point on LG produced at a distance x from G. Its velocity will be composed of that of G together with that due to the rotation. Hence

$$\text{Velocity of O} = u - \omega x.$$

If then O be at such a distance that $x = \frac{u}{\omega}$, it does not move, and the body therefore begins to move round O.

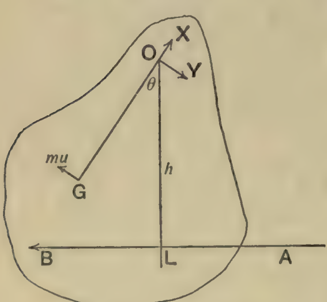
The value of OG is

$$x = \frac{k^2}{h}$$

and depends only on the radius of gyration of the body round G and the distance of the blow from G. If then the body be held at O, the hand will experience no jar when the body is struck. Every one knows by experience that unless a heavy bar is held at a certain point when a blow is delivered by it, the jar on the hand is very unpleasant. The above result explains this fact.

179. If the body is held at some other point than O, a blow will be felt there. The magnitude of the blow will depend on the fixedness with which it is held. If the body is capable of motion about an axis, the axis will experience a blow, the magnitude of which is easily determinable.

Let O denote the axis of suspension, and let h be the distance of the direction of the blow from O, k the radius of gyration through the centre of gravity, and l the distance OG.



After the blow, G will begin to move perpendicularly to OG. Let u be its velocity and ω the angular velocity round O. Then clearly $u = l\omega$. The reaction at O may be decomposed into two parts, X along GO and Y perpendicular to it. Then the momentum mu and couple $m\omega k^2$ are equivalent to X, Y, and P. Hence their moments round O are the same. Therefore, if Q, R denote the components of P along OG and perpendicular,

$$m\omega k^2 + mu \, . \, OG = P \, . \, OL,$$

or

$$m\omega k^2 + m\omega l^2 = P \, . \, h,$$

whence

$$\omega = \frac{Ph}{m(k^2 + l^2)},$$

$$u = \frac{Plh}{m(k^2 + l^2)}.$$

This fully determines the subsequent motion. To find the jerk on O, resolve along and perpendicular to OG.

Then

$$X - Q = 0,$$

$$-Y + R = mu.$$

Hence

$$X = Q,$$

$$Y = R - mu,$$

$$Y = \left(\cos\theta - \frac{lh}{k^2 + l^2}\right)P.$$

If the blow is perpendicular to OG, $\theta = 0$, and the jerk at $O = \left(1 - \frac{lh}{k^2 + l^2}\right)P$. The jerk on the axis is of course in the opposite direction.

180. *Kinetic energy.*—The kinetic energy of motion of a body, like its momentum, depends only on its state at the instant. In finding it, therefore, it is permissible to consider the motion as given by an angular velocity around its instantaneous axis.

Let O denote the instantaneous centre of rotation and P any point of the body at a distance r from the axis of rotation. Let m be the mass of the particle at P, M the whole mass of the body, k_1 its radius of gyration about O and k about a parallel axis through its centre of gravity G. Further, let ω denote the angular velocity.

Then the velocity of P is ωr perpendicular to OP,

$$\therefore \text{its energy} = \tfrac{1}{2}m\omega^2 r^2.$$

Therefore the energy of the whole body

$$= \tfrac{1}{2}\omega^2\Sigma(mr^2),$$

$$= \tfrac{1}{2}Mk_1^2\omega^2.$$

But $k_1^2 = k^2 + OG^2$,

$$\therefore \text{Energy} = \tfrac{1}{2}M(k^2 + OG^2)\omega^2,$$
$$= \tfrac{1}{2}Mk^2\omega^2 + \tfrac{1}{2}M \cdot OG^2 \cdot \omega^2.$$

Now $\omega \cdot OG$ is the velocity of the centre of gravity. Denote it by u. Then

$$\text{Energy} = \tfrac{1}{2}Mk^2\omega^2 + \tfrac{1}{2}Mu^2.$$

We therefore learn this important fact, that the energy of a body consists of two parts—one the same as if the whole mass were collected at the centre of gravity and moved with its velocity. This is the energy of translation. The second is the energy the body would have if it rotated around its centre of gravity without translation. This is the energy of rotation.

EXAMPLES—XIX.

[*Further examples on this chapter will be given at the end of the next; when not known, the moment of inertia is to be taken* = I.]

1. A sphere is rolling along a plane with velocity v. Determine the line of action and magnitude of its momentum; also its energy. Its moment of inertia is mk^2.

2. A rod AB is rotating with constant angular velocity ω, while the point A is moving along a straight line CD with constant velocity u. Determine the magnitude and line of action of the momentum, and the energy at the instants when (1) BAC = 0°, (2) BAC = 90°.

3. Two equal rods AB, BC are freely jointed at B; A is fixed to a point about which it can turn; they are rotating with equal and opposite angular velocities. Determine the momentum and energy of the motion at the instants when (1) ABC = 180°, (2) ABC = 90°, (3) ABC = 0°.

4. Particles of 1 lb., 2 lbs. are fastened to the ends of a rigid rod AB without mass; it is rotating about A, making 4 revolutions a second; A is suddenly set free and B fixed. Find (1) the angular velocity round B afterwards, and (2) the impulsive blow on B.

5. Find the moment of inertia through the centre of gravity of three particles of mass 1, 2, 3 lbs. fastened to a rigid rod without mass at distances of 3 and 4 inches. The system rotates at 10 revolutions per second round the first particle regarded as fixed. Find the energy of the motion and the velocity of translation which would give it the same energy.

6. Find the moment of inertia of three equal particles rigidly con-

nected at the angles of an equilateral triangle about a line perpendicular to the plane.

7. Three equal particles are placed at the angles of an equilateral triangle without mass; the system rotates in its plane about one angle. Find the total momentum and energy.

8. The system in question 6 is struck by a blow P at one angle parallel to the opposite side. Find the subsequent motion and the point about which it begins to move.

9. Find the moment of inertia of eight equal particles at the angles of a cube (1) about a line through the centre parallel to an edge, (2) about a diagonal.

10. The previous system is rotating about a line through the centre of gravity and parallel to an edge with angular velocity ω; this edge is suddenly fixed. Determine the angular velocity afterwards and what proportion of the original energy is lost.

CHAPTER XX

MOMENTS OF INERTIA

181. WHEN the form and distribution of mass of a body are known, it requires in general the methods of the integral calculus to determine its moments of inertia. In cases of irregular bodies, however, even these methods fail of application, and it becomes necessary to determine the moments of inertia by experiment. The method by which this is done is given in the next chapter. In this chapter we deduce the moments of inertia for some important cases which are capable of determination by elementary methods.

We have represented the moment of inertia I as the product of two factors mk^2, in which m denotes the mass of the body and k denotes a length. Now suppose we have two bodies geometrically similar to one another, the question arises—How are their moments of inertia related? To answer this, suppose the body B to be n times the linear magnitude of A. Regard them for a moment as composed of the same number of particles similarly placed, only in B their distances from one another are n times the corresponding distances in A. If then r denote the distance of any particle in A from the axis of rotation, the distance of the corresponding particle in B from its similar axis will be nr. Hence the moments of inertia of the two about similar axes will be

$$m_1r_1^2 + m_2r_2^2 + \ . \ . \ . = mk^2,$$

and $$m_1n^2r_1^2 + m_2n^2r_2^2 + \ . \ . \ . = mn^2k^2 = mk'^2.$$

We therefore learn that $k' = nk$, or *the radius of gyration is proportional to the linear dimensions of the body.*

In the body B, however, the particles will be farther apart than in A, or the density of the body will be less. Suppose now each particle increased to l times its former amount, the whole mass becomes $m' = lm$, and the moment of inertia becomes

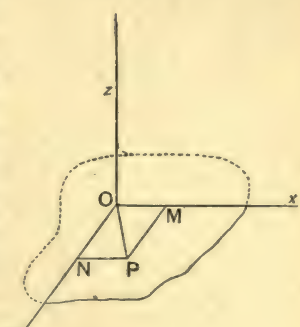
$$lm_1 n^2 r_1^2 + l \, . \, m \, . \, n^2 r_2^2 + \; . \; . \; . = lmk'^2 = m'k'^2.$$

Thus, provided the law of distribution of mass (or density) remain similar in the two bodies, the moments of inertia of the two bodies will be proportional to their total masses and to the squares of their linear dimensions. For instance, if the moment of inertia of one body be I, that of a *similar* body l times as heavy and n times as long will be ln^2I.

Since k is a length, it is clear that the dimensions of moment of inertia are represented by [ML²].

182. When we have to deal with a long thin wire or a very thin plate, we may clearly regard them as, in the first case, a distribution along a *line*, and in the second a distribution over an *area.*

The cases of motion here considered are those in which the body is symmetrical about the axis of rotation, or else symmetrical on both sides of the plane of motion through the centre of inertia. When the bodies then are thin plates, the only axes of rotation we have to consider are (1) an axis perpendicular to the plane of the area, and (2) axes lying in the area. In any case let I denote the moment of inertia of an area about a line perpendicular to the area, and I_1, I_2 moments about two lines perpendicular to each other *in* the area. Then $I = I_1 + I_2$.



For first consider one particle (m) at the point P. Let Oz denote the axis of rotation perpendicular to the area and Ox, Oy the two others in the area. Draw PM, PN perpendicular to Ox, Oy.

Then the moment of inertia of P round O$z = m \, . \, \mathrm{OP}^2$,

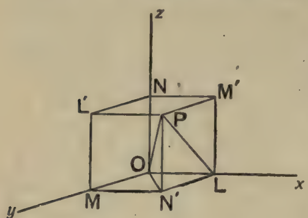
$$= m(\mathrm{PM}^2 + \mathrm{OM}^2) = m\mathrm{PM}^2 + m \, . \, \mathrm{PN}^2,$$

= sum of the moments of inertia round Ox, Oy.

The same is true for all the particles of which the plate is composed. Hence it is true for their sum,

or $$\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2.$$

183. A relation also holds between the moments of inertia of a solid body round any three mutually perpendicular axes through a point. Let Ox, Oy, Oz be three mutually rectangular axes through a point O of a body, and let I_1, I_2, I_3 denote its moments of inertia round Ox, Oy, Oz. Further, let P_1 denote the position of the particle m_1 of the body, and so on, and let $\mathrm{OP}_1 = r_1$, etc. Then



$$\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 = 2(m_1 r_1^{\,2} + m_2 r_2^{\,2} + \ . \ . \ .)$$

For, considering first a single particle at P_1, let i_1, i_2, i_3 denote its moments about the axes. Draw the rectangular parallelopiped, diagonal OP and sides along Ox, Oy, Oz. Then

$$i_1 = m\mathrm{PL}^2 = m(\mathrm{PN}'^2 + \mathrm{LN}'^2) = m(\mathrm{OM}^2 + \mathrm{ON}^2).$$

Similarly $$i_2 = m(\mathrm{ON}^2 + \mathrm{OL}^2),$$

$$i_3 = m(\mathrm{OL}^2 + \mathrm{OM}^2);$$

$$\therefore \ i_1 + i_2 + i_3 = 2m(\mathrm{OL}^2 + \mathrm{OM}^2 + \mathrm{ON}^2),$$

$$= 2m(\mathrm{ON}^2 + \mathrm{ON}'^2),$$

$$= 2m \, . \, \mathrm{OP}^2 = 2mr^2.$$

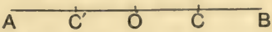
But I_1 is the sum of the i_1 of all the particles in the body, and similarly for I_2, I_3. Hence

$$\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 = 2(m_1 r_1^{\,2} + m_2 r_2^{\,2} + \ . \ . \ .)$$

By means of the foregoing propositions we can now find the values of the moments of inertia for a few simple but important bodies.

184. *Straight wire.*—Let AB denote the wire, O being its middle point, and let $2a$ be its length.

We may regard it as made of two wires AO, OB placed end to end. The moment of the whole about O is the sum



of the moments of the two. Let mk^2 denote the moment of AB about O. Then OB has half the mass of AB and is half the length, and in other respects is similar. Hence the radius of gyration round its centre of gravity $= \frac{1}{2}k$. Therefore its moment of inertia about its centre of gravity C

$$= \frac{1}{2}m\left(\frac{k}{2}\right)^2 = \frac{1}{8}mk^2;$$

$$\therefore\ mk^2 = \frac{m}{2}\left(\text{OC}^2 + \frac{k^2}{4}\right) + \frac{m}{2}\left(\text{OC}'^2 + \frac{k^2}{4}\right) \qquad (\S\,176);$$

$$\therefore\ k^2 = \text{OC}^2 + \frac{k^2}{4},$$

$$\frac{3}{4}k^2 = \frac{a^2}{4},$$

$$k^2 = \frac{a^2}{3},$$

and the moment of inertia of a wire whose length is $2a$ is

$$\tfrac{1}{3}ma^2.$$

Cor. If r_1 denote the distance of a particle m_1 of the wire from O, then the sum of $m_1r_1^2$ between O and B, or $\Sigma_0^a mr^2$, is half the moment of inertia of AB. We hence get the theorem

$$\Sigma_0^a mr^2 = \tfrac{1}{6}ma^2,$$

or, if M denote the whole mass between O and B, $\text{M} = \frac{1}{2}m$ and $\Sigma_0^a mr^2 = \frac{1}{3}\text{M}a^2$.

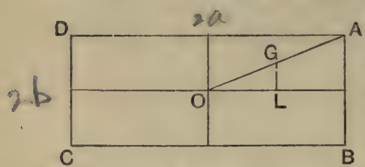
185. *Circular wire.*—In a wire bent into the form of a circle, every particle is at the same distance from the centre. Hence the moment of inertia about an axis through the centre and perpendicular to the plane $= mr^2$, where r is the radius of the circle.

To find the moment of inertia about a diameter, notice that it will be the same round every diameter. If then I denote its magnitude, it follows by the theorem of § 182 and from the preceding result that

$$mr^2 = 2I,$$
$$\therefore I = \tfrac{1}{2}mr^2,$$

or $$\text{The radius of gyration} = \frac{r}{\sqrt{2}}.$$

186. *Rectangular lamina.*—A similar method to that of § 184 will enable us to find the moment of inertia of a rectangular lamina about an axis perpendicular to it. Let $2a$, $2b$ be the length and breadth of the lamina and O its centre. Divide the lamina into four equal parts by lines through O parallel to its sides. These parts will be *similar* to the original lamina, *half* its linear size, and *one-quarter* its mass. The moment of inertia of each about its centre of gravity will therefore be, if mk^2 be that of the original,



$$\frac{m}{4}\left(\frac{k}{2}\right)^2 = \frac{1}{16}mk^2;$$

$$\therefore mk^2 = 4\left(\frac{1}{16}mk^2 + \frac{m}{4}\,.\,OG^2\right),$$

$$\frac{3}{4}mk^2 = m(OL^2 + GL^2),$$

$$\frac{3}{4}mk^2 = m\left(\frac{a^2}{4} + \frac{b^2}{4}\right) \text{ since } OL = \frac{1}{2}a, \text{ etc.};$$

$$\therefore mk^2 = m\frac{a^2 + b^2}{3},$$

or $$\text{The radius of gyration} = \sqrt{\frac{a^2 + b^2}{3}} = \frac{OA}{\sqrt{3}}.$$

That is, the moment is the same as that of a wire of the same mass and whose length is the diagonal of the rectangle.

The moment of inertia about an axis through O parallel to a side is easily found by the same method. For, if mk^2 denote the moment about the axis parallel to DA,

$$mk^2 = 4 \cdot \frac{mk^2}{16} + 4 \cdot \frac{m}{4} \cdot \mathrm{GL}^2,$$

$$\frac{3}{4}mk^2 = m \cdot \left(\frac{b}{2}\right)^2,$$

$$mk^2 = m \cdot \frac{b^2}{3}.$$

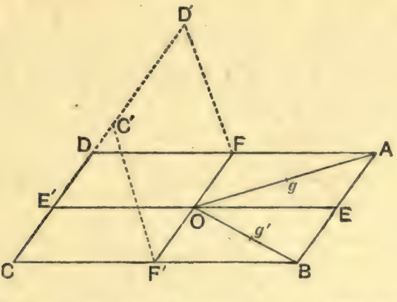
The moment about the axis parallel to the side $2b$ is

$$m\frac{a^2}{3}.$$

We might have first determined these moments, and then obtained that for an axis perpendicular to the plane, viz.—

$$\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2 = m\frac{a^2 + b^2}{3}.$$

187. *Parallelogram, sides* 2a, 2b.—As in the former case, divide the parallelogram into four equal and similar parts,



$$mk^2 = 2(\text{moment of OFAE} + \text{moment of OF}'\text{BE about O}),$$

$$= 2\left\{\frac{m}{4}\left(\frac{k^2}{4} + \mathrm{O}g^2\right) + \frac{m}{4}\left(\frac{k^2}{4} + \mathrm{O}g'^2\right)\right\};$$

$$\therefore\ mk^2 = \frac{mk^2}{4} + \frac{m}{2}(\mathrm{O}g^2 + \mathrm{O}g'^2).$$

Now $$Og = \tfrac{1}{2}OA, \quad Og' = \tfrac{1}{2}OB,$$

$$\therefore \frac{3mk^2}{4} = \frac{m}{8}(OA^2 + OB^2).$$

Now E is the mid point of AB; therefore, by a well-known theorem,*

$$OA^2 + OB^2 = 2OE^2 + 2AE^2,$$

$$\therefore \frac{3}{4}mk^2 = \frac{m}{4}(OE^2 + AE^2) = \frac{m}{4}(a^2 + b^2),$$

$$\therefore \quad mk^2 = m\frac{a^2 + b^2}{3},$$

a similar result to that for a rectangle.

To find the moment round the axis FF′, compare with the case of a rectangle on FF′ and between the same parallels AB, CD. The areas are in both cases the same, and corresponding particles are at the same distance from FF′. Hence the moments are the same for both. If then $2p$ denote the perpendicular distance of the two sides AB, CD from one another, the moment round FF′

$$= m\frac{p^2}{3}.$$

It is to be noticed that rotations round FF′ belong to a class of motion explicitly excluded from our consideration in this book, such a motion not being symmetrical about a plane through O perpendicular to FF′. If, however, we turn round the part CF so as to take the position D′FF′C′, the figure D′FABF′C′ is symmetrical about FF′, and the moment of inertia is the same as before.

188. *Triangular lamina.*—Let ABC be the lamina.

* This theorem being of great use, it may be well to recall the proof. Let D be the middle point of the side BC of a triangle ABC. Join AD and draw AL perpendicular to BC. Suppose ADC is the acute angle. Then

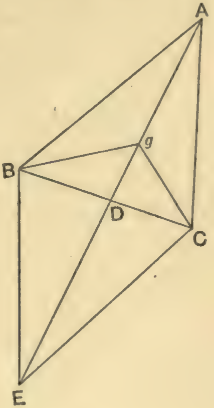
$$AB^2 = BD^2 + AD^2 + 2BD \cdot DL \qquad \text{(Eucl. II 12)};$$
$$AC^2 = CD^2 + AD^2 - 2CD \cdot DL \qquad \text{(Eucl. II 13)};$$

∴ adding and remembering that CD = BD,

$$AB^2 + AC^2 = 2(AD^2 + BD^2).$$

Complete the parallelogram ABEC. Then EBC is a triangle equal to ABC. Let mk^2 be the moment of inertia of ABC about its centre of gravity g. Then moment of inertia of ABEC about D

$$= 2m \cdot \frac{\frac{1}{4}(AB^2 + AC^2)}{3}.$$



But it is also equal to twice that of the triangle ABC, viz. $2(mk^2 + m \cdot Dg^2)$,

$$\therefore m\frac{AB^2 + AC^2}{6} = 2mk^2 + 2m \cdot Dg^2,$$

$$k^2 = \frac{AB^2 + AC^2}{12} - Dg^2.$$

But, since D is the mid point of BC,

$$AB^2 + AC^2 = 2DA^2 + 2BD^2,$$

also $$Bg^2 + Cg^2 = 2BD^2 + 2Dg^2,$$

$$\therefore AB^2 + AC^2 = Bg^2 + Cg^2 + 2DA^2 - 2Dg^2;$$

now $$DA = \frac{3}{2}Ag, \quad Dg = \tfrac{1}{2}Ag,$$

$$\therefore AB^2 + AC^2 = Bg^2 + Cg^2 + 4Ag^2,$$

and $$k^2 = \frac{Bg^2 + Cg^2 + 4Ag^2}{12} - \tfrac{1}{4}Ag^2,$$

$$= \frac{1}{12}(Ag^2 + Bg^2 + Cg^2).$$

This may be put in various forms. Thus

(i.) $$Ag = 2Dg, \text{ etc.},$$
$$\therefore k^2 = \tfrac{1}{3}(Dg^2 + Eg^2 + Fg^2),$$

where D, E, F are the mid points of ABC. The moment of inertia is therefore the same as that of three equal particles $\frac{1}{3}m$ placed at the mid points of the sides.

(ii.) Since $$AB^2 + AC^2 = 2BD^2 + 2DA^2,$$

or $$c^2 + b^2 - \frac{a^2}{2} = \frac{9}{2}Ag^2.$$

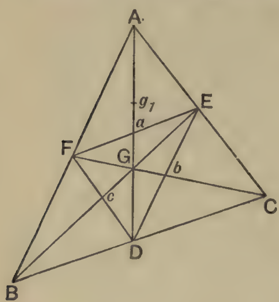
With similar equations for Bg, Cg,

$$\frac{9}{2}(Ag^2 + Bg^2 + Cg^2) = \frac{3}{2}(a^2 + b^2 + c^2),$$

and $$mk^2 = \frac{m}{36}(a^2 + b^2 + c^2).$$

Perhaps a simpler way to find the moment of inertia of a triangle is to use the artifice already employed to find those of a line and a parallelogram.

Let D, E, F denote the mid points of the sides BC, CA, AF. Join DE, EF, FD. Then the triangles AFE, FBD, EDC, DEF are all equal and similar, each being one-half the linear size and one-quarter the mass of the whole triangle ABC.



The moment of inertia of each, therefore, about its centre of gravity $= \frac{1}{4} \cdot \frac{1}{4}$. that of ABC about its centre of gravity. It is necessary first to find the distances of the centre of gravity of each from G.

Let AD meet FE in a.

$$Aa = \tfrac{1}{2}AD, \quad aG = \tfrac{1}{3}aD = \tfrac{1}{6}AD.$$

Also, if g_1 be the centre of gravity of AFE,

$$ag_1 = \tfrac{1}{3}Aa = \tfrac{1}{6}AD;$$

$$\therefore Gg_1 = \tfrac{1}{6}AD + \tfrac{1}{6}AD = \tfrac{1}{3}AD,$$

and $$GD = \tfrac{1}{3}AD.$$

Similarly the centres of gravity of FBD, EDC are $\frac{1}{3}$BE and $\frac{1}{3}$CF from G.

Let now I denote the moment of inertia of ABC round a perpendicular axis through G. Then the moment of inertia of AFE round g_1 is by the above $\frac{1}{16}$I.

$$\therefore I = \frac{1}{16}I + \frac{m}{4} \cdot Gg_1^2 + \frac{1}{16}I + \frac{m}{4} \cdot Gg_2^2 + \frac{1}{16}I + \frac{m}{4}Gg_3^2 + \frac{1}{16}I,$$

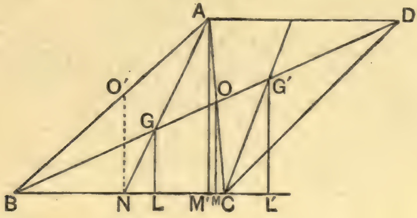
$$= \frac{1}{4}I + \frac{m}{4}(Gg_1^2 + Gg_2^2 + Gg_3^2);$$

or $$I = \frac{m}{3}(Gg_1^2 + Gg_2^2 + Gg_3^2),$$
$$= \frac{m}{3}(GD^2 + GE^2 + GF^2).$$

Hence it is the same as that of three masses $\frac{1}{3}m$ at the mid points of the sides.

The moment about a line through the centre of gravity parallel to the base of the triangle will be required (1) for the case of an isosceles triangle, and (2) in order to find the moments for symmetrical figures made up of triangles arranged in any manner.

Let ABC be the triangle and BC the base.



Complete the parallelogram BACD.

Let G, G′ be the centres of gravity of ABC, ACD and O the intersection of the diagonals.

The triangles are equal in all respects. Therefore their moments round axes parallel to BC or AD through their centres of gravity will be the same. Denote it by I and their mass by m. Draw GL, G′L′, OM, AM′ perpendicular to BC. Then moment of inertia of the parallelogram ABCD round BC

$$= 2m \cdot \left(\frac{OM^2}{3} + OM^2\right) = \frac{8}{3}m \cdot OM^2.$$

But it is also equal to the sum of the moments of the two triangles, and therefore

$$= I + m \cdot GL^2 + I + m \cdot G'L'^2;$$

or, since $GL = \frac{2}{3}OM,\ G'L' = \frac{2}{3}AM' = \frac{4}{3}OM,$

$$\text{The moment} = 2I + m\left(\tfrac{4}{9}OM^2 + \tfrac{16}{9}OM^2\right).$$

Hence

$$2I + \tfrac{20}{9}m \,.\, OM^2 = \tfrac{8}{3}m \,.\, OM^2,$$

and $$I = \tfrac{2}{9}m \,.\, OM^2,$$

or, since $$OM = \tfrac{3}{2} \,.\, GL,$$

$$I = \tfrac{1}{2}m \,.\, GL^2.$$

The moment of inertia of the triangle about BC is

$$\begin{aligned} I + m \,.\, GL^2 &= \tfrac{3}{2}m \,.\, GL^2, \\ &= \tfrac{3}{2} \,.\, \tfrac{4}{9}m \,.\, OM^2, \\ &= \tfrac{2}{3}m \,.\, OM^2. \end{aligned}$$

If now O′ is the middle point of AB and O′N be perpendicular to BC, OM = O′N, whence

$$\text{Moment of inertia round BC} = \frac{2}{3}m \,.\, OM^2 = \frac{m}{3}(OM^2 + O'N^2).$$

That is, it is the same as that of three particles $\frac{1}{3}m$ placed at the mid points of the sides. And this will, therefore, be the case for axes through the centre of gravity, since these particles have the same centre of gravity as the triangle.

Now we have seen the same result holds for axes perpendicular to the plane of the triangle. And the particles together have the same mass and the same centre of gravity as the triangle. Hence we learn that a triangular lamina and a system of particles, each equal to one-third the mass of the lamina and placed rigidly at the mid points of its sides, are not only statically similar, but also kinetically, at least so far as occurs in the cases of motion here considered.*

The moment of inertia of a triangle or any rectilinear lamina about any axis in its plane can be found by supposing it divided up into triangles.

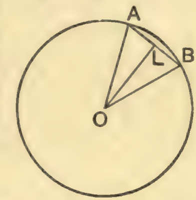
The student should employ the second method to find the moment of inertia about a line parallel to a side.

189. *Circular disc.*—Consider first the case of a lamina of the shape of any regular polygon inscribed in the circle. It may be supposed to be made up of a series of equal isosceles triangles with their vertices all at the centre of the circle. The moment of inertia of the polygon round

* It is so in all cases.

an axis through its centre O perpendicular to its plane is then the same as the sum of those of the triangles round their vertices O.

Let AB be any side of the polygon. Join OA, OB and draw OL perpendicular to AB. Then the moment of inertia of OAB is the same as that of particles of one-third its mass placed at the middle points of OA, AB, BO. Hence its moment of inertia is, if m denote the mass of the triangle,



$$=\frac{1}{3}m\left(\frac{OA^2}{4}+OL^2+\frac{OB^2}{4}\right).$$

If r denote the radius of the circle and p the perpendicular OL from O on a side, this is

$$\tfrac{1}{3}m(\tfrac{1}{2}r^2+p^2).$$

This is the same for every triangle. Hence, if M be the mass of the whole polygon, its moment of inertia

$$=\tfrac{1}{3}(\tfrac{1}{2}r^2+p^2)(m+m+\ldots)=\tfrac{1}{3}M(\tfrac{1}{2}r^2+p^2).$$

This is the moment of inertia, therefore, of any regular polygon, whatever the number of sides.

But if the number of sides be indefinitely increased, the polygon tends more and more to become a circle and $p=r$. Hence in a disc the moment of inertia is

$$\tfrac{1}{3}M(\tfrac{1}{2}r^2+r^2)=\tfrac{1}{2}Mr^2,$$

or the radius of gyration is $\dfrac{r}{\sqrt{2}}$.

To find the moment of inertia round any diameter, note that it must be the same for all. Hence, by the theorem of § 182, if I denote the moment required,

$$\tfrac{1}{2}Mr^2=2I,$$

$$I=M\frac{r^2}{4},$$

or the radius of gyration is half the radius of the circle.

190. *Thin spherical shell.*—Clearly the moments of inertia about any diameter are the same. Hence, by § 183, if I denote its value,

$$3I = 2\Sigma mr^2.$$

Now in this case all the particles of the body are at the same distance from the centre. Hence

$$I = \tfrac{2}{3}r^2\Sigma m,$$
$$= \tfrac{2}{3}Mr^2,$$

or the radius of gyration is $r\sqrt{\tfrac{2}{3}}$.

191. *Rectangular parallelopiped, sides* 2a, 2b, 2c.—Suppose the axis of rotation through the centre of gravity to be perpendicular to the face b, c. The body can be divided in a series of thin laminæ parallel to this face, the radius of gyration of each of them being $\sqrt{\dfrac{b^2+c^2}{3}}$. Hence the radius of gyration of the whole body is the same, and its moment of inertia

$$m\frac{b^2+c^2}{3}.$$

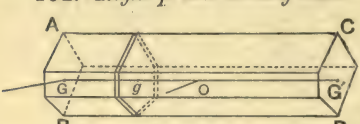
So also for the axis perpendicular to the face a, b it is

$$m\frac{a^2+b^2}{3}.$$

The same result can also be obtained by dividing the solid into eight similar parallelopipeds and using the artifice employed in the case of the parallelograms.

In the cube the moments of inertia about all axes through the centre are equal. This is, however, not proved here (see examples 9, 10). The same is true of the regular tetrahedron.

192. *Right prism on any base.*—Let AB, CD be the two ends, GG′ the axis through the centres of gravity of the ends.



Let $2l$ be the length of the prism, also let k denote the radius of gyration of a lamina of the shape of the ends about GG′.

Firstly, with regard to the moment of inertia about GG′, notice that the prism can be divided in thin laminæ all equal to one another and having the same radius of gyration k. Hence the prism has the same radius of gyration, and if m be its mass, its moment of inertia is

$$mk^2.$$

Secondly, to find the moment of inertia about a line through the centre of gravity O perpendicular to the axis. Let k' denote the radius of gyration of a lamina of the form of the section about an axis through its centre of gravity parallel to the one in question.

Let g be its centre of gravity and m' its mass. Then its moment about the axis through O is

$$m'(k'^2 + \mathrm{O}g^2).$$

Hence the moment for the whole prism is the sum of all these, or

$$\Sigma m'(k'^2 + \mathrm{O}g^2).$$

Here k' is the same for all. Hence

$$\Sigma m'k'^2 = k'^2\Sigma m' = mk'^2,$$

also $\Sigma m' . \mathrm{O}g^2$ is the moment of inertia of the whole mass *supposed collected along the axis* GG′.

It is therefore, by § 184, $\frac{1}{3}ml^2$.

The whole moment of inertia is therefore

$$m(k'^2 + \tfrac{1}{3}l^2).$$

For instance, we have at once for

(1) Circular cylinder, length $2l$, radius r,

$$\text{Round axis, } m\frac{r^2}{2},$$

$$\text{Perpendicular to axis, } m\left(\frac{r^2}{4} + \frac{l^2}{3}\right).$$

(2) Triangular prism, sides a, b, c,

$$\text{Round axis, } m\frac{a^2 + b^2 + c^2}{36},$$

$$\text{Perpendicular to axis and parallel to side } a,\ m\left(\frac{p^2}{18} + \frac{l^2}{3}\right),$$

where p is the perpendicular from A on the base BC of the triangle.

193. *Sphere.*—The moments about any diameter are the same. Hence (§ 183)

$$3\text{I} = 2\Sigma mr^2.$$

Divide the sphere into a very large number of concentric shells, the thickness of each being a small quantity h. All the particles contained in one of these shells are at the same distance from the centre.

Let m' denote the mass of a particle and m that of the whole sphere.

Since the distribution of the particles is uniform, the number in any volume will be proportional to the volume.

Thus
$$\frac{\text{mass of shell}}{m} = \frac{\text{number in shell}}{\text{number in sphere}},$$
$$= \frac{\text{volume of shell}}{\text{volume of sphere}},$$
$$= \frac{4\pi r^2 h}{\frac{4}{3}\pi \text{R}^3} = \frac{3r^2 h}{\text{R}^3},$$

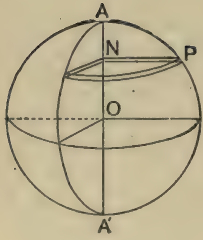
if R denote the radius of the sphere, and r of the shell.

Hence for the shell in question $\Sigma mr^2 = \frac{3m}{\text{R}^3}r^4 h.$

Therefore, adding together all the shells,

$$3\text{I} = 2\frac{3m}{\text{R}^3}\Sigma r^4 h,$$

or
$$\text{I} = 2\frac{m}{\text{R}^3}\Sigma r^4 h.$$



But we may divide the sphere in another way, viz. into a series of discs of the same thickness h perpendicular to the axis of rotation. Consider one of these at a distance ON. Its volume is $\pi\text{PN}^2 . h$ and therefore, as before, its mass

$$= \frac{\pi\text{PN}^2 . h}{\frac{4}{3}\pi\text{R}^3}m = \frac{3}{4}\,\frac{\text{PN}^2 . h}{\text{R}^3}m,$$

and its moment of inertia round ON is therefore

$$\frac{3}{4} \cdot \frac{\text{PN}^2 . h}{\text{R}^3} m . \frac{\text{PN}^2}{2}.$$

Hence for the whole sphere

$$\text{I} = \frac{3}{8}\frac{m}{\text{R}^3}\Sigma \text{PN}^4 . h,$$

the summation extending for all the parallel discs along AA′,

or $$\text{I} = \frac{3}{4}\frac{m}{\text{R}^3}\Sigma \text{PN}^4 . h,$$

the summation extending to discs between O and A.

Now $$\text{PN}^2 = \text{R}^2 - \text{ON}^2 = \text{R}^2 - r^2.$$

Hence $$\text{I} = \frac{3}{4}\frac{m}{\text{R}^3}\Sigma(\text{R}^4 - 2\text{R}^2 r^2 + r^4)h,$$

$$= \frac{3}{4} m\text{R}\Sigma h - \frac{3}{2} \cdot \frac{m}{\text{R}}\Sigma r^2 h + \frac{3}{4}\frac{m}{\text{R}^3}\Sigma r^4 h.$$

But we have already seen that

$$\frac{m}{\text{R}^3}\Sigma r^4 h = \frac{1}{2}\text{I}$$

and $\frac{3}{2}\frac{m}{\text{R}}\Sigma r^2 h$ is the moment for the line OA of mass $\frac{3m}{2\text{R}}$, and therefore

$$= \frac{3}{2} \cdot \frac{m}{\text{R}} \cdot \frac{\text{R}^3}{3} = \frac{1}{2} m\text{R}^2.$$

Hence $$\text{I} = \frac{3}{4} m\text{R}^2 - \frac{1}{2} m\text{R}^2 + \frac{3}{8}\text{I};$$

$$\therefore \frac{5}{8}\text{I} = \frac{1}{4} m\text{R}^2,$$

$$\text{I} = m . \frac{2\text{R}^2}{5},$$

or the radius of gyration of a sphere $= \sqrt{\frac{2}{5}} . \text{R}$.

Cor. It follows that $\Sigma r^4 . h = \frac{\text{R}^5}{5}$

The foregoing method involves the dividing the sphere into an infinitely large number of infinitely small parts. The following proof, however, is perhaps more direct. The radius of gyration will clearly be proportional to the radius of the sphere. Let us then put for the sphere

$$I = m\lambda r^2,$$

where m is the mass, r the radius, and λ some numerical quantity which it is our object to determine.

We have $$I = \tfrac{4}{3}\pi\rho\lambda r^5,$$

where ρ is the density of the material.

Hence the moment of inertia of a spherical shell whose external and internal radii are a, b is

$$\begin{aligned} I' &= \tfrac{4}{3}\pi\rho\lambda a^5 - \tfrac{4}{3}\pi\rho\lambda b^5, \\ &= \tfrac{4}{3}\pi\rho\lambda(a^5 - b^5). \end{aligned}$$

But, if m' denote the mass of the shell,

$$m' = \tfrac{4}{3}\pi\rho(a^3 - b^3).$$

Hence $$\begin{aligned} I' &= m'\lambda\frac{a^5 - b^5}{a^3 - b^3}, \\ &= m'\lambda\frac{a^4 + a^3b + a^2b^2 + ab^3 + b^4}{a^2 + ab + b^2}. \end{aligned}$$

This gives the moment of inertia about a diameter for a shell of any radius and thickness whatever.

Hence, by putting $b = a$ in this, we obtain the moment for a spherical surface distribution of matter. In this case

$$I' = m'\lambda\frac{5a^4}{3a^2} = \frac{5}{3}m'\lambda a^2.$$

But, by § 190, $I' = \tfrac{2}{3}m'a^2$.

Hence, comparing the two,

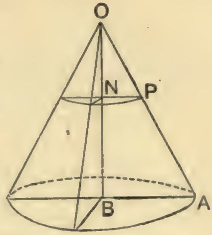
$$\lambda = \tfrac{2}{5},$$

and the moment of inertia of a solid sphere about a diameter is therefore

$$I = m\frac{2a^2}{5}.$$

194. *Right cone.*—Let r be the radius of the base and a the altitude.

I. *About the axis.*—Let mk^2 denote the moment of inertia about the axis. We shall proceed first to express that of the frustum PNBA in terms of its mass and k^2.



This is equal to the difference of the moments of the two cones OAB and OPN. Let $PN = b$ and the mass of $OPN = m'$. Then the cone OPN is similar to OAB, and therefore, by the theorem of § 181, if k' be its radius of gyration,

$$k' = \frac{b}{r}k;$$

also the mass is proportional to the volume. Hence

$$m' = \frac{b^3}{r^3}m,$$

and if M be the mass of the frustum,

$$M = m - m' = \left(1 - \frac{b^3}{r^3}\right)m.$$

Its moment of inertia $= mk^2 - m'k'^2$,

$$= \left(1 - \frac{b^5}{r^5}\right)mk^2,$$

$$= \frac{r^5 - b^5}{r^3 - b^3}M\frac{k^2}{r^2},$$

$$= \frac{r^4 + r^3b + r^2b^2 + rb^3 + b^4}{r^2 + rb + b^2}M\frac{k^2}{r^2}.$$

Suppose now the frustum to become very thin. Then $r = b$ and the frustum becomes a circular disc of radius r In this case the moment of inertia becomes

$$\frac{5}{3}Mk^2.$$

But the moment of inertia of a disc is

$$\frac{1}{2}Mr^2,$$

$$\therefore k^2 = \frac{3}{10}r^2,$$

and the moment of inertia of a cone about its axis is

$$\frac{3}{10}mr^2.$$

II. *About a line through its centre of gravity parallel to the base.*—A similar device will enable us to find the moment of inertia in this case also. Let mk^2 denote its value. Then the moment about a parallel axis through O is

$$m\left(k^2 + \frac{9}{16}a^2\right),$$

and the corresponding moment for the cone OPN is

$$m\frac{b^5}{r^5}\left(k^2 + \frac{9}{16}a^2\right).$$

Hence, as in the former case, the corresponding moment for the frustum is

$$\frac{r^4 + r^3b + r^2b^2 + rb^3 + b^4}{r^2 + rb + b^2}M\left(k^2 + \frac{9}{16}a^2\right).$$

When the frustum becomes a disc this is

$$\frac{5}{3}M\left(k^2 + \frac{9}{16}a^2\right).$$

But the moment of the disc AB about the axis through O is

$$M\left(\frac{r^2}{4} + a^2\right),$$

$$\therefore \frac{5}{3}\left(k^2 + \frac{9}{16}a^2\right) = \frac{r^2}{4} + a^2,$$

whence $$k^2 = \frac{3}{20}\left(r^2 + \frac{a^2}{4}\right)$$

The moment of inertia of a cone about a line through its centre of gravity parallel to the base is therefore

$$\frac{3}{20}m\left(r^2+\frac{a^2}{4}\right).$$

195. The results of this chapter are here collected for reference. They give the moments of inertia about axes through the centre of gravity.

1. Straight line, length $2a$—
   About axis perpendicular to the line . . . . . $m\frac{a^2}{3}$.
2. Circular wire, radius r—
   Axis perpendicular to plane of wire . . . . . mr^2.
   About a diameter . . . $m\frac{r^2}{2}$.
3. Rectangular lamina, sides $2a$, $2b$—
   Perpendicular to plane . . $m\frac{a^2+b^2}{3}$.
   Parallel to side a . . . $m\frac{b^2}{3}$.
4. Triangular lamina, sides a, b, c—
   Same as that of three equal particles at the mid points of the sides.
   Perpendicular to plane . . $m\frac{a^2+b^2+c^2}{36}$.
5. Circular disc, radius r—
   Perpendicular to plane . . $m\frac{r^2}{2}$.
   About diameter . . . $m\frac{r^2}{4}$.
6. Spherical shell, radius r—
   About diameter . . . $m\frac{2r^2}{3}$.

7. Rectangular parallelopiped, sides $2a$, $2b$, $2c$—

About axis perpendicular to b, c . . . $m\frac{b^2+c^2}{3}$.

8. Right prism, length $2l$, and k, k' radii of gyration of a section—

About axis . . . . mk^2.

Perpendicular to axis . . $m(k'^2+\frac{1}{3}l^2)$.

9. Circular cylinder, radius r, length $2l$—

About axis . . . . $m\frac{r^2}{2}$.

Perpendicular to axis . . $m\left(\frac{r^2}{4}+\frac{l^2}{3}\right)$.

10. Sphere, radius r—

About a diameter . . . $m\frac{2r^2}{5}$.

11. Right cone, altitude $= a$, radius of base $= r$—

About axis . . . . $m\frac{3r^2}{10}$.

Perpendicular to axis . . $m\frac{3}{20}\left(r^2+\frac{a^2}{4}\right)$.

EXAMPLES—XX.

[*The moments of inertia, except where otherwise stated, are supposed to be taken about lines through the centre of gravity.*]

1. A uniform wire is bent into a rectangle (sides a, b). Find its moments of inertia about lines (1) perpendicular to its plane, and (2) parallel to its sides.

2. Prove that the moment of inertia of a rod about a line inclined at an angle α to it is $I \sin^2 \alpha$, where I is the moment of inertia about a line perpendicular to it.

3. A uniform wire is bent into an equilateral triangle (side $2a$). Find the moments of inertia about a line perpendicular to its plane and about the line through one angle perpendicular to the opposite side.

4. Find the moment of inertia of a triangle about an axis parallel to the base by the second method of § 188.

5. A wheel and axle are composed of the same material. The wheel is 4 feet radius and 6 inches thick. The axle is 6 inches radius and 4 feet long. Find the radius of gyration about the axis.

6. A fly-wheel weighing $\frac{1}{2}$ ton has the following dimensions—

Rim, of rectangular section, 1 foot broad, 6 inches thick, and outside radius 6 feet.

Axle, a cylinder 2 feet long, radius 6 inches. Eight cylindrical spokes of 4 inches diameter. Find its moment of inertia about the axis, supposing the material uniform throughout.

7. Find the moment of inertia of a square lamina about a diagonal.

8. Show that the moment of inertia of a square lamina about any axis in its plane through its centre is the same.

9. Prove that the moment of inertia about the diagonal of a cube is the same as about a line parallel to a side.

10. Prove that the moments of inertia of a cube about a line parallel to a face are the same.

11. Find the moment of inertia of a solid hemisphere about a line through its centre of gravity parallel to the base.

12. Deduce, by considering the case of a right cone, the moment of inertia of a disc about an axis perpendicular to its plane, when the density at any point is proportional to the distance from the rim.

Find also the moment round any diameter.

13. Find the moment of inertia about the axis of the solid formed by cutting a right cone out of a cylinder of the same height and diameter.

14. Deduce from the foregoing the moment of inertia of a disc about an axis perpendicular to its plane, when the density at any point is proportional to the distance from the centre.

Find also the moment round any diameter.

15. Find the moment of inertia of a hollow circular cylinder about its axis, and about a line perpendicular to the axis. Given length 6 inches, outside diameter 6 inches, inside diameter 4 inches, mass 3 lbs.

16. A spherical cavity is cut out of a solid sphere. Find the moments of inertia about lines (1) through the two centres, and (2) perpendicular to the former. Given radius of inner sphere 1 inch, radius of outer 4 inches, distance of centres 2 inches, mass of *solid* sphere 64 oz.

17. Compare the energies possessed by a straight uniform wire when rotating with the same angular velocity round its centre of gravity and round one end.

18. Two equal cubes are revolving with the same angular velocity, one about its diagonal, the other about a line through its centre of gravity parallel to an edge. Compare the energies of the two.

19. A sphere rolls along a rough plane. Compare its energy with

that of an equal sphere moving with the same velocity along a smooth plane without rolling.

20. The fly-wheel in question 6 makes 120 revolutions per minute. How many foot-pounds of work can be done by bringing it to rest?

21. Determine the energy of rotation of the earth, supposed a uniform sphere of 4000 miles radius and density 5·6.

22. A rod AB (4 feet long) is moving with a velocity of 20 feet per second in a direction perpendicular to its length; the end A is suddenly fixed. Determine the subsequent motion.

Also determine the motion if a point C in it had been fixed where AC = ¾AB.

23. In the previous question, the point C instead of being fixed strikes against an inelastic spherical body of equal mass. Determine the motion just afterwards. Also find the loss of energy.

24. The rod AB moving as above strikes simultaneously at A and B on two particles respectively of the same, and one-half the mass of the rod. Determine the motion just after the impact.

25. A square is moving freely about a diagonal with angular velocity ω, when one of the angular points not in that diagonal becomes fixed. Determine the impulsive pressure on the fixed point and show that the angular velocity afterwards will be $\omega/7$.

26. A triangular lamina ABC is rotating with angular velocity ω about a line through the centre of gravity perpendicular to its plane; A is suddenly fixed. Show that the angular velocity afterwards

$$= \frac{b^2+c^2+a^2}{9(b^2+c^2)-3a^2}\omega.$$

27. A cylinder has a string wound round its middle in a plane perpendicular to its axis; the string is suddenly jerked. Determine the subsequent motion.

Also compare the energies of translation and rotation.

28. If in the previous question the cylinder were rotating with angular velocity ω and the string were attached to a particle at rest of the same mass as the cylinder, determine the alteration in the motion when the string became tight.

29. A square is rotating round a diagonal; it strikes against an inelastic particle of half the mass at a point on the other diagonal half way between the centre and angle. Determine the change of motion.

30. If in the previous case the particle were elastic, and the points of the body where the collision took place followed Newton's law for elastic bodies with coefficient of rebound $= e$, determine the change of motion.

31. A uniform hoop, radius a, mass M, hangs at rest over a perfectly rough peg; a blow P is struck upon the hoop in the horizontal line

through its centre. Show that the initial angular velocity of the hoop is $P/2Ma$.

32. Two equal rods AB, BC are freely jointed at B; BC receives a blow through its centre perpendicular to its length. Determine the initial motion (1) when the rods are in the same line, (2) when ABC is a right angle. Determine in each case the proportion of the energy taken up by each rod.

33. A uniform rod is set in motion by impressing a given impulse on it at one end in a direction at right angles to the rod. Show that the kinetic energy set up in the rod is to that which would result were the rod divided into two equal portions and the parts smoothly jointed together as the ratio 4 : 7.

34. Find where a uniform rod hung up by one end and freely jointed at some point of its length must be struck in order that the lower part may begin to move without rotation.

35. Show that the moment of inertia of a conical surface is the same as that of two rings on it—one half way up and of two-thirds the mass, the other at the base and of one-third the mass.

36. Find the moment of inertia of the surface of a cone about its principal axes at its centre of gravity, by considering it as formed by cutting a similar co-axal cone out of a larger one.

CHAPTER XXI

MOTION UNDER FORCE

196. We have seen in Part I. that force and rate of change of momentum are related in precisely the same way as impulse and change of momentum. Now it has been shown in Chapter XIX that the momentum of a body consists of the momentum or impulse of the whole mass acting at the centre of gravity together with a moment of momentum or impulse-couple round it. These two are measured respectively by Mu and $Mk^2\omega$, where u is the velocity of the centre of gravity and ω the angular velocity round it. When u and ω are changing with the time, let a denote the acceleration of u and $\dot{\omega}$ that of ω. Then the rate of change of momentum is Ma and $Mk^2\dot{\omega}$. Ma is the same as if the mass were collected at the centre of gravity. These will be equivalent to the forces acting on the body.

If a force F act at a point P we have seen that it is equivalent to two forces X, Y at right angles to one another acting at the centre of gravity, together with a couple L round it. Let the motion of the body be given by velocities u, v, accelerations a, a', angular velocity ω and acceleration $\dot{\omega}$. Then, X, Y being in the direction of u, v, we have at once

$$\left.\begin{aligned} Ma &= X \\ Ma' &= Y \\ Mk^2\dot{\omega} &= L \end{aligned}\right\} .$$

These equations hold whether the body is rigid, or whether its parts are capable of relative motion. In case no external forces act on the body the momentum is constant. If the external forces always pass through a fixed line, then the moment of the forces about this line vanishes, and it follows that the angular momentum round it remains constant. This is a very important result. To illustrate its application take the following example—

Supposing the earth a uniform sphere and that in the course of ages its radius has contracted 1/n *of its present value, find the alteration in the length of the day.*

The angular momentum has remained the same always.

Hence, if ω be the angular velocity now and Ω its original angular velocity, this condition gives

$$m \cdot \frac{2r^2}{5}\Omega = m \cdot \frac{2}{5}r^2\left(1-\frac{1}{n}\right)^2\omega,$$

$$\therefore \Omega = \left(1-\frac{1}{n}\right)^2\omega.$$

But, taking the day for the unit of time,

$$1 \text{ day} = 2\pi/\Omega,$$

$\therefore$ the day is shortened by

$$\frac{2\pi}{\Omega}-\frac{2\pi}{\omega}=\frac{2\pi}{\Omega}\left(1-\frac{\Omega}{\omega}\right),$$

$$=\left\{1-\left(1-\frac{1}{n}\right)^2\right\} \text{days},$$

$$=\frac{48}{n} \text{ hours},$$

since $1/n$ is very small.

For instance, if the radius had decreased one mile, n would be 4000, and the shortening of the day would be

$$\frac{48 \times 60 \times 60}{4000} = 43{\cdot}2 \text{ seconds}.$$

197. When forces act on a system of particles, the change in kinetic energy is equal to the work done by the forces. When the particles can change their relative positions, the forces between them can do work and the kinetic energy can change. In the cases, however, here considered the bodies are rigid, and no internal work can be done. Any change of energy is therefore only due to external force, and is measured by the work done by it.

The student will get a clearer insight into the foregoing principles by considering their applications to special cases. The rest of the chapter is therefore devoted to the consideration of examples, many of which are important for their own sakes.

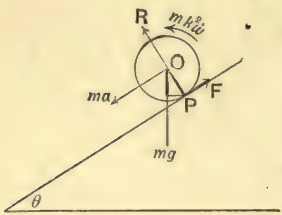
In working examples the student should commence by drawing a diagram, drawing lines to represent the rates of change of momenta and also the forces. He should next determine the geometrical relations between the various velocities and accelerations. By then expressing in the form of equations that the two sets of quantities are equivalent, he will obtain sufficient equations to determine the motion.

198. *A sphere rolls down a rough inclined plane; determine the motion.*

First, suppose the plane so rough that no slipping takes place.

If we desire only to know the velocity at any point, the simplest way will be to equate the energy to the work done by gravity.

Let m denote the mass, r the radius of the sphere, ω its angular velocity at any time, and u the velocity of the centre of gravity down the plane.



The moment of inertia is $\frac{2}{5}mr^2$.

Its energy $=\frac{1}{2}mu^2+\frac{1}{2}\cdot\frac{2}{5}mr^2\omega^2$.

But as the sphere rolls down without slipping, the point of contact P is "the instantaneous centre of rotation" at every instant. Hence the geometrical relation is $r\omega=u$, and the energy $=\frac{1}{2}m(u^2+\frac{2}{5}u^2)=\frac{7}{10}mu^2$.

We might also have obtained this at once by noting that P is the instantaneous centre, and the moment of inertia about it $=m(\frac{2}{5}r^2+r^2)=\frac{7}{5}mr^2$.

The work done by gravity $=mgh$, where h is the vertical height descended. Hence

$$\frac{7}{10}mu^2=mgh,$$

or

$$u^2=\frac{10gh}{7}=2\cdot\frac{5g}{7}\cdot h,$$

and is therefore the same as if the plane had been smooth and gravity diminished in the ratio 5 : 7.

If we desire further to know the forces called into play we must deal with the rates of change of momenta. We work it out independently in this way.

The forces are the weight mg through the centre of gravity, the reaction R at the point of contact, and the friction F acting up the plane. They are represented in the diagram by thick lines.

The momentum rates of change are ma down the plane, $mk^2\dot{\omega}$ round the centre of gravity. They are represented in the diagram by thin lines.

Also, as before, the geometrical relation is $a = r\dot{\omega}$.

Take moments about P. Then

$$ma \cdot r + mk^2\dot{\omega} = mgr \sin\theta,$$

or, since $$k^2 = \tfrac{2}{5}r^2 \text{ and } r\dot{\omega} = a,$$

$$\tfrac{7}{5}mra = mgr \sin\theta,$$

$$a = \tfrac{5}{7}g \cdot \sin\theta.$$

To find F, resolve parallel to the plane, then

$$ma = mg \sin\theta - \text{F},$$

whence $$\text{F} = m(g \sin\theta - a) = \tfrac{2}{7}mg \sin\theta,$$

and R is found by resolving perpendicular to the plane,

$$0 = mg \cos\theta - \text{R},$$

or $$\text{R} = mg \cos\theta.$$

In the above we have supposed the limiting friction so large that the sphere will not slip. Let us now find the condition that this may be true. Let μ be the coefficient of friction. Then, if F_1 denote the largest possible value of F,

$$\text{F}_1 = \mu\text{R}.$$

But here $$\frac{\text{F}}{\text{R}} = \frac{\tfrac{2}{7}mg \sin\theta}{mg \cos\theta} = \frac{2}{7}\tan\theta.$$

Hence $$\tfrac{2}{7}\tan\theta < \mu.$$

Hence, if the inclination of the plane be greater than θ_1, where

$$\tan\theta_1 = \frac{7\mu}{2},$$

the sphere will begin to slip.

It will be interesting to consider the motion in this case.

The point of contact is no longer the instantaneous centre, and consequently $r\dot{\omega}$ does not equal a.

But now F is known, viz. μR.

Consequently the equations of motion become,

$$\left.\begin{array}{lr} \text{Moments about O,} & mk^2\dot{\omega} = \mu\text{R} \cdot r \\ \text{Resolving down plane,} & ma = mg \sin\theta - \mu\text{R} \\ \text{Perpendicular to plane,} & 0 = mg \cos\theta - \text{R} \end{array}\right\} ;$$

or, putting in $k^2 = \frac{2}{5}r^2$,

$$\left.\begin{aligned} \tfrac{2}{5}r\dot{\omega} &= \mu R/m = \mu g \cos\theta \\ r\dot{\omega} &= \tfrac{5}{2}\mu g \cos\theta \end{aligned}\right\};$$

and

$$a = g \sin\theta - \mu g \cos\theta$$

$$\therefore a = (\sin\theta - \mu \cos\theta)g.$$

That is, $r\dot{\omega}$ is a constant acceleration. Hence, after any time t from rest,

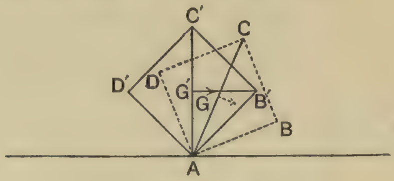
$$r\omega = \tfrac{5}{2}\mu g \cos\theta \, . \, t,$$

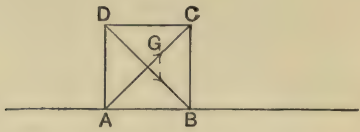
also

$$u = at = (\sin\theta - \mu\cos\theta)gt.$$

Hence $r\omega/u$ is a constant ratio, and they never become equal Therefore the point of contact is never the instantaneous centre, and the sphere continues to slide throughout the motion.

199. *A cube, side* 2a, *rests with one edge on a rough plane, and the opposite edge vertically above the first; it is let fall. Determine if it will rise after striking the plane, and if so the height to which the centre of gravity will rise.*





Let ω be the angular velocity of the cube about the edge when the face AB strikes the plane. Then the moment of inertia about A

$$= m\left(\frac{2a^2}{3} + AG^2\right),$$

$$= m\left(\frac{2}{3} + 2\right)a^2 = \frac{8}{3}ma^2,$$

and

$$\text{The energy} = \frac{1}{2} \cdot \frac{8ma^2}{3} \cdot \omega^2 = \frac{4}{3}ma^2\omega^2.$$

The centre of gravity has fallen from a height AG to a height a. Hence

$$\tfrac{4}{3}ma^2\omega^2 = mg(AG - a),$$

$$= mg(\sqrt{2} - 1)a.$$

At the moment of striking, the momentum is $m \,.\, AG \,.\, \omega$ along GB and $mk^2\omega$ about G.

After impact let ω' be the angular velocity about B. The momentum is then

$$m \,.\, BG \,.\, \omega' \text{ along GC and } mk^2\omega' \text{ about G.}$$

The difference of these is equivalent to a blow acting through B. Its moment, therefore, about B must vanish, or the moments of each about B must be the same. Hence, since $AG = BG = a\sqrt{2}$,

$$ma\sqrt{2}\omega' \times BG + mk^2\omega' = mk^2\omega,$$

or

$$(2a^2 + \tfrac{2}{3}a^2)\omega' = \tfrac{2}{3}a^2\omega,$$

and

$$\omega' = \tfrac{1}{4}\omega.$$

That is, the cube will begin to turn about B with an angular velocity one-quarter of that just before striking.

If h denote the height to which G will rise, we get by the energy

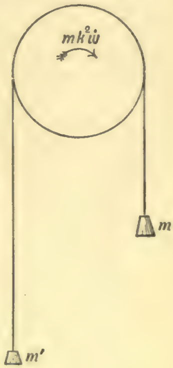
$$\tfrac{4}{3}ma^2\omega'^2 = mg(h - a),$$

$$\therefore \tfrac{1}{16} \cdot \tfrac{4}{3}ma^2\omega^2 = mg(h - a),$$

$$\tfrac{1}{16} mg(\sqrt{2} - 1)a = mg(h - a);$$

$$h = a + \frac{\sqrt{2} - 1}{16}a = \frac{15 + \sqrt{2}}{16}a = \frac{15 + \sqrt{2}}{16\sqrt{2}} \text{ (its former height).}$$

200. *Atwood's machine.*—Two masses are connected by a string which passes over a pulley of radius r. Find the motion when the mass of the pulley is not neglected. Let m, m' be the masses, M the mass of the pulley, and k its radius of gyration. Let t be the tension of the string on the side of m and t' on the side of m'. Suppose also that $m > m'$. Then



$$\text{Acceleration of } m = \frac{\text{force on } m}{m},$$

$$a = \frac{mg - t}{m};$$

so Acceleration on

$$m' \text{ up} = a = \frac{t' - m'g}{m'}.$$

The rate of change of momentum of the pulley is the couple $Mk^2\dot{\omega}$ only. Hence, taking moments about its centre,

$$Mk^2\dot{\omega} = (t - t')r.$$

Further, since there is no slip of the string over the pulley, the acceleration of the string must be the same as that of the *rim* of the pulley. Hence the geometrical relation is

$$r\dot{\omega} = a,$$

therefore $$\mathrm{M}k^2 a = (t - t')r^2.$$

Now $$t - t' = (m - m')g - (m + m')a,$$

$$\therefore \mathrm{M}k^2 a = (m - m')gr^2 - (m + m')ar^2,$$

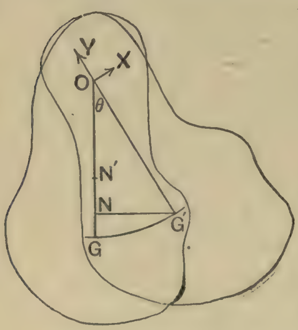
$$\therefore \{\mathrm{M}k^2 + (m + m')r^2\}a = (m - m')gr^2,$$

$$a = \frac{(m - m')r^2}{\mathrm{M}k^2 + (m + m')r^2}g = \frac{(m - m')}{m + m' + \frac{\mathrm{M}k^2}{r^2}}g.$$

If then $\mathrm{M}\frac{k^2}{r^2}$ is comparable with $m + m'$, a considerable error is introduced by neglecting the mass of the pulley, and consequently the value of g as obtained on that supposition (§ 23) will not be correct. (See example 9.)

201. *Motion of a body about an axis under gravity.*—The body in equilibrium will rest with its centre of gravity vertically under the axis, and the reaction on the axis will be equal to the weight of the body.

When in motion it will vibrate about this position of equilibrium. Let O denote the axis of suspension, G the centre of gravity, OG $= h$, and the moment of inertia about G be mk^2.



In any position OG′ draw G′N perpendicular to the vertical through O, and suppose that in the farthest position of G to one side in the motion N is at N′. Then in this position the kinetic energy is zero, and at any other the work done is $mg\mathrm{NN}' = mg(\mathrm{GN}' - \mathrm{GN})$.

The moment of inertia about O is $m(k^2 + h^2)$. Hence, if ω denote the angular velocity when the centre of gravity is at a height GN above its position of equilibrium,

$$\tfrac{1}{2}m(k^2 + h^2)\omega^2 = mg(\mathrm{GN}' - \mathrm{GN}).$$

This gives the velocity in any position.

To find the reaction on the axis we require the acceleration. Now the rate of change of momentum is equivalent to (1) rate of change of $mk^2\omega$, or $mk^2\dot{\omega}$, (2) to rate of change of momentum of the whole mass supposed collected at G′. But this is $mh\dot{\omega}$ perpendicular to OG′ and $m\omega^2 h$ along G′O.

The forces are mg vertically through G′ and the reaction at O. Suppose this given by components X perpendicular to OG′ and Y along G′O. The forces and the rates of change of momenta are equivalent. Hence, if θ denote the inclination of OG′ to the vertical, we have,

resolving along G′O,

$$m\omega^2 h = \mathrm{Y} - mg \cos \theta,$$

resolving perpendicular to G′O,

$$mh\dot{\omega} = \mathrm{X} - mg \sin \theta\,;$$

also, taking moments about O,

$$\begin{aligned} mk^2\dot{\omega} + mh\dot{\omega} \,.\, h &= -mg \,.\, \mathrm{G'N}, \\ &= -mgh \sin \theta. \end{aligned}$$

Hence
$$\begin{aligned} \mathrm{X} &= -\frac{mh}{m(k^2 + h^2)} mgh \sin \theta + mg \sin \theta, \\ &= \frac{k^2}{h^2 + k^2} mg \sin \theta, \\ \mathrm{Y} &= m\omega^2 h + mg \cos \theta. \end{aligned}$$

But
$$\tfrac{1}{2}m(k^2 + h^2)\omega^2 = mg(\mathrm{GN}' - \mathrm{GN}).$$

If α be the greatest amplitude of swing,

$$\mathrm{ON}' = h \cos \alpha,$$
$$\mathrm{ON} = h \cos \theta,$$

and
$$(k^2 + h^2)\omega^2 = 2gh(\cos \theta - \cos \alpha),$$
$$\therefore\ \mathrm{Y} = mg \left\{ \cos \theta + \frac{2h^2}{k^2 + h^2}(\cos \theta - \cos \alpha) \right\}.$$

From these values of X, Y the magnitude and direction of the resultant pressure at O can be determined in any case.

As a special case, take a sphere suspended from a point on its surface, held so that its centre is on the horizontal through the point of support, and then let fall. Here

$$\alpha = 90^\circ, \quad k^2 = \tfrac{2}{5}r^2, \quad h = r.$$

The energy equation gives

$$\tfrac{1}{2}(\tfrac{2}{5} + 1)r^2\omega^2 = g \,.\, r \cos\theta,$$

or
$$r^2\omega^2 = \tfrac{10}{7}gr\cos\theta = \tfrac{10}{7}g \,.\, \text{ON}.$$

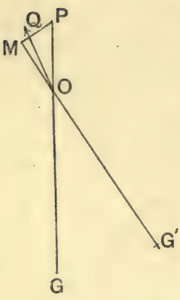
This gives the velocity of the centre at any depth. The reactions are,

$$X = \tfrac{2}{7}mg\sin\theta,$$
$$Y = \tfrac{17}{7}mg\cos\theta.$$

If then the reaction makes an angle ϕ with OG′,

$$\tan\phi = \frac{X}{Y} = \frac{2}{17}\tan\theta.$$

This enables us to draw the direction graphically.



Take any point M in G′O produced, draw MP perpendicular to G′O and meeting the vertical through O in P. Take Q so that $MQ = \tfrac{2}{17}MP$. Then

$$\tan QOM = \frac{QM}{MO} = \frac{2}{17}\frac{PM}{MO} = \frac{2}{17}\tan\theta.$$

Hence OQ is the direction of the resultant pressure.

202. *Pendulum.*—The motion is given by the equation in the preceding article, viz. that obtained by taking moments round O. It is

$$m(k^2 + h^2)\dot{\omega} = mgh\sin\theta.$$

Now $h\dot{\omega}$ is the acceleration of G along the path it describes and $\sin\theta = \dfrac{G'N}{h}$. Hence

$$\text{Acceleration of G along its path} = h\dot{\omega} = \frac{hg}{h^2 + k^2} \,.\, G'N$$

But, comparing this with the acceleration in the case of a simple pendulum as given in § 162, viz.

$$\text{Acceleration} = \frac{g}{l} \, . \, \text{G'N},$$

we see that the two are analogous, and that the rigid body will move in precisely the same manner as a simple pendulum whose length is

$$l = \frac{k^2 + h^2}{h}.$$

This length is called the length of the simple equivalent pendulum. The time of a small vibration is

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{k^2 + h^2}{hg}}.$$

For instance, if the body be a bar suspended at one end, of length $2l$, the time of vibration will be

$$2\pi\sqrt{\frac{\frac{4}{3}l^2}{lg}}.$$

Whereas, if the mass of the bar had been collected at its centre of gravity, the time would have been

$$2\pi\sqrt{\frac{l}{g}}.$$

The first is therefore slower and in the ratio $\frac{2}{\sqrt{3}} = 1{\cdot}15\ldots$

Take a point O′ on the other side of G from O, and such that $\text{GO}' = \frac{k^2}{h}$. Then $\text{OO}' = \frac{k^2 + h^2}{h} = l$, and the motion will be the same as if the whole mass were collected at O′.

Now suppose the body suspended from O′. Then the length of the simple equivalent pendulum is

$$l' = \frac{k^2}{\frac{k^2}{h}} + \frac{k^2}{h} = \frac{k^2 + h^2}{h} = l,$$

or the same as before, and the body will now oscillate in the same way as if the whole mass were collected at O and in the same time as before.

The point O′ is called the centre of oscillation of O. The points are interchangeable.

203. The most accurate method for determining the value of g is by observing the time of vibration of a pendulum. It is not possible to get a perfect simple pendulum—that is, a heavy *particle* at the end of a string without mass. It is therefore necessary to use a rigid body, such as a bar or otherwise, determine the value of k and h and deduce the length of the simple equivalent pendulum. Captain Kater, however, got over the difficulty of accurately determining these by making use of the property of the convertibility of the centres of suspension and oscillation. His pendulum can be swung from either of two points in a line with the centre of gravity, and so fixed as to be very nearly in the relation of centres of suspension and oscillation. On the pendulum also in line with the points is a small movable weight. The pendulum is swung on knife edges alternately from the two points, and the movable weight adjusted until the times of vibration are the same. The points are then accurately in the relation of centres of suspension and oscillation, and the distance between is the accurate length of the simple equivalent pendulum. The time of vibration being accurately determined, the value of g is found with very great exactness.

The centre of oscillation has another property. If the body be struck by a blow perpendicular to OG and through O′, there will be no jerk on the point of suspension. For, by § 179, in this case the jerk on a point distant h from the centre of gravity by a blow at a distance l from it is

$$\left(1 - \frac{hl}{k^2 + h^2}\right)\mathrm{P}.$$

But here $\mathrm{OO}' = l = \dfrac{h^2 + k^2}{h}$, and the jerk is therefore zero.

204. *Torsional vibration of a body suspended by a wire.*—If a mass be suspended from a fixed point by a wire so that the mass is symmetrical about the direction of the wire, it is capable of simple rotational motion about the wire. The

forces brought into play are those caused by the torsion on the wire, and these are proportional to the amount of twist which the wire has.

Suppose the lower end of the wire to be twisted through an angle θ, the forces called into play have a moment round the axis of the wire proportional to θ. Denote it by $\lambda\theta$, where λ is a constant depending on (1) the material of the wire, (2) its length, and (3) its diameter.

Let I denote the moment of inertia of the body about the axis of rotation. Then the moment of rate of change of angular momentum is $I\dot{\omega}$. This is equivalent to the moment of the forces, that is

$$I\dot{\omega} = -\lambda\theta.$$

Now $\dot{\omega}$ is the acceleration of θ. Hence

$$\text{Acceleration of } \theta = -\frac{\lambda}{I}\theta.$$

The motion is therefore a simple harmonic one for large motions as well as small, provided the amplitude is not so great as to give a permanent twist on the wire. The time of vibration is

$$2\pi\sqrt{\frac{I}{\lambda}}.$$

This result can be made use of to determine the value of λ and its dependence on the length and diameter of the wire. The student is advised to carry out himself the experiments here indicated. Thus

(1) Observe the number of vibrations per minute for different lengths of the wire. If they be n_1, n_2, . . ., then the times of vibration are $1/n_1$, $1/n_2$, . . . Then

$$\lambda_1 = 4\pi^2 n_1^2 I, \text{ etc.,}$$

or $$\lambda_1 : \lambda_2 : \ldots = n_1^2 : n_2^2 : \ldots$$

It will then be found that λ is inversely proportional to the length of the wire.

(2) Keeping the length the same, and using wires of the same material but different diameters, make the same experiments; it will be found that λ is proportional to the fourth power of the diameters.

(3) By using bodies in which the value of I is known, the value of λ can be found absolutely, and thus determined for different substances.

205. We have seen in Chapter XX how to calculate the moments of inertia in certain simple cases. In practice, however, we have to deal with bodies whose shape is complicated, and in which the distribution of density may be unknown. In this case the value of their moments of inertia must be determined by experiment.

One of the best ways to do this is to compare the moment in question with that of a body whose moment is known. This can be done as follows. We have seen in the previous article that if a body whose moment of inertia is I is suspended by a wire, the time of vibration is proportional to $\sqrt{I}$, or

$$I = ct^2,$$

where c is a constant depending on the wire only. Suspend the body whose moment of inertia I we require, and observe the time of vibration t.

Next fasten a body of known moment of inertia to the former so that the axis of rotation passes through the centre of gravity, and let I′ denote the known moment of inertia about this axis.

Again observe the time (say t_1). Then the motion is that of *one* mass of moment I + I′, and

$$I + I' = ct_1^2,$$

$$\therefore \frac{I}{I + I'} = \frac{t^2}{t_1^2},$$

$$I = \frac{t^2}{t_1^2 - t^2} I'.$$

Here I′, t, t_1 are known and I can be found.

Or we might first suspend the known body and then the two together. In this case

$$\frac{I'}{I+I'}=\frac{t^2}{t_1^2},$$

$$I=\frac{t_1^2-t^2}{t^2}I',$$

or we might suspend each by itself, and then

$$I=\frac{t^2}{t_1^2}I',$$

but in this experiment it would be necessary to ensure that the suspensions were accurately the same in the two cases.

The student is advised to verify by experiment the moments found for various simple bodies, such as a parallelopiped, taking in this case for the *known* mass two equal weights suspended by threads over the ends of the parallelopiped. Their moment of inertia will be $2ml^2$, where m is the mass of one and $2l$ their distance. The strings must be so thin that no rotations are set up in the weights.

206. The following examples are given in further illustration of the application of the foregoing methods. The student should attempt to solve them himself before reading the solutions given.

EXAMPLE I. *A uniform disc has a weightless string wound round its rim; one end of the string is fixed and the disc is allowed to fall. Determine the motion and the tension of the string.*

Perhaps the easiest way to solve this is to notice that the kinetic energy at any time is equal to the work done by gravity. If u be the velocity of the disc and ω its angular velocity, $u-r\omega=0$ since the point where the string leaves the rim has zero velocity. The above condition then gives

$$\frac{1}{2}mu^2+\frac{1}{2}m\,.\,\frac{r^2}{2}\omega^2=mgh,$$

or, since

$$r\omega=u,$$

$$\frac{3}{4}u^2=gh,$$

$$u^2=2\,.\,\frac{2}{3}g\,.\,h.$$

Hence the disc falls with acceleration $\frac{2}{3}g$.

Next, to find the tension, notice that the forces on the centre of gravity are mg down and T up. Hence

$$mg - \mathrm{T} = m \times \text{acceleration},$$
$$= \frac{2}{3}mg,$$
$$\mathrm{T} = \frac{1}{3}mg = \frac{1}{3}\text{ weight}.$$

It will be instructive to see how the same result is arrived at by starting from the equations of motion.

The rates of change of the momenta are, if a denote the acceleration,

(1) ma down,

(2) $\frac{1}{2}mr^2\dot{\omega}$ round the centre.

The geometrical equation is

$$\dot{\omega}r = a.$$

The forces are

(1) mg down,

(2) T acting up at the rim.

The first set are equivalent to the second. Hence,

$$\left.\begin{array}{ll}\text{resolving vertically,} & ma = mg - \mathrm{T} \\ \text{taking moments round the centre,} & \frac{1}{2}mr^2\dot{\omega} = \mathrm{T}r\end{array}\right\},$$

whence

$$(g-a)r = \frac{1}{2}r^2\dot{\omega} = \frac{1}{2}ra;$$
$$\therefore 3a = 2g,$$
$$a = \frac{2}{3}g,$$

and

$$\mathrm{T} = \frac{1}{3}mg.$$

Example II. *A sphere of radius* r *rolls down a fixed sphere of radius* R *from the highest point. Determine where it will leave the sphere.*

Here the centre of the sphere moves along a circle of radius $\mathrm{R} + r$.

The case is therefore similar to that of Example III in § 149. Hence (see figure of § 149)

$$v^2 = g\,.\,\mathrm{ON}.$$

But here, by the equation of energy,

$$\frac{1}{2}mv^2 + \frac{1}{2}\cdot\frac{2}{5}mr^2\omega^2 = mg\,.\,\mathrm{AN},$$

or, since

$$v = r\omega,$$
$$v^2 = \frac{10}{7}g\,.\,\mathrm{AN} = \frac{10}{7}g(\mathrm{R} + r - \mathrm{ON}),$$
$$\therefore \frac{10}{7}(\mathrm{R} + r - \mathrm{ON}) = \mathrm{ON},$$
$$\mathrm{ON} = \frac{10}{17}(\mathrm{R} + r).$$

If the sphere had been smooth, so that there would have been no rolling, it would have left the lower sphere when $ON = \frac{5}{3}(R + r)$. (See § 149.)

EXAMPLE III. *A mass M is attached to a pulley which is supported by strings which pass round a differential axle. The masses and gyration radii of the axle and pulley are respectively m_1, k_1 ; m_2, k_2. Determine the motion and the tension of the string.*

The easiest way to determine the motion is to use the equation of energy. Let u denote the velocity of M, ω_1, ω_2 the angular velocities of the axle and pulley, and h the distance fallen through from rest.

Then Kinetic energy = work done by gravity,

$$\therefore \frac{1}{2}(M + m_2)u^2 + \frac{1}{2}m_1k_1^2\omega_1^2 + \frac{1}{2}m_2k_2^2\omega_2^2 = (M + m_2)gh.$$

It is now necessary to find the connection between u, ω_1, ω_2.

Let a, b denote the two radii of the axle ($a > b$) and c of the pulley. Then in one second the string has been lengthened $\omega_1 a$ and shortened $\omega_1 b$, *i.e.* on the whole lengthened $\omega_1(a - b)$. Hence the pulley falls $\frac{1}{2}\omega_1(a - b)$ in a second, *or*

$$u = \frac{1}{2}\omega_1(a - b).$$

Again, considering the motion of the point of the pulley at which the string is running on,

$$u + c\omega_2 = \omega_1 a,$$

$$\therefore c\omega_2 = \frac{2a}{a - b}u - u,$$

$$= \frac{a + b}{a - b}u.$$

Hence

$$\left\{ M + m_2 + \frac{4m_1k_1^2}{(a - b)^2} + \frac{m_2k_2^2}{c^2} \cdot \left(\frac{a + b}{a - b}\right)^2 \right\} u^2 = 2(M + m_2)gh.$$

This gives the velocity after falling any distance h. Comparing it with the formula $v^2 = 2as$, it follows that the acceleration is

$$\dot{u} = \frac{M + m_2}{M + m_2 + \frac{4k_1^2}{(a - b)^2}m_1 + \left(\frac{a + b}{a - b}\right)^2 \frac{k_2^2}{c^2}m_2} g.$$

Let T_1, T_2 denote the tensions. Then

$$(M + m_2)g - (T_1 + T_2) = \text{acceleration of mass},$$

$$= (M + m_2)\dot{u},$$

or

$$T_1 + T_2 = (M + m_2)(g - \dot{u}).$$

Again the acceleration of the rotation of the pulley gives

$$m_2k_2^2\dot{\omega}_2 = (T_1 - T_2)c.$$

But $$c\omega_2 = \frac{a+b}{a-b}u \text{ always,}$$

$$\therefore\ c\dot{\omega}_2 = \frac{a+b}{a-b}\dot{u},$$

$$\therefore\ T_1 - T_2 = m_2\frac{k_2^2}{c^2}\cdot\frac{a+b}{a-b}\cdot\frac{M+m_2}{M+m_2+\frac{4k_1^2}{(a-b)^2}m_1+\left(\frac{a+b}{a-b}\right)^2\frac{k_2^2}{c^2}m_2}g.$$

From these two equations T_1 and T_2 may be found.

EXAMPLES—XXI.

1. A cylinder 1 foot long and 6 inches diameter rolls down a rough plane of 1 in 10. Find its velocity after rolling down 20 feet. Also find the least coefficient of friction that it may roll without slipping.

If the coefficient of friction is one-half this, find the velocity in this case.

2. A rod AB 4 feet long can move freely round a pin at A; it is placed with B vertically above A and let go. Find the velocity of B when it arrives at the lowest position.

Also find in that case the pressure on the pin.

3. A sphere is projected so as to roll up a rough inclined plane. Determine how far it will rise.

4. Prove that the kinetic energy stored up in a train of railway carriages moving with velocity v is

$$\left\{W + w\left(1+\frac{\kappa^2}{a^2}\right)\right\}\frac{v^2}{2g}$$

foot-pounds, where w denotes the mass of the wheels and axles, W the mass of the rest of the train in lbs., a denotes the radius of the wheels, and κ the radius of gyration of a pair of wheels about the axis. Determine the acceleration with which the train would run freely down a given incline.

5. A four-wheeled waggon weighs ½ ton without the wheels; each wheel weighs 28 lbs. and is similar in all respects to the fly-wheel in question 6 of Chapter XX, only it is one-third the linear dimensions; the waggon runs down an inclined plane of 1 in 100. Find its velocity after travelling 300 yards.

6. A homogeneous sphere is rotating about a horizontal diameter and is gently placed on a rough horizontal plane, the coefficient of friction being μ. Determine the subsequent motion.

7. Show that a uniform rod of mass m moving in any manner is dynamically equivalent to the system of three particles of masses $\frac{m}{6}$, $\frac{m}{6}$, $\frac{2m}{3}$

placed at its ends and centre of inertia respectively, and connected by stiff weightless wires.

8. If you try to spin a good egg on the table, it partly comes to rest and then moves on again. Why is this ?

9. In an Atwood's machine the wheel consists of a disc 8 oz. in mass ; the suspended masses are 63 and 65 oz. Find the motion and the percentage error caused by neglecting to take account of the inertia of the wheel.

10. A cylinder has a string wound round it in a plane through its centre perpendicular to its axis, and one end of the string is held fast while the cylinder is allowed to fall. How long will it take to fall 16 feet, and what will be its velocity then ? Find also the tension of the string.

11. In the preceding case determine how the free end must be moved so that the cylinder neither rises nor falls.

12. In question 10 the string passes over a pulley without mass and is fixed to a body of the same mass as the cylinder. Determine the motion.

13. A uniform thread whose mass may be neglected passes over a smooth peg, and unwinds itself under gravity from two cylindrical reels freely suspended by the thread. Show that the portion of string unwound increases with uniform acceleration

$$2g\frac{I_1a^2+Ia_1^2}{I_1(a^2+\kappa^2)+I(a_1^2+\kappa_1^2)},$$

where I, a, κ denote moment of inertia, radius, and radius of gyration of a reel. Also find the tension of the string.

14. A sphere rolling on a rough horizontal plane with velocity 10 feet per second comes to the foot of an inclined plane (60°). The impact being inelastic, determine how far the sphere will rise (1) supposing the inclined plane smooth, (2) with coefficient of friction $4\frac{1}{2}$.

15. A cube slips down a rough plane inclined at 15° to the horizon ; the coefficient of friction is less than $\frac{\sqrt{3}}{2}$ tan 15° ; it suddenly strikes against a pin in the plane. Determine the vertical height it has fallen through when the cube just topples over the pin.

16. A sphere (radius r) is placed at the highest point of a horizontal cylinder (radius R) and allowed to fall. Determine the point at which it will leave the cylinder (1) when the cylinder is smooth, (2) perfectly rough.

17. A sphere rolls down the inside of another sphere of double the radius, starting from the end of a horizontal diameter. Find the pressure between them at the lowest point.

18. A uniform cube is free to turn about one edge which is horizontal. Find the length of the edge of the cube, that it may swing to and fro in 1 second.

19. An equilateral triangle is suspended from one angle. Determine the length of the simple equivalent pendulum when it vibrates (1) in its plane, (2) about a line parallel to its base.

20. A thin spherical vessel contains water of twice its mass; it is fastened to a vertical wire and oscillates round it. Compare the times of oscillation when the water is (1) frozen, and (2) is liquid.

21. If the vessel in the preceding question were fastened to a fixed point on its surface and oscillated under gravity, determine the times of vibration under similar circumstances.

22. Four equal uniform rods are jointed to form a square; they are held with one diagonal vertical, the lowest corner resting on a table, and let go. Determine the velocity with which the opposite angles strike one another.

23. A uniform rod is supported by two vertical strings at its ends; one is suddenly cut. Find the change in the tension of the other.

24. Prove that the horse-power consumed in keeping a fly-wheel weighing W lbs. rotating with r revolutions a second about a horizontal axis by means of a couple transmitted by the axle is $2\pi r W b \sin \phi \div 550$, where b denotes the radius of the axle in feet and ϕ is the angle of friction.

Prove that, if left to itself, the wheel will come to rest after

$$\frac{\pi \kappa^2 r^2}{gb \sin \phi} \text{ revolutions in } \frac{2\pi \kappa^2 r}{gb \sin \phi} \text{ seconds,}$$

where κ denotes the radius of gyration of the wheel.

25. A cone is fastened by its vertex to a vertical wire, and oscillates about its axis; it makes 10 complete vibrations a minute. Find the force necessary to be applied at the ends of a diameter of the base to twist the end of the wire through a right angle, having given mass $= 6$ lbs. diameter of base $= 4$ inches, height $= 4$ inches.

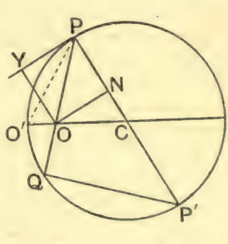
APPENDIX

CHAPTER XVI (*a*)

Central Orbits

155 *a*. We shall devote this chapter to the consideration of certain interesting and important cases of orbits described by particles when moving under forces directed to fixed points. The general question presents itself under two aspects: (1) to determine the law of force required to make a particle describe a given orbit; (2) to determine the orbit when it moves under given forces. The most important and the simplest of such orbits is that in which a circle is described under the action of a force directed to its centre. The solution of this has been given in Chap. XVI. The velocity is constant and the acceleration to the centre v^2/a, where v denotes the velocity and a the radius; or conversely if the force per unit mass be F, the velocity is $\sqrt{(a\text{F})}$, and the period $2\pi\sqrt{(a/\text{F})}$.

But a circle may be described under the action of a force to a point which is not the centre. The discussion of this problem will serve as an example of the problem under the first aspect. It is also important for its applications to problems which follow.



In the rest of this chapter the word "force" will be used shortly for "force per unit mass." It is the same as acceleration.

155 *b*. In the figure let C be the centre of the circle of which the radius is a. Let O be the centre of force and P the position

of the particle at any instant on the circle. Then if v denote the velocity at P, the component acceleration along PC is v^2/a. But this is equal to the component of the central force along PC. If this central force be denoted by F,

$$F \cos OPC = \frac{v^2}{a}.$$

Also by § 153

$$OY \times v = \text{const} = h \text{ (say)},$$

where h is twice the rate at which OP is sweeping out area.

Now

$$OY = PN$$
$$= r \cos OPC,$$
$$\therefore v = \frac{h}{PN}$$
$$= \frac{h}{r \cos OPC}.$$

Hence

$$F = \frac{h^2}{a \,.\, PN^2 \cos OPC}$$
$$= \frac{h^2}{ar^2 \cos^3 OPC}.$$

This may be expressed otherwise, for since

$$\cos OPC = \frac{PQ}{PP'}$$
$$= \frac{PQ}{2a},$$
$$\therefore F = \frac{8h^2a^2}{OP^2 \,.\, PQ^3}.$$

It is to be noted that although F is a central force, its magnitude does not depend on the distance of P from O alone, but also on the direction of OP.

If the law of force is given in general by

$$F = \frac{\mu}{PN^2 \cos OPC},$$

then

$$\mu = \frac{h^2}{a},$$

and the velocity at any point is given by

$$v^2 = aF \cos OPC$$
$$= \frac{\mu a}{PN^2},$$

or $$v = \frac{\sqrt{(\mu a)}}{\text{PN}}.$$

The period (T) is easily found, for h is twice the rate of describing the area, and an area πa^2 is described in time T.

$$\therefore \pi a^2 = \tfrac{1}{2}h \,.\, \text{T},$$

$$\therefore \text{T} = \frac{2\pi a^2}{\sqrt{(\mu a)}}$$

$$= \frac{2\pi}{\sqrt{\mu}} \,.\, a^{\frac{3}{2}}.$$

A specially interesting case is when the centre of force is on the circumference of the circle. Here $\text{O}'\text{P} = \text{PQ} = r$, and

$$\text{F} = \frac{8\mu a^3}{r^5} = \frac{\nu}{r^5},$$

or the force must vary as the inverse fifth power of the distance. In this case, however, it is to be noted that the particle passes through O′ with an infinite velocity, while the momentum in direction CO′ remains finite and in the same direction after the particle has passed through O′, and its velocity again become finite. It follows therefore that after passing O′ the particle will describe an equal circle touching the original orbit at O′, and will complete the original circle only after passing through O′ the second time.

Since $$\text{PN} = r \cos \text{O}'\text{PP}' = \frac{r^2}{2a},$$

$$v^2 = \frac{4\mu a^3}{r^4} = \frac{1}{2} \cdot \frac{\nu}{r^4}.$$

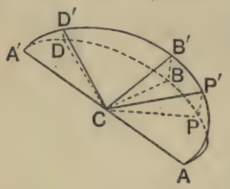
This corresponds to the energy equation. It follows that if the velocity is zero at a distance c under a force of this kind

$$v^2 = 4\mu a^3 \left(\frac{1}{r^4} - \frac{1}{c^4}\right).$$

The converse of the preceding theorems are not necessarily true. The particle will only describe a circle under the specified force if it be properly projected.

155 *c*. The governing condition in a central orbit is (§ 153) that the rate of description of area around the centre of force is constant. Now if any plane curve be projected orthogonally on a second plane, inclined at an angle α to the first, the ratio

of any area of the projection to the area from which it was projected is uniform and equal to cos α. If then a point moves on any curve so as to describe equal areas in equal times round a fixed point, its projection will do the same round the projection of the fixed point. The acceleration and the velocity in the second case will simply be the components in the new plane of the corresponding quantities in the first. As the orthogonal projection of a circle is an ellipse, this remark enables us to solve easily the problem of determining the law of force required to describe an ellipse about any point in the plane of the ellipse, merely by projecting the case of circular motion which has just been solved. We shall, however, confine our attention only to the two cases when the centre of force is (1) at the centre; (2) at a focus.



155 *d*. *Ellipse about the centre.*—Here the original orbit must be a circle described about the centre. Let AB′A′ be the circle, ABA′ the ellipse into which it projects. Let P′ be any point on the circle and P its projection on the ellipse. Then if F′ denote the force in the circle

$$v' = \sqrt{(aF')},$$

$$T = 2\pi\sqrt{\left(\frac{a}{F'}\right)},$$

and F′ acts along P′C. The acceleration of P will be the component of F′ along PC, that is

$$F = \frac{PC}{P'C} \cdot F'$$

$$= \frac{F'}{a} r$$

$$= \mu r \text{ (say).}$$

Hence the force must vary directly as the distance. Also

$$T = 2\pi\sqrt{\left(\frac{a}{F'}\right)}$$

$$= \frac{2\pi}{\sqrt{\mu}}.$$

Hence the periodic time is independent of the size and eccentricity of the ellipse.

Draw in the circle CD′ perpendicular to CP′. Then the velocity of P′ can be represented by CD′. Hence the velocity of P is given by CD, the projection of CD′. That is—

$$v = \frac{\text{CD}}{\text{CD}'} v'$$

$$= \frac{\text{CD}}{a} \sqrt{(a\text{F}')}$$

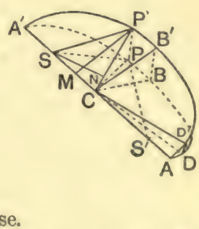
$$= \sqrt{\mu} \, . \, \text{CD},$$

where CD is the semi-diameter in the ellipse conjugate to CP.

This gives a complete solution of the problem.

Conversely, suppose given the centre of force (O), the value of μ, and that a particle is projected from a point P in a given direction with a given velocity (v). Draw from O a line OD parallel to the direction of projection, and of such a length that $\text{OD} = \text{V}/\sqrt{\mu}$. Draw the ellipse with OP, OD for conjugate diameters. The particle will, by what has gone before, proceed to describe this ellipse. Hence we see that in this case, whatever the circumstances of projection, the orbit will be an ellipse when the force varies directly as the distance towards the centre of the force. This is not true, however, when the force is repulsive. In this case it is clear that the orbit must extend to infinity. It will be shown later that it is a hyperbola.

155 *e*. Before treating the case of an ellipse about a focus, it will be necessary to consider a geometrical theorem.



Let, as before, AB′A′ be a circle, and ABA′ its projection. Let a be the angle between the planes.

Take $\text{SC} = \text{S}'\text{C} = \text{BB}' = a \sin a$.

Since $\text{BB}'^2 = \text{CB}'^2 - \text{CB}^2$

$$= a^2 - b^2,$$

it follows that O, O′ are the foci of the ellipse.

Let P′ be any point on the circle, P its projection. Join SP′, SP, CP′, and draw SN perpendicular to CP′. Also let PMP′ be perpendicular to AA′.

Then
$$\begin{aligned} PP' &= P'M \sin \alpha \\ &= CP' \sin P'CM \sin \alpha \\ &= a \sin \alpha \sin P'CM \\ &= SC \sin SCN \\ &= SN. \end{aligned}$$

Thus in the right-angled triangles SPP′, SNP′, SP′ is common, and SN = PP′. Hence

$$SP'N = P'SP$$

and
$$P'N = SP.$$

That is, the projection of SP′ is equal to P′N, and SN is the height of P′ above P; also CN is the height of D′ above D if CD′ is perpendicular to CP.

155 *f*. *Ellipse about a focus.*—Here the case to be projected is the circle described about S, in which, as has been proved,

$$F' = \frac{\mu}{P'N^2 \cos SP'C},$$

$$T' = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}},$$

$$v'^2 = \frac{\mu a}{P'N^2}.$$

Hence projecting,

$$\begin{aligned} F &= F' \cos P'SP \\ &= \frac{\mu}{P'N^2 \cos SP'C} \cos SP'C \\ &= \frac{\mu}{SP^2}, \end{aligned}$$

since by the preceding theorem P′SP = SP′C and SP = P′N. Thus the force must vary inversely as the square of the distance from the centre of force,

$$T = T' = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}},$$

or the period depends only on the length of the major axis.

Draw CD′ perpendicular to CP′. The velocity v' is along CD′. Hence

$$v = \frac{CD}{CD'} \cdot v',$$

or
$$v^2 = \frac{CD^2}{a^2} \cdot \frac{\mu a}{P'N^2}$$
$$= \frac{\mu}{a} \frac{SP . S'P}{SP^2},$$
and
$$S'P = 2a - r \text{ where } SP = r,$$
$$\therefore v^2 = \frac{\mu}{a} \left(\frac{2a - r}{r}\right)$$
$$= \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

This gives the velocity at any point of the orbit.

Since $\frac{1}{2}mv^2$ is the kinetic energy produced by a force $\mu m/r^2$ acting on the particle from a certain distance, we learn incidentally that the work done by a force μ/r^2 is given by

$$\frac{\mu}{r} + \text{const},$$

a result already givèn in Ex. 50, Chap. I. In fact

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

is the energy equation.

Since $v = 0$ if r be put $= 2a$, it follows that the velocity at any point is that due to a fall to the point from rest from a point distant from the focus by a length equal to the major axis of the ellipse.

If a be made infinitely great,

$$v^2 = \frac{2\mu}{r}$$

gives the velocity at P of a particle which has fallen to P from an infinite distance. If v be greater than this it follows that P cannot describe an elliptic orbit, for it will pass again to infinity. If it is equal to this the orbit will be an ellipse whose other focus is at an infinite distance—that is, it will be a parabola.

155 *g*. The methods of the preceding paragraphs are not applicable to hyperbolic orbits. Parabolic orbits about the focus are a case of elliptic orbits, and are traced when the velocity of projection is that due to a fall from an infinite distance. In the case of hyperbolic orbits about the centre it is clear that the force must be repulsive as the path is convex to the centre. It

is natural to suspect from analogy that this orbit requires a repulsive force varying as the distance. In the case of hyperbolic orbits about a focus two cases occur. To describe that branch within which the focus in question lies, that is the path which is everywhere concave to the centre of force, the force must be attractive. To describe that branch outside the focus, that is the path which is everywhere convex to the centre of force, the force must be repulsive. We have seen that an ellipse is described about a focus when the force is attractive, and varies inversely as the square of the distance provided the velocity of projection be less than that due to a fall from infinity. It is then natural to suspect that the concave branch of the hyperbola will be described under the same law of force if the velocity of projection is greater than that due to a fall from infinity, and that the convex branch will be described when the force is repulsive and follows the same law. We shall prove that these suspicions are correct by considering the question under the second of the two aspects referred to in § 155 *a*, and by determining the orbits described when the forces are

(1) Repulsive and varying as the distance.

(2) Attractive and inversely as the square of the distance.

(3) Repulsive and inversely as the square of the distance.

155 *h*. *Force repulsive and varying as the distance* (μr).—The work done under a force of this kind is (§ 97) $\frac{1}{2}\mu r^2 + \text{const.}$ Hence the energy equation gives

$$v^2 = \mu(r^2 \pm c^2),$$

where c is of the dimension of a length.

[If the force is attractive $\quad v^2 = \mu(c^2 - r^2)$]

But $\qquad pv = h$

$$\therefore \mu p^2(r^2 \pm c^2) = p^2v^2 = h^2.$$

But this is the relation between the perpendicular from the centre on the tangent to a hyperbola and the focal distance (r) of its point of contact P. For if CD be conjugate to CP

$$p \,.\, \text{CD} = ab$$

and $\qquad \text{CP}^2 \sim \text{CD}^2 = r^2 \sim \text{CD}^2 = a^2 - b^2.$

Hence $\qquad p^2\{r^2 \pm (a^2 - b^2)\} = a^2b^2.$

Comparing, it follows that the orbit is a hyperbola in which $a^2 - b^2 = c^2$, $a^2b^2 = h^2/\mu$, and

$$v^2 = \mu \,.\, \text{CD}^2.$$

The student should treat the case of the attractive force in the same way.

155 *i*. *Force inversely as the square of the distance.*—It has

been shown in § 155 *f* that the work done under an attractive force of this kind is

$$\frac{\mu}{r} \pm \text{const.}$$

So for a repulsive force the work done is of the same form with sign changed. Hence the energy equation gives

$$\tfrac{1}{2}v^2 = \pm\frac{\mu}{r} \pm \text{const.}$$

Therefore for an attractive force

$$v^2 = \mu\left(\frac{2}{r} \pm \frac{1}{a}\right);$$

for a repulsive force

$$v^2 = \mu\left(\frac{1}{a} - \frac{2}{r}\right);$$

where a is of the dimension of a length.

But $$pv = h.$$

Hence for an attractive force

$$\mu p^2\left(\frac{2}{r} \pm \frac{1}{a}\right) = h^2;$$

and for a repulsive

$$\mu p^2\left(\frac{1}{a} - \frac{2}{r}\right) = h^2.$$

But $$p^2\left(\frac{2}{r} - \frac{1}{a}\right) = \frac{b^2}{a}$$

gives the relation between the perpendicular from the focus of an ellipse on the tangent and the focal distance of the point of contact; also

$$p^2\left(\frac{2}{r} + \frac{1}{a}\right) = \frac{b^2}{a}, \text{ and } p^2\left(\frac{1}{a} - \frac{2}{r}\right) = \frac{b^2}{a}$$

give similar relations for the concave and convex branches respectively of a hyperbola. These statements are easily proved, for if S, S′ be the foci and SY, S′Z denote the perpendiculars on the tangent to an ellipse or hyperbola at a point P,

$$\text{SY}.\text{S}'\text{Z} = b^2$$

and $$\frac{\text{SY}}{\text{S}'\text{Z}} = \frac{\text{SP}}{\text{S}'\text{P}}.$$

Therefore
$$SY^2 \cdot \frac{S'P}{SP} = b^2$$

or
$$p^2 \frac{S'P}{ar} = \frac{b^2}{a} \cdot$$

In the ellipse $S'P + r = 2a$.

Hence
$$p^2\left(\frac{2}{r} - \frac{1}{a}\right) = \frac{b^2}{a} \cdot$$

In the hyperbola for the branch concave to S
$$S'P - r = 2a,$$

and
$$p^2\left(\frac{2}{r} + \frac{1}{a}\right) = \frac{b^2}{a} \cdot$$

In the hyperbola for the branch convex to S
$$r - S'P = 2a$$

and
$$p^2\left(\frac{1}{a} - \frac{2}{r}\right) = \frac{b^2}{a} \cdot$$

The first case belongs to
$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

that is where the velocity is less than that from infinity. The orbit is therefore an ellipse, as has been already found.

The second class belongs to
$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right),$$

that is where the velocity is greater than that from infinity. The orbit is therefore the branch of a hyperbola concave to the centre of force.

The third case belongs to
$$v^2 = \mu\left(\frac{1}{a} - \frac{2}{r}\right),$$

that is where the force is repulsive. The orbit is therefore the branch of a hyperbola convex to the centre of force.

155 *k*. Any point in a central orbit at which the particle is moving perpendicularly to the central force is called an apse, and its distance from the centre of force is then called an apsidal distance. In all cases where the force depends on the

distance alone the orbit must be symmetrical on the two sides of an apse. For suppose the motion of the particle reversed when it comes to an apse, it will travel back over its former path and it will be similar to that which it would have described had it continued in its original direction since the forces at similar points are alike. After leaving one apse the particle may, or may not, arrive at a second apse. If it does the orbit is symmetrical about this also; consequently after passing this second apse it will come again to an apse of the same kind as the first; and so on alternately. Consequently, although an orbit may have several apses there cannot be more than two values of the apsidal distance. Thus, in the ellipse about the centre there are four apses but two apsidal distances, viz. the lengths of the major and minor semi-axes. In the ellipse about the focus there are two apses and two apsidal distances. In the hyperbola about a focus (either attractive or repulsive) there is only one apse and one apsidal distance. Another instance of many apses and two apsidal distances may be found in the complicated orbit discussed in § 154.

155 *l.* The preceding theorems were first published by Newton in 1685, and more fully in the *Principia*[1] in the succeeding year. They are of the greatest importance, for it enabled him to practically demonstrate the truth of the law of gravitation. From the observed motions of the planets, specially of Mars, Kepler had been able to deduce three laws which have since borne his name. They are :—

(1) The planets describe ellipses round the sun in one focus.
(2) The line joining a planet to the sun sweeps over equal areas in equal times.
(3) The squares of the periodic times of the planets are as the cubes of their mean distances from the sun.

Combining these laws of observation with the results obtained in this chapter, we deduce, as Newton did, the following facts :—

From the second law that the planets are acted on by forces directed to the sun.

From the first, that the force varies inversely as the square of the distance.

From the third, that the same cause acts on each planet; in other words "μ," is the same for all the planets.

[1] *Philosophiæ Naturalis Principia Mathematica.*

For by § 155 *f*

$$T^2 = \frac{4\pi^2}{\mu^2} a^3.$$

Hence, as T^2/a^3 is the same for all, so must also μ be.

Kepler's laws do not state the exact truth. They are true only to a first approximation, and this for three reasons :—

(1) The planets and the sun not being exact spheres, the force between them does not pass exactly through their centres of inertia. It is only, however, where bodies are comparatively close, as in the case of the earth and moon, that this produces any appreciable effect.

(2) The sun and planets are all free, the sun is not fixed.

(3) The planets also act on each other and disturb each other's motions round the sun.

155 *m*. In the case of two bodies, say a planet and the sun, moving under each other's attraction, neither can be considered a fixed point. We must consider the modification introduced by this.

We can regard the planets and the sun as collected at their respective centres of inertia. Their mutual gravitation produces a stress between them given by

$$\frac{f . E . S}{r^2},$$

where E, S are the masses of the planet and the sun, r the distance between them, and f the force between two unit masses at unit distance (Exs. 22-24, Chap. III.).

This mutual stress can produce no motion of the common centre of inertia. The bodies will therefore move in relative motion about their centre of inertia as if it were fixed ; and the force on either will therefore always pass through it. They will therefore each describe equal areas in equal times round it. But further, if the masses be E and S, r the distance between them, and R the distance of say E from the centre of inertia

$$\frac{R}{r} = \frac{S}{E+S}.$$

Hence the force on E is

$$\frac{E . S^3}{(E+S)^2} \frac{f}{R^2}.$$

In other words, it varies inversely as the square of the distance from the centre of inertia G.

The path is, therefore, an ellipse about G in the focus. So also with the path of S. This being proved, it is now easy to deduce further that the motion of E *relative* to S is that described in Kepler's laws.

Also the periodic time is

$$\frac{2\pi}{\sqrt{f}}\,\frac{E+S}{S^{\frac{3}{2}}}\,b^{\frac{3}{2}},$$

where b is the mean distance of E from the centre of inertia of the two.

If a is the mean distance of one from the other

$$aS = b(E+S)$$

and
$$T = \frac{2\pi}{\sqrt{f(E+S)}}\,a^{\frac{3}{2}}.$$

That is the same as if the combined mass of the two bodies were condensed at the centre of S for E's motion, and at the centre of E for the motion of S. Kepler's third law is therefore not exactly true. In the cases to which it refers, however, the ratio E/S is extremely small, in the case of the earth 1 : 326,000. The largest ratio is in the case of Jupiter, viz. about 1 : 1050.

EXAMPLES—XVI (*a*).

1. A number of particles are projected from the same point P with the same velocity, and are subjected to the same force. They all move in ellipses: prove that the rate of description of areas is greatest in that ellipse which has its centre in PS.

2. Prove that the velocity with which a particle must be projected upwards from the earth's surface in order to escape from the earth's attraction must be not less than seven miles per second.

3. The period of the moon round the earth is $27\frac{1}{3}$ days nearly; her mean distance from the centre is equal to sixty times the earth's radius. Calculate approximately the value of gravity at the earth's surface.

4. The law of force in a central orbit varies inversely as the square of the distance. A particle is projected from a given point with a given velocity: prove that the periodic time is independent of the direction.

5. In what ratio would the earth's orbital velocity have to be increased in order that it might escape from the solar system, supposing its actual orbit circular?

6. Supposing a cannon ball could be projected horizontally with a

velocity of 6 miles per second, find how long it would be before it returned to the same point, and the greatest distance it would attain from the point of projection.

7. If the velocity of the moon were suddenly halved, determine the new length of the month.

8. Deimos revolves round Mars in $30^h\ 17^m\ 54^s$ at a mean distance of 14,500 miles. Hence determine the ratio of the mass of Mars to the sun.

Phobos revolves round Mars in $7^h\ 39^m\ 14^s$. What is his mean distance from Mars?

9. Two spherical kilogram masses are placed in space at a distance of 5 cm. from one another, and revolve in circles round one another. How long do they take to make a complete revolution? (See Ex. iii. 22.)

10. A body describes a circle about its centre, the law of force being that of the inverse square. The velocity is suddenly increased in the ratio of $\sqrt{3} : \sqrt{2}$. Determine the new orbit.

11. Two centres of force, varying as the distance, of equal absolute intensity, reside in the foci of an ellipse: prove that if properly projected, a particle will describe the ellipse in a time which is $\sqrt{\frac{1}{2}}$ of the period about either centre of force separately.

12. A planet moving round the sun in an ellipse receives at a point in its orbit a sudden velocity in the direction of the normal outwards, which transforms its orbit into a parabola: prove that this added velocity is the same for all points in its orbit.

If this velocity is added at the end of the minor axis, prove that the axis of the parabola will make with the major axis of the ellipse an angle whose sine is equal to the eccentricity.

13. A particle describes a circle (rad. R) round a centre of force varying inversely as the square of the distance. If its direction of motion be altered so as to make an angle β with the radius to the centre, prove that the apsidal distances are $R(1 \pm \cos\beta)$. Thence prove that the eccentricity is $\cos\beta$.

14. Prove that in a nearly circular orbit about a force varying inversely as the square of the distance, if the intensity of the central force be increased by $1/n$th, the periodic time will be diminished by $2/n$ of its amount, n being large.

15. A particle revolves in an ellipse under the action of a force to the centre. When the particle is at the extremity of the minor axis, the centre of force is suddenly transferred to a focus. Show that the new orbit is an ellipse whose axes bisect the angles between the original major axis and the focal distance at the instant.

16. Two particles describe an ellipse and a confocal hyperbola

respectively, under forces in the centre of the same absolute magnitude μ, and the velocities at the apses are the same. Show that the velocity of either particle at a point where the curves intersect is $b\sqrt{2\mu}$ where b is the semi-minor axis of the ellipse.

17. Show that the hodograph of a circle described under a force to a point on the circumference is a parabola.

18. The hodograph of a particle describing an ellipse under the action of a force to the centre can be represented by the ellipse itself.

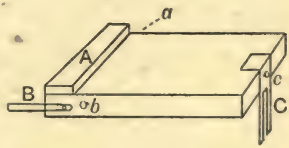
19. The hodograph of a particle describing an ellipse under the action of a force to the focus is a circle.

20. A particle is describing an ellipse under the action of a force to the focus ; a blow is given to it and the subsequent orbit is a circle. Determine the magnitude and direction of the blow.

21. A particle describes a hyperbola under a repulsive force from the centre. Show that if the force and velocity are each resolved into two components parallel to the asymptotes, either component of force varies as the corresponding component of velocity.

NOTE TO § 7

Ballistic Balance.—Actual experiments made by the student himself with this apparatus are so valuable in making him realise the fundamental fact of inertia that I give here a description of a form of the apparatus, which I have found very useful, and which any one may make at the cost of a few pence.



For the carriers take two pieces of hard wood about $6'' \times 4''$ and weighing with fittings exactly 1 lb. or $\frac{1}{2}$ kilogram each. One is shown in perspective in the figure.

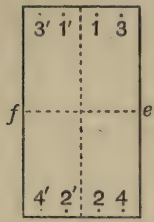
A is a small ledge screwed on to prevent the weights placed on the carrier from moving when the collision takes place.

B is one of the catches ; on the other carrier there is a notch for it to slip into. It will be well to glue on the colliding ends of the carriers a piece of woollen cloth.

C is a forked piece cut from a brass strip. It serves as the pointer.

At a, b, c put in three small screw eyes, ab being parallel to the end of the carrier, c in the middle of the back, and all three in a plane parallel to the top of the carrier.

We shall denote the corresponding screw eyes on the other carrier by a', b', c'. On a large board (say 2′ and 1′, 6″), which is to be fixed to the ceiling of the room, and from which the carriers are to be suspended, put in screw eyes as shown in the second figure at 1, 2, 3, 4, etc. The distance between 1 and 1′ and between 2 and 2′ must be the same as between the line joining ab and the line joining $a'b'$ on the carriers when they are in contact. This is to ensure the plane of the suspensions being vertical when the carriers collide. For the same reason the distance between 12 and 34 must be the same as that between c and the line of ab. The line ef should accurately lie midway between the parallels 3′1′13 and 4′2′24. Small weights should be suspended from e, f for a purpose to be described below.



Tie one end of a long string to say 1, and pass the other end successively through the screw eyes at $a2b13c4$, and let this end be brought to some fixed peg or hook within reach. The carrier is now suspended from the ceiling. By slackening or otherwise the free end the carrier can easily be brought to any desired height above the floor. By pressing on it with the hand it can easily be brought into any desired position as the string smoothly slips through the eyes. The other carrier is suspended in a similar way to 1′2′3′4′. Now let the small weights from ef hang vertically. Stretch on a table or a counter below the carriers a horizontal wire, so as to just touch the strings suspending these weights. This can easily be done by fastening the ends and stretching the wires by two bridges as in a violin. This wire gives the direction in which the carriers must move. Pull aside the suspended weights for similar use on a future occasion.

Take two pieces of cardboard (each about 1′, 6″ × $2\frac{1}{2}$″) and draw on them a scale of lengths (say a centimeter scale). Punch holes along one side and the ends, and put in shoe-eyes (any shoemaker will do this). Pass small wire S-hooks through the top holes and suspend from the stretched wire fixed as above. Connect the end holes of one to the corresponding holes in the other card by *elastic* strings. Tie a piece of string to each of the other ends.

Adjust the carriers so that their pointers straddle over the wire and they move along the wire without touching it. While the carriers hang in contact at rest adjust the card-scales by the strings, so that their zeros are just under the pointers. The elastic strings keep them tight and allow each to be adjusted. Tie threads to the back eyes (cc') of each carrier, pass them through screw eyes at the bridges, or other support, so that the threads when tight are in the same vertical plane as the stretched wire. Bring their ends back in front of the apparatus and see that the catches on the carriers act properly. We are now in a position to make an observation.

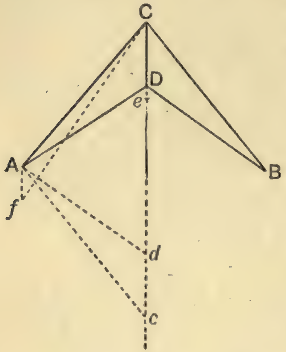
Place given masses on the carriers, remembering to take account of the masses of the carriers themselves. Place the two threads on the table under the first finger of one hand and across one another. With the other hand draw each thread tight until its carrier is pulled back to the desired points on the scale. Lift the finger suddenly and the carriers start simultaneously. The threads are so light that they slip easily through the eyes and produce no appreciable retardation.

One advantage of this suspension is that, although the carriers are easily put out of adjustment, they can be brought back again immediately; whereas, with separate suspensions to each hook it is very difficult to adjust accurately, and any extra tensions due to altering the masses in the carriers throw it out, and another troublesome adjustment is required.

Note to § 91

The essential condition in constructions involving frameworks is that the reactions at the joints shall not involve couples. In other words the joints shall behave practically as if the ties and struts at any point of the system are connected by a single pin. The reactions are in general very much greater than the weights of the ties or struts, so that practically the latter are in equilibrium under the action of force at their ends only. These two forces, therefore, cannot balance unless they each act along the strut or tie. If it is necessary to take into consideration the weights of the parts of the framework, we may suppose half the weight of any part collected at each end, and attached to pivots binding the bars together. This will produce the same action on the pivots and other connections as the actual weights do. The problem is thus reduced to one of the same kind as when the bars are weightless. As graphical methods of determining stresses are of very great importance, and the first example given in § 91 is perhaps difficult, the easier example in iii. 36 is here worked out.

Draw the figure to the scale of say 1 inch per foot. The reactions on the supports at A and B are each 1 ton; at A the forces are parallel to DA, AC, CD. Hence CD represents the weight of 1 ton, and in same proportion AC, AD the other forces. The scale is, therefore, $\frac{1}{3}$ ton per inch.



Hence stress in AC is $\frac{10}{3} = 3\frac{1}{3}$ tons,

in AD is $\frac{8}{3} = 2\frac{2}{3}$ tons.

Draw A*d*, A*c* parallel to DB, *c*B. Clearly A*c* = AC, and the sides of AC*c* are parallel to the forces acting at C, *i.e.* that in CB = that AC = $3\frac{1}{3}$ tons.

The length of C*c* when measured is 15·3 inches,

∴ force along CD = weight of 5·1 tons.

So also D*d* is the resultant of forces in AD, BD. It is 3·1 tons, as it ought to be.

As an example, when the weights are supposed to be too large to be neglected, let us take the same framework without any weight supported at D, and the weight of the "bars" to be 4 lbs. per linear foot.

Thus the problem is the same as if we had

$$20 + 16 = 36 \text{ lbs. at A and B,}$$
$$20 + 20 + 6 = 46 \text{ at C,}$$
$$16 + 16 + 6 = 38 \text{ at D.}$$

The pressure on the wall at A or B is $\frac{1}{2}(80 + 64 + 12) = 78$ lbs. The former two (36 at A and B) only increase the stress on the walls and do not affect the stresses in the framework. We have then to consider only a pressure $78 - 36 = 42$.

In the figure then, now CD represents 42 lbs., and the scale is 14 lbs. per inch.

Hence stresses in AC, CD are 140 lbs. and 112 lbs.

At C we have an applied force 46 lbs. down.

Draw C*e* down and equal $\frac{46}{14}$ inches.

Then the forces at C are *c*A + AC + C*e* + *ec*.

The force along the tie CD is therefore

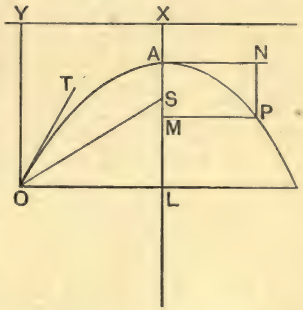
$$ec = Cc - Ce = 14\ (15{\cdot}3 - \tfrac{46}{14}) = 169{\cdot}2 \text{ lbs.}$$

The actual force which AC produces at A is compounded of the 20 lbs. which is half the weight of the bar applied at A and the force CA. Draw Af vertically downwards and equal to $20 \div 14 = 1\frac{3}{7}$ inches. The force which the rod AC exerts at C is then proportional to Cf and Cf = 11·14 inches,

$$\therefore \text{ force} = 11{\cdot}14 \times 14 = 155{\cdot}96 \text{ lbs.}$$

Addition to §§ 140, 141, 142

140 *a.* That the path is a parabola may be shown as follows. Whatever the velocity of projection it may be regarded as being produced by a fall from a certain height. Draw a horizontal line, XY, at a height above the point of projection (O) equal to this height. Then in the subsequent motion the velocity at any point of the path will be that due to a fall to the point from this horizontal line. At some point of its path the motion will be horizontal. After passing through this point the path will clearly be similar to that it had previously traversed. We may then consider the particle as projected horizontally from this point with the velocity due to a fall from XY. Let A be the vertex of the path. The horizontal velocity of projection is $\sqrt{(2g\,.\,AX)}$. If P be the position t seconds later,



$$PM = t\sqrt{(2g\,.\,AX)}$$
$$PN = \text{fall from rest from } AN = \tfrac{1}{2}gt^2,$$
$$\therefore PM^2 = 4AX\,.\,PN,$$

that is, the path is a parabola in which the line XY is the directrix.

Let S be the focus, and O the original point of projection, OT the direction of projection, V the velocity, and β the angle OT makes with the vertical.

Then $$OS = OY = \frac{V^2}{2g};$$

OT bisects SOY,

$$\therefore \text{SOY} = 2\beta,$$
$$\text{SOL} = 90 - 2\beta.$$

Also AL = height due to the vertical component of the velocity

$$= \frac{V^2 \cos^2 \beta}{2g}.$$

The range is $$2\text{OL} = 2\text{OS} \cos \text{SOL}$$
$$= 2\frac{V^2}{g} \sin 2\beta.$$

Since SO depends only on the velocity of projection and not on the direction, the range is clearly greatest when OS = OL or S is at L. In this case OT bisects a right angle and the angle of projection must be 45°

MISCELLANEOUS EXAMPLES

[*Many of the following examples require a knowledge of trigonometry for their solution.*]

1. Two perfectly elastic balls whose masses are m, m' moving in the same direction strike each other. If the hindmost ball is reduced to rest by the blow, show that its velocity must have been more than double that of the other.

2. Particles are placed at A, B, C, and the centres of gravity of A, B and of B, C are known. Show how to determine that of A and C.

3. A heavy particle is placed at a distance of 24 inches from a point of suspension. At what distance must another particle treble its weight be placed so that when connected together they may oscillate in 1 second, the length of the second's pendulum being taken as 39·14 inches.

4. A right cone whose vertical angle is θ and weight W is placed with its vertex on a smooth horizontal plane and slant side vertical. Show that a couple whose arm is the slant side, and of which each force is $\frac{3}{8}W \sin\theta$, will keep the cone in equilibrium.

5. Find the centre of gravity of one of the figures formed by dividing an equilateral and equiangular hexagon by a line joining opposite angular points.

6. Two forces P_1, P_2 act at a point. Find the angle between their directions that the magnitude of their resultant may equal the arithmetical mean between them.

7. The radii of two spheres are r and r_1, and their densities ρ and ρ_1. Find the condition that their centre of gravity may lie in the point of contact when they touch each other.

8. A, B, C are three equal spheres whose common elasticity is $\frac{1}{2}$, and whose centres are in a straight line; B and C are contiguous and at rest; A, moving with a velocity of 1 foot per second, impinges upon B. What will be the positions and velocities of the spheres one second after the impact?

9. Two equal weights P and Q are connected by a thread; P is placed at the bottom of a smooth plane of 30°; the thread is passed over a pulley at the top of the plane and Q hangs freely at its extremity. Compare the time in which P will be drawn to the top with the time in which if separated from Q it would fall down the plane.

10. Parallel forces act at the corners of a triangle, each force being proportional to the length of the opposite side. Show that if the forces all act in the same direction their resultant passes through the centre of the circle inscribed in the triangle.

11 Three balls of equal elasticity and equal radii, the masses of which, taken in order, are as 1 : 2 : 4, rest upon a smooth table, their centres being in a straight line; a velocity is impressed upon the first ball so as to cause it to impinge directly upon the second. Prove that the first ball will bump the second more than once if the coefficient of rebound of the balls be less than $\frac{1}{2}$.

12. ABCDEF is a regular hexagon formed of six rods jointed together; it is suspended from the middle point of AB, and AD, BE are connected by inextensible strings. Find the tension of these strings.

13. If the acceleration of a falling body be the unit of acceleration, the velocity acquired in 1 minute the unit of velocity, and the unit of mass that of a body in which a force of 1 lb. weight would produce an acceleration of 1 yard per minute per minute, find the unit of mass.

14. A railway train of mass 100 tons is moving at 20 miles per hour. What horse-power would be required to impart to it this velocity in five minutes from starting, in addition to overcoming the resistances supposed uniform and equal to 12 lbs. per ton?

15. A projectile is discharged with velocity V at an elevation α, and n seconds afterwards another is discharged after it so as to strike it. Prove that, if V', α' be its velocity and elevation,

$$2VV' \sin(\alpha - \alpha') = n(V \cos \alpha + V' \cos \alpha')g.$$

16. If two small perfectly elastic balls are projected at the same instant with velocities which are as $2 \tan \beta : \sqrt{1 + 4 \tan^2 \beta}$, one up an inclined plane β and the other in the same vertical plane but in a direction making an angle θ with the plane such that $2 \tan \theta = \cot \beta$, prove that they will return to the point of projection at the same instant.

17. Three balls A, B, C of masses proportional to 17, 1, 4 respectively are lying in a straight line on a smooth horizontal table. If A impinge on B at rest, and then B impinge on C at rest, and then B recoils and meets A again, prove that the velocity of B after impact on A = its velocity after A impinged upon it.

18. One end of a string l feet long is fastened to a point on a fixed smooth vertical rod, the other to a small ring of mass m which slides on the rod; another mass m' is fastened to a point of the string and revolves with uniform velocity v in a horizontal circle so that the two parts of the string make angles α and β with the rod (α above β). Prove that $v^2 = \frac{(m+m')\tan\alpha + m\tan\beta}{m'(\operatorname{cosec}\alpha + \operatorname{cosec}\beta)} gl$.

19. A particle is projected from a point on a level plain at the foot of a hill, with a velocity u parallel to the hill in a principal plane and strikes the hill at right angles. Find the distances from the foot of the hill of the point of projection and the point where the particle strikes.

20. Weights $W+w$, $W+2w$, $W+3w$, $W+4w$ are placed at the angular points of a square. Prove that the centre of gravity of these weights is distant $w/(2W+5w)$ of the side of the square from its centre.

21. A ball of elasticity e is projected from a point A in the circumference of a circle, and after impinging at three points on the circumference returns to A. Show that the four tangents of incidence are $e^{\frac{3}{2}}, e^{\frac{1}{2}}, e^{-\frac{1}{2}}, e^{-\frac{3}{2}}$.

22. Bodies are projected vertically downwards from heights h_1, h_2, h_3 with velocities v_1, v_2, v_3 respectively, and they all reach the ground at the same moment. Show that $\frac{h_2-h_3}{v_2-v_3} = \frac{h_3-h_1}{v_3-v_1} = \frac{h_1-h_2}{v_1-v_2}$.

23. A ball falling from the top of a tower had descended a feet when another was dropped at a distance b feet from the top of the tower. Show that, if they reach the ground together, the height of the tower is $\frac{(a+b)^2}{4a}$ feet.

24. A particle is projected horizontally from the top of a tower 64 feet high with a velocity of 50 feet per second. Show that it will strike the ground after 2 seconds at a distance of 100 feet from the foot of the tower measured horizontally, with a velocity of 64 feet per second, at an acute angle whose tangent is $\frac{32}{25}$.

25. A uniform rod CD rests over the rim of a fixed bowl at a point A, the lower end C of the rod being in contact with the inner surface of the bowl; the axis of the bowl, which is in the form of a segment of a sphere of which O is the centre, is vertical. If $2a$ be the length of the rod, and r the radius of the sphere, prove that the position of the rod is determined by the equation $a\cos\phi = 2r\cos(\alpha + 2\phi)$, where α, ϕ are the respective inclinations of OA, CD to the horizon.

26. The centre of gravity of three thin uniform rods, placed end to end so as to form a triangle, is at the centre of the circle inscribed in

the triangle. If a, b, c be the lengths of the rods and λ, μ, ν their respective densities, prove that

$$\lambda : \mu : \nu :: \frac{b+c-a}{a} : \frac{c+a-b}{b} : \frac{a+b-c}{c}.$$

27. A body is supported on a rough plane, inclined to the horizon at an angle α, by a force acting along the plane in a direction at right angles to horizontal lines in the plane. Supposing the greatest magnitude of the force to be n times its least magnitude, prove that the coefficient of friction is $= \frac{n-1}{n+1} \tan \alpha$.

28. A particle descends from rest through a certain space down a rough plane, the inclination of which is α and of which the coefficient of friction is $\tan \epsilon$; it is then projected directly up the plane with a velocity equal to that acquired in its descent. Prove that the ratio of its greatest ascent to that of its descent $= \sin(\alpha - \epsilon) : \sin(\alpha + \epsilon)$.

29. A body is projected at an angle α to the horizon, so as just to clear the summits of two vertical posts of equal height a, at a distance b from one another, the point of projection being in the horizontal line through the lower ends of the posts. Prove that the horizontal range of the body $= 2a \cot \alpha + (4a^2 \cot^2 \alpha + b^2)^{\frac{1}{2}}$.

30. A particle is projected at an angle α to the horizon from a point in a horizontal plane. Prove that when it arrives again at the horizontal plane its distance from the point of projection is greater than at any preceding moment, provided that $\cot \alpha$ is greater than the least value of $\cos \theta \tan \frac{\theta}{2}$.

31. A man, who has just dined at an hotel, stands on the floor of the lift, which is descending with an acceleration f; his feet press the floor with a force equal to his weight just before dinner. Supposing W to have been his weight before he had dined, find the weight of what he has consumed.

32. A ball, of which e is the modulus of elasticity, after dropping through a height h, strikes at a point A a plane inclined to the horizon at an angle α, and afterwards passes through a point B in a horizontal line through A. Find the time of moving from A to B, and show that the problem is impossible if $e < \tan^2 \alpha$.

33. A square picture-frame is suspended by a string attached to a point whose distance from the nearest corner is one-third the side of the square. Find the stresses at the joints.

34. Three equal and equally elastic balls start simultaneously from three corners of a square in the direction of the centre with the same velocity. Find the motion of each after impact, and show that, if one ball be reduced to rest, the elasticity must be perfect.

35. If a ball after falling freely under the action of gravity for 1

second is brought to rest with uniform acceleration in the space of 1 inch, how long will this take?

36. A heavy sphere rests on three pegs A, B, C situated in one horizontal plane. Prove that if P, Q, R be the pressures on the pegs A, B, C respectively,

$$P : Q : R :: \text{area BOC} : \text{area COA} : \text{area AOB},$$

where O is the centre of the circle circumscribing the triangle ABC.

37. A uniform sphere rests upon three equal spheres of the same material placed in contact upon a rough horizontal plane. Prove that the lower spheres will just begin to slide if the coefficient of friction between the spheres and the plane is given by

$$\mu(b^3+3a^3)\{(3b^2+6ab-a^2)^{\frac{1}{2}}+(a+b)\sqrt{3}\}=2ab^3,$$

where a is the radius of one of the lower spheres, and b is the radius of the upper sphere.

38. A uniform circular disc, whose weight is w and radius a, is suspended by three vertical strings attached to three points on the circumference separated by equal intervals; a weight W may be put down anywhere within a concentric circle of radius ma. Prove that the strings will not break if they can support a tension $=\frac{1}{3}(2m\text{W}+\text{W}+w)$.

39. A bowler at cricket delivers the ball at a height of 6 feet above the ground; the ball reaches a height of 10 feet and it strikes the foot of the stumps at a distance 22 yards. Find the velocity of delivery in miles per hour.

40. A particle placed at the centre O of a circle is acted on by forces P, Q, R in the directions OA, OB, OC, where A, B, C are the angular points of a triangle circumscribed about the circle. Prove that if the particle be in equilibrium,

$$P^2 : Q^2 : R^2 :: a(b+c-a) : b(c+a-b) : c(a+b-c),$$

where a, b, c are the sides of the triangle.

41. The moon's distance from the earth is about 239,000 miles, and she revolves once round the earth in about $27\frac{1}{3}$ days. Find her acceleration relatively to the earth with feet and seconds as units.

42. Three equal uniform rods of equal weight are jointed together to form an equilateral triangle, which is suspended from the middle point of a side. Show that the action at one of the upper joints is $\sqrt{13}$ times that at the lower joint.

43. Two equal particles are connected by a string and movable in a smooth vertical circular tube, the string lying in the tube and subtending an angle $2\tan^{-1}2$ at the centre; the particles are just disturbed from the position of equilibrium. Show that the pressure between one particle and the tube changes sign when the radius to

the middle point of the string makes an angle $2 \tan^{-1}\frac{1}{5}$ with the vertical.

44. Two inelastic spheres m, m' are in contact, and m receives a blow through its centre in a direction making an angle α with the line of centres. Show that the kinetic energy generated is less than if m had been free in the ratio of $m + m' \sin^2 \alpha : m + m'$.

45. Three equal balls are placed in contact on a smooth horizontal table; let A, B, C be their centres, AD is the perpendicular to BC, A is withdrawn to a point in DA produced and projected in the direction AD. Show that (1) if the balls be perfectly elastic, A will move back with one-fifth of its original velocity; and (2) that if they be perfectly inelastic, the three will be in a straight line when A has described after impact a distance ten times AD.

46 A picture is hung by a single string of length l which passes through two rings (supposed smooth) at a distance a apart symmetrically attached to it; the ends of the string are fastened to rings which can slide along a horizontal rod at the top of the room. Show that the greatest distance which these rings can be apart on the rod is $a + (l - a) \sin \lambda$, where λ is the angle of friction.

47. A uniform rod of weight W rests in a limiting position of equilibrium in a vertical plane; one end rests on a rough horizontal plane and the other on an equally rough plane inclined at an angle α to the horizon. If $\tan \lambda$ be the coefficient of friction and θ the inclination of the rod to the horizon, prove that $\tan \theta = \frac{\sin (\alpha - 2\lambda)}{2 \sin \lambda \sin (\alpha - \lambda)}$, and find the normal pressure on the inclined plane.

48. A particle is projected in a vertical plane perpendicular to the line of intersection of a given inclined plane with the horizontal plane through the point of projection. Find the range on the given inclined plane, measured from this line of intersection. If the angle of projection be equal to the inclination to the horizon of the inclined plane, prove that the range is $a\left\{\sqrt{\frac{2u^2}{ga}\tan \alpha} - \sec \alpha\right\}$, u being the velocity, α the angle of projection, and a the distance of the point of projection from the above line of intersection.

49. A heavy semicircular disc is in equilibrium in a vertical plane, with its straight edge supported by a peg, and its curved edge resting against a vertical wall, the wall and peg being smooth; a weight is hung from the lower corner of the disc, and the peg occupies the position required for equilibrium if the disc were without weight. Show that if the straight edge be inclined at an angle α to the horizontal,

$$1 + \tan (\tan \alpha) = 0.$$

50. In a tunnel the sparks from the chimney of the engine of a train

are noticed to pass the carriage windows, making an angle $\tan^{-1}\frac{1}{4}$ with the horizon, and they are known to have fallen 4 feet from rest. Assuming that they move in vertical lines and that the effect of the air may be neglected, find the speed of the train.

51. In the system of pulleys in which only one string is employed, if a weight of mass m be more than is necessary to lift a weight of mass M, and the system be left to itself, find the accelerations of m and M.

52. A square lamina in a vertical plane is resting on an inclined plane and is acted on by a force parallel to the plane and acting down it, whose point of application is one of the upper angular points. Show that the lamina will turn over before it slides if $\mu > \frac{1}{2}(1 + \tan\alpha)$, where α is the angle of the plane.

53. The time of quickest descent from a vertical circle, radius b, to another fixed vertical circle in the same plane, radius a, is constant and equal to T. Show that the locus of the centre of the first vertical circle is a circle whose radius $= a + b + \frac{1}{2}g\mathrm{T}^2$.

54. A triangular lamina ABC hangs at rest from the point A. If $\mathrm{AB} = c$, $\mathrm{AC} = b$, and $\triangle$ represent the area of the lamina, prove that the tangent of the inclination of BC to the vertical $= \dfrac{4\triangle}{b^2 \sim c^2}$.

55. A given rectangular plank is placed upon a smooth inclined plane so that its two ends are horizontal; a given insect is placed upon the plank at the middle point of its lower end. Supposing the insect to start off at once up the middle line of the plank and that the plank moves with a given uniform acceleration, find how long it will take the insect to reach the upper end of the plank.

56. AB, BC, CD are equal rods jointed at B, C; A and D rest on a horizontal plane, and B and C are joined to the middle points of CD and AB by equal strings of such a length that the angles at B and C are each 120°. Show that the action at B on BC makes an angle $\tan^{-1}\left(\dfrac{10}{\sqrt{3}}\right)$ with the vertical.

57. A rectangular frame ABCD consists of four bars without weight freely jointed, the bar AD being held fast in a vertical position. If the weight W is placed on the upper horizontal bar AB at a given point and the frame is kept in a rectangular form by a diagonal string AC, find the tension of the string. Show that this tension is unaltered if the weight be placed on the lower bar CD vertically under its former position.

58. A rough circular cylinder of weight W lies with its axis horizontal on a plane whose inclination to the horizon is α, while a man of weight W′ stands with his body vertical upon the cylinder and keeps it at rest. If the man's feet are at A, and a vertical section through

A touch the plane in B and the friction be sufficient to prevent sliding, then the angle θ subtended by AB at the centre of the section will be given by $\frac{\sin(\theta+\alpha)}{\sin \alpha}=1+\frac{W}{W'}$.

59. A weight W_1 is placed on a rough table and has tied to it a light string which hangs over the edge and supports a pulley whose weight is W_2; round this pulley hangs another light string which has attached to it two equal weights W_3, W_4. Find the accelerations of the different parts of the system. Find also the relation between the weights and the coefficient of friction that W_1 may just remain at rest.

60. A mass of 1 gram vibrates through a millimeter on each side of its middle position 256 times per second. Assuming the motion to be simple harmonic, find the maximum force upon the particle in grams weight, taking $g=981$ centimeters per second per second.

61. A man wishes to overturn an upright cylinder standing on a horizontal plane by means of a tension exerted along a string attached to a point of the cylinder, and passing through a smooth ring which is fixed at height b above the horizontal plane, and at a horizontal distance a from the cylinder. If the magnitude of the tension be the least possible, prove that the point of the cylinder to which the string is attached is at a height $(a^2+b^2)/b$ above the horizontal plane.

62. A particle is projected from the lowest point O of a hollow sphere in such a direction and with such velocity that it strikes the sphere at right angles at some point P. If α, θ be the angles which the direction of projection and the line OP respectively make with the horizontal, show that $\tan\alpha=\cot\theta+2\tan\theta$.

63. Show that the difference in the apparent weights of a lb. carried in a train moving at the rate of 60 miles per hour, first west and then east, along the circle of latitude $\cos^{-1}\frac{9}{16}$ is about 5 grains weight; the earth is to be considered spherical, and its attraction on the mass of a lb. to be 32 British absolute units of force.

64. Four rods are hinged at the angular points so as to form a parallelogram ABCD, whose sides are in the ratio of 7 : 1. A and C are joined by a string of such a length that the frame forms a rectangle which is hung up by the angle A. Show that if W be the weight of the frame, the tension of the string is $\frac{1}{2}$W while the action at each of the hinges at B, D is 7W/160. If the frame were hung up by the angle B, what would the tension of the string be?

65. Any triangular lamina ABC has the angular point C fixed, and is capable of free motion about it; a blow is struck at B perpendicular to the plane of the triangle. Show that the instantaneous axis is that trisector of the side AB which is farthest from B.

66. Six thin uniform rods of equal length and each of weight W

are connected by smooth hinge joints at their extremities so as to constitute the six edges of a regular tetrahedron ; one face of the tetrahedron rests on a smooth horizontal plane. Prove that the longitudinal strain of each of the rods at the base is $W/2\sqrt{6}$.

67. A smooth circular tube of mass M has placed within it two equal particles of mass m, which are connected by an elastic string whose natural length is two-thirds of the circumference ; the string is stretched until the particles are in contact, and the tube is placed flat on a smooth horizontal table and left to itself. Show that when the string arrives at its natural length the actual energy of the two particles is to the work done in stretching the string as

$$2(M^2+Mm+m^2) : (M+2m)(2M+m).$$

68. A ladder AB rests against a smooth wall at B, and on a rough horizontal plane at A ; a man whose weight is equal to that of the ladder climbs up it. Prove that the frictions at A in the two extreme cases in which the man is at the two ends of the ladder are in the ratio of 3 : 1. Consider the case when the weight of the ladder can be neglected in comparison with that of the man.

69. Two buckets W and W′ are suspended by a fine inelastic string placed over a fixed pulley ; at the centre of the base of W a frog of weight w is sitting ; at an instant of instantaneous rest of the buckets the frog leaps vertically upwards so as to just arrive at the level of the rim of the bucket. Show that the ratio of the absolute length of the frog's vertical ascent in space to the depth h of the bucket is as $2W'(W+W') : (W+W'+w)^2$, and that the time which elapses before the frog again arrives at the base of the bucket $=2\sqrt{\frac{h}{g}\left(\frac{W+W'}{W'}\right)}$.

70. A smooth heavy particle is projected from the lowest point of a fixed circular arc, whose plane is vertical, up the curve, with a velocity due to the diameter. Prove that if the length of the arc be such that the range of the particle on the horizontal plane through the point of projection is the greatest possible, this range $=a\sqrt{9+6\sqrt{3}}$, where $a=$ radius of the arc.

71. A particle is suspended by three equal strings of length a from three points forming an equilateral triangle of side $2b$ in a horizontal plane. If one string be cut, the tension of each of the other strings is instantaneously changed in the ratio of $3a^2-4b^2 : 2(a^2-b^2)$.

72. Two rods AB, BC of equal weight but of unequal lengths are hinged together at B and their other extremities are attached to two fixed hinges A and C in the same vertical line. Prove that the direction of the reaction at B bisects the straight line AC.

73. Three equal weights are attached to a string of length $4a$, one at its middle point and the others half way between it and the ex-

tremities, which are attached to two points A, B in a horizontal line at a distance $a(\sqrt{3}+1)$ from each other. Find the position of equilibrium.

74. An elastic string of length a has a heavy particle of weight W attached to one end, and the other end is fastened to a peg at a height h from the ground; the particle is dropped from the peg and in its fall breaks the string and reaches the ground with $(1/n)$th of the velocity that it would have had if free. If W′ be the greatest weight which similarly dropped would not break the string and if T be the breaking tension, show that $\frac{2h}{T}=\frac{h}{W'}-\frac{a}{W\left(1-\frac{1}{n^2}\right)}$.

75. The hind-wheel of a carriage has a radius $=b$, and the fore-wheel a radius $=a$, and the horizontal distances between the centres $=c$; a particle of mud leaves the highest point of the hind-wheel and strikes the fore-wheel at the extremity of a horizontal diameter. Show that the rate of travelling

$$=(c-a)\sqrt{\frac{g}{2(2b-a)}}.$$

76. A railway train is running smoothly along a curve at the rate of 60 miles an hour, and in one of the cars a pendulum which would ordinarily oscillate seconds is observed to oscillate 121 times in two minutes. Show that the radius of the curve is $=2\frac{1}{2}$ furlongs nearly.

77. Equal rods AB, BC are jointed at B, and AC is a string of such length that ABC is a right angle; A is a given fixed hinge about which the whole can turn, and C rests on a given horizontal table. Find the action of the joint B and the tension of the string.

78. A ball moving on a smooth horizontal plane impinges in succession on two vertical planes of the same material which are at right angles to each other. Show that the directions of motion before the first impact and after the second impact are parallel.

79. Two small smooth unequal spheres are placed in a fixed hemispherical bowl. When in equilibrium under the action of gravity, find the inclination to the horizon of the line joining their centres.

80. A uniform beam of length $2a$ rests in a horizontal position on two props; the equilibrium is just on the point of being disturbed, (1) if a weight P is suspended from either end of the beam, (2) if a weight W is suspended from the middle point and a weight Q from either end. Determine the weight of the beam and the distance of the props from its extremities.

81. Two unequal rods AC, BC hinged at C, rest in a vertical plane with their extremities on a smooth horizontal plane connected by a string AB. Show that the tension of the string $=W\frac{\cos A\cos B}{\sin C}$, where

W is half the sum of the two weights and A, B, C are the angles of the triangle.

82. Three weights are placed at the three vertices of a given triangle, and it is found that their centre of gravity coincides with the centre of the circle inscribed in the triangle. Determine the ratios of the weights to one another.

83. In the first system of pulleys, if the weights of the n movable pulleys are in G, P with a common ratio $\frac{1}{2}$ commencing from the bottom, prove that $P = \frac{1}{2^n}(W + nw)$ where w = the weight of the lowest pulley.

84. In the third system of pulleys, if the weights of the $n-1$ movable pulleys are in G, P with a common ratio 2 commencing from the bottom, prove that $W = P(2^n - 1) + w(n \,.\, 2^{n-1} - 2^n + 1)$.

85. In the last question, if $w = P$, prove that the mechanical advantage is n times the mechanical advantage of the first system of pulleys when the weights of the pulleys are neglected and there are $(n-1)$ movable pulleys.

86. A uniform rod of weight W is supported in equilibrium by a string of length $2l$ attached to its ends and passing over a smooth peg. If a weight W′ be now attached to one end of the rod, prove that a length $\frac{lW'}{W + W'}$ of the string will slip over the peg.

87. Two balls of equal size and of unequal masses m and mk, where $k > 1$, are projected with equal velocities from the middle points of the opposite ends of a billiard-table of length $2a$ so as to impinge directly. If e be the coefficient of elasticity between the balls and between either ball and the cushion, show that if $e > \frac{1}{2}(k - 1)$, the second collision between the balls will take place at a distance $\frac{a(1+e)^2(k-1)}{e(k+1)}$ from the centre of the table, the radii of the balls being neglected.

88. Two particles of masses m, m' are projected with equal velocities and at the same angle in opposite directions from two points in the same horizontal plane, so that the particles strike one another. If the two coalesce, find the greatest height which the particle attains.

89. A smooth rod of length $2a$ has one end resting on a plane of inclination α to the horizon, and is supported by a horizontal rail parallel to the plane at a distance c from it. Prove that the inclination θ of the rod to the plane is given by $c \sin \alpha = a \sin^2 \theta \cos(\theta - \alpha)$.

Prove also that the equilibrium is stable or unstable according as the rod rests on a point of the plane above or below the foot of the perpendicular from the rail on the plane.

90. Two bodies of equal weights are attached to the ends of a fine inelastic string which hangs over a smooth pulley; one of the bodies is

hard and the other melts so that the decrement of its mass varies as the time. If f, f', f'' be the accelerations of the bodies at the end of times t, t', t'', reckoned from the beginning of motion, prove that

$$\frac{1}{f}\left(\frac{1}{t'}-\frac{1}{t''}\right)+\frac{1}{f'}\left(\frac{1}{t''}-\frac{1}{t}\right)+\frac{1}{f''}\left(\frac{1}{t}-\frac{1}{t'}\right)=0.$$

91. A number of equally elastic balls projected from a point return to the point after striking a vertical wall. Prove that the tangents of their angle of projection are proportional to the squares of their times of flight.

92. A sphere of radius a, whose centre of gravity is at a distance b from its geometrical centre, rests in limiting equilibrium on a rough plane inclined at an angle α to the horizon. If it can be turned through an angle 2θ without disturbing the equilibrium, prove that

$$b\cos\theta = a\sin\alpha.$$

93. A spherical shell, external radius 70 feet, internal radius 35 feet, is suspended by a point in its circumference, and oscillates so as to keep its centre in the same vertical plane. Show that the length of the simple equivalent pendulum is nearly 101 feet.

94. The points A, B, C, D are the angular points of a square; AB and CD are two equal similar rods connected by a string BC equal in length to either rod; the point A receives an impulse in the direction AD. Show that the initial velocity of A is seven times that of D.

95. A perfectly rough and rigid hoop rolling down an inclined plane comes in contact with an obstacle in the shape of a spike. Show that if the radius of the hoop $=r$, the height of the spike $=\frac{1}{2}r$ above the plane, and $V=$ the velocity just before impact. then the condition that the hoop will surmount the spike is that

$$V^2 > \frac{16}{9}gr\left\{1-\sin\left(\alpha+\frac{\pi}{6}\right)\right\},$$ α being the angle of the plane.

96. In the last question show that the hoop will not remain in contact with the spike at all unless $V^2 < \frac{16}{9}gr\sin\left(\alpha+\frac{\pi}{6}\right)$; and if it does, it will leave it when the diameter through the point of contact makes an angle with the horizon

$$=\sin^{-1}\left\{\frac{9}{32}\cdot\frac{V^2}{gr}+\frac{1}{2}\sin^2\left(\alpha+\frac{\pi}{6}\right)\right\}.$$

97. An inelastic rod which can turn about one end fixed in a vertical plane falls from the horizontal position, and when it is inclined at 45° to the horizon impinges at its middle point on a fixed peg. Determine the impulse on the axis.

ANSWERS TO EXAMPLES

ANSWERS

I.

(**1.**) $22\frac{8}{11}$ seconds. (**2.**) 66 feet per second ; 1320 yards per minute. (**3.**) 3657·48 centimeters per minute. (**4.**) 18·317 ; 1582·62. (**5.**) 17,960,011 kilometers per minute. (**6.**) 8 minutes $9\frac{23}{93}$ seconds. (**7.**) 17·338 and 8·669 miles per minute. (**8.**) $13\frac{7}{11}$; $27\frac{3}{11}$. (**9.**) 5 seconds. (**10.**) 35 seconds ; 2310 feet ; 3080 feet. (**11.**) $\frac{10}{11}$ minute ; $1\frac{1}{4}$ minute ; $20\frac{5}{11}$ seconds. (**12.**) 640 feet ; 1120 feet. (**13.**) 12 minutes ; 3168 feet from first. (**14.**) $2\frac{1}{2}$ miles. (**15.**) 60 ; 10 hours 25 minutes. (**16.**) $2166\frac{2}{3}$; $83\frac{1}{3}$ miles per hour. (**17.**) ·628 . . . inch per minute. (**18.**) 1 : 20 : 360. (**19.**) 1 foot. (**20.**) $(1 - d/D)V$. (**21.**) $\frac{22}{5\pi}$ revolutions per minute. (**23.**) 38,400 ; 975·228 . . . (**24.**) 4,976,640,000. (**25.**) Equal. (**26.**) 200 feet per second per second. (**27.**) 90 yards ; $20\frac{2}{5}$ yards. (**28.**) 17 centimeters per second. (**29.**) 8 yards per second per second ; 24 yards per second. (**30.**) $4\frac{1}{2}$ feet per second per second ; 576 feet. (**31.**) $6\frac{1}{4}$ seconds. (**32.**) $5\frac{13}{15}$ feet per second per second. (**33.**) 22 feet per second per second. (**34.**) 21 feet per second ; 30 feet per second per second. (**35.**) 256 feet per second ; 1024 feet. (**36.**) 100 yards. (**37.**) 15,625 feet. (**38.**) At starting-point ; $62\frac{1}{2}$ seconds after first starts ; no. (**39.**) 96 feet per second ; at rest. (**40.**) No. (**41.**) 10 seconds after projection ; 3600 feet below. (**42.**) 75 miles per hour. (**43.**) In $3\frac{1}{3}$ hours ; $38\frac{8}{9}$ miles ; $13\frac{1}{3}$ and $21\frac{2}{3}$ miles per hour. (**44.**) 5 seconds ; 64 feet per second. (**45.**) 450 miles per hour per hour. (**46.**) 484 feet. (**47.**) $172\frac{4}{45}$ feet ; $5\frac{13}{15}$ seconds. (**48.**) 1360 yards.

II.

(**1.**) 5·80078125 tons. (**2.**) 21·5 ; 19·3 ; 10·5 ; 2·6 kilograms. (**3.**) 6·16 . . . $\times 10^{21}$ tons. (**4.**) $55\frac{47}{55}$ square feet. (**5.**) 8 kilograms. (**6.**) 135 lbs. (**7.**) $\frac{56}{275}$ cubic foot ; $\frac{20}{57}$ cubic decimeter. (**8.**) ·8. (**9.**) 7·168.

(10.) 7·623 . . . **(11.)** $6\frac{9}{13}$ gallons. **(12.)** 10·8768. **(13.)** $4\frac{15}{16}$ feet per second; $1\frac{15}{16}$ foot per second in direction of 5·5. **(14.)** In direction of first. **(15.)** $14\frac{2}{7}$ miles per hour. **(16.)** $8\frac{8}{9}$ lbs. **(17.)** $37\frac{1}{3}$. **(18.)** 3 centimeters per second. **(19.)** 11·6 centimeters per second. **(20.)** $26\frac{11}{14}$ feet per second. **(21.)** $128\frac{4}{7}$ feet per second. **(22.)** No; no. **(23.)** $2\frac{11}{13}$; $8\frac{49}{104}$ feet per second; $6\frac{9}{13}$ reversed; $6\frac{45}{104}$ reversed. **(25.)** $\frac{1}{105}$. **(26.)** 2 feet per second; $c=\frac{5}{7}$. **(27.)** $\frac{1}{8}$; one double the other.

III.

(1.) 7 lbs. 9 oz.; 8 lbs. 2 oz.; 7 lbs. **(2.)** 7031·25 feet. **(3.)** $3\frac{1}{3}$ feet per second per second. **(4.)** 2·24. **(5.)** $46{,}296\frac{8}{27}$. **(6.)** $4\frac{11}{16}$; $234\frac{3}{8}$. **(7.)** 13,825. **(8.)** 6 feet per second per second; $1\frac{2}{3}$ second. **(9.)** 14·4 centimeters per second. **(10.)** $2117\frac{1}{2}$ feet. **(12.)** 25 : 4; $1\frac{1}{14}$ inch per second per second. **(13.)** 9600 yards. **(14.)** 400 yards. **(15.)** $\frac{g}{43}$; $\frac{g}{40}$. **(16.)** $\frac{25}{4}$ weight of bullet. **(17.)** 16 feet; .6 seconds. **(20.)** $4h$. **(21.)** 3·72 seconds; 4·54 seconds. **(23.)** $3{\cdot}16 \times 10^4$ lbs. **(24.)** $3{\cdot}1 \times 10^{-11}$. **(25.)** 26·88 poundals. **(28.)** 704 : 9. **(29.)** 1 millimeter per second per second; 800 dynes. **(30.)** $\frac{g}{5}$; weight of 14·4 grams. **(31.)** $21\frac{1}{3}$ feet; $10\frac{2}{3}$ feet, both down. **(32.)** $\frac{2}{5}g$; $50\frac{2}{5}$ lbs. weight. **(33.)** $\frac{2}{7}g$; weight of 36 lbs. **(34.)** 10 seconds; 2 feet per second. **(35.)** $1\frac{1}{4}$ second; 16 feet per second. **(36.)** Weight of $4\frac{4}{5}$ oz. **(37.)** 9 : 10. **(38.)** $\frac{2}{3}$W. **(39.)** $\frac{g}{8}$. **(40.)** 20 seconds. **(45.)** Same as if g were increased to $g+a$. **(46.)** Acceleration of $\mathrm{M}=\frac{\mathrm{M}(m+m')-4mm'}{\mathrm{M}(m+m')+4mm'}g$; acceleration of m and m' relative to $\mathrm{M}=\frac{2\mathrm{M}(m-m')}{\mathrm{M}(m+m')+4mm'}g$; tension of lower string $=\frac{4\mathrm{M}mm'}{\mathrm{M}(m+m')+4mm'}g=$ twice tension of upper string. **(47.)** Acceleration of 20 lbs. $=\frac{2}{9}g$; of 10 lbs. $=\frac{1}{9}g$; tension = weight of $4\frac{4}{9}$ lbs.; acceleration of 20 lbs. $=\frac{2}{3}g$; of 10 lbs. $=\frac{1}{3}g$; tension = weight of $6\frac{2}{3}$ lbs. **(49.)** 1st, $\frac{5g}{17}$ up; 2nd, $\frac{7g}{17}$ up; 3rd, $\frac{7g}{17}$ down; 4th, $\frac{9g}{17}$ down; $\frac{1}{2}$ tension of upper string = tension of either of lower = weight of $1\frac{1}{17}$ lbs.

IV.

(1.) 24,000 foot-pounds; 160 feet per second. **(2.)** $60\frac{1}{2}$ foot-poundals. **(3.)** 1,893,698,400 foot-pounds per second. **(4.)** 10 feet per second; shot can do 162 that of gun. **(6.)** $18\frac{3}{4}$ foot-pounds; $6\frac{1}{4}$ feet; $102\frac{3}{4}$

foot-pounds. (**7.**) 421,372 ; 13,483,904 (taking $g=32{\cdot}2$). (**8.**) 746·25. (**9.**) 6300 foot-pounds ; $\frac{21}{110}$ horse-power. (**10.**) $\frac{25}{256}$. (**11.**) 134,400 foot-pounds ; $20\frac{4}{11}$ minutes. (**12.**) 288. (**13.**) 64. (**14.**) 60 miles per hour. (**15.**) $12\frac{292}{825}$. (**16.**) Weight of $151{,}029\frac{327}{437}$ lbs. (**17.**) 30. (**19.**) 11,520,000. (**21.**) 3300 foot-poundals per minute $=102{\cdot}5$ foot-pounds per minute in London.

V.

(**1.**) F.P.S.: $\frac{\text{Y.T H.}}{\text{C.G.S.}}=4{,}320{,}000:1$; 13,500 : 7 ; 4500 : 7. (**2.**) Inverse to those in (1). (**3.**) 28 : 5 ; 7 : 75 ; 7 : 1500. (**4.**) 13,825 : 1. (**5.**) 1,356,480,000. (**6.**) 21,600. (**7.**) 746·25. (**8.**) $1/4\sqrt{2}$ second. (**9.**) 1/12,800 foot ; 1/640 second. (**10.**) 32 feet ; 1 second. (**11.**) 38,400 yards. (**12.**) $2c/b^2$ feet ; cd/b seconds. (**13.**) 32 lbs. (**14.**) $[ML^{-1}T^{-2}]$. (**16.**) α, β, δ no ; γ yes. (**17.**) First may be right ; second must be wrong.

VI.

(**1.**) 20 feet. (**3.**) 6·47 feet per minute. (**4.**) 2·35 feet per second. (**5.**) Both $=3\pi/\sqrt{2}$ inches per second. (**6.**) ·011 foot per second. (**8.**) $88\sqrt{3}=152{\cdot}41$ feet per second. (**11.**) W. of S. by θ, where $\tan\theta=\frac{1}{3}$. (**13.**) $\text{Sin}\,\theta=\dfrac{1}{6{\cdot}34}$; $v=12.69$. (**15.**) 2 feet 1 inch per second, making θ with BA where $\tan\theta=3/4$; 3 feet ; 48/25 seconds. (**16.**) i. $u\pm v$; ii. $\sqrt{u^2+v^2\pm uv\sqrt{2}}$; iii. $\sqrt{(u^2+v^2)}$. (**17.**) If θ be the angle, $\cos\theta=(au+bv)/(bu+av)$.

VII (a).

(**1.**) $5\sqrt{2}=7{\cdot}07$ . . . (**2.**) 5·77 . . . (**3.**) 8·7 ; 7·21 ; 130·8 poundals ; 5·06. (**4.**) 30·41 . . . (**5.**) 1:732. (**6.**) 7·66 at 22° with the 3 lbs. ; ·96 at 111° 7′ with the 3 lbs. (**7.**) Same as (6). (**9.**) 5·83 . . . F.P.S. units. (**12.**) 13. (**13.**) 256·12 . . . feet per second. (**14.**) 10 ; 26 lbs. weight. (**15.**) Perpendicular to the 6 lbs. (**16.**) 5 : 4. (**18.**) 7·32. (**19.**) 7·96 at angle θ with the 1 produced backwards, where $\tan\theta=1{\cdot}720$. (**20.**) 10 opposite to the 8. (**22.**) 2BC along BC. (**24.**) Line joining mid points of BC, DE. (**25.**) 2·645 lbs. weight. (**26.**) 4 lbs. weight. (**27.**) $\sqrt{106}=10{\cdot}3$ parallel to line joining A to mid point of OB. (**30.**) If A be the destroyed point, O the centre of force, force $=\mu\times$ distance, force at $P=\mu(n\,.\,OP-AP)=\mu(n-1)O'P$, where O′ divides OA so that $AO'=n\,.\,OO'$.

VII (b).

(**4.**) $10\sqrt{3}$; 20. (**5.**) 15 lbs. weight; 3 feet. (**6.**) 30°. (**7.**) 4 lbs. (**8.**) $\sin\alpha/(\sin\beta - \sin\alpha)$ tons weight. (**9.**) $w/\sqrt{35}$; $6w/\sqrt{35}$. (**11.**) ·293W. (**12.**) 12 lbs. weight. (**14.**) Intersection of lines through the mid point of one diagonal parallel to the other. (**18.**) $l = 2a\text{T}\{(4\text{T}^2 - \text{W}^2)^{-\frac{1}{2}} + (4\text{T}^2 - \text{W}'^2)^{-\frac{1}{2}}\}$. (**19.**) Weight of $m\sqrt{2}$; weight of m.

VII (c).

(**2.**) 27·71 . . . feet per second; 1·08 second; ·866 second. (**3.**) 56 feet. (**4.**) If h be the height of the plane, velocity of projection $= \sqrt{gh}$; velocity at meeting of first $= 0$, of second $= \sqrt{gh}$. (**5.**) 400. (**6.**) $180\frac{68}{75}$. (**8.**) 1·8 mile. (**9.**) $264/\sqrt{7}$ feet per second. (**10.**) $64\frac{8}{27}$ lbs. per ton. (**11.**) 12·49 lbs. per ton. (**15.**) 6 lbs. weight. (**17.**) If l be measured in feet, $v = 4\sqrt{l}$ feet per second. (**18.**) ·183 . . . g; ·683 weight; ·965 weight. (**20.**) 17·32 lbs. weight; 8·66 lbs. weight; 0. (**21.**) Change in vertical pressure $= \frac{1}{10}$ weight; $\frac{1}{9}g$. (**23.**) Line passes through lowest point of circle and at 45° to vertical.

VIII.

(**2.**) 5 feet 4 inches. (**3.**) $21\frac{1}{2}$ lbs. weight. (**5.**) At A is W. BC/AB; at B is W. AC/AB. (**6.**) 12 lbs. weight. (**7.**) 6 inches from 27 oz.; $1\frac{5}{7}$ inch farther. (**8.**) $4\frac{1}{3}$; $3\frac{2}{3}$ tons weight. (**9.**) $3\frac{1}{2}$ cwts.; $2\frac{1}{2}$ cwts.; 1 foot. (**10.**) $\frac{1}{3}$BC from B; $\frac{3}{2}$W. (**11.**) $\frac{1}{4}$ length of beam. (**13.**) Bisects line joining 3 to that trisection of opposite side nearer 2. (**14.**) 1; at distances $2 \times$ side from side 8, 7, 6, 5, 2 and $-2 \times$ side from side 5, 6, 7, 8, 8; 0; no centre. (**15.**) Half way between 1, 4 and 2, 3 and 3/10 side from 4, 3; tension = 10 lbs. weight. (**17.**) Yes. (**18.**) If a be a side, A the area, and P the force along a side, force necessary $= 2\text{A} . \text{P}/a^2$. (**19.**) Vertical through A trisects BC; no. (**20.**) 50 lbs. weight. (**21.**) ·763. (**23.**) Through O such that CAO = CBO = 90° − C. (**24.**) 40 lbs. weight parallel to BC and 3 feet from it. (**26.**) Equal to force along BC.

IX.

(**1.**) $4\frac{10}{27}$ feet from end. (**2.**) $16\frac{4}{13}$ inches from first. (**3.**) $\frac{1}{16}$ length from centre. (**4.**) Mid point of line joining 3 to point of trisection of 1, 2; 1 : 1; 1 : 5; 1 : 2. (**7.**) $\frac{1}{n-1}$th of radius from centre. (**8.**) Intersection of diagonals. (**10.**) $8\frac{1}{3}$ inches from B; $11\frac{1}{3}$ inches from C. (**11.**)

2 oz. (**12.**) Centre of inscribed circle. (**13.**) 15 ; 16 ; 17 lbs. (**14.**) 1·4 foot from junction. (**15.**) $(\sqrt{3}-1)/6$ the base from the base. (**16.**) $\frac{1}{2}$ height of given triangle. (**17.**) $\frac{5}{18}$AD from D, the mid point of BC. (**18.**) 60°. (**19.**) ·866W ; 1·322W. (**20.**) i. $x=14$ inches, $y=20$ inches ; ii. $x=\frac{11}{8}$ foot $=y$; iii. $x=\frac{10}{9}$ foot, $y=\frac{1}{3}$ foot ; 1·122 foot from point of contact. (**21.**) $18\frac{1}{4}$ inches from end. (**22.**) ·3535 inch. (**23.**) Back 29 ; front $85\frac{1}{2}$ lbs. weight. (**24.**) 5 inches from point of contact. (**29.**) Distance from vertex $=\frac{45}{56}$ altitude. (**30.**) Same as the others. (**31.**) $\frac{11}{60}$ the height. (**32.**) If α denote area of face opposite to A, then distance of centre of gravity from $\alpha=\frac{1}{3}(\beta+\gamma+\delta)/(\alpha+\beta+\gamma+\delta)\times$ height of A above α. (**34.**) $\tan\alpha=2$. (**39.**) ·634 . . . $\times a$ inches.

X.

(**1.**) **Reaction = tension = $\frac{1}{4}$ (length of rope)/(height) × weight of beam.** (**3.**) W cot α on vertical ; W/sin α on other. (**4.**) $15a$. (**11.**) $\frac{1}{2}l$. (**12.**) W and $\frac{2}{5}$W (W = weight of triangle). (**13.**) Horizontal, and $=\frac{1}{8}$ weight of rods. (**16.**) $2a$; W$\sqrt{3}$; $a\sqrt{3}$; 3W. (**17.**) Tension = one of the weights. (**21.**) W $\cos\alpha/2\sin(\alpha-\beta)$, where α = inclination of rod and β of string to horizontal, AD > $\frac{1}{2}$AB if B is below C, AD < $\frac{1}{3}$AB if B is above C. (**22.**) $\frac{3}{8}$W. (**23.**) $2ra$, where $\sin\alpha=W/2w$; $w>\frac{1}{2}W$. (**24.**) $T+W/\sqrt{3}$; $T-W/2\sqrt{3}$. (**27.**) If W be weight of a sphere and w of the required one, l = length of string, a = radius of spheres, and r of top one,

$$w\left\{\sqrt{\left(\frac{3l^2+6la-a^2}{3r^2+6ra-a^2}\right)}-1\right\}=3W.$$

(**30.**) (1) $\sqrt{2}$ lbs. weight; 2 lbs. weight.—(2) Top bars 2·121 lbs. ; bottom = 3·53 lbs. ; T = 4 lbs. (**31.**) All = W. (**33.**) On strut = 3·464W ; upper inclined rod 3·464W ; lower inclined rod = 2·31W. (**34.**) 7′ bar, a tie, = 7 tons ; 15′ bar, a strut, = 15 tons ; upright, a strut, = 7·7 tons ; back, a tie, = 13 tons. (**36.**) i. side struts = 1·5 ton ; bottom tie = 1·13 ton ; ii. upper strut $3\frac{1}{3}$ tons ; lower strut $2\frac{2}{3}$ tons ; vertical tie 4·96 tons; iii. top half of strut ·78 ton ; lower half 1·66 ton ; cross strut ·5 ton; two ties = ·67 ton ; horizontal tie = ·57 ton.

XI.

(**1.**) 1·87 . . . feet. (**2.**) 1,280,000 foot-poundals. (**3.**) 108. (**4.**) 148·48. (**6.**) 387,360 foot-pounds. (**7.**) 8·925. (**8.**) 26,345,088 foot-pounds ; 4·435 . . . hours. (**9.**) $88n$R/E. (**10.**) $2\sqrt{gr}$; $\sqrt{\{gr(2-\sqrt{2})\}}$; $\sqrt{\{gr(2+\sqrt{2})\}}$. (**11.**) $\frac{1}{2}$ radius ; $\frac{1}{5}$ radius. (**12.**) 16 feet per second. (**13.**) 6 feet per second. (**14.**) $\sqrt{gl}$. (**15.**) $8\sqrt{\{m/(m+M)\}}$ feet per second. (**16.**) $\sqrt{\{u^2+2mg(r-a)/M\}}$. (**17.**) $\sqrt{\{\mu(r^2-a^2)\}}$. (**18.**) $\sqrt{\{u^2+\mu(a^2-r^2)\}}$. (**19.**) 2 × weight of one rod.

XII.

(**2.**) 15 lbs. weight. (**3.**) 7·2 feet. (**4.**) 160 lbs. weight. (**5.**) Gains $\frac{1}{95}$ lb. if tea is placed in heavier pan, loses $\frac{1}{96}$ lb. in other case. (**6.**) Loses on average $\frac{(a-b)^2}{2ab}$ lb. per lb. (**7.**) $34\frac{5}{6}$ inches from fulcrum. (**8.**) 1 inch from end. (**9.**) 50 oz. (**10.**) 4 (inch lb.-weight units) against the body weighed. (**11.**) 78 cwts. (**13.**) 4 cwts.; $4\frac{4}{55}$ horse-power. (**14.**) a, $2a$, $3a$, . . . (**15.**) $\frac{4}{7}$ radius from last string. (**16.**) $4\frac{4}{7}$ inches from first string. (**17.**) W = P. (**18.**) 7 ; 7. (**19.**) 210 lbs.; $\frac{14}{15}$ lb. ; 221 lbs. ; $\frac{1}{5}$ lb. (**22.**) Q = 4P. (**23.**) 30 about. (**24.**) 1·346 inch. (**25.**) 47,040 foot-pounds ; 2 cwts.; 210 feet. (**26.**) 314·16 . . . (**27.**) $g/25$ down ; $7g/25$ up ; 15·36 lbs. weight. (**28.**) $\frac{1}{7}g$ down ; $8\frac{4}{7}$ lbs. weight ; $\frac{5}{7}g$ up ; $1\frac{5}{7}$ lb. weight. (**30.**) $137\frac{1}{2}$ feet per second.

XIII.

(**2.**) $\frac{1}{12}$. (**3.**) 7,392,000 foot-pounds ; $7\frac{7}{15}$ horse-power. (**5.**) ·25 ; ·6. (**6.**) $\frac{10}{3}$ lb.; 30 lbs. (**8.**) 8 feet. (**9.**) 80 feet. (**10.**) $16\sqrt{5}$ feet per second ; 100 feet. (**11.**) $2\epsilon - \alpha$. (**12.**) Yes. (**13.**) $\tan\theta = (\mu W + \mu' W')/(W + W')$. (**16.**) $\mu \,.\, \frac{M+m}{M} g \cos\alpha$. (**19.**) If 2α be angle of cone, θ inclination of plane, $\tan\theta = 4\tan\alpha$; $\tan\alpha = 1/4\sqrt{3}$. (**21.**) 7 feet 1 inch. (**22.**) 50 feet. (**24.**) 45°. (**27.**) $\tan\alpha = \mu/(1-\mu)$.

XIV.

(**1.**) 9·01 feet per second ; tan (angle) = ·289 . . . (**4.**) θ with side, where $\tan\theta = \frac{7}{5}$. (**5.**) 8·610 . . . seconds. (**6.**) 16 lbs. has $\frac{40}{9}\sqrt{2}$ miles per hour; 8 lbs. has $\frac{10}{9}\sqrt{164}$ at angle θ in same direction, where $\tan\theta = 9$.

XV.

(**2.**) u^2/g; $u\sqrt{2}$. (**3.**) 80,000 feet. (**4.**) $32\sqrt{\frac{7}{3}}$ feet per second ; tan (inclination to vertical) = $\sqrt{3}/5$. (**5.**) 15°. (**6.**) 100 yards. (**7.**) 6000 yards. (**10.**) 3 lbs. (**16.**) 45°. (**18.**) $\frac{1}{3}$ previous range and time. (**19.**) $\tan^2\alpha - \frac{2v^2}{gh}\tan\alpha + \frac{2v^2k}{h^2g} + 1 = 0$. (**24.**) $2v^2/g\sqrt{3}$. (**25.**) If θ be inclination to the plane, $2\tan\theta = \cot\alpha$. (**26.**) 30°. (**28.**) $v = 9\sqrt{gr}$.

XVI.

(**1.**) 48. (**2.**) $\frac{48}{\pi}=15$ about. (**4.**) Weight of 18·15 cwts. (**5.**) Weight of $13\frac{4}{9}$ cwts. (**6.**) $\frac{90}{\pi}=28\cdot6$. (**7.**) $\frac{8}{3}$. (**8.**) Weight of 1·71 . . . oz. (**10.**) $l : l' : : Wv^2 : W'v'^2$. (**11.**) $2\frac{2}{3}$ inches. (**12.**) 1·1 inch. (**14.**) 374 : 375 : 376. (**15.**) $\sqrt{(3ga)}$; string becomes slack when sin (inclination) $= v^2/3ag$; $v\cos\alpha > \sqrt{\{ga(3+2\sin\alpha\cos 2\alpha)\}}$. (**16.**) 3967 miles. (**19.**) 22 ; 30 miles per hour. (**20.**) 0 ; $44\sqrt{3}$ feet per second. (**21.**) Weight of $29\frac{1}{4}$ oz.; weight of $\frac{1}{4}\sqrt{(14,657)}$ oz. (**22.**) $\frac{g\cos\alpha}{\omega^2\sin^2\alpha}$; unstable. (**23.**) Between $\frac{\cos\alpha\pm\mu\sin\alpha}{\sin\alpha\mp\mu\cos\alpha}\cdot\frac{g}{\omega^2\sin\alpha}$. (**24.**) Lowest ; highest ; depth below centre $= g/\omega^2$. (**25.**) $\sqrt{(\frac{1}{2}ga)}$; $a\frac{\sqrt{3}}{2}$. (**26.**) $\sqrt{\left\{u^2-\frac{r^2-a^2}{l}g\right\}}$. (**27.**) a ; $u\sqrt{\frac{l}{g}}$. (**28.**) Side $=2r$. (**30.**) 4 feet. (**31.**) $5r$.

XVII.

(**5.**) 428·76. (**6.**) 221·8. (**7.**) 2015. (**9.**) ·9986. (**11.**) 10·8.

XVIII.

(**3.**) $v\sqrt{(2+\sqrt{3})}$. (**4.**) On radius to point at distance v/ω from it. (**6.**) If $2a=$ length of rod, $\sqrt{(u^2+\omega^2a^2)}$; ω^2a ; $u\pm a\omega$; ω^2a. (**7.**) $\sqrt{(u^2+\omega^2a^2)}$; $\sqrt{(u^4/r^2+\omega^4a^2)}$; $u\pm a\omega$; $u^2/r+\omega^2a$.

XIX.

(**1.**) Momentum $=mv$ horizontally at k^2/r above the centre. (**2.**) Momentum $=m\sqrt{(u^2+\omega^2a^2)}$, making θ with CD where $\tan\theta=\omega a/u$, and at a distance $k^2\omega/\sqrt{(u^2+\omega^2a^2)}$ from the centre; momentum $=m(u\pm a\omega)$ parallel to CD at a distance $a+k^2\omega/(u\pm a\omega)$. (**3.**) Momentum $=2ma\omega$ through B perpendicular to ABC; momentum $=ma\omega\sqrt{10}$, making $\tan^{-1}\theta$ with AB distant from B $=a\sqrt{\frac{2}{5}}$; $4ma\omega$ perpendicular to mid point of AB. (**4.**) $16l\pi$ units, where $l=$ length. (**5.**) $43\frac{1}{2}$ lb. inch units ; 2261 foot-poundals ; 27·4 feet per second. (**6.**) ma^2, where side $=a$, mass of one $=m$. (**7.**) Momentum $=ma\omega/\sqrt{3}$; parallel to base and same distance from it as the centre of gravity. (**8.**) Instantaneous centre on perpendicular to base at twice distance of centre of gravity from point. (**9.**) If side $=2a$, whole mass $=$ M, $2Ma^2$; $2Ma^2$. (**10.**) $\frac{1}{2}\omega$; one-half.

XX.

(**1.**) $m\frac{(a+b)^2}{12}$; $m\frac{3a+b}{a+b} \cdot \frac{b^2}{12}$. (**3.**) $\frac{2}{3}ma^2$; $\frac{1}{3}ma^2$. (**5.**) 2·68 . . . feet. (**6.**) 27·5 . . . × mass, foot units. (**7.**) $\frac{1}{12}ma^2$. (**11.**) $\frac{83}{320}mr^2$. (**12.**) $\frac{3}{10}mr^2$; $\frac{3}{20}mr^2$. (**13.**) $\frac{3}{5}mr^2$. (**14.**) $\frac{3}{5}mr^2$. (**15.**) $\frac{39}{2}$; $\frac{75}{4}$ lb. inch units. (**16.**) 409·2 oz. inch units ; 405·2 if axis is through centre of large sphere. (**17.**) 1 : 4. (**18.**) 1 : 1. (**19.**) 7 : 5. (**20.**) 76,057 foot-pounds ($\pi=\frac{22}{7}$). (**21.**) $2{\cdot}036 \times 10^{29}$ foot-pounds. (**22.**) 15 feet per second ; $8\frac{4}{7}$ feet per second. (**23.**) Velocity $=12\frac{8}{11}$ feet per second ; angular velocity $=\frac{60}{11}$; loss $=\frac{60}{121}$ original. (**24.**) Velocity $=8\frac{6}{13}$ feet per second ; angular velocity $=\frac{15}{13}$. (**25.**) $\frac{ma\omega\sqrt{2}}{7}$. (**27.**) If P be magnitude of jerk, velocity $=\mathrm{P}/m$; angular velocity $=\frac{2\mathrm{P}}{mr}$; 1 : 2. (**28.**) Angular velocity after $=\frac{1}{2}\omega$; velocities of centre of cylinder and particle are opposite and both $=\frac{1}{4}r\omega$. (**29.**) Angular velocity $=\frac{2}{3}\omega$; velocity of particle $=\frac{2a\sqrt{2}}{9}\omega$; velocity of centre of gravity $=\frac{a\sqrt{2}}{9}\omega$. (**30.**) Angular velocity $=\frac{2-e}{3}\omega$; velocity of particle $=\frac{\sqrt{2}}{9}(1+e)a\omega$; velocity of centre $=\frac{a\omega(1+e)}{9\sqrt{2}}$. (**32.**) If P be the blow, m the mass of each, and $2a$ the length—(1) angular velocity of AB = angular velocity of BC $=\frac{3\mathrm{P}}{8ma}$; velocity of centre of AB $=\frac{\mathrm{P}}{8m}$; of BC $=\frac{7\mathrm{P}}{8m}$; ratio of energies = 13 : 1 ; (2) angular velocity of BC $=\frac{3\mathrm{P}}{5ma}$; velocity of centre of AB $=\frac{\mathrm{P}}{5m}$; of BC $=\frac{4\mathrm{P}}{5m}$; ratio of energies 19 : 1. (**34.**) If $2a$, $2b$ be lengths of top and bottom parts, it must be struck at $\frac{ab}{a+3b}$ above the centre of the lower part.

XXI.

(**1.**) $16/\sqrt{3}$ feet per second ; $\mu=\frac{1}{9\sqrt{11}}$; $8\sqrt{\frac{5}{3}}$ feet per second. (**2.**) $16\sqrt{3}$ feet per second ; 4 × weight. (**3.**) $\frac{7}{5}$ height if smooth. (**5.**) 23·20 . . . feet per second. (**6.**) Slips for a time $\frac{2}{7}\frac{a\omega}{\mu g}$, then rolls with angular velocity $\frac{2}{7}\omega$. (**9.**) $\frac{g}{66}$; $3\frac{1}{33}$ per cent. (**10.**) $\sqrt{\frac{3}{2}}$ seconds ;

$32\sqrt{\frac{2}{3}}$ feet per second ; $\frac{1}{3}$ weight of cylinder. (**11.**) With acceleration $2g$ up. (**12.**) Both fall with $\frac{1}{2}g$. (**14.**) Vertical height $= \frac{25}{64}$ foot ; $\frac{405}{448}$ foot. (**15.**) 1·885 times a side. (**16.**) At heights above centre $= \frac{2}{3}(R+r)$ and $\frac{7}{17}(R+r)$ in the two cases. (**17.**) $\frac{17}{7}$ the weight. (**18.**) 1·06 . . . length of seconds' pendulum. (**19.**) $\frac{5}{6}$ and $\frac{3}{4}$ height of triangle. (**20.**) $\sqrt{11} : \sqrt{5}$. (**21.**) $\frac{2\pi}{3}\sqrt{\left(\frac{67r}{5g}\right)}$; $\frac{2\pi}{3}\sqrt{\frac{11r}{g}}$. (**22.**) $4\sqrt{(3a\sqrt{2})}$ feet per second, if side $= 2a$ feet. (**23.**) Tension is halved. (**25.**) $\frac{\pi^3}{120}$ poundals.

XVI (a).

(**5.**) 1·414. (**6.**) About 5 days 1 hour ; $3\frac{8}{9}$ earth's radius. (**7.**) ·432 months. (**8.**) 1 : 3,093,500 ; 5800 miles. (**9.**) 1·71 hours. (**10.**) The centre is at the other extremity of the diameter, eccentricity $= \frac{1}{2}$. (**20.**) A blow of $m\sqrt{\mu\left(\frac{1}{SP} - \frac{1}{a}\right)}$ directed in the line to the focus.

THE END

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