THE MODERN MATHEMATICAL SERIES

LUCIEN AUGUSTUS WAIT . . . GENERAL EDITOR

(SENIOR PROFESSOR OF MATHEMATICS IN CORNELL UNIVERSITY)

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ELEMENTARY GEOMETRY

PLANE

BY

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PREFACE

This text-book aims to carry out the spirit of the admirable suggestions made by the Committee on Secondary School studies, appointed by the National Educational Association. While the book speaks for itself, some of its leading features may here be pointed out.

(1) It aims at a combination of Euclidean rigor with modern methods of presentation suitable for beginners in the study of demonstrative geometry; but the rigor is not regarded as consisting so much in excessive formality of expression as in soundness of structural development.

(2) It regards the postulates as a body of fundamental conventions that constitute a definition of Euclidean space, from which (with the definitions of particular figures) other properties of such space are to be unfolded by a series of logical steps.

(3) It regards the postulates of construction as determining or defining the province of elementary as distinguished from higher geometry. Accordingly no hypothetical figure is made the basis of an argument until its construction has been proved to be reducible to the construction postulates; and thus problems, no less than theorems, have their place in the logical development of the subject.

(4) The theorems and problems are arranged in natural groups and subgroups with reference to their underlying principle, thus exhibiting the gradual unfolding of the space relations.

(5) Elementary ideas of logic are introduced comparatively early, so that the student may easily recognize the equiva-

lence of statements that differ only in form, and also distinguish between different statements that may seem to be alike.

(6) The mode of treating ordinary size-relations is purely geometrical. "This method being pure and thoroughly elementary, and involving no abstraction, is surely better suited to the beginner. Indeed, the student is most likely to become a sound geometer who is not introduced to the notion of numerical measures until he has learned that geometry can be developed independently of it altogether. For this notion is subtle, and highly artificial from a purely geometrical point of view, and its rigorous treatment is difficult. The student generally only half comprehends it, so that for him demonstrations lose more in rigor as well as in vividness and objectivity by its use than they gain in apparent simplicity. Moreover, the constant association of number with the geometric magnitudes as one of their properties, tends to obscure the fundamental characteristic of these magnitudes — their continuity."* Words suggestive of measurement, such as length, area, distance, etc., are accordingly not used in the purely geometrical chapters.

(7) The Euclidean doctrine of ratio and proportion is presented in a modernized form, which shows its naturalness and generality, and renders it easier of application than the unsatisfactory numerical theory which is so often allowed to usurp its place, although it is generally conceded by mathematicians that Euclid's treatment of proportion is one of the most admirable and beautiful of his contributions to geometry.

(8) There is a chapter on mensuration, in which measure-numbers are introduced as a natural outgrowth from the general notion of ratio, and the irrational numbers that cor-

* See Report of Conference of School and College Teachers embodied in the Report of the Committee of Ten, p. 113. (Published for the National Educational Association by American Book Company, 1894.)

respond to the ratios of incommensurable magnitudes are given simple logical treatment based on the general theory of ratio, without resorting to the notion of a limit, which has no natural connection with the subject.

(9) The measurement of the circle is based on the correct definition of the length of a curved line (in terms of a straight measuring-unit) given by the best continental writers. Here the idea of a limit is imbedded in the definition; but the existence and uniqueness of the limit must be proved before we can speak of the "length of an arc" so as to make it the subject of our discourse; otherwise we are using a word that has not been completely defined. Similar statements may be made with regard to the area of the circle. As far as the author is aware this plan has not hitherto been followed in any text-book in the English language. It is hoped that this important topic has been presented in a rigorous and simple manner.

(10) Throughout the book there is an endeavor to develop the student's power of invention and generalization, without encouraging looseness, or introducing discouraging difficulties.

These features have received the approval of several experienced educators. Special acknowledgments are due to Professors Wait, Jones, Tanner, and Stecker for assistance and advice.

SUGGESTIONS TO TEACHERS

It is suggested to teachers that the introductory articles be read and discussed in class in an informal way, with the aim of drawing out and clarifying those ideas of space-relations which the students may already possess. Some of the introductory matter can be passed over lightly on first reading, and returned to when necessary. Teachers may exercise their discretion with regard to articles in small print throughout the book.

For a shorter course, any of the following groups of articles may be omitted without breaking the continuity of the subject:—

Book I. 180–186, 195–213, 232–247.
Book II. 2–3, 79–88, 90–107.
Book III. 141–198.
Book IV. 10.

Most of the exercises that are given in immediate connection with the propositions should be solved by the student; but only a few of those placed at the end of sections need be taken on a first reading. They are all carefully graded, and many suggestions are given. The author will be glad to hear from any person who may meet with any error or difficulty.

As some teachers may wish to use the Socratic or heuristic methods of instruction in certain parts of the work, the arrangement and development of the topics are such as to lend themselves easily to these valuable pedagogical methods, without interfering with the more formal presentation that is appropriate to a course in demonstrative geometry. The actual details of any such method are, however, left to individual discretion, as the skillful teacher has usually no difficulty in reconciling the claims of pedagogy and sound reasoning.

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ELEMENTARY GEOMETRY

INTRODUCTION

The Four Fundamental Space Concepts

1. Geometry is that branch of mathematical science which treats of the properties of space.

The space in which we live is divisible into parts. Every portion of matter occupies a part of space. The portion of space occupied by a body, considered separately from the matter which it contains, may be regarded as existing unchanged when the body moves into another portion of space.

2. Any portion of space capable of being occupied by a physical solid is called a *geometrical solid*, or simply a *solid*.

3. The common boundary of two adjoining solids, or of a solid and the surrounding space, is not a solid; it is a second kind of space element, called a *surface*.

4. Any surface is likewise divisible into parts; and the common boundary of two adjoining parts of a surface is not a surface; it is a third kind of space element, called a *line*.

5. Again, any line is divisible into parts; and the common extremity of two adjoining parts of a line is a fourth kind of space element, called a *point*.

A point is not divisible into parts; hence, the point is the simplest space element.

6. A fine tracing point, or a dot on a sheet of paper, gives an approximate representation of the ideal geometric point.

Similarly the lines which we trace on the surface of a sheet of paper give some idea of geometric lines. They are, however, only approximations to ideal geometric lines, no matter how finely they may be traced.

7. It is often convenient to think of a geometric line as traced or generated by a point of a moving body.

The line is then called the *path* of the point.

In the same way a surface may be imagined as generated by a line that is traced on a moving body.

Again, the surface of a moving body may be imagined as tracing or sweeping out a solid portion of space.

We cannot go on, however, and imagine any motion of a solid that will generate any higher space concept.

Hence, the solid is the most comprehensive space concept we can form.

8. Thus whether we begin with the notion of a solid and proceed downwards to the notion of a point, or whether we begin with the point and build up the solid, there are but three steps in the process: from solid to surface, surface to line, line to point; or else from point to line, line to surface, surface to solid.

Accordingly, the space of our experience is said to have *three dimensions*.

A point is said to have no extension and no dimensions; a line is said to be extended in one dimension; a surface to be extended in two dimensions; a solid to be extended in three dimensions.

9. Any combination of points, lines, surfaces, or solids, is called a *geometric figure*.

Two Primary Space Postulates

10. The *postulates* of geometry are fundamental agreements or conventions concerning the starting point and scope of the science.

11. Postulate of space-dimensions. It is commonly agreed that ordinary geometry shall treat only of a space of three dimensions.

We cannot, however, assert that a space of four dimensions could not exist under any conditions. We are not able to form a mental picture of such a space, but it does not follow that no one will ever be able to form such a picture.

12. Postulate of figure-transference. It is also commonly agreed that ordinary geometry shall consider only a space in which figures can be transferred in thought from one position to another without further change.

There is a branch of higher geometry which considers the possible existence of a space in which figures are not transferable without change. (See Art. 34.)

Besides the two postulates just stated, other postulates will be introduced in due course.

Primary Definitions

13. In geometry a *definition* is a statement of what is to be regarded as the fundamental property of a certain class of figures, sufficient to distinguish the class, and also sufficient to furnish a starting point for deriving other properties by logical inference.

The definitions will be introduced wherever occasion arises. The name to be applied to the class of figures so defined will be *italicized* when used for the first time in the definition.

14. Superposable figures. Equal figures. If two figures are such that they can, by transference, be so applied to each other that every point of one falls on some point of the other, point for point, the two figures when so applied are said to be *coincident,* or to be *superposed.* Figures that are capable of superposition are said to be *superposable.*

Superposable figures are also said to be *equal* to each other.

Thus the phrase "equal figures" will always have the same meaning as "superposable figures."

Straightness as a quality of certain lines.

15. The fundamental meaning of straightness as a geometric concept is to be obtained by idealizing our experience.

It is well known that the practical straightness of two rulers, for instance, is tested by observing whether their edges seem to fit each other, no matter how they may be moved or turned. If no want of coincidence could be revealed by any microscope, however powerful, both edges would have the ideal quality of straightness in the geometric sense.

Accordingly the notion of straightness as possessed by certain ideal lines is embodied in the following definition.

16. *Straight lines* are lines of unlimited extent such that any portion of any of them will coincide with any other portion of any of them, however applied, if the extremities of the two portions coincide.

17. It follows from this definition that if two straight lines pass through the same two points, the lines coincide, and may then be regarded as the same line.

This may be conveniently expressed thus:

Only one straight line can pass through the same two points.

18. **Broken line. Curved line.** A line composed of parts of different straight lines is called a *broken line.* A line of which no part is straight is called a *curved line,* or *curve.*

Flatness as a quality of certain surfaces.

19. Again, it is well known that the practical flatness of a surface is tested by observing whether a straightedge fits it, however placed on the surface.

Accordingly the notion of perfect flatness as possessed by certain ideal surfaces is embodied in the following definition.

20. A *flat surface,* or *plane,* is a surface of unlimited extent such that a straight line passing through any two of its points lies wholly in the surface.

21. It follows from this definition, and from the definition of a straight line, that plane surfaces are such that any portion of any of them will coincide with any other portion of any of them, however applied, if the boundaries of the two portions coincide. For the straight line passing through *any* two points of the common boundary must lie in both planes.

22. Figures formed by points and lines traced on a plane surface are called *plane figures.*

The study of plane figures is called *plane geometry,* or geometry of two dimensions. The consideration of all other figures belongs to *solid geometry.*

Boundaries of Separation

23. Two portions of a plane that have a common bounding line are said to be *separated* by that boundary if every line passing from a point of one portion to a point of the other, and not passing out of both portions, passes through some point of the common bounding line.

Two portions of space that have a common bounding surface are said to be *separated* by that boundary if every line passing from a point of one portion to a point of the other, and not passing out of both portions, passes through some point of the common bounding surface.

24. Postulate of separation. Let it be granted that an unlimited straight line on a plane surface divides the whole plane into two portions that are separated from each other by the straight line.

25. Each of the two parts into which a plane is divided by an unlimited straight line is called a *half plane.* Two figures in the same half plane are said to be at the *same side* of the straight line. Two figures, one in each half plane, are said to be at *opposite sides* of the line; and the line is said to pass *between* the two figures.

26. A straight line terminating at a point and extending indefinitely the other way is called an *indefinite half line.* Thus any assigned point on an unlimited straight line divides the line into two indefinite half lines. Two points on the same half line are said to be at the *same side* of the assigned point. Two points, one on each half line, are said to be at opposite sides of the point of separation, and the latter point is said to be *between* the two former.

CLOSED FIGURES

27. A line on a plane surface is said to be *closed* if it separates a finite portion of the plane from the remaining indefinite portion.

A surface is said to be *closed* if it separates a finite portion of space from the remaining indefinite portion.

In both cases the finite portion is said to be *inclosed* by the boundary. All points of the finite portion, not on the boundary, are said to be *within* the figure; and all other points, not on the boundary, are said to be *without* the figure.

Thus any line passing from any point within to any point without a closed figure passes through some point of the boundary.

Hence an unlimited straight line passing through any point within the closed figure passes through at least two points of the boundary.

Roundness as a quality of certain closed figures.

28. The notion of roundness as applied to certain figures is embodied in the following definitions:

A *circle* is a plane closed line such that all straight lines joining any point on this line to a certain point within the figure are equal. This point is called the *center* of the circle.

A *sphere* is a closed surface such that all straight lines joining any point on the surface to a certain point within the figure are equal. This point is called the *center* of the sphere.

In the circle or the sphere a line joining the center to any point on the boundary is called a *radius*.

The property of having equal radii from a certain point to the boundary is called *roundness*.

Thus a circle is a round plane curve; and a sphere is a round surface. A portion of a circle is called an *arc*.

The Construction Postulates

29. The conventions in elementary geometry with regard to the recognized ways of constructing figures on a plane surface are expressed in the following construction postulates:

Let it be granted:

1. That a straight line may be drawn from any one point to any other.

2. That a terminated straight line may be prolonged indefinitely.

3. That a circle may be described on a plane surface with any point of the plane as center, and with a radius equal to any finite straight line.

30. We are thus to be allowed the use of the straightedge for drawing and prolonging lines on a plane surface; and also the use of the compasses for describing circles, and for transferring portions of straight lines.

The lines so drawn will not indeed be true geometrical lines, however finely they may be traced. One of the purposes of the postulates is to make an agreement by which the lines so traced shall be regarded as representing true lines. They will be supposed to have no irregularities, and to cover no portion of the surface, being thought of as mere boundaries.

31. Besides this positive use of the construction postulates, they have also a negative or restrictive use. No construction is to be allowed in elementary plane geometry which cannot be performed by a combination of the primary constructions.

On the Fundamental Conventions

32. Postulates of existence. Besides the postulates that have been formally stated above, or that may hereafter be introduced, there are certain other fundamental conventions, which should be noticed. The foregoing definitions contain implied agreements that the thing defined shall be regarded as existing. These implied agreements are called *postulates of existence.* Thus by means of the definitions we have postulated the existence of points, lines, and surfaces, in general, and also the existence of particular lines and surfaces having the respective qualities of straightness, flatness, closedness, and roundness.

33. Twofold purpose of the postulates. It should be observed that our space postulates, whether expressed or implied, are not of an arbitrary nature, for they are the outcome of our actual space-experience. The postulates, however, go beyond our experience in two ways. In the first place they raise to ideal exactness our ordinary perceptions of space, which are more or less crude. Again, by means of the postulates, we extend to the space outside of our experience the primary notions suggested by our perception of the limited portion of space that we inhabit.

34. The postulates as defining Euclidean space. A space that fulfills the conditions embodied in the postulates and primary definitions is called a *Euclidean space* after the name of Euclid, who wrote the first systematic treatise on geometry. We can never be absolutely certain, at least with our present mode of perception, that our space is of the ideal Euclidean character, but there is no doubt that, for all human needs, it may be regarded as accurately Euclidean.

A perfect system of postulates should embody the primary notions that are necessary and sufficient to distinguish Euclidean space from other kinds of space, and to furnish a starting point from which all its properties could be derived by a chain of reasoning without further resort to experience. Euclid and the ancient geometers did not give close attention to the necessity and sufficiency of their system of conventions. They silently took for granted certain things that do not follow from previously accepted principles; and some of their fundamental conventions are not independent of each other. Modern geometers are not yet entirely agreed on a complete system of mutually independent postulates for Euclidean space, and the full discussion of this question goes beyond the limits set for elementary geometry.

Several different systems of non-Euclidean geometry have been studied, each of which dispenses with one or more of the characteristic properties of Euclidean space. The most celebrated of these systems dispenses with the 'postulate of parallels,' which will be introduced in the proper place. It should be observed that if only the 'postulate of dimensions' is dispensed with, the space is still Euclidean in character, for a Euclidean three-dimensional space may be regarded as existing in a Euclidean space of four or more dimensions, just as a two-dimensional space exists in a three-dimensional one. A Euclidean space of more than three dimensions is called a Euclidean hyper-space.

Primary Magnitude Relations

35. Definitions. A *magnitude* is anything that is divisible into parts.

A magnitude is said to be the *sum* of all its parts.

A magnitude is said to be *greater than* any part of it, and also greater than any other magnitude which is superposable on a part of it.

In the same way a part is said to be *less than* the whole; and any magnitude that is superposable on a part is also said to be less than the whole.

If a magnitude is divided into any two parts, either part is said to be the *difference* of the whole and the other part.

If a magnitude is divided into two superposable parts, each part is said to be *half* of the whole, and the whole is said to be *double* of either part.

36. It will next be shown how to construct the sum and difference of certain magnitudes; and some of the terms introduced above will receive more detailed definition in connection with the special magnitudes to which they are applied.

The simplest magnitudes are straight lines and plane angles. These will be considered in the next section.

PLANE GEOMETRY

BOOK I. — RECTILINEAR FIGURES

LINE-SEGMENTS AND ANGLES

Definitions concerning Line-segments

1. Any portion of a straight line is called a *line-segment.* It is usually designated by two letters, one placed at each extremity.

Any two segments of the same straight line are called *collinear segments.*

If two collinear segments have a common extremity, and are at opposite sides of this common point, they are called *adjacent* collinear segments; and the segment which is composed of them is called their *sum.*

Several collinear segments are said to be *consecutive* when the second is adjacent to the first, the third to the second, and so on without overlapping; and the whole segment composed of them is called their *sum.*

For instance, the segment AE is the sum of the segments AB, BC, CD, DE.

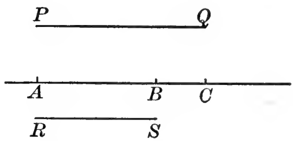
The segment AE is also the sum of the segments AB, BD, DE; and of AC, CE.

Again, the sum of any line-segments, however situated, is the segment obtained by transferring them so as to be consecutive collinear segments, and then taking their sum by the preceding definition.

Comparison of line-segments.

2. Two given line-segments may be compared by transferring one or both so that they may become collinear, have a common extremity, and lie on the same side of this extremity.

E.g., to compare the segments PQ and RS, take an indefinite line, and transfer PQ to the position AC, and RS to the position AB. Then they have a common extremity A. If the other two extremities B and C happen to coincide, the segments PQ and RS are equal. If these extremities do not coincide, and if the three points are in the order A, B, C, then the segment AC is the sum of AB and another segment BC.



In this case AC is said to be *greater than* its part AB, and the latter is said to be *less than* the former. Accordingly, PQ is said to be greater than RS, and RS is said to be less than PQ (Introd. 35).

The other segment BC which when added to the less produces the greater is called the *difference* of the two given segments. It is also called the *excess* of the greater over the less, or the *remainder* obtained by subtracting the less from the greater.

3. The sum of two equal line-segments is called the *double* of either of them; and each of the equal segments is said to be *half* of the whole segment which is composed of them.

4. Hereafter the word *line* when used without a qualifying word will mean *straight line.*

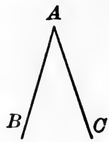
Sometimes a *line-segment* will be called a *line* when there is no possibility of mistaking it for an indefinite line.

The figure which next presents itself is that formed by two lines terminated at the same point.

Definitions concerning Angles

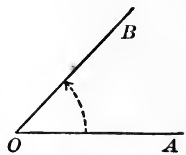
5. An *angle* is the figure formed by two indefinite half lines issuing from the same point. This point is called the *vertex* of the angle, and the half lines are called its *sides.*

An angle is usually designated by three letters, the middle one being placed at the vertex and the other two on the sides; thus the angle of the straight lines AB, AC is called the angle BAC. When, however, there is no other angle having the same vertex, the letter at the vertex is a sufficient designation.



6. A useful notion of an angle may be obtained by the conception of a revolving line.

In the angle AOB, imagine a line at first to coincide with OA and then to revolve about the point O (that is, to coincide in succession with different lines passing through O) until it arrives in the position OB. The revolving line is then said to have turned through the angle AOB.

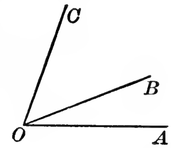


As there are two ways of turning a line from the position OA to the position OB, there are two angles AOB formed by the same two half lines. These are said to be *conjunct angles.*

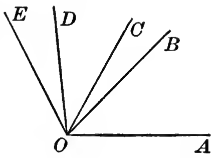
7. Two angles are said to be *equal* (in accordance with the definition of equal figures in general) when either angle may be transferred so as to coincide with the other, *i.e.* so that their sides may be coincident, and so that the two angles in question can then be turned through at the same time by the revolving half line.

When two angles are in coincidence, their conjunct angles are also in coincidence.

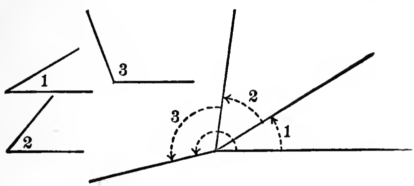
8. Two angles that have the same vertex with one side common, and are situated at opposite sides of this common line, are called *adjacent angles,* and the whole angle formed by the two extreme lines, of which these two angles are parts, is called the *sum* of the two adjacent angles. Thus, the sum of the angles *AOB* and *BOC* is the angle *AOC.*



9. Several angles are said to be *adjacent in succession* when they have a common vertex and are such that the second is adjacent with the first, the third with the second, and so on without overlapping; and the whole angle formed by the two extreme lines, of which these angles are parts, is called the *sum* of the several angles. For instance, the angle *AOE* is the sum of the angles *AOB*, *BOC*, *COD*, and *DOE.* It is likewise the sum of the angles *AOC*, *COD*, and *DOE.*



10. Again, the sum of several angles not adjacent in succession is the angle obtained by transferring them so as to be adjacent in succession, and then taking their sum according to the preceding definition. This process is called the



addition of angles, and the given angles are then said to be *added* or *summed.* It may be stated as the process of letting a revolving line turn successively through angles equal to the given angles (such as 1, 2, 3); the whole angle thus turned through being the required sum of the separate angles.

Comparison of angles.

11. Two angles are compared in regard to magnitude by transferring one or both so that they may have the same vertex, a common side, and lie at the same side of this common line. If the other two sides happen to coincide, the angles are equal. If these sides do not coincide, one of the given angles is equal to the sum of the other given angle and a third angle. The first given angle is then said to be *greater* than the other, and the latter is said to be *less* than the former (Introd. 35).

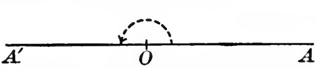
12. The third angle, mentioned above (11), which when added to the less produces the greater, is called the *difference* of the two given angles. It is called also the *excess* of the greater over the less, or the *remainder* obtained by taking the less away from the greater.

13. One angle is said to be the *double* of another, if it is the sum of two angles each equal to the other; and the latter angle is called the *half* of the former.

14. It will be seen from the above definitions of the words equal, sum, difference, greater, less, double, half, when applied to angles, that in comparing the magnitude of different angles nothing is said about the magnitude of their sides. In fact, the sides of an angle may always be thought of as indefinitely prolonged.

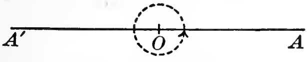
Species of angles.

15. When the revolving half line turns from the position OA into the position OA', the prolongation of OA, it is then said to have turned through a *straight angle.*

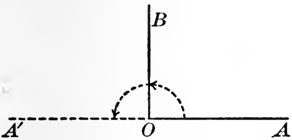


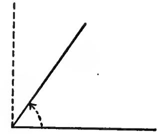
Thus, an angle whose sides are in the same straight line at opposite sides of the vertex, is a *straight angle.*

16. If this revolving line turns through another straight angle, from OA' to OA, so as to complete a revolution, the angle turned through is called a *perigon*.



17. The half of a straight angle is called a *right angle*.

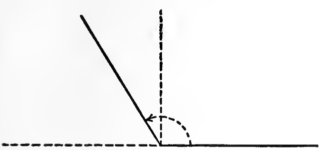


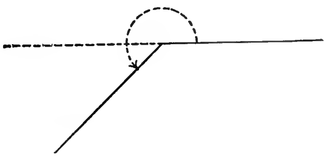


18. An angle less than a right angle is called an *acute angle*.

19. An angle greater than a right angle and less than a straight angle is called an *obtuse angle*.

20. An angle less than a straight angle is called a *concave angle*. Two concave angles are said to be of the *same species* when they are both acute, both right, or both obtuse.





21. An angle greater than a straight angle and less than a perigon is called a *convex angle*.

22. The definitions above given and illustrated (1–21) form the basis of the statements in the next section. Further definitions will be introduced as occasion requires.

Axioms concerning Lines and Angles*

23. An *axiom* is a general statement whose truth can be immediately inferred from the definitions of the terms used.

It is convenient for purpose of reference to designate specially by the term axiom some fundamental statements, relating to the equality and inequality of magnitudes, whose truth can be inferred directly from the above definitions.

We apply these axioms only to those magnitudes for which the appropriate methods of comparison have been already explained.

When other kinds of magnitude are introduced, and when all the terms employed receive precise definitions as applied to such magnitudes, then the appropriate axioms will be stated, and their truth inferred from the definitions.

The first seven of the following axioms relate to the equality of magnitudes, the remaining seven to inequality.

24. Ax. 1. Magnitudes which are equal to the same magnitude are equal to each other.

25. Ax. 2. If equal magnitudes are added respectively to equal magnitudes, the sums are equal.

26. Ax. 3. If equal magnitudes are subtracted respectively from equal magnitudes, the differences are equal.

27. Ax. 4. The doubles of equal magnitudes are equal.

28. Ax. 5. The halves of equal magnitudes are equal.

29. (*a*) Ax. 6. The sum of several magnitudes taken in any order is equal to their sum taken in any other order.

(*b*) Ax. 7. The double of the sum of two magnitudes is equal to the sum of their doubles.

* The student need not dwell on Arts. 23–40 at first reading, but should refer back to them when necessary.

30. Ax. 8. If one magnitude is equal to or greater than a second magnitude, and the second greater than a third, then the first is greater than the third, and also greater than any magnitude equal to the third.

31. (*a*) Ax. 9. If equal magnitudes are added respectively to unequal magnitudes, the sums are unequal, the greater sum arising from the addition of the greater magnitude.

(*b*) Ax. 10. If equal magnitudes are subtracted respectively from unequal magnitudes, the differences are unequal, the greater difference being part of the greater magnitude.

(*c*) Ax. 11. If unequal magnitudes are subtracted respectively from equal magnitudes, the differences are unequal, the greater difference arising from the subtraction of the less magnitude.

32. (*a*) Ax. 12. If unequal magnitudes are added to unequal magnitudes, the sum of the two greater magnitudes is greater than the sum of the two less.

(*b*) Ax. 13. If two magnitudes are unequal, the double of the greater is greater than the double of the less.

(*c*) Ax. 14. If two magnitudes are unequal, the half of the greater is greater than the half of the less.

33. The method by which the truth of any one of these axioms is derived from the definitions is called *immediate inference,* because the whole process consists of only a single step.

We have next to show how a new geometric truth may be derived from definitions, postulates, axioms, or other accepted facts, by a process of *mediate inference,* that is, by a series of steps intermediate between the accepted facts and the new truth.

It is here that geometry, like other sciences, calls in the aid of logic, the science of reasoning.

Some Logical Terms used in Geometry

34. A *theorem* is a statement enunciating a fact whose truth can be inferred from other statements previously accepted as true.

The enunciation of a theorem consists of two parts: the *hypothesis*, or formal statement of the conditions; and the *conclusion*, or that which is asserted to follow necessarily from the hypothesis.

35. The process by which it is made clear, step by step, that the conclusion must be true if the hypothesis is true is called the *demonstration* or *proof* of the theorem. Each step in the demonstration must be authorized by something previously accepted as true.

36. A *corollary* to a theorem is a statement whose truth follows at once from the truth of the theorem, or which can be proved by a similar course of reasoning.

A corollary to a theorem may be used (like the theorem itself) in the proof of a subsequent theorem or corollary.

37. In geometry a theorem relates to a certain kind of figure, and asserts that if the figure possesses a certain property stated in the hypothesis, then it must also possess a certain other property stated in the conclusion.

The theorem will first be stated in general terms so as to apply to a whole class of figures possessing a certain common property. This statement is called the *general enunciation*, or simply the enunciation, of the theorem.

38. For convenience a single representative figure will be drawn; and the assumed hypothesis, with the conclusion to be derived from it, will both be restated with special reference to the particular figure so drawn. Such restatement is called the *special enunciation* of the theorem.

The successive steps of the demonstration will also be explained with regard to this figure, the authority for each advance being quoted until the conclusion is reached.

39. Typical form of geometric theorem. The hypothesis of a geometric theorem can usually be put in the type-form "A is B," and the conclusion in the form "C is D."

Each of these, considered separately, is called a *simple statement*, or a *simple proposition*.

When two simple propositions are brought together, the first being preceded by the word *if* or the word *when*, and the second by the word *then*, they are said to form a *hypothetical proposition*.

In the type-form of hypothetical proposition,

"if A is B, then C is D,"

the assertion is that the second statement (the conclusion) is a necessary consequence of the first (the hypothesis), so that any one who agrees to the first must also accept the second. The hypothesis is sometimes called the *antecedent*, and the conclusion the *consequent*.

It is to be understood that when A and C stand for plural nouns, the plural verb will be used.

All geometric theorems are of the above type-form, or can be put in this form.

For instance, "The halves of equal angles are equal" can be stated in the hypothetical form,

"If two angles are equal, then their halves are equal."

Sometimes it is more convenient not to use the hypothetical form of statement in the general enunciation of a theorem; but the hypothetical form (or something equivalent to it) is always used in the special enunciation.

For instance, in Theorem 2, below, the general enunciation, "All right angles are equal," is an abbreviation of the hypothetical proposition,

"If any two angles are right angles, then they are equal."

In the special enunciation the word *if* is replaced by the word *let;* and the word *then* by the words *to prove:*

"Let AOB and $A'O'B'$ (referring to the figure) be any two right angles. To prove that they are equal."

40. Method of arrangement. The *general enunciation* is placed first, and printed in italics.

Next in order is the *special enunciation*, which consists of two parts: (1) the special statement of the hypothesis, introduced by the word *let*, and preceding the figure to which it refers; (2) the special statement of the conclusion to be demonstrated, introduced by the words *to prove*, following immediately after the figure.

Any *construction* lines that may aid in the proof are next indicated, and are usually dotted in the drawing to distinguish them from the lines mentioned in the hypothesis, which are drawn full.

The successive steps in the *demonstration* leading from hypothesis to conclusion are then made clear with reference to the figure so drawn, the previous authority for each step being quoted, or referred to. The authority may consist in the hypothesis, a definition, a postulate, an axiom, a previous theorem, or corollary.

Theorems concerning Angles

41. Theorem 1. *All straight angles are equal.*

Let the indefinite lines OA, OB be the sides of a straight angle whose vertex is O; and let $O'A'$, $O'B'$ be the sides of a straight angle whose vertex is O'.

To prove that the straight angle formed by the lines OA and OB is equal to the straight angle formed by the lines $O'A'$ and $O'B'$.

Because the first-mentioned angle is a straight angle, therefore BO and OA are in the same straight line [by the definition of a straight angle (15)].

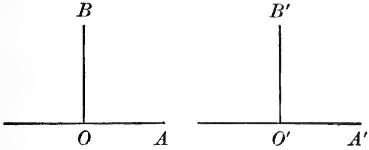
Similarly $B'O'$ and $O'A'$ are in the same straight line.

Then the line BOA can be superposed on the line $B'O'A'$

so that the point O falls on O', by the postulate of transference (Introd. 12), and the definition of straight lines (Introd. 16). The two straight angles will then have their vertices and sides in exact coincidence. Therefore they are equal [by the definition of equal angles (7)].

42. THEOREM 2. *All right angles are equal.*

Let AOB, $A'O'B'$ be any two right angles.



To prove that these angles are equal.

Every right angle is half a straight angle (17, def.).

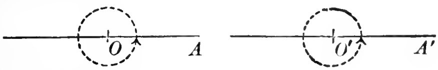
Now all straight angles are equal (theor. 1); and the halves of equal angles are equal (ax. 5, 28).

Therefore all right angles are equal.

43. Cor. *The sum of any two right angles is equal to a straight angle.*

44. THEOREM 3. *All perigons are equal.*

Let a straight line terminated at O and indefinitely extended toward A revolve about its extremity O from the position OA through a perigon into the position OA again. Similarly let a line revolve about O' from the position $O'A'$ through a perigon into the position $O'A'$ again.



To prove that these perigons are equal.

Each perigon is the sum of two straight angles (15, 16).

Now all straight angles are equal (theor. 1); and the sums of equal angles are equal (25, ax. 2).

Therefore the perigons are equal.

45. Cor. *The sum of any four right angles is equal to a perigon.*

COMPLEMENTS, SUPPLEMENTS, ETC.

46. Definitions. If two lines form a right angle each is said to be *perpendicular* to the other.

If two perpendicular lines are prolonged through their intersection, they divide the perigon into four equal parts (17).

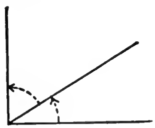
47. When the sum of two angles is a right angle, each is called the *complement* of the other.

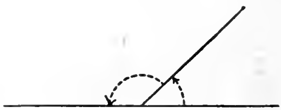
When the sum of two angles is a straight angle, each is called the *supplement* of the other.

When the sum of two angles is a perigon, each is called the *conjunct* of the other.

From the definitions the two following statements are immediate inferences.

48. Two adjacent angles are complemental if their exterior sides form a right angle which includes the two angles.





49. Two adjacent angles are supplemental if their exterior sides form a straight angle.

50. THEOREM 4. *Complements of equal angles are equal.*

Proof. The complement of each angle is obtained by subtracting it from a right angle (def., 47).

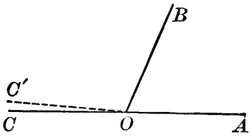
Now all right angles are equal (theor. 2, 42); and if equal angles are subtracted from equal angles, the remainders are equal (ax. 3, 26).

Therefore the complements of equal angles are equal.

51. THEOREM 5. *The supplements of equal angles are equal.* [The proof is left to the student.]

52. THEOREM 6. *If two adjacent angles are supplemental, then their exterior sides are in a straight line.*

Let the adjacent angles AOB and BOC be supplemental.



To prove that OC is the prolongation of OA.

Suppose, if possible, that OC is not the prolongation of OA; and let OC' be that prolongation.

Then the angle BOC' is the supplement of AOB (49).

Therefore BOC' and BOC are equal, being supplements of the same angle (51).

Hence a part of the angle BOC is equal to the whole angle, which is impossible.

Thus the supposition made is proved false, since it leads, by correct reasoning, to an absurdity.

Therefore OC' and OA are not in the same straight line.

In the same way it can be shown that no other line than OC is in a straight line with OA.

Therefore OC and OA are in one straight line.

53. Indirect proof, or proof by exclusion. It may be noticed that the two statements above, "OC is the prolongation of OA," and "OC is not the prolongation of OA," are opposite statements; *i.e.* if either is false, the other is true. Instead of proving the truth of the first statement directly, it was easier to prove the falsity of its opposite, or, in other words, "to exclude the opposite."

The process of proving the truth of a statement indirectly, by proving the falsity of its opposite, is called *indirect proof* or *proof by exclusion*.

54. Converse theorems. There is a close relation between 52 and 49, which will be seen more clearly if the latter is stated in the following form:

If two concave adjacent angles have their exterior sides in a straight line, then the two adjacent angles are supplemental.

Art. 52 differs from this by having the hypothesis and conclusion interchanged. The interchange of hypothesis and conclusion is called *conversion.*

Definition. Two theorems are said to be *converse* to each other when the hypothesis of each is the conclusion of the other. Two converse theorems have the type-forms:

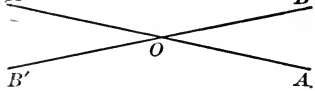
If A is B, then C is D; If C is D, then A is B.

In many cases two converse theorems are both true; but there are cases in which a theorem is true while its converse is not true. The truth of the converse is not a logical consequence of the truth of the original theorem, but always requires separate examination.

55. Definition. Two angles that are situated so that the sides of each are the prolongations of the sides of the other are said to be *vertically opposite angles.*

56. THEOREM 7. *Two vertically opposite angles are equal.*

Let the vertically opposite angles AOB, $A'OB'$ be formed by the straight lines AOA', BOB'.



To prove that the angles AOB and $A'OB'$ are equal.

The angle AOB is the supplement of BOA' (def., 49); and $A'OB'$ is also the supplement of BOA'.

Now supplements of the same angle are equal (51).

Therefore the angles AOB and $A'OB'$ are equal.

TRIANGLES

57. The preceding articles treated of the figure formed by two intersecting lines. The next figure in order of simplicity is that formed by three lines each of which intersects the other two.

58. Definitions. A plane figure formed by three straight lines that inclose a portion of the plane surface is called a *triangle.*

These lines are called the *sides,* and their intersections the *vertices* of the triangle.

Unless otherwise stated, the sides will be taken to mean the segments lying between the vertices; but they may also be thought of as indefinitely prolonged.

The angles formed by the sides, situated toward the interior of the triangle, are called the *interior angles,* or simply the *angles,* of the triangle.

An *exterior angle* of the triangle is the concave angle formed by one of the sides and the prolongation of another.

It is sometimes convenient to regard a triangle as standing on a selected side. We then call that side the *base;* the two angles adjacent to it, the *base angles;* the opposite angle, the *vertical angle;* the sides of this angle, the *two sides* of the triangle; and its vertex, the *vertex* of the triangle.

An *isosceles* triangle is one that has any two of its sides equal.

The vertex common to the equal sides of an isosceles triangle is called the *vertex,* and the side opposite to it is called the *base.*

An *equilateral* triangle is one that has its three sides equal.

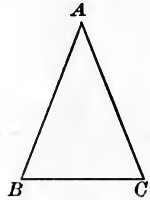
A *scalene* triangle is one that has no two of its sides equal.

Isosceles Triangles

59. Theorem 8. *In a triangle, if two sides are equal, then the angles opposite the equal sides are equal.*

Let the triangle *ABC* have the sides *AB* and *AC* equal.

To prove that the angles *ABC* and *ACB* are equal.



Imagine the triangle turned over on itself so that *AB* takes the position *AC*, and *AC* takes the position *AB*.

Since *AB* and *AC* are equal, the point *B* takes the position of *C*, and *C* takes the position of *B*; hence the new position of the line *BC* coincides with its old position (Introd. 17).

Therefore the angles *ABC* and *ACB*, being superposable, are equal.

60. **Cor. 1.** *If the equal sides AB and AC are extended through B and C, the angles below the base are equal.*

61. **Cor. 2.** *If two angles of a triangle are not equal, then the opposite sides are not equal.*

Proof. Suppose, if possible, that the opposite sides *are* equal.
Then the angles opposite these sides are equal (59).
This is inconsistent with the hypothesis. Therefore the supposition made is false. Hence the opposite sides are *not* equal.

Converse of theorem 8.

62. Theorem 9. *In a triangle, if two angles are equal, then the sides opposite the equal angles are equal.*

Let the triangle *ABC* have the angles *B* and *C* equal.

To prove that the sides *AB* and *AC* are equal.

Imagine the figure turned over on itself so that *B* falls on *C*, and *C* on *B*.

Then the line BA takes the position CA, since the angles B and C are equal.

Similarly the line CA takes the position CB.

Therefore the point A, being the intersection of BA and CA, falls on its former position.

Hence the sides BA and CA, being superposable, are equal.

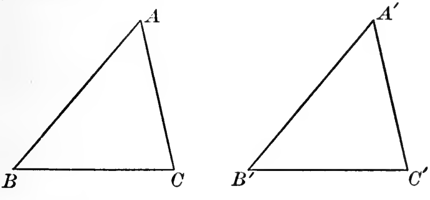
63. Cor. *If two sides of a triangle are not equal, then the opposite angles are not equal.* [Prove by exclusion as in 61.]

Equality of Triangles — Three Primary Cases

Two sides and included angle.

64. Theorem 10. *If two triangles have two sides and the included angle of one respectively equal to two sides and the included angle of the other, the triangles are equal.*

Let the triangles ABC and $A'B'C'$ have the sides AB, AC, and the angle BAC, respectively equal to the sides $A'B'$, $A'C'$, and the angle $B'A'C'$.



To prove that the triangles ABC and $A'B'C'$ are equal.

Imagine the angle A placed on the equal angle A', the side AB taking the position $A'B'$, and AC the position $A'C'$.

Since AB equals $A'B'$, and AC equals $A'C'$, the point B falls on B', and C on C'. Hence the line BC coincides throughout with $B'C'$ (Introd. 17).

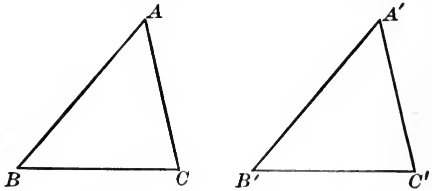
Therefore the triangles, being superposable, are equal.

Note. Sometimes two equal triangles cannot be superposed without turning one of them over. (See 73, 75.)

Two angles and the included side.

65. THEOREM 11. *If two triangles have two angles and the intervening side of one respectively equal to two angles and the intervening side of the other, the triangles are equal.*

Let the triangles ABC and $A'B'C'$ have the angles B, C, and the side BC, respectively equal to the angles B', C', and the side $B'C'$.



To prove that the triangles ABC and $A'B'C'$ are equal.

Imagine the angle B placed on the angle B', the side BC taking the position $B'C'$, and BA the position $B'A'$.

Since BC equals $B'C'$, the point C falls on C'; and since the angles C and C' are equal, the line CA falls on $C'A'$.

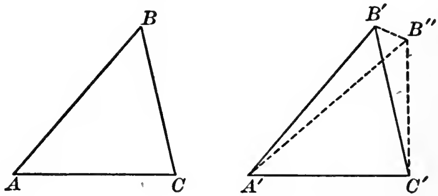
Therefore the point A, the intersection of BA and CA, falls on A', the intersection of $B'A'$ and $C'A'$.

Hence the triangles are superposable and equal.

Three sides.

66. THEOREM 12. *If two triangles have three sides of one equal respectively to three sides of the other, the triangles are equal.*

Let the triangles ABC and $A'B'C'$ have the sides AB, BC, AC equal respectively to $A'B'$, $B'C'$, $A'C'$.



To prove that the triangles ABC and $A'B'C'$ are equal.

Place the triangle ABC so that AC coincides with its equal $A'C'$; the point B falling at the same side as B'.

It is then to be proved that B falls on B'.

This is evident either if AB falls on $A'B'$, or BC on $B'C'$.

Suppose if possible that neither of these coincidences takes place; and first let ABC take the position $A'B''C'$, each of the vertices B' and B'' being without the triangle to which the other belongs. Join $B'B''$.

Then the triangle $A'B'B''$ is isosceles because $A'B'$ equals $A'B''$; hence the angles $A'B'B''$ and $A'B''B'$ are equal (59).

Therefore the angle $C'B'B''$ is less than the angle $C'B''B'$ (the former being less than one of the equal angles and the latter being greater than the other).

But the triangle $C'B'B''$ is isosceles, since the sides $C'B'$ and $C'B''$ are equal; therefore the angles $C'B'B''$ and $C'B''B'$ are equal (59).

Hence these angles are at the same time unequal and equal; which is impossible.

Therefore the supposition made is false.

Next let one of the vertices B' and B'' lie within the triangle to which the other belongs.

It may be proved in a similar manner that this supposition leads to an impossibility. [The student may draw a figure and prove.] Therefore the point B falls on B'.

Hence the triangles ABC and $A'B'C'$ are superposable, and equal.

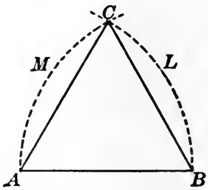
Some Fundamental Constructions

67. A geometrical *problem* is a proposition whose object is the construction of a figure which shall conform to certain prescribed conditions.

The solution consists: (1) in showing how to use the ruler and compass so as to make the required figure; (2) in demonstrating that the figure so constructed satisfies the prescribed conditions; and (3) in discussing what limitations there are on these conditions so that a solution may be possible, and under what circumstances there may be more than one solution.

68. Problem 1. *On a given finite line to construct an equilateral triangle.*

Let AB be the given line on which it is required to construct an equilateral triangle.



With A as center and AB as radius, describe the arc BLC (post. 3, Introd. 29). With B as center and BA as radius, describe the arc AMC. Let the two arcs intersect in C. Draw CA and CB.

Then ABC is an equilateral triangle.

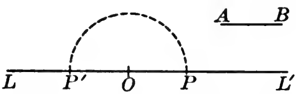
For AC and AB are equal, being radii of the same circle; and BC and AB are equal, being radii of the same circle.

Hence the triangle ABC has its three sides equal; and it is therefore equilateral.

Transference of line-segment.

69. Problem 2. *On a given straight line to lay off a part equal to a given finite straight line.*

Let LL' be the given line from which it is required to lay off a part equal to the given finite line AB.



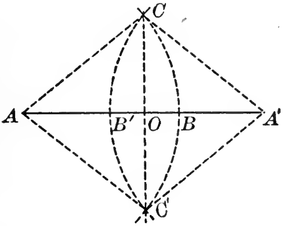
Take any point O on the line LL'; and with O as center and a radius equal to AB, describe a circle cutting the given line in the points P, P'.

Then either of the parts OP, OP' answers the requirements of the problem, since they are each equal to AB.

Bisection of line-segment.

70. PROBLEM 3. *To bisect a given finite straight line; that is, to divide it into two equal parts.*

Let AA' be the given line which it is required to bisect.



With A as center, and any convenient radius AB, describe the arc CBC'. With A' as center and an equal radius, describe the arc $CB'C'$. Let these arcs intersect in C, C'. Draw CC', meeting AA' at O.

Then O is the required mid-point of AA'.

For in the triangles ACC' and $A'CC'$, the sides AC and $A'C$ are equal, since the circles have equal radii. Similarly the sides AC' and $A'C'$ are equal. Also the side CC' is common to the two triangles.

Therefore the angles ACC' and $A'CC'$ are equal (66).

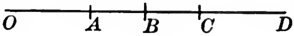
Again, in the triangles ACO and $A'CO$, the sides AC and $A'C$ are equal; the side CO is common; and the included angles ACO and OCA' have been proved equal; therefore AO and OA' are equal (64).

Hence AA' is bisected at O.

NOTE. Observe that if the radius AB is taken too small the circles will not intersect. It is always possible to take such a radius that the circles shall intersect at each side of the given line; this may be inferred from the statements with regard to closed figures (Introd. 27). Experience will show what length of radius is preferable for convenience and accuracy.

71. Cor. I. *Prove that a line-segment has only one mid-point.*

72. Cor. 2. *If three finite lines are such that the difference of the first and second is equal to the difference of the second and third, then the sum of the first and third is equal to double the second.*

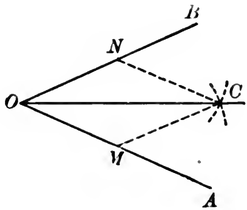


Outline. On an indefinite line lay off OA, OB, OC, equal respectively to the given lines. Then by hypothesis AB and BC are equal. Take CD equal to OA. Prove OD equal to double OB.

Bisection of angle.

73. Problem 4. *To bisect a given angle; that is, to divide it into two equal parts.*

Let AOB be the given angle which it is required to bisect.



Lay off any convenient equal segments OM and ON.

With M, N as centers and any convenient equal radii, describe arcs intersecting at C. Draw OC.

The straight line OC bisects the given angle AOB.

To prove this, draw MC and NC.

The triangles OMC and ONC have their sides respectively equal; therefore the angles MOC and NOC are equal (66).

Ex. 1. A given angle has only one bisector.

Ex. 2. Bisect a given convex angle.

Ex. 3. Bisect a given straight angle.

Ex. 4. Show how to divide a given angle into four, eight, sixteen, . . . equal parts.

Erecting perpendicular.

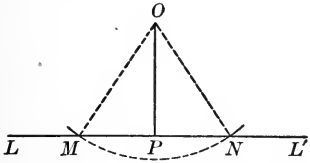
74. Problem 5. *To erect a perpendicular to a given line at a given point of the line.*

[Bisect the straight angle by the method of Problem 4.]

Dropping perpendicular.

75. Problem 6. *To drop a perpendicular to a given line from a given point not on the line.*

Let LL' be the given line, and O the given point from which a perpendicular is to be drawn.



With O as center and any convenient radius, describe an arc cutting the given line in the two points M, N.

Bisect MN at the point P (70). Draw OP.

Then OP is the required perpendicular.

For the triangles MOP and NOP have their sides respectively equal by construction.

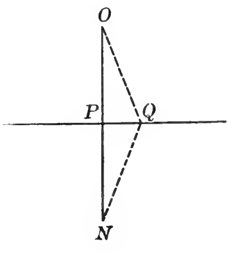
Therefore the angles MPO and NPO are equal (66).

Hence these angles are right angles (17).

Therefore OP is perpendicular to MN.

76. Discussion. There is only one solution to this problem. In other words:

There is only one perpendicular from a given point to a given line.

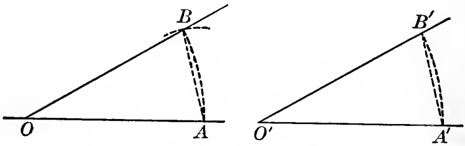


Outline proof. Suppose, if possible, that OQ is a second perpendicular. Prolong OP, making PN equal to OP. Join NQ. Prove that the angles OQP and NQP are equal; and that OQN is a straight line (52). Show the absurdity from Art. 17 of Introduction. Draw conclusion.

Transference of an angle.

77. Problem 7. *At a given point in a given straight line to construct an angle equal to a given angle.*

Let O be the given point in the given line OA, and $A'O'B'$ the given angle. It is required to draw a line OB making the angle AOB equal to $A'O'B'$.



With O and O' as centers, and any convenient equal radii OA and $O'A'$, draw the arcs AB and $A'B'$. Let the latter arc cut the sides of the given angle at the points A', B'. Draw the line $A'B'$.

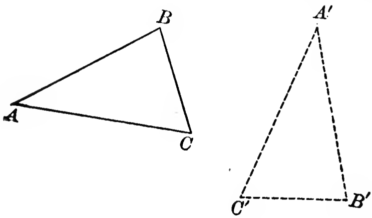
With A as center and a radius equal to $A'B'$, draw an arc cutting the arc AB at the point B. Draw the line OB.

This line OB makes with OA an angle equal to the angle $A'O'B'$. (Prove by 66.)

Note. By this construction the isosceles triangle $O'A'B'$ is transferred to the position OAB. The possibility of such transference shows that there is *some* point (such as B) at which the two arcs intersect.

Transference of a triangle.

78. **Cor.** *To transfer a triangle* (ABC) *to another position* ($A'B'C'$) *so that one side* (AB) *may fall on a given equal line* ($A'B'$) [*by actual construction with ruler and compass*].



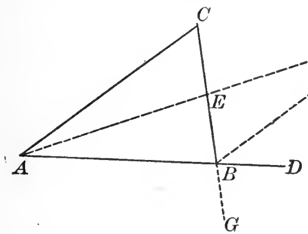
Inequalities relating to Triangles

The following six theorems with their corollaries treat of the fundamental inequalities involving the angles or sides of a triangle.

Exterior and interior angles.

79. Theorem 13. *If any side of a triangle is prolonged, the exterior angle is greater than either of the two opposite interior angles.*

Let the side AB of the triangle ABC be prolonged to D.



To prove that the exterior angle CBD is greater than either of the interior angles BAC and BCA.

Bisect BC at E. Draw AE and prolong it to F, making EF equal to AE. Join BF.

In the triangles AEC and BEF, the sides AE and EC are respectively equal to EF and EB (const.), and the included angles AEC and BEF are equal (56).

Therefore the angles ACE and FBE are equal (64).

Now the angle CBD is greater than its part FBE.

Therefore the angle CBD is greater than ACE (Introd. 35).

In the same way (by bisecting the side AB and making a similar construction) the angle ABG can be proved greater than BAC; hence the vertically opposite angle CBD is also greater than BAC.

Therefore the exterior angle CBD is greater than either of the interior and opposite angles.

80. Cor. *The sum of two angles of a triangle is less than a straight angle.*

Outline. Prove the sum of CAB and CBA less than the sum of CBD and CBA.

Ex. If a triangle has one right angle or obtuse angle, its remaining angles are acute.

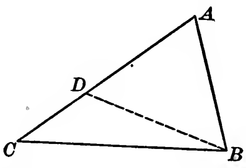
Definitions. A triangle is called a *right triangle* when one of its angles is a right angle; an *obtuse triangle* when one angle is obtuse; an *acute triangle* when all its angles are acute.

Ex. Prove that an equilateral triangle is an acute triangle.

Unequal sides and unequal angles.

81. THEOREM 14. *If one side of a triangle is greater than another, the angle opposite the greater side is greater than the angle opposite the less.*

In the triangle ABC, let the side AC be greater than AB.



To prove that the angle ABC is greater than ACB.

From the greater side lay off a part AD equal to the less side AB. Draw DB.

In the isosceles triangle ABD, the angles ABD and ADB are equal (59).

Now the angle ADB is greater than the interior angle ACB (79).

Therefore ABD is greater than ACB (30, ax. 8).

Hence the whole angle ABC is greater than ACB.

82. Combined statement. For brevity, denote the angles of a triangle by the single capital letters A, B, C, and the opposite sides by the corresponding small letters a, b, c; then, by Theorems 8 and 14 the following statements are true:

(1) If b is equal to c, then B is equal to C;
(2) If b is greater than c, then B is greater than C;
(3) If b is less than c, then B is less than C.

These three statements may be conveniently combined into one general statement as follows:

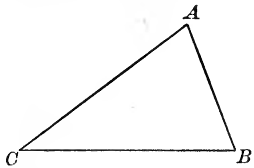
According as one side of a triangle is greater than, equal to, or less than, another side, so is the angle opposite the first side greater than, equal to, or less than, the angle opposite the second side.

Here the word *If* is replaced by the distributive phrase *According as,* and the word *then* by the words *so is.*

Converse of 81.

83. Theorem 15. *If one angle of a triangle is greater than another, then the side opposite the greater angle is greater than the side opposite the less.*

In the triangle ABC, let the angle ABC be greater than ACB.



To prove that the side AC is greater than AB.

The side AC is either equal to, less than, or greater than, the side AB.

Now AC is not equal to AB; for then the angle B would be equal to C (59), contrary to the hypothesis.

Again, AC is not less than AB; for then the angle B would be less than C (81), contrary to the hypothesis.

It only remains that the side AC is greater than AB.

84. Combined statement. By 62 and 83 the following statements are true:

If B is equal to C, then b is equal to c;
If B is greater than C, then b is greater than c;
If B is less than C, then b is less than c.

These three statements are respectively converse to those of Art. 82; and, like them, may be combined into one complete statement as follows:

According as one angle of a triangle is greater than, equal to, or less than, another, so is the side opposite the first angle greater than, equal to, or less than, the side opposite the second angle.

Perpendicular and oblique lines.

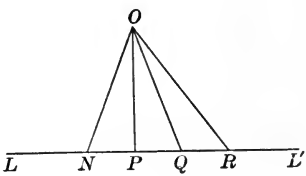
85. THEOREM 16. *Of all the straight lines that can be drawn from a given point to a given line:*

(1) *the perpendicular is the least;*

(2) *any two that make equal angles with the perpendicular are equal;*

(3) *one that makes a greater angle with the perpendicular is greater than one that makes a less angle.*

Let OP be the perpendicular from the given point O to the given line LL'. Let ON, OQ be any lines making equal angles NOP, QOP, with OP. Let OR make with OP the angle POR greater than the angle POQ or NOP.



(1) To prove that OP is less than OQ.

The angle OQP is less than the exterior angle OPL (79).

Now the angles OPL and OPQ are equal, being right angles.

Hence OQP is less than OPQ; therefore the opposite side OP is less than OQ (83).

(2) To prove that the lines *ON*, *OQ* are equal. [Apply 65.]

(3) To prove that *OR* is greater than *OQ* or *ON*.

The angle *OQR* is greater than the right angle *OPR* (79); which equals the right angle *OPN*; which is greater than the interior angle *ORQ* (79).

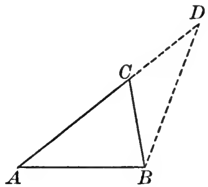
Hence the angle *OQR* is greater than *ORQ* (30); and therefore the side *OR* is greater than *OQ* (83).

86. Cor. *In an isosceles triangle, a line joining the vertex to any point in the base is less than either side; and a line joining the vertex to any point in the base extended is greater than either side.*

Sum of two sides.

87. Theorem 17. *Any side of a triangle is less than the sum of the other two.*

Let *ABC* be a triangle.



To prove that any side *AB* is less than the sum of the other two sides *AC* and *BC*.

Prolong the side *AC* until the prolongation *CD* equals the side *CB*; and draw *BD*.

In the isosceles triangle *BCD*, the angle *CBD* equals *CDB*.

Hence the whole angle *ABD* is greater than the angle *CDB*.

Therefore, in the triangle *ADB*, the side *AD*, opposite the greater angle, is greater than the side *AB* (83).

Now *AD* equals the sum of *AC* and *CD*, which equals the sum of *AC* and *CB*.

Therefore the sum of *AC* and *CB* is greater than *AB*.

88. Cor. 1. *Any side of a triangle is greater than the difference of the other two.*

89. Cor. 2. *Any straight line is less than the sum of the parts of a broken line having the same extremities.*

90. Cor. 3. *If from the ends of a side of a triangle two straight lines are drawn to a point within the triangle, their sum is less than the sum of the other two sides of the triangle.*

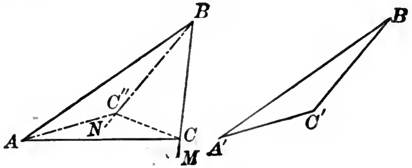
Definition. The sum of the three sides is called the *perimeter* of the triangle.

Ex. The sum of the lines joining any point within a triangle to the three vertices is less than the perimeter of the triangle, and greater than half the perimeter.

A case of unequal triangles.

91. THEOREM 18. *If two triangles have two sides of one respectively equal to two sides of the other, and the included angle in the first greater than the included angle in the second, then the third side of the first is greater than the third side of the second.*

Let the two triangles ABC and $A'B'C'$ have the sides AB, BC respectively equal to $A'B'$, $B'C'$, and the included angle ABC greater than $A'B'C'$.



To prove that the third side AC is greater than $A'C'$.

Draw the line BC'' making the angle ABC'' equal to the less angle B'. Take BC'' equal to $B'C'$; and draw AC'', CC''.

First let the point C'' fall within the triangle ABC. Let M and N be points on the prolongations of BC and BC''.

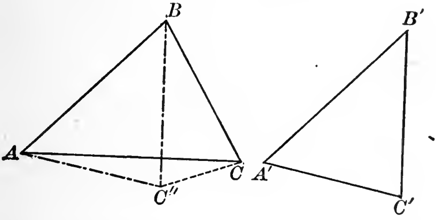
The triangles ABC'' and $A'B'C'$ are equal; and the sides AC'', $A'C'$ are equal (64).

In the isosceles triangle BCC'', the angles $CC''N$ and $C''CM$ below the base are equal (60).

Hence the whole angle $AC''C$, being greater than one of the equal angles, is greater than ACC'', which is a part of the other. Therefore, in the triangle $AC''C$, the side AC, being opposite the greater angle, is greater than AC'' (83).

Therefore AC is greater than $A'C'$.

Next let C'' fall without the triangle ABC.



The proof for the second figure is left to the student; also the consideration of the intermediate case in which C'' falls on AC.

Combined statement.

92. Cor. 1. *If two triangles have two sides of one equal to two sides of the other, then according as the vertical angle of the first is greater than, equal to, or less than, the vertical angle of the second, so is the base of the first greater than, equal to, or less than, the base of the second.* [Combine 64 and 91.]

93. Cor. 2 (*Converse of 91*). *If two triangles have two sides of the first equal respectively to two sides of the second, and the base of the first greater than the base of the second, then the vertical angle of the first is greater than that of the second.* [Prove by exclusion, using 92; see 83.]

Combined statement.

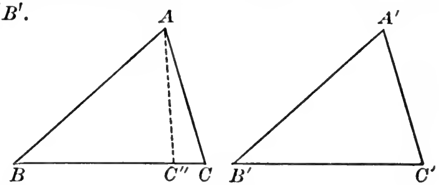
94. Cor. 3. *If two triangles have two sides of the first equal to two sides of the second, then, according as the base of the first is greater than, equal to, or less than, the base of the second, so is the vertical angle of the first greater than, equal to, or less than, the vertical angle of the second.* [66, 93.]

EQUALITY OF TRIANGLES. — TWO SECONDARY CASES

Two angles and the side opposite one.

95. THEOREM 19. *If two triangles have two angles of one respectively equal to two angles of the other, and the sides opposite one pair of angles equal in each triangle, then the triangles are equal.*

Let the two triangles ABC and $A'B'C'$ have the angles B, C, and the side AB respectively equal to the angles B', C', and the side $A'B'$.



To prove that the triangles are equal.

Place the triangle $A'B'C'$ on ABC so that $A'B'$ falls on its equal AB, and the angle $A'B'C'$ on its equal ABC; then the line $B'C'$ falls in the line BC.

Suppose, if possible, that $B'C'$ is not equal to BC, and that the point C' falls on C'' instead of falling on C.

Then the triangles $A'B'C'$ and ABC'' have the sides $A'B'$, $B'C'$, and the included angle $A'B'C'$, respectively equal to AB, BC'', ABC''. Therefore these triangles are equal; and the angles $AC''B$ and $A'C'B'$ are equal (64).

But the angles $A'C'B'$ and ACB are equal by hypothesis.

Therefore the angles $AC''B$ and ACB are equal; which is impossible (79).

Similar reasoning applies if C'' falls at the other side of C.

Thus the supposition that C' does not fall on C is false; hence C' falls on C, and the triangle $A'B'C'$ on ABC.

Therefore the triangles ABC and $A'B'C'$ are equal.

96. Cor. *The two perpendiculars drawn from any point in the bisector of an angle to the sides of the angle are equal.*

Two sides and the angle opposite one.

97. THEOREM 20. *If two triangles have two sides of one respectively equal to two sides of the other, and the angle opposite one of these equal to the corresponding angle in the other triangle, then the angles opposite to the other pair of equal sides are equal or supplemental; and if equal, the triangles are equal.*

Let the two triangles ABC and $A'B'C'$ have the sides AB and BC respectively equal to the sides $A'B'$ and $B'C'$, and the angle BAC equal to the angle $B'A'C'$.

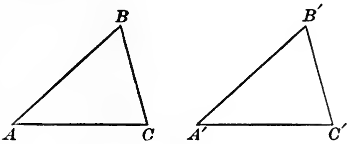


FIG. 1

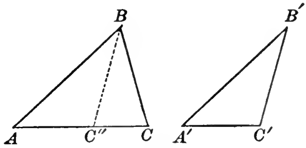


FIG. 2

To prove that the angles BCA and $B'C'A'$ are either equal or supplemental.

The two sides, AC and $A'C'$, are either equal or unequal.

If they are equal (as in fig. 1), the triangles are equal in all their parts (66).

If they are unequal (as in fig. 2), let AC be the greater.

Lay off AC'' equal to $A'C'$; and draw BC''.

In the triangle ABC'' and $A'B'C'$, the sides AB and AC'' and the included angle BAC'' are equal respectively to the sides $A'B'$ and $A'C'$ and the included angle $B'A'C'$. Therefore the angles $BC''A$ and $B'C'A'$ are equal, and the sides BC'' and $B'C'$ are equal (64).

Therefore BC'' equals BC; and hence the angles BCC'' and $BC''C$ are equal (59).

Now $BC''C$ is the supplement of $BC''A$; therefore BCA, being equal to $BC''C$, is equal to the supplement of $BC''A$, and hence equal to the supplement of $B'C'A'$.

98. Cor. 1. *If two triangles have two sides of one respectively equal to two sides of the other, and the angle opposite one of these sides equal to the corresponding angle in the other triangle, then the triangles are equal:*

(1) *If the two angles given equal are right angles or obtuse angles;*

(2) *if the angles opposite to the other two equal sides are both acute, or both obtuse, or if one of them is a right angle;*

(3) *if the side opposite the given angle in each triangle is not less than the other given side.*

99. Cor. 2. *The perpendicular from the vertex of an isosceles triangle to the base bisects both the base and the vertical angle. Conversely, the perpendicular bisector of the base of an isosceles triangle passes through the vertex.*

100. Cor. 3. *If the two perpendiculars drawn from a point to the sides of an angle are equal, then the point is on the bisector of the angle.* (Converse of 96.)

EXERCISES

1. Summarize the five cases of the equality of two triangles.

2. The bisectors of two adjacent supplemental angles are perpendicular to each other.

3. The bisectors of two adjacent conjunct angles are in the same straight line.

4. The bisectors of two vertically opposite angles are in the same straight line.

5. The lines drawn from the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal.

6. The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

7. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

8. If the perpendicular from a vertex to the opposite side bisects that side, then the triangle is isosceles.

9. If two triangles have a common base, and if the vertex of the second triangle is within the first, or on a side of the first, then the vertical angle of the first triangle is less than that of the second.

10. Three equal lines cannot be drawn from a point to a line.

SUMMARY OF TYPES OF INFERENCE

The foregoing propositions have illustrated various methods of drawing conclusions from given premises. It is now time for us to consider some of the essential features of the modes of inference we have been using, and to see the simple logical principles that underlie them. It will be seen that there are a few general type-forms which appear again and again under various modes of expression; and the student will thus early learn to recognize the logical equivalence of certain statements that differ only in form; also to distinguish between different statements that may seem to be alike; and gradually will come to see the legitimate conclusions that can be inferred from any given premises.

101. Related statements. Hereafter when the word "statement" is used without qualification, it will be understood to mean a simple assertion of the form "A is B."

It has been seen that a "theorem" is made up of two such statements placed in a certain relation to each other, the relation of hypothesis (or antecedent) to conclusion (or consequent).

When we say that the theorem is "true," we do not mean that either statement is true in itself, but only that the consequent is true *whenever* the antecedent is true.

This may be conveniently expressed by saying that the truth of the consequent is a necessary result of the truth of the antecedent.

When any two statements are related to each other so that the truth of each is a necessary consequence of the truth of the other, they are said to be *equivalent statements.*

For instance, the two statements,

X is equal to Y,

half X is equal to half Y,

are equivalent. When either is true, so is the other; and hence when either is false, so is the other.

Two statements are said to be *partially equivalent* when the truth of one is a necessary consequence of the truth of the other, but the truth of the latter not a necessary consequence of the truth of the former.

For instance, the two statements,

X is greater than Y,

X is greater than half Y,

are partially equivalent. The truth of the second is a necessary consequence of the truth of the first, but the truth of the first is not a necessary consequence of the truth of the second. When the second is true, the first may be true or false.

Two statements are said to be *independent* when neither is a necessary consequence of the other.

For instance, the two statements,

X is greater than Y,

X is less than double Y,

are independent. When either is true, the other may be true or false.

Two statements are said to be *inconsistent* when they cannot both be true at the same time.

Inconsistency is of two kinds, opposition and partial opposition.

Two statements are said to be *opposite*, if, when either is true the other is false, and when either is false the other is true.

For instance, the two statements,

X is equal to Y,

X is not equal to Y,

are opposite.

Two statements are said to be *partially opposite*, if when either is true the other is false, and when either is false the other may be true or false.

For instance, the two statements,

X is equal to Y,

X is less than Y,

are inconsistent but only partially opposite. They cannot be true at the same time, but they may be false at the same time, for there is a third alternative, X may be greater than Y.

In the preceding case there is no third alternative to the two statements, for X is either equal to Y or not equal to Y.

102. Reductio ad absurdum. It was remarked in 53 that instead of proving the truth of a statement directly it is sometimes easier to prove the falsity of its opposite. Examples of such indirect proof have occurred in 52, 61, 66, 83, 93, 95. The usual way of conducting an indirect proof is to suppose "for the sake of argument" (as in theor. 6) that the *opposite* of the desired conclusion is *true*. Combine this provisional supposition with the given hypothesis; and then show that it leads by correct reasoning to a conclusion which is inconsistent with something previously accepted as true. There must then be an error somewhere. If no flaw exists in the *reasoning*, then the error must lie in the provisional hypothesis. The provisional supposition is then declared to be false; and its opposite (*i.e.* the desired conclusion) is finally pronounced true.

This mode of reasoning is called *reductio ad absurdum*, which may be briefly defined as the process of proving the truth of a statement by reducing its opposite to an absurdity.

103. Conversion. The relation of a theorem to its converse was explained in 54. Examples of pairs of converse theorems have presented themselves in 59 and 62, 81 and 83, 91 and 93, 96 and 100. In no case was the truth of the converse theorem a direct logical inference from the truth of the original theorem, but required separate investigation.

This inquiry is of the same nature as seeking the cause of a given effect in physical science.

In a theorem, we may regard the hypothesis as the cause, and the conclusion as the effect.

The question to be decided is this: Can we from the presence of the effect infer the presence of this particular cause?

The doubt arises from the fact that the same effect may follow from *different* causes.

In order to prove that any particular fact is the true cause it is sufficient to show that if this fact is not present the effect in question does not follow.

Thus from the fact that "C is D" we can infer the fact that "A is B" if we can show that

"when A is *not* B, C is not D."

This would exclude the supposition that "A is not B," leaving the only alternative that "*A is B*."

The method of exclusion is usually the most convenient method of proving the converse of a theorem. This is exemplified in the proofs of 52 and 83.

Conversion is thus not a process of logical inference. The truth or falsity of the result is to be proved by a fresh appeal to geometric facts.

For instance, the truth of 83 was derived by the method of exclusion from the combined statement of 82. The proof consisted in showing that there were only three alternatives, namely, the three hypotheses of Art. 82, then excluding two of these alternatives by showing that each leads to a contradiction of the assumed fact, thus leaving the third alternative as the true one.

By the same method the converse of each of the statements of Art. 82 can be proved by using the other two.

As this method of conversion will be frequently used, it is now convenient to state and prove the general principle involved, once for all, so that it may be available for reference.

104. Rule of conversion. The principle may be stated in general terms as follows: *

"If the hypotheses of a group of demonstrated theorems be exhaustive — that is, form a set of alternatives of which one must be true; and if the conclusions be mutually exclusive — that is, be such that no two of them can be true

* This admirable statement is quoted from the Syllabus of the Association for the Improvement of Geometrical Teaching (London).

at the same time, then the converse of every theorem of the group will necessarily be true."

For instance, suppose the following theorems have been demonstrated:

(1) When A is B, C is D;

(2) When A is B', C is D';

(3) When A is B'', C is D'';

and suppose it is also known that A must have one of the qualities B, B', B''; and that C cannot have more than one of the qualities D, D', D''. We can then prove that the following three converse theorems are all true:

When C is D, A is B;

When C is D', A is B';

When C is D'', A is B''.

Take, for instance, the third one.

Suppose, if possible, that

When C is D'', A is *not* B''.

Then, since the hypotheses of (1), (2), (3) are exhaustive, A must be either B or B'.

Therefore, by (1) and (2), C must be either D or D'.

Then, since the conclusions of (1), (2), (3) are exclusive, C cannot be D''.

Hence, when C is D'', C is not D'', which is impossible.

Therefore the supposition made is false. Hence,

When C is D'', A *is* B''.

Thus the "Rule of Conversion" is established.

Case of two alternatives. In 82 and 84 there are three alternatives. The rule of conversion can likewise be applied to the case of *two* demonstrated theorems whose hypotheses are exhaustive and whose conclusions are exclusive.

Ex. Assuming the truth of the following two theorems:

When A is B, C is D;

When A is not B, C is not D;

apply the rule of conversion to prove the truth of the converse of each.

105. Equivalent theorems. If two theorems are such that each follows logically from the other, the two theorems are said to be *equivalent.*

If one of two equivalent theorems is true, so is the other. Hence, if one of them is false, so is the other.

Two equivalent theorems are not in strictness different theorems, but different ways of enunciating the same theorem.

A simple example of such equivalence is seen on comparing 59 and 61. It will appear in the next Article that each of these theorems is a logical consequence of the other. They are related to each other in a peculiar way. The first asserts that a certain hypothesis leads to a certain conclusion; the second asserts that the opposite of that conclusion leads to the opposite of that hypothesis. Theorems related in this way are said to be *contraposed* to each other.

106. Contraposition. Two theorems are said to be *contraposed* to each other when the hypothesis of each is the opposite of the conclusion of the other. They may be represented by the type-forms:

(1) If A is B, then C is D;

(2) If C is not D, then A is not B.

Each of these theorems is called the *contraposite* of the other.

Equivalence of contraposed theorems. Each of two contraposed theorems follows logically from the other.

For instance, assuming the first to be true, we may infer the truth of the second, by the method of exclusion, as follows:

To prove that

when C is not D, then A is not B.

Suppose, if possible, that

when C is not D, then A *is* B.

Now, when A is B, then C is D, by (1), whose truth was assumed.

Hence, when C is not D, then C is D; which is impossible, since these two statements are *opposite.*

Therefore the supposition made is false. Hence,

when C is not D, then A is *not* B.

In the same way the first theorem may be shown to be a logical consequence of the second.

Two contraposed theorems may thus be regarded as different ways of expressing the logical dependence of one fact or property upon another.

The first form asserts that the property expressed by the statement "A is B" is always accompanied by the property expressed by the statement "C is D."

The second form asserts that the absence of the latter property shows the absence of the former.

Sometimes one form may be more convenient, and sometimes the other, according to the use we wish to make of the fact in question.

Thus contraposition is a process of purely logical inference. It requires no fresh appeal to geometric facts.

107. The four related types. Comparing 59, its converse 62, and their two contraposites (61, 63), we have a group of four theorems of the types:

(1) If A is B, then C is D;

(2) If C is not D, then A is not B;

(3) If C is D, then A is B;

(4) If A is not B, then C is not D.

Of these four theorems

the first and second are contraposed,
the first and third are converse,
the third and fourth are contraposed,
the second and fourth are converse.

We have now to introduce a third term, the word *obverse*, to express the relation that exists between the first theorem and the fourth, and also between the second and the third.

108. Obversion. Two theorems are said to be *obverse* to each other when their hypotheses are opposite, and their conclusions also opposite.

To *obvert* a theorem is to change the hypothesis into its opposite, and the conclusion into its opposite.

The obverse of a theorem is then the contrapositе of its converse. For instance, the converse of (1) is (3), and the contraposite of (3) is (4), which is itself the obverse of (1).

Similarly the converse of (2) is (4), and the contraposite of (4) is (3), which is itself the obverse of (2).

Of the four theorems, the first and second are equivalent, each being a logical consequence of the other; and similarly the third and fourth are equivalent.

109. Logical independence. Two theorems are said to be *logically independent* when neither theorem is a logical consequence of the other; that is, when neither theorem is sufficient to prove the other, even though it may do so when aided by some fresh geometric fact.

The converse theorems, (1) and (3) above, are logically independent, neither being a necessary consequence of the other (103). Similarly the converse theorems, (2) and (4), are independent.

Again, the obverse theorems, (1) and (4), are logically independent, because (1) and (3) are independent and (3) is equivalent to (4). Similarly the obverse theorems, (2) and (3), are independent.

110. The three transformations. We have considered three processes of transforming a known theorem; viz. contraposition, conversion, and obversion. Of these, only the first is a process of logical inference. In the other two cases further geometric facts must be introduced to aid the inquiry into the truth or falsity of the two new theorems. It is only necessary to examine one of the two, however, for they are logically equivalent, being contraposites. Hence it will never be necessary to demonstrate more than two of the four related theorems, care being taken that the two selected are not contraposites; *i.e.* the two selected should be either converse or obverse to each other.

Ex. 1. In which of the theorems 1–20 is the converse not true? In which are the four related types all true? Only two of them true?

Ex. 2. Show that 91 has three converses. [Here the hypothesis has two parts; and the conclusion may be interchanged with the whole hypothesis, or with either of the parts.] Which one of these is true? State the three converses of 'when A is B and C is D, then E is F.'

PARALLEL LINES

The previous sections have treated of the figures formed by two or three intersecting lines. The next plane figure to be considered is that formed by two indefinite lines that never meet.

111. Definitions. Two straight lines lying in one plane, which will never meet however far they are extended both ways, are said to be *parallel.*

A line that intersects two or more other lines (whether parallel or not) is said to be a *transversal* to those lines.

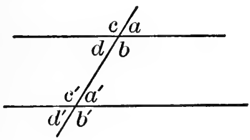
A transversal to two other lines forms with them four interior angles and four exterior angles.

An exterior angle and the non-adjacent interior angle on the same side of the transversal are called *corresponding angles.*

Two non-adjacent interior angles on opposite sides of the transversal are called *alternate angles.*

In the figure there are four pairs of corresponding angles

$$(a, a'), (b, b'), (c, c'), (d, d');$$



and there are two pairs of alternate angles

$$(b, c'), (d, a').$$

The angles made with two lines by a transversal play an important part in the theory of parallel lines; this will be seen in the following theorem, which relates to alternate angles, and lays a foundation for the succeeding problem on the construction of parallel lines.

Use of transversal in proving parallelism.

112. THEOREM 21. *If two lines in one plane are such that a transversal makes a pair of alternate angles equal, then the two lines are parallel.*

Let the lines LL' and MM' be cut by the transversal OP in the points O, P, making the alternate angles LOP and $M'PO$ equal (fig. 1).

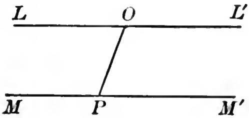


FIG. 1

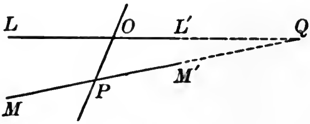


FIG. 2

To prove that the lines LL' and MM' are parallel.

Suppose, if possible, that the lines are not parallel, that is to say that they meet if extended far enough to one side or the other.

Let them meet in the point Q (fig. 2).

Then OPQ is a triangle; and the exterior angle LOP is greater than the interior angle OPQ (79).

This is contrary to the hypothesis; hence the supposition made is false.

Therefore the lines LL' and MM' do not meet, however far they may be extended both ways; that is to say they are parallel.

State the contraposite of this theorem, and compare it with 79.

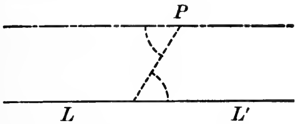
113. **Cor. I.** *If two lines in one plane are such that a transversal makes a pair of corresponding angles equal, then the two lines are parallel.*

Ex. 1. If two lines are cut by a transversal, and if two interior angles on the same side of the transversal are together equal to a straight angle, then the two lines are parallel.

Ex. 2. Two lines perpendicular to the same line are parallel.

Drawing a parallel.

114. PROBLEM 8. *Through a given point to draw a line parallel to a given line.*



Outline. Use problem 7 (Art. 77), and prove by theorem 21 (Art. 112).

115. NOTE. This solution establishes the actual existence of parallel straight lines. It shows that there is at least one line passing through a given point parallel to a given line. The possibility of there being more than one is considered in the next article.

SOME PROPERTIES OF PARALLELS

116. The parallel-postulate. The properties of parallel lines rest upon a fundamental assumption, usually called the *postulate of parallels,* which in its modern form may be stated as follows:

Let it be granted that two intersecting straight lines cannot both be parallel to the same third straight line.

This is equivalent to an agreement that only one line can be drawn through a given point parallel to a given line. In other words, it assumes that there is only one solution to problem 8, viz.:

"Through a given point to draw a line parallel to a given line."

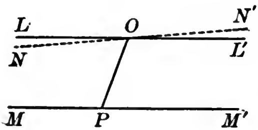
The truth of this postulate cannot be proved from any of the preceding definitions or theorems. It may be regarded as expressing an independent property of Euclidean space; and thus, like the other postulates, it has the nature of a definition.

From the parallel-postulate, by the aid of theorem 21, the following theorem is derived, which is the converse of theorem 21, and establishes the characteristic property of parallel lines.

Equality of alternate angles.

117. THEOREM 22. *If two parallels are cut by a transversal, the alternate angles are equal.*

Let the parallels LL', MM' be met by the transversal OP.



To prove that the alternate angles LOP and $M'PO$ are equal.

Suppose, if possible, that one of them, say LOP, is the greater.

Draw the line NON' cutting off from the greater angle a part NOP equal to the less angle $M'PO$ (77).

Then NON' is parallel to MPM' (112).

Hence, the two intersecting lines are parallel to the same line; contrary to the postulate of parallels.

Therefore the supposition fails; that is to say, neither alternate angle is greater than the other; hence the alternate angles are equal.

Why could 117 not be derived directly from 112? What new geometric fact had to be introduced? Compare the statement at the end of 103.

Contraposite of theorem 22.

118. Cor. I. *If the alternate angles are not equal, the lines are not parallel; and they meet at that side of the transversal at which the smaller angle lies.*

The first part of this corollary is the contraposite of theorem 22, and is, therefore, true by 106. The student may, as an exercise, prove it by the method of exclusion. Compare 112, 118, and their two contraposites. How is the latter part of 118 related to 79?

119. Test of parallelism. Arts. 112 and 118 together show that the equality of the alternate angles is a complete test of parallelism. We could test conclusively whether two lines are parallel or not by comparing the alternate angles which they make with any transversal. If the angles are equal, the lines are parallel; and if the angles are not equal, the lines are not parallel. Such a complete test is sometimes called "a necessary and sufficient condition."

120. Sufficient condition. One statement is said to be a *sufficient condition* of another when the truth of the first is sufficient to insure the truth of the second; or, in other words, when the truth of the second is a necessary consequence of the truth of the first.

For instance, when we say that the statement "A is B" is a sufficient condition of the statement "C is D" we mean only that

When A is B, C is D.

Thus Art. 112 asserts that the equality of the alternate angles is a sufficient condition of parallelism.

121. Necessary condition. One statement is said to be a necessary condition of another when the second cannot be true unless the first is true; or, in other words, when the opposite of the second is a necessary consequence of the opposite of the first.

For instance, when we say that the statement "A is B" is a necessary condition of the statement "C is D" we mean only that

When A is not B, C is not D.

Thus Art. 118 asserts that the equality of the alternate angles is a necessary condition of parallelism.

Arts. 112 and 118 together show that the equality of the alternate angles is both a necessary and a sufficient condition of parallelism.

Ex. 1. If one statement is a sufficient condition of another, show that the latter is a necessary condition of the former.

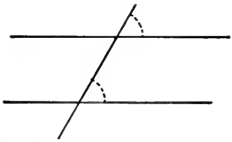
Ex. 2. If one statement is a necessary condition of another, show that the latter is a sufficient condition of the former.

Ex. 3. If one statement is a necessary and sufficient condition of another, then the latter is a necessary and sufficient condition of the former.

Ex. 4. Show that the statement "X equals Y" is a sufficient (but not a necessary) condition of the statement "X is greater than half Y"; and that the latter is a necessary (but not a sufficient) condition of the former.

Equality of corresponding angles.

122. Cor. 2. *If two parallels are cut by a transversal, the corresponding angles are equal; and conversely.*



Interior angles supplemental.

123. Cor. 3. *If two parallels are cut by a transversal, the two interior angles on the same side are together equal to a straight angle; and conversely.*

Interior angles not supplemental.

124. Cor. 4. *If two lines are cut by a transversal, and if the interior angles on the same side of the transversal are together less than a straight angle, then the lines will meet at that side of the transversal at which these angles are.**

What relation does 124 bear to 80? What relation does 124 bear to each of the parts of 123? Which of these four theorems are logical equivalents? Which of them are not logical equivalents?

* The statement in corollary 4 was adopted as a postulate by Euclid, and was placed at the foundation of his theory of parallels. The form stated in 116 was first given in Playfair's edition of Euclid's 'Elements of Geometry' (1813), and has been generally adopted by modern writers. An earlier suggestion of this form of the postulate is found in 'Rudiments of Mathematics,' by W. Ludlam, St. John's College, Cambridge (1794).

Another test of parallelism. Arts. 123 and 124 furnish a useful test as to whether two lines are parallel or not (compare 119). Art. 124 is often used to prove that certain lines will meet if prolonged in a certain way.

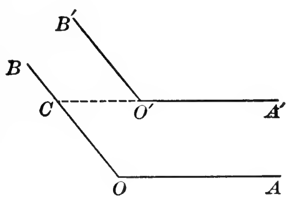
125. Cor. 5. *Lines that are parallel to the same line are parallel to each other.* (Use 122.)

Ex. A line perpendicular to one of two parallels is perpendicular to the other.

Angles having parallel sides.

126. THEOREM 23. *If two angles have the two sides of one respectively parallel to the two sides of the other (parallel lines being at the same side of the line joining the vertices of the angles), then the angles are equal.*

Let AOB, $A'O'B'$ be the angles having OA and $O'A'$ parallel and on the same side of OO'; and similarly for OB and $O'B'$.



To prove that the angles AOB and $A'O'B'$ are equal.

Let $O'A'$ meet OB (extended if necessary) in C.

The angles AOB, $A'O'B'$ are each equal to $A'CB$ (122).

Hence, they are equal to each other.

127 (a). Cor. 1. *If two angles have the two sides of one respectively parallel to the two sides of the other (parallel lines being at opposite sides of the line joining the vertices of the angles), then the angles are equal.*

127 (b). Cor. 2. *If two angles have the two sides of one respectively parallel to the two sides of the other (two of the parallels being at the same side of the line joining the vertices and the other two being on opposite sides of that line), then the angles are supplemental.*

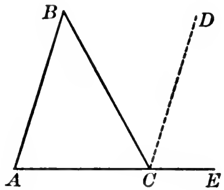
Theory of Parallels applied to Angle-sums

The following two theorems, with their inferences, illustrate how the theory of parallels may be used in the addition and subtraction of angles.

Sum of two angles of a triangle.

128. Theorem 24. *When any side of a triangle is extended, the exterior angle is equal to the sum of the two interior opposite angles.*

Let the side *AC* of the triangle *ABC* be extended to *E*.



To prove that the exterior angle *BCE* is equal to the sum of the opposite interior angles *CAB* and *CBA*.

Draw the line *CD* parallel to *AB* (114).

The angles *ECD* and *CAB* are equal (122).

Also the angles *DCB* and *CBA* are equal (117).

Therefore, by addition of equals, the angle *ECB* is equal to the sum of the angles *CAB* and *CBA*.

Ex. 1. When two lines are met by a transversal, the difference of two corresponding angles is equal to the angle between the two lines.

Ex. 2. The difference between two alternate angles is equal to what angle?

By how much does the sum of the two interior angles at one side of a transversal exceed the sum of the interior angles at the other side?

Ex. 3. In the figure of Art. 87, prove that the angle ADB equals half ACB.

Ex. 4. In the figure of Art. 81, prove that the difference of the angles ABC and ACB equals double DBC.

[Angle ABC equals the sum of ABD and DBC, which equals the sum of ADB and DBC, etc.]

Angle-sum in a triangle.

129. THEOREM 25. *The sum of the three interior angles of a triangle is equal to a straight angle.*

[Use the equality proved in 128; and add to both members the third interior angle.]

Ex. 1. In a right triangle the acute angles are complemental; and in an isosceles right triangle the acute angles are each equal to half a right angle.

Ex. 2. In any isosceles triangle each of the equal angles is equal to the complement of half the third angle.

Ex. 3. In an equilateral triangle each angle is equal to two thirds of a right angle.

Ex. 4. Show how to construct an angle equal to one third of a right angle. Hence show how to trisect a given right angle.

Ex. 5. Trisect a given straight angle.

130. Cor. *If two triangles have two angles of one equal to two angles of the other, then the third angles are equal.*

EXERCISES

1. Through a given point draw a line making with a given line an angle equal to a given angle.

2. If two lines are respectively perpendicular to two other lines, the angles formed by the first pair are respectively equal to the angles formed by the second pair.

3. Two lines perpendicular to two parallel lines, respectively, are parallel.

CONSTRUCTION OF TRIANGLES

131. There is a large class of problems involving the construction of triangles to satisfy certain prescribed conditions involving the sides and angles.

In a triangle the three sides and the three angles are called its six *parts*. For convenience the sides will be denoted by a, b, c and the opposite angles respectively by A, B, C. In a right triangle the side opposite the right angle is called the *hypotenuse*.

The simplest condition that can be imposed on the construction of a triangle is the assignment of certain line segments or angles to which some of the six parts are to be made equal. Those parts which are to be made equal to prescribed segments or angles are said to be *given* or *known*, and the remaining parts, about which nothing is prescribed, are said to be *unknown*.

In each of the five following problems three of the six parts are given, and it is required to find by a geometric construction a triangle answering to the prescribed conditions, and incidentally to determine the three unknown parts of the triangle.

The solution of such a problem has three divisions:

(1) To make the actual construction by means of processes that ultimately involve only the drawing of straight lines and of circular arcs.

(2) To prove by the use of previous propositions that the figure so constructed satisfies the prescribed conditions.

(3) To discuss the solution; *i.e.* to examine what limitations there are on the data so that it may be possible to satisfy the demands; and under what circumstances it will be possible to satisfy them in only one way, or in more than one way; and also to examine certain special and limiting cases.

When the data are such that the demands cannot be satisfied by any triangle, the problem is said to have *no solution.*

If they can be satisfied by one, and only one, type of triangle (*i.e.* if all the triangles fulfilling the stated conditions can be superposed), the problem is said to have a *unique solution.*

If the demands can be satisfied by two or some greater definite number of distinct types of triangles not capable of superposition, the problem is said to have a *determinate but ambiguous* solution.

If the demands can be satisfied by an indefinite number of triangles, the problem is said to have an *indeterminate solution.*

To prepare the way for the solution it is usually best to make a preliminary *analysis* of each problem. This may be described in a general way as follows:

Suppose the problem solved, and the required figure drawn. Mark the parts that are supposed to be equal to "given" lines or angles. Analyze the figure to discover the relations of the known and unknown parts. Draw any lines that may help to bring them into closer relations to each other. Observe which of the problems already solved could be used to construct the various parts subject to the given conditions. This is called "reducing the problem to previous ones."

After this reduction (or analysis) has been made, perform these simpler constructions in their proper order, thus building up the figure step by step. Such building-up process is called a *synthesis.* Then will follow the proof, and the discussion as already stated.

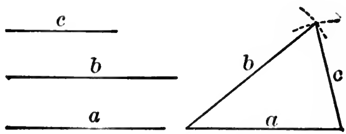
In many cases the preliminary analysis is so simple that it will not be given, but the student should always make such analysis before consulting the synthetic solution.

It is advisable to make some of the actual constructions with ruler and compasses; and it affords better geometrical training to dispense with all other mechanical aids.

Three sides given.

132. Problem 9. *To construct a triangle having its sides equal to three given lines.*

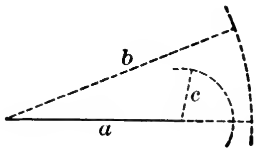
Let a, b, c, be the three given lines.



Draw any line equal to a. With its ends as centers, and radii equal to b, c respectively, describe circles. Join their point of intersection to the ends of a.

The triangle so formed fulfills the given conditions, since it has its sides respectively equal to the given lines.

Discussion. The limitations on the data, in order that a triangle satisfying the demands may exist, are that the sum of any two of the given segments must be greater than the third (87).



Show that this appears in the actual process of construction. (In the figure the sum of a and c is less than b, and the circles do not intersect.) Examine the limiting cases in which the sum of any two of the given lines is equal to the third.

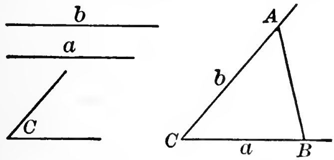
If the other intersection of the two circles had been selected; or if the side b had been taken first, and with its ends as centers, circles had been described with radii a and c; the different triangles so obtained could all be superposed upon the first by suitable movement (66).

Thus there is a unique solution when each of the given segments is less than the sum of the other two; and if any of these conditions be violated there is no solution.

Given two sides and the included angle.

133. PROBLEM 10. *To construct a triangle having two of its sides equal to two given lines, and the included angle equal to a given angle.*

Let a, b be the given lines and C the given angle.



On any line lay off a segment CB equal to a. Through one extremity C of this segment draw a line making an angle equal to the given angle C. On this line lay off from C a segment CA equal to b, and join AB.

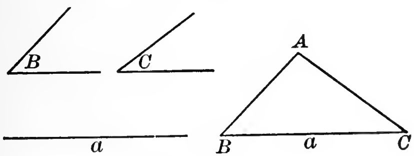
The triangle so formed has the prescribed parts.

Discussion. Show that the only limitation on the data is that the given angle must be less than a straight angle (129).

Show, as in 132, that the solution is unique (64).

Given two angles and the included side.

134. PROBLEM 11. *To construct a triangle having two of its angles equal to two given angles, and the included side equal to a given line.*



Make two applications of 77.

What limitation is there on the sum of the two given angles (129)?

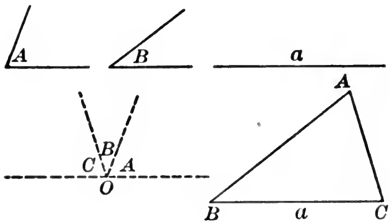
Show that the solution is unique.

Ex. Construct a right triangle being given one of the sides forming the right angle and either of the acute angles.

Given two angles and the side opposite one.

135. PROBLEM 12. *To construct a triangle having two of its angles equal to two given angles, and the side opposite to a specified one of these angles equal to a given line.*

Let A, B be the given angles and a the given line.



First add the angles A and B together, and subtract the sum from a straight angle. (This can be conveniently done by taking any line, then at any point O constructing an angle equal to A, and an adjacent angle equal to B.)

The remaining part of the straight angle is equal to the third angle, C, of the required triangle (129).

Then in the triangle ABC the angles B, C are known, and the included side a. Hence the triangle can be constructed as in 134.

Discussion. When is there no solution (129)? Is there any ambiguity (65, 95)?

Ex. 1. Construct an isosceles triangle, being given its base and opposite angle (129).

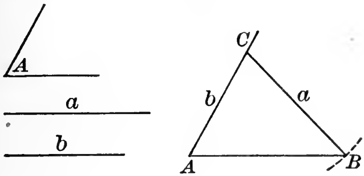
Ex. 2. Construct a right triangle, being given the hypotenuse and one acute angle.

Ex. 3. Draw a line parallel to the base of a triangle so that the portion intercepted between the sides may be equal to a given line.

Given two sides and the angle opposite one.

136. PROBLEM 13. *To construct a triangle having two of its sides equal to two given lines, and the* *angle opposite a specified one of these sides equal to a given angle.*

Let a, b be the given lines and A the given angle.



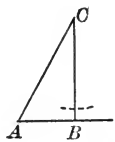
In any line AB take any point A; and draw AC making the angle BAC equal to the given angle (77). Lay off AC equal to that one of the given segments which is to be equal to the side adjacent to the given angle. With C as center and radius equal to the other given segment draw a circle cutting the line AB in the point B; and join CB.

The triangle so formed evidently fulfills the stated conditions.

Discussion. The discussion of this problem falls into three divisions according to the species of the given angle, and each division has a number of cases according to the greater or less magnitude of the two given segments. An examination of the different cases will be found very instructive.

I. *Let the given angle A be acute.*

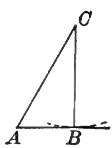
(1) In the process of construction let the segment a be less than the perpendicular p drawn from C to AB.



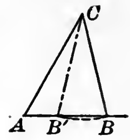
No solution is possible (85).

(2) Let a equal p.

There is one and only one triangle answering the conditions; because any triangle having the specified parts equal to a, b, A, is right angled, and is therefore superposable on the triangle ABC (98).



(3) Let a be greater than p and less than b.

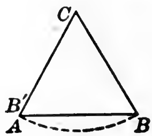


The arc meets AB in two points B, B', both situated on that portion of the line AB which forms one side of the given angle BAC. Hence there are two triangles ABC, $AB'C$, each having the specified parts equal to a, b, A.

The other parts c, B, C and c', B', C' are respectively unequal in the two triangles. In particular the angle B' in the triangle $AB'C$ is the supplement of the angle B in the triangle ABC.

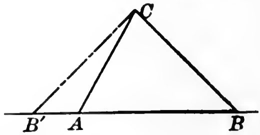
The stated conditions are not satisfied by any other type of triangle; because any other triangle having the specified parts equal to a, b, A, would have the angle opposite the side b equal either to the angle B or to its supplement (97), and is therefore superposable on one or the other of the triangles ABC, $AB'C$ (65).

Thus there are two and only two solutions.



(4) Let a equal b.

The point B' coincides with A. Show that the solution is unique (64, 65).



(5) Let a exceed b.

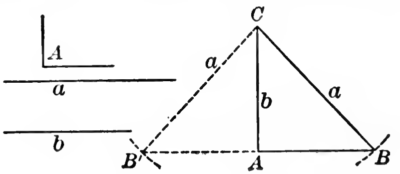
Show that B, B' fall at opposite sides of A, and that the solution is unique.

II. *Let the given angle A be a right angle.*

(1) Let a be less than b, or a equal b. No triangle.

(2) Let a exceed b.

The points B, B' are at opposite sides of A; and the triangles ABC, $AB'C$ each answer the conditions.



Show that these triangles are superposable; that so are all other triangles fulfilling the stated conditions; and that the solution is then unique.

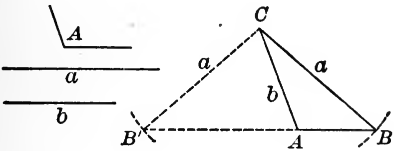
III. *Let the given angle A be obtuse.*

(1) Let a be less than b, or a equal b. No solution.

Show that this conclusion could also be drawn from the fact that there cannot be two obtuse angles in a triangle (80).

(2) Let a exceed b.

The points B, B' are on opposite sides of A, and the triangle $AB'C$ is not a solution.



Any other triangle having the specified parts equal to a, b, A, would have the angle opposite b equal either to B or to its supplement (97). In the former case the triangle would be superposable on ABC. In the latter case the triangle could not exist since the supplement of B is obtuse, and there cannot be two obtuse angles in a triangle.

Hence there is but one type of triangle answering the requirements, and the solution is unique.

Ex. 1. Give a summary of the limitations there are on the data in order that any solution may be possible.

Ex. 2. Summarize the circumstances under which there is a unique solution.

Ex. 3. When is the solution ambiguous? Quote the previous theorem from which it is inferred that there can be no third type of triangle fulfilling the conditions.

137. Determinate and indeterminate solutions. It has perhaps been noticed that the assigned conditions in these five problems of construction of a triangle correspond respectively to the five conditions of equality of two triangles (64–66, 95, 97); and that the determinateness of the solution is in each case tested by applying the corresponding condition of equality.

For instance, the solution of 134 is uniquely determinate, because any two triangles that have two sides and the included angle in each respectively equal, are superposable by the first condition of equality.

Again, the solution of 136 is in one case ambiguously determinate, because any two triangles that have two sides and the angle opposite one of them in each equal have the angles opposite the other equal sides either equal or supplemental, and because there is (in the case referred to) no way of deciding the *species* of the angle in question from previous principles.

In each of these problems three parts were given. If only two parts are assigned, an indefinite number of distinct types of triangles answering the requirements can be constructed.

For instance, if only two sides are given, the included angle can be assumed at pleasure, and an indefinite number of distinct triangles can be formed so as to have the given sides.

Similarly if only two angles are given, the adjacent side can be assumed arbitrarily, and the solution is indeterminate.

The three angles, however, do not constitute three independent data; for then nothing more is given than when only two angles are assigned (as the third angle could be found by subtracting the sum of the first two from a straight angle).

Thus the problem to construct a triangle having its three angles equal to three assigned angles, is either impossible or indeterminate; impossible if the sum of the three given angles is not equal to a straight angle; indeterminate if it is.

EXERCISES

1. Given base, vertical angle, and sum of sides, construct the triangle.

Outline. In the figure of Art. 87, in which AD is the sum of the sides AC and CB, show that the angle ADB is half the vertical angle ACB; and hence that the triangle ADB can be constructed from the data (136). Then show how to construct the triangle ABC.

2. Given base, vertical angle, and difference of sides, construct the triangle.

Outline. In the figure of Art. 81, in which CD is the difference of sides, show that the angle CDB equals the sum of the vertical angle and half its supplement. Construct the triangle CDB by 136, and then the triangle ABC.

3. Given base, difference of sides, and difference of base angles, construct the triangle.

[Use 128, ex. 4.]

4. Given base, an adjacent angle, and the sum (or difference) of the other two sides, construct the triangle.

5. Given the angles and the perimeter, construct the triangle.

Analysis. Prolong base BC both ways, making BB' equal BA, and CC' equal CA. Prove $B'C'$ equal to perimeter; and angle B' equal half B, etc. Give synthesis.

6. Given the angles and the sum (or difference) of two sides, construct the triangle.

7. In the figure of Art. 81, prolong CA until AE equals AB, and draw EB; prove that angle DBE equals the sum of DEB and BDE, and is a right angle.

8. Construct a triangle, being given the base, the sum of sides, and the difference of the base angles. [Use ex. 7.]

9. Construct an equilateral triangle such that the perpendicular from the vertex to the base may be equal to a given line.

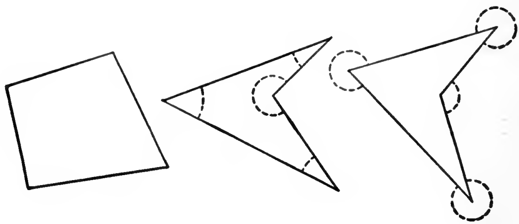
QUADRANGLES

Attention has hitherto been given to various properties of the plane figures formed by two or three straight lines. The figure that next presents itself is that formed by four lines each of which meets the next one in order.

138. Definitions. A plane figure formed by four line-segments that inclose a portion of the plane surface is called a *quadrilateral figure*, or a *quadrangle.*

These line-segments are called the *sides,* and their extremities the *vertices* of the quadrangle.

The angles formed by adjacent sides, and situated toward the interior of the boundary, are called the *interior angles* of the quadrangle, or briefly the *angles.*



The exterior angles conjunct to these will be called for brevity the *conjunct angles.*

A concave angle formed by one side and the prolongation of an adjacent side is called an *exterior angle.*

If all of the conjunct angles are convex (21), the quadrangle is called *convex.*

If one of the conjunct angles is concave, the figure is said to be *concave* at that angle.

In a convex quadrangle all the interior angles are concave, and no side when prolonged traverses the figure; but a concave quadrangle has one of the interior angles convex, and the sides of this angle traverse the figure when prolonged.

A line connecting two non-adjacent vertices is called a *diagonal.*

The sum of the sides is called the *perimeter,* and the sum of the angles the *angle-sum.*

In a convex quadrangle the sum of the exterior angles formed by prolonging each side one way, no two adjacent sides being prolonged through the same vertex, is called the *exterior angle-sum.*

The four sides and four angles are called the eight *parts* of the quadrangle.

Primary relations of parts.

139. Relation 1. *The sum of any three sides is greater than the fourth* (89).

140. **Cor.** *The sum of any two sides is greater than the difference of the other two.*

141. Relation 2. *The angle-sum is equal to a perigon.*

[Divide the quadrangle into two triangles by a diagonal and apply 129.]

142. **Cor. 1.** *A conjunct angle is equal to the sum of the three non-adjacent interior angles.*

143. **Cor. 2.** *Only one of the interior angles in a quadrangle can be convex.*

144. **Cor. 3.** *If two quadrangles have three angles of one equal respectively to three angles of the other, the remaining angles are equal.*

Ex. 1. The sum of the four sides of a quadrangle is greater than the double of either diagonal, and greater than the sum of the diagonals.

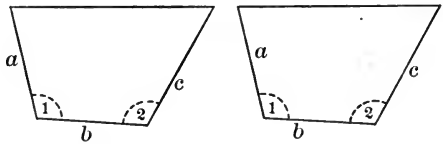
Ex. 2. The sum of the four interior angles is equal to one third of the sum of the four conjunct angles.

Some Conditions of Equality

The following three theorems relate to the equality of two quadrangles under certain conditions.

Three sides and two included angles.

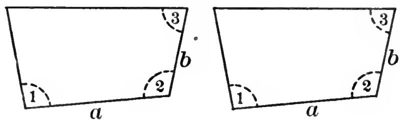
145. Theorem 26. *If two quadrangles have three sides and the two included angles of one equal to the corresponding parts in the other, the figures are equal.*



Outline. Superpose the equal parts, and show that the coincidence of the remaining parts will follow as in 64.

Two adjacent sides and three angles.

146. Theorem 27. *If two quadrangles have two adjacent sides and any three angles of one equal to the corresponding parts in the other, the figures are equal.*

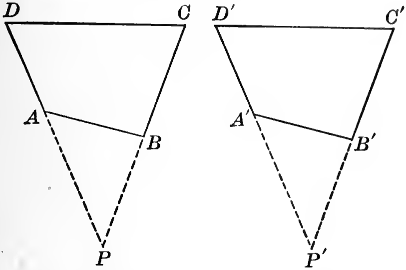


Outline. The remaining angles are equal (144). Superpose the equal parts, and show that the remaining parts will coincide.

Two opposite sides and three angles.

147. Theorem 28. *If two quadrangles have two opposite sides and any three angles of one equal to the corresponding parts in the other, the figures are equal.*

In the quadrangles $ABCD$ and $A'B'C'D'$, let AB equal $A'B'$, CD equal $C'D'$. Also let angle A equal A', B equal B', C equal C', and consequently D equal D' (144).



To prove that the quadrangles are equal.

Prolong BC and AD to meet in P; also $B'C'$ and $A'D'$ to meet in P'.

The angle PCD equals $P'C'D'$, and CDP equals $C'D'P'$.

Therefore the triangles PCD and $P'C'D'$, having a side and the two adjacent angles in each equal, are themselves equal.

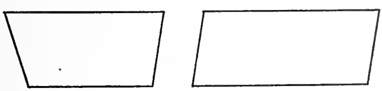
For a similar reason the triangle PBA equals $P'B'A'$.

By subtraction of equals from equals the line BC equals $B'C'$, and AD equals $A'D'$.

Hence the quadrangles are equal by the preceding theorem.

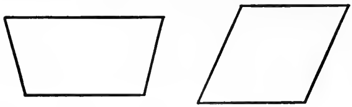
Special Kinds of Quadrangles

148. Definitions. A quadrangle that has a pair of its opposite sides parallel, and the other pair not parallel, is called a *trapezoid.* One that has both pairs of opposite sides parallel is a *parallelogram.*



In contradistinction a quadrangle that has neither pair of sides parallel is called a *trapezium.*

A trapezoid whose opposite non-parallel sides are equal is said to be *isosceles.*

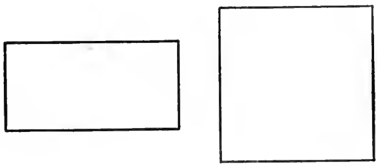


A parallelogram that has two adjacent sides equal is called a *rhombus.*

[It is shown later (154) that the four sides of a rhombus are equal.]

A parallelogram that has one of its angles right is called a *rectangle.*

[It will appear (150) that all the angles of a rectangle are right angles.]



A rectangle that has two adjacent sides equal is a *square.*

[It is shown later (154) that all the sides of a square are equal. A square is at once a rhombus and a rectangle.]

PARALLELOGRAMS AND TRAPEZOIDS

Theorems 29–36 with their corollaries establish the principal properties of parallelograms and trapezoids.

Angles of a parallelogram.

149. THEOREM 29. *Any two consecutive angles of a parallelogram are supplemental; and any two opposite angles are equal.*

[Apply 126 and corollaries, or prove independently by the method of that article.]

150. Cor. *All the angles of a rectangle are right angles.*

Ex. Show that in a trapezoid the sum of a certain pair of angles is equal to the sum of the other pair.

Converse of last theorem.

151. THEOREM 30. *A quadrangle that has both pairs of opposite angles equal is a parallelogram.*

Outline. The sum of two consecutive angles is equal to the sum of the other two, and is therefore equal to a straight angle (141); hence the opposite sides are parallel (113).

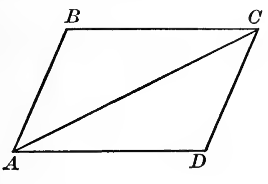
Ex. State and prove the converse of the first part of theorem 29.

152. Cor. *If the sum of one pair of angles of a quadrangle is equal to the sum of the other pair, the figure is a trapezoid.*

Sides of a parallelogram.

153. THEOREM 31. *In a parallelogram the opposite sides are equal, and a diagonal divides the figure into two equal triangles.*

Let *ABCD* be a parallelogram, and *AC* its diagonal.



To prove that *AB* equals *CD*, *BC* equals *AD*, and that the triangle *ABC* equals *ACD*.

Comparing the triangles *ABC* and *CDA*, the angles *BAC* and *ACD* are equal (117); the angle *BCA* equals *DAC*; and the side *AC* is common; therefore the triangles are equal (65), and hence *AB* equals *CD*, and *BC* equals *AD*.

154. Cor. I. *All the sides of a rhombus are equal; and so are all the sides of a square.*

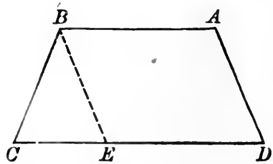
155. **Cor. 2.** *Parallel lines intercept equal segments on parallel lines.*

Ex. 1. Show how to construct a square on a given line as side.

Ex. 2. Show how to construct a square on a given line as diagonal.

Ex. 3. To construct a rhombus, being given one side and one angle.

Ex. 4. To construct a rectangle so that two adjacent sides may be equal to two given lines. Show that there is only one solution.



Ex. 5. In an isosceles trapezoid the equal sides make equal angles with each of the other sides.

Outline. Draw BE parallel to AD. Prove BE, AD, BC all equal. Hence, prove angle ADC equal to BCD.

Converse of theorem 31.

156. THEOREM 32. *In a quadrangle, if the opposite sides are equal, the figure is a parallelogram.*

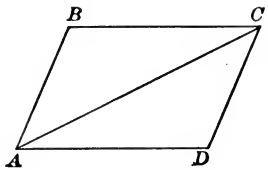
[Draw a diagonal. Prove triangles equal, angles equal, and lines parallel.]

Ex. State and prove the converse of the second part of theorem 31.

Two sides equal and parallel.

157. THEOREM 33. *If a quadrangle has one pair of sides equal and parallel, the figure is a parallelogram.*

Let the quadrangle $ABCD$ have BC equal and parallel to AD.



To prove that $ABCD$ is a parallelogram.

Draw a diagonal AC.

Since the parallel lines BC, AD are met by a transversal AC, the alternate angles BCA, DAC are equal (117).

Then in the triangles *ABC* and *CDA*, the side *BC* equals *AD* (hyp.); the side *AC* is common; and the included angles *BCA* and *CAD* are equal; therefore the triangles are equal (64).

Thus the angles *BAC* and *ACD* (opposite equal sides) are equal; hence *AB* is parallel to *CD* (112).

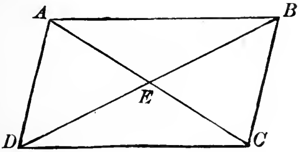
Therefore *ABCD* is a parallelogram (def.).

158. Cor. *The lines joining the adjacent extremities of equal and parallel lines are themselves equal and parallel.*

Ex. If the parallel sides of a trapezoid are equal, it is a parallelogram.

Diagonals.

159. THEOREM 34. *The diagonals of a parallelogram bisect each other.*



[Compare two triangles formed by the diagonals and a pair of opposite sides (as *ABE*, *CDE*); and thus prove the sides opposite equal angles equal.]

160. Cor. 1. *The diagonals of a rhombus bisect each other at right angles.*

161. Cor. 2. *The diagonals of a rhombus bisect its angles.*

Converse of theorem 34.

162. THEOREM 35. *If the diagonals of a quadrangle bisect each other, the figure is a parallelogram.*

[Compare the two pairs of vertically opposite triangles, and apply 64, 112.]

Ex. 1. If the diagonals of a quadrangle bisect each other at right angles, the figure is a rhombus.

Ex. 2. The diagonals of a rectangle are equal.

Ex. 3. Conversely, if the diagonals of a parallelogram are equal, it is a rectangle.

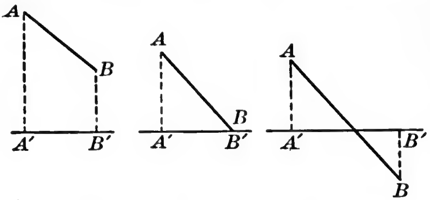
Ex. 4. The diagonals of an isosceles trapezoid are equal.

Ex. 5. Any line drawn through the intersection of the diagonals of a parallelogram divides it into two equal trapezoids. [Apply 145.]

Projections.

163. Definition. The *projection* of a point upon a line is the foot of the perpendicular from the point to the line.

The projection of a line-segment upon a line is the segment between the projections of its extremities.



In these figures the segment $A'B'$ is the projection of AB.

164. Cor. *If two lines are equal and parallel, then their projections upon any other line are equal.*

[Consider first the case in which the third line is parallel to the others. Next, when this is not the case, show that the equal and parallel lines can be made the hypotenuses of equal right triangles whose bases are the projections.]

Ex. 1. Parallel lines that have equal projections on another line are equal.

Ex. 2. Equal lines that have equal projections on another line make equal angles with it, or else they are parallel to it.

Equal trapezoids, or parallelograms.

165. Theorem 36. *Two trapezoids are equal if they have two adjacent sides and any two opposite angles of one equal to the corresponding parts in the other.*

Outline. Show that the remaining angles are respectively equal, and that the figures can be superposed as in 146.

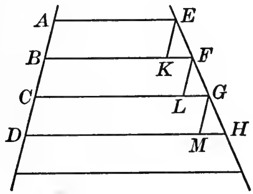
166. Cor. *Two parallelograms are equal if they have two adjacent sides and any angle of one equal to the corresponding parts in the other.*

Ex. Show how to construct a parallelogram, being given two adjacent sides and one angle. Prove that there is always one, and only one, solution.

SERIES OF PARALLELS

167. THEOREM 37. *If a series of parallel lines intercept equal segments on any one transversal, then they intercept equal segments on any other transversal.*

Let the parallels make intercepts AB, BC, CD, etc., on the first transversal, and EF, FG, GH, etc., on the second; and let the first set of intercepts be equal.



To prove that the second set of intercepts are equal.

Draw EK, FL, GM, etc., parallel to the first transversal, and terminated by the successive parallels.

Since EK and AB are parallels intercepted between parallels, they are equal (155). Similarly FL equals BC; GM equals CD; etc.

Now AB, BC, CD, etc., are all equal by hypothesis, therefore EK, FL, GM, etc., are all equal.

Also in the triangles EKF, FLG, GMH, etc., the angles KEF, LFG, MGH, etc., are all equal (122); and the angles EFK, FGL, GHM, etc., are all equal (122).

Therefore these triangles are all equal; and hence EF, FG, GH, etc., are all equal.

168. Definition. A series of parallels (as in 167) that intercept equal segments on any transversal is called a *regular series of parallels.*

169. Cor. 1. *If a regular series of parallels is cut by two transversals, the segments intercepted on consecutive parallels have a common difference.*

170. Cor. 2. *The line drawn through the middle point of one of the non-parallel sides of a trapezoid parallel to the pair of parallel sides bisects the remaining side.*

171. Cor. 3. *The line joining the mid-points of the non-parallel sides of a trapezoid is parallel to the other sides.*

Prove by *reductio ad absurdum*, using cor. 2 and 116.

172. Cor. 4. *The line joining the mid-points of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.*

The three parallels have a common difference (169). Apply 72.

173. Cor. 5. *The line joining the middle points of the sides of a triangle is parallel to the third side; and equal to its half.*

174. Cor. 6. *If one side of a triangle is divided into any number of equal parts, and if through the points of division parallels are drawn to a second side, then these parallels divide the third side into equal parts.* (Prove by 167.)

175. Definitions. A line is said to be *trisected* if it is divided into three equal parts. Thus a line has two points of trisection.

One line is said *to trisect* another when the first line passes through one of the points of trisection of the other.

Ex. 1. Apply 174 to trisect a given line.

Outline. Let AB be the given line. Draw another line AN at a convenient angle. Lay off any three equal successive segments AL, LM, MN. Join NB. By 114 draw through L and M parallels to NB. Prove by 174 that these parallels trisect the line AB.

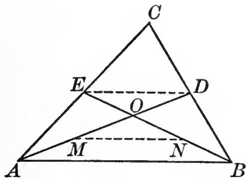
Ex. 2. The lines joining the mid-points of adjacent sides of a quadrangle form a parallelogram (173).

Ex. 3. The mid-points of a pair of opposite sides of a quadrangle and the mid-points of the diagonals are the vertices of a parallelogram.

Medians of a triangle.

176. Definition. In a triangle the line joining any vertex to the mid-point of the opposite side is called a *median line* (or median) of the triangle.

177. THEOREM 38. *Two medians of a triangle trisect each other.*



Outline. In the triangle ABC let the medians AD and BE intersect in O.

Let M and N be the mid-points of AO and BO. Show that MN is equal and parallel to DE (173); hence that MD is bisected at O, and AD trisected at O.

Concurrence of three medians.

178. Cor. *The three medians of a triangle meet in a common point, and trisect each other.*

179. Definition. The point of concurrence of the three medians is called the *median center* of the triangle.

Ex. Construct a triangle whose medians shall be equal to three given lines.

Outline of analysis. Prolong the median COF to G so that FG equals OF. Join AG, BG. Prove $AOBG$ a parallelogram. Show that the triangle AOG has its sides equal respectively to two thirds of the medians; and that the problem is thus reduced to a previous one. Give the complete construction, and proof.

Construction of Quadrangles

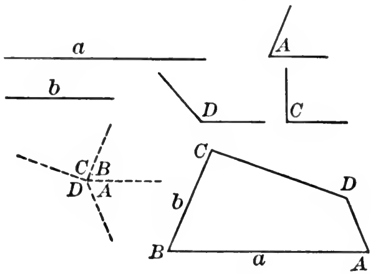
180. The following problems are solved by an extension of the methods exemplified in 131–136. The student should make the usual "preliminary analysis" as described in 131, and also endeavor to perform the synthetic construction before consulting the solution given for any problem.

The vertices of the quadrangle will be lettered A, B, C, D, in order; and the consecutive sides AB, BC, CD, DA will be denoted by the small letters a, b, c, d, in order.

Given two adjacent sides and three angles.

181. Problem 14. *To construct a quadrangle having two adjacent sides equal to two given lines and three angles equal to three given angles, the order in which the five parts are to be taken being specified.*

Let a, b be the given lines, and A, C, D, the given angles.



The fourth angle B equals the conjunct of the sum of A, C, D (141), which is obtained by subtracting the sum of A, C, D from a perigon as shown in the figure.

Lay off AB equal to a; construct angle ABC equal to B; lay off BC equal to b. At C, A draw lines CD, AD making angles BCD, BAD equal respectively to the given angles.

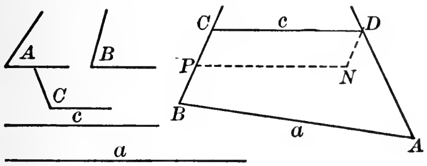
The quadrangle $ABCD$ evidently fulfills the requirements.

Discussion. The limitation on the data is that the sum of the three given angles must be less than a perigon (146).

Given two opposite sides and three angles.

182. Problem 15. *To construct a quadrangle having two opposite sides equal to two given lines, and three angles equal to three given angles.*

Let a, c be the given lines, and A, B, C, the given angles.



Take AB equal to a, and make angles ABC equal to B, BAD equal to A. On BC take any point P; make angle BPN equal to C; take PN equal to c; draw ND parallel to BC, meeting AD in D; and draw DC parallel to NP, meeting BC in C.

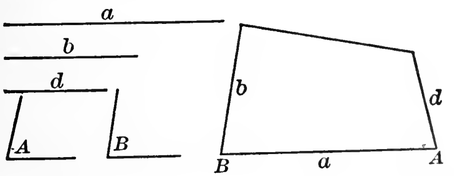
The quadrangle $ABCD$ thus formed has the specified parts equal to the given parts.

Prove that angle BCD equals C, and that side CD equals c.

Discussion. State limitation on data. Prove the solution unique (147).

Given three sides and two included angles.

183. Problem 16. *To construct a quadrangle having three sides equal to three given lines, and the two included angles equal to two given angles.*

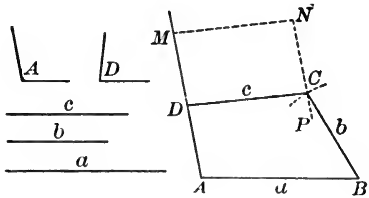


Construct as in 134. State limitation. Prove solution unique (145).

Given three sides and two angles adjacent to fourth.

184. PROBLEM 17. *To construct a quadrangle having three sides equal to three given lines and the two angles adjacent to the fourth side equal to two given angles.*

Let a, b, c be the given lines, and A, D, the given angles.



Take AB equal to a; make angle BAM equal A; take any point M in the line AM, and make the angle AMN equal D; lay off MN equal c, and draw NP parallel to AM. With B as center and radius equal to b, draw an arc cutting NP in C; draw CD parallel to NM.

The quadrangle $ABCD$ has then the prescribed parts.

Prove CD equal to c, and angle ADC equal to D.

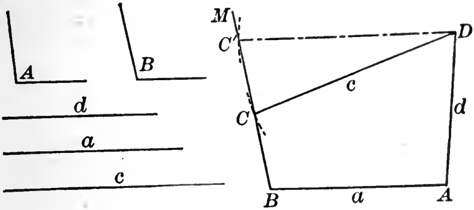
Discussion. In certain cases there are two solutions to the problem, viz. when the arc described with B as center and radius equal to b meets NP in two points both situated within the angle DAB.

Show when there is a unique solution, and when none.

Given three sides and two consecutive angles.

185. PROBLEM 18. *To construct a quadrangle having three sides equal to three given lines and two consecutive angles equal to two given angles, one of the consecutive angles being adjacent to the fourth side, and the other not.*

Let the parts a, c, d, A, B, be given.



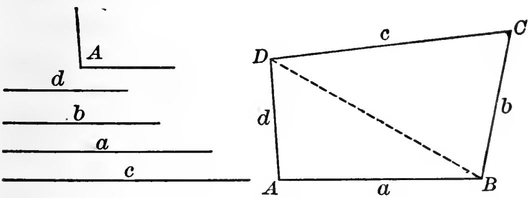
Construct in succession DA, DAB, AB, ABM respectively equal to the given parts. With radius equal to c and center D, describe an arc cutting BM in C, C'.

State when there is no solution, when two solutions, when only one.

Given four sides and an angle.

186. Problem 19. *To construct a quadrangle having four sides equal to four given lines and one angle equal to a given angle.*

Let a, b, c, d be the given lines, and A the given angle.



Let it be required to construct a quadrangle whose sides taken in order may be equal to a, b, c, d, and such that the sides a and d may contain an angle equal to A.

Construct the triangle ABD having AB equal to a, AD equal to d, and the included angle DAB equal to A (133). Next on BD construct the triangle BCD having BC equal to b, and CD equal to c (132).

Show by suitable figures that there may be two solutions, only one solution, or no solution.

NOTE. The case in which three sides and two opposite angles are given is postponed (III, 198).

Ex. 1. Compare the data in the three unique solutions above with the three conditions of equality (145–147).

Ex. 2. Construct a trapezoid being given two adjacent sides, the included angle, and the angle opposite to the latter. Which case does this come under ? Discuss the solution.

EXERCISES

1. Draw a line such that its segment intercepted between two given fixed indefinite lines shall be equal and parallel to a given finite line.

2. Draw a line parallel to the base of a triangle, cutting the sides so that the sum of the two segments adjacent to the base shall be equal to a given line.

Analysis. Let PB, QC be the two segments. Draw PD parallel to QC. Then in the triangle BPD, the angles and the sum of two sides are given (137, ex. 6). Give synthesis.

3. Construct a parallelogram being given two adjacent sides and a diagonal.

4. Construct a parallelogram being given a side and two diagonals.

5. Inscribe a rhombus in a triangle having one of its angles coincident with an angle of the triangle.

6. One angle of a parallelogram is given in position and the point of intersection of the diagonals is given ; construct the parallelogram.

7. If the diagonals of a quadrangle bisect its angles, then it is a rhombus.

8. The perimeter of a quadrangle is greater than the sum of its diagonals.

9. The sum of two sides of a triangle is greater than double the median drawn to the third side ; and the perimeter of the triangle is greater than the sum of the three medians.

10. Given two medians and their included angle, construct the triangle.

POLYGONS

This section considers the figure formed by any number of lines each of which meets the next in order, and generalizes some of the results obtained in the preceding sections.

187. Definitions. A plane figure composed of segments of straight lines that inclose a portion of the plane surface, is called a *polygon*.

These segments are called the *sides*, their extremities the *vertices*, and their sum the *perimeter*, of the polygon.

A line joining any two non-adjacent vertices is called a *diagonal*.

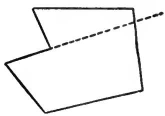
The angles formed by consecutive sides, and situated towards the interior of the boundary, are called the *interior* angles of the polygon.

The exterior angles conjunct to these will be called for brevity the *conjunct angles*.

If all of the conjunct angles are convex, the polygon is called a *convex polygon*.

If one of the conjunct angles is concave, the polygon is said to be *concave* at that angle.

In a convex polygon each of the interior angles is concave, and its exterior conjunct angle is convex.



A concave polygon has at least one of the conjunct angles concave, and the corresponding interior angle convex. The sides of this angle traverse the figure if prolonged.

In any polygon the concave angle formed by one side and the prolongation of an adjacent side is called an *exterior angle* of the polygon.

A polygon whose sides are all equal is *equilateral*, and one whose angles are all equal is *equiangular*.

A polygon which is both equilateral and equiangular is *regular*.

Two polygons that have the sides of one respectively equal to the sides of the other, taken in order, are said to be *mutually equilateral*, or one is said to be equilateral to the other.

Two polygons that have the angles of one respectively equal to the angles of the other, taken in order, are said to be *mutually equiangular*, or one is said to be equiangular to the other.

In two mutually equiangular polygons the vertices of equal angles are said to *correspond;* and the sides joining corresponding vertices are called *corresponding sides.*

The two polygons are said to be *directly equiangular* if the sides of two corresponding angles can be brought into coincidence in such a way that their corresponding sides may coincide, without turning either polygon out of the plane; otherwise the polygons are said to be *obversely equiangular* to each other.

Two equal polygons are said to be *directly superposable* when they can be superposed without turning either polygon out of its plane; otherwise the equal polygons are said to be *obversely superposable.*

The number of interior angles in a polygon is equal to the number of sides.

A polygon of five sides is called a *pentagon,* of six sides a *hexagon,* of seven sides a *heptagon,* of eight sides an *octagon,* of nine sides a *nonagon,* of ten sides a *decagon.* A twelve-sided polygon is called a *dodecagon,* and a fifteen-sided one a *pentadecagon.* In the discussion of general properties, a polygon of n sides is called an *n-gon.*

The sum of the n interior angles is called the *interior angle-sum.*

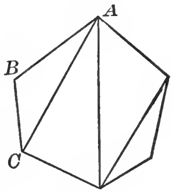
In a convex polygon, the sum of the exterior angles formed by prolonging each side one way, no two adjacent sides being prolonged through the same vertex, is called the *exterior angle-sum.*

General Properties of Polygons

The following preliminary general theorems will be of frequent use in the theory of the polygon.

Division into triangles by diagonals.

188. Theorem 39. *In any n-gon if all possible diagonals are drawn in any manner, except that no two intersect within the polygon, then there will be $n-3$ such diagonals, and the n-gon will be divided into $n-2$ triangles.**



Let the diagonals be drawn as stated. Begin with a diagonal, such as AC, that joins two alternate vertices, and call this the first diagonal. This first diagonal cuts off one triangle from the n-gon and leaves an $(n-1)$-gon. Similarly some second diagonal cuts off a second triangle from this and leaves an $(n-2)$-gon. A third diagonal cuts off a third triangle from the latter and leaves an $(n-3)$-gon, and so on. When $n-3$ diagonals are so drawn, there are $n-3$ triangles cut off, and there is left an $[n-(n-3)]$-gon, that is a 3-gon, or triangle. Thus there are $n-3$ diagonals and $n-3+1$ triangles. Hence the n-gon is divided into $n-2$ triangles.

Note. The student who may not be familiar with algebraic symbols may apply this method of reasoning to the special case of the hexagon or heptagon.

Another mode of proof consists in beginning with a single triangle, and then adding other triangles, so as to form in succession a quadrangle, a pentagon, a hexagon, etc.

* The symbol $n-2$ is read *n minus* 2, and stands for the number which is 2 units less than n.

Interior angle-sum.

189. THEOREM 40. *The sum of the interior angles of any n-gon is equal to $n-2$ straight angles.*

[Use 188 and 129.]

Ex. 1. An internal angle of an equiangular hexagon is equal to twice the angle of an equilateral triangle; and that of a regular octagon is equal to a right angle and a half.

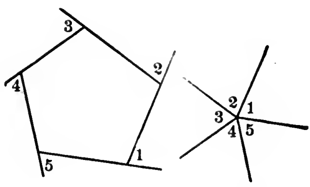
Ex. 2. The angle of a regular dodecagon is equal to five sixths of a straight angle; that is, equal to the angle of a square together with the angle of an equilateral triangle.

Exterior angle-sum.

190. **Cor.** *If each side of a convex n-gon is prolonged one way, no two adjacent sides being extended through the same vertex, then the sum of the exterior angles so formed is equal to a perigon.*

For all the exterior angles with all the interior angles together make up n straight angles; but the interior angles alone make up $n-2$ straight angles; hence the exterior angles are together equal to two straight angles, and therefore equal to a perigon.

Ex. 1. Give an independent proof by applying 126 as indicated in figure.



Ex. 2. An exterior angle of an equiangular hexagon is equal to an interior angle of an equilateral triangle.

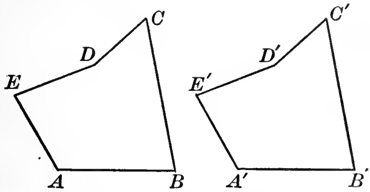
Ex. 3. The exterior angle of a regular dodecagon is equal to half the angle of an equilateral triangle.

EQUALITY OF POLYGONS. — PRIMARY CASES

The following four theorems relate to the primary conditions of equality of two polygons.

Mutually equilateral and equiangular.

191. THEOREM 41. *If two polygons are mutually equiangular, and have the corresponding sides equal, the polygons are equal.*

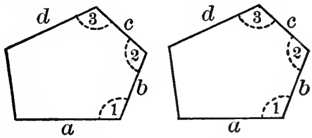


[Show that the polygons are either directly or obversely superposable.]

NOTE. In this theorem more conditions are given than are necessary to insure equality. This will be evident from the next two theorems.

n — 1 sides and n — 2 included angles.

192. THEOREM 42. *If two n-gons have $n-1$ sides of one equal respectively to $n-1$ sides of the other (taken in order), and the $n-2$ interior angles formed by the first set equal to the corresponding angles formed by the second set, then the polygons are equal.*



Outline. Bring the equal parts into coincidence as in 145, and show that this will necessitate the coincidence of the remaining side and the two remaining angles of one polygon with their corresponding parts in the other.

$n-1$ angles and $n-2$ intervening sides.

193. THEOREM 43. *If two n-gons have $n-1$ angles of one equal to the corresponding angles of the other, and the $n-2$ sides situated between the vertices equal to the corresponding sides in the other, then the polygons are equal.*

Outline. Show from the given conditions that $n-2$ consecutive sides can be brought into coincidence with their corresponding parts, and that the equality of the angles necessitates the coincidence of the two remaining sides of one polygon with the corresponding sides of the other.

$n-1$ angles and $n-2$ consecutive sides.

194. THEOREM 44. *If two n-gons have $n-1$ angles of one equal to corresponding angles of the other, and any $n-2$ consecutive sides of one equal to the respective like-placed sides of the other, then the polygons are equal.*

Outline. Show that the remaining angle of one is equal to the remaining angle of the other; and then apply the preceding theorem to prove that the polygons are equal.

CONSTRUCTION OF POLYGONS

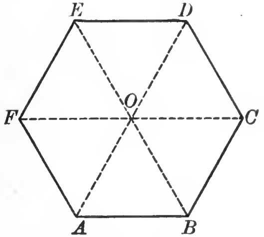
Problems 20–27 are concerned with the construction of a polygon that shall conform to certain prescribed conditions relating to the magnitude and position of some or all of its parts.

SOME REGULAR POLYGONS

Some of the preceding general principles will be used to construct regular polygons of six, eight, and twelve sides, with the aid of previous problems. It would be a good exercise for the student to analyze the constructions down to their simplest elements, namely those authorized in the construction postulates. Some other regular polygons will be considered in Book III.

195. Problem 20. *On a given line to construct a regular hexagon.*

Let AB be the given line on which a regular hexagon is to be constructed.

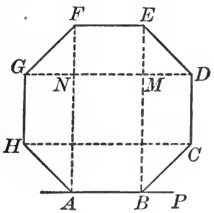


On AB construct the equilateral triangle AOB; continue AO, BO until OD equals AO, OE equals OB; join ED. Draw BC, DC parallel to OD, OB; and draw AF, EF parallel to OE, OA. The figure $ABCDEF$ is a regular hexagon.

Outline proof. Show that each of the triangles whose vertices are at O is equal to the triangle AOB; and hence that the hexagon is equilateral and equiangular.

196. Problem 21. *On a given line to construct a regular octagon.*

Let AB be the given line on which a regular octagon is to be constructed.



Since the angle of a regular octagon is equal to a right angle and a half (189), the figure may be constructed as follows:

Prolong AB to P, and draw BM perpendicular to AB; draw BC bisecting the right angle PBM, and lay off BC equal to

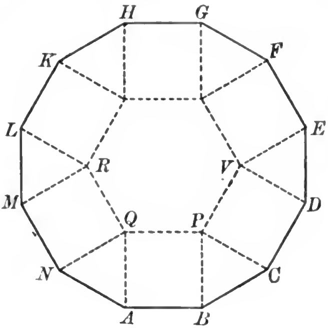
AB. Similarly construct the side *AH*; draw *CD*, *HG* parallel to *BM*, and take *CD* and *HG* each equal to *AB*. Draw *GF*, *DE*, making the angles *NGF*, *MDE* each equal to half a right angle, and connect *FE*.

The figure *ABCDEFGH* is a regular octagon.

Outline proof. Show that each angle is equal to a right angle and a half. Also prove the sides equal.

197. Problem 22. *On a given line to construct a regular dodecagon.*

Let *AB* be the given line on which a regular dodecagon is to be constructed.



Since the angle of a regular dodecagon is equal to the angle of a square plus the angle of an equilateral triangle, the figure may be constructed as follows:

On the given line *AB* construct a square *ABPQ*. On *BP* and *AQ* construct equilateral triangles *BPC*, *AQN*. On *PC* and *QN* describe squares *PCDV*, *QNMR*. On *VD* and *MR* construct equilateral triangles *VDE*, *MRL*; and so on.

The figure *ABCDEFGHKLMN* is a regular dodecagon.

[Prove that the twelve sides are equal, and that each angle is equal to five sixths of a straight angle.]

Ex. To construct a pentagon, being given four angles and three consecutive sides.

TRANSFERENCE OF POLYGONS

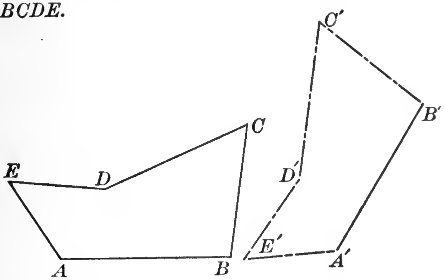
198. To *transfer* a given polygon is to construct another polygon equal to the given one so that certain of its sides or vertices may take assigned positions. This is in accordance with the postulate of figure transference (Introd. 12).

The first polygon is called the *trace* of the second; and any side or vertex of the first is called the trace of the corresponding part of the second.

General transference construction.

199. PROBLEM 23. *To transfer a given polygon so that one of the sides may fall on a given and equal line.*

Let $ABCDE$ be the given polygon of n sides. Let $A'B'$ be equal to AB. It is required to construct on $A'B'$ a polygon equal to $ABCDE$.



Draw $A'E'$, $B'C'$, making angles $B'A'E'$, $A'B'C'$ equal, respectively, to BAE, ABC. Lay off $A'E'$ equal to AE, and $B'C'$ to BC. Continue this process until $n-1$ sides and the $n-2$ included angles have been made equal to their corresponding parts; and then complete the n-gon by joining the unconnected ends of the last two segments.

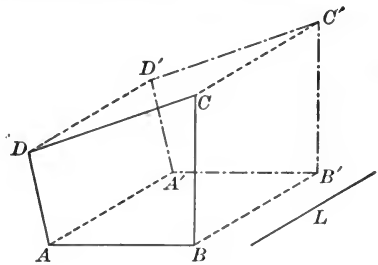
The two polygons are equal (192).

NOTE. If $A'E'$ and $B'C'$ be drawn at the other side of $A'B'$, and the polygon be completed as before, the new polygon is obversely superposable on the given one (187).

Translation construction.

200. PROBLEM 24. *To transfer a given polygon so that corresponding sides in the two positions shall be parallel, and so that the lines joining corresponding vertices may be equal and parallel to one and the same given line.*

Let $ABCD$ be the given polygon, and L the given line.



To construct a polygon $A'B'C'D'$ equal to $ABCD$, so that the sides $A'B'$, $B'C'$, $C'D'$, $D'A'$ may be respectively equal and parallel to AB, BC, CD, DA, and so that AA', BB', CC', DD' may be each equal and parallel to L.

Draw AA', BB', CC', DD' each equal and parallel to L. Join $A'B'$, $B'C'$, $C'D'$, $D'A'$.

Then $A'B'C'D'$ fulfills the given conditions.

[Prove corresponding sides of the polygons equal and parallel and corresponding angles equal.]

NOTE. This kind of transference is called *translation*. The given line is called the *line of translation*. The construction used in this problem is called the *translation construction*.

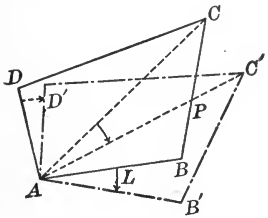
201. Definition. Two equal polygons are said to be *similarly placed* when any side and its corresponding side are parallel and are on the same side of the line joining corresponding extremities.

Thus if a polygon is translated as in 200, the polygon and its trace are similarly placed.

The rotation construction.

202. PROBLEM 25. *To transfer a given polygon so that one vertex may be unchanged, and so that each side may make with its trace an angle equal to one and the same given angle.*

Let $ABCD$ be the given polygon, A the vertex that is to be unchanged, and L the given angle.



To construct an equal polygon $AB'C'D'$ so that the angle between corresponding sides (or their prolongation) shall be equal to L.

Join A to the other vertices. Turn the lines AB, AC, AD in the same sense through an angle equal to L, into the positions AB', AC', AD'. In other words, make the angles BAB', CAC', DAD' each equal to L, and make AB' equal to AB, AC' to AC, AD' to AD. Join $B'C'$, $C'D'$.

Since the angles BAB' and CAC' are equal, the angles BAC and $B'AC'$ are equal; hence the triangles BAC and $B'AC'$ are equal.

Similarly the triangles CAD and $C'AD'$ are equal.

Hence the polygons $ABCD$ and $AB'C'D'$ are equal.

Next, to prove that the angle formed by the prolongations of the sides BC and $B'C'$ equals the angle L.

Let BC meet AC' in P; and let BC prolonged meet $B'C'$ in O (not shown in figure).

The angles ACB and $AC'B'$ have been proved equal; hence the triangles PAC and POC' have two angles of one equal to two angles of the other; therefore, the third angles PAC and POC' are equal.

But the angle PAC equals L; therefore POC' equals the given angle.

Similarly the angle formed by CD and $C'D'$ equals the given angle.

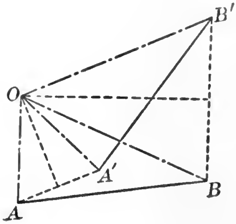
203. Definition. The kind of transference described in 202 is called *rotation*. The fixed vertex is called the *center of rotation;* and the given angle the *angle of rotation.* The construction used is called the *rotation construction.*

204. Again, if *any* point O is taken in the plane of the polygon and joined with the vertices, and if the whole figure is turned about O by a similar construction, then O is called the *center of rotation.*

205. Construction for center of rotation. It will next be shown how to find a center of rotation by means of which a given polygon can be transferred to any other given position. We begin with the simpler problem of rotating a line.

Center of rotation for transferring a line.

206. Problem 26. *Given two equal lines not parallel, and not in the same straight line; to find a point in the plane such that it may be taken as a center of rotation for the purpose of transferring one line into coincidence with the other.*



Let AB and $A'B'$ be the two equal lines.

It is required to find a point O, such that if it be joined to the extremities of the lines, the triangle OAB can be rotated about O into the position $OA'B'$.

Analysis. Suppose the point O satisfies the condition.

Then OAB is directly superposable on $OA'B'$.

Therefore OA is equal to OA', and OB to OB'.

Hence, O lies on the line bisecting AA' at right angles; and also on the line bisecting BB' at right angles. It is therefore to be determined as follows:

Construction. Join AA', BB'. Draw the perpendicular bisector of each of these lines. Let the perpendiculars meet in O. Then O is the required center of rotation.

Prove by showing that OAB is directly superposable on $OA'B'$.

Discussion. There is another solution if A is taken to correspond to B', and B to A'.

If the lines AB and $A'B'$ are parallel as well as equal, and if they are on the same side of the line AA', the two lines bisecting AA', BB' perpendicularly do not meet, and there is no center of rotation. In this case the line AB can be transferred to the position $A'B'$ by translation. Show that there is, however, a center of rotation that transfers B to the position A', and A to the position B'.

Consider the special case in which the mid-points of AA' and BB' coincide at a point M. Show that M is then the required center.

Center of rotation for transferring a polygon.

207. Problem 27. *Given two equal and directly superposable polygons, which are not similarly placed, to find a center of rotation in order to transfer one polygon into coincidence with the other.*

Outline. By the last problem, find a point O about which one side AB may be rotated to coincidence with $A'B'$, in which A corresponds to A', and B to B'. Show that the polygons will then coincide.

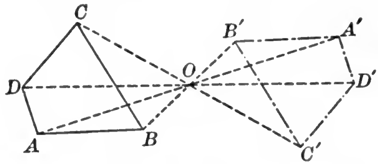
Note. Of two equal and similarly placed polygons, one can be transferred to the position of the other by translation (200).

Ex. Show how to rotate a given square into coincidence with any given equal square. Show that in this case there are four solutions.

Rotation through a straight angle.

208. THEOREM 45. *If a polygon is rotated through a straight angle about any point of its plane, then any side and its trace are parallel, but lie on opposite sides of the line joining corresponding extremities.*

Let $ABCD$ be the polygon, O the center of rotation. Let the lines OA, OB, OC, OD be each turned through a straight angle into the opposite position OA', OB', OC', OD'.



To prove that any side AB and its trace $A'B'$ are parallel, and on opposite sides of the line AA'.

[Compare the triangles OAB and $OA'B'$.]

209. Definition. Two equal polygons are said to be *oppositely placed* when any side and its corresponding side are parallel and are on opposite sides of the line joining corresponding extremities. (See figure in 208.)

210. Cor. 1. *If a polygon is turned through a straight angle, the polygon and its trace are oppositely placed.*

211. Cor. 2. *If two polygons are equal and oppositely placed, the lines joining corresponding points meet in the same point and bisect each other.*

[The diagonals of a parallelogram bisect each other.]

Center of Rotation.

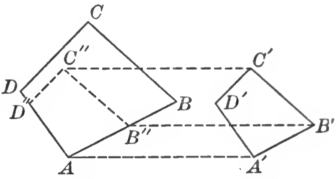
211 (a). Cor. 3. *If two polygons are equal and oppositely placed, one of them can be transferred into the position of the other by rotation through a straight angle about the intersection of the lines joining corresponding vertices.*

Prove by means of 211. Also show that this is the "special case" referred to in the discussion of problem 26.

Equiangular polygons placed in parallelism.

212. THEOREM 46. *If any two polygons are directly equiangular, and if they are placed so as to have a pair of corresponding sides parallel, then each side of one is parallel to the corresponding side of the other.*

Let the equiangular polygons $ABCD$, $A'B'C'D'$ have the corresponding angles A and A', B and B', C and C', D and D', occurring in direct order. Let the sides AB and $A'B'$ be parallel.



To prove that BC is parallel to $B'C'$, CD to $C'D'$, DA to $D'A'$.

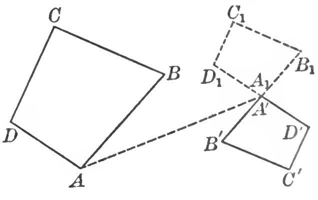
First, let the parallels AB and $A'B'$ lie on the same side of the line AA'.

By the translation construction (prob. 24) transfer the polygon $A'B'C'D'$ (using AA' as the line of translation) so that A' may coincide with A.

Since the sides are equal and parallel to their traces, $A'B'$ falls on the parallel line AB; let it take the position AB''. Since the polygons are directly equiangular, $A'D'$ falls on AD; let it take the position AD''.

Since the angle D'' is equal to D, and B'' to B, it follows that the sides of $ABCD$ are parallel to those of $AB''C''D''$ and therefore to those of $A'B'C'D'$.

Next let the parallels AB and $A'B'$ lie on opposite sides of the line AA'.



Rotate the polygon $A'B'C'D'$ through a straight angle about a point in its plane into the position $A_1B_1C_1D_1$ (208).

The side A_1B_1 is parallel to AB, and on the same side of the line AA_1; hence, by the first part, the sides of $ABCD$ are parallel to those of $A_1B_1C_1D_1$ and therefore to those of $A'B'C'D'$.

213. Cor. *If two polygons are directly equal and have a pair of corresponding sides parallel, the polygons are either similarly placed or oppositely placed.*

Axial Symmetry

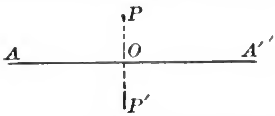
The theory of symmetric figures is of importance in connection with the equality of polygons. We begin with the case of two symmetric points.

214. Definition. Two points are said to be *symmetric* to each other with regard to a certain line, called the *axis of symmetry*, if the line joining the two points is bisected perpendicularly by the axis.

Primary construction.

215. Problem 28. *To construct the symmetric point of a given point with regard to a given axis.*

Let AA' be the given axis, and P the given point.



Draw PO perpendicular to AA' and prolong it to P' so that OP' equals PO.

The points P and P' are symmetric, by definition.

Discussion. Prove that a point has only one symmetrical point with regard to a given axis.

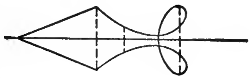
Show that the construction does not apply if the given point lies on the given axis. Show that if P be taken nearer and nearer to the axis, then P' comes nearer and nearer to P.

This fact suggests the definition that follows.

216. Definition. Any point on the axis will be said to have its symmetric point coincident with itself.

217. Definition. The construction just given is called *reflection*. *To reflect* a given point with regard to a given axis is to find its symmetric point.

218. Definition. Any two lines (straight or curved) are said to be *symmetric lines* with regard to a given axis when every point of each line has its symmetric point on the other line.

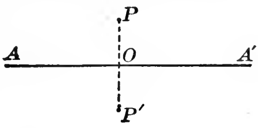


219. Definition. Any two figures are said to be *symmetric figures* with regard to an axis when their bounding lines are symmetric.

Superposition by folding over.

220. THEOREM. 47. *If the portion of the plane at one side of the axis is conceived to be revolved about the axis (or folded over), so that it coincides with the portion at the other side of the axis, then every point of the plane will coincide with its symmetric point.*

Let P, P' be any two symmetric points with regard to AA'.



To prove that P and P' may be made to coincide by revolving the portion of the plane on one side of AA' into coincidence with the other portion.

Let the line PP' meet AA' at O.

By definition PP' is bisected at right angles by AA'.

The revolution of either portion of the plane about OA leaves all points of OA unchanged, and the angles AOP, AOP' remain right angles.

Hence the lines OP and OP' come into coincidence, otherwise the right angles would not be equal.

Then, since OP equals OP', the points P and P' come into coincidence.

221. Cor. 1. *Any figure can be brought into coincidence with its symmetric figure (or reflection) by folding over.*

222. Cor. 2. *If two points are symmetric to two other points respectively, the line-segment joining the first two is symmetric with the line-segment joining the other two.*

223. Cor. 3. *The symmetric figure of a finite straight line is an equal straight line.*

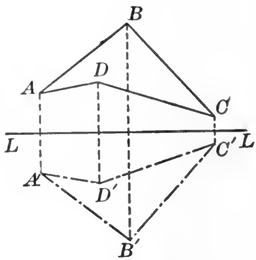
224. Cor. 4. *The symmetric figure of an indefinite straight line is another such line, and the two lines make equal angles with the axis of symmetry.*

225. Cor. 5. *The symmetric figure of a plane angle is an equal plane angle.*

Symmetric polygons.

226. Problem 29. *To construct the figure symmetric to a given polygon with regard to a given axis of symmetry.*

Let $ABCD$ be the given polygon, LL' the given axis of symmetry.



To construct the figure symmetric to $ABCD$ with regard to LL'.

Construct the symmetric points of the vertices with regard to LL' (215).

Join the new points in the same order as their symmetric points are joined.

The polygon $A'B'C'D'$ so formed is symmetric to $ABCD$.

Prove by 222 and definition in 219.

227. **Definition.** The figure symmetric to a given figure is called its *reflection* or *image* with regard to the given axis. As the two figures are obversely superposable, this construction will be called the *obversion construction.*

228. Cor. 1. *If two polygons are obversely equal, it is possible to translate one of them so as to be symmetric to the other with regard to some axis.*

229. Cor. 2. *If two polygons are obversely equal, it is possible to rotate one so as to be symmetric to the other with regard to a given axis.*

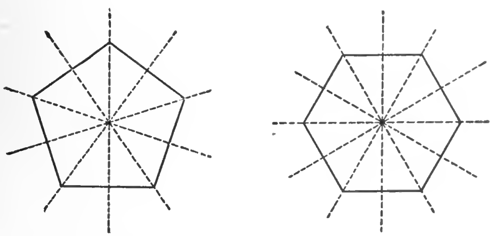
AXIS OF SYMMETRY OF A FIGURE

230. Definition. If a straight line divides a figure into two parts that are symmetric with respect to that line as an axis, the figure is said to be *a symmetric figure,* and the line is called an *axis of symmetry of the figure.*

E.g. in an isosceles triangle the bisector of the vertical angle is an axis of symmetry.

Some figures have two or more axes of symmetry. A rectangle has two axes of symmetry, a rhombus two, and an equilateral triangle three; hence the rectangle and rhombus are said to have *biaxial* symmetry; and the equilateral trangle to have *triaxial* symmetry.

Ex. 1. A square has four axes of symmetry, a regular pentagon five, and a regular hexagon six.



Ex. 2. If two points are symmetric as to an axis, and if each of them is reflected with regard to another axis perpendicular to the first, then the two points and their two reflections are at the vertices of a rectangle; and this rectangle is a symmetric figure with regard to each of the two given axes.

Central Symmetry

231. Definition. Two points are said to be symmetric with regard to a fixed point, called the *center of symmetry,* when the line joining the two points is bisected at the center.

The line joining any point to the center of symmetry is called its *radius of symmetry.*

Primary construction.

232. Problem 30. *To construct the symmetric point of a given point with regard to a given center.*

Let O be the given center, and P the given point.

Draw the radius of symmetry PO and prolong it so that OP' equals PO.

The points P and P' are symmetric by definition.

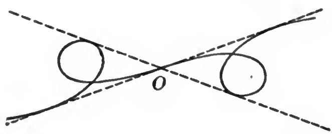
Discussion. Prove that a given point has only one symmetric point with regard to a given center.

Show that if P be taken nearer and nearer to O, then P' comes nearer and nearer to O. This fact suggests the definition that follows:

233. Definition. The center of symmetry will be said to have its symmetric point coincident with itself.

234. Cor. *The point P can be brought into coincidence with its symmetric point P' by revolving the radius of symmetry OP through two right angles into the position OP'.*

235. Definition. Any two lines (straight or curved) are said to be *symmetric lines* with regard to a given center when every point of each has its symmetric point on the other line.

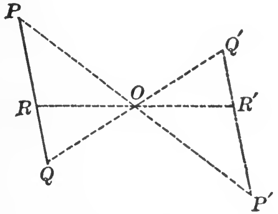


236. Definition. Any two figures are said to be *symmetric* figures with regard to a given center when their bounding lines are symmetric.

Symmetric line-segments.

237. THEOREM 48. *If two points are symmetric to two other points respectively, the line joining the first two is symmetric to the line joining the other two.*

Let O be the center of symmetry; and let P be symmetric to P', and Q to Q'.



To prove that the line PQ is symmetric to $P'Q'$.

Take any point R in PQ. Draw RO and prolong it to meet $P'Q'$ in R'.

Rotate the triangle OPQ about O through a straight angle so that OP falls on OP'; then, by the equality of angles and sides, OQ falls on OQ'; hence PQ falls on $P'Q'$. But OR falls along OR' by the equality of angles. Therefore, the point R falls on R'; hence OR' equals OR, and the points R and R' are symmetric.

Therefore, any point in PQ has its symmetric point in $P'Q'$, and thus the lines PQ and $P'Q'$ are symmetric.

238. Cor. 1. *The symmetric figure of a finite straight line is an equal and parallel line.*

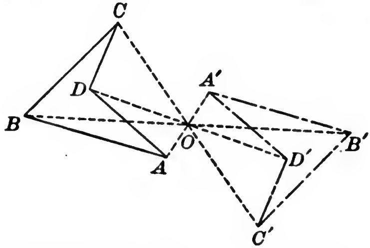
239. Cor. 2. *The symmetric figure of a triangle is another triangle equal and oppositely placed.*

240. Cor. 3. *The symmetric figure of a plane angle is an equal angle, whose sides are parallel to the sides of the first.*

Symmetric polygons.

241. PROBLEM 31: *To construct the symmetric figure of a given polygon with regard to a given center.*

Let $ABCD$ be the given polygon, O the center of symmetry.



To construct the symmetric figure of $ABCD$.

Find the points A', B', C', D', symmetric respectively to the vertices A, B, C, D.

Join these points in the same order as the given vertices are joined. Then $A'B'C'D'$ is symmetric to $ABCD$. [236, 237.]

NOTE. The two polygons are directly equal, but oppositely placed. They can be brought into coincidence by rotating either of them through a straight angle about O.

242. Cor. 1. *If two polygons are directly equal, it is possible to rotate one of them so as to be symmetric to the other with regard to a given center of symmetry.*

243. Cor. 2. *Any two equal and oppositely placed polygons have a center of symmetry.*

244. Definition. A single figure is called *a symmetric figure* as to a certain center, if any point on the boundary has its symmetric point also on the boundary.

E.g. a parallelogram is symmetric as to the intersection of its diagonals.

Ex. 1. An equilateral triangle has no center of symmetry.

Ex. 2. A regular hexagon has a center of symmetry.

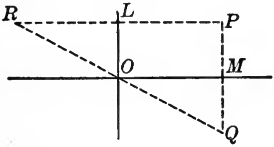
Ex. 3. No pentagon has a center of symmetry.

Ex. 4. Any polygon of an even number of sides whose opposite sides are equal and parallel has a center of symmetry.

Bi-axial related to central symmetry.

245. Theorem 49. *Two points that are symmetric to the same point with regard to two rectangular axes, respectively, are symmetric to each other as to the intersection of these axes.*

Let OL, OM be two axes at right angles. Let the symmetric points of P as to these axes be Q, R, respectively.

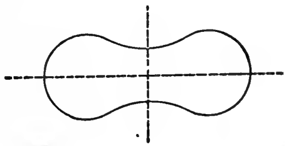


To prove that Q and R are symmetric as to O.

Join OQ, OR. Prove that the triangles RLO and OMQ are equal; that the angles ROL, LOM, MOQ are together equal to a straight angle; and that ROQ is a straight line.

246. Cor. 1. *Two figures that are symmetric to the same figure with regard to two rectangular axes, respectively, are symmetric to each other as to the intersection of these axes.*

247. Cor. 2. *If a single figure has two axes of symmetry at right angles, their intersection is a center of symmetry.*



Ex. If a figure has a center of symmetry and an axis of symmetry, then the center lies on the axis; and there is another axis of symmetry perpendicular to the first.

LOCUS PROBLEMS

248. Many geometric problems are concerned with finding the position of a point so that it may fulfill certain prescribed conditions.

It may happen, however, that the prescribed conditions do not suffice to fix the point entirely, but are sufficient to restrict it to some line or group of lines. Hence arises the idea of a *locus*, which may be defined as follows.

249. Definition. If every point on a certain line or group of lines (straight or curved) satisfies prescribed conditions, and if no other point does so, then that line or group of lines is called the *locus* of the points fulfilling those conditions.

The locus may be conveniently imagined as the path traced by a moving point that continues to satisfy the prescribed conditions.

250. Method of investigation. The investigation of a locus may be divided into an analysis and a synthesis.

Analysis. Take any point and suppose it to satisfy the prescribed conditions. By changing its position subject to these conditions, try to form some idea of the path of the moving point, noting any special positions which it passes through, and thus endeavor to discover what fixed line (straight or curved) is traced by the moving point.

Draw such line and try to demonstrate, by means of the given conditions, that it is actually fixed. When this is done, the analysis is completed.

Synthesis. Draw the logical inference from the preceding that every point satisfying the conditions must lie on the fixed line in question.

The next step is to prove conversely that every point on the line satisfies the prescribed conditions.

It can then be concluded that this line contains all those points (and those only) which fulfill the conditions assigned.

NOTE. The word *line* is here used as an abbreviation for the phrase "line or group of lines (straight or curved)."

251. This method may be briefly summarized thus:

In order to prove that a certain line L is the locus of a point P fulfilling the condition A, it is necessary and sufficient to demonstrate the following two converse propositions:

(1) If P fulfills the condition A, then P lies on the line L;
(2) If P lies on L, then P satisfies A.

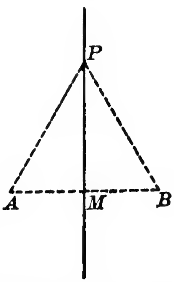
Instead of proving (1) it may sometimes be more convenient to prove its contraposite:

If P does not lie on L, then P does not satisfy A. Show that this proposition is equivalent to (1). See Arts. 105, 106.

252. Definition. For convenience the line-segment connecting two points is sometimes called the *join* of one point to the other.

253. PROBLEM 32. *To find the locus of a point such that its joins to two given points are equal.*

Let A and B be the two given points. Let P be any point such that the lines PA and PB are equal.



To find and construct the locus of P.

Analysis. Taking successive positions of P subject to the condition that PA equals PB, and observing that M the mid-point of AB fulfills this condition, we are led to surmise that the path of P is the straight line joining P to the mid-point of AB.

To prove that this line PM remains actually fixed as P moves.

The triangles PAM and PBM have their sides respectively equal. Therefore the angles PMA and PMB are equal, and each equal to a right angle.

Hence MP is perpendicular to the fixed line AB at its middle point, and is therefore fixed, no matter what position the moving point P takes while subject to the assigned condition.

It follows that P moves along this fixed line.

Synthesis. It has now been proved that every point which satisfies the prescribed condition lies on the perpendicular bisector MP.

It remains to be proved conversely that every point on MP satisfies the condition.

Let P be any point on the perpendicular bisector. Join PA and PB.

The two right triangles have the sides about the right angles respectively equal; therefore the hypotenuses PA and PB are equal.

Hence all those points (and those only) that satisfy the given condition lie on the line MP.

Therefore the perpendicular bisector extended indefinitely both ways is the locus required.

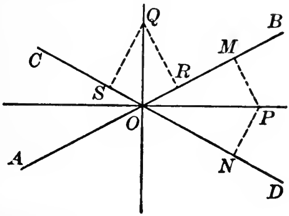
Ex. 1. Find the locus of the vertex of an isosceles triangle whose base is given in magnitude and position.

Ex. 2. Show that a simple construction for bisecting a given line-segment can be derived from the locus problem solved above. (Compare 70.)

Ex. 3. Show that this locus problem also furnishes a solution to the problem of erecting or dropping a perpendicular to a given line.

254. Problem 33. *To find the locus of a point from which the perpendiculars to two given intersecting lines are equal.*

Let AB and CD be the two given lines. Let P be any point such that PM, the perpendicular to AB, is equal to PN, the perpendicular to CD.



To find the locus of P.

Analysis. Taking successive positions of P subject to the given condition, and observing that the moving point can come nearer and nearer to O, we are led to think that O is a point on the locus, and that the straight line OP is the path of P.

To prove that this line is actually fixed in position as P moves.

The triangles POM, PON have two sides of one respectively equal to two sides of the other, and the angles opposite a pair of equal sides are right angles; therefore the angles POM and PON are equal (98).

Hence OP is a bisector of the angle MON, and is therefore a fixed line.

It follows that the point P is on one or other of the bisectors of the angles contained by the two given intersecting lines.

Synthesis. It has now been proved that every point which satisfies the prescribed condition lies on one or other of the two angle-bisectors.

It will next be proved conversely that every point on either of these lines satisfies the condition.

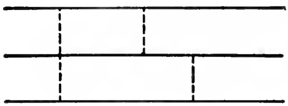
Let Q be any point on either angle-bisector. Draw QR perpendicular to AB, and QS perpendicular to CD.

The two right triangles QOR, QOS have the acute angles QOR and QOS equal, and a common hypotenuse; therefore the sides QR and QS are equal.

Hence the two bisectors of the angles between the two given lines constitute the locus required.

Ex. Find the locus of a point from which the perpendiculars to two given parallel lines are equal.

255. Problem 34. *To find the locus of a point from which the perpendicular to a given line shall be equal to a given line-segment.*



Show that the locus consists of a pair of lines parallel to the given line.

Ex. Find the locus of the mid-points of all the lines drawn from a given point to a given line not passing through the point.

Intersection of Loci

256. Many problems relating to the determination of points satisfying given conditions can be solved by means of the intersection of loci.

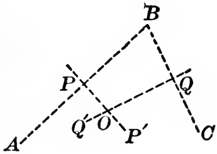
For instance, the problem to determine all the points that satisfy two prescribed conditions A and B may be solved as follows:

Construct the locus of the points satisfying the condition A; and also construct the locus of the points satisfying the condition B.

The points of intersection of the two loci (and these points only) satisfy both the assigned conditions.

257. Problem 35. *To find a point such that its joins to three given points not collinear are equal.*

Let A, B, C be the three given points.



It is required to find a point O, such that OA, OB, OC are all equal.

Draw PP' and QQ' the perpendicular bisectors of the lines AB and BC.

The lines PP' and QQ' intersect, for if they were parallel, then the lines AB and BC, being respectively perpendicular to them, would be in one straight line. Let the lines PP' and QQ' intersect in O.

To prove that the point O, and no other point, satisfies the given conditions.

The line PP' contains all those points and only those, whose joins to A and B are equal (253).

The line QQ' contains all those points, and only those, whose joins to B and C are equal.

Therefore the point common to PP' and QQ', and no other point, has its joins to A, B, and C equal.

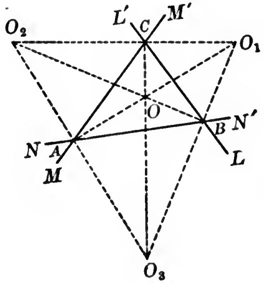
Note. This construction is used later (III. 75) in finding the center of a given circle.

Ex. 1. In a given line find a point whose joins to two given points are equal.

Ex. 2. Find a point from which the perpendiculars to two given intersecting lines are respectively equal to two given line-segments. Four solutions (255).

258. PROBLEM 36. *To find a point from which the perpendiculars to three given lines (forming a triangle) shall be equal.*

Let the three lines LL', MM', NN' form a triangle ABC.



It is required to find a point O, such that the perpendiculars from O to these lines are equal.

[The construction and proof are left to the student. Show that there are four solutions.]

Ex. 1. If two of the three lines are parallel, how many solutions are there?

Ex. 2. On a given line find those points from which the perpendiculars to another given line are equal to an assigned line-segment.

Ex. 3. On a given line how many points are there from which the perpendiculars to two given lines are equal?

THEOREMS ON CONCURRENCE

259. Definition. Three or more lines that meet in a common point (when prolonged if necessary) are said to be *concurrent.*

The principle of the intersection of loci may be used to prove the first two of the following theorems relating to the concurrence of certain lines in a triangle. The third is then derived from the first by the theory of parallels.

Concurrence of perpendicular bisectors of sides.

260. THEOREM 50. *In any triangle the three perpendicular bisectors of the sides are concurrent.*

Outline. Use the construction of 257. Then show that O lies on the perpendicular bisector of the side AC.

Concurrence of angle-bisectors.

261. THEOREM 51. *In any triangle the three bisectors of the interior angles are concurrent.*

Outline. Let the point O in 258 be the intersection of two of the angle-bisectors. Show that O lies on the third angle-bisector.

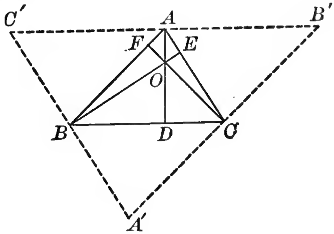
262. **Cor.** *The exterior angle-bisectors through two vertices and the interior angle-bisector through the third vertex are concurrent.*

263. **Definition.** The lines drawn from the vertices of a triangle perpendicular to the opposite sides, respectively, are called the *principal perpendiculars* of the triangle.

Concurrence of principal perpendiculars.

264. THEOREM 52. *The three principal perpendiculars of a triangle are concurrent.*

Let ABC be any triangle. Let AD, BE, and CF be the principal perpendiculars.



To prove that AD, BE, and CF are concurrent.

Through the vertices A, B, C draw lines parallel respectively to the opposite sides, so as to form a second triangle $A'B'C'$.

Outline. Prove by 153 that A is the mid-point of $B'C'$, B the mid-point of $A'C'$, and C of $A'B'$; hence that AD, BE, CF are the perpendicular bisectors of the sides of the new triangle $A'B'C'$. Then draw desired conclusion and quote authority.

On Methods of Analysis

265. When a new theorem or problem is presented for investigation (as in the miscellaneous exercises that follow), we try to discover some connection or relationship between the new proposition and the previous ones with which we are familiar. This relationship is to be discovered by means of a preliminary analysis. The words *analysis* and *synthesis* and the corresponding adjectives *analytic* and *synthetic* are much used in mathematics. In general, analysis means the separation of a whole into its parts, and synthesis means bringing the parts together to make a whole. In geometry the words are used in a more restricted sense. In synthesis we begin with admitted facts, and, by the aid of principles or theorems already accepted and problems already solved, we prove some new theorem or solve some new problem. This is usually the most convenient way of presenting the result when it is once obtained; but the actual discovery is often made in the reverse way by means of an analysis, in which we begin with the conclusion and then examine the different conditions that are necessary or sufficient to lead to the result in question. The analysis of a problem was described in 131, and illustrated in various subsequent articles. The analysis of a theorem is somewhat similar, and may be conducted in two ways, which may be called, respectively, the analysis of antecedents and the analysis of consequents.

266. *Analysis of antecedents.* In this method we examine the antecedent conditions from which the conclusion in question would follow, and then compare these conditions with the given hypothesis. For example, let the conclusion be called 'statement S,' then the analysis of antecedents may be put in the following form: —

The statement S is true, if the statement R is true;
the statement R is true, if the statement Q is true;
the statement Q is true, if the statement P is true;

and so on. If by this method we get back to some antecedent statement A which we know to be true by some principle already accepted, or which would follow from the given hypothesis, then we are warranted in asserting the truth of statement S. The successive steps from A to S can then be presented in the reverse of the order just given, and the proof can be arranged in the usual synthetic form beginning with the hypothesis and ending with the conclusion to be demonstrated.

If, however, we come only to a statement that we know to be false (or do not know to be true), then the statement S may or may not be true, and nothing is proved. A new set of antecedent conditions may then be examined. This method often proves the truth of a theorem; it cannot by itself prove any statement false.

267. *Analysis of consequents.* In this method we examine the consequences that would follow if the theorem were supposed to be true, and then compare these consequences with the hypothesis and other accepted facts. The analysis of consequents may be put in the following form:

> If the statement S is true, then the statement T is true;
> if the statement T is true, then the statement U is true;

and so on. If by this method we arrive at some statement that we know to be false (or inconsistent with the hypothesis), then we conclude that the statement S is false, since it can be reduced to an absurdity.

If, however, we come only to a statement Z that we know to be true (or do not know to be false), then the statement S may or may not be true, and nothing is proved. This method of analysis often proves the falsity of a statement; it cannot by itself prove any statement true, since the steps taken from S to Z are not always reversible; it sometimes, however, points the way to a synthetic proof by reversal of the steps.

268. *Analysis of the opposite.* Either of the two methods of analysis may be applied to the opposite of the statement S. The analysis of antecedents gives a decisive result if we arrive at an antecedent known to be true, for then the opposite of S is true, and S is false. The analysis of consequents gives a decisive result if we arrive at a consequent known to be false, for then the opposite of S is false, and S is true; this case is the familiar *reductio ad absurdum* of which several illustrations have been given (see 102).

EXERCISES ON BOOK I

1. In an isosceles triangle, if a perpendicular is drawn from an extremity of the base to the opposite side, then the angle between this perpendicular and the base is equal to half the vertical angle.

2. In a right triangle, prove that a line can be drawn dividing the right angle into two parts equal respectively to the other angles, and so as to divide the right triangle into two isosceles triangles.

3. In a right triangle the median drawn to the mid-point of the hypotenuse equals half the latter.

4. Through two given points draw two lines forming with a given indefinite line an equilateral triangle. How many solutions are there?

5. The bisectors of the base angles of an isosceles triangle contain an angle equal to an exterior angle of the triangle.

6. The lines joining the adjacent extremities of unequal and parallel lines will meet if prolonged through the extremities of the shorter parallel (124).

7. Construct a right triangle, being given the hypotenuse and the sum (or difference) of the two sides (137, ex. 1). Construct a right triangle, being given one side and the sum (or difference) of the other side and the hypotenuse (137, ex. 3).

8. Lines drawn from two opposite vertices of a parallelogram to the mid-points of a pair of opposite sides trisect a diagonal (167).

9. Lines drawn from any vertex of a parallelogram to the mid-points of the two non-adjacent sides trisect a diagonal (178).

10. If alternate sides of a pentagon are prolonged to meet, then the sum of the five angles so formed is equal to two right angles (190).

11. If alternate sides of a hexagon are prolonged to meet, then the sum of the six angles so formed is equal to four right angles. Consider also the general case of an n-gon.

12. Through a given point draw a line so that the part intercepted between two given parallel lines may be equal to a given line.

13. Through a given point within a given fixed angle draw a line so that the segment between the sides may be bisected at the point (186, ex. 6).

14. Construct a triangle, being given two sides and the median drawn to the mid-point of the third side.

15. Construct a triangle being given one side and the medians to the mid-points of the other two sides.

16. Any line through the intersection of the diagonals of a parallelogram and terminated by opposite sides is bisected at that point.

17. In a given triangle inscribe a parallelogram having one side resting on the base, and having the intersection of its diagonals at a given point.

18. The bisectors of the angles of a parallelogram form a rectangle (123).

19. The bisectors of the angles of a rectangle form a square.

20. In an isosceles triangle the bisector of a base angle, and the bisector of the external angle supplemental to the other base angle, form an angle equal to half the vertical angle.

21. Construct a triangle, being given the angles, and one of the principal perpendiculars.

22. Construct a triangle, being given the mid-points of the three sides (see figure in 264).

23. Draw a parallel to the base of a triangle, so that the intercept may be equal to one of the segments adjacent to the base (compare 186, ex. 5).

24. Draw a parallel to the base of a triangle, so that the intercept may be equal to the sum of the segments adjacent to the base.

25. Draw a parallel to a base of the triangle, so that the intercept may be equal to the difference of the segments adjacent to the base.

26. Given the sum (or difference) of the side and principal perpendicular of an equilateral triangle, construct it (137, ex. 6).

27. Given the sum (or difference) of the side and diagonal of a square, construct it.

28. If the opposite sides of a hexagon are parallel, then its diagonals are concurrent.

29. Given two indefinite lines and a point: to find a point in one of the lines so that the line joining it to the given point may be bisected by the other line. How many solutions are there?

30. If through any vertex of a parallelogram a line is drawn, and if perpendiculars to this line are drawn from the other vertices, then the perpendicular from the vertex opposite the first is equal to the sum or difference of the other two, according as the line passes without or within the parallelogram.

31. In any quadrangle the two lines joining the mid-points of opposite sides, and the line joining the mid-points of the diagonals, all meet in a point and bisect each other (175, ex. 3).

32. If three parallel lines make equal intercepts on a transversal, and if a second transversal cross the first between two of the parallels, then the intercept on the middle parallel equals half the difference of the intercepts on the other two (compare 172).

33. If through the extremities of the base of a triangle whose sides are unequal, lines are drawn to any point in the bisector of the vertical angle, their differences are less than the difference of the sides.

[Let side AB be greater than AC. On AB take AC' equal to AC. Join C' to the point in the bisector.]

34. If the lines in ex. 33 are drawn to any point in the bisector of the external vertical angle, then their sum is greater than the sum of the sides.

35. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shortest side.

36. If a quadrangle is inscribed in a parallelogram and has its opposite vertices symmetric as to the center of symmetry of the parallelogram (244), then the quadrangle is a parallelogram.

37. If a parallelogram is inscribed in a rectangle, having its sides parallel to the diagonals of the rectangle, then two adjacent sides of the parallelogram make equal angles with a side of the rectangle.

38. A billiard ball is placed at any point of a rectangular table. In what direction must it be struck so that it shall return to the first point after being reflected successively at the four sides, the lines of motion, before and after impact, making equal angles with the successive sides of the table?

39. If two sides of a triangle are unequal then the bisector of the angle between them divides the opposite side into unequal segments, the greater segment being adjacent to the greater side.

Outline. In triangle ABC, let AD bisect the angle A. Given AB greater than AC; to prove BD greater than BC. On AB lay off AC' equal to AC, and join $C'D$. By equality and inequality of angles prove angle B less than $BC'D$. Draw conclusion.

40. To inscribe a square in a right triangle.

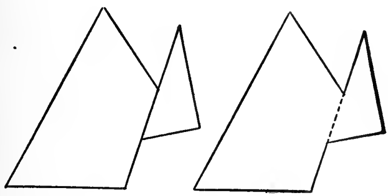
41. To inscribe a square in a rhombus.

42. If two isosceles triangles have equal bases, and if one of the equal sides of the first triangle is greater than one of the equal sides of the second, then the vertical angle of the first triangle is less than the vertical angle of the second.

BOOK II. — EQUIVALENCE OF POLYGONS

GENERAL PRINCIPLES

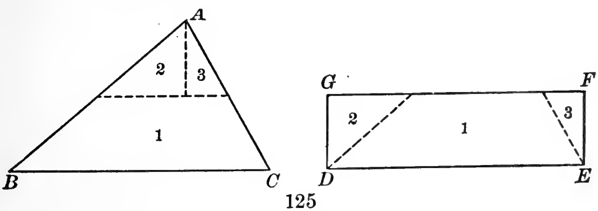
1. Definitions. Two polygons are said to be *joined* when they are brought together, without overlapping, so that a side of one coincides in whole or in part with a side of the other.



When the common portion of the boundaries of two joined polygons is erased (or ignored), the third polygon so formed is called the *sum* of the two original polygons, which are then said to be *added* together.

A polygon is said to be *dissected* when its surface is divided up into any number of smaller polygons by drawing straight lines.

Two polygons are called *equivalent* if their surfaces can be dissected so that each part of one is separately superposable on some part of the other by suitable rearrangement of parts if necessary.



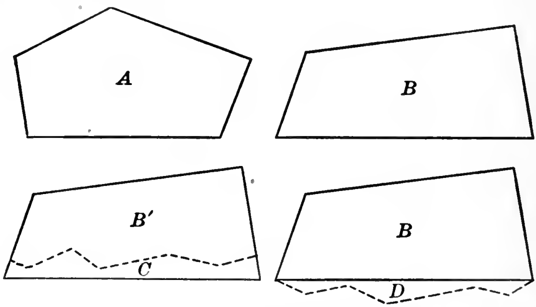
Thus the triangle ABC and the rectangle $DEFG$ are equivalent if the parts marked with corresponding numerals are superposable.

To make this definition of the equivalence of polygons consistent with itself, it is necessary to prove the following two lemmas relating to the permanence of such equivalence.

2. Lemma 1. If one polygon incloses another within its boundary so that the latter is part of the former, it is not possible to dissect the inner polygon and then rearrange and join its parts in such a way as to cover the whole of the outer polygon.

For suppose that this operation is possible; and let the outer polygon be supposed actually covered by the rearranged parts of the inner one. Remove the excess of the outer polygon over the original inner one. Dissect the remaining inner polygon as before and then rearrange the parts so as to cover the outer polygon. Remove the excess again; and repeat the process as often as desired. The excess can be accumulated until it is more than sufficient to cover any polygon however large. But this excess is only a part of the original surface of the inner polygon. Therefore the surface of this finite polygon can be so rearranged as to cover an indefinitely extended surface, which is absurd. Hence the lemma is established.

3. Lemma 2. If two polygons are equivalent for one mode of dissection and superposition, they will be equivalent for all possible modes of dissection and superposition.



Let the polygons A and B be such that there is one way of dissecting them so that every part of A can be fitted on an equal and corresponding part of B, the latter polygon being then just covered by the parts of the former.

Next let the polygon A be dissected in any second way, and let the parts be placed in any order upon B, these smaller polygons being joined so as not to overlap. Should portions of any of them extend over the boundary of B, let the surplus be cut off, and then used to cover any uncovered portion of the surface of B. Continue this process until either

(α) the parts of A are exhausted, leaving a portion of B uncovered, or

(β) the surface of B is covered, leaving a portion of A extending over the boundary of B, or

(γ) the surface of B is just covered by the parts of A without excess or defect.

In case (α) let the second mode of dissection and superposition cover B', leaving a portion C uncovered. Dissect this covering of B' by the second mode and fit the parts back so as to form the original polygon A. Then dissect A by the original mode, and rearrange the parts so as to cover the polygon B (in accordance with the hypothesis). Thus the surface of the inner polygon B' has been rearranged to cover the surface of the outer polygon B, contrary to the preceding lemma; hence case (α) cannot occur.

In case (β) let the second mode of superposition cover a polygon made up of B and the surplus D. Dissect this covering of B by the first mode, and use the parts to form the polygon A (in accordance with the hypothesis); then dissect A by the second mode, and cover the polygon made up of B and D. Thus the surface of the inner polygon B has been rearranged to cover the outer polygon, contrary to the preceding lemma; hence case (β) cannot occur.

Therefore case (γ) is the only one that can occur; that is to say, the polygon B is just covered by the second mode of dissecting A and of superposing the parts on B.

4. Definitions continued. One polygon is said to be *greater* than a second polygon if a portion of the first can be dissected and rearranged so as to cover the second. In the same case the second is said to be *less* than the first.

5. Any polygon which, when added to the less of two given polygons, forms a polygon equivalent to the greater, is called the *difference* of the two given polygons.

6. If two polygons are equivalent, any polygon equivalent to their sum is said to be the *double* of either polygon, and each of the former is said to be equivalent to *half* the latter.

Axioms of Equivalence and Non-equivalence*

From the foregoing definitions, the following statements are direct inferences by means of the principle of superposition:

7. *Polygons which are equivalent to the same polygon are equivalent to each other.*

8. *If a number of polygons are added together in a certain way and order, the sum is equivalent to the sum that would have been obtained if the polygons had been added together in a different way or in a different order.*

For the two resulting polygons can be dissected into superposable parts.

9. *If equivalent polygons are added to equivalent polygons, the sums are equivalent polygons. In particular the doubles of equivalent polygons are equivalent.*

10 (*a*). *If two unequivalent polygons are added respectively to unequivalent polygons, the sum of the two greater polygons is greater than the sum of the two less ones.*

10 (*b*). *If one polygon is greater than a second, the double of the first is greater than the double of the second.*

11. *The halves of equivalent polygons are equivalent.*

Apply indirect proof and use 10 (*b*).

12. *If one polygon is greater than a second, then the half of the first polygon is greater than the half of the second.*

* The student need not dwell on Arts. 7–17 at first reading, but should refer back to them when necessary.

General Theorems relating to Equivalence

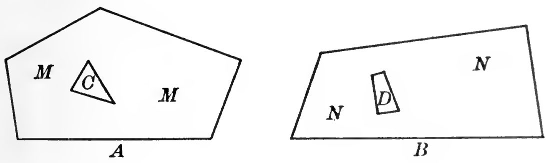
13. Theorem 1. *The double of the sum of two polygons is equivalent to the sum of the doubles of the two polygons.*

For the double of the sum of two polygons A and B is a polygon made up of the four parts A, B, A, B; and the sum of the doubles of A and B is a polygon made up of the four parts A, A, B, B; differing only in the order of arrangement; hence the two sums are equivalent (8).

14. Theorem 2. *The half of the sum of two polygons is equivalent to the sum of the halves of the polygons.*

For the sum of the halves when doubled becomes equivalent to the sum of the two whole polygons (13), and is therefore, by definition, equivalent to half this sum.

15. Theorem 3. *If equivalent polygons are taken away from equivalent polygons, the remaining figures are equivalent.*



Let the polygons A and B be equivalent; and let the polygons C and D be equivalent; then the remaining figures M and N are equivalent.

For, by Lemma 2, A can be fitted on B by any mode of dissection. Choose a mode in which the parts of C are made to cover its equivalent D. Then the parts of M will cover the remaining figure N.

15 (*a*). Ex. Prove the axioms of non-equivalence relating to subtraction. (See ax. 10, 11, p. 17.)

16. Theorem 4. *The double of the difference of two polygons is equivalent to the difference of their doubles.*

For let the difference of the polygons A and B be the polygon C; then the sum of B and C is equivalent to A (5). Therefore the sum of the doubles of B and C is equivalent to the double of A (13); hence the difference between the double of A and the double of B is equivalent to the double of C (5), and is therefore equivalent to double the difference between A and B.

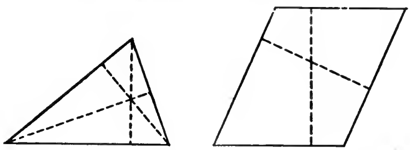
17. Theorem 5. *The half of the difference of two polygons is equivalent to the difference of their halves.*

Use a similar proof, substituting the word *half* for the word *double.*

COMPARISON OF PARALLELOGRAMS

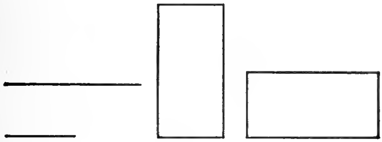
18. Definitions. In a given triangle the line drawn from any vertex perpendicular to the opposite side is called the *altitude* of the triangle with reference to that side taken as *base.* Any side may be so regarded as base and the corresponding perpendicular as the altitude; hence a triangle has three altitudes.

Similarly any side of a parallelogram may be regarded as its *base,* and the line drawn perpendicular to it from any point of the opposite side may be taken as corresponding *altitude.* All such altitudes drawn to the same side are equal, and are also equal to the altitudes drawn to the opposite side. Thus a parallelogram has only two altitudes.



19. In the case of a rectangle, when any particular side is taken as base, either of the adjacent sides is the altitude.

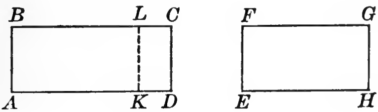
20. A rectangle is completely determined by two adjacent sides; that is to say, all the rectangles whose adjacent sides are equal to two given lines are superposable (I. 166); and, for this reason, each of these rectangles is called the rectangle of the two given lines.



Rectangles of equal altitudes.

21. Theorem 6. *If two rectangles have equal altitudes and unequal bases, that which has the greater base is the greater rectangle.*

Let the rectangles $ABCD$, $EFGH$ have their altitudes AB and EF equal. Let the base AD be greater than the base EH.



To prove that $ABCD$ is greater than $EFGH$.

Lay off AK equal to EH, and complete the rectangle $ABLK$.

This rectangle is equal to $EFGH$ (I. 166). Hence a portion of $ABCD$ will cover $EFGH$. Therefore $ABCD$ is greater than $EFGH$ (4).

22. Cor. 1. *If two rectangles have equal altitudes, then according as the base of the first is greater than, equal to, or less than the base of the second, so is the first rectangle greater than, equal to, or less than the second.* (21; and I. 166.)

23. Cor. 2. *If two rectangles have equal altitudes, then according as the first rectangle is greater than, equal to, or less* *than the second, so is the base of the first greater than, equal to, or less than the base of the second.* (Rule of Conversion, I. 104.)

Ex. Show that 22 and 23 are still true if the words *base* and *altitude* are interchanged throughout.

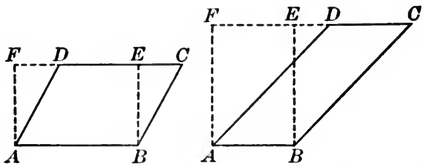
24. Cor. 3. *According as the side of one square is greater than, equal to, or less than the side of another square, so is the first square greater than, equal to, or less than the second; and conversely.*

The student may give an independent proof by superposition; and then state the converse.

Parallelograms and rectangles.

25. THEOREM 7. *A parallelogram is equivalent to the rectangle of its base and altitude.*

Let $ABCD$ be the given parallelogram, having AB for base and AF or BE for altitude.



To prove that $ABCD$ is equivalent to $ABEF$, the rectangle of its base and altitude.

In the triangles AFD and BEC: the side AF equals BE (I. 153); the side AD equals BC; and the angle FAD equals EBC, having parallel sides (I. 126).

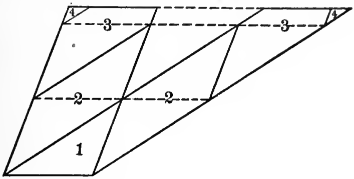
Therefore the triangles AFD and BEC are equal.

Take these equivalents in turn away from the quadrilateral $ABCF$; then the remainders $ABCD$ and $ABEF$ are equivalent (15).

Ex. Prove the theorem for the case in which D and E coincide.

26. **Cor. 1.** *Two parallelograms having equal bases and equal altitudes are equivalent.*

Ex. Show how to dissect any two parallelograms having the same base and equal altitudes, so that the parts may be superposable.



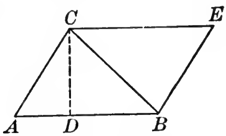
27. **Cor. 2.** *If two parallelograms have equal altitudes, then according as the base of the first is greater than, equal to, or less than the base of the second, so is the first parallelogram greater than, equivalent to, or less than the second.* (Use 22 and 25.)

28. **Cor. 3.** *If two parallelograms have equal altitudes, then according as the first parallelogram is greater than, equivalent to, or less than the second, so is the base of the first greater than, equal to, or less than the base of the second.* (Rule of Conversion.)

Ex. Show that 27 and 28 are still true if the words *base* and *altitude* are interchanged throughout.

Triangles and rectangles.

29. Theorem 8. *A triangle is equivalent to half the rectangle of its base and altitude.*



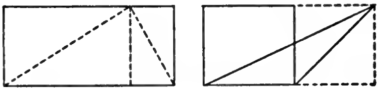
Let ABC be the triangle, having AB for base and CD for altitude.

To prove that *ABC* is equivalent to the rectangle of *AB*, *CD*.

Complete the parallelogram *ABEC*.

The triangle *ABC* is equivalent to half the parallelogram *ABEC* (I. 153); and therefore equivalent to half the rectangle of *AB* and *CD* (25).

Ex. Prove this theorem directly by applying 14, 17 to the adjoining figures.



30. Cor. 1. *A trapezoid is equivalent to the rectangle contained by its altitude and half the sum of its parallel sides.*

31. Cor. 2. *If two triangles have equal altitudes, then according as the base of the first is greater than, equal to, or less than the base of the second, so is the first triangle greater than, equivalent to, or less than the second.* (Use 23, 29.)

32. Cor. 3. *If two triangles have equal altitudes, then according as the first triangle is greater than, equivalent to, or less than the second, so is the base of the first greater than, equivalent to, or less than the base of the second.*

Ex. If there are two equilateral triangles, then according as a side of the first is greater than, equal to, or less than a side of the second, so is the first triangle greater than, equal to, or less than the second.

33. Cor. 4. *Two triangles having the same base, and having their opposite vertices on the same line parallel to the base, are equivalent. Conversely, two equivalent triangles on the same base and at the same side of it are between the same parallels.*

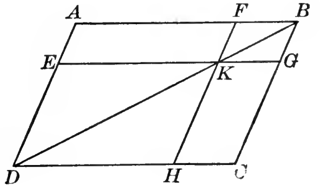
34. Cor. 5. *If a parallelogram and a triangle are upon the same base and between the same parallels, the parallelogram is double the triangle.*

Parallelograms about a diagonal.

35. Definition. If through any point on the diagonal of a parallelogram two lines be drawn parallel to the sides, so as to divide the parallelogram into four smaller parallelograms, the two whose diagonals are portions of the diagonal first mentioned are called the *parallelograms about the diagonal;* and the two which lie one on each side of the diagonal are called the *complements of the parallelograms about the diagonal.*

36. THEOREM 9. *The complements of the parallelograms about the diagonal of a parallelogram are equivalent.*

Let $ABCD$ be the parallelogram, BD its diagonal, K any point on it, FH and EG lines through K parallel to the sides, forming $KGBF$ and $DHKE$ parallelograms about the diagonal, and $KFAE$ and $CGKH$ the complements of these parallelograms.



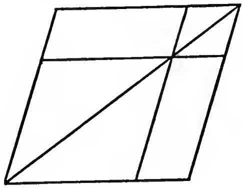
To prove that these complements are equivalent.

Since the diagonal of a parallelogram bisects it (I. 153), the triangle DBA is equivalent to CBD; similarly KBF is equivalent to KGB; and KED to DHK.

Take KBF and KED away from DBA, and take KGB and DHK away from CBD; then the remainders $KFAE$ and $KHCG$ are equivalent (15).

NOTE. This theorem is useful in the construction of equivalent parallelograms (72).

37. THEOREM 10. *Parallelograms about the diagonal of a rhombus are rhombuses, and their complements are equal parallelograms.*



Use 161, 117, 166 of Book I.

38. **Cor.** *Parallelograms about the diagonal of a square are squares, and their complements are equal rectangles.*

EXERCISES

1. If one diagonal of a quadrangle bisects the other, it also bisects the quadrangle.

2. If a parallelogram and a triangle are such that the base and altitude of the parallelogram are respectively equal to half the base and altitude of the triangle, then the parallelogram is equivalent to half the triangle.

3. Lines joining the mid-points of adjacent sides of a quadrangle form a parallelogram equivalent to half the quadrangle.

4. If two triangles stand on the same base and at the same side of it, and if the middle points of the sides are joined, then the joining lines form a parallelogram equivalent to half the difference of the triangles.

5. To construct an isosceles triangle equivalent to a given triangle and standing on the same base.

6. To construct a rhombus equivalent to a given parallelogram and having the same diagonal.

7. A triangle whose base is one of the non-parallel sides of a trapezoid and whose vertex is at the mid-point of the opposite side is equivalent to half the trapezoid.

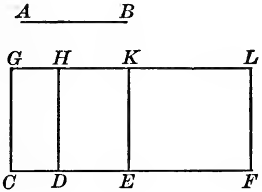
[Through the mid-point in question draw a parallel to the opposite side and complete the parallelogram.]

EQUIVALENCES INVOLVING RECTANGLES

Rectangles of wholes and parts.

39. THEOREM 11. *If there are two lines, one of which is divided into any number of parts at given points, the rectangle of the two given lines is equivalent to the sum of the rectangles of the undivided line and the several parts of the divided line.*

Let AB, CF be the two lines, and let CF be divided at the points D and E into the parts CD, DE, EF.



To prove that the rectangle of AB and CF is equal to the sum of the rectangles of AB, CD; AB, DE; AB, EF.

Draw the line CG perpendicular to CF and equal to AB. Complete the rectangle $CFLG$, and draw DH, EK perpendicular to CF.

The lines DH, EK are equal to CG (I. 153) and therefore equal to AB.

The rectangle CL is equivalent to the sum of the rectangles CH, DK, EL.

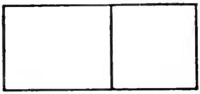
Now CH is the rectangle of CG and CD, that is, of AB and CD; also DK is the rectangle of AB and DE; and EL is the rectangle of AB and EF.

Therefore the theorem is established.

NOTE. Two of the following corollaries are special cases of this theorem, and the third is an extension of it.

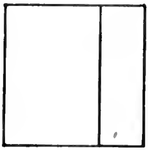
Rectangle of whole line and one part.

40 (*a*). **Cor. 1.** *If a line is divided into any two parts, the rectangle of the whole line and one part is equivalent to the square on that part together with the rectangle of the two parts.*



Square on whole line.

40 (*b*). **Cor. 2.** *If a line is divided into any two parts, the square on the whole line is equivalent to the sum of the rectangles of the whole line and each of the parts.*



Distributive property of rectangles.

41. Cor. 3. *If each of two lines is divided into any number of parts, then the rectangle contained by the whole lines is equivalent to the sum of all the rectangles contained by each part of one and each part of the other.*

[Prove by repeated applications of 39; or else by an independent figure.]

Note. This important principle will be referred to as "the distributive property of rectangles"; it lies at the foundation of many of the subsequent theorems.

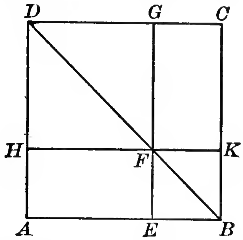
Ex. 1. Show that 39, 40 are special cases of the "distributive property."

Ex. 2. If a line is divided into three parts, then the square on the whole line is equivalent to the sum of the rectangles of the whole line and each of its parts.

Squares on whole and parts.

42. THEOREM 12. *If a line is divided into any two parts, the square on the whole line is equivalent to the sum of the squares of the parts and double the rectangle contained by the parts.*

Let AB be the given line divided at E.



To prove that the square on AB is equivalent to the sum of the squares on AE and EB, and twice the rectangle of AE and EB.

[Use 36, 37, 38.]

Symbolic Proof. Another simple proof may be given by using the distributive property of rectangles. For brevity denote the rectangle of two lines AB and CD by $[AB, CD]$, and the square on AB by the symbol sq. AB. Let the symbol $\Leftrightarrow$ stand for the phrase "is equivalent to"; the sign $+$ for "added to" or "increased by"; and the sign $-$ for "diminished by."

Since $\quad$ sq. $AB \Leftrightarrow [AB, AE] + [AB, EB]$; $\quad$ [40 (b)

and $\quad [AB, AE] \Leftrightarrow$ sq. $AE + [AE, EB]$, $\quad$ [40 (a)

$\quad [AB, EB] \Leftrightarrow$ sq. $EB + [AE, EB]$;

hence $\quad$ sq. $AB \Leftrightarrow$ sq. AE + sq. $EB + 2\,[AE, EB]$.

Square on sum.

43. Cor. 1. *The square on the sum of two lines is equivalent to the sum of their squares and twice their rectangle.*

44. Cor. 2. *The square on any line is equivalent to four times the square on its half.*

Ex. 1. Prove 43 by applying the distributive property to two lines each equal to the sum of the two given lines.

Ex. 2. Prove 44 by applying the distributive property to two equal lines each of which is bisected.

Ex. 3. If a line is divided into three parts, the square on the whole line is equivalent to the sum of the squares on the parts together with twice the rectangles of the parts taken two and two.

Sum of squares on whole and part.

45. Theorem 13. *If a line is divided into any two parts, the sum of the squares on the whole line and one part is equivalent to twice the rectangle of the whole line and that part, together with the square on the other part.*

Let AB be the given line divided at E.

To prove that the sum of the squares of AB and EB is equivalent to twice the rectangle of AB and EB, together with the square on AE.

On AB describe a square, and complete the construction as in the figure of the preceding theorem.

The square DB is equivalent to the sum of the square DF and the rectangles HE and GB. Add to each of these equivalents the square FB. Then the sum of the squares DB and FB is equivalent to the sum of the square DF and the rectangles HB and GB. Now the latter rectangles are each equal to the rectangle of AB and EB. Hence the theorem is proved.

Otherwise:

$$\text{sq. } AB \approx [AB, EB] + [AB, AE], \qquad [40\,(b)$$

$$\approx [AB, EB] + \text{sq. } AE + [AE, EB]. \qquad [40\,(a)$$

Add the square on EB, then

$$\text{sq. } AB + \text{sq. } EB \approx [AB, EB] + \text{sq. } AE + [AE, EB] + \text{sq. } EB,$$

$$\approx [AB, EB] + \text{sq. } AE + [AB, EB], \qquad [40\,(a)$$

$$\approx \text{sq. } AE + 2\,[AB, EB].$$

Square on difference.

46. Cor. *The square on the difference of two lines is equivalent to the sum of their squares diminished by twice their rectangle.*

Square on sum of whole and part.

47. THEOREM 14. *If a line is divided into any two parts, the square on the sum of the whole line and one part is equivalent to four times the rectangle of the whole line and that part, together with the square on the other part.*

Let the line AB be divided at E.

To prove that the square on the sum of AB and EB is equivalent to four times the rectangle of AB and EB, together with the square on AE.

Since sq. $(AB + EB) \backsimeq$ sq. AB + sq. $EB + 2[AB, EB]$; [43

and sq. AB + sq. $EB \backsimeq 2[AB, EB]$ + sq. AE; [45

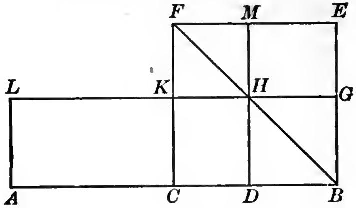
hence sq. $(AB + EB) \backsimeq 4[AB, EB]$ + sq. AE.

48. Cor. *The square on the sum of two segments exceeds the square on their difference by four times their rectangle.*

Rectangles of equal parts and of unequal parts.

49. THEOREM 15. *If a line is divided into two equal parts, and also into two unequal parts, the rectangle of the unequal parts, together with the square on the intermediate part, is equivalent to the square on half the line.*

Let the line AB be divided into equal parts at C, and into unequal parts at D.



To prove that the rectangle of AD and DB together with the square on CD is equivalent to the square on CB.

On CB describe the square $CBEF$. Through D draw DM perpendicular to CB and meeting the diagonal BF in H. Through H draw the line $GHKL$ parallel to AB; and complete the rectangle $ACKL$.

The figures DG, KM are squares; and the rectangles CH, HE are equivalent (36).

Add to each of these the figure DG; then the rectangles CG and DE are equivalent.

Now the rectangles AK and CG are equivalent, because AC equals CB and CK is common.

Therefore the rectangles AK and DE are equivalent.

Add the rectangle CH and also the square KM; then the rectangle AH and the square KM are together equivalent to the square CE.

Now AH is the rectangle of AD and DH, that is of AD and DB; and the square KM is equal to the square on CD.

Therefore the rectangle of AD and DB together with the square on CD is equivalent to the square on CB.

Otherwise:

$$[AD, DB] \backsimeq [AC, DB] + [CD, DB]. \qquad [39$$

Replace AC by its equal CB, and add the square on CD, then

$$[AD, DB] + \text{sq. } CD \backsimeq [CB, DB] + [CD, DB] + \text{sq. } CD,$$

$$\backsimeq [CB, DB] + [CB, CD], \qquad [40\ (a)$$

$$\backsimeq \text{sq. } CB. \qquad [40\ (b)$$

50. Cor. 1. *The rectangle of any two lines, together with the square on half their difference, is equivalent to the square on half their sum.*

Let AD and DB be the lines, AB their sum and CB their half sum.

Take CD' equal to CD. Then DB equals AD', and the difference of the two lines AD and DB is equal to the difference of AD and AD', which is $D'D$.

Therefore CD is half the difference of AD and DB.

Now the rectangle of AD and DB together with the square on CD is equivalent to the square on CB (49).

Hence the rectangle of two lines, together with the square on half their difference, is equivalent to the square on half their sum.

Difference of two squares expressed as a rectangle.

51. Cor. 2. *The rectangle of the sum and difference of two lines is equivalent to the difference of the squares on the lines.*

Let AC, CD be the two lines; then AD is their sum, and DB is their difference.

Hence, by 49, the rectangle of the sum and difference of two lines, together with the square on the less, is equivalent to the square on the greater.

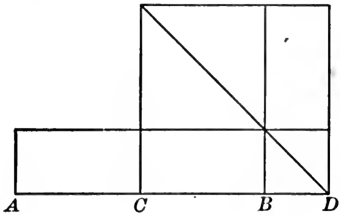
In other words, the rectangle of the sum and difference of two lines is equivalent to the difference of the squares on the lines.

Ex. 1. If there are two given squares, show how to construct a rectangle equivalent to their difference.

Ex. 2. If a line is divided into two equal parts and also into two unequal parts, show that the rectangle of the unequal parts is less than the rectangle of the equal parts.

Modification of 49.

52. THEOREM 16. *If a given line is bisected and then extended to any point, the rectangle contained by the extension and the whole line so extended, together with the square on half the original line, is equivalent to the square on the line between the point of bisection and the point of extension.*



Prove as in theorem preceding.

Otherwise:

$$[AD, BD] \backsimeq [AC, BD] + [CD, BD]. \qquad [39$$

Replace AC by its equal CB, and add the square on CB; then

$$[AD, BD] + \text{sq. } CB \backsimeq [CB, BD] + \text{sq. } CB + [CD, BD],$$

$$\backsimeq [CD, CB] + [CD, BD], \qquad [40\ (a)$$

$$\backsimeq \text{sq. } CD. \qquad [40\ (b)$$

53. Cor. *Show that* CB *is half the difference of* AD, DB; *and that* CD *is half the sum of* AD, DB; *and hence prove again that "the rectangle of two lines together with the square on half their difference is equivalent to the square on half their sum."*

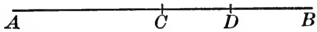
Ex. Prove again that "the rectangle of the sum and difference of two lines is equivalent to the difference of their squares."

[Let CD, CB be the segments, AD their sum, BD their difference.]

Squares on equal parts and on unequal parts.

54. THEOREM 17. *If a line is divided into two equal parts and also into unequal parts, the sum of the squares on the unequal parts is equivalent to double the sum on the squares on the half line and on the intermediate part.*

Let AB be the given line, divided into two equal parts at C and into two unequal parts at D.



To prove that the sum of the squares on AD, DB is equivalent to double the sum of the squares on AC, CD.

By 43 and 46

$$\text{sq. } AD \Leftrightarrow \text{sq. } AC + \text{sq. } CD + 2\,[AC,\ CD],$$

$$\text{sq. } DB \Leftrightarrow \text{sq. } CB + \text{sq. } CD - 2\,[CB,\ CD].$$

Add these equivalents, observing that AC equals CB, and that the equal rectangles disappear since one is added and the other subtracted. Therefore

$$\text{sq. } AD + \text{sq. } DB \Leftrightarrow 2 \text{ sq. } AC + 2 \text{ sq. } CD.$$

55. Cor. 1. *The sum of the squares on any two lines is equivalent to twice the square on half their sum together with twice the square on half their difference.* (See 50.)

56. Cor. 2. *The square on the sum of two lines together with the square on their difference is equivalent to double the sum of the squares on the two lines.*

[Let AC, CD be the given lines, AD their sum, DB their difference.]

Ex. Prove 56 directly from 43 and 46.

Modification of 54.

57. Theorem 18. *If a given line is bisected and then extended to any point, the sum of the squares on the extension and on the whole line so extended is equivalent to twice the square on half the original line, together with twice the square on the line between the point of bisection and the point of extension.*

Let AB be the given line, bisected at C, and then extended to D.

To prove that

$$\text{sq. } AD + \text{sq. } BD \approxeq 2 \text{ sq. } AC + 2 \text{ sq. } CD.$$

Show that the proof of the preceding theorem applies, letter by letter, to this theorem.

Combined statement of theorems 17, 18.

In order to combine these two theorems in one statement an extended meaning will now be given to the phrase "the two segments of a line."

58. *Definition.* If on the line AB the point C is taken between A and B, then the line AB is said to be *divided internally* into the two segments AC, BC.

Again, if the point C is taken on the prolongation of AB, then the line AB is said to be *divided externally* into the two segments AC, BC.

59. *Restatement.* The two theorems may then be restated as follows:

If a given line is bisected and divided unequally (either internally or externally), then the sum of the squares on the unequal parts is equivalent to twice the square on half the original line, together with twice the square on the line between the points of division.

EXERCISES

1. The square on the sum of two lines is greater than the sum of the squares on the two lines.

2. The sum of the squares on two lines is never less than twice their rectangle.

3. If a line is divided into two equal parts and also into two unequal parts, how does the sum of the squares on the equal parts compare with the sum of the squares on the unequal parts?

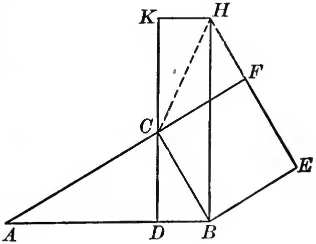
4. If a line is divided into two equal parts and also into two unequal parts, then the sum of the squares on the unequal parts exceeds twice their rectangle by four times the square on the intermediate segment.

EQUIVALENCES IN A TRIANGLE

Relations in a right triangle.

60. THEOREM 19. *In a right triangle the rectangle of the hypotenuse and the projection upon it of one of the other sides is equivalent to the square on that side.*

Let the triangle ABC have the angle C a right angle, and let BD be the projection of the side BC upon the hypotenuse BA.



To prove that the rectangle of AB and DB is equivalent to the square on BC.

On BC describe the square $BCFE$. Prolong EF to meet in H the line BH drawn perpendicular to AB. Complete the rectangle $DBHK$, and join CH.

In the triangles *BCA* and *BEH*, the angles *ABC* and *EBH* are equal, being each complemental to *CBH*; also the angles *BCA* and *BEH* are equal, being right; and the sides *BC* and *BE* are equal, being sides of a square. Hence the side *BA* equals *BH* (I. 65).

Therefore the rectangle *BK* is the rectangle of *AB* and *DB*.

Now this rectangle is double the triangle *CBH*, since they have the same base *BH*, and the same altitude *DB* (29).

Also the square *BF* is double the same triangle, since they have the same base *BC* and the same altitude *BE*.

Therefore the rectangle and square are equivalent (9).

That is to say, the square on the side *BC* is equivalent to the rectangle of the projection *BD* and the hypotenuse *BA*.

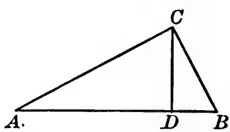
In the same way it can be proved that the square on the side *AC* is equivalent to the rectangle of its projection *AD* and the hypotenuse *BA*.

Theorem of Pythagoras.

61. THEOREM 20. *In any right triangle the square on the hypotenuse is equivalent to the sum of the squares on the other two sides.*

Let *ABC* be a triangle having *C* a right angle.

To prove that the square on the hypotenuse *AB* is equivalent to the sum of the squares on the sides *AC*, *CB*.

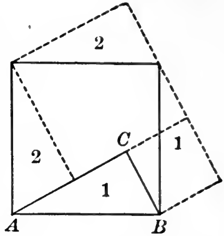


The square on *AB* is equivalent to the sum of the rectangles of *AB* and *AD*, and of *AB* and *BD* [40 (cor. 1)].

Now the rectangle of *AB* and *AD* is equivalent to the square on *AC* (60); and the rectangle of *AB* and *DB* is equivalent to the square on *CB*.

Therefore the square on *AB* is equivalent to the sum of the squares on *AC* and *CB*.

Ex. Show how to dissect the squares on AC and CB so that the parts may cover the square on AB.

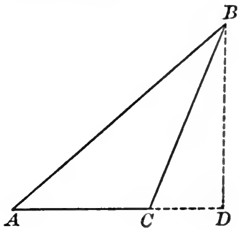


Note. The earliest proof of this celebrated theorem is attributed to Pythagoras (550 B.C.), the founder of the famous Pythagorean School in lower Italy. The theorem itself was, however, probably known as an experimental fact to the ancient Egyptians, a thousand years earlier. It is conjectured that the Pythagorean proof was based on some method of dissection similar to that shown. The proof given by Euclid (300 B.C.) is a combination of 60 and 61. The first part is here enunciated as a separate theorem on account of its great importance.

Relation in an obtuse triangle.

62. Theorem 21. *In an obtuse-angled triangle the square on the side opposite the obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by either of these sides and the projection of the other upon it.*

Let the triangle ABC have the angle C obtuse, and let CD be the projection of the side CB on AC extended.



To prove that the square on AB is greater than the sum of the squares on AC and CB by twice the rectangle of AC and CD.

The square on AD is equivalent to the sum of the squares on AC and CD, together with twice the rectangle of AC and CD (42).

Add to each member of this equivalence the square on BD.

Then the sum of the squares on AD and BD is equivalent to the sum of the squares on AC, CD, and DB together with twice the rectangle of AC and CD.

Now the sum of the squares on AD and DB is equivalent to the square on AB (61); and the sum of the squares on CD and DB is equivalent to the square on CB.

Therefore the square on AB is equivalent to the sum of the squares on AC and CB, together with twice the rectangle of AC and CD.

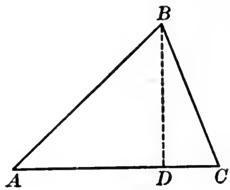
In other words, the square on AB exceeds the sum of the squares on AC and BC by twice the rectangle of AC and CD.

Relations in any triangle.

63. THEOREM 22. *In any triangle the square on the side opposite an acute angle is less than the sum of the squares on the sides containing that angle by twice the rectangle of either of these sides and the projection of the other upon it.*

Let ABC be a triangle having the angle C acute, and let DC be the projection of the side BC upon the side AC.

To prove that the square on AB is less than the sum of the squares on AC and CB by twice the rectangle of AC and DC.



The sum of the squares on AC and DC is equivalent to twice the rectangle of AC and DC together with the square on AD (45).

To each member of this equivalence add the square on DB.

Then the sum of the squares on AC, CD, DB is equivalent to twice the rectangle of AC and DC together with the sum of the squares on AD and DB.

But the sum of the squares on DC and DB is equivalent to the square on BC (61); and the sum of the squares on AD, DB is equivalent to the square on AB.

Therefore, the sum of the squares on AC and CB is equivalent to twice the rectangle of AC and DC together with the square on AB.

That is, the square on AB is less than the sum of the squares on AC and CB by twice the rectangle of AC and DC.

64. Cor. 1. *In any triangle, according as one angle is greater than, equal to, or less than a right angle, so is the square on the opposite side greater than, equivalent to, or less than the sum of the squares on the other two sides.* (61, 62, 63.)

65. Cor. 2. *In any triangle, according as the square on one side is equivalent to, greater than, or less than the sum of the squares on the other two sides, so is the angle opposite the first side greater than, equal to, or less than a right angle.*

Relation involving perpendicular.

66. THEOREM 23. *In any triangle, if a perpendicular is drawn from the vertex to the base, then according as the vertical angle is greater than, equal to, or less than a right angle, so is the rectangle of the segments of the base greater than, equivalent to, or less than the square on the perpendicular.*

Let BD be the perpendicular from the vertex to the base in the triangle ABC.

To prove that the square on BD is equivalent to, greater than, or less than the rectangle of AD and DC according as the angle B is right, acute, or obtuse.

The square on AC is equivalent to, less than, or greater than the sum of the squares on AB and CB according as the angle B is right, acute, or obtuse (64).

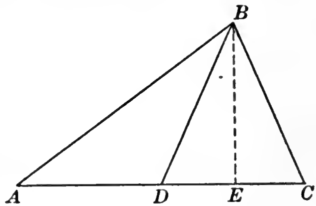
Now the square on AC is equivalent to the sum of the squares on AD and DC with twice the rectangle of AD and DC (42); the square on AB is equivalent to the sum of the squares on AD and DB; and the square on BC is equivalent to the sum of the squares on DC and DB (61).

Rejecting the common sum of squares on AD and DC, it remains that twice the rectangle of AD and DC is equivalent to, less than, or greater than twice the square on DB, according as the angle B is right, acute, or obtuse; whence by taking halves the theorem follows (11, 12).

Relation involving median.

67. THEOREM 24. *In any triangle, if a line is drawn from the vertex to the mid-point of the base, the sum of the squares on the two sides is equivalent to twice the square on half the base, together with twice the square on the median line.*

Let BD be a median line of the triangle ABC.



To prove that the sum of the squares on the sides AB and BC is equivalent to twice the square on DC and twice the square on BD.

Case 1. Let the angles BDA and BDC be equal.

Then BD is perpendicular to AC. Therefore the square on AB is equivalent to the sum of the squares on AD and BD (61); and the square on BC is equivalent to the sum of the squares on BD and DC.

Hence the sum of the squares on AB and BC is equivalent to twice the square on BD, together with twice the square on AD.

Case 2. Let the angles BDA and BDC be unequal.

Then one of them is the greater. Let the angle ADB be greater than BDC. The angle ADB is then obtuse, and BDC is acute.

Let BE be the perpendicular from B to the base AC.

Then in the triangle ADB the square on AB is equivalent to the sum of the squares on AD and BD together with twice the rectangle of AD and DE (62).

Again, in the triangle BDC, the square on BC is equivalent to the sum of the squares on BD and DC diminished by twice the rectangle of DC and DE (63).

Add these equivalences, member to member, and reject the equivalent double rectangles.

Then the sum of the squares on AB and BC is equivalent to twice the square on BD and twice the square on AD.

68. Cor. *The sum of the squares on the four sides of a parallelogram is equivalent to the sum of the squares on the two diagonals.* (Use I. 159; and 67, 44.)

Ex. The sum of the squares on the four sides of a quadrilateral is equivalent to the sum of the squares on the two diagonals, and four times the square on the segment joining the mid-points of the diagonals.

Difference of squares on sides.

69. THEOREM 25. *In any triangle, if a line is drawn from the vertex perpendicular to the base (or base prolonged), then the difference of the squares on the two sides is equivalent to the difference of the squares on the segments of the base.*

[Take the figure of 62, or 63. Apply 61 to each of the right triangles. Then subtract.]

70. Cor. *In an isosceles triangle, if a line is drawn from the vertex to any point of the base (or base prolonged), then the difference of the squares on this line and on one side of the triangle is equivalent to the rectangle of the segments of the base.*

Outline. Let BA, BC be the equal sides, P any point on the base AC. Draw perpendicular BM, which bisects AC at M. Apply 69 to the triangle ABP. Then use 50. Give the proof in the usual form.

EXERCISES

1. In an isosceles right triangle the square on one side is equivalent to half the square on the hypotenuse; and the square on the perpendicular is equivalent to one fourth of the square on the hypotenuse.

2. If the acute angles of a right triangle are respectively equal to one third and two thirds of a right angle, then the squares on the opposite sides are respectively equivalent to one fourth and three fourths of the square on the hypotenuse.

3. In the same case the square on the perpendicular is equivalent to three fourths of the square on the side.

4. In an equilateral triangle the square on the altitude is equivalent to three fourths of the square on the side.

5. In an isosceles triangle, if a perpendicular is drawn from an extremity of the base to the opposite side, then twice the rectangle contained by that side and its segment adjacent to the base is equivalent to the square on the base.

6. In any triangle, if an angle is equal to two thirds of a straight angle, then the square on the side opposite is equivalent to the sum of the squares on the other two sides and the rectangle contained by them.

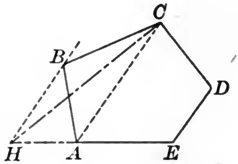
7. If any point is joined to the four vertices of a rectangle, the sum of the squares on the lines drawn to two opposite vertices is equivalent to the sum of the squares on the other two joining lines.

CONSTRUCTION OF EQUIVALENT POLYGONS

Reduction of polygon to equivalent triangle.

71. PROBLEM 1. *To construct a triangle equivalent to a given polygon.*

Let $ABCDE$ be the given polygon.



To construct a triangle equivalent to it.

Draw any diagonal AC connecting the ends of two adjacent sides. Through the intermediate vertex B draw BH parallel to this diagonal to meet one of the sides next in order, say EA, in H; and draw CH.

The triangles CAB and CAH are equivalent (33).

To each add the polygon $ACDE$; then the given polygon $ABCDE$ is equivalent to the polygon $EHCD$.

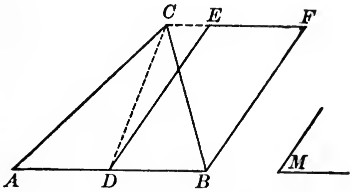
The number of sides of the latter polygon is one less than the number of sides of the given polygon.

Repeating this process a set of equivalent polygons having fewer and fewer sides is obtained; and the process ends when a three-sided figure is reached.

Conversion of triangle into equivalent rectangle.

72. PROBLEM 2. *To construct a parallelogram equivalent to a given triangle, and having an angle equal to a given angle.*

Let ABC be the given triangle, and M the given angle.



To construct a parallelogram equivalent to the triangle ABC, and having an angle equal to M.

Bisect AB at D, and draw DE, making the angle BDE equal to M. Draw CEF parallel to AB, and complete the parallelogram $DBFE$.

This parallelogram is equivalent to the triangle ABC.

To prove this, draw CD.

The parallelogram $DBFE$ is double the triangle BDC, since they have the same base and equal altitudes (34).

The triangles ADC, DBC are equivalent, having equal bases and the same altitude (29). Hence the triangle ABC is double the triangle DBC.

Therefore the parallelogram $DBFE$ is equivalent to the given triangle ABC (9); and it has an angle equal to the given angle.

Ex. 1. To construct a rectangle that shall be equivalent to a given triangle.

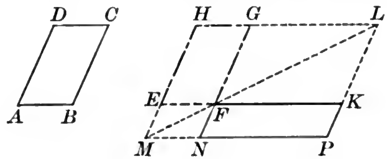
Ex. 2. To construct an isosceles triangle equivalent to a given triangle.

Ex. 3. To construct a parallelogram equivalent to a given parallelogram, and having an angle equal to a given angle.

Conversion of parallelogram into equivalent one.

73. PROBLEM 3. *On a given line to construct a parallelogram equivalent to a given parallelogram, and having its angles equal to the angles of this parallelogram.*

Let $ABCD$ be the given parallelogram and FK the given line.



It is required to construct on FK a parallelogram equivalent to $ABCD$ and having its angles respectively equal to the angles of $ABCD$.

Prolong KF, and lay off FE equal to BA, one of the sides of the given parallelogram.

Transfer the figure $ABCD$ to the position $EFGH$, so that BA may fall on FE (I. 199).

Draw KL parallel to FG to meet HG extended in L. Draw LF, and prolong it to meet HE extended in M. Draw MP parallel to EF, and let it meet the extensions of GF and LK in N and P.

Then $NPKF$ is the required parallelogram.

For it is equivalent to $EFGH$ (36); its angles are equal to the angles of $EFGH$ (I. 126); and it is described on the given line FK.

73 (*a*). Cor. *On a given line to construct a rectangle equivalent to a given rectangle.*

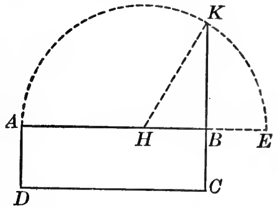
Ex. 1. Given one side of a rectangle and the equivalent square, find the adjacent side.

Ex. 2. On a given line to construct a rectangle equivalent to a given triangle (72, 73).

To "square" a rectangle.

74. Problem 4. *To construct a square equivalent to a given rectangle.*

Let $ABCD$ be the given rectangle.



To construct a square equivalent to it.

Prolong AB to E, making BE equal to BC. Bisect AE at H (I. 70). With H as center and HE as radius describe the arc EKA. Prolong CB to meet this arc in K.

The square constructed on BK is equivalent to the given rectangle.

To prove this, draw the radius HK.

The rectangle of AB and BE with the square on HB is equivalent to the square on HE (49); that is, to the square on HK, which is equivalent to the sum of the squares on HB and BK (61).

Reject the common square on HB. Then the rectangle of AB and BE is equivalent to the square on BK.

Now the rectangle $ABCD$ is the rectangle of AB and BC, that is, of AB and BE.

Therefore the square on BK is equivalent to the given rectangle.

75. Summary. By combining the constructions given in problems 1, 2, 4, a square can be constructed equivalent to any given polygon.

This square is called its *equivalent square;* and the process of construction is called *squaring the polygon.*

Ex. 1. Give the complete construction for squaring a given triangle.

Ex. 2. Also for squaring a given quadrangle.

Ex. 3. If the lines AK, KE are drawn, prove that the angle AKE is equal to the sum of the angles KAE, AEK; and hence that AKE is a right angle.

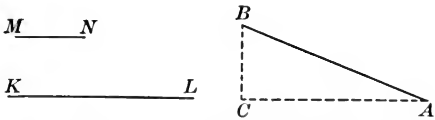
Ex. 4. Use ex. 3 to construct on a given line a rectangle equivalent to a given square.

[Here AB, BK are given, to find BE. Compare with 73, ex. 1.]

Addition of squares.

76. PROBLEM 5. *To construct a square equivalent to the sum of two given squares.*

Let KL, MN be the sides of the given squares.



To construct a square equivalent to the sum of the squares on these lines.

Draw CA equal to KL. Erect CB perpendicular to CA and equal to MN. Draw the line AB.

The square on AB is equivalent to the sum of the squares on KL and MN.

For the square on AB is equivalent to the sum of the squares on AC and CB (61) and hence equivalent to the sum of the squares on KL and MN.

Ex. To construct a square equivalent to the sum of three or more given squares.

Subtraction of squares.

77. PROBLEM 6. *To construct a square equivalent to the difference of two given squares.*

[This is a particular case of I. 136.]

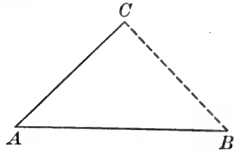
Ex. Show how to construct a square equivalent to the difference of two polygons.

To halve a square.

78. PROBLEM 7. *To construct a square equivalent to half a given square.*

Let AB be the side of the given square.

To construct a square equivalent to half the square on AB.



Draw AC, BC, making the angles BAC, ABC each equal to half a right angle.

Show that the square on AC is half the square on AB.

DIVISION OF A LINE

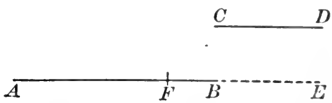
In the following problems a given line is to be divided so that the two parts may fulfill given conditions: *e.g.*, have a given difference; a given sum of squares; etc.

Difference given.

79. PROBLEM 8. *To divide a given line into two parts so that the difference of the parts shall be equal to another given line.*

Let AB, CD be the given lines, of which AB is the greater.

To divide AB into two parts whose difference shall be equal to CD.



Prolong AB, making BE equal to CD. Bisect AE in F. This point F divides AB so that the difference between AF and FB is equal to CD.

For, since AF equals FE, the difference between AF and FB is equal to the difference between FE and FB, which is equal to BE, that is to CD.

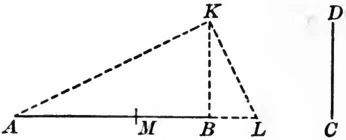
Restriction. The line CD must be less than AB, otherwise there is no solution to the problem.

Difference of squares.

80. PROBLEM 9. *To divide a given line internally so that the difference of the squares on the two segments may be equivalent to a given square.*

Let AB be the given line, and let CD be the side of the given square.

To divide AB so that the difference of the squares on the parts may be equivalent to the square on CD.



Erect BK perpendicular to AB and equal to CD. Draw AK, and erect KL perpendicular to AK, meeting AB extended in L. Bisect AL in M.

Then M divides AB so that the difference of the squares on AM and MB is equivalent to the square on CD.

For the rectangle of AB and BL is equivalent to the square on BK (66). Also the rectangle of AB and BL with the square on MB is equivalent to the square on ML (49).

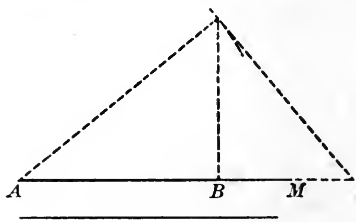
Hence the rectangle of AB and BL is equivalent to the difference of the squares on AM and MB.

Therefore the difference of the squares on AM and MB is equivalent to the square on BK, that is to the square on CD.

Restriction. The given square must be less than the square on the given line.

81. NOTE. This problem may also be solved by drawing KM so as to make the angle AKM equal to MAK, and then proving by 61.

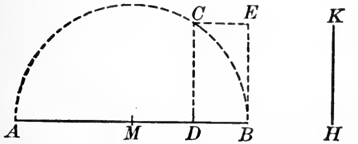
82. Cor. *To divide a given line externally so that the difference of the squares on the two segments shall be equivalent to a given square, when the latter is greater than the square on the given line.*



Rectangle given.

83. PROBLEM 10. *To divide a given line internally so that the rectangle of the two segments may be equivalent to a given square.*

Let AB be the given line, and HK the side of the given square.



To divide AB into two parts AD and DB so that the rectangle of AD and DB may be equivalent to the square on HK.

Bisect AB at M. With M as center and radius MB describe the arc ACB. Erect BE perpendicular to AB and equal to HK. Draw EC parallel to BA, meeting the arc in C. Draw CD perpendicular to AB.

The point D divides AB so that the rectangle of AD and DB is equivalent to the square on HK.

Prove as in 74.

Restriction. The given square must be less than the square on half the given line.

84. NOTE. This problem can also be solved by the use of 51 and 73, etc.

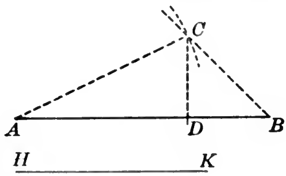
85. PROBLEM 11. *To divide a given line externally so that the rectangle of the two segments shall be equivalent to a given square.*

Outline. Let AB be the given line; draw the perpendicular BC equal to side of given square; take M mid-point of AB; join MC; on AB prolonged lay off MD equal to MC. Prove by 52 that rectangle of AD and BD is equivalent to the square on BC.

Sum of squares given.

86. Problem 12. *To divide a given line internally so that the sum of the squares on the parts may be equivalent to a given square.*

Let AB be the given line, and HK a side of the given square.



To divide AB at D so that the sum of the squares on AD and DB may be equivalent to the square on HK.

Construct the angle ABC equal to half a right angle. With A as center and radius equal to HK, describe an arc cutting BC in C. Draw CD perpendicular to AB.

The point D is the required point of division.

To prove this, draw the line AC.

The angle BCD, being equal to the difference of the angles ADC and DBC, is equal to half a right angle, and is therefore equal to the angle DBC. Hence DC equals DB (I. 62).

Therefore the sum of the squares on AD and DB is equivalent to the sum of the squares on AD and DC; which is equivalent to the square on AC, that is to the square on the given line HK.

Restriction. The given square must be less than the square on the whole line, and greater than twice the square on half the line, otherwise there is no solution.

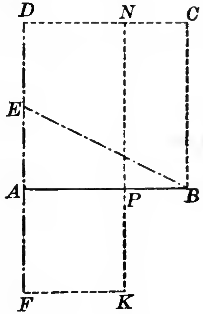
87. Note. This problem can also be solved by means of 55.

88. **Cor.** *To divide a given line externally so that the sum of the squares on the two segments shall be equivalent to a given square, which is greater than the square on the given line.*

Medial section.

89. Problem 13. *To divide a given line into two parts such that the rectangle of the whole line and one part may be equivalent to the square on the other part.*

Let AB be the given line, on which it is required to find a point P such that the rectangle of AB and BP may be equivalent to the square on AP.



On AB construct the square $ABCD$. Bisect AD at E, and join EB. Prolong EA, and take EF equal to EB. On AF construct the square $AFKP$.

Then P is the required point of division.

To prove this, complete the rectangle $FKND$.

Since AD is bisected at E and prolonged to F, therefore the rectangle of DF and AF together with the square on EA is equivalent to the square on EF (52).

Therefore the rectangle DK is equivalent to the difference of the squares on EF and EA; that is, to the difference of the squares on EB and EA; that is, to the square on AB. Hence the rectangle DK is equivalent to the square $ABCD$.

Take away the common part AN; then the square AK is equivalent to the rectangle PC.

Now PC is the rectangle of BC and PB, that is, of AB and PB; and AK is the square on AP.

Therefore the rectangle of AB and PB is equivalent to the square on AP.

Note. When a line is divided as in this problem, it is said to be divided in *medial section*, for reasons that appear later (V. 97). The ancients called this mode of division *sectio aurea*, the golden section, on account of its important applications.

LOCUS PROBLEMS

90. Problem 14. *To find the locus of the vertex of a triangle whose base is given in magnitude and position, and whose equivalent square is given.*

Outline. On the given base construct a rectangle equivalent to double the given square (73 (*a*)). The side parallel to the given line, extended indefinitely, is the required locus. Prove by 29, 33, 31.

Ex. On a given line construct an isosceles triangle equivalent to a given square. [Intersection of loci, I. 253 (ex.)].

91. Problem 15. *To find the locus of the vertex of a triangle whose base is given in magnitude and position, and the difference of the squares on whose sides is equivalent to a given square.*

Outline analysis. Take any point P satisfying the given conditions, and from it draw a perpendicular PQ to the base AB. Show that any point on PQ satisfies the conditions (see 69); and that any point not on PQ does not satisfy them.

Show that the locus is constructed by dividing AB internally or externally at Q so that the difference of the squares on the segments shall be equivalent to the given square (internally as in 80 if the given square is less than the square on the given line, externally as in 82 if greater), and then drawing QP perpendicular.

Ex. Show how to construct a triangle, being given the base, the equivalent square, and the difference of squares on sides. (Intersection of loci.)

MAXIMA AND MINIMA

92. Definition. Of all the magnitudes of a certain class that fulfill prescribed conditions the greatest is called the *maximum,* and the least the *minimum.*

Among the magnitudes that fulfill the prescribed conditions there may be a number of equal magnitudes that are each greater (or less) than any of the others; in such case each of the equal magnitudes will be called a maximum (or minimum).

93. In each of the following theorems it is important to distinguish clearly between the prescribed conditions that define the class of figures considered, and the additional condition that characterizes a maximum (or minimum) figure of the class. The theorems concerning maxima come under one of the following type-forms, which are converse to each other:

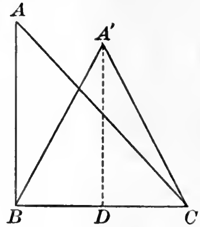
1. Among all the magnitudes that fulfill the set of conditions A, any one that is a maximum fulfills the additional condition B.

2. Among all the magnitudes that fulfill the set of conditions A, any one that fulfills the additional condition B is a maximum.

The first form asserts that the additional condition B is *necessary* for a maximum; the second form asserts that the additional condition B is *sufficient* for a maximum. As we shall be concerned with the complete conditions for a maximum, both of the converse theorems will be considered; which of them is to be proved first will depend on the nature of the prescribed conditions. Similar type-forms apply also to theorems concerning minima.

Greatest triangle having two given sides.

94. Theorem 26. *If two sides of a triangle are given, the triangle is a maximum when the given sides include a right angle.*



Let the triangles ABC and $A'BC$ have the sides AB and BC equal to the sides $A'B$ and BC respectively. Let the angle ABC be a right angle, and $A'BC$ an oblique angle.

To prove the triangle ABC greater than $A'BC$.

[Prove the altitude AB greater than the altitude $A'D$.]

95. Cor. 1. *Conversely, if two sides are given, and if the triangle is a maximum, then the given sides include a right angle.*

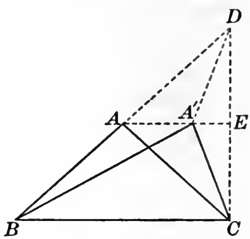
For if not, a greater triangle could be constructed by making the included angle a right angle, contrary to the hypothesis that the given triangle is a maximum.

96. Cor. 2. *Of all parallelograms having given sides, one that is rectangular is a maximum; and conversely.*

Least perimeter in equivalent triangles.

97. Theorem 27. ***Of all equivalent triangles having the same base, that which is isosceles has the least perimeter.***

Let ABC and $A'BC$ be equivalent triangles having the same base BC, the first triangle being isosceles and the second not.



To prove that the perimeter of ABC is less than the perimeter of $A'BC$.

Draw CE perpendicular to AA', and prolong it to meet the prolongation of BA in D. Join $A'D$.

Outline proof. Prove in succession: AA' parallel to BC (33); angle EAD equal to EAC; AD equal to AC; $A'D$ equal to $A'C$; sum of BA and AC equal to BD; which is less than sum of BA' and $A'D$; which equals the sum of BA' and $A'C$. Draw conclusion.

98. **Cor. 1.** *Conversely, of all equivalent triangles having the same base, that which has the least perimeter is isosceles.*

[Prove by *reductio ad absurdum*, as in 95.]

99. **Cor. 2.** *Of all equivalent triangles, one that has a minimum perimeter is equilateral.*

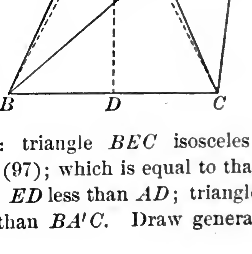
Let ABC be a triangle of minimum perimeter belonging to the given set of equivalent triangles. Take BC as base. Then the sides AC and AB are equal by 98. Similarly, the other pairs of sides are equal. Hence ABC is equilateral.

100. **Cor. 3.** *Conversely, of all equivalent triangles, one that is equilateral has a minimum perimeter.*

Outline. Let ABC be an equilateral triangle belonging to the given set of equivalent triangles. Let $A'B'C'$ be a triangle of minimum perimeter in the set. Then $A'B'C'$ is equilateral (99). Since these two equilateral triangles are equivalent, hence their sides are equal (32, ex.). Therefore ABC is isoperimetric with $A'B'C'$, and has thus the minimum perimeter in the set of equivalent triangles.

Greatest surface in isoperimetric triangles.

101. Theorem 28. *Of all triangles having a given perimeter and a given base, one that is isosceles has a maximum surface.*



Let ABC be an isosceles triangle, and let $A'BC$ be any other triangle having an equal perimeter and the same base BC.

To prove the triangle ABC greater than $A'BC$. Draw AD perpendicular to BC, and $A'E$ parallel to BC; join BE, CE.

Outline proof. Prove in succession: triangle BEC isosceles; perimeter of BEC less than that of $BA'C$ (97); which is equal to that of BAC (hyp.); hence BE less than BA; ED less than AD; triangle BAC greater than BEC; hence greater than $BA'C$. Draw general conclusion.

102. Cor. 1. *Conversely, of all triangles having a given perimeter and a given base, one that has a maximum surface is isosceles.*

103. Cor. 2. *Of all triangles having a given perimeter, one that has a maximum surface is equilateral.* [Compare 102.]

104. Cor. 3. *Conversely, of all triangles having a given perimeter, one that is equilateral has a maximum surface.*

Rectangle of parts greatest.

105. Theorem 29. *If a line is divided into any two parts, the rectangle of the parts is a maximum when the two parts are equal.*

[Divide the line equally and unequally, and show by 49 that the rectangle of the unequal parts is less than that of the equal parts.]

106. Cor. *If a square and a rectangle have equal perimeters, the square is greater than the rectangle.*

Sum of squares least.

107. Theorem 30. *If a line is divided into any two parts, the sum of the squares on the parts is a minimum when the two parts are equal.*

[Show that the sum of the squares on the unequal parts is greater than the sum of the squares on the equal parts (54).]

EXERCISES

1. Divide a given line into two parts such that the square on one of them may be double the square on the other.

2. If A, B, C, and D are four points in order on a line, then the rectangle of AD and BC is equivalent to the sum of the rectangles of AB and CD, and of AD and BC.

3. Divide a given line into three parts so that the sum of the squares on them may be a minimum.

4. Construct a rectangle equivalent to a given square, and having (1) the sum, (2) the difference of adjacent sides equal to a given line.

5. Show that 42 may be regarded as a limiting case of 62; and 45 of 63. Show that 54 and 57 may be regarded as limiting cases of 67.

BOOK III.—THE CIRCLE

FUNDAMENTAL PROPERTIES

1. The circle and its center and radius were defined in Introduction 28; and the postulate relating to drawing a circle was stated in the next article.

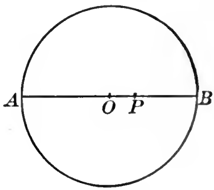
Since the circle is a closed curve, a line drawn from any point within the circle to any point without intersects the circle at least once; and an indefinite line drawn through any point within the circle intersects it at least twice.

That no straight line intersects a circle more than twice will be shown in theorem 4, and corollary.

PROPOSITIONS RELATING TO THE CENTER

2. THEOREM 1. *A circle has only one center.*

Let ABC be a circle described with center O (post. 3).



To prove that there is no other center than O.

Suppose, if possible, that the circle has another center P.

Draw OP, and let it meet the circle in the points A and B.

Since O is a center, therefore OA equals OB. Hence O is the mid-point of AB.

Similarly, P is the mid-point of AB, which is impossible unless O and P coincide.

Therefore a circle has only one center.

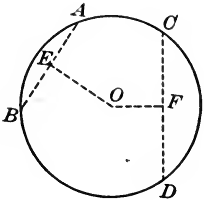
3. THEOREM 2. *If any two points are taken on a circle, the perpendicular bisector of the line joining them passes through the center.*

For since the two lines drawn from the center to the two given points are equal, therefore the center lies on the perpendicular bisector of the line joining the two given points (I. 253).

Ex. What is the locus of the centers of all the circles that pass through two given points?

4. PROBLEM 1. *To find the center of a given circle.*

Let ABC be the given circle of which it is required to find the center.



Take four points A, B, C, and D on the circle, and let them be so situated that the lines AB and CD are not parallel. Through the mid-points of AB and CD draw the perpendiculars EO and FO; these lines are not parallel, for otherwise the lines AB and CD would be parallel. Let the perpendiculars intersect in O.

The point O is the required center.

For the center lies on each of the lines EO, FO (3).

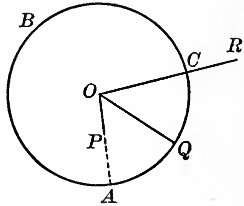
Therefore the center is at O, the intersection of the two perpendiculars.

NOTE. This construction may be regarded as an application of the method of intersection of loci (I. 256).

Ex. Being given any portion of a circle, show how to find the center, and how to complete the circle.

5. Theorem 3. *The line joining any point to the center of a circle is less than, equal to, or greater than the radius, according as the point is within, on, or without, the circle.*

Let ABC be a circle whose center is O. Let P, Q, R be any points within, on, and without, the circle, respectively.



To prove that OP is less than a radius, OQ equal to a radius, and OR greater than a radius.

Extend OP to meet the circle in A, and let OR cut the circle in C.

The lines OA, OQ, and OC are equal, being radii.

But OP is less than OA; and OR is greater than OC.

Therefore OP, OQ, OR are respectively less than, equal to, and greater than, the radius.

6. Cor. 1. *A point is within, on, or without, the circle, according as the line joining it with the center is less than, equal to, or greater than, the radius.*

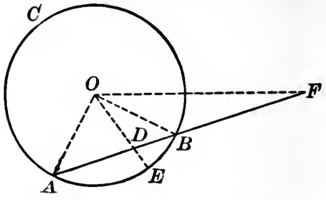
7. Cor. 2. *The locus of a point, such that its join to a given point shall be equal to a given line, is a circle described with the given point as center and the given line as radius.*

Intersections of Line and Circle

8. Theorem 4. *If any two points of a circle are joined by a straight line, all points of this line situated between the given points lie within the circle; and all points in the line extended either way beyond the given points, lie without the circle.*

Let A, B be any two points on the circle; D a point in the line AB, situated between A and B; F any point on the line AB extended either way.

To prove that D is within, and F without, the circle.



Find O, the center of the circle (4). Join OA, OB, OD, OF; and extend OD through D to meet the circle in E.

Since the triangle OAB is isosceles, OD is less than OB, and OF is greater than OB (I. 86).

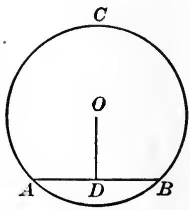
Then since OD is less than the radius, the point D is within the circle; and since OF is greater than the radius, the point F is without the circle (5).

9. Cor. *A straight line cannot meet a circle in more than two points.*

Chords and secants.

10. Definitions. A straight line that meets a circle in two points is called a *secant*. The portion of a secant included within the circle is called a *chord*. A chord that passes through the center is called a *diameter*.

11. THEOREM 5. *The perpendicular from the center to a chord bisects it; and, conversely, the line drawn from the center to the mid-point of a chord is perpendicular to the chord.*



Let AB be any chord of the circle ABC, whose center is O. Let OD be the perpendicular from O to AB.

To prove that AB is bisected at D. (Use I. 99.)

Converse. Let AB be bisected at D.

To prove OD perpendicular to AB. (Use I. 66.)

Ex. 1. The line which bisects perpendicularly one of two parallel chords of a circle also bisects the other perpendicularly.

Ex. 2. If a line intersects two concentric circles, the intercepts between the circles are equal.

Ex. 3. Any diameter of a circle is an axis of symmetry.

12. Theorem 6. *The least line that can be drawn from a given point to a given circle is a segment of the line that passes through the center, and the extremities of this segment are at the same side of the center.*

Let $ABCD$ be a circle whose center is O. Let P be any point, either without the circle (fig. 1), or within the circle (fig. 2). Let the line PO meet the circle in the points A and D, of which A is at the same side of the center as P is. Let PBC be any other secant through P meeting the circle in the points B and C, of which B is at the same side of the mid-point of the chord BC as P is.

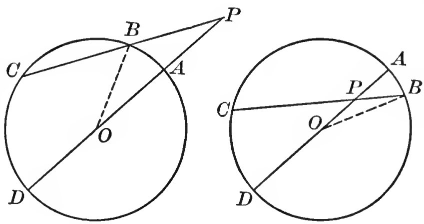


Fig. 1 Fig. 2

To prove that PA is less than PB.

In the triangle POB (fig. 1), the side OP is less than the sum of the sides OB and PB (I. 87).

Taking away the equals OA and OB, it follows that PA is less than PB.

Again, in the triangle POB (fig. 2), the sum of OP and PB is greater than OB, and therefore greater than OA.

Taking away the common part OP, it follows that PB is greater than PA, that is, PA is less than PB.

13. Cor. 1. *The greatest line from any point to the circle is a portion of the same secant as the least line is.*

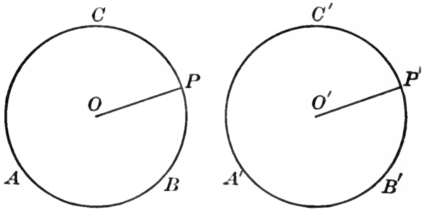
14. Cor. 2. *The greatest chord of a circle is a diameter.*

PROPERTIES OF EQUAL CIRCLES

Conditions for Coincidence

15. Theorem 7. *Circles of equal radii are equal.*

Let ABC, $A'B'C'$ be two circles of equal radii, whose centers are O and O'.



To prove that the circles are equal.

Let P be any point on the circle ABC.

Place the circle ABC upon $A'B'C'$ so that the center O may coincide with the center O'. Let the point P fall on P'.

The line $O'P'$ equals OP, being coincident with it; and OP equals the radius of the circle $A'B'C'$ (hyp.).

Therefore $O'P'$ equals the radius of the circle $A'B'C'$.

Hence P' is on the circle $A'B'C'$ (6).

Thus any point of the circle ABC will fall on the circle $A'B'C'$.

In like manner, any point of the circle $A'B'C'$ will coincide with a point of the circle ABC.

Hence the two circles coincide in all their parts, and are therefore equal.

16. **Cor. 1.** *If two circles are equal, their radii are equal.*

What are the contraposites of 15 and 16?

17. **Cor. 2.** *Two circles which have one point in common, and which have the same center, coincide throughout.*

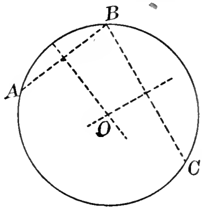
18. Cor. 3. *Two circles which have one point in common, and which do not coincide throughout, have not the same center.*

How is this corollary related to the preceding?

19. Cor. 4. *Two circles which have the same center and have unequal radii have no point in common.*

20. Theorem 8. *Through three given points not in the same straight line, one, and only one, circle can pass.*

Let A, B, C be three given points not in the same straight line.



First, to prove that a circle can pass through A, B, C.

Find the point O such that the three lines OA, OB, OC are equal (I. 257).

The circle described with O as center and a radius equal to OA passes through A, B, C, and thus fulfills the requirements.

Next, to prove that only one circle can pass through A, B, C.

The point O is the only point such that OA, OB, OC are equal (I. 257).

Therefore all the circles passing through A, B, C have the same center and equal radii; and hence they coincide (17).

Therefore there is only one circle passing through A, B, C.

21. Cor. 1. *Two circles having three points in common coincide.*

Show that this is only another statement of the second part of 20.

22. Cor. 2. *Two circles that do not coincide do not meet in more than two points.*

How is this corollary related to the preceding?

23. Cor. 3. *If from any point within a circle more than two lines drawn to the circle are equal, then that point is the center.*

Outline. Show that the opposite of this conclusion would lead to the opposite of I. 257.

24. Cor. 4. *If two triangles be equal, the circle passing through the vertices of one is equal to the circle passing through the vertices of the other.*

25. Cor. 5. *Two circles cannot have a common portion without coinciding throughout.*

Arcs and Central Angles

26. Definitions. An *arc* is part of a circle. Two arcs which together make up the whole circle are said to be *conjunct arcs.* When two conjunct arcs are equal, each is called a *semicircle.* When two conjunct arcs are unequal, the greater is called the *major* conjunct arc and the less the *minor* conjunct arc.

An angle at the center of a circle formed by two radii is called a *central angle.* Of the two conjunct central angles formed by the same two radii, the less is called the *minor* and the other the *major.*

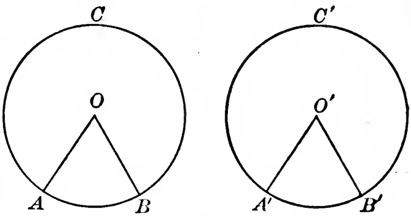
The two conjunct central angles are said to *stand on* the two arcs intercepted by the sides, the minor angle on the minor arc, and the major angle on the major arc. In the same case the minor arc is said *to subtend* the minor angle, and the major arc the major angle. An arc subtending a central right angle is called a *quadrant.*

A *sector* is a figure composed of an arc and the two radii drawn to its extremities. The central angle formed by the radii is called the *angle of the sector;* and the sector is said *to contain* the central angle.

In any two circles, arcs that subtend equal central angles are said to be *similar,* and so are the corresponding sectors.

27. THEOREM 9. *In equal circles, or in the same circle, equal central angles stand on equal arcs.*

Let ABC, $A'B'C'$ be equal circles whose centers are O, O'. Let AOB, $A'O'B'$ be equal central angles.



To prove that the arc AB is equal to the arc $A'B'$.

Let the circle ABC be applied to $A'B'C'$ so that the radius OA coincides with the equal radius $O'A'$, and the angle AOB with its equal angle $A'O'B'$.

Then the radius OB coincides with the equal radius $O'B'$; and the circle ABC with the circle $A'B'C'$ (15).

Hence the arc AB coincides with the arc $A'B'$; and these arcs are therefore equal.

Ex. 1. A diameter divides a circle into two equal arcs; and two diameters at right angles divide it into four equal arcs.

Ex. 2. All quadrants of the same circle are equal; and each is equal to one fourth of the circle.

27 (*a*). Cor. *In equal circles, or in the same circle, sectors which include equal central angles are equal.*

NOTE. As it is evident that the theorems proved for equal circles are also true for the same circle, the words "or in the same circle" will usually be omitted.

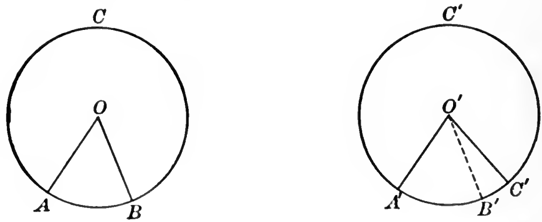
28. Definition. As equal circles have equal radii, any two arcs of equal circles will be called *equiradial arcs;* and any two sectors of equal circles will be called *equiradial sectors.*

29. Comparison of equiradial arcs. Two equiradial arcs are compared in the same way as two line-segments are compared, viz. by transferring one so that an extremity falls on an extremity of the other and so that one of the arcs may coincide with the whole or part of the other. The terms *equal, greater,* and *less* are then applied in accordance with the general definitions (Introd. 35).

The surfaces of two equiradial sectors are compared in the same way as two angles are (I. 11).

30. THEOREM 10. *In equal circles two unequal central angles stand on unequal arcs, the greater angle standing on the greater arc.*

In the equal circles ABC, $A'B'C'$, let the central angle AOB be less than the central angle $A'O'C'$.



To prove that the arc AB is less than the arc $A'C'$.

Draw the line $O'B'$ cutting off from the greater angle $A'O'C'$ a part $A'O'B'$ equal to the less angle AOB (I. 77).

Then the arc $A'B'$ equals the arc AB (27).

Therefore the arc $A'C'$, being greater than its part $A'B'$, is greater than the arc AB.

Combined statement.

31. Cor. I. *In two equal circles, according as a central angle of one is greater than, equal to, or less than, a central angle of the other, so is the arc subtending the first greater than, equal to, or less than, the arc subtending the second.*

32. Cor. 2. *In two equal circles, according as a central angle of one is greater than, equal to, or less than, a central angle of the other, so is the sector containing the first greater than, equal to, or less than, the sector containing the second.*

33. **Cor. 3.** *According as a central angle is greater than, equal to, or less than, a right angle, so is the opposite arc greater than, equal to, or less than, a quadrant ; and conversely.*

34. **Cor. 4.** *According as a central angle is greater than, equal to, or less than, a straight angle, so is the opposite arc greater than, equal to, or less than, a semicircle ; and conversely.*

Converse of 31.

35. THEOREM 11. *In two equal circles, according as an arc of one is greater than, equal to, or less than, an arc of the other, so is the central angle standing on the first greater than, equal to, or less than, the central angle standing on the second.*

Addition of equiradial arcs, or sectors.

36. **Definitions.** The *sum* of two equiradial arcs is the arc obtained by laying off on any equiradial circle two arcs respectively equal to the given arcs, so as to have a common extremity without overlapping. This sum is called the result of *adding* the two arcs.

The sum of three or more equiradial arcs is the result of adding the third arc to the sum of the first two, and so on.

The arc obtained by adding two equal arcs is called the *double* of either, and each of the latter is said to be *half* of the former.

The *difference* of two unequal equiradial arcs is the arc which must be added to the less to produce the greater.

The sum of two or more equiradial sectors is the whole sector obtained by placing them so that their angles are adjacent in succession (I. 9). The words *double, half, difference,* are applied to equiradial sectors in the usual way.

37. **Magnitudes directly comparable.** We have met with four kinds of magnitudes such that two of the same kind may always be directly compared by superposition (without previous dissection). They are: line-segments (I. 2); angles (I. 9); equiradial arcs (29); equiradial sectors.

38. Axioms of equality and inequality. The axioms given in I. 24–32 for line-segments and angles may now be proved for equiradial arcs and sectors as direct inferences from the definitions given in 29, 36.

39. Additional principles of equality. The following additional principles can be easily proved for these four kinds of magnitudes and will be of frequent use:

a. The half of the sum of two magnitudes is equal to the sum of their halves.

b. The double of the difference of two magnitudes is equal to the difference of their doubles.

c. The half of the difference of two magnitudes is equal to the difference of their halves.

These are left as exercises. The corresponding proofs for "equivalence" in II. 13–17 may be consulted.

40. THEOREM 12. *The sum of two or more arcs of equal circles subtends a central angle equal to the sum of the central angles standing on the separate arcs.* [Use 35.]

40 (*a*). Cor. 1. *The difference of two arcs of equal circles subtends a central angle equal to the difference of the central angles standing on the separate arcs.*

Ex. 1. In equal circles, if one central angle is three times another, the arc opposite the first is three times the arc opposite the second.

Ex. 2. Show how to bisect a given arc, or a given sector.

Ex. 3. Show how to trisect a given quadrant, or semicircle, or circle (I. 129, exs.).

41. Definition. Two equiradial arcs are said to be *complemental, supplemental,* or *conjunct,* according as their sum is equal to a quadrant, a semicircle, or a circle.

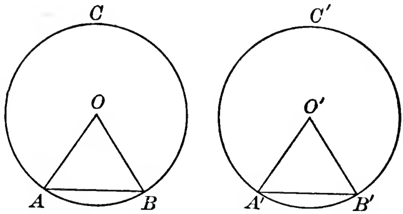
41 (*a*). Cor. 2. *In equal circles, according as two central angles are complemental, supplemental, or conjunct, so are the opposite arcs complemental, supplemental, or conjunct.*

ARCS AND CHORDS

42. Definitions. The line joining the extremities of an arc is called the *chord of the arc,* and the chord is said *to subtend* the arc. Every chord subtends two arcs, one on each side of it. If the chord is not a diameter, the two subtended arcs are unequal, and the greater is called the *major* arc subtended by the chord, and the less the *minor.* When the "arc subtended by a chord" is mentioned without qualification, the minor arc will be understood, that is, the one less than a semicircle.

43. THEOREM 13. *In equal circles, equal arcs have equal chords.*

Let the equal circles ABC, $A'B'C'$ have the arcs AB, and $A'B'$ equal.



To prove that the chords AB and $A'B'$ are equal.

Let O, O' be the centers of the circles. Join OA, OB, $O'A'$, $O'B'$.

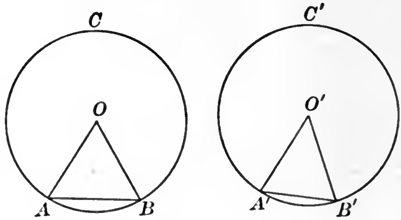
Since the arcs AB, $A'B'$ are equal, therefore, whether these are major or minor arcs, the angles AOB, $A'O'B'$ of the triangles AOB, $A'O'B'$ are equal (35).

Hence these triangles, having two sides and the included angle in each respectively equal, have their bases AB and $A'B'$ also equal.

44. THEOREM 14. *In equal circles, of two unequal minor arcs the greater has the greater chord; and of two unequal major arcs the greater has the less chord.*

Let the equal circles ABC, $A'B'C'$ have the minor arc AB greater than the minor arc $A'B'$.

To prove that the chord AB is greater than the chord $A'B'$.



Let O, O' be the centers of the circles. Join OA, OB, $O'A'$, $O'B'$.

Since the minor arc AB is greater than the minor arc $A'B'$, the central angle AOB is greater than the central angle $A'O'B'$ (35).

Hence, in the triangles AOB, $A'O'B'$, the sides OA, OB are equal to the sides $O'A'$, $O'B'$, and the angle AOB is greater than the angle $A'O'B'$; therefore the base AB is greater than the base $A'B'$ (I. 91).

Next, let the major arc ACB be less than the major arc $A'C'B'$.

To prove that the chord AB is greater than the chord $A'B'$.

Subtracting the unequal major arcs from the equal circles, it follows that the minor arc AB is greater than the minor arc $A'B'$ (38; I. 31).

Therefore, by the preceding case, the chord AB is greater than the chord $A'B'$.

Combined statement.

44 (*a*). **Cor.** *In equal circles, according as a minor arc of one circle is greater than, equal to, or less than, a minor arc of the other, so is the chord of the first arc greater than, equal to, or less than, the chord of the other arc.* [Combine 43, 44.]

Converse statement.

45. Theorem 15. *In equal circles, according as one chord is greater than, equal to, or less than, another chord, so is the minor arc subtended by the first chord greater than, equal to, or less than, the minor arc subtended by the second chord.* [Rule of Conversion.]

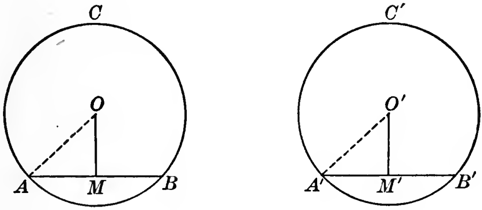
45 (*a*). **Cor.** *In equal circles, according as one chord is greater than, equal to, or less than, another chord, so is the major arc subtending the first chord less than, equal to, or greater than, the major arc subtending the second chord.*

First combine 43 and the second part of 44 into a triple statement; and then apply Rule of Conversion.

Chords and Central Perpendiculars

46. Theorem 16. *In equal circles, the perpendiculars from the centers to equal chords are equal.*

In the equal circles ABC, $A'B'C'$, let OM, $O'M'$ be the perpendiculars from the centers to the two equal chords AB, $A'B'$.



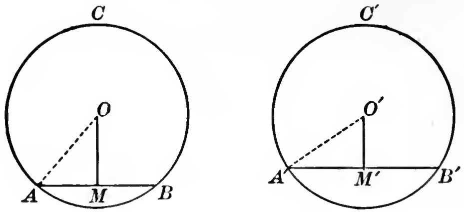
To prove that the perpendiculars OM and $O'M'$ are equal.

The two right triangles OAM, $O'A'M'$ have their hypotenuses equal, and the sides AM, $A'M'$ equal, being halves of equal chords (11).

Therefore the third sides OM and $O'M'$ are equal (I. 98).

47. Theorem 17. *In equal circles, if perpendiculars are drawn from the centers to two unequal chords, the perpendicular drawn to the less chord is the greater.*

In the equal circles ABC, $A'B'C'$, let the chord AB of the first be less than the chord $A'B'$ of the second. Let OM, $O'M'$ be the perpendiculars from the centers to these chords.



To prove that OM is greater than $O'M'$.

The chord AB is less than $A'B'$; and AM, $A'M'$ are halves of AB, $A'B'$ (11).

Therefore AM is less than $A'M'$ (I. 32, ax. 14).

Then, in the right triangles OAM, $O'A'M'$ the hypotenuses are equal, and the side AM of the first is less than the side $A'M'$ of the second.

Now the square on OM is equivalent to the difference of the squares on OA and AM (II. 61); and the square on $O'M'$ is equivalent to the difference of the squares on $O'A'$ and $A'M'$. Subtracting unequals from equals, it follows that the square on OM is greater than the square on $O'M'$ (I. 31, ax. 11, and II. 15 (*a*)). Hence OM is greater than $O'M'$ (II. 24).

Combined statement.

48. Cor. *In equal circles, if perpendiculars are drawn from the centers to any two chords, then according as the first chord is greater than, equal to, or less than, the second chord, so is the perpendicular to the first less than, equal to, or greater than, the perpendicular to the second.*

Converse statement.

49. Theorem 18. *In equal circles, if perpendiculars are drawn from the centers to any two chords, then according as the first perpendicular is less than, equal to, or greater than, the second, so is the first chord greater than, equal to, or less than, the second chord.*

Concerning Order-theorems

50. Order of size of a pair. When two magnitudes A and B of the same kind are compared as to whether A is greater than, equal to, or less than, B, the statement of the result is called the *order of size* of the pair (A, B).

When the two magnitudes are named or written in any order, this order will be called *ascending* when the first is less than the second, *descending* when the first is greater than the second, and *indifferent* when the two magnitudes are equal.

To illustrate the convenience of this phraseology, the two converse propositions in I. 82, 84, each containing a triple statement, may be enunciated thus:

In a triangle, the pair of sides (a, b) and the pair of opposite angles (A, B) are in the same order of size.

This asserts that if either pair is in ascending order, so is the other; if either pair is in descending order, so is the other; and if the order of either pair is indifferent, so is that of the other. Hence this statement includes six different statements.

A theorem that compares the order of size of two pairs of magnitudes is called an *order-theorem*.

As another illustration, I. 94 and its converse may be enunciated as an order-theorem; thus:

If two triangles have two sides of one equal to two sides of the other, then the pair of included angles (A, A') and the pair of opposite sides (a, a') are in the same order of size.

Similarly, the two triple statements in II. 23, 24 may be abbreviated as follows:

If two rectangles have equal altitudes, then the pair of rectangles (R, R') and the pair of bases (b, b') are in the same order of size.

What are the six statements included in this enunciation?

51. Summary of order-theorems in equal circles. All the order-theorems proved for equal circles may be conveniently summarized thus:

In equal circles the following pairs of magnitudes are all in the same order of size:

(1) *Any two central angles* (A, A');
(2) *The two opposite arcs* (a, a');
(3) *The containing sectors* (S, S');
(4) *The subtending chords* (c, c');
(5) *The central perpendiculars* (p', p).

EXERCISES

1. The chords cut off by two equal circles from any line parallel to the line joining their centers are equal.

2. The chords cut off by two equal circles from any line drawn through the mid-point of the line joining their centers are equal.

3. If a line is drawn intersecting two concentric circles, the two parts intercepted between the circles are equal.

4. If two equal chords are drawn in a circle, the two portions intercepted by a concentric circle are equal.

5. Find the locus of the mid-points of equal chords of a circle.

6. Find the locus of the points of trisection of equal chords.

7. Through a given point inside a circle draw the least chord.

ANGLES IN SEGMENTS

52. Definitions. The figure formed by an arc of a circle and its chord is called a *segment of the circle.*

Two segments are said to be *conjunct* when their arcs are conjunct (41). Segments of different circles are said to be *similar* if their arcs are similar (26).

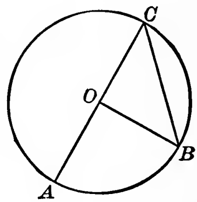
A segment not semicircular is called a *major* or a *minor segment,* according as its arc is a major or a minor arc.

The angle, not convex, formed by any two chords that meet on the circle, is called an *inscribed angle,* and is said to *stand upon* the arc which is between the sides of the angle. In the same case the arc is said to *subtend* the angle.

The angle, not convex, formed by two straight lines drawn from a point in the arc of a segment to the extremities of its chord is called an *angle in the segment.*

Angles standing on the Same Arc

53. Theorem 19. *An inscribed angle is equal to half the central angle standing on the same arc.*



Let BCA be an inscribed angle standing on the arc AB. Let BOA be the central angle standing on the same arc.

To prove the angle ACB equals half the angle AOB.

First let the center O lie on one of the sides of the angle, say AC.

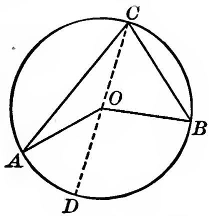
Since the radii OB and OC are equal, the angles OCB and OBC are equal (I. 59).

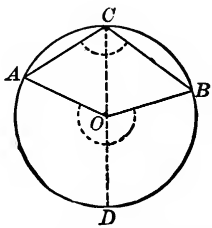
Therefore the angle OCB equals half the sum of the angles OCB and OBC.

Now the sum of the angles OCB and OBC equals the exterior angle AOB (I. 128).

Therefore the angle OCB equals half the angle AOB.

Next let O lie within the angle ACB.

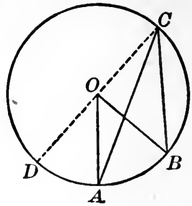




Draw AO, BO, CO. Prolong CO to meet the circle in D.

By the previous case the angle ACO is half the angle AOD, and OCB is half DOB. Hence the whole angle ACB is half the whole angle AOB standing on the same arc (39, *a*).

Lastly let the center O lie without the angle ACB.



Draw AO, BO, CO. Prolong CO to meet the circle in D.

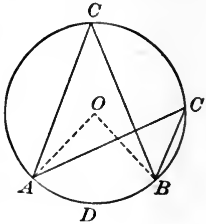
By the first case the angle ACD is half the angle AOD, and BCD is half BOD.

Therefore the remaining angle BCA is half the remaining angle BOA (39, *c*).

Ex. In equal circles, equal inscribed angles stand on equal arcs.

54. THEOREM 20. *Angles in the same segment are equal.*

Let ACB, $AC'B$ be angles in the same segment of the circle $ABC'C$ whose center is O.



To prove that the angles ACB and $AC'B$ are equal.

Draw the lines OA and OB.

Since the angles ACB and $AC'B$ stand on the same arc ADB, they are halves of the same central angle AOB standing on the arc ADB (53).

Therefore the angles ACB and $AC'B$ are equal (I. 28).

NOTE. Since all angles in the same segment are equal, any one of them may be called the angle contained in the segment.

55. Cor. *The angle subtended by the chord of a segment at a point within the segment is greater than the angle in the segment; and the angle subtended at a point without the segment and on the same side of the chord as the segment, is less than the angle in the segment.* (Use I. 79, and ex. 9, p. 44.)

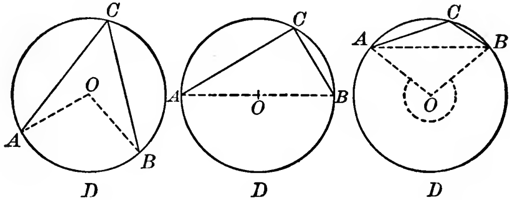
Ex. 1. In equal circles, two segments that contain equal angles are equal. (Prove the arcs equal, and superpose.)

Ex. 2. In any two circles, segments that contain equal angles are similar.

SPECIES OF INSCRIBED ANGLE

56. THEOREM 21. *According as the arc of a segment is greater than, equal to, or less than, a semicircle, so is the angle in the segment less than, equal to, or greater than, a right angle.*

Let ACB be an angle in the segment ABC.



To prove that ACB is less than, equal to, or greater than, a right angle, according as the arc ACB is greater than, equal to, or less than, a semicircle.

Take the center O, and draw OA, OB.

According as the arc ACB is greater than, equal to, or less than, a semicircle, so is the conjunct arc ADB less than, equal to, or greater than, a semicircle, and thus the central angle AOB standing on the arc ADB is less than, equal to, or greater than, a straight angle (34).

Now the angle ACB is half the central angle AOB standing on the same arc (53).

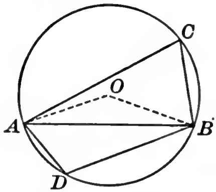
Therefore, according as the arc ACB is greater than, equal to, or less than, a semicircle, the angle ACB is less than, equal to, or greater than, a right angle.

57. THEOREM 22. *According as the angle in a segment is greater than, equal to, or less than, a right angle, so is the arc of the segment less than, equal to, or greater than, a semicircle.* (Rule of Conversion.)

Angles in Conjunct Segments

58. Theorem 23. *The angles in two conjunct segments are supplemental.*

Let AB be the chord of the two conjunct segments ABC and ABD. Let ACB, ADB be angles in these segments, respectively.



To prove that the angles ACB and ADB are supplemental.

The angle ACB is half the central angle AOB standing on the same arc ADB (53).

The angle ADB is half the central angle AOB standing on the same arc ACB.

Now the two central angles AOB together make up a perigon. Therefore their halves, ACB and ADB, together make up a straight angle; and hence are supplemental.

59. Definition. If all the vertices of a polygon are on a circle, the polygon is said to be *inscribed* in the circle, and the circle to be *circumscribed* about the polygon.

60. Cor. 1. *The opposite angles of a quadrangle inscribed in a circle are supplemental.*

61. Cor. 2. *An exterior angle of a quadrangle inscribed in a circle equals the interior opposite angle.*

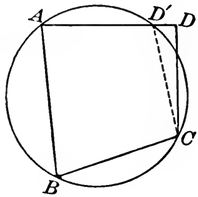
Ex. 1. The angle in a segment is the supplement of half the central angle subtended by the arc of the segment.

Ex. 2. Similar segments of circles contain equal angles.

Ex. 3. Similar segments having equal chords are equal. (Superpose, use 55.)

62. THEOREM 24. *If the opposite angles of a quadrangle are supplemental, a circle may be circumscribed about the quadrangle.*

Let the quadrangle $ABCD$ have the angles ABC and ADC supplemental.



To prove that a circle may be described through the vertices A, B, C, D.

Describe a circle through the three points A, B, C (20).

Suppose, if possible, that it does not pass through the point D. Let it intersect AD in the point D'.

The angle $AD'C$ is supplemental to ABC (58).

Therefore the angles ADC and $AD'C$ are equal (I. 51).

But this equality is impossible (I. 79).

Therefore the supposition is false; hence the circle passing through A, B, C, passes also through D.

EXERCISES

1. If two triangles standing on the same base and at the same side of it have equal vertical angles, then the circle that circumscribes one triangle will circumscribe the other. (Prove similarly to 62.)

2. If two chords intersect within a circle, their included angle is equal to half the sum of the central angles standing on the two arcs intercepted by the sides of the angle.

3. If two chords when extended intersect without the circle, their included angle is equal to half the difference of the central angles standing on the two arcs intercepted by the sides of the angle.

4. The bisectors of all the angles in a given segment pass through a fixed point. The bisectors of all the supplemental angles also pass through a fixed point.

5. The lines joining adjacent extremities of equal chords of a circle are parallel.

6. The lines joining the extremities of parallel chords of a circle are equal.

7. In an inscribed quadrangle, the bisector of an exterior angle and the bisector of the opposite interior angle intersect on the circle.

8. If a parallelogram is circumscribable, then it is a rectangle.

9. If two chords of a circle bisect each other, then their intersection is the center.

10. If a triangle is inscribed in a circle, then the sum of the angles in the three segments exterior to the triangle is equal to a perigon.

11. If a quadrangle is inscribed in a circle, then the sum of the angles in the four segments exterior to the quadrangle is equal to six right angles.

12. If an octagon is inscribed in a circle, then the sum of four alternate angles is equal to the sum of the other four.

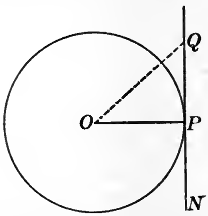
TANGENTS

Conditions for Tangency

The following fundamental theorem leads up to the definition of a tangent line given in the succeeding article. This theorem and the next establish necessary and sufficient conditions for tangency, and lay a foundation for the general theory, and for the problems connected with it.

63. Theorem 25. *Of all the straight lines that can be drawn through a given point on a circle, there is one, and one only, that does not meet the circle again, and this line is perpendicular to the radius drawn to the given point.*

Let P be a point on the circle whose center is O. Let NPQ be the line drawn through P perpendicular to the radius OP.



To prove that P is the only point in which the line NPQ meets the circle.

Take any point Q on the perpendicular NPQ; and draw OQ.

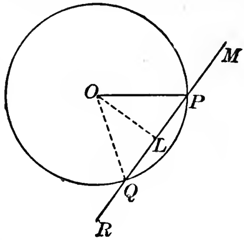
The line OQ is greater than OP (I. 85).

Therefore Q is without the circle (6).

Hence the line NPQ meets the circle in one, and only one, point.

Next, let MPR be any line through P not perpendicular to the radius OP.

To prove that MPR meets the circle in a second point.



Draw OL perpendicular to MPR. On the line LR lay off LQ equal to LP; and draw OQ.

In the triangles OLP, OLQ, the sides PL, LO are respectively equal to the sides QL, LO, and the included angles are equal. Therefore the side OQ equals OP (I. 64).

Hence Q is a point on the circle (6).

Therefore the line MPR meets the circle in a second point Q.

64. Definition. The straight line which meets the circle in a given point and does not meet it again is said to *touch* or be *tangent* to the circle at that point. The point is called the *point of contact* or point of *tangency*.

65. Cor. 1. *Only one tangent can be drawn at a given point on a circle.*

66. Cor. 2. *A tangent is perpendicular to the radius drawn to the point of contact; and a line passing through the extremity of a radius, not perpendicular to it, is not a tangent.*

67. Cor. 3. *The center lies on the perpendicular drawn to any tangent at the point of contact.*

68. Cor. 4. *The perpendicular from the center to any tangent meets it in the point of contact.*

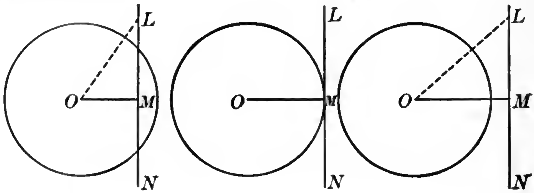
69. Problem 2. *To draw a tangent to a given circle at a given point on the circle.* [Use 63.]

70. Cor. *To a given circle, draw a tangent that shall be parallel to a given line.*

Let the perpendicular from the center to the given line meet the circle in P and Q. The tangents drawn at P and Q are each parallel to the given line. Prove.

71. Theorem 26. *A given line cuts, touches, or does not meet, a given circle, according as the perpendicular to the line from the center is less than, equal to, or greater than, the radius.*

Let O be the center of the given circle; and let LN be the given line. Let OM be the perpendicular from the center to the given line.



First, let OM be less than the radius.

To prove that the line LMN cuts the circle.

On the line LMN lay off ML, MN, each greater than the radius; and draw OL, ON.

Since OL is greater than LM (I. 85) OL is greater than the radius and the point L is without the circle (6). Similarly, the point N is without the circle. But since OM is less than the radius, the point M is within the circle (6). Thus LMN is a secant.

Next, let the perpendicular OM be equal to the radius.

Then the line LMN touches the circle at M (63).

Finally, let OM be greater than the radius.

To prove that the line LMN does not meet the circle.

Let L be any point on the line LMN except the point M.

Since the oblique line OL is greater than OM, it is greater than the radius, and the point L is without the circle.

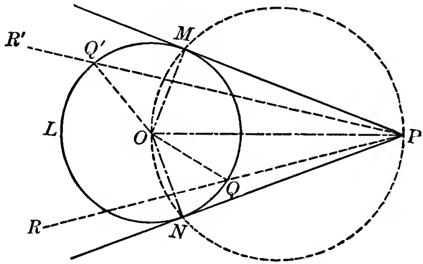
Again, since OM is greater than the radius, the point M is without the circle. Thus LMN is entirely without the circle.

72. **Cor.** *According as a line cuts, touches, or does not meet, a circle, so is the perpendicular to the line from the center less than, equal to, or greater than, the radius.*

Tangents from External Point

73. Theorem 27. *Through a given external point, two, and only two, lines pass that touch a given circle.*

Let LMN be the circle, P the given external point.



First, to prove that two tangents to the circle pass through the point P.

Take the center O, and draw OP. On OP as diameter describe the circle $OMPN$.

Since O is within and P is without the circle LMN, hence part of the circle $OMPN$ is within and part is without the circle LMN; therefore the two circles cut in at least two points. Moreover, they do not cut in more than two points (22). Let M and N be the two points; draw PM, PN, OM.

The angle OMP is a right angle (56).

Therefore PM is tangent at the point M (63).

Similarly, PN is tangent at N.

Therefore two tangents pass through the given point P.

Next, to prove that only two tangents pass through P.

Draw any other line PR meeting the circle LMN in at least one point.

If this point is within the circle $OMPN$ as at Q, the angle OQP is greater than the angle ONP, and hence greater than a right angle (55). Therefore PQR is not tangent at Q (63).

Again, if any line PR' meets the circle LMN in a point outside the circle $OMPN$, as at Q', the angle $OQ'P$ is less than OMP, and hence less than a right angle (55). Therefore $PQ'R'$ is not a tangent at Q' (63).

Hence the lines PM, PN are the only tangents through P.

74. PROBLEM 3. *To draw a tangent to a given circle from a given external point.*

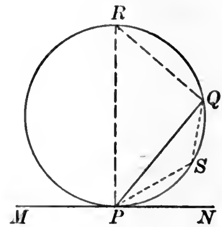
Use the construction and proof given in 73. Two solutions.

75. Cor. *The two tangents to a circle from an external point are equal, and make equal angles with the line joining that point to the center.*

ANGLE OF CHORD AND TANGENT

76. THEOREM 28. *If through any point on a circle a chord and a tangent are drawn, each of the adjacent included angles is equal to the angle in the alternate segment of the circle cut off by the chord.*

Through the point P on the circle PQR, let the chord PQ and the tangent MPN be drawn.



First, to prove that the acute angle NPQ is equal to the angle in the alternate segment PRQ.

Draw the diameter PR; and join RQ.

Since PR is a diameter and PN is a tangent, the angle NPR is a right angle (66).

Therefore the angle NPQ is the complement of QPR.

Now, since PQR is a semicircle, the angle PQR is a right angle (56).

Therefore the angle PRQ is the complement of QPR (I. 129).

Therefore the angles NPQ and PRQ are equal, being complements of the same angle (I. 50).

Hence the angle NPQ is equal to any angle in the segment PRQ (54).

Next, to prove that the obtuse angle MPQ is equal to any angle in the alternate segment PSQ.

The angle MPQ is equal to the supplement of NPQ, and the angle PSQ is equal to the supplement of PRQ (58).

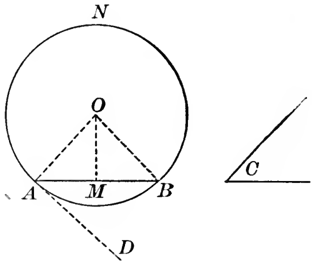
Hence the angles *MPQ*, *PSQ*, being equal to supplements of equal angles, are equal (I. 51).

APPLICATIONS OF THEOREM 28

77. PROBLEM 4. *On a given line, to describe a segment of a circle containing an angle equal to a given angle.*

Let *AB* be the given line, and *C* the given angle.

To describe on *AB* a segment of a circle containing an angle equal to *C*.



Draw *AD*, making the angle *BAD* equal to the given angle *C* (I. 77).

Draw *AO* perpendicular to *AD*; and draw *MO* bisecting *AB* at right angles. Let these lines meet in *O*; and draw *OB*.

The lines *OA* and *OB* are equal (I. 64).

Therefore the circle described with *O* as center, and with radius *OA*, passes through *B*. Let *ABN* be this circle.

The segment *ABN*, alternate to the angle *BAD*, is the required segment.

Since *OAD* is a right angle, *AD* touches the circle (63).

Therefore the angle *BAD* is equal to the angle in the alternate segment *ABN* (76).

Hence the angle in the segment *ABN* is equal to *C*.

78. Ex. Consider the case in which the angle *C* is a right angle.

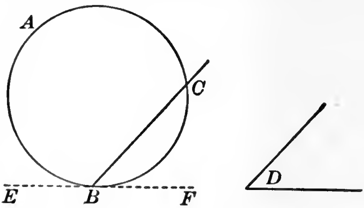
79. Cor. *If the base and vertical angle of a triangle are given, the locus of its vertex consists of the arcs of the two segments described on the base, containing an angle equal to the given vertical angle.*

Show that a triangle on the given base satisfies the requirements if its vertex is on one of these arcs, and not otherwise.

80. PROBLEM 5. *From a given circle, to cut off a segment containing an angle equal to a given angle.*

Let *ABC* be the given circle, and *D* the given angle.

To cut off from the circle *ABC* a segment containing an angle equal to *D*.



Through any point *B* on the circle draw the tangent *EBF* (69). Draw the chord *BC* making the angle *FBC* equal to the given angle *D* (I. 77).

The segment *BAC*, alternate to the angle *FBC*, is the segment required.

Since *BF* is a tangent, the angle *FBC* is equal to the angle in the alternate segment *BÁC* (76).

Therefore the angle in the segment *BAC* is equal to the given angle *D*.

Discussion. How many solutions are there to this problem? How many solutions will there be if the statement of the problem is modified in the following manner: Through a given point on a circle, to draw a line that shall cut off a segment containing an angle equal to a given angle.

EXERCISES

1. All chords of a circle that touch a concentric circle are equal.

2. Through a given point draw a line so that the chord intercepted by a given circle shall be equal to a given line. [Use ex. 1 and art. 74. State the restrictions on the data when the point is within the circle; also when the point is without or on the circle.]

3. The part of any tangent intercepted by two parallel tangents subtends a right angle at the center of the circle.

4. If through the center of a circle two perpendicular lines are drawn to meet any tangent, then the tangents drawn from the two points of intersection are parallel.

5. To draw a tangent to a given circle making a given angle with a given line.

6. Any chord of a circle bisects the angle between the diameter through one extremity and the perpendicular from it on the tangent at the other.

7. Draw a circle through a given point to touch a given line at a given point. [Use 3 and 67.]

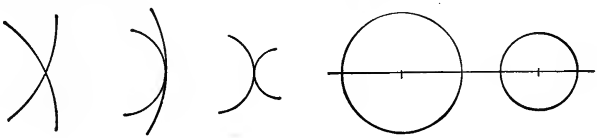
8. If a quadrangle is circumscribed about a circle, the sum of one pair of opposite sides is equal to the sum of the other pair.

9. If a convex quadrangle is such that the sum of one pair of opposite sides is equal to the sum of the other pair, then a circle may be inscribed in it.

TWO CIRCLES

81. Definitions. Two circles are said to *intersect* at a point where they meet if they cross each other at this common point.

Two circles are said to *touch* at a point where they meet if they do not cross each other at this common point; and this point is called the *point of contact.*



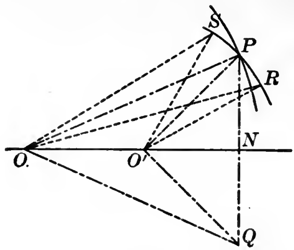
82. The line passing through the centers of two circles is called their *central line.*

Points Common to Two Circles

Common point not on central line.

83. Theorem 29. *If two circles have one common point, not on their central line, then they have a second common point; and the circles intersect at each of these two points.*

Let two circles whose centers are O and O' have the common point P, not on their central line.



First, to prove that they have a second common point.

Draw PN perpendicular to OO', and prolong it to Q making NQ equal to PN. Draw OP, OQ, $O'P$, $O'Q$.

By equality of triangles it follows that OP equals OQ, and $O'P$ equals $O'Q$.

Therefore Q is a point on each of the circles (6).

Moreover, there is no third common point (22).

Next, to prove that the circles *intersect* at each of the points P, Q.

Let R, S be any two points on the circle whose center is O', situated at opposite sides of the point P. Draw OR, $O'R$, OS, $O'S$.

In the triangles $OO'R$ and $OO'P$, the sides OO' and $O'R$ are respectively equal to the sides OO' and $O'P$, and the included angle $OO'R$ is greater than $OO'P$.

Therefore the third side OR is greater than the third side OP (I. 91).

Therefore the point R is without the circle whose center is O (6).

In a similar way it is proved that OS is less than OP.

Therefore the point S is within the circle whose center is O.

Now R and S are *any* two points on the circle whose center is O', situated at opposite sides of P.

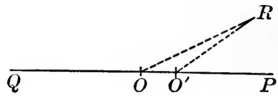
Hence the two circles cross each other at P.

Similarly it can be proved that they cross at Q.

Common point on central line.

84. THEOREM 30. *If two circles have one common point, situated on their central line, then they have no other common point; and the circles touch at this point.*

Let O, O' be the centers of the two circles, and P the common point on the central line OO'.



First, to prove that there is no other common point on the central line.

Suppose, if possible, that Q is another point on the line OO', common to the two circles.

Then the segment PQ is a diameter of each circle. Hence the middle point of PQ is the center of each circle; which is impossible since the centers O, O' do not coincide.

Therefore there is no second common point on the central line.

Next, to prove that there is no second common point not on the central line.

Suppose, if possible, that R is another common point not on the central line. Then there is a third common point R', not on the central line (83).

Since there are three common points, the two circles coincide throughout (21).

This is contrary to the hypothesis; therefore there is no common point not on the central line.

Hence P is the only common point.

Again, to prove that the circles touch at P.

Let R be any other point on the circle whose center is O'. Draw OR and $O'R$.

Since $O'R$ equals $O'P$, therefore the sum of $O'R$ and OO' is equal to OP.

But OR is less than the sum of OO' and $O'R$ (I. 87).

Therefore OR is less than OP.

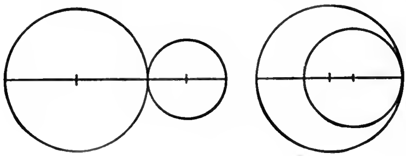
Hence the point R is within the circle whose center is O and whose radius is OP.

Since R is *any* point (other than P) on the circle whose center is O', it follows that the circles do not cross at their common point P. Therefore they touch at this point (81).

Point of contact.

85. Theorem 31. *If two circles touch, then their point of contact is on the central line; and they have no other common point.*

Let there be two circles touching each other at a point.



To prove that the point of contact is on the central line; and that the circles have no other common point.

Suppose, if possible, that the point of contact is not on the central line.

Then, since the circles have a common point not on the central line, they *intersect* at this point (83).

This is contrary to the hypothesis; hence the supposition made is false. Therefore the point of contact is on the central line.

Further, since the two circles have a common point on their central line, it follows that they have no other common point (84).

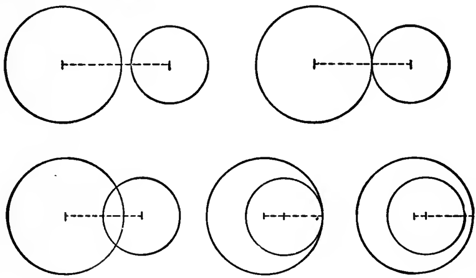
86. **Cor. I.** *If two circles touch each other externally, the line joining their centers is equal to the sum of their radii.*

87. Cor. 2. *If two circles touch, one being internal to the other, the line joining their centers is equal to the difference of their radii.*

88. Cor. 3. *If two circles do not meet, and each is wholly outside the other, the line joining their centers is greater than the sum of their radii.*

89. Cor. 4. *If two circles do not meet, and one is wholly inside the other, the line joining their centers is less than the difference of their radii.*

90. Cor. 5. *If two circles intersect, the line joining their centers is less than the sum of their radii, and greater than the difference of their radii.*



91. Note. The relation of these five cases to each other is well shown by taking first the case in which each circle lies wholly outside the other, and then moving one center toward the other, as successively shown in the figures.

Ex. 1. In the five preceding corollaries, show that the hypotheses are exhaustive, and that the conclusions are mutually exclusive; then apply the rule of conversion (I. 104) to prove the converse of each corollary. *E.g.*,

If the line joining the centers of two circles is less than the difference of the radii, then one circle is wholly within the other.

Ex. 2. Find a point such that its joins to two given points may be equal respectively to two given lines. (Intersection of loci.)

Show when there are two solutions, when only one solution, and when none. Compare I. 132.

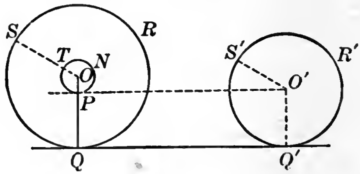
Ex. 3. If two circles touch, they have a common tangent at the point of contact.

TANGENTS COMMON TO TWO CIRCLES

92. PROBLEM 6. *To draw a common tangent to two given circles.*

Let O, O' be the centers of the given circles QRS, $Q'R'S'$, to which it is required to draw a common tangent.

First, let the radii be unequal; and let the circle QRS have the greater radius.

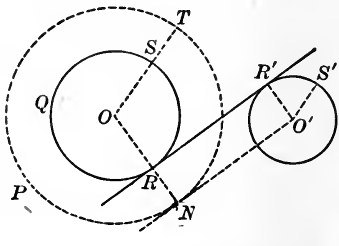


Draw any radii OS and $O'S'$. From the greater OS lay off ST equal to the less $O'S'$. With O as center and OT as radius describe a circle TPN. From the point O' draw a tangent $O'P$ to the latter circle. Join OP and prolong it to meet the circle QRS in Q. Draw $O'Q'$ parallel to OQ, and join QQ'.

The line QQ' touches each circle.

[The proof is left to the student; also the construction of a second common tangent by the same method. How is the method to be modified when the radii are equal? Give construction and proof.]

If the circles have no common point and are external to each other, two other common tangents can be drawn as follows:



Let OS be any radius of either circle. Prolong it so that ST equals the radius of the other circle. With O as center and OT as radius, draw the circle TPN. From the point O' draw $O'N$ tangent to TPN. Draw ON cutting the circle QRS in R. Draw the radius $O'R'$ parallel to OR, and join RR'. Then RR' touches each circle.

[The proof is left to the student; also the construction of a second common tangent.]

Discussion. By successively moving O' toward O, as in 91, show when there are four common tangents, when only three, when only two, when only one, when none.

Definition. A line touching two circles is said to be a *direct* or a *transverse* common tangent according as the two radii drawn to the points of contact are at the same side or at opposite sides of the central line.

In the different cases just mentioned, how many of the common tangents are direct, and how many transverse?

EXERCISES

1. Draw a tangent to one given circle so that the part intercepted by another given circle shall be equal to a given line.

2. Draw a line so that the chords intercepted by two given circles shall be respectively equal to two given lines.

3. If two circles touch, and through the point of contact two lines are drawn cutting the circles again, then the chords joining the other intersections are parallel.

4. If two circles touch, and through the point of contact a line is drawn to cut the circles again, then the tangents at the other intersections are parallel; and the line divides the two circles into arcs that are respectively similar.

5. If two circles touch, and if two parallel diameters are drawn, then an extremity of each diameter and the point of contact are in the same straight line.

6. Describe a circle through a given point and touching a given circle at a given point. (Determine its center by 85 and ex. 5.)

7. Describe a circle to touch a given line, and touching a given circle at a given point. (Draw a diameter perpendicular to the given line, and use ex. 5.)

8. Describe a circle to touch a given circle, and touching a given line at a given point.

9. The two circles described with two sides of a triangle as diameters intersect on the third side.

10. If two equal circles do not intersect, show how to draw a line so that its extremities and points of trisection may be on the two circles. (When are there two solutions; only one solution; no solution?)

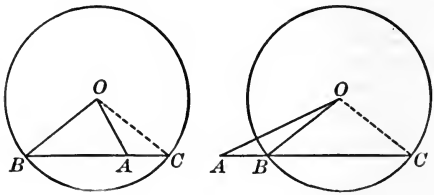
CONCURRENT CHORDS

Rectangle of segments of chord.

93. THEOREM 32. *If a chord of a circle is divided into two segments by a point taken either in the chord or in its prolongation, the rectangle of the two segments is equivalent to the difference of the squares on the radius and on the line joining the given point to the center of the circle.*

Let the chord BC of the circle whose center is O be divided at the point A into the two segments AB and AC.

To prove that the rectangle of AB and AC is equivalent to the difference of the squares on OB and OA.



Draw the radius OC.

Since the triangle OBC is isosceles, and A is any point in the base or its prolongation, therefore the rectangle of the segments AB and AC is equivalent to the difference of the squares on OB and OA (II. 70).

[Consider the special case in which BC passes through O.]

Several concurrent chords.

94 (*a*). **Cor. 1.** *If several chords pass through the same point, the rectangle of the segments of any one chord is equivalent to the rectangle of the segments of any other.*

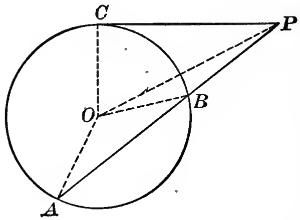
Converse.

94 (*b*). **Cor. 2.** *If two given lines cut each other, either internally or externally, so that the rectangle of the segments of one is equivalent to the rectangle of the segments of the other, then the four extremities of the lines lie on the same circle.*

Chord and tangent.

95. THEOREM 33. *If a chord of a circle is prolonged to any point, then the rectangle of the segments of the chord is equivalent to the square on the tangent drawn from that point to the circle.*

Let the tangent CP and the chord AB meet in P.



To prove that the rectangle of AP and BP is equivalent to the square on CP.

Take the center O, and draw OA, OB, OC, OP.

The rectangle of AP and BP is equivalent to the difference of the squares on OP and OB (93).

But this difference is equivalent to the difference of the squares on OP and OC, which is equivalent to the square on CP (II. 61). Therefore the rectangle of AP and BP is equivalent to the square on CP.

Ex. Show that this may be considered a special case of 94 (*a*).

Converse.

96. **Cor. 1.** *If the rectangle contained by the segments of a chord passing through an external point is equivalent to the square on the line joining that point to a point on the circle, then this line touches the circle.*

Ex. Draw a circle through two given points to touch a given line.

97. **Cor. 2.** *If several circles pass through the same two points, then the tangents drawn to them from any point on the prolongation of their common chord are all equal.*

Ex. If two circles intersect, and if a tangent is drawn to each from a point not on the prolongation of their common chord, then these two tangents are not equal. [Draw a secant through the point and one of the intersections. Use 95.]

EXERCISES

1. If two circles intersect, then the prolongation of their common chord bisects their common tangent.

2. If three circles intersect, then their three chords of intersection meet in a point. [Prove indirectly, using 94.]

3. If two circles touch, and if a third circle intersects them, then the tangent at the point of contact and the two chords of intersection are all concurrent.

4. Draw a circle through two given points to touch a given circle. [Determine the point of contact by means of ex. 3. Two solutions in general. Examine the case in which one of the given points is on the given circle. When is there no solution ?]

INSCRIPTION AND CIRCUMSCRIPTION

98. Definitions. If all the sides of a polygon touch a circle lying within the polygon, then the circle is said to be *inscribed* in the polygon, and the polygon to be *circumscribed* about the circle.

If a circle passes through all the vertices of a polygon lying within the circle, then the polygon is said to be *inscribed* in the circle, and the circle to be *circumscribed* about the polygon. A circle that touches one side of a triangle and the prolongations of the other two is said to be *escribed* to the triangle.

This section will consist chiefly of problems relating to the inscription and circumscription of certain regular polygons to a circle, and of a circle to any regular polygon. In the particular case of the triangle, however, we shall not be restricted to the equilateral triangle.

Circles and Triangles

Circumscribed circle.

99. Problem 7. *To circumscribe a circle about a given triangle.* [Use the method and proof of 20.]

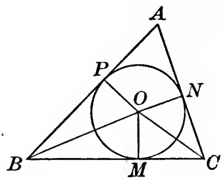
100. Cor. *Every triangle has one, and only one, circumscribed circle.*

Inscribed circle.

101. PROBLEM 8. *To inscribe a circle in a given triangle.*

Let ABC be the triangle in which it is required to inscribe a circle.

Bisect any two of the internal angles, say B and C. Let the bisectors BO and CO meet in O. Draw OM, ON, OP, perpendicular respectively to the sides BC, CA, AB.



The perpendiculars OM, ON, and OP are all equal (I. 258).

Therefore the circle described with O as center and OM as radius, passes through the points M, N, P.

The circle MNP, so described, touches the sides of the triangle at M, N, P, because the angles OMB, ONC, and OPA are right angles by construction (63).

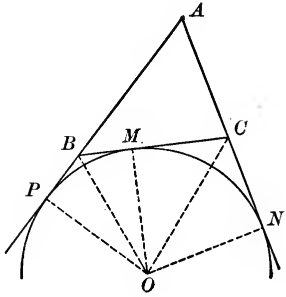
102. **Cor. 1.** *Every triangle has one, and only one, inscribed circle.*

Ex. The inscribed and circumscribed circles of an equilateral triangle are concentric.

Escribed circles.

103. **Cor. 2.** *To escribe a circle to a given triangle.*

[Construct and prove as in 101.]



104. **Cor. 3.** *There is one, and only one, circle touching a given side of any triangle and the prolongations of the other two sides; and there are three, and only three, escribed circles to any given triangle.*

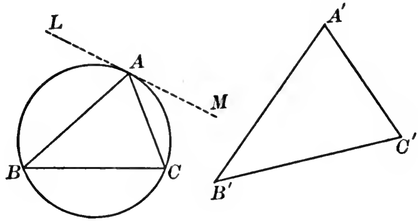
Definitions. The centers of the circumscribed, the inscribed, and the three escribed circles are called respectively the *circum-center*, the *in-center*, and the three *ex-centers* of the triangle.

Ex. 1. The join of two ex-centers passes through a vertex of the triangle; the join of the third ex-center to the in-center passes through the same vertex; and these lines are perpendicular to each other.

Inscribed triangle.

105. Problem 9. *In a given circle, to inscribe a triangle equiangular to a given triangle, and having one vertex at a given point on the circle.*

Let ABC be the given circle, $A'B'C'$ the given triangle, and A the given point.



To inscribe in ABC a triangle equiangular to $A'B'C'$, and having one vertex at A.

At the point A on the circle draw the tangent LAM. Draw the chord AB making the angle LAB equal to C'; draw the chord AC making the angle MAC equal to B'; and join BC.

Then ABC is the required triangle.

The angle LAB equals the angle ACB in the alternate segment of the circle (76). Therefore the angle C equals C'.

Similarly, the angle B equals B'. Hence the remaining angle BAC equals the remaining angle A' (I. 130).

Therefore the inscribed triangle ABC is equiangular to the triangle $A'B'C'$.

Discussion. If the angle LAB had been made equal to B', and MAC to C', another triangle answering the requirements would have been obtained. In both these solutions A corresponds to A'. Two more solutions can be obtained in which A corresponds to B', and two in which A corresponds to C'.

Show that the number of solutions would be reduced to *two* if we should insert in the statement of the problem the additional condition: "the vertex to which the given point is to correspond being previously assigned." In what case would these two solutions reduce to *one?*

Ex. Prove that all triangles inscribed in the same circle and equiangular to each other, are equal.

Circumscribed triangle.

106. Problem 10. *About a given circle to circumscribe a triangle equiangular to a given triangle.*

[Use 72; and I. 126.]

Ex. 1. About a given circle to circumscribe a triangle equiangular to a given triangle, one of the three points of contact being assigned.

[Rotate the given triangle so that one of the sides shall be parallel to the tangent at the given point (I. 202), and then use 72. Three choices of correspondence; and two ways of rotating. Discuss as in 105.]

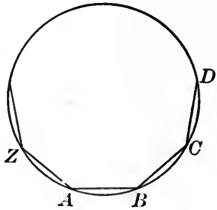
Ex. 2. Prove that all triangles circumscribed about the same circle and equiangular to each other, are equal.

Principles of Inscription and Circumscription

The two following theorems establish the general principles that will be used in the problems of inscribing and circumscribing regular polygons to given circles. These theorems presuppose the division of the circle into a number of equal arcs. This division cannot, however, be actually performed by the constructions of elementary geometry, except in the case of certain special numbers, the chief of which are shown in the succeeding group of problems.

Principle of inscription.

107. THEOREM 34. *If a circle is divided into a number of equal arcs, the chords of these arcs form a regular inscribed polygon.*



Let the circle ABC be divided into a number of equal arcs at the points A, B, C, $\cdots$, Z, and let the chords of these arcs be AB, BC, CD, $\cdots$, ZA.

To prove that the inscribed polygon $ABCD \cdots Z$ is regular.

Since the arcs AB, BC, $\cdots$ are all equal, their chords are all equal (43). Therefore the polygon is equilateral.

Again, the angles of the polygon are equal since they are angles in equal segments of the circle (54). Therefore the polygon is equiangular.

Hence, by definition, the polygon is regular.

108. Definitions. If the extremities of a broken line coincide with the extremities of an arc, and if all its vertices are on the arc, then the broken line is said to be *inscribed* in the arc.

A ***regular*** broken line is one whose sides are equal, and whose successive angles are equal, the equal angles all lying at one side of the line.

109. Cor. *If an arc is divided into a number of equal arcs, then their chords form a regular inscribed broken line.*

Ex. 1. Any equilateral polygon inscribed in a circle is also equiangular.

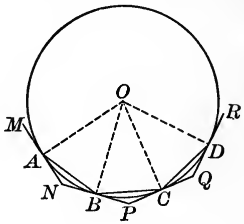
Ex. 2. If an equilateral polygon is not equiangular, it is not circumscribable.

Ex. 3. In any equiangular polygon inscribed in a circle, each side is equal to the next but one; and hence an inscribed equiangular polygon of an odd number of sides is equilateral.

Principle of circumscription.

110. THEOREM 35. *The tangents at the vertices of an inscribed regular polygon form a circumscribed regular polygon.*

Let $A, B, C, D, \cdots$ be the vertices of an inscribed regular polygon; and let tangents $MAN, NBP, PCQ, QDR, \cdots$ be drawn at these vertices.



To prove that the circumscribed polygon so formed is regular.

Find the center O, and draw OA, OB, OC, OD, $\cdots$.

Conceive the figure turned about O until OA coincides with the trace of OB, then the arc AB coincides with its equal arc BC, and the line OB with the trace of OC. Similarly OC coincides with the trace of OD, and so on.

Therefore the tangents at $A, B, C, \cdots$, being perpendicular to the radii, coincide respectively with the traces of the tangents at $B, C, D, \cdots$; and hence the polygon coincides with its trace.

Therefore the angles are all equal, and the sides are all equal; hence the circumscribed polygon is regular.

Ex. If a circumscribed polygon is equiangular, then it is regular. [Prove the central angles AOB and BOC equal, etc.]

CERTAIN REGULAR POLYGONS

111. Division of the Circle. By the two preceding theorems the inscription and circumscription of regular polygons have been reduced to the division of the circle into a given number of equal parts. The next four problems with their corollaries show how to perform the actual division when the given number belongs to one of the four following series:

$$2, 4, 8, 16, 32, \cdots; \qquad 5, 10, 20, 40, 80, \cdots;$$
$$3, 6, 12, 24, 48, \cdots; \qquad 15, 30, 60, 120, 240, \cdots;$$

in each of which the numbers after the first are obtained by successive doubling. These numbers will for convenience be called *Euclid's numbers*, as the problems in question were first systematically treated in Euclid's "Elements of Geometry."

The division of a circle into two equal parts is easily performed by drawing a diameter. As this division does not give an inscribed polygon, we begin with the second number of the first series.

Four, eight, sixteen ⋯ sides.

112. PROBLEM 11. *To inscribe a square in a given circle.*

[Draw two diameters at right angles, and join their extremities.]

113. Cor. 1. *To inscribe a regular octagon in a given circle.*

Outline. Draw two diameters at right angles; bisect the four arcs; draw the eight chords. Prove by 107.

114. Cor. 2. *To inscribe regular polygons of* 16, 32 ⋯ *sides.*

115. Cor. 3. *To circumscribe regular polygons of* 4, 8, 16, 32, ⋯ *sides* (110).

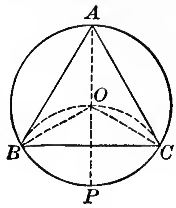
116. Cor. 4. *In any given arc to inscribe a regular broken line of* 2, 4, 8, 16, ⋯ *sides.*

117. Cor. 5. *The inscribed square is equivalent to double the square on the radius, and to half the circumscribed square.*

Three, six, twelve ⋯ sides.

118. PROBLEM 12. *To inscribe an equilateral triangle in a given circle.*

Let *ABC* be the circle in which an equilateral triangle is to be inscribed.



Find the center O, and take any point P on the circle. With center P and radius PO, describe an arc cutting the circle at B and C. Draw PO to meet the circle again in A; and draw AB, BC, CA.

The inscribed triangle ABC is equilateral.

The triangles BOP, COP are each equilateral by construction; therefore the angles BOP, COP are each equal to one third of a straight angle (I. 129).

Therefore their supplements BOA, COA are each equal to two thirds of a straight angle; hence the central angles BOA, COA, and BOC are all equal; and are therefore subtended by equal arcs and equal chords.

Therefore the inscribed triangle ABC is equilateral.

NOTE. This problem could be solved as a special case of problem 9, but the construction would not be so simple as that just given.

119. Cor. 1. *To inscribe a regular hexagon.*

Outline. Bisect the arcs AB, BC, CA, and draw the six chords.

Ex. Prove that the side of a regular inscribed hexagon is equal to the radius of the circle. Hence give another method of inscribing a regular hexagon.

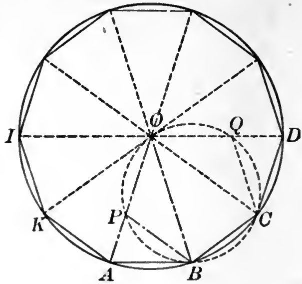
120. Cor. 2. *To inscribe regular polygons of* 12, 24, 48, ⋯ *sides.*

121. Cor. 3. *To circumscribe regular polygons of* 3, 6, 12, 24, ⋯ *sides* (110).

Five, ten, twenty, ⋯ sides.

122. PROBLEM 13. *In a given circle, to inscribe a regular decagon.*

Let ABC be the given circle in which a regular decagon is to be inscribed.



Take the center O, and draw any radius OA. Divide OA at P, so that the rectangle of OA and AP is equivalent to the square on OP (II. 89).

With center A and radius equal to OP, describe an arc cutting the circle in B; and join AB.

Then AB is the side of a regular decagon inscribed in ABC.

To prove this, draw OB and PB; and draw a circle through the points O, P, B.

The square on AB is equivalent to the rectangle of OA and AP, by construction.

Therefore the line AB is tangent to the circle OPB (96).

Hence the angle ABP is equal to the angle POB in the alternate segment (76).

Therefore the whole angle ABO is equal to the sum of the angles POB and PBO, and therefore equal to the exterior angle APB (I. 128).

Now, the angle ABO is equal to the angle BAO (I. 59).

Therefore the angle APB equals the angle BAO.

Hence the opposite sides AB and PB are equal.

Also, AB equals OP by construction; therefore OP equals PB; and hence the opposite angles POB and PBO are equal.

Therefore the angle ABO, which has been proved equal to the sum of POB and PBO, is double the angle POB.

Hence the isosceles triangle OAB has each of the angles at the base equal to double the vertical angle AOB.

Therefore the angle AOB is equal to one fifth of the sum of the three angles of the triangle OAB, that is, equal to one fifth of a straight angle, and hence equal to one tenth of a perigon. Thus the arc AB is a tenth part of the circle (31).

Hence the circle can be divided into ten arcs equal to AB; and the chords of these arcs will form a regular inscribed decagon (107).

123. **Cor. 1.** *To inscribe a regular pentagon in a given circle.* [Join alternate vertices of an inscribed regular decagon (107).]

124. **Cor. 2.** *To inscribe regular polygons of* 20, 40, ⋯ *sides.*

125. **Cor. 3.** *To circumscribe regular polygons of* 5, 10, 20, ⋯ *sides.*

126. **Cor. 4.** *Show that the central angle subtended by the side of an inscribed regular decagon is two fifths of a right angle. Show how to divide a right angle into five equal parts.*

127. **Cor. 5.** *On a given line, to construct a regular decagon.*

[In any circle inscribe a regular decagon. At the extremities of the given line, make angles equal to the angle of the regular decagon, and so on.]

Ex. 1. The triangle OAB is an isosceles triangle having each angle at the base equal to double the vertical angle.

Ex. 2. The triangle POB is an isosceles triangle having each angle at the base equal to one third of the vertical angle.

Ex. 3. If C is the second point of intersection of the circles OPB and ABC, then BC equals AB. If D is the next vertex, and if the circle OPB cuts the radius OD again in Q, then $OPBCQ$ is a regular pentagon.

Ex. 4. Prove that BP prolonged passes through a vertex I of the decagon; and that the difference of BI and BA is equal to the radius.

Ex. 5. Show that OCP is an isosceles triangle, and that the rectangle of OA and PA is equivalent to the difference of the squares on AC and OC; that is to say, the difference of the squares on the sides of the inscribed pentagon and decagon is equivalent to the square on the radius.

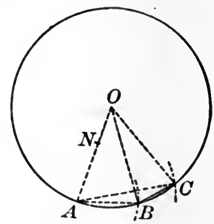
Ex. 6. If a regular pentagon and a regular decagon are inscribed in the same circle, then the apothem of the pentagon equals half the sum of a side of the decagon and a radius of the circle. [BK bisects AP; use I. 72.]

Fifteen, thirty, sixty, . . . sides.

128. Problem 14. *To inscribe a regular quindecagon in a given circle.*

Let *ABC* be the circle in which it is required to inscribe a regular polygon of fifteen sides.

Find the center *O*, and draw a radius *OA*. Divide *OA* at the point *N*, so that the rectangle of *OA* and *ON* is equivalent to the square on *AN* (II. 89).



With *A* as center, and *AN* and *AO* as radii, describe arcs cutting the circle in *B* and *C*; and draw *BC*.

Then the chord *BC* is a side of a regular inscribed quindecagon.

The chord *AB* is a side of a regular inscribed decagon (122); and the chord *AC* is a side of a regular inscribed hexagon (119).

Therefore, if the whole circle were divided into thirty equal parts, the arc *AB* would contain three of them, and the arc *AC* would contain five. Therefore, the arc *BC* would contain two of these parts, and is hence one fifteenth of the whole circle. Thus the circle is divisible into fifteen arcs, each equal to the arc *BC*; therefore the polygon formed by the chords of these arcs is a regular quindecagon (107). Hence the chord *BC* is a side of a regular inscribed quindecagon.

129. **Cor.** *To inscribe regular polygons of* 30, 60 *sides; and to circumscribe regular polygons of* 15, 30, 60 *sides.*

Ex. 1. To divide a right angle into fifteen equal parts.

Ex. 2. To divide the angle of an equilateral triangle into five equal parts.

Ex. 3. To trisect the angle of a regular pentagon.

Note on the numbers of Gauss. The four preceding problems with their corollaries have shown how to divide the circle into *n* equal parts, if *n* is any one of Euclid's numbers (111). From the time of

Euclid no essential advance was made in the problem of dividing the circle until the year 1796, when Gauss* proved the possibility of dividing the circle into n equal parts, if n is any prime number that exceeds a power of 2 by unity. The first four numbers that satisfy this condition are 3, 5, 17, 257. Gauss further proved that the division can be performed if n is the product of any two or more different numbers of this series; the first four numbers that satisfy this condition are 15, 51, 85, 255. Gauss gave the complete analysis for the case of 17 parts, and proved that the problem can be reduced to simpler ones that depend ultimately on the postulates of construction; but the method of proof is beyond the range of elementary geometry.

Inscribed and Circumscribed Circles

The following theorem and its corollaries furnish the basis for the two succeeding problems, which relate to the construction of the inscribed and circumscribed circles of any given regular polygon.

Concurrence of angle-bisectors.

130. Theorem 36. *The lines that bisect the angles of any regular polygon all meet in a point.*

Outline. Let A, B, C, D be consecutive angles of any regular polygon. Bisect the angles A and B; and prove that the bisectors meet at that side of the line, AB, at which the polygon itself is (I. 124). Let the bisectors meet at the point O, and draw OC. Prove that OC bisects the angle C. (This is done by proving that the angle OCB equals OAB, which equals half of the angle A, and hence equals half C.) Prove similarly that OD bisects the angle D; and so on.

131. Cor. 1. *In any regular polygon, the segments of the angle-bisectors intercepted between the vertices and the point of concurrence are all equal.*

132. Cor. 2. *In any regular polygon, the perpendiculars from the intersection of the angle-bisectors to the sides are all equal.*

* "Disquisitiones Arithmeticæ," published in 1801.

Circumscribed circle.

133. PROBLEM 15. *To circumscribe a circle about a given regular polygon.* (Use 130, 131.)

Inscribed circle.

134. PROBLEM 16. *To inscribe a circle in a given regular polygon.* (Use 130, 132.)

135. **Cor.** *The inscribed and circumscribed circles of a regular polygon are concentric.*

Ex. 1. If two regular polygons are equal, then their inscribed circles are equal, and so are their circumscribed circles.

Ex. 2. If two regular polygons of the same number of sides are inscribed in equal circles, then the two polygons are equal.

136. **Definitions.** The common center of the inscribed and circumscribed circles of a regular polygon is called the center of the regular polygon. The angle at the center subtended by any side of the polygon is called the *central angle* of the regular polygon.

In a regular polygon, a line joining the center to any vertex is called a *radius,* and a perpendicular from the center to any of the sides is called an *apothem.* Thus a radius of a regular polygon is a radius of its circumscribed circle, and an apothem is a radius of its inscribed circle.

Ex. 1. If any two regular polygons have the same number of sides, then their central angles are equal.

Ex. 2. If two regular polygons have the same number of sides, and if the radius of one is greater than the radius of the other, then the apothem of the first is greater than the apothem of the second, the side of the first is greater than the side of the second, and the surface of the first is greater than the surface of the second.

[Place the polygons with their centers in coincidence, and so that each radius of the first may fall on a radius of the second; then prove that each side of the first is parallel to a side of the second; etc.]

Ex. 3. If two regular polygons have the same number of sides, then the following pairs of magnitudes are in the same order of size:

The bases (b, b'); the radii (r, r'); the apothems (a, a'); the surfaces (s, s'); the perimeters (p, p').

Equivalent rectangle.

137. THEOREM 37. *A polygon circumscribed about a circle is equivalent to the rectangle contained by the perimeter and half the radius of the circle.*

Ex. A regular polygon is equivalent to the rectangle contained by the perimeter and half the apothem.

MUTUALLY EQUILATERAL POLYGONS

138. THEOREM 38. *If two circles are equal, and if two mutually equilateral polygons are inscribed in them, then the polygons are equal.*

Outline. Compare the respective triangles whose vertices are at the centers and whose bases are corresponding sides of the polygons.

139. THEOREM 39. *If two mutually equilateral polygons are each circumscribable by a circle, then the circles are equal, and the polygons are equal.*

Outline. Suppose the radii unequal. Compare the respective central angles subtended by corresponding sides; see ex. 42 at end of Book I. Reduce to absurdity.

140. Cor. *If in two semicircles are inscribed two broken lines, and if the segments of one broken line are respectively equal to those of the other, taken in order, then the semicircles are equal, and the two figures are superposable.* [Prove as in 139.]

EXERCISES

1. Two regular polygons of the same number of sides circumscribed about equal circles are equal.

2. The center of the inscribed circle of a triangle is the orthocenter of the triangle formed by the centers of the escribed circles.

3. Show how to cut off the corners of an equilateral triangle so as to leave a regular hexagon; also of a square to leave a regular octagon.

4. If a parallelogram is circumscribed to a circle, then it is a rhombus.

5. Show that it is possible to trisect the central angle of a regular n-gon, when n is any one of the first or third series of Euclid's numbers (111).

6. In the same cases show that it is possible to trisect the interior or exterior angle of the regular n-gon.

7. Show that it is possible to divide a right angle into n equal parts, when n is any of Euclid's numbers.

8. Show that it is possible to divide the angle of an equilateral triangle into n equal parts, when n is any number belonging to the first three series of Euclid's numbers.

Note on exs. 5–8. The general problem to trisect a given arbitrary angle is one of the famous problems of antiquity, and has never been solved by methods permitted in elementary geometry. Modern mathematicians have demonstrated that this general problem cannot be analyzed into simpler ones that require only the drawing of straight lines and circles.* Thus the construction cannot be performed by means of only a pair of compasses and an unmarked straightedge. Several general solutions are known which overstep these limitations to a greater or less degree. One of the simplest employs the sliding motion of a straightedge on which two points are marked. There are, however, certain special angles that *can* be trisected by the methods of elementary geometry. (See exs. 5–8 above.)

The still more general problem of dividing a given arbitrary angle into n equal parts can be solved only when n is one of the first series of Euclid's numbers (111 ; I. 73) ; but in the case of certain special angles the problem can be solved for some other values of n (exs. 7, 8).

MAXIMA AND MINIMA †

141. Certain principles of maxima and minima relating to triangles were considered in Book II. 92–108. Similar principles can now be extended to polygons in general, subject to certain given conditions. The theorems here considered fall into five groups according to the nature of the assigned conditions.

* See Klein's "Vorträge über ausgewählte Fragen der Elementar Geometrie." (Translated by Professors Beman and Smith.)

† This topic is discussed here on account of its intimate connection with the properties of the circle, and of inscribed and circumscribed polygons. It may, however, be postponed without inconvenience.

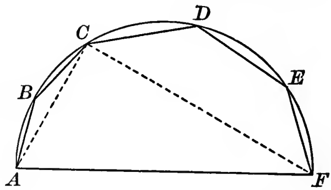
Given Sides

This group of two theorems with their corollaries will show how to make the surface of a polygon a maximum subject to various assigned conditions relating to the magnitude of the sides. In each case the additional condition is to be proved both necessary and sufficient for a maximum; and accordingly each theorem is accompanied by its converse (II. 93).

Greatest polygon with one arbitrary side.

142. Theorem 40. *Among the polygons that have all the sides but one equal respectively to given lines taken in order, any one that is a maximum is circumscribable by a semicircle having the undetermined side as diameter.*

Let the polygon $ABCDEF$ be a maximum subject to the condition that the sides AB, BC, CD, DE, EF are respectively equal to given lines taken in order.



To prove that the semicircle described on AF as diameter passes through all the points B, C, D, E.

Suppose, if possible, that the semicircle does not pass through C; and draw CA, CF.

Then the angle ACF is not a right angle (55, 56).

Hence, by rotating the figures ABC and $FEDC$ about the point C until ACF becomes a right angle, the triangle ACF could be increased (II. 94); and therefore the whole polygon $ABCDEF$ could be increased without changing any of the given sides. This is contrary to the hypothesis that $ABCDEF$ is a maximum under the given conditions.

Hence the semicircle described on AF passes through the point C. Similarly it passes through the other vertices.

143. **Cor.** *Among the polygons that have all the sides but one equal respectively to given lines taken in order, any polygon that is circumscribable by a semicircle having the undetermined side as diameter, is a maximum.*

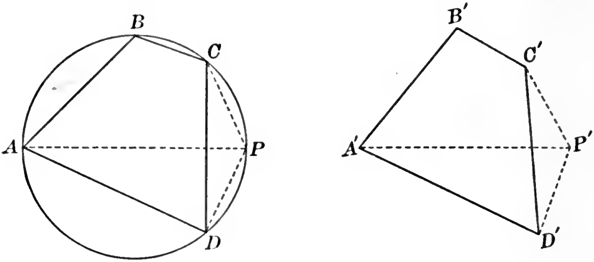
For all polygons that satisfy the given conditions, and the further condition of being circumscribable by a semicircle having the undetermined side as diameter, are equal (140), and are therefore equal to any one that is a maximum (142).

Ex. Show how to enunciate II. 94 so as to make it a particular case of 143.

Greatest polygon with all the sides given.

144. Theorem 41. *Among the polygons that have their sides equal respectively to given lines taken in order, any polygon that is circumscribable by a circle is a maximum.*

Let $ABCD$ and $A'B'C'D'$ be two polygons that satisfy the conditions of having their sides equal respectively to given lines taken in order, and let the former be circumscribable and the latter not.



First to prove that $ABCD$ is greater than $A'B'C'D'$.

Draw the diameter AP; and join CP, DP. On $C'D'$, which equals CD, construct a triangle $C'D'P'$ equal to the triangle CDP; and draw $A'P'$.

The circle whose diameter is $A'P'$ does not pass through all the points B', C', D' (hyp.). Use 143 and add; then subtract the equal triangles.

Next, to prove that $ABCD$ is a *maximum* under the given conditions.

The polygon $ABCD$ is superposable on any other polygon that satisfies the given conditions and the further condition of being circumscribable (139); and it has just been proved greater than any polygon that satisfies the given conditions without satisfying the further condition. Therefore the polygon $ABCD$ is a maximum under the given conditions.

145. Cor. 1. *Among the polygons that have their sides respectively equal to the given lines taken in order, any polygon that is a maximum is circumscribable.* (Indirect proof.)

Equilateral n-gon with given side.

146. Cor. 2. *Of all equilateral polygons having a given side and a given number of sides, one that is regular is a maximum.*

Among the polygons that satisfy the given conditions, one that is equilateral and equiangular is circumscribable; and one that is equilateral and not equiangular is not circumscribable (109, ex. 2); hence one that is equilateral and equiangular is a maximum (144).

147. Cor. 3. *Of all equilateral polygons having a given side and a given number of sides, any polygon that is a maximum is regular.* (Use 145.)

Given Perimeter

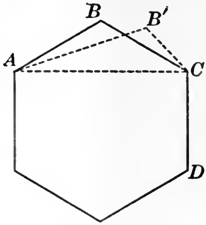
The following theorems show how to make the surface of a polygon a maximum, when the perimeter is given, and when another assigned condition is fulfilled.

The n-gon of greatest surface.

148. Theorem 42. *Among the polygons that have a given perimeter and a given number of sides, one that is a maximum is regular.*

Let the polygon $ABCD\ldots$ be a maximum, subject to the conditions of having a given perimeter and a given number of sides.

First, to prove that $ABCD\ldots$ is an equilateral polygon.



Suppose, if possible, that the two adjacent sides AB and BC are not equal.

On AC as base construct an isosceles triangle $B'AC$ having the sum of the sides $B'A$ and $B'C$ equal to the sum of BA and BC.

The isosceles triangle $B'AC$ is greater than the isoperimetric triangle BAC (II. 101).

Therefore the polygon $AB'CD\ldots$ is isoperimetric with, and greater than, the polygon $ABCD\ldots$; but this is impossible, since $ABCD\ldots$ is one of the greatest of the isoperimetric set, by hypothesis.

Hence the supposition fails, and the adjacent sides AB and BC *are* equal.

Similarly all the sides are equal. Therefore the polygon $ABCD\ldots$ is equilateral.

Next, to prove that the equilateral polygon $ABCD\ldots$ is a regular polygon.

Since $ABCD\ldots$ is an equilateral polygon having a given perimeter (that is to say, a given side) and a given number of sides, and since it is a maximum, hence it is a regular polygon (147).

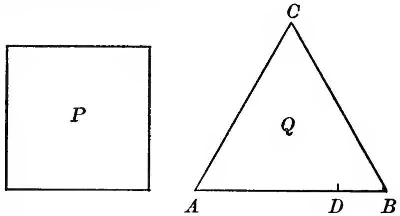
149. Cor. *Among the polygons that have a given perimeter and a given number of sides, one that is regular is a maximum.*

For all polygons that satisfy the given conditions and the additional condition of being regular are equal, and are hence equal to any one that is a maximum.

Regular polygon of greatest surface.

150. THEOREM 43. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater surface.*

Let the polygons P and Q have equal perimeters, and let P have one side more than Q.



To prove that P is greater than Q.

On one of the sides of Q take any point D.

The figure $ADBC$ may be regarded as an irregular polygon having the same number of sides as P; and hence $ADBC$ is less than the isoperimetric regular polygon P (148, 149).

Therefore Q is less than P.

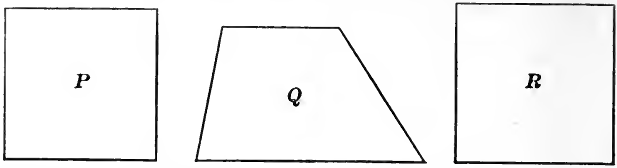
GIVEN SURFACE

The following two theorems show how to make the perimeter of a polygon a minimum when the surface is given and when another assigned condition is fulfilled.

The n-gon of least perimeter.

151. THEOREM 44. *Among the polygons having a given surface, and a given number of sides, one that is regular has a minimum perimeter.*

Let P and Q be two polygons, each having the given surface, and the given number of sides; and let P be regular, and Q not regular.



First, to prove that P has a less perimeter than Q has.

Let R be a regular polygon having the same number of sides, and the same perimeter, as Q has.

Of the two isoperimetric polygons Q and R, the latter, being regular, has the greater surface. Thus the surface of R is greater than that of Q, and hence greater than the surface of P.

Now the regular polygons P and R have the same number of sides, hence the one that has the less surface has the less perimeter (136, ex. 3). Thus the perimeter of P is less than that of R, and hence less than that of Q.

Next, to prove that the perimeter of P is a minimum under the given conditions.

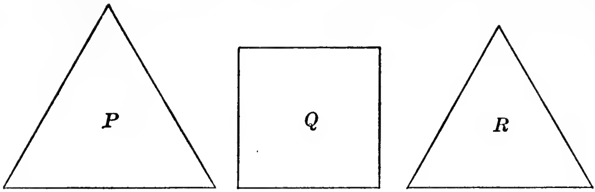
The perimeter of P is equal to that of any regular polygon satisfying the given conditions (136, ex. 3), and has been proved less than the perimeter of any irregular polygon satisfying the given conditions. Hence the perimeter of P is a minimum under the given conditions.

152. Cor. *Among the polygons having a given number of sides and a given surface, one that has a minimum perimeter is regular.* [Use indirect proof.]

Regular polygon of least perimeter.

153. Theorem 45. *If any two regular polygons have equivalent surfaces, then the one that has the greater number of sides has the less perimeter.*

Let P and Q represent any two equivalent regular polygons, and let Q have more sides than P has.



To prove that the perimeter of Q is less than that of P.

Let R be a regular polygon having the same number of sides as P, and the same perimeter as Q.

Then, of the two isoperimetric regular polygons Q and R, the former has the greater number of sides, therefore it has the greater surface (150); thus the surface of R is less than that of Q, and hence less than that of P.

Now the regular polygons P and R have the same number of sides, hence the one that has the less surface has the less perimeter (136, ex. 3); thus the perimeter of R is less than that of P. Hence the perimeter of Q is less than that of P.

Inscribed in Given Circle

154. In this group of theorems one of the stated conditions is that the polygons in question are inscribed in a given circle. The theorems show how to make the surface, or the perimeter, greatest when the polygon is subject to assigned conditions. In each of the two divisions of this group we begin as usual with the case of the triangle and proceed from it to the general polygon.

MAXIMUM SURFACE

Greatest triangle in segment.

155. Theorem 46. *Among the triangles inscribed in the same segment of a circle having their bases coincident with the chord, the triangle that is isosceles is the maximum.*

[The tangent at the mid-point of the arc is parallel to the chord.]

156. **Cor. 1.** *Among the triangles inscribed in the same segment of a circle having their bases coincident with the chord, the triangle that is the maximum is isosceles.*

157. **Cor. 2.** *Of all triangles inscribed in the same circle, one that is a maximum is equilateral.* [Indirect proof.]

158. **Cor. 3.** *Of all triangles inscribed in the same circle, one that is equilateral is a maximum.* [Prove as in 149.]

Greatest n-gon in circle.

159. THEOREM 47. *Among the polygons having a given number of sides and inscribed in a given circle, one that is a maximum is regular.* [Prove as in 157.]

160. **Cor.** *Among the polygons having a given number of sides and inscribed in a given circle, one that is regular is a maximum.* [Prove as in 138 and 149.]

Greatest regular polygon in circle.

161. THEOREM 48. *If any two regular polygons are inscribed in the same circle, then the one that has the greater number of sides has the greater surface.*

Outline. Let there be two regular inscribed polygons P and Q, and let Q have one more side than P has.

To prove that the surface of Q is greater than that of P.

Let AB be one side of P. On the arc AB take any point M; and draw MA, MB. Let the new polygon made up of the polygon P and the triangle MAB be denoted by P'.

Then P' and Q have the same number of sides. Show that Q is greater than P' (160); and hence greater than P.

MAXIMUM PERIMETER

Triangle in given segment.

162. THEOREM 49. *Among the triangles inscribed in a given segment of a circle having their bases coincident with the chord, the triangle that is isosceles has the maximum perimeter.*

Outline. Let ABC and $AB'C$ be triangles inscribed in the same segment; and let ABC be isosceles, having the side AB equal to BC.

To prove that the perimeter of ABC is greater than that of $AB'C$.

Prolong AB to D so that BD equals BC; and prolong AB' to D' so that $B'D'$ equals $B'C$.

Prove that the angles ADC and $AD'C$ are equal, being halves of equal angles; and hence that the four points C, A, D', D, are on the same circle, whose center is B. Then prove AD greater than AD'; etc.

163. Cor. 1. *Among the triangles inscribed in a given segment of a circle having their bases coincident with the chord, the triangle that has the maximum perimeter is isosceles.*

Triangle in given circle.

164. Cor. 2. *Among the triangles inscribed in a given circle, one that has a maximum perimeter is equilateral.*

Prove as in 157, using 163.

165. Cor. 3. *Among the triangles inscribed in a given circle, one that is equilateral has a maximum perimeter.* [See 149.]

The n-gon of greatest perimeter.

166. Theorem 50. *Among the polygons having a given number of sides and inscribed in a given circle, one that has a maximum perimeter is regular.* [See 164.]

167. Cor. *Among the polygons having a given number of sides and inscribed in a given circle, one that is regular has a maximum perimeter.* [Prove as in 149, 165.]

Regular polygon of greatest perimeter.

168. Theorem 51. *If any two regular polygons are inscribed in the same circle, then the one that has the greater number of sides has the greater perimeter.*

Outline. Let there be two regular inscribed polygons P and Q, and let Q have one side more than P has.

To prove that the perimeter of Q is greater than that of P.

Let AB be one side of P. On the arc AB take any point M; and draw MA, MB. Let the new polygon, made up of the polygon P and the triangle MAB, be denoted by P'.

Show that the perimeter of the regular polygon Q is greater than that of P' (167); and that the perimeter of P' is greater than that of P.

Circumscribed about Given Circle

169. In this group of theorems one of the stated conditions is that the polygons in question are circumscribed about a given circle; and it is shown how to make the surface, or the perimeter, a minimum when the polygon is subject to assigned conditions. The first division of this group relates to minimum surface, the second to minimum perimeter. In each case the additional condition should be proved to be both necessary and sufficient for a minimum.

MINIMUM SURFACE

Least triangle about sector.

170. Theorem 52. *If at any point on the arc of a given sector a tangent is drawn to meet the two radii prolonged, then the triangle so formed is the minimum when the tangent is drawn at the mid-point of the arc.*

Outline. Let OAB be the given sector, M the mid-point of the arc AB, and M' any other point of the arc. Suppose M' to be taken on the half arc MB. Let the tangent at M meet the prolongations of the radii OA and OB in the points L and N. Let the tangent at M' meet the same prolongations in the points L' and N'; and let it intersect the preceding tangent in the point I.

Prove that the triangle NIN' is less than LIL'; etc.

One point of contact arbitrary.

171. THEOREM 53. *If all the points of contact but one are assigned, at which the sides of a circumscribed polygon touch a given circle, and if all the points of contact have an assigned order on the circle, then the circumscribed polygon is least when the arbitrary point of contact bisects the arc whose extremities are at the two points of contact adjacent to that arbitrary point.*

Outline. Let O be the center of the given circle. Among the assigned points of contact let P and R be the two which are to be adjacent to the unassigned point, and let Q be any position of the unassigned point on the arc PR. Let the tangents at P and R meet the tangent at Q in the points L and N respectively. Draw OL and ON, meeting the circle in A and B respectively.

Show that the angle LON is half the angle POR, and is hence of constant magnitude, whatever be the position of Q on the arc PR; that the sector AOB is of constant magnitude; and hence that the triangle LON is least when Q is at the mid-point of the arc AB (170); that the pentagon $OPLNR$ is double the triangle LON, and is therefore least when Q is at the mid-point of the arc AB; show that Q is then at the mid-point of the arc PR; and draw conclusion.

The n-gon of least surface.

172. THEOREM 54. *Among all the polygons of a given number of sides circumscribed about a given circle, one that is a minimum is regular.*

Outline. By indirect proof, using 171, show that any point of contact bisects the arc lying between the two adjacent points of contact; and then show that the polygon is regular.

173. **Cor.** *Among all the polygons of a given number of sides circumscribed about a given circle, one that is regular is a minimum.* [Prove as in 149. See 140, ex. 1.]

Regular polygon of least surface.

174. Theorem 55. *If any two regular polygons are circumscribed about a given circle, the one that has the greater number of sides has the less surface.*

Outline. Let there be two regular circumscribed polygons P and Q, and let Q have one more side than P has.

To prove that the surface of Q is less than that of P.

Let T be any one of the vertices of P, and let A and B be the points of contact of the tangents from T. At any point M of the arc AB draw the tangent LMN, meeting TA in L and TB in N.

The tangent LMN cuts off a triangle LTN from the polygon P, leaving a circumscribed polygon which has one side more than P has. Let the new polygon be denoted by P'. Then P' and Q have the same number of sides.

Show that the regular polygon Q is less than P' (172, 173), and hence less than P.

MINIMUM PERIMETER

175. Theorem 56. *Among the polygons of a given number of sides circumscribed about a given circle, one that has a minimum perimeter is regular.*

Outline. The surface of a circumscribed polygon is equivalent to half the rectangle of the perimeter and the radius of the circle (137); hence, when the perimeter is a minimum, the surface is a minimum. Then apply 172.

176. Cor. *Among the polygons of a given number of sides circumscribed about a given circle, one that is regular has a minimum perimeter.* [Prove as in 173.]

Regular polygon of least perimeter.

177. Theorem 57. *If any two regular polygons are circumscribed about a given circle, the one that has the greater number of sides has the less perimeter.* [Prove as in 174, and use 176.]

LOCUS PROBLEMS

Equal tangents to two circles.

178. Problem 17. *To find the locus of a point from which the tangents drawn to two given circles are equal.*

Outline. If the circles intersect, show by means of 97 and ex. that the locus is the extension of the common chord.

If the circles do not intersect, let P be a point from which the tangents PT and PT' drawn to the circles are equal; and let O and O' be the centers. Using II. 61, prove that the difference of the squares on OP and $O'P$ is equivalent to the difference of the squares on the radii OT and $O'T'$. Thus the finding of the locus is reduced to II. 91.

Consider also the case in which the circles touch.

179. Definition. The line which is the locus of a point from which the tangents to two given circles are equal is called the *radical axis* of the two circles.

If the two circles intersect, then their common chord is the radical axis.

180. Cor. 1. *The radical axis of two circles is perpendicular to their central line, and divides it either internally or externally so that the difference of the squares on the segments is equivalent to the difference of the squares on the radii.*

Equal tangents to three circles.

181. Cor. 2. *To find a point from which the tangents to three given circles are equal.*

[Intersection of loci. When is there no solution ?]

182. Cor. 3. *The three radical axes of three circles meet in a point.*

Use 178, 181. Consider separately the case in which there are three chords of intersection that divide each other internally ; and use ex. 2 following Art. 97.

183. Definition. The point of concurrence of the three radical axes of three circles is called the *radical center* of the three circles.

Circles intersecting orthogonally.

184. Definition. If two circles intersect, and if the two tangents drawn at one of their common points are at right angles, then the circles are said to *intersect orthogonally.*

Ex. If two circles intersect orthogonally at one of their common points, then they intersect orthogonally at the other common point.

185. Theorem 58. *If two circles intersect orthogonally, then the tangent drawn to one of them at a common point passes through the center of the other.*

186. Cor. *To find the locus of the center of a circle that cuts a given circle orthogonally at a given point.*

Ex. To describe a circle through a given point, and cutting a given circle orthogonally at a given point.

[Determine the center of the required circle by the intersection of two loci.]

187. Theorem 59. *If two tangents are drawn to a circle from an external point, then the circle described with the point as center, and either of the tangents as radius, cuts the given circle orthogonally.*

188. Cor. *To find the locus of the center of a circle that cuts two given circles orthogonally.*

Show that the required locus is the radical axis.

Ex. To describe a circle cutting three given circles orthogonally.

[Show that there is no solution if each circle intersects the other two.]

Points in a Triangle

189. There is a class of locus problems in which the base and vertical angle of a triangle are given, to find the loci of certain important points connected with the triangle. It has been shown in 79 that the locus of the vertex of a triangle which has a given fixed base and a given vertical angle consists of the arcs of two segments described on the base (one on each side of it) containing an angle equal to the given vertical angle. The problems and corollaries in 190–193 are reducible to the one just mentioned.

190. Problem 18. *To find the locus of the ortho-centers of all the triangles that have a given fixed base and a given vertical angle.*

Outline. Let P be the ortho-center of any triangle ABC standing on the given base BC, and having the vertical angle A equal to the given angle.

Show that the triangle BPC, standing on the given base, has its vertical angle P equal to the supplement of the given angle; hence that the locus of P consists of the two arcs obtained by describing on BC two segments, each containing an angle equal to the supplement of the given angle.

191. **Cor. 1.** *To find the locus of the in-centers of all the triangles that have a given fixed base and a given vertical angle.*

Outline. If P is the in-center, show that the angle BPC is equal to the sum of the given vertical angle and half its supplement, and apply 79 as before.

192. **Cor. 2.** *With the same data, find the locus of each of the three ex-centers.*

Outline. If P is any of these centers, show that the angle BPC can be expressed in terms of A. In two of the cases BPC equals half A; and in the third case BPC equals the complement of half A.

Ex. With the same data, show that the circum-center is fixed.

193. Problem 19. *To find the locus of the median-centers of all the triangles that have a given fixed base and a given vertical angle.*

Outline. Through the median-center P draw parallels to the two sides AB and AC, meeting the base in Q and R.

Prove that Q and R trisect the base. Show that the triangle PQR, standing on the middle segment, has a given fixed base and a given vertical angle. Then apply 79.

194. Problem 20. *Find the locus of the vertices of all the triangles that have a given fixed base, and the sum of the squares on the two sides equivalent to a given square.*

Outline. Show from II. 67, 76–78, that the median AD can be constructed from the data. Show that the locus of A is a circle with D as center, and with radius equal to the line AD so found.

Subtended Angles

195. **Definitions.** The *angle subtended* at a given point by a given line is the angle included by the two lines drawn from the given point to the extremities of the given line.

The *angle subtended* at a given point by a given circle is the angle included between the two tangents drawn from the given point to the circle.

196. Problem 21. *To find the locus of a point at which the angle subtended by a given line is equal to a given angle.* [Compare 79.]

Ex. 1. To find on a given indefinite line a point at which the angle subtended by a given line-segment shall be equal to a given angle.

[When are there two solutions, when only one, and when none ?]

Ex. 2. To find on a given indefinite line a point at which a given line-segment shall subtend the greatest angle. [See ex. following Art. 96. Observe the change in the subtended angle as the point takes different positions on the line. Is there any position at which the subtended angle is least ?]

Ex. 3. To find on a given circle a point at which a given line shall subtend a given angle.

[Discuss the solution as in ex. 1. Show that in certain cases there is no solution unless the given angle is restricted in magnitude.]

Ex. 4. To find on a given circle the points at which the angle subtended by a given line is a maximum or minimum. [97, ex. 4.]

197. Problem 22. *To find the locus of a point at which the angle subtended by a given circle shall be equal to a given angle.*

Outline. Show that the locus is a concentric circle, and that its radius may be constructed as follows: Construct a right triangle having one side equal to the radius of the given circle, and the adjacent acute angle equal to the complement of half the given angle; then the hypotenuse is the required radius.

Ex. In the four exercises of 196, replace the given line-segment by a given circle, and show that similar solutions can be obtained.

Intersection of Loci

198. Each of the following constructions is a combination of two locus problems already solved.

Ex. 1. To construct a triangle, being given:

- (*a*) its base, vertical angle, and altitude (79, I. 255);
- (*b*) its base, vertical angle, and difference of squares on sides (79, II. 91);
- (*c*) its base, vertical angle, and sum of squares on sides;
- (*d*) its base, altitude, and sum of squares on sides (194);
- (*e*) its base, vertical angle, and one side; the base being given in position as well as magnitude.

Ex. 2. To construct a quadrangle, being given two opposite angles, and three sides; the order in which the five parts are to be taken being specified.

Outline. Let the sides AB, BC, CD be given; and also the angles B and D. First construct the triangle ABC (I. 133); and then the triangle ACD (ex. 1, *e*).

[Examine the case in which one of the given angles is convex. Show that there is always a solution when the sum of the two given angles is less than a perigon. Show that there is only one solution.]

EXERCISES

1. If from any point on a given circle a line is drawn equal and parallel to a given line, then the locus of the other extremity consists of two circles each equal to the given circle.

[Take the given line as "line of translation" (I. 200), and translate the center and any radius, thus reducing the locus problem to a previous one (7).]

2. If from any point on a given circle a line is drawn to a given point, and if this line is turned about the given point through a given angle, then the locus of the other extremity of the line so turned consists of two circles each equal to the given one.

[Take the given angle as "angle of rotation" (I. 202), and rotate the circle and any radius about the given point.]

3. To describe three circles of given radii to touch each other externally.

4. To describe three circles of given radii to touch each other so that two may be within the third.

5. In an equilateral triangle the radius of each of the escribed circles is equal to the altitude; and the radii of the circumscribed and inscribed circles are respectively equal to two thirds, and one third of the altitude.

6. If two chords of a circle cut at right angles, then the sum of either pair of opposite arcs is equal to a semicircle.

[Through an extremity of one chord, draw a chord parallel to the other, and join its extremity to the other extremity of the first chord.]

7. If two chords of a circle cut at right angles, then the sum of the squares on the four segments is constant, and equivalent to the square on the diameter.

8. If any chord of a given circle passes through a fixed point, then the rectangle of the segments of the chord is constant.

9. If through a fixed point within a given circle any two chords are drawn at right angles, then the sum of the squares on the two chords is constant (exs. 7, 8).

10. If each of two equal circles has its center on the circumference of the other, then the square on their common chord is equivalent to three times the square on the radius.

11. If two given circles have external contact; show how to draw a line through the point of contact so that the whole intercepted part may be equal to a given line.

[Form an isosceles triangle whose base is the given line and each of whose other sides equals the sum of the radii. Then the angle which the required line makes with the central line equals one of the base angles; prove.]

12. To construct a triangle, being given the vertical angle, one of adjacent sides, and the perpendicular from the vertex to the base.

[Show that the foot of the perpendicular can be found by intersection of loci.]

13. If two circles intersect, then any common tangent subtends, at the common points, angles which are supplemental.

14. If a common tangent is drawn to two circles, and if each point of contact is joined to the two points where the central line meets the corresponding circle, then the two chords so drawn in one circle are respectively parallel to the two chords in the other circle.

15. Find the locus of a point such that the tangent from it to a given circle shall be equal to the line joining it to a given point.

[This is a limiting case of the radical axis of two circles (179) when the radius of one of the circles diminishes so that the circle reduces to a point.]

16. Find a point such that the tangent from it to a given circle shall be equal to each of the lines joining it to two given points.

17. Find the locus of the center of a circle which passes through a given point and cuts a given circle orthogonally.

[In 188 let one of the circles reduce to a point.]

18. Describe a circle through a given point so as to cut two given circles orthogonally.

19. Describe a circle through two given points so as to cut a given circle orthogonally.

20. Given the vertical angle and the altitude of a triangle, prove that the surface is a minimum when the triangle is isosceles (170).

21. Given the vertical angle and altitude of a triangle, when is the base a minimum?

22. The inscribed regular hexagon is equivalent to three fourths of the circumscribed one, to half the circumscribed equilateral triangle, and to double the inscribed one, in the same circle.

BOOK IV. — RATIO AND PROPORTION

1. That relation between two magnitudes which is expressed by the word *ratio* will receive a precise definition after certain preliminary notions are explained.

The principles will be made sufficiently general to apply to any geometric magnitudes for which appropriate methods of comparison have been given, such as two line-segments, two angles, the surfaces of two polygons, etc.

It will not be necessary to restrict our thoughts even to geometric magnitudes. The notion of ratio is applicable to any magnitudes for which the words *equivalent, greater, less, sum, difference,* etc., have a definite and consistent meaning. Such magnitudes are found in the sciences that deal with number, weight, velocity, probability, etc.

MULTIPLES AND MEASURES

Definitions

2. Multiples. If any number of equivalent magnitudes are added together, then their sum is called a *multiple* of any one of them.

If any magnitude P is equivalent to the sum of n magnitudes each equivalent to A, then P is said to be equivalent to *n times* the magnitude A, or to the *nth multiple* of A, which is sometimes denoted by the symbol $n \cdot A$ or nA.

Thus the double of A, previously defined, means the same as twice A, or the second multiple of A. We may regard A itself as *once* A, and call it the *first multiple* of A.

3. Series of multiples. The magnitudes denoted by the symbols

$$A,\ 2A,\ 3A,\ 4A,\ \cdots nA,\ \cdots$$

may be thought of as formed by beginning with the magnitude A and successively adding other magnitudes equivalent to A, as often as desired. The whole set of magnitudes so thought of is called the *series of multiples* of A.

In considering the mutual relations of certain magnitudes, their series of mutiples will play an important part.

4. Magnitudes of the same kind. Two magnitudes will be said to be of the *same kind* when their two series of multiples can be directly compared so as to test the equivalence or non-equivalence of any of them.

In particular two geometric magnitudes A and B will be said to be of the same kind when it is possible to compare the multiples of A with the multiples of B by means of superposition. Such, for instance, are two line-segments, two angles, two equiradial arcs, the surfaces of two polygons; but not a straight line and a curved line, nor two arcs of unequal circles, nor the surfaces of a circle and a polygon.

For a simple example of the comparison of two series of multiples, the student may glance forward to the figure in Art. 12, in which the successive multiples of two line-segments A and B are laid off on an indefinite line, all beginning at the same point O.

Again, the natural numbers $1, 2, 3, 4, \cdots n, \cdots,$

which we have used to indicate the order in a series of multiples, belong to another class of magnitudes called *numerical magnitudes.* The terms *equivalent* (or *equal*), *greater, less, sum, multiple,* etc., have definite meanings when applied to them. Thus, the number 3 has its series of multiples,

$$3, 6, 9, 12, \cdots,$$

and the number n has its series of multiples,

$$n, 2n, 3n, \cdots, pn, \cdots.$$

Any two natural numbers are magnitudes of the same kind, since they, or any of their multiples, are directly comparable.

When any two magnitudes are mentioned together, they are understood to be of the same kind, unless otherwise stated. Numerical magnitudes are denoted by small letters.

5. Common multiple. If it should happen that some multiple of A is also a multiple of B, then this is said to be a *common multiple* of A and B.

It rarely happens that two geometrical magnitudes taken at random have a common multiple; but if they have one, then they have an indefinite number of other common multiples. For instance, if the magnitudes A and B have a common multiple P which is equivalent to $2A$ and also to $3B$, then the double of P is equivalent to $4A$ and also to $6B$, and is therefore a common multiple of A and B.

The least magnitude (if any) which is a common multiple of two given magnitudes is called their *least common multiple.* The other common multiples could be obtained by starting with the least common multiple, and then forming its series of multiples.

In the figure of Art. 12, if A and B have a common multiple, this fact will be shown by the coincidence of two of the points of division; and such coincidence will recur at regular intervals.

6. Submultiples or measures. If one magnitude is a multiple of another, then the latter is said to be a *submultiple* of the former.

Thus the series of submultiples of a magnitude A, in descending order of size, are: one half of A, one third of A, one fourth of A, . . . one nth of A, . . .

A submultiple of A is often called a *measure* of A, because it is contained in A a certain number of times without remainder. It is also called an *aliquot part* of A, because it may be obtained by dividing A into a certain number of equal or equivalent parts. We can use the term that seems most suggestive in any particular connection. The magnitude A itself may be included among the measures of A.

7. Common measure. If it should happen that some measure of A is also a measure of B, then this is said to be a *common measure* of A and B. Two magnitudes that have a common measure are said to be *commensurable.*

It rarely happens that two geometrical magnitudes taken at random have a common measure; but if they have one, then they have an indefinite number of other common measures. For instance, if the magnitudes A and B have a common measure, M, which is contained just three times in A and five times in B, then the half of M is contained just six times in A and ten times in B, and is therefore a common measure of A and B.

The greatest magnitude which is contained a whole number of times in each of two commensurable magnitudes is called their *greatest common measure.* The other common measures could be obtained by starting with the greatest common measure and then taking its series of submultiples.

8. Like multiples and like measures. If there are any two given magnitudes (not necessarily of the same kind), and if two multiples of them are formed by taking each of them the same number of times, then the two resulting magnitudes are called *like multiples* of the two given magnitudes, and the given magnitudes are called *like measures* of the resulting magnitudes.

Thus, if the magnitudes A and X are each taken n times, then the resulting magnitudes nA and nX are called like multiples of A and X, and the magnitudes A and X are called like measures of nA and nX.

Properties of Multiples and Measures

9. The following general statements are immediate inferences from the preceding definitions. They apply to any two magnitudes of the same kind, and are verified by direct comparison. A good illustration is furnished by two line-segments, or two whole numbers.

1. Two magnitudes of the same kind are such that some multiple of one is greater than any given multiple of the other.

2. According as one magnitude is greater than, equivalent to, or less than another, so is any multiple of the first greater than, equivalent to, or less than the like multiple of the other.

3. According as one magnitude is greater than, equivalent to, or less than another, so is any measure of the first greater than, equivalent to, or less than the like measure of the other.

This statement is converse to the preceding; they can be put together in a single order-theorem thus:

The two pairs of magnitudes

$$A,\ B$$

and

$$mA,\ mB$$

are in the same order of size.

4. Any multiple of the sum of two or more magnitudes is equivalent to the sum of like multiples of these magnitudes.

Symbolically: $m(A + B + C) \approx mA + mB + mC.$

5. Any multiple of the difference of two magnitudes is equivalent to the difference of their like multiples.

Symbolically: $m(A - B) \approx mA - mB.$

6. A common measure of two magnitudes is a measure of their sum, and of their difference.

7. A measure of any magnitude is a measure of any multiple of the same magnitude.

8. A multiple of any magnitude is a multiple of any measure of the same magnitude.

9. The mth multiple of the nth multiple of any magnitude is equivalent to the mnth multiple of the same magnitude.

10. The mth multiple of the nth multiple of any magnitude is equivalent to the nth multiple of the mth multiple of the same magnitude.

For instance, the third multiple of the fourth multiple of A is equivalent to the fourth multiple of the third multiple of A, each being equivalent to twelve times A.

11. According as m is greater than, equal to, or less than n, so is the mth multiple of any magnitude greater than, equivalent to, or less than the nth multiple of the same magnitude; and conversely.

These two converse statements may be expressed as an order-theorem thus:

The two pairs of magnitudes

$$m, \ n$$

and

$$mA, \ nA$$

are in the same order of size.

EXAMPLES FOR ILLUSTRATION

10. The following examples are inserted here to illustrate the preceding principles; but they are not essential to the understanding of the subsequent articles.

1. *If the mth multiple of A is equivalent to the nth multiple of B, then the nth submultiple of A is equivalent to the mth submultiple of B.*

Denote the nth submultiple of A by A', and the mth submultiple of B by B', then

$$A \Leftrightarrow nA', \quad B \Leftrightarrow mB'.$$

Take m times each of the first pair, and n times each of the second pair, then

$$mA \Leftrightarrow mnA', \ nB \Leftrightarrow nmB'. \qquad [9\ (2, 9)$$

Now, by hypothesis, mA is equivalent to nB. Hence

$$mnA' \Leftrightarrow nmB' \Leftrightarrow mnB'. \qquad [9\ (10)$$

Take the mnth submultiple of each of these equivalents.

Then $$A' \Leftrightarrow B'. \qquad [9\ (3)$$

Cor. 1. *If two magnitudes have a common multiple, then they are commensurable.*

Cor. 2. *If two magnitudes are incommensurable, then they have no common multiple.*

Ex. Any two whole numbers, m and n, have a common multiple mn, and a common measure unity.

2. *If the nth submultiple of A is equivalent to the mth submultiple of B, then the mth multiple of A is equivalent to the nth multiple of B.*

Outline. With the same notation as before

$$A \Leftrightarrow nA', \quad B \Leftrightarrow mB';$$

and we are given $\quad A' \Leftrightarrow B';$

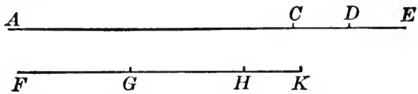
to prove $\quad mA \Leftrightarrow nB.$

Cor. 1. *If two magnitudes are commensurable, then they have a common multiple.*

Cor. 2. *If two magnitudes have no common multiple, then they are incommensurable.*

3. *To find whether two given magnitudes are commensurable or not; and, if so, to find their greatest common measure.*

For convenience, let the magnitudes be represented by the lines AE and FK.



(*a*) From the greater AE take away as many parts as possible each equivalent to the less FK. If there is a remainder, as CE, take away from FK as many parts as possible each equivalent to CE. If there is a second remainder, as HK, take away from the preceding remainder, CE, as many parts as possible, each equivalent to HK, and so on.

It is evident that this process will terminate only when a remainder is obtained which is a measure of the remainder immediately preceding.

(*b*) *If this process terminates, then the two given magnitudes are commensurable, and the last remainder is their greatest common measure.*

First, to prove that the last remainder is a common measure.

Suppose that HK is the last remainder. Then, by hypothesis, HK is a measure of CE, and hence of FH, which is a multiple of CE [9 (7)]; therefore HK is also a measure of FK, which is the sum of FH and HK [9·(6)], and therefore HK is a measure of AC, which is some multiple of FK, by construction; hence, again, HK is a measure of AE, which is the sum of AC and CE; therefore HK is a common measure of FK and AE.

Next, to prove that HK is the greatest common measure of FK and AE.

Every measure of FK is a measure of its multiple AC [9 (7)]; hence every common measure of FK and AE is a common measure of AC and AE, and therefore a measure of their difference CE, and therefore of FH, which is a multiple of CE; hence every common measure of FK and AE is a common measure of FK and FH, and therefore a measure of their difference HK [9 (6)].

Hence no common measure of FK and AE can exceed HK. Therefore HK is the greatest common measure of FK and AE.

(c) *If the two magnitudes have a common measure, then the process in* (a) *will terminate.*

Proof. Any common measure is a measure of each remainder, as shown above. Now any remainder is evidently less than half the second preceding remainder; hence, if the process does not terminate, a remainder will be reached which is less than any assigned magnitude, and therefore less than the greatest common measure; but this is impossible, since the greatest common measure is a measure of every remainder. Therefore the process does terminate if the two given magnitudes have a common measure.

(d) *If the process in* (a) *does not terminate, then the two magnitudes are incommensurable.*

For if they were commensurable the process would terminate (c).

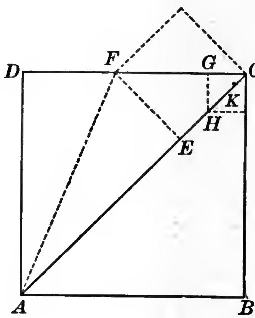
We conclude from (*b*), (*c*), (*d*), that the process in (*a*) is a complete test of commensurability, and that it furnishes the greatest common measure when one exists.

NOTE. The same principles can be used in finding the greatest common measure of two polygons; but the actual process may be difficult in certain cases. In the case of two numerical magnitudes the method is easily applied, and corresponds to the ordinary arithmetical rule.

4. *The side and diagonal of a square are incommensurable.*

Let $ABCD$ be a square, whose side is AB, and diagonal AC.

To prove that AB and AC have no common measure.



On AC lay off a part AE equal to AB. The remainder, EC, is less than AB (I. 88).

Hence AB is contained once in AC with a remainder EC.

Draw EF perpendicular to EC, and join AF. Since AE equals AD, the right triangles AEF and ADF are equal (I. 98). Therefore EF equals FD.

Again, since the right triangle CEF is isosceles, the lines CE, EF, and FD are equal.

Lay off FG equal to FD. Then EC is contained twice in the side DC, with remainder GC.

This remainder GC is for a similar reason contained twice in EC with remainder KC; and so on.

Hence this process of finding the greatest common measure repeats itself indefinitely, and will never terminate.

Therefore AC and AB are incommensurable [10 (3)].

EX. The squares on AC and AB are commensurable.

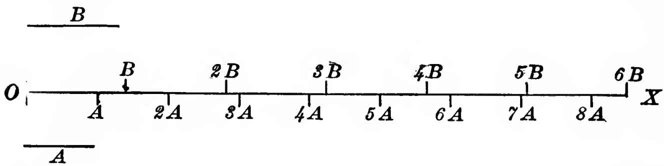
NOTE. Unless two magnitudes are specially selected, they are in all probability incommensurable. Commensurability is a rare exception.

Scale of Relation

11. Definition. If there are two magnitudes, A and B, of the same kind, and if their two series of multiples, namely, $A, 2A, 3A, \cdots$, and $B, 2B, 3B, \cdots$, are supposed written in a single scale in the ascending order of size of all these multiples, the resulting arrangement is called the *scale of relation* of the two magnitudes A and B.

12. Two line-segments. The scale of relation of two given line-segments is easy to construct, and furnishes a good illustration of the definition.

Let the two given lines be A and B.



Take an indefinite line OX, and, beginning at O, lay off successive segments each equal to A. Mark the consecutive points of division with the symbols $A, 2A, 3A, \cdots$. Beginning again at O, lay off successive segments each equal to B; and mark the points of division with the symbols $B, 2B, 3B, \cdots$.

The order in which the symbols occur on the line evidently shows the order of succession of the various multiples of the two given magnitudes, all arranged in one scale in the ascending order of their size. This order of succession constitutes the scale of relation of the two given magnitudes A and B. One use of such a scale is to tell between what two consecutive multiples of B any assigned multiple of A lies. For instance, in the figure $8A$ is greater than $5B$ and less than $6B$. The scale will also show whether A and B have any common multiple. If two of the points of division happen to coincide at any point P, then the line OP is a common multiple of the two given lines. If A and B are incommensurable, they have no common multiple, and none of the points of division will coincide.

13. Two numerical magnitudes. As another illustration, the scale of relation of two given whole numbers may be found by a method similar to that just given. For instance, the scale of relation of the numbers 3 and 14 may be arranged as follows (letting t stand for 3 and f for 14):

t, $2t$, $3t$, $4t$, f, $5t$, $6t$, $7t$, $8t$, $9t$, $2f$, $10t$, $11t$, $12t$, $13t$, $14t = 3f$, $15t$, $16t$, $17t$, $18t$, $4f$, $\cdots$.

RATIO

On the Notion of Ratio

14. Definitions. That relation of two magnitudes of the same kind which is exhibited by the order of succession of their multiples when arranged in one ascending scale is called the *ratio* of one magnitude to the other.

The ratio of a magnitude A to another magnitude B of the same kind is denoted by the symbol $A : B$; and A is called the *antecedent* and B the *consequent* of the ratio.

If there are two other magnitudes of the same kind, X and Y (not necessarily of the same kind as A and B), the definitions of the next two articles furnish a basis for a comparison of the two ratios $A : B$ and $X : Y$, so that the terms *equal, greater,* and *less* may have definite meanings when applied to any two ratios.

The two antecedents are said to be *homologous* terms in the two ratios, and so are the two consequents.

15. Equal ratios. The two ratios $A : B$ and $X : Y$ are said to be *equal ratios,* if, on comparison of the scale of relation of A and B with the scale of relation of X and Y, it is found that the successive multiples of A and B interlie in the same order in the first scale as the corresponding multiples of X and Y do in the second scale.

In the same case the two scales of relation are said to be *similar* to each other.

In applying this definition, the similarity of the scale of A and B with the scale of X and Y can be established in the following manner:

Take any like multiples of the antecedents, say

$$mA, \ mX;$$

and take any like multiples of the consequents, say

$$nB, \ nY;$$

then, to prove that the two scales are similar throughout, we have to show that the two pairs of multiples

$$mA, \ nB$$

and

$$mX, \ nY$$

are in the same order of size, no matter what whole numbers m and n are.

The phrase "in the same order of size" has been defined and fully illustrated in connection with "order-theorems" in III. 50. The student is requested now to review that article, and then to read the simple application of the above definition of equal ratios, which will be found in art. 90, Book V, where it is proved that

If any two rectangles have equal altitudes, then the ratio of the rectangles is equal to the ratio of their bases.

The method may be stated in outline as follows:

Let the rectangles be denoted by R and R', and their bases by b and b', then we have to prove that the ratio $R : R'$ is equal to the ratio $b : b'$.

Of the base b we take any multiple mb; and then show that the rectangle standing on the base mb, and having the given altitude, is equivalent to mR.

Of the base b' we take any multiple nb'; and then show that the rectangle standing on the base nb', and having the given altitude, is equivalent to nR'.

We then quote the order-theorem that any two rectangles of the same altitude are in the same order of size as their bases are; and thence infer that the two pairs of multiples

$$mR,\ nR'$$

and

$$mb,\ nb'$$

are in the *same order* of size, whatever whole numbers m and n may be.

We conclude that the scale of relation of R and R' is similar throughout its whole extent to the scale of relation of b and b'; and hence, by definitions of equal ratios, that

$$R : R' = b : b'.$$

16. Unequal ratios. The ratio $A : B$ is said to be *greater than* the ratio $X : Y$, if, on comparison of the scale of relation of A and B with the scale of relation of X and Y, it is found that some multiple of A has a more advanced position among the multiples of B than the like multiple of X has among the multiples of Y.

In other words, the ratio $A : B$ is said to be greater than the ratio $X : Y$ when it is possible to find any single pair of whole numbers m and n such that

mA is greater than nB, and
mX is not greater than nY,

or else such that

mA is equivalent to nB, and
mX is less than nY.

In the same case the ratio $X : Y$ is said to be *less than* the ratio $A : B$. This is expressed symbolically by

$$A : B > X : Y \quad \text{or} \quad X : Y < A : B.$$

NOTE. A ratio is not to be confused with its number-correspondent, introduced later, in mensuration (VI. 14). It will appear that the general theory of ratio furnishes a natural and logical basis for the science of numerical measurement, which is one of its applications.

Properties of Ratios

17. In the following articles, magnitudes of the same kind will usually be denoted by adjacent letters of the alphabet; and magnitudes that are not necessarily of the same kind will be denoted by non-adjacent letters. For instance, if there are three ratios whose terms are not restricted to be of the same kind, they may be represented by $A:B$, $P:Q$, $X:Y$. A small letter will denote a whole number.

PRINCIPLES OF EQUALITY AND INEQUALITY

18. It has been seen that a ratio is not a magnitude, but a relation between two magnitudes.* Two ratios can, however, be compared with each other by means of the conventions laid down in 15, 16; and it will be proved in theorems 1 and 2 that the definitions of the words *equal, greater,* and *less,* as applied to ratios, lead to principles that correspond to the axioms of equality and inequality.

Principle of equality.

19. Theorem 1. *If two ratios are each equal to the same third ratio, then they are equal to each other.*

Let $A:B=X:Y$,

and $P:Q=X:Y$;

to prove $A:B=P:Q$.

The multiples of A and B have the same inter-order as the multiples of X and Y; and the same thing is true of the multiples of P and Q; therefore, the multiples of A and B have the same inter-order as those of P and Q; thus the scales of relation are similar; hence, by definition (15),

$$A:B=P:Q.$$

* This statement applies even to numerical ratios; the ratio $m:n$ is distinct from the quotient $\frac{m}{n}$, which is its number-correspondent (VI. 14).

Principle of inequality.

20. THEOREM 2. *If two ratios are equal, then any third ratio which is less than one of them is also less than the other.*

Let $A : B = X : Y$,

and $P : Q < X : Y$;

to prove $P : Q < A : B$.

Since $P : Q < X : Y$,

hence, by the definition of unequal ratios, some multiple of P, say the mth, occupies a less advanced position among the multiples of Q than the mth multiple of X does among the multiples of Y; therefore, by the first part of the hypothesis, the mth multiple of P occupies a less advanced position among the multiples of Q than the mth multiple of A does among the multiples of B. Hence, by definition (16),

$$P : Q < A : B.$$

21. Cor. *If two ratios are equal, then any third ratio which is greater than one is also greater than the other.*

RECIPROCAL RATIOS

22. Definition. Two ratios are said to be *reciprocal* to each other when the antecedent of each ratio is the consequent of the other.

Thus, the ratios $A : B$ and $B : A$ are reciprocal to each other.

Reciprocals of equal ratios.

23. THEOREM 3. *If two ratios are equal, then their reciprocal ratios are equal.*

Given $A : B = X : Y$;

to prove $B : A = Y : X$.

From the hypothesis, any multiple of A occupies a position among the multiples of B, similar to that which the like multiple of X occupies among the multiples of Y.

Hence any multiple of B occupies a position among the multiples of A similar to that which the like multiple of Y occupies among the multiples of X.

Therefore the ratio $B : A$ is equal to the ratio $Y : X$ (15).

Reciprocals of unequal ratios.

24. THEOREM 4. *If one ratio is greater than another, then the reciprocal of the first is less than the reciprocal of the second.*

Given $A : B > X : Y$;

to prove $B : A < Y : X$.

From the hypothesis, some multiple of A occupies a more advanced position among the multiples of B than the like multiple of X does among those of Y.

Hence some multiple of B occupies a less advanced position among the multiples of A than the like multiple of Y does among the multiples of X.

Therefore, by definition, the ratio $B : A$ is less than the ratio $Y : X$ (16).

EQUIVALENCE OF ANTECEDENTS OR CONSEQUENTS

This group of theorems is concerned with the comparison of ratios whose antecedents or consequents are equivalent.

Equivalent antecedents and consequents.

25. THEOREM 5. *If two ratios have equivalent antecedents and equivalent consequents, then the ratios are equal.*

Given A equivalent to A', and B equivalent to B';

to prove $A : B = A' : B'$.

Since the multiples of A are equivalent to the like multiples of A' [9 (2)]; and the multiples of B are equivalent to the like multiples of B'; therefore, the multiples of A are distributed among the multiples of B in the same inter-order as the multiples of A' are among those of B'.

Hence, by definition, the ratio $A : B$ equals the ratio $A' : B'$.

26. According to the theorem just proved, a ratio is not altered when either its antecedent or its consequent is replaced by an equivalent magnitude. For this reason two equivalent magnitudes will sometimes be denoted by the same letter; and the symbol of equality (=) will be used as the symbol of equivalence. The symbols for greater than (>) and for less than (<) will also be used.

Equivalent consequents, unequivalent antecedents.

27. THEOREM 6. ***If two ratios have equivalent consequents, then the one that has the greater antecedent is the greater ratio.***

Let each of the consequents be equivalent to a magnitude C; and let the antecedents be A and B.

Given $\qquad A > B;$

to prove $\qquad A : C > B : C.$

Let A exceed B by a magnitude X.

Take a multiple of X that shall exceed C; and let mX be such a multiple. Take the like multiples of A and B. Then mA exceeds mB by mX [9 (5)].

Therefore mA exceeds mB by more than C; or, in other words, mB falls short of mA by more than C.

Next, take the successive multiples of C, and let nC be the first one that does not fall short of mA.

Then either mA is equivalent to nC, or else mA lies between nC and the next lower multiple of C.

But mB falls short of mA by more than C; therefore mB lies in a less advanced interval among the multiples of C than mA does.

Hence the ratio $A : C$ is greater than the ratio $B : C$ (16).

Combined statement.

28. COR. I. *If two ratios have equivalent consequents, then according as the first antecedent is greater than, equivalent to, or less than the second, so is the first ratio greater than, equal to, or less than the second.* (Combination of 25 and 27.)

Converse statement.

29. Cor. 2. *If two ratios have equivalent consequents, then according as the first ratio is greater than, equal to, or less than the second, so is the first antecedent greater than, equivalent to, or less than the second.*

Equivalent antecedents, unequivalent consequents.

30. Theorem 7. *If two ratios have equivalent antecedents, then the one that has the greater consequent is the less ratio.*

Let each of the antecedents be equivalent to A; and let the consequents be B and C.

Given $B > C$;

to prove $A : B < A : C$.

Take B and C as antecedents, and compare them with the same consequent A.

Then, since B is greater than C,

hence, $B : A > C : A$. [27

Therefore, by taking reciprocals,

$A : B < A : C$. [24

Combined statement.

31. Cor. 1. *If two ratios have equivalent antecedents, then according as the first consequent is greater than, equivalent to, or less than the second, so is the first ratio less than, equal to, or greater than the second.* (Combination of 25 and 30.)

Converse statement.

32. Cor. 2. *If two ratios have equivalent antecedents, then according as the first ratio is less than, equal to, or greater than the second, so is the first consequent greater than, equivalent to, or less than the second.*

Note. It follows from 27 and 30 that the scale of relation of two given magnitudes is altered somewhere, if the slightest change is made in either one of the given magnitudes.

Homologous terms of equal ratios compared.

33. THEOREM 8. *If there are two equal ratios, the four magnitudes being of the same kind, then according as the antecedent of the first is greater than, equivalent to, or less than the antecedent of the second, so is the consequent of the first greater than, equivalent to, or less than the consequent of the second.*

Let A, B, C, D, be four magnitudes of the same kind, such that

$$A : B = C : D.$$

To prove that the two pairs

$$A, C$$

and

$$B, D$$

are in the same order of size.

First, let A be equivalent to C.

Then the two equal ratios written above have equivalent antecedents; hence their consequents are equivalent,

i.e. B is equivalent to D. [32

Next, let A be greater than C.

Take A and C as antecedents, and compare them with the same consequent B,

then $A : B > C : B$, [27

but $A : B = C : D$, [hyp.

therefore, by the principle of inequality,

$$C : D > C : B.$$ [20

Since these unequal ratios have the same antecedent, hence the less ratio has the greater consequent,

i.e. B is greater than D. [32

Lastly, let A be less than C.

The student may prove in a similar way that in this case B is less than D.

Thus the two pairs above are always in like order of size.

33 (*a*). Ex. If there are two equal ratios $A : B$ and $X : Y$, then the two pairs

$$A, B$$

and

$$X, Y$$

are in the same order of size.

(Put $m = n = 1$ in the order-statement given in Article 15.)

N.B. This principle, which follows immediately from the definition of equal ratios, is not to be confused with the principle in 33.

RATIOS OF MULTIPLES

34. The next three theorems with their corollaries relate to the comparison of multiples of given magnitudes.

Like multiples of two magnitudes.

35. THEOREM 9. *The ratio of like multiples of two magnitudes is equal to the ratio of the magnitudes.*

Let A and B be two magnitudes of the same kind, and let pA and pB be any like multiples of A and B.

To prove $$pA : pB = A : B.$$

Take the mth multiple of each of these antecedents, and the nth multiple of each consequent; and compare the order of size of the resulting pairs of multiples,

$$m \cdot pA,\ n \cdot pB$$

and

$$m \cdot A,\ n \cdot B.$$

Since the mth multiple of the pth multiple is equivalent to the pth multiple of the mth multiple, the first pair may be written in the form

$$p \cdot mA,\ p \cdot nB. \qquad [9\ (10)$$

Now the members of this pair are like multiples of mA and nB; hence this pair is in the same order of size as the pair

$$mA,\ nB, \qquad [9\ (3)$$

which is the second pair of multiples above.

Hence the above pairs of multiples are in the same order of size whatever m and n are.

Therefore $$pA : pB = A : B.$$

Special case.

36. Cor. 1. *The ratio of two whole numbers is not altered by multiplying both of them by the same number.*

37. Cor. 2. *If* $A : B = X : Y,$

then $pA : pB = qX : qY.$

Multiples of one magnitude.

38. THEOREM 10. *The ratio of the mth multiple of any magnitude to the nth multiple of the same magnitude is equal to the ratio of the number m to the number n.*

To prove $mA : nA = m : n.$

Take p times each antecedent, and q times each consequent, and compare the order of size of the two pairs of multiples

$$p \cdot mA,\ q \cdot nA$$

and

$$p \cdot m,\ q \cdot n.$$

The first pair may be written in the form

$$pm \cdot A,\ qn \cdot A, \qquad [9\ (9)$$

and these two multiples of A are in the same order of size as the pair of numbers

$$pm,\ qn, \qquad [9\ (11)$$

which is the second pair of multiples above.

Hence the two pairs of multiples above are in the same order of size whatever the whole numbers p and q are.

Therefore, by the definition of equal ratios,

$$mA : nA = m : n.$$

Another statement of 38.

39. Cor. *If two magnitudes have a common measure which is contained m times in the first magnitude and n times in the second, then the ratio of the two magnitudes is equal to the ratio of the number m to the number n.*

Equivalent multiples.

40. Theorem 11. *If the ratio of one magnitude to another is equal to the ratio of the number m to the number n, then the nth multiple of the first magnitude is equivalent to the mth multiple of the second.*

Given $$A : B = m : n;$$

to prove $$nA = mB.$$

Take the nth multiple of each antecedent and the mth multiple of each consequent; then, from the hypothesis, the pairs of multiples

$$n \cdot A, \quad m \cdot B$$

and $$n \cdot m, \quad m \cdot n$$

are in the same order of size; but the members of the latter pair are equivalent; therefore the members of the former pair are equivalent.

Hence $$nA = mB.$$

Unequivalent multiples.

41. Cor. 1. *If the ratio of two magnitudes is greater than the ratio of two whole numbers m and n, then the nth multiple of the first magnitude is greater than the mth multiple of the second.*

[Show that the first pair of multiples above are then in descending order (16).]

Combined statement.

42. Cor. 2. *According as* $A : B >=< m : n$,

so is $nA >=< mB$.

Converse statement.

43. Cor. 3. *According as* $nA >=< mB$,

so is $A : B >=< m : n$.

PROPORTION

Properties of a Proportion

44. Definition. A *proportion* is a statement of the equality of two ratios, as $A : B = X : Y$.

These four magnitudes are said to form a proportion, of which A and Y are the *extremes*, and B and X the *means;* and Y is called the *fourth proportional* to the three terms A, B, and X. The proportion is sometimes read thus: A is to B as X is to Y.

The next three theorems are concerned with the establishment of certain general "rules of inference," by which, from a given proportion, certain other proportions can be at once derived. They are the Rules of Equi-multiplication, Alternation, and Composition.

Equi-multiples of homologous terms.

45. Theorem 12. *If two ratios are equal, and if any like multiples of the antecedents are taken, and also any like multiples of the consequents, then the multiple of the first antecedent is to the multiple of the first consequent as the multiple of the second antecedent is to the multiple of the second consequent.*

Given $$A : B = X : Y;$$

to prove $$mA : nB = mX : nY.$$

To compare the latter two ratios, take the pth multiple of each antecedent, and the qth multiple of each consequent, and compare the order of size of the two pairs of resulting multiples

$$p \cdot mA, \quad q \cdot nB$$

and $$p \cdot mX, \quad q \cdot nY.$$

According to 9 (9), these may be written in the form

$$pm \cdot A, \quad qn \cdot B$$

and $$pm \cdot X, \quad qn \cdot Y.$$

Now these two pairs of multiples are in the same order of size, because the ratios $A : B$ and $X : Y$ are equal.

Therefore the former pairs of multiples are in the same order of size, whatever whole numbers p and q may be.

Hence $$mA : nB = mX : nY.$$

Special case.

46. Cor. *Given* $A : B = X : Y$,

and $mA = nB$;

then $mX = nY$.

NOTE. This corollary may be stated in words as follows:

If four magnitudes form a proportion, and if the first is any multiple, or part, or multiple of a part, of the second, then the third is the like multiple, or part, or multiple of a part, of the fourth.

Rule of alternation.

47. THEOREM 13. *If four magnitudes of the same kind form a proportion, then the first is to the third as the second is to the fourth.*

Let A, B, C, D be four magnitudes of the same kind such that
$$A : B = C : D.$$

To prove $$A : C = B : D.$$

Since the ratio of two magnitudes equals the ratio of their like multiples,

hence $$mA : mB = nC : nD. \quad [35$$

Therefore, by comparison of homologous terms in equal ratios, the two pairs

mA, nC

and mB, nD

are in the same order of size. [33

Now m and n are any whole numbers; hence, by definition of equal ratios, $$A : C = B : D.$$

Rule of composition.

48. THEOREM 14. *If four magnitudes form a proportion, then the sum of the first and second is to the second as the sum of the third and fourth is to the fourth.*

Given $A : B = X : Y$;

to prove $A + B : B = X + Y : Y$.

In order to compare the latter two ratios take any like multiples of the antecedents, and any like multiples of the consequents; and then compare the order of size in the two resulting pairs

$$m(A + B),\ nB$$

and $$m(X + Y),\ nY.$$

First, let n be greater than m.

The order of the first pair of multiples is not altered by subtracting mB from each; and the order of the second pair is not altered by subtracting mY from each.

Therefore the above pairs of multiples are in the same order, respectively, as the pairs

$$mA,\ (n - m)B$$

and $$mX,\ (n - m)Y.$$

Now, from the hypothesis, these are in the same order of size; hence the above pairs are in the same order of size.

Next, let n be not greater than m.

Then the pairs of multiples in question are evidently both in descending order.

Therefore $$m(A + B),\ nB$$

and $$m(X + Y),\ nY$$

are always in the same order of size whatever m and n are.

Hence $A + B : B = X + Y : Y$.

Rule of separation.

49. Cor. *In the same case* $A - B : B = X - Y : Y$.

The complete statement and proof are left to the student.

50. It is convenient to insert here the following restatement of theorem 3, to be called the "rule of reciprocation."

51. *If four magnitudes form a proportion, then the second is to the first as the fourth is to the third.*

Two or More Proportions

52. The next three theorems are concerned with rules of inference from two or more proportions. They are the Rules of Combination, of Succession, and of Addition.

Rule of combination.

53. Theorem 15. *If there are any number of equal ratios, all the magnitudes being of the same kind, then as any of the antecedents is to its consequent so is the sum of all the antecedents to the sum of all the consequents.*

Given $A : B = A' : B' = A'' : B''$;

to prove $A : B = A + A' + A'' : B + B' + B''$.

From the hypothesis, and the definition of equal ratios, the three pairs of multiples

$$mA, \quad nB$$
$$mA', \quad nB'$$
$$mA'', \quad nB''$$

are in the same order of size; hence the pair

$$m(A + A' + A''), \ n(B + B' + B'')$$

is also in the same order as any of the preceding pairs, whatever m and n are (axioms I. 25, 32; II. 9, 10; III. 38).

Therefore $A : B = A + A' + A'' : B + B' + B''$.

54. Definitions. A set of ratios will be called *successive* when the consequent of each is the antecedent of the next.

The first antecedent and the last consequent are called the *extremes* of the set. *E.g.*, the ratios $A : B$, $B : C$, $C : D$, $D : E$ are a set of successive ratios, whose extremes are A and E.

Rule of succession.

55. THEOREM 16. *If there are any number of like magnitudes and an equal number of any other like magnitudes, such that the successive ratios in the first set are equivalent respectively to the corresponding successive ratios in the second set, then the ratios of the extremes in the two sets are equal.*

1. Let there be three magnitudes in each set; and let them be

$$A,\ B,\ C$$

and $$X,\ Y,\ Z.$$

Let the successive ratios $A:B$ and $B:C$ be equal to the successive ratios $X:Y$ and $Y:Z$, respectively.

To prove that the ratios of the extremes are equal,

i.e. $$A:C=X:Z.$$

Take any like multiples of the antecedents, and any like multiples of the consequents; and compare the order of size in the two pairs

$$mA,\ nC$$

and $$mX,\ nZ.$$

First, suppose the first pair to be in descending order,

i.e. $$mA>nC;$$

then, comparing each of these magnitudes with the same consequent mB,

$$mA:mB>nC:mB. \qquad [27$$

Now, by hypothesis and rule of equi-multiples,

$$mA:mB=mX:mY,$$

and $$nC:mB=\ nZ:mY. \qquad [45$$

Hence $$mX:mY>\ nZ:mY;$$

therefore, the consequents being identical,

$$mX>nZ. \qquad [29$$

Thus the second pair of the above multiples are also in descending order of size.

Next, suppose the first pair to be in ascending order.

The student may treat this case in a similar way; and also the remaining case, in which the order is indifferent.

It follows that the pairs of multiples

$$mA,\ nC$$

and $$mX,\ nZ$$

are in the same order of size, whatever m and n are.

Therefore $$A : C = X : Z.$$

2. Let there be four magnitudes in each set; namely,

$$A,\ B,\ C,\ D$$

and $$X,\ Y,\ Z,\ W.$$

Let the successive ratios $A : B$, $B : C$, $C : D$ be respectively equal to the successive ratios $X : Y$, $Y : Z$, $Z : W$.

To prove that the ratios of the extremes are equal,

i.e. $$A : D = X : W.$$

The student may prove by using Case 1 twice in succession; and may then generalize.

Applications of rule of succession.

Equivalent consequents in two proportions.

56. Cor. I. *If there are two proportions, and if the two consequents in one proportion are equivalent respectively to the two consequents in the other, then their antecedents form a proportion.*

Given $$A : B = X : Y,$$

and $$A' : B = X' : Y;$$

to prove $$A : A' = X : X'.$$

Outline. Compare the successive ratios in the two sets

$$A,\ B,\ A'$$

and $$X,\ Y,\ X'.$$

NOTE. Observe that the rule of alternation (47) cannot be used in proving (56), for A and X may not be magnitudes of the same kind.

Equivalent antecedents in two proportions.

57. **Cor. 2.** *If there are two proportions, and if the two antecedents in one proportion are equivalent respectively to the two antecedents in the other, then their consequents form a proportion.*

Three terms equivalent in two proportions.

58. **Cor. 3.** *If two proportions have any three terms of one equivalent respectively to the three corresponding terms of the other, then the remaining terms are equivalent.*

Rule of addition.

59. THEOREM 17. ***If there are two proportions, and if the two consequents in one are equivalent respectively to the two consequents in the other, then the sums of corresponding antecedents form a proportion with the same consequents.***

Given $$A : B = X : Y,$$
and $$A' : B = X' : Y;$$
to prove $$A + A' : B = X + X' : Y.$$

Since the two given proportions have their consequents respectively equivalent, hence their antecedents form a proportion; that is, $A : A' = X : X'$, [56

therefore, by the rule of composition,

$$A + A' : A' = X + X' : X'; \qquad [48$$

now, by hypothesis,

$$A' : B = X' : Y;$$

therefore, by the rule of succession,

$$A + A' : B = X + X' : Y. \qquad [55$$

Rule of subtraction.

60. **Cor.** *In the same case*

$$A - A' : B = X - X' : Y.$$

The student may give the complete statement and proof.

BOOK V. — RATIOS OF LINES, POLYGONS, ETC.

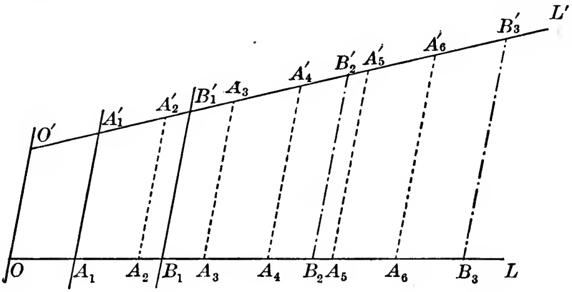
1. The general principles established in Book IV will now be used in comparing particular magnitudes of the same kind, — chiefly segments of lines and surfaces of polygons.

SIMILARLY DIVIDED LINES

Parallel transversals.

2. THEOREM 1. *If two lines are cut by three parallels, any two of the intercepts on one line form a proportion with the corresponding intercepts on the other line.*

Let the lines OL, $O'L'$ be cut by the parallels OO', $A_1A'_1$, $B_1B'_1$ making the three pairs of corresponding intercepts OA_1 and $O'A'_1$, OB_1 and $O'B'_1$, A_1B_1 and $A'_1B'_1$.



First to prove $OA_1 : OB_1 = O'A'_1 : O'B'_1$.

On OL lay off consecutive segments equal to OA_1; and mark their extremities with the symbols A_2, A_3, $\cdots$. Through these points draw parallels to OO' meeting $O'L'$ in A'_2, A'_3, $\cdots$.

Again, on OL lay off consecutive segments equal to OB_1; and mark their extremities B_2, B_3, $\cdots$. Through these points draw parallels to OO' meeting $O'L'$ in B'_2, B'_3, $\cdots$.

Any segment OA_m is equal to the mth multiple of OA_1; and OB_n is equal to the nth multiple of OB_1. Hence the scale of relation of OA_1 and OB_1 is as shown on the line OL. (IV. 11.)

The parallels through A_1, A_2, A_3, $\cdots$ make equal intercepts on the line $O'L'$ (I. 167); thus $O'A'_m$ equals the mth multiple of $O'A'_1$. Similarly, $O'B'_n$ equals the nth multiple of $O'B'_1$. Hence the scale of relation of $O'A'_1$ and $O'B'_1$ is as shown on the line $O'L'$.

Since like multiples evidently occur in the same order in the two scales, hence these scales of relation are everywhere similar; and it follows, by definition of equal ratios, that

$$OA_1 : OB_1 = O'A'_1 : O'B'_1. \qquad \text{[IV. 15}$$

Next to prove that

$$OA_1 : A_1B_1 = O'A'_1 : A'_1B'_1.$$

Since $\qquad OA_1 : OB_1 = O'A'_1 : O'B'_1,$

therefore, by the rule of separation,

$$OA_1 : OB_1 - OA_1 = O'A'_1 : O'B'_1 - O'A'_1, \qquad \text{[IV. 49}$$

i.e. $\qquad OA_1 : A_1B_1 = O'A'_1 : A'_1B'_1.$

3. Cor. 1. *If two lines are cut by any number of parallels the segments of one line taken in order as antecedents form a series of equal ratios with the segments of the other line taken in order as consequents.*

For, by the rule of alternation,

$$OA_1 : O'A'_1 = A_1B_1 : A'_1B'_1 = \cdots. \qquad \text{[IV. 47}$$

Special case.

4. Cor. 2. *If two sides of a triangle are cut by parallels to the third side, the segments of the first side taken in order as antecedents form a series of equal ratios with the segments of the second side taken in order as consequents.*

Definition. Two finite lines are said to be *similarly divided* if the segments of the first taken in order as antecedents form a series of equal ratios with the segments of the second taken in order as consequents.

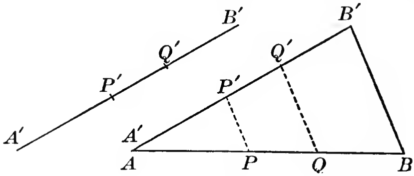
The antecedent and consequent of any one of the equal ratios are called *corresponding segments* of the two lines.

Two points of division are said to *correspond* if the two segments adjacent to the first point correspond respectively to the two segments adjacent to the second point. Two end-points are said to correspond if the segment adjacent to one corresponds to the segment adjacent to the other.

Similar Division

5. Problem 1. *To divide a given line similarly to a given divided line, those end-points which are to correspond being stated.*

Let $A'B'$ be the given line divided at the points P' and Q'; let AB be the other given line which it is required to divide similarly; and let A be that end-point which is to correspond to A'.



Transfer $A'B'$ so that A' may fall on the corresponding point A, and so that the two lines may form a convenient angle. Join $B'B$, and draw $P'P$ and $Q'Q$ parallel to $B'B$.

Then, from the principle of parallel transversals,

$$AP : A'P' = PQ : P'Q' = QB : Q'B'; \qquad [3, 4$$

therefore the lines AB and $A'B'$ are similarly divided.

6. Cor. I. *When two lines are similarly divided, the ratio of two corresponding segments equals the ratio of the whole lines.*

[In a set of equal ratios any antecedent is to its consequent as the sum of the antecedents is to the sum of the consequents (IV. 53).]

7. Cor. 2. *When two lines are similarly divided, the ratio of two corresponding segments is the same whatever be the mode of division.*

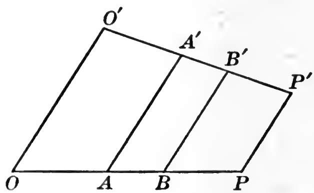
8. *Discussion of Problem* 1. It follows from 7 that when any segment PQ of the first line is given, then the corresponding segment $P'Q'$ of the second line is 'uniquely determined.' Therefore there is only one solution.

Ex. Divide a given line into three parts so that the ratios of the parts may equal the ratios of three given lines, or of given numbers.

Converse of 2.

9. THEOREM 2. *If two similarly divided lines are placed so that the line joining one pair of corresponding points is parallel to the line joining another pair of corresponding points, then the lines joining all pairs of corresponding points are parallel.*

Let the line OP be divided at the points A and B; and let $O'P'$ be divided similarly at A' and B', in such a way that O corresponds to O', A to A', B to B', P to P'. Let OO' be parallel to PP'.



To prove that OO', AA', and BB' are parallel to each other.

Suppose, if possible, that they are not all parallel; and let lines be drawn through A and B parallel to OO' and PP'.

These parallels will divide $O'P'$ similarly to OP; and will therefore pass through the points A' and B'; because there is only one way of dividing the line $O'P'$ similarly to OP (8).

Hence AA' and BB' are parallel to OO' and PP'.

Converse of 4.

10. Cor. *If two similarly divided lines are placed so as to have two corresponding points in coincidence, then the lines joining the other corresponding points are parallel.*

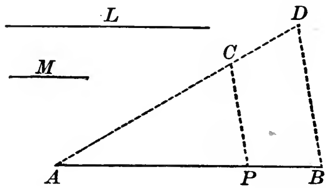
DIVISION IN A GIVEN RATIO

11. Definition. A ratio is said to be *given* when its antecedent and consequent are given magnitudes. Usually the most convenient magnitudes by which a ratio can be assigned are either line-segments or whole numbers. The latter can, however, express only the ratio of commensurable magnitudes.

A line is said to be *divided in a given ratio* at a point when the ratio of the two segments is equal to the given ratio.

Internal division.

12. Problem 2. *To divide a given line internally into two segments whose ratio shall be equal to a given ratio; that end of the line to which the antecedent is to be adjacent being stated.*



Let AB be the given line; and let L and M be the antecedent and consequent of the given ratio.

To find a point P in the line AB such that

$$AP : PB = L : M.$$

Draw AD, making a convenient angle with AB. Lay off AC equal to L, and CD equal to M. Join DB; and draw CP parallel to DB.

Then, from the principle of parallel transversals,

$$AP : PB = L : M. \qquad [4$$

Show that there is only one solution. (8.)

Ex. 1. Divide a given line internally in the ratio 2 : 3.

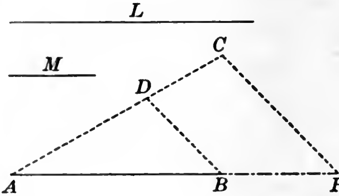
Ex. 2. Find two lines whose ratio shall be equal to a given ratio and whose sum shall be equal to a given line.

Ex. 3. Divide a line into three parts in the ratios 2 : 3 : 4.

External division.

13. Cor. *To divide a given line externally so that the two segments may have a given ratio; that end of the line to which the antecedent is to be adjacent being stated.*

The construction is similar to 12. Show that there is only one solution; and that there is no solution when L and M are equal.



Ex. 1. Find two lines whose ratio shall be equal to a given ratio and whose difference shall be equal to a given line.

Ex. 2. Divide a given line externally in the ratio 3 : 2.

CONVERSION OF LINE-RATIOS

Arts. 14–21 will treat of several problems in the construction of ratios subject to assigned conditions. These problems are classified under the heading "Conversion of line-ratios," which will be explained in Art. 15. They are solved by the principle of "similar division," which has been exemplified in the preceding articles.

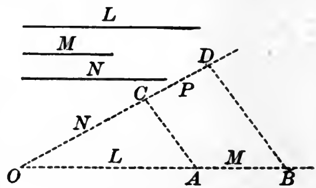
Fourth proportional.

14. Problem 3. *To find a fourth proportional to three given lines.*

Let L, M, and N be the three given lines.

To find a fourth line, P, such that

$$L : M = N : P.$$



Place two indefinite lines OB and OD at any convenient angle. Lay off OA equal to L, AB equal to M, OC equal to N. Join AC; and draw BD parallel to AC.

The line CD is the required fourth proportional.

Since $$OA : AB = OC : CD,$$ [2

therefore, CD is a fourth proportional to the three lines OA, AB, and OC; that is, to the three given lines L, M, and N.

Discussion. All the fourth proportionals to three given magnitudes are equivalent (IV. 58).

Therefore there is only one solution to this problem.

NOTE. Problem 3 may also be stated thus:

Given a line N, to find another line P such that the ratio of N to P may be equal to a given ratio;

or thus:

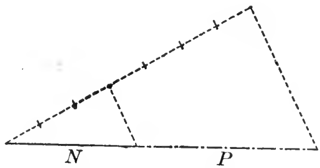
Given the ratio of two lines, and given the antecedent, find the consequent.

Ex. 1. Given a line N, construct another line P such that

$$N : P = 3 : 4.$$

Ex. 2. Given a line P, construct another line N such that

$$N : P = 3 : 4.$$



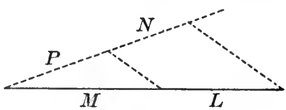
15. Definition. To *convert* a given ratio is to find an equal ratio so as to satisfy stated conditions.

E.g., problem 3 may be enunciated thus:

To convert a given ratio ($L : M$) so that the new antecedent may equal a given line (N).

16. Cor. I. *Convert the line-ratio $L : M$ so that the new consequent may equal P.*

Lay off M, L, P as in figure, and construct as before. Then N is the required antecedent.



Ex. Given a numerical ratio $m : n$, convert it into a line-ratio whose consequent shall be a given line (see 14, ex. 2).

17. Cor. 2. *Given any two line-ratios, convert them so as to have a common consequent.*

NOTE. By this method two ratios can be compared with each other, so as to determine whether the first is greater than, equal to, or less than the second.

Ex. Convert the ratios 3 : 4 and 4 : 5 so as to have a common consequent (IV. 35), and then show which ratio is the greater.

18. Definition. To *enlarge* or *reduce* a given line in a given ratio is to find another line such that the given line is to the new line in the given ratio.

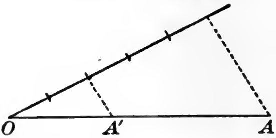
E.g., the line P in 14, ex. 1, is an enlargement of N in the ratio 3 : 4.

An enlargement or reduction is called an *alteration.*

E.g., the line P in 14 is an alteration of N in the ratio $L : M$.

Ex. To reduce a given line OA in the ratio 5 : 2.

Here $OA : OA' = 5 : 2$. Show also that $A'A$ is an enlargement of OA' in the ratio of 2 : 3.



19. Definition. Any set of magnitudes of the same kind are said to be in *continued proportion* when the successive ratios of the set are all equal (IV. 54). When three magnitudes are in continued proportion, the third magnitude is said to be a *third proportional* to the first and second; and the second is said to be a *mean proportional* between the first and third.

Third proportional.

20. PROBLEM 4. *To find a third proportional to two given lines.*

Let L and M be the two given lines.

To find a third line N such that

$$L : M = M : N.$$

[Find a fourth proportional to L, M, and N, by means of 14.]

21. Cor. *Given two lines L and M; find N and P, such that L, M, N, and P may form a continued proportion.*

Ex. Find a third proportional to the numbers 2 and 6. Continue this proportion for two terms more.

COMPOUNDING OF RATIOS

22. Definition. If there is any set of like magnitudes, the first is said to have to the last the *ratio compounded* of the successive ratios of the set (see IV. 54).

E.g., if there are four like magnitudes A, B, C, D, then the ratio of the extremes $A : D$ is compounded of the successive ratios

$$A : B, \quad B : C, \quad C : D.$$

If there is any number of given ratios, whether successive or not, and if there is found a set of magnitudes whose successive ratios are respectively equal to the given ratios, then the extremes of this set are said to have a *ratio compounded* of the given ratios.

E.g., if there are any three ratios

$$A : B, \quad M : N, \quad X : Y;$$

and if there are found any four like magnitudes

$$P, \quad Q, \quad R, \quad S,$$

such that their three successive ratios are respectively equal to the given ratios, then the ratio of their extremes, $P : S$ (which by the preceding definition is compounded of the three successive ratios

$$P : Q, \quad Q : R, \quad R : S),$$

is by the present definition also said to be compounded of the respectively equal ratios

$$A : B, \quad M : N, \quad X : Y.$$

The set of magnitudes just mentioned whose successive ratios are respectively equal to the given ratios are called *auxiliary magnitudes* to the given ratios.

The use of auxiliary magnitudes is illustrated in the next theorem.

Fundamental principle in compounding ratios.

23. THEOREM 3. *If there is any set of ratios and another set severally equal to them, then the ratio compounded of the first set is equal to the ratio compounded of the second set.*

Let the ratios of the first set be

$$A : B,\ P : Q,\ X : Y,$$

and those of the second set

$$A' : B',\ P' : Q',\ X' : Y',$$

the former ratios being respectively equal to the latter.

To prove that the ratio compounded of the first set is equal to that compounded of the second set.

Take an auxiliary set of lines

$$F,\ G,\ H,\ K,$$

such that their successive ratios are respectively equal to the ratios of the first set.

[This may be done by taking an arbitrary line F; then finding G so that $A : B = F : G$; next finding H so that $P : Q = G : H$, and so on (14).]

Similarly take a set of lines

$$F',\ G',\ H',\ K',$$

such that their successive ratios are respectively equal to the ratios of the second set.

Then, by hypothesis and IV. 19, the successive ratios of the set F, G, H, K are respectively equal to the successive ratios of the set F', G', H', K'.

Hence the ratios of their extremes are equal (IV. 55); that is,

$$F : K = F' : K'.$$

But $F : K$ is by definition the ratio compounded of the ratios of the first set; and $F' : K'$ is the ratio compounded of the ratios of the second set.

Hence the theorem is established.

24. NOTE. It follows from this fundamental theorem that in compounding any given ratios it makes no difference what auxiliary magnitudes are chosen provided their successive ratios are respectively equal to the given ratios.

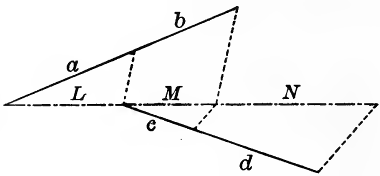
The operation may be conveniently regarded as the performance of several successive 'alterations' (18). If there are any magnitudes of the same kind L, M, N, P, the successive operations of altering L to M, M to N, N to P, give the same result as altering L directly to P. Thus the ratio $L : P$ is appropriately said to be compounded of the successive ratios $L : M$, $M : N$, $N : P$, or of any three ratios equal to these.

Hence any given ratios $a : b$, $c : d$, $e : f$ may be compounded as follows:

Assume any line L, alter it to M in the ratio $a : b$, alter M to N in the ratio $c : d$, alter N to P in the ratio $e : f$, then $L : P$ is equal to the ratio compounded of the three ratios $a : b$, $c : d$, $e : f$.

Compounding line-ratios.

25. Cor. I. *To give a simple construction for compounding the line-ratios $a : b$, $c : d$ into one ratio $L : N$ whose effect shall be the same as their joint effects.*



Compounding numerical ratios.

26. Cor. 2. *If the terms of two ratios are numbers, the ratio compounded of them is equal to the ratio whose antecedent is the product of their antecedents and whose consequent is the product of their consequents.*

[Show that the ratios $m : n$ and $p : q$ have the auxiliary magnitudes

mp, np, nq.]

Ex. Show that the ratio compounded of $A : B$ and $m : n$ is $mA : nB$. [The auxiliary magnitudes are mA, mB, nB (IV. 35, 38).]

Order of compounding.

27. THEOREM 4. *The order in which two given ratios are compounded is indifferent.*

Let there be two ratios $A : B$, $X : Y$.

To prove that the ratio obtained by compounding them is the same in whichever order they be taken.

Take any line L. Alter L to M in the ratio $A : B$. Alter M to N in the ratio $X : Y$. Then the two ratios (taken in the order named) have the auxiliary magnitudes

$$L, \ M, \ N,$$

and hence compound into the ratio $L : N$.

Next take the two ratios in the order $X : Y$, $A : B$.

Take any line L'. Alter L' to M' in the ratio $X : Y$. Alter M' to N' in the ratio $A : B$. Then the two ratios (taken in this order) have the auxiliary magnitudes

$$L', \ M', \ N',$$

and hence compound into the ratio $L' : N'$.

Now it is to be proved that

$$L : N = L' : N'.$$

Find P a fourth proportional such that $L : M = N : P$.

Then the two sets, $\quad M, \ N, \ P,$

and $\quad L', \ M', \ N',$

have their successive ratios respectively equal,

for $\quad L' : M' = X : Y = M : N,$

and $\quad M' : N' = A : B = L : M = N : P.$

Hence the extremes of the two sets are proportional,

i.e. $\quad M : P = L' : N';$ $\quad$ [IV. 55

now $\quad M : P = L : N,$ $\quad$ [alternation

therefore $\quad L : N = L' : N'.$ $\quad$ [IV. 19

Hence the theorem is proved.

Ex. The order of compounding three ratios is indifferent.

Equal ratios compounded with unequal ratios.

28. THEOREM 5. *If one ratio is greater than another, then the ratio compounded of the greater and any third ratio is greater than that compounded of the less and the same third ratio.*

Let $$A : B > P : Q,$$
and let $X : Y$ be any other ratio.

To prove that the ratio compounded of
$$A : B \text{ and } X : Y$$
is greater than the ratio compounded of
$$P : Q \text{ and } X : Y.$$

Take any line L. Alter it to M in the ratio $X : Y$. Alter M to N in the ratio $A : B$.

Then the ratios $X : Y$ and $A : B$ have the auxiliary magnitudes
$$L,\ M,\ N.$$ [22, def.

Again alter M to N' in the ratio $P : Q$.

Then the ratios $X : Y$ and $P : Q$ have the auxiliary magnitudes
$$L,\ M,\ N'.$$

It is now to be proved that
$$L : N > L : N'.$$

Since $A : B > P : Q$, [hyp.

therefore $M : N > M : N'$,

hence $N' > N$, [IV. 32

therefore $L : N > L : N'$. [IV. 30

Hence the required result is proved.

29. Cor. *If* $A : B > P : Q$

and $L : M > X : Y$,

then the ratio compounded of $A : B$ *and* $L : M$ *is greater than that compounded of* $P : Q$ *and* $X : Y$.

[Apply the theorem twice.]

Duplication of a ratio.

30. Definition. When two ratios are equal, the ratio compounded of them is called the *duplicate* of either of them.

When three ratios are equal, the ratio compounded of them is called the *triplicate* of any one of the original ratios.

31. THEOREM 6. *If three magnitudes are proportional, then the ratio of the first to the third is equal to the duplicate of the ratio of the first to the second.*

Given $A : B = B : C$;

to prove $A : C =$ duplicate of $A : B$.

The ratio of $A : C$ is compounded of the successive ratios $A : B$ and $B : C$ (22, def.). But these two ratios are equal.

Therefore the ratio compounded of them is the duplicate of either (30, def.). Hence $A : C$ equals the duplicate of $A : B$.

32. Cor. 1. *If four magnitudes are in continued proportion, the ratio of the first to the fourth is equal to the triplicate of the ratio of the first to the second.*

33. Cor. 2. *To find a ratio equal to the duplicate of a given line-ratio; also of a given numerical ratio.*

34. Cor. 3. *To find the triplicate of a given line-ratio.*

Comparison of duplicate ratios.

35. THEOREM 7. *According as one ratio is greater than, equal to, or less than another, so is the duplicate of the former greater than, equal to, or less than the duplicate of the latter.*

If $A : B = P : Q$,

the ratio compounded of $A : B$ and $A : B$ is equal to the ratio compounded of $P : Q$ and $P : Q$ (23).

If $A : B > P : Q$,

the ratio compounded of $A : B$ and $A : B$ is greater than that compounded of $P : Q$ and $P : Q$ (29).

Hence the theorem is established.

36. Cor. *One ratio is greater than, equal to, or less than another according as the duplicate of the first ratio is greater than, equal to, or less than the duplicate of the second.*

SIMILAR TRIANGLES

This section and the next will treat of similar triangles and similar polygons, respectively. In the following general definitions the word "polygon" will be understood to include "triangle." A former definition is here repeated for convenience.

37. Definitions. Two polygons are said to be *mutually equiangular* if the angles of one, taken in order, are equal respectively to those of the other taken in order. The equal angles are said to *correspond;* and the sides joining the vertices of corresponding angles are called *corresponding sides.*

Two polygons are said to have their *sides proportional* if the sides of one, taken in order as antecedents, form a series of equal ratios with the sides of the other taken in order as consequents.

Two polygons are said to be *similar* if they are mutually equiangular, and if the corresponding sides are proportional.

The ratio of any two corresponding sides is called the *ratio of similitude* of the similar polygons.

E.g., the quadrangles $ABCD$ and $A'B'C'D'$ are similar if the angles A, B, C, D are equal respectively to A', B', C', D', and if $$AB : A'B' = BC : B'C' = CD : C'D' = DA : D'A'.$$

Each of these ratios is equal to the ratio of similitude of the similar quadrangles.

Two similar polygons are said to be *directly* or *obversely* similar according as they are directly or obversely equiangular (I. 187).

Ex. Two regular polygons of the same number of sides are similar.

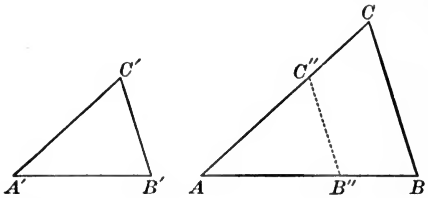
Conditions of Similarity

The next four theorems relate to the conditions of similarity of two triangles.

Angles equal.

38. Theorem 8. *If two triangles are mutually equiangular, then their sides are proportional; and the triangles are similar.*

Let the triangles ABC and $A'B'C'$ be equiangular.



To prove that

$$AB : A'B' = BC : B'C' = CA : C'A'.$$

Apply the triangle $A'B'C'$ to ABC so that A' coincides with A, and $A'B'$ falls on AB; then $A'C'$ falls on AC, because the angles A and A' are equal. Let B' and C' take the respective positions B'' and C'' on the sides AB and AC or else on their prolongations.

Since the angles B and B'' are equal, the lines BC and $B''C''$ are parallel; therefore, by theorem 1,

$$AB : AB'' = AC : AC'',$$

i.e. $$AB : A'B' = AC : A'C'.$$

Similarly by applying the angle B' to the angle B it may be shown that $$AB : A'B' = BC : B'C'.$$

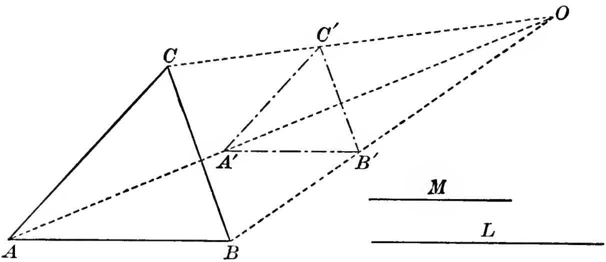
39. **Cor. 1.** *A parallel to one side of a triangle forms with the other two sides a similar triangle.*

40. **Cor. 2.** *Triangles whose sides are parallel, respectively, are similar.*

Construction of similar triangles.

41. PROBLEM 5. *To construct a triangle similar to a given one, and such that the ratio of similitude is equal to a given ratio.*

Let ABC be the given triangle, and $L : M$ the given ratio.



To construct a triangle $A'B'C'$ similar to ABC, and such that

$$AB : A'B' = L : M.$$

Take any point O; and draw OA, OB, OC. Find the fourth proportional to L, M, and OA (14). Lay off OA' equal to this fourth proportional. Draw $A'B'$ parallel to AB, and $A'C'$ parallel to AC; and join $B'C'$.

Then $A'B'C'$ is the required triangle.

Since AB is parallel to $A'B'$, hence

$$OA : OA' = OB : OB'; \qquad [2$$

and since BC is parallel to $B'C'$, then

$$OA : OA' = OC : OC'.$$

Therefore, by equality of ratios,

$$OB : OB' = OC : OC'. \qquad [IV. 19$$

Hence BC is parallel to $B'C'$. [10

Therefore the triangles ABC and $A'B'C'$ are similar.

Moreover, their ratio of similitude, $AB : A'B'$, is equal to $OA : OA'$, and is therefore equal, by construction, to the given ratio $L : M$.

42. Definition. The new triangle is called a *reduction* or *enlargement* of the given one, according as the consequent of the given ratio is less or greater than the antecedent.

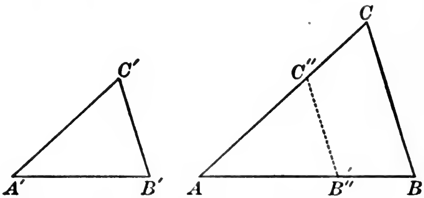
Two similar triangles (or polygons) are said to have the *same shape* or *pattern.* Thus, in a reduction or enlargement, the size is altered, but the shape is preserved.

Sides proportional.

43. THEOREM 9. *If two triangles have their sides proportional, then they are mutually equiangular, and the triangles are similar.*

Let the triangles ABC and $A'B'C'$ have their sides such that

$$AB : A'B' = BC : B'C' = CA : C'A'.$$



To prove that the triangles are similar.

Lay off AB'' equal to $A'B'$; and draw $B''C''$ parallel to BC.

The triangles ABC and $AB''C''$ are mutually equiangular.

Therefore, $\quad AB : AB'' = AC : AC''$; [38

now $\quad A'B : A'B' = AC : A'C'$, [hyp.

and $\quad AB'' = A'B'$, [constr.

hence the two proportions have three corresponding terms respectively equal;

therefore, $\quad AC'' = A'C'$. [IV. 58

Similarly, it may be proved that

$$B''C'' = B'C'.$$

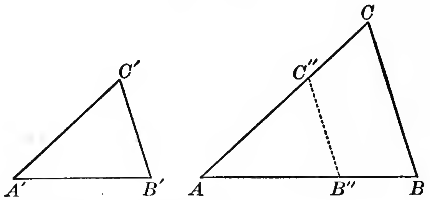
Hence the triangle $A'B'C'$ is equal to $AB''C''$, and therefore similar to ABC.

Two sides and included angle.

44. THEOREM 10. *If two triangles have an angle of one equal to an angle of the other, and the sides about these angles proportional, then the triangles are similar.*

Let the triangles ABC and $A'B'C'$ have the angles A and A' equal, and also have

$$AB : A'B' = AC : A'C'.$$



To prove the triangles similar.

Lay off AB'' equal to $A'B'$; and draw $B''C''$ parallel to BC.

The triangles ABC and $AB''C''$ are similar (38).

Therefore,	$AB : AB'' = AC : AC''$;	
now	$AB : A'B' = AC : A'C'$,	[hyp.
and	$AB'' = A'B'$,	[constr.
therefore,	$AC'' = A'C'$.	[IV. 58

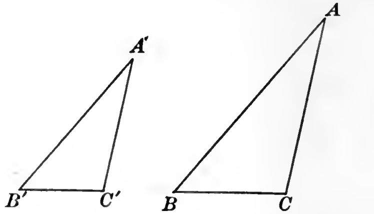
Hence the triangle $A'B'C'$ equals $AB''C''$, and is therefore similar to ABC.

Two sides and a non-included angle.

45. THEOREM 11. *If two triangles have an angle of one equal to an angle of the other, and if the sides about another angle in each are proportional (in such a way that the sides opposite the equal angles correspond), then the third angles are either equal or supplemental.*

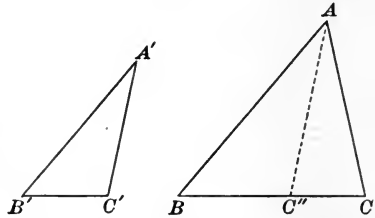
Let the triangles ABC and $A'B'C'$ have the angles B and B' equal, and have the sides about the angles A and A' proportional such that $AB : A'B' = AC : A'C'$.

To prove that the angles C and C' are either equal or supplemental.



The angles A and A' included by the proportional sides are either equal or unequal.

If these angles are equal, as in figure 1, the third angles C and C' are equal (I. 130).



If the angles A and A' are unequal, let A be the greater, as in figure 2.

Draw AC'' cutting off from A a part BAC'' equal to A'.

The triangles BAC'' and $B'A'C'$ are mutually equiangular, since they have two angles of one equal to two angles of the other.

Therefore $AB : AC'' = A'B' : A'C'$; [38

now $AB : AC = A'B' : A'C'$; [hyp.

therefore $AC'' = AC$. [IV. 58

Hence the angle C equals $AC''C$, which equals the supplement of $AC''B$. Now $AC''B$ equals C'; therefore the angle C equals the supplement of C'.

Ex. 1. Summarize the four conditions under which two triangles are similar. Compare them with the five conditions under which two triangles are equal. What two of the latter correspond to the first of the former?

Ex. 2. The ratio of the perimeters of two similar triangles is equal to their ratio of similitude. (Use IV. 53.)

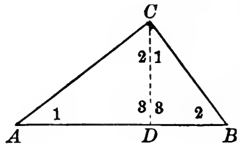
Ex. 3. Prove by the principle of similarity that the line joining the mid-points of two sides of a triangle is parallel to the third side, and equal to half of it.

Applications of Similar Triangles

Right triangle divisible into similar parts.

46. Theorem 12. *In a right triangle, the perpendicular from the vertex of the right angle to the hypotenuse divides the triangle into two parts similar to the whole and to each other.*

Let ABC be a triangle, right angled at C, and let CD be perpendicular to AB.



To prove that the triangles ACD and CBD are similar to ABC and to each other.

[Prove the three triangles equiangular; the corresponding angles being those marked in figure.]

47. Note. The corresponding sides are opposite equal angles. In the triangles ACD and CBD, the side AD of the first corresponds to CD of the second, and the side CD of the first corresponds to DB of the second.

The ratio of similitude of these two triangles equals either of the ratios $AD : CD$, $CD : DB$, $AC : CB$.

MEAN PROPORTIONAL

48. Cor. 1. *In a right triangle the perpendicular on the hypotenuse is a mean proportional between the segments of the hypotenuse.* (19, def.)

49. Cor. 2. *Conversely, if the perpendicular from the vertex to the base is a mean proportional between the segments of the base, then the vertical angle is a right angle.*

50. Cor. 3. *In a right triangle either side is a mean proportional between the hypotenuse and the adjacent segment of the hypotenuse made by the perpendicular.*

[Compare corresponding sides of the triangles ABC and ACD.]

51. Cor. 4. *The segments of the hypotenuse are in the duplicate ratio of the two sides.*

Outline. The ratio $AD : DB$ is compounded of the ratios $AD : CD$ and $CD : DB$, by definition. Now each of these equals $AC : CB$; and the ratio compounded of a ratio and itself is its duplicate ratio (30). Therefore, etc.

Ex. 1. By means of theorem 12 find a third proportional to two given lines.

[Let AD and DC be the two lines.]

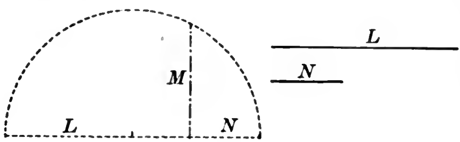
Ex. 2. If in a right triangle one of the sides is double the other, in what ratio does the perpendicular divide the hypotenuse?

Ex. 3. A perpendicular drawn from any point of a circle to a diameter is a mean proportional between the segments of the diameter.

Ex. 4. The radius of a circle is a mean proportional between the segments of a tangent between the point of contact and any pair of parallel tangents.

Construction of mean proportional.

52. Problem 6. *To find a mean proportional between two given lines.*



Use theorem 12, cor. 1, and ex. 3.

Show that there is only one mean proportional. (Use 49.)

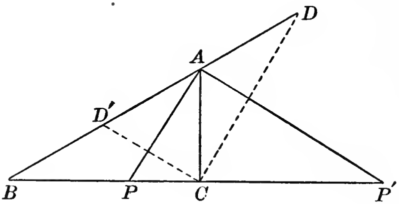
Ex. 1. Give another construction by taking AB equal to the greater of the two given lines and AD equal to the less, and then using theorem 12, cor. 3.

Ex. 2. To find a ratio whose duplicate shall be equal to a given ratio. Show that there is only one solution.

Ex. 3. Show that the mean proportional between two unequal lines is less than half their sum.

Division of base by angle-bisector.

53. THEOREM 13. *If the interior or exterior vertical angle of a triangle is bisected by a line which cuts the base, then the latter is divided internally or externally into segments proportional to the two adjacent sides.*



Let AP and AP' bisect the interior and exterior angles of the triangle ABC, and meet the opposite sides in P and P'.

First, to prove that

$$BA : AC = BP : PC.$$

Draw CD parallel to the bisector PA, meeting the prolongation of the side BA in the point D.

Outline. By hypothesis and the properties of parallels, prove the angles ADC and ACD equal; and AD equal to AC. Then use theorem 1.

Next, to prove that

$$BA : AC = BP' : P'C.$$

Draw CD' parallel to the bisector $P'A$, meeting the side BA in the point D'.

Outline. Prove AD' equal to AC; and use theorem 1.

Converse.

54. Cor. I. *If the base of a triangle is divided internally or externally in the ratio of the sides, the line drawn from the point of division to the vertex bisects the interior or exterior vertical angle.*

For there is only one point P in which BC can be divided internally so that $BP : PC$ shall be equal to a given ratio; and only one point P' in which BC can be divided externally so that $BP' : P'C$ shall be equal to a given ratio (12, 13).

HARMONIC DIVISION

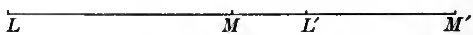
55. Definition. When a line is divided internally and externally into segments having equal ratios, the line is said to be *divided harmonically.*

56. Cor. 2. *In a triangle the bisectors of an interior and its adjacent exterior angle divide the opposite side harmonically.*

57. Cor. 3. *The hypotenuse of a right triangle is cut harmonically by any two lines through the vertex of the right angle, making equal angles with one of the sides.*

Harmonic conjugates. The following corollaries are immediate inferences from the above definition and from certain previous propositions.

58. Cor. 1. *If a line LL' is divided harmonically at the points M and M', then the line MM' is divided harmonically at the points L and L'.*



Outline. Given $LM : ML' = LM' : L'M'$; prove by reciprocation and alternation that $L'M' : ML' = LM' : LM$ (IV. 47, 51).

59. Definition. When the line LL' is divided harmonically at the points M and M', then the four points L, M, L', and M' are said to form a *harmonic* range. The points M and M' are said to be *harmonic conjugates* with regard to the points L and L'; and the points L and L' are said to be harmonic conjugates with regard to the points M and M'.

60. Cor. 2. *Given any three collinear points L, M, L'; to find the harmonic conjugate of M with regard to L and L'.*

[Divide LL' externally in the ratio $LM : ML'$ (13).]

61. Cor. 3. *A point has only one harmonic conjugate with regard to two other collinear points.*

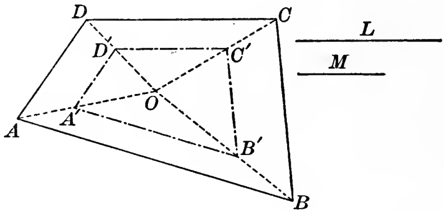
SIMILAR POLYGONS

This section, which treats of the construction and properties of similar polygons, is based on the properties of similar triangles established in the preceding articles.

CONSTRUCTION OF SIMILAR POLYGONS

62. PROBLEM 7. *To construct a polygon similar to a given polygon and such that the ratio of similitude shall be equal to a given ratio.*

Let $ABCD$ be the given polygon, and $L : M$ the given ratio.



To construct a similar polygon $A'B'C'D'$ such that the ratio of two corresponding sides $AB : A'B'$ may be equal to the given ratio $L : M$. (Similar in construction and proof to 41.)

63. NOTE. As in 42, the new polygon is called an *enlargement* or *reduction* of the given one according as the consequent of the given ratio is greater or less than the antecedent.

64. Cor. *If a polygon P is similar to a polygon Q, and if Q is similar to a third polygon R, then P is similar to R; and the ratio of similitude of P to R is equal to the ratio compounded of the two ratios of similitude of P to Q, and Q to R.*

From the hypothesis, the three polygons are mutually equiangular. Let L, L', and L'' be any corresponding sides. Then the ratio $L : L''$ is by definition compounded of the ratios $L : L'$ and $L' : L''$, that is, of the two ratios of similitude. Hence P and R have each pair of corresponding sides in the same ratio, and are therefore similar.

Ex. If a given polygon is first enlarged in a ratio of similitude equal to 2 : 5, and the result reduced as 3 : 1, what is the whole alteration?

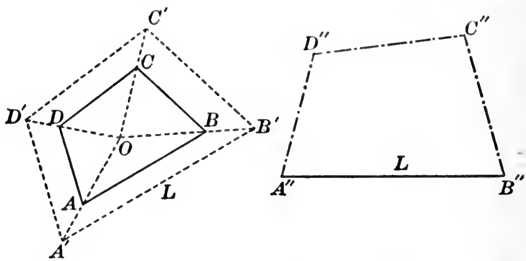
65. Definition. Two broken lines are said to be similar if the segments of one taken in order as antecedents, form a series of equal ratios with the segments of the other taken in order as consequents, and if the angle between two adjacent segments equals the angle between the corresponding segments. The ratio of two corresponding segments is called the *ratio of similitude* of the two similar broken lines.

Ex. To construct a broken line similar to a given broken line and such that they shall have a given ratio of similitude.

Similar polygon on given line.

66. PROBLEM 8. *On a given line to construct a polygon similar to a given polygon, and such that the given line shall correspond to an assigned side of the given polygon.*

Let $ABCD$ be the given polygon, and L the given line.



To construct a polygon on L similar to $ABCD$, and such that L shall correspond to the side AB.

Draw a line $A'B'$ parallel to AB and equal to the given line L. Let the lines AA' and BB' meet at O. Draw $B'C'$ parallel to BC, meeting OC at C'. Draw $A'D'$ parallel to AD, meeting OD at D'; and join $C'D'$.

Outline. Prove $OC' : OC = OD' : OD$; and hence prove $C'D'$ parallel to CD. Show that $ABCD$ and $A'B'C'D'$ are mutually equiangular. Also prove that

$$A'B' : AB = B'C' : BC = C'D' : CD = D'A' : DA.$$

Then transfer $A'B'C'D'$ to the position $A''B''C''D''$ (I. 199).

Properties of Similar Polygons

Transference into parallelism.

67. Theorem 14. *If two polygons are similar, one of them can always be so transferred that the corresponding sides shall be parallel.*

If the polygons are directly similar, turn one of them about a vertex by the rotation construction (I. 202) until a side becomes parallel to its corresponding side.

Each side of one is then parallel to the corresponding side of the other (I. 212).

If the polygons are obversely similar, obvert one of them with regard to a convenient axis (I. 227). The obverse is then directly similar to the given one. Rotate as before.

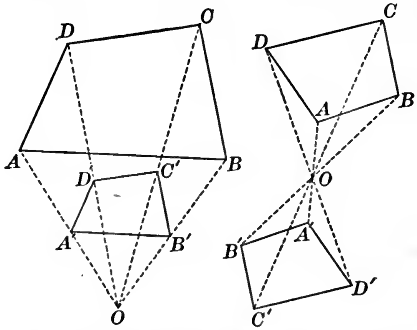
68. Note. As one line can be rotated into parallelism with another in either of two ways, there are two cases to be distinguished, as in the following definition.

69. **Definition**. When two similar polygons are placed so that corresponding sides are parallel, the polygons are said to be *placed in parallelism*. When the parallel sides are at the same side of the line joining corresponding extremities, the polygons are said to be *similarly placed*. When the parallel sides are at opposite sides of the line joining corresponding extremities, the polygons are said to be *oppositely placed*.

Concurrence of certain lines.

70. Theorem 15. *If two similar polygons are placed in parallelism, then the lines joining corresponding vertices are concurrent; except when the polygons are equal and similarly placed, in which case the lines joining corresponding vertices are parallel.*

Let the two similar polygons $ABCD$, $A'B'C'D'$ be so placed that the sides AB, BC, CD, DA are parallel respectively to the homologous sides $A'B'$, $B'C'$, $C'D'$, $D'A'$; and let the polygons be similarly placed in the left-hand figure, and oppositely placed in the right-hand figure.



To prove that the four lines AA', BB', CC', and DD' are concurrent in all cases, except when the polygons are equal and similarly placed.

In the left-hand figure, if the polygons are equal, the quadrangle $ABB'A'$ is a parallelogram; hence AA' is parallel to BB'. Similarly BB' is parallel to CC'; and so on.

If the polygons in the left-hand figure are not equal, prolong AA' to meet BB' in O.

The triangles ABO and $A'B'O$ are similar (38).

Therefore $$AO : A'O = AB : A'B'.$$

That is to say, the line AA' is divided externally by the line BB' in the ratio of similitude of the polygons.

Similarly the same line AA' is divided externally by each of the lines CC' and DD' in the same ratio.

Hence the four lines intersect in the same point (13).

Next take the right-hand figure, in which the similar polygons are oppositely placed. Then by similar reasoning, the line BB' cuts AA' internally in the ratio of similitude.

Similarly AA' is cut internally in the same ratio by CC' and DD'. Hence the lines AA', BB', CC', and DD' intersect in the same point. In this case the proof holds even when the polygons are equal.

71. Definition. The point of concurrence of the lines joining corresponding vertices of two similar polygons placed in parallelism, is called the *center of similitude* of the polygons.

72. Cor. I. *If a line is drawn through the center of similitude to meet two corresponding sides of the polygons, then the segment intercepted between these sides is divided at the center of similitude (internally or externally) in the ratio of similitude.*

Ex. 1. If two similar broken lines are placed with corresponding segments parallel, then the lines joining corresponding vertices are either parallel or concurrent.

Ex. 2. If two similarly divided straight lines are placed parallel, then the lines joining corresponding points of division are either parallel or concurrent.

Ex. 3. If the vertices of a polygon are joined to a given point, and if the joining lines are each divided internally (or externally) in any given ratio, then the lines joining the points of division form a polygon similar to the given one and similarly (or oppositely) placed. (Converse of 72. Compare 62.)

If the given (internal) ratio is equal to $3:5$, show that the ratio of similitude is equal to $8:5$.

Principle of Correspondence

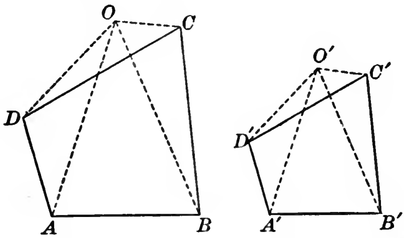
73. Definition. Any two points are said to be *similarly placed* (or to *correspond*) with regard to any two similar polygons, respectively, if the triangles having corresponding sides for bases, and the two points for vertices, are similar in pairs.

Construction of corresponding points.

74. Problem 9. *Given any two similar polygons, and a certain point within, without, or on the boundary of the first polygon; to find a similarly placed point with regard to the second.*

Let $ABCD$, $A'B'C'D'$ be the given similar polygons; and let O be the given point.

To find a point O' such that O and O' shall be similarly placed with regard to the similar polygons.



Join O to two adjacent vertices A and B. Draw $A'O'$, making the angle $B'A'O'$ equal to BAO. Also draw $B'O'$, making the angle $A'B'O'$ equal to ABO. Let these lines intersect in O'.

Then O' corresponds to O.

Since the triangles OAB and $O'A'B'$ are similar, hence the ratio $BO : B'O'$ equals the ratio of similitude of the polygons, and therefore equals the ratio $BC : B'C'$.

Also the angles OBC and $O'B'C'$ are equal, being the differences of angles that are respectively equal.

Hence the triangles OBC and $O'B'C'$ are similar.

Similarly the other triangles whose vertices are at O and O' are similar in pairs.

Therefore O' corresponds to O, by definition.

Discussion. Prove that no other point but O' can correspond to O.

Show that when O comes nearer and nearer to one of the sides, then O' comes nearer and nearer to the corresponding side.

If O coincides with one vertex, prove that O' then coincides with the corresponding vertex.

Show that a similar construction applies when the given point is within the first given polygon.

CORRESPONDING LINES

75. Definition. The lines joining two corresponding points (with regard to two similar polygons) are called *corresponding lines.*

From the definitions of corresponding points and lines, and from the properties of similar triangles, the following corollaries are easily derived.

76. Cor. 1. *In two similar polygons, the lines joining two corresponding points to two corresponding vertices are in the ratio of similitude.* (Use 73, and a property of similar triangles.)

77. Cor. 2. *In two similar polygons, any two corresponding lines are in the ratio of similitude.*

Outline. Let $ABCD$ and $A'B'C'D'$ be the polygons. Let the point P correspond to P', and Q to Q'.

To prove that the ratio $PQ : P'Q'$ is equal to the ratio of similitude.

The angles PAQ and $P'A'Q'$ are equal, being differences of equal angles. Also $AP : A'P' = AQ : A'Q'$, each ratio being equal to the ratio of similitude. Draw conclusion.

78. Cor. 3. *In two similar polygons, two triangles, whose respective vertices are at corresponding points, are similar.*

Use 77, and conditions of similarity.

79. Cor. 4. *If two polygons are similar, then two other polygons, whose respective vertices taken in order are at corresponding points, are similar.*

80. Cor. 5. *If two similar polygons are transferred so as to be similarly placed, then any two corresponding lines become parallel.*

Show that lines joining two corresponding points to two corresponding vertices become parallel, and apply 76 and 78.

81. **Cor. 6.** *If three points are on a straight line, then their corresponding points are on a straight line* (80).

82. **Cor. 7.** *If two lines correspond respectively to two other lines with regard to two similar polygons, then the intersection of the first two lines corresponds to the intersection of the other two.*

Ex. 1. In two similar triangles the feet of the perpendiculars drawn from two corresponding points to the opposite sides are corresponding points; and these perpendiculars are in the ratio of similitude.

Ex. 2. In two similar triangles the ratio of the radii of their inscribed circles is equal to the ratio of similitude; so is the ratio of the radii of their circumscribed circles.

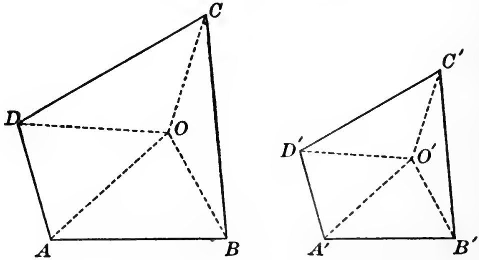
Ex. 3. Given two similar polygons, to find a point such that it *coincides* with its corresponding point.

Analysis. Suppose that P is the required point. Let the two corresponding sides AB and $A'B'$ meet in O. The triangles ABP and $A'B'P$ are equiangular. Prove that the quadrangles $OPAA'$ and $OPBB'$ are each circumscriptible (III. 62). Show that we can determine P by the intersection of the circles described about OAA' and OBB'; and prove that this point is self-correspondent.

Note. This point may be called the *center of similitude* of the two similar polygons. Show that the center of similitude of two similar and similarly (or oppositely) situated polygons is a special case of this.

Similar partition of similar polygons.

83. Problem 10. *To divide two similar polygons into triangles similar in pairs, and having corresponding points for corresponding vertices.*



Outline. Take any point within one of the polygons, and find its correspondent (74). Join these two points with the vertices of the respective polygons. The triangles so formed are similar in pairs, and have the same ratio of similitude as the polygons have.

NOTE. The first polygon may be dissected into triangles in an arbitrary manner, by taking as vertices any number of points inside the polygon. The second polygon can then be divided into similar triangles by joining the respective corresponding points inside that polygon.

84. Cor. *When two similar polygons are divided as in* 83, *then two similarly placed points in a pair of corresponding triangles are also similarly placed with regard to the polygons.*

POLYGONS INSCRIBED IN POLYGONS

85. Definition. One polygon is said to be *inscribed* in another if each vertex of the one is on a side of the other.

Similar polygons in similar polygons.

86. PROBLEM 11. *Given two similar polygons and given any polygon inscribed in the first; to inscribe a similar polygon in the second.*

Outline. Let P and P' be two similar polygons. Let a polygon Q be inscribed in P. To inscribe a similar one in P'.

Let the vertices of Q be A, B, C, ..., all situated on the sides of P. Find their correspondents A', B', C', ..., all situated on the sides of P' (74). The inscribed polygon $A'B'C'$ ... is similar to the inscribed polygon ABC ... (79).

87. Cor. I. *In a given triangle* (P) *to inscribe a triangle similar to a given triangle* (Q).

Outline. Through the vertices of Q draw lines respectively parallel to the sides of P, thus forming a triangle similar to P, and having Q as an inscribed triangle. Then, by means of 86, inscribe in P a triangle similar to Q.

88. **Cor. 2**. *In a given triangle* (ABC) *to inscribe a parallelogram similar to a given parallelogram* ($LMNP$).

Outline. Transfer $LMNP$ so that NP may be parallel to BC. Through L and M draw parallels to AB and AC, thus forming a triangle similar to ABC, and having $LMNP$ as an inscribed parallelogram. Then, by means of 86, inscribe in ABC a parallelogram similar to $LMNP$.

Ex. 1. In a given triangle to inscribe a square.

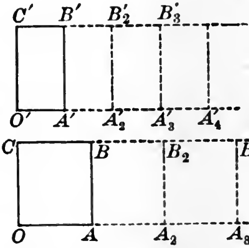
Ex. 2. In a triangle to inscribe a rectangle similar to a given one.

RATIO OF SURFACES OF POLYGONS

89. All the ratios hitherto considered have been ratios of segments of lines. It will now be shown how to compare the surfaces of polygons. The ratio of the surfaces of two polygons will be called the *ratio of the polygons.* We begin with the polygons that are most easily compared, namely, two rectangles of equal altitudes, and thence advance, step by step, to the comparison of polygons in general.

90. Theorem 16. *If two rectangles have equal altitudes, then the ratio of the rectangles is equal to the ratio of their bases.*

Let the rectangles $OABC$, $O'A'B'C'$ have the bases OA, $O'A'$, and the equal altitudes AB, $A'B'$. Let the rectangles be denoted by R, R', and their bases by b, b'.



To prove $R : R' = b : b'$.

Prolong OA and lay off consecutive segments equal to OA. Mark the points of division with the symbols A_2, A_3 ... ; and draw perpendiculars through these points to meet the prolongation of CB. Prolong $O'A'$, and make a similar construction.

The segment OA_m is equal to mb; and the rectangle standing on it is equal to mR.

The segment $O'A'_n$ is equal to nb'; and the rectangle standing on it is equal to nR'.

Since the altitudes are equal, the pairs of magnitudes

$$\text{rect. } OB_m, \text{ rect. } OB'_n$$

and

$$\text{base } OA_m, \text{ base } OA'_n$$

are in the same order of size (II. 22, III. 50), *i.e.*, the pairs of multiples

$$mR, \ nR'$$

and

$$mb, \ nb'$$

are in the same order of size, whatever m and n are; therefore the scale of relation of R and R' is everywhere similar to the scale of relation of b and b'; hence, by definition of equal ratios,

$$R : R' = b : b'.$$

Parallelograms of equal altitudes.

91. Cor. 1. *Two parallelograms of equal altitudes have a ratio equal to the ratio of their bases.*

Triangles of equal altitudes.

92. Cor. 2. *Two triangles of equal altitudes have a ratio equal to the ratio of their bases.*

Cor. 3. *Two parallelograms or triangles of equal bases have a ratio equal to the ratio of their altitudes.*

Ex. 1. Perpendiculars are drawn from any point within an equilateral triangle on the three sides: show that their sum is equal to the altitude of the triangle (IV. 59, 33).

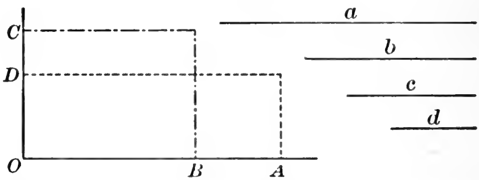
Ex. 2. A quadrangle is divided by its diagonals into four triangles that form a proportion.

Ex. 3. If two triangles have their bases in the same straight line, and their vertices on the same line parallel to the bases, then any other parallel, cutting the sides, cuts off two triangles that form a proportion with the given triangles.

Relation among four proportional lines.

93. THEOREM 17. *If four lines form a proportion, then the rectangle of the extremes is equivalent to the rectangle of the means.*

Let a, b, c, d be four lines such that $a : b = c : d$.



To prove that the rectangle of a and d is equivalent to the rectangle of b and c.

On one of the sides of any right angle lay off OA, OB equal to a, b; and on the other side lay off OC, OD equal to c, d. Complete the rectangles AD and BC.

Compare each of these rectangles with the rectangle BD, which is their common part.

Since rectangles of equal altitudes have a ratio equal to the ratio of their bases, therefore

$$\text{rect. } AD : \text{rect. } BD = OA : OB, \qquad [90$$

and $$\text{rect. } BC : \text{rect. } BD = OC : OD;$$

now $$OA : OB = OC : OD, \qquad [\text{hyp.}$$

therefore, by equality of ratios,

$$\text{rect. } AD : \text{rect. } BD = \text{rect. } BC : \text{rect. } BD;$$

since these equal ratios have a common consequent, hence the rectangles AD and BC are equivalent; that is, the rectangle of the extremes is equivalent to the rectangle of the means.

Special case.

94. Cor. *If three lines are proportional, the rectangle of the extremes is equivalent to the square on the mean.*

Ex. 1. Apply the theorem to prove III. 94.

Ex. 2. Apply the corollary to prove II. 60.

Converse of 93.

95. THEOREM 18. *If two rectangles are equivalent, the sides of one will form the extremes, and the sides of the other the means, of a proportion.*

In figure of theorem 17, if rectangles AD and BC are equivalent, they have equal ratios to rectangle BD (IV. 25). Therefore, etc.

Converse of 94.

96. Cor. *If there are three lines such that the rectangle of the extremes is equivalent to the square on the mean, then the three lines form a proportion.*

EXTREME AND MEAN RATIO

97. Definition. If a given line is divided into two parts such that one of the parts is a mean proportional between the whole line and the other part, the line is said to be divided in *extreme and mean ratio* or in *medial section.*

Application of 94.

98. PROBLEM 12. *To divide a given line in extreme and mean ratio.*

By means of the construction in II. 89, divide the given line so that the rectangle of the whole line and one part is equivalent to the square on the other part.

The line is then divided in extreme and mean ratio; because the latter part is, by 96, a mean proportional between the whole line and the first part.

NOTE. This mode of division is the ancient *sectio aurea* (II. 89).

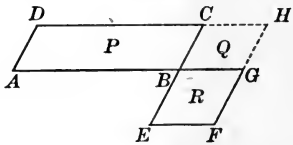
EX. If the radius of a circle is divided in extreme and mean ratio, the greater segment is equal to the side of an inscribed regular decagon (III. 122).

Mutually Equiangular Parallelograms

99. From the comparison of two parallelograms of equal altitudes, or of equal bases (91, 92), we can advance to the comparison of any two mutually equiangular parallelograms. This is done by introducing an intermediate parallelogram having a side in common with each, and then compounding the two successive ratios. The ratio of the two given surfaces is thus expressed as a ratio compounded of two line-ratios by means of the following theorem.

100. Theorem 19. *If two parallelograms are mutually equiangular, then their ratio is equal to the ratio compounded of the ratios of two adjacent sides of the first to the respective adjacent sides of the second.*

Let $ABCD$ and $BEFG$ be two parallelograms that have the angles ABC and EBG equal. Let these parallelograms be noted by P and R.



To prove that the ratio $P : R$ equals the ratio compounded of the two ratios

$AB : BG$ and $CB : BE$.

Place the two parallelograms so that the sides AB and BG are in one line, and so that the equal angles ABC and GBE are vertically opposite. Then the sides CB and BE are in one line (I. 52).

Complete the parallelogram $BGHC$, and denote it by Q.

In the set of three magnitudes P, Q, R, the ratio $P : R$ is, by definition (22), compounded of the successive ratios $P : Q$ and $Q : R$.

Now $\qquad P : Q = AB : BG,$ $\qquad$ [91

and $\qquad Q : R = CB : BE.$

Therefore the ratio of the two parallelograms P and R is equal to the ratio compounded of the ratios of their sides.

Ex. 1. Given two mutually equiangular parallelograms, show how to convert their ratio into a line-ratio, by a construction. (See 25.)

Ex. 2. The sides of one rectangle are to the respective sides of another in the two ratios $3:2$ and $4:5$. Show that the first rectangle is to the second as $6:5$.

101. **Cor. 1**. *The ratio compounded of two line-ratios is equal to the ratio of the rectangle constructed on the antecedents to the rectangle of the consequents.*

102. **Cor. 2**. *If two triangles have an angle of the one equal or supplemental to an angle of the other, the ratio of the triangles is equal to the ratio compounded of the two ratios of the including sides of the one to the including sides of the other.*

[Show that the triangles are halves of mutually equiangular parallelograms.]

Ex. If two triangles have an angle of the one equal to an angle of the other, and if the including sides are respectively as $1:3$ and $1:4$, show that the first triangle is one twelfth of the second.

103. **Cor. 3**. *The ratio of two mutually equiangular parallelograms is equal to the ratio of the two rectangles contained by the adjacent sides respectively.*

Ratio of similar parallelograms.

104. Theorem 20. *If two parallelograms are similar, then their ratio is equal to the duplicate of their ratio of similitude.*

For their ratio is equal to the ratio compounded of the ratios of two adjacent sides of one to the corresponding sides of the other (100).

Now each of these ratios is equal to the ratio of similitude. Hence the ratio of the parallelograms is equal to the duplicate of the ratio of similitude (30).

Ratio of similar triangles.

105. **Cor. 1**. *Two similar triangles have a ratio equal to the duplicate of their ratio of similitude.*

106. Cor. 2. *Two squares have a ratio equal to the duplicate of the ratio of their sides.*

107. Cor. 3. *Two similar triangles have a ratio equal to the ratio of the squares on corresponding sides.*

Ex. If the ratio of similitude of two similar triangles is equal to 3 : 1, how often is the less contained in the greater?

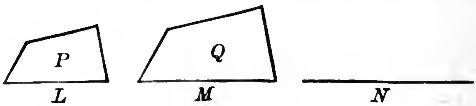
Ratio of Similar Polygons

108. Theorem 21. *Two similar polygons have a ratio equal to the duplicate of their ratio of similitude.* (Apply 83, 105.)

108 (*a*). Cor. 1. *The ratio of two similar polygons is equal to the ratio of the squares on corresponding sides.*

109. Cor. 2. *A polygon is greater than, equal to, or less than a similar polygon, according as a side of the first is greater than, equal to, or less than the corresponding side of the second; and conversely* (II, 24).

110. Cor. 3. *If three lines are proportional, then the first is to the third as any polygon standing on the first is to the similar and similarly situated polygon standing on the second.*



Use 31 and 108.

Surface-ratio converted into line-ratio.

111. Cor. 4. *To find two lines in the ratio of the surfaces of two given similar polygons.*

To two corresponding sides L and M find a third proportional N. Then L and N are the required lines (110).

Ex. 1. To enlarge (or reduce) the surface of a given polygon P in the given ratio $L : N$. (Use 62 and 52, ex. 2.)

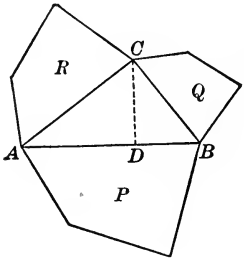
Ex. 2. Show how to double a given polygon, preserving its shape.

Ex. 3. Construct a square equivalent to one third of a given one.

Sum of two similar polygons.

112. THEOREM 22. *In a right triangle, any polygon standing on the hypotenuse is equivalent to the sum of two similar and similarly situated polygons standing on the sides.*

Let ABC be a triangle, right angled at C; and let P, Q, R be similar and similarly situated polygons on AB, BC, CA, respectively.



To prove that P is equivalent to the sum of Q and R.

Draw CD perpendicular to AB.

Then, from the similarity of the triangles ABC and CBD

$$AB : BC = BC : DB. \qquad [46$$

Therefore, by 110,

$$P : Q = AB : DB.$$

Similarly, $\quad P : R = AB : AD.$

Hence, by reciprocation and the rule of addition,

$$P : Q + R = AB : DB + AD. \qquad [IV. 51, 59$$

Now AB equals $DB + AD$.

Therefore P is equivalent to the sum of Q and R (IV. 33).

NOTE. This theorem includes II. 61 as a special case.

EX. Given two similar polygons, show how to construct a third polygon similar to them, and equivalent to their sum.

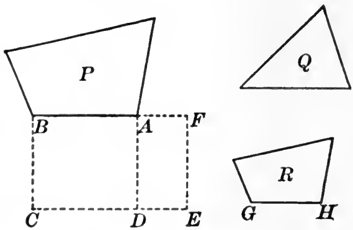
GENERAL CONSTRUCTION OF POLYGONS

Given the shape and size.

113. Problem 13. *To construct a polygon similar to one and equivalent to another given polygon.*

Let P and Q be the two given polygons.

To construct a polygon R similar to P and equivalent to Q.



On AB, a side of P, construct the rectangle $ABCD$ equivalent to P (II. 71–73).

On AD construct the rectangle $ADEF$ equivalent to Q.

Find GH the mean proportional between BA and AF (52).

On GH construct a polygon R similar to P and such that GH and BA are corresponding sides (66).

Then R is the polygon required.

Since $\quad BA : GH = GH : AF,$ [constr.

therefore $\quad BA : AF = P : R,$ [110

hence $\quad BD : AE = P : R.$ [90

Now these equal ratios have equivalent antecedents (by construction); hence they have equivalent consequents.

Thus R is equivalent to AE, and therefore to Q.

Ex. Construct an equilateral triangle equivalent to a given square.

Conversion of a polygon-ratio.

114. Problem 14. *To find two lines whose ratio is equal to the ratio of the surfaces of two given polygons.*

Let P and P' be the given polygons.

To find two lines b and b' such that $P : P' = b : b'$.

As in 113, construct two rectangles R and R' respectively equivalent to P and P', and having equal altitudes (II. 71–3).

Let the bases of the rectangles be denoted by b and b'.

Then $\quad P : P' = R : R' = b : b'.$ [IV. 25, V. 90

Ex. Given any two ratios, $P : Q$ and $R : S$, show how to convert them into line-ratios having equal consequents; and hence show how to determine which of the two given ratios is the greater.

Addition of Ratios

115. Definition. If two ratios have the same consequent, then a third ratio having the same consequent and having its antecedent equivalent to the sum of their antecedents is called the *sum* of the first two ratios.

E.g., the sum of the ratios $A : B$ and $A' : B$ is $A + A' : B$.

The *sum* of any two ratios is the ratio obtained by converting them so as to have equivalent consequents, and then taking the sum by the preceding definition.

To add two ratios.

116. Problem 15. *To construct a line-ratio equal to the sum of any two given ratios, whether of lines, polygons, or whole numbers.*

Outline. If either ratio is not a line-ratio, convert it into a line-ratio (114; 16, ex.). Convert these two line-ratios so as to have a common consequent (17). Then the required ratio has the same consequent, and its antecedent is the sum of the new antecedents.

Ex. Show that the sum of the two numerical ratios $m : n$ and $p : q$ is $mq + np : nq$.

Equals added to equals.

117. Theorem 23. *If equal ratios are added respectively to equal ratios, then the sums are equal.*

Outline. Convert the ratios so that those which are to be added may have common consequents. Let the given equalities then be written,

$$A : B = X : Y,$$
$$A' : B = X' : Y.$$

Then by the rule of addition (IV. 59),

$$A + A' : B = X + X' : Y.$$

Now these ratios are the sums of the given ratios (def., 115). Hence the sums are equal.

Addition commutative.

118. **Cor.** *The sum of two ratios is the same, however they are converted so as to have a common consequent and in whatever order they are taken.*

Ex. Frame a similar definition for the *difference* of two ratios; and show that if equal ratios are subtracted respectively from equal ratios, then the differences are equal.

Equals added to unequals.

119. THEOREM 24. *If one ratio is greater than another, then the sum of the greater, and any third ratio is greater than the sum of the less and the same third ratio.*

Convert all the ratios as before, and apply IV. 28, 29.

120. **Cor.** *If two ratios are respectively greater than two others, then the sum of the two greater ratios is greater than the sum of the two less ratios.*

DISTRIBUTIVE PROPERTY OF RATIOS

Compounding a sum with a third ratio.

121. THEOREM 25. *If the sum of two ratios is compounded with any third ratio, the result is the same as if the two ratios are first separately compounded with the third ratio, and the results then added.*

Convert the first two ratios into line-ratios with a common consequent; let them be $A : B$ and $A' : B$. Convert the third ratio into any line-ratio $R : S$.

To prove that the ratio compounded of $A + A' : B$ and $R : S$ is equal to the sum of the ratio compounded of $A : B$ and $R : S$, and the ratio compounded of $A' : B$ and $R : S$.

Outline. Replace each compound ratio by the ratio of two rectangles (101); the resulting ratios have a common consequent, namely rect. (B, S). To the antecedents apply the distributive property of rectangles (II. 41), *i.e.* that rect. $(A + A', R)$ is equivalent to the sum of rect. (A, R) and rect. (A', R). Draw conclusion by definition in 115.

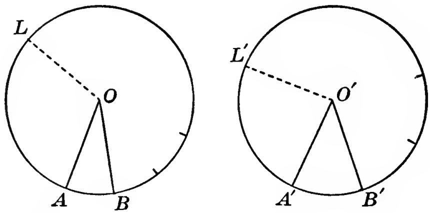
RATIOS IN THE CIRCLE

Arc-ratios and Angle-ratios

Equiradial arcs.

122. Theorem 26. *In two equal circles or in the same circle the ratio of any two central angles is equal to the ratio of their subtending arcs.*

Let AB, $A'B'$ be arcs of equal circles subtending the central angles AOB, $A'O'B'$. Let the arcs be denoted by a, b, and the angles by α, β.



To prove that $\alpha : \beta = a : b.$

Take the arc ABL equal to m times a, and arc $A'B'L'$ equal to n times b, it being understood that either or both of these multiples may exceed a whole circle, and accordingly that the corresponding central angles may exceed a perigon.

The arc ABL is made up of m parts each equal to AB, and each subtending an angle α, hence the arc ma subtends an angle equal to $m\alpha$. Similarly the arc nb subtends an angle equal to $n\beta$.

Hence, from the order-theorem of arcs and central angles, the two pairs of multiples

$$ma,\ nb$$

and

$$m\alpha,\ n\beta$$

are in like order of size whatever m and n are (III. 35, 51).

Therefore, by definition of equal ratios,

$$\alpha : \beta = a : b.$$

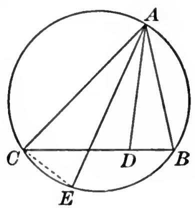
123. Cor. *In two equal circles, or in the same circle, the ratio of any two sectors is equal to the ratio of their arcs, and also equal to the ratio of their angles.*

Inscribed Polygons

Rectangle of sides of inscribed triangle.

124. Theorem 27. *The rectangle of two sides of a triangle is equivalent to the rectangle of two lines drawn from the vertex, making equal angles with these sides respectively, one of the lines being terminated by the base, and the other by the arc of the circumscribed circle below the base.*

Let *ABEC* be the circumscribed circle of the triangle *ABC*. Let the lines *AD*, *AE* be drawn through the vertex *A* making with the sides *AB*, *AC* the equal angles *BAD*, *CAE* respectively. Let *AD* be terminated by the base, and *AE* by the arc *BC* below the base.



To prove that the rectangle of *AB* and *AC* is equivalent to the rectangle of *AD* and *AE*.

[Draw *EC*. Prove the triangles *ACE* and *ADB* mutually equiangular (III. 54); and apply 38, 93.]

125. Cor. 1. *The rectangle of two sides of a triangle is equivalent to the rectangle contained by the diameter of the circumscribed circle and the altitude drawn to the base.*

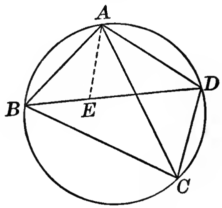
126. Cor. 2. *The rectangle contained by two sides of a triangle is equivalent to the rectangle contained by two lines,* *one of which bisects the vertical angle and is terminated by the circumscribed circle, and the other is the portion of that bisector intercepted between the vertex and the base.*

127. Cor. 3. *The rectangle of two sides of a triangle is equivalent to the square of the line that bisects the vertical angle and is terminated by the base, together with the rectangle of the segments of the base made by that bisector.*

Sides of inscribed quadrangle.

128. THEOREM 28. *If a quadrangle is inscribed in a circle, the rectangle of its diagonals is equivalent to the sum of the two rectangles contained respectively by each pair of opposite sides.*

Let *ABCD* be the inscribed quadrangle.



To prove that the rectangle of *AC* and *BD* is equivalent to the sum of the rectangle of *AB* and *CD* and the rectangle of *AD* and *BC*.

Draw *AE*, making the angle *BAE* equal to the angle *CAD*.

Outline. Prove the triangles *ABE* and *ACD* similar; and rectangle [*AB*, *CD*] equivalent to rectangle [*AC*, *BE*]. Also prove rectangle [*AD*, *BC*] equivalent to rectangle [*AC*, *DE*]. Add, and apply II. 39.

NOTE. This result is called Ptolemy's theorem, after Claudius Ptolemæus of Alexandria, one of the chief geometers among the later Greeks (died 165 A.D.).

Ex. If the three vertices of an equilateral triangle are joined to a point on the circumscribed circle, then one of the joining lines is equal to the sum of the other two.

CIRCUMSCRIPTIBLE POLYGONS

129. THEOREM 29. *If a polygon is circumscriptible, then any similar polygon is also circumscriptible.*

Let the similar polygons be $ABC \cdots$, $A'B'C' \cdots$, and let the former be circumscriptible.

To prove that the latter is also circumscriptible.

Find O, the center of the circumscribing circle of the former; and find its corresponding point O' in the other polygon (74).

Then, by similar triangles,

$$OA : O'A' = OB : O'B' = O'C : O'C' = \cdots.$$

But $$OA = OB = OC = \cdots.$$

Hence $$O'A' = O'B' = O'C' = \cdots.$$

Therefore, the circle described with O' as center and $O'A'$ as radius passes through all the vertices of the polygon $A'B'C' \cdots$. This polygon is therefore circumscriptible.

130. Cor. 1. *If two similar polygons are circumscriptible, the centers of the circumscribed circles are corresponding points, and the ratio of the radii is equal to the ratio of similitude of the polygons* (76).

131. Cor. 2. *If a polygon is such that a circle can be inscribed, then any similar polygon has the same property, and the centers of the inscribed circles are corresponding points.*

132. THEOREM 30. *The perimeters of any two regular n-gons have a ratio equal to the ratio of the radii of their circumscribed circles, and also equal to the ratio of the radii of their inscribed circles.*

Use 130, 131, and 45, ex. 2.

133. Cor. *The ratio of the surfaces of two similar polygons is equal to the ratio of the squares on the radii of their circumscribed circles.*

134. PROBLEM 16. *To inscribe in a given circle a polygon similar to a given circumscriptible polygon.*

Outline. Let $ABCD$ be the given polygon, O the center of its circumscribed circle, and O' the center of the given circle.

Draw radii $O'A'$, $O'B'$, $O'C'$, $O'D'$ parallel to the radii OA, OB, OC, OD.

The inscribed polygon $A'B'C'D'$ is similar to $ABCD$.

Cor. *In a given circle to inscribe a rectangle similar to a given rectangle.*

Ex. 1. In a given semicircle inscribe a rectangle similar to a given rectangle $ABCD$.

[Place the rectangle so that any side AB is parallel to the diameter of the semicircle. Bisect AB in O. Draw OC, OD. In the semicircle draw radii $O'C'$, $O'D'$ parallel to OC, OD. Complete the rectangle $A'B'C'D'$. Prove it similar to $ABCD$. Obtain another solution by placing the side BC parallel to the diameter. When are the two solutions identical?]

Ex. 2. In a given quadrant inscribe a rectangle similar to a given rectangle. How many solutions are possible?

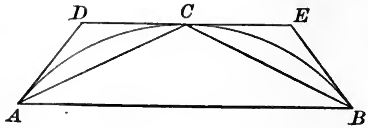
Ex. 3. Given the ratio of two lines, and their mean proportional, find them. [Take two lines in the given ratio, find their mean proportional, enlarge or reduce the figure to suit the condition (42).]

135. Regular n-gons and 2 n-gons. When an inscribed regular n-gon is given, it has been shown that if the arcs are bisected, then the chords of the half arcs form an inscribed regular $2n$-gon, and that the tangents at the vertices of the latter form a circumscribed regular $2n$-gon.

It has also been seen that the perimeter of the inscribed n-gon is less than the perimeter of the inscribed $2n$-gon, that the latter perimeter is less than the perimeter of the circumscribed $2n$-gon, and that this is in turn less than the perimeter of the circumscribed n-gon. It is important, for reasons which will appear later, to obtain more definite relations among these four perimeters. Theorem 31 establishes a simple relation among the first three, and theorem 32 gives a relation among the first, third, and fourth.

136. Theorem 31. *If in a circle are inscribed a regular n-gon and a regular 2 n-gon, and if about the circle is circumscribed a regular 2 n-gon, then the perimeters of these three polygons form a proportion.*

Let AB be the side of an inscribed regular n-gon, and C the mid-point of the arc AB; then the lines AC and BC are sides of an inscribed regular $2n$-gon, and the two tangents at A and B intercept on the tangent at C a line DE equal to the side of a circumscribed regular $2n$-gon. The perimeters of these three polygons are, respectively, $n \cdot AB,\ 2n \cdot AC,\ 2n \cdot DE.$



To prove $\quad n \cdot AB : 2n \cdot AC = 2n \cdot AC : 2n \cdot DE.$

The angles ACD and ABC are equal. Therefore the isosceles triangles ACD and ABC are equiangular (III. 76).

Hence $\quad AB : AC = AC : DC.$

Now $\quad n : 2n = 2n : 4n.$

Hence, by compounding equal ratios,

$$n \cdot AB : 2n \cdot AC = 2n \cdot AC : 4n \cdot DC. \quad [23, 26$$

But $\quad 4n \cdot DC = 2n \cdot DE;$

therefore $\quad n \cdot AB : 2n \cdot AC = 2n \cdot AC : 2n \cdot DE.$

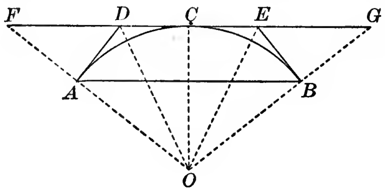
Note. If the perimeters of the inscribed and circumscribed regular n-gons are denoted by p_n, P_n, then this result may be written in the form, $p_n : P_{2n} = P_{2n} : p_{2n}$.

137. Theorem 32. *If in a circle a regular n-gon is inscribed, and if about the circle are circumscribed a regular n-gon and a regular 2 n-gon, then the sum of the perimeters of the first and second of these polygons is to the perimeter of the first as the perimeter of the second is to the semiperimeter of the third.*

Let AB be the side of a regular inscribed n-gon, and draw the lines as in the preceding figure. Join the center O to A, B, D, E. Prolong OA and OB to meet DE prolonged in F and G; then FG is evidently a side of a regular circumscribed n-gon. The perimeters of the inscribed n-gon, the circumscribed n-gon, and the circumscribed $2n$-gon are, respectively,

$$n \cdot AB,\ n \cdot FG,\ 2n \cdot DE.$$

The semiperimeter of the latter polygon is $n \cdot DE$.



To prove $n \cdot AB + n \cdot FG : n \cdot AB = n \cdot FG : n \cdot DE$.

By similar triangles,

$$FG : AB = OF : OA,$$
$$= OF : OC.$$

By equality of right triangles, OD bisects the angle COA.

Hence $$OF : OC = FD : DC.$$ [53

Therefore, by equality of ratios,

$$FG : AB = FD : DC.$$

Hence, by composition,

$$FG + AB : AB = FD + DC : DC,$$ [IV. 48
$$= FC : DC,$$
$$= FG : DE.$$

Therefore, taking nth multiples,

$$n \cdot FG + n \cdot AB : n \cdot AB = n \cdot FG : n \cdot DE.$$

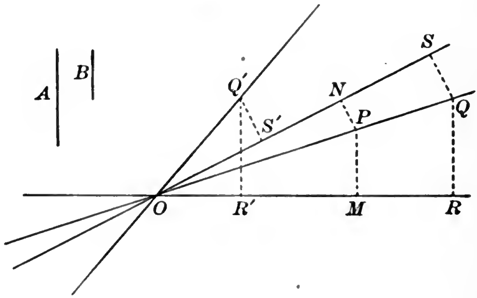
NOTE. With the same notation as in note to 136, this result may be written in the form,

$$p_n + P_n : p_n = P_n : \tfrac{1}{2} P_{2n}.$$

LOCUS PROBLEMS

138. Problem 17. *To find the locus of a point such that the perpendiculars from it to two given lines shall have a given ratio.*

Let OR and OS be the given lines. Let P be a point such that the perpendiculars PM and PN have a given ratio $A : B$.



To find the locus of the point P.

As in I. 254 the point O is a point on the locus. Another point of the locus is obtained by finding Q such that the perpendiculars QR, QS are respectively equal to A, B (I. 257, ex. 2).

Now the line QP must pass through O, because the broken lines RQS and MPN are similar and similarly placed (72, ex. 1). Hence any point P, situated in the angle ROS and satisfying the given condition, lies on the fixed line OQ.

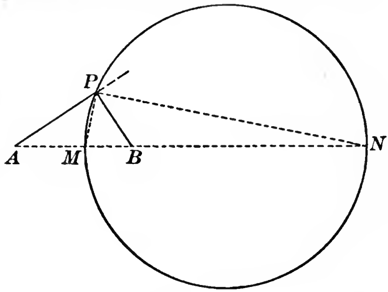
Conversely, any point on the line OQ satisfies the given condition. [The proof is left to the student.]

Show that there is another part of the locus.

Ex. Find a point from which the perpendiculars to three given lines shall have given ratios $L : M : N$. How many solutions are there?

139. Problem 18. *To find the locus of a point such that its joins to two given points shall have a given ratio.*

Let A and B be the given points, and $H:K$ the given ratio. Let P be a point such that $PA:PB = H:K$.



To find the locus of P.

The points M and N which divide AB internally and externally in the ratio $H:K$ are evidently points on the locus.

The line PM bisects the angle APB, because $PA:PB = AM:MB$. [54

Similarly PN bisects the external angle between PA and PB.

Then the angle MPN is a right angle (I. 100, ex. 1).

Hence P is on the circle whose diameter is MN (III. 57).

It follows that any point satisfying the given condition lies on this circle.

Conversely, to prove that every point on this circle satisfies the given condition.

Let P be any point on the circle.

To prove that $\qquad PA:PB = H:K$.

Draw PA' making the angle MPA' equal to MPB, and meeting AB in some point A', which is to be proved coincident with A.

Since MPN is a right angle, hence PN bisects the external angle between PA' and PB. Therefore A' is the harmonic conjugate of B with regard to M and N (59).

But A is also the harmonic conjugate of B with regard to M and N. Therefore A' coincides with A (61).

Hence PM bisects the angle APB; and therefore

$$PA:PB = AM:MB = H:K.$$

Ex. 1. Compare the position of the locus in the three cases

$$H <=> K.$$

Ex. 2. Find the position of a point, whose joins to three given points have given ratios $H:K:L$; show that there may be two solutions, one solution, or none.

EXERCISES

1. The rectangle of two lines is a mean proportional between the squares on the lines.

2. Find a line such that the perpendiculars to it from three given points may have given ratios to each other.

3. A regular polygon inscribed in a circle is a mean proportional between the inscribed and circumscribed circles of half the number of sides.

4. Construct a triangle, being given its base, ratio of sides, and either altitude or vertical angle.

5. Find the locus of a point at which two given circles shall subtend equal angles.

6. Two diagonals of a regular pentagon divide each other in extreme and mean ratio.

Definition. Two points P and P' are said to be *similarly situated* (or to *correspond*) with regard to two circles, whose centers are C and C', when CP and $C'P'$ are parallel and in the ratio of the radii. The correspondence is called direct or transverse according as OP and $O'P'$ are at the same or opposite sides of the central line. The following exercises illustrate the theory of correspondence.

7. The line PP' divides the central line either externally or internally in the ratio of the radii, and each of the points of division is a self-corresponding point (called a *center of similitude*).

8. Two polygons whose respective vertices are all in direct (or transverse) correspondence are similar and have the point S (or S') for center of similitude.

9. The line joining a fixed point to a variable point on a fixed circle is divided in a constant ratio; prove that the locus of the point of division is a circle, and that the two circles have the fixed point as center of similitude.

10. In a given sector OAB inscribe a square, so that two corners may be on the arc AB. [Take any square having a side parallel to AB; circumscribe it by a sector having its radii parallel to OA and OB; then use the principle of correspondence.]

11. A common tangent passes through a center of similitude.

12. Describe a circle through a given point (P) to touch two given lines (OA, OB). [Draw any circle touching the two lines; let it meet OP in P'; then considering P and P' as corresponding points, find the center of the required circle. Two solutions.]

BOOK VI. — MENSURATION

1. Mensuration is the science of measurement. The operation of measuring a magnitude by means of another magnitude of the same kind will be defined after certain preliminary notions are explained. It will be shown to be intimately connected with the theory of ratio set forth in Book IV and applied in Book V.

In Art. 11 of Book IV the scale of relation of two magnitudes of the same kind was explained, and it was shown that any two such magnitudes have a definite ascending order in which their various multiples occur. The theorems of Books IV and V are based on the mere fact that this scale is definite, and their proofs do not require the actual determination of any particular scale of relation.

The determination of a scale (or of a selected portion of it) is, however, important for other purposes, and is the fundamental problem in Mensuration.

ABBREVIATED SCALE

2. If two magnitudes A and B are commensurable, then some of their multiples are equivalent and occupy the same place in the ascending scale of magnitude. If any two multiples mA and nB are known to be equivalent, then the ratio $A : B$ is completely defined, for it is equal to the ratio of two whole numbers $n : m$ (IV. 43), and hence the order of any assigned multiples could be written down.

Any ratio that is equal to the ratio of two whole numbers is called a *rational ratio;* thus the ratio of two commensurable magnitudes is a rational ratio, and the ratio of two incommensurable magnitudes is an irrational ratio.

If two magnitudes A and B are incommensurable, it will be proved presently that their ratio can be sufficiently characterized by assigning the intervals in which merely the *decimal* multiples of the antecedent (A, $10\,A$, $100\,A$, $\cdots$) are found among all the multiples of the consequent (B, $2\,B$, $3\,B$, $\cdots$). Such an arrangement is called the *abbreviated scale* of A and B.

The following is an example of an abbreviated scale:

$2\,B$, A, $3\,B$, $\cdots$ $21\,B$, $10\,A$, $22\,B$, $\cdots$ $215\,B$, $100\,A$, $216\,B$, $\cdots$ $2159\,B$, $1000\,A$, $2160\,B$, $\cdots$.

This exhibits the position of the decimal multiples of the antecedent, showing that

$$\begin{aligned} A &\text{ lies between } 2\,B \text{ and } 3\,B,\\ 10\,A &\text{ lies between } 21\,B \text{ and } 22\,B,\\ 100\,A &\text{ lies between } 215\,B \text{ and } 216\,B,\\ 1000\,A &\text{ lies between } 2159\,B \text{ and } 2160\,B. \end{aligned}$$

The abbreviated scale of P and Q is said to be *similar* to that of A and B if all the like decimal multiples of P and A lie between like multiples of Q and B.

E.g., if the abbreviated scale of P and Q is

$$2\,Q,\ P,\ 3\,Q,\ \cdots,\ 21\,Q,\ 10\,P,\ 22\,Q,\ \cdots,$$

then the abbreviated scale of P and Q is said to be similar to that of A and B written above.

Similar scales.

3. THEOREM 1. *If the abbreviated scale of A and B is similar to the abbreviated scale of P and Q, then the complete scales are similar, that is to say, the ratios $A:B$ and $P:Q$ are equal.*

For suppose, if possible, that the complete scales are somewhere unlike, then, by definition, the two ratios are unequal, say

$$A:B > P:Q.$$

Convert these ratios so as to have a common consequent T (V. 17, 114), and let them become $R : T$ and $S : T$,

then $$R : T > S : T,$$

hence $$R > S. \qquad \text{[IV. 29}$$

Divide T successively into 10, 100, 1000, $\cdots$ parts, until a part (say the one thousandth) is found which is less than the difference between R and S. Take a sufficient number (say m) of these parts, so that m of the parts shall be less than R, and not less than S. Then R contains more than m thousandths of T, while S contains not more than m thousandths of T. Hence

$$R : T > m : 1000,$$

and $$S : T \not> m : 1000;$$

that is $$A : B > m : 1000,$$

and $$P : Q \not> m : 1000;$$

therefore $$1000\, A > mB, \qquad \text{[IV. 41}$$

and $$1000\, P \not> mQ. \qquad \text{[IV. 42}$$

Hence the thousandth multiples of the antecedents occupy different positions in the two scales.

Therefore the abbreviated scales are unlike, which is contrary to the hypothesis. Hence the supposition made is false; that is to say, the complete scales are everywhere alike, and

$$A : B = P : Q.$$

Dissimilar scales.

4. Cor. If $$A : B > P : Q,$$

then some decimal multiple of A occupies a more advanced position among the multiples of B than the like decimal multiple of P occupies among the multiples of Q.

(This is proved in the course of the proof of theorem 1.)

5. NOTE. It follows from 3 and 4 that the abbreviated scale will serve the same purpose as the complete scale, and is sufficient to characterize the corresponding ratio.

Associated Numerical Ratios

6. The abbreviated scale may be used to write down two sets of numerical ratios, such that the ratios of one set are each less than the given ratio, and those of the other set each greater than the given ratio.

E.g., from the abbreviated scale in Art. 2

$$A > 2\,B,$$

hence $$A : B > 2 : 1,$$ [IV. 43

again $$A < 3\,B,$$

hence $$A : B < 3 : 1.$$

In the same way $$A : B > 21 : 10,$$

and $$A : B < 22 : 10.$$

Thus the ratio $A : B$ is greater than each of the numerical ratios $$2 : 1,\ 21 : 10,\ 215 : 100,\ 2159 : 1000,\ \cdots,$$

and less than each of the ratios

$$3 : 1,\ 22 : 10,\ 216 : 100,\ 2160 : 1000,\ \cdots.$$

Each ratio of the first set is called an inferior decimal proximate of the given ratio, and each ratio of the second set a superior decimal proximate. The successive proximates are said to be of the first order, the second order, and so on.

E.g., the ratio $216 : 100$ is the third superior proximate of the ratio $A : B$ above.

A general definition will now be given.

7. Definition. If a certain ratio lies between two numerical ratios whose consequents are each equal to the n^{th} power of 10, and whose antecedents differ by unity, then the less of the two ratios is called the *inferior* (and the greater the *superior*) *decimal proximate*, of the $(n+1)^{\text{st}}$ order, to the ratio that lies between them.

From this definition and Arts. 3, 4 the following corollaries are immediate inferences.

8. Cor. 1. *If two ratios are equal, then their corresponding decimal proximates are equal.*

9. Cor. 2. *If one ratio is greater than another, then some inferior decimal proximate of the first is greater than any inferior decimal proximate of the second.*

10. While the use of decimal proximates is especially applicable to irrational ratios, it is to be observed that rational ratios also have their inferior and superior decimal proximates.

E.g., the ratio $1:3$ has the inferior proximates

$$3:10,\ 33:100,\ 333:1000,\ \cdots,$$

and the superior proximates

$$4:10,\ 34:100,\ 334:1000,\ \cdots.$$

The series of proximates to a certain ratio $A:B$ will terminate if it happens that some decimal multiple of A is exactly equivalent to some multiple of B.

E.g., if the magnitudes A and B mentioned above are such that

$$10000\,A = 21593\,B,$$

then

$$A:B = 21593:10000, \qquad \text{[IV. 43}$$

which is both a rational ratio and a decimal ratio. This ratio would be the last of the series of decimal proximates to the ratio $A:B$.

11. Definition. The ratio of any two magnitudes of the same kind is called a *decimal ratio* if it can be exactly expressed as a numerical ratio whose consequent is a power of 10. If it cannot be so expressed it is called a *non-decimal ratio.*

A non-decimal ratio may be either rational or irrational.

12. Ex. 1. If a given non-decimal ratio is greater than any other given ratio, then some inferior decimal proximate of the first ratio is greater than the second ratio. (The line of proof is as in Arts. 2, 3.)

13. Ex. 2. If a given non-decimal ratio is less than any other given ratio, then some superior decimal proximate of the first ratio is greater than the second.

NUMBER-CORRESPONDENT

14. Definition. If the antecedent of a numerical ratio is divided by its consequent, the quotient is called the *number-correspondent* of the given ratio, or of any ratio equal to it.

E.g., the ratio $10:5$ has the number-correspondent 2; the ratio $5:10$ has the number-correspondent $\frac{5}{10}$ or $\frac{1}{2}$; the ratio $9:5$ has the number-correspondent $\frac{9}{5}$.

If two commensurable magnitudes A and B have the common measure P, and if P is contained m times in A, and n times in B, then

$$A:B = mP:nP = m:n. \qquad \text{[IV. 38}$$

Hence the number-correspondent of the ratio $A:B$ is $\frac{m}{n}$.

Any number that can be expressed as the quotient of two whole numbers is called a *rational number.*

Hence the number-correspondent of the ratio of any two commensurable magnitudes is a rational number.

For this reason such a ratio is called a rational ratio (2).

Comparison of two ratios.

15. THEOREM 2. *According as one rational ratio is greater than, equal to, or less than another rational ratio, so is the number-correspondent of the first greater than, equal to, or less than the number-correspondent of the second.*

Let the two ratios be respectively equal to the numerical ratios $m:n$ and $p:q$.

If $m:n > p:q$,

then $mq > np$, [IV. 41

hence, by division, $\frac{mq}{nq} > \frac{np}{nq}$, therefore, by reducing the fractions to lowest terms,

$$\frac{m}{n} > \frac{p}{q}.$$

If the sign $>$ is replaced by either $<$ or $=$, the proof is similar.

Addition of ratios.

16. THEOREM 3. *The number-correspondent of the sum of two rational ratios is equal to the sum of their number-correspondents.*

For the numerical ratios $m:n$ and $p:q$ are respectively equal to the ratios $mq:nq$, $np:nq$; hence their sum is equal to the ratio

$$mq + np : nq, \qquad \text{[V. 115}$$

whose number-correspondent is $\frac{mq+np}{nq}$, which equals the sum of the numbers $\frac{m}{n}$ and $\frac{p}{q}$.

17. Cor. *The number-correspondent of the difference of two ratios equals the difference of their number-correspondents.*

Compounding ratios.

18. THEOREM 4. *The ratio compounded of two rational ratios has a number-correspondent equal to the product of their number-correspondents.*

For the ratio compounded of the numerical ratios $m:n$ and $p:q$ equals $mp:nq$ (V. 26); and the number-correspondent of this ratio is $\frac{mp}{nq}$, which equals the product of the numbers $\frac{m}{n}$ and $\frac{p}{q}$.

19. It follows from 15, 16, 18 that any two rational ratios can be compared, added, compounded, etc., by means of their number-correspondents. Hence the number-correspondent of a rational ratio is sufficient to characterize it.

IRRATIONAL NUMBERS

20. If A and B are incommensurable, the ratio $A : B$ has no rational number-correspondent. Such a ratio has been shown, however, to have two series of proximate numerical ratios, and each proximate has its own number-correspondent. These number-correspondents collectively characterize the ratio.

By general agreement it is usual to say that the irrational ratio $A : B$ has then an irrational number-correspondent, characterized or defined by the two categories of rational numbers, the decimal proximates, just as the ratio itself is characterized or defined by the order of certain multiples.

These categories of rational numbers are called the two *decimal categories* belonging to the irrational number.

The number-correspondent of any ratio $A : B$ is denoted by the symbol $\frac{A}{B}$.

It is now necessary to give definitions of the words equal, greater, less, sum, product, etc., when applied to the irrational numbers just defined. The definitions, and certain inferences from them, are given in the following articles.

21. Definition. An irrational number is said to be *greater* than, *equal* to, or *less* than another number (whether rational or irrational) according as the ratio to which the first number corresponds is greater than, equal to, or less than the ratio to which the second number corresponds.

22. Theorem 2 may now be restated without restriction:

According as one ratio is greater than, equal to, or less than another ratio, so is the number-correspondent of the first greater than, equal to, or less than the number-correspondent of the second.

I.e., according as $A : B >=< C : D$, so is $\frac{A}{B} >=< \frac{C}{D}$.

23. Cor. 1. *When two irrational numbers are equal, their decimal categories are identical, respectively.*

For the abbreviated scales of their corresponding ratios are similar (def. and 3), hence the decimal proximates are alike (6).

24. Cor. 2. *If two irrational numbers have identical decimal categories, then the irrational numbers are equal.*

For then the decimal proximates are alike, hence the abbreviated scales are alike (6), and hence the corresponding ratios are equal.

25. Cor. 3. *If one irrational number is greater than another, then some inferior decimal proximate of the first is greater than any inferior decimal proximate of the second* (9).

Ex. 1. If an irrational (or a non-decimal number) is greater than any other given number, then some inferior decimal proximate of the first number is greater than the second (12).

Ex. 2. If an irrational number (or a non-decimal number) is less than any other given number, then some superior decimal proximate of the first number is less than the second (13).

26. Definition. The *sum* of two irrational numbers is defined as the number-correspondent of the ratio which is the sum of the two ratios corresponding to the two irrational numbers.

A similar definition applies to the difference of two irrational numbers, and also to the sum (or difference) of a rational and an irrational number.

27. Theorem 3 may now be restated without restriction:

The number-correspondent of the sum of any two ratios is equal to the sum of their number-correspondents.

28. Cor. 1. *The sum of two irrational numbers is greater than the sum of any two numbers that are inferior proximates to them, respectively, and less than the sum of any two superior proximates.* (Use 22, 27; and V. 120.)

Addition commutative.

29. Cor. 2. *The addition of numbers is a commutative operation, i.e. the sum of any two or more numbers is the same, in whatever order they may be taken* (V. 118).

30. Definition. The *product* of two irrational numbers (or of a rational number and an irrational number) is defined as the number-correspondent of that ratio which is compounded of the ratios corresponding to the given numbers.

31. Theorem 4 may now be stated without restriction:

The ratio compounded of any two or more ratios has a number-correspondent equal to the product of their number-correspondents.

The process of finding the product of two or more numbers is called *multiplication.*

Multiplication commutative.

32. Cor. 1. *Multiplication is a commutative operation, i.e. the product of any two or more numbers is the same, in whatever order they may be taken.* (Use definition, and V. 27.)

Multiplication distributive.

33. Cor. 2. *Multiplication is distributive as to addition, i.e. the product of any number by the sum of any other numbers is equal to the sum of the products of the first number by the other numbers separately* (V. 121).

MEASURE-NUMBER

34. Definitions. The ratio which any magnitude bears to a standard magnitude of the same kind is called the *measure-ratio* of the first magnitude.

The number-correspondent of the measure-ratio of any magnitude is called the *measure-number* of that magnitude.

The measure-number of a magnitude is rational or irrational according as the magnitude is or is not commensurable with the standard magnitude (14).

If M is any magnitude, and S the standard magnitude of the same kind, then the measure-ratio of M, and the measure-number of M, are respectively

$$M : S \text{ and } \frac{M}{S}.$$

The measure-number of a straight line is called its *length* with reference to the standard line.

The measure-number of a polygon is called its *area* with reference to the standard polygon.

The square described on the standard line is usually taken as the standard polygon.

The universal standard of line-magnitude adopted by scientific men is the *meter*. It is the largest dimension of a certain standard bar of platinum when taken at the temperature of melting ice. This standard bar is carefully preserved in the Paris observatory.

The first three decimal multiples of the meter are denoted by prefixes formed from the Greek words for 10, 100, 1000.

NAME	MAGNITUDE	ABBREVIATION
meter	standard	m.
dekameter	10 meters	Dm.
hektometer	100 meters	Hm.
kilometer	1000 meters	Km.

The first three decimal submultiples of the meter are denoted by prefixes formed from the Latin words for 10, 100, 1000.

NAME	MAGNITUDE	ABBREVIATION
decimeter	one tenth meter	dm.
centimeter	one hundredth meter	cm.
millimeter	one thousandth meter	mm.

A straightedge on which are marked divisions equal to a meter and to its decimal submultiples is called a *measuring-line.*

FRAGMENT OF METER RULE

(Graduated to millimeters, numbered in centimeters; 10cm. = 1dm.)

With such a measuring-line the successive decimal proximates to the measure-number of any other accessible line can be found as follows:

Apply the meter in succession as often as it will go until the remainder is less than a meter. Suppose the meter goes 3 times. Then 3 is the first inferior proximate, and 4 the first superior proximate, to the measure-number of the given line.

Next, to the remainder apply the decimeter as often as it will go until there is a remainder less than a decimeter. Suppose the decimeter goes 5 times. Then the proximates of the second order are

$$3 + \tfrac{5}{10},\ 3 + \tfrac{6}{10}.$$

Again, to the last remainder apply the centimeter until the remainder is less than a centimeter. Suppose it goes 8 times. Then the proximates of the third order are

$$3 + \tfrac{5}{10} + \tfrac{8}{100},\ 3 + \tfrac{5}{10} + \tfrac{9}{100}.$$

Next, to the last remainder apply the millimeter, and suppose it goes twice with a remainder less than a millimeter. Then the proximates of the fourth order are

$$3 + \tfrac{5}{10} + \tfrac{8}{100} + \tfrac{2}{1000},\ 3 + \tfrac{5}{10} + \tfrac{8}{100} + \tfrac{3}{1000}.$$

Again, to the last remainder apply the tenth of the millimeter; suppose it goes four times with a remainder less than the divisor. Then the proximates of the fifth order are

$$3+\tfrac{5}{10}+\tfrac{8}{100}+\tfrac{2}{1000}+\tfrac{1}{10000},\ 3+\tfrac{5}{10}+\tfrac{8}{100}+\tfrac{2}{1000}+\tfrac{2}{10000},$$

or, in the decimal notation, 3.5821, 3.5822. The error of either of these last proximates is less than one tenth of a millimeter, *i.e.* one ten-thousandth of a meter.

MEASUREMENT OF RECTANGLES

35. Theorem 5. *If the standard polygon is the square described on the standard line, then the measure-number of a rectangle equals the product of the measure-numbers of two adjacent sides.*

Let l be the standard line; S the standard square whose side is l; R the rectangle whose adjacent sides are the lines a and b.

The surface-ratio $R:S$ equals the ratio compounded of the line-ratios $a:l$ and $b:l$ (V. 100).

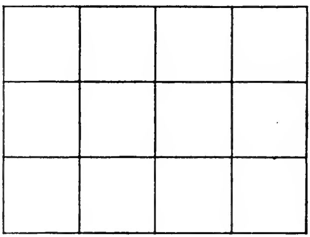
Therefore, by 31, the number-correspondent of $R:S$ equals the product of the number-correspondents of $a:l$ and $b:l$; that is to say

$$\frac{R}{S}=\frac{a}{l}\times\frac{b}{l},$$

i.e. the measure-number of R equals the product of the measure-numbers of its adjacent sides a and b.

Ex. 1. Find the measure-number of a rectangle whose sides are 3 and 4 centimeters respectively.

Taking the centimeter as standard line and the square centimeter as standard surface, it is evident from the figure that the measure-number of the rectangle is 12, which agrees with the theorem. This method of proof does not apply when either of the sides is incommensurable with the standard line.



Ex. 2. Find the measure-number of a rectangle whose sides are 2 meters and 1 decimeter.

In terms of the meter the sides are 2, $\frac{1}{10}$; hence the area equals $2 \cdot \frac{1}{10}$, or $\frac{1}{5}$ of the square meter.

When the decimeter is used as standard line, the measure-numbers of the sides are 20, 1; and the area equals $20 \cdot 1$, or 20 square decimeters.

Ex. 3. The sides of a rectangle are 2.21 m. 14 cm.; find its area. *Answer*, .3094 sq. m., or 30.94 sq. dm., or 3094 sq. cm.

Ex. 4. A rectangle contains 3 sq. m., one side is 5 cm., find the other side.

Ex. 5. What theorem in Book II corresponds to the following algebraic theorem: $a(b+c+d) = ab + ac + ad$?

36. Cor. 1. *The measure-number of a square equals the second power of the measure-number of its side.*

37. Note. For this reason the second power of a number is often called its square; and the number whose second power is equal to the given number is called the square root of the given number.

Ex. State what theorems in Book II correspond to the following algebraic theorems: $a(a+b) = a^2 + ab$; $(a+b)^2 = a^2 + b^2 + 2ab$.

38. Cor. 2. *The measure-number of the side of a square equals the square root of the measure-number of the square itself.*

Ex. A square contains two square meters, find its side. *Answer*, $\sqrt{2} = 1.4142 \ldots m$.

39. Cor. 3. *In a right triangle the measure-number of the hypotenuse equals the square root of the sum of the squares of the measure-numbers of the other two sides; and the measure-number of one of the perpendicular sides equals the square root of the difference of the squares of the measure-numbers of the other two sides* (II. 61).

In symbols, if the lengths of the perpendicular sides are a, b, and of the hypotenuse c, then $c^2 = a^2 + b^2$, $a^2 = c^2 - b^2$.

DIRECTED LINES

40. The line joining two points A and B may be regarded as reaching either from A to B or from B to A. A segment having the initial point A and the terminal point B is denoted by AB, and the segment having the initial point B and the terminal point A is denoted by BA.

The two segments AB and BA are said to be equal in magnitude and opposite in *direction* or *sense.*

Any two collinear segments AB and CD may be compared by imagining CD to slide, without turning out of its line, until the initial point C falls on the initial point A. If the terminal points are then on the same side of the common initial point, the two segments are said to have the *same sense.* If not they are said to have *opposite sense.*

Similarly any indefinite line may be regarded as traced in either of two opposite senses or directions. The sense in which it is supposed to be traced is indicated by the order of naming its leading letters.

Any segment of a directed indefinite line is called a *forward* or a *backward* segment according as its sense is similar or opposite to that of the indefinite line.

For instance, AB is a forward segment of the line $L'L$, and BA is a backward segment.

All forward segments of the same or different lines are said to be of the *same quality,* and so are all backward segments; but any forward segment and any backward segment are said to be of *opposite quality.*

The ratio of any two segments of opposite quality will be represented by a negative number.

A forward segment is commonly taken as the standard, and then any forward segment has a positive measure-number, and any backward segment has a negative measure-number.

The *distance* from a point A to another point B is defined as the measure-number of the segment AB.

A point on a directed indefinite line is said to divide it into a *forward* part and a *backward* part, which are distinguished by the fact that a segment reaching from any point of the latter to any point of the former is a forward segment. If two parallel directed lines are cut by a transversal, and if their forward parts are at the same side of the transversal, then the parallels are said to be *similar in direction;* but if the forward parts are at opposite sides of the transversal, then the parallels are said to be *opposite in direction.*

Addition of segments.

41. Two collinear segments are added by sliding one of them so that its initial point falls on the terminal point of the first. The segment reaching from the initial point of the first to the terminal point of the second is called the *sum* of the two segments.

From this definition it follows that the sum of the collinear segments AB and BC is AC, no matter in what order the three points come on the line.

The measure-number of the segment AB will be denoted by the symbol AB.

Hence, AB and BA have opposite algebraic signs.

That is, $$AB = -BA; \quad BA = -AB.$$

Addition of measure-numbers.

42. The meaning just given to the addition of segments corresponds to the algebraic addition of their measure-numbers.

E.g., if $AB = 7$ and $BC = {}^{-}3$, then $AC = 7 + ({}^{-}3) = 4$.

This principle may be stated in general terms thus:

The sum of two or more collinear segments has a measure-number equal to the algebraic sum of the measure-numbers of the several segments.

E.g., $$AB + BC + CD = AD,$$
$$AB + BC + CA = AA = 0.$$

A segment is subtracted by adding its opposite.

E.g., $$AC - BC = AC + CB = AB,$$
$$DA - DB = DA + BD = BD + DA = BA.$$

MEASUREMENT OF TRIANGLES

43. Algebraic relations. One advantage of the conventions just laid down is that by taking account of the sense of collinear segments, two different geometric theorems can often be made to correspond to one algebraic statement. This is illustrated in some of the following examples:

Ex. 1. The lengths of the sides of a triangle are 8, 10, 5; find the segments of the base made by the perpendicular to the third side from the opposite vertex, and also the length of this perpendicular.

When the angle ACB is obtuse, the relation between the measure-numbers of the sides and of the projections is furnished by II. 62, of which the corresponding algebraic statement is

$$AB^2 = AC^2 + BC^2 + 2\,AC \cdot CD,$$

in which AB^2 stands for the second power of the measure-number of the side AB, and $AC \cdot CD$ for the product of the measure-numbers of the lines AC and CD, this product being the measure-number of the rectangle contained by these two lines.

Again, when the angle ACB is acute, the appropriate relation is furnished by II. 63, of which the corresponding algebraic statement is

$$AB^2 = AC^2 + BC^2 - 2\,AC \cdot DC.$$

Now it will be seen that, when account is taken of the *sense* of the segments CD and DC, the two algebraic statements are identical, for the second could be derived from the first by replacing CD by its equivalent $-DC$. Either of these equivalent statements may be taken to apply to all cases, attention being paid to the proper signs to be given to the segments CD, DC, and AC. We shall use the latter form, and shall take AC as positive. Then DC is positive when D and A are at the same side of C; and DC is negative when D and A are at opposite sides of C.

When AB, AC, and BC are given, then DC is found correctly both in magnitude and sign by solving the above equation.

Then AD, the other segment of the base, is found from the equation

$$AD + DC = AC,$$

which is true irrespective of the order of the points A, D, C.

Substituting the numbers given above, DC is to be found from the equation

$$5^2 = 8^2 + 10^2 - 2 \cdot 10 \cdot DC.$$

Hence
$$DC = \frac{64 + 100 - 25}{20} = 6.95 \text{ m}.$$

It may be observed that since DC and AC have the same sign, hence D and A are at the same side of C, and the angle ACB is acute.

Since $AD + 6.95 = 10$, hence $AD = 3.05$ m.

Again, the altitude BD is given by

$$BD = \sqrt{\overline{BC}^2 - \overline{DC}^2} = \sqrt{64 - 48.3025}$$
$$= \sqrt{15.6975} = 3.962 \text{ m}.$$

And the area is given by $\triangle = \frac{1}{2} AC \cdot BD = 19.81$ sq. m.

Ex. 2. The lengths of the sides of a triangle are a, b, c; find the perpendicular to the side b and the area. As in the last example,

$$c^2 = a^2 + b^2 - 2b \cdot DC,$$

hence $DC = \dfrac{a^2 + b^2 - c^2}{2b}$,

and
$$BD^2 = a^2 - \frac{(a^2 + b^2 - c^2)^2}{4b^2} = \frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4b^2}$$
$$= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4b^2}$$
$$= \frac{[(a + b)^2 - c^2][c^2 - (a - b)^2]}{4b^2}$$
$$= \frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4b^2}.$$

Now let s be the semi-perimeter; then $a + b + c = 2s$, and $a + b - c = 2(s - c)$. Similarly $c + a - b = 2(s - b)$, $c - a + b = 2(s - a)$.

Hence
$$BD^2 = \frac{16\, s(s - a)(s - b)(s - c)}{4b^2},$$

and $$BD = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

This is the length of the perpendicular on the side b. The other two perpendiculars can be written down by algebraic symmetry.

The area is found from the relation

$$\triangle = \tfrac{1}{2} b \cdot BD = \sqrt{s(s-a)(s-b)(s-c)}.$$

Thus the area of a triangle equals the square root of the continued product of the semi-perimeter and the differences between the semi-perimeter and each side in turn. [Heron's Rule (110 B.C.).]

Ex. 3. The lengths of the sides of a triangle are a, b, c. Find the lengths of the three medians.

In the figure of II. 67, let the lengths of the sides opposite the angles A, B, C, be a, b, c; and let the length of the median BD be m.

then $$a^2 + c^2 = 2\,m^2 + 2\,DC^2 = 2\,m^2 + 2\left(\frac{b}{2}\right)^2.$$

Therefore $$m = \tfrac{1}{2}\sqrt{2\,a^2 + 2\,c^2 - b^2}.$$

By algebraic symmetry the other two medians are

$$\tfrac{1}{2}\sqrt{2\,b^2 + 2\,c^2 - a^2}, \qquad \tfrac{1}{2}\sqrt{2\,a^2 + 2\,b^2 - c^2}.$$

Ex. 4. Find the lengths of the three bisectors of the angles.

The bisector CD of the angle C divides c in the ratio $a : b$; hence the segments are

$$AD = \frac{a}{a+b}\cdot c, \qquad DB = \frac{b}{a+b}\cdot c.$$

Now $$AC \cdot CB = AD \cdot DB + CD^2;$$ [V. 127

hence $$ab = \frac{ab}{(a+b)^2}\cdot c^2 + CD^2,$$

therefore $$CD^2 = ab\left[1 - \frac{c^2}{(a+b)^2}\right].$$

The lengths of the two other bisectors can be written by symmetry.

Ex. 5. Find the radius of the circumscribed circle.

From V. 125, $$a \cdot b = 2\,R \cdot p,$$

thus $$R = \frac{ab}{2\,p} = \frac{abc}{2\,pc} = \frac{abc}{4\,\triangle},$$

where $\triangle$ stands for the area of the triangle.

Ex. 6. Find the radius of the inscribed circle.

Let O be the center of the inscribed circle and r its radius.

Then $$OAB + OAC + OBC = ABC,$$

hence $\frac{1}{2} r \cdot AB + \frac{1}{2} r \cdot AC + \frac{1}{2} r \cdot BC = \triangle,$ [III. 101

$$r(AB + AC + BC) = 2\triangle,$$

therefore $$r = \frac{2\triangle}{a + b + c}.$$

Ex. 7. Prove that the radii of the escribed circles are

$$r_1 = \frac{2\triangle}{b + c - a}, \quad r_2 = \frac{2\triangle}{a - b + c}, \quad r_3 = \frac{2\triangle}{a + b - c}.$$ [III. 103

Ex. 8. To compute the two parts of a line whose length is a, when divided in extreme and mean ratio.

In the figure of II. 89, let $AB = a$. Then $BE = \frac{1}{2}\sqrt{5} \cdot a$;

and $AP = \frac{1}{2}(\sqrt{5} - 1)a$; $PB = \frac{1}{2}(3 - \sqrt{5})a$.

MEASUREMENT OF REGULAR POLYGONS

44. General relations. Take a circle of radius r; and let the sides of the regular inscribed and circumscribed n-gons be denoted by s_n, S_n; and the apothem of the former by a_n. (No special symbol is needed for the apothem of the latter, since it is always equal to r.)

Among the four numbers, s_n, S_n, a_n, r, there are two simple general relations:

Since the apothems of two regular n-gons are in the ratio of similitude, hence

$$\frac{s_n}{S_n} = \frac{a_n}{r}; \tag{1}$$

and, since the apothem bisects the side perpendicularly, hence r, a_n, $\frac{1}{2}s_n$, are the lengths of the sides of a right triangle, therefore

$$r^2 = a^2_n + \tfrac{1}{4}s^2_n. \tag{2}$$

45. Special relations. In the case of the simpler polygons, the figure usually furnishes some special relation between two of the lengths s_n, a_n, S_n, r. This special relation together with the two general relations stated above will be sufficient to express any three of these lengths in terms of the fourth.

In the following examples, s_n, a_n, S_n, are each expressed in terms of r; the values of n are taken in order of simplicity.

Ex. 1. For $n=6$: show from a figure that $s_6 = r$; hence, by (2), that $a_6 = \frac{1}{2}\sqrt{3}\,r$; and, by (3), that $S_6 = \frac{2}{3}\sqrt{3}\,r$.

Ex. 2. For $n=4$: show from a figure that $a_4 = \frac{1}{2}s_4$; hence, by (2), that $s_4 = \sqrt{2}\,r$; also that $S_4 = 2r$.

Ex. 3. For $n=3$: show that $a_3 = \frac{1}{2}r$; $s_3 = \sqrt{3}\,r$; $S_3 = 2\sqrt{3}\,r$.

Ex. 4. For $n=10$: show (III. 122, V. 98, and VI. 43, ex. 8) that $s_{10} = \frac{1}{2}(\sqrt{5}-1)\,r$; hence that $a_{10} = \frac{1}{4}\sqrt{10+2\sqrt{5}}\cdot r$, $S_{10} = \dfrac{2(\sqrt{5}-1)}{\sqrt{10+2\sqrt{5}}}\cdot r$.

Ex. 5. For $n=5$: show from the figure of III. 122 that BK bisects AP perpendicularly, and $2a_5 = r + s_{10}$; hence that $a_5 = \frac{1}{4}(\sqrt{5}+1)\,r$, $s_5 = \frac{1}{2}\sqrt{10-2\sqrt{5}}\cdot r$, $S_5 = \dfrac{2\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}\cdot r$.

Ex. 6. For $n=15$: in figure of III. 128, $AC = s_6$, $AB = s_{10}$, $BC = s_{15}$. Let AO meet circle again in D. Prove $BD = 2a_{10}$, $CD = 2a_6$, $AD = 2r$. Show by V. 128 that $2r\cdot s_{15} + 2a_6\cdot s_{10} = 2a_{10}\cdot s_6$, and hence that

$$s_{15} = (a_{10}s_6 - a_6 s_{10})\cdot\frac{1}{r} = \tfrac{1}{4}[\sqrt{10+2\sqrt{5}} - \sqrt{3}(\sqrt{5}-1)]\cdot r.$$

Then show how to find a_{15}, S_{15}.

46. Regular 2 n-gon. The next step is to show how to proceed from any of the above regular polygons to another of double the number of sides.

Let s_{2n}, S_{2n} be the sides of the inscribed and circumscribed $2n$-gon; then by V. 136, 137, the following two relations exist between the four numbers s_n, S_n, s_{2n}, S_{2n}:

$$\frac{s_n + S_n}{s_n} = \frac{S_n}{S_{2n}}; \tag{3}$$

$$\frac{s_n}{s_{2n}} = \frac{s_{2n}}{\frac{1}{2}S_{2n}}, \tag{4}$$

i.e. $$s_n S_{2n} = 2\,s_{2n}^2.$$

Now from (1), Art. 44, $\dfrac{s_{2n}}{S_{2n}} = \dfrac{a_{2n}}{r}$, whence (4) becomes, by eliminating S_{2n},

$$r\cdot s_n = 2\,s_{2n}\cdot a_{2n}. \tag{5}$$

This can also be proved directly from the figure of V. 137.

Ex. 7. For $2n = 8$: put $n = 4$ in (3), and solve for S_8. Use the values of s_4 and S_4 found in ex. 2. The reduced result is $S_8 = 2(\sqrt{2} - 1) \cdot r$. Then show, from (4), that $s_8 = \sqrt{2 - \sqrt{2}} \cdot r$, and, from (1), that $a_8 = \dfrac{\sqrt{2 - \sqrt{2}}}{2(\sqrt{2} - 1)} \cdot r$.

Ex. 8. Show that $s_{12} = \sqrt{2 - \sqrt{3}} \cdot r$.

47. Apothem in terms of side. It is often convenient to know the value of a_n in terms of s_n. In the above examples a_n and s_n are each expressed in terms of r. Hence a_n can be expressed in terms of s_n, when n is 3, 4, 5, 6, 8, etc.

$$a_3 = \tfrac{1}{6}\sqrt{3} \cdot s_3;\ a_4 = \tfrac{1}{2} s_4;\ a_6 = \tfrac{1}{2}\sqrt{3} \cdot s_6;\ a_8 = \tfrac{1}{2}(\sqrt{2} + 1)\, s_8;$$

$$a_5 = \frac{\sqrt{5} + 1}{2\sqrt{10 - 2\sqrt{5}}}\, s_5;\quad a_{10} = \frac{2\sqrt{10 + 2\sqrt{5}}}{\sqrt{5} - 1}\, s_{10}.$$

48. Area in terms of side. A regular n-gon is equivalent to the sum of n triangles, each having its base equal to the side, and its altitude equal to the apothem. Hence the area A_n is given by

$$A_n = \tfrac{1}{2} n \cdot s_n a_n.$$

Therefore, by 47,

$$A_3 = \tfrac{1}{2}\sqrt{3}\, s_3^2;\ A_4 = s_4^2;\ A_6 = \tfrac{3}{2}\sqrt{3}\, s_6^2;$$

$$A_8 = (\sqrt{2} + 1)\, s_8^2;\ A_5 = \frac{5(\sqrt{5} + 1)}{4\sqrt{10 - 2\sqrt{5}}}\, s_5^2.$$

Ex. The regular pentagon is about 1.72 times the square on its side. The regular hexagon is about 2.6 times the square on its side.

MEASUREMENT OF THE CIRCLE

49. Hitherto we have been concerned with the measurement of figures bounded by straight lines. To lead up to the measurement of the circle, it is necessary to give some elementary principles relating to variables and their limits.

Variables and Limits

50. Definitions. A number which takes a series of different values in succession is called a *variable.*

E.g., the population of a city in successive years; the number of seconds between sunrise and sunset on successive days; the perimeter of a regular polygon inscribed in a given circle, when the number of sides is 3, 4, 5, 6, ⋯ in succession.

When the law of change of a variable is such that its successive values approach nearer and nearer to a certain fixed number so that the difference between the latter and the variable can become and remain smaller than any assigned number, then the fixed number is called the *limit* of the variable in question.

E.g., the series of fractions, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{11}{12}$, ⋯ (in which each term is derived from the preceding by adding two units to numerator and denominator), approaches unity as a limit; for by continuing the series far enough under the same law, a term will be reached that differs from unity by less than any assigned number, however small; for instance, if the assigned number is $\frac{1}{1000}$, we can continue the series up to the term $\frac{1001}{1002}$, which differs from unity by less than the assigned number.

Again, the series of numbers 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ⋯ (in which each term is half the preceding), tends toward zero as a limit; for the series can be continued until a term is reached which is less than any assigned number.

Two variables are said to be *related* when one depends on the other, so that when the value of one is known, the value of the other can be found.

E.g., the length of the side of a square and its area are related variables. If the side takes the series of values 1, 2, 3, 4, ⋯, then the area takes the series of values 1, 4, 9, 16, ⋯, each term in the latter series being the second power of the corresponding term in the former series.

ELEMENTARY PRINCIPLES OF LIMITS

51. PRINCIPLE 1. *If two variable numbers are so related that they remain always equal, and if one of them approaches a limit, then the other approaches the same limit.*

For the two equal variables are at all stages represented by one number, and this number has only one limit.

52. PRINCIPLE 2. *If two finite related variables are such that their quotient approaches unity as a limit, then their difference approaches zero as a limit.*

Let the two variables be represented by the measure-numbers of the finite lines OA and OB.

By hypothesis, the measure-number of the ratio $OA : OB$ approaches unity as a limit; hence the points A and B can come as near together as desired. Therefore, the difference of the two variables approaches zero as a limit.

53. Cor. *If two finite related variables are such that their difference approaches zero as a limit, then their quotient approaches unity as a limit.*

54. PRINCIPLE 3. *If a variable continually increases, and never exceeds a certain fixed number, then the variable approaches some limit not greater than the fixed number.*

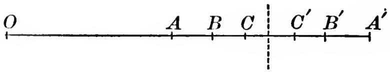
For if the variable has no limit, it must (since it continually increases) ultimately exceed any assigned number.

55. Cor. *If a variable continually decreases, and never becomes less than a certain fixed number, then the variable approaches some limit, not less than the fixed number.*

Ex. If a variable increases toward a certain limit, and if the variable always exceeds a certain fixed number, then the limit exceeds this fixed number.

56. Principle 4. *If one of two related variables is always less than the other, and if the former continually increases, and the latter continually decreases, so that their difference approaches zero as a limit, then the two variables have a common limit which lies between them.*

Let any successive values of the first variable be represented by the measure-numbers of the lines OA, OB, OC ...; and let the corresponding values of the second variable be represented by the measure-numbers of the lines OA', OB', OC' ..., all measured from the same point O.



Since the first variable continually increases and by hypothesis remains less than OC', hence the first variable has some limit (54). Since the second variable continually decreases and remains greater than OC, hence the second variable has some limit (55). These two limits are equal; for if not, the difference of the two variables could not be made less than the difference of the two limits, contrary to the hypothesis. Hence the two variables have a common limit which lies between them.

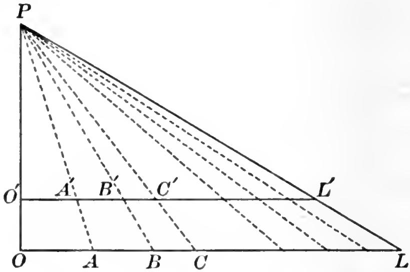
57. Principle 5. *If there are any two variables, one of which is never greater, and the other never less, than a certain fixed number L, and if the difference of the two variables tends to zero as a limit, then the two variables approach L as a common limit.*

From the hypothesis, the difference of either variable from L is not greater than the difference of the two variables, and will therefore become and remain less than any assigned number. Hence by definition each variable tends to L as a limit.

58. **Cor.** *If two variables have a common limit, then any third variable that always lies between them has the same limit.*

59. Principle 6. *If while approaching their limits the ratio of two related variables remains constant, the ratio of their limits equals the same constant.*

Take two parallel lines, OL, $O'L'$ and let any successive values of the first variable be represented by the measure-numbers of the lines OA, OB, OC, $\cdots$; and let the corresponding values of the second variable be represented by the measure-numbers of the lines $O'A'$, $O'B'$, $O'C'$, $\cdots$.



By hypothesis $\qquad \dfrac{OA}{O'A'} = \dfrac{OB}{O'B'} = \dfrac{OC}{O'C'} = \cdots.$

Hence the two lines, OL, $O'L'$, are similarly divided at the points A, B, C, $\cdots$, and A', B', C', $\cdots$.

Therefore the lines OO', AA', BB', $\cdots$ meet in a point P (V. 72, ex. 2). Thus if a line starts in the position PO, and turns about P, it will in its successive positions mark off on OL and $O'L'$ corresponding values of the two variables.

Let the first variable have the limit OL, and let PL meet $O'L'$ in the point L'.

Then $O'L'$ is the limit of the second variable. For since OL is the limit of the first variable, hence the revolving line can come as close to PL as desired, therefore its intersection with $O'L'$ can come as close to L' as desired; thus $O'L'$ is the limit of the second variable.

Therefore the ratio of the limits is the same as the ratio of the variables.

Length of a Circle

60. The length of a straight line has been defined as its measure-number in terms of a certain standard straight line.

Such measurement presupposes the possibility of the superposition of the standard line, or of some of its submultiples, on the line to be measured. Hence the word "length" has as yet no meaning when applied to a curved line. The phrase "length of a circle" will now be given a precise definition.

61. Definition. If in a circle a series of convex polygons of 3, 4, 5, ..., sides are inscribed, the limit approached by the length of the successive perimeters, as the number of sides is continually increased and each side tends to zero as a limit, is called the *length of the circle.*

62. To justify this definition it is necessary to prove that this series of perimeters has a limit, and that this limit is the same by whatever law the sides tend to zero.

It will first be proved that there is a limit when the successive inscribed polygons are regular and when the number of sides is continually doubled. It will then be proved that the same limit is obtained whatever the law of inscription may be.

63. Theorem 6. *The lengths of the perimeters of two similar regular polygons, one inscribed, the other circumscribed, to a given circle, tend to a common limit, when the number of sides is continually doubled.*

The ratio of the perimeters of these two polygons equals the ratio of their apothems (V. 132), and hence equals the ratio of the apothem of the inscribed polygon to the radius.

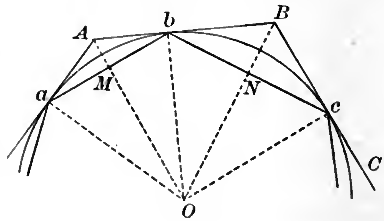
But the latter ratio tends to unity as a limit (53), hence the former ratio tends to unity as a limit (51). Therefore the difference of the perimeters tends to zero as a limit (52).

Now as the number of sides is continually doubled the circumscribed perimeter continually diminishes and the inscribed perimeter continually increases (V. 135).

Therefore the two variable perimeters have a common limit which lies between them (56).

64. THEOREM 7. *The length of the perimeter of any convex polygon, inscribed or circumscribed, tends to one and the same limit, by whatever law each side tends to zero as a limit.*

Let abc be a convex inscribed polygon, ABC the convex circumscribed polygon formed by the tangents drawn at the points $a, b, c, \cdots$. Let p and P be the perimeters of these polygons. Let L be the limit obtained when the law is that of the preceding theorem.



To prove that p and P have the common limit L by whatever law the sides of each polygon tend to zero as a limit.

The inscribed perimeter p is less than any of the circumscribed perimeters considered in the preceding theorem, hence p never exceeds the limit L of those perimeters.

Again, the circumscribed perimeter P is greater than any of the inscribed perimeters considered in the preceding theorem; hence P never becomes less than the limit L of those inscribed perimeters.

It will next be proved that p can come as near to P as desired. Let the length of the radius be denoted by R.

Since OA bisects ab at right angles, hence

$$\frac{aA + Ab}{ab} = \frac{aA}{aM} = \frac{R}{OM};$$

also $$\frac{bB + Bc}{bc} = \frac{R}{ON}, \text{ and so on.} \qquad [V. 46$$

By combining the numerators and denominators of the fractions on the left, another fraction is formed which (by a theorem in algebra) lies between the greatest and least of these fractions. Therefore the fraction $\frac{P}{p}$ lies between the greatest and least of the fractions $\frac{R}{OM}, \frac{R}{ON}, \cdots$

Now, when each side of the polygons is continually diminished, the fractions $\frac{R}{OM}$, $\frac{R}{ON}$, $\cdots$ each tend to unity as a limit (53).

Hence the fraction $\frac{P}{p}$, which lies between two of them, tends to unity as a limit (58); and therefore the difference $P-p$ tends to zero as a limit (52).

But P never becomes less than L, and p never becomes greater than L; hence L is the common limit of P and p (57).

65. NOTE. It should be observed that, under the most general law of approach now supposed, it is not necessary that P should continually decrease, nor that p should continually increase. Hence 57 has been used instead of 56, which was properly employed in the last proposition.

66. THEOREM 8. *The lengths of any two circles have the same ratio as the radii.*

Outline. In the two circles inscribe similar regular polygons; and apply V. 132. Imagine the number of sides to be increased; and apply 59, 61.

67. **Cor. 1.** *The ratio of the length of the circle to that of the diameter is the same for all circles.*

Apply alternation to 66.

NOTE. The number-correspondent of this constant ratio is denoted by the Greek letter π. Thus, π is the quotient of the length of the circle by the length of the diameter. The length of a circle is called the *circumference.*

68. **Cor. 2.** *If R is the length of the radius, and C the circumference, then* $$C = 2\pi \cdot R.$$

COMPUTATION OF THE NUMBER π

69. The successive decimal proximates to the ratio of the circumference to the diameter may be computed as follows:

Take the perimeters of some regular inscribed and the corresponding circumscribed polygons as found in 45. Compute the perimeters of regular inscribed and circumscribed polygons of double the number of sides by 46. From these in turn compute the perimeters of polygons of double the number of sides; and so on. These successive perimeters will be closer and closer approximations to the length of the circle. *E.g.*, if a decimal proximate of the fourth order is required, continue the process until the expressions for the inscribed and circumscribed perimeters agree to the third decimal place.

For convenience take the diameter as standard line; then its measure-number is unity.

The perimeters of the inscribed and circumscribed squares and octagons have been found to be

$$p_4 = 4\sqrt{2}\cdot r = 4\sqrt{2}\cdot \tfrac{1}{2} \doteq 2.8284271,$$

$$P_4 = 8\,r = 4,$$

$$p_8 = 8\sqrt{2-\sqrt{2}}\cdot r = 3.0614675,$$

$$P_8 = 16(\sqrt{2}-1)\cdot r = 3.3137085.$$

Now, to compute P_{16}, use the result of V. 137, viz.

$$\frac{p_n + P_n}{p_n} = \frac{2\,P_n}{P_{2n}},$$

which gives
$$P_{2n} = \frac{2\,p_n P_n}{p_n + P_n},$$

hence
$$P_{16} = \frac{2\,p_8 P_8}{p_8 + P_8} = 3.1825979.$$

To compute p_{16}, use the result of V. 136, viz.

$$\frac{p_n}{p_{2n}} = \frac{p_{2n}}{P_{2n}},$$

i.e.
$$p_{2n}^{\;2} = p_n\cdot P_{2n},$$

hence
$$p_{16} = \sqrt{p_8\cdot P_{16}} = 3.1214452.$$

For polygons of 32 sides

$$P_{32} = \frac{2\,p_{16}P_{16}}{p_{16} + P_{16}} = 3.1517249,$$

$$p_{32} = \sqrt{p_{16}\cdot P_{32}} = 3.1365485.$$

The results obtained by continuing this process for eight more steps are shown in the following table:

Number of sides	Perimeter of inscribed polygon	Perimeter of circumscribed polygon
4	2.8284271	4.0000000
8	3.0614675	3.3137085
16	3.1214452	3.1825979
32	3.1365485	3.1517249
64	3.1403312	3.1441184
128	3.1412773	3.1422236
256	3.1415138	3.1417504
512	3.1415729	3.1416321
1024	3.1415877	3.1416025
2048	3.1415914	3.1415951
4096	3.1415923	3.1415933
8192	3.1415926	3.1415928

The last numbers show that the length of the circle whose diameter is unity lies between 3.1415926 and 3.1415928. Hence, the value $\pi = 3.1415927$ has an error of less than one unit in the seventh decimal place.

Archimedes (250 B.C.) obtained the value $\frac{22}{7}$, which is correct to two decimal places. Metius of Holland (1600 A.D.) gave $\frac{355}{113}$, correct to six places. More recently by methods of the Calculus, π has been computed to several hundred figures. Lambert (1750 A.D.) proved that π is an irrational number (14). Lindemann (1882) proved it transcendental, *i.e.* not expressible by a finite combination of radicals.

LENGTH OF A CIRCULAR ARC

70. Definition. The length of a circular arc is defined as the limit to which tend the perimeters of any inscribed (or circumscribed) convex broken line when each side tends to zero as a limit.

The existence of a unique limit is proved by the method employed in the two preceding theorems, *i.e.* by first considering the case of a regular inscribed broken line, and the corresponding circumscribed line, the number of sides being continually doubled; and from this case passing to the most general law of approach.

71. Cor. 1. *The length of any arc of a circle is greater than the length of its chord.* (Use I. 89, and VI. 55, ex.)

72. Cor. 2. *The length of any arc of a circle is less than the length of any broken line exterior to it and having the same extremities.*

For a continually decreasing series of circumscribed broken lines can be constructed, all less than the given broken line, hence their limit is less than the same line.

73. Cor. 3. *Equal arcs have equal lengths.*

74. Cor. 4. *In equal circles, according as one arc is greater than, equal to, or less than another, so is the length of the first arc greater than, equal to, or less than the length of the second.*

75. Cor. 5. *In equal circles, the ratio of any two arcs is equal to the ratio of the lengths of the arcs.*

Take any equimultiples of the antecedents, and any equimultiples of the consequents, and apply 74.

Area of a Circle

76. The definition of the measure-number of a polygon in terms of a standard polygon presupposes the possibility of the superposition of their parts by some mode of dissection.

As this is not possible when the boundary of the figure to be measured is a curved line, it becomes necessary to give a precise definition to the phrase "area of a circle."

77. Definition. The limit to which the area of a polygon inscribed in a circle tends, when each side tends to zero as a limit, is called the *area* of the circle.

To justify this definition, it is necessary to prove that there is such a limit, and that its value is the same by whatever law each side approaches zero.

78. THEOREM 9. *The areas of two similar regular polygons, one inscribed, the other circumscribed, to a given circle, tend to a common limit when the number of sides is continually doubled.*

The ratio of the areas equals the ratio of the squares of their apothems, and hence equals the ratio of the square of the apothem of the inscribed polygon to the square of the radius. (V. 132, 133; III. 136.) Conclude the proof as in 64.

79. THEOREM 10. *The area of any convex polygon, inscribed or circumscribed, tends to the same limit by whatever law each side tends to zero as a limit.*

Use the figure of art. 64; and let a and A be the areas of the two polygons. Let S be the limit obtained when the law is that of the preceding theorem.

Show as in 64 that a never becomes greater than S, and that A never becomes less than S. Prove that

$$\frac{OaAb}{Oab}=\frac{R^2}{OM^2};\quad \frac{ObBc}{Obc}=\frac{R^2}{ON^2},\quad \cdots. \qquad [\text{V. } 107$$

Hence prove that $\frac{A}{a}$ tends to unity as a limit; and finally that S is the common limit of A and a.

80. THEOREM 11. *The area of a circle equals the product of the circumference by half the radius.*

Draw a regular circumscribed polygon.

Its apothem is equal to the radius. Hence the area A of the polygon equals the product of its perimeter P by half the radius; that is,

$$A=\tfrac{1}{2}R\cdot P, \qquad [\text{III. } 137$$

or

$$\frac{A}{P}=\frac{R}{2}.$$

On continually increasing the number of sides, A tends to the area of the circle (S), and P to the length (C). [79, 64

Since the quotient of the variables A and P is constant, the quotient of their limits equals the same constant (59).

Therefore

$$\frac{S}{C}=\frac{R}{2}, \text{ and } S=\tfrac{1}{2}R\cdot C.$$

81. Cor. $S = 2\pi R \cdot \frac{1}{2} R = \pi R^2$.

82. Definition. The *area* of a sector of a circle is defined as the limit to which the area of an inscribed polygonal sector tends, when each side of the corresponding inscribed broken line tends to zero.

The existence of this limit is established as in 78, 79.

The area of a segment of a circle may be defined in a similar way. It is equal either to the difference or the sum of the areas of the corresponding sector and the corresponding triangle, according as the arc of the segment is less or greater than a semicircle.

83. Theorem 12. *The area of a circular sector equals the product of the length of its arc by half the radius.* (Prove as in 80.)

Ex. Show how to find the area of any portion of a plane bounded by either straight lines or by arcs of circles.

Measurement of Angles

84. The standard unit for measuring angles is the right angle.

As it is too large for convenient use, a certain fraction of it, called a *degree*, is employed in practice.

A degree is defined as $\frac{1}{90}$ of a right angle.

Hence a right angle equals 90 degrees, written $90°$; a straight angle equals $180°$; and a perigon equals $360°$.

Ex. 1. How many degrees are there in the sum of the angles of a triangle? In the angle of an equilateral triangle? In each of the angles of an isosceles right triangle?

Ex. 2. How many degrees are there in the angle of a regular pentagon? A regular hexagon? A regular octagon?

The sixtieth part of a degree is called a *minute*, written $1'$; and the sixtieth part of a minute is called a *second*, written $1''$.

E.g. the seventh part of a right angle equals $12°\ 51'\ 25\frac{5}{7}''$.