ELEMENTARY GEOMETRY

PRACTICAL AND THEORETICAL

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ELEMENTARY GEOMETRY

PRACTICAL AND THEORETICAL

BY

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PREFACE.

THE aim of the authors of the present work has been to produce a book which will help to make Geometry an attractive subject to the average British boy or girl.

The new schedule of geometry recently adopted by Cambridge has been taken as a basis of operations. These regulations will affect candidates for the Previous Examination after March 1904.

It has been found easy to follow this schedule closely and at the same time to have regard to the reformed schedules of various other examinations, such as Oxford and Cambridge Locals, Oxford Responsions, together with the examinations of the University of London, and the Civil Service Commissioners. The reports of the British Association and of the Mathematical Association have been very helpful.

The book opens with a course of **experimental work**; great pains have been taken to make the exercises perfectly explicit and free from ambiguity. The beginner is taught to use instruments, to measure accurately lines and angles (this will in future be regarded as an indispensable part of geometrical work), to construct and recognize the simpler plane *and solid* figures, to solve problems by drawing to scale. At the same time he is led to discover many geometrical truths which are proved later; he should be encouraged to put into words and make notes of any such discoveries. There is much in this part which will be useful revision work for more advanced pupils.

Then follows the course of **Theoretical Geometry**, which is divided into four 'books.' The experimental method is still prominent, in the shape of exercises leading up to propositions.

The sequence of **theorems** is Euclidean in form, but greatly simplified by the omission of non-essentials, and by the use of hypothetical constructions. There is reason to hope that it is now possible to adopt a sequence (not differing very greatly from that of Euclid) which will be generally accepted for some time to come.

The treatment of **problems** is practical, though proofs are given; for this part of the subject the present work is designed to fulfil the purposes of a book on geometrical drawing.

Among the **exercises**, some are experimental and lead up to future propositions, some are graphical and numerical illustrations of known propositions, some are 'riders' of the ordinary type*. In a great number of the earlier exercises the figures are given. There is a collection of exercises on plotting **loci** and **envelopes**; a subject which is found interesting, and introduces the learner to other curves than the circle and straight line.

Book I. deals with the **subject-matter of Euclid I. 1—34**; angles at a point, parallels, angles of polygons, the triangle, the parallelogram, sub-division of straight lines, the earliest constructions and loci.

Book II. treats of **area**. The notion of area is enforced by a large number of exercises to be worked on squared paper, the use of **coordinates** being explained incidentally. Euclid's second book appears in a new garb as **geometrical illustrations of algebraical identities.**

* We are indebted to the kindness of Mr R. Levett and of Messrs Swan Sonnenschein and Co. for permission to use a few of the riders from *The Elements of Plane Geometry* issued under the auspices of the A. I. G. T.

Book III.—the **circle**; relieved of a great number of useless propositions. In addition to the topics usually treated, there are sections on the **mensuration** of the circle, a knowledge of which is generally assumed in works on solid geometry.

Book IV.—**similarity**. Here again much of Euclid VI. is omitted, as not really illustrating the subject of similar figures. Euclid's definition of proportion has gone, and is replaced by an easy algebraic treatment applicable (as is now permitted) to commensurable magnitudes only.

On the whole, the authors believe that with two-thirds of the number of *theorems*, more ground is covered than by Euclid I.—IV. and VI.

References have generally been given in the proof of propositions; it is not supposed however that pupils will be required to quote references. Their presence in a book can be justified only on the ground that they may help a reader to follow the argument.

The authors desire to express their gratitude to many friends, whose criticisms have been both salutary and encouraging.

C. G.
CAMBRIDGE, *August*, 1903. A. W. S.

An appendix on the pentagon group of constructions is now added.

July, 1906.

Revision papers have been added at the end of the book. For permission to print certain items we are indebted to the courtesy of H.M. Stationery Office, the Oxford Local Examinations Delegacy, the Cambridge Local Examinations Syndicate, the Joint Board, the University of London and the Board of Management of the Common Entrance Examination.

C. G.
December, 1916. A. W. S.

PREFACE TO SECOND EDITION.

IN this edition the first four theorems of Book II. (areas of parallelogram and triangle) have been rewritten and compressed into three theorems, the enunciations now following the arrangement of the Cambridge Syllabus. The proofs of III. 6 and 7 have also been rewritten.

In the first edition references were given, as a rule, in the proofs of theorems; but in some cases an easy step was left to the reader, by the insertion of (why?). This is now deleted from theorems, and the reference is given in all such cases.

An additional set of exercises on drawing to scale has been inserted.

A very full table of contents now appears: this, in fact, was added in an earlier reprint.

Other minor changes have been made (e.g. new figure for I. 3, II. 7, IV. 1).

For the convenience of users of the first edition, it has been arranged that there is no change in the numbering of pages or exercises.

C. G.
A. W. S.

April, 1909.

PREFACE TO THIRD EDITION.

IN this edition no changes have been made in the numbering of pages or of exercises. The most important change is that exercises of a theoretical character (riders) have been marked thus †Ex. 326, and exercises intended for discussion in class are distinguished thus ¶Ex. 30.

In order to economise time some of the drawing exercises in the later part of the book have been slightly changed so that they now require only a description of the method of performing the construction instead of requiring that it shall actually be performed.

C. G.
A. W. S.

December, 1911.

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BOOK IV. SIMILARITY.

APPENDIX I.

†APPENDIX II. THE PENTAGON.

PART I.

EXPERIMENTAL GEOMETRY.

INSTRUMENTS.

The following instruments will be required :—

A hard pencil (HH).

A ruler about 6″ long (or more) graduated in inches and tenths of an inch and also in cm. and mm.

A set square (60°); its longest side should be at least 6″ long.

A semi-circular protractor.

A pair of compasses (with a hard pencil point).

The pencil should have a chisel-point.

The compass pencil may have a chisel-point, or may be sharpened in the ordinary way.

In testing the equality of two lengths or in transferring lengths, compasses should always be used.

Exercises distinguished by a paragraph sign thus : ¶ *Ex.* 27, *are intended for discussion in class.*

Exercises of a theoretical character (riders) are marked with a dagger thus : † *Ex.* 323.

EXPERIMENTAL GEOMETRY.

Straight Lines.

In stating the length of a line, remember to give the unit; the following abbreviations may be used :—*in.* for *inch*; *cm.* for *centimetre*; *mm.* for *millimetre.*

In Ex. 1–163, all lengths measured in inches are to be given to the nearest tenth of an inch, all lengths measured in centimetres to the nearest millimetre.

Always give your answers in decimals.

Ex. 1. Measure the lengths AB, CD, EF, GH in fig. 1

(i) in inches,

(ii) in centimetres.

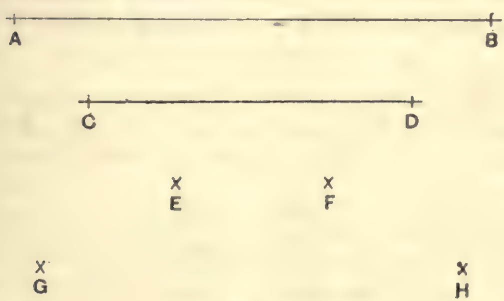


fig. 1.

Ex. 2. Measure in inches and centimetres the lengths of the edges of your wooden blocks.

Ex. 3. Measure in inches the lengths AB, BC, CD in fig. 2; arrange your results in tabular form and add them together.

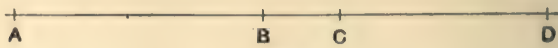


fig. 2.

AB = in.

BC = in.

CD = in.

AB + BC + CD = in.

Check by measuring AD.

Ex. 4. Repeat Ex. 3, using centimetres instead of inches.

Ex. 5. Repeat Ex. 3, for fig. 3, (i) using centimetres, (ii) using inches.

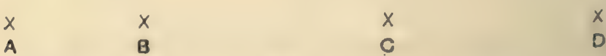


fig. 3.

Ex. 6. Measure in centimetres the lengths AB, BC in fig. 4, and find their difference; arrange your results in tabular form.

fig. 4.

AB = cm.

BC = cm.

AB − BC = cm.

Check by measuring AC.

Ex. 7. Repeat Ex. 6, using inches instead of centimetres.

Ex. 8. Repeat Ex. 6, for fig. 5, (i) using inches, (ii) using centimetres.

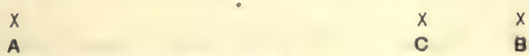


fig. 5.

Ex. 9. Measure in inches, and also in centimetres, the length of the paper you are using.

Your ruler is probably too short to measure directly; divide the length into two (or more) parts by making a pencil mark on the edge, and add these lengths together.

Ex. 10. Measure the breadth of your paper in inches and also in centimetres.

Ex. 11. Draw a straight line about 6 in. long and cut off a part AB = 2 in., a part BC = 1·5 in., and a part CD = 1·8 in.; find the length of AD by adding these lengths; check by measuring AD. [Make a table as in Ex. 3.]

Ex. 12. Repeat Ex. 11, with

(i)	AB = 2·7 cm.,	BC = 9·6 cm.,	CD = 1·3 cm.
(ii)	AB = 5·2 cm.,	BC = 3·9 cm.,	CD = 2·8 cm.
(iii)	AB = ·7 in.,	BC = 2·6 in.,	CD = 2·4 in.
(iv)	AB = ·8 cm.,	BC = ·5 cm.,	CD = 2·4 cm.
(v)	AB = 1·8 in.,	BC = 2·9 in.,	CD = ·6 in.

Ex. 13. A man walks 3·2 miles due north and then 1·5 miles due south, how far is he from his starting point? Draw a plan (1 mile being represented by 1 inch) and find the distance by measurement.

Ex. 14. A man walks 5·4 miles due west and then 8·2 miles due east, how far is he from his starting point? (Represent 1 mile by 1 centimetre.)

Ex. 15. A man walks 7·3 miles due south, then 12·7 miles due north, then 1·1 miles due south, how far is he from his starting point? (Represent 1 mile by 1 centimetre.)

Ex. 16. Draw a straight line, guess its middle point and mark it by a short cross-line; test your guess by measuring the two parts.

Ex. 17. Repeat Ex. 16, three or four times with lines of various lengths. Show by a table how far you are wrong.

Ex. 18. Draw a straight line of 10·6 cm.; bisect it by calculating the length of half the line and measuring off that length from one end of the line, then measure the remaining part.

When told to draw a line of some given length, you should draw a line a little too long and cut off a part equal to the given length as in fig. 6. You should also write the length of the line against it, being careful to state the unit.

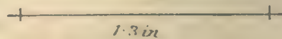


fig. 6.

Ex. 19. Draw a straight line 3·2 in. long, bisect it as in Ex. 18.

Ex. 20. Draw a straight line 2·7 in. long, bisect it as in Ex. 18.

Ex. 21. Draw straight lines of the following lengths, bisect each of them: (i) 7·6 cm., (ii) 10·5 cm., (iii) 4·1 in., (iv) ·9 in., (v) 5·8 cm., (vi) 11·3 cm.

A good practical method of bisecting a straight line (AB) is as follows:—measure off with dividers equal lengths (AC, BD) from each end of the line (these lengths should be very nearly half the length of the line) and bisect the remaining portion (CD) by eye.

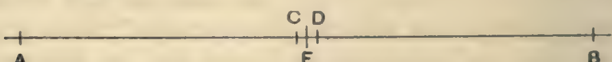


fig. 7.

Ex. 22. Draw three or four straight lines and bisect them with your dividers (as explained above); verify by measuring each part of the line (remember to write its length against each part)

Ex. 23. Open your dividers 1 cm., apply them to the inch scale and so find the number of inches in 1 centimetre.

Ex. 24. Find the number of inches in 10 cm. as in Ex. 23; hence express 1 cm. in inches. Arrange your results in tabular form.

Ex. 25. Find the number of centimetres in 5 in. as in Ex. 23; hence find the number of centimetres in 1 inch.

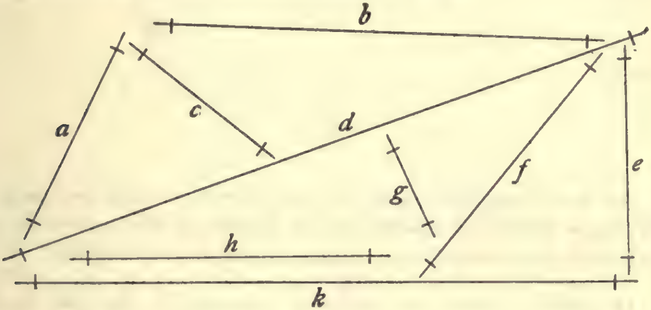


fig. 8.

Ex. 26. Guess the lengths of the lines in fig. 8 (i) in inches, (ii) in centimetres; verify by measurement. Make a table thus:—

Line	Guessed	Measured
a		
b		

Angles.

If you hold one arm of your dividers firm and turn the other about the hinge, the two arms may be said to form an **angle**.

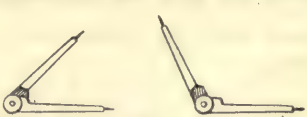


fig. 9.

In the same way if two straight lines OA, OB are drawn from a point O, they are said to form an **angle** at O. O is called the **vertex** of the angle, and OA, OB its **arms.**

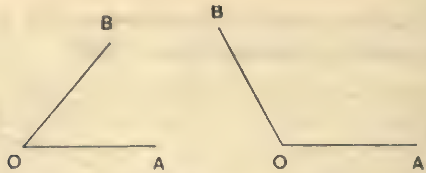


fig. 10.

An angle may be denoted by three letters; thus we speak of the angle AOB, the middle letter denoting the vertex of the angle and the outside letters denoting points on its arms.

If there is only one angle at a point O, we call it the angle O.

Sometimes an angle is denoted by a small letter placed in it; thus in the figure we have two angles *a* and *b*.

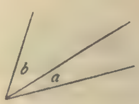


fig. 11.

$\angle$ is the abbreviation for *angle.*

Two angles AOB, CXD (see figs. 10 and 12), are said to be **equal** when they can be made to fit on one another exactly (i.e. when they are such that, if CXD be cut out and placed so that X is on O and XC along OA, then XD is along OB). It is important to notice that it is not necessary for the *arms* of the one angle to be equal to those of the other, in fact *the size of an angle does not depend on the lengths of its arms.*

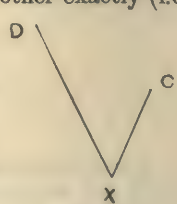


fig. 12.

¶Ex. 27. Draw an angle on your paper and open your dividers to the same angle.

¶Ex. 28. Which is the greater angle in fig. 13? Test by making on tracing paper an angle equal to one of the angles and fitting the trace on the other.

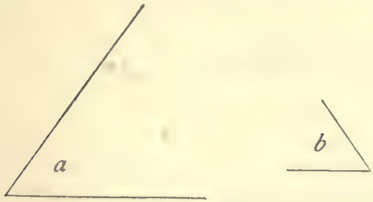


fig. 13.

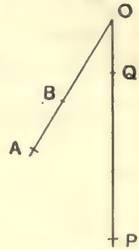


fig. 14.

¶Ex. 29. Name the angle at O in fig. 14 in as many different ways as you can.

¶Ex. 30. Take a piece of paper and fold it, you will get something like fig. 15, fold it again so that the edge OB fits on the edge OA; now open the paper; you have four angles made by the creases, as in fig. 16; they are all equal for when folded they fitted on one another. Such angles are called **right angles**. An angle less than a right angle is called an **acute** angle. An angle greater than a right angle is called an **obtuse** angle.

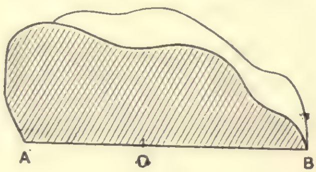


fig. 15.

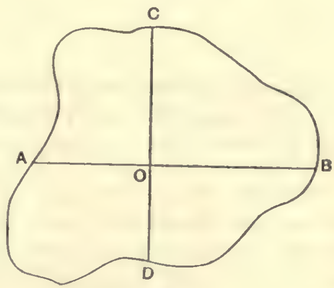


fig. 16.

¶Ex. 31. Make a right angle BOC as in Ex. 30, cut it out and fold so that OB falls on OC. Does the crease (OE) bisect ∠ BOC? (i.e. are ∠s BOE, EOC equal?) What fraction of a right angle is each of the ∠s BOE, EOC?

¶Ex. 32. If the ∠ BOE of Ex. 31 were bisected by folding, what fraction of a right angle would be obtained?

If a right angle is divided into 90 equal angles, each of these angles is called a **degree**.

25° is the abbreviation for "25 degrees."

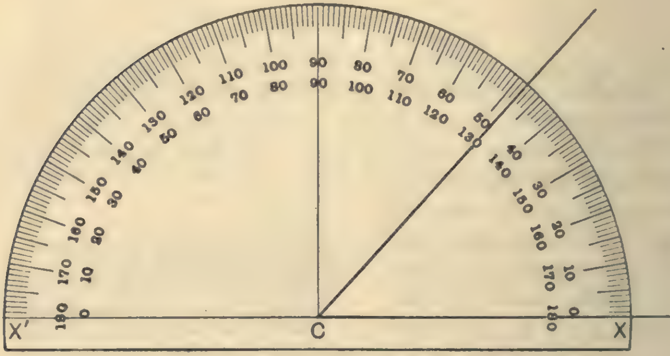


fig. 17.

Fig. 17 represents a **protractor**; if each graduation on the edge were joined to C, we should get a set of angles at C each of which would be an angle of one degree.

¶Ex. 33. What fractions of a right angle are the angles between the hands of a clock at the following times:—(i) 3.0, (ii) 1.0, (iii) 10.0, (iv) 5.0, (v) 8.0? State in each case whether the angle is acute, right, or obtuse.

¶Ex. 34. Find the number of degrees in each of the angles in Ex. 33. [Use the results of that Ex.]

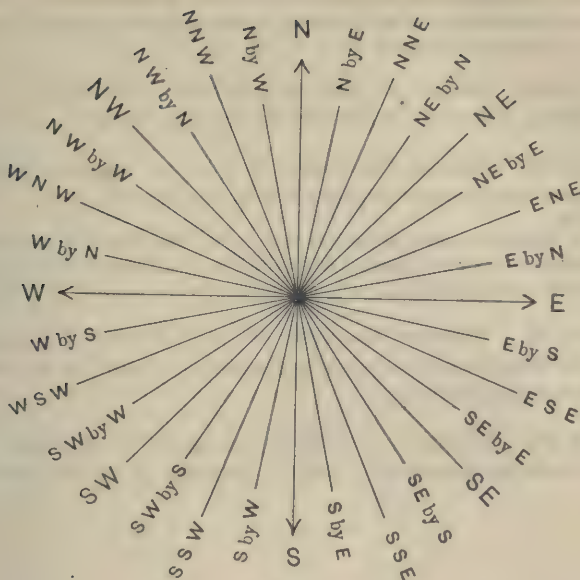


fig. 18.

¶Ex. 35. Fig. 18 shows the points of the compass; what are the angles between (i) N and E, (ii) W and S W, (iii) W and W N W, (iv) E and E by S, (v) N E and N N W, (vi) S W and S E?

To measure an angle, place the protractor so that its centre C is at the vertex of the angle and its base, CX, along one arm of the angle; then note under which graduation the other arm passes; thus in fig. 17, the angle = 48°.

In using a protractor such as that in fig. 17, care must be taken to choose the right set of numbers; e.g. if the one arm of the angle to be measured lies along CX, the set of numbers to be used is obviously the one in which the numbers increase as the line turns round C from CX towards CX′.

You should also check your measurement by noticing whether the angle is acute or obtuse.

When you measure an angle in a figure that you have drawn (or make an angle to a given measure), always indicate in your figure the number of degrees, as in fig. 19.

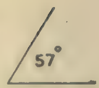


fig. 19.

Ex. 36. Cut out of paper a right angle, bisect it by folding, and measure the two angles thus formed.

Ex. 37. Measure the angles of your set square (i) directly, (ii) by making a copy on paper and measuring the copy.

It is difficult to draw a straight line right to the corner of a set square; it is better to draw the lines to within half a centimetre of the corner and afterwards produce them (i.e. prolong them) with the ruler till they meet.

Ex. 38. Measure the angles of your models—this may be done either directly, or more accurately by copying the angles and measuring the copy.

Ex. 39. Measure ∠s AOB, BOC in fig. 20; add; and check your result by measuring ∠ AOC. (Arrange in tabular form.)

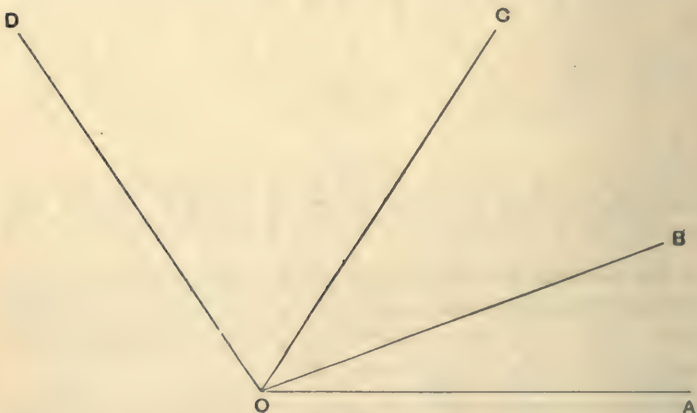


fig. 20.

Ex. 40. Measure ∠s AOC, COD, AOD in fig. 20. Check your results.

Ex. 41. Measure ∠s AOB, BOD, AOD in fig. 20. Check your results.

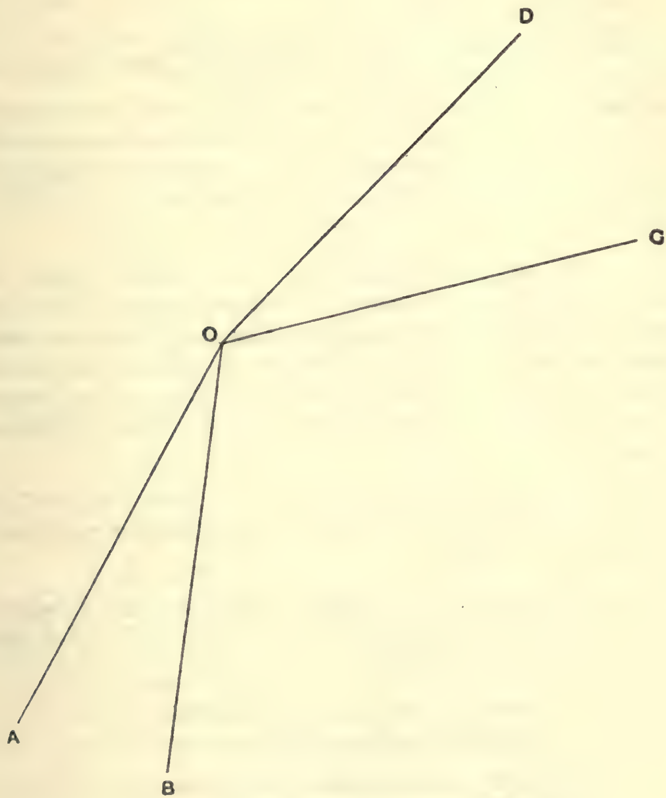


fig. 21.

Ex. 42. Repeat the last three exercises for fig. 21.

Ex. 43. Draw a circle (radius about 2·5 in.), cut off equal parts from its circumference (this can be done by stepping off with compasses or dividers). Join OA, OB, OF. Measure ∠s AOB, AOF. Is ∠ AOF = 5 times ∠ AOB?

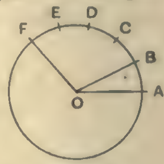


fig. 22.

To make an angle to a given measure. Suppose that you have a line AB and that at the point A you wish to make an angle of 73°. Place the protractor so that its centre is at A and its base along AB, mark the 73° graduation with your dividers (only a small prick should be made), and join this point to A. (Remember to write 73° in the angle.)

Ex. 44. Make a copy of the smallest angle of your set square and bisect it as follows:—measure the angle with your protractor, calculate the number of degrees in half the angle, mark off this number (as explained above) and join to the vertex. Verify by measuring each half. (This will be referred to as the method of bisecting an angle *by means of the protractor.*)

Ex. 45. Make angles of 20°, 35°, 64°, 130°, 157°, 176° (let them point in different directions). State whether each one is acute, right, or obtuse.

Ex. 46. Make the following angles and bisect each by means of the protractor, 24°, 78°, 152°, 65°, 111°. (Let them point in different directions.)

¶Ex. 47. Draw an acute angle AOB; produce AO to C; what kind of angle is BOC? (*freehand*)

¶Ex. 48. Draw an obtuse angle BOC; produce CO to A; what kind of angle is AOB? (*freehand*)

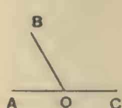


fig. 23.

¶Ex. 49. Make ∠ AOB = 42°; produce AO to C. By how much is ∠ AOB less than a right angle? By how much is ∠ BOC greater than a right angle?

¶Ex. 50. (i) Make $\angle$ AOB = 65°; produce AO to C; measure $\angle$ BOC; what is the sum of $\angle$s AOB, BOC?

(ii) Repeat (i) with $\angle$ AOB = 77°.

(iii) Repeat (i) with $\angle$ AOB = 123°.

Compare the results of (i), (ii), (iii); how many right angles are there in each sum?

¶Ex. 51. If, in fig. 23, $\angle$ AOB = 57°, what is $\angle$ BOC? Check by drawing and measuring.

¶Ex. 52. (i) If, in fig. 23, $\angle$ BOC = 137°, what is $\angle$ AOB?

(ii) „ „ $\angle$ BOC = 93° „ „ $\angle$ AOB?

(iii) „ „ $\angle$ AOB = 5° „ „ $\angle$ BOC?

¶Ex. 53. Draw a straight line OB; on opposite sides of OB make the two angles AOB = 42°, BOC = 129°. What is their sum? Is AOC a straight line?

¶Ex. 54. Repeat Ex. 53, with

(i) $\angle$ AOB = 42°, $\angle$ BOC = 138°.

(ii) $\angle$ AOB = 90°, $\angle$ BOC = 90°.

(iii) $\angle$ AOB = 73°, $\angle$ BOC = 113°.

(iv) $\angle$ AOB = 113°, $\angle$ BOC = 76°.

¶Ex. 55. What connection must there be between the two angles in the last Ex. in order that AOC may be straight?

¶Ex. 56. Make an $\angle$ AOB = 36°; produce AO to C; make $\angle$ COD = 36°; calculate $\angle$ BOC; is BOD a straight line in your figure? Give a reason.

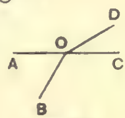


fig. 24.

¶Ex. 57. From a point O in a straight line AB, draw two lines OC, OD (see fig. 25); measure the three angles; what is their sum?

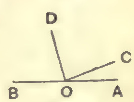


fig. 25.

¶Ex. 58. Repeat Ex. 57, with AOB drawn in a different direction.

¶Ex. 59. Draw fig. 26 making $\angle$ BOC = 67° and $\angle$ B′O′D′ = 29°. What is the sum of the four angles?

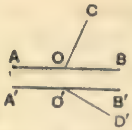


fig. 26.

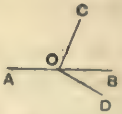


fig. 27.

¶Ex. 60. Draw fig. 27 making $\angle$ BOC = 67° and $\angle$ BOD = 29°. What is the sum of the four angles at O? Give a reason.

¶Ex. 61. From a point O in a straight line AB, draw straight lines OC, OD, OE, OF, OG as in fig. 28. Measure the angles AOC, COD, &c. What is their sum?

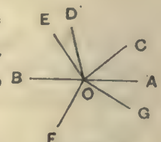


fig. 28.

¶Ex. 62. From a point O, draw a set of straight lines as in fig. 29, measure the angles so formed. What is their sum? How many right angles is the sum equal to?

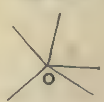


fig. 29.

Ex. 63. From a point O, draw a set of straight lines as in fig. 29. Guess the size of the angles so formed; verify by measurement. Make a table thus:—

Angle	Guessed	Measured
a	45°	47°
b	27°	153°
c		

Ex. 64. Draw two straight lines as in fig. 30; measure all the angles.

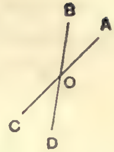


fig. 30.

Ex. 65. Make $\angle$ AOB $= 47°$; produce AO to C and BO to D; measure all the angles.

Ex. 66. Repeat Ex. 65 with $\angle$ AOB $= 166°$.

¶Ex. 67. In fig. 30, if $\angle$ AOB $= 73°$, what are the remaining angles? Verify by drawing.

¶Ex. 68. (i) In fig. 30, if $\angle$ AOD $= 132°$, what are the remaining angles?

(ii) In fig. 30, if $\angle$ COD $= 58°$, what are the remaining angles?

(iii) In fig. 30, if $\angle$ BOC $= 97°$, what are the remaining angles?

Regular Polygons.

Ex. 69. Describe a circle of radius 5 cm.; at its centre O draw two lines at right angles to cut the circle at A, B, C, D. Join AB, BC, CD, DA. Measure each of these lines and each of the angles ABC, BCD, CDA, DAB.

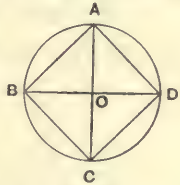


fig. 31.

A **square** has all its sides equal and all its angles right angles.

Ex. 70. Describe a circle of radius 5 cm.; at its centre make a set of angles each equal to 60° $\left(\text{i.e. } \frac{360°}{6}\right)$; join the points where the arms cut the circle; the figure you obtain is a **hexagon** (6-gon), and it is said to be **inscribed** in the circle. What do you notice about its sides and angles?

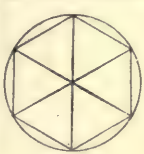


fig. 32.

A figure bounded by equal straight lines, which has all its angles equal, is called a **regular polygon.**

A figure of	3	sides	is	called	a	**triangle** (△).
„ „	4	„	„	„		**quadrilateral** (4-gon).
„ „	5	„	„	„		**pentagon** (5-gon).
„ „	6	„	„	„		**hexagon** (6-gon).
„ „	7	„	„	„		**heptagon** (7-gon).
„ „	8	„	„	„		**octagon** (8-gon).

The corners of a triangle or polygon are called its **vertices.**

The **perimeter** of a figure is the sum of its sides.

Ex. 71. What is the perimeter of a regular 6-gon, each of whose sides is 2·7 in. long?

Ex. 72. In a circle of radius 5 cm. make a regular pentagon (5-gon) as in Ex. 70; the angles you make at the centre must all be equal and there will be five of them; what is each angle?

Ex. 73. Calculate the angle at the centre for each of the following regular polygons; inscribe each in a circle of radius 5 cm.

(i) 8-gon, (ii) 9-gon, (iii) triangle, (iv) 10-gon, (v) 16-gon.

Ex. 74. Make a table of the results of Ex. 73.

Regular Polygons			
Number of sides	Angle at centre	Length of side	Perimeter
3	120°		
4	90°		
5			

Ex. 75. Explain in your own words a simple construction for a regular hexagon depending on the fact you discovered in Ex. 70, that each side of the hexagon was equal to the radius of the circle.

Pattern Drawing

Ex. 76. Copy fig. 33, taking 5 cm. for the radius of the large circle. The dotted lines are at right angles to one another. How will you find the centres of the small circles?

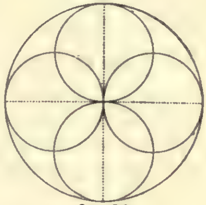


fig. 33.

If you describe only part of a circle, the curve you make is called an **arc** of the circle.

Ex. 77. Copy fig. 34, taking 5 cm. for the radius of the circle. The six points on the circle are the vertices of a regular hexagon (see Ex. 75); each of these points is the centre of one of the arcs.

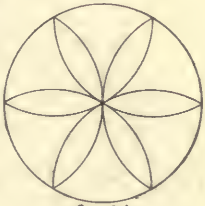


fig. 34.

Ex. 78. Copy fig. 35, taking 5 cm. for the radius of the circle. The centres of the arcs are the midpoints of the sides of a square inscribed in the circle.

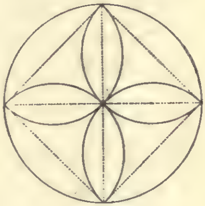


fig. 35.

Ex. 79. Copy fig. 36, taking 5 cm. for the radius of the circle. The angles between the dotted lines are equal; what size is each of these angles? The centres of the arcs are the midpoints of the dotted lines.


fig. 36.

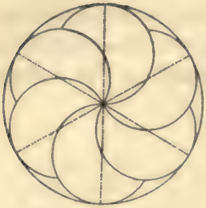


fig. 37.

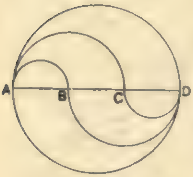


fig. 38.

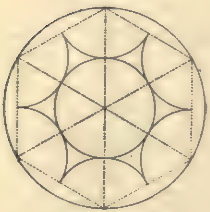


fig. 39.

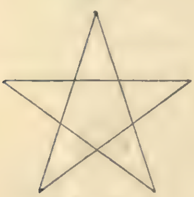


fig. 40.

Ex. 80. Copy fig. 37, taking 5 cm. for the radius of the circle. Where are the centres of the arcs?

A straight line drawn through the centre of a circle to meet the circumference both ways is called a **diameter**.

The two parts into which a diameter divides a circle are called **semicircles**.

Ex. 81. Copy fig. 38, taking AD = 9 cm. AD is a diameter of a circle and is divided into three equal parts at B and C; semicircles are described on AB, AC, CD, BD as diameters.

Ex. 82. Draw a figure showing the points of the compass. See fig. 18.

Ex. 83. Copy fig. 39, taking 5 cm. for the radius of the large circle. The radius of the small circle is half that of the large circle; the centres of the arcs are the vertices of the regular hexagon.

Ex. 84. Copy fig. 40. The points of the star are the vertices of a regular pentagon.

Triangles.

Ex. 85. Draw a triangle (each side being at least 2·5 in. long). Measure all its angles; find the sum of its angles.

Ex. 86. Repeat Ex. 85 three or four times with triangles of different shapes.

When told to construct a figure to given measurements, first make a rough sketch of the figure on a small scale and write the given measurements on the sketch.

Ex. 87. Make an angle ABC = 74°; cut off from its arms BC = 3·2 in., BA = 2·8 in.; join AC. Measure the remaining side and angles of the triangle ABC.

In all cases where triangles or quadrilaterals are to be constructed to given measurements, measure the remaining sides (in inches if the given sides are measured in inches, in centimetres if the given sides are measured in centimetres); also measure the angles, and find their sum.

Ex. 88. Construct triangles to the following measurements:—

(i)	$\angle$ ABC = 80°,	AB = 2·2 in.,	BC = 2·9 in.
§(ii)	$\angle$ B = 28°,	AB = 7·3 cm.,	BC = 12·1 cm.
(iii)	$\angle$ A = 42°,	AB = 3·7 in.,	CA = 3·7 in.
§(iv)	$\angle$ B = 126°,	AB = 6·1 cm.,	BC = 6·1 cm.
(v)	$\angle$ C = 90°,	BC = 3·9 in.,	CA = 2·8 in.
(vi)	BC = 6·7 cm.,	$\angle$ C = 48°,	CA = 9·0 cm.
(vii)	AB = 4·7 in.,	BC = 2·9 in.,	$\angle$ B = 32°.
§(viii)	CA = 2·6 in.,	AB = 3·3 in.,	$\angle$ A = 162°.
(ix)	$\angle$ C = 79°,	CA = 4·7 cm.,	BC = 6·1 cm.
(x)	AB = 4·6 cm.,	CA = 8·7 cm.,	$\angle$ A = 58°.

Ex. 89. Draw a straight line AB 9 cm. long, at A make an angle BAC = 60°, at B make an angle ABC = 40°, produce AC, BC to cut at C. Measure the remaining sides and angle of the triangle ABC. What is the sum of the three angles?

Ex. 90. Construct triangles to the following measurements:—
(In case the construction is impossible with the given measurements, try to explain why it is impossible.)

(i)	AB = 8·3 cm.,	$\angle$ A = 45°,	$\angle$ B = 72°.
§(ii)	AB = 3·9 in.,	$\angle$ A = 39°,	$\angle$ B = 39°.

§ These will be enough exercises of this type unless much practice is needed.

(iii) $\angle B = 90^\circ$, $BC = 7\cdot2$ cm., $\angle C = 42^\circ$.

§(iv) $\angle C = 116^\circ$, $CA = 1\cdot8$ in., $\angle A = 78^\circ$.

(v) $\angle A = 60^\circ$, $\angle C = 60^\circ$, $AC = 6\cdot5$ cm.

(vi) $\angle B = 33^\circ$, $\angle C = 113^\circ$, $BC = 6\cdot9$ cm.

(vii) $\angle A = 73^\circ$, $\angle B = 24^\circ$, $AB = 3\cdot2$ in.

(viii) $CA = 9\cdot2$ cm., $\angle C = 31^\circ$, $\angle A = 59^\circ$.

§(ix) $AB = 2\cdot8$ in., $\angle A = 50^\circ$, $\angle B = 130^\circ$.

(x) $AB = 12\cdot1$ cm., $\angle A = 27^\circ$, $\angle B = 37^\circ$.

Ex. 91. Construct triangles to the following measurements:—

(i) $BC = 10\cdot8$ cm., $\angle A = 90^\circ$, $\angle C = 60^\circ$.

(ii) $CA = 9\cdot0$ cm., $\angle C = 48^\circ$, $\angle B = 57^\circ$.

Ex. 92. Construct quadrilaterals ABCD to the following measurements:—

(Here it is especially important that, before beginning the construction, a rough sketch should be made showing the given parts.

Note that the letters must be taken in order round the quadrilateral; e.g. the quadrilateral in fig. 41 is called ABCD and *not* ABDC.)

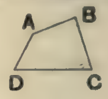


fig. 41.

(i) $AB = 6\cdot3$ cm., $\angle B = 82^\circ$, $BC = 8\cdot2$ cm., $\angle C = 90^\circ$, $CD = 7\cdot7$ cm.

(ii) $AB = 3\cdot4$ in., $BC = 2\cdot2$ in., $AD = 2\cdot9$ in., $\angle A = 68^\circ$, $\angle B = 86^\circ$.

(iii) $\angle B = 116^\circ$, $BC = 1\cdot4$ in., $\angle C = 99^\circ$, $CD = 1\cdot9$ in., $\angle D = 92^\circ$.

(iv) $\angle A = 67^\circ$, $\angle B = 113^\circ$, $\angle D = 46^\circ$, $AB = 5\cdot3$ cm., $AD = 8\cdot6$ cm.

(v) $\angle B = 122^\circ$, $\angle C = 130^\circ$, $\angle D = 130^\circ$, $BC = CD = 1\cdot6$ in.

(vi) $AD = 3\cdot0$ in., $\angle D = 118^\circ$, $\angle DAC = 27^\circ$, $\angle BAC = 35^\circ$, $AB = 2\cdot4$ in.

(vii) $AC = 5\cdot6$ cm., $\angle BAC = 58^\circ$, $\angle DAC = 69^\circ$, $\angle BCA = 58^\circ$, $\angle DCA = 69^\circ$.

(viii) $AB = 1\cdot9$ in., $BD = 1\cdot7$ in., $CD = 2\cdot0$ in., $\angle ABD = 118^\circ$, $\angle BDC = 23^\circ$.

§ These will be enough exercises of this type unless much practice is needed.

(ix) AB = CD = 5·8 cm., AD = 4·7 cm., $\angle$ A = 72°, $\angle$ BDC = 46°.

(x) AB = 6·3 cm., CD = 5·4 cm., $\angle$ BAC = 64°, $\angle$ ACD = 59°, $\angle$ D = 76°.

(xi) AB = 5·2 cm., AC = 6·8 cm., AD = 5·6 cm., $\angle$ BAC = 106°, $\angle$ BAD = 122°.

(xii) $\angle$ ABD = $\angle$ ADB = 50°, $\angle$ C = 68°, BC = 2·3 in., CD = 3·0 in.

(xiii) AC = 11·0 cm., AB = 5·9 cm., BD = 7·4 cm., $\angle$ BAC = 22°, $\angle$ ABD = 68°.

¶Ex. 93. Take a point O on your paper and mark a number of points each of which is 2 in. from O. [To do this most easily, open your dividers 2 in., place one point at O, and mark points with the other.] The pattern you obtain is a circle; all the points 2 in. from O are on this circle.

¶Ex. 94. How does a gardener mark out a circular bed?

Ex. 95. Draw a figure to represent the area commanded by a gun which can fire a distance of 5 miles in any direction. (Represent 1 mile by 1 cm.)

Ex. 96. Two forts are situated 7 miles apart; the guns in each have a range of 5 miles; draw a figure showing the area in which an enemy is exposed to the fire of both forts. (Represent 1 mile by 1 cm.)

Ex. 97. A circular grass plot 70 feet in radius is watered by a man standing at a fixed point on the edge with a hose which can throw water a distance of 90 feet; show the area that can be watered. (Represent 10 feet by 1 cm.)

What is the distance between the two points on the edge of the grass which the water can only just reach?

¶Ex. 98. Mark two points A, B, 3 in. apart.

(i) On what curve do all the points lie which are 2·7 in. from A?

(ii) On what curve do all the points lie which are 2·2 in. from B?

(iii) Is there a point which is 2·7 in. from A and also 2·2 in. from B?

(iv) Is there more than one such point?

Ex. 99. A and B are two points 7·4 cm. apart; find, as in Ex. 98, a point which is 5·7 cm. from A and 3·5 cm. from B.

Ex. 100. Repeat Ex. 99, without drawing the whole circles. See fig. 42.

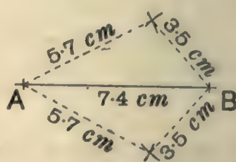


fig. 42.

Ex. 101. (i) Construct a triangle, the lengths of whose sides are 12·1 cm., 8·2 cm., 6·1 cm. See Ex. 100.

(ii) In how many points do your construction circles intersect?

(iii) How many triangles can you construct with their sides of the given lengths? Are these triangles **congruent** (i.e. could they be made to fit on one another exactly)?

Ex. 102. Construct triangles to the following measurements:—

(It is best to draw the longest side first.)

§(i)	BC = 8·9 cm.,	CA = 8·3 cm.,	AB = 6·7 cm.
(ii)	BC = 6·9 cm.,	CA = 11·4 cm.,	AB = 5·8 cm.
§(iii)	BC = 5·3 cm.,	CA = 8·3 cm.,	AB = 2·5 cm.
(iv)	BC = 3·9 in.,	CA = 2·5 in.,	AB = 2·5 in.
(v)	BC = 3·2 in.,	CA = 3·2 in.,	AB = 1·8 in.
(vi)	BC = 6·6 cm.,	CA = 6·6 cm.,	AB = 9·3 cm.
(vii)	BC = 6·9 cm.,	CA = 6·9 cm.,	AB = 6·9 cm.
(viii)	BC = 6·5 cm.,	CA = 9·6 cm.,	AB = 7·2 cm.
§(ix)	BC = 2·1 in.,	CA = 1·1 in.,	AB = 3·2 in.
§(x)	BC = 4·1 in.,	CA = 4·1 in.,	AB = 4·1 in.

§ These will be enough exercises of this type unless much practice is needed.

A triangle which has two of its sides equal is called an **isosceles** triangle (ἴσος equal, σκέλος a leg).

A triangle which has all its sides equal is called an **equilateral** triangle (*aequus* equal, *latus* a side).

A triangle which has no two of its sides equal is called a **scalene** triangle (σκαληνός lame or uneven).

¶Ex. 103. Which of the triangles in Ex. 102 are isosceles, and which are equilateral?

¶Ex. 104. Make a triangle of strips of cardboard, its sides being 4 in., 5 in., 6 in. long.

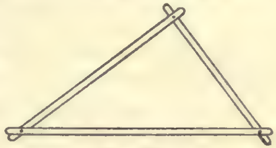


fig. 43.

To do this, cut out strips about $\frac{1}{2}$ in. longer than the given lengths, pierce holes at the given distances apart and hinge the strips together by means of string, or gut with knots, or by means of "eyes" such as a shoemaker uses.

Can the shape of the triangle be altered without bending or straining the sides?

¶Ex. 105. Make a quadrilateral of strips of cardboard, its sides being 3 in., 3·5 in., 4·5 in., 6 in. long.

Can its shape be altered without bending or straining?

Could it be made rigid by a strip joining two opposite corners?

The straight line joining opposite corners of a quadrilateral is called a **diagonal**.

¶Ex. 106. Repeat Ex. 105 with a pentagon each of whose sides is 3 in. long. How many additional strips must be put in to make the frame-work rigid?

Ex. 107. Construct quadrilaterals ABCD to the following measurements:—

(i) AB = 2·3 in., BC = 2·1 in., CD = 3·3 in., DA = 1·5 in., BD = 3·4 in.

(ii) AB = CD = 6·4 cm., BC = DA = 3·7 cm., BD = 5·7 cm.

(iii) AB = AD = 1·9 in., CB = CD = 2·9 in., BD = 2·5 in.

(iv) $AB = BC = CD = DA = 5{\cdot}1$ cm., $AC = 9{\cdot}2$ cm.

(v) $AB = 3{\cdot}8$ in., $BC = 1{\cdot}7$ in., $CD = 1{\cdot}0$ in., $DA = 4{\cdot}9$ in., $\angle B = 146°$.

(vi) $AB = 5{\cdot}3$ cm., $BC = 6{\cdot}3$ cm., $CD = 6{\cdot}7$ cm., $\angle B = 70°$, $\angle C = 48°$.

(vii) $AB = 2{\cdot}7$ cm., $BC = 7{\cdot}5$ cm., $AD = 8{\cdot}4$ cm., $\angle C = 98°$, $\angle DBC = 28°$.

(viii) $BC = CD = 2{\cdot}4$ in., $BD = 1{\cdot}9$ in., $\angle ABD = \angle ADB = 67°$.

(ix) $AB = 9{\cdot}3$ cm., $BC = DA = 6{\cdot}7$ cm., $\angle A = 111°$, $\angle B = 28°$.

Ex. 108. Construct pentagons ABCDE to the following measurements :—

(i) $AB = 2{\cdot}0$ in., $BC = 2{\cdot}2$ in., $CD = 1{\cdot}7$ in., $DE = 2{\cdot}2$ in., $EA = 2{\cdot}5$ in., $\angle B = 111°$, $\angle C = 149°$.

(ii) $AB = 1{\cdot}7$ in., $BC = 1{\cdot}0$ in., $CD = 2{\cdot}2$ in., $DE = 3{\cdot}4$ in., $EA = 0{\cdot}5$ in., $\angle A = 126°$, $\angle B = 137°$.

(iii) $AB = 5$ cm., $BC = 3{\cdot}7$ cm., $CD = 3{\cdot}6$ cm., $DE = 4{\cdot}3$ cm., $EA = 3{\cdot}8$ cm., $AC = 6{\cdot}4$ cm., $AD = 6{\cdot}7$ cm.

(iv) $AB = BC = CD = DE = EA = 5{\cdot}0$ cm., $AC = BE = 8{\cdot}1$ cm.

Pyramids.—The Tetrahedron.

Figs. 44, 45 represent a **tetrahedron**, i.e. a **solid** bounded by four **faces** (*τετρα-* four-, *ἕδρα* a seat, a base).

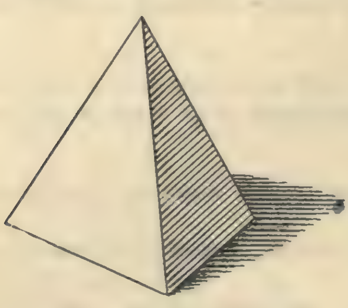


fig. 44.

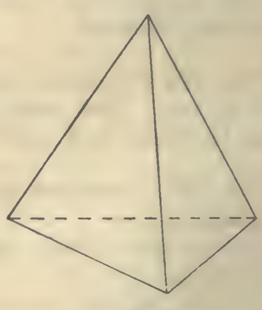


fig. 45.

¶Ex. 109. Make a tetrahedron of thin cardboard (or thick paper); fig. 46 represents what you will have to cut out (this will be referred to as the **net** of the tetrahedron); each of the small triangles is equilateral (their sides should be 4 in. long); the paper is to be creased (not cut) along the dotted lines, and the edges fastened with stamp-edging.

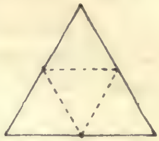


fig. 46.

¶Ex. 110. How many corners has a tetrahedron?

¶Ex. 111. How many edges meet at each corner?

¶Ex. 112. What is the total number of edges?

¶Ex. 113. Can you explain why the total number of edges is not equal to the number of corners multiplied by the number of edges at each corner?

¶Ex. 114. What is the greatest number of faces you can see at one time?

Ex. 115. Make sketches of your model in three or four different positions.

Figs. 47, 48 represent a **square pyramid** (i.e. a pyramid on a square base).

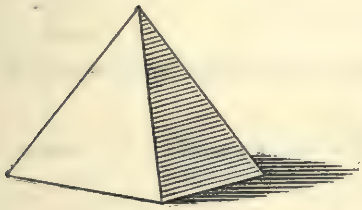


fig. 47.

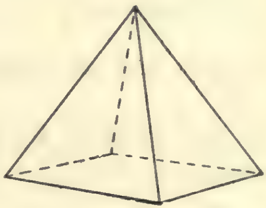


fig. 48.

¶Ex. 116. Make a square pyramid (fig. 49 represents its net); make each side of the square 2 in. long and the equal sides of each triangle 2·5 in. long.

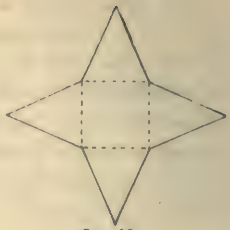


fig. 49.

¶Ex. 117. How many corners has a square pyramid?

¶Ex. 118. How many edges?

¶Ex. 119. What is the greatest number of faces you can see at one time?

Ex. 120. Make sketches of your model in three or four different positions.

Ex. 121. Draw the net of a regular hexagonal pyramid, and make a rough sketch of the solid figure.

TRIANGLES (*continued*).

¶Ex. 122. What is the sum of the angles of a triangle?

¶Ex. 123. Cut out a paper triangle; mark its angles; tear off the corners and fit them together with their vertices at one point, as in fig. 50.

What relation between the angles of a triangle is suggested by this experiment?

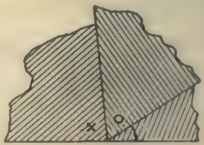


fig. 50.

¶Ex. 124. Cut out a paper quadrilateral and proceed as in Ex. 123.

¶Ex. 125. If two angles of a triangle are 54°, 76°, what is the third angle?

¶Ex. 126. If two angles of a triangle are 27°, 117°, what is the third angle?

¶Ex. 127. If two angles of a triangle are 23°, 31°, what is the third angle?

¶Ex. 128. If two angles of a triangle are 65°, 132°, what is the third angle?

¶Ex. 129. If the angles of a triangle are all equal, what is the number of degrees in each?

¶Ex. 130. If one angle of a triangle is 36°, and the other two angles are equal, find the other two angles.

¶Ex. 131. Repeat Ex. 130 with the given angle (i) 90°, (ii) 132°, (iii) 108°.

¶Ex. 132. In fig. 51, triangle ABC has $\angle$ A = 90°, AD is drawn perpendicular to BC. If $\angle$ B = 27°, find the angles marked x, y, z.

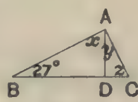


fig. 51.

¶Ex. 133. Repeat Ex. 132 with (i) $\angle$ B = 54°, (ii) $\angle$ B = 33°, (iii) $\angle$ B = 45°.

¶Ex. 134. A triangle ABC has $\angle$ A = 75°, $\angle$ B = 36°; if AD is drawn perpendicular to BC, find each angle in the figure.

¶Ex. 135. Would it be possible to have triangles with angles of (i) 90°, 60°, 30°, (ii) 77°, 84°, 20°, (iii) 59°, 60°, 61°, (iv) 135°, 22°, 22°, (v) 73°, 73°, 33°, (vi) 54°, 54°, 72°?

¶Ex. 136. (i) Give two sets of angles which would do for the angles of a triangle.

(ii) Give two sets which would not do.

Ex. 137. Construct a triangle ABC, having $\angle$ A = 76°; $\angle$ B = 54°, BC = 2·8 in. What is $\angle$ C?

First find $\angle$ C by calculation, then construct the triangle as though BC, $\angle$ B and $\angle$ C were given.

Measure $\angle$ A; this will be a means of testing the accuracy of your drawing.

Ex. 138. Construct triangles to the following measurements:—

(i) BC = 8·0 cm., $\angle$ A = 77°, $\angle$ B = 46°.

§(ii) AB = 7·3 cm., $\angle$ B = $\angle$ C = 57°.

(iii) $\angle$ B = 114°, $\angle$ C = 33°, AC = 9·4 cm.

§(iv) $\angle$ C = $\angle$ A = 60°, AB = 2·7 in.

(v) AB = 4·3 cm., $\angle$ A = 57°, $\angle$ C = 33°.

§(vi) BC = 1·1 in., $\angle$ A = 14°, $\angle$ C = 52°.

¶Ex. 139. Draw a quadrilateral ABCD; join AC.

(i) What is the sum of the angles of $\triangle$ ABC?

(ii) " " " " " $\triangle$ ADC?

(iii) " " " " " the quadrilateral?

¶Ex. 140. If three of the angles of a quadrilateral are 110°, 60°, 80°, what is the fourth angle?

¶Ex. 141. Repeat Ex. 140 with angles of (i) 75°, 105°, 75°, (ii) 90°, 90°, 90°, (iii) 123°, 79°, 35°.

¶Ex. 142. If two angles of a quadrilateral are 117° and 56°, and the other two angles are equal, what are the other two angles?

¶Ex. 143. If the four angles of a quadrilateral are all equal, what is the number of degrees in each?

¶Ex. 144. Draw a pentagon ABCDE freehand; join AC and AD. What is the sum of the angles of the pentagon?

¶Ex. 145. If the five angles of a pentagon are all equal, what is the number of degrees in each?

Ex. 146. Construct a triangle ABC having BC = 6 in., CA = 5 in., AB = 4 in.

Construct a triangle A′B′C′ having B′C′ = 6 cm., C′A′ = 5 cm., A′B′ = 4 cm.

Measure and compare the angles of the two triangles.

§ These will be enough exercises of this type unless much practice is needed.

Ex. 147. Construct a triangle ABC having $BC = 4$ in., $\angle B = 90°$, $\angle C = 30°$.

Construct a triangle $A'B'C'$ having $B'C' = 2$ in., $\angle B' = 90°$, $\angle C' = 30°$.

Measure and compare the sides of the two triangles.

Ex. 148. Construct a triangle ABC having $BC = 9$ cm., $\angle B = 18°$, $\angle C = 35°$.

Construct a triangle $A'B'C'$ having $B'C' = 6$ cm., $\angle B' = 18°$, $\angle C' = 35°$.

Measure and compare the sides of the two triangles.

Ex. 149. Draw any triangle. Without using a graduated ruler, draw three straight lines respectively double the lengths of the sides of the triangle; with these three lines as sides construct a triangle. Compare the angles of the two triangles.

¶Ex. 150. How many triangles of different sizes can you make which have their angles 30°, 60° and 90°?

Figures which are of the same shape (even though of different sizes) are called **similar figures.**

¶Ex. 151. Which of the following pairs of figures are of necessity similar :—(i) two circles, (ii) two right-angled triangles, (iii) two isosceles triangles, (iv) two equilateral triangles, (v) two squares, (vi) two rectangles, (vii) two right-angled isosceles triangles, (viii) two regular hexagons, (ix) two spheres, (x) two cubes?

¶Ex. 152. What is a triangle called which has two of its sides equal? What do you know about the angles of such a triangle?

¶Ex. 153. What is a triangle called which has all its sides equal? What do you know about the angles of such a triangle?

Ex. 154. Sketch a right-angled triangle (i.e. a triangle which has *one* of its angles a right angle).

What kind of angles are the other two? Give a reason.

¶**Ex. 155.** Try to make a triangle on a base of 1·5 in. having the angles at the ends of the base each right angles.

¶**Ex. 156.** Draw an obtuse-angled triangle freehand (i.e. a triangle which has *one* of its angles obtuse).

What kind of angles are the other two? Give a reason.

¶**Ex. 157.** Try to make a triangle on a base of 2 in. having angles of 120°, 60° at the ends of the base.

How could you have foretold the result of your experiment?

¶**Ex. 158.** Sketch a triangle which is neither right-angled nor obtuse-angled. What do you note about its angles? What would you call such a triangle?

¶**Ex. 159.** Can you draw a right-angled isosceles triangle? What will its other angles be?

¶**Ex. 160.** Can you draw an obtuse-angled isosceles triangle?

¶**Ex. 161.** Can you draw an isosceles triangle with the equal angles obtuse?

¶**Ex. 162.** Which of the following combinations of angles are possible for a triangle?

(i) Right, acute, acute. (ii) Right, acute, obtuse.
(iii) Acute, acute, acute. (iv) Obtuse, obtuse, acute.
(v) Right, right, acute. (vi) Acute, acute, obtuse.

Ex. 163. Make a table showing in column A whether the triangles in fig. 52, are acute-, right-, or obtuse-angled, and in column B whether they are equilateral, isosceles, or scalene.

Triangle numbered	A	B
1		
2		
3		

So far you have only measured to one place of decimals in inches or centimetres, but you will often need to measure more

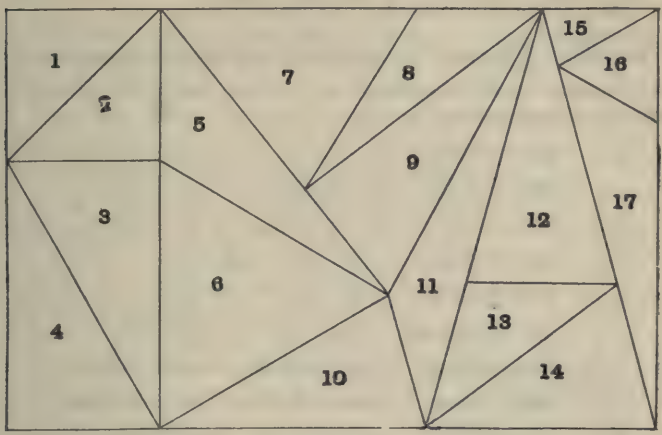


fig. 52.

accurately. To do this you must imagine each tenth of an inch (or centimetre) divided again into 10 equal parts.

The line AB is more than 1·2 in. and less than 1·3 in.;

fig. 53.

if its length is almost exactly half-way between these measurements you will say it is 1·25 in.;

if it is a little more than half-way you will say it is 1·26 in.;

if it is about a third of the way you will say it is 1·23 in.;

if it is about two-thirds of the way you will say it is 1·27 in. and so on.

With a little practice you ought to get this figure nearly accurate.

In the same way you can measure angles to within less than a degree.

¶Ex. 164. (i) What fraction of an inch does a figure in the second place of decimals represent?

(ii) What fraction of an inch is ·03?

Ex. 165. Construct triangles to the following measurements:

(All lengths should be measured to 2 decimal places and all angles to within one-fifth of a degree*.)

(i) BC = 3·18 in., AB = 3·18 in., $\angle$ B = 33·5°.

(ii) BC = 2·39 in., CA = 2·44 in., $\angle$ C = 63·5°.

(iii) AB = 2·82 in., AC = 2·77 in., $\angle$ A = 137°.

(iv) AB = 3·00 in., $\angle$ A = 61°, $\angle$ B = 59°.

(v) BC = 3·52 in., $\angle$ B = 25°, $\angle$ C = 23°.

(vi) AC = 10·65 cm., $\angle$ A = 54·5°, $\angle$ C = 36°.

(vii) BC = 6·40 cm., CA = 9·05 cm., AB = 7·63 cm.

(viii) BC = 7·69 cm., CA = 9·30 cm., AB = 5·30 cm.

(ix) BC = 4·53 in., CA = 2·68 in., AB = 2·02 in.

(x) AB = 2·71 in., $\angle$ B = 55·5°, $\angle$ C = 67·5°.

(xi) $\angle$ A = 24°, $\angle$ C = 47·5°, BC = 3·04 cm.

(xii) $\angle$ A = 133°, BC = 10·73 cm., $\angle$ B = 23·5°.

(xiii) $\angle$ C = 90°, BC = 1·00 in., CA = 2·00 in.

(xiv) BC = 4·09 cm., CA = 3·31 cm., AB = 7·54 cm.

(xv) $\angle$ A = 90·5°, $\angle$ B = 78°, BC = 3·54 in.

(xvi) AB = 2·99 in., $\angle$ B = 127·5°, $\angle$ C = 53·5°.

(xvii) AB = 2·92 in., $\angle$ B = 59°, AC = 2·39 in.

(xviii) $\angle$ B = 33·5°, BC = 2·61 in., CA = 1·54 in.

(xix) CB = 2·16 in., CA = 2·64 in., $\angle$ B = 64·5°.

(xx) $\angle$ A = 24°, AB = 7·76 cm., BC = 2·87 cm.

* This can be done with a well graduated protractor of 2-inch radius, with a smaller protractor it is difficult.

Parallels and Perpendiculars.

¶**Ex. 166.** Give instances of parallel straight lines (e.g. the flooring boards of a room, the edges of your paper).

¶**Ex. 167.** Draw with your ruler two straight lines as nearly parallel as you can judge; draw a straight line cutting them as in fig. 54; measure the angles marked. These are called **corresponding** angles. Are they equal?

fig. 54.

¶**Ex. 168.** Repeat Ex. 167 two or three times drawing the cutting line in different directions.

¶**Ex. 169.** Draw two straight lines which are *not* parallel and proceed as in Ex. 167. Are the angles equal?

¶**Ex. 170.** Draw a straight line AB (see fig. 55). In AB take a point C; through C draw CD making $\angle$ BCD $= 90°$ (use your set square); through A draw AE making $\angle$ BAE $= 90°$. Are AE and CD parallel?

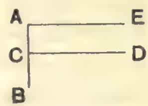


fig. 55.

¶**Ex. 171.** In the figure you obtained in the last Ex. draw two more straight lines at right angles to CD; measure the part of each of these three straight lines cut off between AE and CD; are these parts equal?

Would these three parts be equal if the lines all made different angles with CD?

¶**Ex. 172.** Repeat Ex. 170 with $\angle$ BCD $=$ $\angle$ BAE $= 60°$ (use your set square); draw three straight lines at right angles to CD; measure the parts cut off between AE and CD.

¶**Ex. 173.** Repeat Ex. 170 with $\angle$ BCD $=$ $\angle$ BAE $= 30°$ (use your set square); measure as in Ex. 172.

In the course of Ex. 166–173, you should have observed the following properties of parallel straight lines :—

(i) they do not meet however far they are produced in either direction.

(ii) if a straight line cuts them, corresponding angles are equal.

(iii) parallel straight lines are everywhere equidistant.

To draw a parallel to a given line QR through a given point P by means of a set square and a straight edge.

It is important that the straight edge should not be bevelled (if it is bevelled the set square will slip over it); in the figures below a ruler with an unbevelled edge is represented, but the base of the protractor or the edge of another set square will do equally well.

Place a set square so that one of its edges lies along the given line QR (as at (i)); hold it in that position and place the straight (unbevelled) edge in contact with it; now hold the straight edge firmly and slide the set square along it. The edge which originally lay along QR will always be parallel to QR. Slide the set square till this edge passes through P (as at (ii)), hold it firmly and rule the line.

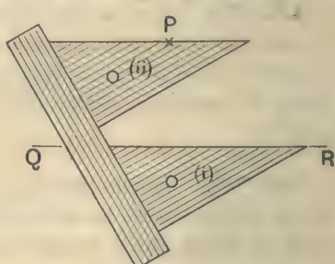


fig. 56.

This method of drawing parallels suggests an explanation of the term *corresponding* angles.

Ex. 174. Draw a straight line QR and mark a point P; through P draw a parallel to QR.

Ex. 175. Repeat Ex. 174 several times using the different edges of the set square. (See fig. 57, and Ex. 170.)

Ex. **176.** Near the middle of your paper draw an equilateral triangle with its sides 1 in. long; through each vertex draw a line parallel to the opposite side.

If the angle between two straight lines is a right angle the straight lines are said to be at right angles to one another or **perpendicular** to one another.

To draw through a given point P a straight line perpendicular to a given straight line QR.

The difficulty of drawing a line right to the corner of a set square can be overcome as follows :—

Place a set square so that one of the edges containing the right angle lies along the given line QR (as at (i)); place the straight edge in contact with the side opposite the right angle; now hold the straight edge firmly and slide the set square along it; the edge which lay along QR will always be parallel to QR and the other edge containing the right angle will always be perpendicular to QR. Slide the set square till this other edge passes through P; then draw the perpendicular.

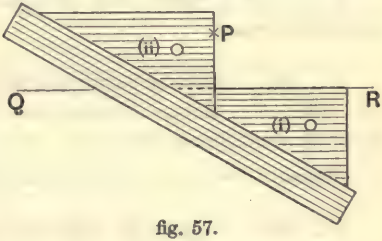


fig. 57.

Ex. **177.** Through a given point in a straight line draw a perpendicular to that line.

Ex. **178.** Draw an acute-angled triangle; from each vertex draw a perpendicular to the opposite side.

Ex. **179.** Repeat Ex. 178 with an obtuse-angled triangle. (You will find it necessary to produce two of the sides.)

Ex. **180.** Describe a circle, take any two points A, B upon it, join AB; from the centre draw a perpendicular to AB; measure the two parts of AB.

Ex. 181. Draw an acute-angled triangle; from the middle point of each side draw a straight line at right angles to that side.

Ex. 182. Repeat Ex. 181 with an obtuse-angled triangle.

Parallelogram, Rectangle, Square, Rhombus.

Ex. 183. Make an angle $ABC = 65°$, cut off $BA = 2{\cdot}2$ in., $BC = 1{\cdot}8$ in.; through A draw AD parallel to BC, through C draw CD parallel to BA.

A four-sided figure with its opposite sides parallel is called a **parallelogram.**

Ex. 184. Make a parallelogram two of whose adjacent sides (i.e. sides next to one another) are 6·3 cm. and 5·1 cm., the angle between them being 34°.

Measure the other sides and angles.

Ex. 185. Repeat Ex. 184 with the following measurements: 10·4 cm., 2·6 cm., 116°.

Ex. 186. Repeat Ex. 184 with the following measurements: 10·4 cm., 2·6 cm., 64°.

¶Ex. 187. Draw a parallelogram two of whose sides are 3·7 in., and 0·8 in., and one of whose angles is 168°.

Are its opposite sides and angles equal?

It will be proved later on that the opposite sides and angles of a parallelogram are always equal.

¶Ex. 188. Construct a quadrilateral ABCD having $AB = CD = 4{\cdot}7$ cm., $AD = BC = 7{\cdot}2$ cm., and $\angle A = 85°$. Is it a parallelogram?

¶Ex. 189. Make a parallelogram of strips of cardboard, one pair of sides being 5 in. long and the other pair 3 in.

¶Ex. 190. Open one of the acute angles of the framework you have just made until it is a right angle; examine the other angles.

A parallelogram which has one of its angles a right angle is called a **rectangle.**

Ex. 191. Draw a rectangle having sides = 7·3 cm. and 3·7 cm. Measure all its angles.

Ex. 192. Draw a parallelogram having sides = 9·2 cm. and 4·3 cm., and one angle = 125°. Draw its diagonals, and measure their parts.

Ex. 193. Repeat the last Ex. with the following measurements, 8·6 cm., 6·8 cm., 68°; test any facts you noted in that Ex.

Ex. 194. Draw a parallelogram and measure the angles between its diagonals; are any of them equal? Give a reason.

Ex. 195. Draw a rectangle having sides = 3·5 in. and 2·3 in. Measure its diagonals.

Ex. 196. Repeat the last Ex. with the following measurements, (i) 8·6 cm., 11·2 cm., (ii) 14·3 cm., 2·8 cm.

A rectangle which has two adjacent sides equal is called a **square.**

Ex. 197. Draw a square having one side = 5·6 cm. Measure all its sides and angles.

Ex. 198. Draw a square having each side = 3·2 in. Measure its diagonals and the angles between them.

¶Ex. 199. Explain how you would test by folding whether a pocket handkerchief is square.

¶Ex. 200. Make a paper square by folding.

A parallelogram which has two adjacent sides equal is called a **rhombus.**

Ex. 201. Draw a rhombus having one side = 2·2 in. and one angle = 54°. Measure the sides, angles, diagonals, and the angles between the diagonals.

Ex. 202. Repeat Ex. 201 making one side = 6·8 cm. and one angle = 105°.

In the course of Ex. 183–202, you should have observed the following properties:—

(i) The opposite sides and angles of a parallelogram are equal.

(ii) The diagonals of a parallelogram bisect one another.

The above properties, (i) and (ii), must be true for a rectangle, square, and rhombus, since these are particular cases of a parallelogram (i.e. special kinds of parallelogram).

(iii) All the angles of a rectangle are right angles.

(iv) The diagonals of a rectangle are equal.

(iii) and (iv) must be true for a square, since a square is a particular case of a rectangle.

(v) The diagonals of a square intersect at right angles.

(vi) The diagonals of a rhombus intersect at right angles.

Since a square may be regarded as a particular case of a rhombus, (v) might have been deduced from (vi).

Ex. 203. Copy the table given below; indicate for which figures the given properties are always true by inserting the words "yes" or "no" in the corresponding spaces.

	Opposite sides and angles equal	Diagonals bisect one another	Angles at corners right angles	Diagonals equal	Diagonals at right angles	Adjacent sides equal
Parallelogram						
Rectangle						
Square						
Rhombus						

A **square inch** is a square whose sides are one inch long.

Ex. 204. Draw a square inch, and measure its diagonals.

Ex. 205. Draw a square ABCD having its sides 3 in. long; divide AB and BC into inches and through the points of division draw parallels to the sides of the square. Into how many square inches is ABCD divided?

Ex. 206. Repeat Ex. 205 with a square 5 in. long.

Ex. 207. Draw a square having its sides 6 cm. long; divide it up into square centimetres; how many are there?

Ex. 208. Describe a rectangle ABCD having AB=5 in., BC=4 in.; divide it up into square inches; how many are there?

Ex. 209. Describe a rectangle 6 cm. by 3 cm.; divide it up into square centimetres; how many are there?

Cube, Cuboid, and Prism.

Figs. 58, 59 represent a **cube** (i.e. a solid bounded by six equal squares).

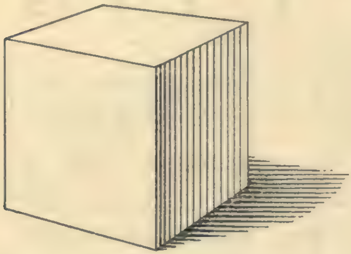


fig. 58.

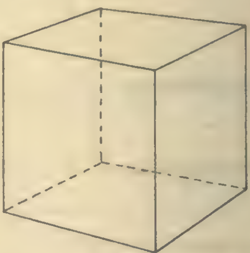


fig. 59.

¶Ex. 210. Make a cube of thin cardboard; its net is given in fig. 60 (see Ex. 109); each edge should be 2 in. long.

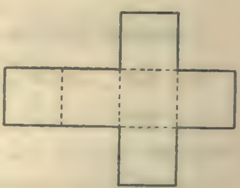


fig. 60.

¶Ex. 211. How many corners has a cube?

¶Ex. 212. How many edges has a cube?

¶Ex. 213. How many edges meet at each corner?

¶Ex. 214. Is the number of edges equal to the number of corners multiplied by the number of edges which meet at each corner? Give a reason.

¶Ex. 215. How many edges has each face?

¶Ex. 216. Is the number of edges equal to the number of faces multiplied by the number of edges belonging to each face? Give a reason.

¶Ex. 217. Is the number of angles equal to the number of faces multiplied by the number of angles belonging to each face? Give a reason.

¶Ex. 218. What is the greatest number of faces, edges, and corners you can see at one time?

Ex. 219. Make sketches of a cube from three different points of view.

Figs. 61, 62 represent a **cuboid** or **rectangular block** (i.e. a solid like a brick).

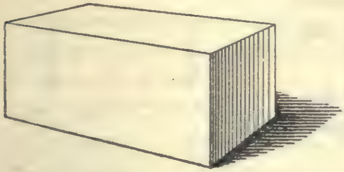


fig. 61.

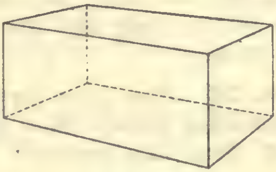


fig. 62.

¶Ex. 220. How does a cuboid differ from a cube?

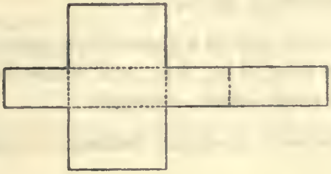


fig. 63.

¶Ex. 221. Make a cuboid of thin cardboard; its net is given in fig. 63; it should measure 3 in. by 1·9 in. by 1·3 in.

¶Ex. 222. Choose one edge of the cuboid; how many other edges are equal to this edge?

Figs. 64, 65 represent a regular **three-sided prism.**

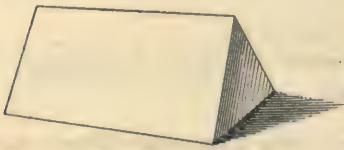


fig. 64.

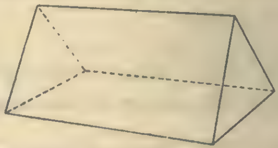


fig. 65.

¶Ex. 223. What sort of figures are the ends of the prism in fig. 64? What are the sides?

¶Ex. 224. Make a regular three-sided prism; its net is given in fig. 66; the short lines should each be 2 in. long, and the long ones 3·5 in.

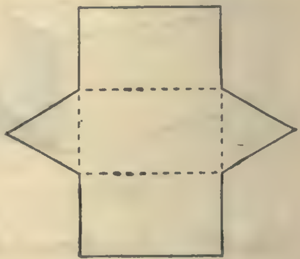


fig. 66.

¶Ex. 225. How many edges has a three-sided prism? How many faces? How many corners?

¶Ex. 226. What is the greatest number of faces, edges, and corners you can see at one time?

¶Ex. 227. Make sketches of your model from three different points of view.

¶Ex. 228. Draw the net of a three-sided prism whose ends are triangles with sides 3 in., 3 in., 1 in. and whose length is 1·5 in.

Such a prism is often called a **wedge.**

Figs. 67, 68 represent a regular **hexagonal prism.**

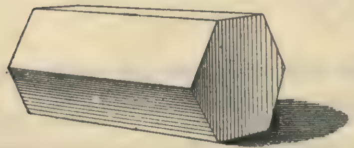


fig. 67.

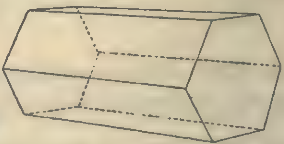


fig. 68.

¶Ex. 229. What sort of figures are the ends of the prism in fig. 67? What are the sides?

¶Ex. 230. Draw its net.

¶Ex. 231. What is the number of edges, faces, and corners?

¶Ex. 232. What is the greatest number of edges, faces, and corners you can see at one time?

¶Ex. 233. Make sketches of a regular hexagonal prism from three different points of view.

Drawing to Scale.

When drawing a map, or plan, to scale you should always begin by making a rough sketch showing the given dimensions, and then work from the sketch.

The **bearing** of a place A from a second place B is the point of the compass towards which a man at B would be facing if he were looking in the direction of A.

By "N. 10° W." or "10° W. of N." is meant the direction in which you would be looking if you first faced due north and then turned through an angle of 10° towards the west.

Ex. 234. A is 2·5 miles W. of B, and C is 4·5 miles S. of A. What is the distance from B to C? What is the bearing of B from C, and of C from B? (Scale 1 mile to 1 inch.)

Ex. 235. G is 7·5 miles S. of H, and 10 miles W. of K. What is the distance and bearing of K from H? (Scale 1 mile to 1 cm.)

Ex. 236. X is 17·5 miles N.W. of Y, Y is 23 miles N.E. of Z. What is the distance and bearing of X from Z? (Scale 10 miles to 1 inch.)

Ex. 237. P is 64 miles W. of Q, R is due N. of Q; if PR is 72 miles, what is QR? What is the bearing of P from R? (Scale 10 miles to 1 cm.)

Ex. 238. Draw a plan of a room 30 ft. by 22 ft.; find the distances between opposite corners. (Scale 2 ft. to 1 cm.)

Ex. 239. Exeter is 48 miles W. of Dorchester, and Barnstaple is 35 miles N.W. of Exeter. What is the distance and bearing of Barnstaple from Dorchester? (Scale 10 miles to 1 in.)

Ex. 240. Rugby is 44 miles N. of Oxford, and Reading is 24 miles S. 30° E. of Oxford. Find the distance from Rugby to Reading. (Scale 10 miles to 1 in.)

Ex. 241. Southampton is 72 miles S. 53° W. of London, Gloucester is 75° W. of N. from London, and 29° W. of N. from Southampton. Find the distance between Southampton and Gloucester. (Scale 10 miles to 1 cm.)

In the following exercises, use any suitable scale; always state what scale you use.

Ex. 242. Draw a plan of a rectangular field 380 yards by 270 yards. What is the distance between the opposite corners?

Ex. 243. The legs of a pair of compasses are 10 cm. long. I open them to an angle of 35°. What is the distance between the compass points?

Ex. 244. Two blockhouses are known to be 1000 yards apart, and one of them is due E. of the other. A party of the enemy are observed by one blockhouse in a N.W. direction, and at the same time by the other in a N.E. direction. How far are the enemy from each blockhouse?

Ex. 245. A and B are two buoys 800 yards apart, B due N. of A. A vessel passes close to B, and steering due E., observes that after 5 minutes the bearing of A is 57° W. of S. Find the distance the vessel has moved.

Ex. 246. Stafford is 27 miles from Derby and the same distance from Shrewsbury, and the three towns are in a straight line. Birmingham is 40 miles from Shrewsbury and 35 from Derby. How far is Stafford from Birmingham?

Ex. 247. A buoy is moored by a cable 55 feet long; at low tide the distance between the extreme positions the buoy can occupy is 100 feet. What will be the distance between the extreme positions when the water is 24 feet higher?

Ex. 248. Two ships sail from a port, one due N. at 15 miles an hour, the other E.N.E.; at the end of half an hour they are in line with a lighthouse which is 11 miles due E. of the port. At what rate does the second ship sail?

Ex. 249. A donkey is tethered to a point 20 feet from a long straight hedge; he can reach a distance of 35 feet from the point to which he is tethered. How much of the hedge can he nibble?

Ex. 250. A is a lighthouse. B and C are two ships 3·5 miles apart. B is due north of A, C due east of B, and C north-east of A. Find the distance of both ships from the lighthouse.

Ex. 251. A man standing on the bank of a river sees a tree on the far bank in a direction 20° W. of N. He walks 200 yards along the bank and finds that its direction is now N.E. If the river flows east and west, find its breadth.

Ex. 252. A ferry-boat is moored by a rope 30 yards long to a point in the middle of a river. The rope is kept taut by the current. What angle does it turn through as the boat crosses the river, whose width is 30 yards?

Ex. 253. The case of a grandfather clock is 16 inches wide; the pendulum is hung in the middle of the case and its length is 39 inches. Assuming that the end of the pendulum swings to within 3 inches of each side of the case, find the angle through which it swings.

Ex. 254. Brixham is 4·6 miles N.E. of Dartmouth, Torquay is 4 miles N. of Brixham, Totnes is 7·4 miles S. 75° W. of Torquay; what is the distance and bearing of Totnes from Dartmouth?

Ex. 255. From G go 9 miles W. to H, from H go 12 miles N. to A, from A go 17 miles W. to R. What is the distance from G to R?

Ex. 256. A is 12 miles N. of H, D is 24 miles S. of H, O is due W. of A and OH is 42; find OD and OA.

Ex. 257. $XT = 19$ miles, $MX = 11$ miles, $MT = 17{\cdot}5$ miles; how far is M from the line XT?

Heights and Distances*.

If a man who is looking at a tower through a telescope holds the telescope horizontally, and then raises (or "elevates") the end of it till he is looking at the top of the tower, the angle he has turned the telescope through is called the **angle of elevation** of the top of the tower.

If a man standing on the edge of a cliff looks through a horizontal telescope and then lowers (or "depresses") the end of it till he is looking straight at a boat, the angle he has turned the telescope through is called the **angle of depression** of the boat.

Remember that the angle of elevation and the angle of depression are always angles at the observer's eye.

If O is an observer and A and P two points (see fig. 14), the angle AOP is said to be the **angle subtended** at O by AP.

Ex. 258. In fig. 51, name the angles subtended (i) by BD at A, (ii) by AD at B, (iii) by AC at B.

Ex. 259. A vertical flagstaff 50 feet high stands on a horizontal plane. Find the angles of elevation of the top and middle point of the flagstaff from a point on the horizontal plane 15 feet from the foot of the flagstaff.

Ex. 260. The angle of elevation of the top of the spire of Salisbury Cathedral at a point 1410 feet from its base was found to be 16°. What is the height of the spire?

* For further exercises on heights and distances see p. 59.

Ex. 261. A torpedo boat passes at a distance of 100 yards from a fort the guns of which are 100 feet above sea-level; to what angle should the guns be depressed so that they may point straight at the torpedo boat?

Ex. 262. From the top of Snowdon the Menai Bridge can be seen, the angle of depression being 4°. The height of Snowdon is 3560 feet. How far away is the Menai Bridge?

Ex. 263. From a point A the top of a church tower is just visible over the roof of a house 50 feet high. If the distance from A to the foot of the tower is known to be 160 yards, and from A to the foot of the house 60 yards, find in feet the height of the tower, and the angle of elevation of its top as seen from A.

Ex. 264. A flagstaff stands on the top of a tower. At a distance of 40 feet from the base of the tower, the angle of elevation of the top of the tower is found to be $23\frac{1}{2}°$, and the flagstaff subtends an angle of $25\frac{1}{2}°$. Find the length of the flagstaff and the height of the tower.

Ex. 265. At two points on opposite sides of a poplar the angles of elevation of its top are 39° and 48°. If the distance between the points is 150 feet, what is the height of the tree?

Ex. 266. From the top of a mast 80 feet high the angle of depression of a buoy is 24°. From the deck it is $5\frac{1}{2}°$. Find the distance of the buoy from the ship.

Ex. 267. At a window 15 feet from the ground a flagstaff subtends an angle of 43°; if the angle of depression of its foot is 11°, find its height.

Ex. 268. A man observes the angle of elevation of the top of a spire to be 23°; he walks 40 yards towards it and then finds the angle to be 29°. What is the height of the spire?

Ex. 269. An observer in a balloon, one mile high, observes the angle of depression of a church to be 35°. After ascending vertically for 20 minutes, he observes the angle of depression to be now $55\frac{1}{2}°$. Find the rate of ascent in miles per hour.

Ex. 270. An observer finds that the line joining two forts A and B subtends a right angle at a point C; from C he walks 100 yards towards B and finds that AB now subtends an angle of 107°; find the distance of A from the two points of observation.

Ex. 271. A man on the top of a hill sees a level road in the valley running straight away from him. He notices two consecutive mile-stones on the road, and finds their angles of depression to be 30° and 13° respectively. Find the height of the hill (i) as a decimal of a mile, (ii) in feet.

How to copy a given rectilinear figure.

A **rectilinear** figure is a figure made up of straight lines.

An exact copy of a given rectilinear figure may be made in various ways.

1st method. Suppose that it is required to copy a pentagon ABCDE (as in fig. 69). First copy side AB; then $\angle$ ABC; then side BC; then $\angle$ BCD; etc. You will not find it necessary to copy *all* the sides and angles.

Ex. **272.** Draw a good-sized quadrilateral; copy it by Method I. If you have tracing paper, make the copy on this; then see if it fits the original.

Ex. **273.** Repeat Ex. 272, with an (irregular) pentagon.

2nd method. A simpler way is to prick holes through the different vertices of the given figure on to a sheet of paper below; then join the holes on the second sheet by means of straight lines.

3rd method. Place a sheet of **tracing paper** over the given figure, and mark on the tracing paper the positions of the different vertices. Then join up with straight lines.

4th method—by **intersecting arcs.**

To copy ABCDE by this method (see fig. 69). Make A′B′ = AB.

With centre A′ and radius equal to AC describe an arc of a circle.

With centre B′ and radius equal to BC describe an arc of a circle.

Let these arcs intersect at C′. Then C′ is the copy of C.

Similarly, fix D′ by means of the distances A′D′ and B′D′; fix E′ by means of the distances A′E′ and B′E′.

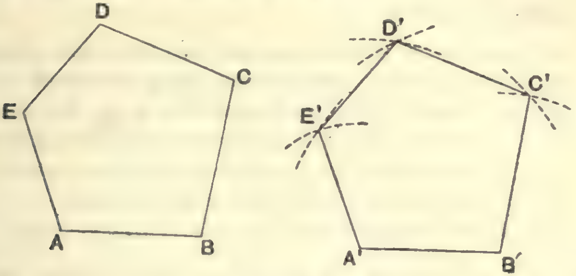


fig. 69.

The five vertices A′B′C′D′E′ are now fixed, and the copy may be completed by joining up.

In Ex. 274—276 the copies should be made on **tracing paper** if possible; the copies can then be fitted on to the originals.

Ex. **274.** Draw, and copy (i) a quadrilateral, (ii) a pentagon, by the method of intersecting arcs. If tracing paper is not used, the copy may be checked by comparing its angles with those of the original.

Ex. **275.** By intersecting arcs, copy figs. 45 and 48.

Ex. **276.** By intersecting arcs, copy the part of fig. 52 which consists of triangles 1, 2, 3, 4, 5, 6.

Symmetry.

¶Ex. 277. Fold a piece of paper once; cut the folded sheet into any pattern you please; then open it out (see fig. 70).

The figure you obtain is said to be **symmetrical about the line of folding.** This line is called an **axis** of symmetry.

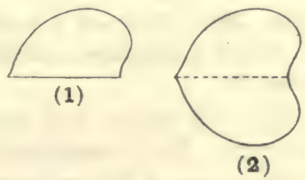


fig. 70.

¶Ex. 278. Make sketches of the symmetrical figures produced when the folded sheet is cut into the following shapes. (Give names if possible.)

(i) a rt. $\angle^d$ $\triangle$ with its shortest side along the crease.
(ii) an isosceles $\triangle$ with its base along the crease.
(iii) a scalene $\triangle$ with its longest side along the crease.
(iv) an obtuse $\angle^d$ $\triangle$ with its shortest side along the crease.
(v) a semi-circle with its diameter along the crease.
(vi) a rectangle with one side along the crease.
(vii) a parallelogram with one side along the crease.

¶Ex. 279. Which of the following figures possess an axis of symmetry? (You may find that in some cases there is more than one axis.) In each case make a sketch showing the axis (or axes), if there is symmetry. (i) isosceles $\triangle$, (ii) equilateral $\triangle$, (iii) square, (iv) rectangle, (v) parallelogram, (vi) rhombus, (vii) regular 5-gon, (viii) regular 6-gon, (ix) circle, (x) a semi-circle, (xi) a figure consisting of 2 unequal circles, (xii) a figure consisting of 2 equal circles.

¶Ex. 280. Fold a piece of paper twice (as in Ex. 30), so that the two creases are at right $\angle$s; cut the folded sheet into any shape, being careful to cut away all of the original edge of the paper. On opening the paper you will find that you have made a figure with two axes of symmetry at right angles.

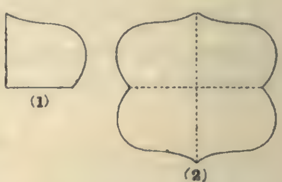


fig. 71.

¶Ex. 281. Cut out a paper parallelogram (be careful not to make it a rhombus). Fold it about a diagonal; do the two halves fit?

You will notice that the parallelogram has no axis of symmetry. Yet it certainly has symmetry of some kind.

The nature of this symmetry will be made clear by the following exercise.

¶Ex. **282.** Draw a parallelogram. Through O, the intersection of the diagonals, draw a number of straight lines, meeting the boundary of the parallelogram.

Suppose that one of these lines meets the boundary in P and P′. Notice that PP′ is bisected at O. This is the case for each of the lines. In fact, every straight line drawn through O to meet the boundary in two points is bisected at O.

The parallelogram is therefore said to be **symmetrical about the point O.** O is called the **centre** of symmetry.

¶Ex. **283.** Which of the figures in Ex. 279 are symmetrical about a centre?

¶Ex. **284.** Fasten a sheet of paper to the desk (or to a drawing board), and on it draw a parallelogram. Drive a pin through the centre of the parallelogram into the desk. With a knife, cut out the parallelogram. When it is cut free from the sheet of paper, turn it round the pin and see if you can bring it into a position where it exactly fits the hole from which it was cut; what angle must it be turned through to fit in this manner?

¶Ex. **285.** Has figure 71 (2) central symmetry?

¶Ex. **286.** Describe the symmetry of the following capital letters:—

A, C, H, I, O, S, X, Z.

Solids may have symmetry. The human body is more or less symmetrical about a plane. Consider the reflexion in a mirror of the interior of a room. The objects in the room together with their reflexions form a symmetrical whole; the surface of the mirror is the plane of symmetry.

¶Ex. **287.** Give 4 instances of solids possessing planes of symmetry.

¶Ex. **288.** Fold a sheet of paper once. Prick a number of holes through the double paper, forming any pattern. On opening the paper you will find that the pin-holes have marked out a symmetrical figure.

Join **corresponding points** as in fig. 72. Notice that when

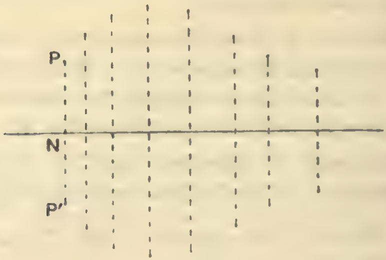


fig. 72.

the figure was folded NP′ fitted on to NP. This shows that NP′ = NP.

The line joining any pair of corresponding points, in a figure which is symmetrical about an axis, is bisected by and perpendicular to the axis of symmetry.

¶Ex. 289. If a point P lies on the axis of symmetry, where is the corresponding point P′?

¶Ex. 290. Draw freehand any curve (such as APB in fig. 73); and rule a straight line XY. Mark a number of points on the

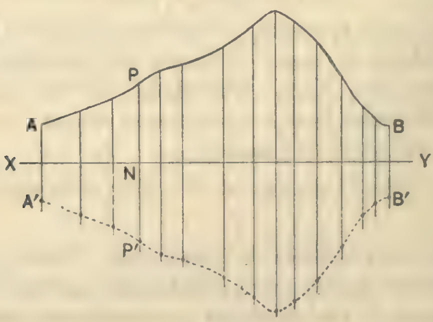


fig. 73.

curve; draw perpendiculars to the line (e.g. PN); produce to an equal distance below the line (e.g. $NP' = PN$). Draw a curve, freehand, through the points thus obtained.

¶Ex. 291. What points would you describe as "corresponding" in the case of a figure with a centre, but no axis of symmetry?

¶Ex. 292. By a method similar to that of Ex. 290 construct a curve symmetrical about a centre.

Points, Lines, Surfaces, Solids.

This should be taken viva voce; the definitions are not intended to be learnt.

In Ex. 109, 116, 210, 221, 224 you have made some solids. The term does not refer to the stuff of which the solids are made, but to the space occupied—geometry deals with size and shape, and not with material, colour, hardness, temperature, &c.

Any body, such as a brick, a sheet of cardboard or paper, a planet, a drop of water, the water of a lake, the air inside a football, the flame of a candle, a smoke-ring, is called a **solid** in the geometrical sense of the word.

¶Ex. 293. Has a brick any length? Has it any breadth? Has it any thickness?

A solid is bounded by one or more **surfaces**.

¶Ex. 294. Which of the solids mentioned above is bounded by one surface only?

¶Ex. 295. A bottle is filled partly with water and partly with oil; the water and oil do not mix; the boundary between them is neither water nor oil, it is not a body but a surface. Has it any thickness?

¶Ex. 296. Consider the boundary between the water of a calm lake and the air. Is it water or is it air? Has it any thickness? Has it any length? Has it any breadth?

¶**Ex. 297.** Suppose the end of the lake is formed by a wall built up out of the water; what would you call the boundary which separates the wall from the air and water? Has it any thickness? Has it any length? Has it any breadth?

A surface has length and breadth, but no thickness.

¶**Ex. 298.** Part of the surface of the wall is wet and part dry; is the boundary between these two parts wet or dry? Has it any thickness? Has it any length? Has it any breadth?

This boundary is really the intersection (or cutting place) of the air-water surface and the wall surface.

The intersection of two surfaces is a **line**. A line has length but no breadth or thickness.

We cannot represent a line on paper except by a mark of some breadth; but, in order that a mark may be a good representation of a line, it should be made as narrow as possible.

¶**Ex. 299.** Take a model of a cube; what are its edges? Have they any length, breadth, or thickness?

¶**Ex. 300.** If you painted part of your paper black, would the boundary between the black and the white have any width?

¶**Ex. 301.** If part of the wall in Ex. 297 were painted red and the rest painted black, would the boundary between the two parts be red or black?

¶**Ex. 302.** Suppose that the red and black paint were continued below the water as well as above, the line bounding the red and black would be partly wet and partly dry; has the boundary between the wet and dry parts of this line any length?

The intersection of two lines is a **point**. A point has neither length, breadth, nor thickness, but it has position.

We cannot represent a point on paper except by a mark of some size; the best way to mark a point is to draw two fine lines through the point.

We have now considered in turn a solid, a surface, a line, and a point. We can also consider them in the reverse order.

A **point** has position but no magnitude.

If a point moves, its path is a **line** (it is said to **generate** a line).

A pencil point when moved over a sheet of paper leaves a streak behind, showing the line it has generated (of course it is not really a line because it has some thickness).

If a line moves, as a rule it generates a **surface**.

A piece of chalk when laid flat on the blackboard and moved sideways leaves a whitened surface behind it. Consider what would have happened if it had moved along its length.

If a surface moves, as a rule it generates a **solid**.

The rising surface of water in a dock generates a (geometrical) solid.

¶Ex. 303. Does a flat piece of paper moved along a flat desk generate a solid?

A **straight line** cannot be defined satisfactorily in a simple way; the idea of a straight line however is familiar to everyone.

¶Ex. 304. How can you roughly test the straightness of (i) a billiard cue, (ii) a railway tunnel, (iii) a metal tube?

¶Ex. 305. How does a gardener obtain a straight line?

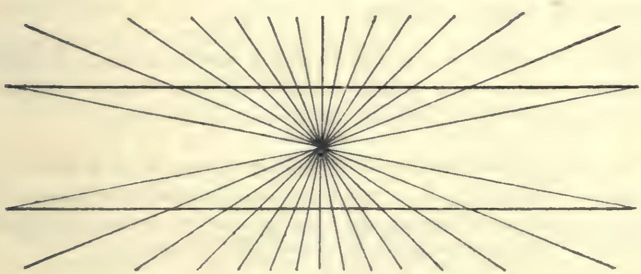


fig. 74.

¶Ex. 306. Test whether the two thick lines in fig. 74 are straight.

Make a careful tracing of one line; move the tracing along and see if it can be made to fit on the line everywhere else; turn the tracing over and try again. If it is impossible to find a position in which they do not fit on one another, then the line must be straight.

The above assumes that the paper is plane.

¶Ex. 307. Test the straightness of the lines in fig. 74 by means of a stretched thread.

Ex. 304–7 lead us to a conclusion which may be stated in various ways as follows:—

(i) *Two straight lines cannot enclose a space.*

(ii) *Two straight lines cannot intersect in more than one point.*

(iii) *If two straight lines have two points in common, they must coincide.*

(iv) *One straight line, and one only, can be drawn through two given points.*

(v) *Two points determine a straight line.*

A surface which is such that the straight line joining every pair of points in it lies wholly in the surface is called a **plane surface**, or, briefly, a **plane**.

¶Ex. 308. Push a straight knitting needle through an apple; does the straight line joining the two points where the needle cuts the surface lie wholly in the surface of the apple?

¶Ex. 309. Test whether the surface of your desk is plane.

Place a straight edge (e.g. the edge of your ruler or set square) against the surface and see if the straight edge touches the surface all along its length; if it does so in all positions, the surface is plane.

¶Ex. 310. Is the lid of your instrument box plane?

¶Ex. 311. Is the glass of your watch plane?

¶Ex. 312. Are the faces of your cuboid plane?

¶Ex. 313. Could you find two points on the surface of a garden roller such that the straight line joining them lies wholly in the surface? Is the surface plane?

Parallel straight lines are defined to be straight lines *in the same plane* which do not meet however far they are produced in either direction.

¶Ex. 314. Can you explain why the words in italics are necessary?

¶Ex. 315. A five-barred gate is half-open; there is one of the gate-posts which the line of the top bar does not meet; is the top bar parallel to this post?

¶Ex. 316. Give instances of pairs of straight lines which are not parallel but do not meet however far they are produced.

¶Ex. 317. Would a set of telegraph poles along the side of a straight road be parallel to one another? Would they be parallel if the road were crooked?

¶Ex. 318. Are the upright edges of a box parallel?

HEIGHTS AND DISTANCES (*Continued from p.* 50).

Ex. 318*a*. The shadow of a tree is 30 feet long when the sun's altitude is 59°; find, by drawing, the height of the tree, taking a scale of 1 inch to 10 feet.

Ex. 318*b*. A telegraph pole standing upright on level ground is 23·6 feet high and is partly supported by a wire attached to the top of the pole at one end and fixed to the ground at the other so that its inclination to the pole is 54° 22′.

Find the length of the wire.

Ex. 318*c*. The angle of elevation of the top of a tower on level ground is read off on a theodolite. Find the height of the tower from the following data:

Reading of theodolite = 15°.

Height of theodolite telescope above ground = 3′ 6″.

Distance of theodolite from foot of tower = 372 yards.

Ex. 318*d*. From a ship at sea the top of Aconcagua has an angle of elevation of 18°. The ship moves out to sea a distance of 5 nautical miles further away from the mountain. The angle of elevation of the top of the mountain is now 13°. Find the height of Aconcagua above sea level in feet. (1 nautical mile = 6080 ft.)

Ex. 318*e*. Two observations are made to find the height of a certain monument. From the first station the angle of elevation of the top is found to be 32° and from the second station, which is 27 yards from the first and exactly between it and the foot of the monument, the angle of elevation is 43°. If the telescope of the theodolite with which these observations are made is 3 feet above the ground, what is the height of the monument in feet?

Ex. 318*f*. Wishing to find the height of a cliff I fix two marks A and B on the same level in line with the foot of the cliff. From A the angle of elevation of the top of the cliff is 37° and from B the angle of elevation is 23° 30′. If A and B are 120 feet apart, calculate the height of the cliff.

Ex. 318*g*. From a point on a battleship 30 ft. above the water, a Torpedo Boat Destroyer is observed steaming away in a straight line. The angle of depression of the bow is observed to be 11°, and that of the stern to be 21°. Find the length of the T. B. D.

Ex. 318*h*. From the top of a mast 70 feet high, two buoys are observed due N. at angles of depression 57° and 37°; find the distance between the buoys to the nearest foot.

Ex. 318*i*. The angles of depression of two boats in a line with the foot of a cliff are 25° 16′ and 38° 39′ as observed by a man at the top of the cliff. If the man is 250 feet above sea-level, find how far apart the boats are.

Ex. 318*k*. A torpedo boat is steering N. 14° E., and from the torpedo boat a lighthouse is observed lying due N. If the speed of the vessel is 15 knots and it passes the lighthouse 40 minutes after the time of observation, find the clearance between the vessel and the lighthouse, and its distance from the lighthouse at the first observation.

Ex. 318*l*. A landmark bears N. 32° W. from a ship. After the ship has sailed 7·2 miles N. 22° E. the landmark is observed to bear N. 71° W. How far is it then from the ship?

Ex. 318*m*. The position of an inaccessible point C is required. From A and B, the ends of a base line 200 yards long, the following bearings are taken:

From A { Bearing of B is N. 70° 30′ E.
" " C is N. 30° 20′ E.
From B " " C is N. 59° 40′ W.

Find the distances of C from A and B.

Ex. 318*n*. A ship observes a light bearing N. 52° E. at a distance of 5 miles. She then steams due S. 6 miles, and again observes the light. What does she find the bearing and distance of the light to be at the second observation?

Ex. 318*o*. An Admiral signals to his cruiser squadron (bearing N. 40° W. 50 miles from him) to meet him at a place N. 50° E., 70 miles from his present position. Find bearing and distance of the meeting place from the cruisers.

Ex. 318*p*. A is 1 mile due W. of B.
From A, C bears N. 28° W. and D bears N. 33° E.
From B, C bears N. 34° W. and D bears N. 9° W.
Find the distance and bearing of D from C.

Ex. 318*q*. It is required to find the distance between Stokes Bay Pier and a buoy from the following readings:

Bearing of Stokes Bay Pier from Ryde Pier, N. 9° E.
" " Buoy from Ryde Pier N. 36° W.
" " Buoy from Stokes Bay Pier S. 79° W.
Known distance from Stokes Bay Pier to Ryde Pier, 2·29 m.

Ex. 318*r*. A lies 7 miles N. 32° W. of B; C is 5 miles S. 67° E. of A. Find the distance and bearing of C from B.

Ex. 318*s*. Two rocks A, B are seven miles apart, one being due East of the other. How many miles from each of them is a ship from which it is observed that A bears S. 24° W. and B bears S. 35° E.?

Ex. 318*t*. From a ship at sea the following observations are made: Dover bears N. 16° E., and Boulogne S. 81° E. From the chart it is found that Dover is 26 miles N. 24° W. of Boulogne. Find the distance of the ship from Dover.

Ex. 318*u*. O and P are points on a straight stretch of shore. P is 4·5 miles N. 74° E. of O. From O a ship at sea bears S. 58° E., and from P the ship bears S. 32° W. Find the distance of the ship from P, and also its distance from the nearest point of the shore.

Ex. 318*v*. A ship steaming due E. at 9·15 knots through the Straits of Gibraltar observes that a point on the Rock bears N. 35° E.; 40 minutes later the same point bears due N.; how far is she from the point at the second observation?

Ex. 318*w*. A ship is observed to be 3 miles N. 28° E. from a coast-guard station, and to be steaming N. 72° W. After 15 minutes the ship bears N. 36° W. At what rate is she steaming?

Ex. 318*x*. C and D are inaccessible objects. A and B are points 100 yards apart, B due East of A.

From A the bearing of C is due North.
" A " " D is N. 46° E.
" B " " C is N. 63° W.
" B " " D is N. 10° W.

Find (i) distance of C from B,
(ii) distance of D from B,
(iii) distance of C from D.

PART II.

THEORETICAL GEOMETRY.

BOOK I.

We are now going to prove *theoretically* that certain geometrical statements are always true.

By using instruments we have been led to *assume* that certain statements are true. For instance, by measuring the angles of a large number of isosceles triangles we were led to *assume* that two angles of such a triangle are always equal; we now need something more than this, we must prove that this is true for every isosceles triangle whether it is possible to measure its angles or not.

Theoretical proof has two advantages over *verification by measurement*,

(i) Measurement is at best only approximate.

(ii) It is impossible to measure every case.

In theoretical geometry, we must never assume that things are equal because they look equal or because our instruments lead us to suppose them equal, and we must never make a statement unless we have a sound reason for it. The reasons which we use will in some cases depend on facts which we shall have already proved in the course of our theoretical work, in some cases on the definitions, in others on self-evident truths (called **axioms**).

It is impossible to state here all the axioms we shall employ, but we may give two examples.

Things which are equal to the same thing are equal to one another.

[John is the same height as James, and William is the same height as James; therefore John is the same height as William.]

If equals be added to equals the sums are equal.

[If two boys each have five shillings and are each presented with another shilling, the amounts which they then have must be equal.]

ANGLES AT A POINT.

Points, lines, surfaces, etc. The formal definitions are given later; for the present, the general ideas obtained from the introduction are sufficiently definite. (See pp. 55—58.)

DEF. When two straight lines are drawn from a point they are said to form, or contain, an **angle**. The point is called the **vertex** of the angle, and the straight lines are called the **arms** of the angle.

The size of an angle does not depend on the lengths of its arms. (See Ex. 27, 28.)

DEF. When three straight lines are drawn from a point, if one of them is regarded as lying between the other two, the angles which this line makes with the other two are called **adjacent** angles (e.g. $\angle^s$ a and b in fig. 11).

DEF. When one straight line stands on another straight line and makes the adjacent angles equal, each of the angles is called a **right angle**; and the two straight lines are said to be at right angles or **perpendicular** to one another.

We shall assume that all right angles are equal.

¶**Ex. 319.** How would you test the accuracy of the right angle of your set square?

DEF. An angle less than a right angle is said to be **acute**.

DEF. An angle greater than a right angle is said to be **obtuse**.

Revise Ex. 47—50, 57, 58.

¶Ex. **320.** A, B, C, D are four towns in order on a straight road; a man walks from A to B and then on from B to D; another man walks from A to C and then on from C to D; have they walked the same distance?

¶Ex. **321.** If in fig. 75 a straight line OP revolves about O from the position OB to the position OA, and then on to the position OC; and if another straight line OQ revolves about O from the position OB to the position OD and then on to OC; will OP and OQ have turned through the same angle?

Theorem 1.

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

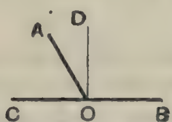


fig. 75.

Data The st. line AO meets the st. line BC at O.

To prove that $\angle BOA + \angle AOC = 2$ rt. $\angle$s.

Construction Draw OD to represent the line through O perpendicular to BC.

Proof

$$\angle BOA = \angle BOD + \angle DOA,$$
$$\angle AOC = \angle DOC - \angle DOA,$$
$$\therefore \angle BOA + \angle AOC = \angle BOD + \angle DOC$$
$$= 2 \text{ rt. } \angle\text{s.} \qquad \textit{Constr.}$$

Q. E. D.

Cor. **If any number of straight lines meet at a point, the sum of all the angles made by consecutive lines is equal to four right angles.**

Revise Ex. 51, 52.

¶†Ex. **322.** If two straight lines AOB, COD intersect at O and ∠AOC is a right angle, prove that the other angles at O are right angles.

¶†Ex. **323.** If △ABC has ∠ABC = ∠ACB, prove that the exterior angles formed by producing the base both ways must be equal to one another (i.e. prove that ∠ABD = ∠ACE). (See fig. 76.)

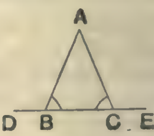


fig. 76.

When angles or lines are given or made equal it is well to indicate the fact in your figure by putting the same mark in each.

†Ex. **324.** In △ABC, ∠ABC = ∠ACB and AB and AC are produced to X and Y, prove that ∠CBX = ∠BCY. (See fig. 77.)

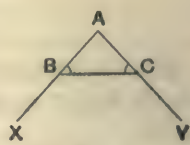


fig. 77.

Def. If a straight line or angle is divided into two equal parts it is said to be **bisected.**

Ex. **325.** If in fig. 75, ∠BOA = 110°, and OP is drawn bisecting ∠BOA and OQ bisecting ∠AOC; what are ∠s POA, AOQ? What is their sum?

†Ex. **326.** Three straight lines OA, OB, OC are drawn from a point O; OP is drawn bisecting ∠BOA, and OQ bisecting ∠AOC. Prove that

∠POQ = ½∠BOC.

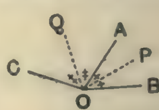


fig. 78.

†Ex. **327.** **If a straight line stands on another straight line, prove that the bisectors of the two adjacent angles so formed are at right angles to one another.** (See Ex. 325, 326.)

Ex. **328.** Prove the corollary to Theorem 1. See Ex. 59—62.

Def. When the sum of two angles is equal to two right angles, each is called the **supplement** of the other, or is said to be **supplementary** to the other.

¶Ex. **329.** Name the supplements of ∠ABC and ∠BCY in fig. 77. Name the supplements of ∠AOB, ∠COD, and ∠AOC in fig. 75.

Ex. **330.** State Theorem 1, introducing the term "supplementary."

Ex. **331.** In fig. 75, show how to obtain another supplement of ∠AOB.

†Ex. **332.** If two angles are equal, their supplements are equal.

Revise Ex. 53—55.

Theorem 2.

[Converse of Theorem 1.]

If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.

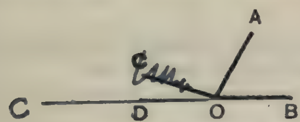


fig. 79.

Data The sum of the adjacent $\angle$s BOA, AOC $= 2$ rt. $\angle$s.

To prove that BOC is a straight line.

Construction Produce BO to D.

Proof Since AO meets the st. line BD at O,

$\therefore \angle$ BOA $+ \angle$ AOD $= 2$ rt. $\angle$s. I. 1.

But $\angle$ BOA $+ \angle$ AOC $= 2$ rt. $\angle$s, *Data*

$\therefore \angle$ BOA $+ \angle$ AOD $= \angle$ BOA $+ \angle$ AOC,

$\therefore \angle$ AOD $= \angle$ AOC,

$\therefore$ OC coincides with OD.

Now BOD is a st. line, *Constr.*

$\therefore$ BOC is a st. line.

Q. E. D.

†Ex. **333.** From a point A in a straight line AB, straight lines AC and AD are drawn at right angles to AB on opposite sides of it; prove that CAD is a straight line.

†Ex. **334.** From a point O in a straight line AOC, OB and OD are drawn on opposite sides of AC so that $\angle AOB = \angle COD$; prove that BOD is a straight line.

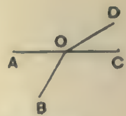


fig. 80.

†Ex. **335.** Three straight lines OB, OA, OC are drawn from a point (see fig. 78), OP bisects $\angle BOA$, OQ bisects $\angle AOC$; prove that, if $\angle POQ$ is a right angle, BOC is a straight line.

†Ex. **336.** Two straight lines XOX′, YOY′ intersect at right angles; OP bisects $\angle XOY$, OQ bisects $\angle X'OY'$. Is POQ a straight line? [Find the sum of $\angle^s$ POY, YOX′, X′OQ.]

Revise Ex. 64—66.

¶Ex. **337.** If a straight line rotates about its middle point, do the two parts of the straight line turn through equal angles?

If the line rotates about any other point, are the angles equal?

¶Ex. **338.** ABCD are four points in order on a straight line; if $AC = BD$ then $AB = CD$.

¶Ex. **339.** If two straight lines AOB, COD intersect at O (see fig. 81) what is the sum of $\angle^s$ AOB, BOC? What is the sum of $\angle^s$ BOC, COD?

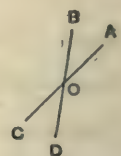


fig. 81.

Def. The opposite angles made by two intersecting straight lines are called **vertically opposite angles** (*vertically* opposite because they have the same vertex).

¶Ex. **340.** Name two pairs of vertically opposite angles in fig. 81.

THEOREM 3.

If two straight lines intersect, the vertically opposite angles are equal.

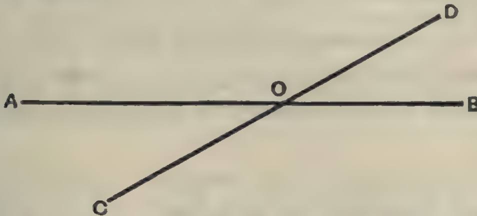


fig. 82.

Data The two st. lines AOB, COD intersect at O.

To prove that $\angle$ AOD = vert. opp. $\angle$ BOC,

$\angle$ AOC = vert. opp. $\angle$ BOD.

Proof Since st. line OD stands on st. line AB,

$\therefore$ $\angle$ AOD + $\angle$ DOB = 2 rt. $\angle$s, I. 1.

and since st. line OB stands on st. line CD,

$\therefore$ $\angle$ DOB + $\angle$ BOC = 2 rt. $\angle$s, I. 1.

$\therefore$ $\angle$ AOD + $\angle$ DOB = $\angle$ DOB + $\angle$ BOC,

$\therefore$ $\angle$ AOD = $\angle$ BOC.

Sim[ly] $\angle$ AOC = $\angle$ BOD. Q. E. D.

Revise Ex. 67, 68.

†Ex. **341.** Write out in full the proof that $\angle$ AOC = $\angle$ BOD in I. 3.

¶Ex. **342.** Draw a triangle and produce every side both ways; number all the angles in the figure, using the same numbers for angles that are equal.

†Ex. **343.** In fig. 83, prove that

(i) if $\angle b = \angle f$, then $\angle c = \angle f$.

(ii) if $\angle c = \angle f$, then $\angle d = \angle e$.

(iii) if $\angle d + \angle f = 2$ rt. $\angle$s, then $\angle b = \angle f$.

(iv) if $\angle g = \angle c$, then $\angle d = \angle h$.

(v) if $\angle h = \angle a$, then $\angle e = \angle d$.

(vi) if $\angle a = \angle e$, then $\angle b = \angle g$.

(vii) if $\angle c = \angle f$, then $\angle d + \angle f = 2$rt. $\angle$s.

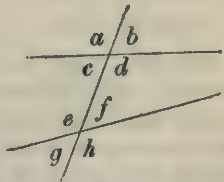


fig. 83.

†Ex. 344. If two straight lines AOC, BOD intersect at O and OX bisects $\angle$AOB, then XO produced bisects $\angle$COD.

†Ex. 345. **The bisectors of a pair of vertically opposite angles are in one and the same straight line.**

Parallel Straight Lines.

Def. **Parallel straight lines** are straight lines in the same plane, which do not meet however far they are produced in either direction.

Def. In the figure two straight lines are cut by a third straight line; $\angle^s$ c and f are called **alternate** angles, $\angle^s$ b and f **corresponding** angles (sometimes $\angle^s$ b and f are spoken of as "an exterior angle and the interior opposite angle on the same side of the cutting line").

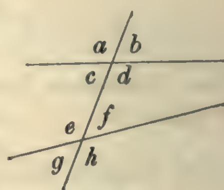


fig. 83.

¶Ex. 346. Name another pair of alternate angles in fig. 83.

¶Ex. 347. Name another pair of corresponding angles.

¶Ex. 348. What are the names of the following pairs:

(i) c, f, (ii) b, f, (iii) h, d, (iv) a, d, (v) c, g, (vi) e, f, (vii) e, a, (viii) e, d?

†Ex. 349. Prove that if a straight line cuts two other straight lines and makes a pair of alternate angles equal, then a pair of corresponding angles are equal.

[That is, in fig. 83, prove that if $\angle c = \angle f$, then $\angle b = \angle f$.]

†Ex. 350. In fig. 83, prove that, if $\angle c = \angle f$, then $\angle d + \angle f = 2$ rt. $\angle^s$. State this formally as in Ex. 349. ($\angle^s$ d and f are interior angles on the same side of the cutting line.)

Revise Ex. 167.

¶Ex. 351. Draw two parallel straight lines and a line cutting them; measure a pair of alternate angles.

¶Ex. 352. Take a strip of paper about two inches wide with parallel sides, cut it across as in fig. 84; measure the angles so formed with your protractor, noting which are equal, and test whether the two pieces can be made to coincide (i.e. fit on one another exactly).

A First Treatment of Parallels (for Beginners).

The strict treatment of parallels given on pages 71, 72 may be found difficult for beginners. The following treatment, based upon the equality of corresponding angles, is recommended as more suitable for a first reading of theoretical geometry; it must not however be regarded as a satisfactory proof.

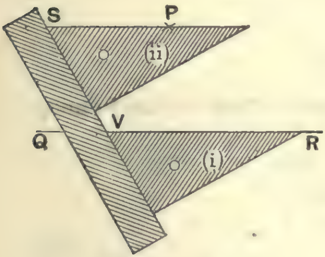


fig. 83 (*a*).

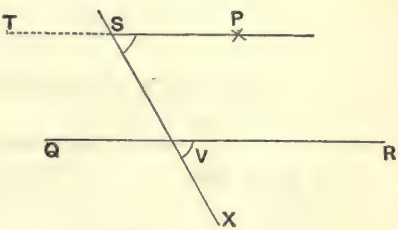


fig. 83 (*b*).

On page 36 was explained the set-square method of drawing through P a parallel to QR, fig. 83 (*a*). Figure 83 (*b*) shews the lines with the set-square removed.

It will be seen at once that the *corresponding* angles PSV, RVX were covered by the same angle of the set-square, and must be equal. Thus, the actual method of drawing parallel lines suggests that

When a straight line cuts two other straight lines, if a pair of corresponding angles are equal, then the two straight lines are parallel.

From this it is easy to deduce

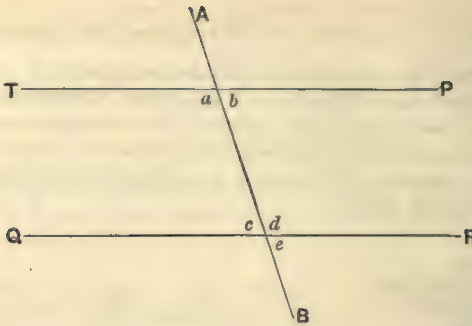
Theorem A.

When a straight line cuts two other straight lines, if

(2) **a pair of alternate angles are equal,**

or (3) **a pair of interior angles on the same side of the cutting line are together equal to two right angles (supplementary),**

then the two straight lines are parallel.



(2) *Data* The st. line AB cuts the two st. lines TP, QR forming the $\angle$s a, b, c, d, e;

$\angle b =$ alternate $\angle c$.

To prove that TP, QR are parallel.

Proof $\angle c =$ vert. opp. $\angle e$.

But $\angle b = \angle c$. *Data*

$\therefore \angle b = \angle e$,

and these are corresponding angles,

$\therefore$ TP, QR are parallel.

(3) *Data* $\angle b + \angle d = 2$ rt. $\angle$s.

To prove that TP, QR are parallel.

$\angle e + \angle d = 2$ rt. $\angle$s. I. 1.

But $\angle b + \angle d = 2$ rt. $\angle$s. *Data*

$\therefore \angle e + \angle d = \angle b + \angle d$.

$\therefore \angle e = \angle b$,

and these are corresponding angles,

$\therefore$ TP, QR are parallel. Q. E. D.

After this point the class may return to the ordinary treatment at the middle of page 73; and deal with the converse theorem. But it is probably a mistake to lay any stress, in a first reading, upon the difficulties connected with the parallel theorem and its converse.

The above presentation is easily seen to be open to objection; in fact we have virtually *assumed* Th. 4 (2). But no harm is likely to result from adopting this treatment of parallels with beginners, so long as it is clearly understood to be provisional.

THEOREM 4.*

(1) When a straight line cuts two other straight lines, if a pair of alternate angles are equal, then the two straight lines are parallel.

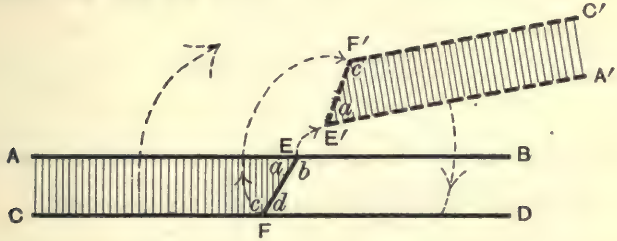


fig. 84.

(1) *Data* The st. line EF cuts the two st. lines AB, CD at E, F, forming the ∠s a, b, c, d; and ∠a = alternate ∠d.

To prove that AB, CD are parallel.

Proof

$\angle a + \angle b = 2$ rt. ∠s, I. 1.
$\angle c + \angle d = 2$ rt. ∠s, I. 1.
$\therefore \angle a + \angle b = \angle c + \angle d.$
But $\angle a = \angle d.$ *Data*
$\therefore \angle b = \angle c.$

Take up the part AEFC, call it A′E′F′C′; and, turning it round in its own plane, apply it to the part DFEB so that E′ falls on F and E′A′ along FD.

$\because \angle a = \angle d,$ *Data*
∴ E′F′ falls along FE,
and ∵ E′F′ = FE (being the same line),
∴ F′ falls on E,
again $\because \angle c = \angle b,$ *Proved*
∴ F′C′ falls along EB.

* The proof of this theorem should be omitted at a first reading.

Now if EB and FD meet when produced towards B and D, F′C′ and E′A′ must also meet when produced towards C′ and A′, i.e. FC and EA must also meet when produced towards C and A.

∴ if AB, CD meet when produced in one direction, they will also meet when produced in the other direction; but this is impossible, for two st. lines cannot enclose a space.

∴ AB, CD cannot meet however far they are produced in either direction.

∴ AB and CD are parallel.

Q. E. D.

When a straight line cuts two other straight lines, if

(2) a pair of corresponding angles are equal,

or (3) a pair of interior angles on the same side of the cutting line are together equal to two right angles,

then the two straight lines are parallel.

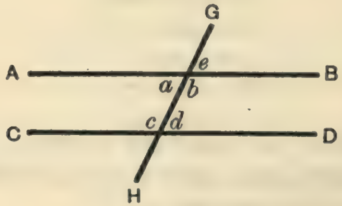


fig. 85.

(2) *Data* The st. line GH cuts the two st. lines AB, CD forming the $\angle$s a, b, c, d, e.

$\angle e =$ corresp. $\angle d$.

To prove that AB, CD are parallel.

Proof $\angle e =$ vert. opp. $\angle a$. I. 3.

But $\angle e = \angle d$, *Data*

$\therefore \angle a = \angle d$,

and these are alternate angles,

$\therefore$ AB, CD are parallel. by (1).

(3) *Data* $\angle b + \angle d = 2$ rt. $\angle$s.

To prove that AB, CD are parallel.

Proof $\angle b + \angle a = 2$ rt. $\angle$s. I. 1.

But $\angle b + \angle d = 2$ rt. $\angle$s. *Data*

$\therefore \angle b + \angle a = \angle b + \angle d$,

$\therefore \angle a = \angle d$,

and these are alternate angles,

$\therefore$ AB, CD are parallel. by (1).

Q. E. D.

COR. **If each of two straight lines is perpendicular to a third straight line, the two straight lines are parallel to one another.**

†Ex. **353.** Prove the corollary.

†Ex. **354.** Prove that the straight lines in fig. 83 would be parallel (i) if $\angle a = \angle h$, or (ii) if $\angle b + \angle h = 2$ rt. $\angle$s.

DEF. A plane figure bounded by three straight lines is called a **triangle**.

DEF. A plane figure bounded by four straight lines is called a **quadrilateral**.

DEF. The straight lines which join opposite corners of a quadrilateral are called its **diagonals**.

DEF. A quadrilateral with its opposite sides parallel is called a **parallelogram**.

†Ex. **355.** ABCD is a quadrilateral, its diagonal AC is drawn; prove that, if $\angle$BAC $=$ $\angle$ACD and $\angle$DAC $=$ $\angle$ACB, ABCD is a parallelogram.

PLAYFAIR'S AXIOM. Through a given point one straight line, and one only, can be drawn parallel to a given straight line.

THEOREM 5.

[CONVERSE OF THEOREM 4.]

If a straight line cuts two parallel straight lines,

(1) alternate angles are equal,

(2) corresponding angles are equal,

(3) the interior angles on the same side of the cutting line are together equal to two right angles.

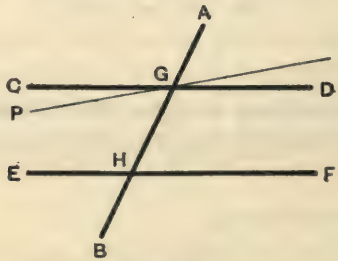


fig. 86.

Data AB cuts the parallel st. lines CD, EF at G, H.

To prove that (1) $\angle$ CGH = alt. $\angle$ GHF,

(2) $\angle$ AGD = corresp. $\angle$ GHF,

(3) $\angle$ DGH + $\angle$ GHF = 2 rt $\angle$ s.

(1) *Construction* If $\angle$ CGH is not equal to $\angle$ GHF, suppose GP drawn so that $\angle$ PGH = $\angle$ GHF.

Proof $\because$ $\angle$ PGH = alt. $\angle$ GHF,

$\therefore$ PG is || to EF. I. 4.

$\therefore$ the two straight lines PG, CG which pass through the point G are both || to EF.

But this is impossible. *Playfair's Axiom*

$\therefore \angle$ CGH cannot be unequal to $\angle$ GHF,

$\therefore \angle \text{CGH} = \angle \text{GHF}$.

(2) Since, by (1), $\angle \text{CGH} = \angle \text{GHF}$

and $\angle \text{CGH} =$ vert. opp. $\angle$ AGD,

$\therefore \angle \text{AGD} = \angle \text{GHF}$.

(3) Since GH stands on CD,

$\therefore \angle \text{DGH} + \angle \text{CGH} = 2$ rt. $\angle$s, I. 1.

and, by (1), $\angle \text{CGH} = \angle \text{GHF}$,

$\therefore \angle \text{DGH} + \angle \text{GHF} = 2$ rt $\angle$s. Q. E. D.

Ex. 356. Copy fig. 86, omitting the line PG. If $\angle \text{AGD} = 72°$, find all the angles in the figure, giving your reasons; make a table.

†**Ex. 357.** Prove case (2) of Theorem 5 from first principles [i.e. without assuming case (1)].

†**Ex. 358.** Prove case (3) of Theorem 5 from first principles [i.e. without assuming cases (1) or (2)].

†**Ex. 359.** In fig. 87 there are two pairs of parallel lines; prove that the following pairs of angles are equal:—(i) b, l, (ii) f, k, (iii) m, s, (iv) f, h, (v) r, l, (vi) s, h, (vii) s, q, (viii) s, k, (ix) s, a, (x) g, l.

[State your reasons carefully.

e.g. WX, YZ are ∥ and ST cuts them,

$\therefore \angle q = \angle f$ (corresponding angles).]

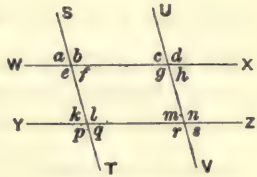


fig. 87.

Ex. 360. What do you know about the sums of (i) $\angle^s f, g$, (ii) $\angle^s f, l$, (iii) $\angle^s m, n$, in fig. 87? Give your reasons.

Ex. 361. Draw a parallelogram ABCD, join AC, and produce BC to E; what pairs of angles in the figure are equal? Give your reasons.

†**Ex. 362.** A triangle ABC has $\angle \text{B} = \angle \text{C}$, and DE is drawn parallel to BC; prove that $\angle \text{ADE} = \angle \text{AED}$.

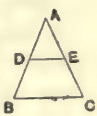


fig. 88.

†**Ex. 363. If a straight line is perpendicular to one of two parallel straight lines, it is also perpendicular to the other.**

†**Ex. 364.** The opposite angles of a parallelogram are equal. [See Ex. 360.]

†**Ex. 365.** What is the sum of the angles of a parallelogram?
Hence find the sum of the angles of a triangle.

†**Ex. 366.** If one angle of a parallelogram is a right angle, prove that all its angles must be right angles.

Note on a Theorem and its Converse.

The enunciation of a theorem can generally be divided into two parts (1) the **data** or **hypothesis**, (2) the **conclusion**.

If data and conclusion are interchanged a second theorem is obtained which is called the **converse** of the first theorem.

For example, we proved

in I. 4, that, if $\angle a = \angle d$ (data), then AB, CD are $\parallel$ (conclusion);
in I. 5, that, if AB, CD are $\parallel$ (data), then $\angle a = \angle d$ (conclusion).

The data of I. 4 is the conclusion of I. 5, and the conclusion of I. 4 is the data of I. 5; so that I. 5 is the converse of I. 4 (and I. 4 is the converse of I. 5).

It must not be assumed that the converses of all true theorems are true; e.g. "if two angles are vertically opposite, they are equal" is a true theorem, but its converse "if two angles are equal, they are vertically opposite" is not a true theorem.

¶**Ex. 367.** State the converses of the following: are they true?

(i) If two sides of a triangle are equal, then two angles of the triangle are equal.

(ii) If a triangle has one of its angles a right angle, two of its angles are acute.

(iii) London Bridge is a stone bridge.

(iv) A nigger is a man with woolly hair.

Theorem 6.

Straight lines which are parallel to the same straight line are parallel to one another.

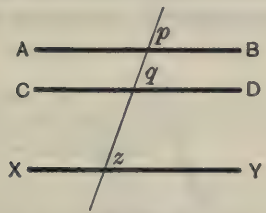


fig. 89.

Data AB, CD are each $\parallel$ to XY.

To prove that AB is $\parallel$ to CD.

Construction Draw a st. line cutting AB, CD, XY and forming with them corresponding $\angle$s p, q, z respectively.

Proof $\because$ AB is $\parallel$ to XY,

$\therefore \angle p =$ corresp. $\angle z$. I. 5.

Again $\because$ CD is $\parallel$ to XY,

$\therefore \angle q =$ corresp. $\angle z$, I. 5.

$\therefore \angle p = \angle q$.

Now these are corresponding angles,

$\therefore$ AB is $\parallel$ to CD. I. 4.

Q. E. D.

Ex. **368.** Prove I. 6 by means of Playfair's Axiom.
[Suppose AB and CD to meet.]

¶Ex. **369.** Are the theorems true which you obtain (i) by substituting "perpendicular" for "parallel" in I. 6, (ii) by substituting "equal" for "parallel" in I. 6?

THEOREM 7. †

If straight lines are drawn from a point parallel to the arms of an angle, the angle between those straight lines is equal or supplementary to the given angle.

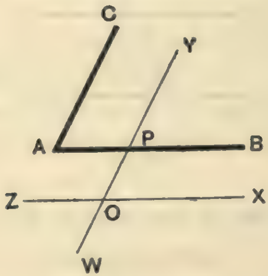


fig. 90.

Data BAC is an angle.

From O, OX is drawn ∥ to AB and in the same sense* as AB,
and OY is drawn ∥ to AC and in the same sense as AC;
XO, YO are produced to Z, W respectively.

To prove that $\angle$ XOY = $\angle$ ZOW = $\angle$ BAC,
$\angle$ YOZ = $\angle$ WOX = supplement of $\angle$ BAC.

* A straight line may be generated by the motion of a point, and the point may move in either of two opposite directions or **senses**; thus, in fig. 90, the line AB may be generated by a point moving from A to B or from B to A, and the line OX by a point moving from O to X or from X to O. If a point moves from A to B and another from O to X we say that they move in the same sense, or AB and OX *have the same sense*; but if the one moves from A to B and the other from X to O they move in opposite senses, or AB and XO *have opposite senses.*

Proof Let WY cut AB at P,

then $\angle$ XOY = corresp. $\angle$ BPY, I. 5.

and $\angle$ BAC = corresp. $\angle$ BPY, I. 5

$\therefore$ $\angle$ XOY = $\angle$ BAC.

But $\angle$ ZOW = vert. opp. $\angle$ XOY,

$\therefore$ $\angle$ ZOW = $\angle$ BAC.

Again $\angle$ YOZ = $\angle$ XOW = supplement of $\angle$ XOY

= supplement of $\angle$ BAC.

Q. E. D.

†Ex. 370. **If straight lines are drawn from a point perpendicular to the arms of an angle, the angle between those straight lines is equal or supplementary to the given angle.**

(Take BAC as the given angle, through A draw straight lines parallel to the given perpendiculars; first prove that the angle between these lines is equal or supplementary to $\angle$ BAC.)

Theorem 8.

The sum of the angles of a triangle is equal to two right angles.

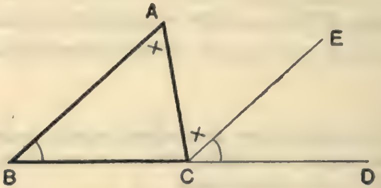


fig. 91.

Data ABC is a triangle.

To prove that $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

Construction Produce BC to D.

Through C draw CE $\parallel$ to BA.

Proof Since AC cuts the $\parallel$s AB, CE,

$\therefore \angle A =$ alt. $\angle ACE$.

And since BC cuts the $\parallel$s AB, CE,

$\therefore \angle B =$ corresp. $\angle ECD$,

$\therefore \angle A + \angle B = \angle ACE + \angle ECD$.

Add $\angle ACB$ to each side,

$\therefore \angle A + \angle B + \angle ACB = \angle ACB + \angle ACE + \angle ECD$

$= 2$ rt. $\angle$s (for BCD is a st. line),

$\therefore$ sum of $\angle$s of $\triangle ABC = 2$ rt. $\angle$s.

Q. E. D.

Cor. 1. **If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.** (Proof as above.)

Cor. 2. If one side of a triangle is produced, the exterior angle so formed is greater than either of the interior opposite angles.

COR. 3. Any two angles of a triangle are together less than two right angles.

COR. 4. Every triangle has at least two of its angles acute.

COR. 5. **If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.**

COR. 6. **The sum of the angles of a quadrilateral is equal to four right angles.** (Draw a diagonal.)

Revise Ex. 127—136.

†Ex. **371.** Write out the full proof of Cor. 1.

†Ex. **372.** Prove I. 8 by drawing through A a straight line PAQ parallel to BC.

†Ex. **373.** In a triangle ABC, $\angle A = \angle B$; prove that if BC is produced to D, $\angle DCA = 2\angle B$.

†Ex. **374.** Prove Cor. 6.

†Ex. **375.** What is the sum of the angles of a pentagon?
[Join one vertex to the two opposite vertices.]

Ex. **376.** If, in fig. 91, $\angle A = 58^\circ$ and $\angle ACD = 100^\circ$, find all the other angles.

Ex. **377.** In a quadrilateral ABCD, $\angle A = 77^\circ$, $\angle B = 88^\circ$, $\angle C = 99^\circ$; find $\angle D$.

Ex. **378.** In a quadrilateral ABCD, $\angle A = 37^\circ$, $\angle B = 111^\circ$, and $\angle C = \angle D$; find $\angle C$ and $\angle D$.

Ex. **379.** If the exterior angles formed by producing the base of a triangle both ways are 105° and 112°, find all the angles of the triangle.

†Ex. **380.** If one angle of a triangle is a right angle, the other two angles must be acute.

†Ex. **381.** If one angle of a triangle is obtuse, the other two angles must be acute.

DEF. A triangle which has one of its angles an obtuse angle is called an **obtuse-angled** triangle.

DEF. A triangle which has one of its angles a right angle is called a **right-angled** triangle.

The side opposite the right angle is called the **hypotenuse**.

DEF. A triangle which has *all* its angles acute angles is called an **acute-angled** triangle.

In Ex. 378—9, we have seen that **every triangle must have at least two of its angles acute.**

DEF. A triangle which has two of its sides equal is called an **isosceles** triangle.

DEF. A triangle which has all its sides equal is called an **equilateral** triangle.

DEF. A triangle which has no two of its sides equal is called a **scalene** triangle.

DEF. A triangle which has all its angles equal is said to be **equiangular.**

Revise Ex. 159—163.

¶Ex. **382.** If two of the angles of a triangle are 67° and 79°, what is the third angle? What are the exterior angles, formed by producing the sides in order (see fig. 93)? What is their sum?

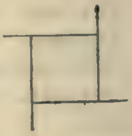


fig. 92.

¶Ex. **383.** Produce the sides of a square* in order (see fig. 92); what is the sum of the exterior angles?

¶Ex. **384.** In fig. 93 the sides of a triangle are produced in order; what are the following sums: (i) $\angle a+\angle x$, (ii) $\angle b+\angle y$, (iii) $\angle c+\angle z$, (iv) $\angle a+\angle b+\angle c$?

Hence find $\angle x+\angle y+\angle z$.

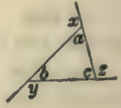


fig. 93.

¶Ex. **385.** Which angles are equal to the following sums:

(i) $\angle b+\angle c$, (ii) $\angle c+\angle a$, (iii) $\angle a+\angle b$?

Hence find $\angle x+\angle y+\angle z$.

¶Ex. **386.** If a yacht sails from A round the pentagon BCDEF back to A, what angles does it turn through at B, C, D, E, F?

When it gets back to A, it has headed towards every point of the compass; what then is the sum of the angles through which it has turned?

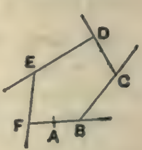


fig. 94.

¶Ex. **387.** Draw a figure to show which angles a yacht turns through in sailing round a triangular course. What is the sum of these angles?

* In Ex. 383—689, the following properties of a square may be assumed: (i) all its sides are equal and (ii) all its angles are right angles.

DEF. A plane figure bounded by straight lines is called a polygon.

THEOREM 9.

If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.

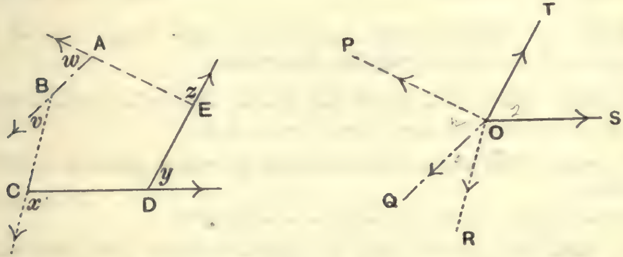


fig. 95.

Data ABCDE is a convex polygon; its sides are produced in order and form the exterior angles, w, v, x, y, z.

To prove that $\angle w + \angle v + \angle x + \angle y + \angle z = 4$ rt. $\angle$s.

Construction Through any point O draw OP, OQ, OR, OS, OT $\parallel$ to and in the same sense as EA, AB, BC, CD, DE respectively.

Proof Since OP, OQ are respectively $\parallel$ to and in the same sense as EA, AB,

$\therefore \angle w = \angle$ POQ, I. 7.

Sim^{ly} $\angle v = \angle$ QOR,

$\angle x = \angle$ ROS,

$\angle y = \angle$ SOT,

$\angle z = \angle$ TOP,

$\therefore \angle w + \angle v + \angle x + \angle y + \angle z =$ sum of $\angle$s at O

$= 4$ rt. $\angle$s. I. 1 *Cor.*

Q. E. D.

COR. **The sum of the interior angles of any convex polygon together with four right angles is equal to twice as many right angles as the polygon has sides.**

Ex. **388.** Three of the exterior angles of a quadrilateral are 79°, 117°, 65°; find the other exterior angle and all the interior angles.

†Ex. **389.** Prove the corollary for a pentagon

(i) by considering the sum of the exterior and interior angles at each corner, and the sum of all the exterior angles;

(ii) by joining a point O inside the pentagon to each corner, and considering the sums of the angles of the triangles so formed and the sum of the angles at the point O.

Def. A polygon which has all its sides equal and all its angles equal is called a **regular polygon.**

¶Ex. **390.** What is the size of each exterior angle of a regular octagon (8-gon)? Hence find the size of each interior angle.

Ex. **391.** What are the exterior angles of regular polygons of 12, 10, 5, 3 sides?

Hence find the interior angles of these polygons.

Ex. **392.** The exterior angle of a regular polygon is 60°, how many sides has the polygon?

Ex. **393.** How many sides have the regular polygons whose exterior angles are (i) 10°, (ii) 1°, (iii) $2\frac{1}{2}$°?

Ex. **394.** Is it possible to have regular polygons whose exterior angles are (i) 15°, (ii) 7°, (iii) 11°, (iv) 6°, (v) 5°, (vi) 4°?

¶Ex. **395.** Is it possible to have regular polygons whose exterior angles are obtuse?

Ex. **396.** Is it possible to have regular polygons whose interior angles are (i) 108°, (ii) 120°, (iii) 130°, (iv) 144°, (v) 60°? (Think of the exterior angles.)

In the cases which are possible, find the number of sides.

Ex. **397.** Make a table showing the exterior and interior angles of regular polygons of 3, 4, 5...10 sides.

Draw a graph showing horizontally the number of sides and vertically the number of degrees in the angles.

Ex. **398.** Construct a regular pentagon having each side 2 in. long.

(Calculate its angles, draw AB = 2 in., at B make $\angle$ABC = the angle of the regular pentagon, cut off BC = 2 in., &c., &c.)

Ex. **399.** Construct a regular octagon having each side 2 in. long.

Ex. **400.** Construct a regular 12-gon having each side 1·5 in. long.

CONGRUENT TRIANGLES.

If two figures when applied to one another can be made to **coincide** (i.e. fit exactly) they must be equal in all respects.

This method of testing equality is known as the method of **superposition**.

¶**Ex. 401.** How did you test the equality of two angles? (See Ex. 28.)

¶**Ex. 402.** How would you test whether two cricket bats were of the same length?

Figures which are equal in all respects are said to be **congruent**.

The sign $\equiv$ is used to denote that figures are congruent.

¶**Ex. 403.** Draw a triangle DEF having DE = 3 in., DF = 2 in., $\angle$D = 26°; on tracing paper draw a triangle ABC having AB = 3 in., AC = 2 in., $\angle$A = 30°.

Apply $\triangle$ABC to $\triangle$DEF so that A falls on D; put a pin through these two points; turn $\triangle$ABC round until AB falls along DE.

B falls on E. Why is this?

Does AC fall along DF?

(Keep the $\triangle$ABC for the next Ex.)

¶**Ex. 404.** Draw a triangle DEF having DE = 3 in., DF = 2 in., $\angle$D = 30°.

Apply $\triangle$ABC (made in the last Ex.) to $\triangle$DEF so that A falls on D; put a pin through these two points; turn $\triangle$ABC round until AB falls along DE.

B falls on E. Why is this?

AC falls along DF. Why is this?

C falls on F. Why is this?

Do the triangles coincide altogether?

Theorem 10.

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

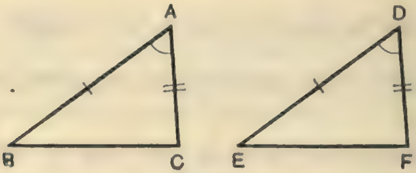


fig. 96.

Data ABC, DEF are two triangles which have AB = DE, AC = DF, and included $\angle$ BAC = included $\angle$ EDF.

To prove that $\triangle$ ABC $\equiv$ $\triangle$ DEF.

Proof Apply $\triangle$ ABC to $\triangle$ DEF so that A falls on D, and AB falls along DE.

$\because$ AB = DE,

$\therefore$ B falls on E.

Again $\because$ $\angle$ BAC = $\angle$ EDF,

$\therefore$ AC falls along DF.

And $\because$ AC = DF,

$\therefore$ C falls on F,

$\therefore$ $\triangle$ ABC coincides with $\triangle$ DEF,

$\therefore$ $\triangle$ ABC $\equiv$ $\triangle$ DEF.

Q. E. D.

N.B. It must be carefully noted that the congruence of the triangles cannot be inferred unless the equal angles are the angles *included* (or contained) by the sides which are given equal.

Ex. 405. Make a list of all the equal sides and angles in $\triangle^s$ ABC and DEF of I. 10. Say which were given equal and which were proved equal.

†Ex. **406.** Draw two triangles PQR, XYZ and mark QR = XY, RP = YZ, and ∠Q = ∠Z. Would this theorem prove the triangles congruent? Give two reasons.

†Ex. **407.** ABCD is a square, E is the mid-point of AB; equal lengths AP and BQ are cut off from AD and BC. Join EP and EQ. Prove that △AEP ≡ △BEQ.

Write down all the pairs of lines and angles in these triangles which you have proved equal.

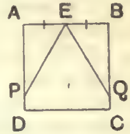


fig. 97.

†Ex. **408.** ABCD is a square, E is the mid-point of AB; join CE and DE. Prove that △AED ≡ △BEC.

Write down all the pairs of lines and angles in these triangles which you have proved equal.

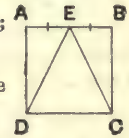


fig. 98.

†Ex. **409.** PQRS is a quadrilateral in which PQ = SR, ∠Q = ∠R, and O is the mid-point of QR. Prove that OP = OS.

[You must first join OP and OS, and mark in your figure all the parts that are given equal; you will then see that you want to prove that △OQP ≡ △ORS.]

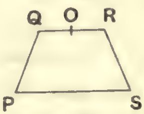


fig. 99.

†Ex. **410.** ABCD is a square; E, F, G are the mid-points of AB, BC, CD respectively. Join EF and FG and prove them equal.

[Which are the two triangles that you must prove equal?]

†Ex. **411.** ABC, DEF are two triangles which are equal in all respects; X is the mid-point of BC, Y is the mid-point of EF. Prove that AX = DY, and ∠AXB = ∠DYE.

[You will of course have to join AX and DY.]

†Ex. **412.** The equal sides QP, RP of an isosceles triangle PQR are produced to S, T so that PS = PT; prove that TQ = SR.

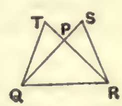


fig. 100.

†Ex. **413.** D is the mid-point of the side BC of a △ABC, AD is produced to E so that DE = AD. Prove that AB = EC and that AB, EC are parallel.

[First prove a pair of triangles congruent.]

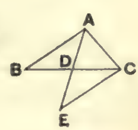


fig. 101.

†Ex. **414.** Show that the distance between G and H, the opposite corners of a house, can be found as follows. At a point P set up a post; step off HP and an equal distance PN, taking care to keep in a straight line with the post and the corner H; step off GP and an equal distance PM, M being in the same straight line as G and P. Measure MN; this must be equal to GH.

Draw a ground plan and prove that MN = GH.

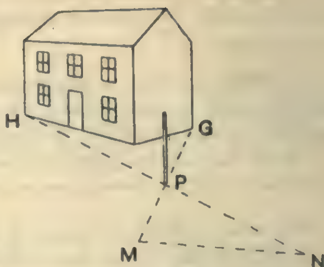


fig. 102.

†Ex. **415.** W is the mid-point of a straight line YZ, WX is drawn at right angles to YZ. Prove that XY = XZ.

[A line which is a side of each of two triangles is said to be **common** to the two triangles.]

fig. 103.

†Ex. **416.** The bisector of the angle between the equal sides of an isosceles triangle is perpendicular to the base.

[Let XYZ be an isosceles triangle, having XY = XZ; let XW bisect ∠YXZ and let it meet YZ at W; prove ∠XWY = ∠XWZ. See fig. 103.]

†Ex. **417.** XYZ is an isosceles triangle having XY = XZ; prove that ∠Y = ∠Z.

[Draw XW the bisector of ∠YXZ.]

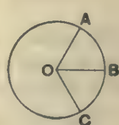


fig. 104.

†Ex. **418.** OA, OB, OC are three radii of a circle. If ∠AOB = ∠COB, prove that BO bisects AC.

†Ex. **419.** In fig. 105, AB and DC are equal and parallel; prove that AD = BC.

[Join BD. Since AB is parallel to DC ∴ ∠? = ∠?.]

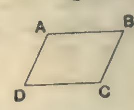


fig. 105.

†Ex. **420.** The equal sides AB, AC of an isosceles triangle ABC are produced to X and Y respectively; BX is made equal to CY (see fig. 77). If ∠CBX = ∠BCY, prove that CX = BY.

[By the side of your figure make sketches of the triangles BCX, CBY.]

†Ex. **421.** XY is a straight line, XP and YQ are drawn at right angles to XY and XP is made equal to YQ. Prove that $\angle$PYX = $\angle$QXY.

†Ex. **422.** ABCD is a quadrilateral in which AB = CD, AD = BC and $\angle$A = $\angle$C; prove that ABCD is a parallelogram.

[Join BD.]

†Ex. **423.** If the diagonals of a quadrilateral bisect one another it must be a parallelogram.

¶Ex. **424.** In two $\triangle^s$ ABC, DEF, $\angle$A = $\angle$D, $\angle$B = $\angle$E; prove that $\angle$C = $\angle$F.

¶Ex. **425.** Draw a triangle DEF having EF = 3·7 in., $\angle$E = 35°, $\angle$F = 64°; on tracing paper draw a triangle ABC having BC = 3·7 in., $\angle$B = 35°, $\angle$C = 64°. Apply $\triangle$ABC to $\triangle$DEF so that B falls on E, and BC falls along EF. Do the two triangles coincide?

Theorem 11.

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

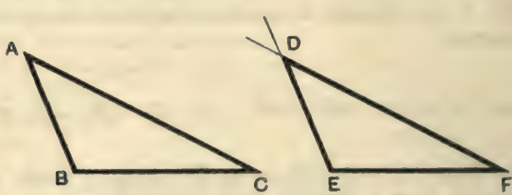


fig. 106.

Data ABC, DEF are two triangles which have BC = EF and two angles of the one equal to the two corresponding angles of the other.

To prove that $\triangle ABC \equiv \triangle DEF$.

Proof Since two angles of $\triangle ABC$ are respectively equal to two angles of $\triangle DEF$,

$\therefore$ the third angle of $\triangle ABC$ = the third angle of $\triangle DEF$, I. 8, *Cor.* 5.

$\therefore \angle A = \angle D, \ \angle B = \angle E$, and $\angle C = \angle F$.

Apply $\triangle ABC$ to $\triangle DEF$ so that B falls on E, and BC falls along EF.

$\because BC = EF$,

$\therefore$ C falls on F.

Now $\angle B = \angle E$,

$\therefore$ BA falls along ED,

$\therefore$ A falls somewhere along ED or ED produced.

Again $\angle C = \angle F$,

$\therefore$ CA falls along FD,

$\therefore$ A falls somewhere along FD or FD produced,

$\therefore$ A falls on D,

$\therefore \triangle ABC$ coincides with $\triangle DEF$,

$\therefore \triangle ABC \equiv \triangle DEF$. Q. E. D.

Ex. **426.** Make a list of all the equal sides and angles in $\triangle^s$ ABC, DEF of I. 11.

†Ex. **427.** Draw two $\triangle^s$ GHK, XYZ, and mark $GH = XY$, $\angle H = \angle Y$, and $\angle K = \angle X$; are the triangles congruent?

†Ex. **428.** ABCD is a square, E is the mid-point of AB; at E make $\angle AEP = 60°$ and $\angle BEQ = 60°$; let EP, EQ cut AD, BC at P and Q respectively. Prove that $AP = BQ$. (See fig. 97.)

†Ex. **429.** In a $\triangle XYZ$, $\angle Y = \angle Z$; XW is drawn so that $\angle X$ is bisected; prove that $XY = XZ$. (See fig. 103.)

†Ex. **430.** If the bisector of an angle of a triangle cuts the opposite side at right angles, the triangle must be isosceles.

[Let XYZ be a triangle; and let XW, the bisector of $\angle X$, cut YZ at right angles at W; prove that $XY = XZ$. See fig. 103.]

†Ex. **431.** ABC, DEF are two triangles which are equal in all respects; AP, DQ are drawn perpendicular to BC, EF respectively. Prove that $AP = DQ$.

†Ex. **432.** $\triangle ABC \equiv \triangle DEF$. AG, DH are the bisectors of $\angle A$, $\angle D$ and meet the opposite sides in G, H. Prove that $AG = DH$.

†Ex. **433.** The following method may be used to find the breadth of a river. Choose a place where the river is straight, note some conspicuous object T (e.g. a tree) on the edge of the other bank; from a point O opposite T measure a distance OS along the bank; put a stick in the ground at S; walk on to a point P such that $SP = OS$; from P walk at right angles to the river till you are in the same straight line as S and T. PQ is equal to the breadth of the river. Prove this.

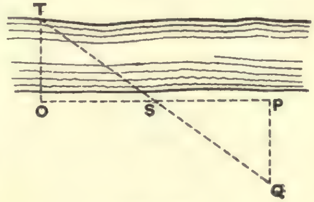


fig. 107.

†Ex. **434.** The perpendiculars drawn to the arms of an angle from any point on the bisector of the angle are equal to one another.

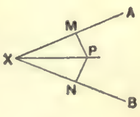


fig. 108.

†Ex. **435.** ABCD is a parallelogram, prove that $AB = CD$.

[Join AC and use I. 5.]

†Ex. **436.** If the diagonal PR of a quadrilateral PQRS bisects the angles at P and R, prove that the quadrilateral has two pairs of equal sides.

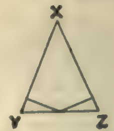


fig. 109.

†Ex. **437.** A triangle XYZ has $\angle Y = \angle Z$; prove that the perpendiculars from the mid-point of YZ to XY and XZ are equal to one another.

†Ex. **438.** A triangle ABC has $\angle B = \angle C$; prove that the perpendiculars from B and C on the opposite sides are equal to one another.

fig. 110.

†Ex. **439.** A triangle ABC has $AB = AC$; prove that the perpendiculars from B and C on the opposite sides are equal to one another.

†Ex. **440.** The diagonal AC of a quadrilateral ABCD bisects the angle A and $\angle ABC = \angle ADC$; does $BC = CD$?

¶Ex. **441.** Draw two or three isosceles triangles; measure their angles.

THEOREM 12.

If two sides of a triangle are equal, the angles opposite to these sides are equal.

fig. 111.

Data ABC is a triangle which has AB = AC.

To prove that $\angle C = \angle B$.

Construction Draw AD to represent the bisector of $\angle$ BAC.
Let it cut BC at D.

Proof In the $\triangle$s ABD, ACD

$$\left\{\begin{array}{ll} AB = AC, & \textit{Data} \\ AD \text{ is common,} & \\ \angle BAD = \angle CAD \text{ (included } \angle\text{s)}, & \textit{Constr.} \end{array}\right.$$

$\therefore \triangle ABD \equiv \triangle ACD$, I. 10.

$\therefore \angle B = \angle C$.

Q. E. D.

The phrase "the sides" of an isosceles triangle is often used to mean the equal sides, "the base" to mean the other side, "the vertex" to mean the point at which the equal sides meet, and "the vertical angle" to mean the angle at the vertex.

Ex. **442.** State the converse of this theorem.

Ex. **443.** In a triangle XYZ, XY = XZ; find the angles of the triangle in the following cases: (i) $\angle Y = 74°$, (ii) $\angle X = 36°$, (iii) $\angle X = 142°$, (iv) $\angle Y = 13°$, (v) $\angle Z = 97°$, (vi) $\angle Z = 45°$.

†Ex. **444.** Each base angle of an isosceles triangle must be acute.

Ex. **445.** Find the angles of an isosceles triangle in which each of the base angles is half of the vertical angle.

Ex. **446.** Find the angles of an isosceles triangle in which each of the base angles is double of the vertical angle.

†Ex. **447.** **Prove that a triangle which is equilateral is also equiangular.** (See definition, p. 82.)

[If PQR is an equilateral triangle, $\because QP = QR \therefore \angle ? = \angle ?$.]

Ex. **448.** In a triangle ABC, AB = 9·2 cm., $\angle$C = 82°, AC = 9·2 cm.; AB, AC are produced to D, E respectively. Find all the angles in the figure.

†Ex. **449.** ABC is an isosceles triangle; the equal sides AB, AC are produced to X, Y respectively. Prove that $\angle XBC = \angle YCB$.

State the converse of this theorem.

†Ex. **450.** EDA, FDA are two isosceles triangles on opposite sides of the same base DA; prove that $\angle EDF = \angle EAF$. See fig. 123.

†Ex. **451.** EDA, FDA are two isosceles triangles on the same side of the same base DA; prove that $\angle EDF = \angle EAF$.

†Ex. **452.** Through the vertex P of an isosceles triangle PQR a straight line XPY is drawn parallel to QR; prove that $\angle QPX = \angle RPY$.

†Ex. **453.** From the mid-point O of a straight line AB a straight line OC is drawn; if OC = OA, $\angle$ACB is a right angle.

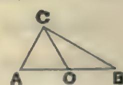


fig. 112.

†Ex. **454.** In fig. 113, $\triangle$ABC is isosceles and BP = CQ; prove that $\angle APQ = \angle AQP$.

[First prove AP = AQ.]

fig. 113.

†Ex. **455.** The perpendicular from the vertex of an isosceles triangle to the base bisects the base.

†Ex. **456.** The perpendiculars to the equal sides of an isosceles triangle from the mid-point of the base are equal. (See fig. 109.)

†Ex. **457.** The perpendiculars from the ends of the base of an isosceles triangle to the opposite sides are equal. (See fig. 110.)

†Ex. **458.** The straight lines joining the mid-point of the base of an isosceles triangle to the mid-points of the sides are equal.


fig. 114.

†Ex. **459.** If A, B are the mid-points of the equal sides XY, XZ of an isosceles triangle, prove that AZ = BY.

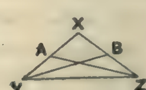


fig. 115.

†Ex. **460.** The bisectors of the base angles of an isosceles triangle are equal.

fig. 116.

†Ex. **461.** At the ends of the base BC of an isosceles triangle ABC, perpendiculars are drawn to the base to meet the equal sides produced; prove that these perpendiculars are equal.

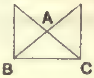


fig. 117.

†Ex. **462.** XYZ is an isosceles triangle (XY=XZ), the bisectors of ∠X and ∠Z meet at O; prove that OY bisects ∠Y.

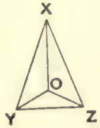


fig. 118.

†Ex. **463.** The angle between a diagonal and a side of a square is 45°.

†Ex. **464.** OA, OB are radii of a circle, AO is produced to P; prove that ∠BOP=2∠BAP.

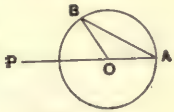


fig. 119.

†Ex. **465.** In fig. 119, prove that the perpendicular from O to AB bisects AB.

†Ex. **466.** If a four-sided figure has all its sides equal, its opposite angles are equal.

[Draw a diagonal.]

¶Ex. **467.** Draw a line BC, at B and C make equal angles CBA, BCA so as to form a triangle ABC. Measure AB and AC.

¶Ex. **468.** Repeat Ex. 467 two or three times with other lines and angles.

Theorem 13.

[Converse of Theorem 12.]

If two angles of a triangle are equal, the sides opposite to these angles are equal.

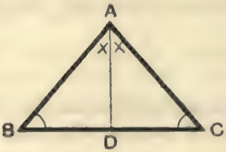


fig. 120.

Data ABC is a triangle which has $\angle B = \angle C$.

To prove that $AC = AB$.

Construction Draw AD to represent the bisector of $\angle BAC$. Let it cut BC at D.

Proof In the $\triangle$s ABD, ACD,

$$\left\{\begin{array}{l} \angle B = \angle C, \\ \angle BAD = \angle CAD, \\ \quad AD \text{ is common,} \end{array}\right. \qquad \begin{array}{r} \textit{Data} \\ \textit{Constr.} \\ \, \end{array}$$

$$\therefore \triangle ABD \equiv \triangle ACD, \qquad \text{I. 11.}$$

$$\therefore AB = AC.$$

Q. E. D.

†Ex. **469.** **Prove that if a triangle PQR is equiangular, it must also be equilateral.**

[$\angle Q = \angle R$, $\therefore$ side ? = side ?.]

†Ex. **470.** The sides AB, AC of a triangle are produced to XY; prove that, if $\angle XBC = \angle YCB$, $\triangle ABC$ is isosceles. (See fig. 77.)

†Ex. **471.** A straight line drawn parallel to the base of an isosceles triangle to cut the equal sides forms another isosceles triangle.

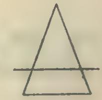


fig. 121.

†**Ex. 472.** XYZ is an isosceles triangle; the bisectors of the equal angles (Y, Z) meet at O; prove that $\triangle$OYZ is also isosceles. (See fig. 118.)

†**Ex. 473.** From Q and R, the extremities of the base of an isosceles triangle PQR, perpendiculars are drawn to the opposite sides. If these perpendiculars intersect at X, prove that XQ=XR.

†**Ex. 474.** XYZ is an isosceles triangle (XY=XZ), the bisectors of $\angle$Y and $\angle$Z meet at O; prove that OX bisects $\angle$X.

†**Ex. 475.** If through any point in the bisector of an angle a line is drawn parallel to either of the arms of the angle, the triangle thus formed is isosceles.

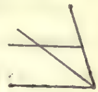


fig. 122.

†**Ex. 476.** ABCD is a quadrilateral in which AB=AD, and $\angle$B=$\angle$D; prove that CB=CD.

[Draw a diagonal.]

†**Ex. 477.** In the base BC of a triangle ABC, points P, Q are taken such that $\angle$BAP=$\angle$CAQ; if AP=AQ, prove $\triangle$ABC is isosceles.

†**Ex. 478.** In a quadrilateral ABCD, $\angle^s$ A, B are equal and obtuse, and AB is parallel to CD; prove that AD=BC.

[Produce DA, CB till they meet.]

†**Ex. 479.** If the $\angle^s$ G, H of a triangle FGH are each double of $\angle$F, and if the bisector of $\angle$G meets FH in K, prove that FK=GK=GH.

¶**Ex. 479 a.** If one side of a triangle is *double* another, is the angle opposite the former double the angle opposite the latter?

In order to answer this question, take the following instances:

(1) Consider a triangle whose angles are 45°, 45°, 90°.

(2) Consider a triangle whose angles are 30°, 60°, 90°.

(3) Draw $\triangle$ABC in which AB=8·2 cms., BC=4·1 cms., CA=6 cms. Measure the angles. Is C double A?

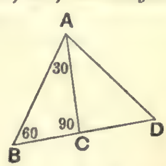
(4) Draw $\triangle$ABC in which A=82°, B=41°, BC=3″. Measure the remaining sides. Is BC double CA?

(5) Prove that in $\triangle$ABC whose angles are 30°, 60°, 90°, the *longest side* AB is double the *shortest* BC.

[Make $\angle$CAD=30°
and produce BC to meet AD in D.

How many degrees in $\angle$D?

What kind of a triangle is ABD?]



Theorem 14.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

Data ABC, DEF are two triangles which have BC = EF, CA = FD, and AB = DE.

To prove that $\triangle$ABC $\equiv$ $\triangle$DEF.

Proof Apply $\triangle$ABC to $\triangle$DEF so that B falls on E and BC falls along EF but so that A and D are on opposite sides of EF; let A′ be the point on which A falls. Join DA′.

Since BC = EF, C will fall on F.

Case I *When* DA′ *cuts* EF.

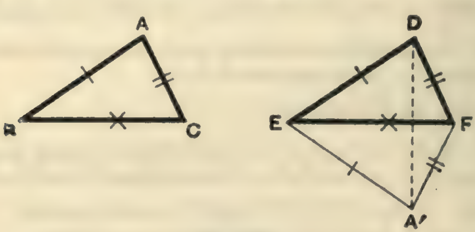


fig. 123.

In $\triangle$EDA′, ED = EA′ (i.e. BA),

$\therefore$ $\angle$EA′D = $\angle$EDA′. I. 12.

In $\triangle$FDA′, FD = FA′ (i.e. CA),

$\therefore$ $\angle$FA′D = $\angle$FDA′, I. 12.

$\therefore$ $\angle$EA′D + $\angle$FA′D = $\angle$EDA′ + $\angle$FDA′.

i.e. $\angle$EA′F = $\angle$EDF,

i.e. $\angle$BAC = $\angle$EDF,

$\therefore$ in $\triangle$s ABC, DEF,

AB = DE, *Data*

AC = DF, *Data*

$\angle$BAC = $\angle$EDF (included $\angle$s). *Proved*

$\therefore$ $\triangle$ABC $\equiv$ $\triangle$DEF. I. 10.

CASE II. *When* DA′ *passes through one end of* EF, *say* F.

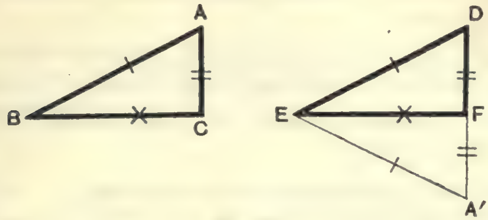


fig. 124.

In $\triangle$EDA′, ED = EA′ (i.e. BA),

$\therefore$ $\angle$EA′D = $\angle$EDA′, I. 12.

i.e. $\angle$BAC = $\angle$EDF,

$\therefore$ as in Case I. $\triangle$ABC $\equiv$ $\triangle$DEF.

CASE III. *When* DA′ *does not cut* EF.

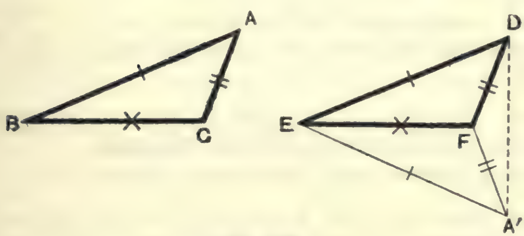


fig. 125.

As in Case I. $\angle$EA′D = $\angle$EDA′,

and $\angle$FA′D = $\angle$FDA′,

$\therefore$ $\angle$EA′D − $\angle$FA′D = $\angle$EDA′ − $\angle$FDA′,

i.e. $\angle$EA′F = $\angle$EDF,

i.e. $\angle$BAC = $\angle$EDF,

$\therefore$ as in Case I. $\triangle$ABC $\equiv$ $\triangle$DEF.

Q. E. D.

Ex. **480.** State the converse of this theorem. Is it true?

†Ex. **481.** If, in a quadrilateral ABCD, AB = AD, CB = CD, prove that AC bisects ∠A and ∠C.

†Ex. **482.** PQ and RS are two equal chords of a circle whose centre is O. Prove that ∠POQ = ∠ROS.

(A **chord** of a circle is a straight line joining any two points on the circle.)

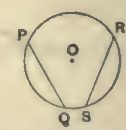


fig. 126.

†Ex. **483.** AB is a chord of a circle whose centre is O; C is the mid-point of the chord AB. Show that OC is perpendicular to AB.

†Ex. **484.** If the opposite sides of a quadrilateral are equal, it is a parallelogram.

[Draw a diagonal, and use I. 4.]

†Ex. **485.** Equal lengths AB, AC are cut off from the arms of an angle BAC; on BC a triangle BCD is drawn having BD = CD. Show that AD bisects ∠BAC.

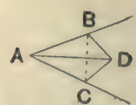


fig. 127.

†Ex. **486.** The bisectors of the equal angles Y, Z of an isosceles triangle XYZ meet at O. Prove that XO bisects ∠X.

†Ex. **487.** EDA, FDA are two isosceles triangles on opposite sides of the same base DA; prove that EF bisects DA at right angles.

[First prove △DEF ≡ △AEF; see fig. 123.]

†Ex. **488.** EDA, FDA are two isosceles triangles on the same base DA and on the same side of it; prove that EF produced bisects DA at right angles.

†Ex. **489.** In a quadrilateral ABCD, AD = BC and the diagonals AC, BD are equal; prove that ∠ADC = ∠BCD.

Also prove that, if AC, BD intersect at O, △OCD is isosceles.

†Ex. **490.** **Two circles intersect at X, Y; prove that XY is bisected at right angles by the straight line joining the centres of the two circles.**

[Join the centres of the circles to X and Y.]

THEOREM 15.

If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.

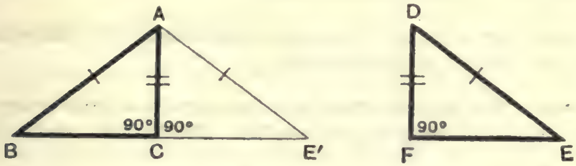


fig. 128.

Data ABC, DEF are two triangles which have $\angle$s C, F right $\angle$s, AB = DE, and AC = DF.

To prove that $\triangle$ABC $\equiv$ $\triangle$DEF.

Proof Apply $\triangle$DEF to $\triangle$ABC so that D falls on A and DF along AC, but so that E and B are on opposite sides of AC; let E′ be the point on which E falls.

Since DF = AC, F will fall on C.

Since $\angle$s ACB, ACE′ (i.e. DFE) are two rt. $\angle$s, *Data*

BCE′ is a st. line. I. 2.

$\therefore$ ABE′ is a $\triangle$.

In this $\triangle$, AB = AE′ (i.e. DE) *Data*

$\therefore$ $\angle$E′ = $\angle$B. I. 12.

Now in the $\triangle$s ABC, AE′C,

$\angle$B = $\angle$E′, *Proved*

$\angle$ACB = $\angle$ACE′, *Data*

AB = AE′, *Data*

$\therefore$ $\triangle$ABC $\equiv$ $\triangle$AE′C, I. 11.

$\therefore$ $\triangle$ABC $\equiv$ $\triangle$DEF.

Q. E. D.

†Ex. **491.** In fig. 97, given that E is the mid-point of AB and EP = EQ, prove that $\triangle AEP \equiv \triangle BEQ$.

†Ex. **492.** In a triangle XYZ, XY = XZ, and XW is drawn at right angles to YZ: prove that $\triangle XYW \equiv \triangle XZW$. (Use I. 15.)

†Ex. **493.** Perpendiculars are drawn from a point P to two straight lines XA, XB which intersect at a point X; prove that, if the perpendiculars are equal, PX bisects $\angle AXB$. (See fig. 108.)

†Ex. **494.** AB is a chord of a circle whose centre is O. Show that the perpendicular from O on AB bisects AB.

†Ex. **495.** The perpendiculars from the centre of a circle on two equal chords of the circle are equal to one another. (See fig. 126; use Ex. 494.)

†Ex. **496.** In fig. 129, PM, QN are drawn perpendicular to the diameter AOB, O being the centre of the circle; show that, if PM = QN, then $\angle POM = \angle QON$.

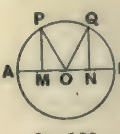


fig. 129.

†Ex. **497.** If the perpendiculars from the mid-point of the base of a triangle to the other two sides are equal, the triangle is isosceles. (See fig. 109.)

†Ex. **498.** If the perpendiculars from two corners of a triangle to the opposite sides are equal, the triangle is isosceles. (See fig. 110.)

†Ex. **499.** From the vertices A, X of two triangles ABC, XYZ, lines AD, XW are drawn perpendicular to BC, YZ respectively. If AD = XW, AB = XY, and AC = XZ, prove that the triangles ABC, XYZ are congruent, provided they are both acute-angled, or both obtuse-angled.

†Ex. **500.** With the same notation as in Ex. 499, prove that, if AD = XW, AB = XY, and BC = YZ, the triangles are congruent.

CONSTRUCTIONS.

Hitherto we have constructed our figures with the help of graduated instruments. We shall now make certain constructions with the aid of nothing but a straight edge (not graduated) and a pair of compasses.

We shall use the straight edge

(i) for drawing the straight line passing through any two given points,

(ii) for producing any straight line already drawn.

We shall use the compasses

(i) for describing circles with any given point as centre and radius equal to any given straight line,

(ii) for the transference of distances; i.e. for cutting off from one straight line a part equal to another straight line. [(ii) is really included in (i).]

By means of theorems which we have already proved, we shall show that our constructions are accurate.

In the exercises, when you are asked to construct a figure, you should always explain your construction in words. You need not give a proof unless you are directed to do so.

In the earlier constructions the figures are shown with
given lines—thick,
construction lines—fine,
required lines—of medium thickness,
lines needed only for the proof—broken.

In making constructions, only the necessary parts of construction circles should be drawn even though "the circle" is spoken of.

Revise Ex. 98—102.

To construct a triangle having its sides equal to three given straight lines.

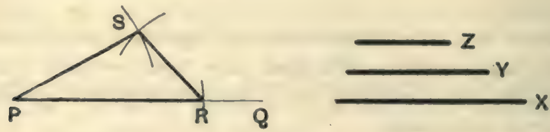


fig. 130.

Let X, Y, Z be the three given straight lines.

Construction Draw a straight line PQ.
From PQ cut off a part PR = X.
With centre P and radius = Y describe a circle.
With centre R and radius = Z describe a circle.
Let the circles intersect at S.
Join PS, RS.
Then PRS is the required triangle.

NOTE. It is best to draw the longest line first.

It should be observed that the construction is impossible if one of the given straight lines is greater than the sum of the other two. (Why?)

Ex. **501.** Draw a large* triangle and construct a congruent triangle.

Ex. **502.** Construct a triangle having its sides equal to the lines *b, d, h* of fig. 8.

Ex. **503.** Draw a straight line (about 3 in. long); on it describe an equilateral triangle. Measure its angles.

Ex. **504.** Construct an isosceles triangle of base 5 cm. and sides 10 cm. Measure the vertical angle.

†Ex. **505.** Draw an angle ABC; complete the parallelogram of which AB, BC are adjacent sides. [On AC construct $\triangle ACD \equiv \triangle CAB$.] Give proof.

Ex. **506.** Make an angle of 60° (without protractor or set square).

Ex. **507.** Make an angle of 120° (without protractor or set square).

Revise Ex. 274—276.

* Constructions should always be made on a large scale; an error of ·5 mm. is less important in a large figure than in a small one. In this case let the shortest side be at least 3 in. long.

Through a point O in a straight line OX to draw a straight line OY so that $\angle$XOY may be equal to a given angle BAC.

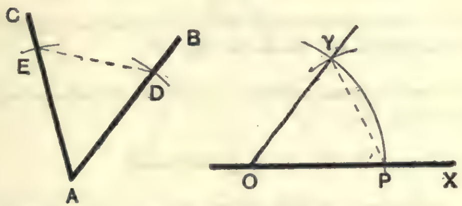


fig. 131.

Construction With centre A and any radius describe a circle cutting AB, AC at D, E respectively.

With centre O and the same radius describe a circle PY cutting OX at P.

With centre P and radius = DE describe a circle cutting the circle PY at Y.

Join OY.

Then $\angle$XOY = $\angle$BAC.

Proof Join DE and PY.

In the $\triangle^s$ OPY, ADE,

$$\begin{cases} OP = AD, & \textit{Constr.} \\ OY = AE, & \text{''} \\ PY = DE, & \text{''} \end{cases}$$

$\therefore$ $\triangle POY \equiv \triangle ADE$, I. 14

$\therefore$ $\angle POY = \angle DAE$,

i.e. $\angle XOY = \angle BAC$.

[*The protractor must not be used in Ex.* 508—518.]

Ex. **508.** Draw an acute angle and construct an equal angle*.

Ex. **509.** Draw an obtuse angle and make a copy of it.

Ex. **510.** Draw an acute angle ABC; at C make an angle BCD = ∠ABC. Let BA, CD intersect at O. Measure OB, OC.

Ex **511.** Draw a triangle ABC; at a point O make a copy of its angles in the manner of fig. 50.

Ex. **512.** Repeat Ex. 511 for a quadrilateral.

Ex. **513.** Draw two straight lines and an angle. Construct a triangle having two sides and the included angle equal respectively to these lines and angle.

Ex. **514.** Construct a triangle ABC having given BC, ∠B and ∠C.

Ex. **515.** Construct a triangle ABC having given BC, ∠A and ∠B.

Ex. **516.** Draw a straight line EF and mark a point G (about 2 in. from the line); through G draw a line parallel to EF.

[Draw any line through G cutting EF at H; make ∠HGC = ∠GHF; see fig. 86.]

Ex. **517.** Repeat Ex. 516, using corresponding instead of alternate angles.

Ex. **518.** Draw a large polygon and make a copy of it, using the first method described on p. 50.

Revise "Symmetry" pp. 51—55.

Ex. **519.** Cut out an angle of paper; bisect it by folding as in Ex. 31.

* It is convenient to draw the angle on tracing paper so as to compare it with the angle made equal to it.

To bisect a given angle.

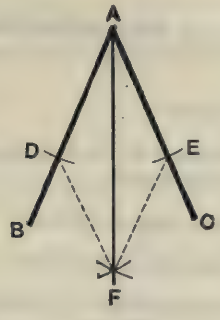


fig. 132.

Let BAC be the given angle.

Construction From AB, AC cut off equal lengths AD, AE.

With centres D and E and any convenient radius describe equal circles intersecting at F.

Join AF.

Then AF bisects $\angle$ BAC.

Proof Join DF and EF.

In the $\triangle^s$ ADF, AEF,

$$\begin{cases} AD = AE, & \textit{Constr.} \\ DF = EF, & " \\ AF \text{ is common.} & \end{cases}$$

$\therefore \triangle ADF \equiv \triangle AEF,$ I. 14.

$\therefore$ AF bisects $\angle$ BAC.

"*Any convenient radius.*" If it is found that the equal circles do not intersect, the radius chosen is not convenient, for the construction breaks down; it is necessary to take a larger radius so that the circles may intersect.

¶Ex. **520.** If fig. 132 were folded about AF, what points would coincide? What lines?

¶Ex. **521.** Make two equal angles and bisect them; in one case join the vertex to the nearer point at which the equal circles intersect, in the other to the further point.

Which gives the better result?

¶Ex. **522.** Is there any case in which one point of intersection would coincide with the vertex of the angle?

Ex. **523.** Draw an acute angle and bisect it. Check by measurement.

Ex. **524.** Draw an obtuse angle and bisect it. Check by measurement.

Ex. **525.** Quadrisect a given angle (i.e. divide it into four equal parts).

Ex. **526.** Draw an angle of 87° and bisect it (1) by means of the protractor, (2) as explained above.

Do the results agree? (This will test the accuracy of your protractor.)

Ex. **527.** Construct angles of 15°, 30° and 150° (without protractor).

Ex. **528.** Draw a large triangle and bisect each of its angles.

Ex. **529.** Construct an isosceles triangle, bisect its vertical angle and measure the parts into which the base is divided.

Ex. **530.** Draw a triangle whose sides are 5 cm., 10 cm., 12 cm. Bisect the greatest angle and measure the parts into which the opposite side is divided.

¶Ex. **531.** Draw a straight line AB on tracing paper; fold it so that A falls on B; measure the parts into which AB is divided by the crease and the angles the crease makes with AB.

Revise Ex. 288—290.

To draw the perpendicular bisector of a given straight line.

To bisect a given straight line.

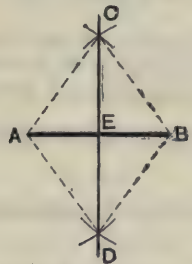


fig. 133.

Let AB be the given straight line.

Construction With centres A and B and any convenient radius describe equal circles intersecting at C and D.

Join CD and let it cut AB at E.

Then CD is the perpendicular bisector of AB, and E is the mid-point of AB.

Proof Join AC, AD, BC, BD.

In the $\triangle^s$ ACD, BCD,

$$\begin{cases} AC = BC, & \textit{Constr.} \\ AD = BD, & \text{"} \\ CD \text{ is common,} & \end{cases}$$

$$\therefore \triangle ACD \equiv \triangle BCD, \qquad \text{I. 14.}$$

$$\therefore \angle ACD = \angle BCD.$$

In the $\triangle^s$ ACE, BCE,

$$\begin{cases} AC = BC, & \textit{Constr.} \\ CE \text{ is common,} & \\ \angle ACE = \angle BCE, & \textit{Proved} \end{cases}$$

$$\therefore \triangle ACE \equiv \triangle BCE, \qquad \text{I. 10.}$$

$$\therefore AE = BE,$$

and $\angle^s$ CEA, CEB are equal and are therefore rt. $\angle^s$, *Def.*

$\therefore$ CD bisects AB at right angles.

¶Ex. 532. Describe the symmetry of fig. 133.

Ex. 533. Draw a straight line and bisect it.

Ex. 534. Quadrisect a given straight line.

¶Ex. 535. Draw a straight line AB and its perpendicular bisector CD. Take any point P in CD and measure PA and PB. Take three other points on CD and measure their distances from A and B.

Ex. 536. Draw a large acute-angled triangle; draw the perpendicular bisectors of its three sides.

Ex. 537. Repeat Ex. 536 for (i) a right-angled triangle, (ii) an obtuse-angled triangle.

Ex. 538. Draw any chord of a circle and its perpendicular bisector.

DEF. The straight line joining a vertex of a triangle to the mid-point of the opposite side is called a **median.**

Ex. 539. Draw a large triangle; and draw its three medians. Are the angles bisected?

¶Ex. 540. Call one of the short edges of your paper AB; construct its perpendicular bisector by folding. Fold the paper again so that the new crease may pass through A, and B may fall on the old crease; mark the point C on which B falls and join CA, CB. What kind of triangle is ABC?

Ex. 541. Draw a large obtuse angle (very nearly 180°) and bisect it.

To draw a straight line perpendicular to a given straight line AB from a given point P in AB.

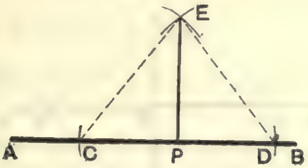


fig. 134.

Construction From PA, PB cut off equal lengths PC, PD.

With centres C and D and any convenient radius describe equal circles intersecting at E.

Join PE.

Then PE is $\perp$ to AB.

Proof Join CE, DE.

In the $\triangle^s$ CPE, DPE,

$$\left\{\begin{array}{l} PC = PD, \\ CE = DE \text{ (radii of equal } \odot^s\text{)}, \\ PE \text{ is common.} \end{array}\right. \qquad \textit{Constr.}$$

$\therefore \triangle CPE \equiv \triangle DPE$, I. 14.

$\therefore \angle EPC = \angle EPD$,

$\therefore$ PE is $\perp$ to AB. *Def.*

[*The protractor and set square must not be used in Ex.* 542—555.]

Ex. **542.** Draw a straight line, and a straight line at right angles to it. Test with set square.

Ex. **543.** Draw an isosceles triangle; at the ends of the base erect perpendiculars and produce the sides to meet them (see fig. 117). Measure all the lines in the figure.

Ex. **544.** Construct angles of 45° and 75°.

Ex. **545.** Draw a chord AB of a circle, at A and B erect perpendiculars to cut the circle at P and Q respectively. Measure AP, BQ.

†Ex. **546.** Make an angle AXB; from XA, XB cut off equal lengths XM, XN; from M, N draw MP, NP at right angles to XA, XB respectively; join PX. Prove that PX bisects $\angle$AXB. Check by measurement.

[See fig. 108.]

To draw a straight line perpendicular to a given straight line AB from a given point P outside AB.

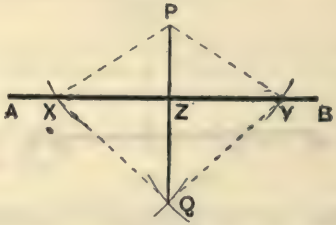


fig. 135.

Construction With centre P and any convenient radius describe a circle cutting AB at X and Y.

With centres X and Y and any convenient radius describe equal circles intersecting at Q.

Join PQ cutting AB at Z.

Then PZ is $\perp$ to AB.

Proof Join XP, XQ, YP, YQ.

In the $\triangle^s$ PQX, PQY,

$$\begin{cases} PX = PY \text{ (radii of a } \odot\text{)}, \\ QX = QY \text{ (radii of equal } \odot^s\text{)}, \\ PQ \text{ is common.} \end{cases}$$

$\therefore \triangle PQX \equiv \triangle PQY$, I. 14.

$\therefore \angle XPQ = \angle YPQ$.

We can now prove that

$\triangle PXZ \equiv \triangle PYZ$, (give the three reasons)

$\therefore \angle PZX = \angle PZY$,

$\therefore$ PZ is $\perp$ to AB.

Ex. **547.** Draw a large acute-angled triangle; from each vertex draw a perpendicular to the opposite side.

Ex. **548.** Repeat Ex. 547 with a right-angled triangle.

Ex. **549.** Repeat Ex. 547 with an obtuse-angled triangle. [You will have to produce two of the sides.]

Ex. 550. Draw an acute angle and bisect it; from any point on the bisector drop perpendiculars on the arms of the angle; measure the perpendiculars.

Ex. 551. Repeat Ex. 550 for an obtuse angle.

Ex. 552. From the centre of a circle drop a perpendicular on a chord of the circle.

Ex. 553. Cut out of paper an acute-angled triangle; by folding construct the perpendiculars from each vertex to the opposite side.

Ex. 554. Cut out a paper triangle ABC ($\angle^s$ B and C being acute); by folding construct AD perpendicular to BC. Again fold so that A, B and C all fall on D.

Ex. 555. Cut out of paper an equilateral triangle ABC (see Ex. 540). Construct two of the perpendiculars from the vertices to the opposite sides; let them intersect at O. Fold so that A falls on O, and then so that B and C fall on O. What is the resulting figure?

Construction of Triangles from given data.

We have seen how to construct triangles having given

(i) the three sides (Ex. 99–102, and p. 104);

(ii) two sides and the included angle (Ex. 87, 88, 513);

(iii) one side and two angles (Ex. 89, 90, 514, 515).

I. 14, 10, 11 prove that if a set of triangles were constructed from the same data, such as those given above, they would all be congruent.

In Ex. 146–150, we saw that, given the angles, it is possible to construct an unlimited number of different triangles.

If two angles of a triangle are given, the third angle is known; hence the three angles do not constitute more than two data.

We have still to consider the case in which two sides are given and an angle not included by these sides.

¶Ex. 556. Construct a triangle ABC having given BC = 2·4 in., CA = 1·8 in., and ∠ B = 32°.

First make BC = 2·4 in. and ∠ CBD = 32°.

A must lie somewhere on BD, and must be 1·8 in. from C.

Where do all the points lie which are 1·8 in. from C?

How many points are there which are on BD and also 1·8 in. from C?

We see that it is possible to construct two unequal triangles which satisfy the given conditions. This case is therefore called the **ambiguous case**.

Ex. **557.** Construct triangles to the following data:—

(i) BC = 8·7 cm., CA = 5·3 cm., ∠B = 29°;
(ii) BC = 7·3 cm., CA = 9·0 cm., ∠A = 53°;
(iii) AB = 3·9 in., AC = 2·6 in., ∠C = 68°;
(iv) AB = 2·2 in., BC = 3·7 in., ∠A = 90°;
(v) AC = 5·3 cm., BC = 10 cm., ∠B = 32°;
(vi) AC = 1·6 in., BC = 4·7 in., ∠B = 26°.

†Ex. **558.** Prove (theoretically) that the two triangles obtained in Ex. 557 (iv) are congruent.

We may summarise the cases of congruence of triangles as follows:—

Data	*Conclusion*	*Theorem*
3 sides	All the triangles are congruent	I. 14
2 sides and included angle	All the triangles are congruent	I. 10
2 sides and an angle not included	Two triangles are generally possible (ambiguous case)	Ex. 867
1 side and 2 angles	All the triangles are congruent	I. 11
3 angles	All the triangles have the same shape, but not necessarily the same size	—

MISCELLANEOUS EXERCISES.

CONSTRUCTIONS.

Ex. 559. Construct angles of (i) 135°; (ii) 105°; (iii) $22\frac{1}{2}$° (without protractor or set square).

Ex. 560. Show how to describe an isosceles triangle on a given straight line, having each of its equal sides double the base.

Are the base angles double the vertical angle?

Ex. 561. Describe a circle and on it take three points A, B, C; join BC, CA, AB. Bisect angle BAC and draw the perpendicular bisector of BC. Produce the two bisectors to meet.

Ex. 562. Having given two angles of a triangle, construct the third angle (without protractor).

Ex. 563. Draw an isosceles triangle ABC; on the side AB describe an isosceles triangle having its angles equal to the angles of the triangle ABC (without protractor).

Ex. 564. Show how to describe a right-angled triangle having given its hypotenuse and one acute angle.

Ex. 565. Construct a triangle ABC having AB = 3 in., BC = 5 in., and the median to BC = 2·5 in. Measure CA.

Ex. 566. Construct a triangle ABC having given AB = 10 cm., AC = 8 cm., and the perpendicular from A to BC = 7·5 cm. Measure BC. Is there any ambiguity?

[First draw the line of the base, and the perpendicular.]

Ex. 567. Construct a triangle ABC having given AB = 11·5 cm., BC = 4·5 cm., and the perpendicular from A to BC = 8·5 cm. Measure AC. Is there any ambiguity?

Ex. 568. Show how to construct a quadrilateral having given its sides and one of its angles.

Ex. 569. Four of the sides, taken in order, of an equiangular hexagon are 1, 3, 3, 2 inches respectively: construct the hexagon and measure the remaining sides.

[What are the angles of an equiangular hexagon?]

†**Ex. 570.** Show how to construct an isosceles triangle having given the base and the perpendicular from the vertex to the base. Give a proof.

[See Ex. 455.]

†Ex. **571.** A, B are two points on opposite sides of a straight line CD; in CD find a point P such that $\angle APC = \angle BPD$. Give a proof.

†Ex. **572.** A, B are two points on the same side of a straight line CD; in CD find a point P such that $\angle APC = \angle BPD$. Give a proof.

[From A draw AN perpendicular to CD and produce it to A′ so that NA′ = NA; if P is any point in CD, $\angle^s$ APN and A′PN can be proved equal; in fact, A and A′ are symmetrical points with regard to CD.]

Ex. **573.** Show how to construct an isosceles triangle on a given base, having given the sum of the vertical angle and one of the base angles.

Ex. **574.** Construct a triangle, having one angle four times each of the other two. Find the ratio of the longest side to the shortest.

[First calculate the angles.]

†Ex. **575.** Show how to construct an isosceles triangle on a given base, having its vertical angle equal to a given angle. Give a proof.

†Ex. **576.** Show how to construct an equilateral triangle with a given line as median. Give a proof.

†Ex. **577.** Through one vertex of a given triangle draw a straight line cutting the opposite side, so that the perpendiculars upon the line from the other two vertices may be equal. Give a proof.

[See Ex. 670.]

†Ex. **578.** From a given point, outside a given straight line, draw a line making with the given line an angle equal to a given angle. (Without protractor.) Give a proof.

[Use parallels.]

†Ex. **579.** Through a given point P draw a straight line to cut off equal parts from the arms of a given angle XOY. Give a proof.

[Use parallels.]

Ex. **580.** Draw a triangle ABC in which $\angle B$ is less than $\angle C$. Show how to find a point P in AB such that PB = PC.

†Ex. **581.** In the equal sides AB, AC of an isosceles triangle ABC show how to find points X, Y such that BX = XY = YC. Give a proof.

Theorems.

Ex. **582.** How many diagonals can be drawn through one vertex of (i) a quadrilateral, (ii) a hexagon, (iii) a *n*-gon?

Ex. **583.** How many different diagonals can be drawn in (i) a quadrilateral, (ii) a hexagon, (iii) a *n*-gon?

†Ex. **584. The bisectors of the four angles formed by two intersecting straight lines are two straight lines at right angles to one another.**

†Ex. **585.** If the bisector of an exterior angle of a triangle is parallel to one side, the triangle is isosceles.

†Ex. **586.** The internal bisectors of two angles of a triangle can never be at right angles to one another.

†Ex. **587.** AB, CD are two parallel straight lines drawn in the same sense, and P is any point between them. Prove that $\angle BPD = \angle ABP + \angle CDP$.

†Ex. **588.** ABC is an isosceles triangle ($AB = AC$). A straight line is drawn at right angles to the base and cuts the sides or sides produced in D and E. Prove that $\triangle ADE$ is isosceles.

†Ex. **589.** From the extremities of the base of an isosceles triangle straight lines are drawn perpendicular to the opposite sides; show that the angles which they make with the base are each equal to half the vertical angle.

†Ex. **590.** The medians of an equilateral triangle are equal.

†Ex. **591.** The bisector of the angle A of a triangle ABC meets BC in D, and BC is produced to E. Prove that $\angle ABC + \angle ACE = 2\angle ADC$.

†Ex. **592.** From a point O in a straight line XY, equal straight lines OP, OQ are drawn on opposite sides of XY so that $\angle YOP = \angle YOQ$. Prove that $\triangle PXY \equiv \triangle QXY$.

†Ex. **593.** The sides AB, AC of a triangle are bisected in D, E; and BE, CD are produced to F, G, so that $EF = BE$ and $DG = CD$. Prove that FAG is a straight line.

†Ex. **594.** If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, show that they contain an angle equal to an exterior angle at the base of the triangle.

†Ex. **595.** The bisectors of the angles B, C of a triangle ABC intersect at I; prove that $\angle BIC = 90° + \frac{1}{2}\angle A$.

†Ex. **596.** XYZ is an isosceles right-angled triangle ($XY = XZ$); YR bisects $\angle Y$ and meets XZ at R; RN is drawn perpendicular to YZ. Prove that $RN = XR$.

†Ex. **597.** The perpendiculars from the vertices to the opposite sides of an equilateral triangle are equal to one another.

†Ex. 598. If two of the bisectors of the angles of a triangle meet at a point I the perpendiculars from I to the sides are all equal.

†Ex. 599. The perpendicular bisectors of two sides of a triangle meet at a point which is equidistant from the vertices of the triangle.

†Ex. 600. In the equal sides PQ, PR of an isosceles triangle PQR points X, Y are taken equidistant from P; QY, RX intersect at Z. Prove that $\triangle^s$ ZQR, ZXY are isosceles.

†Ex. 601. ABC is a triangle right-angled at A; AD is drawn perpendicular to BC. Prove that the angles of the triangles ABC, DBA are respectively equal.

†Ex. 602. From a point O in a straight line XOX′ two equal straight lines OP, OQ are drawn so that $\angle$ POQ is a right angle. PM and QN are drawn perpendicular to XX′. Prove that PM = ON.

†Ex. 603. If points P, Q, R are taken in the sides AB, BC, CA of an equilateral triangle such that AP = BQ = CR, prove that PQR is equilateral.

†Ex. 604. ABC is an equilateral triangle; DBC is an isosceles triangle on the same base BC and on the same side of it, and $\angle$ BDC = $\frac{1}{2}$ $\angle$ BAC. Prove that AD = BC.

Ex. 605. How many sides has the polygon, the sum of whose interior angles is three times the sum of its exterior angles?

[What is the sum of all the exterior and interior angles? What is the sum of an exterior angle and the corresponding interior angle?]

†Ex. 606. If two isosceles triangles have equal vertical angles and if the perpendiculars from the vertices to the bases are equal, the triangles are congruent.

†Ex. 607. If, in two quadrilaterals ABCD, PQRS,

AB = PQ, BC = QR, CD = RS, $\angle$ B = $\angle$ Q, and $\angle$ C = $\angle$ R,

the quadrilaterals are congruent.

Prove this (i) by superposition (see I. 10 and 11);
(ii) by joining BD and QS and proving triangles congruent.

†Ex. 608. If two quadrilaterals have the sides of the one equal respectively to the sides of the other taken in order, and have also one angle of the one equal to the corresponding angle of the other, the quadrilaterals are congruent.

[Draw a diagonal of each quadrilateral, and prove triangles congruent.]

†Ex. 609. If points X, Y, Z are taken in the sides BC, CA, AB of an equilateral triangle, such that $\angle$ BAX = $\angle$ CBY = $\angle$ ACZ, prove that, unless AX, BY, CZ pass through one point, they form another equilateral triangle.

†Ex. 610. If points X, Y, Z are taken in the sides BC, CA, AB of any triangle, such that $\angle$ BAX = $\angle$ CBY = ACZ, prove that, unless AX, BY, CZ pass through one point, they form a triangle whose angles are equal to the angles of the triangle ABC.

†Ex. 611. If AA′, BB′, CC′ are diameters of a circle, prove

$$\triangle ABC \equiv \triangle A'B'C'.$$

†Ex. 612. On the sides AB, BC of a triangle ABC, squares ABFG, BCED are described (on the opposite sides to the triangle); prove that

$$\triangle ABD \equiv \triangle FBC.$$

†Ex. 613. On the sides of any triangle ABC, equilateral triangles BCD, CAE, ABF are described (all pointing outwards); prove that AD, BE, CF are all equal.

†Ex. 614. The side BC of a triangle ABC is produced to D; $\angle$ ACB is bisected by the straight line CE which cuts AB at E. A straight line is drawn through E parallel to BC, cutting AC at F and the bisector of $\angle$ ACD at G. Prove that EF = FG.

†Ex. 615. ABC, DBC are two congruent triangles on opposite sides of the same base BC; prove that either AD is bisected at right angles by BC, or AD and BC bisect one another.

†Ex. 616. In a triangle ABC, the bisector of the angle A and the perpendicular bisector of BC intersect at a point D; from D, DX, DY are drawn perpendicular to the sides AB, AC produced if necessary. Prove that

$$AX = AY \text{ and } BX = CY.$$

[Join BD, CD.]

Inequalities.*

¶Ex. 617. Draw a scalene triangle, measure its sides and arrange them in order of magnitude. Under each side in your table write the opposite angle and its measure, thus:—

Sides	AC = 5·8 in.	BC = 4·3 in.	AB = 3·2 in.
Angles	$\angle$B =	$\angle$A =	$\angle$C =

Are the angles now in order of magnitude?

¶Ex. 618. In fig. 136, AD = AC; if $\angle A = 88°$, find $\angle$ADC and $\angle$ACD. What is the sum of $\angle$B and $\angle$DCB?

* This section, pp. 119—132, may be omitted at a first reading.

The sign > means "is greater than."

The sign < means "is less than."

These signs are easily distinguished if it is borne in mind that the greater quantity is placed at the greater end of the sign.

THEOREM 16.

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

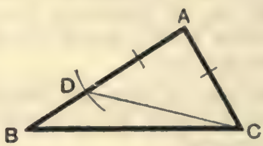


fig. 136.

Data ABC is a triangle in which AB > AC.

To prove that $\angle$ ACB > $\angle$ B.

Construction From AB, the greater side, cut off AD = AC.

Join CD.

Proof In $\triangle$ ACD, AD = AC,

∴ $\angle$ ACD = $\angle$ ADC. I. 12.

But since the side BD of the $\triangle$ DBC is produced to A,

∴ ext. $\angle$ ADC > int. opp. $\angle$ B, I. 8, *Cor.* 2.

∴ $\angle$ ACD > $\angle$ B.

But $\angle$ ACB > its part $\angle$ ACD,

∴ $\angle$ ACB > $\angle$ B.

Q. E. D.

¶Ex. 619. In a $\triangle$ABC, BC = 7 cm., CA = 6·7 cm., AB = 7·5 cm.; which is the greatest angle of the triangle? Which is the least angle? Verify by drawing.

¶Ex. 620. If one side of a triangle is known to be the greatest side, the angle opposite that side must be the greatest angle. (Notice that I. 16 only compares *two* angles; here we are comparing *three.*)

†Ex. 621. The angles at the ends of the greatest side of a triangle are acute.

†Ex. 622. In a parallelogram ABCD, AB > AD; prove that

$$\angle ADB > \angle BDC.$$

[What angle is equal to $\angle$BDC?]

†Ex. 623. In a quadrilateral ABCD, AB is the shortest side and CD is the longest side; prove that $\angle$B > $\angle$D, and $\angle$A > $\angle$C.

[Draw a diagonal.]

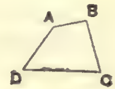


fig. 137.

†Ex. 624. Assuming that the diagonals of a parallelogram ABCD bisect one another, prove that, if BD > AC, then $\angle$DAB is obtuse.

[Let the diagonals intersect at O, then OB > OA and OD > OA; what follows?]

†Ex. 625. Prove Theorem 16 by means of the following construction:—from AB cut off AD = AC, bisect $\angle$BAC by AE, join DE.

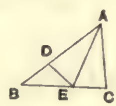


fig. 138.

Theorem 17.

[Converse of Theorem 16.]

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

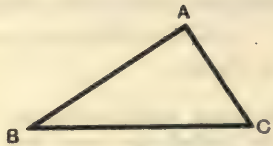


fig. 139.

Data ABC is a triangle in which $\angle C > \angle B$.

To prove that $AB > AC$.

Proof

Either (i) $AB > AC$,
or (ii) $AB = AC$,
or (iii) $AB < AC$.
If, as in (iii), $AB < AC$,
then $\angle C < \angle B$, I. 16.
which is impossible. *Data*
If, as in (ii), $AB = AC$,
then $\angle C = \angle B$, I. 12.
which is impossible. *Data*
$\therefore$ AB must be $> AC$.

Q. E. D.

Note. The method of proof adopted in the above theorem is called **reductio ad absurdum.**

¶Ex. **626.** In a $\triangle ABC$, $\angle A = 68°$ and $\angle B = 28°$. Which is the greatest side of the triangle? Which is the shortest side?

Ex. 627. Repeat Ex. 626 with $\angle B = 34°$, $\angle C = 73°$.

Ex. 628. Draw accurately a triangle whose sides measure 5 cm., 7 cm., 9 cm.; guess the number of degrees in each angle, and verify your guesses by measurement.

†**Ex. 629.** **In a right-angled triangle, the hypotenuse is the longest side.**

†**Ex. 630.** **The side opposite the obtuse angle of an obtuse-angled triangle is the greatest side.**

†**Ex. 631.** If one angle of a triangle is known to be the greatest angle, the side opposite to it must be the greatest side.

†**Ex. 632.** If ON is drawn perpendicular to a straight line AB, and O is joined to a point P in AB, prove that $ON < OP$.

Ex. 633. The side BA of a triangle ABC is produced to E so that $AE = AC$; if $\angle BAC = 86°$ and $\angle ACB = 52°$, find all the angles in the figure.

Ex. 634. In the last Ex. prove that $BE > BC$.

†**Ex. 635.** AD is drawn perpendicular to BC the opposite side of a triangle ABC; prove that $AB > BD$ and $AC > CD$.

Hence show that $AB + AC > BC$.

[There will be two cases.]

†**Ex. 636.** The bisectors of the angles B, C of a triangle ABC intersect at O. Prove that, if $AB > AC$, $OB > OC$.

†**Ex. 637.** If the perpendiculars from B, C to the opposite sides of the triangle ABC intersect at a point X inside the triangle, and if $AB > AC$, prove that $XB > XC$.

†**Ex. 638.** The sides AB, AC of a triangle are produced, and the bisectors of the external angles at B, C intersect at E. Prove that, if $AB > AC$, $EB < EC$.

†**Ex. 639.** A straight line cuts the equal sides AB, AC of an isosceles triangle ABC in X, Y and cuts the base BC produced towards C. Prove that $AY > AX$.

†**Ex. 640.** Prove that the straight line joining the vertex of an isosceles triangle to any point in the base produced is greater than either of the equal sides.

†**Ex. 641.** Prove that the straight line joining the vertex of an isosceles triangle to any point in the base is less than either of the equal sides of the triangle.

THEOREM 18. †

Any two sides of a triangle are together greater than the third side.

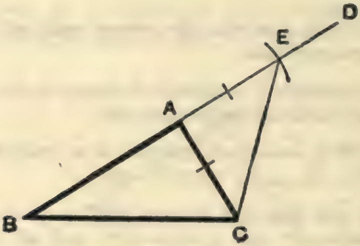


fig. 140.

Data ABC is a triangle.

To prove that (1) $BA + AC > BC$,

(2) $CB + BA > CA$,

(3) $AC + CB > AB$.

(1) *Construction* Produce BA to D.

From AD cut off $AE = AC$.

Join CE.

Proof In the $\triangle AEC$, $AE = AC$, *Constr.*

$\therefore \angle ACE = \angle AEC$, I. 12.

But $\angle BCE >$ its part $\angle ACE$,

$\therefore \angle BCE > \angle AEC$,

$\therefore$ in the $\triangle EBC$, $\angle BCE > \angle BEC$,

$\therefore BE > BC$ I. 17.

i.e. $BA + AE > BC$,

$\therefore BA + AC > BC$, *Constr.*

(2) Sim^{ly} $CB + BA > CA$,

(3) and $AC + CB > AB$.

Q. E. D.

†Ex. **642.** Prove this theorem by drawing AD the bisector of $\angle$A, and applying I. 17 to the two triangles thus formed.

†Ex. **643.** The difference between any two sides of a triangle is less than the third side.

Prove this (i) by means of the same construction as in fig. 136.

(ii) by means of the result of I. 18.

¶Ex. **644.** Why would it be impossible to form a triangle with three rods whose lengths are 7 in., 4 in., and 2 in.?

¶Ex. **645.** If you had four rods of lengths 2 in., 3 in., 4 in., and 6 in. with which sets of three of these would it be possible to form triangles?

†Ex. **646.** S is a point inside a triangle PQR such that PS = PQ; the bisector of $\angle$QPS cuts QR at T. Prove that QT = TS.

Hence from $\triangle$STR prove that RQ > RS.

†Ex. **647.** Any three sides of a quadrilateral are together greater than the fourth side.

[Draw a diagonal.]

†Ex. **648.** If D is any point in the side AC of a triangle ABC, prove that BA + AC > BD + DC.

†Ex. **649.** If O is any point inside a triangle ABC, prove that BA + AC > BO + OC.

[Produce BO to cut AC.]

†Ex. **650.** Any chord of a circle which does not pass through the centre is less than a diameter.

[Join the ends of the chord to the centre.]

†Ex. **651.** In fig. 141, O is the centre of the circle and POA is a straight line; prove that PA > PB.

[Join OB.]

†Ex. **652.** In fig. 141, prove that PC < PB.

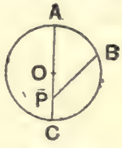


fig. 141.

Tie a piece of elastic to the ends of the arms of your dividers so as to form a triangle; notice that the more the dividers are opened the more the elastic is stretched; or, in other words, the greater the angle between the sides of the triangle the greater the base.

THEOREM 19.* †

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the triangle which has the greater included angle has the greater third side.

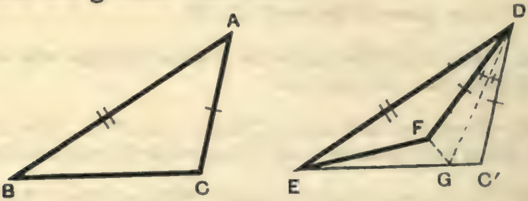


fig. 142.

Data ABC, DEF are two triangles which have AB = DE, and AC = DF but ∠ BAC > ∠ EDF.

To prove that BC > EF.

Proof Apply △ABC to △DEF so that A falls on D and AB falls along DE; then B falls on E (for AB = DE).

Since ∠ BAC > ∠ EDF,

∴ AC falls outside ∠ EDF.

Let C′ be the point on which C falls.

CASE I. If EFC′ is a st. line, EC′ > EF,

i.e. BC > EF.

CASE II. If EFC′ is not a st. line.

Construction Draw DG to represent the bisector of ∠ FDC′; let DG cut EC′ at G.

Proof In △s DGF, DGC′,

$\left\{\begin{array}{l} \text{DF} = \text{DC}' \text{ (i.e. AC)}, \\ \text{DG is common}, \\ \angle\,\text{FDG} = \angle\,\text{C}'\text{DG (included } \angle\text{s)}, \end{array}\right.$ *Data* / *Constr.*

∴ the triangles are congruent, I. 10.

∴ GF = GC′.

Now, in △EFG, EG + GF > EF. I. 18.

But GF = GC′, *Proved*

∴ EG + GC′ > EF,

i.e. EC′ > EF,

i.e. BC > EF. Q. E. D.

* This proposition may be omitted.

Ex. **653.** Draw a figure for I. 19 in which AC, DF are greater than AB, DE. Does the proof hold for this figure?

†Ex. **654.** A, B, C, D, are four points on a circle whose centre is O, such that $\angle AOB > \angle COD$; prove that chord AB > chord CD.

Also state the converse. Is it true?

†Ex. **655.** If, in fig. 155, a point P′ is taken not in the straight line PN, prove that P′A, P′B must be unequal.

[Join P′N.]

†Ex. **656.** In a quadrilateral ABCD, AD = BC and $\angle ADC > \angle BCD$; prove that AC > BD.

†Ex. **657.** Equal lengths YS, ZT are cut off from the sides YX, ZX of a triangle XYZ; prove that, if XY > XZ, YT > ZS.

†Ex. **658.** The sides XY, XZ of a triangle XYZ are produced to S, T so that YS = ZT; prove that, if XY > XZ, ZS > YT.

THEOREM 20.* †

[CONVERSE OF THEOREM 19.]

If two triangles have two sides of the one equal to two sides of the other, each to each, and the third sides unequal, the triangle which has the greater third side has the greater included angle.

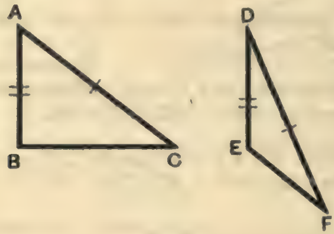


fig. 143.

Data ABC, DEF are two triangles which have AB = DE and AC = DF but BC > EF.

To prove that ∠ BAC > ∠ EDF.

Proof Either (i) ∠ BAC > ∠ EDF,
or (ii) ∠ BAC = ∠ EDF,
or (iii) ∠ BAC < ∠ EDF.
If, as in (iii), ∠ BAC < ∠ EDF,
then BC < EF, I. 19.
which is impossible. *Data*
If, as in (ii), ∠ BAC = ∠ EDF,
then BC = EF, I. 10.
which is impossible. *Data*
∴ ∠ BAC must be > ∠ EDF.

Q. E. D.

* This proposition may be omitted.

†Ex. 659. In a triangle ABC, AB > AC; D is the mid-point of BC. Prove that ∠ ADC is acute.

†Ex. 660. P is any point in the median AD of a triangle ABC; prove that, if AB > AC, PB > PC. (Use Ex. 659.)

†Ex. 661. Equal lengths YS, ZT are cut off from the sides YX, ZX of a triangle XYZ; prove that, if YT > ZS, XY > XZ.

†Ex. 662. State and prove the converse of Ex. 658.

†Ex. 663. In a circle ABCD whose centre is O, the chord AB > the chord CD; prove that ∠AOB > ∠COD.

†Ex. 664. In a quadrilateral ABCD, AD = BC, but AC > BD; prove that ∠ADC > ∠BCD.

†Ex. 665. In a quadrilateral ABCD, AD = BC, but AB < CD; prove that ∠DAC > ∠ACB.

†Ex. 666. In a quadrilateral ABCD, AD = BC, and ∠ADC > ∠BCD; prove that ∠ABC > ∠BAD.

¶Ex. 667. Draw a straight line AB, and draw ON perpendicular to AB (see fig. 144); from O draw six or seven straight lines to meet AB. Measure all these lines.

THEOREM 21.

Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.

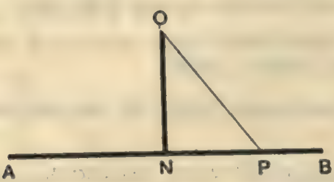


fig. 144.

Data AB is a straight line and O a point outside it; ON is drawn $\perp$ to AB meeting it at N.

To prove that ON $<$ any other st. line that can be drawn from O to AB.

Construction Draw any other st. line from O to meet AB at P.

Proof In the $\triangle$ONP,

$\angle N + \angle P < 2$ rt. $\angle$s, I. 8, *Cor.* 3.

and $\angle N = 1$ rt. $\angle$,

$\therefore \angle P < 1$ rt. $\angle$,

$\therefore \angle P < \angle N$,

$\therefore ON < OP$, I. 17.

Sim^{ly} ON may be proved less than any other st. line drawn from O to meet AB.

$\therefore$ ON is the shortest of all such lines.

Q. E. D.

NOTE. Since the perpendicular is the shortest line that can be drawn from a given point to a given line, it is called the **distance** of the point from the line.

†Ex. 668. In fig. 144, prove that OB > OP.

Ex. 669. Is it possible, in fig. 144, to draw from O to AB a straight line equal to OP?

†Ex. 670. The extremities of a given straight line are equidistant from any straight line drawn through its middle point.

†Ex. 671. If the bisectors of two angles of a triangle are produced to meet, their point of intersection is equally distant from the three sides of the triangle.

Miscellaneous Exercises.

Ex. 672. How many triangles can be formed, two of whose sides are 3 in. and 4 in. long and the third side an exact number of inches?

†Ex. 673. ABC, APQC, are a triangle and a convex quadrilateral on the same base AC, P and Q being inside the triangle; prove that the perimeter of the triangle is greater than that of the quadrilateral.

[Produce AP, PQ to meet BC and use I. 18.]

†Ex. 674. O is a point inside a triangle ABC; prove that ∠BOC > ∠BAC.

[Produce BO to cut AC.]

†Ex. 675. The sum of a median of a triangle and half the side bisected is greater than half the sum of the other two sides.

†Ex. 676. Two sides of a triangle are together greater than twice the median drawn through their point of intersection.

[Use the construction and figure of Ex. 413.]

†Ex. 677. O is a point inside a quadrilateral ABCD; prove that

$$OA + OB + OC + OD$$

cannot be less than AC + BD.

†Ex. 678. The sum of the distances of any point O from the vertices of a triangle ABC is greater than half the perimeter of the triangle.

[The perimeter of a figure is the sum of its sides. Apply I. 18 to △s OBC, OCA, OAB in turn and add up the results.]

†Ex. 679. The sum of the distances from the vertices of a triangle of any point within the triangle is less than the perimeter of the triangle.

[Apply Ex. 649 three times.]

Would this be true for a point outside the triangle?

†Ex. **680.** The sum of the diagonals of a quadrilateral is greater than half its perimeter.

†Ex. **681.** The sum of the diagonals of a quadrilateral is less than its perimeter.

†Ex. **682.** The sum of the medians of a triangle is less than its perimeter.

[Use Ex. 676.]

†Ex. **683.** The sum of the distances of any point from the angular points of a polygon is greater than half its perimeter.

†Ex. **684.** In a triangle ABC, D is the mid-point of BC; if AD < BD, △ ABC must be obtuse-angled.

†Ex. **685.** Find the position of P within a quadrilateral ABCD, for which PA + PB + PC + PD is least. Give a proof.

[See Ex. 677.]

†Ex. **686.** ABC, DBC are two triangles on the same base BC, and AD is parallel to BC. If the triangle ABC is isosceles its perimeter is less than that of the triangle DBC.

[Produce BA to E so that AE = AB. Join DE.]

†Ex. **687.** P is any point in the median AD of a triangle ABC; prove that, if AB > AC, PB > PC.

†Ex. **688.** In a quadrilateral ABCD, ∠ BCA > ∠ DAC; prove that ∠ ADB > ∠ DBC.

†Ex. **689.** O is a point within an equilateral triangle ABC; if ∠ OAB > ∠ OAC, ∠ OCB > ∠ OBC.

Parallelograms.

Def. A quadrilateral with its opposite sides parallel is called a parallelogram.

Revise Ex. 183—203.

THEOREM 22.

(1) The opposite angles of a parallelogram are equal.

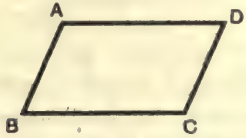


fig. 145.

Data ABCD is a parallelogram.

To prove that $\angle A = \angle C$, $\angle B = \angle D$.

Proof Since AD and BC are ||, and AB meets them,

$\therefore \angle A + \angle B = 2$ rt. $\angle$s. I. 5.

Sim^{ly} $\angle B + \angle C = 2$ rt. $\angle$s. I. 5.

$\therefore \angle A + \angle B = \angle B + \angle C$,

$\therefore \angle A = \angle C$.

Sim^{ly} $\angle B = \angle D$.

Q. E. D.

(2) The opposite sides of a parallelogram are equal.

(3) Each diagonal bisects the parallelogram.

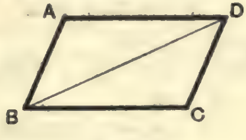


fig. 146.

Data ABCD is a parallelogram, and BD one of its diagonals.

To prove that $AB = CD$, $AD = CB$, and that BD bisects the parallelogram.

Proof Since AD is || to BC and BD meets them,

$\angle ADB =$ alt. $\angle CBD$. I. 5.

Since AB is || to CD and BD meets them,

$\angle ABD =$ alt. $\angle CDB$. I. 5.

$\therefore$ in $\triangle$s ABD, CDB,

$\left\{\begin{array}{l}\angle ADB = \angle CBD, \\ \angle ABD = \angle CDB, \\ BD \text{ is common,}\end{array}\right.$

$\therefore \triangle ABD \equiv \triangle CDB,$ I. 11.

$\therefore AB = CD, \quad AD = CB.$

And since $\triangle ABD \equiv \triangle CDB$,

BD bisects the parallelogram.

Sim^{ly} AC bisects the parallelogram. Q. E. D.

(4) **The diagonals of a parallelogram bisect one another.**

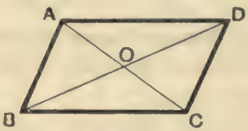


fig. 147.

Data ABCD is a parallelogram; its diagonals AC, BD intersect at O.

To prove that $OA = OC$ and $OD = OB$.

Proof Since AD is || to BC and BD cuts them,

$\therefore \angle ADO = \angle CBO,$

$\therefore$ in $\triangle$s OAD, OCB

$\left\{\begin{array}{l}\angle ADO = \angle CBO, \\ \angle AOD = \text{vert. opp. } \angle COB, \\ AD = CB,\end{array}\right.$ I. 22 (2).

$\therefore$ the $\triangle$s are congruent, I. 11.

$\therefore OA = OC$ and $OD = OB$. Q. E. D.

Cor. 1. **If two straight lines are parallel, all points on either line are equidistant from the other.**

Cor. 2. **If a parallelogram has one of its angles a right angle, all its angles must be right angles.**

Cor. 3. **If one pair of adjacent sides of a parallelogram are equal, all its sides are equal.**

†Ex. **690.** Prove Cor. 1. (See note to I. 21.)

†Ex. **691.** Prove Cor. 2.

†Ex. **692.** Prove Cor. 3.

DEF. A parallelogram which has one of its angles a right angle is called a **rectangle**.

Cor. 2 proves that all the angles of a rectangle are right angles.

DEF. A rectangle which has two adjacent sides equal is called a **square**.

Cor. 3 proves that all the sides of a square are equal to one another. Again, since a square is a rectangle, all its angles are right angles.

DEF. A parallelogram which has two adjacent sides equal is called a **rhombus**.

Cor. 3 proves that all the sides of a rhombus are equal to one another.

Revise p. 40 and Ex. 203.

DEF. A quadrilateral which has only one pair of sides parallel is called a **trapezium**.

DEF. A trapezium in which the sides which are not parallel are equal to one another is called an **isosceles trapezium**.

†Ex. **693.** Draw an isosceles triangle ABC and a line parallel to the base cutting the sides in D, E; prove that DECB is an isosceles trapezium.

¶Ex. **694.** In fig. 195, what lines are equal to (i) PQ, (ii) QR? Give a reason.

¶Ex. **695.** In fig. 199, what are the lengths of SV, VT, ST, ZY, RV?

Ex. **696.** Draw a parallelogram ABCD; from AB, AD cut off equal lengths AX, AY; through X, Y draw parallels to the sides. Indicate in your figure what lines and angles are equal. (*Freehand*)

†Ex. **697.** In fig. 167, ABCD is a parallelogram and PBCQ is a rectangle; prove that $\triangle BPA \equiv \triangle CQD$.

†Ex. **698.** The bisectors of two adjacent angles of a parallelogram are at right angles to one another.

†Ex. **699.** The bisectors of two opposite angles of a parallelogram are parallel.

†Ex. **700.** Any straight line drawn through O, in fig. 147, and terminated by the sides of the parallelogram is bisected at O.

†Ex. **701.** ABCD is an isosceles trapezium (AD = BC); prove that

$$\angle C = \angle D.$$

[Through B draw a parallel to AD.]

†Ex. **702.** If in Ex. 701 E, F are the mid-points of AB, CD, then EF is perpendicular to AB. [Join AF, BF.]

THEOREM 23. †

[CONVERSES OF THEOREM 22.]

(1) **A quadrilateral is a parallelogram if both pairs of opposite angles are equal.**

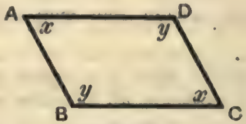


fig. 148.

Data ABCD is a quadrilateral in which

$\angle A = \angle C = \angle x$ (say) and $\angle B = \angle D = \angle y$ (say).

To prove that ABCD is a parallelogram.

Proof The sum of the angles of a quadrilateral is equal to 4 rt. $\angle$s, I. 9, *Cor.*

$\therefore 2\angle x + 2\angle y = 4$ rt. $\angle$s,

$\therefore \angle x + \angle y = 2$ rt. $\angle$s,

$\therefore \angle A + \angle B = 2$ rt. $\angle$s,

$\therefore$ AD is $\parallel$ to BC. I. 4.

Also $\angle A + \angle D = 2$ rt. $\angle$s,

$\therefore$ AB is $\parallel$ to DC, I. 4.

$\therefore$ ABCD is a $\parallel^{\text{ogram}}$. Q. E. D.

(2) **A quadrilateral is a parallelogram if one pair of opposite sides are equal and parallel.**

(Draw a diagonal and prove the two triangles congruent.)

(3) **A quadrilateral is a parallelogram if both pairs of opposite sides are equal.**

(Draw a diagonal and prove the two triangles congruent.)

(4) **A quadrilateral is a parallelogram if its diagonals bisect one another.**

(Prove two opposite triangles congruent.)

COR. **If equal perpendiculars are erected on the same side of a straight line, the straight line joining their extremities is parallel to the given line.**

†Ex. **703.** Prove I. 23 (2).

†Ex. **704.** Prove I. 23 (3).

†Ex. **705.** Prove I. 23 (4).

†Ex. **706.** Prove the Corollary.

†Ex. **707.** The straight line joining the mid-points of two opposite sides of a parallelogram is parallel to the other two sides.

†Ex. **708.** ABCD is a parallelogram; AB, CD are bisected at X, Y respectively; prove that BXDY is a parallelogram.

†Ex. **709.** If the diagonals of a quadrilateral are equal and bisect one another at right angles, the quadrilateral must be a square.

†Ex. **710.** Two straight lines bisect one another at right angles; prove that they are the diagonals of a rhombus.

†Ex. **711.** If the diagonals of a parallelogram are equal, it must be a rectangle.

†Ex. **712.** An equilateral four-sided figure with one of its angles a right angle must be a square.

†Ex. **713.** In a quadrilateral ABCD, $\angle A = \angle B$ and $\angle C = \angle D$; prove that ABCD is an isosceles trapezium. In what case would it be a parallelogram?

Revise Ex. 516, 517.

Through a given point to draw a straight line parallel to a given straight line.

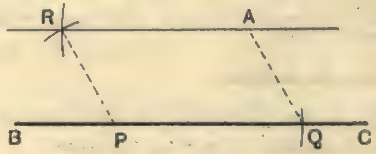


fig. 149.

Let A be the given point and BC the given straight line.

Construction In BC take any point P and cut off any length PQ.

With centre A and radius PQ describe a circle.

With centre P and radius AQ describe a circle.

Let the circles intersect at R.

Join AR.

Then AR is || to BC.

Proof Join AQ and PR.

In the quadrilateral ARPQ

$$\begin{cases} AR = QP, \\ AQ = RP, \end{cases}$$

∴ ARPQ is a ||$^{\text{ogram}}$. I. 23 (3).

∴ AR is || to BC.

The set square method of drawing parallels is the most practical (see p. 36).

Ex. **714.** Show how to construct, without using set square, a parallelogram having given two adjacent sides and the angle between them.

Ex. **715.** Show how to construct a square on a given straight line.

Ex. **716.** Show how to construct a rectangle on a given straight line, having each of its shorter sides equal to half the given line.

†Ex. **717.** Show how to construct a rhombus on a given straight line, having one of its angles = 60° (without protractor or set square). Give a proof.

Ex. **718.** Construct a parallelogram having two sides and a diagonal equal to 5 cm., 12 cm., 13 cm. respectively. Measure the other diagonal.

Ex. **719.** Construct a rectangle having one side of 2·5 in. and a diagonal of 4 in. Measure the sides.

Ex. **720.** Construct a parallelogram with diagonals of 3 in. and 5 in. intersecting at an angle of 53°. Measure the shortest side.

Ex. **721.** Construct a rectangle with a diagonal of 7 cm., the angle between the diagonals being 120°. Measure the shortest side.

Ex. **722.** Construct a rhombus with diagonals of 4 in. and 2 in. Measure the side.

Ex. **723.** Construct a square whose diagonal is 3 in. long. Measure its side.

Ex. **724.** Construct an isosceles trapezium whose sides are 4 in., 3 in., 1·5 in., 1·5 in. Measure its acute angles.

fig. 150.

To draw a straight line parallel to a given straight line and at a given distance from it.

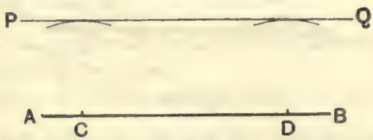


fig. 151.

Let AB be the given straight line and ·5 in. the given distance.

Construction In AB take any two points C, D, as far apart as possible.

With C, D as centres and radius of ·5 in. describe two circles.

With a ruler draw a common tangent PQ to the two circles.

Then PQ is parallel to AB.

Proof This must be postponed, as it depends on a theorem in Book III.

Ex. **725.** On a base 3 in. long construct a parallelogram of height 1·2 in. with an angle of 55°. Measure the other side.

Ex. **726.** Construct a rhombus whose side is 7·3 cm., the distance between a pair of opposite sides being 5·6 cm. Measure its acute angle.

¶Ex. **727.** Draw a straight line and cut off from it two equal parts AC, CE; through A, C, E draw three parallel straight lines and draw a line cutting them at B, D, F; measure BD, DF. (See fig. 152.)

¶Ex. **728.** Draw a straight line and mark off equal parts PR, RQ; join P, Q, and R to a point O; draw a straight line (not parallel to PQ) to cut OP, OQ, OR at p, q, r; is $pr=qr$?

THEOREM 24.

If there are three or more parallel straight lines, and the intercepts made by them on any one straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

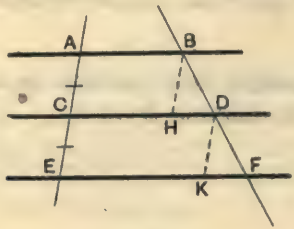


fig. 152.

Data The parallels AB, CD, EF are cut by the straight lines ACE, BDF, and the intercepts AC, CE are equal.

To prove that the corresponding intercepts BD, DF are equal.

Construction Through B draw BH || to ACE to meet CD at H.

Through D draw DK || to ACE to meet EF at K.

Proof [$\triangle$s BHD, DKF must be proved congruent].

AH is a $\|^{\text{ogram}}$, $\therefore$ AC = BH, I. 22.

CK is a $\|^{\text{ogram}}$, $\therefore$ CE = DK.

But AC = CE, *Data*

$\therefore$ BH = DK.

Now CD is || to EF,

$\therefore \angle$ BDH = corresp. $\angle$ DFK. I. 5.

Again BH, DK are || (each || to ACE),

$\therefore \angle$ DBH = corresp. $\angle$ FDK, I. 5.

$\therefore$ in $\triangle$s BHD, DKF

$$\begin{cases} \angle BDH = \angle DFK, \\ \angle DBH = \angle FDK, \\ BH = DK, \end{cases}$$

$\therefore$ the $\triangle$s are congruent,

$\therefore$ BD = DF. I. 11.

Q. E. D.

†Ex. **729.** In fig. 152, if AB, CD are parallel and AC = CE and BD = DF, prove that EF is parallel to CD.

[Use *reductio ad absurdum.*]

†Ex. **730.** The straight line drawn through the mid-point of one side of a triangle parallel to the base bisects the other side.

[Let A, B coincide in fig. 152.]

†Ex. **731. The straight line joining the mid-points of the sides of a triangle is parallel to the base.**

[Prove this (i) by *reductio ad absurdum*;

(ii) directly, with the following construction:—

Let ABC be the triangle; D, E the mid-points of AB, AC. Produce DE to F so that EF = DE. Join CF.]

†Ex. **732. The straight line joining the mid-points of the sides of a triangle is equal to half the base.**

[Join the mid-point of the base to the mid-point of one of the sides.]

†Ex. **733.** The straight lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

†Ex. **734.** Given the three mid-points of the sides of a triangle, construct the triangle. Give a proof.

†Ex. **735.** If AD = $\frac{1}{4}$AB and AE = $\frac{1}{4}$AC, prove that DE is parallel to BC and equal to a quarter of BC.

†Ex. **736.** If the mid-points of the adjacent sides of a quadrilateral are joined, the figure thus formed is a parallelogram.

[Draw a diagonal of the quadrilateral.]

†Ex. **737.** The straight lines joining the mid-points of opposite sides of a quadrilateral bisect one another.

Ex. **738.** Draw a straight line 4 in. long; divide it into seven equal parts by calculating the length of one part and stepping off with dividers.

To divide a given straight line into five equal parts.

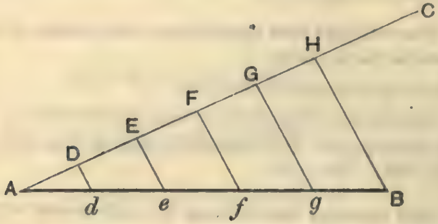


fig. 153.

Let AB be the given straight line.

Construction Through A draw AC making any angle with AB.

From AC cut off any part AD.

From DC cut off parts DE, EF, FG, GH, equal to AD, so that AH is five times AD.

Join BH.

Through D, E, F, G draw st. lines ∥ to BH.

Then AB is divided into 5 equal parts.

Proof AD = DE = ..., *Constr.*

and D*d*, E*e*, ..., HB are all parallel. *Constr.*

∴ A*d* = *de* = ..., I. 24.

∴ AB is divided into 5 equal parts.

The graduated ruler must not be used in the constructions of Ex. 739—747.

Ex. **739.** **Divide a given straight line AB into five equal parts by means of the following construction:—**

As in fig. 153, draw AC and cut off equal parts AD, DE, EF, FG, GH; through B draw BK parallel to HA and cut off from it BP, PQ, QR, RS, ST each equal to AD. Join GP, FQ, ... These lines divide AB into five equal parts.

Give a proof.

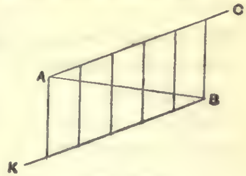


fig. 154.

Ex. **740.** Trisect a given straight line by eye; check by making the construction.

Ex. **741.** Divide a straight line of 10 cm. into six equal parts; measure the parts. Give a proof.

Ex. **742.** **From a given straight line cut off a part equal to $\frac{2}{9}$ of the whole line.**

Ex. **743.** Divide a straight line decimally (i.e. into ten equal parts).

Ex. **744.** Construct a line equal to (i) $1\frac{1}{3}$, (ii) 1·2 of a given line.

Ex. **745.** **Divide a straight line of 13·3 cm. in the ratio of 3 : 4.**

[Divide the straight line (AB) into seven (i.e $3+4$) equal parts; if D is the third point of division from A, AD contains three parts and DB contains four parts, $\therefore \frac{AD}{DB}=\frac{3}{4}$.]

Ex. **746.** Divide a straight line in the ratio of 5 : 3.

Ex. **747.** Divide a straight line 10 cm. long so that the ratio of the two parts may be $\frac{4}{7}$.

Loci.

Mark two points A and B, 2 inches apart. Mark a point 3 inches from A and also 3 inches from B: then a point 4 inches from A and B.

In a similar way mark about 10 points equidistant from A and B; some above and some below AB.

Notice what pattern this set of points seems to form. Draw a line passing through all of them.

Find a point on AB equidistant from A and B; this belongs to the set of points.

The pattern formed by all possible points equidistant from two fixed points A and B is called the **locus** of points equidistant from A and B.

¶Ex. **748.** What is the locus of points at a distance of 1 inch from a fixed point O?

¶Ex. **749.** Draw a straight line right across your paper. Construct the locus of points distant 1 inch from this line.

(Do this either by marking a number of such points; or, if you can, without actually marking the points. Remember that the distance is reckoned perpendicular to the line.)

¶Ex. **750.** A bicyclist is riding straight along a level road. What is the locus of the hub of the back wheel?

¶Ex. **751.** What is the locus of the tip of the hand of a clock?

¶Ex. **752.** What is the locus of a man's hand as he works the handle of a common pump?

¶Ex. **753.** A stone is thrown into still water and causes a ripple to spread outwards. What is the locus of the points which the ripple reaches after one second?

¶Ex. **754.** Sound travels about 1100 feet in a second. A gun is fired; what is the locus of all the people who hear the sound 1 second later.

¶Ex. **755.** A round ruler rolls down a sloping plank; what is the locus of the centre of one of the ends of the ruler?

¶Ex. **756.** A man walks along a straight road, so that he is always equidistant from the two sides of the road. What is his locus?

¶Ex. **757.** A runner runs round a circular racing-track, always keeping one yard from the inner edge. What is his locus?

¶Ex. **758.** Two coins are placed on a table with their edges in contact. One of them is held firm, and the other rolls round the circumference of the fixed coin. What is the locus of the centre of the moving coin? Would the locus be the same if there were slipping at the point of contact?

¶Ex. **759.** What is the locus of a door-handle as the door opens?

¶Ex. **760.** What is the locus of a clock-weight as the clock runs down?

¶Ex. **761.** Slide your set-square round on your paper, so that the right angle always remains at a fixed point. What are the loci of the other two vertices?

The above exercises suggest the following alternative definition of a locus.

Def. If a point moves so as to satisfy certain conditions the **path** traced out by the point is called its **locus**.

Ex. **762.** A man stands on the middle rung of a ladder against a wall. The ladder slips down; find the locus of the man's feet.

(Do this by drawing two straight lines at right angles to represent the wall and the ground; take a length of, say, 4 inches to represent the ladder; draw a considerable number of different positions of the ladder as it slips down; and mark the middle points. This is called **plotting** a locus.

The exercise is done more easily by drawing the ladder (the line of 4 inches) on transparent **tracing-paper**; then bring the ends of the ladder on to the two lines of the paper below; and prick through the middle point.)

¶Ex. **763.** Draw two unlimited lines, intersecting near the middle of your paper at an angle of 60°. By eye, mark a point equidistant from the two lines. Mark a number of such points, say 20, in various positions. The pattern formed should be *two* straight lines. How are these lines related to the original lines? How are they related to one another?

¶Ex. **764.** (On squared paper.) Draw a pair of lines at right angles (OX, OY); plot a series of points each of which is twice as far from OX as from OY. What is the locus? (Keep your figure for the next Ex.)

Ex. **765.** Using the figure of Ex. 764, plot the locus of points 3 times as far from OX as from OY; also the locus of points $\frac{1}{2}$ as far from OX as from OY.

Ex. **766.** (On squared paper.) Plot the locus of a point which moves so that the sum of its distances from two lines at right angles is always 4 inches.

Ex. **767.** (On squared paper.) Plot the locus of a point which moves so that the difference of its distances from two lines at right angles is always 1 inch.

Ex. **768.** Draw a line, and mark a point O 2 inches distant from the line. Let P be a point moving along the line. Experimentally, plot the locus of the mid-point of OP.

Ex. **769.** A point O is 3 cm. from the centre of a circle of radius 5 cm. Plot the locus of the mid-point of OP, when P moves round the circumference of the circle.

THEOREM 25.

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

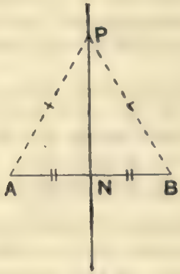


fig. 155.

Data P is any one position of a point which is always equidistant from two fixed points A and B.

To prove that P lies on the perpendicular bisector of AB.

Construction Join AB; let N be the middle point of AB. Join NP.

Proof In the $\triangle$s ANP, BNP,

$$\begin{cases} AP = BP, & \textit{Data} \\ AN = BN, & \textit{Constr.} \\ PN \text{ is common,} & \end{cases}$$

$\therefore$ the triangles are congruent, I. 14.

$\therefore$ $\angle ANP = \angle BNP$,

$\therefore$ PN is $\perp$ to AB,

$\therefore$ P lies on the perpendicular bisector of AB.

Sim[ly] it may be shown that any other point equidistant from A and B lies on the perpendicular bisector of AB.

Q. E. D.

NOTE. It will be noticed that N is a point on the locus.

†Ex. 770. Prove that any point on the perpendicular bisector of a line AB is equidistant from A, B.

THEOREM 26.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

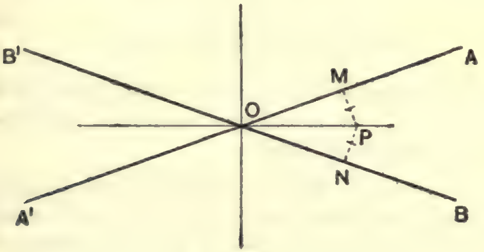


fig. 156.

Data AOA′, BOB′ are two intersecting straight lines; P is any one position (in ∠ AOB, say) of a point which is always equidistant from AOA′, BOB′.

To prove that P lies on one of the bisectors of the angles formed by AOA′, BOB′.

Construction Draw PM, PN ⊥ to AA′, BB′ respectively.
Join OP.

Proof In the rt. ∠d △s POM, PON,

∠s M and N are rt. ∠s, *Constr.*
OP is common,
PM = PN, *Data*

∴ the triangles are congruent, I. 15.
∴ ∠ POM = ∠ PON,
∴ P lies on the bisector of ∠ AOB (or ∠ A′OB′).

Sim^ly, if P be taken in ∠ AOB′ or ∠ A′OB, it may be shown that the point lies on the bisector of ∠ AOB′ (or ∠ A′OB).

Q. E. D.

†Ex. **771.** Prove that any point on the bisector of an angle is equidistant from the arms of that angle.

†Ex. **772.** Prove formally that the locus of points at a distance of 1 inch from a given line, on one side of it, is a parallel line. (Take two such points, and show that the line joining them is parallel to the given line.)

†Ex. **773.** O is a fixed point. P moves along a fixed line; Q is in OP produced, and PQ = OP. Prove that the locus of Q is a parallel line.

Intersection of Loci.

Draw two unlimited straight lines AOA′, BOB′, intersecting at an angle of 45°. It is required to find a point (or points) distant 1 inch from each line.

First draw the locus of points distant 1 inch from AOA′; this consists of a pair of lines parallel to AOA′ and distant 1 inch from it. The points we are in search of must certainly lie somewhere upon this locus.

Next draw the locus of points distant 1 inch from BOB′. The required points must lie upon this locus also.

The two loci will be found to intersect in four points. These are the points required.

Measure the distance from O of these points.

Ex. **774.** Draw two unlimited straight lines intersecting at an angle of 80°. Find a point (or points) distant 2 cm. from the one line and 4 cm. from the other.

Ex. **775.** Draw an unlimited straight line and mark a point O 2 inches from the line. Find a point (or points) 3 inches from O and 3 inches from the line. (What is the locus of points 3 inches from O? What is the locus of points 3 inches from the line? Draw these loci.) Measure the distance between the two points found.

Ex. **776.** In Ex. 775 find two points distant 4 inches from O and from the line. Measure the distance between them.

Ex. **777.** In Ex. 775 find as many points as you can distant 1 inch from both point and line.

Ex. **778.** Given two points A, B 3 inches apart, find a point (or points) distant 4 inches from A and 5 inches from B.

Ex. 779. Make an angle of 45°; on one of the arms mark a point A 3 inches from the vertex of the angle. Find a point (or points) equidistant from the arms of the angle, and 2 inches from A. Measure distance between the two points found.

Ex. 780. Draw a circle of radius 5 cm. and mark a point A 7 cm. from centre of circle. Find two points on the circle 3 cm. from A, and measure the distance between them.

Ex. 781. Construct a quadrilateral ABCD, having AB = 6 cm., BC = 13 cm., CD = 10 cm., $\angle$ ABC = 70°, $\angle$ BCD = 60°.

On diagonal BD (produced if necessary), find a point

(1) equidistant from A and C,

(2) equidistant from AB and AD,

(3) equidistant from AB and DC.

In each case measure the equal distances.

Ex. 782. Find two points on the base of an equilateral triangle (side 3 inches) distant 2·7 inches from the vertex. Measure distance between them.

Ex. 783. Find a point on the base of an equilateral triangle (side 10 cm.) which is 4 cm. from one side. Measure the two parts into which it divides the base.

Ex. 784. On the side AB of an isosceles triangle ABC (base BC = 2 ins., $\angle$ A = 36°), find a point P equidistant from the base and the other side AC. Measure AP, and the equal distances.

†**Ex. 785.** In Ex. 784 *prove* that AP = CP = CB.

Ex. 786. Find a point on the base of a scalene triangle equidistant from the two sides. Is this the middle point of the base?

Ex. 787. Draw a circle of radius 2 ins.; a diameter; and a parallel line at a distance of 3 ins. Find a point (or points) in the circle equidistant from the two lines. Measure distance between these points.

Ex. 788. Draw a circle, a diameter AB, and a chord AC through A. Find a point P on the circle equidistant from AB and AC. Measure PB and PC.

Ex. 789. In Ex. 788, find a point on the circle equidistant from AB and CA produced.

Ex. 790. Draw $\triangle$ABC having AB = 2·8 ins., AC = 4·6 ins., BC = 4·6 ins. Find a point (or points) equidistant from AB and AC, and 1 inch from BC. Measure distance between points.

Ex. 791. Using the triangle of Ex. 790, find a point (or points) equidistant from AB and AC, and also equidistant from B and C. Test the equidistance by measurement.

Ex. 792. In triangle of Ex. 790, find a point (or points) 2 inches from A, and equidistant from B, C. Measure the distance between them.

Ex. 793. Draw a triangle ABC; find a point O which is equidistant from B, C; and also equidistant from C, A. Test by drawing circle with centre O to pass through A, B, C.

Ex. 794. Two lines XOX′, YOY′ intersect at O, making an angle of 25°. A lies on OX, and OA = 7 cm. Through A is drawn AB parallel to YOY′. Find a point (or points) equidistant from XOX′ and YOY′; and also equidistant from AB and YOY′. Draw the equal distances and measure them.

Ex. 795. Draw a triangle ABC. Inside the triangle find a point P which is equidistant from AB and BC; and also equidistant from BC and CA. From P draw perpendiculars to the three sides; with P as centre and one of the perpendiculars as radius draw a circle.

Ex. 796. A river with straight banks is crossed, slantwise, by a straight weir. Draw a figure representing the position of a boat which finds itself at the same distance from the weir and the two banks.

†**Ex. 797.** P is a moving point on a fixed line AB; O is a fixed point outside the line. P is joined to O, and PO is produced to Q so that OQ = PO. Prove that the locus of Q is a line parallel to AB. (See Ex. 772.)

Ex. 798. Use the locus of Ex. 797 to solve the following problem. O is a point in the angle formed by two lines AB, AC. Through O draw a line, terminated by AB, AC, and bisected at O.

Ex. 799. Draw a figure like fig. 157, making radius of circle 2 ins., CO = 3 ins., CN = 5 ins. Through O draw a line (or lines), terminated by AB and the circle, and bisected at O. (See Ex. 797.)

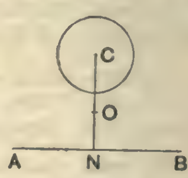


fig. 157.

Ex. 800. A town X is 2 miles from a straight railway; but the two stations nearest to X are each 3 miles from X. Find the distance between the two stations.

Construction of Triangles, etc. by means of Loci.

In Exs. 801—811 *accurate figures need not be drawn unless technical skill is required.*

Ex. **801.** Construct $\triangle$ABC, given

(i) base BC = 14 cm., height = 9 cm., $\angle$B = 65°. Measure AB.

(ii) AB = 59 mm., AC = 88 mm., height AD = 49 mm. (Draw height first.) Measure base BC.

(iii) BC = 4 in., $\angle$B = 80°, median CN = 4 in. Measure BA.

(iv) base BC = 12 cm., height AD = 4 cm., median AL = 5 cm. Measure AB, AC.

Ex. **802.** Construct a right-angled triangle, given

(i) longest side = 10 cm., another side = 5 cm. Measure the smallest angle.

(ii) side opposite right angle = 4 in., another side = 3 inches. Measure the third side.

Ex. **803.** Construct a right-angled triangle ABC, given $\angle$ A = 90°, AB = 7 cm., distance of A from BC = 2·5 cm. Measure the smallest angle.

Ex. **804.** Construct an isosceles triangle having each of the equal sides twice the height. Measure the vertical angle.

Ex. **805.** Construct a triangle, given height = 2 in., angles at the extremities of the base = 40° and 60°. Find length of base.

Ex. **806.** Construct an isosceles triangle, given the height and the angle at the vertex (without protractor).

Ex. **807.** Construct a parallelogram ABCD, given

AB = 12 cm., AD = 10 cm., distance between AB, DC = 8 cm.

Measure the acute angle.

Ex. **808.** Construct a rhombus, given that the distance between the parallel sides is half the length of a side. Measure the acute angle.

Ex. **809.** Construct a quadrilateral ABCD, given diagonal AC = 9 cm., diagonal BD = 10 cm., distances of B, D from AC 5 cm. and 4 cm. respectively, side AB = 7 cm. Measure CD.

Ex. **810.** Construct a trapezium ABCD, given base AB = 10 cm., height = 4 cm., AD = 4·5 cm., BC = 4·2 cm. Measure angles A and B. (There are 4 cases.)

Ex. **811.** Construct a trapezium ABCD, given base AB = 3·5 in., height = 1·7 in., diagonals AC, BD = 2·5, 3·5 ins. respectively. Measure CD.

CO-ORDINATES.

Take a piece of squared paper; near the middle draw two straight lines intersecting at right angles (XOX, YOY in fig. 158). These will be called **axes**; the point O where they intersect will be called the **origin**.

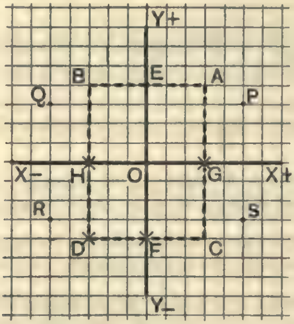


fig. 158.

In order to arrive at the point A, starting from the origin O, one may travel 3 divisions along towards X+, *to the right*, and then 4 divisions *upwards*. Accordingly the point A is fixed by the two numbers (3, 4). These two numbers are called the **co-ordinates** of the point A.

Ex. **312.** Mark on a sheet of squared paper

(i) the points (3, 5), (3, 10), (8, 10), (8, 5).

(ii) the points (1, 2), (2, 4), (3, 6), (4, 8), (5, 10).

(iii) the points (4, 3), (4, 2), (4, 1), (4, 0), (4, −1), (4, −2).

(iv) the points (6, 6), (4, 6), (2, 6), (0, 6), (−2, 6).

To reach B (fig. 158) from O, one may travel 3 divisions along towards X− *to the left,* and then 4 divisions upwards. Accordingly the point B is fixed by the co-ordinates $(-3, 4)$.

To reach C from O, go 3 divisions along to the right, then 4 divisions *downwards.* C is therefore $(3, -4)$.

N.B. To the right is reckoned +; to the left, −.
Upwards is reckoned +; downwards −.

To get from O to E, it is not necessary to travel *along* at all; the journey is simply 4 divisions *upwards.* Accordingly, E is the point $(0, 4)$.

Ex. **813.** Write down the co-ordinates of the following points in fig. 158: D, F, G, H, O, P, Q, R, S.

Ex. **814.** Plot (i.e. mark on squared paper) the following points: (5, 0), (4, 3), (3, 4), (0, 5), (−3, 4), (−4, 3), (−5, 0), (−4, −3), (−3, −4), (0, −5), (3, −4), (4, −3), (5, 0).

Ex. **815.** Plot the points: (8, 16), (6, 9), (4, 4), (2, 1), (0, 0), (−2, 1), (−4, 4), (−6, 9), (−8, 16).

Ex. **816.** Plot the points: (0, 0), (2, 0), (−2, 0), (0, 13), (1, −10), (8, 6), (−8, −6), (−3, −5). (The constellation of Orion.)

Ex. **817.** Plot the points: (−12, −2), (−8, 0), (−4, 0), (0, 0), (3, −2), (7, 0), (5, 4). (The Great Bear.)

Ex. **818.** (Inch paper.) Find the co-ordinates of two points each of which is 3 inches from (0, 0) and (2, 2).

Ex. **819.** (Inch paper.) Find the co-ordinates of all the points which are 2 inches from the origin and 1 inch from the x-axis (XOX).

Ex. **820.** (Inch paper.) Find the co-ordinates of all the points which are equidistant from the two axes and 3 inches from the origin.

Ex. **821.** (Inch paper.) Find the co-ordinates of a point which is equidistant from

(i) $(2, -1)$, $(1, 3)$, $(-2, 0)$,

(ii) $(2, 3)$, $(2, -1)$, $(-2, -1)$.

(iii) $(2, 3)$, $(2, -1)$, $(-2, -2)$.

Ex. **822.** (Inch paper.) Find the co-ordinates of a point inside the triangle given in Ex. 821 (i), and equidistant from its three sides.

Ex. **823.** Repeat Ex. 822 for the triangles given in Ex. 821 (ii) and (iii).

MISCELLANEOUS EXERCISES.

CONSTRUCTIONS.

Ex. **824.** A ship is sailing due N. at 8 miles an hour. At 3 o'clock a lighthouse is observed to be N.E. and after 90 minutes it is observed to bear $7\frac{1}{2}°$ S. of E. How far is the ship from the lighthouse at the second observation, and at what time (to the nearest minute) was the ship nearest to the lighthouse?

Ex. **825.** Is it possible to make a pavement consisting of equal equilateral triangles?

Is it possible to do so with equal regular figures of (i) 4, (ii) 5, (iii) 6, (iv) 7 sides?

Ex. **826.** A triangle ABC has $\angle B = 60°$, $BC = 8$ cm.; what is the least possible size for the side CA? What is the greatest possible size for $\angle C$?

†Ex. **827.** Draw a triangle ABC and show how to find points P, Q in AB, AC such that PQ is parallel to the base BC and $= \frac{1}{3}BC$. Give a proof.

[Trisect the base and draw a parallel to one of the sides.]

†Ex. **828.** In OX, OY show how to find points A, B such that $\angle OAB = 3\angle OBA$. Give a proof.

[Make an angle equal to the sum of these angles.]

†Ex. **829.** A and B are two fixed points in two unlimited parallel straight lines; show how to find points P and Q in these lines such that APBQ is a rhombus. Give a proof.

†Ex. **830.** Prove the following construction for bisecting the angle BAC:—With centre A describe two circles, one cutting AB, AC in D, E, and the other cutting them in F, G respectively; join DG, EF, intersecting in H; join AH.

†Ex. **831.** A, B are two points on opposite sides of a straight line CD; show how to find a point P in CD so that $\angle APC = \angle BPC$. Give a proof.

†Ex. **832.** Show how to construct a rhombus PQRS having its diagonal PR in a given straight line and its sides PQ, QR, RS passing through three given points L, M, N respectively. Give a proof.

†Ex. **833.** A and B are two given points on the same side of a straight line CD; show how to find the point in CD the difference of whose distances from A and B is greatest.

Also show how to find the point for which the difference is least.

†Ex. **834.** A and B are two points on the same side of a straight line CD; show how to find the point P in CD for which AP + PB is least. Give a proof.

†Ex. **835.** Show how to describe a rhombus having two of its sides along the sides AB, AC of a given triangle ABC and one vertex in the base of the triangle. Give a proof.

†Ex. **836.** Show how to draw a straight line equal and parallel to a given straight line and having its ends on two given straight lines. Give a proof.

¶Ex. **837. To trisect a given angle.**

Much time was devoted to this famous problem by the Greeks and the geometers of the Middle Ages; it has now been shown that it is impossible with only the aid of a pair of compasses and a straight edge (ungraduated).

In fig. 160, DE = the radius of the circle; prove that $\angle BDE = \frac{1}{3} \angle ABC$.

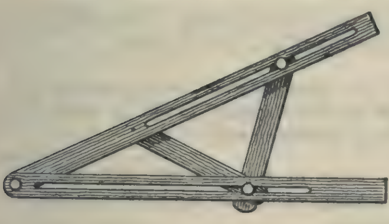


fig. 159.

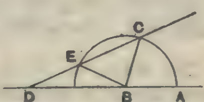


fig. 160.

Fig. 159 shows a simple form of trisector; the instrument is opened until the angle between the rods corresponding to BA and BC can be made to coincide with the given angle; then the angle between the long rods (corresponding to D) is one-third of the given angle.

With a ruler, marked on its edge in two places, and a pair of compasses, it is possible to trisect an angle as follows:—

Let ABC be the angle. With B as centre and radius = the distance between the two marks describe a circle cutting BC at C; place the ruler so that its edge passes through C and has one mark on AB produced, the other on the circle (this must be done by trial, a pin stuck through the paper at C will help); rule the line DEC, then $\angle D = \frac{1}{3} \angle ABC$.

Theorems.

Ex. 838. The gable end of a house is in the form of a pentagon, of which the three angles at the ridge and eaves are equal to each other: show that each of these angles is equal to twice the angle of an equilateral triangle.

†Ex. 839. If on the sides of an equilateral triangle three other equilateral triangles are described, show that the complete figure thus formed will be (i) a triangle, (ii) equilateral.

†Ex. 840. Two isosceles triangles are on the same base: prove that the straight line joining their vertices bisects the base at right angles.

†Ex. 841. Two triangles ABC, DCB stand on the same base BC and on the same side of it; prove that AD is parallel to BC if AB = DC and AC = DB.

†Ex. 842. In the diagonal AC of a parallelogram ABCD points P, Q are taken such that AP = CQ; prove that BPDQ is a parallelogram.

†Ex. 843. ABCD, ABXY are two parallelograms on the same base and on the same side of it. Prove that CDYX is a parallelogram.

†Ex. 844. The diagonal AC of a parallelogram ABCD is produced to E, so that CE = CA; through E, EF is drawn parallel to CB to meet DC produced in F. Prove that ABFC is a parallelogram.

†Ex. 845. E, F, G, H are points in the sides AB, BC, CD, DA respectively of a parallelogram ABCD, such that AH = CF and AE = CG: show that EFGH is a parallelogram.

†Ex. 846. C is the mid-point of AB; from A, B, C perpendiculars AX, BY, CZ are drawn to a given straight line. Prove that, if A and B are both on the same side of the line, AX + BY = 2CZ.

What relation is there between AX, BY, CZ when A and B are on opposite sides of the line?

†Ex. 847. If the bisectors of the base angles of an isosceles triangle ABC meet the opposite sides in E and F, EF is parallel to the base of the triangle.

†Ex. 848. In a quadrilateral ABCD, AB = CD and $\angle$B = $\angle$C; prove that AD is parallel to BC.

†Ex. 849. Prove that the diagonals of an isosceles trapezium are equal.

†Ex. **850.** ABCD is a quadrilateral, such that $\angle A = \angle B$ and $\angle C = \angle D$; prove that $AD = BC$.

†Ex. **851.** The figure formed by joining the mid-points of the sides of a rectangle is a rhombus.

†Ex. **852.** The medians BE, CF of a triangle ABC intersect at G; GB, GC are bisected at H, K respectively. Prove that HKEF is a parallelogram. Hence prove that G is a point of trisection of BE and CF.

†Ex. **853.** The diagonal AC of a parallelogram ABCD is produced to E, so that $CE = CA$; through E and B, EF, BF are drawn parallel to CB, AC respectively. Prove that ABFC is a parallelogram.

†Ex. **854.** T, V are the mid-points of the opposite sides PQ, RS of a parallelogram PQRS. Prove that ST, QV trisect PR.

†Ex. **855.** Any straight line drawn from the vertex to the base of a triangle is bisected by the line joining the mid-points of the sides.

†Ex. **856.** The sides AB, AC of a triangle ABC are produced to X, Y respectively, so that $BX = CY = BC$; BY, CX intersect at Z. Prove that $\angle BZX + \frac{1}{2}\angle BAC = 90°$.

†Ex. **857.** ABCD is a parallelogram and $AD = 2AB$; AB is produced both ways to E, F so that $EA = AB = BF$. Prove that CE, DF intersect at right angles.

†Ex. **858.** In a triangle whose angles are 90°, 60°, 30° the longest side is double the shortest.

[Complete an equilateral triangle.]

†Ex. **859.** In a right-angled triangle, the distance of the vertex from the mid-point of the hypotenuse is equal to half the hypotenuse.

[Join the mid-point of the hypotenuse to the mid-point of one of the sides.]

†Ex. **860.** Given in position the right angle of a right-angled triangle and the length of the hypotenuse, find the locus of the mid-point of the hypotenuse. (See Ex. 859.)

†Ex. **861.** ABCD is a square; from A lines are drawn to the mid-points of BC, CD; from C lines are drawn to the mid-points of DA, AB. Prove that these lines enclose a rhombus.

†Ex. **862.** ABC is an equilateral triangle and D is any point in AB; on the side of AD remote from C an equilateral triangle ADE is described; prove that BE=CD.

†Ex. **863.** In a triangle ABC, BE and CF are drawn to cut the opposite sides in E and F; prove that BE and CF cannot bisect one another.

†Ex. **864.** If P be any point in the external bisector of the angle A of a triangle ABC, AB+AC<PB+PC.

†Ex. **865.** ABC is an acute-angled triangle, whose least side is BC. With B as centre, and BC as radius, a circle is drawn cutting AB, AC at D, E respectively. Show that, if AD=DE, ∠ABC=2∠BAC.

†Ex. **866.** ABC is an isosceles triangle (AB=AC); a straight line is drawn cutting AB, BC, and AC produced in D, E, F respectively. Prove that, if DE=EF, BD=CF.

†Ex. **867.** If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to two equal sides equal, the angles opposite the other equal sides are either equal or supplementary; and in the former case the triangles are congruent.

†Ex. **867 a.** A quadrilateral ABCD, that has AB=AD and BC=DC, is called a **kite**. Use Th. I. 25 to prove that the diagonals of a kite are at right angles.

†Ex. **867 b.** If two circles cut at P, Q, use I. 25 to prove that the line joining their centres bisects PQ at right angles.

BOOK II.

AREA.

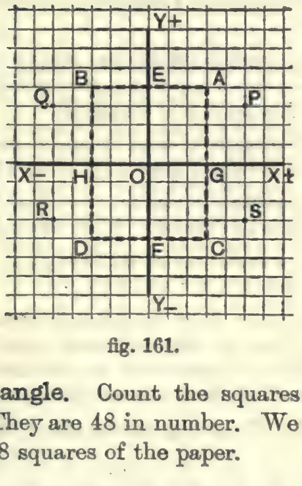


fig. 161.

Area of rectangle. Count the squares in the rectangle ABDC (fig. 161). They are 48 in number. We say, then, that the **area** of ABDC is 48 squares of the paper.

Ex. **868.** In each of the following exercises plot the points mentioned, join them up in the order given, and find the number of squares in the area.

(i) (1, 16), (9, 16), (9, 1), (1, 1).

(ii) (−6, 2), (2, 2), (2, −13), (−6, −13).

(iii) (0, 0), (8, 0), (8, −15), (0, −15).

(iv) (10, 20), (−10, 20), (−10, −20), (10, −20).

So far, we have taken the **unit of length** to be one division of the paper, and the **unit of area** to be one square of the paper.

If we wish to use the inch for unit of length, we shall need paper ruled in squares 1 inch each way. On inch paper there are generally finer lines at distances of $\frac{1}{10}$ inch. The paper will show larger squares and smaller squares; the larger squares 1 inch each way, and therefore of area 1 sq. inch; the smaller squares $\frac{1}{10}$ inch each way. Paper ruled like this will be referred to as **inch paper.**

Ex. **869.** On inch paper, draw a square inch. (Use the lines of the paper to guide your drawing.)

Ex. **870.** On inch paper draw rectangles whose areas, in square inches, are 6, 9, 16, 4, 2, $2\frac{1}{4}$, 1.

Ex. **871.** Draw two rectangles of different shape so that the area of each shall be 12 sq. inches. See whether the two rectangles have the same perimeter (the perimeter is the sum of the sides).

Ex. **872.** Count the number of small squares in one square inch. What fraction of a square inch is each of these small squares? What decimal?

Ex. **873.** Mark out a square containing 25 of these small squares. What decimal of a square inch is this square? What fraction?

Ex. **874.** Mark out a square containing 64 small squares. What decimal of a square inch is this?

Ex. **875.** On inch paper, draw the rectangle whose corners are (2, 15), (7, 15), (7, 2), (2, 2). (Take the side of a small square for unit of length.) How many hundredths of a square inch are contained in this rectangle? How many square inches? (Always express your answer in decimals.)

Ex. **876.** Repeat Ex. 875, taking, instead of the points there mentioned, the following:—

(i) $(-1, 10)$, $(14, 10)$, $(14, -10)$, $(-1, -10)$.

(ii) $(0, 0)$, $(0, 12)$, $(11, 12)$, $(11, 0)$.

(iii) $(-3, 7)$, $(14, 7)$, $(14, -3)$, $(-3, -3)$.

You will probably have noticed that the most convenient way of counting the number of squares in a rectangle is as follows:—count how many squares there are in one row, and multiply by the number of rows. Or, we may say: count the number of divisions in the length, and multiply by the number of divisions in the breadth. Use this plan in the following exercises:

Ex. **877.** How many squares are contained in a rectangle drawn on squared paper, the length being 30 divisions and the breadth 20?

Ex. **878.** On inch paper draw a rectangle 55 tenths in length and 33 tenths in breadth. How many hundredths of a square inch are there in the area? How many square inches?

Ex. **879.** Repeat Ex. 878 with the following numbers for length and breadth respectively:

(i) 40, 25, (ii) 125, 80, (iii) 23, 17, (iv) 125, 8.

Hitherto we have dealt only with rectangles whose dimensions are expressed by whole numbers. We will now see whether the same rule will hold for rectangles whose dimensions are not expressed by whole numbers.

On inch paper draw a rectangle 5·3 inches long and 4·7 inches broad. Count the number of tenths of an inch in the length and breadth. Hence find the number of hundredths of a square inch in the area. Reduce this to square inches; the result should be 24·91 sq. inches. Now multiply together the numbers of inches in the length and breadth: $5\cdot3 \times 4\cdot7$. The result is again 24·91.

Why are these two results the same? The reason is as follows:—

$$\frac{53 \times 47}{100} = \frac{53}{10} \times \frac{47}{10} = 5\cdot3 \times 4\cdot7.$$

We may now state the rule for the area of any rectangle:—

To find the number of square units in the area of a rectangle, multiply together the numbers of units in the length and breadth of the rectangle.

Ex. **880.** What is the corresponding rule for calculating the area of a square?

Ex. **881.** Find the area of a rectangle,

(i) 16·7 ins. by 14·3 ins.
(ii) 10 mm. by 10 mm., in square mm. and also in sq. cm.
(iii) 21·6 cm. by 14·5 cm., in sq. cm. and also in sq. mm.
(iv) 7 kilometres 423 metres by 1 km. 275 m., in sq. km. and also in sq. m.
(v) a inches by b inches.
(vi) x cm. by $2x$ cm.

Ex. **882.** Find the area of a square whose side is (i) 70 yards, (ii) 69 yds. Say in each case whether the square is greater or less than an acre.

Ex. **883.** Find the areas of squares of side (i) 2 inches, (ii) 1 foot (in sq. ins.), (iii) 1 yd. (in sq. ins.), (iv) a cm., (v) $2x$ ins.

¶Ex. **884.** Draw a figure to show that if the side of one square is 3 times the side of another square, the area of the one square is 9 times the area of the other. ***(Freehand)***

Ex. **885.** Find (i) in sq. ins., (ii) in sq. cm., the area of the rectangle which encloses the print on this page. Hence find the number of sq. cm. in a sq. inch (to 1 place of decimals).

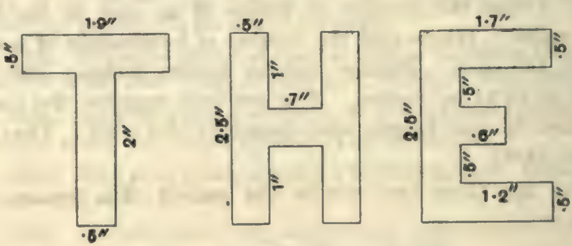


fig. 162.

Ex. **886.** Make freehand sketches of the given figures (fig. 162). In each case find the area.

Ex. **887.** Find the other dimension of a rectangle, given

(i)	area $=140$ sq. ft.,	one dimension $=35$ ft.
(ii)	area $=1$ sq. ft.,	one dimension $=6$ ins.
(iii)	area $=30\frac{1}{4}$ sq. yds.,	one dimension $=5\frac{1}{2}$ yds.
(iv)	area $=1$ acre ($=4840$ sq. yds.),	one dimension $=22$ yds.
(v)	area $=2x^2$ sq. ins.,	one dimension $=x$ ins.

Ex. **888.** How many bricks 9 in. by 4 in. are required to cover a floor 34 ft. long by 17 ft. wide?

Area of right-angled triangle. By drawing a diagonal of a rectangle we divide the rectangle into two equal right-angled triangles. Hence the area of a right-angled triangle may be found by regarding it as half a certain rectangle.

Ex. **889.** Find the number of squares contained by a triangle whose corners are

(i) (0, 0), (0, 2), (6, 0). (Complete the rectangle.)

(ii) (2, 5), (17, 5), (17, 10).

(iii) (5, −5), (−5, −5), (−5, 5).

(iv) (5, −5), (−5, −5), (5, 5).

Ex. **890.** Find the areas of right-angled triangles in which the sides containing the right angle are (i) 2″, 3″, (ii) 6·5 cm., 4·4 cm., (iii) 4·32″, 3·71″, (iv) 112 mm., 45 mm. (in sq. mm. and also in sq. cm.).

Area of any rectilinear figure (on squared paper). With the aid of rectangles and right-angled triangles we can find the area of any figure contained by straight lines (i.e. any **rectilinear** figure). This way is especially convenient when one side of the figure runs along a line of the squared paper.

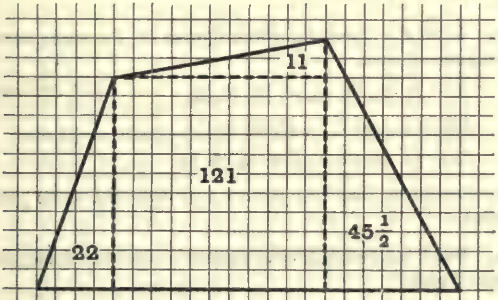


fig. 163.

Fig. 163 shows how a 4-sided figure may be divided up into rectangles and right-angled triangles; the number inside each rectangle and triangle indicates the number of squares it contains; and the complete area is 199½ or 199·5 squares.

Ex. **891.** Measure the size of the small squares in fig. 163; hence find the area of the 4-sided figure in sq. inches.

Ex. **892.** Find the area (in squares of your paper) of each of the following figures by dividing up the figures into rectangles and right-angled triangles:

(i) (2, 1), (11, 1), (8, 6), (2, 6).

(ii) (1, 2), (1, 10), (6, 13), (6, 2).

(iii) (5, 0), (3, 4), (−5, 4), (−6, 0).

(iv) (0, 6), (−3, 2), (−3, −2), (0, −3).

(v) (0, 0), (1, 4), (6, 0).

(vi) (1, 4), (6, 3), (1, −3).

(vii) (−4, −3), (−3, 3), (5, 6), (10, −3).

(viii) (3, 5), (−3, 2), (−5, −3), (3, −7).

(ix) (3, 0), (0, 6), (−3, 0), (0, −6).

(x) (2, 5), (5, 2), (5, −2), (2, −5), (−2, −5), (−5, −2), (−5, 2), (−2, 5). Are all the sides of this figure equal?

(xi) (3, 4), (4, 3), (4, −3), (3, −4), (−3, −4), (−4, −3), (−4, 3), (−3, 4).

(xii) (5, 0), (4, 3), (3, 4), (0, 5), (−3, 4), (−4, 3), (−5, 0), (−4, −3), (−3, −4), (0, −5), (3, −4), (4, −3).

Ex. **893.** Draw the three following figures on the same axes; find the area and perimeter of each.

(i) (1, 1), (1, 6), (6, 6), (6, 1).

(ii) (1, 1), (4, 5), (9, 5), (6, 1).

(iii) (1, 1), (5, 4), (10, 4), (6, 1).

(This exercise shows that two figures may have the same perimeter and different areas.)

Ex. **894.** Draw the two following figures on the same axes; find the area and perimeter of each.

(i) (0, 0), (7, 0), (9, 5), (2, 5).

(ii) (0, 0), (7, 0), (3, 5), (−4, 5).

(This exercise shows that two figures may have the same area and different perimeters.)

Ex. **895.** Find the area of

(i) (1, 0), (1, 8), (4, 14), (2, 14), (0, 10), (−2, 14), (−4, 14), (−1, 8), (−1, 0).

(ii) (5, 7), (−4, 7), (−5, 5), (1, 5), (−5, −7), (5, −7), (6, −5), (−1, −5).

If there is no side of the figure which coincides with a line of the paper (ABCD in fig. 164), it is generally convenient to draw lines *outside* the figure, parallel to the axes, thus making up a rectangle (PQRS); the area required can then be found by subtracting a certain number of right-angled triangles from the rectangle.

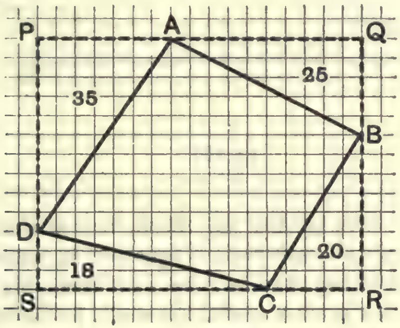


fig. 164.

Thus in fig. 164

$$\text{ABCD} = \text{PQRS} - \text{AQB} - \text{BRC} - \text{CSD} - \text{DPA}$$
$$= 221 - 25 - 20 - 18 - 35$$
$$= 123.$$

Ex. 896. Find the areas of the following figures:—

(i) (1, 1), (16, 5), (9, 14).

(ii) (6, 3), (12, 9), (3, 11).

(iii) (10, −20), (20, −24), (12, 4).

(iv) (0, 0), (9, −1), (7, 6), (2, 5).

(v) (1, 0), (6, 1), (5, 6), (0, 5).

(vi) (3, 0), (7, 3), (4, 7), (0, 4).

(vii) (4, 0), (10, 4), (6, 10), (0, 6).

(viii) (5, 0), (0, 5), (−5, 0), (0, −5).

Area of a curvilinear figure. This cannot be found *exactly* by the method of counting squares: the approximate value however is easily calculated as follows.

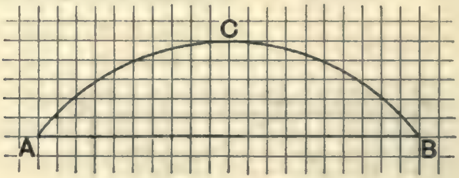


fig. 165.

To find the area of the fig. ACBA, notice that the curved boundary ACB *cuts through* various squares; in counting squares we have to decide what is to be done with these broken squares. The following rule gives a useful approach to the true value :— *If the broken square is more than half a complete square, count* 1; *if less than half a square, count* 0.

Counting up the squares in ACBA on this system, we find that the area is 72 squares. As each of the above squares is $\frac{1}{100}$ sq. inch, the area is ·72 sq. inches.

Ex. 897. On inch paper draw a circle of radius 1 inch; find its area as above, and reduce to square inches. (The counting can be shortened in various ways; e.g. by dividing the circle into 4 quarters by radii.)

Ex. 898. Find the area of circles of radii 2, and 3 inches. Calculate, to 2 places, how many times each of these circles contains the 1-inch circle of Ex. 897.

Ex. 899. Plot the graph $y = 6 - \frac{x^2}{6}$, and find the area contained between the curve and the x-axis.

DEF. Any side of a **parallelogram** may be taken as the **base**. The perpendicular distance between the base and the opposite (parallel) side is called the **height**, or **altitude**.

Thus in fig. 166 if BC be taken as base, MN (which may be drawn from any point of the base) is the height (or altitude). If AB be taken as base, GH is the height.

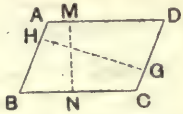


fig. 166.

¶Ex. **900.** In fig. 166 what is the height if CD be taken as base? if AD be taken?

Ex. **901.** Prove that the altitudes of a rhombus are equal.

Area of parallelogram. Take a sheet of paper (a rectangle) and call the corners P, B, C, Q; BC being one of the longer sides (fig. 167). Mark a point A on the side PQ. Join BA, and cut (or tear) off the right-angled triangle PBA. You now have two pieces of paper; you will find that you can fit them together to make a parallelogram (ABCD in fig. 167).

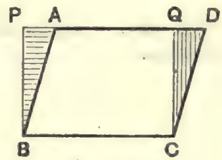


fig. 167.

Notice (i) that the rectangle you had at first and the parallelogram you have now made, are composed of the same paper, and *therefore have the same area.*

(ii) that the rectangle and the parallelogram are on the same base BC, and both lie between the same pair of parallel lines BC and PAQD. Or, we may say that they have *the same height.*

¶Ex. **902.** Make a paper parallelogram with sides of 6 and 4 ins. and an angle of 60°. Cut the parallelogram into two pieces which you can fit together to make up a rectangle. Find its area.

Ex. **903.** Repeat Ex. 902 with sides of 12 and 6 cm. and angle of 60°.

Ex. 904. Draw a parallelogram ABCD, having AB = 13 cm., BC = 16 cm., angle B = 70°; on the same base draw a rectangle of equal area; find the area. Measure the two altitudes of the parallelogram and calculate the products BC . MN and AB . GH (see fig. 166).

Ex. 905. On base 2 inches draw a parallelogram of angle 50° and height 4 inches. On the same base construct a rectangle of the same area; and find the area. Also calculate the products BC . MN and AB . GH as in Ex. 904.

Ex. 906. Repeat Ex. 905 with the same base and height, but with angle of 75°.

DEF. Figures which are equal in area are said to be **equivalent**.

Notice that congruent figures are necessarily equivalent; but that equivalent figures are not necessarily congruent.

¶**Ex. 907.** Give the sides of a pair of equivalent rectangles, which are not congruent.

Theorem 1.

Parallelograms on the same base and between the same parallels (or, of the same altitude) are equivalent.

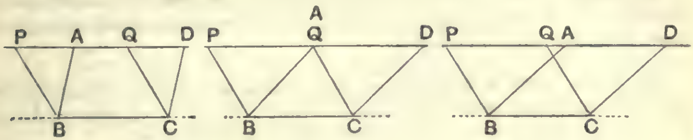


fig. 168.

Data ABCD, PBCQ are ||ograms on the same base BC, and between the same parallels BC, PD.

To prove that ABCD and PBCQ are equivalent.

Proof In the $\triangle$s PBA, QCD,

$\angle$ BAP = corresp. $\angle$ CDQ ($\because$ BA, CD are ||), I. 5.

$\angle$ BPA = corresp. $\angle$ CQD ($\because$ BP, CQ are ||), I. 5.

BA = CD (opp. sides of ||ogram ABCD), I. 22.

$\therefore$ the triangles are congruent. I. 11.

Now if $\triangle$ PBA is subtracted from figure PBCD, ||ogram BD is left; and if $\triangle$ QCD is subtracted from figure PBCD, ||ogram BQ is left.

Hence the ||ograms are equivalent.

Q. E. D.

Cor. 1. Parallelograms on equal bases and of the same altitude are equivalent.

(For they can be so placed as to be on the same base and between the same parallels.)

Cor. 2. **The area of a parallelogram is measured by the product of the base and the altitude.**

(For the ||ogram is equivalent to a rectangle on the same base and of the same altitude, whose area = base $\times$ altitude.)

Ex. **908.** Find the area of a parallelogram of sides 2 ins. and 3 ins. and of angle 30°.

Ex. **909.** Draw a rectangle on base 12 cm. and of altitude 10 cm.; on the same base construct an equivalent parallelogram of angle 60°; and measure its longer diagonal.

Ex. **910.** Show how to construct a parallelogram equivalent to a given rectangle, on the same base and having one of its angles equal to a given angle (without using protractor).

Ex. **911.** Draw a rectangle of base 4 ins., and height 3 ins.: on the same base make an equivalent parallelogram with a pair of sides of 5 ins. Measure the angle between the base and the shorter diagonal.

Ex. **912.** Show how to construct on the same base as a given rectangle an equivalent parallelogram having its other side equal to a given straight line (without using scale). Is this always possible?

Ex. **913.** Draw a rectangle whose base is double its height; on the same base construct an equivalent rhombus and measure its acute angle.

Ex. **914.** Transform a rectangle of base 4·53 cm. and height 2·97 cm. into an equivalent parallelogram having a diagonal of 8·45 cm. Measure the angle between the base and that diagonal.

Ex. **915.** Transform a parallelogram of sides 2 and 1 ins. and angle 80° into an equivalent parallelogram of sides 2 and 2·5 ins. Measure acute angle of the latter.

Ex. **916.** Transform a parallelogram of sides 8·3 and 12·4 cm. and angle 12° into an equivalent rhombus of sides 8·3 cm. Measure angle of rhombus.

Ex. **917.** Repeat Ex. 916, making side of rhombus 12·4 cm.

Ex. **918.** Transform a parallelogram of base 2·34 ins., height 2·56 ins. and angle 67° into an equivalent parallelogram on the same base with angle 60°. Measure the other side of the latter.

Ex. **919.** Transform a given parallelogram into an equivalent parallelogram with one of its angles = a given angle (without using protractor).

Ex. **920.** Make parallelogram ABCD, with AB = 2·5 ins., AD = 3 ins., angle A = 60°. Transform this into an equivalent parallelogram with sides of 2 ins., and 4 ins.; measure acute angle of the latter.

(First, keeping the same base AB, make equivalent parallelogram ABEF having AE = 4 ins. Next, taking AE for base, construct an equivalent parallelogram with sides 2 and 4 ins.)

Ex. **921.** Show how to make a parallelogram equivalent to a given rectangle, having its sides equal to two given lines. Is this always possible?

Ex. **922.** Construct a parallelogram of sides 9 and 8 cm. and angle 20°; make an equivalent rhombus of side 6 cm. and measure its angle.

Ex. **923.** Repeat Ex. 922, with angle 30° instead of angle 20°.

Ex. **924.** What is the locus of the intersection of the diagonals of a parallelogram whose base is fixed and area constant?

In calculating the area of a parallelogram by means of II. 1 (area = base × height), you will notice that the product may be formed in two different ways; e.g. in fig. 169 we may take either BC . MN or AB . GH; these two products should be equal, being both equal to the area. In practice it will be found that the two results do not generally agree exactly; (what is the reason for this?). The difference however should not be greater than 1 or 2 per cent. In order to get the best possible result for the area, calculate both products and take the average.

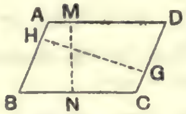


fig. 169.

Ex. **925.** Find the area of each of the following parallelograms, taking the average of two results as explained above.

(i) Sides 3·5 and 4·5 ins., angle of 70°.

(ii) Sides 12·7 and 14·5 cm., angle of 120°.

(iii) Sides 10 and 6 cm., angle of 30° (in this case one of the altitudes will fall partly outside the parallelogram; produce a side).

(iv) Sides 5·53 and 1·61 ins., angle of 160°.

(v) Diagonals 3·7 and 2·2 ins., angle between diagonals 55°.

(vi) Equal diagonals of 3·2 ins., angle between diagonals 150°.

(vii) Sides 6·6 and 8·8 cm., a diagonal of 11 cm.

Ex. **926.** Find the area of a rhombus of side 2 inches and angle 30°.

Ex. **927.** Find (correct to $\frac{1}{100}$ inch) the height of a rectangle whose area is 10 sq. ins. and whose base = 3·16 ins.

Ex. **928.** Draw a parallelogram of area 24 sq. cm., base 6 cm. and angle 75°. Measure the other sides.

Ex. **929.** Draw a parallelogram of area 12 sq. ins., sides of 4 and 3·5 ins. Measure its acute angle.

Ex. **930.** Draw a rhombus of area 24 sq. cm. and side 5 cm. Measure its acute angle.

Ex. **931.** Draw a parallelogram of area 15 sq. ins., base 5 ins. and diagonal 4 ins. Measure the acute angle.

Area of Triangle.

Def. Any side of a triangle may be taken as **base.** The line drawn perpendicular to the base from the opposite vertex is called the **height**, or **altitude.**

There will be three different altitudes according to the side which is taken as base.

¶Ex. **932.** Draw an acute-angled triangle and draw the three altitudes. (*Freehand.*)

¶Ex. **933.** Repeat Ex. 932 for a right-angled triangle. (*Freehand.*)

¶Ex. **934.** Repeat Ex. 932 for an obtuse-angled triangle. (*Freehand.*)

¶Ex. **935.** In what case are two of the altitudes of a triangle equal?

¶Ex. **936.** In what case are all three altitudes equal?

¶Ex. **937.** In what case do some of the altitudes fall outside the triangle?

¶Ex. **938.** By making rough sketches, try whether you can find a triangle (1) in which one (and only one) altitude falls outside, (2) in which all three altitudes fall outside.

Theorem 2.

Triangles on the same base and between the same parallels (or, of the same altitude) are equivalent.

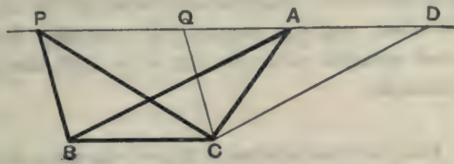


fig. 170.

Data ABC, PBC are $\triangle$s on the same base BC, and between the same parallels BC, PA.

To prove that ABC, PBC are equivalent.

Construction Complete the $\|^{\text{ograms}}$ ABCD, PBCQ by drawing CD, CQ $\|$ to BA, BP respectively, to meet PA (produced if necessary) in D, Q.

Then $\triangle ABC = \frac{1}{2} \|^{\text{ogram}}$ ABCD, I. 22 (3).

and $\triangle PBC = \frac{1}{2} \|^{\text{ogram}}$ PBCQ. I. 22 (3).

But $\|^{\text{ograms}}$ ABCD, PBCQ are equivalent, being on the same base and between the same parallels. II. 1.

$$\therefore \triangle ABC = \triangle PBC.$$

Q. E. D.

Cor. 1. Triangles on equal bases and of the same altitude are equivalent.

(For they can be so placed as to be on the same base and between the same parallels.)

Cor. 2. **The area of a triangle is measured by half the product of the base and the altitude.**

†Ex. **939.** **Prove Cor. 2.**

†Ex. **940.** Prove that, in general, the area of a triangle is less than half the product of two of its sides.

†Ex. **941.** Prove that the area of a right-angled triangle is half the product of the sides which contain the right angle.

Since any one of the three sides may be taken for base, there are three different ways of forming the product of a base and the corresponding altitude. Thus the area may be calculated in three different ways; and of course, theoretically, the result is the same in each case. Practically, none of the measurements will be quite exact, and the results will generally differ slightly. To get the best possible value for the area **take the average of the three results.**

Ex. **942.** Find, to three significant figures, the areas of the following triangles, taking the average of three results in each case:

(i) sides 3, 4, 4·5 ins.

(ii) sides 6, 8, 9 cm.

(iii) sides 3, 4, 5 ins.

(iv) sides 6, 8, 10 cm.

(v) sides 2, 3, 4·5 ins.

(vi) sides 4, 7, 10 cm.

(vii) sides 3, 4 ins., included $\angle 120°$.

(viii) $BC = 7·2$ cm., $\angle B = 20°$, $\angle C = 40°$.

Ex. **943.** Make a copy of your set-square and find its area (i) in sq. inches, (ii) in sq. cm.

Ex. **944.** (On inch paper.) The vertices of a triangle are the points $(2, 0)$, $(-1, 2)$, $(-2, -2)$. Find the area (i) by measuring sides and altitudes, (ii) as on p. 165.

Ex. **945.** (On inch paper.) Repeat Ex. 944 with the following vertices:

(i) $(-1, 2)$, $(0, -1)$, $(2, -2)$.

(ii) $(-2, -2)$, $(1, 1)$, $(3, 0)$.

Ex. **946.** Find the area of an equilateral triangle of side (i) 1 inch, (ii) 2 inches. Find the ratio of the greater area to the smaller.

Ex. **947.** Find the surface (i.e. the sum of the areas of all the faces):

(i) of the tetrahedron in Ex. 109.

(ii) of the square pyramid in Ex. 116.

(iii) of the cube in Ex. 210.

(iv) of the cuboid in Ex. 221.

(v) of the 3-sided prism in Ex. 224.

Ex. **948.** Find the combined area of the walls and roof of the house in fig. 102; take width of house = 8 yds., depth (front to back) = 4 yds., height of front wall = 6 yds., height of roof-ridge above ground = $7\frac{1}{2}$ yds. Neglect doors and windows.

Ex. **949.** Find the area (i) in sq. inches, (ii) in sq. cm., of the triangle whose vertices are ACD in fig. 20.

†Ex. **950.** Prove that the area of a rhombus is half the product of its diagonals.

†Ex. **951.** **D is the mid-point of the base BC of a triangle ABC; prove that triangles ABD, ACD are equivalent.**

†Ex. **952.** ABCD is a parallelogram; P, Q the mid-points of AB, AD. Prove that $\triangle APQ = \frac{1}{8}$ of ABCD. (Join PD, BD.)

†Ex. **953.** The base BC of $\triangle ABC$ is divided at D so that $BD = \frac{1}{3}BC$; prove that $\triangle ABD = \frac{1}{3}\triangle ABC$.

†Ex. **954.** The base BC of $\triangle ABC$ is divided at D so that $BD = \frac{3}{7}BC$; prove that $\triangle ABD = \frac{3}{4}\triangle ACD$.

†Ex. **955.** **The ratio of the areas of triangles of the same height is equal to the ratio of their bases.**

†Ex. **956.** The ratio of the areas of triangles on the same base is equal to the ratio of their heights.

†Ex. 957. ABCD is a quadrilateral and the diagonal AC bisects the diagonal BD. Prove that AC divides the quadrilateral into equivalent triangles (fig. 171).

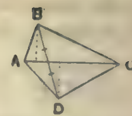


fig. 171.

†Ex. 958. E is the mid-point of the diagonal AC of a quadrilateral ABCD. Prove that the quadrilaterals ABED, CBED are equivalent.

†Ex. 959. E is a point on the median AD of $\triangle$ ABC; prove that $\triangle$ ABE = $\triangle$ ACE.

†Ex. 960. D is a point on the base BC of $\triangle$ ABC; E is the mid-point of AD; prove that $\triangle$ EBC = $\frac{1}{2}$ $\triangle$ ABC.

†Ex. 961. Divide a triangle into 4 equivalent triangles. (*Freehand*)

Ex. 962. The base of a triangle is a fixed line of length 3 inches, and the vertex moves so that the area of the triangle is always 6 sq. ins. What is the altitude? What is the locus of the vertex?

†Ex. 963. Prove that the locus of the vertex of a triangle of fixed base and constant area is a pair of straight lines parallel to the base.

Ex. 964. Draw a scalene triangle, and transform it into an equivalent isosceles triangle on the same base. (Keep the base fixed; where must the vertex be in order that the triangle may be isosceles? Where must the vertex be in order that the triangle may be equivalent to given triangle?) (*Freehand.*)

Ex. 965. Show how to transform a given triangle

(i) into an equivalent right-angled triangle.

(ii) into an equivalent triangle on the same base, having one side of 2 inches. Is this always possible?

(iii) into an equivalent triangle with an angle of 60°.

(iv) into an equivalent triangle having one angle = a given angle (without protractor).

(v) into an equivalent right-angled triangle with one of the sides about the right angle equal to 5 cm. (First make one side 5 cm.; then take this as base and make the triangle right-angled.)

(vi) into an equivalent isosceles triangle with base equal to a given line.

Ex. 966. Transform an equilateral triangle of side 3 ins. into an equivalent triangle with a side of 4 ins., and an angle of 60° adjacent to that side. Measure the other side adjacent to the 60° angle.

Ex. **967.** Transform a given triangle into an equivalent triangle with its vertex (i) on a given line, (ii) one inch from a given line, (iii) one inch from a given point, (iv) equidistant from two given intersecting lines.

¶Ex. **968.** Transform a given quadrilateral ABCD into an equivalent quadrilateral ABCD′, so that the three vertices A, B, C may be unchanged, and $\angle$ BAD′ = 170°.

¶Ex. **969.** Repeat Ex. 968, making $\angle$ BAD′ = 180°. What kind of figure is produced?

†Ex. **970.** In fig. 172 PA is parallel to BC.

Prove that $\triangle$ POB = $\triangle$ AOC.

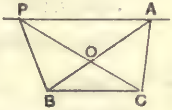


fig. 172.

†Ex. **971.** A line parallel to the base BC of $\triangle$ ABC cuts the sides AB, AC in D, E respectively. Prove that $\triangle$ ABE = $\triangle$ ACD.

†Ex. **972.** F is any point on the base BC of $\triangle$ ABC : E is the mid-point of BC. ED is drawn parallel to AF. Prove that $\triangle$ DFC = $\frac{1}{2}\triangle$ ABC. (Join AE.) Fig. 173.

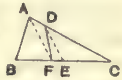


fig. 173.

†Ex. **973.** Draw a line through a given point of a side of a triangle to bisect the area of the triangle. (See Ex. 972.) Verify your construction by measuring and calculating areas.

Area of any rectilinear figure. This may be determined in various ways.

METHOD I. By dividing up the figure into triangles.

METHOD II. Perhaps the most convenient method is that of constructing a single triangle equivalent to the given figure, as follows:

To construct a triangle equivalent to a given quadrilateral ABCD.

Construction Join CA. Through D draw DD′ ∥ CA, meeting BA produced in D′.

Join CD′.

Then △BCD′ = quadrilateral ABCD.

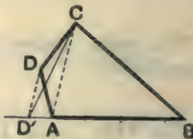


fig. 174.

Proof △ACD′ = △ACD. (Why?)

Add to each △ACB.

∴ △BCD′ = quadrilateral ABCD.

In a similar way a pentagon may be reduced, first to an equivalent quadrilateral and then to an equivalent triangle: and so for figures of more sides. The area of the triangle can then be found as already explained. A convenient method of dealing with the pentagon is shown in fig. 175.

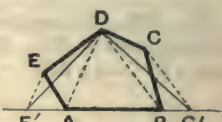


fig. 175.

†Ex. 974. Explain the construction of fig. 175 and prove that

△C′DE′ = figure ABCDE.

†Ex. 975. Given a quadrilateral ABCD, construct an equivalent triangle on base AB having ∠A in common with the quadrilateral. (*Freehand*)

Ex. 976. Construct a triangle whose area is equal to the sum of the areas of two given triangles. (First transform one triangle till it has a side equal to a side of the other triangle; then fit the triangles together to form a quadrilateral, and consider how to reduce the sum to a single triangle.)

Ex. 977. Construct a triangle equivalent to the difference of two given triangles.

Ex. 978. Find the area of a quadrilateral ABCD, when

(i) DA = 1 in., $\angle$A = 100°, AB = 2·3 ins., $\angle$B = 64°, BC = 1·5 ins.

(ii) AB = 5·7 cm., BC = 5·2 cm., CD = 1·7 cm., DA = 3·9 cm., $\angle$A = 76°.

Ex. 979. Find the area of a pentagon ABCDE, given AB = 6·5 cm., BC = 2·4 cm., CD = DE = 4 cm., EA = 2·5 cm., $\angle$A = 80°, $\angle$B = 133°.

Ex. 980. Find the area of a regular hexagon inscribed in a circle of radius 2 ins.

Ex. 981. Find the area of a regular pentagon of side 6 cm.

Ex. 982. Find the areas of the 4-gons and 5-gons in Ex. 107 (i), (ii), 108 (i), (ii).

Ex. 983. Find the area of a **trapezium** ABCD (fig. 176), given AB = 3 ins., height = 2 ins., $\angle$A = 70°, $\angle$B = 50°. (Divide into 2 $\triangle$s, and notice that their heights DE, BF are equal.)

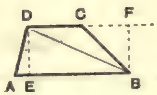


fig. 176.

Ex. 984. Find the area of a trapezium ABCD, given

(i) AB = 7·5 cm., height = 4 cm., AD = 5 cm., BC = 4·3 cm., $\angle$A obtuse, $\angle$B acute.

(ii) AB = 3·6 ins., CD = 2·5 ins., height = 1·3 ins., $\angle$A = 60°.

(iii) Same dimensions as in (ii) except that $\angle$A = 80°.

(iv) AB = 5 cm., AD = 4 cm., BD = 5 cm., $\angle$DBC = $\angle$BDC.

†**Ex. 985.** In fig. 177 E is the mid-point of BC, PQ is $\parallel$ to AD. Prove that trapezium ABCD = $\parallel$ogram APQD.

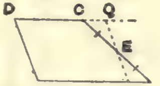


fig. 177.

†**Ex. 986. Prove that the area of a trapezium is equal to half the product of the height and the sum of the two parallel sides (see Ex. 985).**

¶**Ex. 987.** Cut out of paper two congruent trapezia, and fit them together to make up a parallelogram. Hence prove Ex. 986.

METHOD III. This method is used by *land-surveyors* and depends on the following principle. It is required to find the area of the field ABCDEFG (fig. 178). The field is treated as a polygon, the sides of the polygon being chosen so that the small irregularities may roughly compensate one another. The longest diagonal AE is chosen as **base-line.** In AE points L, M, N, P are determined, namely the points where the perpendiculars from the corners meet AE. The field is thus divided up into right-angled triangles, trapezia and rectangles, whose areas can be calculated as soon as the necessary measurements have been made. The surveyor now measures with a chain the different distances along the base-line, AL, AM, AN, AP, AE; also the distances to the different corners, right and left of the base-line*, namely, LB, MC, MG, NF, PD. These measurements are recorded in the Field-Book in the following form:—

Yards.

	To E	
	600	
240	460	
	360	50
240	300	120
200	100	
From	A	go North

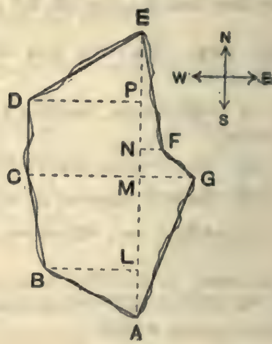


fig. 178.

* The distances at right angles to the base-line are called **offsets**; in practice they are never allowed to exceed a few yards, on account of the difficulty of determining accurately the feet of the perpendiculars.

This record is to be read *upwards*. In the middle column are set down the distances from A of the different points on the base-line; on the right and left are set down the offsets as they occur; e.g. L is 100 yards North of A, and B is 200 yds. to the left of L; and so on.

Ex. **988.** On inch paper draw a plan of the field represented in fig. 178 from the measurements given (scale, 1 inch to represent 100 yards); calculate its area in square yards.

Ex. **989.** Give the coordinates of the corners of the field in fig. 178, taking AE as axis of y and A as origin.

Ex. **990.** Draw a plan and find the area of the field in the following survey:— (*Freehand*)

Yards

	To D	
	400	
	340	50
70	300	
90	200	
30	100	50
From	A	go North

In practice, distances are measured with a **chain** of 100 **links**. The length of the surveyors' chain is the same as the length of a cricket-pitch, namely 22 yards. A square whose side is 1 chain has area 22^2 or 484 sq. yards. Now an acre contains 4840 sq. yards; hence **10 sq. chains = 1 acre.**

Ex. 991. Draw plans and find the area (in acres) of the fields whose dimensions are recorded below: (*Freehand*)

(i)

Links

	To B	
	800	
	600	400
500	300	
100	200	
From	A	go East

(ii)

Links

	To B	
	1100	
400	1000	
	800	800
400	600	
From	A	go S.E.

(iii)

Links

	To B	
	800	
150	700	
100	500	200
	400	350
300	300	
100	200	
From	A	go N.W.

Ex. 992. Draw a plan of a field whose corners are represented by the points ABCDO in fig. 20; choose the longest diagonal as base-line and draw offsets; enter measurements as for Field-Book (taking 1 inch to represent 100 yards); find the area of the field in square yards.

Also find the area by constructing a single equivalent triangle.

THEOREM 3.

Equivalent triangles which have equal bases in the same straight line, and are on the same side of it, are between the same parallels.

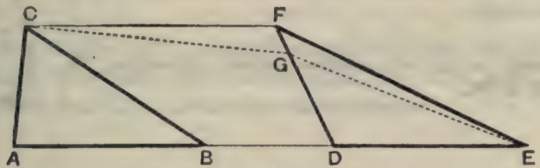


fig. 179.

Data ABC, DEF are equivalent triangles on equal bases AB, DE, these being in a straight line, and C and F being on the same side of AE.

To prove that CF is parallel to AE.

Construction Join CF.

If possible, draw a line CG || to AE, distinct from CF, meeting FD (produced if necessary) in G. Join EG.

Proof Since AB = DE, and CG is || to AE,

$\therefore \triangle ABC = \triangle DEG$. II. 2.

But $\triangle ABC = \triangle DEF$, *Data*

$\therefore \triangle DEF = \triangle DEG$,

$\therefore$ F coincides with G, and CF with CG,

$\therefore$ CF is || to AE.

Q. E. D.

COR. 1. Equivalent triangles on the same or equal bases are of the same altitude.

COR. 2. Equivalent triangles on the same base and on the same side of it are between the same parallels.

†Ex. **993 a.** Give another proof of Cor. 1.

Ex. **993.** What is the converse of the above Theorem?

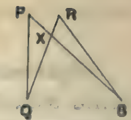


fig. 180.

†Ex. **994.** D E **are the mid-points of the sides** AB, AC **of a triangle** ABC; **prove that** DE **is parallel to** BC. (Join DC, EB.)

†Ex. **995.** In fig. 180 △PXQ = △RXS; prove that PR is parallel to QS.

fig. 181.

†Ex. **996.** In fig. 181 △AEB = △ADC; prove that DE is parallel to BC.

THEOREM 4. †

If a triangle and a parallelogram stand on the same base and between the same parallels, the area of the triangle is half that of the parallelogram.

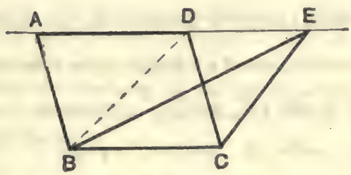


fig. 182.

Data $\triangle$ EBC and $||^{\text{ogram}}$ ABCD stand on the same base BC and between the same parallels BC, AE.

To prove that $\triangle \text{EBC} = \frac{1}{2}||^{\text{ogram}}\,\text{ABCD}$.

Construction Join BD.

Proof Since AE is $||$ to BC,

$\therefore \triangle \text{EBC} = \triangle \text{DBC}$, II. 2.

and $\triangle \text{DBC} = \frac{1}{2}||^{\text{ogram}}\,\text{ABCD}$, I 22.

$\therefore \triangle \text{EBC} = \frac{1}{2}||^{\text{ogram}}\,\text{ABCD}$.

Q. E. D.

†Ex. **997.** Construct a rectangle equal to a given triangle. Give a proof.

†Ex. **998.** F, E are the mid-points of the sides AD, BC of a parallelogram ABCD; P is any point in FE. Prove that $\triangle$APB $= \frac{1}{2}$ABCD.

†Ex. **999.** P, Q are any points upon adjacent sides AB, BC of a parallelogram ABCD; prove that $\triangle$CDP $=$ $\triangle$ADQ.

†Ex. **1000.** AB, CD are parallel sides of a trapezium ABCD; E is the mid-point of AD; prove that $\triangle$BEC $= \frac{1}{2}$ trapezium. (Through E draw line parallel to BC.)

†Ex. **1001.** O is a point inside a parallelogram ABCD; prove that

$$\triangle OAB + \triangle OCD = \tfrac{1}{2} ABCD.$$

Miscellaneous Exercises on Area.

Ex. **1002.** Find the area of a triangle whose sides are

(i) $y=2x+2,\quad y=\dfrac{x-2}{2},\quad y=2-x.$

(ii) $y=2x+2,\quad y=2-x,\quad y=0.$

(iii) $x=0,\quad y=1-\dfrac{x}{2},\quad y=x-1.$

Ex. **1003.** The area of a parallelogram of angle 30° is half the area of a rectangle with the same sides.

†Ex. **1004***. O is any point on the diagonal BD of a parallelogram ABCD. EOF, GOH are parallel to AB, BC respectively. Prove that parallelogram AO = parallelogram CO.

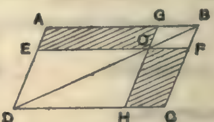


fig. 183.

†Ex. **1005.** Any straight line drawn through the centre of a parallelogram (i.e. through the intersection of the diagonals) bisects the parallelogram.

Ex. **1006.** Show how to divide a parallelogram into three equal parallelograms.

Ex. **1007.** Show how to bisect a parallelogram by a straight line drawn perpendicular to a side.

* This exercise appears in old books on Geometry as a proposition, and was used by Euclid in the proof of later propositions. It was enunciated as follows: "The complements of the parallelograms which are about the diagonal of any parallelogram are equal."

†Ex. **1008.** E is any point on the diagonal AC of a parallelogram ABCD. Prove that $\triangle ABE = \triangle ADE$.

†Ex. **1009.** Produce the median BD of a triangle ABC to E, making DE = DB. Prove that $\triangle EBC = \triangle ABC$.

†Ex. **1010.** P, Q are the mid-points of the sides BC, AD of the trapezium ABCD; EPF, GQH are drawn perpendicular to the base. Prove that trapezium = rectangle GF. (See fig. 184.)

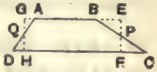


fig. 184.

†Ex. **1011.** L, M are the mid-points of the parallel sides AB, CD of a trapezium ABCD. Prove that LM bisects the trapezium.

†Ex. **1012.** In Ex. 1011 O is the mid-point of LM; prove that any line through O which cuts AB, CD (not produced) bisects the trapezium.

†Ex. **1013.** Prove that the area of the parallelogram formed by joining the mid-points of the sides of *any* quadrilateral ABCD (see Ex. 736) is half the area of the quadrilateral.

†Ex. **1014.** The medians BD, CE of $\triangle ABC$ intersect at G; prove that quadrilateral ADGE = $\triangle BGC$. (Add to each a certain triangle.)

The Theorem of Pythagoras.

Fig. 185 represents an isosceles right-angled triangle with squares described upon each of the sides. The dotted lines divide up the squares into right-angled triangles, each of which is obviously equal to the original triangle. This sub-division shows that the square on the hypotenuse of the above right-angled triangle is equal to the sum of the squares on the sides containing the right angle. (A tiled pavement often shows this fact very clearly.)

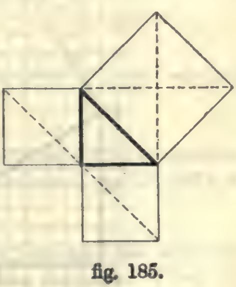


fig. 185.

¶ Ex. **1015.** Construct a right-angled triangle with sides of 3 cm. and 4 cm. containing the right angle. Construct squares on these two sides, and upon the hypotenuse. Measure the length of the hypotenuse, and ascertain whether or no the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle. See fig. 186.

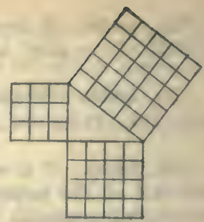


fig. 186.

Ex. **1016.** Repeat Ex. 1015 taking 4·3 cm. and 6·5 cm. as the sides containing the right angle.

Ex. **1017.** Draw a good-sized scalene right-angled triangle ABC, right-angled at A. Measure the three sides and calculate the areas of the squares upon them. Add together the areas of the two smaller squares, and arrange your results like this—

AB = ...cm., sq. on AB = ...sq. cm.,
AC = ...cm., sq. on AC = ...sq. cm.,
sum of sqq. on AB, AC = ...sq. cm.,
BC = ...cm., sq. on BC = ...sq. cm.

Ex. **1018.** Repeat Ex. 1017 with a different right-angled triangle.

Ex. **1019.** Repeat Ex. 1017 making $\angle A = 60°$ instead of 90°.

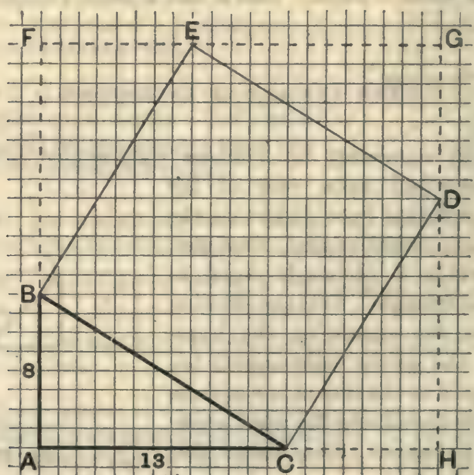


fig. 187.

Ex. **1020.** In fig. 187 find (in squares of the paper) the area of the square BD by first finding the area of the square AG and then deducting the four triangles at the corners. Also calculate the areas of the squares on AB and AC, and see whether these add up to the square on BC.

Ex. **1021.** Repeat Ex. 1020 (drawing your own figure on squared paper) with different numbers instead of 8 and 13.

The above exercises lead up to the fact that

"In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares on the other two sides."

This famous theorem was discovered by Pythagoras (B.C. 570—500). Before proving it, the pupil may try the following experiment.

Ex. **1022.** Draw (on paper or, better, on thin cardboard) a right-angled triangle and the squares on the three sides (see fig. 188). Choose one of the two smaller squares and cut it up in the following manner. First find the centre of the square by drawing the diagonals. Then, through the centre, make a cut across the square parallel to BC, the hypotenuse, and a second cut perpendicular to BC. It will be found that the four pieces of this square together with the other small square exactly make up the square on the hypotenuse.
(Perigal's dissection.)

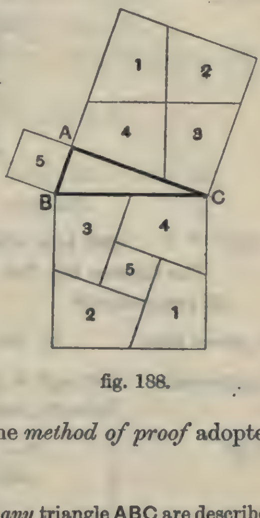


fig. 188.

The following exercises lead up to the *method of proof* adopted for the theorem of Pythagoras.

†Ex. **1023.** On two of the sides AB, BC of *any* triangle ABC are described squares ABFG, BCED (as in fig. 189); prove that triangles BCF, BDA are congruent; and that CF = DA.

†Ex. **1024.** On the sides of any triangle ABC are described equilateral triangles BCD, CAE, ABF, their vertices pointing outwards. Prove that AD = BE = CF.

Theorem 5.

[The Theorem of Pythagoras.]

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

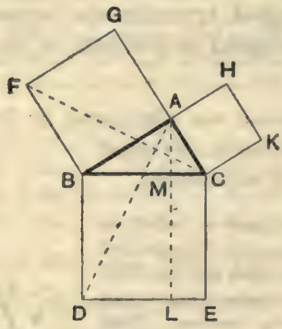


fig. 189.

Data ABC is a triangle, right-angled at A.

The figures BE, CH, AF are squares described upon BC, CA, AB respectively.

To prove that sq. BE = sq. CH + sq. AF.

Construction Through A draw AL ∥ to BD (or CE).
Join CF, AD.

Proof

rt. ∠ CBD = rt. ∠ FBA,
add to each ∠ ABC,
∴ ∠ ABD = ∠ FBC.
Hence, in △s ABD, FBC

△ABD ≡ △FBC

∠ ABD = ∠ FBC,
AB = FB (sides of a square),
BD = BC,
∴ △ABD ≡ △FBC. I. 10.

$\triangle$ FBC $= \frac{1}{2}$ sq. AF
$\triangle$ ABD $= \frac{1}{2}$ rect. BL

Since each of the angles BAC, BAG is a right angle

$\therefore$ CAG is a st. line, I. 2.

and this line is $\parallel$ to BF.

$\therefore$ $\triangle$ FBC and sq. AF are on the same base BF, and between the same parallels BF, CG,

$\therefore$ $\triangle$ FBC $= \frac{1}{2}$ sq. AF. II. 4.

Again $\triangle$ ABD and rect. BL are on the same base BD and between the same parallels BD, AL,

$\therefore$ $\triangle$ ABD $= \frac{1}{2}$ rect. BL. II. 4.

sq. AF = rect. BL
sq. CH = rect. CL
$\therefore$ sq. AF + sq. CH
= sq. BE

But $\triangle$ FBC $\equiv$ $\triangle$ ABD. *Proved*

$\therefore$ sq. AF = rect. BL.

In a similar way, by joining BK, AE, it may be shown that

sq. CH = rect. CL.

Hence

sq. AF + sq. CH = rect. BL + rect. CL
= sq. BE.

Q. E. D.

An alternative method of proof is indicated below; the pupil should work out for himself the actual details of the proof.

The figs. AF, GE are two squares placed side by side.

Mark off AC = GK and join BC.

Then BAC is a rt. $\angle^{d}$ $\triangle$ and AF, GE are equal to the squares on the sides containing the right angle.

Produce GF to D so that FD = GK.

Join BD, DE, EC.

Prove that $\triangle^{s}$ BAC, CKE, DHE, BFD are congruent.

Prove that fig. CD is a square, namely the square on the hypotenuse of $\triangle$ BAC.

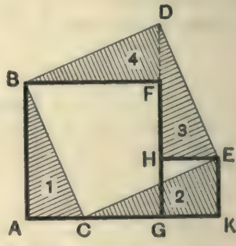


fig. 190.

From the figure AKEHFB subtract the triangles (1) and (2) and fit them on to (3) and (4), thus making up the sq. CD.

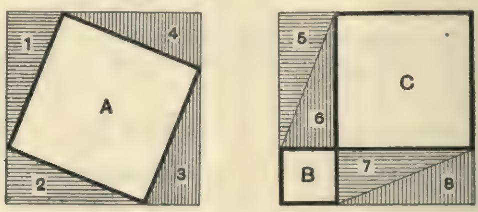


fig. 191.

Another proof of the theorem is shown in fig. 191.

The triangles marked 1, 2, 3, 4, 5, 6, 7, 8 are all congruent right-angled triangles. A is the square on the hypotenuse of one of these triangles, B and C are the squares on the sides containing the right angle. A little consideration will make it evident that A = B + C.

Ex. 1025. What is the side of a square whose area is 4 sq. in.; 9 sq. in.; 16 sq. cm.; 17 sq. cm.; 2 sq. in.; 5 sq. miles; a^2 sq. in.; b sq. cm.?

Ex. 1026. What is the square on the hypotenuse of a right-angled triangle if the sides containing the right angle are 6 cm. and 8 cm.? Hence calculate the length of the hypotenuse. Verify by drawing.

Note on "error per cent." In cases where a result is obtained both by calculation and by drawing, it will generally be found that there is a slight disagreement. To see whether this disagreement, or "error," is serious, it is necessary to reduce it to a percentage. Thus, the calculation in Ex. 1026 would be as follows:—

$$\begin{aligned}\text{sum of sqq. on sides} &= (6^2 + 8^2) \text{ sq. cm.}\\ &= (36 + 64) \text{ sq. cm.}\\ &= 100 \text{ sq. cm.,}\\ \therefore \text{ sq. on hypotenuse} &= 100 \text{ sq. cm.,}\\ \therefore \text{ hypotenuse} &= \sqrt{100} \text{ cm.}\\ &= 10 \text{ cm. (by calculation).}\end{aligned}$$

$$\begin{aligned}\text{Suppose that we find hypotenuse} &= 9{\cdot}95 \text{ cm. (by drawing),}\\ \text{error} &= 0{\cdot}05 \text{ in } 10\\ &= 0{\cdot}5 \text{ in } 100\\ &= 0{\cdot}5 \text{ per cent.}\end{aligned}$$

N.B. (1) **It is not necessary to calculate the "error per cent." to more than one significant figure.**

(2) **Do not be satisfied until your error is less than 1 per cent.**

Work the following exercises (i) *by calculation,* (ii) *by drawing, in every case making a rough estimate of the error per cent. Every calculation is to be "to three significant figures."*

Ex. 1027. Find the hypotenuse of a right-angled triangle when the sides containing the right angle are

(i) 5 cm., 12 cm.,	(ii) 4·5 in., 6 in.,
(iii) 7·8 cm., 9·4 cm.,	(iv) 2·34 in., 4·65 in.
(v) $4\frac{1}{4}$ miles, $5\frac{3}{4}$ miles,	(vi) 65 mm., 83·5 mm.

Ex. 1028. Find the remaining side and the area of a right-angled triangle, given the hypotenuse and one side, as follows:—

(i) hyp. = 15 cm., side = 12 cm.; (ii) hyp. = 6 in., side = 4 in.;

(iii) hyp. = 8 in., side = 4 in.; (iv) hyp. = 160 mm., side = 100 mm.;

(v) hyp. = 143 mm., side = 71·5 mm.

Ex. 1029. A flag-staff 40 ft. high is held up by several 50 ft. ropes; each rope is fastened at one end to the top of the flag-staff, and at the other end to a peg in the ground. Find the distance between the peg and the foot of the flag-staff.

Ex. 1030. Find the diagonal of a rectangle whose sides are (i) 4 in. and 6 in., (ii) 9 cm. and 11 cm.

Ex. 1031. Find the remaining side and the area of a rectangle, given (i) diagonal = 10 cm., one side = 7 cm.; (ii) diagonal = 4·63 in., one side = 3·47 in.

Ex. 1032. Find the diagonal of a square whose side is (i) 1 in., (ii) 5 cm., (iii) 6·72 cm.

Ex. 1033. Find the side and area of a square whose diagonal is (i) 2 in., (ii) 10 cm., (iii) 14·14 cm.

Ex. 1034. Find the side of a rhombus whose diagonals are

(i) 16 cm., 12 cm.; (ii) 6 in., 4 in.

Ex. 1035. Find the altitude of an isosceles triangle, given (i) base = 4 in., side = 5 in., (ii) base = 64 mm., side = 40 mm.

Ex. 1036. Find the altitude of an equilateral triangle of side 10 cm.

Ex. 1037. In fig. 192, ABCD represents a square of side 3 in.; AE = AH = CF = CG = 1 in. Prove that EFGH is a rectangle; find its perimeter and diagonal.

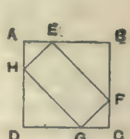


fig. 192.

Ex. 1038. Find how far a traveller is from his starting point after the following journeys:—(i) first 10 miles N., then 8 miles E., (ii) first 8 miles E., then 10 miles N., (iii) 43 km. S.W. and 32 km. S.E., (iv) 14 miles S., 10 miles E., 4 miles N. (try to complete a right-angled triangle having the required line for hypotenuse), (v) 4 miles E., 6 miles N., 3 miles E., 1 mile N.

Ex. 1039. (Inch paper.) If the coordinates of a point P are (1, 1) and of Q (2, 3), find the distance PQ. (PQ is the diagonal of a certain rectangle.)

Ex. **1040.** (Inch paper.) In each of the following cases find the distance between the pair of points whose coordinates are given:—(i) (2, 1) and (1, 3); (ii) (0, 0) and (3, 1); (iii) (2, 0) and (0, 3); (iv) $(-1, -1)$ and (2, 1); (v) $(-2, 2)$ and $(1, -2)$; (vi) (0·4, 1·3) and (2·3, 0·4); (vii) $(-0{\cdot}9, 0{\cdot}4)$ and $(1{\cdot}6, -0{\cdot}7)$.

Ex. **1041.** Find the lengths of the sides of the triangle whose vertices are $(2, -2)$, $(0, -3)$ and $(-2, 1)$.

Ex. **1042.** Newhaven is 90 miles N. of Havre, and 50 miles E. of Portsmouth. How far is it from Portsmouth to Havre?

Ex. **1043.** St Albans is 32 miles N. of Leatherhead, and Leatherhead is 52 miles from Oxford. Oxford is due W. of St Albans; how far is Oxford from St Albans?

Ex. **1044.** A ship's head is pointed N., and it is steaming at 15 miles per hour. At the same time it is being carried E. by a current at the rate of 4 miles per hour. How far does it actually go in an hour, and in what direction?

Ex. **1045.** Two men are conversing across a street 30 feet wide from the windows of their respective rooms. Their heads are 15 ft. and 30 ft. from the level of the pavement. How far must their voices carry?

Ex. **1046.** A man, standing on the top of a vertical cliff 700 ft. high, estimates the distance from him of a boat out at sea to be 1500 ft. How far is the boat from the foot of the cliff?

Ex. **1047.** A ladder 60 ft. long is placed against a wall with its foot 20 ft. from the foot of the wall. How high will the top of the ladder be?

Ex. **1048.** A field ABCD is right-angled at B and D. AB = 400 yards, AD = 300 yards, the diagonal AC = 500 yards. Find the area of the field.

Ex. **1049.** Find the distance between the summits of two columns, 60 and 40 ft. high respectively, and 30 ft. apart.

Ex. **1050.** An English battery (A) finds that a Boer gun is due N., at a range of 4000 yards. A second English battery (B) arrives, and takes up a pre-arranged position 1000 yards E. of A. A signals to B the range and direction in which it finds the enemy's gun. Find the range and direction in which B must fire.

Ex. **1051.** What is the hypotenuse of a right-angled triangle whose sides are a and b in.?

Ex. 1052. What is the remaining side of a right-angled triangle which has hypotenuse $=x$ in. and one side $=y$ in.?

If further practice is needed, the reader may solve, by calculation, Ex. 234—239, 242, 244, 247, 249, 255, 256.

Ex. 1053. Given two squares of different sizes, show how to construct a square equal to the sum of the two squares. (Will the side of the new square be equal to the sum of the sides of the old squares?)

Ex. 1054. Construct a square equal to the sum of the squares BD, AG in fig. 187, and measure the side of the resulting square in inches.

Ex. 1055. Given two squares of different sizes, show how to construct a square equal to the difference of the two given squares.

Ex. 1056. Construct a square equal to the difference of the squares BD, AG in fig. 187, and measure the side of the resulting square in inches.

Ex. 1057. Draw three squares of different sizes and construct a square equal to the sum of the three squares. (Begin by adding together two of the squares and then adding in the third.)

Ex. 1058. Make a square to have twice the area of square BD in fig. 187.

Square-roots found graphically. The square on a side of 1 inch is 1 square inch. The square on a side of 2 inches is 4 square inches. What is the side of a square of 2 square inches? Clearly $\sqrt{2}$ inches. Such a square may be constructed by adding together two 1 inch squares. If the side of the resulting square be measured in inches and decimals of an inch, we shall have an approximate numerical value of $\sqrt{2}$.

(*The following exercises are most easily done on inch paper.*)

Ex. 1059. Construct a square of area 2 sq. in. Hence find $\sqrt{2}$ to two places of decimals; check by squaring.

Ex. 1060. Construct a square of area 5 sq. in. (by adding together squares of area 1 and 4 sq. in.). Hence find $\sqrt{5}$; check.

Ex. 1061. As in Ex. 1060 find graphically $\sqrt{10}$, $\sqrt{8}$, $\sqrt{13}$, checking your result in each case.

In the preceding set of exercises a number of square roots have been found graphically. There were, however, gaps in the series, e.g. $\sqrt{3}$ did not appear. The square roots of all integers may be found in succession by the following construction, which is most easily performed on accurate inch paper.

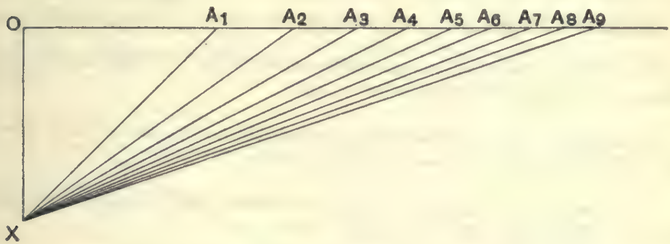


fig. 193.

Draw $OX = 1$ inch and draw at O a line of unlimited length perpendicular to OX. Mark off $OA_1 = OX = 1$. Then $A_1X = \sqrt{2}$.

Mark off $OA_2 = A_1X = \sqrt{2}$.

Then
$$\begin{aligned} A_2X^2 &= OA_2^2 + OX^2 \\ &= (\sqrt{2})^2 + (1)^2 \\ &= 2 + 1 \\ &= 3, \\ \therefore\ A_2X &= \sqrt{3}. \end{aligned}$$

$$\text{Mark off } OA_3 = A_2X = \sqrt{3},$$
$$OA_4 = A_3X,$$
$$OA_5 = A_4X, \text{ \&c.}$$

We now have

$$OA_1 = \sqrt{1},\ OA_2 = \sqrt{2},\ OA_3 = \sqrt{3},\ OA_4 = \sqrt{4},\ OA_5 = \sqrt{5},\ \text{\&c.},$$

and, by measurement, these square roots may be determined.

†Ex. **1062.** Prove I. 15 by means of Pythagoras' theorem.

†Ex. **1063.** AD is the altitude of a triangle ABC. Prove that

$$AB^2 - AC^2 = BD^2 - CD^2.$$

Ex. 1064. In Ex. 1063 let $AB = 3$ in., $AC = 2$ in., $BC = 3$ in. Calculate $BD^2 - CD^2$. Hence find $BD - CD$.

$$[BD^2 - CD^2 = (BD - CD)(BD + CD) = (BD - CD)\,BC.]$$

Knowing $BD - CD$ and $BD + CD$, you may now find BD and CD. Hence find AD. Hence find area of $\triangle ABC$. Verify all your calculations by drawing.

Ex. 1065. Repeat Ex. 1064, taking $AB = 3$ in., $AC = 2$ in., $BC = 4$ in.

†**Ex. 1066.** PQR is a triangle, right-angled at Q. On QR a point S is taken. Prove that $PS^2 + QR^2 = PR^2 + QS^2$.

†**Ex. 1067.** ABC is a triangle, right-angled at A. On AB, AC respectively points X, Y are taken. Prove that $BY^2 + CX^2 = XY^2 + BC^2$.

†**Ex. 1068.** The diagonals of a quadrilateral ABCD intersect at right angles. Show that $AB^2 + CD^2 = BC^2 + DA^2$.

†**Ex. 1069.** O is a point inside a rectangle ABCD. Prove that

$$OA^2 + OC^2 = OB^2 + OD^2.$$

(Draw perpendiculars from O to the sides of the rectangle.)

(*The following 3-dimensional exercises give further practice in the use of Pythagoras' Theorem.*)

Ex. 1069 a. The edges of a certain cuboid (rectangular block) are 3″, 4″, 6″; find the diagonals of the faces.

Ex. 1069 b. A room is 18 ft. long, 14 ft. wide, 10 ft. high. Find the diagonals of the walls. Find the diagonal of the floor.

Ex. 1069 c. Find the length of a string stretched across the room in the preceding exercise, from one corner of the floor to the opposite corner of the ceiling.

Ex. 1069 d. Find the diagonal of the face of a cubic decimetre. Also find the diagonal of the cube.

Ex. 1069 e. Find the slant side of a cone of (i) height 5″, base-radius 3″; (ii) height 4·6 cm., base-radius 7·5 cm.; (iii) height 55 mm., base-diameter 46 mm.

Ex. 1069 f. Find the height of a cone of (i) slant side 10″, base-radius 4″; (ii) slant side 5·8 m., base-diameter 11 m.

Ex. 1069 g. Find the base-radius of a cone of (i) slant side 7 ft., height 5 ft.; (ii) slant side 11·3 cm., height 57 millimetres.

THEOREM 6.†

[CONVERSE OF PYTHAGORAS' THEOREM.]

If a triangle is such that the square on one side is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.

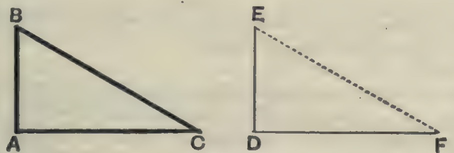


fig. 194.

Data The triangle ABC is such that $BC^2 = AB^2 + AC^2$.

To prove that $\angle BAC$ is a right angle.

Construction Construct a $\triangle DEF$, to have $DE = AB$, $DF = AC$, and $\angle EDF$ a rt. $\angle$.

Proof Since $\angle EDF$ is a right angle, *Constr.*

$$\therefore EF^2 = DE^2 + DF^2$$
$$= AB^2 + AC^2 \qquad \textit{Constr.}$$
$$= BC^2, \qquad \textit{Data}$$
$$\therefore EF = BC.$$

Hence, in the $\triangle$s ABC, DEF,

$$\begin{cases} AB = DE, & \textit{Constr.} \\ AC = DF, & \textit{Constr.} \\ BC = EF, & \textit{Proved} \end{cases}$$

$\therefore$ the triangles are congruent,

$$\therefore \angle BAC = \angle EDF.$$

Now $\angle EDF$ is a right angle, *Constr.*

$\therefore$ $\angle BAC$ is a right angle.

Q. E. D.

Ex. **1070.** Are the triangles right-angled whose sides are

(i) 8, 17, 15; (ii) 12, 36, 34; (iii) 25·5, 25·7, 3·2;

(iv) $4n$, $4n^2-1$, $4n^2+1$; (v) m^2+n^2, m^2-n^2, $2mn$; (vi) a, b, $a+b$?

Ex. **1071.** Bristol is 71 miles due W. of Reading; Reading is 55 miles from Northampton; Northampton is 92 miles from Bristol. Ascertain whether Northampton is due N. of Reading.

Ex. **1072.** Ascertain (i) by measurement and calculation, (ii) by constructing the triangle, whether a right-angled triangle could be made having for sides the lines d, h, k in fig. 8.

Ex. **1073.** Ascertain, by considering the lengths of the sides, whether the triangle of Ex. 821 (i) is right-angled.

Ex. **1074.** Perform, and prove, the following construction for erecting a perpendicular to a given straight line AB at its extremity A. Along AB mark off AC = 3 units. On AC as base construct a triangle ACD, having AD = 4, CD = 5. Then AD is perpendicular to AB. (Ancient Egyptian method.)

ILLUSTRATION OF ALGEBRAICAL IDENTITIES BY MEANS OF GEOMETRICAL FIGURES.

It has been shown that the area of a rectangle 4 inches long and 3 inches broad is 4×3 sq. inches.

In the same way the area of a rectangle a inches long and b inches broad is ab sq. inches.

(*Caution.* Notice carefully the form of the above statement :—area $= 4 \times 3$ sq. inches. Never say, 4 inches $\times$ 3 inches; which is nonsense. It is impossible to multiply *by a length*—such as 3 inches. The statement :—*area of rectangle = length × breadth* is really a convenient but inaccurate way of abbreviating the following statement :—*the number of units of area in a rectangle is equal to the product of the numbers of units of length in the length and breadth of the rectangle.*)

¶Ex. **1075.** What is the area of a rectangle

(i) x cm. long, y cm. broad;
(ii) $2x$ cm. long, $2y$ cm. broad;
(iii) a cm. long, a cm. broad (a square)?

¶Ex. **1076.** What is the area of a square whose side is x inches?

Ex. **1077.** Write out the accurate form of the statement of which the following is a convenient abbreviation :—area of square = square of its side.

¶Ex. **1078.** Find an expression for the area of each of the following rectangles (do not remove the brackets):—

(i) $(a+b)$ inches long, k inches broad;
(ii) $(a+b)$ cm. long, $(c+d)$ cm. broad;
(iii) $(a+b)$ cm. long, $(a-b)$ cm. broad.

¶Ex. **1079.** What is the area of a square whose side is $(a+b)$ inches? Is the answer equal to (a^2+b^2) sq. inches?

¶Ex. **1080.** What is the area of a square whose side is $(a-b)$ inches? Is the answer equal to (a^2-b^2) sq. inches?

¶Ex. **1081.** Simplify the following expressions by removing brackets :—

(i) $(a+b)(c+d)$, (ii) $(a+b)^2$, (iii) $(a-b)^2$,
(iv) $(a+b)^2+(a-b)^2$, (v) $(a+b)^2-(a-b)^2$.

(A) Geometrical illustration of the identity

$$(a+b)\,k \equiv ak + bk.$$

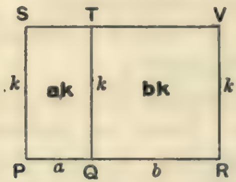


fig. 195.

Let PQ $= a$ units of length, QR $= b$ units of length.

Then PR $= (a+b)$ units of length.

At P, Q, R erect equal perpendiculars PS, QT, RV; the length of each being k units of length.

Then STV is a straight line $\parallel$ to PQR I. 23, *Cor.* and all the figures are rectangles.

Rect. PV $= (a+b)\,k$ units of area.

Rect. PT $= \quad ak$ " " "

Rect. QV $= \quad bk$ " " "

But rect. PV $=$ rect. PT $+$ rect. QV,

$$\therefore (a+b)\,k \equiv ak + bk.$$

¶Ex. **1082.** In the above proof, why would it have been wrong to say, PR $= ab$ units of length, instead of $(a+b)$?

Ex. **1083.** Give geometrical illustrations of the following identities (i.e. draw figures and give explanations):

(i) $(a+b+c)\,k \equiv ak+bk+ck$,

(ii) $(a-b)\,k \equiv ak-bk$,

(iii) $ab \equiv ba$.

(B) **Geometrical illustration of the identity**

$$(a+b)(c+d) \equiv ac+bc+ad+bd.$$

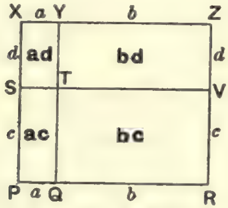


fig. 196.

In the figure, all the angles are right angles and all the figures rectangular.

Also PQ, QR, PS, SX are respectively a, b, c, d units of length.

Then PR $= (a+b)$ units of length, PX $= (c+d)$ units of length.

Rect. PZ $= (a+b)(c+d)$ units of area.
Rect. PT $=$ ac ,, ,, ,,
Rect. QV $=$ bc ,, ,, ,,
Rect. SY $=$ ad ,, ,, ,,
Rect. TZ $=$ bd ,, ,, ,,

But rect. PZ is the sum of rectangles PT, QV, SY, TZ,

$$\therefore (a+b)(c+d) \equiv ac+bc+ad+bd.$$

(C) Geometrical illustration of the identity

$$(a+b)^2 \equiv a^2 + b^2 + 2ab.$$

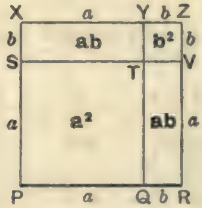


fig. 197.

Let PQ $= a$ units of length, QR $= b$ units of length.

Then PR $= (a+b)$ units of length.

On PR construct the square PRZX.

From PX cut off PS $=$ PQ $= a$ units of length.

Through Q draw QTY $\parallel$ to PX.

Through S draw STV $\parallel$ to PR.

Then all the angles formed are right angles, and all the figures rectangular.

Also PT is a square. (Why?)

TZ is a square $\left\{\begin{array}{l} \text{Again RZ} = (a+b) \text{ units of length,} \\ \quad \text{and RV} = \text{PS} = a \quad \text{,,} \quad \text{,,} \quad \text{,,} \\ \quad \therefore \text{VZ} = \quad b \quad \text{,,} \quad \text{,,} \quad \text{,,} \\ \quad \text{and YZ} = \text{QR} = b \quad \text{,,} \quad \text{,,} \quad \text{,,} \\ \quad \therefore \text{TZ is a square.} \end{array}\right.$

Sq. PZ $= (a+b)^2$ units of area.

Sq. PT $= \quad a^2$,, ,, ,,

Sq. TZ $= \quad b^2$,, ,, ,,

Rect. SY $= \quad ab$,, ,, ,,

Rect. QV $= \quad ab$,, ,, ,,

But sq. PZ $=$ sq. PT $+$ sq. TZ $+$ rect. SY $+$ rect. QV,

$$\therefore\ (a+b)^2 \equiv a^2 + b^2 + 2ab.$$

Ex. 1084. State the above result in words.

Ex. 1085. Prove algebraically that

$$(a+b+c)^2 \equiv a^2+b^2+c^2+2bc+2ca+2ab;$$

also give a geometrical illustration of the identity. (It will be enough to draw a figure, and mark the lengths and areas.)

Ex. 1086. Illustrate the identities (i) $(2x)^2 \equiv 4x^2$, (ii) $(2a)(3b) \equiv 6ab$.

Numerical cases of identities may be illustrated on squared paper. For instance, to illustrate the identity

$$(x+4)(x+6) \equiv x^2+10x+24.$$

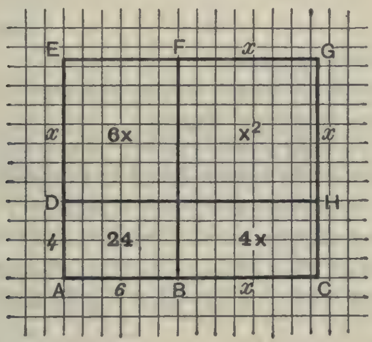


fig. 198.

In fig. 198 AB $= 6$ units of length,

AD $= 4$ units of length,

BC $=$ DE $= x$ units of length (any length).

The numbers inside the rectangles denote the areas. It is now obvious how the figure illustrates the given identity.

Ex. 1087. By means of figures, illustrate the following identities:—

(i) $(x+5)(x+9) \equiv x^2+14x+45$,

(ii) $(y+7)^2 \equiv y^2+14y+49$,

(iii) $5(x+12) \equiv 5x+60$,

(iv) $5(x-12) \equiv 5x-60$ (when $x>12$),

(v) $5(12-x) \equiv 60-5x$ (when $x<12$),

(vi) $a(b+16) \equiv ab+16a$.

(D) Geometrical illustration of the identity

$$(a-b)^2 \equiv a^2 + b^2 - 2ab.$$

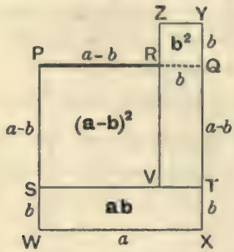


fig. 199.

Let $PQ = a$ units of length.
From PQ cut off the length QR, containing b units.
Then $PR = (a-b)$ units of length.
On PQ construct the square PQXW; its area is a^2 units of area.
On QR construct the square QRZY as in the figure.
The area of this square is b^2 units of area.
Then the whole figure contains $(a^2 + b^2)$ units of area.
From PW cut off $PS = PR = (a-b)$ units of length.
Then $SW = PW - PS = a - (a-b)$ units of length
$= \quad b \quad$ " " "
Through S draw ST ‖ to PQ; produce ZR to meet ST in V.
All the figures so formed are rectangular.
Also figure SR is a square, and contains $(a-b)^2$ units of area.
Rect. WT contains ab units of area.
Lastly, in rect. VY, side $YZ = QR = b$ units of length,
and side $YT = YQ + QT$
$= RQ + PS$
$= b + (a-b)$ units of length
$= \quad a \quad$ " " "
∴ Rect. VY contains ab units of area.
Now sq. SR = whole fig. − rect. WT − rect. VY,

$$\therefore\ (a-b)^2 \equiv (a^2+b^2) - ab - ab$$
$$\equiv a^2 + b^2 - 2ab.$$

Ex. 1086. State the above result in words.

(E) Geometrical illustration of the identity

$$a^2 - b^2 \equiv (a+b)(a-b).$$

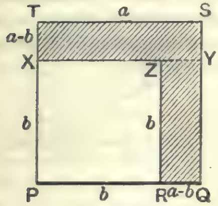


fig. 200.

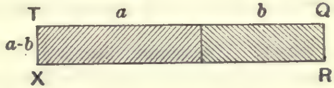


fig. 201.

Let PQ $= a$ units of length.

On PQ construct the square PS; its area is a^2 units of area.

From PQ cut off the length PR, containing b units of length.

From PT cut off PX = PR; through X draw XY || to PQ.

Through R draw RZ || to PT to meet XY in Z.

All the figures so formed are rectangular.

Also PZ is the square on PR; its area is b^2 units of area.

If sq. PZ is subtracted from sq. PS, there remains the shaded part of the figure.

The area of the shaded part is therefore $(a^2 - b^2)$ units of area.

Now this part is composed of the rectangles XS and RY.

These rectangles have the same breadth, namely $(a - b)$ units of length. (Why?)

They might therefore be placed end to end, so as to form a single rectangle (as shown above on the right).

The length of this single rectangle = TS + ZR $= (a + b)$ units of length, and the area of this rectangle $= (a+b)(a-b)$ units of area,

$$\therefore a^2 - b^2 \equiv (a+b)(a-b).$$

Ex. 1089. State the above result in words.

Express each of the following theorems (Ex. 1090—1093) as an algebraical identity; prove the identity.

Ex. **1090. If there are two straight lines, one of which is divided into any number of parts (x, y, z say) while the other is of length a, then the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided straight line and the several parts of the divided line.** (Draw a figure.)

Ex. **1091.** If a straight line is divided into any two parts (x and y), the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts. (Draw a figure.)

Ex. **1092.** If a straight line is divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts. (Draw a figure.)

Ex. **1093. If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.** (Draw a figure.)

Ex. **1094.** What algebraical identity is suggested by fig. 202? (Take $AO=OB=a$, $OP=b$.)

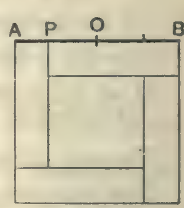


fig. 202.

Ex. **1095.** Express and prove algebraically:—If a straight line is divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole line and the first part.

Ex. **1096.** Prove that the square on the difference of the sides of a right-angled triangle, together with twice the rectangle contained by the sides, is equal to the square on the hypotenuse. (Use Algebra.)

Ex. **1097.** If a straight line AB (length $2x$) is bisected at O and also divided unequally at a point P (distant y from O), what are the lengths of the two unequal parts AP, PB? **Prove algebraically that the rectangle contained by the unequal parts, together with the square on the line between the points of section (OP), is equal to the square on half the original line.**

Ex. **1098.** Show that in the above exercise AO is half the sum of AP, PB; and that OP is half the difference of AP, PB. (Most easily proved by Algebra.)

Ex. **1099.** If a straight line AB (length $2x$) is bisected at O, and produced to any point P ($OP=y$) the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the original line, is equal to the square on the straight line made up of the half and the part produced.

Ex. **1100. If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole and that part, together with the square on the other part.** (Draw figure.)

Ex. **1101.** If a straight line AB is bisected at O and also divided unequally at a point P (as in Ex. 1097), the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section (OP).

Ex. **1102. If a straight line is bisected and produced to any point (as in Ex. 1099), the sum of the squares on the whole line thus produced and on the part produced, is twice the sum of the squares on half the original line, and on the line made up of the half and the part produced.**

Ex. **1103.** Four points A, B, C, D are taken in order on a straight line; prove algebraically that AB . CD + BC . AD = AC . BD. (Take $AB=x$, $BC=y$, $CD=z$.)

Verify numerically.

Ex. **1104.** If a straight line is bisected and also divided unequally (as in Ex. 1097) the squares on the two unequal parts are together equal to twice the rectangle contained by these parts together with four times the square on the line between the points of section.

Projections

Def. If from the extremities of a line AB perpendiculars AM, BN are drawn to a straight line CD, then MN is called the **projection** of AB upon CD (figs. 203, 204).

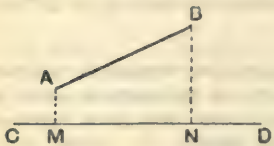


fig. 203.

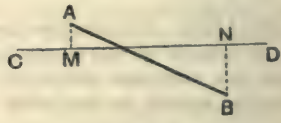


fig. 204.

¶Ex. **1105.** In fig. 189 name the projection of AB upon DE; of AE upon BC; of AC upon AL.

¶Ex. **1106.** In fig. 208 name the projection of AC upon BN; of BC upon NC.

¶Ex. **1107.** (On squared paper.) What is the length of the projection (i) upon the axis of x, (ii) upon the axis of y, of the straight lines whose extremities are the points

(*a*) (2, 3) and (6, 6).
(*b*) (2, 4) and (6, 7).
(*c*) (0, 0) and (4, 3).
(*d*) $(-1, -3)$ and (3, 0).
(*e*) $(-5, 0)$ and $(-1, 3)$.
(*f*) (1, 1) and (5, 1).
(*g*) $(0, -2)$ and (0, 2).

†Ex. **1108. Prove that the projections on the same straight line of equal and parallel straight lines are equal.** (See fig. 205.)

†Ex. **1109.** O is the mid-point of AB; the projections of A, B, O upon any line are P, Q, T. Prove that PT = QT.

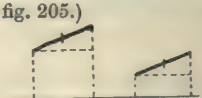


fig. 205.

¶Ex. **1110.** Measure the projection of a line of length 10 cm. when it makes with the line upon which it is projected the following angles:—15°, 30°, 45°, 60°, 75°, 90°. Draw a graph.

¶Ex. **1111.** In what case is the projection of a line equal to the line itself?

¶Ex. **1112.** In what case is the projection of a line zero?

Ex. **1113.** Prove that, if the slope of a line is 60°, its projection is half the line.

[Consider an equilateral triangle.]

Ex. **1114.** A pedestrian first ascends at an angle of 12° for 2000 yards and then descends at an angle of 9° for 1000 yards. How much higher is he than when he started? What horizontal distance has he travelled (i.e. what is the projection of his journey on the horizontal)?

Ex. **1115.** The projections of a line of length l upon two lines at right angles are x, y. Prove that $x^2+y^2=l^2$.

¶Ex. **1116.** How does the projection of a line of given length alter as the slope of the line becomes more and more steep?

NOTE. It may be necessary to produce the line upon which we project, e.g. if required to project AB upon CD in fig. 206, we must produce CD.

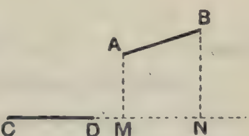


fig. 206.

EXTENSION OF PYTHAGORAS' THEOREM.

BAC, BAC_1, BAC_2 (fig. 207) are triangles respectively right-angled, acute-angled, and obtuse-angled at A.

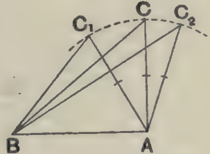


fig. 207.

Also $AC = AC_1 = AC_2$.

By I. 19 $BC_1 < BC$ and $BC_2 > BC$.

Now $BC^2 = CA^2 + AB^2$,

$\therefore$ $BC_1^2 = C_1A^2 + AB^2$ – some area,

and $BC_2^2 = C_2A^2 + AB^2$ + some area.

The precise value of the quantity referred to as "some area" is given in the two following theorems.

THEOREM 7.

In an obtuse-angled triangle, the square on the side opposite to the *obtuse* angle is equal to the sum of the squares on the sides containing the obtuse angle *plus* twice the rectangle contained by one of those sides and the projection on it of the other.

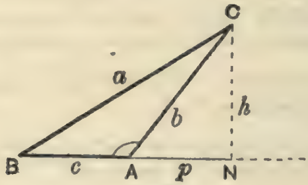


fig. 208.

Data The △ABC has ∠BAC obtuse.

CN is the perpendicular from C upon BA (produced),

∴ AN is the projection of AC upon BA.

Let BC $= a$ units, CA $= b$ units, AB $= c$ units, AN $= p$ units, CN $= h$ units.

To prove that $BC^2 = CA^2 + AB^2 + 2AB \cdot AN$,

i.e. that $a^2 = b^2 + c^2 + 2cp$.

Proof Since △BNC is right-angled,

$\therefore BC^2 = BN^2 + NC^2$, *Pythagoras*

i.e. $a^2 = (c+p)^2 + h^2$

$= c^2 + 2cp + p^2 + h^2$.

But △ANC is right-angled,

$\therefore p^2 + h^2 = b^2$, *Pythagoras*

$\therefore a^2 = c^2 + 2cp + b^2$,

i.e. $BC^2 = AB^2 + 2AB \cdot AN + AC^2$.

Q. E. D.

THEOREM 8.

In any triangle, the square on the side opposite to an *acute* angle is equal to the sum of the squares on the sides containing that acute angle *minus* twice the rectangle contained by one of those sides and the projection on it of the other.

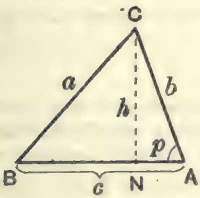


fig. 209.

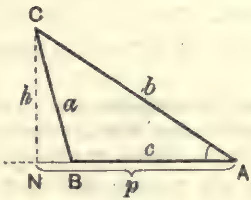


fig. 210.

Data The $\triangle$ ABC has $\angle$ BAC acute.

CN is the perpendicular from C upon AB (or AB produced),

$\therefore$ AN is the projection of AC upon AB.

Let BC $= a$ units, CA $= b$ units, AB $= c$ units, AN $= p$ units, CN $= h$ units.

To prove that $BC^2 = CA^2 + AB^2 - 2AB \cdot AN$,

i.e. that $a^2 = b^2 + c^2 - 2cp$.

Proof Since $\triangle$ BNC is right-angled,

$\therefore BC^2 = BN^2 + NC^2$, *Pythagoras*

i.e. in fig. 209, $a^2 = (c-p)^2 + h^2$,

in fig. 210, $a^2 = (p-c)^2 + h^2$,

$\therefore$ in both figures,

$a^2 = c^2 - 2cp + p^2 + h^2$.

But $\triangle$ ANC is right-angled,

$\therefore p^2 + h^2 = b^2$, *Pythagoras*

$\therefore a^2 = c^2 - 2cp + b^2$,

i.e. $BC^2 = AB^2 - 2AB \cdot AN + AC^2$.

Q. E. D.

†Ex. **1117.** Write out the proof of II. 8 for the case in which ∠ B is a right angle. What does the theorem become?

Ex. **1118.** Verify the truth of II. 7, 8 by drawing and measurement.

Ex. **1119.** What is the area of the rectangle referred to in the enunciation of II. 7, 8 for the following cases:—

(i) $c=5$ cm., $b=4$ cm., ∠ BAC$=120°$ (by drawing);

(ii) $c=5$ cm., $b=4$ cm., ∠ BAC$=60°$ (by drawing);

(iii) $c=3$ in., $b=2$ in., $a=4$ in. (by calculation; check by drawing);

(iv) $c=3$ in., $b=2$ in., $a=2$ in. („ „)?

Ex. **1120.** By comparing the square on one side with the sum of the squares on the two other sides, determine whether triangles having the following sides are acute-, obtuse-, or right-angled (check by drawing):—

(i) 3, 4, 6; (ii) 3, 4, 3; (iii) 2, 3, 5; (iv) 2, 3, 4; (v) 12, 13, 5.

Ex. **1121.** Given four sticks of lengths 2, 3, 4, 5 feet, how many triangles can be made by using three sticks at a time? Find out whether each triangle is acute-, obtuse-, or right-angled.

Ex. **1122.** Calculate BC when

AB$=10$ cm., AC$=8$ cm., ∠ A$=60°$. (See Ex. 1118.)

Ex. **1123.** Calculate BC when

AB$=10$ cm., AC$=8$ cm., ∠ A$=120°$.

Ex. **1124.** Bristol is 26 miles E. of Cardiff; Reading is 70 miles E. of Bristol; Naseby is due N. of Reading and 95 miles from Bristol. Calculate the distance from Cardiff to Naseby, and check by measurement.

Ex. **1125.** Brighton is 48 miles S. of London; Hertford is 20 miles N. of London; Shoeburyness is due E. of London, and 64 miles from Brighton. How far is it from Hertford? Verify graphically.

Revise Ex. 256.

¶Ex. **1126.** Suppose that ∠A in fig. 208 becomes larger and larger till BAC is a straight line. What does II. 7 become in this case?

¶Ex. **1127.** Suppose that $\angle A$ in fig. 209 becomes smaller and smaller till C is on BA. What does II. 8 become in this case?

†Ex. **1128.** In the trapezium ABCD (fig. 211), prove that $AC^2 + BD^2 = AD^2 + BC^2 + 2AB \cdot CD$.

(Apply II. 9 to $\triangle^s$ ACD and BCD.)

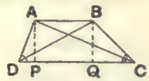


fig. 211.

†Ex. **1129.** D is a point on the base BC of an isosceles $\triangle$ ABC. Prove that $AB^2 = AD^2 + BD \cdot CD$.

(Let O be mid-point of BC, and suppose that D lies between B and O. Then

$$BD = BO - OD, \quad CD = CO + OD = BO + OD.)$$

†Ex. **1130.** ABC is an isosceles $\triangle$ $(AB = AC)$; BN is an altitude. Prove that $2AC \cdot CN = BC^2$.

†Ex. **1131.** BE, CF are altitudes of an acute-angled $\triangle$ ABC. Prove that $AE \cdot AC = AF \cdot AB$.

(Write down two different expressions for BC^2.)

†Ex. **1132.** In the figure of Ex. 1131, $BC^2 = AB \cdot FB + AC \cdot EC$.

†Ex. **1133.** **The sum of the squares on the two sides of a triangle ABC is equal to twice the sum of the squares on the median AD, and half the base. (Apollonius' theorem.)**

(Draw AN $\perp$ to BC; apply II. 7, 8 to $\triangle^s$ ABD, ACD.)

Ex. **1134.** Use Apollonius' theorem to calculate the lengths of the three medians in a triangle whose sides are 4, 6, 7.

Ex. **1135.** Repeat Ex. 1134, with sides 4, 5, 7.

Ex. **1136.** Calculate the base of a triangle whose sides are 8 cm. and 16 cm., and whose median is 12 cm. Verify graphically.

Revise Ex. 246.

†Ex. **1137.** The base BC of an isosceles $\triangle$ ABC is produced to D, so that $CD = BC$; prove that $AD^2 = AC^2 + 2BC^2$.

†Ex. 1138. A side PR of an isosceles $\triangle$ PQR is produced to S so that $RS = PR$: prove that $QS^2 = 2QR^2 + PR^2$.

†Ex. 1139. The base AD of a triangle OAD is trisected in B, C. Prove that $OA^2 + 2OD^2 = 3OC^2 + 6CD^2$.

(Apply Apollonius' theorem to $\triangle^s$ OAC, OBD ; then eliminate OB^2.)

†Ex. 1140. In the figure of Ex. 1139, $OA^2 + OD^2 = OB^2 + OC^2 + 4BC^2$.

†Ex. 1141. A point moves so that the sum of the squares of its distances from two fixed points A, B remains constant; prove that its locus is a circle, having for centre the mid-point of AB.

†Ex. 1142. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

†Ex. 1143. In any quadrilateral the sum of the squares on the four sides exceeds the sum of the squares on the diagonals by four times the square on the straight line joining the mid-points of the diagonals.

(Let E, F be the mid-points of AC, BD ; apply Apollonius' theorem to $\triangle^s$ BAD, BCD and AFC.)

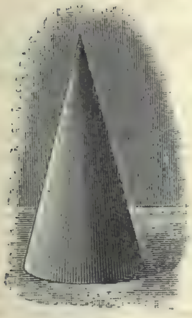
†Ex. 1144. The sum of the squares on the diagonals of a quadrilateral is equal to twice the sum of the squares on the lines joining the mid-points of opposite sides. (See Ex. 736 and 1142.)

†Ex. 1145. In a triangle, three times the sum of the squares on the sides = four times the sum of the squares on the medians.

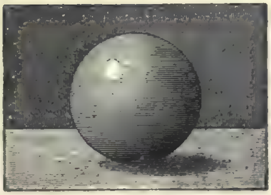
¶Ex. 1146. What does Apollonius' theorem become if the vertex moves down (i) on to the base, (ii) on to the base produced?

BOOK III.

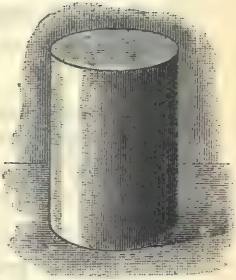
THE CIRCLE.



CONE.



SPHERE.



CYLINDER.

SECTION I. PRELIMINARY.

DEF. A **circle** is a line, lying in a plane, such that all points in the line are equidistant from a certain fixed point, called the **centre** of the circle.

In view of what has been said already about loci we may give the following alternative definition of a circle :—

DEF. A **circle** is the locus of points in a plane that lie at a fixed distance from a fixed point (the centre). The fixed distance is called the **radius** of the circle.

The word "circle" has been defined above to mean a certain kind of curved line. The term is, however, often used to indicate the part of the plane inside this line. If any doubt exists as to the meaning, the line is called the **circumference** of the circle.

Two circles are said to be **equal** if they have equal radii.

If one of two equal circles is applied to the other so that the centres coincide, then the circumferences also will coincide.

Point and circle. A point may be either outside a circle, on the circle or inside the circle. The point will lie outside the circle if its distance from the centre > the radius; it will lie on the circle if its distance = the radius; it will lie inside the circle if the distance < the radius.

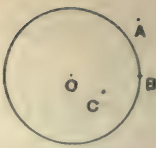


fig. 212.

Straight line and circle. A straight line cannot cut a circle in more than two points. In fact, an unlimited straight line may

(i) cut a circle in two points, e.g. AB or CD in fig. 213. In this case the part of the line which lies inside the circle is called a **chord** of the circle.

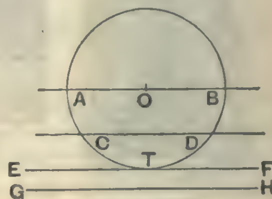


fig. 213.

(ii) The line may meet the circle in one point only; thus EF meets the circle in T. In this case the line is said to **touch** the circle; it is called a **tangent**; T is called the **point of contact** of the tangent.

The tangent lies entirely outside the circle and has one point, and one only, in common with the circle. It is obvious that there is one and only one tangent which touches the circle at a given point.

(iii) The line may lie entirely outside the circle, and have no point in common with the circle, e.g. GH in fig. 213.

A chord may be said to be the straight line joining two points on a circle. If the chord passes through the centre it is called a **diameter,** e.g. AOB in fig. 213.

The length of a diameter is twice the length of the radius; all diameters are equal.

A chord divides the circumference into two parts called **arcs.** If the arcs are unequal, the less is called the **minor** arc and the greater the **major** arc.

Three letters are needed to name an arc completely; e.g. in fig. 213, CTD is a minor arc, CBD a major arc.

A diameter divides the circumference into two equal arcs, each of which is called a **semicircle.**

It will be proved below that the two semicircles are equal.

The term "semicircle" like the term "circle" is used in two different senses; sometimes in the sense of an arc (as in the definition); sometimes as the part of the plane bounded by a semi-circumference and the corresponding diameter.

A **segment** of a circle is the part of the plane bounded by an arc and its chord (fig. 214). A **sector** of a circle is the part of the plane bounded by two radii, and the arc which they intercept (fig. 214).

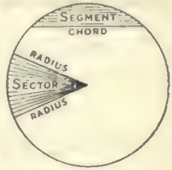


fig. 214.

¶Ex. **1147.** A circular hoop is cut into two pieces; what is each called?

¶Ex. **1148.** A penny is cut into two pieces by a straight cut; what is the shape of each piece?

¶Ex. **1149.** What geometrical figure has the shape of an open fan?

¶Ex. **1150.** A certain gun in a fort has a range of 5 miles, and can be pointed in any direction from 15° E. of N. to 15° W. of N. What is the shape of the area commanded by the gun?

Section II. Chord and Centre.

Symmetry of the circle. From what has been said about symmetry (Ex. 277 onwards) it will be seen that the circle is symmetrical about any diameter, and is also symmetrical about the centre.

¶Ex. **1151.** Draw a circle of about 3 in. radius; draw a set of parallel chords (about 10); bisect each chord by eye. What is the locus of the mid-points of the chords? (*Freehand.*)

¶Ex. **1152.** Draw a circle and a diameter. This is an axis of symmetry. Mark four pairs of corresponding points. Is there any case in which a pair of corresponding points coincide? (*Freehand.*)

¶Ex. **1153.** What symmetry is possessed by (i) a sector, (ii) a segment, (iii) an arc, of a circle?

Theorem 1.

A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord;

Conversely, **the perpendicular to a chord from the centre bisects the chord.**

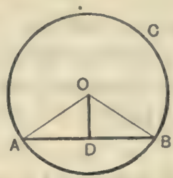


fig. 215.

(1) *Data* OD is a straight line joining O, the centre of ⊙ABC, to D, the mid-point of the chord AB.

To prove that OD is ⊥ to AB.

Construction Join OA, OB.

Proof In the △s OAD, OBD

$\left\{\begin{array}{l} OA = OB \text{ (radii)}, \\ OD \text{ is common}, \\ AD = BD. \end{array}\right.$ *Data*

∴ the triangles are congruent, I. 14.

∴ ∠ODA = ∠ODB,

∴ OD is ⊥ to AB.

(2) Converse Theorem.

Data OD is a straight line drawn from O, the centre of ⊙ABC, to meet the chord AB at right angles in D.

To prove that AD = BD.

Construction Join OA, OB.

Proof In the right-angled $\triangle$s OAD, OBD

$$\left\{\begin{array}{l} \angle\text{s ODA, ODB are rt. } \angle\text{s}, \\ \text{OA} = \text{OB (radii)}, \\ \text{OD is common}, \end{array}\right. \qquad \textit{Data}$$

$\therefore$ the triangles are congruent, I. 15.

$\therefore$ AD = BD.

Q. E. D.

COR. A straight line drawn through the mid-point of a chord of a circle at right angles to the chord will, if produced, pass through the centre of the circle.

(For only one perpendicular can be drawn to a given line at a given point in it.)

To find the centre of a given circle.

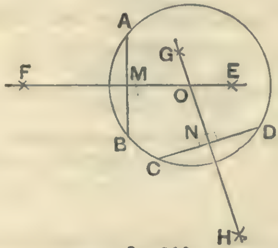


fig. 216.

Construction Draw any two chords AB, CD (not parallel).
Draw EMF to bisect AB at right angles,
and GNH to bisect CD at right angles.
Let these straight lines meet at O.
Then O is the centre of the circle.

Proof Since EMF bisects chord AB at right angles
∴ the centre must lie somewhere on EMF. I. 25.
Similarly the centre must lie somewhere on GNH.
Hence the centre is at O, the point of intersection of EMF and GNH.

¶Ex. **1154.** Why is it necessary that the chords AB, CD should not be parallel?

To complete a circle of which an arc is given.

Find the centre of the circle as in the preceding construction.

Ex. **1155.** With a fine-pointed pencil trace round part of the edge of a penny, so as to obtain an arc of a circle. (Take care to keep the pencil perpendicular to the paper.) Complete the circle by finding the centre.

Ex. **1156.** By the method described in Ex. 1155, examine how far the curved edge of your protractor differs from a true semicircle.

¶Ex. **1157.** Describe five circles (in the same figure) to pass through two given points A, B, 6 cm. apart. (The centre must be equidistant from A and B; what is the locus of points equidistant from A and B?)

Ex. **1158.** Describe a circle to pass through two given points A, B, 6 cm. apart, and to have a radius of 5 cm. Measure the distance of the centre from AB.

¶Ex. **1159.** What is the locus of the centres of circles which pass through two given points?

THEOREM 2.

There is one circle, and one only, which passes through three given points not in a straight line.

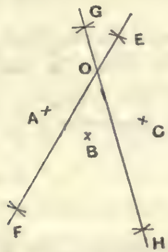


fig. 217.

Data A, B, C are three points not in a straight line.

To prove that one circle, and one only, can be drawn to pass through A, B and C.

Proof It is only necessary to show that there is one point (and one only) equidistant from A, B, and C.

Now the locus of all points equidistant from A and B is FE, the perpendicular bisector of AB; I. 25.
and the locus of all points equidistant from B and C is HG, the perpendicular bisector of BC. I. 25.

These bisectors, not being parallel, will intersect.

Let the point of intersection be O.

The point O is equidistant from A and B; also from B and C;

∴ O is equidistant from A, B and C;
and there is no other point equidistant from A, B and C.

Hence a circle with centre O and radius OA will pass through A, B and C;
and there is no other circle passing through A, B and C.

Q. E. D.

COR. 1. **Two circles cannot intersect in more than two points.**

For if the two circles have three points in common, they have the same centre and radius, and therefore coincide.

COR. 2. **The perpendicular bisectors of AB, BC, and CA meet in a point.**

¶**Ex. 1160.** How would the proof of III. 2 fail if A, B, C were in a straight line?

Ex. 1161. Prove Cor. 2. (Let two of the bisectors meet at a point O; then prove that O lies on the third bisector.)

DEF. If a circle passes through all the vertices of a polygon, the circle is said to be **circumscribed** about the polygon; and the polygon is said to be **inscribed** in the circle (fig. 218).

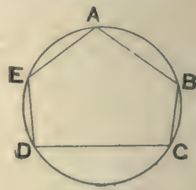


fig. 218.

DEF. If a circle touches all the sides of a polygon, the circle is said to be **inscribed** in the polygon; and the polygon is said to be **circumscribed** about the circle (fig. 219).

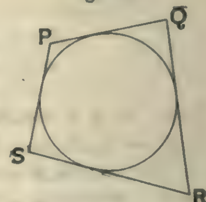


fig. 219.

To circumscribe a circle about a given triangle.

This is the same problem as that of describing a circle to pass through three given points, namely the three vertices of the triangle (see III. 2).

DEF. The centre of the circle circumscribed about a triangle is called the **circumcentre** of the triangle.

Notice that, though the perpendicular bisectors of all three sides pass through the circumcentre, yet it is not necessary to draw more than *two* of these bisectors in order to find the centre.

Ex. 1162. (Inch paper.) Draw a circle to pass through the points (0, 3), (2, 0), (−1, 0), and measure its radius.

Does this circle pass through

(i) (0, −3), (ii) (1, 3), (iii) (0, $-\frac{2}{3}$)?

Ex. 1163. (Inch paper.) Draw the circumcircle of the triangle whose vertices are (0, 2), (4, 0), (−1, 0), and find its radius.

Does this circle pass through

(i) (0, −2), (ii) (0, −3), (iii) (1·5, 3)?

Ex. **1164.** (Inch paper.) Find the circumradius, and the coordinates of the circumcentre of $(0, 1)$, $(3, 0)$, $(-3, 0)$.

Ex. **1165.** (Inch paper.) Find the circumradius, and the coordinates of the circumcentre of each of the triangles in Ex. 821 (i), (ii), (iii).

Ex. **1165 a.** Find the circumradii of $\triangle^s$ (i)—(vi) in Ex. 942.

Ex. **1166.** Mark four points (at random) on plain paper, and find out whether it is possible to draw a circle through all four.

Ex. **1167.** (Inch paper.) Can a circle be drawn through the four points

(i) $(2, 0)$, $(0, 2)$, $(-2, 0)$, $(0, -2)$;

(ii) $(2, 0)$, $(0, 1)$, $(-2, 0)$, $(0, -1)$;

(iii) $(2, 0)$, $(0, 2)$, $(-2, 0)$, $(0, 1)$?

¶Ex. **1168.** Can a circle be circumscribed about a rectangle?

¶Ex. **1169.** Draw a parallelogram (not rectangular) and try if a circle can be circumscribed about it.

Ex. **1170.** Draw an acute-angled scalene triangle ABC (no side to be less than 3 inches). Draw the circumscribing circle. Find P, Q, R the middle points of the sides. Draw the circle which passes through P, Q, R. Find the ratio of the radius of the greater circle to that of the less;

i.e. $$\frac{\text{greater radius}}{\text{smaller radius}}.$$

Ex. **1171.** Repeat Ex. 1170 with a right-angled triangle.

Ex. **1172.** Draw a scalene triangle, and on its three sides construct equilateral triangles, pointing outwards. Draw the circumcircles of these equilateral triangles; they should all pass through a certain point inside the triangle.

Ex. **1173.** Draw four straight lines, such that each line meets the three other lines. Four triangles are thus formed. Draw the circumcircles of these triangles; they should meet in a point.

†Ex. **1174.** If a chord cuts two concentric circles in A, B; C, D, then $AC = BD$. (Draw perpendicular from centre on to chord.)

†Ex. **1175.** From a point O outside a circle two equal lines OP, OQ are drawn to the circumference. Prove that the bisector of $\angle$ POQ passes through the centre of the circle. (Join PQ.)

Ex. **1176.** O is a point 4 inches from the centre of a circle of radius 2 inches. Show how to construct with O as vertex an isosceles triangle having for base a chord of the circle, and a vertical angle of 50°.

(Freehand)

Ex. **1177.** If a polygon is such that the perpendicular bisectors of all the sides meet in a point, a circle can be circumscribed round the polygon.

Section III*. Arcs, Angles, Chords

Theorem 3.

In equal circles (or, in the same circle)

(1) if two arcs subtend equal angles at the centres, they are equal.

(2) *Conversely*, if two arcs are equal, they subtend equal angles at the centres.

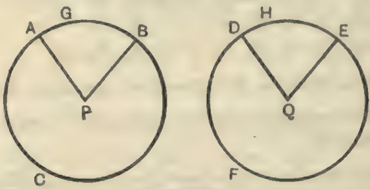


fig. 220.

(1) *Data* ABC, DEF are equal ⊙s.

The arcs AGB, DHE subtend equal ∠s APB, DQE at the centres P, Q.

To prove that arc AGB = arc DHE.

Proof Apply ⊙DEF to ⊙ABC, so that centre Q may fall on centre P.

Since the ⊙s are equal, the circumference of ⊙DEF falls on the circumference of ⊙ABC.

Make ⊙DEF revolve about the centre till QD falls along PA.

Then, since ∠DQE = ∠APB . . . . . . *Data*

QE falls along PB, and since the circumferences coincide, D coincides with A, and E with B.

∴ arc DHE coincides with arc AGB.

∴ arc DHE = arc AGB.

(2) Converse Theorem.

Data arc AGB = arc DHE.

To prove that ∠s APB, DQE, subtended by these arcs at the centres, are equal.

* This section may be omitted at first reading, with the exception of Theorem 5 and the exercises which follow (pp. 235—237).

Proof Apply ⊙DEF to ⊙ABC, so that centre Q may fall on centre P.

Since the ⊙s are equal, the circumference of ⊙DEF falls on the circumference of ⊙ABC.

Make ⊙DEF revolve about the centre till D coincides with A.

Then, since arc DHE = arc AGB *Data*

E coincides with B.

∴ QD coincides with PA, and QE with PB,

∴ ∠DQE = ∠APB.

Q. E. D.

Cor. Equal angles at the centre determine equal sectors.

Note on the case of "the same circle."

The above proposition is proved for *equal* circles. To see that it applies to arcs and angles in the *same* circle, let the arcs AB, PQ of circle ABPQ (fig. 221 i.) subtend equal angles AOB, POQ at the centre. To prove that arc AB = arc PQ.

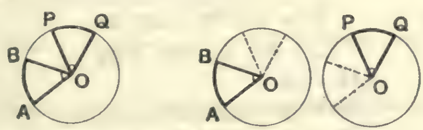


fig. 221 i. ii. iii.

Fig. i. may be regarded as consisting of the two circles in figs. ii., iii. superposed. But these are equal ⊙s,

∴ arc AB = arc PQ.

†Ex. **1178.** Show how to bisect a given arc of a circle. Give a proof.

†Ex. **1179.** P, A, B are points on a circle whose centre is O; PA = PB. Prove that P is the mid-point of arc AB; and that OP bisects AB.

†Ex. **1180.** PQ, PR are a chord and a diameter meeting at a point P in the circumference. Prove that the radius drawn parallel to PQ bisects the arc QR.

†Ex. **1181.** P is a point on the circumference equidistant from the radii OA, OB. Prove that arc AP = arc BP.

Theorem 4.

In equal circles (or, in the same circle)

(1) if two chords are equal, they cut off equal arcs.

(2) *Conversely*, if two arcs are equal, the chords of the arcs are equal.

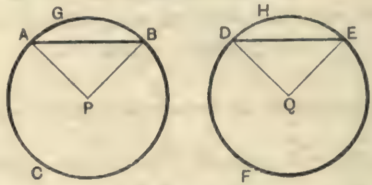


fig. 222.

(1) *Data* ABC, DEF are equal ⊙s; their centres are P and Q.
Chord AB = chord DE.

To prove that arc AGB = arc DHE, and arc ACB = arc DFE.

Construction Join PA, PB; QD, QE.

Proof In the △s APB, DQE

$$\begin{cases} AB = DE & \textit{Data} \\ AP = DQ \text{ (radii of equal ⊙s)} & \\ BP = EQ \text{ (radii of equal ⊙s)} & \end{cases}$$

∴ the triangles are congruent, I. 14.

∴ ∠ APB = ∠ DQE,

∴ arc AGB = arc DHE, III. 3.

Again, whole circumference of ⊙ ABC = whole circumference of ⊙ DEF.

∴ the remaining arc ACB = the remaining arc DFE.

(2) Converse Theorem.

Data arc AGB = arc DHE.

To prove that chord AB = chord DE.

Construction Join PA, PB; QD, QE.

Proof Since arc AGB = arc DHE, *Data*

$\therefore \angle APB = \angle DQE$ III. 3.

$\therefore$ in the $\triangle$s APB, DQE

$$\begin{cases} AP = DQ, \\ BP = EQ, \\ \angle APB = \angle DQE. \end{cases}$$

$\therefore$ the triangles are congruent, I. 10.

$\therefore$ chord AB = chord DE.

Q. E. D.

†Ex. **1182.** A quadrilateral ABCD is inscribed in a circle, and AB = CD. Prove that AC = BD.

†Ex. **1183.** Prove the converse, in III. 4, by superposition. Also try to prove the direct theorem by superposition, and point out where such a proof fails.

To place in a circle a chord of given length.

Adjust the compasses to the given length. With a point A on the circle as centre draw an arc cutting the circle in B. Then AB will be the chord required.

Ex. **1184.** Place in a circle, end to end, 6 chords each equal to the radius.

Ex. **1185.** Place in a circle, end to end, 12 chords each equal to $\frac{1}{2}$ the radius.

Ex. **1186.** Draw a circle of radius 5 cm. Place in the circle a number of chords of length 8 cm. Plot the locus of their middle points.

Ex. **1187.** Show how to construct an isosceles triangle, given that the base is 7 cm. and the radius of the circumscribing circle is 5 cm. (Which will you draw first—the base, or the circle?)

Ex. **1188.** Construct a triangle, given BC = 3 in., $\angle B = 30°$, radius of circumscribing circle = 2 ins. Measure AC and $\angle A$.

†Ex. **1189.** In a circle are placed, end to end, equal chords PQ, QR, RS, ST. Prove that PR = QS = RT.

To inscribe a regular hexagon in a circle.

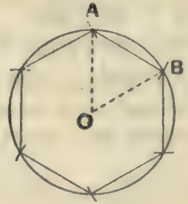


fig. 223.

In the circle place a chord AB, equal to the radius.

Join A, B to O, the centre.

Then $\triangle$ OAB is equilateral,

$\therefore \angle AOB = 60°$.

Place end to end in the circle 6 chords each equal to the radius.

Each chord subtends 60° at the centre,

$\therefore$ the total angle subtended by the 6 chords is 360°.

In other words, the 6 chords form a closed hexagon inscribed in the circle.

Since each side of the hexagon = the radius,

the hexagon is equilateral;

and since each angle of the hexagon = 120°, the hexagon is equiangular,

$\therefore$ the hexagon is regular.

†Ex. **1190.** The side of an isosceles triangle of vertical angle 120° is equal to the radius of the circumcircle.

Ex. **1191.** Find the area of a regular 6-gon inscribed in a circle of radius 2 in.

Revise "Regular polygons," Ex. 69—74.

Ex. **1192.** Find the perimeter and area of a regular 8-gon inscribed in a circle of radius 2 in.

Circumference of Circle.

Consider any circular object, such as a penny, a round tin, a garden-roller, a bucket, a running track. Measure the circumference and the diameter; how many times does the circumference contain the diameter? Work out your answer to three significant figures

Methods of measuring the circumference:—

(i) Put a small spot of ink on the edge of a penny; roll the penny along a sheet of paper, and measure the distance between the ink spots left on the paper.

(ii) Wrap a piece of paper tightly round a cylinder; prick through two thicknesses of the paper; unroll the paper and measure the distance between the pin-holes.

(iii) Wrap cotton round a cylinder several times, say 10 times; measure the length of cotton used, and divide by 10.

In measuring the diameter, make sure that you are measuring the greatest width.

Ex. **1198.** Find the value of the quotient $\dfrac{\text{circumference}}{\text{diameter}}$ for several circular objects of different sizes, and take the average of your answers.

Theory shows that the value of this quotient (or *ratio*) is the same for all circles; it has been worked out to 700 places of decimals and begins thus

$$3{\cdot}1415926535\ldots\ldots$$

For the sake of brevity this number is denoted by the Greek letter π; a useful approximation for π is $\frac{22}{7}$.

The ratios of the perimeters (or circumferences) of regular polygons to the diameters of their circumscribing circles are shown in the following table:—

Table showing the perimeters of regular polygons inscribed in a circle of radius 5 *cm.*

No. of sides	Perimeter in centimetres	Ratio of perimeter to diameter
3	25·98	2·598
4	28·29	2·829
5	29·39	2·939
6	30·00	3·000
7	30·38	3·038
8	30·61	3·061
9	30·78	3·078
10	30·90	3·090

It will be noticed that the ratio increases with the number of sides, being always less than π. If the number of sides is very great, the ratio is very nearly equal to π. E.g. for a polygon of 384 sides the ratio is 3·14156.......

Ex. **1194.** By how much per cent. does the perimeter of a regular decagon inscribed in a circle differ from the perimeter of the circle?

We have seen that

$$\text{circumference of circle} = \text{diameter} \times \pi$$
$$= \text{radius} \times 2\pi$$
$$= 2\pi r, \text{ where } r \text{ is the radius.}$$

Ex. **1195.** Calculate the circumference of a circle whose radius is (i) 7 in., (ii) 14 cm., (iii) 35 miles. (Take $\pi = \frac{22}{7}$.)

Ex. **1196.** Calculate the circumference of a circle whose diameter is (i) 70 ft., (ii) 21 mm., (iii) 49 miles.

Ex. 1197. Calculate to three significant figures the circumference of a half-penny (diameter 1 inch).

Ex. 1198. Calculate to three significant figures the circumference of the earth, measured round the equator, taking radius = 3963 miles.

Ex. 1199. Calculate to three significant figures the circumference of a circle whose radius is 5 cm., and compare your result with the perimeters of regular polygons in the table on page 232.

Ex. 1200. How far does a wheel roll in one revolution if its diameter is 28 in.?

Ex. 1201. In fig. 224, AD is divided into three equal parts and all the arcs are semicircles; show that the four curved lines which connect A with D are of equal length.

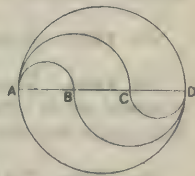


fig. 224.

Ex. 1202. The driving-wheel of an engine is 6 ft. high, and each of the smaller wheels is 3·5 ft. high; how many turns does a small wheel make for one turn of the large wheel?

Ex. 1203. Calculate the radius of a circle of circumference (i) 22 ft., (ii) 40 ft. (to three significant figures).

Ex. 1204. A bicycle wheel makes 7200 turns in an hour while the cyclist is riding 10 miles an hour: what is the diameter of the wheel (to the nearest inch)?

Ex. 1205. Calculate the circumference of a circle whose diameter is (i) 4·35 in., (ii) 617 mm.

Ex. 1206. Calculate the circumference of a circle whose radius is (i) 0·346 yards, (ii) 21·7 in.

Ex. 1207. Calculate the radius of a circle whose circumference is (i) 478 miles, (ii) 27·5 ft.

†**Ex. 1208.** Prove that the circumference of a circle is > three times the diameter, by inscribing a hexagon in the circle.

†**Ex. 1209.** Prove that the circumference is < four times the diameter by circumscribing a square round the circle.

DEF. If an arc of a circle subtends, say, 35° at the centre, it is called **an arc of** 35°.

Ex. 1210. What fractions of a circumference are arcs of 90°, 60°, 120°, 1°, 35°, 300°?

Ex. **1211.** Calculate the length of an arc of 60° in a circle of radius 7 cm. What is the length of the chord of this arc? Find, to three significant figures, the ratio $\frac{\text{arc}}{\text{chord}}$: also the difference of arc and chord.

Ex. **1212.** Repeat Ex. 1211 for a circle of radius 2·57 in.

Ex. **1213.** Draw a circle of any radius; mark an arc of 40°; calculate the length of the arc, and measure the chord; then find ratio $\frac{\text{arc}}{\text{chord}}$ to three significant figures.

Ex. **1214.** Repeat Ex. 1213, with an arc of 80°.

Ex. **1215.** The circumference of a circle is 7·82 in. and the length of a certain arc is 1·25 in. What decimal of the circumference is the arc? What angle does the arc subtend at the centre?

Ex. **1216.** The radius of a circle is 10 cm.; a piece of string as long as the radius is laid along an arc of the circle; what angle does it subtend at the centre? Also find the angle subtended at the centre by a *chord* of 10 cm.

Ex. **1217.** In a circle of radius 3 in., what is the chord of an arc of 6 in.? (Calculate the angle at the centre; then draw the figure and measure.)

Ex. **1218.** Draw an arc of a circle (any radius and angle). Calculate its length, and test the accuracy of the following *approximate* rule:—"To find the length of an arc, from eight times the chord of half the arc subtract the chord of the whole arc, and divide the result by three." (It will be necessary to *measure* the length of the chords.)

Ex. **1219.** Find the length of the minor and major arcs cut off from a circle of radius 7 cm. by a chord of 7 cm.

Ex. **1220.** Find the lengths of the two arcs cut from a circle of diameter 4·37 in. by a chord of 4 in. (Measure the angle at the centre.)

THEOREM 5.

In equal circles (or, in the same circle)

(1) equal chords are equidistant from the centres.

(2) *Conversely*, chords that are equidistant from the centres are equal.

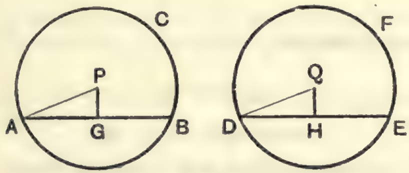


fig. 225.

(1) *Data* ABC, DEF are equal circles; their centres are P and Q.
Chord AB = chord DE.
PG, QH are perpendiculars from the centres P, Q upon the chords AB, DE.

To prove that PG = QH.

Construction Join PA, QD.

Proof Since PG is $\perp$ to AB,

$\therefore$ AG = BG, III. 1.

$\therefore$ AG = $\frac{1}{2}$AB.

Sim^{ly} DH = $\frac{1}{2}$DE.

But AB = DE, *Data*

$\therefore$ AG = DH.

In the right-angled $\triangle^s$ APG, DQH,

$\angle^s$ G and H are rt. $\angle^s$, *Constr.*
AP = DQ, *Data*
AG = DH, *Proved*

$\therefore$ the triangles are congruent, I. 15.

$\therefore$ PG = QH.

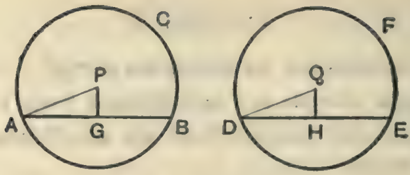


fig. 225.

(2) Converse Theorem.

Data $PG = QH$.

To prove that chord $AB =$ chord DE.

Proof In the right-angled $\triangle^s$ APG, DQH,

$$\left\{\begin{array}{ll} \angle^s\ G \text{ and } H \text{ are rt. } \angle^s, & \textit{Constr.} \\ AP = DQ, & \textit{Data} \\ PG = QH, & \textit{Data} \end{array}\right.$$

$\therefore$ the triangles are congruent, I. 15.

$\therefore AG = DH$.

But $AB = 2AG$, $DE = 2DH$,

$\therefore AB = DE$.

Q. E. D.

†**Ex. 1221.** Prove III. 5 by means of Pythagoras' theorem.

Ex. 1222. Calculate the distances from the centre of a circle (radius 5 cm.) of chords whose lengths are (i) 8 cm., (ii) 6 cm., (iii) 5 cm.

Ex. 1223. Calculate the lengths of chords of a circle (radius 2·5 in.) whose distances from the centre are (i) 2 in., (ii) 1·5 in., (iii) 1 in.

Ex. 1224. Find the locus of the mid-points of chords 6 cm. in length in a circle of radius 5 cm.

†Ex. **1225.** Prove that the locus of the middle points of a set of equal chords of a circle is a concentric circle.

Ex. **1226.** A chord CD of a circle, whose centre is O, is bisected at N by a diameter AB. $OA = OB = 5$ cm., $ON = 4$ cm. Calculate CD, CA, CB.

Ex. **1227.** The lengths of two parallel chords of a circle of radius 6 cm. are 10 cm. and 6 cm. respectively. Calculate the distance between the chords. (There are two cases.)

Ex. **1228.** Calculate the length of (i) the longest, (ii) the shortest chord of a circle, radius r, through a point distant d from the centre (see Ex. 1238).

Ex. **1229.** Calculate the radius of a circle, given that a chord 3 in. long is 2 in. from the centre.

Ex. **1230.** What is the radius of a circle when a chord of length $2l$ is at distance d from the centre?

Ex. **1231.** Given that a chord 12 cm. long is distant 2·5 cm. from the centre, calculate (i) the length of a chord distant 5 cm. from centre, (ii) the distance from the centre of a chord 6 cm. long.

†Ex. **1232.** If two chords make equal angles with the diameter through their point of intersection, they are equal.

[Prove that they are equidistant from the centre.]

†Ex. **1233.** A straight line is drawn cutting two equal circles and parallel to the line joining their centres; prove that the chords intercepted by the two circles are equal.

†Ex. **1234.** A straight line is drawn cutting two equal circles, and passing through the point midway between their centres. Prove that the chords intercepted by the two circles are equal.

Ex. **1235.** Show how to draw a chord of a circle (i) equal and parallel to a given chord, (ii) equal and perpendicular to a given chord, (iii) equal to a given chord and parallel to a given line.

†Ex. **1236. If two chords are at unequal distances from the centre, the nearer chord is longer than the more remote.**

†Ex. **1237.** **State and prove the converse of Ex. 1236.**

†Ex. **1238.** The shortest chord that can be drawn through a point inside a circle is that which is perpendicular to the diameter through the point.

[Prove that it is furthest from the centre.]

Ex. **1238 a.** A wooden ball of 4″ radius is planed down till there is a flat circular face of radius 2″. If the block is now made to stand on the flat face, how high will it stand?

Ex. **1238 b.** The distance from the centre of the earth of the plane of the Arctic circle is 3700 miles (to the nearest 100 miles); the radius of the earth is 4000 miles. Find the radius of the Arctic circle.

Ex. **1238 c.** A ball of radius 4 cm. floats in water immersed to the depth of $\frac{1}{4}$ of its diameter. Calculate the circumference of the water-line circle.

Ex. **1238 d.** The diameter of an orange is 4″, and the thickness of the rind is $\frac{1}{4}$″. A piece is sliced off just grazing the flesh; find the radius of the piece.

Section IV. The Tangent.

The meaning of the term tangent has been explained on p. 218. It may be defined as follows:—

Def. A **tangent** to a circle is a straight line which, however far it may be produced, has one point, and one only, in common with the circle.

The tangent is said to **touch** the circle; the common point is called the **point of contact.**

We shall assume that at a given point on a circle there is one tangent and one only.

THEOREM 6.

The tangent at any point of a circle and the radius through the point are perpendicular to one another.

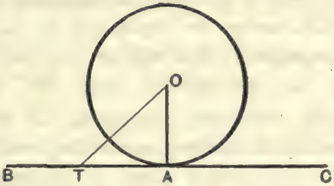


fig. 226.

Data O is the centre of ⊙; A is a point on the circumference; BC is the tangent at A.

To prove that BC and OA are ⊥ to one another.

Construction If OA be not ⊥ to BC, draw OT ⊥ to BC.

Proof Since ∠OTA is a rt. ∠, *Constr.*

∴ OT < OA, I. 21.

∴ T is inside the circle,

∴ the tangent AT, if produced, will cut the circle in another point.

This is impossible, *Def.*

∴ OA *is* ⊥ to BC,

∴ the tangent at A and the radius through A are ⊥ to one another.

Q. E. D.

COR. A straight line drawn through the point of contact of a tangent at right angles to the tangent will, if produced, pass through the centre of the circle.

To draw the tangent to a circle at a given point on the circle.

Join the point to the centre, and draw a straight line through the point perpendicular to the radius.

The proper method of **drawing a tangent to a circle from an external point** cannot be explained at the present stage, as it depends on a proposition that has not yet been proved. In the meantime it will be sufficient to draw the tangent from an external point with the ruler (by eye). It is not possible to distinguish the point of contact accurately without further construction; to find this point, drop a perpendicular upon the tangent from the centre; the foot of this perpendicular is the point of contact.

This method is accurate enough for many purposes; the student is warned, however, that it would not be accepted in most examinations. The correct construction is given on page 262.

†Ex. **1239. Prove that the two tangents drawn to a circle from a point A are (i) equal, (ii) equally inclined to AO.** (Fig. 227.)

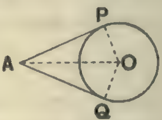


fig. 227.

Ex. **1240.** P is 4 in. distant from O, the centre of a circle of radius 3 in. From P draw a tangent with your ruler. Determine T, the point of contact, (i) by eye, (ii) by drawing a perpendicular from O.

Calculate PT, the length of the tangent (using Pythagoras' theorem). Verify by measurement.

Ex. **1241.** Calculate the lengths of the tangents to a circle of radius r from a point distant d from the centre when (i) $r=6$ cm., $d=8$ cm.; (ii) $r=1$ in., $d=5$ in.

Ex. **1242.** At a point A of a circle (radius r, centre O) is drawn a tangent AP of length l; find OP.

Ex. **1243.** At a point P on the circumference of a circle of radius 4 cm. is drawn a tangent PT 3 cm. in length. Find the locus of T as P moves round the ⊙.

Ex. **1244.** Two circles, of radii 3 and 2 in., are concentric. Calculate the length of a chord of the outer circle which touches the inner.

Ex. **1245.** Prove that all chords 8 cm. long of a circle of radius 5 cm. touch a certain concentric circle; find its radius.

†Ex. **1246.** All chords of a circle which touch an interior concentric circle are equal, and are bisected at the point of contact.

†Ex. **1247.** **PQRS is a quadrilateral circumscribed about a circle. Prove that** $PQ+RS=QR+SP$. (See fig. 219.)

†Ex. **1248.** Draw a circle and circumscribe a parallelogram about it. Prove that the parallelogram is necessarily a rhombus (use Ex. 1247).

†Ex. **1249.** Prove that the point of intersection of the diagonals of a rhombus is equidistant from the four sides.

¶Ex. **1250.** Draw a quadrilateral ABCD. What is the locus of the centres of ⊙ˢ touching AB, BC; touching BC, CD? Draw a circle to touch AB, BC and CD. Does it touch DA? What relation must hold between the sides of a quadrilateral in order that it may be possible to inscribe a circle in it?

¶Ex. **1251.** Construct a quadrilateral ABCD, having the sum of one pair of opposite sides = the sum of the other pair of opposite sides (e.g. $AB=2$ in., $BC=3$ in., $CD=4$ in., $DA=3$ in.). Draw a circle to touch three of the sides. Does it touch the fourth side? Measure the radius.

¶Ex. **1252.** Repeat Ex. 1251, using the same sides, but altering the shape of the quadrilateral. Inscribe a circle in it. Is the radius the same as in Ex. 1251?

†Ex. **1253.** ABCDEF is an irregular hexagon circumscribed about a circle; prove that $AB+CD+EF=BC+DE+FA$.

†Ex. **1254.** Two parallel tangents meet a third tangent at U, V; prove that UV subtends a right angle at the centre.

†Ex. **1255.** The angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral are supplementary.

¶Ex. **1256.** What is the locus of the centres of circles touching two lines which cross at an angle of 60°? (Remember that two lines form four angles at a point.) Draw a number of such circles.

¶Ex. **1257.** What is the locus of the centres of circles of radius 1 in. which touch a given line? Hence draw a circle which has a radius of 1 in. and touches two given lines inclined at an angle of 60°.

Ex. **1258.** Draw four circles of radius 3 cm. to touch two straight lines which cross at an angle of 140°.

†Ex. **1259.** A is a point outside a circle, of centre O. With centre O and radius OA describe a circle. Let OA cut the smaller circle in B. Draw BC perpendicular to OB, cutting the larger circle in P, Q. Let OP, OQ cut the smaller circle in S, T. Prove that AS, AT are tangents to the smaller circle. (This is Euclid's construction for tangents from an external point.)

†Ex. **1260.** A chord makes equal angles with the tangents at its extremities.

Ex. **1261.** Each of the tangents, TA, TB, at the ends of a certain chord AB is equal to the chord; find the angle between the tangents, and the angle subtended at the centre by the chord.

†Ex. **1262.** In fig. 227, the angles PAQ, POQ are supplementary.

Ex. **1263.** Show how to draw a tangent to a given circle (i) parallel to a given line, (ii) perpendicular to a given line, (iii) making a given angle with a given line.

Ex. **1264.** Show how to draw two tangents to a circle (i) at right angles, (ii) at an angle of 120°, (iii) at a given angle (without protractor).

†Ex. **1265.** The area of any polygon circumscribing a circle is equal to half the product of the radius of the circle, and the perimeter of the polygon. (Divide the polygon into triangles, with the centre for vertex.)

To inscribe a circle in a given triangle.

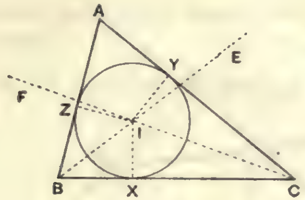


fig. 228.

Construction It is necessary to find a point equidistant from the three straight lines AB, BC, CA.

Draw BE, CF to bisect the angles ABC, ACB respectively.

Let these lines intersect at I.

Then I is the centre of the inscribed circle.

Proof Every point on BE is equidistant from AB and BC, and every point on CF is equidistant from BC, CA. I. 26.

Therefore I is equidistant from AB, BC and CA.

From I draw IX, IY, IZ ⊥ to BC, CA, AB respectively.

Then IX = IY = IZ.

Therefore a circle described with I as centre and IX as radius will pass through X, Y, Z. Also BC, CA, AB will be tangents at X, Y, Z. (Why?)

This circle is the **inscribed** circle of the triangle ABC.

Ex. **1266.** Draw the inscribed circle of a triangle whose sides are (i) 5, 6, 7 in., (ii) 8, 6, 8 cm. Measure the radii of the circles.

†Ex. **1267. The bisectors of the three angles of a triangle meet in a point.**

(Join IA, and prove that IA bisects ∠A.)

The escribed circles of a triangle.

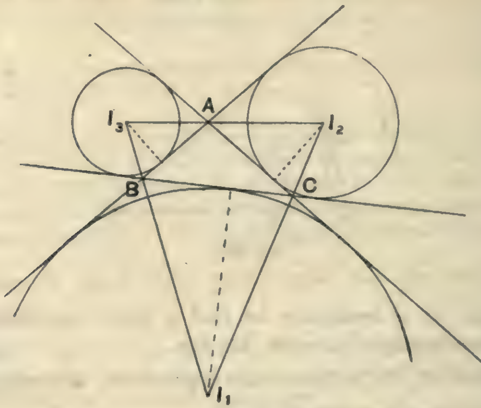


fig. 229.

Draw BI_1, CI_1 to bisect the angles exterior to ABC and BCA.

Then I_1 is equidistant from AB (produced), BC and AC (produced).

Drop a perpendicular from I_1 to BC. A circle drawn with I_1 as centre and this perpendicular as radius will touch the side BC and the sides AB, AC produced. This circle is called an **escribed** circle of the triangle. There are three such circles (see fig. 229).

Ex. **1268.** Draw the inscribed and escribed circles of a triangle whose sides are 3, 4, 5 in. Measure the radii.

†Ex. **1269.** Prove that the internal bisector of $\angle A$ and the external bisectors of $\angle^s$ B and C meet in a point.

†Ex. **1270.** Prove that AII_1 is a straight line. (I is the centre of the inscribed circle.)

Ex. **1271.** Verify, by drawing, that the circle drawn through the mid-points of the sides of a triangle touches the inscribed and each of the escribed circles.

It has been shown that, in general, four circles can be drawn to touch three unlimited straight lines, namely the inscribed and escribed circles of the triangle which the three lines enclose.

¶**Ex. 1272.** How many circles can be drawn to touch two parallel straight lines and a third straight line cutting them.

¶**Ex. 1273.** How many circles can be drawn to touch three straight lines which intersect in a point?

¶**Ex. 1274.** How many circles can be drawn to touch three parallel straight lines?

SECTION V. CONTACT OF CIRCLES.

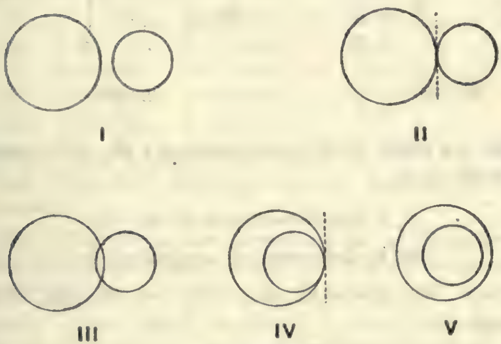


fig. 230.

The different relative positions which are possible for two circles are shown in fig. 230.

In Cases II and IV the circles are said to **touch, externally** in Case II, **internally** in Case IV. The formal definition of contact of circles is as follows:—

DEF. If two circles touch the same line at the same point, they are said to touch one another.

THEOREM 7.

If two circles touch, the point of contact lies in the straight line through the centres.

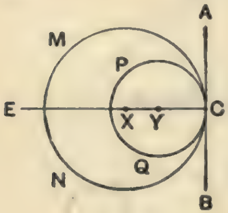


fig. 231.

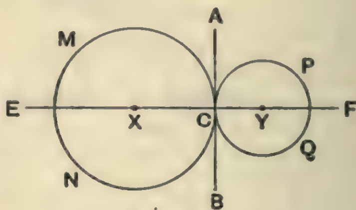


fig. 232.

Data The ⊙s CMN, CPQ touch internally (fig. 231) or externally (fig. 232) at C.

X, Y are the centres of the ⊙s.

AB is the common tangent at C.

To prove that XY produced (fig. 231), or XY (fig. 232) passes through C.

Construction Join XC, YC.

Proof Since CA is the tangent at C to ⊙ CMN, and CX the radius through C,

∴ ∠ XCA is a rt. ∠, III. 6.

Sim^{ly} ∠ YCA is a rt. ∠,

∴ if the ⊙s touch internally, XYC is a straight line,

and if the ⊙s touch externally, ∠ XCA + ∠ YCA = 2 rt. ∠s.

∴ XCY is a straight line. I. 2.

Q. E. D.

COR. If two circles touch externally the distance between their centres is equal to the sum of their radii; if they touch internally the distance between their centres is equal to the difference of their radii.

¶Ex. **1275.** Draw a figure showing the different relative positions possible for two equal circles.

¶Ex. **1276.** Describe in words each of the relative positions shown in fig. 230.

¶Ex. **1277.** Describe the relative position of the two circles in each of the following cases (d is the distance between the centres, R and r are the radii). Do this, if you can, without drawing the circles.

(i)	$d=4{\cdot}1$ cm.,	$R=2{\cdot}1$ cm.,	$r=1{\cdot}4$ cm.
(ii)	$d=0{\cdot}7$ cm.,	$R=2{\cdot}2$ cm.,	$r=1{\cdot}2$ cm.
(iii)	$d=3{\cdot}4$ cm.,	$R=2{\cdot}0$ cm.,	$r=1{\cdot}4$ cm.
(iv)	$d=0{\cdot}8$ cm.,	$R=2{\cdot}1$ cm.,	$r=1{\cdot}3$ cm.
(v)	$d=0$ cm.,	$R=1{\cdot}9$ cm.,	$r=1{\cdot}2$ cm.
(vi)	$d=1{\cdot}5$ cm.,	$R=2{\cdot}0$ cm.,	$r=1{\cdot}5$ cm.
(vii)	$d=2{\cdot}5$ cm.,	$R=1{\cdot}7$ cm.,	$r=1{\cdot}7$ cm.

Ex. **1278.** What is the distance between the centres of two circles of radii 15 and 14 in. (i) if they have external contact, (ii) if they have internal contact?

Ex. **1279.** Show how to draw three circles having for centres the vertices of an equilateral triangle of side 2 in., so that each circle may touch the two others externally.

Ex. **1280.** Three circles, of radii 1, 2, 3 in., touch externally, each circle touching the other two. What are the distances between the centres? Draw the circles.

CONSTRUCTION OF CIRCLES TO SATISFY GIVEN CONDITIONS.

¶Ex. **1281. What is the locus of the centres of all circles of radius 1 in., which touch externally a fixed circle of radius 2 in.? Draw the locus, and draw a number of the touching circles.**

¶Ex. **1282.** If required to draw a circle to touch a given circle *at a given point*, where would you look for the centre of the touching circle? **What is the locus of the centres of circles touching a given circle at a given point?** Draw a number of such circles, some enclosing the given circle, some inside it, some external to it.

¶Ex. **1283.** **What is the locus of the centres of circles which touch a given line at a given point?**

¶Ex. **1284.** What is the locus of the centres of circles of radius 1 in., touching a given circle of radius 2 in., and lying inside it? Draw a number of such circles.

¶Ex. **1285.** Repeat Ex. 1284 with 1 in. radius for the touching circles, and 3 in. radius for the fixed circle.

¶Ex. **1286.** Draw a number of circles of radius 3 in. to touch a circle of radius 2 in. and enclose it.

¶Ex. **1287.** Draw a number of circles of radius 4 in. to touch a given circle of radius 2 in. and enclose it.

¶Ex. **1288.** **What is the locus of centres of circles of given radius passing through a given point?**

¶Ex. **1289.** What is the locus of centres of circles (i) passing through two given points, (ii) touching two given lines?

Each of the following problems is to be solved by finding the centre of the required circle, (generally by the intersection of loci). Some of the group have been solved already; they are recapitulated below for the sake of completeness. In several cases a numerical instance is given which should be attempted first, the radius of the resulting circle being measured.

Ex. **1290.** Draw a circle (or circles) to satisfy the following conditions:—

(i) To pass through three given points (solved already).

(ii) Of given radius, to pass through two given points (solved already).

(iii) Of given radius, to pass through a given point and touch a given line, e.g. take radius 2 in. and a point distant 1 in. from the line. (What is the locus of centres of 2 in. circles passing through given point? touching given line?) When is the general problem impossible?

(iv) To touch a given line AB at a given point P, and to pass through a given point Q outside the line. (What is the locus of centres of ⊙s touching line at P? passing through P and Q? Let $PQ = 3$ cm., $\angle QPA = 30°$.)

(v) To touch a given circle at a given point P, and to pass through a given point Q not on the circle. In what case is this impossible?

(vi) To touch a given line AB at P, and also to touch a given line CD, not parallel to AB. (What is the locus of centres of circles touching AB and CD?)

(vii) Of given radius, to pass through a given point P and touch a given circle, e.g. let given radius = 4 cm., radius of given circle = 3 cm., distance of P from centre of given circle = 5 cm. (Compare (iii).)

(viii) Of given radius, to touch a given circle at a given point (how many solutions are there?).

(ix) To touch three given lines (solved already).

(x) Of given radius to touch two given lines, e.g. let the lines intersect at an angle of 60°, and radius = 1 in. (How many solutions are there?)

(xi) Of given radius, to touch a given line and a given circle (e.g. given radius = 3 cm., radius of given circle = 5 cm., distance of line from centre of circle = 6 cm.). What is the condition that the general problem may be possible?

(xii) To touch three equal circles (*a*) so as to enclose them all, (*b*) so as to enclose none of them. (Begin by drawing a circle through the three centres.)

(xiii) Of given radius, to touch two given circles (e.g. let given radius = 2 in., radii of given circles = 1 in., 1·5 in., distance between centres = 3·5 in.).

Ex. **1291.** In a semicircle of radius 5 cm. inscribe a circle of radius 2 cm. Measure the parts into which the diameter of the semicircle is divided by the point of contact. See fig. 233.

fig. 233.

Ex. **1292.** Draw four circles of radius 2 in., touching a fixed circle of radius 1 in., and also touching a straight line 2 in. distant from the centre of the fixed circle.

Ex. **1293.** Show how to inscribe a circle in a sector of 60° of a circle whose radius is 4 in.

Ex. **1294.** Show how to draw three equal circles, each touching the other two; and how to circumscribe a fourth circle round the other three.

†Ex. **1295.** Prove that, if circles are described with centres A, B, C (fig. 228) and radii AY, BZ, CX, the three circles touch.

†Ex. **1296.** A variable circle (centre O) touches externally each of two fixed circles (centres A, B). Prove that the difference of AO, BO remains constant.

†Ex. **1297.** If two circles touch and a line is drawn through the point of contact to meet the circles again at P and Q, the tangents at P and Q are parallel. (Draw the common tangent at the point of contact.)

†Ex. **1298.** If two circles touch externally at A and are touched at P, Q by a line PQ, then PQ subtends a right angle at A. Also PQ is bisected by the common tangent at A.

†Ex. **1299.** Prove that, in Ex. 1298, the circle on PQ as diameter passes through A and touches the line of centres.

†Ex. **1300.** **Two circles intersect at A, B; prove that the line of centres bisects AB (the common chord) at right angles.** (See III. 1 Cor.)

What kind of symmetry has the above figure?

Ex. **1301.** Find the distances between the centres of two circles, their radii being 5 and 7 cm. and their common chord 8 cm. (There are two cases.)

Section VI. Angle Properties.

Reflex angles. Take your dividers and open them slowly. The angle between the legs is first an acute angle, then a right angle, then an obtuse angle. When the dividers are opened out flat, the angle has become two right angles (180°). If the dividers are opened still further the angle of opening is greater than 180° and is called a **reflex angle.**

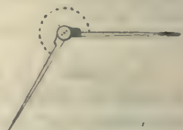


fig. 234.

Def. A **reflex angle** is an angle greater than two right angles and less than four right angles. Fig. 235 shows two straight lines OA, OB forming a reflex angle (marked), and also an obtuse angle (unmarked).

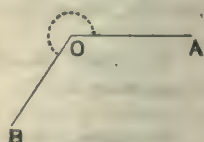


fig. 235.

¶Ex. **1302.** Account for the necessity of the phrase "less than four right angles" in the above definition.

¶Ex. **1303.** Open a book to form a reflex angle.

¶Ex. **1304.** What is the sum of the reflex angle a and the acute angle b in fig. 236? If $\angle b = 36°$, what is $\angle a$?

¶Ex. **1305.** What kind of angle is subtended at the centre of a circle by a major arc?

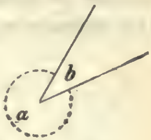


fig. 236.

¶Ex. **1306.** Draw a quadrilateral having one angle reflex. Prove that the sum of the four angles is 360°.

¶Ex. **1307.** Is it possible for (i) a four-sided figure, (ii) a five-sided figure to have *two* of its angles reflex?

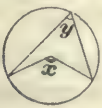


fig. 237.

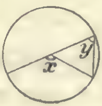


fig. 238.

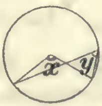


fig. 239.

¶Ex. **1308.** Draw a figure like fig. 237, making the radius of the circle about 2 in. Measure angles x and y.

¶Ex. **1309.** Do the same for figs. 238, 239, 240. What relation do you notice between the angle x and the angle y in the four experiments?

¶Ex. **1310.** Draw a circle of radius 5 cm.: place in it a chord AB of length 9·5 cm. Mark four points P, Q, R, S in the major arc. Make the necessary joins and measure the angles APB, AQB, ARB, ASB. What relation do you notice between these angles? Can you connect this with the results of Ex. 1308, 1309?

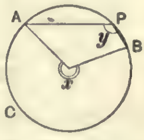


fig. 240.

¶Ex. **1311.** In the figure of Ex. 1310 mark three points X, Y, Z in the minor arc; measure the angles AXB, AYB, AZB.

¶Ex. **1312.** Draw a circle and a diameter. Mark four points on the circle, at random. Measure the angle subtended by the diameter at each of these points.

†Ex. **1313.** A side BA of an isosceles triangle ABC is produced, through the vertex A, to a point D. Prove that $\angle DAC = 2\angle ABC = 2\angle ACB$.

THEOREM 8.

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

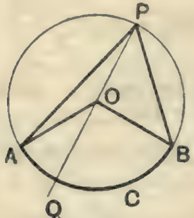


fig. 241.

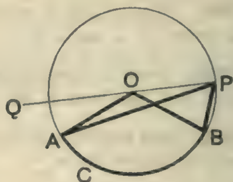


fig. 242.

Data The arc ACB of ⊙ACB subtends ∠AOB at the centre O; and subtends ∠APB at P, any point on the remaining part of the circumference.

To prove that $\angle AOB = 2\angle APB$.

Construction Join PO, and produce to Q.

Proof. CASE I. *When the centre* O *is inside* ∠APB.

In △AOP, OA = OP (radii)

$\therefore \angle OPA = \angle OAP$. I. 12.

Now ∠QOA is an exterior ∠ of △AOP,

$\therefore \angle QOA = \angle OPA + \angle OAP$ I. 8, *Cor.* 1.

$= 2\angle OPA$.

Sim^{ly} $\angle QOB = 2\angle OPB$,

$\therefore \angle QOA + \angle QOB = 2(\angle OPA + \angle OPB)$,

$\therefore \angle AOB = 2\angle APB$.

CASE II. *When the centre* O *is outside* ∠APB.

As before, $\angle QOB = 2\angle OPB$,

and $\angle QOA = 2\angle OPA$,

$\therefore \angle QOB - \angle QOA = 2(\angle OPB - \angle OPA)$,

$\therefore \angle AOB = 2\angle APB$.

Q. E. D.

†Ex. **1314.** Prove the above theorem for the case in which ACB is a major arc, and the angle subtended at the centre a reflex angle (see fig. 240). What kind of angle is ∠APB in this case?

†Ex. **1315.** Prove the above theorem for the case in which O lies on AP (see fig. 238).

†Ex. **1316.** Prove that in fig 243

$$\angle a = \angle b.$$

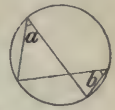


fig. 243.

†Ex. **1317.** If the two circles in figs. 241 and 242 are equal, and the arcs ACB are equal, prove that the angles APB are equal.

Ex. **1318.** Draw a figure for the case of III. 8 in which arc ACB is a semicircle. What does ∠AOB become in this case? What does ∠APB become?

Ex. **1319.** Find the magnitude of all the marked angles in fig. 244. What is the sum of the angles at the centre? of ∠s ACB and ADB? of ∠s CAD and CBD?

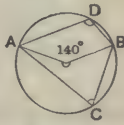


fig. 244.

Def. A **segment** of a circle is the part of the plane bounded by an arc and its chord.

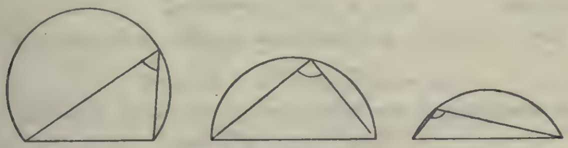


fig. 245.

Def. An **angle in a segment** of a circle is an angle subtended by the chord of the segment at a point on the arc (fig. 245).

Def. A segment is called a **major** segment or a **minor** segment according as its arc is a major or a minor arc. It is obvious that a major segment of a circle is greater than the semi-circle (considered as an area) and that a minor segment is less than the semi-circle.

¶Ex. **1320.** Show by a figure that a minor *sector* of a circle can be divided into a segment and a triangle. What is the corresponding theorem for a major sector? Is there any figure which is at the same time a sector and a segment?

THEOREM 9.

Angles in the same segment of a circle are equal.

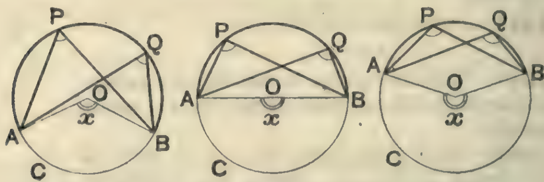


fig. 246. fig. 247. fig. 248.

Data $\angle$s APB, AQB are two $\angle$s in the same segment APQB of $\odot$APB. (Three figures are drawn, for the three cases in which the segment >, = or < a semi-circle.)

To prove that $\angle$APB = $\angle$AQB.

Construction Join A, B to the centre.
Let x be the $\angle$ subtended at the centre by arc ACB.

Proof In each figure $\angle x = 2\angle$APB,
for these angles are subtended by the same arc ACB. III. 8.
Sim^{ly} $\angle x = 2\angle$AQB,
$\therefore$ $\angle$APB = $\angle$AQB.

Q. E. D.

NOTE. Since all the angles in a segment are equal, we may in future speak of *the* angle in a segment when we mean the magnitude of any angle in the segment.

¶**Ex. 1321.** Are $\angle^s$ PAQ, PBQ in fig. 248 equal? Give a reason.

Ex. 1322. Find the angle in a segment of a circle, the chord of the segment being 6 cm. and the height 2 cm.

Ex. 1323. Repeat Ex. 1322 with chord = 4 ins. and height = 2 ins.

Ex. 1324. Repeat Ex. 1322 with chord = 5·43 cm., height = 8·61 cm.

THEOREM 10.

The angle in a major segment is acute; the angle in a semi-circle is a right angle; and the angle in a minor segment is obtuse.

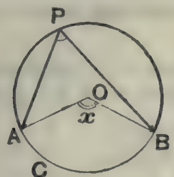


fig. 249.

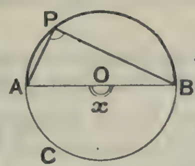


fig. 250.

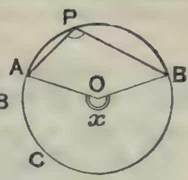


fig. 251.

CASE I.

Data APB is a major segment.

To prove that $\angle$ APB is acute.

Proof Since APB is a major segment,

$\therefore$ arc ACB is a minor arc,

$\therefore \angle x < 2$ rt. $\angle$s.

But $\angle \text{APB} = \frac{1}{2} \angle x$. III. 9.

$\therefore \angle \text{APB} < 1$ rt. $\angle$.

CASE II.

Data APB is a semi-circle.

To prove that $\angle$ APB is a rt. $\angle$.

Proof Since APB is a semi-circle, so also is ACB,

$\therefore \angle x = 2$ rt. $\angle$s,

$\therefore \angle \text{APB} = 1$ rt. $\angle$.

CASE III.

Data APB is a minor segment.

To prove that $\angle$ APB is obtuse.

Proof Since APB is a minor segment,

$\therefore$ arc ACB is a major arc,

$\therefore \angle x > 2$ rt. $\angle$s,

$\therefore \angle \text{APB} > 1$ rt. $\angle$.

Q. E. D.

Ex. **1325.** A regular hexagon is inscribed in a circle. What is the angle in each of the segments of the circle which lie outside the hexagon?

Ex. **1326.** Repeat Ex. 1325 for the case of (i) a square, (ii) an equilateral $\triangle$, (iii) a regular n-gon.

†Ex. **1327.** A, B, C, D are points on a circle; the diagonals of ABCD meet at X; prove that $\triangle^s$ ABX, DCX are equiangular; as also $\triangle^s$ BCX, ADX.

†Ex. **1328.** Through X, a point outside a circle, XAB, XCD are drawn to cut the circle in A, B; C, D. Prove that $\triangle^s$ XAD, XCB are equiangular.

Ex. **1329.** Copy fig. 252 (on an enlarged scale); join BC. Find all the angles of the quadrilateral ABCD; and prove that two of its sides are equal.

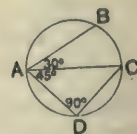


fig. 252.

†Ex. **1330.** Prove the following construction for erecting a perpendicular to a given straight line AB at its extremity B. With centres A, B describe arcs of equal circles, cutting at C. With centre C and radius CA describe a circle. Produce AC to meet this circle again in D; then BD is $\perp$ to AB.

†Ex. **1331.** The circle described on a side of an isosceles triangle as diameter bisects the base.

†Ex. **1332.** The circles drawn on two sides of a triangle as diameters intersect on the base.

†Ex. **1333.** The four circles drawn with the sides of a rhombus for diameters have one point in common.

†Ex. **1334.** Two circles intersect at P, Q. Through P diameters PA, PB of the two circles are drawn. Show that AQ, QB are in the same straight line. (Join QP.)

†Ex. **1335.** AD is $\perp$ to the base BC of $\triangle$ABC; AE is a diameter of the circumscribing circle. Prove that $\triangle^s$ ABD, AEC are equiangular; as also $\triangle^s$ ACD, AEB.

†Ex. **1336.** The bisector of A, the vertical angle of $\triangle$ABC, meets the base in D and the circumscribing circle in E. Prove that $\triangle^s$ ABD, AEC are equiangular. Also prove that $\triangle^s$ ACD, AEB are equiangular.

¶Ex. **1337.** Draw four straight lines roughly in the shape of ACBD (fig. 253), making $\angle C = \angle D = 30°$. Draw a circle round ACB; notice whether it passes through D.

THEOREM 11.

[CONVERSE OF THEOREM 9.]

If the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.

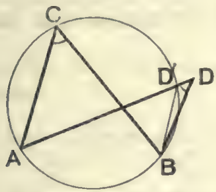


fig. 253.

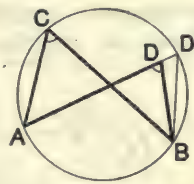


fig. 254.

Data The line joining AB subtends equal ∠s at the points C, D, which lie on the same side of AB.

To prove that the four points A, B, C, D lie on a ⊙.

Construction Draw ⊙ to pass through A, B and C.

It must be shown that this ⊙ passes through D.

Proof If ⊙ABC does not pass through D, it must cut AD (or AD produced) in some other point D′.

Join BD′.

Then ∠AD′B = ∠ACB (in same segment). III. 9.

But ∠ADB = ∠ACB, *Data*

∴ ∠AD′B = ∠ADB.

But this is impossible, for one of the ∠s is an exterior ∠ of △DD′B, and the other is an interior opposite ∠ of the same △.

Hence ⊙ABC must pass through D,

i.e. A, B, C, D lie on a ⊙. Q. E. D.

DEF. Points which lie on the same circle are said to be **concyclic.**

†Ex. **1338.** BE, CF are altitudes of the triangle ABC; prove that B, F, E, C are concyclic. Sketch in the circle.

¶Ex. **1339.** Draw a circle (radius about 3 in.); take four points A, B, C, D upon it. By measurement, find the sum of the angles BAD, BCD; also of the angles ABC, ADC.

Theorem 12.

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

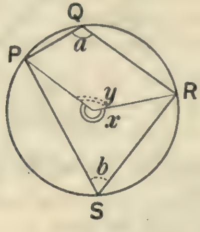


fig. 255.

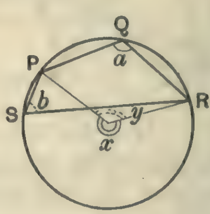


fig. 256.

Data PQRS is a quadrilateral inscribed in ⊙ PQR*.

To prove that (1) $\angle$ PQR + $\angle$ PSR = 2 rt. $\angle$s,

(2) $\angle$ SPQ + $\angle$ SRQ = 2 rt. $\angle$s.

Construction Join P and R to the centre of ⊙.

Proof $\angle a = \frac{1}{2} \angle x$, III. 8.

$\angle b = \frac{1}{2} \angle y$, III. 8.

$\therefore \angle a + \angle b = \frac{1}{2}(\angle x + \angle y)$.

But $\angle x + \angle y = 4$ rt. $\angle$s,

$\therefore \angle a + \angle b = 2$ rt. $\angle$s,

i.e. $\angle$ PQR + $\angle$ PSR = 2 rt. $\angle$s.

Sim^ly it may be shown that $\angle$ SPQ + $\angle$ SRQ = 2 rt. $\angle$s.

Q. E. D.

* The two figures represent the two cases in which the centre is (i) inside, (ii) outside the quadrilateral. The same proof applies to both.

Ex. **1340.** From the given angles, find all the angles in fig. 257.

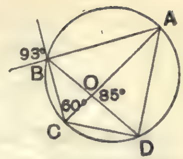


fig. 257.

Ex. **1341.** Repeat Ex. 1340, taking $\angle B = 71°$, $\angle BCO = 36°$, $\angle AOD = 108°$. Prove that in this case AD is ∥ to BC.

†Ex. **1342.** **The side PQ of a quadrilateral PQRS, inscribed in a circle, is produced to T. Prove that the exterior $\angle RQT$ = the interior opposite $\angle PSR$.**

†Ex. **1343.** If a parallelogram can be inscribed in a circle, it must be a rectangle.

†Ex. **1344.** If a trapezium can be inscribed in a circle, it must be isosceles.

†Ex. **1345.** The sides BA, CD of a quadrilateral ABCD, inscribed in a circle, are produced to meet at O; prove that $\triangle^s$ OAD, OCB are equiangular.

†Ex. **1346.** ABCD is a quadrilateral inscribed in a circle, having $\angle A = 60°$; O is the centre of the circle. Prove that

$$\angle OBD + \angle ODB = \angle CBD + \angle CDB.$$

¶Ex. **1347.** What is the relation between the angles subtended by a chord at a point in its minor arc, and at a point in its major arc?

¶Ex. **1348.** Draw a quadrilateral ABCD, having $\angle A + \angle C = 180°$. Draw a circle to pass through ABC; notice whether it passes through D.

THEOREM 13.

[CONVERSE OF THEOREM 12.]

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

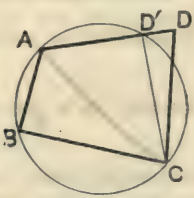


fig. 258.

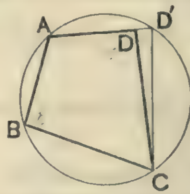


fig. 259.

Data The $\angle$s ABC, ADC of the quadrilateral ABCD are supplementary.

To prove that A, B, C, D are concyclic.

Construction Draw $\odot$ to pass through A, B, C.
It must be shown that this $\odot$ passes through D.

Proof If $\odot$ABC does not pass through D, it must cut AD (or AD produced) in some other point D′.

Join CD′.

Then $\angle AD'C + \angle ABC = 2$ rt. $\angle$s. III. 12.

But $\angle ADC + \angle ABC = 2$ rt. $\angle$s, *Data*

$\therefore \angle AD'C + \angle ABC = \angle ADC + \angle ABC$,

$\therefore \angle AD'C = \angle ADC$.

But this is impossible, for one of the $\angle$s is an exterior $\angle$ of $\triangle DD'C$, and the other is an interior opposite $\angle$ of the same $\triangle$.

Hence $\odot$ABC must pass through D,
i.e. A, B, C, D are concyclic.

Q. E. D.

DEF. If a quadrilateral is such that a circle can be circumscribed round it, the quadrilateral is said to be **cyclic.**

†Ex. **1349.** BE, CF, two altitudes of $\triangle$ABC, intersect at H. Prove that AEHF is a cyclic quadrilateral. Sketch in the circle.

Ex. **1350.** ABC, DBC are two congruent triangles on opposite sides of the base BC. Under what circumstances are A, B, C, D concyclic?

†Ex. **1351.** ABCD is a parallelogram. A circle drawn through A, B, cuts AD, BC (produced if necessary) in E, F respectively. Prove that E, F, C, D are concyclic.

†Ex. **1352.** ABCD is a quadrilateral inscribed in a circle. DA, CB are produced to meet at E; AB, DC to meet at F. Prove that, if a circle can be drawn through AEFC, then EF is the diameter of this circle; and BD is the diameter of $\odot$ABCD.

†Ex. **1353.** The straight lines bisecting the angles of any convex quadrilateral form a cyclic quadrilateral.

For further exercises on the subject-matter of the above section see end of section IX.

SECTION VII. CONSTRUCTION OF TANGENTS.

¶Ex. **1354.** Stick two pins into the paper 2 in. apart at A and B; place the set-square on the paper so that the sides containing the 60° are in contact with the pins; mark the point where the vertex of the angle rests. Now slide the set-square about, keeping the same two sides against the pins, and plot the locus of the 60° vertex. What is the locus? are A, B points in the locus? Complete the circle, and measure the angle subtended by AB at a point in the minor arc.

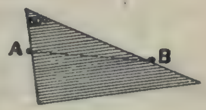


fig. 260.

¶Ex. **1355.** Repeat the experiment of Ex. 1354 with the 30° vertex.

¶Ex. **1356.** Repeat the experiment of Ex. 1354 with the 90° vertex.

¶Ex. **1357.** What is the locus of points at which a given line subtends a right angle?

Ex. **1358.** O is the centre of a circle and Q is a point outside the circle. Construct the locus of points at which OQ subtends a right angle. Find two points A, B on the first circle, so that $\angle$OAQ = $\angle$OBQ = 90°. Prove that QA is a tangent to the first circle.

To draw tangents to a given circle ABC from a given point T outside the circle.

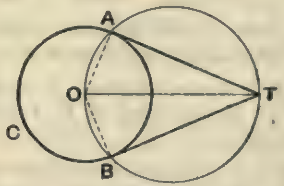


fig. 261.

Construction Join T to O, the centre of ⊙ ABC.

On OT as diameter describe a ⊙ cutting the given circle in A, B.

Join TA, TB.

These lines are tangents.

Proof Join OA, OB.

Since OT is the diameter of ⊙ OAT,

∴ ∠ OAT is a right angle,

∴ AT, being ⊥ to radius OA, is the tangent at A.

Similarly BT is the tangent at B.

Ex. **1359.** Draw tangents to a circle of radius 2 ins. from a point 1 in. outside the circle; calculate and measure the length of the tangents.

Ex. **1360.** Draw a circle of radius 3 cm. and mark a point T distant 7 cm. from the centre. Find where the tangents from T meet the circle (i) by the method of p. 240, (ii) as above. Calculate the length of the tangents, and ascertain which method gives greater accuracy.

Ex. **1361.** Find the angle between the tangents to a circle from a point whose distance from the centre is equal to a diameter.

Ex. **1362.** Through a point 2 in. outside a circle of radius 2 in. draw a line to pass at a distance of 1 in. from the centre. Measure and calculate the part inside the circle.

Common Tangents to Two Circles.

Def. A straight line which touches two circles is called a **common tangent** to the two circles.

Fig. 262 shows that when the circles do not intersect there are four common tangents.

If the two circles lie on the same side of a common tangent, it is called an **exterior** common tangent; thus AB, CD (fig. 262) are exterior common tangents. If the two circles lie on opposite sides of a common tangent, it is called an **interior** common tangent; thus EF, GH are interior common tangents.

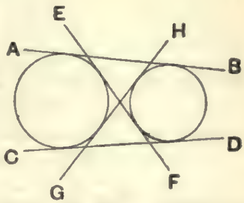


fig. 262.

Ex. **1363.** Draw sketches to show how many common tangents can be drawn in cases II., III., IV., V., of fig. 230; in each case state the number of exterior and of interior common tangents.

¶Ex. **1364.** Draw the tangents to a circle (centre A; radius 1 in.) from a point B (AB = 3 in. fig. 263). Draw, parallel to each tangent, a line $\frac{1}{2}$ in. from the tangent, these lines not to cut the circle. With centres A and B draw circles touching these two lines. Show that the difference of the radii of these circles is equal to the radius of the original circle.

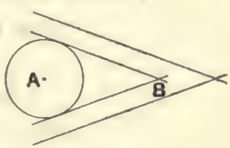


fig. 263.

To construct an exterior common tangent to two unequal circles.

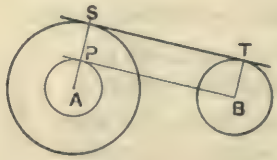


fig. 264.

[*Analysis* Let A, B be the centres of the larger and smaller circles respectively; R, r their radii.

Suppose that ST is an exterior common tangent, touching the ⊙s at S, T.

Join AS, BT. Then ∠s AST, BTS are right angles,

∴ AS is ∥ to BT.

Through B draw BP ∥ to TS, meeting AS in P.

Then BTSP is a rectangle. (Why?)

∴ $PS = BT = r$.

And $AP = AS - PS = R - r$.

Also ∠ APB is a right angle. (Why?)

∴ BP is a tangent from B to a circle round A, whose radius is $R - r$.

The foregoing analysis suggests the following construction.]

Construction With centre A describe a circle having for radius the *difference* of the given radii.

From B draw a tangent BP to this circle.

Join AP and produce it to meet the larger ⊙ in S.

Through B draw BT ∥ to AS to meet the smaller circle in T.

Join ST.

Then this line is a common tangent to the two ⊙s.

Proof PS is equal and $\parallel$ to BT (why?),

$\therefore$ STBP is a parallelogram,

and $\angle$ SPB is a right angle (why?),

$\therefore$ STBP is a rectangle,

$\therefore$ angles at S and T are right angles,

$\therefore$ ST is a tangent to each circle.

Ex. **1365.** Draw two circles of radii 1·5 in. and 0·5 in., the centres 2·5 ins. apart. Draw the two exterior common tangents.

Measure and calculate the length of these tangents (i.e. the distance between the points of contact). [Use right-angled $\triangle$ APB.]

†Ex. **1366.** Where does the above method fail when the two circles are equal? Give a construction (with proof) for the exterior common tangents in this case.

To construct an interior common tangent to two circles.

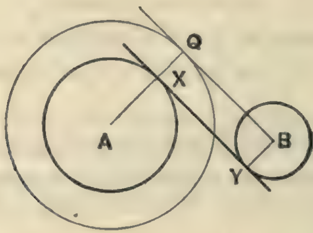


fig. 265.

[*Analysis* Suppose that XY is an interior common tangent, touching the $\odot^s$ at X, Y.

Join AX, BY. Then $\angle^s$ AXY, BYX are right angles.

$\therefore$ AX is $\parallel$ to BY.

Through B draw BQ $\parallel$ YX, meeting AX produced in Q.

Then BYXQ is a rectangle.

$\therefore$ $QX = BY = r$.

And $AQ = AX + XQ = R + r$.

Also $\angle$ AQB is a right angle,

$\therefore$ BQ is a tangent from B to a circle round A, whose radius is $R + r$.

Hence the following construction.]

Construction With centre A describe a circle having for radius the *sum* of the given radii.

From B draw a tangent BQ to this circle.

Join AQ; let this line cut the (A) circle in X.

Through B draw BY $\parallel$ QA to meet the (B) circle in Y.

Join XY.

Then this line is a common tangent to the two $\odot^s$.

Proof (i) Prove that BYXQ is a rectangle.

(ii) Prove that XY is a tangent to the (A) circle at X, and to the (B) circle at Y.

Ex. 1367. Draw the two circles of Ex. 1365, and draw the interior common tangents. Measure and calculate the length of these tangents.

¶**Ex. 1368.** Draw two equal circles, not intersecting. Draw the interior common tangents by the above method. Can you suggest an easier method for this special case?

Ex. 1369. In the following exercises R, r denote the radii of the circles, d the distance between their centres. For each pair of circles calculate the lengths of possible common tangents. (*Freehand.*)

(i)	R = 5 cm.,	r = 3 cm.,	d = 8 cm.
(ii)	R = 5 cm.,	r = 3 cm.,	d = 7 cm.
(iii)	R = 3 in.,	r = 1 in.,	d = 2 in.
(iv)	R = 3 in.,	r = 1 in.,	d = 1 in.
(v)	R = 3·52 cm.,	r = 1·41 cm.,	d = 6·29 cm.

¶**Ex. 1370.** If the radius of the smaller circle diminishes till the circle becomes a point, what becomes of the four common tangents?

Ex. 1371. The diameters of the wheels of an old-fashioned bicycle are 4 ft. and 1 ft., and the distance between the points where the wheels touch the ground is 2½ ft. Calculate the distance between the centres of the wheels; check by drawing.

Section VIII. Constructions depending on Angle Properties.

¶Ex. **1372.** Draw a line of 2 ins.; on this line as base draw a triangle with a vertical angle of 40°.

(You will find that it is practically impossible to draw the vertical angle directly: first draw the angles at the ends of the base. What is their sum? Notice that many different triangles may be drawn with the given vertical angle.)

¶Ex. **1373.** Draw a line of 2 ins.; on this line as base, and on the same side of it, draw a number of triangles (about 10) having a vertical angle of 40°. What is the locus of their vertices? Complete the curve of which this locus is a part. Is it possible for the vertex to coincide with an end of the base, in an extreme case? Does the curve pass through the ends of the base?

¶Ex. **1374.** Repeat Ex. 1373 with base 2 in. and vertical angle 140°. Compare this with the locus obtained in Ex. 1373.

¶Ex. **1375.** What locus would be obtained if Ex. 1373 were repeated with an angle of 90°?

¶Ex. **1376.** (Tracing paper.) Draw, on tracing paper, two straight lines intersecting at P. On your drawing paper mark two points A, B. Move your tracing paper about so that the one line may always pass through A, and the other through B. Plot the locus of P by pricking through.

The foregoing exercises will have prepared the reader for the following statement:—

The locus of points (on one side of a given straight line) at which the line subtends a constant angle is an arc of a circle, the given line being the chord of the arc.

¶Ex. **1377.** Upon what theorem does the truth of this statement depend?

¶Ex. **1378.** What kind of arc is obtained if the angle is (i) acute, (ii) a right angle, (iii) obtuse?

¶Ex. **1379.** If the constant angle is 45°, what angle is subtended by the given line at the centre of the circle? Use this suggestion in order to draw the locus of points at which a line of 5 cm. subtends 45°, without actually determining any of the points.

Ex. 1380. Show how to construct the locus of points at which a given line subtends an angle of 30°. Prove that in this case the radius of the circle is equal to the given line.

Ex. 1381. Show how to construct the locus of points at which a given line subtends a given angle.

†**Ex. 1382.** On a chord of 3·5 ins. construct a segment of a circle to contain an angle of 70°. Measure the radius.

Ex. 1383. Repeat Ex. 1382 with chord of 7·24 cm. and angle of 110°.

Ex. 1384. Repeat Ex. 1382 with chord of 3 in. and angle of 120°.

†**Ex. 1385.** Prove that the locus of the mid-points of chords of a circle which are drawn through a fixed point is a circle.

†**Ex. 1386.** Of all triangles of given base and vertical angle, the isosceles triangle has greatest area.

†**Ex. 1387.** P is a variable point on an arc AB. AP is produced to Q so that PQ = PB. Prove that the locus of Q is a circular arc.

To construct a triangle with given base, given altitude, and given vertical angle.

Let the base be 7 cm.; the altitude 6·5 cm.; the vertical angle 46°.

Draw the given base.

Draw the locus of points at which the given base subtends 46°.

Draw the locus of points distant 6·5 cm. from the given base (produced if necessary).

The intersections of these loci will be the required positions of the vertex.

How many solutions are there to this problem?

Measure the base angles of the triangle.

Ex. 1388. Construct a triangle having

(i) base = 4 in., altitude = 1 in., vertical angle = 90°.

(ii) base = 10 cm., altitude = 2 cm., vertical angle = 120°.

(iii) base = 8 cm., altitude = 5 cm., vertical angle = 90°.

(iv) base = 3·5 in., altitude = 1 in., vertical angle = 54°.

In each case measure the base angles.

Ex. **1389.** (Without protractor.) Construct a triangle, given the base, vertical angle and altitude.

Ex. **1390.** Show how to construct a triangle of given base, vertical angle and median.

Ex. **1391.** Show how to construct a triangle, given the base, the vertical angle and the area.

Ex. **1392.** Show how to construct quadrilateral ABCD, given AB = 5·4 cm., AC = 9·5 cm., AD = 5·6 cm., $\angle$BAD = 113°, $\angle$BCD = 70°.

Ex. **1393.** Show how to construct a cyclic quadrilateral ABCD, given AB = 1·6 in., BC = 3·0 in., CD = 4·9 in., $\angle$B = 125°.

Why are only four measurements given for the construction of this quadrilateral?

Ex. **1394.** Show how to construct a quadrilateral ABCD, given that AB = 6·1 cm., BC = 11·4 cm., CA = 11·7 cm., AD = 5·1 cm., $\angle$BDC = 76°.

Ex. **1395.** Show how to construct a parallelogram with base 2·8 in. and height 2 in., the angle (subtended by the base) between the diagonals being 80°. (Try to find the centre of the parallelogram.)

To inscribe in a given circle a triangle with given angles.

Let the radius of the circle be 2 in. and the angles of the required triangle 40°, 60° and 80°.

[*Analysis* Draw a sketch of the required figure; join the vertices of the triangle to the centre of the circle. What are the angles subtended at the centre by the three sides?

Knowing these three angles at the centre, it is easy to draw the required figure.]

Ex. **1396.** Draw the figure described above; measure the sides of the triangle. State the construction formally, and give a proof.

Ex. **1397.** Inscribe in a circle of radius 5 cm. a triangle of angles 30°, 80°, 70°. Measure the sides.

Ex. **1398.** Inscribe in a circle of radius 3·5 in. a triangle with angles 50°, 40°. Measure the sides.

Ex. 1399. Inscribe in a circle of radius 4 cm. an isosceles triangle having each of the angles at the base double the angle at the vertex. Measure the base.

Ex. 1400. Inscribe in a circle of radius 2·5 in. a triangle having two of its angles 35° and 40°. Measure the sides.

Ex. 1401. (Without protractor.) Inscribe in a circle of radius 6 cm. a triangle equiangular with a given triangle.

Ex. 1402. Copy fig. 266 on an enlarged scale; making the radius of the circle 2 in. Check the angles marked, and measure AC.

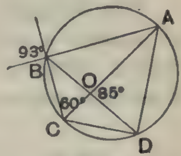


fig. 266.

To circumscribe about a given circle a triangle with given angles.

Let the radius of the given circle be 2·4 in.: the angles of the required triangle 45°, 70°, 65°.

[*Analysis* Draw a sketch of the required figure (fig. 267).

Join the centre O to L, M, N the points of contact of the sides.

If the angles at O can be calculated it will be easy to draw the figure.

Now ∠s AMO, ANO are right angles,

∴ ∠s MAN, MON are supplementary.

Hence calculate ∠ MON, and similarly the other angles at O.]

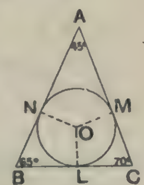


fig. 267.

Ex. 1403. Draw the figure described above. Measure the longest side of the triangle. State the construction formally, and give a proof.

Ex. 1404. Circumscribe about a circle of radius 5 cm. a triangle of angles 30°, 60°, 90°. Measure the longest side.

Ex. 1405. Circumscribe about a circle of radius 3 cm. an isosceles right-angled triangle. Measure the longest side.

Ex. 1406. Circumscribe about a circle of radius 4 cm. a parallelogram having an angle of 70°. Measure the sides of the parallelogram, and prove that it is a rhombus.

Ex. 1407. (Without protractor.) Circumscribe about a circle of radius 2·6 in. a triangle equiangular to a given triangle.

Ex. 1408. (Without protractor.) Circumscribe about a circle of radius 5 cm. a triangle having its sides parallel to three given straight lines.

Section IX. "Alternate Segment."

Theorem 14.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

Data AB touches ⊙ CDE in C; the chord CD is drawn through C, meeting ⊙ again in D.

To prove that (1) ∠ BCD = ∠ in alternate segment CED,

(2) ∠ ACD = ∠ in alternate segment CFD (fig. 269).

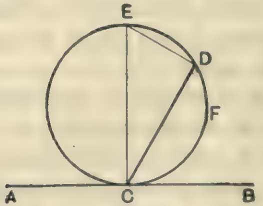


fig. 268.

(1) *Construction* Through C draw CE ⊥ to AB, meeting ⊙ in E. Join CE, DE.

Proof Since CE is drawn ⊥ to tangent AB, at its point of contact C,

∴ CE passes through centre of ⊙, and is a diameter, III. 6, *Cor.*

∴ ∠ CDE is a rt. ∠, III. 10.

∴ in △ CDE, ∠ CED + ∠ DCE = 1 rt. ∠. I. 8.

Now ∠ BCD + ∠ DCE = 1 rt. ∠. *Constr.*

∴ ∠ BCD + ∠ DCE = ∠ CED + ∠ DCE,

∴ ∠ BCD = ∠ CED.

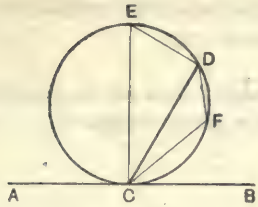


fig. 269.

(2) *Construction* Take any point F in arc CFD; join CF, DF.

$\angle$ BCD + $\angle$ ACD = 2 rt. $\angle$ s.

Also, since CFDE is a quadrilateral inscribed in a circle,

$\angle$ CED + $\angle$ CFD = 2 rt. $\angle$ s, III. 12.

$\therefore$ $\angle$ BCD + $\angle$ ACD = $\angle$ CED + $\angle$ CFD.

But $\angle$ BCD = $\angle$ CED, *Proved*

$\therefore$ $\angle$ ACD = $\angle$ CFD.

Q. E. D.

¶**Ex. 1409.** In fig. 269 point out an angle equal to $\angle$ BCF.

¶**Ex. 1410.** Taking CE as the chord (fig. 269), what is the segment alternate to $\angle$ ACE?

¶**Ex. 1411.** Find all the angles of fig. 269, supposing that $\angle$ BCD = 60°, and that $\angle$ FCD = 20°. What angles do the chords ED, CD, FC subtend at the centre?

Ex. 1412. Find all the angles of fig. 270.

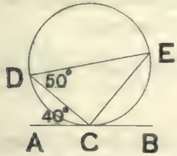


fig. 270.

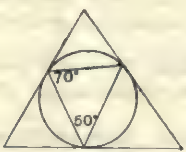


fig. 271.

Ex. 1413. Find all the angles of fig. 271.

¶**Ex. 1414.** Draw the tangent at a given point on a circle without finding (or using) the centre of the circle.

(*For further exercises on "Alternate segment" see Ex.* 1434—1438.)

III. 14 provides alternative methods of dealing with the constructions of section VIII.

On a given straight line AB to construct a segment of a circle to contain a given angle X.

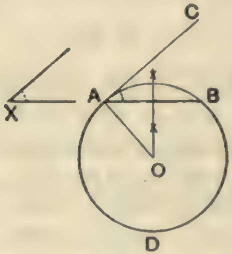


fig. 272.

Construction At A make ∠ BAC = ∠ X.

Construct a circle to pass through A and B, and to touch AC at A.

The segment ADB is the segment required.

Proof ∠ X = ∠ CAB (between tangent AC and chord AB)
= ∠ in alternate segment ADB.

Ex. **1415.** Show how to construct on a given straight line a segment of a circle to contain a given obtuse angle. (*Freehand.*)

Ex. **1416.** Show how to construct on a given base an isosceles triangle with a given vertical angle. (*Freehand.*)

Ex. **1417.** Show how to construct on a given base a triangle of given vertical angle and given median. Is this always possible? (*Freehand.*)

In a given circle to inscribe a triangle equiangular to a given triangle XYZ.

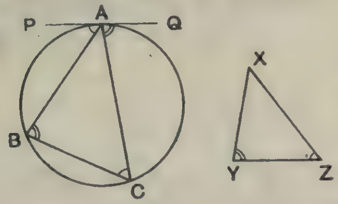


fig. 273.

[*Analysis* Suppose that the problem has been solved; and that ABC is the required $\triangle$.

Let PAQ be the tangent at A.

Then $\angle PAB = \angle ACB$ (in alternate segment)

$= \angle Z$,

and $\angle QAC = \angle ABC$ (in alternate segment)

$= \angle Y$.]

Hence:—

Construction At any point A on the given circle draw the tangent PAQ.

Make $\angle PAB = \angle Z$; let AB cut $\odot$ in B.

Make $\angle QAC = \angle Y$; let AC cut $\odot$ in C.

Join BC.

Then ABC is a triangle equiangular to $\triangle$ XYZ, inscribed in the given circle.

†Ex. **1418.** Give the proof of the above construction.

Ex. **1419.** (Without protractor.) In a circle of radius 3 in. inscribe a triangle equiangular to a given obtuse-angled triangle. Test the accuracy of the angles.

Ex. **1420.** In a circle of radius 2 in. inscribe a triangle having its sides parallel to three given straight lines.

TANGENT AS LIMIT OF CHORD.

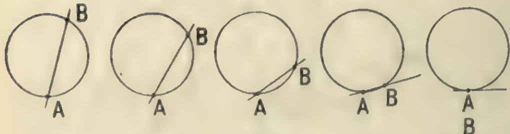


fig. 274.

Figs. 274 (i—iv) show four positions of a chord AB (produced both ways). Looking at the figures from left to right, the chord is seen to be turning about the point A; as it turns, the second point of intersection, B, comes nearer and nearer to A until in fig. v, B has coincided with A, and the chord has become the tangent at A.

A tangent therefore may be regarded as the **limit** of a chord whose two points of intersection with the circle have come to coincide.

Fig. 275 suggests another way in which the chord may approach its limiting position—the tangent.

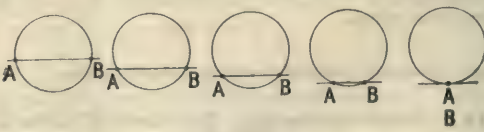


fig. 275.

Looking at the tangent from this point of view, it is interesting to see that the angle in a segment of a circle develops into the angle between the chord and the tangent at its extremity. This is shown by fig. 276.

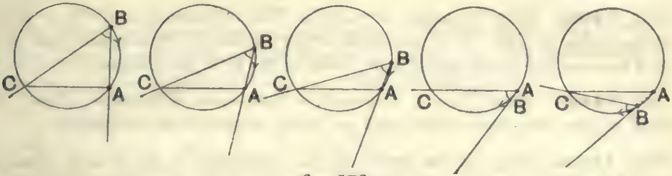


fig. 276.

¶Ex. **1421.** In fig. 275, what becomes of the theorem that "the perpendicular from the centre on a chord bisects the chord" when B comes to coincide with A?

¶Ex. **1422.** Prove III. 6 by means of fig. 275.

¶Ex. **1423.** In fig. 275, if O is the centre of the circle, what do the angles OAB, OBA become in the limiting case?

¶Ex. **1424.** What is the limiting form of Ex. 1300 when the circles touch?

Miscellaneous Exercises on Sections VI., VIII. and IX.

†Ex. **1425.** Through P, Q, the points of intersection of two circles, are drawn chords APB, CQD; prove that AC is ∥ to BD. [Join PQ.]

What does this theorem become if A, C are made to coincide?

†Ex. **1426.** Through P, Q, the points of intersection of two circles, are drawn parallel chords APB, CQD; prove that AB = CD.

†Ex. **1427.** If two opposite sides of a cyclic quadrilateral are equal, the other two are parallel.

†Ex. **1428.** Each of two equal circles passes through the centre of the other: AB is their common chord. Through A is drawn a line cutting the two circles again in C, D; prove that △BCD is equilateral.

†Ex. **1429.** ABC is an equilateral triangle inscribed in a circle; P is any point on the minor arc BC. Prove that PA = PB + PC. [Make PX = PB. Then prove XA = PC.]

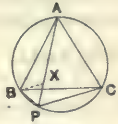


fig. 277.

†Ex. **1430.** In fig. 229, B, C, I_1, and the centre of the inscribed circle are concyclic.

†Ex. **1431.** From a point on the diagonal of a square, lines PR, QS are drawn parallel to the sides, P, Q, R, S being on the sides. Prove that these four points are concyclic.

†Ex. **1432.** O is the centre of a circle, CD a diameter, and AB a chord perpendicular to CD. If B is joined to any point E in CD, and BE produced to meet the circle again in F, then A, O, E, F are concyclic.

Ex. **1433.** Show how to construct a right-angled triangle, given the radius of the inscribed circle, and an acute angle of the triangle.

†Ex. **1434.** Two circles touch at A. Through A are drawn straight lines PAQ, RAS; cutting the circles in P, Q and R, S. Prove that PR is parallel to QS. (Draw tangent at A. Compare Ex. 1425.)

†Ex. **1435.** Two circles cut at P, Q. A, a point on the one circle, is joined to P, Q; and these lines are produced to meet the other circle in B, C. Prove that BC is parallel to the tangent at A. (Compare Ex. 1425.)

†Ex. **1436.** A chord AB of a circle bisects the angle between the diameter through A, and the perpendicular from A upon the tangent at B.

†Ex. **1437.** ABCD is a cyclic quadrilateral, whose diagonals intersect at E: a circle is drawn through A, B and E. Prove that the tangent to this circle at E is parallel to CD.

†Ex. **1438.** AB, AC are two chords of a circle; BD is drawn parallel to the tangent at A, to meet AC in D; prove that $\angle ABD$ is equal or supplementary to $\angle BCD$. Hence show that the circle through B, C and D touches AB at B.

Section X. Arcs and Angles at the Circumference.

¶Ex. **1439.** Draw a circle of radius 2·5 in.; draw a diameter OP_5 and a tangent AOB as in fig. 278. Divide $\angle AOP_5$ into five equal parts; also $\angle BOP_5$. Measure the chords OP_1, P_1P_2, ... etc. What angle does P_2P_3 subtend at the centre of the circle? Prove that OP_1P_2... etc. are the vertices of a regular decagon.

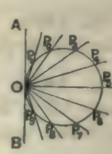


fig. 278.

¶Ex. **1440.** In the fig. of Ex. 1439 draw any straight line cutting across the set of lines OP_1, OP_2, OP_3, etc. Is this line divided into equal parts?

¶Ex. **1441.** Take a point O, 1 in. from the centre of a circle of radius 2·5 in.; draw through O a diameter and a set of chords making angles of 18° with one another. Find by measurement whether these chords divide the circumference into equal arcs.

¶Ex. **1442.** Would the circumference be divided into equal arcs if the point O in Ex. 1441 were taken at the centre? How many arcs would there be?

†Ex. **1443.** **Prove that equal arcs or chords of a circle subtend equal (or supplementary) angles at a point on the circumference.** Draw a figure to illustrate the case of supplementary angles.

Prove the converse.

NOTE. *In the following exercises (Ex.* 1445—1462) *the student is advised to make use of "the angle subtended at the circumference."*

†Ex. **1444.** Draw a regular pentagon ABCDE in a circle. Prove that the angle A is trisected by AC, AD.

†Ex. **1445.** ABCDE is a regular pentagon.

(i) Prove that AB is parallel to EC. (Join AC.)

(ii) At what angle do BD, CE, intersect?

(iii) Prove that △ ACD is isosceles, and that each of its base angles is double its vertical angle.

(iv) If BD, CE meet at X, prove that △s CXD, CDE are equiangular.

(v) Prove that the tangent to the circle at A is parallel to BE. [Use III. 14.]

†Ex. **1446.** AB, CD are parallel chords of a circle. Prove that arc AC = arc BD.

†Ex. **1447.** On a circle are marked off equal arcs AC, BD. Prove that AD is parallel or equal to CB.

†Ex. **1448.** AOB, COD are two chords of a circle, intersecting at right angles. Show that arc AC + arc BD = arc CB + arc DA.

†Ex. **1449.** Through a given point draw a chord of a given circle so that the minor segment cut off may be the least possible.

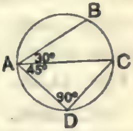


fig. 279.

†Ex. **1450.** Prove that in fig. 278

$$\text{arc } OP_1 = \text{arc } P_1P_2.$$

Ex. **1451.** In fig. 279 what fractions of the circumference are the arcs AB, BC, CD, DA, BCD?

Ex. **1452.** In fig. 280 what fractions of the circumference are the arcs AB, BC, CD, DA?

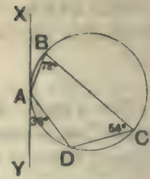


fig. 280.

Ex. **1453.** ABC and ADEFG are respectively an equilateral triangle and a regular pentagon inscribed in a circle. What fraction of the circumference is the arc BD?

Ex. **1454.** PQRS is a quadrilateral inscribed in a circle; the two diagonals intersect at A. PQ is an arc of 30° (see p. 233), QR 100°, RS 70°. Find all the angles in the figure.

Ex. **1455.** In the figure of Ex. 1454 find two pairs of equiangular triangles.

Ex. **1456.** If in fig. 268 arc CD = 2 arc DE, what is ∠BCD?

†Ex. **1457.** In fig. 268, the bisector of ∠BCD bisects arc CD.

Ex. **1458.** If in fig. 268 arc ED were $\frac{1}{2}$ arc DC, what would be the magnitude of ∠BCD?

Ex. **1459.** The two tangents OA, OB from a point O are inclined at an angle of 48°. How many degrees are there in the minor and major arc AB respectively? What is the ratio of the major to the minor arc?

†Ex. **1460.** P is a variable point on an arc AB. Prove that the bisector of ∠APB always passes through a fixed point.

[Begin by finding the probable position of the fixed point by experiment.]

†Ex. **1461.** A, B, C are three points on a circle. The bisector of ∠ABC meets the circle again at D. DE is drawn ∥ to AB and meets the circle again at E. Prove that DE = BC.

†Ex. **1462.** A tangent is drawn at one end of an arc; and from the midpoint of the arc perpendiculars are drawn to the tangent, and the chord of the arc. Prove that they are equal.

Regular Polygons*.

Def. A polygon which is both equilateral and equiangular is said to be **regular.**

¶Ex. **1463.** What is the name for a quadrilateral that is (i) equilateral and not equiangular, (ii) equiangular and not equilateral, (iii) regular?

¶Ex. **1464.** Draw a hexagon that is equiangular but not equilateral.

¶Ex. **1465.** Is there any polygon which is necessarily regular if it is *either* (i) equilateral, *or* (ii) equiangular?

* The section on regular polygons should be omitted at a first reading.

THEOREM 15. †

If the circumference of a circle be divided into n equal arcs, (1) the points of division are the vertices of a regular n-gon inscribed in the circle; (2) if tangents be drawn to the circle at these points, these tangents are the sides of a regular n-gon circumscribed about the circle.

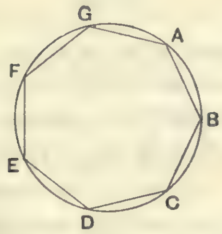


fig. 281.

(1) *Data* The circumference is divided into n equal arcs at the points A, B, C, D, E, F, G.

The chords AB, BC, etc. are drawn forming the inscribed n-gon ABCDEFG.

To prove that ABCDEFG is regular.

Proof Since arcs AB, BC, etc. are equal,

∴ chords AB, BC, etc. are equal, III. 4.

∴ ABCDEFG is equilateral.

Again, arc GA = arc BC, *Data*

∴ adding arc AB to both,

arc GAB = arc ABC,

∴ ∠ GAB = ∠ ABC, these angles being contained in equal arcs.

Sim[ly] it may be shown that all the ∠s of the polygon are equal; i.e. that ABCDEFG is equiangular.

∴ ABCDEFG, being equilateral and equiangular, is regular.

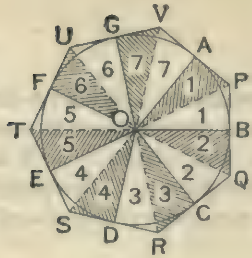


fig. 282.

(2) *Data* The tangents at A, B, C, etc. are drawn forming the circumscribed n-gon PQRSTUV.

To prove that PQRSTUV is regular.

Construction Join P, Q, R, etc. to the centre O.

Proof Show that

(i) adjacent $\triangle$s, similarly numbered, are congruent.

(ii) adjacent $\triangle$s, differently numbered, are congruent

(by (i) $\angle POB = \frac{1}{2} \angle AOB$, $\angle QOB = \frac{1}{2} \angle COB$,
$\therefore \angle POB = \angle QOB$).

(iii) all the numbered $\triangle$s are congruent.

(iv) PQRSTUV is equilateral.

(v) PQRSTUV is equiangular.

$\therefore$ PQRSTUV is regular.

Q. E. D.

†Ex. 1465 (a). **(Alternative proof of Th. 15 (2).)**

Join ED, DC.

Prove that

(1) $\triangle$s ESD, DRC are isosceles,

(2) $\angle EDS = \angle CDR$ (by means of angles in alternate segments),

(3) $\triangle$s ESD, DRC are congruent.

$\therefore$ etc.

Ex. **1466.** Construct a regular pentagon of side 2 in. (see Ex. 398); draw the circumscribed and inscribed circles and measure their radii.

Ex. **1467.** Repeat Ex. 1466 with a regular octagon of side 2 in. (Without protractor.)

Ex. **1468.** Find the perimeter and area of a regular 6-gon circumscribed about a circle of radius 5 cm.

†Ex. **1469.** Prove that an equilateral polygon inscribed in a circle must also be equiangular.

Ex. **1470.** Is the converse of Ex. 1469 true?

†Ex. **1471.** Prove that an equiangular polygon circumscribed about a circle must also be equilateral.

Ex. **1472.** Is the converse of Ex. 1471 true?

Ex. **1473.** The area of the square circumscribed about a circle is twice the area of the square inscribed in the same circle.

Ex. **1474.** Prove that the area of the regular hexagon inscribed in a circle is twice the area of the inscribed equilateral triangle. Verify this fact by cutting a regular hexagon out of paper, and folding it.

Ex. **1475.** The side of an equilateral triangle circumscribed about a circle is twice the side of an inscribed equilateral triangle.

†Ex. **1476.** The exterior angle of a regular n-gon is equal to the angle which a side subtends at the centre.

†Ex. **1477.** The lines joining a vertex of a regular n-gon to the other vertices divide the angle into $(n-2)$ equal parts.

SECTION XI. AREA OF CIRCLE.

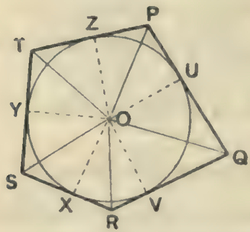


fig. 283.

Let PQRST be a polygon (not necessarily regular) circumscribing a circle.

Join the vertices of the polygon to the centre of the circle. The circle is thus divided into a number of triangles, having for bases the sides of the polygon, and for vertex the centre of the circle.

Draw perpendiculars from the centre to the sides of the polygon. These meet the sides at their points of contact and are radii of the circle. Thus the triangles OPQ, OQR, etc. are all of height equal to the radius of the circle.

Let r be the radius of the circle, a, b, c, d, e the sides of the polygon ($PQ = a$, $QR = b$, etc.).

The area of $\triangle OPQ$ is $\frac{1}{2}ar$; $\triangle OQR = \frac{1}{2}br$, etc.

$$\begin{aligned} \therefore \text{ area of polygon} &= \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr + \tfrac{1}{2}dr + \tfrac{1}{2}er \\ &= \tfrac{1}{2}r(a + b + c + d + e) \\ &= \tfrac{1}{2} \text{ radius} \times \text{perimeter of polygon.} \end{aligned}$$

This is true for any polygon circumscribing the circle.

If we draw a polygon of a very great number of sides, it is difficult to distinguish it from the circle itself. The area of the polygon approaches closer and closer to the area of the circle; and the perimeter of the polygon to the circumference of the circle. Hence we conclude that

$$\textbf{area of a circle} = \tfrac{1}{2}\text{ radius} \times \text{circumference of circle}$$
$$= \tfrac{1}{2} r \times 2\pi r$$
$$= \pi r^2.$$

[*In the following exercises it will generally be sufficient if answers are given correct to three significant figures.*]

Ex. **1478.** Calculate the area of a circle whose radius is 1 inch. Also draw the circle on inch paper and find the area by counting the squares.

Ex. **1479.** Repeat Ex. 1478 for a circle of radius 2 in. Check your result by squared paper.

Ex. **1480.** The radius of one circle is twice the radius of another; how many times does the area of the greater contain the area of the smaller? Fig. 284 shows that the area of the greater is more than double the area of the smaller. Find the area of the shaded part of fig. 284, taking the diameter of the small circles to be 1 cm.

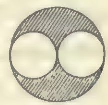


fig. 284.

Ex. **1481.** Find the ratio of the area of a circle to the area of the circumscribing square.

Ex. **1482.** Squares are inscribed and circumscribed to a circle (fig. 285); how many times does the circumscribed square contain the inscribed square?

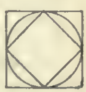


fig. 285.

Ex. **1483.** What is the ratio of the area of the circle to the area of the inscribed square?

Ex. **1484.** Find the area of a circle, given (i) radius $= 5{\cdot}72$ cm., (ii) diameter $= 1$ in. (the size of a halfpenny), (iii) $r = 0{\cdot}59$ in.

Ex. **1485.** Find, in square inches, the area of one side of a penny.

Ex. **1486.** Draw an equilateral triangle of side 10 cm. and its circumscribing circle; make the necessary measurements and calculate the area of the circle. Find the ratio of the area of the circle to that of the triangle.

Ex. **1487.** Find the ratio of the area of a circle to the area of the inscribed regular hexagon. (Compare result with those of Ex. 1483 and 1486.)

Ex. **1488.** In the centre of a circular pond of radius 100 yards is a circular island of radius 20 yards. Find the area of the surface of the water.

Ex. **1489.** Find whether the area in Ex. 1488 is greater or less than the area of a circular sheet of water of 80 yards radius.

Ex. **1490.** The radius of the inside edge of a circular running track is a feet; and the width of the track is b feet; find the area of the track.

Ex. **1491.** From a point P, on the larger of two concentric circles, a tangent PT is drawn to the smaller. Show that area of the circular ring between the circles is $\pi \cdot PT^2$.

Ex. **1492.** Show how to draw a circle equal to (i) the sum, (ii) the difference of two given circles.

Ex. **1493.** Calculate the radius of a circle whose area is 1 sq. in.

Ex. **1494.** Calculate the diameter of a circular field whose area is 1 acre (=4840 sq. yards).

Ex. **1495.** Let A = area of circle, c = circumference, r = radius, d = diameter.

(i) Express c in terms of r,
(ii) .... c .... d,
(iii) .... A .... r,
(iv) .... A .... d,
(v) .... r .... c,
(vi) .... d .... c,
(vii) .... r .... A,
(viii) .... d .... A,
(ix) .... A .... c,
(x) .... c .... A.

Ex. **1496.** Find the radius and circumference of a circle whose area is (i) 6 sq. in., (ii) 765 sq. cm.

Ex. **1497.** Calculate the area of a circle whose circumference is 25,000 miles. (Find r first.)

Ex. **1498.** Prove that in fig. 224 the three portions into which the circle is divided by the curved lines are of equal area.

†Ex. **1499.** Prove that if circles are described with the hypotenuse and the two sides of a right-angled triangle for diameters, the area of the greatest is the sum of the areas of the other two.

†Ex. **1500.** In fig. 286 ∠ BAC is a right angle, and the curves are semicircles. Prove that the two shaded areas are together equal to the triangle.

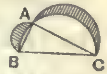


fig. 286.

Area of sector of circle.

If through the centre of a circle were drawn 360 radii making equal angles with one another, 360 angles of 1 degree would be formed at the centre of the circle. The area of the circle would be divided into 360 equal sectors. A sector of angle 1° has therefore $\frac{1}{360}$ of the area of the circle; and a sector of angle, say, 53° contains $\frac{53}{360}$ of the area of the circle.

Ex. **1501.** Find the area of a sector of 40° in a circle of radius 5 in.

Ex. **1502.** Find the area of a sector of 87° in a circle of radius 12·4 cm.

Ex. **1503.** Find the areas of the two sectors into which a circle of diameter 12·5 inches is divided by two radii inclined at an angle of 60°.

Ex. **1504.** Calculate the area of a sector whose chord is 3 in. in a circle of radius 4 in. (find the angle by measurement).

Ex. **1505. Prove that the area of a sector of a circle is half the product of the radius and the arc of the sector.**

Area of segment of circle.

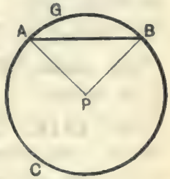


fig. 287.

In fig. 287,

segment AGB = sector PAGB − triangle PAB.

Ex. **1506.** Find the areas of the two segments into which a circle radius 10 cm. is divided by a chord of 10 cm.

Ex. **1507.** Repeat Ex. 1506 with the same circle and a chord of 20 cm.

Ex. **1508.** Repeat Ex. 1506 with a chord that subtends 90° at the centre.

Ex. **1509.** Find the area of a segment whose chord is 12 cm. and height 3 cm. Also find the ratio of the segment to the rectangle of the same base and height.

Ex. **1510.** Find the area of a segment of base 10 cm. and height 5 cm.

Ex. **1511.** Find the area of a segment of base 4 cm. and height 8 cm.

Ex. **1512.** A square is inscribed in a circle of radius 2 in. Find the area of a segment cut off by a side of the square.

Ex. **1513.** From a point outside a circle of radius 10 cm., a pair of tangents are drawn to the circle; the angle between the tangents is 120°. Find the area included between the two tangents and the circumference.

Section XII. Further Examples of Loci.

Ex. **1514.** Plot the locus of points the sum of whose distances from two fixed points remains constant.

(Mark two points S, H, say, 4 in. apart. Suppose that the point P moves so that $SP+HP=5$ in. Then the following are among the possible pairs of values:

SP	4·5	4·0	3·5	3·0	2·5	2·0	1·5	1·0	0·5
HP	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0	4·5

Plot all the points corresponding to all these distances, by means of intersecting arcs. Why were not values such as $SP=4{\cdot}7$, $HP=0{\cdot}3$ included in the above table? Draw a neat curve, free-hand, through all these points. The locus is an oval curve called an **ellipse**.)

¶Ex. **1515.** What kinds of symmetry are possessed by an ellipse?

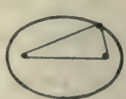


fig. 288.

Ex. **1516.** Describe an ellipse mechanically as follows. Stick two pins into the paper about 4 in. apart; make a loop of fine string, gut or cotton and place it round the pins (see fig. 288). Keep the loop extended by means of the point of a pencil, and move the point round the pins. It will, of course, describe an ellipse.

Ex. **1517.** Plot the locus of points the difference of whose distances from two fixed points remains constant.

(For example, let the two fixed points S, H be 4 in. apart, and let the constant difference be 2 in. Make a table as in Ex. 1514. Remember to make $SP > HP$ for some points, $HP > SP$ for other points.)

This curve is called a **hyperbola.**

Ex. **1518.** Plot the locus of points the product of whose distances from two fixed points remains constant.

(For example, mark two points S, H exactly 4 in. apart. First, to plot the locus $SP \cdot HP = 5$.

Fill up the blanks in the following table:

SP	5	4·8	4	3	$\sqrt{5}$	2				
HP							3	4	4·8	5

Secondly, plot the locus $SP \cdot HP = 4$; thirdly, plot the locus $SP \cdot HP = 3$. All three loci should be drawn in the same figure.

The first locus will be found to resemble a dumb-bell, the second a figure of 8; the third consists of two separate ovals.)

Ex. **1519.** Plot the locus of a point which moves so that the ratio of its distances from two fixed points remains constant.

(For example, let the two fixed points S, H be taken 3 in. apart; and let $\frac{SP}{HP} = 2$.)

Ex. **1520.** OP is a variable chord passing through a fixed point O on a circle; OP is produced to Q so that $PQ = OP$; find the locus of Q.

Ex. **1521.** A point moves so that its distance from a fixed point S is always equal to its distance from a fixed line MN: find its locus.

(This is best done on inch paper. Take the point S 2 in. distant from the line MN. Then plot points as follows. What is the locus of points distant 3 in. from MN? distant 3 in. from S? The intersection of these two loci gives two positions of the required point. Similarly find other points.)

The curve obtained is called a **parabola.** It is the same curve as would be obtained by plotting the graph $y = \frac{x^2}{4} + 1$, taking for axis of x the line MN, and for axis of y the perpendicular from S to MN. It is remarkable as being the curve described by a projectile, e.g. a stone or a cricket-ball. Certain comets move in parabolic orbits, the sun being situated at the point S.

Ex. **1522.** A point moves in a plane subject to the condition that its distance from a fixed point S is always in a fixed ratio to its distance from a fixed straight line MN. Plot the curve described.

(i) Let the distance from S be always half the distance from MN. Take S 3 in. from MN.

(ii) Let the distance from S be always twice the distance from MN. Take S 3 in. from MN.

These curves will be recognized as having been obtained already.

Ex. **1523.** Plot the locus of a point on the connecting-rod of a steam-engine.

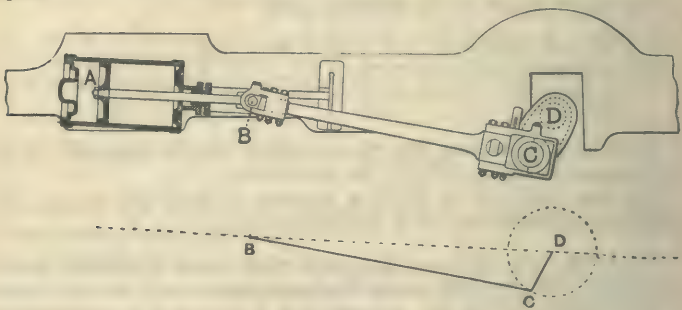


fig. 289.

(The upper diagram in fig. 289 represents the cylinder, piston-rod (AB), connecting-rod (BC), and crank (CD) of a locomotive. In the lower diagram the different parts are reduced to lines. B moves to and fro along a straight line, C moves round a circle. Take BC = 10 cm., CD = 3 cm. Plot the locus of a point P on BC, where BP is (i) 1 cm., (ii) 5 cm., (iii) 9 cm. This may be done, either by drawing a large number of different positions of BC; or, much more easily, by means of **tracing paper.** Draw BD and the circle on your drawing paper, BC on tracing paper. Keep the two ends of BC on the straight line and circle respectively, and prick through the different positions of P.)

Ex. **1524.** A rod moves so that it always passes through a fixed point while one end always lies on a fixed circle. Plot the locus of the other end.

(Tracing paper should be used. A great variety of curves may be obtained by varying the position of point and circle, and the length of the rod. It will be seen that this exercise applies to the locus of a point on the piston-rod of an oscillating cylinder; also to the locus of a point on the stay-bar of a casement window.)

Ex. **1525.** The ends of a rod slide on two wires which cross at right angles. Find the locus of a point on the rod.

(Represent the rod by a line of 10 cm.; take the point 3 cm. from one end of the rod; also plot the locus of the mid-point. Use tracing paper.)

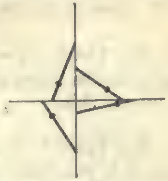


fig. 290.

Ex. **1526.** Two points A, B of a straight line move along two lines intersecting at right angles. Plot the locus of a point P, in AB produced. [Tracing paper.]

Ex. **1527.** Draw two intersecting lines. On tracing paper mark three points A, B, C. Make A slide along one line and B along the other; plot the locus of C.

Ex. **1528.** Draw two equal circles of radius 4 cm., their centres being 10 cm. apart. The two ends of a line PQ, 10 cm. in length, slide one along each circle. Plot the locus of the mid-point of PQ; also of a point 1 cm. from P.

(Most quickly done with tracing paper. It is easy to construct a model machine to describe the curve.)

Ex. **1529.** Draw two circles. On tracing paper mark three points A, B, C. Make A slide along one circle, B along another, and plot the locus of C. (Experiment with different circles and arrangements of points. You will find that in at least one case the locus-curve shrinks to a single point.)

Ex. **1530.** OA, AP are two rods jointed at A. OA revolves about a hinge at O, and AP revolves twice as fast as OA, in the same direction. Find the path of a point on AP. (Make OA = 2 in., AP = 2 in. Plot the locus of

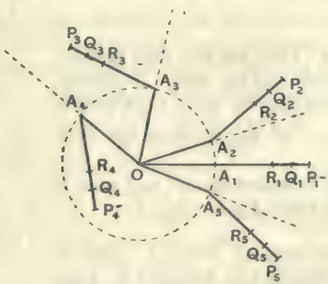


fig. 291.

P; also of Q and R, taking AQ = 1 in., AR = $\frac{1}{2}$ in. To draw the different positions of the rod, notice that when OA has turned through, say, 30°, AP has turned through 60° and therefore makes an angle of 30° with OA produced.)

The loci are different forms of the **limaçon**; the locus of Q is heart-shaped, and is called a **cardioid**. The locus of P has a small loop in it.

Ex. **1531.** Repeat Ex. 1530, with the difference that, as OA revolves, AP remains parallel to its original position.

Ex. **1532.** Two equal rods OA, AQ, jointed as in Ex. 1530, revolve in *opposite* directions at the same rate. Find the locus of Q and of the mid-point of AQ.

Ex. **1533.** O is a fixed point on a circle of radius 1 in. OP, a variable chord, is produced to Q, PQ being a fixed length; also PQ′ (= PQ) is marked off along PO. Plot the locus of Q and Q′ when PQ is (i) $2\frac{1}{2}$ in., (ii) 2 in., (iii) $1\frac{1}{2}$ in.

(Draw a long line on tracing paper, and on it mark P, Q and Q′.)

Ex. **1534.** Through a fixed point S is drawn a variable line SP to meet a fixed line MN in P. From P a fixed length PQ is measured off along SP (or SP produced). Find the locus of Q.

(Use tracing paper. Take S 1 in. from MN. Plot the locus of Q

(i) when PQ = 1 in., measured from P away from S,

(ii) when PQ = 1 in., measured from P towards S,

(iii) when PQ = 2 in., measured from P towards S.)

The curves obtained are different forms of the **conchoid**.

Ex. **1535.** A company of soldiers are extended in a straight line. At a given signal, they all begin to move towards a certain definite point, at the regulation pace. Are they in a straight line after 3 minutes? If not, what curve do they form?

Ex. **1536.** XOX′, YOY′ are two fixed straight lines, C is a fixed point (see fig. 292). A variable line PQ is drawn through C to meet XOX′, YOY′ in P, Q respectively. Plot the locus of the mid-point of PQ.

(Let XOX′, YOY′ intersect at 60°, and take C on the bisector of ∠XOY, 5 cm. from O.)

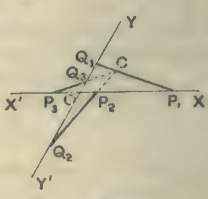


fig. 292.

Ex. **1537.** (Inch paper.) Draw a circle of radius 2 in. and a straight line distant 6 in. from the centre of the circle. P is a variable point on the circle; Q is the mid-point of PN, the perpendicular from P upon the line. Plot the locus of Q.

ENVELOPES.

We have seen that a set of points, plotted in any regular way, marks out a curve which is called the locus of the points.

In a rather similar manner, a set of lines (straight or curved) drawn in any regular way, marks out a curve which is called the **envelope** of the lines. Each of the lines touches the envelope.

Let a piece of paper be cut out in the shape of a circle, and a point **S** marked on it. Then fold the paper so that the circumference of the circle may pass through **S**. If this is done many times, the creases left on the paper will envelope an ellipse (fig. 293).

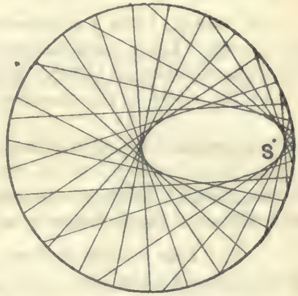


fig. 293.

Ex. **1538.** Take a piece of cardboard with one edge straight; drive a pin through the cardboard into the paper underneath; then turn the cardboard round the pin, and in each position use the straight edge of the cardboard to rule a line. What is the envelope of these lines?

Ex. **1539.** One edge of a flat ruler is made to pass through a fixed point, and lines are drawn with the other edge. Find their envelope.

Ex. **1540.** Prove that the envelope of straight lines which lie at a constant distance from a fixed point is a circle.

Ex. **1541.** Find the envelope of equal circles whose centres lie on a fixed straight line.

Ex. **1542.** Find the envelope of a set of equal circles whose centres are on a fixed circle when the radius of the equal circles is (i) less than, (ii) equal to, (iii) greater than, the radius of the fixed circle.

Ex. **1543.** Draw a straight line **MN** and drive a pin into your paper at a point **S** $\frac{1}{2}$ in. from **MN** (see fig. 294). Keep the short edge (**AB**) of your set-square pressed against the pin, and keep the right angle (**B**) on the line **MN**. Rule along **BC**; and thus plot the envelope of **BC**, as the set-square slides on the paper. (Lines must of course be drawn with the set-square placed on the left of **S**, as well as on the right.)

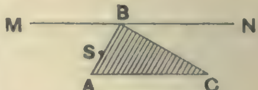


fig. 294.

Ex. **1544.** Repeat Ex. 1543 using the 30° angle instead of the right angle, and putting the pin 1 in. from **MN**.

Ex. **1545.** Draw a circle of radius 5 cm. and mark a point **S** 4 cm. from the centre. Let a variable line **SP** meet the circle in **P** and let **PQ** be drawn perpendicular to **SP**. Find the envelope of **PQ**. (The part of **PQ** *inside* the circle is the important part.)

Ex. **1546.** Repeat Ex. 1545 with the point **S** on the circle.

Ex. **1547.** Find the envelope of circles passing through a fixed point **O**, and having their centres on a fixed circle.

(i) Take radius of fixed circle = 4 cm., distance of **O** from centre of fixed circle = 3·2 cm.

(ii) Take radius of fixed circle = 4 cm., distance of **O** from centre of fixed circle = 4 cm.

(iii) Take radius of fixed circle = 3 cm., distance of **O** from centre of fixed circle = 5 cm.

Ex. **1548.** Find the envelope of circles passing through a fixed point, and having their centres on a fixed straight line.

Ex. **1549.** Plot the envelope of a straight line of constant length whose ends slide upon two fixed lines at right angles.

MISCELLANEOUS EXERCISES.

Ex. **1550.** (Without protractor.) Trisect an arc of 90°.

Ex. **1551.** (Without protractor.) Trisect a given semicircular arc.

†Ex. **1552.** There are two fixed concentric circles; AB is a variable diameter of the one, and P a variable point on the other. Prove that $AP^2 + BP^2$ remains constant.

[Use Apollonius' theorem, Ex. 1133.]

Ex. **1553.** In a circle of radius 2·5 in. inscribe an isosceles triangle of vertical angle 40°. Measure its base.

†Ex. **1554.** Points A, P, B, Q, C, R are taken in order on a circle so that arc AP = arc BQ = arc CR. Prove that the triangles ABC, PQR are congruent.

Ex. **1555.** The railway from P to Q consists of a circular arc AB and two tangents PA, BQ. AB is an arc of 28° of a circle whose radius is ½ mile; PA = 1 mile, BQ = ½ mile. Draw the railway, on a scale of 2 inches to the mile, and measure the distance from P to Q as the crow flies. Also calculate the distance as the train goes.

†Ex. **1556.** From a point P on a circle, a line PQ of constant length is drawn parallel to a fixed line. Plot the locus of Q, as P moves round the circle. Having discovered experimentally the shape of the locus, prove it theoretically.

†Ex. **1557.** YZ is the projection of a diameter of a circle upon a chord AB; prove that AY = BZ.

†Ex. **1558.** Through two given points P, Q on a circle draw a pair of equal and parallel chords. Give a proof.

†Ex. **1559.** AOB, COD are two variable chords of a circle, which are always at right angles and pass through a fixed point O. Prove that $AB^2 + CD^2$ remains constant.

†Ex. **1560.** Through A, a point inside a circle (centre O), is drawn a diameter BAOC; P is any point on the circle. Prove that AC > AP > AB.

Ex. **1561.** What is the length of (i) the shortest, (ii) the longest chord of a circle of radius r, drawn through a point distant d from the centre?

Ex. **1562.** Two chords of a circle are at distances from its centre equal to $\frac{3}{5}$ and $\frac{4}{5}$ of its radius. Find how many times the shorter chord is contained in three times the longer chord.

Ex. **1563.** The star-hexagon in fig. 295 is formed by producing the sides of the regular hexagon. Prove that the area of the star-hexagon is twice that of the hexagon.

fig. 295.

†Ex. **1564.** Chords AP, BQ are drawn $\perp$ to a chord AB at its extremities. Prove that AP = BQ.

†Ex. **1565.** The line joining the centre of a circle to the point of intersection of two tangents is the perpendicular bisector of the line joining the points of contact of the tangents.

†Ex. **1566.** Find the locus of the point of intersection of tangents to a circle which meet at an angle of 60°.

Ex. **1567.** Show how to construct a right-angled triangle, given that the radius of the inscribed circle is 2 cm. and that one of the sides about the right angle is 5 cm.

Ex. **1568.** Construct an isosceles triangle, given the radius of the inscribed circle, and the base.

†Ex. **1569.** A is a point outside a given circle (centre O, radius r). With centre O and radius $2r$ describe a circle; with centre A and radius AO describe a circle; let these two circles intersect at B, C. Let OB, OC cut the given circle at D, E. Prove that AD, AE are tangents to the given circle.

†Ex. **1570.** A circle is drawn having its centre on a side AC (produced) of an isosceles triangle, and touching the equal side AB at B. BC is produced to meet the circle at D. Prove that the radius of the circle through D is perpendicular to AC.

Ex. **1571.** Find the angles subtended at the centre of a circle by the three segments into which any tangent is divided by the sides (produced if necessary) of a circumscribed square.

†Ex. **1572.** An interior common tangent of two circles cuts the two exterior common tangents in A, B. Prove that AB is equal to the length intercepted on an exterior tangent between the points of contact.

†Ex. **1573.** The radius of the circumcircle of an equilateral triangle is twice the radius of the in-circle.

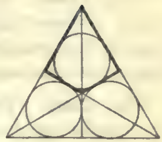


fig. 296.

Ex. 1574. Show how to inscribe three equal circles to touch one another in an equilateral triangle, of side 6 in. (fig. 296).

Ex. 1575. Show how to inscribe in a square, of side 6 in., four equal circles, each circle to touch two others.

†**Ex. 1576.** Two circles touch externally at E; AB, CD are parallel diameters drawn in the same sense (see page 78, footnote); prove that AE, ED are in the same straight line; as also are BE, EC.

†**Ex. 1577.** Two circles touch at A; T is any point on the tangent at A; from T are drawn tangents TP, TQ to the two circles. Prove that TP = TQ. What is the locus of points from which equal tangents can be drawn to two circles in contact?

†**Ex. 1578.** S is the circumcentre of a triangle ABC, and AD is an altitude. Prove that $\angle$BAD = $\angle$CAS.

†**Ex. 1579.** Through a given point on the circumference of a circle draw a chord which shall be bisected by a given chord. Give a proof.

Ex. 1580. From the given angles, find all the angles of fig. 297.

Draw the figure, making the radius of the circle 2 in. Check the marked angles, and measure CD.

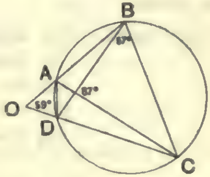


fig. 297.

†**Ex. 1581.** Two circles intersect at B, C; P is a variable point on one of them. PB, PC (produced if necessary) meet the other circle at Q, R. Prove that QR is of constant length.

[Show that it subtends a constant angle at B.]

Ex. 1582. Show how to find a point O inside $\triangle$ABC so that

$$\angle \text{AOB} = 150°, \quad \angle \text{AOC} = 130°.$$

Ex. 1583. Show how to find a point O inside $\triangle$ABC, such that the three sides subtend equal angles at O.

Ex. 1584. Show how to construct a triangle, having given the vertical angle, the altitude and the bisector of the vertical angle (terminated by the base).

†**Ex. 1585.** A is one of the points of intersection of two circles whose centres are C, D. Through A is drawn a line PAQ, cutting the circles again in P, Q. PC, QD are produced to meet at R. Prove that the locus of R is a circle through C and D.

†Ex. **1586.** A, C are two fixed points, one upon each of two circles which intersect at B, D. Through B is drawn a variable chord PBQ, cutting the two circles in P, Q. PA, QC (produced if necessary) meet at R. Prove that the locus of R is a circle.

†Ex. **1587.** Two equal circles cut at A, B; a straight line PAQ meets the circles again in P, Q. Prove that BP = BQ. [Consider the angles subtended by the two chords.]

†Ex. **1588.** C is a variable point on a semicircle whose diameter is AB, centre O; CD is drawn ⊥ to AB; OX is the radius ⊥ to AB. On OC a point M is taken so that OM = CD. Prove that the locus of M is part of a circle whose diameter is OX.

†Ex. **1589.** ABC, DCB are two congruent triangles on the same side of the base BC. Prove that A, B, C, D are concyclic.

†Ex. **1590.** D, E, F are the mid-points of the sides of BC, CA, AB of △ABC; AL is an altitude. Prove that D, E, F, L are concyclic (see Ex. 1589).

†Ex. **1591. Prove that the circle through the mid-points of the sides of a triangle also passes through the feet of the altitudes** (see Ex. 1590).

†Ex. **1592.** The altitudes BE, CF of △ABC intersect at H; prove

(i) that AEHF is a cyclic quadrilateral,
(ii) that ∠FAH = ∠FEH,
(iii) that ∠FEH = ∠FCB,
(iv) that, if AH is produced to meet BC in D, AFDC is cyclic,
(v) that AD is ⊥ to CB.

Hence: **The three altitudes of a triangle meet in a point**; which is called the **orthocentre.**

†Ex. **1593.** In fig. 298 AD is ⊥ to BC and BE is ⊥ to CA; S is the centre of the circle. Show that

BF = AH,

and that AB, FH bisect one another.

[Prove AHBF a parallelogram.]

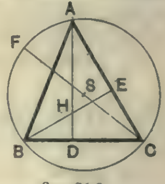


fig. 298.

†Ex. **1594.** BE, CF, two altitudes of △ABC, intersect at H. BE produced meets the circumcircle in K. Prove that E is the mid-point of HK.

[Show that BFEC is a cyclic quadrilateral, ∴ ∠FCE = ∠FBE. But ∠KCE = ∠FBE (why?), ∴ etc.]

†Ex. **1595.** I is the centre of the inscribed circle of $\triangle$ABC; I_1 is the centre of the circle escribed outside BC. Prove that $BICI_1$ is cyclic.

†Ex. **1596.** An escribed circle of $\triangle$ABC touches BC externally at D, and touches AB, AC produced at F, E respectively; O is the centre of the circle. Prove that

$$\text{(i) } \angle BOC = \tfrac{1}{2}\angle FOE = 90^\circ - \frac{A}{2},$$

$$\text{(ii) } 2AE = 2AF = BC + CA + AB.$$

†Ex. **1597.** Prove that

$$\angle BIC = 90^\circ + \frac{A}{2},$$

where I is the inscribed centre of $\triangle$ABC.

Hence find the locus of the inscribed centre of a triangle, whose base and vertical angle are given.

†Ex. **1598.** I is the centre of the inscribed circle of $\triangle$ABC; AI produced meets the circumcircle in P; prove that PB = PC = PI.

†Ex. **1599.** **P is any point on circumcircle of $\triangle$ABC. PL, PM, PN are $\perp$ to BC, CA, AB respectively. Prove that**

(i) $\angle PNL = 180^\circ - \angle PBC$,

(ii) $\angle PNM = \angle PAM$,

(iii) $\angle PNL + \angle PNM = 180^\circ$,

(iv) LNM **is a straight line.**

Verify this result by drawing.

LNM is called Simson's line.

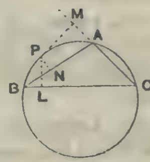


fig. 299.

†Ex. **1600.** ABCDEF is a regular hexagon; prove that BF is trisected by AC, AE.

†Ex. **1601.** In fig. 300, BC is $\perp$ to PA. Prove that PA bisects $\angle$QPR.

†Ex. **1602.** Through A, a point of intersection of two circles, lines BAC, DAE are drawn, B, D being points on the one circle, C, E on the other. Prove that the angle between DB and CE (produced if necessary) is the same as the angle between the tangents at A.

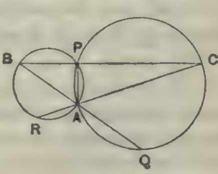


fig. 300.

†Ex. **1603.** Two circles touch internally at A; BC, a chord of the larger circle, touches the smaller at D; prove that AD bisects $\angle$BAC.

[Let BC meet the tangent at A in T.]

†Ex. **1604.** A radius of one circle is the diameter of another; prove that any straight line drawn from the point of contact to the outer circle is bisected by the inner circle.

†Ex. **1605.** In fig. 301 AB is a tangent;

$$OD = DA = AB.$$

BD cuts the circumference at E. Prove that arc AE is $\frac{1}{12}$ and arc EF $\frac{1}{24}$ of the circumference.

fig. 301.

†Ex. **1606.** Join O, the circumcentre of a triangle, to the vertices A, B, C. Through A draw lines ∥ to OB, OC; through B lines ∥ to OC, OA; through C lines ∥ to OA, OB. Prove that these lines form an equilateral hexagon; that each angle of the hexagon is equal to one other angle, and double an angle of the triangle.

Ex. **1607.** Power is being transmitted from one shaft to another parallel shaft by means of a belt passing over two wheels. The radii of the wheels are 2 ft. and 1 ft. and the distance between the shafts is 6 ft. Assuming the belt to be taut, find its length (i) when it does not cross between the shafts, (ii) when it does cross.

†Ex. **1608.** PQ is a chord bisected by a diameter AB of a circle (centre O). PG bisects the ∠OPQ. Prove that it bisects the semi-circle on which Q lies.

†Ex. **1609.** If through C, the mid-point of an arc AB, two chords are drawn, the first cutting the chord AB in D and the circle in E, the second cutting the chord in F and the circle in G, then the quadrilateral DFGE is cyclic.

†Ex. **1610.** P is a point on an arc AB. Prove that the bisector of ∠APB and the perpendicular bisector of the chord AB meet on the circle.

Ex. **1611.** P, Q are two points on a circle; AB is a diameter. AP, AQ are produced to meet the tangents at B in X, Y. Prove that △s APQ, AYX are equiangular; and that P, Q, Y, X are concyclic.

†Ex. **1612.** In fig. 302 the angles at O are all equal; and OA = AB = BC = CD = DE. Prove that O, A, B, C, D, E are concyclic.

†Ex. **1613.** From a point A on a circle, two chords are drawn on opposite sides of the diameter through A. Prove that the line joining the mid-points of the minor arc of these chords cuts the chords at points equidistant from A.

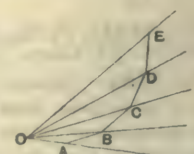


fig. 302.

†Ex. **1614.** Two equal chords of a circle intersect; prove that the segments of the one chord are respectively equal to the segments of the other.

Ex. **1615.** In fig. 303 O is the centre of the arc AB; and Q is the centre of the arc BC; ∠ADC is a right angle. DA = 3, DC = 5, DQ = 3. Find OA and QC; and draw the figure. (Let OD = x.)

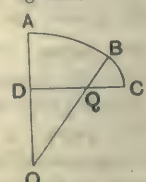


fig. 303.

BOOK IV.

SIMILARITY.

RATIO AND PROPORTION.

To **measure** a length is to find how many times it contains another length called the **unit** of length.

The unit of length may be an inch, a centimetre, a millimetre, a mile, a light-year[1], or any length you choose. Hence the importance of always stating your unit.

If you have two lines, one 4 in. long, the other 5 in., you say that the first is $\frac{4}{5}$ of the second.

The **ratio** of a length XY to a length PQ is the quotient

$$\frac{\text{measure of XY}}{\text{measure of PQ}},$$

the two measurements being made with respect to the same unit of length.

The practical way then, to find the ratio of two lengths, is to measure them in inches or centimetres or any other convenient unit, and divide.

The ratio of a to b is written $\frac{a}{b}$, or a/b, or $a : b$, or $a \div b$.

[1] Astronomers sometimes express the distances of the fixed stars in terms of the distance traversed by light in a year. This distance is called a light-year, and is 63,368 times the distance of the earth from the sun. The nearest star is α Centauri, whose distance is 4·26 light-years.

¶ Ex. **1616.** Find the ratio $\frac{PQ}{RS}$ (fig. 304); measure (i) in inches, (ii) in centimetres. Work out the ratios to three significant figures. Why might you expect your results to differ?

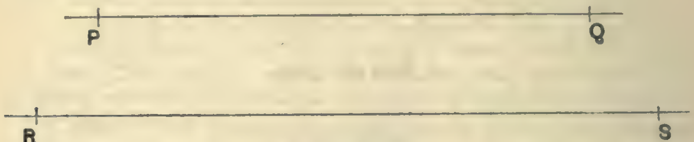


fig. 304.

Ex. **1617.** Find the ratio $\frac{RS}{PQ}$ as in Ex. 1616.

If the lengths are determined approximately, the ratios can be calculated only approximately.

If you measure two lines and find their lengths to be 5·82 in., and 3·65 in., the last figure in each case is doubtful; you are not sure, for example, that the second length is not nearer to 3·64 in. or 3·66 in.

Now $\frac{5\cdot83}{3\cdot64}=1\cdot602-$, and $\frac{5\cdot81}{3\cdot66}=1\cdot587+$.

You see that the results differ in this instance by ·015 (i.e. about 1 %).

As a general rule, work out ratios to three significant figures.

Ex. **1618.** Express the following ratios as decimals:—

(i) $\frac{72\cdot5}{819}$, (ii) $\frac{5\cdot64}{2\cdot15}$, (iii) $\frac{0\cdot361}{462}$, (iv) 9310 : 3·35, (v) ·0128 : ·00637.

Hitherto we have only considered the ratio of two lengths. In the case of other magnitudes, ratio may be defined as follows :—

DEF. The **ratio** of one magnitude to another of the same kind is the quotient obtained by dividing the numerical measure of the first by that of the second, the unit being the same in each case.

The ratio of two magnitudes is independent of the unit chosen.

For example, the ratio of a length of 5 yds. to a length of 2 yds. is $5:2$; if these lengths are measured in feet the measures are 15 and 6, and the ratio is $15:6$. Now we know that $5:2=15:6$.

Def. If $a:b=c:d$, the four magnitudes a, b, c, d are said to be **in proportion.**

¶Ex. **1619.** Are the following statements correct?

(i) 3 yds. : 1 yd. = 3 shillings : 1 shilling.

(ii) 3 yds. : 3 shillings = 1 yd. : 1 shilling.

¶Ex. **1620.** Fill in the missing terms in the following:—

(i) $\frac{2}{3}=\frac{\ }{6}$, (iv) $5:2=7:\ $,

(ii) $\frac{\ }{6}=\frac{1}{2}$, (v) $\frac{\ }{2p}=\frac{3}{2}$,

(iii) $7:\ =3:11$, (vi) $\frac{a}{b}=\frac{\ }{d}$.

The following algebraical processes will be used in the course of Book IV.

I. If $$\frac{a}{b}=\frac{c}{d},$$

then $$\frac{a}{b}\times bd=\frac{c}{d}\times bd,$$

$$\therefore\ ad=bc$$

[e.g. $\frac{3}{5}=\frac{9}{15}$, $\therefore\ 3\times 15=5\times 9$].

II. Conversely if $$ad=bc,$$

then $$\frac{ad}{bd}=\frac{bc}{bd},$$

$$\therefore\ \frac{a}{b}=\frac{c}{d}.$$

III. If $$\frac{a}{b}=\frac{c}{d},$$

$$\therefore\ ad=bc,$$

$$\therefore\ \frac{ad}{cd}=\frac{bc}{cd},$$

$$\therefore\ \frac{a}{c}=\frac{b}{d}.$$

IV. If $$\frac{a}{b}=\frac{c}{d},$$

$$\therefore\ ad=bc,$$

$$\therefore\ \frac{ad}{ac}=\frac{bc}{ac},$$

$$\therefore\ \frac{d}{c}=\frac{b}{a}.$$

V. If $$\frac{a}{b}=\frac{c}{d},$$

$$\therefore\ \frac{a}{b}\pm 1=\frac{c}{d}\pm 1,$$

$$\therefore\ \frac{a\pm b}{b}=\frac{c\pm d}{d}.$$

VI. If $$\frac{a}{x}=\frac{b}{y}=\frac{c}{z}=\ldots=k,$$

then $$\frac{a+b+c+\ldots}{x+y+z+\ldots}=k.$$

Ex. **1621.** Draw two straight lines SVT and XZY.

Prove fully that, if $\frac{SV}{ST}=\frac{XZ}{XY}$, then

(i) $\frac{ST}{SV}=\frac{XY}{XZ}$, (ii) $\frac{VT}{SV}=\frac{ZY}{XZ}$, (iii) $\frac{VT}{ST}=\frac{ZY}{XY}$.

What rectangle properties can be obtained from the above results by clearing of fractions?

Ex. **1622.** State and prove the converses of the properties proved in Ex. 1621.

¶Ex. **1623.** From each of the following rectangle properties deduce a ratio property:

(i) $AB \, . \, CD = PQ \, . \, QR$,

(ii) $XY^2 = XZ \, . \, XW$.

Ex. **1624.** In fig. 4, find what fraction AC is of AB.

Internal and External Division *.

If in a straight line AB a point P is taken, AB is said to be divided internally in the ratio $\frac{PA}{PB}$ (i.e. the ratio of the distances of P from the ends of the line). In the same way, if in AB produced a point P is taken, AB is said to be divided externally in the ratio $\frac{PA}{PB}$ (i.e. the ratio of the distances of P from the ends of the line).

In the latter case, it must be carefully noted that the ratio is not $\frac{AB}{BP}$. Suppose the points A, B connected by an elastic string; take hold of the string at a point P and, always keeping the three points in a straight line, vary the position of P; whether P is in AB or AB produced, the ratio in which AB is divided is always the ratio of the lengths of the two parts of the string.

* The discussion of cases of external division may be postponed.

¶**Ex. 1625.** In fig. 305, name the ratios in which (i) H divides AB, (ii) A divides BH, (iii) C divides KA.

¶**Ex. 1626.** In fig. 317, what lines are divided (i) by D in the ratio $\frac{BD}{DC}$, (ii) by Z in the ratio $\frac{ZY}{ZW}$, (iii) by B in the ratio $\frac{BC}{BD}$?

Proportional Division of Straight Lines.

Revise pp. 142, 143.

¶**Ex. 1627.** Draw a triangle ABC and draw HK parallel to BC (see fig. 305). What fraction is AH of AB? What fraction is AK of AC?

[Express these fractions as decimals.]

Ex. 1628. In the figure of Ex. 1627, calculate

$$\text{(i)}\ \frac{AH}{HB},\ \frac{AK}{KC},\qquad \text{(ii)}\ \frac{HB}{AB},\ \frac{KC}{AC}.$$

¶**Ex. 1629.** (On inch paper.) Mark A (1, 2), B (1, 0), C (2, 0); draw the triangle ABC. In AB mark the point H (1, 0·7), through H draw HK parallel to BC cutting AC at K. The horizontal lines of the paper divide AC into 20 equal parts (why are they equal?); how many of these parts does AK contain? What are the values of $\frac{AK}{AC}$, $\frac{AH}{AB}$?

Ex. 1630. (On inch paper.) Repeat Ex. 1629 with A (1, 1), B (1, 0), C (3, 0), H (1, 0·3).

THEOREM 1.

If a straight line HK drawn parallel to the base BC of a triangle ABC cuts AB, AC in H, K respectively, then $\frac{AH}{AB}=\frac{AK}{AC}$.

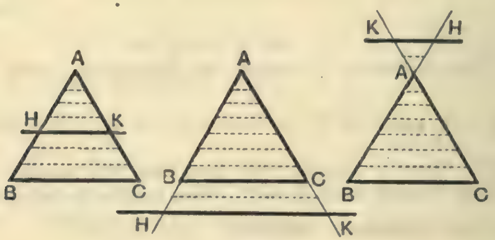


fig. 305.

Proof Suppose that $\frac{AH}{AB}=\frac{p}{q}$, where p and q are integers. Then if AB is divided into q equal parts, AH contains p of these parts.

Through the points of division draw parallels to BC.

Now AB is divided into equal parts.

$\therefore$ these parallels divide AC into equal parts; I. 24.

AC contains q of these parts, and AK contains p of these parts.

$$\therefore \frac{AK}{AC}=\frac{p}{q},$$

$$\therefore \frac{AH}{AB}=\frac{AK}{AC}.$$

Q. E. D.

COR. 1. **If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally.**

To prove that $\dfrac{AH}{HB} = \dfrac{AK}{KC}$.

*First proof** In the figure, AB is divided into q equal parts;

AH contains p of these equal parts;

$\therefore$ HB „ $q-p$ „ „

$$\therefore \frac{AH}{HB} = \frac{p}{q-p}.$$

$$\text{Sim}^{\text{ly}}\ \frac{AK}{KC} = \frac{p}{q-p}.$$

$$\therefore \frac{AH}{HB} = \frac{AK}{KC}.$$

Q. E. D.

*Second proof** Since $\dfrac{AH}{AB} = \dfrac{AK}{AC}$, *Proved*

$$\therefore \frac{AB}{AH} = \frac{AC}{AK},$$

$$\therefore \frac{AB}{AH} - 1 = \frac{AC}{AK} - 1,$$

$$\therefore \frac{AB - AH}{AH} = \frac{AC - AK}{AK},$$

$$\text{i.e. } \frac{HB}{AH} = \frac{KC}{AK},$$

$$\therefore \frac{AH}{HB} = \frac{AK}{KC}.$$

Q. E. D.

* These proofs apply to the first figure: see Ex. 1631.

Cor. 2. **If two straight lines are cut by a series of parallel straight lines, the intercepts on the one have to one another the same ratios as the corresponding intercepts on the other.**

†Ex. **1631.** Write out the two proofs of Cor. 1 for the second and third figures of page 306.

†Ex. **1632.** Triangles of the same height are to one another as their bases.

[Suppose one base is $\frac{3}{5}$ of the other.]

Ex. **1633.** Divide a given straight line so that one part is $\frac{3}{8}$ of the whole line.

Ex. **1634.** Divide a given straight line in the ratio $\frac{2}{5}$ (i.e. so that the ratio of the two parts $=\frac{2}{5}$).

Ex. **1635. Show how to divide a given straight line AB in the ratio of two given straight lines p, q.**

[Through A draw AC, from AC cut off AD $=p$, DE $=q$; join BE; draw a line through D to divide AB in the ratio $\frac{AD}{DE}$; in what direction must this line be drawn?]

Ex. **1636.** Find the value of x, when $\frac{4 \cdot 2}{2 \cdot 5}=\frac{3 \cdot 7}{x}$, (i) graphically, (ii) by calculation.

[Make an $\angle$POQ; from OP cut off OD $=4 \cdot 2$ in., DE $=2 \cdot 5$ in.; from OQ cut off OF $=3 \cdot 7$; draw EG $\parallel$ to DF. Which is the required length?]

Ex. **1637.** Find, both graphically and by calculation, the value of x in the following cases:

(i) $\frac{2 \cdot 25}{3 \cdot 05}=\frac{3 \cdot 05}{x}$, (ii) $\frac{\cdot 935}{x}=\frac{1 \cdot 225}{5 \cdot 75}$,

(iii) $x : 2 \cdot 63 = 5 \cdot 05 : 2 \cdot 84$, (iv) $8 \cdot 36 : \cdot 025 = x : \cdot 037$.

Def. If x is such a magnitude that $\frac{a}{b}=\frac{c}{x}$ (or $a : b = c : x$), x is called the **fourth proportional** to the three magnitudes a, b, c.

To find the fourth proportional to three given straight lines.

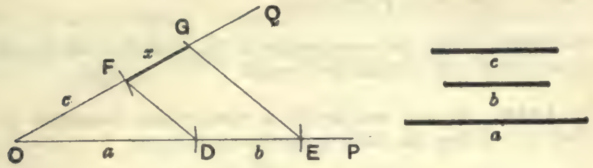


fig. 307.

Let a, b, c be the three given straight lines.

Construction Make an angle POQ.

From OP cut off OD $= a$, and DE $= b$.

From OQ cut off OF $= c$.

Join DF.

Through E draw EG || to DF, cutting OQ in G.

Then FG (x) is the fourth proportional to a, b, c.

Proof Since FD is || to EG,

$$\therefore \frac{a}{b} = \frac{c}{x}, \qquad \text{IV. 1.}$$

DEF. If x is such a magnitude that $\frac{a}{b} = \frac{b}{x}$ (or $a : b = b : x$), x is called the **third proportional** to the two magnitudes a, b.

NOTE. If x is the third proportional to a, b, it is also the fourth proportional to a, b, b.

Ex. **1638.** **Show how to find the third proportional to two given straight lines.**

[See note above.]

Ex. **1639.** Find graphically the fourth proportional to 3, 4, 5. Check by calculation.

Ex. **1640.** Find graphically the third proportional to 6·32, 8·95. Check by calculation.

†Ex. **1641. Justify the following construction for finding the fourth proportional to** p, q, r:—Make an $\angle$BAC; from AB cut off AX $=p$, AY $=q$; from AC cut off AZ $=r$; join XZ, and draw YW $\parallel$ to XZ. Then AW is the fourth proportional.

Ex. **1642.** Using the construction of Ex. 1641, find the fourth proportional to 1, 1·41, 4·23. Check your result.

Ex. **1643.** Explain and justify a construction, analogous to that of Ex. 1641, for finding the third proportional to p, q.

Ex. **1644.** Using the construction of Ex. 1643, find the third proportional to 1, 1·73. Check your result.

Ex. **1645.** Given that the circumference of a circle of 1 in. radius is 6·28 in., find graphically the circumferences of circles whose radii are (i) 3·28 cm., (ii) 16·7 in., (iii) 8·37 miles, (iv) 4·28 km.

Also find the radii of circles whose circumferences are (i) 3·36 in., (ii) 12·35 in., (iii) 8·66 cm., (iv) 11 yards.

Ex. **1646.** (On inch paper.) Mark four points A (1, 1), B (1, 4), C (4, 1), D (3, 3); join AB, and mark P (1, 2). Produce AC, BD to meet at V; join VP; let it cut CD at Q. Find $\frac{\text{AP}}{\text{PB}}$ and $\frac{\text{CQ}}{\text{QD}}$; are they equal?

Ex. **1647.** Make a copy of the points A, B, C, D, P in Ex. 1646, by pricking through. Divide CD at R so that $\frac{\text{AP}}{\text{PB}} = \frac{\text{CR}}{\text{RD}}$.

[Begin by dividing CB in the required ratio.]

Ex. **1648.** Draw a straight line AB, on it take two points P, Q; draw another straight line CD; divide CD similarly to AB. (*Freehand*)

Theorem 2.

[Converse of Theorem 1.]

If H, K are points in the sides AB, AC of a triangle ABC, such that $\frac{AH}{AB} = \frac{AK}{AC}$, then HK is parallel to BC.

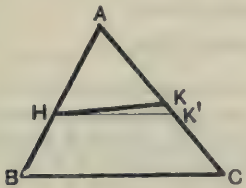


fig. 308.

Construction Draw HK′ parallel to BC.

To prove that HK and HK′ coincide.

Proof Since HK′ is ∥ to BC.

$$\therefore \frac{AH}{AB} = \frac{AK'}{AC}. \qquad \text{IV. 1.}$$

$$\text{But } \frac{AH}{AB} = \frac{AK}{AC}. \qquad \textit{Data}$$

$$\therefore \frac{AK'}{AC} = \frac{AK}{AC},$$

$$\therefore AK' = AK,$$

∴ K and K′ coincide,

∴ HK and HK′ coincide,

∴ HK is parallel to BC.

Q. E. D.

Cor. 1. If $\frac{AB}{AH} = \frac{AC}{AK}$, then HK and BC are parallel.

Cor. 2. **If a straight line divides the sides of a triangle proportionally, it is parallel to the base of the triangle**

†Ex. **1649.** Prove Cor. 1 without assuming IV. 2.

†Ex. **1650.** Prove Cor. 2 without assuming IV. 2.

†Ex. **1651.** O is a point inside a quadrilateral ABCD; OA, OB, OC, OD are divided at A′, B′, C′, D′

so that $$\frac{OA'}{OA}=\frac{OB'}{OB}=\frac{OC'}{OC}=\frac{OD'}{OD}=\frac{2}{3}.$$

Prove that A′B′ is parallel to AB.

Also prove that $\angle D'A'B'=\angle DAB$.

†Ex. **1652.** Draw a large quadrilateral ABCD; in it take a point O, and join OA, OB, OC, OD; in OA take a point A′, through A′ draw A′B′ parallel to AB to cut OB at B′, through B′ draw B′C′ parallel to BC to cut OC at C′, through C′ draw C′D′ parallel to CD to cut OD at D′. Prove that D′A′, DA are parallel. Are they parallel in your figure? Keep your figure for Ex. 1653.

Ex. **1653.** In the figure of Ex. 1652, calculate

$$\frac{A'B'}{AB},\ \frac{B'C'}{BC},\ \frac{C'D'}{CD},\ \frac{D'A'}{DA}.$$

Ex. **1654.** Repeat Ex. 1652 for (i) a triangle, (ii) a pentagon.

†Ex. **1655.** A variable line, drawn through a fixed point O, cuts two fixed parallel straight lines at P, Q; prove that OP : OQ is constant.

†Ex. **1656.** O is a fixed point and P moves along a fixed line. OP is divided at Q (internally or externally) in a fixed ratio. Find the locus of Q.

†Ex. **1657.** D is a point in the side AB of $\triangle$ABC; DE is drawn parallel to BC and cuts AC at E; EF is drawn parallel to AB and cuts BC at F. Prove that AD : DB = BF : FC.

†Ex. **1658.** D is a point in the side AB of $\triangle$ABC; DE is drawn parallel to BC and cuts AC at E; CF is drawn parallel to EB and cuts AB produced at F. Prove that AD : AB = AB : AF.

†Ex. **1659.** AD, BC are the parallel sides of a trapezium; prove that a line drawn parallel to these sides cuts the other sides proportionally.

†Ex. **1660.** From a point E in the common base AB of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G; show that FG is parallel to CD.

†Ex. **1661.** In three straight lines OAP, OBQ, OCR the points are chosen so that AB is parallel to PQ, and BC parallel to QR. Prove that AC is parallel to PR.

†Ex. **1662.** AB, DC are the parallel sides of a trapezium. P, Q are points on AD, BC, so that AP/PD = BQ/QC. Prove that PQ is ∥ to AB and DC. (Use *reductio ad absurdum.*)

SIMILAR TRIANGLES.

DEF. Polygons which are equiangular to one another and have their corresponding sides proportional are called **similar** polygons.

¶Ex. **1663.** Draw a quadrilateral ABCD; draw a straight line parallel to CD to cut BC at P and AD at Q. Prove that ABCD, ABPQ are equiangular. Are they similar?

¶Ex. **1664.** Draw a quadrilateral ABCD having AB = 3 in., BC = 2 in., CD = 3 in., DA = 2 in., $\angle B = 30°$; draw a quadrilateral PQRS having PQ = 6 cm., QR = 4 cm., RS = 6 cm., SP = 4 cm., $\angle Q = 90°$. Are ABCD, PQRS similar?

¶Ex. **1665.** Draw a quadrilateral XYZW having XY = 3 in., YZ = 2 in., ZW = 1 in., WX = 4 in., $\angle Y = 120°$. Outside XYZW describe a quadrilateral X′Y′Z′W′ having its sides parallel to the sides of XYZW and 1 in. away from them. Are the two quadrilaterals similar? Find

$$\frac{X'Y'}{XY},\ \frac{Y'Z'}{YZ},\ \frac{Z'W'}{ZW},\ \frac{W'X'}{WX}.$$

¶Ex. **1666.** Draw two equiangular triangles; find the ratios of their corresponding sides.

Revise Ex. 146—151.

Theorem 3.

If two triangles are equiangular, their corresponding sides are proportional.

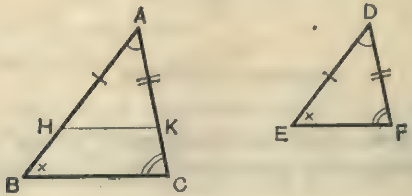


fig. 309.

Data ABC, DEF are two triangles which have

$\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. (See I. 8, Cor. 5.)

To prove that $\dfrac{BC}{EF} = \dfrac{CA}{FD} = \dfrac{AB}{DE}$.

Construction From AB cut off AH = DE,

From AC cut off AK = DF.

Join HK.

Proof In the $\triangle$s AHK, DEF,

$$\left\{\begin{array}{ll} AH = DE, & \textit{Constr.} \\ AK = DF, & \textit{Constr.} \\ \angle A = \angle D, & \textit{Data} \end{array}\right.$$

$\therefore \triangle AHK \equiv \triangle DEF$, I. 10.

$\therefore \angle AHK = \angle E$,

$= \angle B$. *Data*

$\therefore$ HK is $\parallel$ to BC, I. 4.

$$\therefore \frac{AH}{AB} = \frac{AK}{AC}, \qquad \text{IV. 1.}$$

$$\therefore \frac{DE}{AB} = \frac{DF}{AC}.$$

Sim^{ly} by cutting off lengths from BA, BC,

$$\frac{ED}{BA} = \frac{EF}{BC},$$

$$\therefore \frac{EF}{BC}, \ \frac{FD}{CA}, \ \frac{DE}{AB} \text{ are all equal.}$$

$$\therefore \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}.$$

Q. E. D.

†Ex. **1667.** Write out the complete proof that $\frac{ED}{BA} = \frac{EF}{BC}$.

Ex. **1668.** ABC is a triangle having BC = 3 in., CA = 4 in., AB = 5 in.; DEF is an equiangular triangle having EF = 2·2 in. Calculate DE, DF and check by measurement.

Ex. **1669.** Repeat Ex. 1668 with BC = 5·8 cm., CA = 7·7 cm., AB = 8·3 cm., EF = 1·8 in.

¶Ex. **1670.** If P is any point on either arm of an angle XOY, and PN is drawn perpendicular to the other arm, $\frac{PN}{OP}$ has the same value for all positions of P.

[Take several different positions of P and prove that $\frac{PN}{OP} = \frac{P_1N_1}{OP_1} = \ldots$]

$\frac{PN}{OP}$ is the **sine** of $\angle$XOY; this exercise might have been stated as follows:—the sine of an angle depends only on the magnitude of the angle.

¶Ex. **1671.** Prove that the **cosine** $\left(\frac{ON}{OP}\right)$ and **tangent** $\left(\frac{PN}{ON}\right)$ of an angle depend only on the magnitude of the angle.

Ex. **1672.** On a base 4 in. long draw a quadrilateral; on a base 3 in. long construct a similar quadrilateral. Calculate the ratio of each pair of corresponding sides.

[Draw a diagonal of the first quadrilateral.]

†Ex. **1673.** PQRS is a quadrilateral inscribed in a circle whose diagonals intersect at X; prove that the $\triangle^s$ XPS, XQR are equiangular. Write down the three equal ratios of corresponding sides.

†Ex. **1674.** In the figure of Ex. 1673, prove that $\frac{PQ}{SR} = \frac{XP}{XS}$.

[If you colour PQ, SR red, and XP, XS blue, you will see which two triangles you require.]

†Ex. **1675.** XYZW is a cyclic quadrilateral; XY, WZ produced intersect at a point P outside the circle; prove that $\frac{PY}{PW} = \frac{PZ}{PX}$.

†Ex. **1676.** ABC is a triangle right-angled at A; prove that the altitude AD divides the triangle into two triangles which are similar to $\triangle$ ABC. Write down the ratio properties you obtain from the similarity of $\triangle^s$ BDA, BAC.

[See Ex. 132—134.]

†Ex. **1677.** The altitude QN of a triangle PQR right-angled at Q cuts RP in N; prove that $\frac{QN}{RN} = \frac{PN}{QN}$.

[Find two equiangular triangles; colour the given lines; see Ex. 1674.]

†Ex. **1678.** XYZ is a triangle inscribed in a circle, XN is an altitude of the triangle, and XD a diameter of the circle; prove that

$$XY : XD = XN : XZ.$$

†Ex. **1679.** XYZ is a triangle inscribed in a circle; the bisector of $\angle X$ meets YZ in P, and the circle in Q; prove that XY : XQ = XP : XZ.

†Ex. **1680.** PQRS is a quadrilateral inscribed in a circle; PT is drawn so that $\angle$ SPT = $\angle$ QPR. (See fig. 310.) Prove that (i) SP : PR = ST : QR,
(ii) SP : PT = SR : TQ.

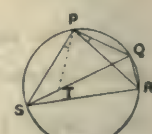


fig. 310.

†Ex. **1681.** Three straight lines are drawn from a point O; they are cut by a pair of parallel lines at X, Y, Z and X′, Y′, Z′. Prove that XY : YZ = X′Y′ : Y′Z′.

On a given straight line to construct a figure similar to a given rectilinear figure. (First Method.) †

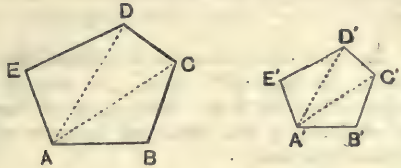


fig. 311.

Let ABCDE be the given figure and A′B′ the given straight line

Construction Join AC, AD.

On A′B′ make $\triangle$ A′B′C′ equiangular to $\triangle$ ABC.

On A′C′ make $\triangle$ A′C′D′ equiangular to $\triangle$ ACD.

On A′D′ make $\triangle$ A′D′E′ equiangular to $\triangle$ ADE.

Then A′B′C′D′E′ is similar to ABCDE.

Proof This may be divided into two parts:

(i) the proof that the figures are equiangular; this is left to the student.

(ii) the proof that $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$.

Since $\triangle^s$ ABC, A′B′C′ are equiangular, *Constr.*

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}. \qquad \text{IV. 3.}$$

Since $\triangle^s$ ACD, A′C′D′ are equiangular, *Constr.*

$$\therefore \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'}. \qquad \text{IV. 3.}$$

Again since $\triangle^s$ ADE, A′D′E′ are equiangular, *Constr.*

$$\therefore \frac{AD}{A'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}, \qquad \text{IV. 3.}$$

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$$

Ex. **1682.** On inch-paper, mark the points O (0, 0), P (3, 0), Q (5, 2), R (4, 5), S (1, 4); join OP, PQ, QR, RS, SO. On plain paper, draw O′P′ = 1·5 in.; on O′P′ describe a similar polygon. Check by measuring the angles and finding the ratios of corresponding sides. (Keep your figures for the next exercise.)

Ex. **1683.** On inch-paper, describe a polygon similar to OPQRS of Ex. 1682, having its base O′P′ = 1·5 in. Do this by halving the co-ordinates of the points O, P, Q, R, S. Make a copy on tracing paper of the smaller polygon obtained in Ex. 1682, and compare with the pólygon obtained in the present exercise.

Ex. **1684.** On inch-paper, mark the points A (1, 0), B (4, 0), C (1, 3), D (3, 4); join AB, BC, CD, DA. On plain paper draw A′B′ = 2·5 in.; on A′B′ describe a figure similar to ABCD. Check by calculating the ratios of corresponding sides.

Ex. **1685.** Draw a pentagon ABCDE; draw A′B′ ∥ to AB; on A′B′ construct a pentagon similar to ABCDE. (This should be done with set-square and straight edge only.)

Revise Ex. 146.

Ex. **1686.** Draw four parallel lines AP, BQ, CR, DS; draw two straight lines ABCD, PQRS to cut them. With AB, BC, CD as sides, describe a triangle; with PQ, QR, RS describe a triangle. Measure and compare the angles of the two triangles.

THEOREM 4.

[CONVERSE OF THEOREM 3.]

If, in two triangles ABC, DEF, $\frac{BC}{EF}=\frac{CA}{FD}=\frac{AB}{DE}$, then the triangles are equiangular.

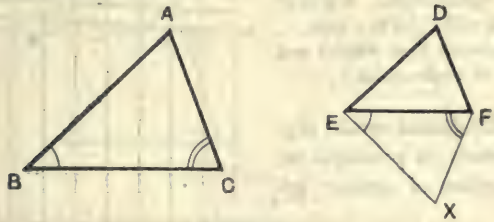


fig. 312.

Construction Make $\angle FEX = \angle B$ and $\angle EFX = \angle C$, X and D being on opposite sides of EF.

Proof In the $\triangle$s ABC, XEF,

$$\begin{cases}\angle B = \angle FEX,\\ \angle C = \angle EFX,\end{cases}$$

∴ the third angles are equal,

and the triangles are equiangular.

$$\therefore \frac{BC}{EF}=\frac{CA}{FX}=\frac{AB}{XE}. \qquad \text{IV. 3.}$$

$$\text{But } \frac{BC}{EF}=\frac{CA}{FD}=\frac{AB}{DE}, \qquad \textit{Data}$$

$$\therefore \frac{CA}{FX}=\frac{CA}{FD} \text{ and } \frac{AB}{XE}=\frac{AB}{DE},$$

∴ FX = FD and XE = DE.

In the $\triangle$s XEF, DEF,

$$\begin{cases}XE = DE,\\ FX = FD,\\ \text{and EF is common,}\end{cases}$$

$$\therefore \triangle XEF \equiv \triangle DEF. \qquad \text{I. 14.}$$

But the $\triangle$s ABC, XEF are equiangular,

∴ the $\triangle$s ABC, DEF are equiangular.

Q. E. D.

†Ex. **1687.** Draw a quadrilateral ABCD; join AC. Make an angle XOY; from OX cut off OP = AB, OQ = BC, OR = CD, OS = DA, OT = CA; through P, Q, ... draw a set of parallel lines cutting OY in P′, Q′, ... Construct a quadrilateral A′B′C′D′ having A′B′ = OP′, B′C′ = OQ′,.... Prove and verify that ABCD and A′B′C′D′ are equiangular.

The **diagonal scale** (fig. 313), depends in principle on the properties of similar triangles.

¶Ex. **1688.** Are the triangles whose corners are marked 0, *d*, 10 and 0, *c*, 6 equiangular?

¶Ex. **1689.** What fraction is the distance between the points 6, *c* of the distance between 10, *d*?

The distance between 10, *d* is ·1 in.; what is the distance between 6, *c*?

¶Ex. **1690.** What are the distances between the points (i) *a*, 6, (ii) 6, *c*, (iii) *c*, *b*?

What is the whole distance between *a*, *b*?

¶Ex. **1691.** Draw a triangle ABC; make an $\angle XOY = \angle A$; from OX, OY cut off OP = $\frac{2}{3}$AB, OQ = $\frac{2}{3}$AC; join PQ; measure $\angle$s P, Q, and compare them with $\angle$s B, C.

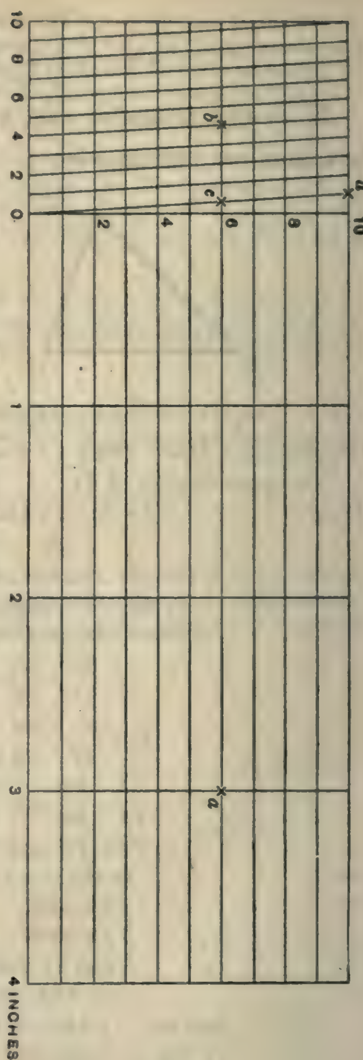


fig. 313.

THEOREM 5.

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, the triangles are similar.

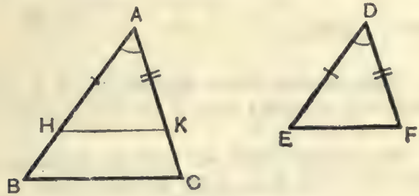


fig. 314.

Data ABC, DEF are two triangles which have $\angle A = \angle D$, and

$$\frac{AB}{DE} = \frac{AC}{DF}.$$

To prove that the $\triangle$s ABC, DEF are similar.

Construction From AB cut off AH = DE.

From AC cut off AK = DF.

Join HK.

Proof In $\triangle$s AHK, DEF,

$$\begin{cases} AH = DE, & \textit{Constr.} \\ AK = DF, & \textit{Constr.} \\ \angle A = \angle D, & \textit{Data} \end{cases}$$

$\therefore \triangle AHK \equiv \triangle DEF.$ I. 10.

Now $\dfrac{AB}{DE} = \dfrac{AC}{DF}.$

$\therefore \dfrac{AB}{AH} = \dfrac{AC}{AK},$

$\therefore$ HK is $\parallel$ to BC. IV. 2, *Cor.* 1.

$\angle H = \angle B$ and $\angle K = \angle C$,

$\therefore$ $\triangle$s AHK, BCA are equiangular.

Hence $\triangle$s DEF, ABC are equiangular, and therefore have their corresponding sides proportional. IV. 3.

$\therefore$ $\triangle$s DEF, ABC are similar. Q. E. D.

NOTE. In IV. 3 and 5, if DE$>$AB and DF$>$AC, H, K lie in AB, AC produced; the proofs hold equally well for these cases.

†Ex. **1692.** S is a point in the side PQ of $\triangle$ PQR; ST is drawn parallel to QR and of such a length that ST : QR = PS : PQ. Prove that T lies in PR.

[Prove $\angle$SPT = $\angle$QPR.]

Ex. **1693.** (Inch paper.) Prove that the points (0, 0), (2, 1), (5, 2·5) are in a straight line. In what ratio is the line divided?

†Ex. **1694.** In a triangle ABC, AD is drawn perpendicular to the base; if BD : DA = DA : DC, prove that $\triangle$ ABC is right-angled.

†Ex. **1695.** AX, DY are medians of the two similar triangles ABC, DEF; prove that they make equal angles with BC, EF, and that AX : DY = AB : DE. (Compare Ex. 411.)

†Ex. **1696.** The bases, BC, EF, of two similar triangles, ABC, DEF, are divided in the same ratio at X, Y. Prove that AX : DY = BC : EF.

Fig. 315 represents a pair of **proportional compasses.** AB = AC and AH = AK,

$$\therefore \frac{AH}{AB} = \frac{AK}{AC}, \text{ and } \angle BAC = \angle HAK,$$

$\therefore$ $\triangle^s$ ABC, AHK are similar.

Hence $\frac{HK}{BC} = \frac{AH}{AB}$, which is constant for any fixed position of the hinge. In fig. 315 the hinge is adjusted so that $\frac{AH}{AB} = \frac{1}{2}$; thus, whatever the angle to which the compasses are opened, HK = $\frac{1}{2}$ BC.

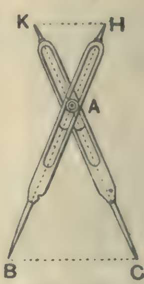


fig. 315.

¶**Ex. 1697.** On bases of 5 in. and 3 in. describe two similar triangles; calculate their areas, and find the ratio of their areas. Is it 5 : 3?

What is the ratio of their altitudes?

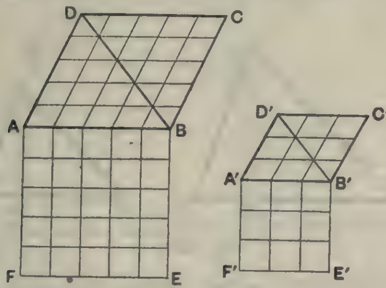


fig. 316.

In fig. 316,

$$\triangle ABD = \tfrac{1}{2} \parallel^{\text{ogram}} ABCD, \text{ and } \triangle A'B'D' = \tfrac{1}{2} \parallel^{\text{ogram}} A'B'C'D'.$$

The parallelograms ABCD, A′B′C′D′ are divided up into congruent parallelograms; the squares are divided up into congruent squares.

$$\therefore \frac{\triangle ABD}{\triangle A'B'D'} = \frac{\frac{1}{2}ABCD}{\frac{1}{2}A'B'C'D'} = \frac{ABCD}{A'B'C'D'} = \frac{25 \text{ small } \parallel^{\text{ograms}}}{9 \text{ small } \parallel^{\text{ograms}}} = \frac{25}{9}.$$

But $$\frac{\text{sq. } AE}{\text{sq. } A'E'} = \frac{25 \text{ small squares}}{9 \text{ small squares}} = \frac{25}{9},$$

$$\therefore \frac{\triangle ABD}{\triangle A'B'D'} = \frac{\text{square on } AB}{\text{square on } A'B'}.$$

†**Ex. 1698.** The ratio of corresponding altitudes of similar triangles is equal to the ratio of corresponding sides.

THEOREM 6.

The ratio of the areas of similar triangles is equal to the ratio of the squares on corresponding sides.

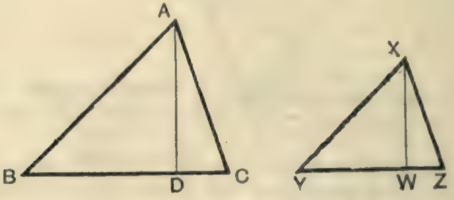


fig. 317.

Data ABC, XYZ are two similar triangles.

To prove that $\dfrac{\triangle ABC}{\triangle XYZ} = \dfrac{BC^2}{YZ^2}$.

Construction Draw AD $\perp$ to BC,
and XW $\perp$ to YZ.

Proof $\triangle ABC = \frac{1}{2}BC \cdot AD$, II. 2.

and $\triangle XYZ = \frac{1}{2}YZ \cdot XW$,

$\therefore \dfrac{\triangle ABC}{\triangle XYZ} = \dfrac{BC \cdot AD}{YZ \cdot XW}$.

[It remains to prove that $\dfrac{AD}{XW} = \dfrac{BC}{YZ}$].

Now in the $\triangle$s ABD, XYW,

$\begin{cases} \angle B = \angle Y, \\ \angle D = \angle W \text{ (rt. } \angle\text{s)}, \end{cases}$ Data

$\therefore$ the third angles are equal,
and the $\triangle$s are equiangular,

$\therefore \dfrac{AD}{XW} = \dfrac{AB}{XY}$. IV. 3.

But $\dfrac{AB}{XY} = \dfrac{BC}{YZ}$, Data

$\therefore \dfrac{AD}{XW} = \dfrac{BC}{YZ}$.

$$\text{But } \frac{\triangle \mathsf{ABC}}{\triangle \mathsf{XYZ}} = \frac{\mathsf{BC}}{\mathsf{YZ}} \cdot \frac{\mathsf{AD}}{\mathsf{XW}} \qquad \textit{Proved}$$

$$= \frac{\mathsf{BC}}{\mathsf{YZ}} \cdot \frac{\mathsf{BC}}{\mathsf{YZ}}$$

$$= \frac{\mathsf{BC}^2}{\mathsf{YZ}^2}.$$

Q. E. D.

¶Ex. **1699.** What is the ratio of the areas of two similar triangles on bases of 3 in. and 4 in.?

¶Ex. **1700.** The area of a triangle with a base of 12 cm. is 60 sq. cm.; find the area of a similar triangle with a base of 9 cm.

What is the area of a similar triangle on a base of 9 in.?

¶Ex. **1701.** The areas of two similar triangles are 100 sq. cm. and 64 sq. cm.; the base of the greater is 7 cm.; find the base of the smaller.

Ex. **1702.** The areas of two similar triangles are 97·5 sq. cm. and 75·3 sq. cm.; the base of the first is 17·2 cm.; find the base of the second.

¶Ex. **1703.** The sides of a triangle ABC are 7·2 in., 3·5 in., 5·7 in.; the sides of a triangle DEF are 7·2 cm., 3·5 cm., 5·7 cm.; find the ratio of the area of the first triangle to that of the second.

¶Ex. **1704.** Find the ratio of the bases of two similar triangles one of which has double the area of the other.

Show how to draw two such triangles, without using a graduated ruler.

Ex. **1705.** Describe equilateral triangles on the side and diagonal of a square; find the ratio of their areas. (*Freehand.*)

Ex. **1706.** Show how to draw a straight line parallel to the base of a triangle to bisect the triangle.

Ex. **1707.** Describe equilateral triangles on the sides of a right-angled triangle whose sides are 1·5 in., 2 in., 2·5 in. What connection is there between the areas of the three equilateral triangles? (*Freehand*)

†Ex. **1708.** Prove that, if similar triangles are described on the three sides of a right-angled triangle, the area of the triangle described on the hypotenuse is equal to the sum of the other two triangles.

†Ex. **1709.** ABC, DEF are two triangles in which $\angle B = \angle E$; prove that $\triangle ABC : \triangle DEF = AB . BC : DE . EF$.

[Draw AX $\perp$ to BC, and DY $\perp$ to EF.]

Ex. **1710.** What is the ratio of the areas of two circles whose radii are R, r? 3 in., 2 in.?

¶Ex. **1711.** Draw two similar quadrilaterals ABCD, PQRS; calculate their areas (join AC, PR); find the ratio of their areas, and compare this with the ratio of corresponding sides.

Rectangle properties.

†Ex. **1712.** XYZ is a triangle inscribed in a circle, XN is an altitude of the triangle and XD a diameter of the circle; prove that $\frac{XY}{XN} = \frac{YD}{NZ}$. Express this as a result clear of fractions. What two rectangles are thus proved equal?

†Ex. **1713.** With the same construction as in Ex. 1712, prove that

$$XZ . NY = XN . ZD.$$

[You will have to pick out two equal ratios from two equiangular triangles. If you colour XZ, NY red and XN, ZD blue you will see which are the triangles.]

†Ex. **1714.** ABCD is a quadrilateral inscribed in a circle; its diagonals intersect at X. Prove that (i) $AX . BC = AD . BX$, (ii) $AX . XC = BX . XD$.

†Ex. **1715.** ABCD is a quadrilateral inscribed in a circle; AB, DC produced intersect at Y. Prove that

(i) YA . BD = YD . CA, (ii) YA . YB = YC . YD.

†Ex. **1716.** The rectangle contained by two sides of a triangle is equal to the rectangle contained by the diameter of the circumcircle and the altitude drawn to the base.

[Draw the diameter through the vertex at which the two sides intersect.]

†Ex. **1717.** The bisector of the angle A of △ ABC meets the base in P and the circumcircle in Q. Prove that the rectangle contained by the sides AB, AC = rect. AP . AQ.

†Ex. **1718.** In Ex. 1680, prove that PQ . SR = PR . TQ.

†Ex. **1719. The sum of the rectangles contained by opposite sides of a cyclic quadrilateral is equal to the rectangle contained by its diagonals. (Ptolemy's theorem.)**

[Use the construction of Ex. 1680.]

¶Ex. **1720.** Draw a circle of radius 7 cm.; mark a point P 3 cm. from the centre O; through P draw five or six chords APB, CPD, ... Measure their segments and calculate the products PA . PB; PC . PD; .... Take the mean of your results and estimate by how much per cent. each result differs from the mean. (Make a table.)

¶Ex. **1721.** Draw a circle of radius 7 cm. and mark a point P 10 cm. from the centre O; through P draw a number of chords of the circle, and proceed as in Ex. 1720.

[Remember that if P is in the chord AB produced, PA, PB are still regarded as the segments into which P divides AB; you must calculate PA . PB, not PA . AB.]

¶Ex. **1722.** What will be the position of the chord in Ex. 1721 when the two segments are equal?

THEOREM 7 (i).

If AB, CD, two chords of a circle, intersect at a point P inside the circle, then PA . PB = PC . PD.

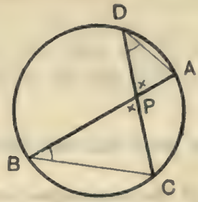


fig. 318.

Construction Join BC, AD.

Proof In the $\triangle$s PAD, PCB,

$$\begin{cases} \angle APD = \angle CPB \text{ (vert. opp.)} \\ \angle B = \angle D \text{ (in the same segment),} \end{cases}$$

$\therefore$ the third angles are equal,

and the $\triangle$s are equiangular,

$$\therefore \frac{PA}{PC} = \frac{PD}{PB}, \qquad \text{IV. 3.}$$

$$\therefore PA . PB = PC . PD.$$

Q. E. D.

To calculate the area of the rectangle PA . PB in IV. 7 (i).

Suppose EPF is the chord bisected at P.

Then $PA . PB = PE . PF = PE^2 = OE^2 - OP^2$.

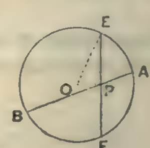


fig. 319.

THEOREM 7 (ii).

If AB, CD, two chords of a circle, intersect at a point P outside the circle, then PA . PB = PC . PD.

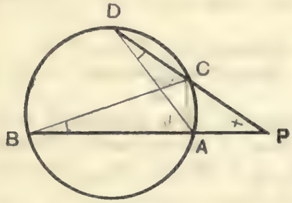


fig. 320.

Construction Join BC, AD.

Proof In the $\triangle$s PAD, PCB,

$$\begin{cases} \angle P \text{ is common,} \\ \angle B = \angle D \text{ (in the same segment)} \end{cases}$$

$\therefore$ the third angles are equal,

and the $\triangle$s are equiangular,

$$\therefore \frac{PA}{PC} = \frac{PD}{PB}, \qquad \text{IV. 3.}$$

$$\therefore PA . PB = PC . PD.$$

Q. E. D.

NOTE. Theorems 7 (i) and 7 (ii) are really two different cases of the same theorem; notice that the proofs are nearly identical. For alternative proofs, not depending on similarity, see Appendix I, pages 354, 355.

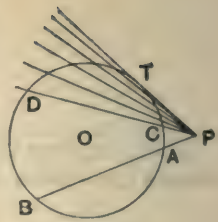


fig. 321.

†Ex. **1723.** **If PT is a tangent to a circle and AB a chord of the circle passing through P, then $PT^2 = PA . PB$.** (See fig. 321.)

To calculate the area of the rectangle PA. PB in IV. 7 (ii).

Use the fact that

$$PA . PB = PT^2 = OP^2 - OT^2.$$

¶Ex. **1724.** What becomes of IV. 7 when P is a point *on* the circle? When P is the centre?

Ex. **1725.** Calculate (and check graphically) the areas of the rectangles contained by the segments of chords passing through P when (i) $r=5$ in., $OP=3$ in., (ii) $r=5$ cm., $OP=13$ cm., (iii) $r=3{\cdot}7$ in., $OP=2{\cdot}3$ in., (iv) $r=2{\cdot}9$ in., $OP=3{\cdot}3$ in.

Ex. **1726.** Find an expression for the areas in Ex. 1725, r being the radius, and d the distance OP (i) when $d<r$, (ii) when $d>r$. Explain fully.

Ex. **1727.** Draw two straight lines APB, CPD intersecting at P; make $PA=4$ cm., $PB=6$ cm., $PC=3$ cm. Describe a circle through ABC, cutting CP produced in D. Calculate PD, and check by measurement.

What would be the result if the exercise were repeated with the same lengths, but a different angle between APB, CPD?

Ex. **1728.** From a point P draw two straight lines PAB, PC; make $PA=4$ cm., $PB=9$ cm., $PC=6$ cm. Describe a circle through ABC; let it cut PC again at D. Calculate PD, and check by measurement.

†Ex. **1729.** APB, CPD intersect at P; and the lengths PA, PB, PC, PD are so chosen that $PA . PB = PC . PD$. Prove that A, B, C, D are concyclic. (Draw ⊙ through ABC; let it cut CP produced in D′.) Make up a numerical instance, and draw a figure. What relation does this exercise bear to IV. 7 (i)?

†Ex. **1730.** State and prove the converse of IV. 7 (ii).

†Ex. **1731.** P is a point outside a circle ABC and straight lines PAB, PC are drawn (A, B, C being on the circle); prove that, if $PA . PB = PC^2$, PC is the tangent at C.

[Use *reductio ad absurdum.*]

†Ex. 1732. ABC is a triangle right-angled at A; AD is drawn perpendicular to BC; prove that $AD^2 = BD \cdot DC$.

[Produce AD to cut the circumcircle of △ABC.]

†Ex. 1733. If the common chord of two intersecting circles be produced to any point T, the tangents to the circles from T are equal to one another.

†Ex. 1734. The common chord of two intersecting circles bisects their common tangents.

†Ex. 1735. The altitudes BE, CF of a triangle ABC intersect at H, prove that

(i) $BH \cdot HE = CH \cdot HF$, (ii) $AF \cdot AB = AE \cdot AC$, (iii) $BH \cdot BE = BF \cdot BA$.

†Ex. 1736. Two circles intersect at A, B; T is any point in AB, or AB produced; TCD, TEF are drawn cutting the one circle in C, D, the other in E, F. Prove that C, D, E, F are concyclic.

Ex. 1737. ABC is a triangle right-angled at A; AD is an altitude of the triangle. Prove that △s ABD, CDA are equiangular. Write down the three equal ratios; and, by taking them in pairs, deduce the corresponding rectangle properties.

Def. If x is such a quantity that $a : x = x : b$, then x is called the **mean proportional** between a and b.

¶Ex. 1738. Prove that, if x is the mean proportional between a and b, $x^2 = ab$.

¶Ex. 1739. Find the mean proportional between

(i) 4 and 9, (ii) 1 and 100, (iii) $\frac{1}{2}$ and 2,

(iv) $\frac{3}{4}$ and $\frac{4}{3}$, (v) 1 and 2, (vi) 2 and 4.

To find the mean proportional between two given straight lines.

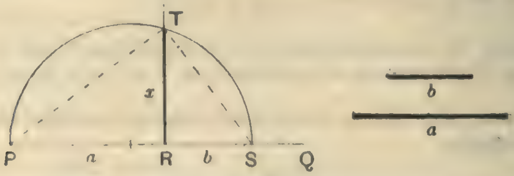


fig. 322.

Let a, b be the two given straight lines.

Construction Draw a straight line PQ.

From PQ cut off PR $= a$, and RS $= b$.

On PS as diameter describe a semicircle.

Through R draw RT $\perp$ to PS to cut the semicircle at T.

Then RT (x) is the mean proportional between a, b.

Proof Join PT, TS.

$\triangle^s$ PRT, TRS are equiangular. (Why?)

$\therefore$ RP : RT = RT : RS,

$\therefore a : x = x : b,$

$\therefore x$ is the mean proportional between a and b.

†Ex. **1740.** Prove the above construction by completing the circle, and producing TR to meet the circle in T′.

Ex. **1741.** (On inch paper.) Find graphically the mean proportionals between (i) 1 and 4, (ii) 1 and 3, (iii) 1·5 and 2·5, (iv) 1·3 and 1·7.

Check by calculation.

NOTE. If $\frac{a}{x} = \frac{x}{b}$, $x^2 = ab$, and therefore $x = \sqrt{ab}$; thus the mean proportional between two numbers is the square root of the product.

Ex. **1742.** (On inch paper.) Find the square roots of (i) 2, (ii) 3, (iii) 6, (iv) 7.

[Find the mean proportionals between (i) 1 and 2, (iii) 2 and 3.]

Ex. **1743.** Draw a triangle; and construct an equivalent rectangle.
[What is the formula for the area of a triangle?]

To describe a square equivalent to a given rectilinear figure.

Construction (i) Reduce the figure to a triangle (see p. 178).

(ii) Convert the triangle into a rectangle.

(iii) Find the mean proportional between the sides of the rectangle.

This will be the side of the required square.

Proof If a, b are the sides of the rectangle, x the side of the equivalent square, then

$$\text{area of rectangle} = ab = x^2.$$

Ex. **1744.** (On inch paper.) Find the side of the square equivalent to the triangles whose angular points are

(i) (1, 0), (5, 0), (4, 3),

(ii) (0, 0), (0, 2), (5, 0·5),

(iii) (0, 0), (3, 1), (2, 3).

Ex. **1745.** Construct a square equivalent to a regular hexagon of side 2 in.; measure the side of the square.

Ex. **1746.** Repeat Ex. 1745 for a regular octagon of side 2 in.

Ex. **1747.** Find the side of a square equivalent to the quadrilateral ABCD, when

(i) DA = 1 in., ∠A = 100°, AB = 2·3 in., ∠B = 64°, BC = 1·5 in.

(ii) AB = 5·7 cm., BC = 5·2 cm., CD = 1·7 cm., DA = 3·9 cm., ∠A = 76°.

†Ex. **1748.** In fig. 322, prove that (i) $PT^2 = PR \,.\, PS$, (ii) $ST^2 = SR \,.\, SP$.

†Ex. **1749.** Prove Pythagoras' theorem by drawing the altitude to the hypotenuse and using similar triangles (see Ex. 1748).

¶Ex. **1750.** Draw a large scalene triangle ABC; draw the bisector of ∠A and let it cut BC at D. Calculate AB : AC and DB : DC.

¶Ex. **1751.** Repeat Ex. 1750 with a triangle of different shape.

¶Ex. **1752.** Draw a large scalene triangle ABC; draw the bisector of the external angle at A; let it cut the base produced at D. Calculate AB : AC and the ratio in which D divides the base BC (see p. 304).

¶Ex. **1753.** Repeat Ex. 1752 with a triangle of different shape.

THEOREM 8 (i).

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

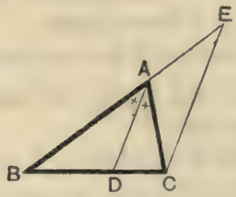


fig. 323.

Data ABC is a triangle,

AD bisects $\angle$ BAC internally and cuts BC at D.

To prove that $\frac{DB}{DC} = \frac{AB}{AC}$.

Construction Through C draw CE $\parallel$ to DA to cut BA produced at E.

Proof Since DA is $\parallel$ to CE,

$\therefore \frac{DB}{DC} = \frac{AB}{AE}$. IV. 1.

[It remains to prove that AE = AC].

$\because$ DA is $\parallel$ to CE,

$\therefore \angle$ BAD = corresp. $\angle$ AEC, I. 5.

and $\angle$ DAC = alt. $\angle$ ACE. I. 5.

But $\angle$ BAD = $\angle$ DAC, *Data*

$\therefore \angle$ AEC = $\angle$ ACE,

$\therefore$ AE = AC, I. 13.

$\therefore \frac{DB}{DC} = \frac{AB}{AC}$.

Q. E. D.

†Ex. **1754.** State and prove the converse of this theorem. [Use *reductio ad absurdum.*]

THEOREM 8 (ii).

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.

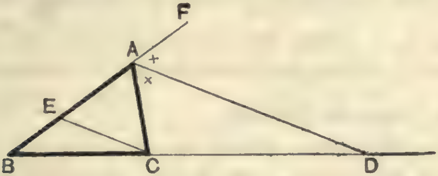


fig. 324.

Data ABC is a triangle,

AD bisects $\angle$ BAC externally (i.e. AD bisects $\angle$ CAF) and cuts BC produced at D.

To prove that $\frac{DB}{DC} = \frac{AB}{AC}$.

Construction Through C draw CE $\parallel$ to DA to cut BA at E.

Proof Since DA is $\parallel$ to CE,

$\therefore \frac{DB}{DC} = \frac{AB}{AE}$. IV. 1.

[It remains to prove that AE = AC].

$\because$ DA is $\parallel$ to CE,

$\therefore \angle$ FAD = corresp. $\angle$ AEC, I. 5.

and $\angle$ DAC = alt. $\angle$ ACE. I. 5.

But $\angle$ FAD = $\angle$ DAC, *Data*

$\therefore \angle$ AEC = $\angle$ ACE,

$\therefore$ AE = AC, I. 13.

$\therefore \frac{DB}{DC} = \frac{AB}{AC}$.

Q. E. D.

NOTE. There is a very close analogy between theorems 8 (i) and 8 (ii); notice that the proofs are nearly identical.

†Ex. **1755.** State and prove the converse of this theorem.

Ex. 1756. In a $\triangle$ABC, BC = 3·5 in., CA = 3 in., AB = 4 in. and the internal bisector of $\angle$A cuts BC at D; calculate BD, DC; check by drawing.

Ex. 1757. The internal bisector of $\angle$B of $\triangle$ABC cuts the opposite side in E; find EA, EC when BC = 8·9 cm., CA = 11·5 cm., AB = 4·7 cm.

Ex. 1758. In a $\triangle$ABC, BC = 3·5 in., CA = 3 in., AB = 4 in. and the external bisector of $\angle$A cuts the base produced at D; calculate BD, DC.

Ex. 1759. Repeat Ex. 1758 with

(i) BC = 5·2 in., CA = 4·1 in., AB = 4·5 in.,

(ii) BC = 11·5 cm., CA = 4·7 cm., AB = 8·9 cm.

Ex. 1760. Calculate the distance between the points in which AB, a side of a $\triangle$ABC, is cut by the bisectors of $\angle$C, having given that BC = 6·9 cm., CA = 11·4 cm., AB = 5·8 cm.

†**Ex. 1761.** The base BC of a triangle ABC is bisected at D. DE, DF bisect $\angle^s$ ADC, ADB, meeting AC, AB in E, F. Prove that EF is ∥ to BC.

†**Ex. 1762.** Prove that the bisectors of an angle of a triangle divide the base internally and externally in the same ratio.

Ex. 1763. The internal and external bisectors of the $\angle$P of a $\triangle$PQR cut the base at X, Y respectively; what is $\angle$XPY?

†**Ex. 1764. A point P moves so that the ratio of its distances from two fixed points Q, R is constant; prove that the locus of P is a circle. (The Circle of Apollonius.)**

[Draw the internal and external bisectors of $\angle$P, and use Ex. 1763.]

†**Ex. 1765.** O is a point inside a triangle ABC. The bisectors of $\angle^s$ BOC, COA, AOB meet BC, CA, AB in P, Q, R respectively. Prove that

$$\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = 1.$$

Revise Ex. 1651.

THEOREM 9. †

If the straight lines joining a point to the vertices of a given polygon are divided (all internally or all externally) in the same ratio the points of division are the vertices of a similar polygon.

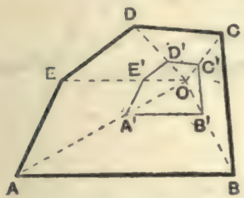


fig. 325.

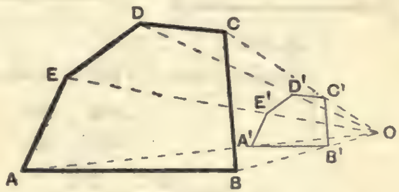


fig. 326.

Data ABCDE is a polygon; the st. lines joining a pt. O to its vertices are all divided in the same ratio at A′, B′, C′, D′, E′.

To prove that $\angle A = \angle A'$, $\angle B = \angle B'$, ...,

$$\text{and } \frac{A'B'}{AB} = \frac{B'C'}{BC} = \ldots.$$

Proof Since OA, OB, ... are divided at A′, B′, ... in the same ratio, it follows that

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \ldots = k \text{ (say).}$$

In the $\triangle$s OA′B′, OAB

$$\because \frac{OA'}{OA} = \frac{OB'}{OB},$$

and $\angle$ AOB is common,

$\therefore$ $\triangle$s OA′B′, OAB are similar, IV. 5.

$\therefore \angle OA'B' = \angle OAB.$

Sim^{ly} $\angle OA'E' = \angle OAE$,

$\therefore \angle B'A'E' = \angle BAE.$

Sim^{ly} the other $\angle$ s of A′B′C′D′E′ are equal to the corresponding $\angle$ s of ABCDE.

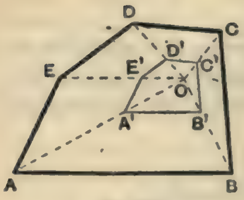


fig. 325.

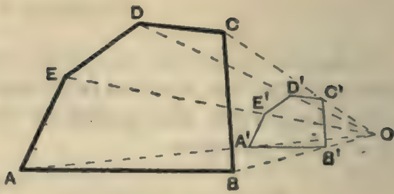


fig. 326.

Again, since $\triangle$s OA′B′, OAB are similar, *Proved*

$$\therefore \frac{A'B'}{AB} = \frac{OA'}{OA} = k.$$

$$\text{Sim}^{\text{ly}}\ \frac{B'C'}{BC},\ \frac{C'D'}{CD},\ \ldots,\ \text{each} = k,$$

$$\therefore \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \ldots.$$

$\therefore$ ABCDE, A′B′C′D′E′ are similar.

Q. E. D.

NOTE. This theorem is the principle of the magic lantern; every part of the figure is magnified outwards from a point.

¶Ex. **1766.** Draw a figure to show that equiangular pentagons are not necessarily similar.

¶Ex. **1767.** Draw a figure to show that a pentagon whose sides taken in order are halves of the sides of another pentagon is not necessarily similar to the other pentagon.

¶Ex. **1768.** A rectangular picture frame is made of wood 1 in. wide; are the inside and outside of the frame similar rectangles?

¶Ex. **1769.** Draw a figure for IV. 9 for the case in which O coincides with B.

¶Ex. **1770.** Draw a figure for IV. 9 for the case in which O is on AB.

†Ex. **1771.** Assuming that the polygons ABCDE in figs 325, 326 are congruent, and that the ratio of division is the same for the two figures, prove that the two polygons A′B′C′D′E′ are congruent.

On a given straight line to construct a figure similar to a given rectilinear figure. (Second Method.) † [See p. 317.]

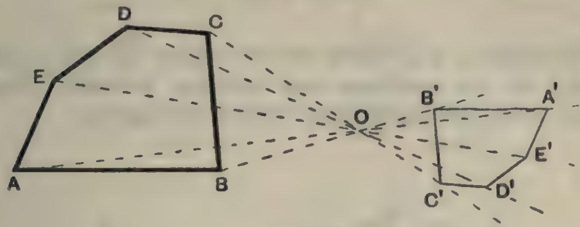


fig. 327.

Construction Let ABCDE be the given figure, A′B′ the given straight line (see figs. 325, 326, 327.)

Place A′B′ parallel to AB, and produce AA′, BB′ to meet at O; join OC, OD, OE.

Divide OC, OD, OE at C′, D′, E′ in the same ratio as OA and OB are divided. [This is most easily done by drawing parallels.]

Join B′C′, C′D′, D′E′, E′A′.

Then A′B′C′D′E′ is similar to ABCDE. IV. 9.

NOTE. The method (used in Ex. 1683) of dividing coordinates in a given ratio is substantially the same as the above.

Ex. **1772.** (On inch paper.) Mark the points (0, 0), (3, 0), (3, 3), (1, 4), (0, 3); join them in order. On the line (1, 1), (2, 1) describe a similar pentagon by the method just explained. From your figure, read off the coordinates of its vertices.

Ex. **1773.** Repeat Ex. 1772, with (0, 0), (·5, 0), (·7, ·3), (·1, ·6) as the corners of the given figure, and (1, 1), (3·2, 1) as the ends of the given line.

Ex. **1774.** Repeat Ex. 1772, with (−2, −2), (2, −2), (3, 3), (−1, 2) as the corners of the given figure, and (−1, −1), (1, −1) as the ends of the given line.

Ex. **1775.** (On inch paper.) Draw the triangle ABC, A (2, 0), B (2, 3), C (0, 1); on PQ, P (3, 3), Q (3, 0·2), as base describe a triangle similar to △ABC. Find the coordinates of the vertex.

[Take O as the point of intersection of AP, BQ.]

THEOREM 10. †

If a polygon is divided into triangles by lines joining a point to its vertices, any similar polygon can be divided into corresponding similar triangles.

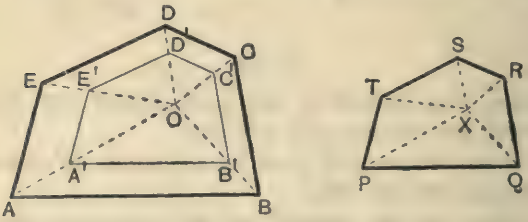


fig. 328.

Data ABCDE, PQRST are two equiangular polygons which have

$$\frac{PQ}{AB}=\frac{QR}{BC}=\frac{RS}{CD}=\frac{ST}{DE}=\frac{TP}{EA}=k \text{ (say)}.$$

ABCDE is divided into $\triangle$s by lines joining its vertices to a pt. O.

To prove that there is a point X such that the $\triangle$s formed by joining X to the vertices of PQRST are similar to the corresponding $\triangle$s into which ABCDE is divided.

Construction Divide OA, OB, ... at A′, B′, ... so that

$$\frac{OA'}{OA}=\frac{OB'}{OB}=\ldots=k.$$

Join A′B′, B′C′, ...

Proof [First to prove A′B′C′D′E, PQRST congruent].

$$\text{Since } \frac{OA'}{OA} = \frac{OB'}{OB} = \ldots, \qquad \textit{Constr.}$$

$\triangle$s $OA'B'$, $OB'C'$, ... are similar to $\triangle$s OAB, OBC, ... respectively. *Proved in* IV. 9.

For the same reason A′B′C′D′E′ is equiangular to ABCDE, IV. 9.

but ABCDE is equiangular to PQRST, *Data*

∴ A′B′C′D′E′ „ „ PQRST.

Again, since

$$\frac{OA'}{OA} = \frac{OB'}{OB} = \ldots,$$

$$\frac{A'B'}{AB} = \frac{OA'}{OA} = k. \qquad \text{IV. 9.}$$

$$\text{But } \frac{PQ}{AB} = k,$$

$$\therefore \frac{A'B'}{AB} = \frac{PQ}{AB}.$$

$$\therefore A'B' = PQ.$$

$$\text{Sim}^{\text{ly}}\ B'C' = QR,\ C'D' = RS, \ldots,$$

∴ A′B′C′D′E′, PQRST have all their corresponding angles and sides equal and are therefore congruent.

Apply A′B′C′D′E′O to PQRST; since A′B′C′D′E′, PQRST are congruent, they must coincide; let X be the point on which O falls.

Join XP, XQ, ...

Then $\triangle XPQ \equiv \triangle OA'B'$.

But $\triangle$s $OA'B'$, OAB are similar,

∴ $\triangle$s XPQ, OAB are similar.

Likewise the other pairs of corresponding $\triangle$s in the two polygons are similar.

Q. E. D.

NOTE. The practical way to find the point X is to make

$$\angle QPX = \angle BAO \text{ and } \angle PQX = \angle ABO.$$

O and X are called **corresponding** points.

COR. **If in two similar figures whose sides are in the ratio $1:k$, O_1, O_2 correspond to X_1, X_2, then $O_1O_2 : X_1X_2 = 1:k$.**

Ex. **1776.** (Inch paper.) O is a point inside a triangle ABC. A is $(-3, 3)$, B is $(-2, -1)$, C is $(2, -2)$, O is $(-1, 0)$. PQR is a similar triangle; P is $(-1{\cdot}5, 1{\cdot}5)$, Q is $(-1, -0{\cdot}5)$, R is $(1, -1)$. Find the coordinates of the point X which corresponds to O.

†Ex. **1777.** O is the circumcentre of $\triangle$ABC; X is the "corresponding" point in a similar triangle PQR. Prove that X is the circumcentre of $\triangle$PQR.

Ex. **1778.** Construct $\triangle$ABC, given $\angle A = 70°$, $\angle B = 45°$, $\angle C = 65°$, and altitude AD = 8 cm. Measure BC.

[First construct $\triangle A'B'C'$ having its angles equal to the given angles; draw the altitude $A'D'$. Magnify $\triangle$ABC in the ratio AD : $A'D'$.]

Ex. **1779.** Construct $\triangle$ABC, given $\angle A = 45°$, $\angle B = 25°$, $\angle C = 110°$, and median BM = 7·5 cm. Measure BC. [See note to Ex. 1778.]

Ex. **1780.** Show how to describe a triangle, having given its angles and its perimeter.

Ex. **1781.** Show how to describe a triangle, having given its angles and the difference of two of its sides.

Ex. **1782.** Show how to inscribe in a given triangle a triangle which has its sides parallel to the sides of a given triangle.

Ex. **1783.** Show how to inscribe a square in a given triangle.

Ex. **1784.** Show how to inscribe a square in a given sector of a circle.

Ex. **1785.** Show how to inscribe an equilateral triangle in a given triangle.

Ex. **1786.** Show how to describe a circle to touch two given straight lines and pass through a given point.

Ex. **1787.** Show how to inscribe a regular octagon in a given square.

THEOREM 11.†

The ratio of the areas of similar polygons is equal to the ratio of the squares on corresponding sides.

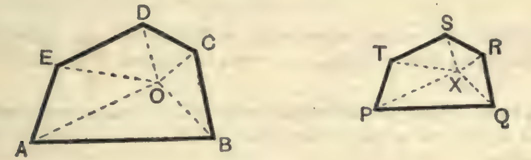


fig. 329.

Data ABCDE, PQRST are two similar polygons; let $\frac{AB}{PQ} = k$.

To prove that $$\frac{\text{area of ABCDE}}{\text{area of PQRST}} = \frac{AB^2}{PQ^2}.$$

Construction In ABCDE take any point O.
Let X be the corresponding point in PQRST. IV. 10.
Join OA, OB, ...; XP, XQ, ...

Proof Since O, X are corresponding points,
$\therefore \triangle$s OAB, XPQ are similar,

$$\therefore \frac{\triangle OAB}{\triangle XPQ} = \frac{AB^2}{PQ^2} = k^2.$$

$$\text{Sim}^{ly} \frac{\triangle OBC}{\triangle XQR} = \frac{BC^2}{QR^2} = k^2,$$

....

$$\therefore \triangle OAB = k^2 . \triangle XPQ,$$
$$\text{and } \triangle OBC = k^2 . \triangle XQR,$$

....

$$\therefore \triangle OAB + \triangle OBC + \ldots = k^2 \{\triangle XPQ + \triangle XQR + \ldots\},$$
$$\therefore ABCDE = k^2 . PQRST,$$
$$\therefore \frac{ABCDE}{PQRST} = k^2 = \frac{AB^2}{PQ^2}.$$

Q. E. D.

¶Ex. **1788.** What is the ratio of the area of a room to the area by which it is represented on a plan whose scale is 1 in. to 1 ft.?

¶Ex. **1789.** On a map whose scale is 1 mile to 1 in., a piece of land is represented by an area of 20 sq. in.; what is the area of the land?

¶Ex. **1790.** On a map whose scale is 2 miles to 1 in., a piece of land is represented by an area of 24 sq. in.; what is the area of the land?

Ex. **1791.** What is the acreage of a field which is represented by an area of 3 sq. in. on a map whose scale is 25 in. to the mile? (640 acres = 1 sq. mile.)

Ex. **1792.** What areas represent a field of 1 acre on maps in which 1 mile is represented by (i) 1 in., (ii) $\frac{1}{2}$ in., (iii) 6 in., (iv) 2·5 in.?

If the field were square, what would be the length of a line representing a side of the field?

Ex. **1793.** Two similar windows are glazed with small lozenge-shaped panes of glass, these panes being all identical in size and shape. The heights of the windows are 10 ft. and 15 ft. The number of panes in the smaller window is 1200; what is the number in the larger?

Ex. **1794.** A figure described on the hypotenuse of a right-angled triangle is equal to the sum of the similar figures described on the sides of the triangle. (This is a generalisation of Pythagoras' theorem.)

Ex. **1795.** Similar figures are described on the side and diagonal of a square; prove that the ratio of their areas is 1 : 2.

Ex. **1796.** Similar figures are described on the side and altitude of an equilateral triangle; prove that the ratio of their areas is 4 : 3.

To construct a figure equivalent to a given figure A and similar to another figure B. †

Construction Reduce both figures to squares (see p. 333).

Let a and b be the sides of these squares.

Let l be a side of the figure B.

Construct a length x so that $b : a = l : x$.

On x describe a figure C similar to B; the side x of C corresponding to the side l of B.

Proof The area of C : area of B $= x^2 : l^2$

$$= a^2 : b^2$$

$$= \text{area of A} : \text{area of B},$$

$\therefore$ area of C $=$ area of A.

Ex. **1797.** Show how to construct an equilateral triangle equivalent to a given square.

Ex. **1798.** Show how to construct an equilateral triangle equivalent to a given triangle.

Ex. **1799.** Show how to construct a rectangle having its sides in a given ratio and equivalent to a given square.

MISCELLANEOUS EXERCISES.

†Ex. **1800.** One of the parallel sides of a trapezium is double the other; show that the diagonals trisect one another.

†Ex. **1801.** A straight line drawn parallel to the parallel sides of a trapezium divides the other two sides (or those sides produced) proportionally.

†Ex. **1802.** Find the locus of a point which moves so that the ratio of its distances from two intersecting straight lines is constant.

Ex. **1803.** Show how to draw through a given point within a given angle a straight line to be terminated by the arms of the angle, and divided in a given ratio (say $\frac{2}{3}$) at the given point.

†Ex. **1804. Prove that two medians of a triangle trisect one another. Hence prove that the three medians pass through one point.**

†Ex. **1805.** The bisectors of the equal angles of two similar triangles are to one another as the bases of the triangles.

†Ex. **1806.** In two similar triangles, the parts lying within the triangle of the perpendicular bisectors of corresponding sides have the same ratio as the corresponding sides of the triangle.

†Ex. **1807.** ABC, DEF are two similar triangles; P, Q are any two points in AB, AC; X, Y are the corresponding points in DE, DF. Prove that PQ : XY = AB : DE.

†Ex. **1808.** The sides AC, BD of two triangles ABC, DBC on the same base BC and between the same parallels meet at E; prove that a parallel to BC through E, meeting AB, CD, is bisected at E.

Ex. **1809.** Show how to divide a parallelogram into five equivalent parts by lines drawn through an angular point.

†Ex. **1810.** Show how to divide a given line into two parts such that their mean proportional is equal to a given line. Is this always possible?

Ex. **1811.** Show how to construct a rectangle equivalent to a given square, and having its perimeter equal to a given line. [See Ex. 1810.]

†Ex. **1812.** A common tangent to two circles cuts the line of centres externally or internally in the ratio of the radii.

Ex. **1813.** Show how to construct on a given base a triangle having given the vertical angle and the ratio of the two sides.

Ex. **1814.** Show how to construct a triangle having given the vertical angle, the ratio of the sides containing the angle, and the altitude drawn to the base.

†Ex. **1815.** TP, TQ are tangents to a circle whose centre is C, CT cuts PQ in N; prove that $CN \cdot CT = CP^2$.

†Ex. **1816.** In fig. 318, prove that $\triangle PBC : \triangle PAD = BC^2 : AD^2$. Is the same property true for fig. 319?

†Ex. **1817.** In fig. 318, prove that $PB \cdot PC : PA \cdot PD = BC^2 : AD^2$.

†Ex. **1818.** ABCDE is a regular pentagon; BE, AD intersect at F; prove that EF is a third proportional to AD, AE.

Ex. **1819.** In fig. 295, the area of the regular hexagon obtained by joining the vertices of the star is three times that of the small hexagon.

†Ex. **1820.** In fig. 320, PQ is drawn parallel to AD to meet BC produced in Q; prove that PQ is a mean proportional between QB, QC.

†**Ex. 1821.** The angle BAC of a $\triangle$ABC is bisected by AD, which cuts BC in D; DE, DF are drawn parallel to AB, AC and cut AC, AB at E, F respectively. Prove that $BF : CE = AB^2 : AC^2$.

†**Ex. 1822.** ABC is a triangle right-angled at A; AD is drawn perpendicular to BC and produced to E so that DE is a third proportional to AD, DB; prove that $\triangle ABD = \triangle CDE$, and $\triangle ABD$ is a mean proportional between $\triangle^s$ ADC, BDE.

†**Ex. 1823.** Two circles touch externally at P; Q, R are the points of contact of one of their common tangents. Prove that QR is a mean proportional between their diameters.

[Draw the common tangent at P, let it cut QR at S; join S to the centres of the two circles.]

†**Ex. 1824.** Two church spires stand on a level plain; a man walks on the plain so that he always sees the tops of the spires at equal angles of elevation. Prove that his locus is a circle.

†**Ex. 1825.** The rectangle contained by two sides of a triangle is equal to the square on the bisector of the angle between those sides together with the rectangle contained by the segments of the base. [See Ex. 1717.]

†**Ex. 1826.** The tangent to a circle at P cuts two parallel tangents at Q, R; prove that the rectangle QP . PR is equal to the square on a radius of the circle.

†**Ex. 1827.** ABCD is a quadrilateral. If the bisectors of $\angle^s$ A, C meet on BD, then the bisectors of $\angle^s$ B, D meet on AC.

Ex. **1828.** Prove the validity of the following method of **solving a quadratic equation** graphically:—

Suppose that $ax^2 + bx + c = 0$ is the equation; on squared paper, mark the origin, from OX cut off $OP = a$, from P draw a perpendicular PQ upwards of length b, from Q draw to the left $QR = c$ (regard must be paid to the signs of a, b, c; e.g. if b is negative PQ will be drawn downwards); on OR describe a semicircle cutting PQ at S, T; the roots of the equation are

$$-\frac{PS}{OP} \text{ and } -\frac{PT}{OP}.$$

[Consider $\triangle^s$ OPS, SQR.]

Ex. 1829. Solve the following equations graphically as in Ex. 1828, and check by calculation :—

$$\text{(i)}\quad 2x^2 + 5x + 1 = 0,$$

$$\text{(ii)}\quad x^2 + 3x - 2 = 0,$$

$$\text{(iii)}\quad 2x^2 - x + 1 = 0.$$

†Ex. **1837.** Find a point P in the arc AB of a circle such that chord AP is three times the chord PB.

†Ex. **1838.** Show how to draw through a given point D in the side AB of a triangle ABC a straight line DPQ cutting AC in P and BC produced in Q so that PQ is twice DP.

†Ex. **1830.** A straight line AB is divided internally at C; equilateral triangles ACD, CBE are described on the same side of AB; DE and AB produced meet at F. Prove that FB : BC = FC : CA.

†Ex. **1831.** ABC is an equilateral triangle and from any point D in AB straight lines DK and DL are drawn parallel to BC and AC respectively. Find the ratio of the perimeter of the parallelogram DLCK to the perimeter of the triangle ABC.

†Ex. **1832.** If from each of the angular points of a quadrilateral perpendiculars are let fall upon the diagonals, the feet of these perpendiculars are the angular points of a similar quadrilateral.

†Ex. **1833.** ABCD is a parallelogram, P is a point in AC produced; BC, BA are produced to cut the straight line through P and D in Q, R respectively. Prove that PD is a mean proportional between PQ and PR.

†Ex. **1834.** ABCD is a quadrilateral inscribed in a circle of which AC is a diameter; from any point P in AC, PQ and PR are drawn perpendicular to CD and AB respectively. Prove that DQ : PR = DC : BC.

†Ex. **1835.** Two circles ABC, ADE touch internally at A; through A straight lines ABD, ACE are drawn to cut the circles. Prove that AB . DE = AD . BC.

†Ex. **1836.** In the sides AD, CB of a quadrilateral ABCD points P, Q are taken so that AP : PD = CQ : QB. Prove that △ADQ + △BPC = ABCD.

†Ex. **1839.** Show how to draw through a given point O a straight line to cut two given straight lines in P and Q respectively so that OP : PQ is equal to a given ratio.

†Ex. **1840.** O is a fixed point inside a circle, P is a variable point on the circle; what is the locus of the mid-point of OP?

†Ex. **1841.** ABCD is a quadrilateral; through A, B draw parallel straight lines to cut CD in X, Y so that $CX = DY$. [X and Y are both to be between C and D, or one in CD produced and the other in DC produced.]

†Ex. **1842.** Show how to construct a triangle having given the lengths of two of its sides and the length of the bisector (terminated by the base) of the angle between them.

†Ex. **1843.** From any point X in a chord PR of a circle, XY is drawn perpendicular to the diameter PQ, prove that $PX : PY = PQ : PR$.

†Ex. **1844.** Through the vertex A, of a triangle ABC, DAE is drawn parallel to BC and AD is made equal to AE; CD cuts AB at X and BE cuts AC at Y; prove XY parallel to BC.

†Ex. **1845.** ABCD is a parallelogram; a straight line through A cuts BD in O, BC in P, DC in Q. Prove that AO is a mean proportional between OP and OQ.

†Ex. **1846.** A triangle PQR is inscribed in a circle and the tangent to the circle at the other end of the diameter through P cuts the sides PQ, PR produced at H, K respectively; prove that the $\triangle$s PKH, PQR are similar.

†Ex. **1847.** Two circles ACB, ADB intersect at A, B; AC, AD touch the circles ADB, ACB respectively at A; prove that AB is a mean proportional between BC and BD.

†Ex. **1848.** A variable circle moves so as always to touch two fixed circles; prove that the straight line joining the points of contact cuts the line of centres of the fixed circle in one of two fixed points.

†Ex. **1849.** ABC is an equilateral triangle and D is any point in BC. On BC produced points E and F are taken such that AB bisects the angle EAD and AC bisects the angle DAF. Show that the triangles ABE and ACF are similar and that $BE . CF = BC^2$.

†Ex. **1850.** (i) In a $\triangle ABC$, $AB = \frac{1}{2}AC$, CX is drawn perpendicular to the internal bisector of the $\angle BAC$; prove that AX is bisected by BC.

(ii) State and prove an analogous theorem for the external bisector of the $\angle BAC$.

†Ex. **1851.** Two circles touch one another externally at A, BA and AC are diameters of the circles; BD is a chord of the first circle which touches the second at X, and CE is a chord of the second which touches the first at Y. Prove that $BD . CE = 4DX . EY$.

†Ex. 1852. Two straight lines AOB, COD intersect at O; prove that, if OA : OB = OC : OD, then the $\triangle$s AOD, BOC are equivalent.

†Ex. 1853. The sides AB, AD of the rhombus ABCD are bisected in E, F respectively. Prove that the area of the triangle CEF is three-eighths of the area of the rhombus.

†Ex. 1854. ABC is a triangle right-angled at A, the altitude AD is produced to E so that DE is a third proportional to AD, DC; prove that $\triangle$s BDE, ADC are equal in area.

†Ex. 1855. Two circles ABC, AB′C′, whose centres are O and O′, touch externally at A; BAB′ is a straight line; prove that the triangles OAB′, O′AB are equal in area.

†Ex. 1856. ABC is a triangle, and BC is divided at D so that $BD^2 = BC . DC$. A line DE parallel to AC meets AB in E. Show that the triangles DBE, ACD are equal in area.

†Ex. 1857. PA, PB are the two tangents from P to a circle whose centre is O; prove that $\triangle PAB : \triangle OAB = PA^2 : OA^2$.

†Ex. 1858. Two triangles ABC, DEF have $\angle$ A and $\angle$ D supplementary and the sides about these angles proportional, prove that the ratio of the areas of these triangles is equal to $AB^2 : DE^2$.

†Ex. 1859. Through the vertices of a triangle ABC, parallel straight lines are drawn to meet the opposite sides of the triangle in points α, β, γ; prove that $\triangle \alpha\beta\gamma = 2 \triangle ABC$.

Ex. 1860. Through the vertices A, B, C of an equilateral triangle straight lines are drawn perpendicular to the sides AB, BC, CA respectively, so as to form another equilateral triangle. Compare the areas of the two triangles.

†Ex. 1861. A square BCDE is described on the base BC of a triangle ABC, and on the side opposite to A. If AD, AE cut BC in F, G respectively, prove that FG is the base of a square inscribed in the triangle ABC.

†Ex. 1862. Prove that the rectangle contained by the hypotenuses of two similar right-angled triangles is equal to the sum of the rectangles contained by the other pairs of corresponding sides.

†Ex. 1863. The sides AB, AC of a triangle are bisected at D and E respectively; prove that, if the circle ADE intersect the line BC, and P be a point of intersection, then AP is a mean proportional between BP and CP.

†Ex. **1864.** Circles are described on the sides of a right-angled triangle ABC as diameters, and through the right angle A a straight line APQR is drawn cutting the three circles in P, Q, R respectively. Show that AP = QR.

†Ex. **1865.** The bisector of the angle BAC of a triangle ABC meets the side BC at D. The circle described about the triangle BAD meets CA again at E, and the circle described about the triangle CAD meets BA again at F. Show that BF is equal to CE.

†Ex. **1866.** D, E, F are points in the sides BC, CA, AB of a $\triangle$ABC such that AD = BE = CF. From any point O within the $\triangle$ABC, OP, OQ, OR are drawn parallel to AD, BE, CF to meet BC, CA, AB in P, Q, R respectively. Show that OP + OQ + OR = AD.

†Ex. **1867.** ABCD is a quadrilateral with the angles at A and C right angles. If BK and DN are drawn perpendicular to AC, prove that AN = CK.

†Ex. **1868.** The angle BAC of a triangle is bisected by a straight line which meets the base BC in D; a straight line drawn through D at right angles to AD meets AB in E and AC in F. Prove that EB : CF = BD : DC.

†Ex. **1869.** If the tangents at the ends of one diagonal of a cyclic quadrilateral intersect on the other diagonal produced, the rectangle contained by one pair of opposite sides is equal to that contained by the other pair.

†Ex. **1870.** Two circles ABC, APQ (of which APQ is the smaller) touch internally at A; BC a chord of the larger touches the smaller at R; AB, AC cut the circle APQ at P and Q respectively. Prove that AP : AQ = BR : RC.

†Ex. **1871.** AB is a fixed chord of a circle; CD is the diameter perpendicular to AB; P is a variable point on the circle; AP, BP cut CD (produced if necessary) in X, Y; if O is the centre of the circle, prove that OX . OY is constant.

†Ex. **1872.** Any point P is taken within a parallelogram ABDC; PM and PN are drawn respectively parallel to the sides AC and AB and terminated by AB and AC; NP produced meets BD in E; AE is joined meeting PM in P′; P′Q is drawn parallel to AB meeting the diagonal AD in Q. Prove that AQ : AD = parallelogram AMPN : parallelogram ABDC.

†Ex. **1873.** A straight line HK is drawn parallel to the base BC of a triangle ABC to cut AB, AC in H, K respectively; BK, HC intersect at X, AX cuts HK, BC at Y, Z respectively. Prove that YX : XZ = AY : AZ.

†Ex. **1874.** ABCDEFG is a regular heptagon; BG cuts AC, AD in X, Y respectively; prove that AX . AC = AY . AD.

†Ex. **1875.** P, Q, R, S are four consecutive corners of a regular polygon; PR, QS intersect at X; prove that QR is a mean proportional between PR and RX.

†Ex. **1876.** Two straight lines BGE, CGF intersect at G so that GE $=\frac{1}{3}$BE and GF $=\frac{1}{3}$CF; BF and CE are produced to meet at A; prove that BF $=$ FA and CE $=$ EA.

†Ex. **1877.** In two circles ABC, DEF, $\angle$ BAC $=$ $\angle$ EDF, prove that the ratio of the chords BC, EF is equal to the ratio of the diameters of the circles.

†Ex. **1878.** ABCDEF is a hexagon with its opposite sides parallel, CF is parallel to AB (and DE), and AD is parallel to BC (and EF); prove that BE must be parallel to CD (and AF).

APPENDIX I.

EUCLID II. 14*.

To describe a square equal to a given rectangle ABCD.

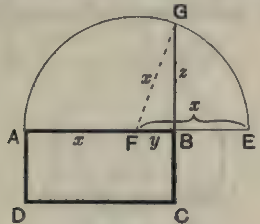


fig. 330.

Construction Produce AB to E, so that $BE = BC$.

Bisect AE at F.

With centre F and radius FA describe a semicircle AGE.

Produce CB to meet the semicircle at G.

Then, if a square is described on BG, this square is equal to rect. AC.

Proof Since F is the centre of $\odot$ AGE,

$$\therefore FA = FG = FE = x.$$

Let $FB = y$, $BG = z$.

Then $AB = x + y$,

$$BC = BE = x - y,$$

$\therefore$ area of rect. $AC = AB \cdot BC = (x+y)(x-y) = x^2 - y^2$.

Again, since $\triangle$ FBG is rt. $\angle^{d}$ at B,

$$\therefore x^2 = y^2 + z^2, \qquad \textit{Pythagoras}$$

$$\therefore x^2 - y^2 = z^2,$$

$\therefore$ rect. $AC =$ square on BG.

For Exercises see p. 333.

* The two propositions given below have been treated, in the present work, as applications of the theory of similar figures. For examinations in which only the first three books are required, an independent proof of these propositions is desirable: the proofs in the Appendix are substantially those of Euclid.

Euclid III. 35, 36.

If two chords of a circle intersect, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Case I. *Let the chords* AB, CD *intersect at* P, *a point inside the circle.*

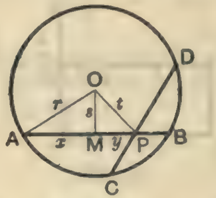


fig. 331.

To prove that rect. PA . PB = rect. PC . PD.

Construction From O, the centre of the ⊙, draw OM ⊥ to AB.
Join OA, OP.

Proof Since OM is ⊥ to chord AB,

$\therefore$ AM = BM = x, say. III. 1.

Let PM = y, OA = r, OM = s, OP = t.

Then PA = $x+y$, PB = $x-y$,

$\therefore$ rect. PA . PB = $(x+y)(x-y)$
$= x^2 - y^2$.

Now △ OMA is rt. ∠ᵈ at M,

$\therefore x^2 + s^2 = r^2$. *Pythagoras*

Sim^{ly} $y^2 + s^2 = t^2$,

$\therefore$ subtracting, $x^2 - y^2 = r^2 - t^2$,

$\therefore$ rect. PA . PB = $r^2 - t^2$
= radius^2 − OP^2.

Sim^{ly} by drawing a perpendicular to chord CD it may be shown that

rect. PC . PD = radius^2 − OP^2,

$\therefore$ rect. PA . PB = rect. PC . PD.

Q. E. D.

CASE II. *Let the chords* AB, CD *intersect at* P, *a point outside the circle.*

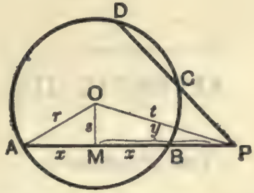


fig. 332.

To prove that rect. PA . PB = rect. PC . PD.

Construction Draw OM ⊥ to AB.

Proof As in Case I., AM = BM = x,

$$PA = y + x, \; PB = y - x,$$

$$\therefore \text{rect. } PA \,.\, PB = (y+x)(y-x)$$
$$= y^2 - x^2.$$

Again, as in Case I.,

$$y^2 + s^2 = t^2,$$
$$x^2 + s^2 = r^2,$$
$$\therefore y^2 - x^2 = t^2 - r^2,$$
$$\therefore \text{rect. } PA \,.\, PB = t^2 - r^2$$
$$= OP^2 - \text{radius}^2.$$

Sim[ly] it may be shown that

$$\text{rect. } PC \,.\, PD = OP^2 - \text{radius}^2,$$
$$\therefore \text{rect. } PA \,.\, PB = \text{rect. } PC \,.\, PD.$$

Q. E. D.

For the discussion of the case in which C, D *in fig.* 332 *coincide, and* PCD *becomes a tangent, see Ex.* 1723. *Exercises on the above theorem will be found on page* 330.

APPENDIX II.†

The Pentagon.

To divide a given straight line into two parts such that the square on the greater part may be equal to the rectangle contained by the whole line and the smaller part.

[*Analysis.* Let the whole line contain a units of length.

Let the ratio of the greater part to the whole line be $x : 1$.

Then the greater part contains ax units; and the smaller $a - ax$ units.

The square on the greater part contains a^2x^2 units of area and the rectangle contained by the whole line and the smaller part contains $a(a - ax)$ units of area,

$$\therefore \ a^2x^2 = a^2 - a^2x,$$

$$\therefore \ x^2 = 1 - x,$$

$$\therefore \ x^2 + x - 1 = 0.$$

Solving this equation, we find

$$x = \pm\frac{\sqrt{5}}{2} - \frac{1}{2}.$$

For the present* we reject the lower sign, which would give a negative value for x; and we are left with

$$x = \frac{\sqrt{5}}{2} - \frac{1}{2} = 0{\cdot}618\ldots.]$$

* It will be seen below (p. 358) that a meaning can be found for the negative value of x.

In order to construct this length with ungraduated ruler and compass only, we proceed as follows :—

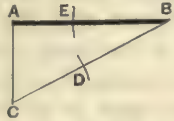


fig. 333.

Let AB be the given straight line.

Construction At A erect AC ⊥ to AB, and equal to $\frac{1}{2}$AB.

Join CB.

From CB cut off CD = CA.

From BA cut off BE = BD.

Then AB is divided as required.

Proof

$$BC^2 = AB^2 + AC^2.$$

But AB $= a$ and AC $= \frac{1}{2}a$,

$$\therefore\ BC^2 = a^2 + \tfrac{1}{4}a^2 = \tfrac{5}{4}a^2.$$

$$\therefore\ BC = \sqrt{\tfrac{5}{4}}\,a = \frac{\sqrt{5}}{2}\,a.$$

$$\therefore\ BE = BD = \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)a.$$

To verify that this length satisfies the given conditions.

$$BE^2 = \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^2 a^2 = \left(\frac{5}{4} + \frac{1}{4} - \frac{\sqrt{5}}{2}\right)a^2 = \left(1\tfrac{1}{2} - \frac{\sqrt{5}}{2}\right)a^2.$$

$$AE = a - \left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)a = \left(1\tfrac{1}{2} - \frac{\sqrt{5}}{2}\right)a.$$

$$\therefore\ AE\,.\,AB = \left(1\tfrac{1}{2} - \frac{\sqrt{5}}{2}\right)a \times a = BE^2.$$

Extreme and mean ratio. The relation $AE \cdot AB = BE^2$ may be written $AE : BE = BE : AB$. Thus the straight line AB has been divided so that **the larger part is the mean proportional between the smaller part and the whole line.** In other words, the larger part is the mean, while the smaller part and the whole line are the extremes of a proportion. For this reason, a line divided as above is said to be **divided in extreme and mean ratio.** This method of dividing a line is also known as **medial section.**

NOTE. The solution $x = -\frac{\sqrt{5}}{2} - \frac{1}{2}$ was rejected. Strictly speaking, however, it is a second solution of the problem. The fact that this value of x is negative indicates that BE must be measured from B *in the other direction*—away from A rather than towards A—as BE′ in fig. 334.

fig. 334.

Ex. **1890.** With ruler and compass, divide a straight line one decimetre long in extreme and mean ratio. Calculate the correct lengths for the two parts, and estimate the percentage error in your drawing.

Ex. **1891.** Devise a geometrical construction for dividing a line externally as in the above note (fig. 334).

†Ex. **1892.** Prove that, if E′ is constructed as in the note (fig. 334), then $AB \cdot AE' = BE'^2$; and hence that the line AB is divided externally in extreme and mean ratio.

†Ex. **1893.** Prove that if AB is divided externally in extreme and mean ratio at E′, then AE′ is divided internally in extreme and mean ratio at B.

Ex. **1894.** Show how to divide a straight line AB at C so that

(i) $AB \cdot AC = 2CB^2$,

(ii) $2AB \cdot AC = CB^2$,

(iii) $AC^2 = 2CB^2$.

To construct an isosceles triangle such that each of the base angles is twice the vertical angle.

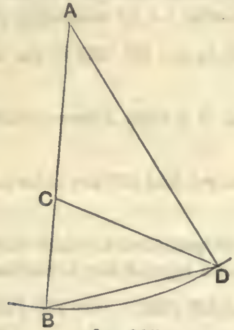


fig. 335.

Construction Draw a straight line AB of any length.

Divide AB at C so that $AB \,.\, BC = AC^2$.

With centre A and radius AB describe a circle.

In this circle place a chord $BD = AC$.

Join AD.

Then ABD is an isosceles $\triangle$ having $\angle B = \angle D = 2 \angle A$.

Proof Join CD.

Since $BC \,.\, BA = AC^2 = BD^2$,

$$\therefore \quad BC : BD = BD : BA.$$

Thus, in the $\triangle^s$ BCD, BDA, the $\angle B$ is common and the sides about the common angle are proportional.

$\therefore$ $\triangle^s$ are similar. IV. 5.

But $\triangle BDA$ is isosceles ($\because AB = AD$),

$\therefore$ $\triangle BCD$ is isosceles,

$\therefore$ $CD = BD = CA$.

$\therefore$ $\angle CDA = \angle A$.

Now $\angle BCD$ (ext. $\angle$ of $\triangle CAD$)

$= \angle A + \angle CDA$

$= 2 \angle A$,

$\therefore$ $\angle B = 2 \angle A$.

Ex. 1895. Perform the above construction. Calculate what should be the magnitudes of the angles of the triangle, and verify that your figure agrees with your calculation. (To save time, it will be best to divide AB in the required manner *arithmetically*, i.e. by measuring off the right length.)

†Ex. **1896.** Show that, in fig. 335, BD is the side of a regular decagon inscribed in the circle.

†Ex. **1897.** Show that, if ⊙ACD is drawn, BD will be a tangent to that circle.

†Ex. **1898.** Prove that AC and CD are sides of a regular pentagon inscribed in ⊙ACD.

†Ex. **1899.** Let DC be produced to meet the circle of fig. 335 in E. Prove that BE is the side of a regular 5-gon inscribed in ⊙A.

†Ex. **1900.** Prove that AE = EC. (See Ex. 1899.)

†Ex. **1901.** Prove that AE is ∥ to BD. (See Ex. 1899.)

†Ex. **1902.** Prove that △s AED, CAD are similar. (See Ex. 1899.)

†Ex. **1903.** Prove that DE is divided in extreme and mean ratio at C. (See Ex. 1899.)

†Ex. **1904.** Prove that, if ⊙ABD is drawn, BD is the side of a regular pentagon inscribed in the ⊙.

†Ex. **1905.** Let the bisectors of ∠s B, D meet ⊙ABD in F, G. Prove that AGBDF is a regular pentagon.

To describe a regular pentagon.

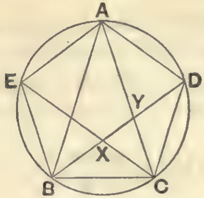


fig. 336.

Construction Construct an isosceles $\triangle$ ABC with each of its base angles twice the vertical angle.

Draw the circumscribing $\odot$ of $\triangle$ ABC.

Then BC is a side of a regular 5-gon inscribed in $\odot$ ABC.

Proof Since $\angle ABC = \angle ACB = 2 \angle BAC$,

$\therefore \quad \angle BAC = \frac{1}{5}$ of 2 rt. $\angle s = 36°$.

$\therefore$ BC subtends 36° at the circumference and 72° at the centre.

$\therefore$ BC is a side of a regular 5-gon inscribed in the $\odot$

The pentagon may now be completed. (How?)

Practical method of describing a regular pentagon.

The above method is interesting theoretically, but inconvenient in practice. The practical method is as follows.

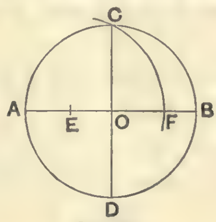


fig. 337.

Draw AOB, COD, two perpendicular diameters of a circle.

Bisect OA at E.

With centre E and radius EC describe a $\odot$ cutting OB in F.

Then CF is equal to a chord of a regular pentagon inscribed in the $\odot$ O.

(The proof of this needs some knowledge of Trigonometry.)

†Ex. 1906. Prove that in fig. 336 AB, CE divide each other in extreme and mean ratio.

†Ex. 1907. In fig. 336, show that $\triangle$ DCX is similar to $\triangle$ ABC.

†Ex. 1908. Show that $\triangle$ CXY is similar to $\triangle$ ABC.

†Ex. 1909. Prove that BY is divided in medial section at X.

†Ex. 1910. Prove that BY is the mean proportional between BX and BD.

To prove that $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$ *.

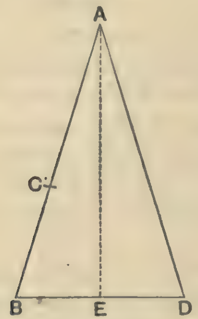


fig. 338.

Let ABD be an isosceles $\triangle$ having $\angle B = \angle D = 2 \angle A$ (see page 359); let AC = BD as in fig. 335, and let AE be drawn to bisect BD at rt. $\angle$s.

Then AB is divided in extreme and mean ratio at C.

Thus, if $AB = a$, $AC = \dfrac{\sqrt{5}-1}{2} a$ (see p. 356).

$$\text{Now } \angle BAD = 36^\circ \text{ (p. 361)},$$

$$\therefore \angle BAE = 18^\circ,$$

$$\therefore \sin 18^\circ = \frac{BE}{AB} = \frac{BD}{2AB} = \frac{\sqrt{5}-1}{4}.$$

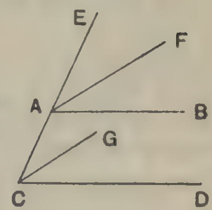
Ex. 1911. Calculate sin 18° as a decimal; and verify the value by measurement.

* See Ex. 1670, p. 315.

REVISION PAPERS

PAPER I (ON BOOK I).

†1. In the given figure AB is parallel to CD; and AF and CG are the bisectors of $\angle$ BAE and $\angle$ DCA. Prove that AF and CG are parallel.



2. ABC and DEF are two triangles. If the following facts hold, are the triangles congruent (give reasons for your answers):

(i) AB = DE, AC = DF, $\angle A = \angle D$;

(ii) AB = DF, AC = DE, $\angle A = \angle D$;

(iii) AB = EF, AC = DF, $\angle A = \angle F$;

(iv) DE = BC, $\angle A = \angle F$, $\angle B = \angle D$;

(v) BC = DE, $\angle A = \angle E$, $\angle B = \angle F$?

†3. ABCD and ABPQ are a parallelogram and a rectangle on opposite sides of a straight line AB; join DQ, CP: prove that CDQP is a parallelogram.

†4. The triangles ABC and A′BC are on the same side of their common base BC, and the angle A′BC equals the angle ACB, and the angle A′CB equals the angle ABC; also AB and A′C intersect in O. Prove that the triangles AOC and A′OB are congruent.

5. In a triangle AB = 12 cm., BC = 9 cm. and the perpendicular from B to AC is 5·7 cm. Show that there are two triangles that fulfil these conditions and draw them both. State how the two triangles are related.

6. A tower, whose base is a circle of diameter 40 feet, is surmounted by a spire. The distance of the shadow of the point of the spire from the nearest point of the base is measured to be 33 feet, while the line joining the top of the spire with its shadow makes an angle of 60° with the ground. Draw a sketch to scale, and measure to the nearest foot the height of the top of the spire above the ground.

PAPER II (ON BOOK I).

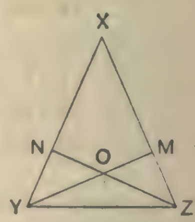
†1. AB and CD are two parallel straight lines and a straight line is drawn to cut them in E and F. If B and D are both on the same side of EF, prove that the bisectors of ∠BEF and ∠DFE are at right angles to one another.

2. How many sides has a regular figure the angle of which contains 162°?

Is it possible for a regular figure to have angles of 130°?

†3. ABCD is a parallelogram (not rectangular), and AL and CM are the perpendiculars from A and C on to the diagonal BD. Prove (otherwise than by a mere appeal to symmetry) that ALCM is a parallelogram.

†4. In the given figure YM is perpendicular to XZ, ZN is perpendicular to XY, and YM and ZN intersect at O, also OY = OZ. Prove that XY = XZ.



5. Construct a parallelogram ABCD whose sides AB, AD are 5·4″ and 3″, and the distance between AB and CD is 2·6″. Measure the angle ADC.

Find a point P in AB which is equidistant from D and B.

Measure PB. State your construction.

6. Find, by drawing, the length of the shadow of a man 6 feet high, when the altitude of the sun is 57°.

PAPER III (ON BOOK I).

1. Give careful definitions of the following, and draw simple figures to make your definitions more clear:—Supplementary angles; angle of depression.

Explain, with sketches, the meaning of prism; triangular pyramid.

†2. ABC is an isosceles triangle (AB = AC); through C is drawn CD at right angles to BC; CD cuts BA produced at D. Prove that ACD is an isosceles triangle.

†3. ABCD is a parallelogram; E is the mid-point of AB; CE and DA are produced to meet at F. What angles in the figure are equal to angles ECD and ECB? Give reasons. Also prove AF = AD.

4. Is each of the following statements true for any parallelogram? If not, state in each case a kind of quadrilateral for which it *is* true. No proofs are required. (*a*) The diagonals bisect one another. (*b*) The diagonals bisect the angles. (*c*) The opposite angles together make two right angles. (*d*) The diagonals are equal.

†5. PQR is an isosceles triangle having PQ = PR. A straight line is drawn perpendicular to QR and cuts PQ, PR (one of them produced) in X, Y. Prove that the triangle PXY is isosceles.

6. Two points of land, A, B, on the shore are 2·8 miles apart, A being S. 71° W. of B. A ship at sea observes A to bear N. 17° E., and B to bear N. 42° E. Find the distance of the ship from A and from B.

If the ship's course is N. 50° E., at what distance will she pass B?

PAPER IV (ON BOOK I).

1. Draw a figure of a cuboid showing three of its faces. If you placed a cuboid with one of its faces vertical, how many of its faces (i) must be vertical, (ii) might be vertical? If you placed it with one of its edges horizontal, how many of its edges (i) must be horizontal, (ii) might be horizontal, (iii) must be vertical? How many diagonals has a cuboid?

†2. Prove that, if all the sides of a quadrilateral are equal, the figure is a parallelogram and its diagonals cut at right angles.

†3. D is the middle point of the side BC of a triangle ABC. If DA is equal to half BC, prove that the angle BAC is equal to the sum of the angles B and C.

4. A and B are two points on paper. What is the locus of the point C under the following conditions:—*firstly*, when C is restricted to being in the plane of the paper; *secondly*, when C may be anywhere in space?—(i) Angle ACB is 90°. (ii) C is equidistant from A and B. (iii) Angle CAB is 20°. (iv) C is always 1 inch from AB (which may be produced indefinitely in both directions).

5. Construct a triangle whose base is 7·3 cms. long, with vertex 3 cms. away from the base line and 4 cms. away from the middle point of the base; measure the sides of the triangle.

6. A ship steaming N. 55° W. at 18 knots sights a lighthouse bearing N. 42° W., distant 3·5 miles at noon. Find, by drawing, how near the ship will pass to the lighthouse, if she keeps on her course. Find also at what time (to the nearest minute) she will pass the lighthouse.

PAPER V (ON BOOK I).

†1. If the bisector of an exterior angle of a triangle is parallel to one of the sides, prove that the triangle is isosceles.

2. Draw freehand diagram of

(i) a quadrilateral with only two sides parallel which has equal diagonals,

(ii) any other quadrilateral which has equal diagonals.

What is the name of the quadrilateral (i)?

3. What is the name of the geometrical solid whose surface is traced out (i) by one arm of a pair of dividers being rotated about the other when the latter is kept vertical? (ii) by one edge of the piece of paper on which you are writing being rotated about the opposite edge?

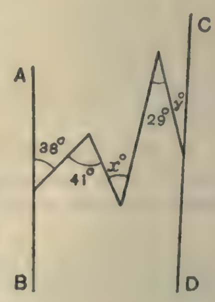
†4. ABCD is a parallelogram; DA and DC are produced to X and Y respectively so that AX = DA and CY = DC; XB and BY are drawn. Prove that XBY is a straight line.

5. P, Q are two points 6 cms. apart. PS is a straight line making an angle of 40° with PQ. Find a point (or points) equidistant from P and Q and 4 cms. from PS. Measure the distance of your point (or points) from P. Is this problem always possible whatever the angle SPQ?

6. Three ships, A, B and C, start together from a port. A proceeds due North; B, N.E., and C, East. If the ships always keep in a line, A going 20 knots and B 12 knots, what is C's speed?

PAPER VI (ON BOOK I).

1. In the figure, which is not drawn to scale, find y if AB and CD are parallel and x is 18; also prove that AB and CD will meet if produced towards B and D (whatever x may be), provided that $x+y$ is greater than 32.



2. ABC is a triangle in which ABC is 50° and ACB is 70°. CB is produced beyond B to D, so that BD = BA, and BC is produced beyond C to E, so that CE = CA. Determine from theoretical considerations the angles of the triangle ADE.

Construct the triangle ABC when its perimeter is 3″.

†3. ABC is an isosceles triangle, the equal sides AB, AC are produced to D, E respectively; the bisectors of $\angle$ DBC, $\angle$ ECB intersect at F. Prove that $\triangle$ FCB is isosceles.

4. A right cylinder of diameter 6·8 cm. and height 7·6 cm. is divided into two parts by a plane through its centre at 28° to its base: measure the length of the section.

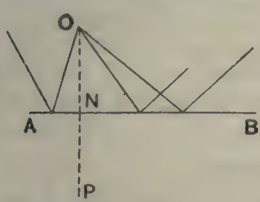
†5. PS, the bisector of the angle P of a triangle PQR, cuts QR at S; through S, ST and SU are drawn parallel to PR and PQ, thus forming a quadrilateral TPUS: prove that the sides of TPUS are all equal to one another

6. A captive balloon is observed from two positions A and B on a horizontal plane, A being due north of the balloon and B due south of it. A and B are 2 miles apart. From A the angle of elevation of the balloon is 27° and from B it is 18°. Find by drawing the height of the balloon.

PAPER VII (ON BOOKS I, II).

†1. CX, the bisector of an exterior angle of $\triangle$ ABC, which is not isosceles, meets AB in X. Prove that $\angle$ AXC is equal to half the difference between the angles A and B.

2. Draw XOX′, YOY′, two straight lines intersecting at O, so that $\angle$XOY = 50°. Make OX = 4 in., OY = 3 in. Find a point P equidistant from X, Y, and at the same time equidistant from XOX′, YOY′. Find another such point, Q. Explain (in two or three lines) how these points are found. Measure OP, OQ in inches.



†3. Rays of light proceeding from a point O fall on a mirror AB and are reflected, making an equal angle with the mirror. Prove that the reflected rays, if produced backwards, would all be found to pass through a point P, such that ON = PN and OP is perpendicular to AB.

4. ABC is a triangle having a fixed base BC, 5 cm. long, and a moveable vertex A. What is the locus of A

(i) when ABC is isosceles (AB = AC)?

(ii) when ABC has a fixed area (= 10 sq. cm.)?

(iii) when the median AM has a fixed length (= 6 cm.)?

5. Draw a parallelogram having sides of 4 cm. and 6·5 cm. and an angle of 75°. Find its area.

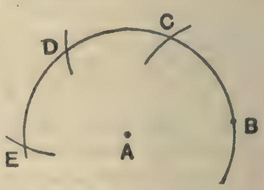
†6. Call the corners of a rectangular sheet of paper A, B, C, D (AB being a long side of the rectangle); if it were folded along the diagonal AC, then AB and CD would cut at a point we will call O. Make a freehand sketch of the figure you would obtain and prove triangles ADO, BCO equal in area.

7. The range of a gun is $2\frac{1}{2}$ miles. If it is stationed $1\frac{1}{2}$ miles from a straight road, what length of the road can it command?

PAPER VIII (on Books I, II).

1. A destroyer, steaming N. 10° E. 30 knots, sights a cruiser, 11 miles off, bearing N. 62° E. Half an hour later the cruiser is 5 miles off, bearing N. 70° E. Find the course and speed of the cruiser. If the destroyer then alter course to N. 70° E., how far astern of the cruiser will she pass?

†2. With centre A and radius AB a circle is drawn. With centre B and equal radius an arc is drawn intersecting the first circle at C. Similarly from centre C the point D is determined, and from centre D the point E. Prove that BAE is a straight line.



3. OA is the vertical line which is the junction between two walls of a room, OB and OC the horizontal lines running along the junction between the walls and the floor. What is the locus of the point P which moves about in the room under the following conditions:—(i) so as to be always 5 feet above the floor? (ii) so that the angle AOP is always 50°? (iii) so as to be always equidistant from OB and OC? (iv) so as to be 4 feet from O and 2 feet from the plane AOC?

4. How is the area of a parallelogram measured?

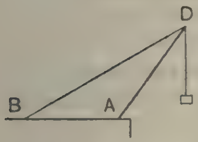
Construct a rhombus whose area is 8 sq. ins. and whose sides are each 3·2″ long.

5. The area of a triangle is 24 square inches, the altitude is 8″; find the length of the base, and on it describe a parallelogram, the area of which shall be 48 square inches.

†6. A four-sided field is to be divided into two parts of equal area; prove the accuracy of the following construction. Draw a quadrilateral ABCD to represent the field; draw the diagonal AC; find E, the mid-point of AC; join BE, DE; then the areas ABED and CBED are equal.

†7. HVQ is a triangle right-angled at V, HVT is a triangle on the opposite side of HV having ∠ THV a right angle; prove that the squares on HT, HQ are together equal to the squares on TV, VQ.

PAPER IX (on Books I, II).



1. In the shear-legs shown in the figure AD is 30 feet, BD is 50 feet, and the angle BAD is 130°. The load is supported by a chain passing over a pulley at D and controlled by a winch at A. If the end B of the tie-rod BD is moved away from A until D is brought vertically over A, find (1) the distance through which B is moved, and (2) the length of chain which must be let out so that the load remains at the same height above the ground.

2. A, B are points 3″ apart on an unlimited straight line; *state* carefully and fully the locus of the following points:—

(i) points equidistant from A and B, (ii) points 3″ from AB, (iii) the middle points of chords of a circle (centre C), which are parallel to AB, (iv) the middle points of chords of a circle (centre C), which are equal to AB, (v) the centres of circles of radius 3″ which pass through A, (vi) points at which AB subtends a right angle, (vii) the centres of circles passing through A and B, (viii) the centres of spheres passing through A and B.

3. (i) How many faces has a prism on a six-sided base? (ii) How many vertices has a cone? (iii) How many edges has a pyramid on a base of 5 sides?

4. ABC is a triangle having AC = 7·2″, BC = 9·6″. AX, BY are drawn perpendicular to BC, AC respectively. If AX = 2·4″, find length of BY.

5. Draw a parallelogram of base 8 cm., angle 70°, and area 56 sq. cm. Transform this parallelogram into an equivalent rhombus on the same base. Measure the acute angle of the rhombus.

†6. Draw a quadrilateral ABCD having the angles at A and D both acute; from B and C draw BE and CF perpendicular to AD. Prove that the area of the quadrilateral ABCD is equal to the sum of the areas of the triangles ABF and ECD.

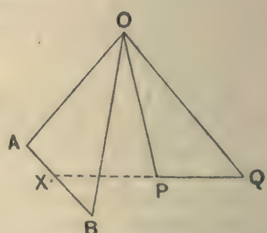
7. A volcanic mountain is in the shape of a cone 4000 ft. high: the base is a circle of 8000 ft. radius. Calculate (to the nearest tenth of a mile) the length of a rack-and-pinion railway which takes the shortest way to the top.

PAPER X (ON BOOKS I, II).

1. CD and EF are two given parallel straight lines, and A is a given point in CD. B is a given point on the side of EF remote from A. It is required to determine the position of a point P in CD, such that, when the straight line PB is drawn crossing EF at Q, then PQ may be equal to AP. Prove that the perpendicular distance of the line PB from the point A is equal to the distance between the parallel lines. Hence solve the problem, and show that there are in general two possible positions for the point P.

Draw the figure, making the perpendicular distance between the parallel lines equal to 3 cm., AB = 8 cm. and angle DAB = 60°. Determine from your drawing the two possible lengths of PQ.

†2. The triangles OAB, OPQ are congruent. QP produced meets AB in X. Prove that OX bisects the angle AXP.



3. How many edges has (i) a cube; (ii) a cuboid; (iii) a square pyramid?

Draw a freehand sketch of each.

4. Construct a triangle, given BC = 9·2 cms.; CA = 8·2 cms.; AB = 10 cms.

On AB construct an equivalent isosceles triangle. Measure the equal sides and find the area.

5. A triangular field ABC has to be divided into four parts which are to be equal to one another in area. Draw any triangle to represent the field and show how to divide it so that the given conditions may be satisfied. Give a proof.

6. Prove that the area of a trapezium is obtained by multiplying half the sum of the parallel sides by the altitude.

Draw a trapezium having its parallel sides 8 cms. and 6 cms., and altitude 5 cms., and one of its acute angles 68°. Transform the trapezium into an equivalent triangle. Describe your construction briefly and show how the above rule for finding the area of a trapezium follows directly from your new figure.

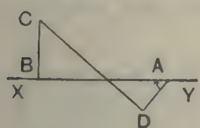
†7. In the right-angled triangle ABC, BC is the hypotenuse and the side AB is double the side AC. A square is described on BC and is divided into two rectangles by a line through A perpendicular to BC. Prove that one rectangle is four times the other.

PAPER XI (ON BOOKS I, II).

1. A vessel steaming at uniform speed finds that the bearings of a lighthouse at 3, 4 and 5 P.M. are N. 20° E., N. 25° W., and N. 50° W. respectively. Its distance from the lighthouse at 4 P.M. is 10 miles. Find, by drawing, the ship's course.

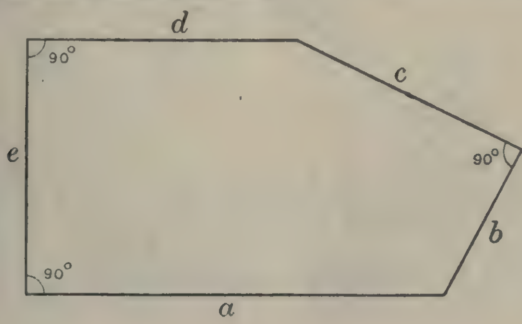
2. Show that it is usually possible to draw two circles each of which touches two given sides of a given triangle (produced if necessary) and has its centre on the third side, but that under certain circumstances only one such circle can be drawn.

3. The figure shows three jointed rods, BC having a length of 6 cm., CD of 10 cm., and DA of 3·6 cm. B moves in a slot XY and BC is *always perpendicular* to XY. The rod AD revolves in the plane of the paper about the point A, which is fixed.



Draw the figure full size, with C at its greatest possible distance from A. Now imagine AD to revolve clockwise at the rate of 60° per second, and show the position of B at the end of each second. Give the greatest and the least distances of C from A and the range of movement possible for B. Show in a table the distance of B from its original position at the end of each second, and illustrate by a graph.

4. The figure represents a field to scale, 1 centimetre denoting a chain. Estimate the area of the field in acres.



†5. A straight line is drawn parallel to the base BC of a triangle ABC cutting AB at X and AC at Y; prove (i) that triangles XBC, YBC are equal in area, and (ii) that triangles ABY, ACX are equal in area.

6. By means of a sketch-figure, with a very brief explanation, illustrate the identity

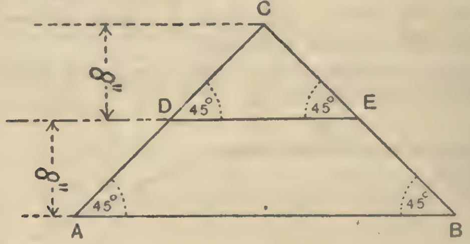
$$(a+x)(a+y) \equiv a^2+ax+ay+xy.$$

7. A shelter trench of rectangular section is 3 ft. wide and 4 ft. deep; the earth excavated is piled up in front as a rampart; if the vertical section is a right-angled isosceles triangle, how high is the rampart?

PAPER XII (on Books I, II).

1. If the line joining two points P, Q is bisected perpendicularly by a given straight line, then Q is said to be the *image* of P in the given straight line. Given a point and a straight line, show how to find the image of the point using compasses only (no proof is required).

2. A corner shelf ABC is to be made from a board and to consist of two pieces ABED and CDE glued together along DE. The depth 8″ of each piece is to be the same as the breadth of the board. Determine the greatest breadth AB of the shelf. What is the shortest length of board which will suffice for the job?



3. (i) A shot is fired from an airship high overhead. Assuming that sound travels through the atmosphere at a uniform rate of 1100 feet per second, what is the solid-locus of points at which the report will be heard in one second?

(ii) What is the space-locus of points equally distant from two given points?

†4. E and F are the middle points of AD and BC, the sides of a parallelogram ABCD. Prove that the lines BE, DF divide the diagonal AC into three equal parts.

5. In a field in the form of a quadrilateral ABCD, B is due North of A and D is due East of A. Also AB = 7·5 chains, BC = 8·4 chains, CD = 1·3 chains, DA = 4 chains. Find the area of the field in acres.

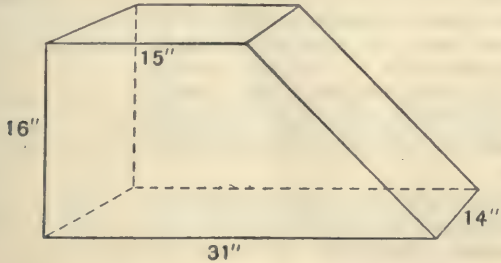
†6. Assuming that the medians of a triangle ABC pass through one point, prove that the six triangles into which they divide the triangle ABC are equal in area to one another.

7. Find, in centimetres, the base-radius of a cone of slant side 3 decimetres and height 12 cm.

PAPER XIII (on Books I—III).

†1. If two pairs of straight railway lines cross one another, prove that the figure they enclose is a rhombus. [You are to assume that the perpendicular distance between one pair of lines is the same as the perpendicular distance between the other pair of lines.]

2. The figure represents a coal-box. Find the volume of the solid figure shown.



3. Treasure is known to be buried in a field 20 yards from a straight hedge, and 30 yards from a cairn, this being inside the field and 40 yards from the hedge. Show that it may be in either of two positions. Find the distance apart of these positions (i) by measurement; (ii) by calculation.

4. A model boat sails in a straight line across a circular pond, towards a point 50 yards away. The greatest distance across the pond is 70 yards. How near to the centre of the pond will the boat go? What will be the boat's least distance from the point on the pond's edge exactly opposite the starting-point? [Both answers by calculation.]

5. Describe a triangle with sides 4·5, 6, 7·5 in.; find the centres of the inscribed and circumscribed circles, and measure the distance between them.

†6. ABC is a triangle having the sides AB, AC equal; perpendiculars drawn to AC at A and to BC at B meet at D. Prove that AD bisects the angle between CD and BD produced.

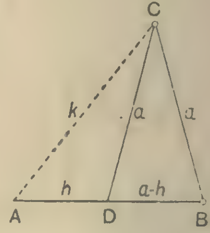
7. A circle whose centre is O is touched internally at A by a circle of half its radius. A radius OQ of the former circle cuts the smaller circle at P. Prove that arc AQ = arc AP.

PAPER XIV (on Books I—III).

†1. Prove that the line joining the middle points H, K of the sides AB, AC of a triangle ABC is parallel to BC.

2. A ship is situated 4·5 miles from a straight shore. Two piers are respectively 6 miles and 8·9 miles from the ship. Calculate the distance between the piers.

3. The figure shows three equal bars AB, BC, CD, jointed at B and C. The three are placed on a table, and the bar AB is kept fixed while the point D is gradually moved along AB from A to B, the joint C moving in consequence across the table. Prove that, if the straight line AC is drawn, in all positions of D, the triangle ADC has one of its angles double of another.



If each bar is of length a, obtain an expression for the length k of AC when D has been moved a distance h from A towards B. Also calculate k when $h = \frac{1}{2}a$, taking $a = 10$ cm.

4. In a circle a chord 24 in. long is 5 in. distant from the centre. Calculate (i) the radius of the circle; (ii) the length of a chord which is 10 in. distant from the centre.

5. A triangle ABC is inscribed in a circle, centre O, and radius 4″.

If the angles of the triangle are A = 72°, B = 55°, C = 53°, what are the angles BOC, COA, AOB? Hence find the sides of the triangle by drawing.

†6. AB is a chord of a circle whose centre is O, and AB is parallel to the tangent at P. If the tangents at A and P intersect at T, prove that the angles POA, TAB are equal to one another.

7. RS is a fixed chord of the circle RLNS; a chord LN of given length is placed in the arc RNS, and RN and SL meet in O. Show that the magnitude of the angle ROS is independent of the position of the chord LN, in the arc.

PAPER XV (ON BOOKS I—III).

†1. Draw a triangle ABC, bisect AB at D; draw DE parallel to BC and let it cut AC at E; prove that E is the mid-point of AC.

2. ABC is a right-angled triangle. The angle A is 90°. A circle is circumscribed round the triangle. Its radius is found to be 6″. AN is drawn perpendicular to the base BC. O is the mid-point of BC and ON = 2·4″. *Calculate* the lengths of AN, AB, and AC.

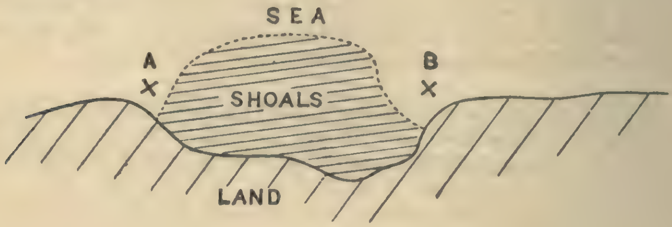
†3. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair of opposite sides.

4. Construct a triangle ADE such that AD, DE, EA measure 5, 6·1, 9·7 cm. respectively. Construct the circumscribing circle and the circle escribed to DE. Measure the radii of the circles and the distance between their centres.

5. ABC is a tangent to a circle at B, BD is a diameter and BE, BF are chords such that ∠ABE = 20°, ∠CBF = 60°; DE, DF, EF are joined. Find all the angles of the figure.

†6. A triangle ABC is right-angled at A, O is the mid-point of BC, and AP is drawn perpendicular to BC: prove that the angle OAP is equal to the difference between the angles at B and C.

7. In order to avoid the shoals shown in the figure, the navigator is instructed to take bearings of the fixed objects A and B and to take care that the angle subtended by AB never exceeds 130°. Explain the reason for this instruction.



PAPER XVI (ON BOOKS I—III).

†1. ABCDE is a five-sided figure in which BC, CD are respectively equal to AE, DE and $\angle BCD = \angle DEA$. Prove that $AC = BE$.

2. ABCD is a rectangle in which $AB = 4$ in., $BC = 6$ in. A circle with A as centre passes through the middle point of BC and cuts AD at F. Calculate the length of CF.

3. ABC is a triangle in which AB is 7 in., BC is 5 in., CA is 3 in. The circle whose centre is A and radius is AC cuts BC again in D. Prove that ACD is an equilateral triangle.

4. In playing with coins of the same size a boy observed that he could arrange six coins round a centre one, each touching the centre one and two others. Show the possibility of this by drawing a careful figure in which each circle has a radius of 2 cm. State clearly how you determine the centres of the circles and what help you get in this construction from considerations of symmetry; then justify by general reasoning the method you have adopted.

5. P and Q are two points on the circumference of a circle, and the tangents to the circle at P and Q intersect at an angle of 56°. What fraction of the whole circumference is the minor arc PQ? and what is the ratio of the major arc PQ to the minor arc PQ?

†6. AOB is a diameter of a circle; through A and B parallel chords of the circle are drawn. Prove that these chords are equal.

†7. Two circles cut one another in the points A and B. Through A any line is drawn which cuts the circles again in the points P, Q and the tangents at P, Q cut in T. Prove that the four points B, P, T, Q are on a circle.

PAPER XVII (ON BOOKS I—III).

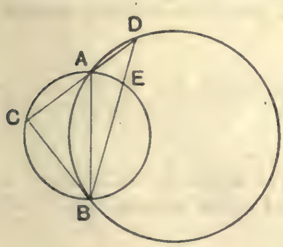
†1. PQR is a triangle, and S is the mid-point of QR. From S, ST is drawn parallel to QP, meeting PR in T, and SU parallel to RP, meeting PQ in U. Prove $SU = RT$ and also $SU = TP$.

2. Construct (without any calculation) a square which shall be equal in area to the difference between the areas of two squares whose sides are 7 and 4 cm.

†3. PQR is a triangle right-angled at Q, S is the mid-point of PQ; prove that $PR^2 = RS^2 + 3QS^2$.

4. A, B, C, D are four points on a circle of which O is the centre. AC is a diameter and $\angle BAC = 35°$, $\angle DBC = 40°$. Find $\angle ODC$, $\angle ODB$, giving your reasons briefly.

5. The line CD measures 14 cm.; with centres C and D describe circles of radii 3 and 7 cm. respectively. Construct one of the interior common tangents, and measure the perpendiculars upon this from the nearer points at which the line joining the centres cuts the circumferences.

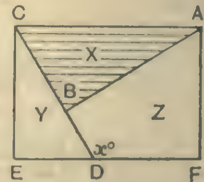


†6. A triangle ACE is inscribed in a circle ABCDEF; prove that the sum of the angles ABC, CDE, EFA is equal to four right angles.

†7. In the figure, BC touches the circle ABD. Show that CE touches the circle ADE at E. (You may assume the converse of the "alternate segment" theorem.)

PAPER XVIII (ON BOOKS I—III).

1. Take a line AB, 9 cm. long (Fig.), and through B draw CD at right angles, making BC = 6 cm., and BD = 3 cm. Join AC and complete the rectangle AFDEC. Denote by x the number of degrees in the angle BDF, and write in each angle of the figure its value in degrees.



You are told that if the parts of the figure marked Y and Z were cut out they could be placed against the part marked X so that the three parts would form a square.

Give the area of the square.

2. AB and XY are unlimited parallel straight lines 2 cm. apart; AB is 8 cm. long and C is its middle point. P is a point on XY, PN is perpendicular to AB, and CN is x cm. long. Find expressions for AN, BN, AP, BP, and simplify the last two as far as possible. Hence find an equation for x when AP is three times BP; solve it, and test the accuracy of your result by drawing a figure to scale.

†3. A straight line AB is produced to C, so that AC = 3AB; on BC an equilateral triangle BCD is described. Prove that the square on AD is seven times the square on AB.

4. What would be the radius of a circle in which an arc 11″ in length subtended an angle of $31\frac{1}{2}°$ at the circumference?

5. Construct a quadrilateral OPQR, given OP = 6 cm., OR = 5 cm., angle O = 74°, angle P = 83°, and angle R = 97°. Draw a circle to pass through O, P, and R.

†6. ABC is a triangle. Points X, Y are taken in AC, BC respectively such that the angle XYC is equal to angle BAC. Prove that the angle XYA = the angle ABX.

†7. O is the centre of the inscribed circle of a triangle ABC, and AO is produced to meet at D the circle circumscribed to the triangle. Show that

DB = DC = DO.

PAPER XIX (on Books I—IV).

1. A rectangular sheet of paper ABCD 12 in. by 10 in. is folded along XY, a line 4 inches from the shorter side BC. Find by calculation to three significant figures, and illustrate by rough sketches, the shortest distance of A from C: (*a*) before folding. (*b*) when the two parts of the sheet are at right angles.

2. On a fixed line AB, 8 cm. long as base, construct a triangle ABC, whose area is 24 sq. cm., such that the vertical angle is 63°. Measure (1) the smallest angle, (2) the radius of the circumcircle of the triangle.

3. A position X lies 4000 yards N. 68° E. of Y, whilst Z lies 3000 yards due S. of Y. Find the distance and bearing from X of a position which is equidistant from the three positions X, Y and Z.

†4. ABC is a triangle inscribed in a circle; the bisector of the angle BAC meets the circumference in D. A circle described with centre D and radius DC cuts AD in E. Prove that BE bisects angle ABC.

†5. AB is a chord of a circle and AD the tangent at A. A chord QP is drawn parallel to AB, meeting the tangent AD at D. Prove that the triangles DPA and AQB are equiangular.

6. A pendulum swings through an angle of 10° on either side of the vertical: calculate the length of a scratch made on the clock-case by the back of the pendulum-weight, given that the pendulum is 4′ 4″ long. If the pendulum were to swing through twice as large an angle, would the scratch be twice as long? Would the distance between the ends of the scratch be doubled?

†7. The sides BA, CD of a cyclic quadrilateral ABCD are produced to meet in O; the internal bisector of the angle BOC meets AD in L and BC in M. Prove that

$$AL : LD = MC : BM.$$

PAPER XX (on Books I—IV).

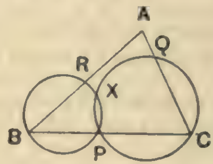
1. A room is 20 ft. long, 16 ft. wide, and 12 ft. high. A string is stretched diagonally from one corner of the floor to the opposite corner of the ceiling. By drawing and measurement determine approximately in degrees the inclinations of the string (i) to the floor of the room, (ii) to one of the longer sides of the floor.

2. Draw two straight lines making an angle of 60° with one another and intersecting at O. On one of the lines take the two points X, Y on opposite sides of O such that XO = 2 in., OY = 4·5 in. Draw the circle through the points X and Y which will cut the other line in two points equidistant from O. Measure the distance of each of these points from O.

3. Construct a triangle ABC in which $a=3''$, $c=4''$, and $B=29°$. Draw its inscribed circle, and also the escribed circle which touches AC between A and C. Measure the radius of each circle, and show theoretically that the line joining their centres must pass through B.

4. What is the length of the edge of the largest equilateral triangular piece of paper which, when lying perfectly flat, will just float on the surface of a hemispherical bowl, filled with water, of 6·2 cm. radius?

†5. Show that the four points A, Q, X, R lie on a circle.



†6. QR is a chord of a circle, TR is the tangent at R; a straight line through Q perpendicular to this tangent meets it in T and the circumference of the circle again in P; PM is the perpendicular from P on QR. Prove that the angles QPM, TPR, TMR, TRM are all equal.

†7. In a triangle PQR, PQ = PR = 2 inches, and QR = 1 inch. In the side PQ a point S is taken such that QS = $\frac{1}{2}$ inch. Prove that the triangle QRS is isosceles.

PAPER XXI (ON BOOKS I—IV).

1. A sphere of 6″ diameter rests on the top of an open hollow cylinder whose inner diameter is 4″. To what distance will the sphere project above the top of the cylinder?

2. State, without actually performing any construction, how you would solve the problem of drawing two tangents to a circle, which should include a given angle and intersect upon a given straight line. How many solutions of the problem would you expect to get?

3. AB is a fixed line. Through A a line AC is drawn, of length 2·4 in., making an angle of 40° with AB. Draw the figure to full scale, and construct a circle to touch AB at A and to pass through C. Explain your construction. Measure the radius of this circle. Verify by calculation.

4. Two circles of radii 4 cm. and 7 cm. have their centres 9 cm. apart. Calculate the length of the common tangent to the two circles.

†5. AB is arc of a circle and C its middle point. Prove that the angle ABC is one-quarter of the angle which the arc AB subtends at the centre of the circle.

†6. ABC is a right-angled triangle in which C = 90°. A square is described on AB so as to be on the opposite side of AB from C. The diagonals of the square intersect in D. Prove that CD bisects the angle C.

†7. ABCD is a parallelogram. From any point E in the diagonal AC, EH is drawn parallel to AD to meet DC at H, and EF parallel to DC to meet BC at F. Prove that the triangles ABD, EFH are similar.

PAPER XXII (ON BOOKS I—IV).

1. A penny falls into a cup whose shape is an exact hemisphere of radius 5 cm. If the penny lies symmetrically at the bottom and its diameter is 3 cm., calculate how far below the penny the lowest point of the cup is.

2. A, B, C are 3 landmarks. B is 200 yards due East of A, and C is 200 yards N. 26° E. of B. An observer in a ship which is due North of B, observes that AC subtends an angle of 90° at his eye. Find, by drawing, the distance of the ship from A, B, and C.

3. Draw a circle of radius 7 cm. and a chord PQ distant 4 cm. from the centre.

Now draw a circle of radius 5 cm. to touch your first circle internally, and also to touch PQ. State your construction.

†4. To two circles, centres O and O′, an internal and an external common tangent are drawn, meeting in P. Prove that P lies on the circle on OO′ as diameter.

5. ABCD is a circle: AC, BD meet in X. Given that ∠ABD = 33°, ∠ADB = 27°, ∠BAC = 45°, calculate the angles BXC, ACD, ABC, showing your reasoning clearly but shortly.

6. AB is a breakwater, 2000 yards long, B being due East of A. The breakwater subtends an angle of 50° at each of two ships, x and y. If x bears N. 10° E. from A, and y is 800 yards to the Eastward of x, find the distance of each ship from the breakwater. [Scale 400 yards to 1 inch.]

7. The height of the Great Pyramid is 149 metres; an exact model of the pyramid is made of height 1·49 metres, its side faces being triangles similar to the side faces of the pyramid. What is the ratio of the total slant surface of the pyramid to that of the model?

PAPER XXIII (ON BOOKS I—IV).

1. A dirty football was found to leave a circle of mud, 11″ in circumference, when bouncing; if the radius of the ball was $6\frac{1}{4}$″, find the depth to which it was squashed in by the impact.

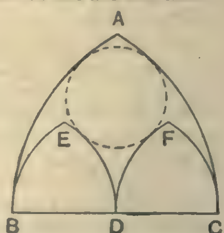
2. Draw a circle of 4 cm. radius to touch two circles of radii 3 cm. and 2 cm. respectively, whose centres are 6 cm. apart. The 3-cm. circle is to lie entirely inside the 4-cm. circle, and the 2-cm. circle is to lie entirely outside.

3. P is a point on the circumference of a circle of centre O and radius $1\frac{1}{2}$ in. Q is taken so that $\angle$ POQ $= 40^\circ$ and OQ $= 3$ in. Construct a circle to touch the given circle at P and to pass through Q. Measure the radius of this circle.

†4. If AB is a tangent to a circle of radius 5″, where A is any point on the circumference and B is 12″ from A, find the locus of B as A moves round the circle.

5. O and P are points 1000 yards apart, P being due East of O. At Q the line OP subtends an angle of 63°. If Q is 450 yards from the line OP, draw a figure to scale, and find the distance and bearing of Q from O.

6. BAC is an equilateral arch, B being the centre of the arc AC and C the centre of arc BA; BED, CFD are similar arches, B being the centre of DE and C the centre of DF and D of BE and CF. What is the locus of the centres of circles touching (i) arcs AB and AC, (ii) arcs DE and DF, (iii) arcs BA and DF?



Hence explain how to construct with your instruments a circle (shown dotted in the figure) which will touch the arcs BA, AC, DE, DF.

Draw the figure carefully, taking BC 5 inches long.

†7. In any triangle ABC, P is a point in BC such that BP is one-third of BC. Join AP and take on it a point Q such that AQ is one-third of AP. Then prove that the area of the triangle ABQ is one-ninth of that of ABC.

What is the ratio of the areas of the triangles ABQ and ACQ? Give your reason.

PAPER XXIV (ON BOOKS I—IV).

1. A paper pyramid on a square base is made as follows. On each side of a square of side 3 inches is constructed an isosceles triangle of height 5 inches, the triangles lying outside the square A 4-pointed star is thus formed, which is cut out of paper. By folding the triangles upwards a pyramid is formed. Find its height, and the length of each of its sloping edges.

†2. ABC is a triangle inscribed in a circle and the tangents at B and C meet in T. Prove that, if through T a straight line is drawn parallel to the tangent at A meeting AB, AC produced in F and G, then T is the mid-point of FG.

3. OY, OX are two straight lines at right angles. On OX two points A, B are marked so that $OA = 1''$, $OB = 3''$. By construction find a point (or points) on OY at which AB subtends an angle of 25°. Explain your construction and measure the distance of the point (or points) from O. Find, by drawing *or otherwise*, the position of the point on OY at which AB subtends the greatest possible angle.

†4. Two chords of a circle AB, CD intersect at a point X.

If XB = XD, show that AB = CD, and that ACBD is a trapezium.

†5. A given point D lies between two given straight lines AB and AC. Find a construction for a line through D terminated by AB and AC, such that D is one of its points of trisection. Prove also that there are two such lines.

6. Draw two straight lines AB, AC enclosing an angle of 48°. Take a point D in AB such that AD = 2·6 in. Construct a circle DEF to touch AB in D and also to touch AC.

Construct another circle to touch AB, AC and also to touch the circle DEF. State the steps of this construction.

7. The mouth of a stable bucket (Fig.) is 13 inches in diameter, the base $8\frac{1}{2}$ inches in diameter, and the slant side measures 9 inches. Draw a vertical section through the axis of the bucket, and find by calculation the height of the bucket and the height of the cone formed by producing the slant sides beyond the base.

Assuming the volume of a cone to be a third of the product of the base and the height, find how many gallons the bucket will hold. (1 cubic foot $= 6\frac{1}{4}$ gallons.)

PAPER XXV (ON BOOKS I—IV).

1. A cubical block of edge 4 ft. rests on a table; the base is ABCD, and P, Q, R, S are the corners above A, B, C, D respectively. If the edge CD is raised 2 ft., AB remaining on the table, find by drawing to scale the height of R above the table, and the inclination of AR to the table.

2. A paper cone (like an electric light reflector) is slit down straight from the vertex to the base, and opened out flat; sketch the figure produced, and name it.

†3. ABC is a triangle inscribed in a circle. BD, CE are drawn perpendicular to AC, AB, and are produced to cut the circle in F and G. Prove that FG is parallel to DE.

†4. The side BC of a triangle ABC is divided at D so that BD$=$2DC; AD is bisected at E; and CE meets AB in F. Prove that CE$=$2EF.

5. Draw a straight line AB of length 5 cm. Find a point P at which AB subtends an angle of 54° and such that AP is 4 cm. Measure the distance PB.

6. From a point P outside a circle of radius a, are drawn the tangent PQ (of length x), and the straight line PAB through the centre cutting the circle in points A and B, A being nearer to P and PA being of length y. Write down the relation connecting the lengths of the lines PQ, PA, PB, and express it in terms of x, y, a.

If the circle is taken as representing a section of the earth through its centre, PQ will be the range of vision of a person situated at a height y above the surface. Take $a = 4000$ miles, use the relation in the approximate form $2ay = x^2$, and find in miles and in feet to what height it is necessary to ascend in order to have a range of vision of 50 miles.

PAPER XXVI (ON BOOKS I—IV).

1. What is the locus of centres of

(*a*) circles which touch a fixed line PQ at a fixed point P;

(*b*) circles of radius 3″ which touch a fixed line PQ?

Also of the following points:—

(*c*) the points of contact of tangents drawn from a fixed point to a fixed sphere;

(*d*) points on the earth which are 3000 nautical miles N. of the equator?

2. The figure represents a bridge, whose span AB is 80 ft., supported on an arch in the form of an arc of a circle. AC = 16 ft. = BD. Let r ft. denote the radius of the circular arc and b ft. the height of the roadway AB above the highest point of the arch. Find an algebraic equation connecting r with b. Then use it to calculate (i) the value of b when $r = 65$, (ii) the value of r when $b = 1$.

3. A stick, 4′ long, is leant up against a cylindrical wooden roller of 18″ diameter. The axis of the cylinder is perpendicular to the vertical plane in which the stick lies. The point where the stick touches the ground is 3′ away from the point of contact of the cylinder with the ground. *Without* drawing to scale, find (i) the distance between the two points of contact which the stick makes respectively with the ground and the roller, (ii) the distance of the axis of the roller from the point of contact of the stick with the ground.

†4. ABCD is any parallelogram. From A a straight line is drawn cutting BC in E and BD in F. Prove that AF : FE = BC : CE.

†5. The side BC of an equilateral triangle ABC is produced to D so that CD = BC. Prove that the perpendiculars to AC drawn through B and C respectively trisect AD.

6. Construct a triangle ABC, in which BC is 2″, the angle BAC is 60°, and the sides AB and AC are in the ratio 3 : 4.

PAPER XXVII (ON BOOKS I—IV).

†1. AOC, BOD are chords of a circle; the tangents at A and B meet at P; the tangents at C and D meet at Q. Prove that the sum of the angles P and Q is twice the angle BOC.

2. AB is a diameter of a circle of radius 5 cm. Draw a chord CD of the circle perpendicular to AB and 6 cm. in length. Also, through O, the point of intersection of AB and CD, draw a chord of the circle 8 cm. long. State the steps of your construction.

†3. ABC is a triangle. O is the middle point of BC, and AO is produced to T. The lines bisecting internally the angles BOT, COT cut externally the sides AB, AC in D, E. Prove that DE is parallel to BC.

4. Construct a square equal in area to an equilateral triangle of side 3 inches. Measure the side of the square.

†5. D is the middle point of the base BC of a triangle ABC, E is a point in AC such that the angle ADE is equal to the angle ABC. EF is drawn parallel to BC and meeting AD in F. Prove that the rectangle AF . FD is equal to the square on EF.

6. Draw a circle of radius 5 cm. and take a point O at a distance of 10 cm. from its centre. From O draw a line cutting the circle in P and Q such that P is the middle point of OQ.

PAPER XXVIII (ON BOOKS I—IV).

1. A and B are two forts 5 miles apart. The effective range of A's guns is $3\frac{1}{2}$ miles, and of B's, 3 miles. Draw the circles bounding the area covered by the two forts, and let C be one of the points of intersection of these circles. An enemy's ship comes to C so as to be able to bombard a town lying between A and B without being within the range of the guns from either fort. By measurement, find how far C is from the coast-line. Measure the angles CAB and CBA and hence calculate approximately the number of square miles covered by the zone of effective fire from the two forts.

†2. O is the centre of a circle of 2 in. radius, A is a point 3 in. from O, AP is a tangent from A. If OP is produced to Q so that PQ = 2AP, prove that the circle whose centre is A and radius AQ will touch the given circle.

†3. In a triangle ABC, AB = AC and $\angle$ A is a right angle; if the bisector of $\angle$ C cuts AB in D, prove that $BD^2 = 2DA^2$.

4. Show how to construct a triangle similar to and double the area of a given triangle.

5. A point R is taken on the side AB of a triangle ABC of area z, so that $AR = x \, . \, AB$, where $x > 1/2$. RQ is drawn parallel to BC to meet AC at Q, RH parallel to AC to meet BC at H, and QK parallel to AB to meet BC at K. Prove that the areas of ARQ and BRH are x^2z and $(1-x)^2z$ respectively (notice that they are similar to ABC), and find in similar form the area of CQK; use these results to find the area of QRHK. Verify your result for QRHK by giving x the values 1 and 1/2.

†6. AB is an arc of a circle, of radius 4 inches, subtending an angle of 45° at its centre C. Let the tangents at A and B meet at T, and produce CA and BT to meet at S. Prove that AS = AT = TB, and, denoting each of these equal lengths by x inches, *calculate* the value of x.

Now suppose that AT and STB represent two railway lines crossing each other at T. The points A and B are connected by a loop-line represented by the arc AB of radius 400 yards. Determine in yards the distances of the points A and B from the crossing T.

PAPER XXIX (on Books I—IV).

1. The two equal circles, centres P and Q, are so drawn that each passes through the centre of the other: they intersect at A and B. The radius of each circle is r. Prove that (i) arc AQB subtends an angle of 120° at P. (ii) the sector of arc AQB and centre P is of area $\frac{\pi r^2}{3}$. (iii) $AB = r\sqrt{3}$. (iv) area of $\triangle PAB = \frac{r^2\sqrt{3}}{4}$. (v) the area common to the two circles $= r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$. (vi) the ratio of this common area to the area of each circle is 0·39.

†2. ABCD is a quadrilateral in a circle. One side BC is produced to E. Prove that the bisectors of the angles BAD, DCE meet on the circumference.

3. Draw a straight line OBC, making OB = 2·5 cm., OC = 6·4 cm. Through O draw a line OA making the angle AOB = 42°. Then draw a circle passing through B and C and touching OA. (Describe the steps of your construction.)

†4. The bisector of the angle A of a triangle ABC meets BC at D; and DE, DF are drawn respectively perpendicular to the external bisectors of the angles B, C, to meet AB, AC produced at E, F respectively. Prove that EF is parallel to BC.

5. B, C, D are three points in order on a straight line, such that BC = 2″, and CD = 4″. Construct a triangle ABC, such that AB + AC = 5″ and the bisector of the external angle at A passes through D.

PAPER XXX (ON BOOKS I—IV).

1. Draw a circle whose diameter is 7 cm. long, and a line 2·5 cm. distant from the centre. Mark off on the line a point which is 6 cm. distant from the centre, and then describe a circle touching the line at this point and also touching the circle.

2. C is the middle point of a straight line AB, 12 cm. long. On AC, CB and AB semicircles are described. What is the radius of the circle which can be described in the space enclosed by the three semicircles touching all three of them?

†3. Draw any triangle ABC. It is required to inscribe in this triangle an equilateral triangle one side of which is parallel to AB, and the opposite vertex lies on AB.

Show how this can be done by employing the properties of similar triangles.

†4. At two points A, B of a straight line perpendiculars AC, BD are erected and AD, BC meet in a point E; from E a perpendicular EF is drawn to AB. Prove that

$$\frac{1}{EF} = \frac{1}{AC} + \frac{1}{BD}.$$

†5. ABCD is a rhombus; a straight line through C meets AB and AD, both produced, at P and Q respectively. Prove that $PB : DQ = AP^2 : AQ^2$.

†6. AD is the bisector of the angle BAC of the triangle ABC, and F is the middle point of AB; also AD and CF intersect in P, and PH is parallel to AB cutting BC in H. Prove that

$$\frac{PH}{BH} = \frac{AC}{BC}.$$

PAPER XXXI (ON BOOKS I—IV).

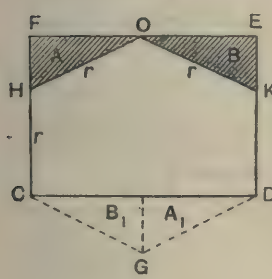
†1. O, A, B, C, D are points on the circumference of a circle such that the angles AOB, BOC, COD are equal. Prove that the angle between the chords AC, OB is equal to that between AD, OC.

†2. Through the vertices B, C of a triangle ABC two parallel lines BL and CM are drawn, meeting any straight line through A in L and M respectively. If LO is drawn parallel to AC and meets BC in O, prove that OM is parallel to AB.

†3. Prove that, if circles are described passing through two given points A and B and cutting a given circle in P and Q, the chord PQ cuts AB in a fixed point.

4. PX, PY are two straight lines intersecting at an angle 45°: A, B are points on PX such that PA = AB = 5 cm. Construct in one figure the points on PY at which the segment AB subtends angles 20°, 30°, 40°, explaining your method. How would you find the point K on PY at which AB subtends the greatest angle? Construct this point in any way you please, and measure this greatest angle.

5. CDEF is a rectangle (Fig.) in which CD $= a$ and CF $= b$. A circle, whose centre is at O, the middle point of EF, is described to cut CF at H and DE at K. If the radius r of the circle is such that CH = HO, express r in terms of a and b.



Suppose that the two right-angled triangles A and B are cut away from the rectangle and placed in the positions A_1 and B_1, thus converting the rectangle into an equilateral hexagon. Show that if $a/2 = b/\sqrt{3}$, the resulting hexagon is regular, i.e. has also its angles all equal.

PAPER XXXII (ON BOOKS I—IV).

1. ABC is any triangle. Show how to inscribe a square PQRS in the triangle so that P lies on AB, Q on AC, and the side RS on BC.

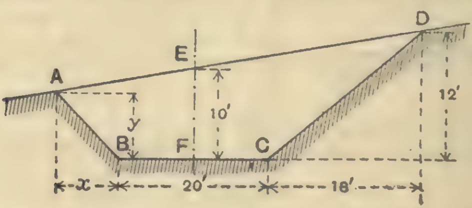
2. Draw a circle 3 inches in diameter and place in it a chord AB 2·5 inches in length; draw the diameter BC and produce BC to D so that D is 1 inch distant from the circle; and through D draw DE perpendicular to BD. Then draw a circle touching DE and also touching the former circle at A. State the steps of your construction.

†3. If ABCD is a cyclic quadrilateral and AB, DC be produced to meet at Q, and BC, AD to meet at R, prove that QP, RP, the bisectors of the angles BQC, CRD, are at right angles to one another.

†4. A straight line is drawn parallel to the side BC of a triangle ABC cutting AC in P and AB in Q; BP cuts CQ in T. Prove that AT produced bisects BC.

5. In Fig. ABCD is a cross-section showing a railway cutting made in ground, the surface of which slopes in a direction at right angles to the cutting as shown by the line AED. BC is the trace of the horizontal plane on which the track will be laid, and EF is a vertical centre line bisecting BC. The side AB of the cutting is to have the same slope to the horizontal as the side CD.

Calculate the dimensions x and y. Find the volume, in cubic yards, of the earth which must be excavated per chain length of track.



List of Definitions.

Acute angle, obtuse angle, reflex angle. An angle less than a right angle is said to be acute; an angle greater than a right angle and less than two right angles is said to be obtuse (p. 64); an angle greater than two and less than four right angles is said to be reflex. (p. 250.)

Acute-angled triangle. A triangle which has all its angles acute is called an acute-angled triangle. (p. 82.)

Adjacent angles. When three straight lines are drawn from a point, if one of them is regarded as lying between the other two, the angles which this line makes with the other two are called adjacent angles. (p. 64.)

Alternate angles, corresponding angles. (See p. 70.)

Altitude. See **triangle, parallelogram.**

Angle. When two straight lines are drawn from a point, they are said to form, or contain, an angle. The point is called the **vertex** of the angle, and the straight lines are called the **arms** of the angle. (p. 64.)

Angle in a segment. An angle in a segment of a circle is the angle subtended by the chord of the segment at a point on the arc. (p. 253.)

Angle of elevation, of depression. (See p. 48.)

Arc of a circle. (See p. 218.)

Base. See **triangle, parallelogram.**

Chord of a circle. (See p. 218.)

Circle. A circle is a line, lying in a plane, such that all points in the line are equidistant from a certain fixed point, called the **centre** of the circle. The fixed distance is called the **radius** of the circle. (p. 217.)

Circumcentre. The centre of a circle circumscribed about a triangle is called the circumcentre of the triangle. (p. 224.)

Circumference of a circle. (See p. 215.)

Circumscribed polygon. If a circle touches all the sides of a polygon, it is said to be inscribed in the polygon; and the polygon is said to be circumscribed about the circle. (p. 224.)

Common tangents, exterior and interior. (See p. 263.)

Concyclic. Points which lie on the same circle are said to be concyclic. (p. 257.)

Cone. (See p. 215.)

Congruent. Figures which are equal in all respects are said to be congruent. (p. 85.)

Contact of circles. If two circles touch the same line at the same point, they are said to touch one another. (p. 245.)

Converse. (See p. 76.)

Coordinates. (See p. 152.)

Cube. (See p. 42.)

Cuboid. (See p. 43.)

Cyclic quadrilateral. If a quadrilateral is such that a circle can be circumscribed about it, the quadrilateral is said to be cyclic. (p. 261.)

Cylinder. (See p. 217.)

Diagonal. See **quadrilateral.**

Diameter of circle. (See p. 218.)

Envelope. If a line moves so as to satisfy certain conditions, the curve which its different positions mark out is called its envelope. (See p. 293.)

Equilateral triangle. A triangle which has all its sides equal is called an equilateral triangle. (p. 82.)

Equivalent. Figures which are equal in area are said to be equivalent. (p. 168.)

Escribed circles of a triangle. (See p. 244.)

Fourth proportional. If x is such a magnitude that $a : b = c : x$, then x is called the fourth proportional to the three magnitudes a, b, c. (p. 309.)

Height. See **triangle, parallelogram.**

Heptagon. See **pentagon.**

Hexagon. See **pentagon.**

Hypotenuse. See **right-angled triangle.**

Inscribed polygon. If a circle passes through all the vertices of a polygon, the circle is said to be circumscribed about the polygon; and the polygon is said to be inscribed in the circle. (p. 224.)

Isosceles triangle. A triangle which has two of its sides equal is called an isosceles triangle. (p. 82.)

Line. The boundary between any two parts of a surface is called a line. A line has length but no breadth or thickness.

Locus. If a point moves so as to satisfy certain conditions, the path traced out by the point is called its locus. (p. 144.)

Major arc, minor arc. (See p. 218.)

Major segment, minor segment. (See p. 253.)

Mean proportional. If x is such a magnitude that $a : x = x : b$, then x is called the mean proportional between a and b. (p. 331.)

Median. See **triangle.**

Net. (See p. 27.)

Obtuse angle. See **acute angle.**

Obtuse-angled triangle. A triangle which has one of its angles an obtuse angle is called an obtuse-angled triangle. (p. 81.)

Octagon. See **pentagon.**

Parallel straight lines are straight lines in the same plane, which do not meet however far they are produced in either direction. (p. 70.)

Parallelogram. A quadrilateral with its opposite sides parallel is called a parallelogram. (p. 73.)

Any side of a parallelogram may be taken as the **base.** The perpendicular distance between the base and the opposite (parallel) side is called the **height,** or **altitude.** (p. 167.)

Pentagon, hexagon, heptagon, octagon, etc.—a polygon of 5, 6, 7, 8, ... sides; 5-gon, 6-gon, 7-gon, 8-gon.... (p. 18.)

Perimeter. The perimeter of a figure is the sum of its sides. (p. 18.)

Perpendicular. See **right angle.**

Plane. A surface which is such that the straight line joining every pair of points in it lies wholly in the surface is called a plane surface, or, briefly, a plane.

Point. The boundary between any two parts of a line is called a point. A point has no length, breadth, or thickness, but it has position.

Polygon. A plane figure bounded by straight lines is called a polygon, or, a **rectilinear** figure. (p. 83.)

Prism. (See p. 44.)

Projection. (See p. 210.)

Proportion. (See pp. 302, 303.)

Pyramid. (See p. 27.)

Quadrilateral. A plane figure bounded by four straight lines is called a quadrilateral. (p. 73.)

The straight lines which join opposite corners of a quadrilateral are called its **diagonals.** (p. 73.)

Radius. See **circle.**

Ratio. (See pp. 302, 303.)

Rectangle. A parallelogram which has one of its angles a right angle is called a rectangle. (p. 135.)

Rectilinear figure. A figure contained by straight lines.

Reductio ad absurdum. (See p. 122.)

Reflex angle. See **acute angle.**

Regular polygon. A polygon which has all its sides equal and all its angles equal is called a regular polygon. (p. 84.)

Rhombus. A parallelogram which has two adjacent sides equal is called a rhombus. (p. 135.)

Right angle, perpendicular. When one straight line stands on another straight line and makes the adjacent angles equal, each of the angles is called a right angle; and the two straight lines are said to be at right angles, or perpendicular to one another. (p. 64.)

Right-angled triangle. A triangle which has one of its angles a right angle is called a right-angled triangle.

The side opposite the right angle is called the **hypotenuse.** (p. 81.)

Scalene triangle. A triangle which has no two of its sides equal is called a scalene triangle. (p. 82.)

Sector of a circle. (See p. 219.)

Segment of a circle. (See p. 219.)

Semicircle. (See p. 219.)

Similar. Figures which are equiangular to one another and have their corresponding sides proportional are said to be similar. (p. 313.)

Solid. Any limited portion of space is called a solid. A solid has length, breadth and thickness. (pp. 55—59.)

Sphere. (See p. 217.)

Square. A rectangle which has two adjacent sides equal is called a square. (p. 135.)

Straight line. If a line is such that any part, however placed, lies wholly on any other part if its extremities are made to fall on that other part, the line is called a straight line.

Supplementary angles. When the sum of two angles is equal to two right angles, each is called the supplement of the other, or is said to be supplementary to the other. (p. 66.)

Surface. The boundary between two parts of space is called a surface. A surface has length and breadth but no thickness.

Symmetry. (See p. 51.)

Tangent. A tangent to a circle is a straight line which, however far it may be produced, has one point, and one only, in common with the circle.

The tangent is said to **touch** the circle; the common point is called the **point of contact.** (p. 238.)

Tetrahedron. (See pp. 26—27.)

Third proportional. If x is such a magnitude that $a:b=b:x$, then x is called the third proportional to the two magnitudes a, b. (p. 309.)

Trapezium. A quadrilateral which has only one pair of sides parallel is called a trapezium. A trapezium in which the sides that are not parallel are equal is called an **isosceles** trapezium. (p. 135.)

Triangle. A plane figure bounded by three straight lines is called a triangle. (p. 73.)

Any side of a triangle may be taken as **base.** The line drawn perpendicular to the base from the opposite vertex is called the height, or **altitude.** (p. 172.)

The straight line joining a vertex of a triangle to the mid-point of the opposite side is called a **median.** (p. 110.)

Vertically opposite angles. The opposite angles made by two intersecting straight lines are called vertically opposite angles (*vertically* opposite because they have the same vertex). (p. 68.)

Vertices. The corners of a triangle or polygon are called its vertices. (p. 16.)

Wedge. A 3-sided prism. (See p. 44.)

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